Teacher Noticing and Use of Language as a Resource in the Teaching and Learning of Mathematics

by

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Abstract

With an increasing resource-oriented perspective in mathematics education research focusing on language, the notion of language as a resource has been of interest to many researchers in the field. However, not many studies focused on understanding the notion from the perspectives of experienced mathematics teachers. Likewise, while teachers are expected to use mathematical language in teaching, there was little focus in understanding the specific knowledge teachers have in relation to language (particularly the mathematics register) and how they attend to language in the mathematics classroom. Hence, my intent in this research is two-fold. Firstly, I seek to understand the existing state of how teachers are noticing and using language as a resource in the mathematics classroom through the lenses of language-related dilemmas and language-related orientations. Secondly, I hope to explicate teachers' knowledge and potential usage of the mathematics register through the four dimensions of the Mathematics Register Knowledge Quartet.

A task-based interview protocol was designed for this study. Eleven experienced mathematics teachers were asked to reflect upon what they noticed and how they would respond to a series of tasks, designed to illustrate situations which might lead to language-related dilemmas and challenges in using the mathematics register. In addressing the first research focus, the lenses of language-related dilemmas and language-related orientations were used in the analysis to account for how teachers notice and use language in the mathematics classroom. Specifically, two main categories of language use emerged from the analysis, namely as a resource for developing mathematical understanding, and as a resource for mathematics talk. The findings were discussed through the cases of two teachers who primarily notice and use language, as described by the two categories respectively. In addressing the second research focus, an exemplification of teachers' knowledge (or lack of knowledge) of the mathematics register in relation to mathematics teaching and learning was presented based on an analysis of their responses to three tasks, using the Mathematics Register Knowledge Quartet. Consequently, their knowledge (or lack of knowledge) of the mathematics register provided insights into how language might be attended to in the teaching and learning of related concepts.

Keywords: language as resource; mathematics register; teachers’ noticing and use of language; language-related dilemmas; language-related orientations; knowledge of the mathematics register

Dedication

To all whom have inspired to go on and supported me during this Ph.D. journey.

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Chapter 1.

My Motivation and an Overview

> *… mathematics education begins in language, it advances and stumbles because of language, and its outcomes are often assessed in language.*
>
> *(Durkin, 1991, p. 3)*

What is the role of language in relation to the teaching and learning of mathematics? Reflecting on my own teaching experience, this is a recurring question for me as a mathematics teacher in Singapore for more than a decade. While I might not have specifically attended to the question then, I recalled two rather disparate comments that relate to the role of language in relation to the teaching and learning of mathematics. "It is a language problem" – a comment which I often heard from some colleagues when discussing about why students faced difficulties in word problems. "We (teachers) should use appropriate mathematical language when teaching mathematics" – another comment which I also heard previously while attending department meetings and professional development workshops led by other colleagues. In other words, language appeared to have two seemingly contrasting roles in the context of mathematics teaching and learning – as a *problem* and as a *resource*.

1.1. Language as a Problem or as a Resource?

Why does the role of language in relation to the teaching and learning of mathematics matter to me (as a mathematics teacher from Singapore)? Perhaps, some background about my own education and teaching background here is needed to provide the context for my question.

Being a multi-racial society, Singapore has four official languages – English, Malay, Mandarin and Tamil. Other than English, the other three languages correspond to the three major ethnic groups in Singapore. Since the nation's independence in 1965, English is deemed as the nation's *lingua franca*. It is also officially taken as the first language and used as the main medium of instruction in the Singapore education system. As Singapore has a bilingual language education policy, learning a second language (Mother-Tongue languages which correspond to ethnicity) is compulsory for most K–12 students. For example, I am a Singaporean who is ethnically registered as a Chinese. Hence, as a student, it is a requirement for me to learn English as a first language and Mandarin as a second language in school. Moreover, except for the second language classes, all other academic subjects – including mathematics – are taught and learnt in English, the main medium of instruction, in Singapore.

Since I started teaching in schools, it had always been intriguing how often I would hear comments such as “it is a language problem” or “a student is not doing well in Math as he/she is bad in English” from my fellow mathematics teacher colleagues. These comments would usually be made in response either to why a student was not able to understand and solve a certain word problem or to why he/she was unable to perform (in terms of attaining good results) in mathematics. In most cases, such students would be those with weak English language foundation due to a lack of family support to learn English at home before officially starting school. They might also be the international students who did not have the chance to learn English, prior to starting or continuing their formal education in Singapore at whatever age.

At times, I wondered if these comments were convenient excuses to explain why these students were not doing well in mathematics, given their weak language abilities. At other times, I wondered, if language is indeed a problem, then whose responsibilities would it be to resolve this problem? Would it be the mathematics teachers or the language teachers and how would they resolve this problem? Yet, upon reflecting on my own experiences with language both as a mathematics learner and a mathematics teacher, I began to question if language can be a resource, instead of purely deemed as a problem, in the context of the mathematics classroom.

1.1.1. My Experience with Language as a Mathematics Student

Recalling my own learning experiences as a student, language being deemed as a problem in the mathematics classroom was probably not baseless, I wondered if my teachers had similarly used the ‘language problem’ comment to describe me as a learner. Considering how my own foundation and knowledge in English language was only acquired formally through school, I did face some difficulties in learning mathematics due to a weaker grasp of the language as an elementary student.

One of my biggest struggles was with the use of phrases such as "more / less than", "twice as much / twice more". I was often confused with whether a or b is the number with the bigger numerical value in statements such as "a is x more/less than b". Differentiating the meanings between statements such as "y times as much as x" and "y times more x" in the context of mathematics word problems was yet another challenge for me. However, by putting in extra effort and time to practise and become familiar with the keywords and phrasing used in different mathematics statements and problems, mathematics gradually became my best subject in school. In retrospect, learning the language used in the mathematical context (in other words, the English mathematics register) seemed to have played an important part in helping me learn and do mathematics.

1.1.2. My Experience with Language as a Mathematics Teacher

Prior to becoming a teacher, I was basically schooled into thinking that mathematics is a scientific subject. Hence, I was certainly not cognizant of any possible interplay between teaching and learning of mathematics with language. The disconnect between mathematics and (English) language perhaps even widened with my teacher training experience when I was trained to teach both mathematics and (English) language[1] at the secondary level. As academic subjects, the basis of teaching mathematics and of teaching language appeared to be in great contrast and of different nature. Teaching mathematics generally focused on the need for precision, facts and formulae (like the objective sciences), whereas teaching language focused on being fluent in spoken and written expressions through the use of words and can be ambiguous (like the subjective arts). Hence, I always thought that my role as a mathematics teacher would be rather distinct from my role as an English language teacher. At the very least, I did not recall my instructors explicitly mentioning or focusing on the role of language in mathematics teaching and learning during any of my pre-service methods courses. Consequently, I used to wonder if there would even be a need for me as a mathematics teacher to deal with problems which arose due to language in the context of the mathematics classroom.

[1] English language is taught as a first language subject to all grades K–10 students in the Singapore education system, and not as a second or foreign language, as it is the nation's *lingua franca*.

Interestingly, the more I learned about mathematics teaching and learning, the more I realised how I was not only a mathematics teacher, but concurrently a mathematics–language teacher in the context of the mathematics classroom. While language is not explicitly stated as a learning goal in the Singapore mathematics curriculum, it has always been implicitly embedded as 'communication'[2] – one of the processes necessary in supporting the central problem-solving focus of the mathematics curriculum framework since 1990 (Kaur, 2014). While attending in-service professional development workshops subsequently, I also noticed an increasing emphasis and push for teachers to use 'proper' and more precise mathematical language in their classrooms. I sometimes wonder if these were consequences of the Singapore Ministry of Education's efforts to resolve the 'language problem' in the mathematics classroom. But they certainly helped to illuminate the importance of language as a resource in mathematics teaching and learning when used appropriately.

The evolution of how fractions have been read or verbalised by Singapore teachers would probably be one such example. As a student, my teachers would read the fraction $\frac{9}{8}$ as "nine *over* eight". When I first started teaching, I was advised to read $\frac{9}{8}$ as "nine *out of* eight". instead. For some reason, there was a strong push towards strengthening students' understanding of the part–whole relationship in fractions then. In recent years, I have been encouraged to use proper fraction names to read fractions, in this case $\frac{9}{8}$ is read as "nine-eighths". Notably, the first two ways of verbalising fractions are limiting in their own ways. If I say $\frac{9}{8}$ as "nine *over* eight", it tends to lead students to see the fraction $\frac{9}{8}$ as being made up of two numbers, instead of being a number itself. The line in between the numerator and the denominator is also often seen as just a line of separation, instead of being associated with the operation of division. If I say "nine *out of* eight", students' perception of $\frac{9}{8}$ as being made up of two separate numbers may remain unresolved as well. In addition, this way of reading fractions may over-emphasise the part–whole relationship and hinder students' understanding of improper fractions – where the number of parts is greater than the number of parts in the whole. In contrast,

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[2] Coincidentally or not, the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics (NCTM), 1989), published around the same time, also highlighted communication as a process standard in helping students acquire and apply mathematical content knowledge (and language) in the mathematics classroom (NCTM, n.d.).

by using the proper fraction name, such as nine-eighths[3] in this case, seems to be most helpful in reinforcing that a fraction is a number, while not placing any particular emphasis on a specific understanding of fractions.

1.1.3. My Exposure to Language in Mathematics Education Research

From my experiences as both a mathematics learner and teacher, I became particularly curious about the phenomenon of language being seemingly embedded as both a problem and a resource in the context of the mathematics teaching and learning. Yet, I also wondered if there could be a greater emphasis towards seeing language as a resource, rather than a problem, in the context of the mathematics classroom. Coincidentally, I have had many opportunities to learn about research focusing on language in mathematics education in my Ph.D. studies. Particularly, I was greatly inspired how language and thought are intricately connected (Vygotsky, 1934/1986), the notion of a mathematics register (Halliday, 1975) and the many other ideas which hint at the important role of language in mathematics education.

My research interest on the role of language in mathematics education is further piqued by how Durkin (1991) claimed that, "mathematics education begins and proceeds in language" (p. 3) which has its "ambiguities and inconsistencies" (p. 14). Interestingly, Barwell (2021) made a relatively similar claim that, "we cannot do without language, with all its slipperiness, in mathematics" (p. x) in his foreword for the recently published book *Classroom Research on Mathematics and Language* (Planas et al., 2021). Despite being thirty years apart, both claims seem to suggest that research on language in mathematics education will continue to be a central area of interest for many more years to come.

1.2. Outline of the Thesis

Motivated by a bigger interest in understanding the role of language in the mathematics education, this thesis documents my Ph.D. research journey in exploring the phenomenon of language as a resource (rather than a problem) in mathematics teaching

[3] Notably, $\frac{9}{8}$ can also be seen as *nine eighths* (without the hyphen) to mean $9\left(\frac{1}{8}\right)$ (Hewitt & Pimm, 2021).

and learning. But, as a great part of my interest stems from my experience as a teacher, I choose to focus on how this phenomenon can be addressed from the perspective of teachers who have been teaching in the mathematics classroom.

To set the context for my research, I review how research on the role of language in mathematics education has since shifted from an initially deficit-oriented perspective (problem) to a more resource-oriented perspective (resource) in Chapter 2. I start with a broad overview of the development in research that has focused on language in mathematics education – from how this area of research first gained attention in the field of mathematics education to the three theoretical perspectives of language (cognitive, discursive and sociopolitical) which have primarily framed this area of research. Next, I narrow and zoom into the notion of language as a resource. Specifically, I look into how language, framed by the notion of the mathematics register, is essential as a resource in the development of mathematical knowledge (Pimm, 1987, 1991), through the lens of research. The mathematics register is discussed in relation to its two (though certainly not exclusive) particular, yet very complementary, roles, namely as a resource for mathematical thinking (and communication); and as a resource for mathematics teaching (and learning).

As my research is an exploration of the phenomenon of language as a resource from the perspective of teachers, I choose to dwell more deeply into understanding three theoretical constructs which have been used in research to study teachers' use of language in mathematics education in Chapter 3. The three theoretical constructs are teachers' language-related dilemmas (Adler, 1996, 2002), language-related orientations (Prediger, 2019; Prediger et al., 2019) and knowledge of the mathematics register (Lane et al., 2019). Specifically, I describe what the intent, the definition(s) and application (including the possible limitations) of each theoretical construct within the research in which it has been situated. This is followed by my research questions and how these three constructs are relevant in framing my exploration towards addressing the research questions.

Chapter 4 focuses on describing my method in probing teachers' use of language in this thesis. I discuss how I come to choose interviews with teachers as the main source of data for my research. Here, I share the development of the interview protocol through a series of pilot trials and the design of the eight reflection tasks which is a critical aspect of the interview protocol. This is followed by a description of the actual data collection process – from the recruitment of the eleven participants to the conducting and transcription of the interviews. The chapter ends with an explanation of how the data is organised and analysed with two foci, in relation to the three theoretical constructs.

With respect to the first focus of the analysis, I apply the lens of language-related dilemmas (Adler, 1996, 2002) and language-related orientations (Prediger, 2019; Prediger et al., 2019) to the data, as I attempt to understand how language is noticed and used as a resource by the participants in their classrooms. Two main categories emerge from the analysis, namely language as a resource for developing mathematical understanding, and language as a resource for mathematics talk. Instead of discussing the findings in general, I present the findings through the cases of two teachers who best represent each category in Chapters 5 and 6 respectively. In Chapter 5, I illustrate the case of Karen as one teacher who primarily deems language as a resource for developing mathematical understanding, through providing an account of her use of language in her classroom. This is followed by a discussion of her experience with and in managing language-related dilemmas, which further sheds light on her orientations towards the use of language in her classroom. Likewise, in Chapter 6, I illustrate the case of Lena as one teacher who primarily deems language as a resource for mathematics talk. This is followed by a discussion of her language-related orientations, which is useful in substantiating why she does not seem to face any tension when faced with language-related dilemmas in the teaching and learning of mathematics.

With respect to the second focus of the analysis, I attempt to explicate teachers' knowledge of the register through the lens of the Mathematics Register Knowledge Quartet (Lane et al., 2019). In Chapter 7, I discuss how teachers' knowledge of the mathematics register may look like or be lacking in the four dimensions of the Knowledge Quartet. Through an analysis of the participants' responses to three specific tasks, it is evident how their level of knowledge of content-specific mathematics register may differ by teachers and has implications on what they notice and how they use language as a resource in their teaching.

Finally, in Chapter 8, I discuss the findings from the two foci in response to my two research questions respectively. This is followed by a discussion of the contribution and implications to mathematics education research that my study has brought forth. I close the thesis with a reflection of my research journey and possible next steps to further this research.

Chapter 2.

A Shift towards Understanding Language as a Resource in Mathematics Education Research

In this chapter, I provide a selective literature review of existing research relating to the role of language in mathematics education to situate my research interest and focus. In section 2.1, I look at how language has since become an important sub-field in mathematics education research. In section 2.2, I describe the three theoretical perspectives or lenses to which the topic of language has been typically researched in mathematics education. In section 2.3, I highlight how the notion of language as a resource has become increasingly studied in research. In particular, I focus on my discussion on how the mathematics register is essential both as a resource in the development of mathematical thinking (and communication), and as a resource for mathematics teaching (and learning).

2.1. An Overview of the Development of Language as a Research Focus in Mathematics Education

While language as a research focus is certainly not new in the field of human sciences (e.g., linguistics, philosophy, sociology, psychology), language as a research focus in mathematics education is a relatively young sub-field (Sfard, 2021). However it is undoubtedly a growing one in the recent decades, as evidenced by the increasing number of special issues in mathematics education journals (e.g., FLM, *18*(1), 1998; ZDM, *46*(6), 2014; JMB, *40*(PA), 2015; ZDM, *50*(6), 2018; RME, *21*(2), 2019), and edited volumes (e.g., Durkin & Shire, 1991; Moschkovich, 2010; Moschkovich et al., 2018; Planas et al., 2021) dedicated to this area of research.

2.1.1. The Two Significant Works

Notably, it is suggested that this area of research started to gain more attention in the field of mathematics education with the 1979 paper “Language and Mathematical Education” by Austin and Howson, one of the first with a clearer focus in this area (Morgan et al., 2014; Planas & Schütte, 2018). Austin and Howson’s attempt to consolidate research related to the “interaction of language and mathematical education” through an annotated bibliography at that time has presumably “placed language in the complexity on the research and development agenda in mathematics education” (Adler, 2002, p. 9). According to their article, the question is no longer whether language has a role in the teaching and learning of mathematics. Rather, research should start focusing on understanding how language interacts with mathematics education (Austin & Howson, 1979), something which many researchers have since endeavoured to carry out in relation to these “two inseparable worlds: language (in use) and mathematics (education)” (Barwell, 2021, p. xi) even if they own varying perspectives of these two worlds and the respective interactions or relationships.

Following the 1979 article, several researchers (e.g., Adler, 2002; Morgan et al., 2014; Sfard, 2021; Schütte, 2018) further proposed that the book *Speaking mathematically: Communication in mathematics classrooms* by David Pimm (1987) is one key trigger (which reinforced the need) for the prominent language-turn in mathematics education research. In his book, Pimm discussed the “often-heard claim that ‘mathematics is a language’” (p. 2) and explored the significance and implications of such a claim in the context of mathematics teaching and learning. By providing an in-depth analysis of how language both in its spoken and its written form had been used in mathematics classrooms and how it should be used for the purpose of communicating mathematically, the book is deemed to have laid the foundation for subsequent research in this sub-field and its expansion. Pimm further highlighted how the learning of mathematics is analogous to the learning of a foreign language as being able to speak and write either in mathematics or a foreign language are both non-instinctive and thus need to be taught and learnt. Yet, he reminded the mathematics education field that mathematics is not a natural language though it has to be communicated linguistically through or in a natural language, such as English and Mandarin. As such, he brought attention towards the notion of the *mathematics register*, first coined by Halliday (1975), to represent more accurately what is commonly known or metaphorically referred to as the ‘mathematical language’.

2.1.2. The Two Significant Events and an Enabler

Certainly, the presence of these two works, *Language and mathematical education* (Austin & Howson, 1979) and *Speaking mathematically* (Pimm, 1987), are not the sole reasons why attention started to be given to language in the context of mathematics education research. According to Sfard (2021), there are at least two other “disruptive occurrences” (p. 42) – a move away from monolingual classrooms as the norm; and a shift away from the belief that mathematics practices are universal – in the educational landscape which have contributed to this growing focus. Firstly, with a move away from monolingual classrooms as the norm due to the effects of globalisation, classrooms everywhere are becoming increasingly multilingual. While English or other national languages may have been made the official *lingua franca* for the teaching and learning mathematics in many classrooms, it may not necessarily be the native language of the communities within the classrooms. Having a myriad of languages among students (and even teachers) has thus raised the visibility of language as a contributing factor to the processes (and usually the success) of mathematics learning in such (mathematics) classrooms, resulting in a need or an interest to understand what the multilingual (bilingual included) context means for mathematics education.

An immediate consequence (Chronaki & Planas, 2018; Morgan et al., 2021; Schütte, 2018) of this increased visibility of language in mathematics learning processes seemed to be a strong focus in deficit-oriented research[4] which focused on addressing or resolving the (language) deficits or cultural differences of children in relation to the respective *lingua franca* for mathematics learning (e.g., Cummins, 1979, as cited in Schütte, 2018). Fortunately, research on language in mathematics education has since moved towards a more resource-oriented perspective (e.g., Adler, 2000, 2002; Moschkovich, 2002, 2015; Planas & Setati-Phakeng, 2014). Specifically, the multiple languages and diverse culture of learners are now increasingly deemed as a sociocultural resource which supports the social interactions needed for mathematics learning within mathematics classrooms (cf. Barwell’s, 2018, argument for language as *sources of meaning* rather than as a *resource*). Among the various developments in language becoming a research focus in mathematics education, this is probably one particularly interesting and important reminder for me as a researcher (and my future research). I realised how I similarly took on a deficit-oriented mode of looking at my interest initially as I was thinking of solving the ‘language problem’ in my mathematics classrooms as a teacher. But, as a researcher, I needed to be aware of this possible trap

[4] This seemed aligned with how a more deficit model of thinking tends to permeate education when explaining success and failure (see Jorgensen, 2018, for an example).

and instead adopt a more resource-oriented understanding towards the ‘language problem’ by exploring what and how language can be (better) perceived and used as a resource for mathematics teaching and learning.

Secondly, a gradual (though not complete to date, cf. Schütte, 2018) shift away from the belief that mathematics practices are universal (or that there is one universal mathematics language) has started to emerge due to the increased popularity of cross-cultural studies since the early 20th century (Sfard, 2021). In particular, educators and researchers became more cognizant that the presumably universal mathematics language (and practices) can be (very) different when situated within different cultures and native languages. This additionally led to the emergence of a special area of research, namely ethnomathematics (D’Ambrosio, 2004, as cited in Sfard, 2021), which has since provided much evidence that “mathematics is sensitive to cultural idiosyncrasies, including those related to language” (Schütte, 2018, p. 27; cf. the Sapir–Whorf hypothesis as cited in Sfard, 2021). On a related note, Walkerdine (1988) argued how mathematical meaning, which is often neither universal nor neutral, is produced through young children’s lived experiences of everyday practices, and heavily influenced by the dynamics of power relationships, such as that of a mother and a child.

This gradual shift also conveniently fed into the “social-turn” in the mid–1980s, which acknowledges social activity (interactions) as the source or stimulus to the individual’s mathematical meaning, thinking and reasoning processes (Lerman, 2000, as cited in Schütte, 2018; Walkerdine, 1988) and subsequently the “sociopolitical-turn” in the 2000s with similar (though not necessarily congruent) views of sociocultural and sociopolitical discourses as sources of knowledge, power and identity (Stinson & Walshaw, 2017). With the greater role of the socially and culturally interactions and discourses coming into the picture of mathematics learning, it is unsurprising that language (and all forms of communication[5]) has become one key object of study that mathematics education researchers attend to (Morgan et al., 2014; Schütte, 2018).

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[5] In their recent review article “Mathematics education research on language and on communication including some distinctions”, Planas and Pimm (2023) reiterated how the distinction between language and communication has generally been unclear in mathematics education research. They argued for a need for the field to be cognizant of how language and communication are not necessarily the same constructs and should be treated as possibly different constructs in research.

Admittedly, while there is a need and growing interest in research on language in mathematics education, both Barwell (2021) and Sfard (2021) unanimously suggest that it may not have been easily accomplished without the necessary and timely technological advancements in terms of recording devices (from the initial huge and bulky tape recorders with only audio-recording functions to the now handy handphones with audio and video-recording functions). These increasingly sophisticated recording devices come with affordances in capturing and replaying “live” social interactions and discourses (happening in the classrooms or any social learning settings). In other words, the documentation of “live” and complete language data for analysis and discussion is now made possible through detailed transcriptions of the recorded events. Researchers no longer need to rely on their own memories or field notes to provide analysis and interpretation of a past event. The timeless recordings also afford the possibilities of close analyses and multiple interpretations of the same event when used by different researchers to feed different research agendas (Sfard, 2021). With such easy access to rich language data, it is no wonder why language data can be heavily tapped on as a source to illuminate and support many studies, either quantitatively and/or qualitatively, within the field of mathematics education research for at least half a century now (Pimm, 2018, 2021).

2.2. The Varying Roles of Language in Mathematics Education

Though language in mathematics education is a relatively young sub-field, much work has been done in this area of research for at least the last half of a century, ranging from language’s nature and role in teaching and learning mathematics to its sociopolitical dimensions in mathematics education (Barwell, 2021). Simultaneously, this mass of research has resulted in varying conceptualisations and interpretations of language and its interactions with mathematics education even in studies when the same terms or approaches are used (Planas & Schütte, 2018). For instance, though the word *discourse* (commonly used as a synonym to *language*) has grown to be one of the most frequently used terms in recent years, Ryves (2011) has found the concept to be unclear in its use in mathematics education (cf. Planas & Schütte, 2018). While there may seem to be a lack of agreement in terms of language as a construct in mathematics education, and more clarity is likely required in moving future research, one can also argue that this is probably reflective of the “complexity, diversity and contention” (Planas & Schütte, 2018, p. 973) that language has brought to mathematics education research. Undeniably, such “complexity, diversity and contention” have helped open up many possibilities in terms of research within this sub-field and will certainly continue to do so in the years to come.

Despite the presence of much “complexity, diversity and contention”, several survey papers (Barwell et al., 2017; Barwell et al., 2019; Morgan et al., 2014; Planas & Schütte, 2018) suggest that this sub-field does converge towards (one of) three distinct but interrelated lenses – namely sociopolitical, discursive/interactionist and cognitive – in its study of the intricate interactions between mathematics and language situated within social and/or multilingual contexts and how language functions as a resource in these contexts. Correspondingly, depending on the lens taken, it is noticed that different theories (out of mathematics education), such as critical theories, discursive theories and linguistics theories, have been adapted to understand better how language interacts with mathematics education (Barwell, 2018, 2021).

Specifically, research with the sociopolitical lens typically focuses on the wider sociopolitical role of language where it becomes important to understand language in terms of its inherent relationships with power and privilege in relation to mathematics teaching and learning, or even the possibly political mathematics curriculum (Barwell et al., 2019). Through this lens, researchers (e.g., Planas & Setati-Phakeng, 2014) consider language as a resource to be inseparable from the social, political and cultural contexts in which it is situated, where the use and meaning of words and structure of language are not fixed or neutral. This leads to issues relating to identities and positioning due to social and political inequalities such as race, gender or even class (Walkerdine, 1988). In other words, the macro level rather than the micro level of the role of language in mathematics education is of utmost concern for research lying within this subgroup. Generally, since every classroom is situated within a unique sociocultural context and influenced by the political agenda of the respective education landscape, it is noted how elements of the sociopolitical lens can be nuanced implicitly in all research on language in mathematics education, regardless of the primary research lens (Planas & Schütte, 2018).

On the contrary, research with the discursive/interactionist lens generally focuses on the mathematics discourses and classroom interactions which reside at the micro level of language interactions with mathematics education. The interest of researchers (e.g., Moschkovich, 2002) here lies in how discursive practices and interactions play a role in the meaning-making process of learning mathematics. In particular, the main attention is given to the subjective use of language by different individuals (due to sociocultural diversity) during mathematical discourses and interactions and how personal meanings are integrated to form the resultant co-constructed and shared mathematical understanding (Planas & Schütte, 2018). Moreover, the nature of the participation of the individuals in the discourses and interactions and how the different meanings arising from the participation result in greater access to mathematical understanding are both of concern. In particular, Barwell (2018, 2021) proposed that such emphasis on the development of mathematical understanding itself as a discursive (language) activity situated within classroom interactions seems to refer to language as a resource in a broader and metaphorical manner, where the focus resides in how learners participate through language activities to get access to mathematics.

As for research with the cognitive lens, the focus is on language (including the mathematics register) as a tool or resource (Adler, 2000) which mediates the development of mathematical thinking and understanding. Instead of putting an emphasis on the shared mathematical meaning that emerges from social discourses and interactions like research with the discursive/interactionist lens, the cognitive lens primarily examines how mathematical meaning and understanding is produced or constructed in and with language (Planas & Schütte, 2018). Through this lens, language is often viewed as having a certain material presence which allows it to be metaphorically referred to as a tool to think about and develop mathematical understanding; or a resource for teaching and learning as it possesses potential to make mathematical objects, ideas and concepts more accessible and understandable (Barwell, 2018, 2021).

Notably, a bulk of the research on language in mathematics education has origins or intentional focus in multilingual classrooms as a key context of study, regardless of the different lens they may use in the study. However, as a final note for this section, researchers in this sub-field should be increasingly cognizant of the inherent nature of language diversity in all social learning contexts. Simply put, it is not hard to realise that mathematics classrooms are becoming multilingual almost everywhere – a claim similarly made by Morgan (2007, as cited in Morgan et al., 2021). Hence, it is probably a necessary move to focus future research beyond the multilingual contexts but rather onto the language(s) itself as the object of study (Barwell, 2021).

2.3. Language as a Resource with a Focus on Mathematics Register (*vis-à-vis* Mathematics Language)

While recognising the different lens that can be used to study language in mathematics education, the cognitive lens resonates with what and how I perceive the role of language to be in relation to mathematics teaching and learning. In particularly, I am interested in how language, framed by the notion of the *mathematics register*, can mediate the development of mathematical meaning. Here, I elaborate on how and why the *mathematics register* is often commonly referred to as the mathematical language, as well as how it has been considered as a tool for thinking (and communication), and thus can be viewed as a resource for teaching (and learning) within a more cognitive lens of research.

More often than not, I have noticed how both teachers and students are often encouraged to use the appropriate 'mathematical language' when expressing their thoughts as they explain or make sense of mathematical ideas or concepts. This is an intriguing thought, as it seems to be implying that mathematics is a language, analogous to natural languages such as English, Mandarin, etc. However, this is a claim that many mathematics education researchers (e.g., Wheeler, 1983; Pimm, 1987) will probably disagree with. Yet, although mathematics is not a language in itself, it needs to be communicated in or through a natural language (for instance, English, in many curriculums), in order for mathematical ideas to be expressed. This is probably why mathematics is often referred to as a language, except we tend to forget that it is actually used at most in a metaphorical sense.

Markedly, Pimm (1987) proposed Halliday's (1975) notion of a *mathematics register* as a more apt representation of the so-called "mathematical language". Notably, as in many ambiguous constructs within the research literature on language in mathematics education, the *mathematics register* is often synonymously referred to as the *formal or school mathematical language* (e.g. Austin & Howson, 1979; Adler, 2002;

Moschkovich, 2021); and the *specialised language or vocabulary of mathematics* (e.g., Austin & Howson, 1979; Morgan, 2021; Andrews et al., 2021).

Specifically, a mathematics register involves a unique use (in both spoken and written forms) of words and structures in a natural language, such as English, to express "the set of meanings that is appropriate to" the mathematics discipline, or in other words, "the mathematical use of natural language" (Halliday, 1975, p. 65). And it is important to note that even though the mathematics register may "allow us to 'linguify' mathematics symbols" (Pimm, 2021, p. 27), especially for the sake of spoken communication, it does not include any written mathematical notations or symbols (not to mention images such as geometrical diagrams or graphs) as those certainly do not belong in any natural language in the first place. As such, Pimm (2021) argued against the notion of a multimodal register (to include notations and images other than words) proposed by O'Halloran (2015), as it defies the fact that a register has to be based in a certain natural language. In other words, "a register is language-specific", and influenced by culture (Pimm, 2021, p. 30). Written notations and images, on the contrary, are usually more "universally" understood and not specific to any natural language nor culture. If there is really a need to refine the notion of a mathematics register, Pimm would instead consider including specific multimodal aspects such as the idea of "(conceptual) gestures" as part of the mathematics register (cf. the notion of mathematics *communication* register proposed by Planas & Pimm, 2023). Although gestures are similarly not language-specific, at least, "many gestures are cultural rather than global" (p. 30) and thus more closely related to a certain natural language in his opinion.

As mentioned, mathematics is not a natural language in itself. For the same reason, the mathematics register does not pre-exist within any natural language, it needs to be developed before mathematical meanings can be expressed through a natural language. In the case of the mathematics register in English – its development started in the sixteenth century with the publication of the first mathematical textbooks in English by Robert Recorde (Pimm, 1991) and took many centuries to become the current "relatively rich" mathematics register (Durkin, 1991, p. 13). But once developed, it has allowed the articulation of mathematics in both spoken and written forms with respect to the natural language (English, in this case) it is developed in or from (Pimm, 2021). As such, the development of a mathematics register in any natural language[6] is certainly not an easy process (see e.g., Barton et al., 1998; Roberts, 1998, for the need, attempts and issues faced in the development of a mathematics register in Indigenous languages). It needs to be in tandem with the ways of thinking in mathematics, influenced by the cultural historical developments of mathematics concepts or sometimes changes to the natural language as well (cf. Vygotsky's, 1934/1986, notion of language as a cultural tool).

To illustrate further the complexity involved in the process of developing a register which fully or effectively communicates all or even most mathematical meanings in a natural language, the following are some methods which have been discussed in detail by Pimm (1987) in Chapter 4 of *Speaking Mathematically*:

- Creation or coining of new specialised (unlikely to be used outside of mathematics) terms with mathematical meanings, such as *parallelogram* and *hypotenuse*.
- Reinterpretation of existing words (can be nouns and verbs) from the everyday language in English, usually in a metaphorical or analogous sense, for instance, from the *face* of a human being to the *face* of a geometric solid; from *real* or *imaginary* objects to *real* or *imaginary* numbers, etc. Within this method, changes in the grammatical categories of words may sometimes be observed. For example, the word *diagonal* is reinterpreted from being a property of an object as an adjective in everyday language to being the object itself as a noun used in mathematics.
- Formation of composite words and phrases to convey certain mathematical meanings, such as *square-root* and *simultaneous equations*.

Moreover, metaphors and/or analogies play a significant role in the above methods especially when words or phrases are reinterpreted or created for the register, such as in the words *face* and *slope*. In particular, two types of metaphorical language are noted to be common in the mathematics register – extra-mathematical metaphors and structural metaphors. Extra-mathematical metaphors refer to those which use real-world objects or activities to express a mathematical idea. An example is how the *slope* of the line is used to refer to gradient. Structural metaphors refer to those which make

[6] Pimm (2021) reminded how English mathematics register is not the only one in mathematics education with "seldom any acknowledgement" given to the presence of many other mathematics registers residing within other natural languages (p. 30). Similarly, I have always presumed it to be the most widely used.

use of metaphors which already reside within the mathematics register. An example is the extension of the gradient of a point on a curve from the gradient of a line as a point technically has no gradient (Pimm, 1987).

2.3.1. A Tool for Mathematical Thinking (and Communication): A More Theoretical Perspective

It should be noted that the term 'register' is certainly not unique to mathematics, as it can be used to represent the specific set of meanings found only in any academic discipline (e.g., Science, Art, etc) through or in a natural language (Halliday, 1975). Yet it is an important notion in every academic discipline as "ways of thinking and communicating" of ideas in each discipline will require the use of the particular register of that discipline according to Halliday (as cited in Wilkinson, 2015, p. 2). Similarly, in the case of the mathematics register, it is developed to serve the function of thinking about (and communicating in spoken or written forms) mathematical ideas and meanings (Pimm, 1987, 1991; Wilkinson, 2015), that is, the mathematics register can be considered as a tool for thinking (and communicating)[7] mathematics.

However, being a complex notion, the register is often misunderstood as simply a collection or addition of highly technical vocabulary terms in the language to describe objects in the discipline (Halliday, 1975). In the same manner, the mathematics register is certainly not just a collection of mathematics-related words or terms, as what many teachers and students may think it is (with a surface understanding of the "mathematical language"). Instead, it is important to understand that the mathematics register further determines how these words or terms are used or structured within the natural language to form unique phrases or clauses that can precisely represent both explicit and implicit mathematical meanings or relationships (Schleppegrell, 2007; Wilkinson, 2015).

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[7] My interpretation of "a tool for thinking and communicating" aligns more closely with Vygotsky's (1934/1986) notion of language as a tool for the development of thought. According to Vygotsky, communication exists as the "external speech", a "direct expression" of one's thoughts either in the spoken or written form (p. 256) and which continues to interact with the development of one's thoughts as meanings are being mediated. Hence communication here varies in terms of its nuance to how it is usually interpreted through the discursive/interactionist lens, mentioned in the previous section (cf. Sfard's, 2008, attempt to marry both the cognitive and discursive aspects of language with her commognitive framework).

In fact, research has shown that the mathematics register plays an important part in developing and acquiring mathematical ideas or concepts (e.g., Sigley & Wilkinson, 2015; Uptegrove, 2015). For instance, the interdependency between the development of mathematical understanding and the use of the register to express this understanding was seen in a case study of a middle-school student (Sigley & Wilkinson, 2015). Uptegrove (2015) similarly observed how students tend to tap on or need to acquire different representations (including the register) to communicate their ideas as they make sense of mathematical ideas or develop mathematical understanding in a longitudinal study. Specifically, as mathematical ideas or concepts become more abstract and generalised, the need for knowledge and fluency in the mathematics register also increases. As students further refine their ways of thinking and communicating through a "richer" register, they concurrently develop a deeper mathematical understanding.

Although the interdependency mentioned may seem like a "chicken and egg" problem, it fundamentally supports the argument that mathematical concepts and the mathematics register should be acquired and developed together or "in concert" (Wilkinson, 2015, p. 5). This resonates with Vygotsky's (1934/1986) works which have elaborated at length the intricate connections between thought (mathematical thinking) and (mathematics) language. In particular, while the mathematics register is a tool which is essential in the mediation of the systemic or scientific concepts required for the development of mathematical knowledge, it also resides within the realm of systemic or scientific concepts as it represents certain mathematical "word meanings", in Vygotsky's terms. As such, students will only develop true mathematical concepts when they are able to acquire and internalise the register as a tool for their own mathematical thinking.

To illustrate further the importance of the mathematics register as a tool for mathematical thinking, students have been observed to develop only surface understanding or even misunderstandings of mathematics concepts when they lack awareness and knowledge of the register. The most common form of mathematical misunderstanding arises from the confusion or ambiguities between the mathematics register with the everyday language, particularly in cases where words in the mathematics register are reinterpreted or "borrowed" from everyday language (Pimm, 1987). For example, there is a *right* angle (= 90°) in the mathematics register, but students (including mine) may query if there are *wrong* or *left* angles based on their knowledge of how the antonym(s) of *right* can be *wrong* or *left* in the everyday language (cf. Long's, 2011, example of straight angle). Even simple words such as *some*, *any* and *all* may be confusing for students as these words may be used to mean almost opposing ideas (e.g., generic vs general) in the mathematics register and everyday language (Mason & Pimm, 1984; Pimm, 1987). For instance, *any* often means *every* mathematical object that belongs to a group in the mathematics register while *any* generally means only one or some of a specific group of objects in everyday language.

While the mathematics register is an important tool for the development of mathematical thinking, it is, however, not something students are able to acquire from everyday experiences spontaneously, according to Vygotsky's (1934/1986) distinction between the development of scientific and spontaneous concepts. Hence it is essential for teachers to provide the necessary mediation by using it appropriately as a resource for teaching (and learning), (e.g., through multiple activities which "encourage students' communication and shared use of the mathematics register" as suggested by Wilkinson, 2015, p. 5), so that students can acquire and develop ways of mathematical thinking by internalising the mathematics register as a tool for thinking (and communication)[8].

2.3.2. A Resource for Mathematics Teaching (and Learning): A More Practical Perspective

Based on how the mathematics register is an important tool for mathematical thinking, it is certainly a rich resource[9] for mathematics teaching (and learning) with appropriate mediation. However, this is only possible if teachers have actually developed an understanding of "the forms and the meanings and ways of seeing enshrined in the mathematics register" (Pimm, 1987, p. 207) or have internalised the mathematics register as a tool for thinking for themselves (Wilkinson, 2018). Unfortunately, this is not necessarily true for most, if not all, teachers.

[8] Notably, if students are able to internalise the mathematics register as a tool for thinking, they will likely have developed the necessary *mathematical communicative competence* for communication, that is, the ability to communicate ideas or make meaning in and with the mathematics register in the context of mathematics discussions (Pimm, 1987).

[9] I chose to refer to the mathematics register as a *resource* rather than *tool* in the context of mathematics teaching and learning, as the word *resource* is more commonly used in existing literature (e.g., Adler, 2000; Planas, 2018) focusing on the role of language in mathematics education.

For instance, in a study (Lane et al., 2019) on pre-service teachers' mathematics register proficiency, it was found that these teachers generally do not have sufficient knowledge and understanding of the mathematics register. They were not familiar with the mathematical vocabulary or were using them incorrectly, such as forgetting the proper term of improper fractions (and used "top-heavy fractions"); mixing up the terms *expression* and *equation*; and not understanding the meaning of the phrase "uniqueness of multiplicative inverse" (p. 798). Moreover, these teachers demonstrated a lack of awareness of the significance and role of the mathematics register in the teaching and learning of mathematics. They were also not comfortable or fluent with the use of the mathematics register and shared their preference for the everyday language in their practices, while only using the mathematics register to introduce the lessons. Similar findings were observed in the case study of six early career teachers' understandings of the register (Turner et al., 2019), which resulted in variations in terms of their practices in using the mathematics register as a resource for teaching. While both studies presented what could be seen as deficit-oriented narratives of pre-service teachers' mathematics register proficiency, they also highlighted the importance for the mathematics register to be used as a resource in mathematics *teacher* education.

On the same note, without a clear understanding of what the mathematics register entails, teachers may also face tensions when attempting to use the register as a resource for the teaching and learning of mathematics. Specifically, Adler (2002) identified three possible teaching dilemmas which are related to the tensions teachers may face when using the register as a resource for teaching. They are namely the *dilemma of code-switching*, the *dilemma of mediation* and the *dilemma of transparency* (further elaborated upon in sub-section 3.1.2). While these dilemmas were surfaced from Adler's (2002) work in multilingual classrooms, she acknowledged that these are probably the same dilemmas faced by teachers in all mathematics classrooms (e.g., Herbel-Eisenmann et al., 2015; Turner et al., 2019) in their attempt to use the register as a resource for teaching and learning. However, the context of multilingual classrooms is likely to further complicate or intensify these dilemmas, especially in cases where the register does not reside in the first or native languages of the learners and/or teachers. Notably, the use of the mathematics register in English often presupposes the user's ability in the English language. As a result, in countries where English may be the second or even third language, learners may find it even more challenging to fully understand and acquire the English mathematics register (cf. Barwell et al., 2017). Correspondingly, teachers also find it difficult to teach mathematics by only using the English mathematics register and may more likely be faced with the dilemma of code-switching between English and a more familiar native language (usually with a less developed mathematics register) in their teaching, (e.g., Moschkovich, 2002; Setati, 1998; Sikhondze & Goosen, 2010).

As such, there is a need to help teachers develop their own mathematics register proficiency before they can better use the register or more broadly language as a resource for teaching and learning and manage language-related dilemmas such as those identified by Adler (2002). And this is possible if targeted support has been provided for teachers in this aspect, as shown in the one-year study that Herbel-Eisenmann et al. (2015) carried out with a group of secondary mathematics teachers. As the teachers were interested in improving the mathematics discourse in the classroom, they had many opportunities to discuss the mathematics register which led to shifts and expansion in their own knowledge and understanding of the mathematics register. In particular, their discussions shifted from a focus on only mathematical vocabulary (a small aspect of the register) initially to also paying attention to the other aspects of the mathematics register such as the reinterpretations of existing words from the everyday language and the use of phrases to convey and compact certain mathematical meanings. Likewise, Zazkis (2000) saw positive results with the use of code-switching as a teaching tool with pre-service teachers to address the dilemmas of mediation and transparency with regard to the use of everyday language and the mathematics register in her own classroom. Not only were the gap between everyday language and mathematics register bridged, the pre-service teachers also became more cognizant of and appreciative of the role of the mathematics register in mathematics teaching and learning.

Yet, research on professional development programs for teachers, specific to the notion of developing teachers' mathematics register proficiency, seems to be generally limited. However, more researchers have started to look into the development of mathematics teachers' pedagogical content knowledge and expertise in relation to the use of language in mathematics classrooms. In particular, there has been increasing emphasis on promoting teacher expertise in response to the notion of *language-*

responsive mathematics teaching within the literature (e.g., Prediger, 2019; Adler, 2021).

For example, Prediger (2019) investigated how teachers' expertise in "integrating mathematics and language learning" (p. 368) can be improved through a theoretical framework, in which she proposed specific learning needs for language-responsive mathematics teaching. The needs include the abilities to identify, support and develop students' language demands (including an understanding of mathematical word use) relevant to specific mathematics learning; and a shift in teachers' orientations (further expanded in Prediger et al., 2019) such as the accountability for students' language learning in mathematics; and the viewing of language learning as integrative, rather than an add-on, in mathematics teaching and learning. Similar efforts in the development of teachers' competence using Adler's (2021) mathematics teaching framework which is designed to help teachers plan and reflect on their lessons. Motivated by the dilemmas she identified (Adler, 2002), intentional focus has been given to the developing of teachers' competence in the meditation of word use in mathematical explanations within the framework as it is deemed as key to "producing a coherent lesson, and leading towards the mediation of scientific concepts" (Adler, 2021, p. 83).

While the focus of language-responsive mathematics teaching is not explicitly related to the mathematics register, there are elements focusing on how teachers should develop students' understanding of mathematical terms and word use. This implicitly points towards the register as an important resource in mathematics teaching and learning, in relation to its role as a tool for mathematical thinking. Yet the register will remain a theoretical notion if teachers do not have the necessary awareness and knowledge of what it entails, including the possible problems that will arise and how to meaningfully use it as a resource for teaching. It is thus important for researchers and mathematics educators to be cognizant of the existing state of teachers' knowledge, orientations and dilemmas in relation to language (in particular, the mathematics register) in mathematics education before the mathematics register can be positioned meaningfully as a resource for mathematics teaching and learning.

Chapter 3.

Three Theoretical Constructs to Understand Language as a Resource in Mathematics Classrooms

Based on the literature review, it is noted that language, in particular, the mathematics register (Halliday, 1975; Pimm, 1987), is increasingly discussed in terms of its role as an important resource in the mathematics classroom. This is largely attributed to how it functions as a tool for thinking, from the cognitive perspective (Vygotsky, 1934/1986) and, thus, a useful resource for mathematics teaching (and learning). However, before language can be meaningfully noticed and used as a resource, teachers need to have the necessary knowledge, orientations, coupled with an awareness of the possible language-related dilemmas in the mathematics classroom.

Particularly, within my research, I have chosen to consider three theoretical constructs which have been used to study teachers' use of language in mathematics education. The three theoretical constructs are, namely, teachers' language-related dilemmas (Adler, 1996, 2002), language-related orientations (Prediger, 2019; Prediger et al., 2019) and knowledge of the mathematics register (Lane et al., 2019). Though the three constructs may focus on a different aspect in understanding how teachers attend to language in their classrooms, they are similarly motivated by the assumption that language is an important resource in relation to mathematics education.

In sections 3.1 to 3.3, I describe the intent, the specific construct and the application (including the possible limitations) of each theoretical construct within the research in which it has been situated respectively. In section 3.4, I share my research questions and how these three constructs are connected in a complementary manner to frame the analysis of the data.

3.1. Interpreting Teachers' Language-Related Dilemmas

The notion of language-related teaching issues or dilemmas was one key contribution by Jill Adler (1996, 2002) as part of her dissertation research work (Adler, 1996), which was later published as a book (Adler, 2002). Through analysing six teachers' articulated and tacit knowledge of their practices in the contexts of multilingual classrooms, Adler (1996,

2002) identified and elaborated on three key language-related dilemmas – the dilemma of code-switching, the dilemma of mediation and the dilemma of transparency (further elaborated upon in sub-section 3.1.2) – which can explain the tensions that teachers may face when using language as a resource for mathematics teaching.

3.1.1. Intent: What Research Phenomenon Does It Address?

Adler's (1996) work was situated at a time of mass political and educational changes in South Africa in the mid-1990s. Classrooms were becoming increasingly multilingual where the language of instruction (English) was generally not the language of students (and even teachers), thus leading to the challenge of having to learn both English and mathematics in most situations. Concurrently, curriculum initiatives in mathematics education had led to the need for learner-centred classrooms where communication was being introduced and promoted as a key process skill in the learning of mathematics. Yet communication as a process for learning in mathematics classroom was often unclear in terms of its role and impact. This was further complicated by the assumption that students had the necessary communicative competence (in terms of language ability and knowledge about what to and how to engage in mathematical discourse) required in learning to talk mathematics and learning mathematics from talk. Such a setting in the educational landscape of South Africa has resulted in the need to understand the three-dimensional dynamics of teaching and learning in multilingual mathematics classrooms, specifically in relation to the access to the language of instruction (English), mathematical discourse (as educated discourse in learning to talk mathematics) and classroom discourse (as educational discourse for learning mathematics from talk).

Adler (1996, 2002) was also motivated by her work with teachers in multilingual classrooms where she often observed how issues and challenges due to the interactions between language and mathematics teaching and learning were often raised but remained unresolved. Thus, her study (Adler, 1996) started with the aim of examining secondary mathematics teachers' theoretical and practical knowledge as a way to investigate the increasingly complex three-dimensional dynamics of teaching and learning mathematics set in multilingual classrooms. In particular, it focused on addressing the research question, "What is secondary teachers' knowledge (both conscious and tacit) of their practices in multilingual classrooms?" (p. 4)

The research question was further unpacked into sub-questions where *knowledge* was operationalised in terms of what teachers say and do in their practice in relation to the three-dimensional dynamics of teaching mathematics in multilingual classrooms. The sub-questions also considered *practice* in three specific situations[10] with obvious interplay between language of instruction, language of mathematics (educated discourse) and language of the classroom (educational discourse). The situations were namely code-switching between language of instruction and language of students; mediating between formal and informal language use in the development of scientific concepts; and using language as an invisible or visible resource in teaching and learning mathematics.

3.1.2. Construct: What Are the Details and Possible Scope of Analysis?

In order to examine teachers' knowledge of their practices which is argued to be situated and mediated in the social and cultural context of teaching and learning, the study (Adler, 1996) generated and was guided by a social theory of mind. As such, the study was broadly framed by Vygotsky's sociocultural and sociohistorical theory (1978, 1986, as cited in Adler, 2002) and Lave and Wenger's social practice theory (1991, as cited in Adler, 2002) in interpreting issues relating to the teachers' practices.

But there were two specific theoretical frames that were noted to have been developed and used in the study – one that was used to map teachers' articulated knowledge of their practice; and one that was used to discuss and elaborate teaching dilemmas teachers faced in relation to the interactions between language and mathematics teaching and learning. Adler proposed that the use of both frames, together with the broad perspective of the social theory of mind, provided the necessary language for the description, analysis and explanations of teachers' knowledge in the multilingual mathematics classroom. Notably, the second frame – the notion of teaching dilemmas – emerged at her initial data analysis phase. Hence, it was not originally intended but became a useful and the central notion in elucidating teachers' knowledge in relation to the complexity of the teaching practice in the study.

[10] This was interesting as the notion of the three language-related dilemmas were the findings from the study, yet they also seemed to frame the three specific situations.

A Map of Teachers' Articulated Knowledge

To analyse and make sense of individual teachers' articulated knowledge (i.e., data of what they said), Adler created a mapping framework with six categories of description, based on how she perceived curriculum as being relational in terms of the possible interactions which may occur between learner, teacher and knowledge (of mathematics and language in this case) in context. Hence, within the map (see Figure 3.1), there are five interrelated broad curriculum categories - mathematical knowledge, teacher, learner, context, pedagogy (adapted from Christiansen and Walter's, 1986, relational conception of curriculum, as cited in Adler, 1996) and; a category on language (based on Vygotsky's, 1986, notion of language as mediator).

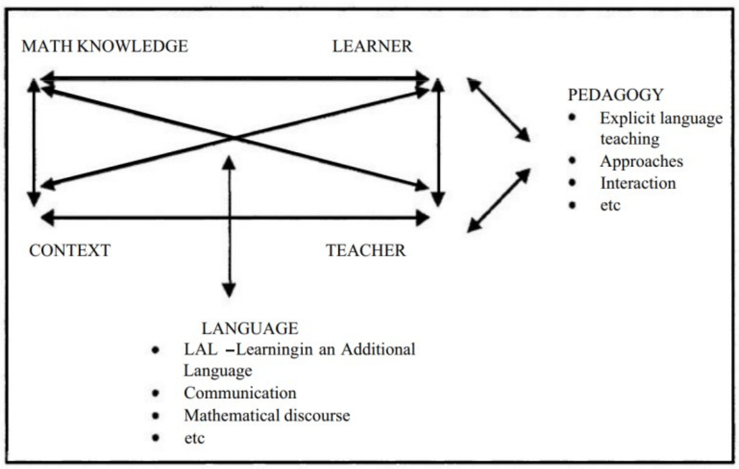

Figure 3.1. Framework for individual teacher map
Source: Adler, 2002, p. 58.

These categories of description allowed the coding of the relevant teacher utterances accordingly to form individual teacher maps, which provided a quick snapshot of their knowledge and beliefs of mathematics (and the role of language in teaching mathematics). It also allowed a quantitative analysis of the comparisons (for all six teachers in this case) in terms of language-related commonalities and divergences; presences and silences within and across these teachers' articulated knowledge of their practice, which gave rise to the notion of dilemmas that was visibly present in the different teachers' articulated knowledge of their practice in the study.

A Notion of Language-Related Dilemmas in Mathematics Classrooms

The language of dilemmas was first developed by Berlak and Berlak (1981, as cited by Adler, 1996, 2002) to capture the "contradictions that reside in the situation, in the individual and in the larger society" (Adler, 2002, p. 51). In particular, Berlak and Berlak had wanted to provide a language that can account for teachers' tensions in the context of schooling, residing at a structure-agency level of understanding. Subsequently, Lampert (1985) argued for a more personal and practical focus in relation to the language of dilemmas and discussed dilemmas from the teacher's perspectives. In particular, she referred to practical (teaching) dilemmas as situations of tensions which teachers may face in their teaching practice when there seems to be "no one 'right' solution" (Adler, 2002, p. 49) that can resolve the tensions, from the perspective of teachers.

As mentioned, the language of dilemmas was not intended as a frame for Adler's (1996, 2002) study initially. However, based on the analysis of the teachers' individual maps, she noticed that all teachers in the study raised concerns with the need to communicate in the language of instruction (English) and difficulty with mathematical discourse (mathematics register and symbolic form). It was further noted that issues, tensions and challenges formed most part of the data, thus giving rise to the notion that all teachers face and manage dilemmas in the complexity of their teaching. Hence, she added further dimensions to the existing conception of teaching dilemmas (from Lampert and the Berlaks) and developed a language for language-related dilemmas to better describe and explain teachers' knowledge in the context of teaching in multilingual mathematics classrooms.

In particular, Adler identified the three following key language-related dilemmas in relation to three specific language issues – the issues of code-switching, mediating student group discussion and explicit language teaching – which recurred in the data for all teachers.

- The *dilemma of code-switching*[11] where teachers (usually of English Language Learners) need to decide whether to change the language of instruction to develop students' mathematical understanding when decisions made to

[11] cf. the dilemma of code-switching which was extended in Zazkis' (2000) study where the focus of code-switching is between the use of the mathematics register and everyday language, rather than across languages, in a relatively more monolingual mathematics classroom.

change the language of instruction may compromise the learning of the mathematics register (and also English);

- The *dilemma of mediation* where teachers (usually those who adopt more learner-centred pedagogies) need to decide whether to intervene to validate students' meanings using everyday language during group discussions or presentations, when decisions made to intervene may compromise their opportunities to develop mathematical communicative competence (in relation to the mathematics register);
- The *dilemma of transparency* where teachers need to decide whether to explicitly teach the mathematics register when the decisions made to teach the language explicitly (i.e., language as a visible resource) may compromise the development of student mathematical understanding (i.e., language as an invisible resource).

She further elaborated how each of these dilemmas might manifest differently for teachers situated within varying multilingual contexts through the analytic narrative vignettes of three particular classroom episodes which were significant to the study.

3.1.3. Application: How Has the Framework Been Used in Research?

From a comparison of the maps (see Figure 3.1) of the individual teachers in her study, Adler (1996, 2002) surfaced the notion of language-related dilemmas. It is interesting as the findings from her study resulted in a bigger theoretical frame which allowed the analysis and explanation of teachers' knowledge within the complexities of teaching mathematics (particularly in response to the interactions between language and mathematics) through how they experience and manage these dilemmas. As such, her notion of language-related dilemmas has often been quoted in research (beyond the multilingual context) which focused on understanding teachers' knowledge, practices and tensions related to language and mathematics education (e.g., Herbel-Eisenmann et al., 2015; Turner et al., 2019). Specifically, Turner et al. (2019) used Adler's notion of language-related dilemmas to frame their study of early career teachers' mathematics teaching practices, in relation to the issues and challenges brought about by language. Other than noting similar dilemmas, they also attempted to extend the language of dilemmas across the teachers' practices in their study, though I argue that it can be easily traced back to Adler's original notion of dilemmas[12].

In addition, Adler (1996, 2002) proposed that the language-related dilemmas can become sources of praxis for development of teachers' knowledge, particularly in relation to the use of language (or more specifically the mathematics register) as a resource for mathematics teaching. From her research, she noticed how the teachers in her narratives learnt to identify with and transcend beyond the dilemmas as they become better at managing these dilemmas. As such, she suggested the possibility of tapping on teaching dilemmas as a resource in (mathematics) teacher education.[13] This was concurred by Turner et al. (2019) as they similarly argued how the language of dilemmas can possibly inform the type of support or professional development teachers may need with regard to the use of the language as a resource for mathematics teaching.

A particular example of how this can be possibly done is seen in Zazkis' (2000) attempt to turn the dilemma of code-switching into a pedagogical tool for pre-service teachers to learn the mathematics register. Specifically, she introduced a fictional student, Simon, who was unable to use the appropriate mathematical language (i.e., the mathematics register) to express his mathematical concepts. Students (pre-service teachers) in her course were asked conscientiously to interpret and rephrase what Simon said using appropriate mathematics language. As a result, they became more aware of the need to speak mathematically in order to think mathematically. Hence, she suggested that it may be useful to first make language visible in teacher-training courses so as to enable teachers to use the language effectively as a *transparent resource* (Adler, 2000) for mathematics teaching and learning.

Notably, one limitation of using the notion of language-related dilemmas (with an intentional linguistic focus) to analyse and explain teachers' mathematical knowledge for teaching may seem to be the reduced emphasis on understanding the mathematics

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[12] Turner et al. (2019) discussed their findings in terms of when and how to introduce vocabulary (cf. dilemma of transparency); when, how and why to intervene with students' use of mathematical vocabulary (cf. dilemma of mediation); and the role of multiple languages in the mathematics classroom (cf. dilemma of code-switching).

[13] Adler (2021) has since developed a mathematics teaching framework with a specific focus of developing teachers' competence in the meditation of word use in mathematical explanations. This was motivated by the dilemma of mediation she identified in this study (Adler, 1996, 2002).

content and teaching involved. However, based on how language (specifically the mathematics register) has been shown to be an important tool for the development of mathematical thinking in research, I argue that the focus on language as a resource may in fact enhance the understanding of what good mathematics teaching and learning entails in the field.

3.2. Understanding Teachers' Orientations towards Language-Responsive Teaching

In an attempt to deconstruct the necessary teacher expertise for language-responsive mathematics teaching, Susanne Prediger (2019) highlighted teacher orientations (further expanded in Prediger et al., 2019) as one key construct which influences what (the *categories*, in Prediger's terms) they notice in terms of the demands of a language-responsive mathematics classroom, and consequently shape their (productive or unproductive) practices in coping with these demands.

3.2.1. Intent: What Research Phenomenon Does It Address?

Situated within a bigger study (Prediger, 2019) which seeks to provide more clarity in terms of the necessary teachers' expertise required for language-responsive mathematics teaching, the study (Prediger et al., 2019) aimed to examine the connections between teachers' *language-related orientations* and *diagnostics categories* they notice when looking at students' written explanations. Notably, these are two of the five constructs which conceptualise the necessary teacher expertise (based on the ideas of Bromme, 1992, as cited in Prediger, 2019) for language-responsive teaching. The five constructs, which are intricately connected, are as follows (see Prediger, 2019, for a full discussion of the framework and its applications):

- *Jobs* defined as the demands or tasks mathematics teachers have to manage when situated within the context of language-responsive teaching (namely, noticing, demanding, supporting, developing language in the context of mathematics teaching, as well as, identifying mathematically relevant language demands);
- *Practices* defined as what teachers consistently do or say, in coping with the jobs, which typically draw on or are influenced by the following:

- *Pedagogical tools* refer to the concrete strategies, tools, etc that teachers use in coping with different jobs;
- *Orientations* refer to teachers' beliefs which guide how they perceive and prioritise the jobs of language-responsive mathematics teaching;
- *Categories* refer to the specific conceptual knowledge elements teachers use to decide what to notice or think about in response to the jobs and are likely to hinge on their orientations.

Specifically, Prediger and her team (2019) argued that while research (mainly qualitative and small scale) has shown how teachers' practices in language-responsive mathematics classrooms are likely guided or influenced by their orientations and categories for noticing (though not necessarily language-specific), there is a need for strong quantitative evidence to support these claims and connections. Hence using a quantitative approach, their study sought to address the following research questions:

1. What language-related orientations do mathematics teachers hold and how do the orientations correlate with each other?
2. What personal diagnostic categories do teachers activate when noticing and evaluating students' written explanations of a mathematical concept?
3. How are the language-related orientations and the personal diagnostic categories connected to each other? (p. 106)

3.2.2. Construct: What Are the Details and Possible Scope of Analysis?

In order to address the research questions, two specific theoretical constructs were used to frame the study (Prediger et al., 2019) – teachers' *language-related orientations* and teachers' *language-related diagnostic categories* – in relation to the specific demands of a language-responsive mathematics classroom. Particularly, as the task in the study involved students' written explanations to a mathematics problem, two situational demands (or *jobs*, in Prediger's terms) – identifying mathematically relevant language demands; noticing and evaluating language resources and needs in students' utterances and written work – have been identified as what a teacher needs to cope with in such situations. Correspondingly, the two constructs – language-related orientations and diagnostic categories – were identified and defined based on how they will likely influence teachers' practices in managing these two demands.

Language-related orientations

Research (Bromme, 1992; Schoenfeld, 2010, as cited in Prediger et al., 2019) has shown that teachers' orientations often influence their practices. Specifically, orientations here refer to the "content-related and more general beliefs that implicitly or explicitly guide the teachers' perceptions (e.g. beliefs about the content or students' learning processes)" (Prediger, 2019, p. 370), adapted from Schoenfeld's (2010) construct of orientations. Thus, in relation to a language-responsive mathematics classroom, a teacher's language-related orientations are likely to lead to different focus or treatment, in terms of pedagogical approaches and actions, of language as a resource for his/her mathematics teaching. In particular, Prediger et al. identified the following five language-related orientations (based on past research) as being crucial in influencing teachers' practices when responding to the demands of language-responsive mathematics teaching, mentioned above.

- O1: *Language as a learning goal* in subject-matter classrooms which considers the extent which teachers assume responsibility for language learning as a goal in their mathematics classrooms;
- O2: *Striving for pushing rather than reducing language* which considers how teachers react in terms of language demands within their mathematics classrooms.
- O3: *Focus on the discourse level rather than on the word level only* which considers if teachers focus on involving students in discourse practices to learn the language (and the mathematics) or only teaching new vocabulary (e.g. mathematical terms).
- O4: *Integrative perspectives instead of additives only* which considers if teachers see the learning of language as being integrated in the learning of mathematics or something beyond the learning of mathematics.
- O5: *Conceptual understanding before procedures* which considers if teachers focus on developing conceptual understanding which necessitates the use of language (i.e., the mathematics register) as a resource for mathematics teaching.

Notably, research (as cited in Prediger, 2019; Prediger et al., 2019) has shown that teachers who hold all the above five orientations generally take more effective actions within language-responsive mathematics classrooms. Specifically, only O1, O2 and O4 were originally discussed in Prediger's (2019) framework as important orientations which guide teachers' practices for language-responsive mathematics teaching. While O3 and O5 may be more specific orientations in relation to this study

(Prediger et al., 2019), they are also seemingly important in relation to teacher expertise for language-responsive mathematics teaching.

Language-related diagnostics categories

The construct of categories was first introduced by Bromme (1992, 2001, as cited in Prediger et al., 2019) as the "conceptual, non-propositional knowledge elements that filter and focus the categorial perception and the thinking of the teacher when coping with situational demands" (Prediger et al., 2019, p. 103). In the context of language-responsive mathematics teaching, it refers to the specific knowledge or personal categories (can be mathematical or language aspects) which provide the basis for teachers to notice[14] and evaluate certain language demands in their teaching practices as being important. Typically, the language aspect categories teachers choose to notice and evaluate are closely tied to their language-related orientations and the interplay of both constructs will consequently influence teachers' language-responsive (or not) practices in their mathematics classrooms. For example (specific to this study), conceptual knowledge, procedural knowledge, etc. are categories which teachers consider in terms of the mathematical aspects of what to notice; whereas word level/technical terms, discourse practices which explain meaning, etc. are categories which teachers consider in terms of the language aspects of what to notice.

3.2.3. Application: How Has the Framework Been Used in Research?

As the Prediger et al. (2019) study took a quantitative approach with a large sample of 223 secondary mathematics teachers, both constructs were operationalised such that quantitative data could be generated and analysed. In particular, data with regard to teachers' language-related orientations were based on their six-point, Likert-scaled responses to statements which corresponded to the respective orientations. Data with regard to the diagnostics categories were generated and coded based on teachers' diagnostic practices – the teachers were given an open-ended task where they had to provide the criteria (two for mathematical aspects and two for language aspects) they will use to evaluate a set of students' written work to a particular problem.

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[14] cf. the notion of noticing by van Es and Sherin (2002, 2021) which focuses on the act of noticing rather than what is being noticed.

The findings suggest that the teachers generally hold productive language-related orientations, especially O1 and O4, indicating that they see language both as a learning goal and integrative part of learning mathematics. Yet, it is peculiar that most will still choose to reduce rather than push for language (O2) in the teaching of mathematics. In addition, it is noted that their diagnostic categories for noticing and evaluating language demands in the mathematics classrooms are varied and seemingly unproductive as most tend to highlight the surface language aspects, such as grammar, which are not crucial to the learning of mathematics. By applying statistical analysis to both sets of data, Prediger et al. (2019) further report that the unproductive categories, and consequently unproductive practices, seem to correlate to the less desired language-related orientations. Based on their findings, they argue that while teachers may be aware of the practices required for language-responsive mathematics teaching, they need to broaden their language-related orientations for language to be meaningfully used as a resource in teaching. However, they are also cognizant of the limitation of the study as only one task (designed based on one particular mathematical concept) was used to collect and generate teachers' diagnostic categories.

3.3. Assessing Teachers' Knowledge of the Mathematics Register

In a study to understand pre-service teachers' knowledge of the mathematics register, Lane et al. (2019) adapted the Knowledge Quartet (Rowland et al., 2005) and conceptualised a framework to map aspects of teachers' mathematics register proficiency. The framework can also be used to identify the gaps in teachers' knowledge of the mathematics register, with respect to the dimensions of the Knowledge Quartet.

3.3.1. Intent: What Research Phenomenon Does It Address?

The study (Lane et al., 2019) was situated in an Irish university. The team of researchers, Ciara Lane, Niamh O'Meara and Richard Walsh, was interested in analysing pre-service teachers' ability in using the mathematics register (i.e., their mathematics-register proficiency) to facilitate a peer-teaching segment of a teachers' education course. In particular, they argued that, while there has been much research done in terms of pre-service teachers' mathematical knowledge for teaching (e.g., Ball et al., 2008), little attention has been given to the aspect of mathematics language as being part of knowledge for teaching, though it is deemed as an essential component of mathematics teacher education (Cramer, 2004, as cited in Lane et al., 2019). Hence, they were motivated to provide a glimpse into this aspect of pre-service teachers' mathematical knowledge for teaching, specifically in terms of mathematics-register proficiency, through their study.

Moreover, from the team's preliminary analysis of the data from the peer-teaching videos and post-teaching interviews with the pre-service teachers in the study, they observed that these teachers generally lacked fluency in the use of the mathematics register and tended to use the mathematical language without precision. As such, Lane et al. wanted to explain further their observations by identifying the gaps in terms of these teachers' knowledge of the mathematics register, which will lead to implications on their mathematical knowledge for teaching. In particular, the study focused on addressing the following research questions:

1. What gaps can be identified in pre-service mathematics teachers' mathematics-register proficiency?
2. How do these gaps relate to the pre-service teachers' mathematical knowledge for teaching? (p. 795)

Notably, the study was only able to look into the gaps in the teachers' knowledge of the register, considering the small scope of the study – a small number of participants situated within a single course context. The researchers noted that the limited data would not be able to provide a full or fair assessment of the teachers' mathematics register proficiency, according to the researchers' original intent.

3.3.2. Construct: What Are the Details and Possible Scope of Analysis?

As Lane et al. (2019) were interested in analysing pre-service teachers' mathematics register proficiency, an aspect of mathematical knowledge for teaching in practice, they adapted the Knowledge Quartet which was first conceptualised by Rowland et al. (2005) for this purpose.

The Knowledge Quartet

Based on a grounded-theory approach in the analysis of 24 videotaped lessons by pre-service teachers, Rowland et al. (2005) conceptualised the Knowledge Quartet. The quartet was designed with the intent of providing a framework to analyse and discuss pre-service teachers' mathematical content knowledge based on classroom observations. These researchers claimed that the Knowledge Quartet "provides a means of reflecting on teaching and teacher knowledge, with a view to developing both" (p. 257), which will, in turn, help fuel and frame productive conversations between the teacher and the observer (e.g., a mentor) in relation to pre-service teacher education. Notably, the quartet can also be used as a tool to think about mathematics teaching practices with a focus on the teaching specific subject matter content, thus reflective of teachers' subject-matter knowledge (SMK) as well as pedagogical-content knowledge (PCK) (Shulman, 1986). As such, although the quartet emerged from a study involving pre-service elementary teachers, it has also been applied to studies beyond the primary context and the initial teacher education context in other works of this group of researchers (e.g. Rowland et al., 2011; Rowland, 2012, as cited in Lane et al., 2019).

Specifically, the Knowledge Quartet is made up of four dimensions/units which were the result from a synthesis of 19 codes identified from the data. A brief description of what the four dimensions entails is as follows (see Rowland et al., 2005, pp. 265–266, for the complete construct):

- *Foundation*, which focuses on teachers' mathematical knowledge and the corresponding beliefs and understanding of their knowledge from their own learning experiences (from school and teacher training), including the why and how of teaching mathematics.
- *Transformation*, which focuses on teachers' *knowledge-in-action* through their planning and actual teaching, in terms of how their own mathematical knowledge is transformed into pedagogical strategies and mathematical representations to help students develop mathematical understanding.
- *Connection*, which focuses on teachers' awareness and coherency of the connections within and across topics and lessons, including the "relative complexity and cognitive demands" of different mathematical concepts and procedures - which are also part of their *knowledge-in-action*.
- *Contingency*, which focuses on teachers' *knowledge-in-interaction* observed through their ability to "think on their feet" and respond appropriately to unexpected classroom events, e.g. a student's response which is not planned for, or change their lesson flow, when necessary.

Interestingly, it is noted that Rowland et al. seem to suggest language as a "less fundamentally mathematical aspect" (p. 260) of mathematics teachers' knowledge for teaching, in terms of how it will be analysed through the Knowledge Quartet. However, teachers' "choice of representation", which certainly involves the use of the mathematics register, is one of the contributory codes within the dimension of transformation. Also, the mathematics register, though based in language, is not necessarily a less mathematical aspect of this knowledge. Teachers will need to be aware and able to transform it into an important tool for the development of mathematical thinking for the students.

Considering Mathematics Register as a Focus in the Knowledge Quartet

In contrast, Lane et al. (2019) claimed that the mathematics register is an important aspect of teachers' mathematical knowledge for teaching, and thus an integral component of the respective dimensions within the Knowledge Quartet. They adapted the Knowledge Quartet and conceptualised one with a specific focus on the mathematics register as "the knowledge component" with respect to the four dimensions. While the researchers have similarly adopted Halliday's (1975) notion of the mathematics register, it is noted that they have included the use of symbols within their construct of the mathematics register (cf. O'Halloran, 2015, and Pimm, 2021) in their conceptualisation.

In particular, Lane et al. (2019) "redefined" how teachers' knowledge of the mathematics register will look like in relation to the four dimensions of the Knowledge Quartet. A full illustration of their version of the Mathematics Register Knowledge Quartet with examples can be found in their study (see Table 1 in Lane et al., 2019, p. 793). However, their framework did not explicitly connect their definitions of the respective categories with the original Knowledge Quartet. Hence, I made an attempt to map their "redefined categories" in terms of what they attend to within the respective dimensions described in the original quartet (Rowland et al., 2005) for greater clarity (see Table 3.1).

Notably, both Knowledge Quartets by Rowland et al. (2005) and Lane et al. (2019)[15] were primarily developed for the purpose of observing, analysing and

[15] As the study by Lane et al. (2019) is rather recent, the Mathematics Register Knowledge Quartet has not been found to be applied to other studies.

discussing pre-service teachers' knowledge for teaching and their teaching practices. However, I see value in how the constructs within the four dimensions can also be used to "assess" and understand experienced teachers' knowledge for teaching (particularly of the mathematics register), as well as, how and why their practices are shaped in a certain manner.

Table 3.1. Mapping of the mathematics register to the Knowledge Quartet

Dimension	Category as defined in the quartet (Rowland et al., 2005, p. 265 – 266)	Category as defined with a focus on the mathematics register (Lane et al., 2019, p. 793)
Foundation	Mathematical knowledge and understanding	Knowledge and understanding of the mathematics register, especially mathematical terminology and vocabulary
	Beliefs and awareness of the knowledge	Awareness of differences between the everyday register and the mathematics register
Transformation	Deliberation in planning	Evidence of planning for mathematical language in a classroom setting
	Choice of representation for teaching	Use of representations and analogies that elicit mathematical meaning for students
Connection	Coherence in terms of connections within and across topics and lessons	Consistency in mathematics register, especially terminology and vocabulary within lessons and between lessons and across different mathematics topics
	Awareness of the relative complexity and cognitive demands	Awareness of difficulties students may experience with the mathematics register
Contingency	Ability to "think on their feet" and respond appropriately	Ability to interpret students' register in line with the mathematics register
		Ability to facilitate an adherence to the mathematics register during classroom interactions

3.3.3. Application: How Has the Framework Been Used in Research?

In responding to the two research questions of their study, Lane et al. (2019) used the Mathematics Register Knowledge Quartet as a frame to identify the gaps in pre-service teachers' knowledge of the mathematics register with respect to the different dimensions. They paid special attention to the inaccuracies in terms of teachers' use of the mathematics register from the data collected (included videos, interviews and reflections in relation to the ten-minute peer teaching segments that the pre-service teachers did as part of a teacher training course). However, as the teaching segments were only ten-minutes long, they were only able to report findings from the *Foundation* and *Transformation* dimensions of their framework, due to a lack of evidence which were relevant to the *Connection* and *Contingency* dimensions. Although the findings from the *Foundation* and *Transformation* dimensions will probably provide a glimpse into the other two dimensions, it is a pity as the findings may be more compelling with evidence with regard to the teachers' knowledge-in-action as well as knowledge-in-interaction in using the mathematics register as a resource for teaching.

Particularly based on their data, Lane et al. (2019) found that the teachers in the study generally did not have sufficient or inaccurate knowledge and understanding of the mathematics register within the *Foundation* dimension. As such, in terms of the *Transformation* dimension, it was observed that the teachers were not sufficiently prepared to transform their knowledge and beliefs of the register into mathematical explanations which were clear and conceptually accurate in their planning and actual practice. More often than not, the teachers would use everyday language, rather than the mathematics register, in their explanations as it is deemed to "simplify" the mathematics. Lane et al. suggested that their findings were indicative of a possible lack in terms of these teachers' mathematics-register proficiency, which needs to be better supported and developed during initial teacher education. Otherwise, it will be questionable as to whether they are able to exemplify the use of the mathematics register and consequently, help students develop a proficiency towards using the register as a tool for mathematical thinking.

While Lane et al. (2019) did not explicitly discuss any limitations of their Mathematics Register Knowledge Quartet, Rowland (2012) emphasised the need to be cognizant that the knowledge for teaching may be analysed differently using the Knowledge Quartet, depending on the scope and demands of mathematics the teacher is working with (e.g., elementary, secondary or advanced).

3.4. My Research Questions

What is the role of language in relation to the teaching and learning of mathematics? This is the question that has started my research. It is also one that has been asked by many other researchers in the mathematics education, as evident in the literature review (Chapter 2). However, it is probably not a question that I can fully address through this thesis. Reflection of my own experiences with language as a teacher and what I have learnt as an early researcher led me to redefine my focus in exploring the phenomenon of language as a resource in mathematics education instead. Similarly, one may argue that this phenomenon has long been assumed as an important aspect of the role of language in the field. Most researchers have thus moved beyond asking the question of whether language can be a resource in mathematics education. Instead, the question they are asking is how better to help teachers use language as a resource through research focusing on the development of mathematics teachers' pedagogical content knowledge and expertise in relation to the use of language in mathematics classrooms (e.g., Prediger, 2019; Adler, 2021).

While I agree that it is a very much needed move to design better professional development programs in relation to the use of language as a resource in mathematics education, research on investigating what teachers actually notice in relation to language and how they use language as a resource in their classrooms seem to be lacking. In particular, I wonder what teachers, particularly those who probably have not specifically thought of language as a resource for mathematics teaching and learning, would say. Moreover, while the three theoretical constructs – teachers' language-related dilemmas, language-related orientations and knowledge of the mathematics register – have been used to study teachers' use of language in mathematics education, they each focus on a very specific aspect that inform how teachers notice and use of language in the mathematics classroom. Though my intent in this study is not to combine the three theoretical constructs into a new or massive construct, I see value in adopting a complementary approach in co-ordinating the theoretical constructs to provide a more "networked understanding" to the phenomenon of how and why teachers are using (or not) language as a resource in the mathematics classroom (Bikner-Ahsbahs & Prediger, 2010, p. 495). Hence, I ask the following research questions:

1. How do teachers notice[16] and use language as a resource for mathematics teaching and learning? In particular, how do their language-related dilemmas and orientations influence their noticing and use of language?

2. How are teachers' knowledge and potential usage of the mathematics register featured through their responses to teaching situations designed with language-related issues?

With these questions, I hope to inform the existing state of how teachers are attending to language as a resource in their classroom, which appear to be lacking in recent research. Correspondingly, the findings may explain why some mathematics teachers or educators do not see value or need to be thinking about using language as a resource in the mathematics classrooms.

[16] The notion of *notice* I am using here is a very broad one, in terms of what they become aware of and attend to. While it may not be specifically aligned to any existing notion of noticing in research, I consider my notion of noticing to be closer to Mason's (2002) notion of professional noticing (p. 30), than van Es and Sherin's (2002, 2021) notion of noticing.

Chapter 4.

The Method to Probe Teachers' Use of Language in the Mathematics Classroom

To address the research questions, an overall qualitative approach was taken to collect and analyse data. The data was collected through clinical, semi-structured interviews (Zazkis & Hazzan, 1998), with eleven experienced mathematics teachers who had taught or were still teaching mathematics in classrooms, ranging from elementary to tertiary levels. My aim was to gather data which could probe (and provide justifications for) information with regard to how teachers notice and use language in the mathematics classroom. While some may argue that such data should have been more objectively generated through observing teachers' practice in the actual setting of the classroom (coupled with follow-up discussions of their practice), this was not a viable option at the point when I was collecting data. The COVID-19 pandemic situation has resulted in restrictions in terms of classroom visits and face-to-face interactions I could have had with teachers. Moreover, with changes in lesson structures (e.g., the switch to remote learning and shorter face-to-face time in school) due to the pandemic, classroom observations would probably not be as reflective of what the actual classroom site would look like in a pre-/post-pandemic situation. As such, the clinical interview method was chosen as the closest way in which I could observe the teachers in the most "naturalistic" manner (Ginsburg, 1981) outside of their natural teaching practice sites – the mathematics classroom.

In this chapter, I describe the development and the design of the method for my research (including the rationale behind it), as well as the data collection and analysis process. In section 4.1, I elaborate on how the interview protocol for the main study was developed through a pilot study. In section 4.2, I share the design and intent of tasks used in the interview protocol. In sections 4.3 and 4.4, I first elaborate on how the data was collected and how the data was organised and analysed respectively.

4.1. Developing the Interview Protocol

As the interview is the sole instrument deployed in this study, I needed to ensure that its design was targeted in gathering data which would help address my research questions. In order to do that, I conducted a small pilot study which went through three iterations prior to deciding on the final and refined interview protocol.

4.1.1. The First Iteration

To address the research questions, I required data from the interview which could be reflective of how teachers use language in the mathematics classroom. Specifically, I hoped to find out about how the teachers typically use language (with attention given to the mathematics register in particular) in mathematics teaching and learning. As such, the first version of my semi-structured interview design took reference from what Adler (1996) did in her dissertation research. In her research, she had an initial in-depth interview with the teachers individually to elicit teachers' articulated knowledge in mathematics classrooms. The data which she gathered from those interviews saw the emergence of her notion of the three different language-related dilemmas. As my only data source similarly hinged on teachers' articulated knowledge about how they would use language in their classrooms, I thought it would be worthwhile adapting her interview design to my own research. My assumption was that an in-depth interview would perhaps lead to data which would help inform my research. By getting teachers' responses to how they would manage situations involving language-related dilemmas, I would possibly find evidence in terms of teachers' knowledge of and orientations towards the use of language in their mathematics classrooms.

The first version of the interview was thus designed to comprise ten open-ended questions (see Appendix A) which sought to prompt teachers to respond in the following two broad areas:

- Information about their background, experience and beliefs in teaching and learning mathematics (from questions 1 to 5). This would likely contribute to the knowledge of and orientation towards the use of language in their respective teaching practices. In addition, question 10 was included in the interview to elicit their knowledge of the mathematics register which frames the notion of mathematics language I am attending to in my research.

- Information regarding their experience with and in managing language-related dilemmas, including the three types of dilemmas proposed by Adler (1996, 2002), though not exclusive (from questions 6 to 9). This was intended to shed light on how teachers may use language in their mathematics classrooms.

This set of questions were then piloted with Teacher A, a peer (a current and experienced teacher) in the same Ph.D. program with me. Based on the first pilot trial, it turned out that the data was not as focused as I had envisioned it to be. While I was able to get a glimpse of Teacher A's background and experience in teaching and learning mathematics from the interview, the generic nature of the questions resulted in data which lacked specificity in terms of the actual actions which she would take in her use of language to manage language-related dilemmas. In the post-interview discussion with Teacher A, she also provided feedback that, since the questions were mostly general, she found it difficult to recall or relate to specific experiences on-the-spot during the interview. In most parts of the interview, she could only share generic comments to the questions. This data was certainly not sufficient in distilling specific evidence relating to her experience, knowledge and orientations in terms of using language in the mathematics classroom. Consequently, she suggested that perhaps having some specific incidents involving language-related tensions may be helpful to trigger deeper conversations (and thus data) which would serve the purpose of my research.

4.1.2. The Second Iteration

While reflecting upon why the data from the first version of the interview did not turn out as expected, I re-looked at Adler's (1996) design. What was essentially absent in my design, but had worked well for her research, were the lesson observations in actual classroom sites and follow-up reflective interviews she was able to carry out in addition to the initial interviews. These additional data sources provided the specificities which very much complemented her initial interview data. However, as mentioned earlier, gathering data at actual classroom sites was not a possible option at the time of my data collection (due to the pandemic). Hence, with Teacher A's suggestion in mind, I started to consider the inclusion of specific accounts (in the form of dialogues) to substitute what may be observed with regard to teachers' use of language in an actual mathematics teaching and learning setting.

Moreover, according to Mason (2002), the use of specific accounts can also become “entries into or pointers to experiences, which constitute the actual data” (p. 99). In this case, the specific accounts would serve as mirrors (using Mason’s terms) for teachers in my study to reflect upon their own practice and experiences in terms of how they would use language in their mathematics classrooms. Based on what they would attend to or notice (or not) about the use of language in the specific accounts, I was hopeful that the inclusion of specific accounts would help provide the evidence which had been missed with my first version of interview design. Specifically, the inclusion of specific accounts served as reflection tasks (Zazkis & Hazzan, 1998) for teachers to reflect upon and talk about their use of language when faced with language-related dilemmas in the mathematics classroom. Coupled with question prompts, these tasks would also enable me to probe further into teachers’ experiences and beliefs in the use of language in a setting outside of traditional teaching practice sites – to which I had no access.

Hence, two specific accounts, written in the form of classroom-based dialogues (see section 4.2), were created as the reflection tasks for the interview. Embedded within each task was some form of language-related tensions which might lead to language-related dilemmas for teachers in the mathematics classroom. The tasks were intended as triggers for reflection, together with the following set of question prompts.

- What do you notice in this dialogue? Specifically, what do you notice about what the teacher did?
- Are there any parts that jumped out to you or surprised you? Why?
- If you were the teacher, would you have done something similar or something different? Why? How will you continue this conversation?
- Did you experience something similar before in your classroom? What would you do in such situations?

Again, I invited Teacher A for a second interview to pilot the two reflection tasks and to gather her feedback on these tasks which replaced the broader questions (see questions 4 – 9 in Appendix A) she was asked in the previous iteration. Befittingly, the tasks and prompts proved to be helpful in eliciting data which was more specific to her experience with and in managing language-related dilemmas. I was also able to gain more insights in terms of her use and perception of language (both the mathematics and the everyday registers) in her mathematics classroom. However, I noticed that the question prompts needed to be clearer in terms of guiding Teacher A towards focusing on the phenomenon of interest presented in these tasks. Moreover, due to the way the prompts were phrased, unintentional attention had been brought to the teacher character in the tasks. There was a general tendency for Teacher A to focus first on the "critique" of the teacher character's actions, prior to reflecting about her own experiences in the use of language. This seemed to reflect how it tends to be easier to criticize others than to reflect upon one's own actions. As such, during the interview, I had constantly to remind her to step back and reflect upon the original intent of the tasks – what she noticed in terms of language use and related dilemmas; and how she would manage them.

Yet from the interviewee's point of view, something that bothered Teacher A in this trial was the length of the tasks and the myriad of language-related issues that were loaded within them. While it became easier for her to recall and relate to specific experiences with the tasks as the backdrop for reflection, she found it difficult to focus and dwell deeply in the discussion of a particular aspect of language-related issues. As such, there may be missed opportunities for in-depth reflection on the part of future interviewees if they were to feel obliged to discuss the full myriad of issues noticed in the tasks. Consequently, a key suggestion which arose from this iteration was the need to shorten the tasks and sharpen the focus of each task by embedding only one or two aspects of language-related issues. This resonates with Mason's (2002) emphasis on having brief-but-vivid accounts to describe instances of a certain phenomenon for research as long as they are significant enough to trigger similar experiences. Ultimately, "it is not the incident itself which is of particular interest, but the incidents which come to mind in the reader, for the reader can only work on their own experiences" (p. 50). With greater brevity in each task, it would also help reduce the cognitive load on the interviewees and in turn enhance the quality of their reflections and the conversations around it.

4.1.3. The Third and Final Iteration

For the third iteration, I refined and also created more reflection tasks with a focus of making them brief-but-vivid. In addition, I reviewed the set of question prompts that accompanied the reflection tasks so as to bring interviewees' attention to the intent of having these tasks, that is to reflect upon their own experiences rather than to critique the tasks or the characters' actions. The revised set of question prompts and the rationale for re-wording or omitting some of the initial prompts is presented in Table 4.1.

Table 4.1. Revision of question prompts for the third iteration

Question Prompts (Original)	Question Prompts (Revised)	Revision Made	Rationale
1. What do you notice in this dialogue? Specifically, what do you notice about what the teacher did?	What do you notice in what the students are saying in this dialogue?	Rephrased to focus on students' use of language in the accounts.	To reduce focus on the critique of teacher character's actions.
2. Are there any parts that jumped out to you or surprised you? Why?	---	Omitted to reduce probability of interviewees focusing on other aspects of the accounts which may be more significant to them.	To prevent the conversation away from being steered away from the focus of language-related dilemmas and challenges.
3. If you were the teacher, would you have done something similar or something different? Why? How will you continue this conversation?	How will you respond if you were a teacher in this situation? Why will you do/say that? • What do you notice about what the teacher did or may have done (if given additional teacher's responses / hints of teacher's influence)?	Rephrased to probe actions based on interviewees' experiences. Additional prompt added for accounts with teacher characters.	To shift focus from a possible critique of teacher character to the interviewee taking the role of the teacher in the account.
4. Did you experience something similar before in your classroom? What would you do in such situations?	Have you experienced something similar before in your classroom? Can you share what happened? What did you do then?	Rephrased to probe specific incidents from interviewees' experiences.	To reduce the possibility of interviewees responding generically to the use of language as a resource.

The revised set of tasks and question prompts was then piloted with two other peers (similarly experienced teachers) – Teacher B and Teacher C – from my program. The main intent of this final trial was to observe if the tasks were effective triggers for the intended language-related dilemmas, and if the prompts were useful in drawing out the interviewees' reflections and experiences of those dilemmas. As Teacher A was already familiar with the purpose of my research, her perspective as an interviewee might have been skewed by the prior two post-interview discussions. Hence, I thought it would be better to gather feedback from other sources to enhance my interview protocol.

Table 4.2. Interview protocol for the main study

<table>
<tr><th>Question Sets</th><th>Intended Data</th></tr>
<tr><td>1. Could you briefly share about your own academic background with regard to teaching and learning mathematics?</td><td>Information regarding teachers' academic background and experience in teaching and learning mathematics.</td></tr>
<tr><td>2. <u>Reflection Tasks' Prompts</u>
a. What do you notice in what the students are saying in this dialogue?
• What do you notice about the language the students are using in this dialogue?
• Why do you think the students say that? / What do you think the students are thinking about?
b. How would you respond if you were a teacher in this situation? Why will you do/say that?
• If you were a teacher in this situation,
o Will you step in to modify the language the students are using? (For tasks in Categories 2 and 3)
o Will you step in to teach the formal/ mathematical terms? (For all tasks, especially Category 1)
o Will you switch between formal and informal use of language or between mathematical and everyday usage of terms? (For tasks in Categories 2 and 3)
• If no (to any of the above) ⇒ Why not?
• If yes (to any of the above) ⇒ When?
c. Have you experienced something similar before in your classroom? Can you share what happened? What did you do then?
• Can you think of other instances when students have used everyday language to connect mathematical ideas or everyday words which may be used differently in the mathematical context? (For tasks in Category 4)
• What are some examples of language (e.g., specific words/ ways of explaining) you usually use in teaching this topic? When/How do you use the language in your teaching?</td><td>Teachers' reflections and deeper discussions centered around language-related dilemmas (through attending to the reflection tasks, though not exclusive) – the key data to addressing my research questions.</td></tr>
<tr><td>3. Just wondering, have you heard about the term "mathematics register"? What do you know about it? Can you elaborate?</td><td>Information regarding teachers' experience with and/or understanding of the mathematics register.</td></tr>
</table>

From the interviews with both Teachers B and C, most of the tasks proved to be generally effective triggers for the intended language-related dilemmas. Some feedback from the teachers was useful towards the further refinements of the final set of tasks used for my main study. However, the phrasing of the revised prompts (in Table 4.1) still seemed to be unclear, in terms of getting the interviewees to focus on the language aspects in the accounts when I first used them in my interview with Teacher B. As a result, the aim in probing her experiences of language use and language-related dilemmas through the tasks was sidelined as she focused on the other pedagogical and conceptual issues present in the different accounts. Perhaps those were the more significant connections she made in relation to her experiences, but that might not be the most relevant data in terms of my research.

Hence, I decided to try a slightly nuanced and expanded set of prompts (see Question Set 2 in Table 4.2) with Teacher C. Other than drawing her attention to the language-related dilemmas in the accounts, the more specific prompts were also added, in order to probe the specific actions she would take when responding to different dilemmas. This change proved to be much needed as the pilot interview with Teacher C noticeably produced more sharing of experiences and discussions pertaining to her use of language in her mathematics classroom – data which would inform my study.

On the whole, I was able to gain many insights from the interviews conducted in the pilot study. These had helped me improve and streamline my interview protocol, in terms of its approach and questions. The final and actual version of the interview protocol, intended to be clinical and semi-structured, comprised of three sets of questions. The question sets and their respective intended data are listed in Table 4.2.

4.2. Designing the Reflection Tasks

Since the reflection tasks (Zazkis & Hazzan, 1998) eventually formed the basis in my interview protocol, they were designed in order to illuminate the situations when teachers would have to make connections to their experiences of using language (Mason, 2002). In particular, these tasks depicted specific classroom-based accounts which would help teachers relate to some form of language-related dilemmas in the context of a mathematics classroom. Each account typically presented a different pair of fictional student characters (and sometimes a fictional teacher character, Ms. Wilson) discussing a specific mathematical concept in the context of an elementary or a secondary mathematics classroom. For ease of reference, Ms. Wilson was designed to be a fictional teacher character who can teach mathematics at both elementary and secondary levels. When choosing the mathematical concepts which could associate with possible language-related dilemmas, I used relevant research literature and my own experiences. I also ensured that a variety of concepts were selected across the different content foci (numbers, fractions, geometry and graphs) and grade levels to cater to the different teaching experiences of my research participants. As the focus of my study resides heavily on the use of language in teaching and learning mathematics, I chose to present the accounts in the form of dialogues, rather than in prose. It would certainly be more vivid for the teachers if they could "hear" (or see) what was being said in these accounts.

4.2.1. Learning from the Pilot Tasks: A Reflection

As mentioned in sub-section 4.1.2, two reflection tasks (Table 4.3 and Table 4.4) were created for the second iteration of the pilot study with Teacher A. As she was a secondary teacher, I selected two mathematical concepts – transformation of graphs and graphs of rational functions – which are generally covered at the secondary level. Each task was then written as an illustration of a possible discussion students may have around the respective concept.

In my first attempt at creating the tasks for the interview, I thought it would be best if I could fit multiple language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2) in each task to expand the scope of Teacher A's reflections. Firstly, with the intent of prompting her response and reflection to the *dilemma of mediation*, the student characters portrayed in the two tasks were mainly using the everyday language, with occasional use of the mathematics register, to express mathematical ideas in their discussion. For example, in task B1, as the student characters were discussing the transformation of graphs, they used phrases like "move up and down" and "skinnier or fatter" to describe changes to the characteristics of the graph after it was translated and stretched.

Table 4.3. Pilot task B1

Context: A secondary mathematics classroom – a lesson on transformation of graphs where students were working on an activity to explore vertical and horizontal translations, stretches, and reflections.		**Embedded Language-related Dilemmas (with reference to the underlined parts)**
Dean:	This seems to cause the graph to move up and down right? Like if you see these two, the shape looks the same, I think, but it's at a higher position.	Use of everyday language to express mathematical ideas ➔ Dilemma of mediation
Asha:	And this one is left and right? They have the same shape too.	
Dean:	Oh yes, that's it. One type of transformation is movement, shifting, up down left right? One down, two more to go. What about these? One of them looks fatter or broader like it's been stretched sideways somehow.	
Asha:	Hmm, what about this, they look skinnier, that's not stretching.	
Dean:	Maybe it's stretched upwards instead of sideways, that's a stretch right?	Multiple meanings for *stretch* - everyday vs mathematical ➔ Dilemma of mediation
Asha:	Ohhh, that's a possibility, just like how we stretch before gym class?	
[Ms. Wilson happens to walk by.]		
Ms. Wilson:	I thought I heard something interesting going on here. Tell me what you were discussing?	
Asha:	Well, Ms. Wilson, we were just talking about these two graphs, they look, hmm, skinnier or fatter, literally, when compared to the first one.	
Dean:	I think there's some kind of stretching or enlargement going on but Asha made a good point that it doesn't quite work for the skinny one, stretch is usually bigger, so we are not sure. But we also say that maybe if it has to be bigger, it can be taller, so it's a stretch upwards instead of sideways.	

Context: A secondary mathematics classroom – a lesson on transformation of graphs where students were working on an activity to explore vertical and horizontal translations, stretches, and reflections.		Embedded Language-related Dilemmas (with reference to the underlined parts)
Ms. Wilson:	Very good, you both have raised very good and valid points. Let's begin with what Asha said, skinnier or fatter, you were describing the shape of the graph. So we realised that the shape changes for this transformation but not for these other graphs, which have been translated or shifted or moved vertically or horizontally. To better understand how the shape changes, we need to look at how the expressions of the functions change. Take for example, y equal to x-squared, y equal to two x-squared and y equal to half x-squared.	Teacher's attempt to code-switch between everyday language and mathematics register ➔ Dilemma of code-switching
Dean:	Half x square is fatter than x square but two x square is skinnier or narrower.	
Asha:	Half x square is stretched sidesway but two x square is stretched upwards. That's quite different because usually, half and two, they are opposite of each other.	Ambiguity in the use of the word *opposite* which may lead to possible misconception in mathematical understanding ➔ Dilemma of mediation
Ms. Wilson:	Ah, can you elaborate on what you mean by opposite, Asha?	
Asha:	Hmm, hmm like half should make it smaller but two should make it bigger?	

Table 4.4. Pilot task B2

Context: A secondary mathematics classroom – a lesson on graphs of rational functions where students were working on an activity to discover the characteristics of graphs of rational functions.		Embedded Language-related Dilemmas (with reference to the underlined parts)
Ethel:	Look at these graphs, they all have these sort of lines which the graphs go very near to. They can be straight up or lying down or slanted?	Use of everyday language to express mathematical ideas but shows awareness of the mathematics register (annotated in bold print) ➔ Dilemma of mediation
Theo:	I see them too. I think we saw something similar in logarithms but it was always straight up and the graphs were not supposed to touch them. Did they have a name?	
Ethel:	Oh yes, I think **Ms. Wilson used a special term**. Eeks, but I can't remember. But, are these the same thing? There's like more of them, there is a slanted one, and they can cut each other, I mean, **intersect each other**.	

<table>
<tr><th colspan="2">Context: A secondary mathematics classroom – a lesson on graphs of rational functions where students were working on an activity to discover the characteristics of graphs of rational functions.</th><th>Embedded Language-related Dilemmas (with reference to the underlined parts)</th></tr>
<tr><td>Theo:</td><td>Hmm, I am not sure, shall we ask Ms. Wilson?</td><td rowspan="6"></td></tr>
<tr><td colspan="2">[Ms. Wilson walks over to join Ethel and Theo.]</td></tr>
<tr><td>Theo:</td><td>Ms. Wilson, we have a question. Are all these lines, [pointing] like those in logarithms? But there's more types here, so we are not sure.</td></tr>
<tr><td>Ms. Wilson:</td><td>Hmm, what do you mean by lines that are like those in logarithms?</td></tr>
<tr><td>Ethel:</td><td>We meant those lines that the graphs go closer and closer to?</td></tr>
<tr><td>Theo:</td><td>And can never touch or meet the lines. We know there's a name for it but we don't remember.</td></tr>
<tr><td>Ms. Wilson:</td><td>Ah, I see, you are talking about asymptotes, a-symp-totes.</td><td rowspan="5">Teacher's attempt to teach mathematics register explicitly but subtly as she tried to connect to the mathematical ideas proposed by the students ➔ Dilemma of transparency</td></tr>
<tr><td>Ethel:</td><td>Yes yes, that's what it is, asymptotes! So are these also asymptotes? Do they have to be straight up only like in logarithms? Are these other types also asymptotes?</td></tr>
<tr><td>Ms. Wilson:</td><td>Yes these lines are all asymptotes. Both of you made the good observation that the graphs are approaching these lines, asymptotes, going closer and closer without touching, or rather, intersecting them. They can be vertical, what you mean by straight up, horizontal and also oblique, for those diagonal lines.</td></tr>
<tr><td>Theo:</td><td>Why do the graphs go closer and closer to these so-called asymptotes? Will the graphs never ever intersect them? We don't think they will.</td></tr>
<tr><td>Ms. Wilson:</td><td>Very good question, Theo. Hmm, [pause], we might want to be a little careful here to say that the graphs will never ever intersect the asymptote. Although yes many graphs do not intersect their asymptotes. But, but, let me try this, it may be a little complex but let's try it together ok? For this graph [points to a particular graph], which part of the graph is going closer and closer to the two asymptotes? What do you notice about the values of x or y?</td></tr>
</table>

Secondly, with the intent of prompting her response and reflection to the *dilemma of code-switching*, the teacher character in both tasks was portrayed to use a combination of the everyday language and the mathematics register. These actions were made explicit in their interactions with the student characters when helping them understand the concept and also learn the mathematics register. For instance, in task B1, to describe the horizontal and vertical translations of the graph, the teacher character used the word "translated" interchangeably with "shifted" or "moved" – words that were used by the student characters. Thirdly, to prompt her response and reflection to the *dilemma of transparency*, there were exemplification of instances where the teacher character placed more emphasis on the mathematics register and resulted in some form of explicit teaching of terms. For example, in task B2, the teacher character was seen to be focusing more on introducing relevant terms in the mathematics register such as *asymptote, approaching, intersecting, vertical, horizontal* and *oblique*.

Last but not least, there were also intentional language-related challenges specific to the use of the mathematic register. Referencing Pimm's (1987) work, these challenges highlighted the difficulties students may face when learning and using words from the mathematics register and were intended to probe teachers' use of language in such situations. A particular challenge resides in how many mathematical terms tend to have multiple meanings (context-dependent) as they are reinterpretations of existing words in the everyday language. For example, in task B1, a confusion on the use of the word *stretch* was depicted. The word *stretch* was intentionally chosen as it is commonly used (in Singapore) when describing transformation of graphs. But it is also a common word used in the everyday context, such as to *stretch* before exercise or to *stretch* a rubber band. However, there are slight differences in how this word is used in the different contexts. In the everyday context, *stretch* tends to denote an extension or enlargement (of an object) but in the mathematical context, *stretch* may instead represent a compression or contraction (of a graph) instead.

While the use of reflection tasks proved to be useful in eliciting data relevant for my study, Teacher A found the multitude of language-related dilemmas and challenges embedded in the tasks rather distracting. Hence the key feedback from the pilot study was to shorten the tasks and sharpen the focus of each task by embedding only one or two aspects of language-related issues. In addition, Teacher A commented that the word *stretch* is not commonly used as part of the mathematics register here (in Canada).

Rather teachers would mostly use *compression* and *enlargement* to more precisely describe similar transformations. In other words, this may not be an appropriate example for teachers (particularly my research participants) to relate to their experiences involving words with multiple meanings. Based on the feedback from the second pilot, I decided to discard task B1 as a possible task for my study. I also shortened task B2 before including it as a task (Task 5 – Graphs of rational functions) in the main study.

4.2.2. Tasks and Design Considerations for the Main Study

Considering the feedback from the pilot study, I created a set of brief-but-vivid reflection tasks, which spanned across a range of topics and grades, for the actual study. The intent of these tasks is to provide specific incidents as "mirrors" for teachers to reflect upon their own practice and experiences (Mason, 2002) in terms of how language would be used in their mathematics classrooms. The tasks were generally motivated by situations where there were instances of language-related dilemmas (Adler, 1996, 2002) and possible challenges that students may face when learning and using the mathematics register (Pimm, 1987). In addition, a couple of tasks also adapted some of the examples discussed in Zazkis' (2000) work focusing on the code-switching of everyday language and the mathematical register to express mathematical ideas. In all, I created a total of eight tasks which can be placed in four broad categories, as follows:

A) the dilemma of transparency;

B) the dilemma of mediation amid a *presence* of student confusion;

C) the dilemma of mediation amid an *absence* of student confusion;

D) the challenge in using mathematics register words with multiple meanings.

In each of the tasks presented below, certain words or phrases are underlined[17] as they relate to the specific language-related dilemmas and/or challenges which were intentionally embedded to evoke teachers' reflections during the interviews. As the dilemma of code-switching (Adler, 1996, 2002; Zazkis, 2000) was not explicitly used to frame any of the categories, it was embedded as one of the question prompts (see 2b in

[17] The tasks were presented to the participants without the underlines, so as to get an unbiased representation of what they noticed (or not).

Table 4.2). In other words, teachers would also be prompted to reflect on this dilemma during the interview when they attend to a particular task. I now turn to the description and the intent of tasks in each category (including the context of the accounts) in greater detail. An overview of the reflection tasks and their corresponding language-related foci is summarised in Table 4.5.

Table 4.5. Overview of the tasks used in the main study

Task	Level	Concept Involved	Specific Areas of Language Use: Mathematics Register	Specific Areas of Language Use: Everyday Language	Specific Language-Related Dilemmas / Challenges
1	Elementary	Sum of angles in a triangle	Euclidean space Plane Interior angles Degree Sum	(Units omitted) Add up	Dilemma of transparency
2	Secondary	Prime factorisation	Prime number Factors Factorise Divided by	Split Goes into	Dilemma of mediation
3	Secondary	Slope of linear functions	Slope Steep Small number	Gentle Small number	Dilemma of mediation
4	Elementary	Fractions	Numerator Denominator	Top number Bottom number	Dilemma of mediation
5	Secondary	Graphs of rational functions	Asymptote Vertical Horizontal Oblique Intersect Approach	Straight up Lying down Slanted Cut / meet / touch Go closer	Dilemma of mediation
6	Elementary	Diagonals of a polygon	Diagonal Horizontal Pentagon	Diagonal(ly)	Multiple meanings of *diagonal*
7	Elementary	Division	Divide Even(ly) Odd	Evenly	Multiple meanings of *evenly*
8	Secondary	Operations with integers	Negative Positive Plus Minus	Negative Positive Plus Minus	Multiple meanings of *the "–" symbol*

Category A: The dilemma of transparency

Within this category, the intent was to depict scenarios where the dilemma of transparency may arise for teachers as shown in Task 1. As the category was very specific in terms of its focus, I decided one task was sufficient. When necessary, I could possibly prompt teachers to consider similar incidents in different contexts.

Task 1 (Sum of angles in a triangle)

Context: **An elementary mathematics classroom – a lesson on angles of triangles where students were working on an activity to find the angle sum of a triangle.**

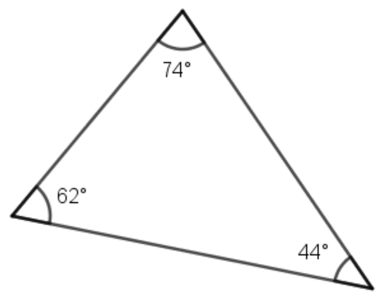

Figure 4.1. Diagram of triangle given in Task 1

Janet: *[Pointing to angles in the triangle]* These three add up to one hundred and eighty (180)!

Silas: Wait a minute, what do you mean?

Janet: Oh, I'm saying these three angles *[pointing to interior angles in the triangle]* in the triangle add up to one hundred and eighty <u>degrees</u> (180°).

Silas: Ah, I see, you are saying that the <u>SUM</u> *[said with an emphasis]* of the three <u>IN-TE-RI-OR</u> [said with an emphasis] angles of the triangle is one hundred and eighty degrees (180°).

Ms. Wilson: *[Interrupted after having heard the conversation]* And yes, to be more precise, you should say the SUM *[said with an emphasis]* of the three IN-TE-RI-OR *[said with an emphasis]* angles of the triangle on a <u>PLANE</u> *[said with an emphasis]* in an <u>EU-CLI-DE-AN S-PACE</u> *[said with an emphasis]* is one hundred and eighty degrees (180°).

In Task 1, the student characters, Janet and Silas, were discussing their observation of the sum of the interior angles of triangles on a plane (in the Euclidean space). They demonstrated an understanding of the concept as illustrated by their observation and were using appropriate and precise register words such as sum and interior angles. Yet, the teacher character, Ms. Wilson, intentionally brought in two other specialised terms from the mathematics register – plane and Euclidean space – to the discussion. While she seemed to be hinting that these words were more precise and should be used in their observation, she did not provide any further explanation of the meaning of those words in relation to the discussion.

Specifically, Task 1 was intended to prompt teachers to respond to the dilemma of transparency by first discussing if they would act the same way as what Ms. Wilson did in the account. This would likely lead to a deeper reflection on situations when they may be more explicit in teaching the mathematics register; and when they would not.

Category B: The dilemma of mediation amid a presence of student confusion

Within this category, the intent was to depict scenarios where the dilemma of mediation may arise for teachers, particularly when students' usage of everyday language or imprecise terms have resulted in confusion in the learning process. In both Tasks 2 and 3, the student characters were seen to be using mainly everyday language or imprecise terms in their discussion. As a consequence, their use of language had led to some obvious confusion or disagreement in the discussion due to a lack of understanding in terms of register words used.

Task 2 (Prime factorisation)

Context: **A secondary mathematics classroom – a lesson on prime factorisation where students were working on an activity to determine the prime factorisation of 180.**

Flor: One hundred and eighty equals four times five times nine ($180 = 4 \times 5 \times 9$).

Vish: Hmm, are they *[pointing to the factors]* all <u>prime numbers</u>? I know four and nine are not, so we have to <u>split them further</u>. As for five, it should be a prime number because <u>nothing goes into it</u>, right?

Flor: Oh yes, there's <u>twos in four</u> and <u>threes in nine</u>.

Vish: Huh? What do you mean? Where are the two and three?

Flor: Because <u>two goes into four</u>. I mean, <u>four can be divided by two</u>.

Vish: But <u>nine can be divided by two</u> too.….

In Task 2, the student characters, Flor and Vish, were discussing the prime factorisation of 180. Flor used language such as “twos in four” and “two goes into four” (more colloquial expressions) to describe the operation of division, which was not understood by Vish. When Flor switched to the use of *divided by* to explain what she meant, Vish again had a different understanding of the phrase as they appeared to be thinking about *divided by* in different number systems. For Flor, *divided by* was understood as divisibility without remainder when working with whole numbers; for Vish, *divided by* was understood as the operation of dividing rational numbers.

Specifically, Task 2 was intended to prompt teachers to respond to the dilemma of mediation by thinking about whether (and how) they would mediate the use of language in this situation. For example, they may choose to provide more clarity and precision in the use of *divided by* to address the difference in students’ understanding; or address the ambiguity brought by the use of *goes into*. It would also lead them to reflect about the factors influencing their decision to mediate (or not) in similar situations.

Task 3 (Slope of linear functions)

Context: **A secondary mathematics classroom – a lesson on the slope of linear functions where students were engaging in a discussion of the concept of slope, given some graphs of linear functions.**

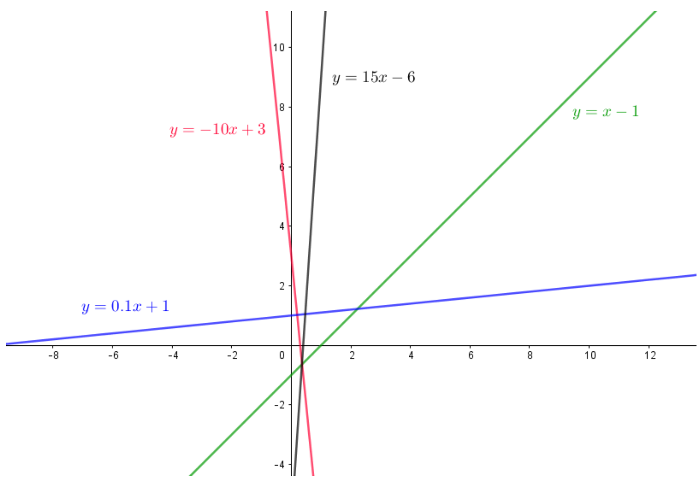

Figure 4.2. Diagram of graphs given in Task 3

Paul: If <u>the number</u> is very big, the line is very steep. If it is very <u>small</u>, it is very gentle.

Lyn: What do you mean by very small? This line *[pointing to a graph with a negative slope of -10]* has <u>a number that is very very small</u> and it looks very very steep.

Similarly, in Task 3, the student characters, Paul and Lyn, appeared to have understood the use of *small number* differently due to a lack of precision in terms of their language used. For Paul, a *small number* was a small whole number; for Lyn, a *small number* was a small integer. As a result, it led to a confusion in the discussion as to why the line with slope of -10 was "very very steep".

Again, Task 3 was intended to prompt teachers to respond to the dilemma of mediation by thinking about whether (and how) they would mediate the use of language in this situation. However, the focus was slightly different from that in Task 2. In response to student understanding of the use of *small number*, I hoped to probe further the specific actions they would take. For example, would they introduce "new" register words such as *absolute value* into the discussion? Would they also code-switch when explaining what *absolute value* in relation to the different *small numbers* that Paul and Lyn were saying? Other than the dilemma of mediation, this task may possibly lead to a discussion of the other language-related dilemmas.

Category C: The dilemma of mediation amid an absence of student confusion

Within this category, the intent was to depict scenarios where the dilemma of mediation may arise for teachers although there may not seem to be any confusion during the learning process due to students' usage of everyday language or imprecise terms. In both Tasks 4 and 5, the student characters were similarly seen to be using mainly everyday language or imprecise terms in their discussion. However, unlike Tasks 2 and 3, there were apparently no confusion in their discussions as they seemed to have shared understanding of what they were saying in relation to the mathematical concepts.

Task 4 (Fractions)

Context: **An elementary mathematics classroom – a lesson on fractions (equal-sized parts of whole) where students were working with fraction strips to show the fraction $\frac{3}{5}$ [as in the diagram below].**

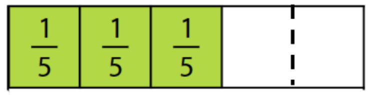

Figure 4.3. Diagram of fraction strips given in Task 4

Nodo: Because the <u>bottom number</u> is five, we need to use the green piece (which denotes the fraction $\frac{1}{5}$).

Vick: And we need three of them to get the <u>top number</u> three.

In Task 4, the student characters, Nodo and Vick, appeared to have no difficulty understanding each other's use of top and bottom numbers in the context of fractions. In fact, it seemed like they shared a common understanding of the *top number* as referring to the numerator and the *bottom number* as referring to the denominator.

Specifically, Task 4 was intended to prompt teachers to respond to the dilemma of mediation by thinking about whether (and how) they would mediate the use of language in situations when there was no apparent confusion among students. Although the usage of the *top number* and the *bottom number* in this case did not seem to be causing any obvious confusions or misconceptions yet, a discussion of whether they were still appropriate might surface in the interview. In particular, teachers may point out how students might go away thinking that a fraction is a representation of a mathematical object that is made up of two numbers, rather than representing a number itself, if such language use was not mediated. Consequently, this task may provide evidence regarding teachers' language-related orientations in the use of language for mathematics teaching and learning.

Task 5 (Graphs of rational functions)

Context: **A secondary mathematics classroom – a lesson on graphs of rational functions focusing on the characteristics of asymptotes where students were discussing the characteristics of graphs of rational functions, given a set of such graphs.**

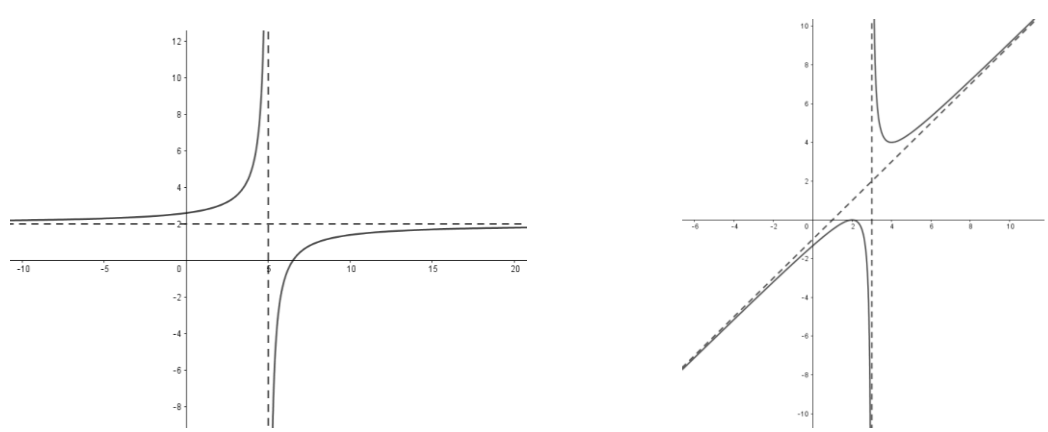

Figure 4.4. Diagram of graphs given in Task 5

Ethel: Look at these graphs, they all have these sort of lines *[pointing to the dotted lines]* which the graphs go very near to. They can be straight up or lying down?

Theo: I see them too. I think Ms. Wilson used a special term. Eek, but I can't remember. But, are these the same thing? There's like more of them, there's a slanted one, and they can cut each other. Ms. Wilson, we have a question. These lines that the graphs go closer and closer to but can never touch or meet the lines. We know there's a name for it but ....

Ms. Wilson: Ah, you are talking about asymptotes, A-SYMP-TOTES *[said with an emphasis]*. Yes these lines are all asymptotes. Both of you made the good observation that the graphs are A-PPROA-CHING *[said with an emphasis]* these lines, asymptotes, going closer and closer without touching, or rather, IN-TER-SEC-TING *[said with an emphasis]* them. They can be VER-TI-CAL *[said with an emphasis]*, what you mean by straight up, HO-RI-ZON-TAL *[said with an emphasis],* what you mean by lying down and also OB-LIQUE *[said with an emphasis],* for those slanted lines.

In Task 5, the two student characters, Ethel and Theo, were discussing their observations of the graphs of rational functions and were particularly focused on the characteristics of the asymptotes. They described the orientation of the asymptotes using everyday language such as straight up, lying down and slanted. They also described how the graphs "go closer and closer but never touch or meet the lines" to refer to the tendency for graphs to be approaching their asymptotes without intersecting them. However, they have forgotten about the specialised term *asymptote*. The teacher character, Ms. Wilson, was then brought in to illustrate a possible response to students' imprecise use of everyday language by introducing and emphasising words from the mathematics register such as approaching, intersecting and oblique. In order not to cloud what the participants would say in terms of what they would do at first, this task would be intentionally shown in two parts – the dialogue between the student characters, followed by the teacher character's response.

Specifically, Task 5 is intended to primarily observe how the teachers would mediate the use of language, and whether they would relate to what Ms. Wilson did as a response in this situation; or disagree with her decision to correct students' use of everyday language even when students are able to understand one another. In addition, the common misconception that graphs can never intersect their asymptotes was embedded in this task although it was not a primary intent. Some teachers may be drawn to the use of the word *never* and discuss the misconception which may result. Though there may be others who would not notice this as the word *never* is often used loosely in everyday conversation without a true extreme meaning. Thus, this little addition would also make the task an interesting one to gather data in terms of what teachers would attend to (and not attend to) in relation to language use in situations with no immediate impact on student understanding.

Category D: The challenge of mathematics register words with multiple meanings

Within this category, the intent was to depict scenarios where certain words (in the mathematics register) with multiple meanings in different contexts may result in confusion, and thus misconceptions for students.

Task 6 (Diagonals of a polygon)

Context: **An elementary mathematics classroom – a lesson on diagonals of a polygon where students were working on an activity to identify the number of diagonals in a given regular pentagon.**

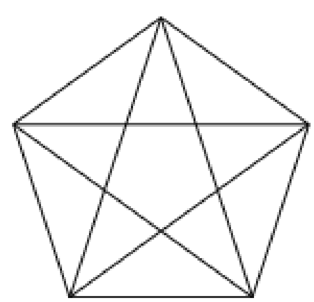

Figure 4.5. Diagram of the pentagon given in Task 6

Aria: So how many diagonals do we have here *[pointing to the pentagon]*? Hmm, one … two … three … FOUR *[counting only the slanted diagonals in the pentagon]*! There are four diagonals in this pentagon.

Bert: But there are five lines connecting the corners of the pentagon. Is this not a diagonal *[pointing to the horizontal diagonal]*?

Aria: Yes, it's not because it's horizontal!

In Task 6, the two student characters, Aria and Bert, appeared to have a disagreement in terms of the number of diagonals a given pentagon had. This account was adapted from Pimm's (1987) illustration of students' common confusion with the two different meanings of the word *diagonal*, depending on the context it is used. On the one hand, *diagonal* is an adjective, describing an orientation that are neither horizontal nor vertical in the everyday context. It is commonly used in the mathematics context to describe the orientations of mathematical objects as well. On the other hand, *diagonal* is also a noun, defining a specific mathematical object – a line segment connecting two non-adjacent vertices of a polygon. The confusion with the two meanings of *diagonal* tends to be exacerbated as diagonals of polygon are often portrayed to lie diagonally on the page although they definitely need not be.

Specifically, Task 6 was intended to bring teachers' attention to the possible challenges students may face while learning about *diagonal* (of a polygon). It may prompt them to reflect on how they would teach this mathematical concept and how they can be more explicit when differentiating between the two meanings of the word *diagonal*. There may also be further discussion of other words with multiple meanings that may confuse students.

Task 7 (Division)

Context: **An elementary mathematics classroom – a lesson on division where students were discussing about $14 \div 2$.**

Ben: Ok, <u>two divides fourteen evenly</u>.

Gina: No? The answer is seven which is <u>odd</u>!

Similarly, in Task 7, the two student characters, Ben and Gina, appeared to have understood the use of the phrase *divides evenly* in two different manners. On the one hand, *divides evenly* evokes the idea of equal sharing where the objects are evenly divided into groups with each group having the same number (can be odd or even) of objects. On the other hand, *divides evenly* may be interpreted as dividing objects into groups where the number of objects in each group is an even number as the word "evenly" may be relatable to the property of even numbers.

Specifically, Task 7 was intended to bring teachers' attention to the dual understanding of the phrase *divides evenly* although both may reside in the mathematical context. It may prompt them to reflect on how they can be more explicit and careful in their choice of words. For example, the use of phrases such as, "divides equally" or "divides with no remainder", may be less ambiguous in terms of expressing the same mathematical idea. There may also be further discussion of other common phrases with multiple meanings that may confuse students.

Task 8 (Operations with integers)

Context: **A secondary mathematics classroom – a lesson on operations with integers where students were discussing the answer to $-2-3$.**

Ken: Hmm, <u>minus two minus three is … minus five</u>!

Tala: No, you are wrong. <u>Negative and negative become positive</u>, so the answer should be plus and not minus ….

In Task 8, the two student characters, Ken and Tala, appeared to have a disagreement in terms of the answer to "$-2-3$" due to the different meanings that the "$-$" symbol can represent. Notably, the challenge of multiple meanings is also present in the use and reading of mathematical symbols. In this case, the "$-$" symbol can be seen as either an operation of subtraction or a property of a number, pointing to a negative number. Particularly while working with operations involving negative numbers, students tend to struggle when distinguishing the two meanings of the "$-$" symbol. This task was thus designed to illustrate the situation where Ken interpreted the "$-$" symbol as the subtraction operation of two numbers; while Tala interpreted it as two negative numbers and related it to the rule that the product of two negative numbers is positive.

Specifically, Task 8 was intended to bring teachers' attention to the possible dual or multiple meanings behind symbols. It may prompt them to reflect on whether they would differentiate the way they read certain symbols; and how they could be more intentional when describing subtractions involving negative numbers. Although symbols are not considered as being part of the mathematics register (Pimm, 1987), the need to use language to verbalise these symbols is inevitable in the mathematics classrooms. Hence, the task intended to probe teachers' experiences in this aspect of language use.

4.3. Interviewing the Participants

The research participants comprised eleven experienced mathematics teachers, who were my peers in the Mathematics Education Ph.D. program at Simon Fraser University. All of them have had experience or are still teaching in (English-medium) mathematics classrooms currently. All of them are effectively fluent in English and a few of them speak other languages due to different cultural and educational backgrounds. However, their teaching contexts differ in terms of the student grade level and range from the elementary level to the tertiary level. Being an international student in Canada, I have limited access to actual schools, classrooms and thus teachers in this country. Thus, I decided to reach out to my peers (who are mostly experienced mathematics teachers) in my program at Simon Fraser University as they formed a convenience sample for my research. An overview of the teachers' pseudonyms, their language backgrounds and their primary teaching experiences is summarised in Table 4.6.

Table 4.6. Overview of participants' language and teaching backgrounds

Teacher's Pseudonym	Language Background	Teaching Background		
		Elementary	Secondary	Tertiary
Alicia	monolingual		✓	
Cass	monolingual			✓
Evie	Bi/multi-lingual		✓	
Faye	monolingual	✓		
Joey	Bi/multi-lingual		✓	
Karen	monolingual			✓
Lena	monolingual	✓		
Mindy	monolingual			✓
Nadia	Bi/multi-lingual		✓	
Simon	monolingual		✓	
Sofia[18]	monolingual	✓	✓	

Perhaps one may argue that my research sample would bring possible biases to the data in terms of possible influences to their teaching practices due to their scholarly profiles. However, considering how their exposure to research on language in mathematics education varies and teaching experiences are generally different, I would argue otherwise. In particular, the data which I was gathering focused on eliciting their knowledge and experiences in relation to how they use language by reflecting on their teaching practices. It was not focused on how much they understood about the use of language or how it might be used as a resource in mathematics education, which would then draw on their scholarly knowledge. As such, it drew on their teaching experiences

[18] Sofia was first trained and taught as an elementary teacher. But she has been teaching in the secondary mathematics classroom for the past decade.

and practices to a great extent. In fact, I thought it would be interesting if their scholarly experiences could prompt them to reflect more deeply during the interviews and bring other thoughtful perspectives to the data.

All the interviews were conducted online via Zoom, a video conferencing platform, as face-to-face interviews were not possible with the imposed social distancing restrictions due to the pandemic. Each interview typically took about 1 to 1.5 hours and was video-recorded with the permission of the participants for purpose of my research. The interview protocol (see Table 4.2) was followed in sequence, with question set 2 being repeated for the respective tasks. However, the sequence of tasks was arranged differently according to the grade levels taught by the interviewees. This was to ensure the familiarity of the mathematical content discussed in the tasks for the different groups of interviewees, and that the first few tasks would be more likely incidents which they could connect with in their classrooms. In particular, the following list shows the sequences of the tasks that were planned for the different groups of interviewees.

- Tertiary Group
    - Task 3 → Task 2 → Task 5 → Task 6 → Task 7 → Task 1 → Task 8 → Task 4
- Secondary Group
    - Task 2 → Task 3 → Task 8 → Task 4 → Task 6 → Task 7 → Task 1 → Task 5
- Elementary Group
    - Task 4 → Task 8 → Task 2 → Task 6 → Task 7 → Task 1 → Task 3 → Task 5

Notably, not all of the interviewees went through all the tasks planned for the interview. Some tasks at the end of each sequence were dropped for some of the interviewees who were not familiar or felt uncomfortable with the respective mathematical content involved. Some were also dropped due to a lack of time or an increase in interviewees' fatigue level during the interviews.

During the interviews, interviewees were asked to elaborate when there was a need to clarify further their ideas or when I saw opportunities to probe further into their experiences, based on their responses. I also made use of the annotations of specific language-related dilemmas and challenges embedded within the different tasks as a reference to prompt the interviewers if they did not notice any in their responses. In addition, I made a conscious attempt to rephrase what I heard and interpreted as a form of feedback to the interviewees for validation. This is to reduce the possibility of a monopolised interpretation of the data by the interviewer and increase the objectiveness of the interview data (Kvale, 2016).

4.4. Organising and Analysing the Data

A two-step approach is taken to pre-examine and prepare the relevant interview recordings for further analysis. First, I listened to each interview recording and took down notes relevant to the following areas:

- the teacher's academic background and experience in teaching and learning mathematics to provide the context in which he/she comes with. This data primarily drew from their responses to question set 1 (see Table 4.2);
- what the teacher noticed in terms of language use and his/her corresponding actions/ reactions in terms of language use in specific teaching and learning situations. This data was mainly taken from his/her responses to question set 2 (in relation to the different tasks they attended to). In instances where the tasks led to a spinoff to his/her personal experiences and interactions with language use in their classrooms, that would be described as well;
- the teacher's experience with and/or understanding of the mathematics register, which was drawn from his/her responses to question set 3.

Based on the notes, I was able to identify segments of the interviews to be transcribed in entirety for analysis in greater depth when considering my research questions and the corresponding theoretical constructs. In doing the transcription of the selected interview segments, the following annotations were used to bring clarity to the final transcripts:

- parentheses ( ) were used for notes or words which are not present on the recording, e.g. (symbols) for mathematical expressions made, (inaudible) for expressions which are intelligible;
- brackets [ ] were used for nonverbal expressions, e.g. [pause] for pauses more than five seconds; [gestures] for certain gestures which were essential to understand what was being said;
- long dashes (—) was used to indicate sudden change or interruptions in the responses.

In addition, to ensure a smoother read of the ideas shared and prevent an overload for data analysis, the following were omitted in the transcripts:

- filler markers such as: uh, ah, er;
- brief pauses which were less than five seconds; other paralinguistics such as tone, pitch, body gestures (unless it added meaning to what they were saying).

4.4.1. A Two-Foci Analysis

To address my two research questions, I adopted a two-foci approach towards the analysis of the transcribed data to be examined further. The first focus looks at how teachers are generally noticing and using language as a resource in their classrooms, through the lenses of language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2) and language-related orientations (Prediger et al., 2019 – see sub-section 3.2.2). The second focus looks at exemplifying teachers' knowledge of the mathematics register through the Mathematics Register Knowledge Quartet (Lane et al., 2019 – see sub-section 3.3.2).

Teachers' use of language as a resource

With respect to the first focus, I was interested in responses relating to what and how the participants attended to the use of language (particularly the mathematics register) in the different tasks presented to them and analysed them through the lenses of language-related dilemmas and language-related orientations respectively. Specifically, I identified quotes which corresponded to what and how the participants would respond when faced with the different language-related dilemmas, and the reasons which would be indicative of the language-related orientations they seemed to lean towards. Table 4.7 shows the indicators specific to the different dilemmas and orientations, with examples of quotes identified in this part of the analysis. Often, the quotes identified were noted to overlap across the two lenses of dilemmas and orientations. Subsequently, the quotes identified using the two guiding constructs were categorised to surface common themes relating to what the participants attend to in relation to language and how they would use language as a resource for mathematics teaching and learning.

Table 4.7. Indicators used and examples of quotes identified during the analysis

Theoretical Constructs	Indicators Used	Examples of Quotes Identified
Language-related Dilemmas	Decisions to code-switch (or not) with students' register (everyday and/or mathematical)	"I would say […] when we talk about fractions, the top number is the numerator and the bottom number is the denominator."
	Decisions to mediate (or not) students' use of language	"I would definitely pick out the word diagonal and say you're both using the word diagonal but I think you're talking about different diagonals."
	Decisions to teach / introduce (or not) the mathematics register explicitly	"I think language can be really valuable too, if it's introduced in that good way, it can be so good, but if you're just throwing terms at them and be like, you have to use it? Oh, come on!"
Language-related Orientations	Instances where language is considered as a learning goal	"Sometimes defining a term they haven't heard of gives them a language, with which they can speak about the thing that they're trying to understand."
	Instances where language use is pushed or reduced	"That definitely would be beneficial […] to go in and just clarify and encourage Ken to use the word *negative* instead of minus two."
	Instances where the focus on language is at a discourse level or word level	"I'm not really noticing tons more for language, other than divided, times, like equals, just those basic words."
	Instances where language learning is integrated or an add-on	"If they can attach the name to them, the proper mathematical terminology, that's a bonus."
	Instances where the learning focus is on conceptual or procedural understanding	"I'm noticing is that they're focusing on like procedure there. To me, there isn't necessarily understanding behind this."

Based on the analysis of the data, I observed two main categories in which language has been noticed and used in the mathematics classrooms, namely as a resource for developing mathematical understanding (Category A), and as a resource for mathematics talk (Category B). Participants in Category A mostly consider language (particularly the mathematics register) as a resource for teaching and learning as it can help to construct mathematical meaning and lead to deeper mathematical understanding. In their responses, they typically attended to and would mediate the use of the mathematics register with the intent of clarifying and deepening students' mathematical understanding. An example is noted in Karen's response to Task 6 (Diagonals of a polygon):

> I would definitely pick out the word diagonal and say you're both using the word *diagonal* but I think you're talking about different diagonals […] in mathematics we talk about things really clearly because it gives us ways to talk about things and know that we're talking about the same things. So we talk about specific contexts and we define words within those contexts so that when we're talking about polygons, we all know what the word diagonal means in this context.

Participants in Category B primarily deem language as a resource to engage students in classroom discourse and interactions to learn mathematics. To them, language in the mathematics classroom has a broader connotation and the mathematics register would not necessarily be of a greater significance, as compared to the everyday or colloquial register. Thus, they typically attended to the meaning being articulated, rather than the specific registers used. An example is noted in Lena's response to Task 1 (Sum of angles in a triangle):

> […] she doesn't need to say the word interior in that moment because she's pointing to it. Just like in the first one, does she really actually have to say angles because she's pointing to them? I mean it's better that she does, don't get me wrong, but the understanding, she's communicating her understanding here of what's happening effectively by pointing.

Notably, most, if not all, participants did not show tendency towards only one specific category. As such, I grouped the participants in the category where they showed greater and more consistent tendency in terms of how they would notice and use language in the mathematics classroom, as shown in Table 4.8.

Table 4.8. Grouping of participants according to how they would tend to notice and use language in the mathematics classroom

Category A	Category B	Mixed Tendencies towards Both Categories
Evie	Faye	Alicia
Joey	Lena*	Cass
Karen*		Mindy
Sofia		Nadia
		Simon

* Karen and Lena were chosen to represent teachers in Category A and Category B respectively in my analysis and discussion of the two categories.

However, there were some participants whom I was unable to group them under either category as they showed mixed tendency towards the two categories of language use, based on the interview notes. For instance, these participants might seem to tend towards Category A when responding to some tasks and simultaneously tend towards Category B when responding to the other tasks. These participants were, thus, placed in a separate group, as having mixed tendencies towards both categories.

Moreover, to avoid a blurring of the distinction across the categories with a possibly convoluted and general discussion by considering all participants in this part of the analysis, I decided to focus my analysis and discussion within the two main categories of language use which I observed in the preliminary analysis. In particular, I chose two participants (one from each category respectively), Karen and Lena, to analyse in greater depth and present my findings to the first focus through their perspectives instead. Karen was chosen to represent teachers in Category A and Lena was chosen to represent teachers in Category B. They were also chosen as both participants have similar language backgrounds – effectively monolingual in English only – and the interviews with both of them provided more data for me to dwell deeper in terms of the analysis and discussion of the findings for the first focus.

In analysing the interview data (fully transcribed) for these two teachers, I adopted Mason's (2002) approach of *account-of* and *accounting-for*, as the method to analyse how they notice and use language in their mathematics classrooms. An *account-of* is a description of a phenomenon of interest in an objective manner with minimum "evaluation, judgements and explanation" on the part of the observer; while *accounting-for* provides the "explanation, theorising" (p. 40) of the phenomenon of interest. The distinction between *account-of* and *accounting-for* allowed me to be more "impartial" (in Mason's terms) in my analysis of the phenomenon of interest – teachers' noticing and use of language as a resource in mathematics classrooms – without clouding the details of what they noticed and responses with my own value judgements or expectations from the onset.

Specifically, I created an *account-of* what each teacher noticed in terms of language use (including the mathematics register) and their corresponding actions/ reactions in terms of language use in specific teaching and learning situations. Through creating the *account-of* each teacher, I hoped better to unravel how he/she would likely use language in his/her mathematics classroom, based on the interview data. As I needed to stay objective and not cloud the account-of each teacher with my own interpretations and analysis while creating the *account-of* each teacher, I described the entirety of the interview, regardless of the relevance of the data to my research question. Notably, it was a difficult process, hence I chose to include some exact words or phrases from the interview data within each teacher's *account-of*, when a rephrasing of these words or phrases used by the teacher might misconstrue his/her original meaning. Subsequently, the *accounts-of* were then analysed with the intent of *accounting-for* why they would use language in their classrooms through understanding their experiences with language-related dilemmas and their language-related orientations respectively.

As the tasks for the interviews were designed with language-related dilemmas and challenges in mind, my intent was to first use Adler's (1996, 2002) *notion of language-related dilemmas* (see sub-section 3.1.2) to frame the analysis of what the two teachers deemed as dilemmas when thinking about the use of the mathematics register as a resource for teaching in the mathematics classroom. This was coupled with Zazkis' (2000) extension of the dilemma of code-switching between languages to code-switching between everyday language and the mathematics register as it is more relevant in discussing the framing of the dilemma of code-switching in my study. I next attempted to account for the two teachers' language-related orientations, which are specific to their beliefs and approach towards the role of language in mathematics teaching and learning, in relation to their experiences with language-related dilemmas and how they would manage the dilemmas. While doing this part of the analysis, I took reference to Prediger et al.'s (2019) *construct of language-related orientations* (see sub-section 3.2.2) expanded from the framework focusing on understanding teachers' expertise for language-responsive teaching (Prediger, 2019). Notably, I paid more attention to language as framed by the mathematics register being the tool for thinking and a resource for teaching though Prediger's notion of language is not specifically defined in the manner.

However, this approach of analysis did not work well for Lena, unlike the case of Karen. A major reason was due to how Lena attended to language in a more general sense, with a lack of emphasis on the mathematics register, which conflicted with my notion of language as framed by the mathematics register. Hence, I decided to flip the order of the analysis to account for her language-related orientations first before accounting for her use of language in relation to language-related dilemmas.

Teachers' knowledge of the mathematics register

While attending to the second focus which is to exemplify teachers' knowledge of the mathematics register, I first looked at all the participants' responses according to tasks. Among the eight tasks (see sub-section 4.2.2), the responses to three of them appeared to be most varied and interesting in terms of what the participants noticed about language use and their corresponding articulated *knowledge-in-action* and *knowledge-in-interaction* they would take in response to the tasks. The three tasks were Task 4 (Fractions), Task 6 (Diagonals of a Polygon) and Task 8 (Operations with Integers). All eleven participants' responses to these three tasks were thus transcribed for further analysis, using Lane et al.'s (2019) adapted framework of the Knowledge Quartet (Rowland et al., 2005) focusing on teachers' knowledge of the mathematics register.

In particular, I identified quotes which are indicative of the participants' knowledge of the mathematics register within each of the four dimensions of the Mathematics Register Knowledge Quartet (see sub-section 3.3.2) respectively. Again, most quotes identified are not unique to only one dimension, but often relevant to other dimensions of the Knowledge Quartet. The following response by Joey to Task 4 (Fractions) is an example of a quote which is indicative of knowledge residing at both the *Foundation* and the *Contingency* dimensions.

> […] there is no reference to that relationship, part-whole. So it is like that they are separate but procedurally. But they will see that the bottom, it should be divided to that one in the top, it should be divide. […] so it feels like they didn't build a meaning for this part-whole relationship here […] they don't develop this relationship, they see them as separate numbers.

When commenting on the student characters' use of *top number* and the *bottom number* to refer to the numerator and the denominator of a fraction respectively, Joey showed her understanding of the mathematics register surrounding the concept of fractions within the *Foundation* dimension. Her response was also indicative of her ability to interpret students' register and inferred their understanding of fractions within the *Contingency* dimension. The quotes presented in the findings eventually were selected such that all the eleven participants were represented at least once in the discussion of their knowledge of the mathematics register played out within the respective dimensions of the Knowledge Quartet.

4.5. Summary

In this chapter, I shared and explained the design of the method that framed my research. Specifically, I presented a detailed narrative of the iterative processes which I had undergone to develop the interview protocol and design the reflection tasks as they formed the basis of my research method. I also described the actual data collection and the two-part data analysis process.

In the subsequent parts of this thesis, I discuss the findings to the two-part analysis in sequence. The two main categories in which language has been noticed and used in the mathematics classrooms – as a resource for developing mathematical understanding, and as a resource for mathematics talk – are discussed in Chapters 5 (case of Karen) and 6 (case of Lena) respectively. The exemplification of teachers' knowledge of the mathematics register is discussed in Chapter 7. Specific snippets from the interview transcripts which substantiate the discussion of the findings are included in the chapters, when necessary.

Chapter 5.

Language as a Resource for Developing Mathematical Understanding

In this chapter, I discuss how language may be deemed by some teachers as a resource for teaching and learning, as it can help develop (conceptual) understanding in mathematics. To this group of teachers, language, particularly the mathematics register, needs to be used and taught meaningfully and timely in mathematics classrooms. They see language as a means for students to make sense of mathematical concepts and develop deeper understanding. Amongst the research participants, there were four of them who primarily noticed and would use language as a resource in their classrooms with this intent. However, the extent to which language is framed by the mathematics register varied for the different teachers.

Here, I choose Karen as a case to illustrate how teachers who fall in this category would likely notice and use language in mathematics classrooms. She is one interesting teacher as the language in her mathematics classroom would typically require a good mix of the mathematics register and the everyday language – where these two registers are necessary to complement each other in helping students develop mathematical understanding. In section 5.1, a summary of Karen's background and experience of teaching and learning mathematics is first presented. This is followed by an *account-of* and an *accounting-for* Karen's use of language in her mathematics classroom in relation to language-related dilemmas and language-related orientations in sections 5.2 and 5.3 respectively. All quotations used in this chapter were taken from the interview with Karen. Phrases which related specifically to arguments made in the analysis were indicated in bold font within long quotations.

5.1. The Case of Karen

Karen is a university instructor who has taught mathematics courses at the undergraduate level for about ten years. Her strong interests in mathematics started when she was a child. She loved working on mathematics problems as they felt like puzzles to her. She mentioned how she "probably skipped a lot of the math anxiety" when growing up and had always felt success as a mathematics learner. She continued to take her mathematics education to the tertiary level as she enjoyed discussing mathematics (over other subjects) with people of similar interests. Her brief experience in teaching mathematics to undergraduates, while doing her Master's in Applied Mathematics, further motivated her decision to become a mathematics teacher. While teaching at the university level tends to be of a more teacher-centred or lecture-styled approach, she had the opportunity to adapt a more student-centred or student-led approach (for which she also expressed a preference) in one of the universities at which she taught. To do that, she designed interactive activities and provided space for students to "guide the discussion and ask questions" as they made sense of mathematical concepts in small groups.

Though Karen has minimal opportunities working with K–12 students as a university instructor, she did have some experience teaching and interacting with children of these ages through a one-month enrichment program at a private school previously. When asked how different it was teaching younger children mathematics, she mentioned that she was surprised that it was "not too different actually". Although there needed to be more consideration in planning the types of activities that would be suitable for younger children, she realised that there were "so many of the same interactions" which would also occur in her undergraduate mathematics classroom. Those children, like her undergraduate students, would ask questions of similar nature while working on mathematics activities. When her activities were designed in a way which matched the level of those children, she found that they were also able to engage with the activities and connect or understand the mathematical ideas involved.

5.2. *Account-of* Karen's Use of Language in her Mathematics Classroom

During the interview with Karen, she and I managed to discuss all the eight tasks (Task 3 → Task 2 → Task 5 → Task 6 → Task 7 → Task 1 → Task 8 → Task 4, see sub-section 4.2.2 for the tasks) and her understanding of the mathematics register. As she mainly teaches mathematics at the undergraduate level, I thought it was necessary to manage her expectations of the tasks to be discussed. As such, she was informed that the tasks focused on mathematical concepts specific to either the elementary or the secondary levels only, prior to discussing the tasks. In this section, I present an account-

of what Karen noticed in terms of language use, and her corresponding actions and reactions to the different tasks in sequence, as well as her articulated understanding of the mathematics register.

In Task 3 (Slope of linear functions), Karen first noticed how the two student characters, Paul and Lyn, were engaging with the graphs and each other's ideas though there seemed to be "a discrepancy of what small means". She elaborated that, in the task, Paul seemed to be suggesting a conjecture about the slope of the line graphs based on the coefficients of the x-terms in the linear functions given, though he did not explicitly use terms such as *slope* or *coefficients*. She also highlighted that Paul was using language that she likes to use in her teaching – "simple" language which is "supposed to be clear". Hence, to her, Paul had made a "clear" conjecture with "if the number is something, then this, and if it is this, then it is this" and she would commend him for making "a great guess". She added that, typically, she would use "conjecture" with her undergraduate students, but she might not use it in the case of a secondary classroom. In contrast, she found it interesting that Lyn questioned Paul about his conjecture and provided a counterexample of a steep line graph with a negative slope. She ended with a comment that "they (Paul and Lyn) don't agree with what small means, (it) sounds like they both know what steep is".

When prompted on whether the two student characters were referring to the term *number* similarly (or not), she shared that she was unsure of what Paul might be thinking, based on the one statement he said. She explained that there might be two possibilities – Paul had made the conjecture by only considering "a couple of" line graphs "like the green one ($y = x - 1$) and the blue one ($y = 0.1x + 1$)", which happened to have positive slope, in his conjecture, or he might be thinking of "small as being close to zero". In contrast, she commented that Lyn seemed "pretty clear" in her concept of "very small" numbers as she had "an idea of small extending towards negative infinity".

In terms of her actions in such a situation, Karen would "ideally" ask both students to explain "what it means for a number to be small" and discuss if -10 is smaller than 0.1, which she deemed as "the hard question". Alternatively, she might get them to first look at the line graphs with positive slope to identify the smallest slope before discussing "what it means to be small" again with the line graph that has a slope of -10. When asked if she would use student-used language such as *small* or *number* or bring in terms such as *absolute value*, *slope*, *positive* or *negative* for the discussion at this point, she commented that she would make it a point to "definitely use the words that they've used" first. She would want the students to make sense of the concept of "small" in relation to positive and negative numbers themselves before introducing *absolute value* or talking about the *direction* and the *magnitude* of a number. She further mentioned that her decision to introduce "new terms" would depend on the level of students' understanding and needs as well. She would generally not introduce or define a new term if it would not help students who were still "struggling with the ideas". However, if "defining a term they haven't heard of gives them a language with which they can speak about the thing that they're trying to understand", she would more certainly do so. She would even "use different modalities to explain it" such that the term (*absolute value*, in this case) could become useful to help students grasp other concepts in the discussion. In response to whether her decision might depend on students' grade level, she shared that she would likely still mention the term *absolute value* in passing to younger students, and her focus would be on how she could explain the concept "as simply as possible", rather than "overwhelm them" with new terms.

In Task 2 (Prime factorisation), Karen first noticed how the student characters, Flor and Vish, were working together to factorise 180 into prime factors and appeared to be having "a good discussion". To her, Flor seemed to "know what's going on" and was able to "split it (180) into three factors that do multiply to give one hundred eighty", which was "a really good start". As for Vish, she noticed that he brought in "the terminology of prime" when he questioned whether all three factors were prime. While Karen mentioned that Vish did not explicitly define what a prime number is when he identified which factors were prime (or not), she subsequently commented that he did provide the definition of a prime number with the phrase "nothing goes into it".[19] She thus wondered why Vish became confused and did not understand Flor's various descriptions of how four and nine can "break into" their corresponding prime factors, since Vish was the one who introduced the idea of prime. She attributed Vish's confusion to Flor not using "those words" – "you can factorise four into primes" – and a possible "disconnect about whole numbers and decimal numbers".

[19] Mathematically, this definition for a prime number is flawed as a prime number does have two factors, one and itself.

While Karen would hope that Flor and Vish could continue the good discussion and figure out the confusion themselves, she shared how she would have made use of their confusion to have a discussion involving the concepts of whole numbers, decimal numbers, divisibility and indivisibility with them if they were to request for help. She added that students at the secondary level “should be okay with those words”. She would also want to have a more elaborated discussion with them on what prime numbers are, what “nothing goes into it” means, why prime factorisation only involves “prime, whole numbers” as factors and not decimal numbers. When prompted further on whether the phrase “nothing goes into it” might be more colloquial and mathematically ambiguous, she acknowledged that slightly but mentioned she would use it anyway. She also added that she would say “goes nicely into” when referring to (whole number) factors.

Moreover, in response to whether she would step in to correct students’ colloquial expressions, such as “goes into” and “split them further” in this case, she shared that it would depend on her goal for the students. At that point of the discussion, she would likely not do that as “it’s really important that students can use words they have to describe things”. She commented that, at the problem-solving stage, she would not “be too picky with the words” as it would be more important for students to be able to communicate with one another and collectively figure out the mathematical concepts first. At this stage, she would only introduce words, such as absolute value (in the previous task) and factor (in this task), when they mean “useful things” and can help students “talk about things” in their discussion. When the students were past the problem-solving stage and needed to present their ideas to a class or in writing, she would then attend to the clarity of their language use and “talk about the language that would be used to be communicated to a wider audience at a clearer level”.

In Task 5 (Graphs of rational functions), Karen first noticed how the two student characters, Ethel and Theo, were “noticing the difference between the dotted lines and the other lines” and listing the various properties of the dotted lines (asymptotes), based on their reading of the graphs. To her, it seemed that the students knew “what’s happening, they just don’t have the word for it, they’re actually reaching for the word” or the “classification” for the dotted lines in the task. She immediately shared that, unlike the previous tasks, it would be a “more clear-cut” situation where “terms in math show up”. Hence, she would step in and say, “that’s an asymptote”, as it would be “helpful to talk about it”. Other than introducing or reminding students of the term *asymptote* they seemed to be looking for, she added that she would ask students to define *asymptote* and clarify if the properties they had listed are true for all asymptotes.

In response to what the teacher character, Ms. Wilson, said in the task, Karen first expressed hesitance in doing what Ms. Wilson did, which was to correct “things they (students) have said that aren’t wrong, just maybe aren’t as precise as they could be”. She might even use terms such as “these guys” to refer to the dotted lines prior to introducing the specific mathematics term *asymptote* to “make math more approachable to students”. Depending on her rapport with the students, she would probably introduce mathematics terms gradually (instead of at once, like Ms. Wilson) while code-switching with what the students said while emphasising that what they said was not wrong. She added that “tone matters quite a bit” when introducing new or proper terms to students, so she would not be correcting students “unless they’re actually wrong”. Based on what students said, she would instead use gesturing or questioning to understand better what students were thinking first, before introducing the terms.

She further elaborated on her reluctance to “gate keep” the use of language or be “privileging certain terms over another without any obvious benefit”. She explained, “words are right because we've decided they're right, as mathematicians”. As long as what the students said meant the same idea, she would lean towards a more “casual” way of communicating in her classroom, while introducing formal language when necessary. For example, she would introduce terms when students either asked for them (e.g., asymptote) or when the terms help to define or make ideas clearer or simpler (e.g., approaching, vertical, horizontal) for presentations or written work. To her, “that’s how terms in math show up” and they are not “defined for fun”. As this was the second time Karen mentioned the use of more precise use of language during students’ presentation and written work, as compared with students’ discussions, I asked if that was her preference to how students use language in her classroom. To that, she commented that her preference would depend on “how formal (oral and written) it is and who your audience is”. She elaborated that she would often tell students that there are three “different levels of convincing people”, which include themselves, a friend and an enemy. She explained that language use would need to be increasingly precise from everyday or colloquial to more formal or mathematical when convincing oneself to a friend to an enemy. She gave an example on how it would be “too much” to “throw in nine different terms into a paragraph that is meant for someone who's not in the mathematical community".

When asked if there had been instances when she had taught formal mathematical terms explicitly (similar to what Ms. Wilson did), Karen shared what she did when teaching differential equations that day. At the start of her class, she told the students that she had to teach them a few terms, including *general solution*, *particular solution* and *arbitrary constant*, as they would be used in the subsequent lessons. However, she added that she would always "preface" the teaching of the terms or words in her class. She explained, "words sometimes exist without reasons" in the mathematics classroom and students might not know why they needed to know them. In that class, her reason to students was that she could not be "saying the solution that comes up when you're solving a differential equation and it has a constant in it" whenever she referred to the *general solution*. She further mentioned that if she had to be "pedantic about it … there's a mathematical community where you (students) should know the word that other people are using". As a teacher, she shared that it is important for her students to "feel comfortable in the mathematical community", rather than "feel alienated by words they cannot use" or not know.

In Task 6 (Diagonals of a polygon), Karen first noticed the differences in how the student characters, Aria and Bert, were thinking of *diagonals*. She explained that Aria had associated slanted lines with diagonals and hence horizontal lines are different, while Bert associated diagonals with "lines connecting corners", which included the horizontal diagonal. She wondered what might have happened in class prior to the dialogue. In response, she shared how she would first clarify what the two students meant when they used the word *diagonal* as they seemed to be using it differently. She added that, typically, diagonals are associated with squares, and thus "always show up as slanted". Hence, she would "draw that square on an angle" and ask if the vertical and horizontal diagonals were diagonals, where she predicted that Aria might disagree.

As the task was situated in an elementary classroom, she shared that she might explain to Aria more explicitly as younger children tend to "pick things up really easily" as they might not have as many (mis)pre-conceptions as the older students. Thus, she might say to Aria, "a lot of people have looked at this and all agreed that when we talk about the diagonals of polygons, we look at the lines that connect corner to corner". She would also add that a diagonal is defined “as a very specific thing” in mathematics, which is different from “what a diagonal is in everyday life”. She continued to mention that, with younger students, she would not mind if they “call that (diagonal) the D-line” or whatever they wanted at first in their discussion. However, they would need to be clear how “that (the D-line) is the thing connecting two corners and doesn't necessarily have to be slanted” though others would “call it a diagonal”. She shared that it would then be a good teaching moment to highlight the need for “certain language” in different contexts, like “in mathematics […] because it gives us ways to talk about things and know that we're talking about the same things”. She further explained that, while colloquial language could be used, “in math we do things very precisely and sometimes we define things that have a slightly different meaning, or it could be even one that contrasts”. Hence, she stressed the importance of being “clear about the language we’re using with each other because that's the basis of mathematics”.

In Task 7 (Division), Karen first commented on the similarity of the task with Task 6 (Diagonals of a polygon), where “*even* is a word that has a meaning in mathematics”. She elaborated that the student characters, Ben and Gina, appeared to be confused with each other as they were thinking of the word *even(ly)* differently. In response, she would first ask Ben to “reframe” what he meant by “two divides fourteen evenly”, focusing on his meaning of *evenly*. She commented that Ben was probably thinking of evenly in terms of “an idea of symmetry” where “two goes into fourteen, and you got a seven out of it which is a nice number or a good number, a whole number”, though she also wondered if Ben would mention *factors*. She shared that she would also explain to Gina that she was not wrong though she had misinterpreted Ben’s use of “evenly” as referring to “even and odd” numbers in mathematics. She would then stress to the students the need to be “cautious about how” evenly can be used to mean different things. When asked if she might mention to the students how “even and odd” can have different meanings within different mathematical contexts, such as functions, Karen commented that it would be age dependent. In relation to this task set in the elementary classroom, she would not mention the concept of even and odd functions as the focus was on “numbers and dividing things”.

In Task 1 (Sum of angles in a triangle), Karen first commented that the student character, Janet, was initially “pointing and using gestures” before “using more language” to communicate clearly her observation about the sum of angles in the given triangle. In contrast, she noticed that the other student character, Silas, was either seeking more clarity or trying to be “very pedantic”, to an extent of being “aggressive or showing off”, about getting Janet to be clearer in what she was saying. She continued to comment that perhaps the students were trying to practise using the words “they’ve been taught […] which is great”. Disagreeing with what the teacher character, Ms. Wilson, said, Karen shared that she would instead commend the students for using “great words” to communicate their observations and “absolutely not say a plane and the Euclidean space”. She explained, unless the concept of spaces has been discussed prior to this task, she would not consider mentioning *plane* or *Euclidean space* as it would not “actually help them clarify anything”. In her understanding, the default for elementary mathematics is the Euclidean space and these students would unlikely have worked with things that were not on flat planes. She continued to comment that she would not even say those words when communicating with her peers typically, let alone to these students in the context of an elementary mathematics classroom.

Karen also elaborated that she might only mention these words in an elementary classroom if she was doing some enrichment activities such as getting students to draw and discuss the angle sum of triangles on both a flat piece of paper and a balloon. She argued that, in that case, “names” for different spaces would probably be needed for the discussion to be “really clear”. In the context of this task, she would pick “interior” as the only specific term she would want the students to be using, as it would be needed to “classify the different types of angles” – the interior and the exterior angles. She added that, while *sum* is a word that would be useful for students to know as it is frequently used, she did not think that “*add up* is any less than *sum*”. Hence it would not matter if the students had said “add up” instead of “sum”. She further questioned if more words would “always” bring more clarity to “comprehension” in different contexts. To her, “language has to show up when there's a need for it. Otherwise, it's just extra words.”

When asked if she would consider introducing additional language as exposure for students who are interested and ready to learn more, Karen shared that she would do that “quite easily and happily” though “it is really a subjective thing”. She added that it would be “important to tune things […] to the (students’) level of understanding and ability to communicate and put (things) into context” in the learning process. For example, she might mention “Euclidean space” to very young students as some kind of “anticipatory language” which would have “value” in helping them “understand or realise that there's things beyond or like new things that are going to be coming up". In contrast, she might not expect or mention certain language when students (even if they are older) were still struggling with the concepts to be learnt. She argued that, learning mathematics "isn't about memorising things and using all the right words." While she acknowledged that "words are good for the community and to be very clear about things" as she could communicate with "a mathematician across the world without having to first define what a group is", she commented that one could learn and "do math with all the wrong words", particularly in the context of an elementary classroom when two students were discussing and learning a new concept. To her, as long as the concepts are right and there is mutual understanding of the language used, there might not be a need to use "(certain) language as an actual necessary component of communication".

In Task 8 (Operations with integers), Karen first noticed how the two student characters, Ken and Tala, were focusing on different ideas when given the problem $-2-3=?$. She commented that, Ken was doing an additive operation of two integers, while Tala was "fixated on the negatives" and recalling "a forgotten rule about negative signs". When asked if she would be concerned with Ken's use of *minus* to refer to both the operation and the sign of a negative number, Karen replied that she would not be "super picky" about that unless Ken was having difficulty in understanding the problem or concept, like Tala in this task. In terms of her actions in such a situation, Karen would first clarify Tala's understanding of what she meant by "negative and negative become positive" and perhaps use a number line to differentiate the operation of subtraction with the operation of multiplication which Tala was likely thinking about. She would also point out the difference between how the "$-$" sign can represent two different things – minus (an operation) and negative (a property) – though she would not "emphasise" that the "$-$" sign must be read in a certain way when the students were having a discussion.

In Task 4 (Fractions), Karen's first reaction was that the dialogue between the student characters, Nodo and Vick sounded "great". With an assumption that the students were aware that each green piece represented one-fifth, she commented that they were not "just randomly picking out" three pieces but basing it on the "top number three". When asked if she would correct the use of *top* and *bottom numbers* with *numerator* and *denominator*, she replied that she would not "care about language they use when they're talking about it". However, she elaborated that she would model the use of the terms or write them on the board after the students had understood the concept of fractions or if she were formally to explain the concept of fractions at some point in the lesson. She added that she would typically expect the students to know the words but not expect them to use it. She explained that words are arbitrary. For instance, if the words *numerator* and *denominator* had not been "decided" (by mathematicians) to be the terms used in describing fractions, she thought that saying *top* and *bottom numbers* would be "fine too because it's pretty clear what they are", unlike the case of a *diagonal* which may not always be *slanted*. Though, at that point, she started to wonder "Is that right? Should I? Should I be?" before sharing her thoughts that as a teacher, her "first job is to get them (students) to do math […] second job or later job is to get them to be precise, so that they can communicate to an audience".

When probed for her thoughts on whether using a precise word would be more indicative of conceptual understanding, Karen proposed a counterargument, based on her experience teaching at the university level. Going back to her example of the word "group", she shared how it might be totally possible for students to use the word in "mimicking or describing something without really getting it". She supported her argument by recalling how often she would meet students "who just use words, and they don't use them properly". For example, her students could have used the word *derivative* without connecting it to the slope of a function at a particular point. She also had students who used "the (wrong) word *derivate* instead of *differentiate*" in relation to the concept of derivative. She also had students who had given her "super-formal, intense, paragraph proofs" which were unnecessary when "all you (they) had to say was the number was positive". Hence, she shared how she would often (prefer to) ask her students to explain or re-describe what they understood about the concepts and the formal terms "using easy language or accessible language". She added that, if her students could explain in a way that even "a twelve-year-old would understand", then she could be sure that they understood the concepts involved.

Lastly, in response to what she knew about the mathematics register, Karen associated the mathematics register "with a collection of words (like denominator, numerator, group) or the language that exists within like the mathematical community or world". She reiterated her example when she taught terms related to differential equations to her university students. She commented that she had to do that in that instance because she was mindful of her role "specifically to prepare them (students)" to engage with other mathematicians and mathematical resources within a very short time duration of the course. However, if she were to teach elementary students whom she would "have for a year", she would probably "not be super-precise" about language use generally. But she would introduce language when it is "helpful" to students without being "overwhelming".

Karen added that she would also search and share the etymology of the words or break down the meaning of the words with students. She recalled a class she had on eigenvalues and eigenvectors. With the help of a German student, she was able to explain what "eigen"[20] is originally in German and her students were then able to make the connection as to why the word *eigenvector* is related to "the idea of it being its own vector". With this example, she stressed that if words could be "introduced in a good way", then "language can be really valuable". She continued to comment that she does not believe in "throwing terms at them (students)" and expecting students to use the terms without any understanding or sense of what the words mean. Towards the end of the interview, Karen shared her stand about the mathematics register and said, "I think I will introduce any word at any point, as long as it's properly motivated by why that word exists or has to exist". She gave the example of why the word *vertex* is required as it would be "awkward to say *point* (act of pointing) at the *point* (an informal way of referring to a *vertex*)". Relating back to the tasks in the interview, she highlighted *asymptote* as being one such word since "too many words" would be needed to fully describe an *asymptote.* In comparison, between *numerator* and *top number*, she ended with the comment, "I don't know. I'd use top number for a while."

5.3. *Accounting-for* Karen's Use of Language in her Mathematics Classroom

In *accounting-for* Karen's use of language in teaching and learning mathematics, I analysed her *account-of* through the lens of language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2) and language-related orientations (Prediger et al., 2019 – see sub-section 3.2.2) respectively. Here, I first present an analysis of her experience with and in managing language-related dilemmas. This is followed by an analysis of her

[20] "*eigen"* (2023) is a word borrowed from German, with a meaning "to own".

possible language-related orientations in relation to her experience with and in managing language-related dilemmas.

5.3.1. Karen's Experience with and in Managing Language-Related Dilemmas

In this sub-section, I account for Karen's use of language in the mathematics classroom by discussing what appeared to be (or not) language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2) to her in the mathematics classroom, and her corresponding actions to and experience with managing these dilemmas.

The dilemma of code-switching

From the account-of Karen's responses to the various tasks, she did not appear to face any obvious tension in relation to the dilemma of code-switching. From her interview, it is evident that she would use both the mathematics register and everyday language, and would not hesitate to code-switch between the two registers whenever necessary, in the teaching and learning of mathematics in her classroom.

Specific to how language should be used in the mathematics classroom, she often stressed the importance of meeting students at their level of comfort and readiness in using language to learn mathematics. Based on her responses to most of the tasks, if students were using mostly everyday language to make sense of ideas in a discussion, she would "definitely use the words that they've used" first while engaging with their ideas so long as the concepts are not totally incorrect. For example, although she was aware that the phrase "goes into" is a more colloquial expression for division, she would not hesitate to use the phrase in response to the student characters in Task 2 (Prime factorisation). She even suggested that she might use "goes nicely into" when referring to the concept of factors if it would help students communicate with one another while figuring out the concepts. Moreover, she articulated in Task 5 (Graphs of rational functions) that she might even refer to the asymptotes as "these guys" when talking to students, instead of introducing the term *asymptote* immediately. She explained that by introducing or talking about new or difficult concepts using everyday or informal language first would help to "make math more approachable to students".

However, when students seemed ready for the register, or if it would help with clarifying or deepening their understanding of the concepts, Karen would certainly introduce or model the use of the mathematics register while making connections to and code-switching with the language students used. An example is noted in her response to Task 4 (Fractions).

> I think, after this activity, if everyone was done with it and they figured it out, I think I would write on the board, and I would say, okay, **so when we talk about fractions, the top number is the numerator and the bottom number is the denominator.** And **because then I can say when I use the word like the denominator's five, so I would model it**.

Yet, within the same task, it was interesting to note how Karen seemingly began to have some doubts about her choice of actions. She seemed to wonder if the use of solely everyday language would be enough to help students make sense of mathematical ideas. While reiterating her stand of not correcting students' use of top and bottom numbers immediately when referring to the numerator and denominator of a fraction and only code-switch later, she began to question if that was the right action or not. Would a greater emphasis on the use of the mathematics register have made any difference in students' learning?

> And I wouldn't correct them to use bottom number and top number as denominator and numerator. **Is that right? Should I? Should I be?**

The dilemma of mediation

The dilemma of mediation appeared to be most apparent to Karen, as suggested by what she noticed in relation to students' language use (in terms of everyday language and/or mathematics register) in the various tasks. While she was able to notice the possible dilemmas of mediation embedded within most tasks (particularly Tasks 2 to 5), she generally did not seem to face much tension when thinking about how she would respond to these dilemmas. For the most parts of her account-of, she demonstrated clarity in deciding when and how much she would (or would not) mediate students' use of language when faced with possible dilemmas of mediation. Her choice of actions resided largely on the level of students' understanding of mathematical concepts and students' needs for language to communicate mathematical ideas. The only instance in the account-of which might suggest the presence of some tension in relation to the dilemma of mediation was noted in Task 4 (Fractions) where she started questioning if correcting the use of *bottom number* and *top number* would be necessary (see the dilemma of code-switching).

Mediation with respect to students' level of understanding

Primarily, the level of students' understanding of mathematical concepts would likely be a key consideration in Karen's decision to mediate (or not) students' use of language in her classroom. She would typically mediate the use of language in situations which she deemed as necessary to address students' confusion or disagreements. She would also choose to introduce or encourage the use of specific language or the mathematics register if it would help to clarify and deepen students' understanding of the concepts to be learnt. Notably, the extent to which she would mediate for language would be dependent on students' age and grade level.

Karen would likely mediate if a confusion were to arise among students due to the use of everyday language (which was ambiguous) rather than the mathematics register. An example was illustrated in Task 2 (Prime factorisation) when she observed how the student character, Vish, was not understanding what another student character, Flor, meant by the statement, "there's twos in four and threes in nine". When Flor rephrased and said that she meant to say four can be divided by two and nine can be divided by three (where the quotients are whole numbers), Vish was confused. To Vish, nine can also be divided by two though the quotient is a decimal number.

> Flor is trying to describe two goes into four, which means four can be divided by two so they're trying to describe that basically two times two is four. So, **you can factorise four into primes, but they're not using those words.** So Vish is confused and saying 'but nine can be divided by two' [...] there's **this disconnect about maybe the whole numbers and decimal numbers.**

To address the confusion, Karen would choose to have a discussion with the students about the concepts involved while introducing and reinforcing terms such as *factorise*, *divisibility*, *whole numbers* and *decimal numbers* from the mathematics register. Her rationale was that the use of these terms would bring more clarity to what Vish and Flor were thinking about respectively. Considering the context of the task, she would also expect them to understand and use these terms instead "they should be okay with those words" at the secondary level.

Karen would also tend to mediate in situations when disagreements occurred due to students' use of the same word(s) with different meanings. In such situations, she would choose to intervene as there would be a need to resolve the disagreements and help students reach a common (and correct) understanding of the concepts discussed. For instance, in Task 3 (Slope of linear functions), she noticed that both student characters were using the exact same words *small* and *number* with clearly different understanding of the two words. She attributed the difference in understanding of what *small* was to the possibility that the student characters might be thinking about the different number systems. For Paul, a small number might be referring to a small *whole number* (where small means close to zero); for Lyn, a small number referred to a small *integer* (where small means going towards negative infinity).

> Lyn is pretty clear and that when they're saying very small and they're calling negative ten very small, it's like they have an idea of **small extending towards negative infinity**, the more negative you get the more small something is […] but Paul might not. Paul might see **small as being close to zero**.

As this difference in understanding may result in misconceptions in relation to the steepness of a slope, she shared that she would lean towards introducing the idea of *absolute value* – both the concept and term – after clarifying with the student characters regarding their understanding of a small number in this case.

> […] **sometimes defining a term they haven't heard of gives them a language, with which they can speak about the thing that they're trying to understand** […] if they're really stuck on something and it's likely the absolute value thing. Like **it will be really helpful to bring that in and say here's a useful thing that we can use** and it's the idea of the *absolute value*.

She explained that, by introducing *absolute value*, it would help to clarify and enhance students' understanding that the steepness of slope is dependent on the absolute value (of the coefficient of the x-term) rather than the small number(s) to which they were referring. It would also provide them with a language where students could speak about or communicate their ideas in a more precise way and better understand one another.

Other than the instance of how a *small number* might be interpreted differently by students, disagreements might also occur when the same word has different meanings when used in everyday language and the mathematics register respectively. Karen noticed such disagreements in Task 6 (Diagonals of a polygon) and Task 7 (Division).

For Task 6, she noticed that both student characters, Aria and Bert, were using the same word *diagonal* in their conversation. But they were having some disagreement as they were using the word *diagonal* differently. For Aria, she had associated *diagonal* with the (more) everyday meaning of slanted lines, whereas for Bert, he had associated *diagonal* with the mathematical meaning of lines connecting non-adjacent vertices.

> I would **definitely pick out the word *diagonal*** and say you're both using the word *diagonal,* but I think **you're talking about different diagonals** [...] in mathematics we talk about things really clearly because it gives us ways to talk about things and know that we're talking about the same things. So, we talk about **specific contexts and we define words within those contexts** so that when we're talking about polygons, we all know what the word diagonal means in this context.

In response to the disagreement, Karen would highlight to Aria that the word *diagonal* means something more specific in the context of mathematics which is different from the everyday meaning. She further elaborated that, since the task was situated at the elementary level, she would point out the different meanings to Aria directly. Her rationale was that elementary students might not need as much convincing as older students who might have developed a certain fixed way of thinking about diagonals of a polygon having to always be lying diagonally. But, if Aria were to be a secondary student, Karen would additionally "take a square and turn it on its side" to address the misconception.

Similarly, for Task 7 (Division), Karen would mediate to distinguish the dual meanings of the word *even(ly)* as the two student characters, Ben and Gina, were again using the same word but referring to its meaning in different contexts. To elicit the difference in their understanding of the word *even*, Karen would encourage them to rephrase and explain their ideas in their own words. She would then affirm both understandings as correct and caution the students that they had used the word *even(ly)* to mean different things – a more everyday meaning of balancing or equal sharing and a specific mathematical meaning in relation to even and odd numbers. Although she is aware that the word *even* has another meaning in the context of even functions, she would not mention it to the elementary students in this task. She commented that, if the students "were older and they were closer to getting towards the even, odd functions", she might then "allude to the fact that they can mean different things even in different mathematical contexts".

In contrast, when students' understanding of the mathematical concepts were clearly not wrong, Karen would most certainly not correct their use of language even if it was less or not precise. For example, she was, in fact, pleasantly surprised at how the student characters, Nodo and Vick, in Task 4 (Fractions) could discuss fractions in the way they did, given the context set at the elementary level. Though they were clearly not using the terms *numerator* and *denominator* when referring to the fraction, she commented that they were seemingly on the same page and able to explain what they were doing with their use of top number and bottom number. Hence, she felt that there was no need to mediate their use of the language as there were no obvious misconceptions at that point.

> Okay, because the bottom number is five, we need to use the green piece, which is the one fifth, and we need three of them to get the top number three. Yeah, **I mean, that sounds great** […] That's amazing. Wow. They are in elementary grade. And **I'm assuming they know that the green one is one fifth like they're not just randomly picking them out.** I mean three of them, take the top number three. Yeah, so it means you need three of them.

Mediation with respect to students' needs for language

A secondary consideration influencing Karen's decision to mediate (or not) students' use of language in her classroom would be students' needs for language to communicate mathematical ideas. As such, she would be more inclined to mediate and "teach" or bring in the mathematics register if students had reached a certain level of understanding, and requested or required certain language to progress further in their discussion. The extent of such mediation would be dependent on how useful it would be for students to be learning the language, including new register words, at the particular state of learning and for future use. Two such instances were noted in her responses to Task 5 (Graphs of rational functions) and Task 1 (Sum of angles in a triangle).

In Task 5, Karen mentioned that she would certainly provide students with the term *asymptote* because the student characters showed awareness of the concept and had requested for the term in the task. She would thus make use of the opportunity to (re-)introduce the term and re-affirm or further their understanding of *asymptote*, in terms of both the definition and the properties. But unlike the teacher character, Ms. Wilson, Karen would be hesitant to introduce the other terms (such as *vertical*, *horizontal*) all at once. She would instead choose to bring these terms in gradually and selectively. For example, she shared how she would be more ready to introduce or use the terms *vertical* and *horizontal*, rather than *oblique.* To her, *vertical* and *horizontal* are simpler words which are also used commonly in everyday contexts, while there did not seem to be more advantage of using *oblique* instead of *slanted*. She would also choose to introduce and model the use of the term *approaching* as “it means that things are getting closer together in a really simple way” and “it makes sense”. However, she emphasised that “there’s nothing wrong with saying getting closer and closer without touching”[21] and would not insist for students to use *approaching*. She explained her stand in not “privileging certain terms over another without any obvious benefit”. Moreover, she was also mindful how “tone matters quite a bit” when introducing new (and precise) terms to students. As such, she would have preferred using a more suggesting tone rather than a correcting tone (like what Ms. Wilson did) when the students were actually not wrong in their thinking but imprecise in their articulation of the ideas.

Besides, her responses consistently revealed that Karen would unlikely over-mediate for specific language use or insist for students to be always using the most precise language. In response to what the teacher character, Ms. Wilson, did in Task 1 (Sum of angles in a triangle), she disagreed with what Ms. Wilson had said. She added that she would “absolutely not say a plane and the Euclidean space”, since elementary students would not “ever talk about anything that is not a flat plane”. Thus, the terms *plane* and *Euclidean space* would not be relevant or helpful in clarifying students’ understanding in that context. By contrast, the one term she would choose to introduce would be *interior* as “we use *interior* a lot and that’s like a good word for them to pick up”. She explained that the emphasis on the use of *interior* could actually deepen students’ understanding of the difference between *interior* and exterior angles.

> The word *interior* has a meaning to them (students) because they can look at exterior and say, ‘oh, but this is different’. And *interior* helps because they (students) have **a contrasting thing, where they need to classify the different types of angles**. But **they don't have to classify Euclidean versus other**, like that’s not a thing they're thinking about, so they don't need that. Yeah, so I think the **language has to show up when there’s a need for it. Otherwise, it’s just extra words.**

[21] Mathematically, this description of *asymptote* is flawed as there are functions that can cross their horizontal asymptotes (e.g., $f(x) = \frac{x}{x^2+1}$, $f(x) = \frac{\sin x}{x}$).

Additionally, Karen mentioned a few times during the interview that she would mediate students' language use differently, depending on the phase of mathematical discourse (problem solving *vis-à-vis* presentation) and the mode of mathematical discourse (oral *vis-à-vis* written) they were engaged in. In other words, the audience and context in which language is used would matter a lot in her decisions to mediate (or not) students' use of language to talk about mathematics. Typically, she would not be particular about students' language use when they were engaged in the process of problem solving – a time when students were talking to each other and making sense of the concept collectively. As such, the emphasis should be placed on the content and the process of mathematics talk. If she were to intervene for the purpose of 'teach(ing)' language at a point when they were completely making sense and understanding one another, she did not think that it would necessarily value-add to the students' discussion. Instead, it might disrupt or interfere with the flow in which they were focusing on.

> I'm actually **really cautious about picking apart when they're trying to like problem solve** [...] So if it is just the problem solving, I wouldn't want to be too picky about the words they're using. Because I think **the interesting thing is whether they can both agree on what's happening** [...] because I want to put the emphasis on them, like **doing it all in their brains more so than like having the exact thing (word)**.

But after the problem-solving phase, she would more likely mediate at the point when students needed to present or share their ideas to a larger audience or in written form. In those contexts, ideas shared would generally need to be more conclusive, and thus require great clarity and precision in the language use. She explained that it would be important to help the audience (classmates) understand the ideas even if they did not participate in the discussion previously.

> And then, I think there is a next level, like if I asked them to present this to a class or write something up. Then we talk about, **maybe the language that would be used to be communicated to a wider audience at a clearer level** [...] **if they were writing this down and they wrote down lying down. I might say, great, we can use the word horizontal, to be clear for lying down**, and really emphasise that the audience matters.

She also elaborated how there would be "different levels of convincing people". In other words, the level of precision in language use would perhaps differ according to who the students were communicating with – themselves, a friend or an enemy.

> And if **you're convincing yourself, use whatever words that make sense to you**, and that's fine. If **you're convincing a friend**, you can assume that they are on your side and maybe **you don't need to be perfectly precise**. But then if **you're convincing an enemy, you have to be as precise as possible and use very good words** [...] To someone who's never met any of these terms before, I would actually prefer that you use slanted, because if you throw in like nine different terms into a paragraph that is meant for someone who's not in the mathematical community, that's too much for them. So, I always emphasise who the audiences and whether that's oral or written. **It's how formal it is and who your audience is.**

The dilemma of transparency

While Karen noticed most of situations which might lead to possible dilemmas of transparency within the tasks (particularly in Tasks 1 and 5), she did not seem to face much tension when discussing her actions in response to the respective situations. She was rather decisive and articulate in her responses, in relation to when and why she would (or would not) teach the mathematics register explicitly or make language visible in the teaching and learning of mathematics. If a decision to teach language explicitly were to value-add to the development of mathematical understanding and the ease of mathematical communication, she would generally be more inclined to do so. Notably, the discussion in this section closely connects with the discussion of the dilemmas of mediation and code-switching as her decisions in managing the other dilemmas often involved the explicit teaching of language. As such, some examples discussed in relation to the other two dilemmas previously, might have been mentioned in this section again, but with a focus specific to the dilemma of transparency.

Language made visible for mathematical understanding

From the account-of Karen's use of language, it was evident that she would more readily teach the mathematics register explicitly (or make language a visible resource) if there were to be no compromise with the development of students' mathematical understanding (where language should be an invisible resource). In other words, language is likely a transparent resource (Adler, 1996, 2002) in Karen's mathematics classroom as she would strive to keep a good balance between the visibility and invisibility of language as a resource for mathematics teaching and learning.

When asked if she would ever teach language (particularly the mathematics register) explicitly like what Ms. Wilson did in Task 1 (Sum of angles in a triangle) and

Task 5 (Graphs of rational functions), Karen's response was almost a no. In her opinion, such an approach would be overwhelming for students. It would also not be a meaningful pedagogical move in helping students develop mathematical understanding. She argued that, simply introducing the mathematics register explicitly without involving students in the sense-making process of these terms (like what Ms. Wilson did) would unlikely help students develop any mathematical understanding of the concepts involved. Instead, Karen would choose to teach the mathematics register explicitly when students had acquired an adequate level of understanding in relation to the concepts discussed. For instance, in Task 3 (Slope of linear functions), she would introduce *absolute value* to enhance the student characters' understanding that the steepness of slope is dependent on the absolute value of the coefficient of the x-term, after they had clarified what each meant when saying *small number* (see the dilemma of mediation).

Moreover, Karen would selectively bring in new or illuminate certain language if they were useful in deepening students' understanding of the concepts. An example was noted in how she would mediate the use of some specific terms (and not all the terms introduced by Ms. Wilson) in response to Task 1 (Sum of angles in a triangle). To her, the use of the term *interior* was deemed as necessary as it highlights the presence of two types of angles that can be defined at a vertex of a polygon – a characteristic which is usually neglected. The contrasting nature of the words *in*-terior and *ex*-terior (both within everyday language and the register) would also help to accentuate the distinction between the two types of angles at a vertex of a polygon.

On the contrary, she disagreed with the introduction of *Euclidean space* and argued that it was an unnecessary language-use moment in the context of the task. In her opinion, mentioning *Euclidean space* did not and would not enhance students' understanding in relation to the angle sum of a triangle, which was the key concept in the context of the task. Additionally, students at elementary level (and perhaps even up to secondary level) would almost always be interacting with only Euclidean geometry, and rarely exposed to the existence of the non-Euclidean space.

> […] **generally elementary grade doesn't ever talk about anything that is not a flat plane**. So, it's not helpful. **It doesn't actually help them (students) clarify anything** […] Like I would never say, 'Oh, a Euclidean space' […] that's the default.

However, she would probably "talk about how mathematics takes place on certain spaces" after students had been exposed to related concepts in class. For example, if the students had previously contrasted the angle sum of a triangle drawn on a piece of paper and on a balloon. Even then, she would not necessarily mention *Euclidean space*, simply for the sake of using the mathematics register. On a similar note, she would not emphasise the use of *sum*, rather than *add up* in this task. While she deemed that *sum* is a useful register word for students to know, she did not think that "add up is any less than sum".

Another example where Karen would introduce a specific word with the intent of deepening students' understanding was noted in her response to Task 2 (Prime factorisation). In this case, she mentioned that she would likely introduce the word *factor* as it would be useful in clarifying the differences in how the student characters (Flor and Vish) were using the phrase *divided by*. Flor seemed to refer to *divided by* as being divisible, where the remainder is zero upon division. Vish referenced *divided by* with a more computational point of view, where two numbers can be divided to give an answer, with no condition imposed on the remainder. As such, Karen commented how introducing the word *factor* would make the concept of having no remainder clearer as the use of *divided by* may not necessarily imply that the remainder has to be zero.

Beyond introducing or teaching register words when deemed necessary, Karen also shared how she would sometimes bring in the etymology of the words to deepen students' understanding of mathematical concepts. This was illustrated in her own classroom example when she had to teach *eigen*values and *eigen*vectors – generally deemed as "weird" words by students. In that example, she had a German student to share how *eigen* was originally derived from German, meaning "own" with the class. She then noticed how the students were subsequently more appreciative of the mathematical meaning *eigen*values and *eigen*vectors, in terms of why an eigenvector is always mapped to a scalar multiple of itself with any transformation. In short, the etymology of the word *eigen* provided a connection for students to correspondingly make sense of and use the new word (and concept), which would otherwise be "weird", in a meaningful way.

> That one is cool. Eigenvectors? **When you teach eigenvectors, that's a weird, weird word for people**, like eigenvalues and eigenvectors. **Eigen means almost like unique to each.** I actually had a class one

> time where one of the students was German, and she explained what eigen meant in German and it was so beautiful. **It actually helped (other students) so much because then they had this word that meant something**, **like that had a connection to things. And then they could use it** and it wasn't just a title […] someone (a student) was like, 'oh, this is a vector that when you do anything to it, it maps to itself, it never changes direction. If I multiply that vector by any matrix or transform in any way, it stays there, you know, it stays there.' And so **the idea of it being its own vector is like an eigenvector and it feels so (amazing)** […] like all the students are like, 'Oh!' […] **So I think language can be really valuable too, if it's introduced in that good way, it can be so good, but if you're just throwing terms at them and be like, you have to use it? Oh, come on!**

Interestingly, while Karen would explicitly teach certain language to deepen understanding, her intent seemed to be primarily placed on building students' awareness of the proper language. She would not be overly concerned if students did not use the proper language in their verbal discussions even after she had used or taught the language. For instance, to help the two student characters reconcile their two different interpretations of the "–" symbol in Task 8 (Operations with integers), she would certainly highlight how the two ways of reading the "–" symbol had been used interchangeably. She would also point out how they should be read and use differently when representing an operation (minus) and a property of a number (negative) respectively through a discussion with the students.

> **'What do you (Tala) mean negative and negative become positive?'** […] I'd want them to explain it a little bit. And I'd be, 'okay, so what you're talking about is two negative numbers, and when you multiply them together, it becomes a positive.' I think I'd probably (say), 'like we'll talk about that another time, that's a different thing' […] I would want to draw negative two minus positive three equals something, and I would also write negative two plus negative three equals something and show that those are actually asking for the same thing. And **I would want to point out that one of them is minus, one of them is negative**. […] I think, minus and negative are things that I would make sure they know the difference between. Yeah, but I **wouldn't be super picky about what they use** when they're talking about it with each other.

However, as she said, she would not insist that the students read the "–" symbol in the correct or precise way amid their discussion. In other words, while she would make visible to students the precise language in referencing the "–" symbol, the precise language need not be visible in students' conversations or discussions.

Karen's stand was similar in her response to Task 4 (Fractions). She would introduce the words *numerator* and *denominator* as she would expect students to know but not necessarily use the words in their discussion.

> **I will use words, numerator and denominator, and I'll expect them to know it. But I won't expect them to use it** and not with each other, especially when they're talking like this (in the task). I wouldn't request it of them. And I wouldn't correct them to use bottom number and top number as denominator and numerator.

To her, words are arbitrary. In this instance, the words *numerator* and *denominator* had been "decided" (by mathematicians) to be the terms used in describing fractions. Otherwise, saying top and bottom numbers might just be "fine too because it's pretty clear what they are".[22] From her responses to both Task 8 (Operations with integers) and Task 4 (Fractions), Karen appeared to view language (particularly the mathematics register) as a semi-visible resource in students' learning process. The visibility of the mathematics register is needed when it enhances understanding and awareness; and not necessarily needed when it does not compromise understanding and communication of mathematical ideas.

Perhaps, Karen's non-insistence of students using the mathematics register might be a consequence of her teaching experience as a university lecturer. She had probably met many undergraduate students who used chunks of terms in the mathematics register, which might be inappropriate, or which they might not fully understand. In particular, she mentioned:

> […] this is another reason that I like using easy language or like accessible language. It's because sometimes, especially in university […] There are a ton of students who write the dumbest things, and they use all the words. They use derivative and they talk about differentiating things and it's all wrong. And they don't know what they're talking about when they use the word **derivate, instead of differentiate** because they're trying to talk about a derivative, like, that's the wrong word, it's not actually a word in this context. So sometimes **they can hide behind those words**. as well though. Sometimes they can say things that are fine, but they don't really know what a derivative is, **they're just using the word because they know it means something that matters** or that they should talk about in this context.

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[22] This argument might not work well in the case of a complex fraction such as $\frac{\frac{2}{3}}{\frac{3}{4}}$.

As such, her preference would be for students to explain the ideas clearly in their own words, using everyday language and not necessarily using terms in the mathematics register. She seemed to think that, if students could articulate their understanding in the simplest way possible, it would show that they had a clear understanding of the concepts. Again, her following comment seemed to suggest that Karen would delicately balance the use of language as both a visible resource (in terms of learning new terminology) and an invisible resource (in terms of showing mathematics understanding) in her own classroom.

> I think that's also why I tend to be, 'use your (students') language, use your words. Can you describe this thing to me, in the words, like a twelve-year-old would understand? And that's how I know that you actually understand this because I've told you what the thing is. **I've given you the terminology for it. But now that you have the terminology for it, I need you to go back and re-describe what it is for me in your words.'**

Language made visible for mathematical communication

Beyond making language visible to help students develop better mathematical understanding, Karen was also cognizant of situations where she would have to talk about or teach the mathematics register "*more*" explicitly first. Specifically, language would be needed for ease of communicating mathematical concepts or ideas and for students to become familiar and comfortable with the language of the mathematics community.

For example, she shared how she had previously introduced the terms *general solution* and *particular solution* explicitly at the beginning of a class on differential equations. Yet, she made it a point to explain to students why she needed to teach the terms explicitly. As the terms were essential key words which would be frequently used throughout the topic of differential equations, (re-)explaining what the term *general solution* meant each time she had to refer to it would probably not be effective in the teaching and learning process.

> 'I'm going to teach you a bunch of words, because I'm going to use those words for the next couple of weeks. Because I'm tired of saying the solution that comes up when you're solving a differential equation and it has a constant in it. **I'm going to say *general solution*, but you need to know that.**' [...] So, I expect them to be able to **look at a problem that says find the general solution and they have to know how to do that**. I think I have a specific awareness in this case,

> where I don't control their environment. **They're going to be engaging with other mathematical resources and part of my job there is specifically to prepare them to do that**, on a very short time scale.

Moreover, she seemed to think that it would be her responsibility (as a university lecturer) to teach her undergraduate students the necessary register words in each topic to help them understand and gain access to other mathematical materials or resources. However, she repeatedly emphasised the need to provide students with a reason or purpose for learning certain words or when language must be made a visible resource. Her stand remained that students should not feel burdened or "alienated by" the language needed to learn mathematics. In other words, language should be an invisible resource which can develop, rather than hinder mathematical understanding.

5.3.2. Karen's Orientations towards Language in Teaching and Learning

In this sub-section, I account for Karen's use of language in the mathematics classroom, by discussing her possible language-related orientations (see sub-section 3.2.2), based on her experience with and actions in managing the three language-related dilemmas. The five orientations, as proposed in the framework (Prediger et al., 2019), are discussed in sequence.

O1: Language as a learning goal in subject-matter classrooms

Although Karen emphasised the need to focus and prioritise students' development of mathematical understanding before language, language is presumedly still a learning goal in her mathematics classroom. In particular, her actions in managing the three dilemmas collectively suggest that learning to speak mathematically would be a part of learning mathematics in her classroom.

First and foremost, she demonstrates cognizance in how speaking mathematically can be rather different from speaking (a language) in the everyday context through her responses. But students may often not be aware of the differences, especially in the use of words which have different or contrasting meanings in the mathematics and everyday context. Hence, she would be inclined to teach them how to speak mathematically and ensure clarity in the understanding and communication of mathematical ideas. This is evident based on the instances when she would mediate the use of language while teaching and discussing mathematical concepts in situations involving the dilemma of mediation. For example, she would focus on mediating language when it would help students to clarify their disagreements, or to deepen their understanding in relation to mathematical concepts. She would also introduce or teach the use of mathematics register when students need them to communicate better mathematical ideas. In addition, her preference or expectations for students to be speaking or using language more precisely in formal settings and with a larger audience suggests that her students will likely be learning language alongside learning mathematics, in preparation for the use of language in those situations.

Moreover, as discussed in the dilemma of transparency, Karen shows an inclination towards thinking that it is her responsibility to teach language to some extent. For instance, in her own classroom, she had taught the terms *general solution* and *particular solution* explicitly, for the reason that these were the necessary key words which would enable students to understand and gain access to the mathematics concepts in the topic. Consequently, she would likely assume the responsibility to help students develop both understanding of mathematical concepts and the corresponding language (in particular, the mathematics register) required to talk about the concepts.

O2: Striving for pushing rather than reducing language

Based on how she managed the different dilemmas, it is mostly apparent that Karen would unlikely reduce language in her mathematics classroom. However, she would also not blindly push for the use of certain words or language without reason if there were to be no need for it. To her, “the language has to show up when there's a need for it. Otherwise, it's just extra words”. Aligned with how she would use language as a transparent resource (see dilemma of transparency), she would likely strive for a balanced and appropriate push of language when it is motivated by students’ need for understanding and their need for communication.

Karen would push for language when it helps to deepen students’ understanding or to resolve confusion between students. For example, in Task 1 (Sum of angles in a triangle), she would push for students to use the term *interior* as it is important to differentiate between interior angles and exterior angles. But she would not push for the use of *Euclidean space* since it would not “actually help them (students) clarify anything”. She would also push for language when it adds clarity to the communication of mathematics ideas between students and herself or among the students themselves. For example, in Task 2 (Prime factorisation), she mentioned that she would introduce mathematics register terms such as *factorise*, *divisibility*, *whole numbers* and *decimal numbers*, as it would bring more clarity to what each student character was thinking about. Towards the end of the interview, she added the example of the words *point* and *vertex* to illustrate when she would push for language to enhance clarity in communication. She explained she would push for the use of *vertex* as it would be “awkward” to say “*point* (act of pointing) at the *point* (an informal way of referring to a *vertex*)”.

> **I think I will introduce any word at any point, as long as it’s properly motivated by why that word exists or has to exist.** So if you want to call something a point, great. **But if you keep saying things like *point* at the *point*, eventually you’ll be, like, ‘yeah, let’s just call it something different (vertex) so that we have a nice easy word for it**.’

Moreover, Karen appeared to have different considerations in relation to the extent to which she would push for language. For instance, she mentioned a couple of times that she would prefer to introduce words and ideas that are appropriate for the specific grade level.

> **If they were older** and they were closer to getting towards the even, odd functions, **I might allude to the fact that they (the term *even*) can mean different things even in different mathematical contexts. But, at this moment when we’re talking about numbers and dividing things** [...] **definitely not in elementary grade.**

Besides students’ grade level, she would also consider the state of students’ level of understanding of the concept in the moment. If students were at a stage when they were still struggling to make sense of the idea, she would not introduce new words as it is unlikely to help reduce their confusion.

> It depends on who it is, I think, and **if they’re struggling with the ideas and they don’t (understand)** [...] **introducing a new term won't help**.

Another consideration she highlighted was the amount of time she would have with the students. For instance, would it be one full year or just a month or a week? Depending on the amount of time she would have with the students, she would moderate and scaffold the teaching of language accordingly so that it would not be overwhelming for the students. She also mentioned about the need to always provide a reason or purpose for pushing for certain words so that students do not feel burdened or “alienated by” the language needed to learn mathematics.

> Like if they’re (students) in elementary and **I know that I have them for a year**, I’m not going to be super precise about the things that they need to do. But I think your idea of ‘when do you bring it in?’ It’s still a really interesting idea and I think for me, **it’s always what is not going to be overwhelming**, like I always wanted to be helpful to them.

Consequently, Karen further talked about how she would sometimes intentionally use everyday terms that students can better relate to or are more familiar with. She might even avoid (reduce) register terms she would not deem as necessary, depending on some of her considerations. For instance, in Task 3 (Slope of rational functions), she noticed that the student characters were able to make conjectures which she would like to affirm. But she “*wouldn't say conjecture*” or push for the use of *conjecture*, based on the consideration that they were in the secondary level and not her undergraduate students. Instead, she would rather say “*that's a great guess*”. She felt that, in the context of a secondary classroom, *conjecture* might sound like an unnecessarily big word as compared to *guess* which is similar in meaning and a word that students would encounter and use in their daily conversations.

Notably, Karen’s primary focus in pushing for language in her classroom appeared to be for students to have an increased awareness of the mathematics register and not for an increased usage of the mathematics register. Her reason was that while it is beneficial for students to be aware of the register, she did not want to be the gatekeeper of language use in learning mathematics. Specifically, she would not want to limit or control the way students can use language to access and make sense of mathematics. Instead, she would try to balance the push for language such that language functions as a resource with which she could utilise to create an environment where students can be comfortable and willing to use language to learn mathematics. For example, when thinking about the terms *add up* and *sum* in Task 1 (Sum of angles in a triangle), she shared that she would certainly want students to know the word *sum* and its meaning since it is often used in mathematics. But she would be perfectly fine with students saying *add up* rather than *sum* in their discussions as the articulated meaning would still be clear and *add up* are words that students use all the time, both in their daily lives and in the mathematics classroom. Similarly, she did not see a big difference between using *slanted* and *oblique*, and therefore would not mediate if students were to use *slanted* instead of *oblique*. She reiterated that:

> […] **it can't be correcting unless they're actually wrong**. If they're saying something that's wrong, that you have to correct […] So I'm really hesitant about gatekeeping. I think when I'm teaching math, **I don't want to gate keep and say these are the correct ways and those are the wrong ways, like privileging of certain terms over another without any obvious benefit**.

O3: Focus on the discourse level rather than on the word level only

In general, Karen seems to be oriented towards the use of language at the discourse level rather than word level, as the use of language resides more in terms of having students to engage in discussion to learn mathematics (and the language when needed). This is evident in how Karen would tend to adopt a pedagogical approach which involves having dialogues and discussions with and among students in both her description of her own teaching experience and her responses to the different dilemmas. In other words, she would usually involve students in discourse practices to learn mathematics. For example, in her response to Task 3 (Slope of linear functions), she mentioned:

> **I'm going to ask what it means for a number to be small. And ask both students to answer that. Or, ideally, just to talk about it with each other and to define it clearly** and talk about 'is negative ten (-10) smaller than zero point one (0.1)?' […] and figure out what it means to be small.

While engaging students to learn mathematics through such discourse practices, students' ability to express and explain their mathematical ideas and thinking, in other words, their language, would likely be developed as well. However, Karen tended to disagree with the argument that students must always be speaking formally (using only the mathematics register) in the mathematics classroom. She would typically not mediate students' use of specific words or everyday language as long as they were using it in ways which sufficiently communicate their ideas and attain common understanding with their audience. In fact, she would prefer students to use their own words rather than mathematics register words if they do not fully understand the meaning behind the mathematics register words. (see dilemma of transparency).

For instance, in Task 5 (Graphs of rational functions), when student characters did not use the word *asymptote* to describe their observations, she was not overly concerned about their lack of the word. She was instead actively looking out for evidence showing how they were able to describe the characteristics of *asymptote* in their own words. To her, they were clearly showing understanding of the concept of *asymptote*. In particular, she said, "*it sounds to me like they know what's happening, they just don't have the word for it*". This similarly illustrates how Karen would likely put more focus on the discourse level, rather than on the word level in relation to attending to and using language in her classroom.

> So they're (students) identifying that they (asymptotes) are showing up in different orientations. And they're identifying that the **graphs go closer and closer to but don't touch or meet the lines (asymptotes). So they've identified that property**.[23]

She even mentioned, in relation to Task 6 (Diagonals of a polygon), that it would be totally fine with her if her students were to decide to invent a new word (instead of using the right mathematics register word) to represent an existing concept in their discussion. However, it would be on the basis that the students had shown a clear understanding of the concept.

> So if you want to call it the *D-line*, go ahead, but then **be really clear about what a *D-line* is, that is the thing connecting two corners and doesn't necessarily have to be slanted.**

Karen elaborated that the *diagonal* could be called the *D-line* or whatsoever if the students had unanimously agreed to do so, provided they clearly understood the properties of the *D-line*. In that case, rather than correcting the use of the *D-line*, she would instead model the use of the mathematics register by code-switching between students' language and the register when necessary to help them make the connection of their language to the register.

Perhaps, the way Karen attended to all the tasks during the interview is an additional point which supports Karen's language-related orientation that focuses on the discourse level rather than the word level. In most of the responses, she would typically share what she noticed in terms of the meaning or ideas the student characters were

[23] Again, this argument is flawed (as mentioned in footnote 21).

conveying first, based on what they said in totality, rather than the specific words they used (or not used).

O4: Integrative perspectives instead of additives only

Based on how Karen would mediate the use of language (see dilemma of mediation) and teach language explicitly (see dilemma of transparency) while teaching mathematics, she appears to adopt a more integrative rather than additive orientation when attending to language as a resource in her classroom. To her, it is highly probable that the learning of mathematical concepts is developed through language, and the learning of language is developed through the learning of mathematical concepts.

On the one hand, it is noted from the description of her teaching experience that Karen adopts a pedagogical approach which involves having dialogues and conversations with students. She expects students to talk about mathematical concepts and make sense of them through discussions. In other words, language inevitably plays a significant role in students' learning of mathematics. Consequently, students' ability to express and explain their mathematical ideas and thinking, using language, will be developed as well.

On the other hand, she indicated that the learning of the mathematics register needs to be motivated by and integrated in the learning of the mathematics. An example was illustrated in her response to Task 5 (Graphs of rational functions) which she commented as a "clear-cut" situation where "terms in math show up". In that task, the student characters knew "what's happening" in relation to the concept of *asymptote* but did not "have the word for it". As the students were "actually reaching for the word", she would provide them with the intent of helping them perhaps recall or learn the specialised term. She also commented that the term *asymptote* is a useful term to learn, as without it, "it's too many words to describe what an asymptote is". However, she would not stop with only introducing the term, and instead make use of the opportunity to ask students to define *asymptote* and clarify their understanding with respect to the properties they had listed. Integrated within the learning of the concept, she would also mediate the use of terms such as *horizontal* and *vertical* which would help students explain mathematical concepts more succinctly and precisely.

In addition, Karen expresses a view that language should be a resource that allows students access to mathematics in a comfortable manner, rather than be a deterrence. In relation to the dilemma of code-switching, she mentioned she would code-switch to relate better to students' language and help them feel comfortable with learning mathematics. She would also not over-mediate and push for language use all the time (O2). These again suggest an orientation of being more integrative rather than additive in the learning of language in relation to the learning of mathematics. From Karen's perspective, the learning or use of language has to be integrated within the learning of mathematics, rather than be taught or positioned as an additional component. Imposing language on students for the sake of language alone will make students "feel alienated" in the learning of mathematics.

O5: Conceptual understanding before procedures

As mentioned in the previous orientation (O4), Karen's preferred mathematics classroom will always be one where students are interacting with one another and her (the teacher) to talk about mathematics as they make sense of the concepts together. This suggests that learning only procedures or getting the right answers will unlikely be prioritised in her classroom.

> I leave **a lot of space for students to guide the discussion and ask questions** and you know request certain examples or things. And a lot of it is […] allow students to ask the next questions that are going to lead to the next topics, so it's **very, very interactive**.

Moreover, the various examples discussed when accounting for her use of language through the lens of dilemmas suggest that Karen places a strong focus on developing conceptual understanding in her mathematics classroom. The most apparent piece of evidence probably comes from her response to Task 8 (Operations with integers). As the task was focused on determining the answer to the problem $-2-3$, some teachers could have quickly zoomed in to correct the wrong procedures and focused on teaching how to get to the answer of -5. Clearly, that was not the focus for Karen, as evident from her response. Instead, she would first ask the student character, Tala, to clarify what she meant by "negative and negative become positive". She would also use a number line to discuss and differentiate the operation of subtraction with the operation of multiplication that Tala was likely thinking about.

Another example that possibly illuminates Karen's focus on conceptual understanding before procedures is Task 7 (Division). Similarly, she would ask the student characters to explain and discuss their different understanding of the word *evenly* before highlighting the dual meaning of the word *even*. In both instances, there would likely be back-and-forth discussions between students and her to reach a common understanding of the concepts or at least reach a better understanding of the problem. In other words, her attempt to focus on developing conceptual understanding though questioning and use of different modes of representations will also necessitate the use of language as a resource in her teaching.

Karen's focus on developing conceptual understanding in her classroom is perhaps also evident through how she uses language as a resource in her teaching. She would often share with students the etymology of words or break down the words into the root words or parts (for compound words). She would also take time to talk about what the words may mean in everyday context. All in all, she would use language to help students make sense of the register and its mathematical meaning while students make sense of new concepts. This suggests that in her classroom, the use (and teaching) of language becomes necessary and important if and only if it helps students develop conceptual understanding.

5.4. Summary

In this chapter, I presented the case of Karen to illustrate an example of a teacher who would use language primarily as a resource to develop mathematical understanding. Karen shows a high tendency in constantly seeking the balance between the visibility and invisibility of language when used as a resource in her classroom. In other words, for Karen, language can be deemed as a transparent resource (Adler, 1996, 2002) which is important in the development of mathematical understanding in her classroom.

In *accounting-for* how Karen would notice and use language through the lens of language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2), it was evident that she would mostly not face any tension though she was cognizant of the different dilemmas. With the exception of the one instance when she began to question her stand in not mediating students' use of language when they were not wrong, she was generally clear in her considerations of when to code-switch, mediate or teach language (particularly the mathematics register) in her classroom. Broadly, her key considerations were students' level of understanding and their need for language in communicating mathematical ideas. Consequently, she would typically decide to make language visible (in other words, mediate for the use of the mathematics register) if and only if the visibility of language value-adds to the development of mathematical understanding and communication.

In addition, Karen's deliberate actions in managing language-related dilemmas collectively substantiated her strong language-related orientations (Prediger, 2019 – see sub-section 3.2.2). With a focus on developing conceptual understanding through discourse practices in her classroom, the use of (and the learning to use) language, particularly the mathematics register, as a resource would be inevitable in the teaching and learning of mathematics. While she would not reduce the use of the register to disadvantage the development of mathematical understanding, she would also not overly push for its use but instead adopt an integrative perspective towards the use of language in her classroom.

In the next chapter, I present a discussion of my findings in relation to the other main category in how some teachers would notice and use language in mathematics classrooms. This is illustrated through the case of Lena who would primarily deem language as a resource for mathematics talk which may not necessarily be framed by the mathematics register.

Chapter 6.

Language as a Resource for Mathematics Talk – Mathematics Register is not Key

In this chapter, I discuss how language may be deemed by some teachers as a resource to engage students in talk to learn mathematics. To this group of teachers, mathematics talk is important as a learning process, but the language use within mathematics talk does not necessarily need to encompass the use of the mathematics register. In other words, the use of everyday language to talk about mathematics will be deemed as sufficient if mathematical ideas can be expressed and understood by students (and teachers). Amongst the participants I interviewed, Lena appeared to be one teacher who mostly noticed and used language in this way. Hence, I present her as a case to illustrate how language can be used for mathematics talk, yet without a necessary focus on the use of the mathematics register. Again, a summary of Lena's background and experience of teaching and learning mathematics is first presented in section 6.1. This is followed by an *account-of* and an *accounting-for* Lena's use of language in her mathematics classroom in sections 6.2 and 6.3 respectively. All quotations used in this chapter were taken from the interview with Lena. Similarly, phrases which related specifically to arguments made in the analysis were indicated in bold font within long quotations.

6.1. The Case of Lena

Prior to starting her Ph.D. program, Lena had taught in elementary schools (grades K–6) for more than fifteen years and worked as an instructional coach for a couple of years. Her own academic background and teaching experience resided in general elementary education and mathematics was amongst one of the core subjects she had to teach. As an instructional coach, she worked with grades K–9 teachers to help them develop professionally, depending on what they had wanted to work on. As such, her background as an elementary generalist teacher used to have limited focus on mathematics and mathematics education. Moreover, she mentioned numerous times that she was "not very good at math" during the interview and thus would struggle in discussions which required higher-level mathematical content. She further attributed her lower academic achievement in mathematics (as compared with other subjects) to her “experiences with math starting in grade six (which) were very negative”.

Her interest in mathematics education (and subsequent focus on mathematics education research) started to grow after attending a professional development session on building thinking classrooms (Liljedahl, 2020) in 2016. The session motivated her to re-think how mathematics could be taught and learnt differently by providing opportunities or “experiences” for students to learn and construct mathematical understanding through collective problem solving and discussions with peers. In fact, she mentioned how her experience as a mathematics student “would have been very different” if mathematics had been taught to her in that manner. Consequently, she started to teach mathematics differently from how she had always experienced learning mathematics in a teacher-centred classroom where the teacher was constantly “telling”. Her mathematics classroom became one which centred around the “idea of providing students with an experience first, and then teaching after”. Students in her mathematics classroom had since been working in groups to problem solve and learn mathematics together, with her guidance and teaching, when needed.

6.2. *Account-of* Lena’s Use of Language in her Mathematics Classroom

During the interview with Lena, she and I managed to discuss a total of six tasks (Task 4 → Task 8 → Task 2 → Task 6 → Task 7 → Task 1) and her understanding of the mathematics register. Tasks 3 and 5 were dropped as they involved more advanced mathematical content and thus situations with which Lena was not familiar. She also asked to clarify the mathematics concepts in some tasks (2, 6 and 8) so that she could “focus in more on the language” when attending to the tasks. In this section, I present an *account-of* what she noticed in terms of language use, and her corresponding actions and reactions to the different tasks in sequence, as well as her articulated understanding of the mathematics register.

In Task 4 (Fractions), Lena first noticed how the student characters, Nodo and Vick, were using “top number” and “bottom number” in their dialogue, and did not seem to have the formal mathematics terminology “numerator” and “denominator” at that point. She then shared her thoughts about figuring out a process of how the student characters had possibly come to represent the fraction, using fraction strips and the white strip of paper. When prompted further on whether students understood what they were required to do from what they were saying, she mentioned that she had to “make some assumptions” as “there’s not enough (information)”. Her first assumption was that the students appeared to be doing the right thing as they used the green piece (which denoted the fraction $\frac{1}{5}$) even though “it’s not specifically connecting necessarily to the fact that, that strip is the equivalent of five of those green pieces put together”. She further assumed that they had likely understood the meaning of numerator and denominator in a fraction, though “it’s just not explicit” in their speech.

In terms of her actions in such a situation, she would first step in with the intent of clarifying her above assumptions regarding what Nodo and Vick understood. To her, if she could ask questions to “push them to explicitly explain” and “verbalise” what they were thinking, she would be able to ascertain that they understood the meaning of numerator and denominator in a fraction. When asked if she would then connect their use of “bottom number” and “top number” to “denominator” and “numerator”, her response was that she would expose students to these words and be intentional in modelling the use of these words. She also shared how she would use the similarity in pronunciation of the words, “bottomed” (ending with a D-sound) and “denominator” (starting with a D-sound) to help students remember that “the denominator goes bottomed”. However, she would not insist that students used these terms though she would affirm them if they could remember and use the words in their explanations. She explained that while “the vocabulary is important”, “the understanding behind those words is more important than the use of the words themselves”. She further mentioned how “numerator” and “denominator” are words which are not commonly used in real life. As such, the use of such proper mathematical terminology by students would be “a bonus”, and not a must in her classroom. She specifically mentioned that, “unless my curriculum specifically says they have to know it, in real life, I don't necessarily think that the terminology itself is super-useful” in this situation.

In Task 8 (Operations with integers), Lena first noticed how the student characters, Ken and Tala, were “focusing on procedures” in their dialogue and seemed unsure about what they were doing. She added that Tala seemed to be “drawing from a rule they’ve been told” based on what Tala said. She also noticed how Ken and Tala were using “minus” and “negative” respectively when referring to the “$-$” symbol in the expression “$-2-3$” and one was more mathematically correct than the other, though not entirely.

In terms of her actions as a teacher in this situation, she would question Ken and Tala to explain how they came to different answers from “$-2-3$” and what they were thinking about. For Ken, her guess was he probably understood the mathematical task here, based on the correct answer he got. He was just not using the appropriate terminology, hence “the polishing of the terminology would be beneficial for Ken” in this case. As such, she would rephrase what he said and “be more formal about that” with Ken to distinguish between the naming of a negative integer and a subtraction operation to avoid confusion. In particular, she would say, “it is *minus* two, but a better way of saying that is *negative* two”, and “for the minus three, I would also use the word *subtract* instead of *minus* again to clarify, being that there's two minuses here, one is in regard to negative and one in regard to subtraction”. As for Tala, who seemed to be applying a wrong rule, Lena would focus on pointing out that, “it isn't a negative and a negative, it's technically a negative being subtracted by a positive” and that “the operation that’s happening is subtraction”. In this case, she would work on helping Tala see that “the three isn’t negative”.

In Task 2 (Prime factorisation), Lena first noticed the use of terminology “prime numbers” and “split” by the student character, Vish. To her, Vish understood the concept of prime numbers, as he used phrases such as “split them further” for the numbers four and nine which are not prime, and “nothing goes into it”[24] for the prime number five. She then added that there were not “tons more for language” to be noticed in this dialogue except for the “basic words” such as “divided, times, equals” used by the student characters, Vish and Flo. After making the comment that “maybe because my focus isn’t language, I’m going towards the meaning immediately”, she highlighted her surprise in how Vish did not connect to what Flo meant by “there’s twos in four and threes in nine”, when he was the one who first pointed out that four and nine are not prime.

In this situation, she would ask Flo to explain further, hoping that Vish would then be able to catch on to what she meant by “there’s twos in four and threes in nine”. To

[24] Again, this argument is flawed (as mentioned in footnote 19).

her, they seemed to be “on the right track […] and close to the way there” and probably “making a minor error” somewhere. In particular, she would question what Vish meant by “nine can be divided by two”. While “technically it can, it just doesn’t give you an even number or a whole number”, she suspected that Vish had made a verbal error as he probably meant to say “three” instead of “two”. She also added that she “would use the word factor in there, in association with how it should be correctly used […] again modeling that language”. When prompted if she noticed any ambiguity of the phrase “goes into”, she shared that it is an English phrase which has commonly been used for division and did not confuse students in her experience, though she acknowledged that it was ambiguous at the end.

In Task 6 (Diagonals of a polygon), Lena noticed that the student characters, Aria and Bert, were using words including *diagonals, lines, connecting, corners, pentagon* and *horizontal*, which were specific to the discussion of diagonals where they were having. She also mentioned that there were “a lot of pointing that’s important here” – the gesturing which accompanied the use of language in the dialogue. She added that the pointing suggested to her that Aria was looking at what were on “the inside of the shape”. But she was not sure if Bert was referring to the diagonals on “the internal part of the shape” or the edges of the pentagon instead when he said, “there’s five lines are connecting the corners”.

In this instance, she would ask Aria to show her and Bert where the four diagonals were by pointing or tracing those lines, rather than explaining in words. She would concurrently ask Bert to show her which five lines he was referring to. If he was indeed thinking about the five edges, she would correct him by introducing those lines as being “on the outside edges of the pentagon” and thus not diagonals. As for Aria’s claim, that the horizontal diagonal is not a diagonal, Lena would then explain to Aria that diagonals in mathematics need not be “actually on a diagonal, like it’s not on a slope” – a more everyday interpretation of diagonals. She further commented that Aria would likely not be the only one in the class with this misconception. This might even be “a teachable moment” for her class if she could find a student who understood the word diagonal mathematically to explain to the class before stepping in to reiterate the mathematical meaning of diagonals.

When asked if there were other similar examples of words with multiple meanings in everyday and mathematical contexts, like the word *diagonal*, she could not think of any. However, she shared how she had previously made use of the everyday meanings of “faces” and “edges” to help students figure out that “vertices” is “the math name for corners” in the context of 3D objects, without first defining these terms. As “faces” and “edges” are common English words, her students were mostly able to identify the “faces” and “edges” in the context of 3D objects, and thus deduce that the “corners have to be the vertices”. She also added how she would be “planting seeds” by introducing and using proper terminology more consistently in connection to students’ descriptions in everyday language, when necessary. Two examples she gave were “vertical and horizontal” as students “often use up or down” instead. As students would “need to know them eventually”, she would use the proper terminology even when they were not specifically taught or required at a certain grade level.

In Task 7 (Division), what Lena noticed immediately was how the student characters Ben and Gina were “using the words, even and odd […] for different things” – Ben was focusing on the division process and Tala was focusing on the answer (seven being odd). According to Lena, though they were both correct in the use of the words from their respective descriptions, they were not understanding each other in relation to the same mathematical context and needed clarification on how they were using the words. As such, she would first affirm what Ben and Gina were saying respectively before getting them to explain what they meant in their use of “evenly” and “odd”. She also mentioned that she would address Ben’s “weird” way of saying “two divides fourteen[25]” as “that’s backwards” in terms of what was being divided and would result in a fraction. Hence, to ensure that both Ben and Gina understood what were going on with their respective use of “evenly” and “odd”, she would then say, “seven is odd, but do you understand what Ben is saying when fourteen divides evenly? If you divide fourteen by two, it ends up with an even number like it divides without a remainder”.

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[25] As an afterthought, “two divides fourteen” is an expression which would commonly not be used in an elementary classroom though this example was inspired by an example in work with pre-service elementary teachers (Zazkis, 2002). The expression “two divides fourteen” in Task 7 (Division) was intended to be analogous to “$2|14$”, where 2 is a divisor of 14, in relation to the concept of divisibility in number theory.

She further commented that students often do not understand the meaning of the words “even” and “odd”, as they tend to be told “if it ends with this, this or this in the one’s spot, it's even and if it ends with this, this, or this in the one’s spot, it's odd”. Instead of telling them about “even” and “odd” (which she had done in the past), she would now provide students with “an experience to develop the understanding”. Specifically, she would connect the idea of sharing things “fairly” (something that children do in real life) to division and how even means “distributes equally and odd means that it doesn’t”.

In Task 1 (Sum of angles in a triangle), Lena noticed that the student characters, Janet and Silas, both understood what they needed to do in the task. While Janet was not using words such as “angles” or “interior”, “she's communicating her understanding here of what's happening effectively by pointing”. To Lena, she was clearly showing her understanding of the three angles she was considering with pointing gestures although it would be better if she had used specific terminology as well. When focusing on what Silas was saying, Lena shared that she was amused as “Silas could be the teacher”. In this case, “Silas’ job” was to help Janet be more specific in her explanations. In response to what the teacher character, Ms. Wilson, said, Lena commented, “okay, it's more specific, which is great. Who says this? Like, really?”. She explained that her priority as a teacher is for students to have and show clear understanding. Hence, while what Ms. Wilson said might be helpful for Silas in becoming even more “specific in his language”, “it’s secondary to the understanding” which Janet was already showing. She further mentioned how “a lot of mathematics occurs on paper, and is individual quite often, it isn't something that's necessarily verbalised out loud unless you're working with other people in the context of a classroom”. With that, she reiterated that while the ability to use proper terminology and language is “obviously the ideal”, to her as a teacher, that “isn’t necessarily the first priority”.

Lastly, in response to what she knew about the mathematics register, Lena shared that she would think of it as “the language that’s used in mathematics”. And the mathematics register would include common words such as “diagonals” with specific mathematical meanings which differed from everyday usage; and symbols – “different things in mathematics that represent ideas”. She added how she had read that the mathematics register “can be its own language” and how it could prevent people from accessing or working with them if they are unable to read them. Referring to Pimm’s

(1987) book which she had read, she recalled an example of how by reading "×" as "of" instead of "times" helped her make sense of why multiplication could result in a smaller (instead of larger) answer in the context of fractions. Yet, when asked if teachers' and students' understanding of mathematics would be deeper with a greater familiarity of the mathematics register, she disagreed. To her, "understanding language, and what it means, doesn't necessarily mean you're going to understand processes that are involved in mathematics" and thus these were "two different things". She further explained that, while the mathematics register would allow students easier access to mathematics, being able to "label things with names and use those names doesn't necessarily mean we understand the mathematics or the processes that underlie them".

6.3. *Accounting-for* Lena's Use of Language in her Mathematics Classroom

Like Karen, I first analysed and accounted for what Lena noticed in terms of language use and her corresponding actions and reactions in specific teaching and learning situations through the lens of language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2). However, I had difficulty accounting for her use of language in terms of language-related dilemmas as she did not seem to face tensions when using language (particularly the mathematics register) in her classroom. This was probably due to her primary focus on students' meaning and understanding through their explanations, which does not necessarily need to include the mathematics register. In other words, thinking about how the mathematics register can be used for teaching in mathematics classrooms was not a key concern from Lena's perspective.

While I could hastily conclude that Lena did not face any language-related dilemmas in her mathematics classroom for that reason, it might not be a fair analysis as the task design in my study was centred around the use of the mathematics register and everyday language, leading to different language-related dilemmas. Hence, I decided to analyse her account-of through the lens of language-related orientations (Prediger et al., 2019 – see sub-section 3.2.2). I also decided to take a step back and paid attention to the general construct of language as well, when trying to understand her language-related orientations. With less emphasis on language framed by the mathematics register, I was able to account for her use of language through the lens of orientations.

The analysis of her language-related orientations consequently helped explain why she did not seem to face any language-related dilemmas.

Thus, in accounting for Lena's use of language in teaching and learning mathematics, I first present the analysis of her language-related orientations. This is followed by the analysis of her (non-existent) experience with language-related dilemmas, which seemed to be a corollary of her language-related orientations.

6.3.1. Lena's Orientations towards Language in Teaching and Learning

In this sub-section, I account for Lena's use of language in the mathematics classroom, using the lens of language-related orientations (see sub-section 3.2.2). The five orientations, as proposed in the framework (Prediger et al., 2019), are discussed in sequence.

O1: Language as a learning goal in subject-matter classrooms

Language appears to be a learning goal to Lena in her mathematics classroom, but it is certainly not the primary learning goal for her students. When sharing about her teaching experiences, she described her teaching approach as one that would be experience-based where she would seek to provide opportunities for students to make sense of mathematical concepts. This might be through working through tasks or problem solving in groups, which thus indicated that her students would likely be engaged in mathematics talk with their peers to learn mathematics. For example, a task which she had done with grade two students, who were learning about odd and even numbers, she described the following:

> I gave them bags full of stuff, like counters and things, and they had to decide if they could share it evenly, like fairly with a friend. And if you can share it fairly, the numbers are even, and then we tried to link to 'Okay now keep track. What are you noticing? What numbers? How many are there that are even?'. And then trying to get them to notice the patterns. 'How can we tell which ones are even, based on the numbers?'

Based on her description, students would need to work in groups to explain and discuss what they noticed in the experience, and in the process learnt what odd and even numbers are. As such, she would generally be demanding students to use language to make sense of the mathematical concepts in her classroom, and it is plausible to assume that students might also learn language while learning mathematics.

Moreover, in her responses to what she would do in response to the tasks during the interviews, Lena would mostly start by asking students questions to check their understanding. Her actions, which aligned to her experience-based teaching approach, would again require students to use language to explain their thinking, as seen in her response to Task 8 (Operations with integers).

> So, if I came in as the teacher, I would try and **question them about why each thought what they did.** So 'Ken, why do you think that minus two minus three is minus five? How do you know? Why do you think that?'

Yet, during the interview, there were several instances when she stressed how being able to use "the proper terminology and language" would be "ideal but […] isn't necessarily the first priority" for her as a teacher. In her opinion, there was also no correlation between students' familiarity with the mathematics register and their understanding of mathematics. Hence, while language seemed to be a means for Lena and her students to mathematical understanding, she would unlikely demand students to learn "the proper terminology and language", or the mathematics register in her mathematics classroom. In short, she would be more inclined towards seeing language (particularly the mathematics register) as a bonus learning goal and not a key learning goal. In her classroom, the key learning goal would always be mathematical understanding.

O2: Striving for pushing rather than reducing language

From the interview, there were several examples of evidence which supported how Lena would likely push for general language use, but probably neither push nor reduce specific language or register use, in her mathematics classroom. From the perspective of general or more everyday language use, Lena would probably push for it due to her experience-based teaching approach. As mentioned, her approach required students to first discuss and work in groups, and thus use language to learn mathematics. She would also step in with questions to prompt further discussions or refined explanations, again requiring using language, whenever necessary. One such example was noted in her response to Task 4, where she mentioned how she would ask Nodo and Vick questions to "push them to explicitly explain" and "verbalise" what they were thinking. To her, this was a necessary step to take before she could ascertain if they had understood the meaning of 'numerator' and 'denominator' in a fraction. Hence, in a way, Lena would likely focus on the use of discussions to facilitate the learning of mathematics, but not necessarily focus on promoting the use of mathematical register in her classroom.

Consequently, her orientation in this aspect seemed to be more nuanced when it came to the perspective of mathematics register or specific language use. From this perspective, it seemed that there was no strong orientation in terms of whether she would push or reduce such language use. During the interview, Lena repeatedly mentioned how she would model and use words in the mathematics register in connection with students' language. One example was noted in how she would introduce "numerator" and "denominator" in Task 4 (Fractions).

> I'd say, 'Oh you know in math, we have a name for that top number, and it's called a numerator [...] with the bottom number, the math name for a bottom number is a denominator'.

However, she elaborated that she would not insist that students use these words nor correct their use of top and bottom numbers in their discussions. Her modelling of the use of the register was primarily for exposure to and awareness of the proper terminology when referring to the numerator and denominator. Again, she placed greater emphasis on the development of students' mathematical understanding rather than the development of their language ability in mathematics (linked to O1).

> To me, **what's more important is the understanding** of what they are and what they mean. And if they can attach the name to them, the proper mathematical terminology, that's a bonus.

Hence, as long as the students could explain their understanding of fractions clearly, it would not matter to Lena if they were using "top number" and "bottom number" instead. Although she might not push for students to use the register words, she would make it a point to acknowledge and affirm them when they use it. For instance, in Task 4 (Fractions), if Vick were to use the word "numerator" subsequently in his explanation to others, she would encourage him by saying, "Nice job, I love the way you remember the word *numerator*!". She explained that as "kids respond well to positive feedback", her affirmation (rather than insistence) might motivate students to remember and use the words in their explanations more often.

While Lena would unlikely push for students' use of the mathematics register or specific language use in her classroom, she similarly would not reduce such language use, when necessary. For example, in response to Task 6 (Diagonals of a polygon), she suggested that she would introduce the word 'edge' to Bert whom she noticed as having difficulty in differentiating between edges and diagonals.

> [...] but even if it was grade two, I would introduce the [drawing the outline of the pentagon in the air] if Bert is tracing the outside edges, I would say, 'Oh, you know the lines that you're referring to right here?' [...] like it would be edges.

To Lena, that situation constituted one which would necessitate the use of specific mathematics register, rather than reducing it, as it would have helped Bert clarify his (mis)understanding. Interestingly, she also shared her inclination not to reduce the use of the mathematics register in a seemingly contrasting situation – when students clearly demonstrated understanding and were able to explain what they meant. This was evident in her response to Task 8 (Operations with integers), where she mentioned that "the polishing of the terminology would be beneficial for Ken, if Ken already understands what's going on". Her concern was that his inappropriate use of terminology which might lead to the potential confusion between the operation of subtracting two, as compared to the property of the number being negative two. In this situation, she would choose to "be more formal about that" with Ken to distinguish between the naming of a negative integer and a subtraction operation to avoid confusion.

> [...] for Ken, that definitely would be beneficial as the teacher to kind of go in and just clarify and encourage Ken to **use the word *negative* instead of minus two**. And, I would do it for the minus three, I would also use the word subtract instead of minus again to clarify, being that there's two minuses here, one is in regard to negative and one in regard to subtraction.

While Lena's motivation may differ, there would be situations where she would certainly not reduce and use the mathematics register in her teaching. To her, these moments would provide the opportunities for her to "plant those seeds and be intentional about using those words", even when they might not be required in the curriculum.

O3: Focus on the discourse level rather than on the word level only

With an experience-based teaching approach, Lena's classroom would generally be more discourse-oriented, where students would be involved in discourse practices to learn the language together with the mathematics (cf. O1). This suggests that Lena's language-related orientation would probably be more focused on the discourse level rather than the word level only. For instance, to help Ben and Gina, who were confused with the terms "odd" and "evenly" in Task 7 (Division), she would also prompt a discussion for them to explain so that they could understand each other. To her, this would be a better learning experience for students, in contrast to her direct teaching of the terms "evenly" and "odd" without involving any student discussion. In the event when the direct teaching of terms was inevitable, she would introduce or teach the specific mathematical terms through explanation with the context of the class discussion, and not simply taught as vocabulary (as discussed in O2).

While Lena's general teaching approach supported her likely focus on the discourse level. rather than the word level when using language in her classroom, it was interesting to note how her orientation in this aspect, at a more personal level, could possibly be contradictory to her practice. Throughout the interview, when asked what she noticed about language in the different tasks, she would always attend to the use of specific words first before commenting on the dialogue as a whole. This hinted at a focus on word level rather than discourse level in terms of her orientation. Moreover, in her response to Task 1 (Sum of angles in a triangle), she made a comment which suggested that her focus on incorporating discourse as part of her practice may not be a true reflection of her personal belief regarding the use of language in teaching and learning mathematics.

> […] a lot of mathematics occurs on paper, and is individual quite often, **it isn't something that's necessarily verbalised out loud** unless you're working with other people in the context of a classroom…

By highlighting that mathematics was often done individually, with pen and paper, and would not necessitate the use of verbal language, she seemed to be contradicting her focus on discourse practices in her classroom. Perhaps, in her experiences, discourse was hardly a common practice in the discipline of mathematics, and thus there would be no need for a discourse-focused orientation when learning mathematics.

O4: Integrative perspectives instead of additives only

Considering the more general construct of language, Lena seemed to have a more integrative perspective towards language use in her classroom. With a discourse-oriented classroom to learn mathematics, learning to talk about mathematics (and thus language) would naturally be a part of learning mathematics, instead of something additional that students need to learn outside her classroom.

However, when attending to language as framed by the mathematics register in the analysis, Lena's additive rather than integrative perspective towards language use in the classroom became evident, as illustrated by her responses to some questions during the interview. As mentioned in O2, she would not insist on students' use of the mathematics register in their discussions or explanations. When prompted to explain why, she had two reasons. One reason was the lack of practical usage for the mathematics register words in everyday life. The other reason was her disbelief in how these words would be useful in developing or enhancing students' mathematical understanding. In her response to whether she would emphasise the use of *numerator* and *denominator* in Task 4 (Fractions), she specifically questioned and doubted the practicality of these terms in everyday life, as well as helping students learn mathematics.

> […] unless you're teaching this to little kids, **when do you ever use the words**, numerator, denominator?" […] unless my curriculum specifically says they have to know it, in real life, **I don't necessarily think that the terminology itself is super useful**.

In addition, she often referred to the mathematics register words as names or labels for mathematical objects or concepts during the interview. For example, vertex is "just the **math name** for a corner on the shape", and "the **math name** for a bottom number is a denominator". This suggests how she seemingly perceived formal terminology or the mathematics register as being peripheral to the corresponding mathematical objects or concepts which they describe. As such, Lena would likely see the learning of the mathematics register as additive rather than integrative in the learning of mathematics. Towards the end of the interview, she mentioned:

> […] just because we are able to **better label things with names and use those names doesn't necessarily mean we understand the mathematics** or the processes that underlie them.

Her opinion that there was no correlation between a greater familiarity with the mathematics register and student understanding of mathematics further substantiates her additive rather than integrative perspective towards the language use in her classroom (cf. O1 and O2 on how she would focus on students' understanding rather than the language usage).

O5: Conceptual understanding before procedures

In the interview, Lena would almost immediately attend to what the student characters understood (or not) in relation to the different mathematics concepts discussed in the tasks, after highlighting what she noticed in terms of terminology used. For example, in Task 2 (Prime factorisation), she commented how she was "going towards the meaning immediately" and elaborated on what she thought Vish and Flo understood about prime numbers and division and why they were seemingly not understanding one another. In that instance, she mentioned how she would ask questions (as in most tasks) and get students to elaborate on their thinking to help them develop understanding.

Moreover, she would attempt to differentiate the types of understanding – procedural versus conceptual – when describing her thoughts on students' understanding. For instance, in Task 8 (Operations with integers), she highlighted how Ken and Tala were "focusing on procedures" and thus seemed unsure about what they were doing.

> [...] the first thing that I'm noticing is that **they're focusing on like procedure** there. To me, **there isn't necessarily understanding behind this**, they're trying to draw from a rule that they've been told.

While this example might suggest how Lena likely valued conceptual understanding more than procedural understanding, she rarely related or connected "understanding" of any kind to the use of mathematical language throughout the interview.

6.3.2. Lena's Experience with and in Managing Language-Related Dilemmas

In this sub-section, I account for Lena's use of language in the mathematics classroom by discussing why she did not seem to face any language-related dilemmas (see sub-section 3.3.2), as proposed by Adler (1996, 2002).

The dilemma of code-switching

In general, Lena did not seem to face the dilemma of code-switching, as she did not express any concerns or struggles in changing between students' everyday language and the mathematics register. In fact, she mentioned how she would always connect the usage of everyday language with the mathematics register. For instance, she mentioned how she would be introducing and using proper terminology more consistently in connection to students' descriptions in everyday language, when necessary. In particular, she described how she would "go back and forth a bit with them, with what they (students) commonly use" or code-switch between everyday language and the mathematics register. Considering her key learning goal (O1) of mathematical understanding, rather than language (particularly the mathematics register), she would unlikely be overly concerned with how using everyday language may compromise the learning of the mathematics register. It was thus not surprising that she did not seem to face the dilemma of code-switching as there would not be any tensions between the use of everyday language and the mathematics register in her classroom, as long as "they (students) can explain their understanding in a way that's clear".

The dilemma of mediation

Similarly, Lena did not seem to experience the dilemma of mediation. As evident in her actions/reactions to all the tasks, she would not hesitate to mediate and validate students' meanings regardless of how they were articulated. For example, she was not overly concerned with students' use of top and bottom numbers instead of numerator and denominator when describing fractions in Task 4 (Fractions). Rather, her focus resided on whether they were able to explain the meaning or their understanding behind those words. In other words, even when students were not explicit or specific in their articulation of ideas, Lena would affirm their ideas as long as they were clear (to her and their peers). She would unlikely mediate for further explanation or the use of the mathematics register as she would not see the need to do that, as apparent in her reaction to what Janet said and did in Task 1 (Sum of angles in a triangle).

> […] **she doesn't need to say the word interior** in that moment **because she's pointing to it**. Just like in the first one, does she really actually have to say angles because she's pointing to them? I mean it's better that she does, don't get me wrong, but the understanding, **she's communicating her understanding here of what's happening effectively by pointing.**

Notably, Lena had acknowledged Janet's pointing as a clear demonstration of her understanding and did not see a need for both Silas and Ms. Wilson to push for a more explicit explanation in this case. Her response seemed aligned to her orientation that would not push for the use of mathematics register (O2) and greater emphasis on discourse instead of only word level (O3).

In addition, based on her response specific to the mathematics register at the end of the interview, she mentioned how exposure to the mathematics register would help students access mathematics. Yet she did not view the mathematics register as a resource that would help deepen students' understanding. While she might not face the dilemma of mediation with regard to the usage of the mathematics register in students' articulation of ideas, she shared her tension in thinking about the role of the register in learning mathematics.

> It's a weird thing because on one hand, **it doesn't necessarily make it mean that it's deeper**, but **if they don't have enough of it, then it's going to block them out and they're not going to be able to access it at all, or on a very superficial level**, possibly.

The dilemma of transparency

Finally, Lena did not experience the dilemma of transparency in her classroom as she would usually teach or introduce the mathematics register *after* she had ascertained students' understanding. For instance, in Task 8 (Operations with integers), she would rephrase what Ken said to remind him of the need to distinguish between the naming of a negative integer and a subtraction operation to avoid confusion, since he already has understood the concept. Similarly, she would only bring in the terms *numerator* and *denominator* in Task 4 (Fractions), on the condition that the students had demonstrated understanding of the "top number" and "bottom number". Moreover, when certain terms in the mathematics register are more intuitive in nature, Lena mentioned how she saw no need explicitly to teach or define them. For example, in one of the lesson examples she shared about "faces" and "edges", she highlighted how students were intuitively able to identify and understanding the meaning of "faces" and "edges" in the context of 3D–objects by making connections to their everyday usage of these words even though she did not "define what those are".

Looking at the situations when Lena would teach or introduce the mathematics register explicitly, her decision to whether and when to teach the mathematics register is rather clear. She would do so only when there is minimal interference to the development of students' understanding. Her decision could be a result of her additive orientation (O4) towards the learning of the mathematics register and her orientation towards seeing the learning of the mathematics register as a bonus, rather than key learning goal (O1). In other words, the introduction or teaching of mathematics register would tend to be an add-on which would unlikely compromise the development of students' mathematical understanding in her classroom. Consequently, there would be no visible tension for Lena, in terms of the dilemma of transparency, as students' understanding would always take priority.

6.4. Summary

In this chapter, I presented the case of Lena to illustrate an example of a teacher who would use language primarily as a resource to promote and engage students in talk to learn mathematics. Lena believes strongly in the importance of engaging students in collective problem solving and discussions to learn and construct mathematical understanding. Hence, the use of language for mathematical discussions in her classroom would be inevitable.

In *accounting-for* how she would notice and use language through the analysis of her orientations, it was evident that she would primarily focus on students' meaning and understanding through their explanations. While she would certainly push for mathematical discussions in her classroom, she would not necessarily emphasise or integrate language framed by the mathematics register in her teaching, as that would be deemed as a bonus. Her language-related orientations also explained why she tended to notice the use of specific register words such as *prime*, *diagonals* and *interior* in relation to language use in the different tasks. Notably, her noticing of language use generally stopped at the word level and it was generally not directly connected to her noticing of students' understanding throughout the interview. As such, she would mostly not attend to or mediate how the mathematics register (or the "math name", in Lena's words) was being used or not used in the explanations, unless further prompted to do so.

In other words, she would not be overly concerned when thinking about how the mathematics register can be used for teaching in mathematics classrooms. Consequently, her orientations towards the use of the mathematics register substantiated why Lena would unlikely face any language-related dilemmas in her mathematics classroom. To her, decisions regarding code-switching, mediating or teaching language would be made only after she ascertained students' understanding and when there is no compromise to the development of students' understanding, the priority in her classroom. Therefore, there would not be any tensions for Lena, when faced with the respective dilemmas.

In the next chapter, I turn to the findings in relation to the second focus of my analysis – to exemplify teachers' knowledge of the mathematics register in relation to using the mathematics register as a resource in their classrooms.

Chapter 7.

An Exemplification of Teachers' Knowledge of the Mathematics Register through Three Tasks

In this chapter, I provide an exemplification of teachers' knowledge of the mathematics register through analysing all eleven[26] participants' responses to three specific tasks with respect to the four dimensions of the Mathematics Register Knowledge Quartet (Lane et al., 2019 – see sub-section 3.3.2). In particular, I chose to examine, in greater depth, the teachers' responses to three tasks (see sub-section 4.2.2) – Task 6 (Diagonals of a polygon), Task 8 (Operations with integers) and Task 4 (Fractions). Notably, the participants' responses to these tasks were the most varied, in terms of what they attended to in the respective teaching situations designed with language-related issues, and their corresponding articulated *knowledge-in-action* and *knowledge-in-interaction* in relation to the mathematics register within each task. The three tasks also provide a greater scope for discussion about teachers' knowledge of the mathematics register in relation to two broad areas of mathematics – geometry and numbers – the content focus in most elementary and secondary mathematics classrooms. While Task 8 (Operations with integers) and Task 4 (Fractions) both focus on the use of mathematics register in numbers, they are two separate topics which are based at different grade levels.

In the following sections, I discuss how teachers' knowledge of the mathematics register may look like or be lacking in the four dimensions of the Knowledge Quartet with respect to each of the three tasks. The intent here is not to evaluate or to judge teachers' knowledge of the mathematics register, but rather to explore the possible relationships between teachers' knowledge and what they would notice and attend to, in terms of the use of the mathematics register for mathematics teaching and learning. For ease of reference, the three tasks (described and discussed in detail in sub-section 4.2.2) are repeated at the beginning of each section to set the context for the discussion and to distinguish the names of the student characters from the participants. The discussion is substantiated with the participants' responses to the respective tasks, where relevant.

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[26] Refer to Table 4.6 in section 4.3 for the list of the eleven participants mentioned and quoted in this chapter.

Like the findings of Rowland et al. (2005), certain responses were noted to overlap in the discussion of two or more dimensions of the Knowledge Quartet as knowledge in one dimension may relate to or build on knowledge in another dimension. While the examples discussed in each section are specific to concepts related to the presented task, the discussion appears to have some resemblance across the three tasks, particularly in terms of how the participants' knowledge of the mathematics register played out within the respective dimensions.

7.1. Diagonal or Diagonal(ly)?

Task 6 (Diagonals of a polygon) was designed to illuminate the confusion students may have with the two different meanings of the word *diagonal*, depending on the context it is used. Specifically, the student characters in the task (as follows), Aria and Bert, associated the word *diagonal* to the two different meanings respectively. In the task, Aria counted only four diagonals in the pentagon and rejected the diagonal which was horizontally oriented as she was thinking of the everyday meaning of *diagonal* (as an orientation). In contrast, Bert appeared to be using the mathematical meaning of *diagonal* (as the line segment connecting two non-adjacent vertices of the given pentagon).

Context: **An elementary mathematics classroom – a lesson on diagonals of a polygon where students were working on an activity to identify the number of diagonals in a given regular pentagon.**

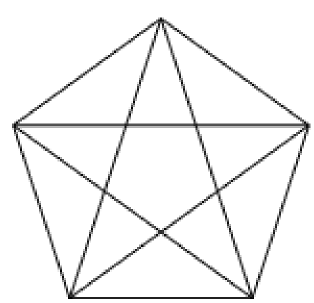

Figure 7.1. Diagram of the pentagon given in Task 6

Aria: So how many diagonals do we have here *[pointing to the pentagon]*? Hmm, one … two … three … FOUR *[counting only the slanted diagonals in the pentagon]*! There are four diagonals in this pentagon.

Bert: But there are five lines connecting the corners of the pentagon. Is this not a diagonal *[pointing to the horizontal diagonal]*?

Aria: Yes, it's not because it's horizontal!

In this section, I discuss how teachers' knowledge of the word *diagonal* in the mathematics register may be (or not be) demonstrated within the four dimensions of the Knowledge Quartet (see sub-section 3.1.2) respectively.

7.1.1. The *Foundation* Dimension

From the responses to this task, all interviewees seemed to be cognizant that the term *diagonal* (of a polygon) resides in the mathematics register as a noun with a specific meaning in the mathematical context, that differs from how the term *diagonal(ly)* is more commonly used as an adjective in the everyday context. They were generally able to describe what a diagonal is in relation to the pentagon example discussed in the task. However, some possible gaps were observed in their knowledge of *diagonal* as a noun in the mathematics register, within the *Foundation* dimension.

Knowledge and understanding of the mathematics register

Notably, the precision in how the interviewees defined a diagonal (of a polygon) varied. Some interviewees (e.g., Joey and Mindy) were able to articulate the definition more precisely as a line segment connecting non-adjacent vertices of a polygon, though they have not used the terms *non-adjacent* and *vertices* explicitly. But inferences could be made to these terms. For instance, they used terms or words like "next to each other" or "consecutive" and "not next to each other" or "opposite" when referring to the *adjacency* and *non-adjacency* of the vertices respectively. And the term "corners" was often used in place of *vertices*.

In contrast, some other interviewees were not as clear in their definitions. They often missed out on the non-adjacency of the vertices in defining a *diagonal*. This is a key aspect which distinguishes the *diagonals* from the *edges* of polygons, as *edges* are also lines which connect vertices. Instead, they focused on explicating how *diagonals* are lines that connect vertices *in* the polygons. For example, Lena described *diagonals* as line segments connecting the vertices that are in "the internal part of the shape", while *edges* are the lines lying on the "outside" of the shape[27]. Similarly, Cass described a *diagonal* as "anything" that "crosses the interior and connects two corners". Though such descriptions or understanding of *diagonals* may seem sufficient in the case of convex polygons where all diagonals lie on the interior of the polygons, they become inadequate in the case of concave polygons. For concave polygons, some *diagonals* may lie on the exterior of the polygons (as illustrated in Figure 7.2) or some may lie on both the interior and exterior of the polygons (as illustrated in Figure 7.3).

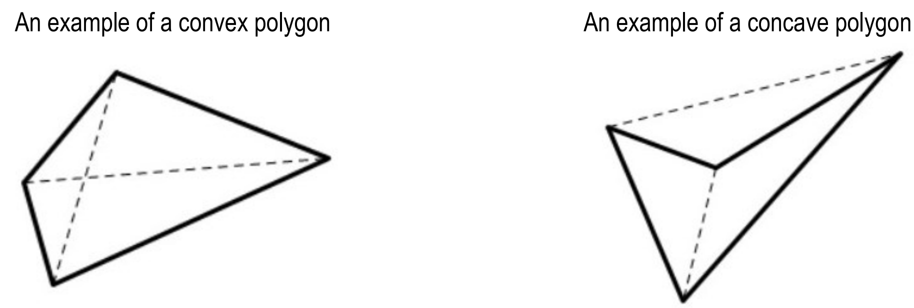

Figure 7.2. Examples of diagonals of convex and concave polygons
Note: Diagonals of the polygons are indicated by the dotted lines.

Moreover, it was noted that the term *diagonal lines* was often used by some interviewees when they were referring to *diagonals.* This may have inevitably reinforced the understanding that diagonals are always diagonally orientated, which is not necessarily true in the mathematics register (also illustrated in the example of the pentagon in the task). When describing *diagonal lines* (in the everyday context), these participants also associated these lines with the concept of slope, though some were more accurate in their descriptions of the association than others. For instance, Lena described diagonal lines as lines with slope (in a more everyday sense, in other words, inclined), while Mindy characterised diagonal lines as lines that "have a slope, other than zero, or undefined" to differentiate them from horizontal and vertical lines respectively.

Awareness of differences between the everyday and the mathematics registers

As a word in everyday context, *diagonal(ly)* is commonly used as an adjective to describe an oblique direction, such as *diagonally* across the junction. This everyday definition and usage of *diagonal(ly)* often gets extended by students and teachers to the

[27] Note that "the internal part of the shape" and the "outside" of the shape are flawed descriptions of the "interior" and the "exterior" regions bounded by the outline of a polygon.

mathematical context. Again, this may have reinforced how *diagonal* refers to a property of lines, rather than to a specific line, when used in the mathematical context. Most of the interviewees were aware and able to relate this meaning of *diagonal* to Aria's possible misunderstanding of *diagonal* (of a polygon) as having to be oriented diagonally in the task.

However, not all interviewees noticed this connection (and confusion) between the everyday meaning and mathematical meaning of *diagonal* immediately. They mostly agreed that it might be a possible reason for Aria's misconception, upon further prompting. Interestingly, one interviewee, Joey (a non-native English speaker), reflected and shared that she did not actually make that connection to "how diagonal is used in the daily life". She continued to explain that in her native language, Turkish, there are, in fact, two distinctive words for the two different meanings of *diagonal*. In Turkish, *köşegen* is used to refer to a diagonal of a polygon in the mathematics register and *çapraz* is used to refer to the orientation or direction in both everyday and mathematical contexts. As such, to Joey, the connection was not apparent initially.

In contrast, another interviewee, Simon, wondered if the everyday–mathematical connection for the meaning of *diagonal* might have been an inverse connection instead. He raised the possibility of how the "real-world appreciation of being on a diagonal" as a slanted orientation might instead be a consequence of how "we just always see it with rectangles that are sitting horizontal" in the everyday context. Such experiences with rectangles in the everyday context seemed to mirror how rectangles would usually be presented in the mathematics classroom as well.

Reflecting on his point, I became curious enough to look up the etymology of the word *diagonal.* The noun *diagonal* was borrowed from the Latin word *diagōnālis* which means "slanting line", and first used in the mid-1500s to describe "extending as a line from one angle to another not adjacent" in geometry (Harper, n.d.; Oxford University Press, n.d.). In other words, the word *diagonal* was first used in the mathematics register before it was more loosely used in everyday context as "having an oblique direction like the diagonal of a square" in the 1700s (Oxford University Press, n.d.). In retrospect, the inverse mathematical–everyday connection in relation to the meaning of diagonal, as proposed by Simon, is highly probable.

7.1.2. The *Transformation* Dimension

The responses to this task provided me a glimpse into the interviewees' "knowledge-in-action" – their knowledge in the *Transformation* dimension – in relation to the concept of *diagonal* (of a polygon). As mentioned in the discussion of the *Foundation* dimension, all the interviewees were aware of dual meanings of the word *diagonal*, though there were observable differences in terms of the depth of understanding. With that knowledge as the foundation, there was much discussion around pedagogical strategies and the use of representations and analogies in relation to how they would teach the concept of *diagonal* (of a polygon) to students. Collectively, in relation to how they would respond to the two students in the task, most of the interviewees mentioned that they would rotate the given pentagon to help them (especially Aria) visualise and notice that the diagonals which were originally oriented diagonally can also be oriented horizontally and vertically. Primarily, most of their articulated "knowledge-in-action" related to how they would be developing students' understanding and appreciation for the word *diagonal* in the mathematics register, notwithstanding differences in their approaches.

Evidence of planning for mathematical language

In thinking about how they would help students understand the meaning of the word *diagonal* in the mathematics register (in contrast to its meaning in the everyday context), some interviewees mentioned how they would be intentional in motivating and facilitating discussions around the usage of the word. For example, Karen said she would "definitely pick out the word *diagonal*" and tell the two students in the task that, although they were "both using the word *diagonal*", they were "talking about different diagonals". She would guide them to "figure out what these two different diagonals are […] in specific contexts" and focus the discussion on the two different meanings of *diagonal*.

To Karen, it is important for the students to know if they are "talking about the same things" (or not), as there is a need for students and teachers to be "clear about the language we're using with each other because that's like the basis of mathematics". She elaborated that this task would provide "a really good opportunity to talk about language […] like when and why you choose certain language" in the mathematics classroom. However, she also shared that her greater emphasis would be on "the importance of agreeing on terms" used by students, rather than using the specific term – diagonal, in this case. In other words, it would be perfectly fine with her if her students were to invent a new word for *diagonal* (as the noun in the mathematics register). They could use a term like the "D-line", on the precursor that there is clarity about "what a D-line is, that is, the thing connecting two corners and doesn't necessarily have to be slanted".

Apart from being aware and certainly addressing the confusion of the dual meanings of *diagonal* with the students in the task when it arose, Evie commented that greater attention should be placed in the initial planning phase when thinking about the teaching of *diagonal* (of a polygon). She highlighted how this awareness of students' possible confusion with *diagonal* should translate to more intentional task design and choice of words used in the explanations. For instance, she would deliberately "introduce diagonal using one of these unconventional polygons" instead of the usual "square or rectangle" to pre-empt the possibility of the misconception that diagonals in polygons must always be oriented diagonally. Moreover, she would "minimise the use of these types of language" in her explanations and "alleviate these types of language issues" which might have resulted in the confusion. She further commented, even "if they come up as well, it's just a matter of revisiting to ensure that the proper mathematical language is secure" once students have a good grasp of the meaning of *diagonal* in the mathematics register.

In addition, examples of how the interviewees would plan for mathematical language in response to this task did not only revolve around the term *diagonal*. For instance, Lena suggested how she would be explicit in using the words "inside" and "outside"[28] to help students differentiate between *diagonals* (that connect non-adjacent vertices) and *edges* (that connect adjacent vertices). While her suggestion was built on a gap in a flawed understanding of diagonals (as diagonals need not always lie on the interior of the polygon – see Figures 7.2 and 7.3), it showed her attempt to plan for specific language use in her classroom.

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[28] See footnote 27.

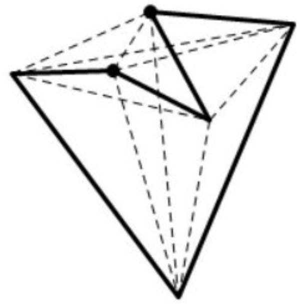

Figure 7.3. Another example of diagonals in a concave polygon
Note: Diagonals of the polygon are indicated by the dotted lines.

Another instance was shared by Mindy, who would ask students questions such as “Do the corners have to be next to each other? Or can diagonals only occur when the corners are not next to each other?”. She mentioned that she would deliberately choose to use “not next to each other” to highlight the non-adjacency condition in the definition of diagonals in this case. She noted that the students (in the task) were in the elementary grade and, thus, may not have learnt or understood terms such as *adjacent* or *consecutive* yet. Arguably, Mindy’s choice of ““next to each other” may also not be ideal as well. For example, in the concave polygon illustrated in Figure 7.3, there are two vertices (in bold) which appear to be “next to each other” but are clearly non-adjacent.

Use of representations and analogies

Several interviewees (e.g., Simon and Evie) attributed the confusion of *diagonal* presented in the task to how *diagonal* (of a polygon) would typically be taught and learnt. They mentioned how students’ first encounters with diagonals are usually associated with squares or rectangles that are oriented vertically or horizontally. This way of defining diagonals through such shapes and orientations tends to reinforce the (mis)conception of diagonals as being oriented diagonally. In order to reduce such (mis)conception, they would instead introduce the concept of diagonal with various (more complex) polygons. With polygons (other than squares and rectangles) that are oriented differently, it would expose students to diagonals which are not necessarily oriented diagonally, but also oriented horizontally and vertically as well. In addition, some interviewees shared the use of real-life examples to help students understand that diagonals need not be oriented diagonally. For example, Cass would “call out, draw out the baseball diamond[29], because the diagonals on a baseball diamond are vertical and horizontal". Faye suggested how the classroom can be considered as a polygon and she would ask students to "race around the classroom for a bit, doing a diagonal in our classroom with the classroom shape". She explained that this would better reinforce the non-adjacency condition in the definition of diagonals as students could "only go from one corner to the other" and not "from a side to side".

Interestingly, Cass suggested that the misconception that *diagonal* (of a polygon) cannot be oriented horizontally or vertically may be attributed to how younger students may think that diagonals can only be "one thing" and "not able to be another". She wondered if Aria may be thinking that "it's a horizontal so it's not a diagonal because horizontal is a different word than diagonal". In response to this, Cass proposed the use of an analogy that "you're allowed to be two things". She would explain to the students that a diagonal can also be a horizontal like how a student can be a second grader and a dancer at the same time. While her suggestion (and corresponding analogy) may seem obvious in the context of mathematical objects which tend to be characterised by more than one property, I wonder if some students (including mine) might have certain misconceptions due to similar reasons. For instance, I had encountered students who often thought that an equilateral triangle is not an acute-angled triangle. This might be a consequence of them thinking that the classifications of triangles by angles and sides are different "things".

7.1.3. The *Connection* Dimension

With the interview data alone, it was difficult to determine fully if the interviewees would use the mathematics register consistently within and between lessons and across topics – relating to their knowledge in the *Connection* dimension. The tasks also spread across a range of topics (and grade levels) which were not necessarily connected. However, inconsistency in the use of the mathematics register (within specific topics) was still noticeable in some of their responses. Moreover, even though it may be more task-specific, most of the interviewees were observed to demonstrate awareness of the

[29] The word *diamond* does not reside in the mathematics register. However, it is often used informally to refer to the *rhombus* at the elementary level. In the case of the "baseball diamond" or rather the baseball field, it would be more appropriately described to be of the shape of a square oriented at an angle in the mathematics context.

possible difficulties students might have in relation to the specific task concepts and mathematics register (e.g., the concept of diagonal and use of the word *diagonal* in this task) by relating to their own teaching experiences.

Consistency in mathematics register

While most of the interviewees were observed to distinguish the dual usage of the word *diagonal* in their responses to the task, a couple of interviewees used “diagonal line” and “diagonal” more loosely and interchangeably, without clear differentiation. For example, Faye used “diagonal lines” when she was referring to the diagonals of a polygon. This may have implications on students’ understanding as the use of the term “diagonal line” tends to evoke the image of slanted line, rather than the mathematical object which connects non-adjacent vertices of a polygon. On the same note, Nadia used “a diagonal” when referencing the orientation of a line. She mentioned that how students tend to think that “this is a diagonal” as she used her arm as a slanted line; and “this is not a diagonal” as she then used it as a horizontal or vertical line.

Aside from how the word *diagonal* was used (with consistency or not) by the interviewees, all the interviewees mentioned other geometrical concepts such as *vertices* and *edges* during the interviews. It was again notable that some of them were more inconsistent than others in their own usage of the mathematics register during the interviews. There were quite a few instances where words such as *corners* and *sides* (everyday language) were used in place of or alternated with *vertices* and *edges* (mathematics register). This observed inconsistency may subconsciously translate to and be evident in their actual teaching as well.

Awareness of students’ difficulties

From the interviews, it was evident that all the interviewees were either already aware or became aware (through the interviews) of the confusion (and thus the likely difficulties) students may have with *diagonal* as a word in the mathematics register. Other than the possibility of being confused with how *diagonal(ly)* is used in everyday language, many interviewees associated the difficulty with the meaning of *diagonal* in the mathematics register with how Aria and Bert might have been taught. Most of the interviewees mentioned how the students in the task might only have experiences with diagonals of “upright” squares and rectangles, hence developed the thinking that diagonals must be oriented diagonally. Thus, they highlighted the importance of rotating the pentagon in the task and using other examples of polygons that have diagonals which are oriented horizontally and/or vertically.

While this might not be directly relevant to students' difficulties with *diagonal* as a word in the mathematics register and perhaps a little peculiar, Cass raised the possibility that some students may have difficulty accepting that a diagonal (of a polygon) can be oriented horizontally or vertically. As mentioned in the *Transformation* dimension, she ascribed this difficulty to a broader assumption that younger students may have a slightly more distorted understanding with mathematical objects having specific names or properties. In other words, "if it's one thing, it's not able to be another".

7.1.4. The *Contingency* Dimension

Through how the interviewees responded to what Aria and Bert were saying in the task, I was able to 'observe' their ability to respond to the simulated (certainly unplanned-for) classroom situation relating to the teaching and learning of *diagonal* (of a polygon), both as a concept and a word. Most of them demonstrated an ability (to varying levels of precision) to interpret what the students were saying in relation to diagonal as a term in the mathematics register. However, it is notable that some would likely decide to facilitate an adherence to the mathematics register more than others in their interactions with students when discussing the issue of the *diagonal*.

Ability to interpret students' register in line with the mathematics register

Based on Aria's response, most interviewees were able immediately to rationalise that her understanding of *diagonal* (of a polygon) might have pertained to only line segments that are diagonally oriented. As such, their first response to Aria would be to rotate the pentagon and ask her if the number of diagonals have changed. They argued that this would likely prompt Aria to realise that it is not only about the orientation of the diagonals.

In addition, some of the interviewees questioned if Bert fully understood the concept of diagonals with his statement, "there's five lines connecting the corners of the pentagon". For example, Lena commented that she would clarify with Bert about what he meant by the statement. She explained that she wondered "if what Bert is seeing is actually the five sides of the pentagon versus the diagonals that are the internal part of the shape".[30] In other words, she was not certain if Bert is referring to the five edges of the pentagon or the five diagonals of the pentagon. In comparison, Alicia would perhaps propose a counter-argument here. She attended to the way the question was phrased and eliminated the possibility that both students were referring to the edges of the pentagon in their conversation at any point. Specifically, she mentioned, "it's the number of diagonals in a given pentagon so I don't think they would have gone there", that is, thinking about the edges in this case.

Ability to facilitate an adherence to the mathematics register

Notably, the word *diagonal* not only resides in the mathematics register, *diagonal(ly)* is also commonly used in everyday language. As such, students may be familiar with the word in their everyday communication and, hence, use it as well in the mathematics classroom. Hence, to most interviewees, the utmost concern would not be reinforcing the use of the word *diagonal* (and other specialised terms such as *vertices* and *edges*) though they would still model the use of these specialised terms during classroom interactions. For instance, Faye mentioned her preference of "letting them (her students) hear the language that they're going to hear". But her intent was more for exposure rather than a strict adherence to the mathematics register, as she added, "they don't need to be tested on it right now, but they might as well hear it".

Instead, many of the interviewees were more concerned about what students meant or understood by using the word *diagonal*, when thinking about how to respond to Aria in the task. Indeed, as suggested by the responses, demonstrating an ability to adhere to the mathematics register in this dimension would go beyond the use of the specialised vocabulary used in mathematics. It should also hinge on the clarity and precision of the mathematical meaning being communicated through these (or other) words and how prepared teachers may be in responding to the different (and perhaps flawed) meanings communicated by their students. In this regard, some interviewees articulated how they would ask students to elaborate and refine their understanding of the concept of diagonal first before preparing them to use the mathematics register in a more precise manner. For instance, Mindy mentioned how she would focus "less on getting them to do a really precise definition of a diagonal", but refine their understanding

[30] See footnote 27.

based on the students' existing definition of diagonal. She believed that, by "getting them to refine those, more so out of the colloquial and more into precise terms", students would be more "ready to start hearing about" and using the mathematics register.

However, it was rather surprising (and a little sad) to note that, out of the eleven interviewees, only one interviewee, Joey, mentioned that she would emphasise the importance of the non-adjacency condition in the definition of *diagonal* (of a polygon). She shared that she would explicitly include that condition when formally defining diagonals of a polygon with the whole class. Her reason for doing so was that some students might still "think that those adjacent ones also hold for diagonals too" if there were no explicit class discussion around the non-adjacency condition. Yet, it was interesting how she expressed reluctance when asked if she would correct Bert directly, since he did not mention the non-adjacency condition in his statement too. She explained that, based on what Bert was saying, "he understood that the diagonals are (connecting) the non-adjacent corners", even though his statement might have been ambiguous. Thus, she would not want to correct "his description and try to include that 'non-adjacent' thing". Similarly, to her, her students' understanding of the concept would precede their provision of a precise definition, especially when they were still trying to make sense of the concept at a more individual level or with their peers. In a way, her argument seemed to be in line with Karen's idea of students using an invented term like "D-line" in their register, as long as the intended meaning is aligned to the use of *diagonal* as a noun in the mathematics register.

7.2. Minus or Negative?

Task 8 (Operations with integers) was designed to illuminate the confusion students may have with the multiple meanings behind the use of the "–" symbol. As mentioned in the design of the task (see sub-section 4.2.2), the verbalising or naming of mathematical symbols through spoken language takes place in the mathematics classrooms all the time though symbols are not considered as being part of the mathematics register (Pimm, 1987). Hence, the task was specifically chosen to exemplify teachers' knowledge of the mathematics register when there is an interplay between mathematical symbols (the basis of representation for most mathematical objects) and language (Pimm, 1995). Specifically, in the task (as follows), a different naming or interpretation of the "–" symbol (in this case) had led to ambiguity in its meaning and resulted in some confusion and disagreement between the two student characters, Ken and Tala.

Context: **A secondary mathematics classroom – a lesson on operations with integers where students were discussing the answer to $-2-3$.**

Ken: Hmm, minus two minus three is … minus five!

Tala: No, you are wrong. Negative and negative become positive, so the answer should be plus and not minus ….

In this section, I discuss how teachers' knowledge of the "–" symbol and its interplay with language (as situated within the mathematics register) may be (or not be) demonstrated within the four dimensions of the Knowledge Quartet (see sub-section 3.1.2) respectively.

7.2.1. The *Foundation* Dimension

From the responses to this task, it was evident that all the interviewees were cognizant that the "–" symbol are used to represent both the operation of subtraction and the sign of an integer when representing a negative number. However, some differences, in terms of the depth of their understanding in relation to the mathematics concepts and register involved, were observed.

Knowledge and understanding of the mathematics register

In general, all the interviewees were aware of different ways of naming the "–" symbol and the corresponding implications in meaning in relation to the operation of integers. They mentioned how *subtract* (or, more commonly, *minus*) should be used when referring to the operation and *negative* should be used when referring to the sign of a number.

In discussing how to help students understand the difference between the dual usage of the "–" symbol, Cass further elaborated on the importance of explicating the inverse relationships between addition and subtraction, and between positive and negative numbers. Specifically, to help Ken and Tala understand the subtraction operation when working with integers, she would remind them that "minus is the same thing as plus a negative". In other words, she would tap on her knowledge and understanding that subtraction is the inverse operation of addition, and the additive inverse of a positive number is a negative number in her teaching. This response suggests that Cass is likely a teacher who has a good understanding of concepts involved when working on operations with integers.

Awareness of differences between the everyday and the mathematics registers

Other than in the mathematics register, terms such as minus, subtract(ion) and negative are also commonly used in everyday conversations when one needs to verbalise the "–" symbol. As the everyday usage of these terms is mostly analogous to their usage in a mathematical context, discussing the interviewees' awareness of the differences between the everyday and the mathematics registers did not seem relevant in this task. Moreover, all the interviewees were aware of the distinction between how minus or subtract(ion) would be used when referring to the "–" symbol as an operation, and how negative would be used when referring to the "–" symbol as a property of a number.

Perhaps, one term, which is borrowed from the more everyday context, is "take away". This term was used by a couple of the interviewees in place of the "–" symbol or the operation of subtraction. Often, "take away" is used almost synonymously to mean or explain minus or subtract(ion) in the mathematics context, especially at the lower grades. For example, Nadia talked about how "subtraction means we're taking away" when she shared how she would explain the difference between subtraction and multiplication of integers to Tala in this case. The act of "taking away" is also probably how most younger students first encounter and understand as the meaning of subtraction. Similarly, Cass brought up how "saying minus helps them [Ken] because it's taking away" when discussing what Ken said in the task. However, she seemed concerned with the use of "take away" as she added that "I don't like take away either, but if they went from take away to minus, they got better". Indeed, the conceptualisation of subtraction goes beyond that one action of "taking away". Thus, Cass would be more mindful with the use of *take away* and prefer her students to use *minus* instead, to highlight the difference between the corresponding underlying meanings.

7.2.2. The *Transformation* Dimension

When presented the task, the first comment from almost all interviewees was how the confusion illustrated by Tala in this task is common amongst students, even in the classrooms of those who teach at the high-school level. They attributed it to the reason that Tala had likely committed to her memory a procedural trick or rule (e.g., “you get a plus when you have two negatives” in Karen’s words), without understanding what it really means. Consequently, Tala (and students in general) would struggle to differentiate between the subtraction of a positive number from a negative number ($-2 - 3$) and the multiplication of two negative numbers ($(-2) \times (-3)$), in terms of both the symbolic representations and the verbal descriptions. In other words, to most of the interviewees, Ken and Tala were likely thinking about different operations and thus disagreed on a solution. However, only three interviewees, Simon, Cass and Joey, noticed how using *minus* and *negative* interchangeably, with respect to the “–” symbol, might have added to the confusion in the task, without receiving additional prompting.

In response to what were noticed as the causes for the confusion in the task, the interviewees shared approaches to how they would plan to overcome the confusion between Ken and Tala. Generally, their articulated “knowledge-in-action” hinged on the development of a more conceptual understanding of operations of integers with students (in relation to what Tala said). Correspondingly, they would be more intentional in thinking about what they would say or the representations they would choose to help students better understand the concept. As such, the confusion due to the different ways of naming the “–” symbol (in relation to what Ken said) would likely be resolved as a by-product in the process.

Evidence of planning for mathematical language

To help Tala understand that the operation in the mathematical statement $-2 - 3$ is subtraction and not multiplication, some of the interviewees shared that they would be more deliberate in highlighting the difference between operation with integers and operation with whole numbers. For example, to set the context of the operation, Nadia would emphasise the difference between integers and whole numbers, that is, integers can be positive or negative while whole numbers[31] are only positive. When referring to the mathematical statement $-2 - 3$, she would intentionally say “negative two subtract a positive three ($-2 - (+3)$),” to help students attend to the “+” sign which is typically omitted when representing positive integers. She explained that this would help students

[31] Typically, students first learn whole numbers without any association with sign differences. In other words, the positive (“+”) sign is always omitted in the expression of whole numbers. Similarly, the positive (“+”) sign is generally omitted from the positive integers in the operation of integers.

distinguish the difference between this statement and the mathematical statement $-2 - (-3)$, which would then be read or said as “negative two subtract negative three”.

Similarly, Cass shared how she would explicitly teach and discuss the equivalence between a subtracting positive number and adding a negative number (also mentioned in the *Foundation* dimension). She explained that students generally find the operation of addition easier to understand. With reference to Task 8, she would ask Ken and Tala, “what operation will be between my negative two and my negative three?” and hope that they would understand and reply with “plus (add)”. In other words, she would instead emphasise how the mathematical statement $-2 - 3$ can be understood and read as “negative two plus negative three ($-2 + (-3)$)”. In her view, this would better help students identify and associate the minus sign with subtraction rather than multiplication and she might even deliberately “drill that into them”. Notably, these two mathematical statements are equivalent in terms of the value of the results, but Cass had fundamentally changed the operation of the original statement from a subtraction to an addition. While the addition operation is generally easier for students to grasp, I wonder if this might have implications on students’ understanding of the subtraction operation, in terms of what it means and how it is used.

In addition, Mindy commented that teachers “can’t just keep telling them (students) tips and tricks forever” and “avoiding the register” in their teaching. Her comment seemingly echoed the responses that Nadia and Cass had made in relation to this task. While she did not elaborate specifically on how she would respond to Ken and Tala, she noticed the lack of precision in their use of “minus … minus” and “negative … negative” respectively. She also added how this “lack of awareness around their own precision” in the use of language was common and perhaps the root of the problem. She often found herself guilty of using the two words *minus* and *negative* interchangeably as well. In response to that, she believed teachers should be “aware of what the mathematics register is and how we use it”. She explained that this would help teachers “better prepare students so that they can use it (mathematics register)”. Otherwise, students would be in a situation where they are “never going to know it” and “they're never going to learn it”. In the long run, students might easily “get confused (in the mathematics classroom), either because they don't know the language or because their understanding of the language is not strong enough”. In other words, she seemed to be suggesting that there must be a conscious effort for teachers to know and use the mathematics register in their teaching to aid students' learning of mathematical concepts in general.

Use of representations and analogies

Unanimously, all the interviewees hinted that concepts involving negative numbers and operations involving negative numbers are infamous as stumbling blocks for many students, not to mention how both concepts are represented by the same "–" symbol. Hence, other than paying special attention to what they would say when teaching the concept, they also proposed a range of representations and analogies to help students understand the concepts involved.

A common representation or tool, which was deemed useful by many interviewees, was the number line. They would use the number line to help students visualise the position of negative and positive numbers (i.e., to the left and right of zero respectively). They would also use the number line to help students visualise the operation of subtraction and addition in a more concrete manner through the actions of moving left and right respectively. In addition, the use of manipulatives was also mentioned by several interviewees. For instance, in asking Ken to explain what he meant in the task, Alicia would "definitely use manipulatives or something visual and get him to explain it in a different way". She would attempt to ask Ken to explain his thinking by either using the number line or coloured tiles representing the different integers involved (where "red is negative and black is positive"). Similarly, Faye mentioned how she would ask both Ken and Tala to draw something to explain their understanding. Through the process of drawing their own representations, she hoped that they would be able to see what they meant visually, and perhaps realise where they might have gone wrong.

Beside the use of visual representations and manipulatives, several participants shared the common analogies which can be used to help students visualise and make sense of the concepts involved in this task. For example, Lena brought up how the notion of piles and holes[32] can be used analogically to illustrate positive and negative numbers respectively where "negative two would be two holes". This analogy was also

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[32] The use of piles and holes to represent positive and negative integers was first introduced and popularised by James Tanton, a research mathematician. An elaboration and discussion of this analogy was published on his website *Thinking Mathematics!* (Tanton, 2013).

mentioned in passing by Faye. In contrast, Cass shared how she always used the analogy of going "up and down" to represent the act of "adding and subtracting". Thus, in the case of $-2 + (-3)$ or "negative two plus negative three, which she expressed preference for as the interpretation to $-2 - 3$, she would also use the analogy of "we went down two and we went down three more" in her explanation to students.

As a contrasting note to the range of representations and analogies mentioned by the participants, I wonder if the use of alternative symbols to denote the negative sign would have better differentiated it from the "–" symbol used to denote the subtraction operation. Notably, other notations had been used historically to denote the negative sign. Some such notations included the "←" and "→" symbols and a raised "+" and "–" symbols to denote the positive and negative signs respectively (Cajori, 1993). For example, in place of $+3$ and -3 to denote positive and negative three respectively, $\leftarrow 3$ and $\rightarrow 3$ or ${}^{+}3$ or ${}^{-}3$ were used instead. Apparently, the use of raised (or in some cases, lowered) "+" and "–" symbols can still be found in some mathematics textbooks (e.g., Brown et al, 2000, p. 36) and online worksheet resources to help students distinguish the signs from the operations. This also reminds me of how the subtraction operation and the negative sign have always been (intentionally) denoted by two different keys or buttons on the scientific calculator. Perhaps, a difference in notation would have indeed helped to decrease the possibility of students' confusion behind the dual uses (and naming) of the "$-$" symbol.

7.2.3. The *Connection* Dimension

As mentioned in sub-section 7.1.3, the nature of these task-based interviews was not able fully to explicate the interviewees' knowledge in the *Connection* dimension, especially in terms of their use of the mathematics register across topics and lessons. I was only able to make observations with regard to their own use of the register and their awareness of students' difficulties with respect to mathematics concepts (and register) within the task (in this case, operation with integers).

Consistency in mathematics register

When responding to this task, all the interviewees were observed to be generally consistent in using the term *negative* when referring to a negative integer and the terms *minus* or *subtract* when referring to the operation of subtraction. However, I was not able to find further evidence to support how (in terms of consistency) most of them would use these terms beyond the context of this task. Perhaps, only one interviewee, Sofia, showed a glimpse into her consistency in use of the register across topics. When discussing how students' difficulties with negative numbers tend to extend to numbers with negative exponents, she was consistent and clear in how she was using the term *negative* in her response. Specifically, she said, "about negative exponents, they're not negative numbers necessarily, right? Two to the power of negative three is not a negative number."

Awareness of students' difficulties

Notably, all the interviewees were able to recognise and relate to the difficulty the student characters had in relation to the use of the "$-$" symbol in the task. For example, Alicia's comment of "what confuses kids so much is minus two and negative two" suggested how students are generally confused if "minus two and negative two" are the same or different when working with integers.

Similarly, Sofia shared an awareness of students having difficulty working with negative numbers and subtraction. She commented how "grade nine is where they (students) first started to struggle with subtraction, and negative integers". Basing off her own experience, she elaborated that students generally found it confusing due to the didactical sequence they might have learnt concepts involving operation of integers. Typically, in grade eight, they would start with learning the addition of negative numbers. In the case of the mathematical statement $-2-3$ discussed in the task, Sofia shared how students would have first encountered the same statement in an equivalent form using the addition operation, "negative two plus and then a bracket negative three ($-2+(-3)$)". But, when they progressed forward to the next grade, it would have become $-2-3$, as "now you don't have brackets and minus three can be negative three". She further commented that during the transition, students might simply be told, "Ok, we're entering this world now where minus three and negative three are kind of meshing together". In other words, Sofia appeared to be cognizant of how the connection in terms of the equivalence of the two mathematical statements might likely not have been made explicit in any way to the students in the teaching and learning process, and thus led to students' difficulties with subtraction of integers.

Moreover, Sofia continued to share another difficulty students might have with the meanings behind the "–" symbol with the example of how "two to the negative three (2^{-3}) is not a negative number", but a positive rational number. Particularly, she highlighted how the result may not necessarily be negative numbers in the case of negative exponents. As such, she stressed the importance of students having to understand the context, as well, when they were interpreting the meaning(s) behind the "–" symbol being used. Consequently, her various responses to this task support how Sofia generally shows a deeper level of awareness (as compared to the other participants), in relation to what students might often struggle with, when working with the "–" symbol, that can have different meanings and results in different contexts.

In contrast to how most of the interviewees focused on students' difficulty in differentiating between the multiple meanings of the "–" symbol, Cass provided a slightly different perspective in relation to why Ken would have read $-2 - 3 = -5$ as "minus two minus three is minus five", instead of "negative two subtract or minus three is negative five". She attributed this lack of precision in students' language to how they might have developed a habit or preference to saying *minus* whenever they see the "–" symbol. She argued from Ken's perspective that "*minuses* have been drilled into their head since grade three or two or one or whenever you start with *minuses*". She further commented how students tend to bring with them the "years and years of experience" of using *minus* in place of the "–" symbol. Consequently, she would often "still get students using *minuses* even at college algebra" level. She also had prior experiences with students who would "mix *minuses* with *negatives*". For instance, they might say "minus two and minus three is negative five, because that negative five is by itself", in place of what Ken had said in the task. She continued to suggest that these students might likely have interpreted and read the "–" symbol in -5 as negative because it is used in a single numerical answer. Correspondingly, the other two "–" symbols might be interpreted as the operation of subtraction and hence read as *minus* because the mathematical statement $-2 - 3$ is seen as a combination of mathematical objects where an operation would be required. Broadly, Cass's responses to this task suggests an added awareness which might be required in relation to the possible difficulties students might experience with the interplay of the mathematics symbols and the mathematics register in this case.

7.2.4. The *Contingency* Dimension

Similarly, the nature of the task-based interviews provided me with a glimpse of how the participants might demonstrate their knowledge of the mathematics register in responding to a simulated classroom situation highlighting the interplay of mathematical symbols and language in this case. From their responses, most of the interviewees were able to interpret and suggest what Ken and Tala were likely thinking with their uses of *minus* and *negative* to represent the "–" symbol. What seemed rather different from the previous discussion on *diagonals* was that more interviewees would be more concerned about the use of *minus* and *negative* and the corresponding implied meanings in this task. Hence, there were more discussions on when and why they would facilitate a greater adherence to the mathematics register during their interactions with students.

Ability to interpret students' register in line with the mathematics register

Most interviewees were able to infer the instances when Ken and Tala were referring to or thinking about negative numbers, though they used the word *minus* instead of *negative* in the task, particularly with reference to the numerical solution (-5). However, the use of the phrase "negative and negative" by Tala raised some ambiguity in terms of what she might be thinking amongst a couple of the interviewees.

For instance, when Joey was sharing what she noticed in the task, she wondered if Tala was seeing the two numbers "as a multiplication case" or rather "in an additive relationship" with reference to "this rule […] which holds for multiplication". Her uncertainty arose "because of this 'and'" in the phrase "negative *and* negative" that Tala said. She further explained that it was ambiguous because Tala did not say "negative times negative". To Joey, the imprecise use of "and" has reduced the likelihood that Tala was thinking of multiplication rather than addition in the task.

Similarly, Lena questioned what Tala meant by saying "negative and negative", with an interpretation slightly different from that of Joey. Lena first noticed that Tala was likely "drawing from a rule they've been told". However, Lena did not specify which rule she was referring to. There was also no mention of multiplication, or anything related to multiplication in her response to this task. As such, it seemed unlikely that Lena was on the same page as Joey who was concerned about the possible confusion between multiplication and addition due to the use of "and" in a supposedly multiplication rule.

Instead, Lena seemed concerned with Tala's likely perception of the mathematical statement $-2-3$ as a string of two negative numbers due to the use of "and", seemingly in a more colloquial manner. She argued that "technically it isn't a negative *and* a negative", but should rather be "a negative *and* a positive and the operation that's happening is subtraction".

As such, the ambiguity in the use of *and* in the phrase "negative and negative" could be due to a lack of precision in the intended operation (noticed by Joey) or a more colloquial use of language (noticed by Lena). Interestingly, while I did not intend for this ambiguity to surface, as part of the design of the task, the responses from Joey and Lena suggest how teachers may possibly attend to the specific phrasing of rules as it may inform them of possible student misconceptions as well.

Ability to facilitate an adherence to the mathematics register

In comparison with Task 6 (Diagonals of a polygon) discussed in the previous section, this task seemed to have provided a context where the interviewees were generally more concerned with facilitating an adherence to how the "$-$" symbol was read or verbalised, in relation to its meaning. Collectively, most of the interviewees would prefer students to use *subtract* (or *minus*) when referring to the subtraction operation and use *negative* when referring to the sign of a negative number. However, the extent to which and how they would facilitate the adherence to their preferred register would vary, based on different considerations.

A couple of interviewees seemed more concerned with the use of *negative* and *minus* interchangeably. They shared that they would want to correct students immediately if they were using *minus* to refer to negative numbers, like what Ken did in the task. For example, Simon said that he "would try to steer Ken into a more accurate way of representing that". Reflecting upon his own experience, he had met grade nine students who were still stuck with the thinking that "a string of integers in brackets like negative two in brackets, negative four in brackets ($(-2)(-4)$)" is equivalent to "negative two minus four ($-2-4$)". He attributed such a misconception to "the unappreciation of a negative compared to an operation earlier on". As such, he argued that there might be negative repercussions "if we just always take negatives as minuses", like what Ken did. Simon's concern was similarly reflected in the interview with Alicia. Quoting Alicia, "I worry if I don't correct the minus two minus three is minus five now, then it carries forward". In other words, she seemed to suggest that the imprecise use of *minus* in place of *negative* might carry forward to other related topics or lead to other misconceptions. Consequently, both Simon and Alicia would more likely facilitate an adherence to the mathematics register if students were to use *negative* and *minus* interchangeably when referring to the negative sign in classroom interactions.

By contrast, both Karen and Faye mentioned that they would not be particular with how Ken was using *minus* to represent a negative number in the task. To them, Ken seemed to be demonstrating a certain level of understanding in relation to the concepts involved. Hence, they would not correct his language use or expect him strictly to adhere to the mathematics register at this point. However, they shared some exceptions where they would facilitate a closer adherence to the mathematics register. For instance, Karen would correct students' imprecise use of *minus* and *negative* in relation to the meaning represented by the "–" symbol if they were showing signs of confusion or misconceptions. Thus, she would likely correct or clarify with Tala at this point, rather than Ken. To Karen, Tala seemed to be in a greater state of confusion with her use of "negative and negative". As for Faye, she would place a greater emphasis on the precise use of the different terms if students were presenting in a more formal context or to a larger audience. Specifically, she said, "if they're going to do a presentation at a math conference, then depending on who they're talking to [...] I would encourage them to use negative and positive".

On the same note, there were others who mentioned that they would generally not be particular with students' use of language at first, when putting themselves in the role of the teacher in the task. They would consider the context and state of students' learning during the interactions. For instance, Cass shared that she would not "correct them (students) talking to each other, especially if they're trying to learn it" and developing understanding of the concepts. But once students gain sufficient understanding of the concepts, she would "start correcting the words more". She gave the example of students "just working through a worksheet maybe", in preparation for a topic test. Yet, she stressed that she would always pay special attention to her own adherence to the mathematics register with the belief that students would gradually pick up the use of the register in the process. Specific to the concepts in this task, she would hope to see her students becoming more precise in the reading and verbalisation of the "–" symbol (from using *take away* to using *minus* and *eventually* using subtract) when it means the subtraction operation in their interactions. Likewise, Lena commented that she would only emphasise the use of the proper terms after her students acquire understanding of the concepts. In relation to this task, her response suggested that she would more likely facilitate an adherence to the mathematics register with Ken, and not Tala at this point. In particular, she mentioned that “the polishing of the terminology would be beneficial for Ken because if Ken already understood what’s going on”. As for Tala, the focus to Lena would be to address Tala’s (mis)conception of the mathematical statement $-2-3$ first, as discussed in the earlier aspect of this dimension. In short, both Cass and Lena would less likely facilitate an adherence to the mathematics register in contexts when students were still in the process of making sense of the concepts in their interactions.

7.3. Numerator (Denominator) or Top (Bottom) Number?

Task 4 (Fractions) was designed to illuminate the possible implications on students’ understanding of fractions by using the *top number* and the *bottom number* when referring to the numerator and the denominator of a fraction respectively. In the task (as follows), the two student characters, Nodo and Vick, seemed to understand what each other meant by the *top number* and the *bottom number* when discussing about how they should represent the fraction $\frac{3}{5}$, using the fraction strips. In other words, they appeared to have a common student mathematics register, which is more colloquial, in relation to fractions.

Context: **An elementary mathematics classroom – a lesson on fractions (equal-sized parts of whole) where students were working with fraction strips to show the fraction $\frac{3}{5}$ [as in the diagram below].**

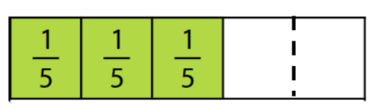

Figure 7.4. Diagram of fraction strips given in Task 4

Nodo: Because the bottom number is five, we need to use the green piece (which denotes the fraction $\frac{1}{5}$).

Vick: And we need three of them to get the top number three.

As compared with the tasks discussed in the previous two sections, there did not seem to be any apparent misconception or confusion due to the use of the student mathematics register (which differs from the formal mathematics register) in this case. However, the students' understanding of the concept of fractions, due to the use of colloquial terms in their register, was up for interpretation. For example, there might be a possibility that the students did not understand or view the fraction as a number[33] – an important aspect in fraction concepts which tends to be neglected – since they appeared to be thinking of fractions as a mathematical object made up of two separate numbers, as evident from their use of the *top number* and the *bottom number* in the discussion.

In this section, I discuss how teachers' knowledge of fractions and the related mathematics register may be (or not be) demonstrated within the four dimensions of the Knowledge Quartet (see sub-section 3.1.2) respectively.

7.3.1. The *Foundation* Dimension

From the responses to this task, it was evident that all the interviewees were cognizant of the mathematics register surrounding the concept of fractions. For example, they were mostly using terms such as *numerator*, *denominator*, *parts* and *whole* during the interviews. However, there were (again) differences observed in relation to their depth of understanding, as well as how they described the concept of fraction within the mathematics register.

Knowledge and understanding of the mathematics register

All the interviewees showed an understanding of what the words *numerator* and *denominator* referred to or meant in relation to a fraction representation. Most of the interviewees described the two words in relation to their position in the fraction representation, analogous to the top and bottom notion that Nodo and Vick were ascribing to in the task. This explained how the interviewees were all able to make sense of what Nodo and Vick meant with the *top number* and the *bottom number* in the task.

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[33] The concept of fraction as a number may not be widely accepted as some may still consider a fraction to be a representation of a number, rather than a number itself. For example, the *Concise Oxford Dictionary of Mathematics* (Oxford Reference, 2021) did not explicitly define fraction as a number but described its representation, whereas the *Encyclopedia of Mathematics* ("Fraction", 2013) defined fraction as a number.

The part–whole relationship in fractions was also often referred to when they elaborated on the meaning of the two words. For example, Mindy would define a fraction as "pieces of a whole, where the whole is the *denominator* and the number of pieces you have is the *numerator*". Alicia would perhaps add that the *denominator* represents "how many pieces the whole [is] divided into". However, their explanations, which mainly focused on the definitions of *numerator* and *denominator* in the case of proper fractions, might not be totally applicable to other types of fractions such as improper fractions or compound fractions.

Moreover, only two interviewees, Faye and Evie, explicitly mentioned the idea that fractions should be seen as numbers. Specifically, Faye articulated that she would want her students to "think of one-fifth as the number, not like the five as a number and the one is a number".[34] This concept has implications on how students may understand what the numerator and the denominator mean. For example, Evie highlighted that, "it is clear here from this discourse that fraction is not seen as a number, it's seen as something separate." Indeed, due to the symbolic representation of fractions, it is not uncommon for students to develop the misconception that a fraction is not a number, but a mathematical object made up of two numbers, namely the *numerator* and the *denominator* or the *top* and *bottom numbers* in the words of Nodo and Vick.

Awareness of differences between the everyday and the mathematics registers

Most of the interviewees zoomed in on the student characters' use of the *top* and *bottom numbers* to refer to the *numerator* and the *denominator* respectively. Seemingly, the former appeared to be taken from everyday or colloquial language to reference the latter which resides in the mathematics register, by focusing on their positions in the fractional representation. While most of the interviewees noticed this discrepancy between the everyday and the mathematics registers, only Faye and Evie (as discussed above) addressed explicitly how it might result in students thinking that a fraction is a mathematical object made up two numbers instead of it as being a number itself.

Interestingly, several interviewees hinted at how there seemed to be a disconnect in terms of how fractions are used or talked about within the everyday

[34] Mathematically, it is not wrong to interpret "the five as a number and the one is a number" when seeing fraction as division of two numbers.

register and the mathematics register. For example, Lena mentioned how “there’s a gap between how we talk about fractions in real life, versus the terminology”, when asked if she would be concerned with students using terms such as *top* and *bottom numbers*, instead of *numerator* and *denominator* in her mathematics classroom. She highlighted how terms such as *numerator, denominator,* specific to the concept of fractions, are rarely (or almost never) mentioned or heard outside of the mathematics classroom.

Lena’s viewpoint was similarly shared by Evie, who added that perhaps even *top number* and *bottom number* did not seem to be common terms that would be heard in the everyday context. To Evie, these terms seemed more concept-related and, hence, she wondered “where this language (*top number* and *bottom number)* comes from”. She highly doubted that it would be “out of their natural home experience” and elaborated that parents would more likely be using names of specific fractions (e.g., a half or a third) if they were to teach or talk to their children about fractions. One might more likely hear a phrase such as “cut that apple in *half*”, than a discussion on the *top* and *bottom numbers* of a fraction, in the everyday context. As such, she shared that, to her as a teacher, “it could be interesting to have a deeper understanding of where this language would have entered into their (Nodo’s and Vick’s) mathematics register”.

7.3.2. The *Transformation* Dimension

As mentioned in the *Foundation* dimension, most of the interviewees focused on the use of *top* and *bottom numbers* as being indicative that Nodo and Vick likely understood the fractional representation. Hence, there was not much in-depth discussion of the approaches (nor representations and analogies) to how they would specifically plan for the teaching of fractions. Primarily, most of their articulated “knowledge-in-action” related to how they would introduce the mathematics register, specifically *numerator* and *denominator*, in relation to the task.

Evidence of planning for mathematical language

Apart from Lena and Evie, a few other interviewees also pointed out that *numerator* and *denominator* are words which are not commonly used in the everyday context, even within conversations involving fractions. Quoting Joey, these words would likely seem to be “foreign language, maybe for the students?” Moreover, Faye shared how the length and spelling of these words (as well as other specialised terms in the mathematics register) made them seem like “hard words” or “bigger adult words” to students, in general.

Specifically, two interviewees shared strategies on how they would plan to help students learn this “foreign language” or rather terms in the mathematics register, though there was no obvious connections made to the concept of fractions. For example, Lena shared how she might make use of the similarity in the pronunciation of the words. In relation to this task, she gave an example how she might use the D-sound to relate how “the denominator (starting with a D-sound) goes bottom-ed (ending with a D-sound)”. As for Faye, she shared how she would intentionally include motivational strategies or incentives to make the learning of the mathematics register “a really exciting thing” for students. For instance, she might pose the learning and use of new terms such as *numerator* and *denominator* as a challenge to students and “make the whole big deal about it”. From her experience, students would typically be excited about and “enjoy using correct terms” as they would feel “really smart”. Sometimes, an incentive such as “a ticket out of the door to recess” might be included in the challenge. In other words, students would be able to go for recess if they could say the “correct terms”.

Use of representations and analogies

As the concept of fractions is often taught with representations and analogies in most classrooms, I was surprised that most interviewees did not go into that discussion. Instead, the interviews revolved around whether and how they would teach words in the register. In retrospect, I wondered if it was due to the design of the task, or due to how the connection between the learning of language and the learning of mathematics concepts might have gone unnoticed by the interviewees in my study (and perhaps by most teachers).

Nonetheless, there was one interviewee, Alicia, who elaborated how she would use other representations to help students develop deeper understanding of the part–whole relationship in the concept of fractions. In her response to the task, she felt that the brief dialogue between Nodo and Vick was insufficient to inform her about their understanding of fractions. Other than asking the two students to explain what they meant, with reference to “what’s the part and what’s the whole”, she mentioned the importance of using different (and atypical) representations to reinforce students’ understanding of the part–whole relationship. She further described how she would “switch up the wholes and switch up the parts”, by using different polygons to represent the whole and the parts of the whole, instead of the typical pie or number strip. For instance, she would use the regular hexagon as the whole and consider its fractional parts in the form of different polygons such as triangles and trapezoids. She would also “play around” with the whole where reference unit for the whole might change from the hexagon to the trapezoid (which is a half of the hexagon) or even be made up of a combination of the hexagon and the trapezoid. Correspondingly, a triangle that started off being one-sixth of the hexagon would now become one-ninth of the hexagon-trapezoid combo.

7.3.3. The *Connection* Dimension

Like the previous two tasks, the responses to this task alone were insufficient in ascertaining the interviewees’ use of the mathematics register across topics and lessons. However, some observations relating to their consistency in the mathematics register relating to the concept of fractions could still be made. Similarly, there was also discussion relating to what they noticed and identified as students’ difficulties with the register.

Consistency in mathematics register

Notably, the interviewees seemed to be less consistent in terms of their use of the mathematics register while relating to the concepts of fractions. Many interviewees mentioned (implicitly and explicitly) that they would not mind the student mathematics register presented in the task and even code-switch between the student mathematics register and mathematics register. For example, Cass would “call it the bottom number” as she introduced the concept of the denominator. Subsequently, while she would use the term *denominator* in her teaching, she would constantly remind students that it “means the bottom number”, until they become “comfortable with the math”.

In contrast, only Faye, who also explicitly mentioned the idea that fractions should be seen as numbers (see sub-section 7.3.1), shared that she was not comfortable with Nodo and Vick using *top* and *bottom numbers*, in the context of fractions. The student register, in this case, would likely be inconsistent with their prior or future understanding of rational numbers.

Awareness of students' difficulties

In relation to why Nodo and Vick might have chosen to use *top* and *bottom numbers*, rather than *numerator* and *denominator*, in this task, all the interviewees were able to make connections to at least one of the following three difficulties – the unfamiliarity with *numerator* and *denominator*, the perceived lack of usefulness of *numerator* and *denominator*, and the challenge with describing *numerator* and *denominator* other than as numbers.

Firstly, the unfamiliarity with the terms were pointed out by Joey and Evie. Notably, the terms *numerator* and *denominator* are rarely used in the everyday context and very specific to the concept of fractions in the mathematical context. As such, students would mostly likely think of them as analogous to unfamiliar "foreign language", unlike terms such as *subtract* or *negative* discussed in the previous task. Secondly, some interviewees (e.g., Alicia and Lena) voiced their opinion regarding the perceived lack of usefulness of these terms, although they valued the precision of the mathematics register. To them, learning the terms might not necessarily be helpful in developing students' understanding. Possibly because the part–whole relationship is not obvious in the terms *numerator* and *denominator*, Lena even commented how she would not "necessarily think that the terminology itself is super-useful".

Lastly, Faye shared her struggled attempt to avoid referring to *numerator* and *denominator* as two separate numbers. She commented how "the language here is tricky" and questioned how teachers could name and refer to *numerator* and *denominator* without using the word number. Specifically, she asked, "one-fifth is a number, so what should we call that bottom five?" Consequently, it made me reflect about how I might have also described the one and five as numbers in the fractional representation for one-fifth previously, and perhaps, led to my students having problems understanding why one-fifth is a number too.

7.3.4. The *Contingency* Dimension

Intentionally, the design of this task focused the interviewees' attention to the student mathematics register (*top* and *bottom numbers*) and how they interpreted it with the formal mathematics register (*numerator* and *denominator*). In responding to what they would do as a teacher in this task, the extent to which they would adhere to the mathematics register during their interactions with students varied, pending different considerations.

Ability to interpret students' register in line with the mathematics register

Generally, all the interviewees were able to interpret how Nodo and Vick were referring to the *numerator* and the *denominator* when they used *the top number* and *the bottom number* respectively. As mentioned in the *Foundation* dimension, most of the interviewees similarly defined or described the *numerator* and the *denominator* in relation to their position in a fractional representation.

However, two interviewees further shared their inference of the two students' understanding of fractions, based on their more colloquial mathematics register. For instance, Joey felt that the description of the fraction three-fifths as "bottom number is five, top number is three" indicated a rather procedural understanding of fractions. In other words, Nodo and Vick might be thinking that, "the only relationship between the numbers is where they stay, top or bottom". Moreover, the lack of "reference to that part–whole relationship" seemed to reinforce the possibility that they were seeing the *numerator* and the *denominator* as separate numbers. This inference was also shared by Evie, as evident in her comment, "it is clear here from this discourse that fraction is not seen as a number, it's seen as something separate".

Ability to facilitate an adherence to the mathematics register

Notably, there were similarities in terms of how the interviewees would facilitate an adherence (or not) to the mathematics register across all three tasks. As before, many interviewees shared that they would model the use of the mathematics register, but they would not necessarily correct the students' usage in this task. They would tend to consider if the two students were likely still at the early stages of learning about the fraction concepts and/or the readiness of students at a younger age. Moreover, they generally agreed that mathematics register would be acquired gradually. For instance, Faye commented how "the class would eventually pick up on that". Similarly, Simon suggested that the mathematics register "would be something to develop and grow" as students become more familiar with fraction concepts but "not at the elementary level" necessarily.

Consequently, most of the participants would likely push for a greater adherence to the mathematics register after the students developed a more conceptual understanding of fractions. For example, Cass would “step in and actually emphasise the correct word because they have the conceptual understanding”. She would also encourage them to “practise using it”, since “they're both there and they both can practise at the same time”. Cass’s actions would probably resonate with Nadia who strongly believed that, in order to “build our knowledge base, we have to know the language as well”. Nadia further explained, “if they don't use these terms, how are they going to get comfortable with them?”

Unlike the previous two tasks, a strict adherence to the mathematics register was not preferred by some interviewees in response to this task. The reason that they commonly shared was that the terms are specialised terms which only exist in the mathematics register. For example, Joey was doubtful about the connection between the students’ usage of the mathematics register and their understanding of the concepts. Attributing to how the terms *numerator* and *denominator* “are so mathematical, and it’s not used in daily life that much”, she was not confident that students who “say denominator” would actually demonstrate a different (or higher) level of understanding as compared with “student(s) who would describe it as a bottom number”. On the same note, Lena was less inclined to push for the use of the terms *numerator* and *denominator*, “unless my curriculum specifically says they have to know it”, because “in real life, I don’t necessarily think that the terminology itself is super-useful”.

7.4. Summary

In this chapter, I exemplified the four dimensions of the Mathematics Register Knowledge Quartet (see sub-section 3.1.2 in relation to Lane et al., 2019) through a discussion of teachers’ responses to three specific tasks. Collectively, all the teachers demonstrated an understanding of the mathematics register specific to the concepts of the three tasks within the *Foundation* and *Connection* dimensions. However, the discussion in this chapter illustrated some of the differences (and gaps) in their knowledge of the mathematics register, across the different tasks. Moreover, how their knowledge was translated into actions, that is the “knowledge-in-action”, regardless of whether they were planned for (the *Transformation* dimension) or unplanned for (the *Contingency* dimension), also varied in terms of the considerations they might have. In other words, there are implications in terms of how teachers may notice and attend to language in their classrooms. This would be discussed further in the next chapter, in relation to my second research question.

In the next and final chapter, I address the research questions that motivated this thesis, as I seek to consolidate the findings in my study. I also discuss the contributions, possible implications and next steps for research in relation to my research.

Chapter 8.

Using Language as a Resource in the Mathematics Classroom: Insights and Reflections

The focus of my research resided in understanding the phenomenon of language as a resource for mathematics teaching and learning from the perspectives of experienced mathematics teachers. Data for this study was collected through task-based interviews with eleven mathematics teachers. Specifically, they were asked to reflect upon what they noticed and how they would respond to a series of (up to eight) tasks which were designed to illustrate situations embedded with potential language-related dilemmas and challenges in using the mathematics register. In this chapter, I consolidate the findings from my research. Based on the findings elaborated in Chapters 5 to 7, I first address the two research questions (declared in section 3.4) that motivated my study in sections 8.1 and 8.2 respectively:

1. How do teachers notice and use language as a resource for mathematics teaching and learning? In particular, how do their language-related dilemmas and orientations influence their noticing and use of language?
2. How are teachers' knowledge and potential usage of the mathematics register featured through their responses to teaching situations designed with language-related issues?

This is followed by a discussion of the contribution and implications from my research and suggestions of next steps for research in section 8.3. Finally, in section 8.4, I conclude my thesis with a short reflection of my own learning in this journey of understanding the role of language in mathematics education.

8.1. A Response to Research Question 1

Collectively, the data analysis in Chapters 5 and 6 supports how the language-related dilemmas (Adler, 1996, 2002 – see sub-section 3.1.2) and language-related orientations (Prediger et al., 2019 – see sub-section 3.2.2) are two useful and intricately connected theoretical constructs which can be used in a complementary manner to account for how and why teachers may notice and use language differently as a resource in their mathematics classrooms. In Chapter 5, it is evident that a teacher's deliberate actions in managing language-related dilemmas can provide insights on his/her language-related orientations. Conversely, in Chapter 6, a teacher's language-related orientations can explain why he/she may or may not face any tensions in situations of language-related dilemmas.

Correspondingly, both constructs helped to provide a networked understanding of how teachers notice and use language as a resource for mathematics teaching and learning. Specifically, teachers' views fall into two main categories. On the one hand, some teachers view language as a resource for the development of mathematical thinking and understanding, focusing on how mathematical meaning is constructed in and with language (labelled as Category A in sub-section 4.4.1 and elaborated in Chapter 5). On the other hand, some teachers view language as a resource for discourses and classroom interactions, focusing on how students use language to talk during mathematical discussions (labelled as Category B, elaborated in Chapter 6). Notably, these two views of language as a resource in mathematics classrooms by the teachers respectively coincide with two of the three lenses that research in the sub-field of language in mathematics education has converged on – the cognitive lens and the discursive/interactionist lens (see section 2.2).

In Category A (language as a resource for developing mathematical understanding), the primary concern is students' use of language to acquire and develop understanding of mathematical concepts. Language functions as a resource for mathematical thinking and learning, and both the mathematics register and students' everyday register play important roles. In particular, code-switching between the mathematics register and students' everyday register (Adler, 1996, 2002; Zazkis, 2000) is a resource (rather than a dilemma) that teachers use to help students make connections between their (developing) understanding of the mathematical concepts and the conventional interpretation in mathematics. These teachers also mediate and introduce the mathematics register as a visible resource whenever it helps students clarify confusion or gain a more in depth understanding of the mathematical concepts. For example, Karen mentioned that she would introduce the idea of absolute value (both the concept and the term), in order to help students differentiate between their two different understandings of small 'numbers' – one as small whole numbers and the other as small integers. In this situation, the specific terms from the mathematics register are taught in context, in terms of what it meant, how it is used and how it may relate to other terms, instead of only as vocabulary.

Additionally, there are situations when language is utilised as a transparent resource, where the mathematics register is taught explicitly (making language visible) without compromising the development of students' understanding (Adler, 1996, 2000, 2002). For instance, when teaching differential equations, Karen would bring in the necessary terms for students to work with right from the start, such as particular solutions and general solutions. With the knowledge of the mathematics register, students are better able to understand and access other related mathematical materials or resources to develop their understanding. Thus, these teachers view and use language (particularly the mathematics register) as a resource for both teaching and learning of mathematics. Their pedagogical strategies encompass language considerations in terms of what they would say and introduce to students in relation to the mathematical concepts and how students would be constructing the meaning of mathematical concepts with their language use.

In Category B (language as a resource for mathematics talk), the main emphasis is on students' participation and engagement in language activities to learn mathematics. For instance, with her experiential approach, Lena would create opportunities for students to participate and engage in learning mathematics through language activities such as collective problem solving and discussions with peers. Thus, language functions as a resource for students to talk about mathematics to learn mathematics. In other words, students are expected to explain their ideas and understanding in their own words and share it with others eventually to co-construct mathematical meaning together. These teachers are mostly concerned with students' ability to articulate their thinking clearly and coherently and be understood by others.

Notably, since teachers with this view are focused on how language impacts the unfolding of interactions or discussions and not how language can impact the sense-making of mathematical ideas, the mathematics register does not play a significant role or create any dilemmas in their view of language as a resource. To them, their primary goal is for students to engage in some form of mathematics talk with one another. It is a bonus if students are able to use the mathematics register during the interactions and discussions. Consequently, the act of code-switching between students' everyday register and mathematics register is not frequently viewed and used as a resource as teachers would only bring in the mathematics register to help resolve misunderstanding among students or to avoid confusion on what is being discussed. Also, these teachers tend to introduce the mathematics register *after* students have (more or less) developed their understanding of the mathematical concepts. In other words, the language (particularly the mathematics register) is not operating as a resource which students use to learn and think about mathematical concepts. Instead, it is perceived as names or labels that are attached to mathematical objects or ideas which students may or may not need to be aware of or use. Overall, teachers in this category view and use language as a resource for mathematical communication. Their pedagogical strategies are focused mainly on planning for tasks that encourage students to engage in mathematical talk or to explain their ideas through language.

Beyond the two main categories, another small but observable difference in how teachers may view and use language as a resource manifested through how they attended to language throughout the interviews. Some teachers focused mainly on the word-level usage while other teachers attended to language usage at large. The teachers who attended to language at the word-level usage tended to zoom in on the specific words from the mathematics register that were present in the tasks and centred their observations around them. For example, for Task 2 (Prime factorisation), Lena commented that, “I'm not really noticing tons more for language, other than divided, times, like equals, just those basic words”. This suggests that these teachers likely associate and view the notion of language in the mathematics classroom as mainly words from the mathematics register. Teachers, who attended to language usage at large, paid attention to both what and how the students were saying. In other words, they attended to both the word and discourse levels of language usage. They also considered the possible meanings that students were trying to convey and how successful they were in doing so. To them, the notion of language in the mathematics classroom is associated with both the mathematics register and the appropriate use of the mathematics register to express mathematical ideas. Moreover, although all the teachers eventually attended to language in some way (since it was the focus of my study), one very interesting observation I consistently made as the interviewer throughout all the interviews was that the teachers did not usually attend to language unless specifically asked to do so.

While I mainly highlighted the differences in how teachers may notice and use language in my findings, from the analysis, I also noted a couple of other common themes in relation to how they may deem language as a resource. Firstly, the idea of accessibility to mathematics through language came up briefly in many teachers' discussions. While they introduce and model the use of the mathematics register, they typically do not insist on students' use of the register at all times. By creating awareness of the mathematics register, these teachers use language as a resource to help students be familiar and comfortable with the language of mathematics and hopefully feel less alienated from the mathematical community. Perhaps, this observation has some connection with the third lens of research in the sub-field of language in mathematics education – the sociopolitical lens – which was not discussed in relation to the two views of language as a resource in my research.

Another common theme lies in the extent of emphasis that the teachers place on language. To most (if not all) teachers, language tends to take a secondary role as a resource rather than a primary one in their mathematics classrooms. They mostly attend to what students are thinking or understanding before noticing how students are saying and communicating their ideas. This is evident when considering some aspects of language-related orientations (Prediger et al., 2019 – see sub-section 3.2.2) of the teachers. Specifically, with respect to the orientation on perceiving language as a learning goal, almost all teachers expressed that their main goal and responsibility was to help students learn mathematics, to develop mathematical understanding, before developing students' mathematical communication abilities. Even in the case of Karen, who sees and uses language as a resource for developing mathematical understanding, she made one comment which specifically substantiated this theme:

> it is my job to get them to be precise, but my first job is to get them to do math. And then my second job or later job is to get them to be precise, so that they can communicate to an audience.

Consequently, most teachers tend to be less inclined to push for language use (especially the mathematics register) in their mathematics classrooms, though they would generally address the language demands appropriately when they notice them.

8.2. A Response to Research Question 2

The data analysis in Chapter 7 featured teachers' knowledge and potential use of the mathematics register, based on their responses to three tasks with language-related issues (selected from the eight tasks designed for this study), using the Mathematics Register Knowledge Quartet (Lane et al., 2019 – see sub-section 3.3.2). Specifically, it was observed that the teachers' knowledge of the mathematics register resides at different levels, in relation to the four dimensions of the Knowledge Quartet, thus leading to differences in their "knowledge-in-action" or how they potentially attend to or use the register in their respective mathematics classrooms.

Broadly speaking, within the *Foundation* dimension, all the teachers demonstrated an understanding of the mathematics register, particularly in relation to the specialised mathematical terms used in the respective topics. However, the teachers' understanding of these terms vary in terms of both precision and depth. Only a few teachers were noted to be more precise in their definitions of mathematical terms like *diagonal* (of a polygon) or *fractions*. These teachers were able to explain that a *diagonal* (of a polygon) is a line segment connecting any two *non-adjacent* vertices, instead of just any two vertices, or how fractions should be seen as numbers. Yet, there were also others who may emphasise how diagonals lie on the interior of polygons, which is certainly not true in the case of concave polygons. The difference in their depth of understanding was reflected in how some teachers were better able to make connections across related terms in the mathematics register than others. An example was illustrated by how Mindy connected the characterisation of *diagonal lines* with the concept of slope. Generally, there were also variations in the teachers' level of awareness of differences between the everyday and the mathematics registers. For instance, not all teachers recognised the differences in the everyday meaning and mathematical meaning of *diagonal* immediately. In contrast, some teachers even showed awareness of the differences through their concerns about how the differences between the two registers may contribute to students' misconceptions. A few examples were noted in how these teachers compared the use of *take away* instead of *subtraction* and the use of *top* and *bottom numbers* instead of *numerator* and *denominator* in their responses.

Consequently, the differences in the teachers' knowledge in the *Foundation* dimension led to differences observed in their articulated *knowledge-in-action* (Rowland et al., 2005) – how they attend to and use the mathematics register in their teaching – as evident in both the *Connection* and *Transformation* dimensions. In particular, the level of teachers' awareness of difficulties students may experience with the mathematics register (the *Connection* dimension) seems to be closely linked to their level of awareness of differences between the everyday and mathematics register (the *Foundation* dimension). For example, in the case of fractions, teachers who recognised the uncommon use of the terms *numerator* and *denominator* in the everyday context, were also the ones who pointed out how students would likely struggle to understand and use them these terms. On the same note, teachers who struggled to refer to numerator and denominator without using the word *number* were the ones who explicated students' difficulty in viewing a fraction as a number, instead of as a mathematical object made up of two numbers. While it was not intended as an outcome of my research, all the teachers seemed to have become more aware of the different difficulties students may have with the mathematics register, because of what they were prompted to attend to during the task-based interviews. However, the structure of the single-session, task-based interview with each teacher did not provide me with sufficient data to comment fully on the teachers' level of consistency in their use of the mathematics register within and between lessons and across topics, within the *Connection* dimension. Nonetheless, I was able to observe how most of the teachers were relatively inconsistent in their own use of the mathematics register during the interviews. As they did not appear to be conscious of their own inconsistencies in using the mathematics register during a research interview which clearly focused on language, I wondered if these inconsistencies might translate to their own classroom practices.

In relation to the *Transformation* dimension, while suggesting the various pedagogical strategies and activities in response to the tasks, the teachers tended to focus on the planning for understanding of the mathematical meaning behind the terms first before planning for the actual usage of the mathematical language. The pedagogical strategies and activities suggested to help students develop understanding of the mathematical concept, and thus the mathematics register, generally built on their knowledge in the *Foundation* dimension. For instance, motivated by the awareness that many students would misconstrue diagonal (of a polygon) as having to be diagonally oriented, several teachers proposed to the use of regular polygons which are not oriented in the usual way. Moreover, the teachers remarked that, besides teaching the mathematics register explicitly, they would create opportunities for students to discuss the meanings and potential differences when using the different register. Several teachers also emphasised how they would deliberately use the everyday register at times to help students understand or better relate to the mathematics register. Consequently, they also shared a range of representations and analogies which are useful in helping students understand the mathematics concepts or register. These representations and analogies include the use of atypical examples, real-life applications, mathematical manipulatives and the physical environment.

By contrast, it is perhaps not entirely clear how the teachers' *knowledge-in-interaction* (within the *Contingency* dimension) looks like in the actual classroom due to the nature of my data. However, I argue that the tasks provide simulated classroom situations which are similarly unplanned for and allow for the teachers to demonstrate their *knowledge-in-interaction* through their responses. In particular, all the teachers were observed to demonstrate a reasonable ability to interpret students' register in line with the mathematics register. Unsurprisingly, teachers with greater precision and depth in their understanding of the everyday and mathematics registers (*Foundation* dimension) were better able to provide explanations of what the students may be thinking based on what they said. For example, teachers who made the connection between the dual meanings of diagonal were quick to pick up what the student characters in the task were confused about, whereas other teachers needed more explicit prompting to realise that. In addition, as I was intentional in asking the teachers if they would correct students' use of the everyday register during the interviews, I was able to make inferences about their ability to facilitate an adherence to the mathematics register. In general, most of the teachers would model the use of the mathematics register in their classrooms. However, they may not necessarily correct students' mathematics or everyday register as strict adherence to the formal mathematics register is not their priority as mathematics teachers. Instead, their emphasis resides on students' understanding and ability to explain their ideas clearly and simply. Teaching decisions to demand for adherence of the mathematics register depend on the specific mathematics concepts, students' level of readiness for the register, the stage of the learning process (e.g., in discussion or presenting phases) and the audience involved.

For example, the use of *minus* and *negative* interchangeably (to refer to the negative sign) was raised by some teachers as one situation where they are more likely to correct students immediately. In this case, the demand for adherence is deemed as necessary, as the lack of precision in language use may potentially led to students' confusion when working with integers.

Finally, the teachers' knowledge of the mathematics register across the four dimensions of the Knowledge Quartet appears to be reflective of what they generally know or understand of the *mathematics register*. To end my interviews with all the participants, I asked if they have heard about or know about the term *mathematics register* (see Table 4.2) with the intent of eliciting their general understanding of the *mathematics register*, which basically frames my research. Notably, teachers (e.g., Karen and Evie), who posit that the *mathematics register* represents the language used by the mathematics community to talk about mathematics or the way to express the "set of ideas that is specific to the field of mathematics" with "its own language structure", tend to demonstrate a more precise and deeper understanding in the *Foundation* dimension of the Knowledge Quartet. Consequently, their attention to language tends to be broader as they are more deliberate in their planning for students to use language to acquire and develop understanding of mathematical concepts, as well as for communication. In comparison, teachers (e.g., Lena and Faye), who associate the mathematics register with mainly mathematical vocabulary and terminologies or view it as a convention to "label things with names", tend to have more limited understanding in the *Foundation* dimension. Therefore, their attention to language resides at only the word level and centres around the acquisition of new vocabulary (deemed necessary for communication) when they plan for mathematical language in their teaching.

8.3. A Contribution and Some Implications to Mathematics (Teacher) Education Research

Through my research, I have shared findings on how teachers attend to language, particularly in relation to how they notice and use language as a resource for teaching and learning of mathematics. My research has attempted to address the gap in research literature on teachers' existing views on language as a resource in mathematics education. In particular, my research has provided many instances of how experienced teachers notice and respond to language differently in content-specific teaching situations involving language-related dilemmas (Adler, 1996, 2002) or challenges students may face with the mathematics register (Pimm, 1987). Specific to the Mathematics Register Knowledge Quartet (Lane et al., 2019), the discussion in Chapter 7 has expanded the description and understanding of Lane et al.'s Knowledge Quartet through the illustration of content-specific examples in all four dimensions. As my intent was neither to evaluate nor to judge teachers' knowledge of the mathematics register (which would be a more deficit-oriented discussion), my research has also provided an example of how the use of the Knowledge Quartet can create opportunities for researchers to understand teachers' need with regard to using language as a resource for mathematics teaching and learning.

Moreover, by adopting a complementary approach in co-ordinating the use of the three theoretical constructs – teachers' language-related dilemmas (Adler, 1996, 2002), language-related orientations (Prediger et al., 2019) and knowledge of the mathematics register (Lane et al., 2019) – to account for teachers' use of language as a resource, the findings also highlighted the relationships between teachers' practice, orientations and knowledge relating to language (particularly the mathematics register) as a resource in mathematics education. Correspondingly, through analysing teachers' language-related practices, orientations and knowledge, my research also provides data from teachers' perspectives to concretise the theorisation of language as a resource, which has been argued by Planas (2018) as a necessary move in mathematics education.

Additionally, due to the various restrictions and consequences of the COVID-19 pandemic situation (see Chapter 4), my options for data collection were limited. While I might not have actual classroom observation data to substantiate directly how teachers notice and use language in the classroom, my research method – the use of task-based interviews – has provided a novel alternative to understand the phenomenon through teachers' articulated practices, orientations and knowledge in relation to language as a resource in their mathematics classrooms. In particular, the use of reflection tasks designed deliberately to illuminate situations when teachers can connect to their experiences of using language (Mason, 2002) has proven to be an effective method in prompting teachers' reflection of their own knowledge and practices in relation to the use of language during the interviews. Additionally, using fictitious but relatable classroom scenarios as the basis of discussion in the interview may even allow for deeper introspection on the part of the teachers, as they are less likely to be concerned about how the interviewer may be evaluating their actual practices (e.g., using classroom observation data).

Overall, my research contributes towards a better understanding of the existing state of teachers' language-related practices, orientations and knowledge of the mathematics register. This, in turn, helps to inform the design of teachers' professional development (PD) programs focusing on language-and learner-responsive mathematical teaching (e.g., Adler, 2021; Planas et al., 2022). More specifically, the reflection tasks that I have designed for this study can potentially contribute to the design of these PD programs in two aspects. The tasks can be used as a tool to surface teachers' pre-existing orientation and knowledge before PD sessions as it should not be assumed that all teachers consider language as a resource. Consequently, more targeted PD sessions can be designed to address teachers' gaps or support teachers' needs. The tasks can also be used during PD sessions to encourage reflections and discussions around noticing and using language as a resource for mathematics teaching and learning, and consequently build teachers' knowledge of the mathematics register.

Finally, my research also sheds light on some broader implications in terms of current mathematics teacher education. Particularly, the lack of emphasis on the role of language and the mathematics register in mathematics teacher education (at all levels) stood out as one apparent observation which I made across all the interviews. Most, if not all, the participants mentioned that they were introduced to or became more aware of the notion of mathematics register through learning about and participating in my research. Hence, there appears to be a compelling need to increase mathematics teachers' awareness and knowledge of the mathematics register, so as to bring their attention to the importance of viewing and using language as a resource for mathematics teaching and learning. Only with greater awareness and knowledge can mathematics teachers better attend to and notice their own use of language and students' use of language in their classrooms. In other words, mathematics teacher education needs to expand its focus in terms of PD programs to include those that are specifically designed with a focus on thinking about or using language as a resource for mathematics teaching and learning.

Perhaps, one other interesting implication, which emerged as an after-thought from my research, is the difference in teachers' perceptions towards language as a resource for teaching (or teachers) and language as a resource for learning (or learners). Notably, amongst the teachers I interviewed for my study, there are teachers who choose to be the ones modeling the proper use of mathematics register, and often not insist on students' use in the same way. As such, these teachers seem to treat language as mainly a resource for themselves in enhancing their own teaching and not one which necessarily translates into a resource for students. In comparison, there are also teachers who model the use of the register and similarly expect students to use proper language as they develop mathematical understanding and communicate with one another. These teachers are perhaps the ones who are more likely to perceive language as both a resource for their own teaching and their students' learning. In other words, language is considered as a resource which influences these teachers' pedagogical strategies and decisions made in their mathematics classroom. Concurrently, language is also considered as a resource which helps students develop their understanding of mathematical concepts and communication skills in the learning process.

8.4. A Closing Reflection and Some Next Steps

Like most of the teachers in my study, I had similarly not heard of the mathematics register, nor seriously considered how language can be a resource in the mathematics classroom as a teacher. As I shared my motivation at the start of this thesis, language was, instead, often deemed as a problem to most teachers (including myself) in my teaching context. However, the decision to embark on a Mathematics Education Ph.D. program has provided me an opportunity to learn about and reconsider the role of language in mathematics education. In particular, I began to glean more clarity and knowledge in relation to what Durkin (1991) meant by "mathematics education begins in language" (p. 3) in the epigraph I chose for my research. I was also greatly inspired by the potential of language as a resource after I learnt about the notion of the *mathematics register* and Vygotsky's theories on language and thought. Consequently, language as a resource became a central theme in my research journey.

As I looked back at this entire journey so far, it has certainly not been an easy or smooth one from conceptualising the study to writing this thesis. Not to mention, a bulk of it took place during a time (of uncertainty) when many things and activities that we have always taken for granted almost came to a halt or had to change. Yet, it was also a journey which has helped me grow as an early researcher. For instance, the central theme of language as a resource propelled me to conscientiously stay away from applying or adopting the deficit lens in my research, though I must admit that it was a challenge, considering how I similarly considered language as a problem as a mathematics teacher initially. However, the many interactions with researchers who shared similar research interests were certainly valuable in helping me rethink and refine my research and focus whenever I was in doubt. During the data collection process, I was also intrigued and glad at how my research has begun to prompt more teachers to attend to the role of language in mathematics education. Specifically, two of the participants, Karen and Nadia, shared, at the end of their interviews, how they were beginning to reflect on their own use of language and also the need to do so.

Moreover, as I reflect on what I have chosen to focus on and address through this thesis, I am cognizant of some gaps in my research. In particular, my two research questions did not explicitly address the relationship between teachers' language-related orientations and their knowledge of the mathematics register. Although I argue that they are inherently connected, especially in terms of how the *Foundation* dimension of the Knowledge Quartet and the orientations were defined in relation to beliefs, it may be worth spending some time to explicate their connections through further analysis. Additionally, many of the inferences I made in relation to the respective teachers' orientations and knowledge in this thesis can perhaps be further substantiated with actual classroom observations. For instance, it was challenging to ascertain fully if the teachers would focus on developing conceptual understanding before procedures (orientation O5 – see sub-section 3.2.2) or use the mathematics register consistently within and between lessons and across topics (in the *Connection* dimension – see subsection 3.3.2), based on a single-session interview.

While this final paragraph marks the end of this thesis, it clearly does not signal an end to this research journey. As I proceed to the next phase of my research, I hope to look beyond teachers' language-related dilemmas, orientations and knowledge of the mathematics register and consider other factors (e.g., the notion of access to the language of mathematics) which might influence how teachers notice and use language as a resource. I also look forward to expanding the scope of my data by including other topics in design of the reflection tasks and interviewing more teachers, including those teaching in other educational contexts (e.g., Singapore). Last but not least, I certainly intend to use what I have learnt from my research to design PD courses for teachers to engage in discussions and further their perspectives and knowledge relating to the use of language as a resource for both mathematics teaching and mathematics learning.

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Appendix A.

Interview Questions (Pilot Study – First Iteration)

1. Could you share briefly about your own academic background (prior to and within math education)?
2. Could you share briefly about your experience in teaching? What about the teaching of mathematics specifically? This can include the schools, levels and students you have taught before.
3. With a certain group of students in mind, could you describe briefly a typical mathematics lesson you would have with them? You may want to share some information on the students, the activities (both teacher and student) and the strategies or resources you use.
4. What do you consider to be the most important tasks for you in the teaching of mathematics to this group of students? Why are these important to you as a mathematics teacher?
5. What do you consider as rewards/achievements in teaching this group of students? Correspondingly, what are some problems or challenges you face in teaching mathematics to this group of students? Why and how do you overcome these challenges?
6. Getting back to the focus on language in mathematics education, what do you consider to be language-related issues that arise in your teaching and student learning? Why do you think these are language-related issues? How do you try to address or overcome these issues?
7. What are your views on the teaching and learning of the mathematics language/ register in the classroom? Do you find yourself teaching it explicitly? Or do you think students need to have mastery of the mathematics register for the development of mathematical concepts?
8. If I were to ask you to tell me something that you will be very concerned about when you are listening or when you are in a discussion with your students, what would that be? Or supposed in a discussion with your students, you happen to hear that the students are not using formal mathematical language, what would you do?
9. Being in a multilingual classroom, have you faced any situations when you need to change the language of instruction or switch between the mathematics language and the everyday language? What did you do and why?
10. In your opinion, what constitutes the mathematics language or the mathematics register? What do you notice in terms of your own knowledge of the mathematics language /register? How do you develop your own knowledge in this area?

Appendix B.

Interview Protocol for Main Study

1. Could you briefly share about your own academic background with regard to teaching and learning mathematics?
2. <u>Reflection Tasks' Prompts</u>
    a. What do you notice in what the students are saying in this dialogue?
        - What do you notice about the language the students are using in this dialogue?
        - Why do you think the students say that? / What do you think the students are thinking about?

    b. How would you respond if you were a teacher in this situation? Why will you do/say that?
        - If you were a teacher in this situation,
            - Will you step in to modify the language the students are using? *(for tasks in Categories 2 and 3)*
            - Will you step in to teach the formal/ mathematical terms? (for all tasks, especially Category 1)
            - Will you switch between formal and informal use of language or between mathematical and everyday usage of terms? *(for tasks in Categories 2 and 3)*
        - If no (to any of the above) ⇒ Why not?
        - If yes (to any of the above) ⇒ When?

    c. Have you experienced something similar before in your classroom? Can you share what happened? What did you do then?
        - Can you think of other instances when students have used everyday language to connect mathematical ideas or everyday words which may be used differently in the mathematical context? *(for tasks in Category 4)*
        - What are some examples of language (e.g. specific words/ ways of explaining) you usually use in teaching this topic? When/How do you use the language in your teaching?
3. Just wondering, have you heard about the term "mathematics register"? What do you know about it? Can you elaborate?