# Explicitly Representing Vector Bundles over Elliptic Curves

by

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# Abstract

Algebraic vector bundles are a construction useful for studying the geometry of varieties; they are objects which associate a vector space to each point of the variety in a "polynomial" fashion. These bundles can be explicitly represented via *transition matrices*, which encode how the vector spaces vary as one moves along the variety. In 1957, Sir Michael Atiyah showed that every indecomposable bundle over a smooth elliptic curve was determined by a point on the curve, and two invariants; the rank and degree. However, his work is not entirely explicit—using his results, we obtain explicit representations of the bundles in terms of transition matrices. As an application, we present a constructive proof of global generation for certain indecomposable bundles over elliptic curves.

**Keywords:** Vector Bundles; Elliptic Curves; 14H52; 14H60

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# Introduction

Algebraic geometry is the study of solutions to systems of polynomial equations, by interpreting such systems as topological spaces and examining the underlying geometry. Spaces defined by polynomial equations in this way are called *algebraic varieties*. There are many interesting families of algebraic varieties—among these are *elliptic curves* (Definition 2.1). They are smooth complete curves of genus one, together with a marked point which we call the *base point*. Elliptic curves form one of the first non-trivial classes of varieties, making them a useful testing ground for conjectures. They are famous for their application in the proof of Fermat's Last Theorem, as well as their use in cryptography.

One way to study the geometry of an algebraic variety is by using vector bundles (Definition 2.21)—objects which associate a vector space to every point of the variety in a "polynomial" fashion. These allow one to probe subvarieties of the base variety—particularly those with nice properties, such as complete intersections. There are two invariants which can be associated to any vector bundle: the *rank*, which is the dimension of the fibres, and the *degree* (Definition 2.32), which roughly encodes how twisted the bundle is, see Figure 1.1.

The Möbius strip is an example of a rank one vector bundle over the projective real line (a circle), see Figure 1.1. Topologically (i.e. up to homeomorphism), this is the unique non-trivial vector bundle over the circle. Algebraically however, there is a distinction between clockwise and counter-clockwise twists, and the degree essentially tracks the number of twists. Note that in the interest of visualisation, we have truncated the vector spaces—they are meant to continue on infinitely.

Unlike for vector spaces, in general there are rank two or higher bundles which cannot be decomposed into a direct sum of rank one vector bundles—these are called *indecomposable* bundles. In the case of the projective line however, there is a well-known result of Grothendieck that every vector bundle decomposes into line bundles (bundles of rank one) [5].

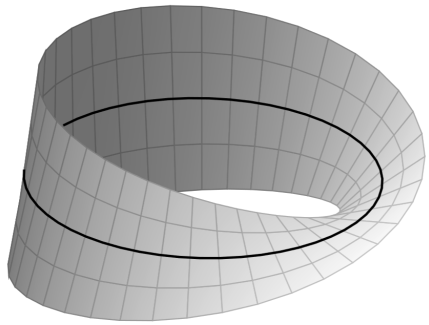

Figure 1.1: A rank one vector bundle.

**Theorem 1.1.** *Every vector bundle over $\mathbb{P}^1$ decomposes as a direct sum of line bundles. More precisely, for any vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ of rank $r$, we have*

$$\mathcal{E} \cong \mathcal{O}_X(k_1) \oplus \mathcal{O}_X(k_2) \oplus \cdots \mathcal{O}_X(k_r)$$

*where $k_1 \geq k_2 \geq \cdots \geq k_r \in \mathbb{Z}$.*

Hazewinkel and Martin provided an elementary proof of this result by providing so-called *transition matrices* for vector bundles over the projective line [9]. Transition matrices encode how the vector spaces associated to each point vary as one moves along the variety. The use of transition matrices is one of the standard ways to explicitly work with vector bundles.

**Proposition 1.2.** *Every vector bundle over $\mathbb{P}^1$ can be represented by a single transition matrix of the form*

$$M_{01} = \begin{pmatrix} x^{k_1} & 0 & 0 & \cdots & 0 \\ 0 & x^{k_2} & 0 & & 0 \\ 0 & 0 & x^{k_3} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{k_r} \end{pmatrix}$$

*where $k_1 \geq k_2 \geq \cdots \geq k_r \in \mathbb{Z}$ and $r$ is the rank of the bundle.*

Classically, vector bundles have been challenging to study—even over projective $n$-space, there are many unanswered questions regarding indecomposable vector bundles [8]. For instance, there is a classical result which shows that there is always an indecomposable bundle of rank $n-1$ over $\mathbb{P}^n$ where $n$ is odd [8]. Additionally, a famous conjecture of Hartshorne states that there are no indecomposable bundles of rank two when the dimension of the space is sufficiently large [6, Section 6]:

**Conjecture 1.3** (Hartshorne)**.** *If $n \geq 7$, then there are no indecomposable vector bundles of rank two over $\mathbb{P}^n$. In other words, every vector bundle of rank two decomposes into a direct sum of line bundles.*[1]

These suggest that the indecomposability of vector bundles over projective space is generally very erratic and hard to predict. However, for elliptic curves, Atiyah classified indecomposable vector bundles in his 1957 paper [1]:

**Theorem 1.4** (Atiyah)**.** *For any rank $r$ and degree $d$, there exist indecomposable vector bundles $\mathcal{E}$ of rank $r$ and degree $d$ over an elliptic curve. Furthermore, there is an isomorphism*

$$\mathcal{E} \cong \mathcal{E}(r, d) \otimes \mathcal{L}$$

*where $\mathcal{E}(r, d)$ is a unique vector bundle of rank $r$ and degree $d$, and $\mathcal{L}$ is a degree zero line bundle.*

This shows that there are bundles of arbitrary rank which do not decompose into a direct sum of line bundles, unlike Theorem 1.1. Moreover, since degree zero line bundles over an elliptic curve are in correspondence with points of the curve (Lemmas 2.8, 2.30), this shows that every indecomposable bundle over an elliptic curve is completely determined by its rank, degree, and a point on the curve. Atiyah constructs the distinguished bundles $\mathcal{E}(r, d)$ inductively via certain non-split extensions (Definition 2.27). Our main contribution in this thesis is to make this construction explicit—we provide transition matrices representing the indecomposable bundles $\mathcal{E}(r, d)$, and extend this to further find transition matrices which represent any indecomposable bundle over an elliptic curve.

**Construction 1.5** (Z.)**.** *Let $X$ be an elliptic curve over an algebraically closed field $\mathbb{K}$ with char $\mathbb{K} \neq 2$, and let $P \in X$. For any rank $r$ and degree $d$, we provide an open cover $U_1, U_2$ of $X$ (depending on $P$) and a matrix $M_P(r, d)$ such that the vector bundle given by $M_P(r, d)$ is isomorphic to the indecomposable bundle $\mathcal{E}(r, d) \otimes \mathcal{O}(P - O)$ (Definition 2.29).*

Many applications of vector bundles come from understanding the *global sections* of the bundle: regular functions mapping each point on the variety to a vector in its corresponding vector space. The sections can be used to understand subvarieties of the underlying variety, or find mappings of the variety into projective space or Grassmannians. Such mappings are possible when the global sections *globally generate* the bundle (Definition 2.44, Figure 1.2). In Figure 1.2, we provide an example of global sections on a line bundle over a curve. They *globally generate* the bundle if they span every fibre of the bundle. For instance, at the fibre indicated by the dashed line, the section $s_2$ does not span, but $s_1$ does, so that

[1] Hartshorne's conjecture is also commonly referred to as a statement about the existence of so-called complete intersections of small co-dimension. The connection can be found in [6, Section 6].

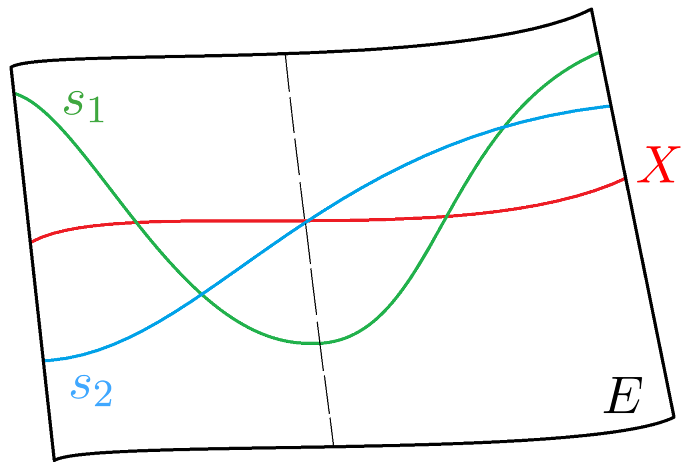


Figure 1.2: An example of global sections on a line bundle over a curve $X$.

together they globally generate this bundle. Using Riemann-Roch (Theorem 2.33), one can prove that a line bundle over an elliptic curve is globally generated if and only if its degree is at least two. There is an analogous argument which can be used to extend this result to arbitrary rank indecomposable bundles over an elliptic curve:

**Theorem 1.6** (Z.)**.** *Let $X$ be any elliptic curve over an algebraically closed field $\mathbb{K}$ (no restriction on characteristic), and let $\mathcal{E}$ be a non-trivial indecomposable vector bundle over $X$ of rank $r$ and degree $d$. Then $\mathcal{E}$ is globally generated if and only if*

$$d \geq r + 1.$$

As an application of our transition matrices, we provide an explicit proof of this result for the distinguished bundles $\mathcal{E}(r, d)$. Additionally, we have included software to compute global sections of a bundle associated to any upper triangular transition matrix over a suitable open cover.

Our study of vector bundles over elliptic curves was partially motivated by the following problem:

**Problem 1.7.** *Given an ample vector bundle $\mathcal{E}$ over an elliptic curve $X$, what is the smallest $m$ such that $\mathcal{E}^{\otimes m} \otimes \omega_X$ is globally generated, where $\omega_X$ is a canonical line bundle over $X$ (in this case $\omega_X \cong \mathcal{O}_X$, the structure sheaf of $X$. See [7, Chapter 2, Section 2]).*

There is a conjecture by Fujita in 1985 which addresses the case of line bundles [4]:

**Conjecture 1.8** (Fujita's Freeness Conjecture)**.** *If $X$ is a projective variety, and $\mathcal{L}$ is an ample line bundle over $X$, then $\mathcal{L}^{\otimes m} \otimes \omega_X$ is globally generated if $m \geq \dim X + 1$, where $\omega_X$ denotes a canonical line bundle over $X$.*

Via our construction, we were able to compute various examples which provide evidence towards an extension of this conjecture:

**Conjecture 1.9.** *If $X$ is a projective variety and $\mathcal{E}$ is an indecomposable ample vector bundle of rank $r$ over $X$, then $\mathcal{E}^{\otimes m} \otimes \omega_X$ is globally generated if*

$$m \geq \dim X + \text{rank } \mathcal{E}$$

We note that in the case of $X$ an elliptic curve, there is an ample bundle $\mathcal{E}$ over $X$ for which the bound is sharp.

In Chapter 2, we will cover the necessary preliminaries to discuss the central results, including a review of the results obtained by Atiyah in [1]. Chapter 3 contains the construction of the transition matrices. Chapter 4 consists of some of the immediate applications of the transition matrices, including an explicit proof of global generation for the distinguished bundles, as well as a tool for computing global sections. In the Appendix, we have included the code for software to compute global sections given a transition matrix, implemented in SageMath.

# Chapter 2

# Preliminaries

In this chapter, we will review or introduce the necessary tools to describe and prove our central results. We assume the reader is familiar with the basics of algebraic geometry via e.g. [7, Chapter 1] or [10, Chapter I]. We also assume the reader is familiar with some of the basics of sheaves of $\mathcal{O}_X$-modules from the beginning of [7, Chapter 2, Section 5]. Throughout this chapter, we assume $\mathbb{K}$ is an algebraically closed field and char $\mathbb{K} \neq 2$.

## 2.1 Elliptic Curves

**Definition 2.1.** An elliptic curve over $\mathbb{K}$ is a smooth irreducible projective (i.e. a variety in $\mathbb{P}^n_{\mathbb{K}}$ for some $n$) curve of genus one, together with a marked point which we denote by $O$.[1]

From Proposition 1.7 in [12, Chapter 3], as long as char $\mathbb{K} \neq 2$, every elliptic curve can be represented in *Legendre form*: as the zero locus in $\mathbb{P}^2$ of the polynomial

$$x_2^2 x_0 = x_1(x_1 - x_0)(x_1 - \lambda x_0) \tag{2.1}$$

where $\lambda \in \mathbb{K}\backslash\{0, 1\}$. In this case, the marked point is $O = (0 : 0 : 1)$. For the rest of this chapter, as well as in Chapter 3, we will assume any given elliptic curve is in Legendre form, though we note that for the general proof of global generation (Theorem 4.1), this is not necessary.

From this representation, we have a natural open covering of $X$ given by the standard affine open covering of $\mathbb{P}^2$. Recall that

$$\mathbb{P}^2 = D_+(x_0) \cup D_+(x_1) \cup D_+(x_2) \tag{2.2}$$

[1] Some authors do not insist that elliptic curves come with a marked point.

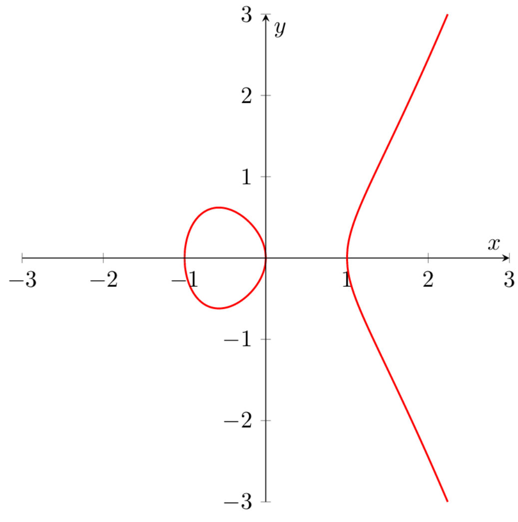


Figure 2.1: The real zero locus of an elliptic curve in Legendre form, on an affine patch of $\mathbb{P}^2$.

where $D_+(x_i)$ are the distinguished open sets where $x_i \neq 0$. We set

$$\begin{aligned} U_0 &= D_+(x_0) \cap X \\ U_2 &= D_+(x_2) \cap X \end{aligned} \tag{2.3}$$

In other words, $U_0$ is all of $X$ except for the point $O$, and $U_2$ is all of $X$ except the three points $(1:0:0), (1:1:0)$, and $(1:\lambda:0)$. Therefore, we have that $U_0 \cup U_2$ is an open (affine) cover of $X$. For our discussion, it is useful to understand the structure sheaf of $X$ explicitly on this open cover. First, on the open set $U_0$, we have that $x_0 \neq 0$, and hence we define

$$x := \frac{x_1}{x_0},\ y := \frac{x_2}{x_0}. \tag{2.4}$$

Dividing (2.1) by $x_0^3$ yields

$$y^2 = x(x-1)(x-\lambda),$$

so that the ring of regular functions on $U_0$ is[2]

$$\mathcal{O}_X(U_0) = \mathbb{K}[x,y]/\langle y^2 - x(x-1)(x-\lambda)\rangle.$$

[2]Here we are adopting Hartshorne's notation for the ring associated to an open set over a sheaf. Other conventional notations are $\Gamma(\mathcal{O}_{U_0})$ or $H^0(U_0, \mathcal{O}_X)$.

On $U_2$, we have $x_2 \neq 0$, so we have the two monomials

$$xy^{-1} = \frac{x_1}{x_2},\ y^{-1} = \frac{x_0}{x_2},$$

and if we divide (2.1) by $x_2^3$, then we obtain

$$y^{-1} = xy^{-1}(xy^{-1} - y^{-1})(xy^{-1} - \lambda y^{-1}),$$

so that the ring of regular functions on $U_2$ is

$$\mathcal{O}_X(U_2) = \mathbb{K}[xy^{-1}, y^{-1}]/\langle y^{-1} - xy^{-1}(xy^{-1} - y^{-1})(xy^{-1} - \lambda y^{-1})\rangle.$$

Finally, on the intersection $U_0 \cap U_2$, we simply note that $y$ becomes invertible. Therefore, in summary we have

$$\begin{aligned} \mathcal{O}_X(U_0) &= \mathbb{K}[x, y]/\langle y^2 - x(x-1)(x-\lambda)\rangle \\ \mathcal{O}_X(U_2) &= \mathbb{K}[xy^{-1}, y^{-1}]/\langle y^{-1} - xy^{-1}(xy^{-1} - y^{-1})(xy^{-1} - \lambda y^{-1})\rangle \\ \mathcal{O}_X(U_0 \cap U_2) &= \mathbb{K}[x, y^{\pm 1}]/\langle y^2 - x(x-1)(x-\lambda)\rangle. \end{aligned} \tag{2.5}$$

## 2.2 Divisors and Valuations

Vital to our central construction will be the use of valuations. We will also cover divisors, as Cartier divisors yield a useful way to represent line bundles. We will do so on smooth projective curves, although we note that every definition we give can be extended appropriately to higher dimensional projective varieties. One may find more information on discrete valuations from [2, Chapter 9], and for a discussion on divisors, see [7, Chapter 2, Section 6] or [10, Chapter III, Section 1].

**Definition 2.2.** Let $R$ be a commutative ring with unity. Then a discrete valuation on $R$ is a map $\nu : R \to \mathbb{Z} \cup \{\infty\}$ satisfying

1. $\nu(r) = \infty$ if and only if $r = 0$
2. $\nu(rs) = \nu(r) + \nu(s)$
3. $\nu(r + s) \geq \min\{\nu(r), \nu(s)\}$

Here, we are defining $\infty + a = \infty$ and $\min\{\infty, a\} = a$ for any $a \in \mathbb{Z}$.[3]

**Example 2.3.** The simplest example of a non-trivial discrete valuation is on $\mathbb{K}[x]$: if we set

$$\nu(f) = -\deg(f),\ \nu(0) = \infty,$$

[3]Some authors exclude 0 from the domain of $\nu$, to avoid dealing with $\infty$.

then $\nu$ is a discrete valuation on $\mathbb{K}[x]$. $\triangle$

Given a smooth projective curve $X$, the order of vanishing of a function on a point $x \in X$ forms a valuation on the local rings $\mathcal{O}_{X,x}$—the rings of rational functions which are regular (non-singular) at $x$. Consequently, we also get a valuation on the ring of rational functions $\mathbb{K}(X)$: Let $x \in X$ be a point on the curve. Since $X$ is smooth, each local ring $\mathcal{O}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$ which is principal since $X$ is a curve. Given a non-zero rational function $f = \frac{p}{q} \in \mathbb{K}(X)^*$, where $p$ and $q$ are elements of the local ring $\mathcal{O}_{X,x}$, we have

$$p = t^k p', \ q = t^\ell q'$$

for some units $p', q' \in \mathcal{O}_{X,x}$ and where $t$ is a generator of $\mathfrak{m}_x$.

**Definition 2.4.** The *order of vanishing of $f$ at $x$* as

$$\mathrm{ord}_x(f) = k - \ell.$$

If $\mathrm{ord}_x(f)$ is strictly positive, we say *$f$ vanishes at $x$*; if strictly negative, we say *$f$ has a pole at $x$*. The order of vanishing at $x$ forms a valuation, and we will often instead denote it as

$$\nu_x(f) := \mathrm{ord}_x(f), \ \nu_x(0) := \infty$$

To understand the geometry of a projective curve, it is useful to study rational functions defined on the variety; and in particular, on the vanishing of these functions along the variety. This is captured via *divisors*.

**Definition 2.5.** Let $X$ be a smooth projective curve. A *Weil divisor on $X$* is a formal sum of points

$$D = \sum_{x \in X} n_x \cdot x, \ n_x \in \mathbb{Z}$$

where all but finitely many $n_x = 0$. The *degree* of a divisor is the sum of the coefficients

$$\deg D = \sum_{x \in X} n_x.$$

Given a non-zero rational function $f \in \mathbb{K}(X)^*$, we define the *principal divisor*

$$\mathrm{div}\ f = \sum_{x \in X} \nu_x(f) \cdot x.$$

Given an open set $U \subset X$, we can restrict a divisor to only points in $U$, which we denote by $D|_U$. We say a divisor $D$ is *effective* if all of its coefficients are non-negative, and denote this by $D \geq 0$.

More generally, divisors over a smooth projective variety are formal sums of co-dimension one subvarieties, but in the case of a curve, a co-dimension one subvariety is just a finite union of points on the curve. This partially motivates our restriction to curves, since we obtain a useful invariant in the degree (the number of points counted with multiplicity).

**Remark 2.6.** Principal divisors are indeed Weil divisors, since a rational function on a curve has only finitely many points where it vanishes or contains a pole (Lemma 6.1 in [7, Chapter 2]). Furthermore, principal divisors have degree zero (Proposition 6.4 in [7, Chapter 2]).

**Definition 2.7.** Two divisors $D_1$ and $D_2$ are said to be *linearly equivalent*, denoted by $D_1 \sim D_2$, if there exists $f \in \mathbb{K}(X)^*$ such that

$$D_1 - D_2 = \text{div } f.$$

A *linear equivalence class* $\overline{D}$ is the set of all divisors linearly equivalent to $D$. By Remark 2.6 above, we can see that the degree of all divisors in a class are equal, and hence we define the *degree* of a linear equivalence class to be the degree of any representative.

Linear equivalence classes of divisors over elliptic curves are particularly well-behaved.

**Lemma 2.8.** *If $X$ is an elliptic curve and $P, Q \in X$, then $P \sim Q$[4] if and only if $P = Q$. Furthermore, any degree zero divisor on $X$ is linearly equivalent to $P - O$ for a unique point $P \in X$.*

*Proof.* This is Lemma 3.3 and Proposition 3.4 (a) in [12, Chapter 3]. $\square$

**Corollary 2.9.** *If $X$ is an elliptic curve, then every degree one divisor is linearly equivalent to one of the form $P$ for some unique $P \in X$.*

*Proof.* Suppose $D$ is a Weil divisor of degree one. Then $D - O$ is of degree zero, and hence by Lemma 2.8 above, we have that it is linearly equivalent to $P - O$ for some unique $P \in X$, so that $D \sim P$. $\square$

When $X$ is an elliptic curve, with the open covering $U_0, U_2$ as in (2.3), we can find special elements in $\mathcal{O}_X(U_0 \cap U_2)$ which attain all values for the valuation $\nu_O$. Recall from

[4]Here, we are interpreting the points $P$ and $Q$ as the divisors $1 \cdot P$ and $1 \cdot Q$. In [12], this is denoted $[P]$—but in the interest of simplifying notation, we exclude this.

(2.5) that $\mathcal{O}_X(U_0 \cap U_2) = \mathbb{K}[x, y^{\pm 1}]/\langle y^2 - x(x-1)(x-\lambda)\rangle$. Set

$$\omega_k = \begin{cases} \overline{y^{-k/3}} & \text{if } k \equiv 0 \mod 3 \\ \overline{xy^{-(k+2)/3}} & \text{if } k \equiv 1 \mod 3 \\ \overline{x^2y^{-(k+4)/3}} & \text{if } k \equiv 2 \mod 3 \end{cases} \tag{2.6}$$

**Lemma 2.10.** *We have $\nu_O(\omega_k) = k$.*

*Proof.* In Example 3.3 in [12, Chapter 2], Silverman computes the order of vanishing of the coordinate functions along an elliptic curve in Legendre form. We find that $\nu_O(\overline{x}) = -2$ and $\nu_O(\overline{y}) = -3$, so that

$$\nu_O(\omega_k) = \begin{cases} -3\left(\frac{-k}{3}\right) & \text{if } k \equiv 0 \mod 3 \\ -2 - 3\left(\frac{-(k+2)}{3}\right) & \text{if } k \equiv 1 \mod 3 \\ -4 - 3\left(\frac{-(k+4)}{3}\right) & \text{if } k \equiv 2 \mod 3 \end{cases}$$

which evaluates to $k$ in all cases. □

Later, when we are explicitly working with global sections of vector bundles such as in Theorem 3.19 or Theorem 4.8, the interactions between the rings of regular functions on the open cover (2.3) will be important to understand. In particular, we want to understand how the rings behave with respect to the valuation at $O$. Recall that $\mathcal{O}_X(U)$ is the ring consisting of rational functions which are regular everywhere on $U$.

**Lemma 2.11.** *1. The ring $\mathcal{O}_X(U_0)$ contains $\omega_k$ for $k \leq -2$.*

*2. There are no elements of valuation $-1$ at $O$ in $\mathcal{O}_X(U_0)$. In particular, the element $\omega_{-1}$ is not in the ring.*

*3. If $f \in \mathcal{O}_X(U_0)$, then $\nu_O(f) \leq 0$.*

*Proof.* In the construction of $\omega_k$ (2.6), taking $k$ sufficiently negative will make the exponent in $\overline{y}$ non-negative. We separate these into the three cases defining $\omega_k$:

1. If $k \equiv 0 \mod 3$ and $k \leq -2$, then in fact $k \leq -3$, and so $\omega_k$ is in fact a positive exponent of $\overline{y}$.

2. If $k \equiv 1 \mod 3$ and $k \leq -2$, then the exponent in $\overline{y}$ is non-negative.

3. If $k \equiv 2 \mod 3$ and $k \leq -2$, then in fact $k \leq -4$, so that the exponent in $\overline{y}$ is non-negative.

Therefore, for $k \leq -2$, the $\omega_k$ are monomials in $\overline{x}$ and $\overline{y}$ with non-negative exponents. Recall from 2.5 that

$$\mathcal{O}_X(U_0) = \mathbb{K}[x, y]/\langle y^2 - x(x-1)(x-\lambda)\rangle,$$

establishing (1).

Now suppose an element $f \in \mathcal{O}_X(U_0)$ has valuation $-1$ at $O$. Since it is regular on $U_0$ and is a rational function, it must therefore vanish at exactly one point $P \in X$, so that $\operatorname{div} f = P - O$ and $P \neq O$. This contradicts Lemma 2.8 and establishes (2).

Finally, we prove (3) by the contrapositive: Suppose $f \in \mathbb{K}(X)$ is such that $\nu_O(f) > 0$. Then $f$ must have a pole at some point $P \neq O$, and therefore $P \in U_0$. Hence $f$ is not regular on $U_0$ and we conclude that $f \notin \mathcal{O}_X(U_0)$. □

**Lemma 2.12.** *The ring $\mathcal{O}_X(U_2)$ consists of all elements of $f \in \mathcal{O}_X(U_0 \cap U_2)$ satisfying $\nu_O(f) \geq 0$. In particular, $\mathcal{O}_X(U_2)$ does not contain $\omega_{-1}$.*

*Proof.* A rational function in $\mathcal{O}_X(U_0 \cap U_2)$ lies in $\mathcal{O}_X(U_2)$ if and only if it is regular at the one missing point, $O$. In other words, if and only if the rational function has non-negative valuation at $O$. □

**Lemma 2.13.** *We have that*

$$\mathcal{O}_X(U_0) \cap \mathcal{O}_X(U_2) = \mathbb{K}.$$

*Proof.* If an element lies in $\mathcal{O}_X(U_0) \cap \mathcal{O}_X(U_2)$, then it is regular everywhere, but the only globally regular functions on a projective variety are constant (Theorem 3.4 in [7, Chapter 1]). □

These lemmas will allow us to interact with the rings without having to deal with equivalence classes modulo the ideal. Instead we can exclusively work with the special elements $\omega_k$. For our explicit proof of global generation, we will need the ability to cancel elements of the same valuation.

**Lemma 2.14.** *Let $X$ be a smooth projective curve over a field $\mathbb{K}$, and let $x \in X$. Suppose $f, g \in \mathbb{K}(X)^*$ are non-zero rational functions such that $\nu_x(f) = \nu_x(g) = k$. Then there exists a constant $c \in \mathbb{K}$ such that $\nu_x(f - cg) > k$.*

*Proof.* Let $x \in X$. Since $X$ is smooth, the local ring $\mathcal{O}_{X,x}$ is a PID with maximal ideal $\mathfrak{m}_x = \langle t_x \rangle$. Then $f = ut_x^k$ and $g = vt_x^k$ for $u, v$ units. We know that $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{K}$, so the images $\overline{u}, \overline{v}$ are just some elements in $\mathbb{K}$. In particular, there is some other constant $c \in \mathbb{K}$ such that $\overline{u} - c\overline{v} = 0$. Therefore $u - cv \in \mathfrak{m}_x$, so that $u - cv = wt_x$ for some $w \in \mathcal{O}_{X,x}$.

Hence

$$f - cg = (u - cv)t_x^k = wt_x^{k+1},$$

so that $\nu_x(f - cg) \geq k + 1$. □

**Lemma 2.15.** *For any integer $m \geq 1$, and $f \in \mathcal{O}_X(U_0 \cap U_2)$, there is an element $g \in \mathcal{O}_X(U_0)$ such that $f - \omega_1^m g \in \mathcal{O}_X(U_2)$. Furthermore, if $m \geq 2$, then we can further enforce that $\nu_O(f - \omega_1^m g) > 0$.*

*Proof.* If the valuation of $f$ at $O$ is non-negative then it is in $\mathcal{O}_X(U_2)$ already by Lemma 2.12, so set $g = 0$. We proceed by reverse induction on $k := \nu_O(f)$. The base case is $k = 0$, which follows from the above. Now let $k \leq -1$, and suppose that if $f' \in \mathcal{O}_X(U_0 \cap U_2)$ satisfies $\nu_O(f') > k$, then there exists $g'$ such that $f' - \omega_1^m g' \in \mathcal{O}_X(U_2)$.

Set $g_0 = \omega_{-m+k}$, so that $\nu_O(g_0) = -m + k$ by Lemma 2.10. Since $m \geq 1$ and $k \leq -1$, we have that $\nu_O(g_0) \leq -2$, so that $g_0 \in \mathcal{O}_X(U_0)$ by Lemma 2.11. Furthermore, by the properties of valuations, we have $\nu_O(\omega_1^m g_0) = k$, so that we can apply Lemma 2.14 to find a constant $c \in \mathbb{K}$ such that

$$\nu_O(f - c\omega_1^m g_0) > k.$$

Now apply induction to obtain $g'$, and set $g = g' + cg_0$.

If $m \geq 2$, then we can also cancel valuation zero terms, by taking $g = c\omega_{-m}$ for an appropriate $c \in \mathbb{K}$. This will be in $\mathcal{O}_X(U_0)$ by Lemma 2.11 since $m \geq 2$. □

**Corollary 2.16.** *Let $q \in \mathbb{Z}^+$, $m \geq 1$ and suppose*

$$\boldsymbol{f} \in \mathcal{O}_X(U_0 \cap U_2)^q.$$

*Then there exists a vector $\boldsymbol{g} \in \mathcal{O}_X(U_0)^q$ such that*

$$\boldsymbol{f} - \omega_1^m I_q \boldsymbol{g} \in \mathcal{O}_X(U_2)^q,$$

*where $I_q$ is $q \times q$ identity matrix. Furthermore, if the $i$-th entry of $\boldsymbol{f}$ is already in $\mathcal{O}_X(U_2)$, then the $i$-th entry of $\boldsymbol{f} - \omega_1^m I_q \boldsymbol{g}$ is precisely that same entry.*

*Proof.* This is just the vector form of Lemma 2.15 above. The only addition is the final sentence, which follows from the proof of the lemma. Namely, if an entry already lies in $\mathcal{O}_X(U_2)$, we took $g = 0$ in the proof, so that taking the difference does not affect that entry. □

Finally, in order to explicitly represent line bundles, we will need an alternative representation of Weil divisors, coming from *locally principal divisors*.

**Definition 2.17.** Let $X$ be a smooth projective curve. A *Cartier divisor on $X$* is an equivalence class[5] $D$ which can be represented by the data $\{(U_i, f_i)\}_{i=0}^n$, where the $\{U_i\}_{i=0}^n$ form an open cover of $X$ and $f_i \in \mathcal{O}_X(U_i)^*$. Furthermore, for any indices $i, j$, we have that the quotients

$$\frac{f_i}{f_j} \in \mathcal{O}_X(U_i \cap U_j)^*.$$

Note that the quotients being invertible further implies that

$$\text{div } \frac{f_i}{f_j}|_{U_i \cap U_j} = \text{div } f_i|_{U_i \cap U_j} - \text{div } f_j|_{U_i \cap U_j} = 0,$$

so that the divisors of the functions $f_i$ are equal on intersections of open sets in the cover. Starting from a Cartier divisor $D = \{(U_i, f_i)\}_{i=0}^n$, we can construct a Weil divisor as follows: Any point $x \in X$ lies in some $U_i$. Set $n_x := \nu_x(f_i)$. It is possible that $x$ may lie in another $U_j$ for $i \neq j$, but by the above observation, we have that $\nu_x(f_i) = \nu_x(f_j)$ for all $x \in U_i \cap U_j$, so that this is well-defined. Since there are only finitely many $f_i$, and they only have finitely many zeroes and poles, only finitely many of the $n_x$ will be non-zero. This motivates the idea that Cartier divisors are locally principal Weil divisors—they locally look principal, and globally glue together to give a Weil divisor since they agree on intersections.

**Proposition 2.18.** *If $X$ is a smooth projective curve,[6] then every Weil divisor can be represented by a Cartier divisor.*

*Proof.* This is Proposition 6.11 and Remark 6.11.2 from [7, Chapter 2]. □

**Example 2.19.** In the case of an elliptic curve $X$, having an explicit form for the divisor $O$ as a Cartier divisor will be particularly useful. Let $X$ be in Legendre form, and let $U_0, U_2$ be the open cover of $X$ from (2.3). Example 3.3 in [12, Chapter 2] shows that

$$\begin{aligned}
&\text{div } \overline{x} = 2 \cdot (1 : 0 : 0) - 2 \cdot O, \\
&\text{div } \overline{y} = (1 : 0 : 0) + (1 : 1 : 0) + (1 : \lambda : 0) - 3 \cdot O,
\end{aligned}$$

so from (2.6), we have

$$\text{div } \omega_1 = \text{div } \overline{xy^{-1}} = O + (1 : 0 : 0) - (1 : 1 : 0) - (1 : \lambda : 0). \tag{2.7}$$

In particular, $(\text{div } \omega_1)|_{U_0 \cap U_2} = 0$, so that

$$D = \{(U_0, 1), (U_2, \omega_1)\}$$

[5]Formally, Cartier divisors are global sections of the so-called sheaf of total quotient rings—which is meant to be the sheaf equivalent of the function field. See [7, Section 6].

[6]This more generally applies to varieties whose local rings are all unique factorization domains.

is a Cartier divisor. By construction, $\nu_x(1) = 0$ for all $x \in U_0$, $\nu_x(\omega_1) = 0$ for all $x \in U_2 \backslash \{O\}$, and $\nu_O(\omega_1) = 1$. Therefore the Weil divisor corresponding to this Cartier divisor is $O$. $\triangle$

## 2.3 Sheaves

Our results are primarily concerned with *vector bundles* over elliptic curves. However, there is a correspondence between vector bundles and so-called locally free sheaves; and it is easier to work with the sheaves instead. As such, we omit the definition of vector bundles here. One can find the definition of vector bundles in [3, Chapter 6, Section 0] or [11, Chapter VI, Section 1], together with their correspondence with locally free sheaves.

In this section, we recall a few definitions regarding sheaves, following [7, Chapter 2]. In addition, we will present some ways to explicitly work with vector bundles via transition matrices. Throughout this discussion, we will assume $X$ is a projective variety unless otherwise specified.

**Definition 2.20.** Given a point $x \in X$ and a sheaf $\mathcal{F}$ over $X$, the *stalk at $x$ of $\mathcal{F}$* is the set

$$\mathcal{F}_x = \{(U, s) \mid U \subset X \text{ open containing } x,\ s \in \mathcal{F}(U)\}/\sim$$

where $(U, s) \sim (V, t)$ if and only if there exists $W \subset U \cap V$ containing $x$ such that $s|_W = t|_W$. This is otherwise known as the *direct limit* of $\mathcal{F}(U)$ via the restriction maps $\rho$, which is denoted by $\varinjlim_\rho \mathcal{F}(U)$. Given a section $s \in \mathcal{F}(U)$ and $x \in U$, we denote the *image of $s$ at the stalk of $x$* as the equivalence class $s_x = \overline{(U, s)}$.

**Definition 2.21.** A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be *locally free of rank $r$* if there exists an open cover $\{U_i\}_{i \in I}$ of $X$ such that for each $i \in I$, we have an isomorphism $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^r$. As mentioned above, we will refer to a *vector bundle of rank $r$* as being a locally free sheaf of rank $r$. A line bundle is a locally free sheaf of rank 1.

**Definition 2.22.** Given a vector bundle $\mathcal{E}$, $\mathcal{L}$ is said to be a *subbundle of $\mathcal{E}$* if $\mathcal{L}$ is a locally free subsheaf of $\mathcal{E}$, and the quotient sheaf $\mathcal{E}/\mathcal{L}$ is also locally free.

There is another equivalent characterization of locally free sheaves:

**Lemma 2.23.** *A sheaf $\mathcal{F}$ is locally free of rank $r$ if and only if $\mathcal{F}_x \cong \mathcal{O}_{X,x}^r$ for any $x \in X$.*

*Proof.* This is (b) of Exercise 5.7 in [7, Chapter 2]. $\square$

**Definition 2.24.** The *trivial vector bundle of rank $r$* is the sheaf

$$\mathcal{I}_r = \overbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}^{r \text{ copies}}$$

**Proposition 2.25.** *Let $\mathcal{E}$ be a vector bundle of rank $r$. The sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ defined by*

$$\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)(U) = \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{O}_X(U))$$

*is locally free. We call this the dual of $\mathcal{E}$, and denote it by $\mathcal{E}^*$.*

*Proof.* This is Exercise 5.1 in [7, Chapter 2]. □

**Proposition 2.26.** *Let $\mathcal{E}$ be a vector bundle of rank $r$. The sheaf $\bigwedge^r \mathcal{E}$ defined by the sheafification of the presheaf*

$$\left(\bigwedge^r \mathcal{E}\right)(U) = \bigwedge^r \mathcal{E}(U)$$

*is a line bundle.*

*Proof.* Fix $x \in X$. By Exercise 2.20 in [2, Chapter 2], direct limits commute with tensor products and quotients. Therefore they commute with exterior products, and so we have

$$\varinjlim_{\wedge^r \rho} \bigwedge^r \mathcal{E}(U) = \bigwedge^r \varinjlim_{\rho} \mathcal{E}(U) = \bigwedge^r \mathcal{E}_x = \bigwedge^r \mathcal{O}_{X,x}^r = \mathcal{O}_{X,x},$$

where $\rho$ denotes the restriction maps given by $\mathcal{E}$. Since sheafifying does not change the stalks, we therefore have

$$\left(\bigwedge^r \mathcal{E}\right)_x = \mathcal{O}_{X,x}.$$

Thus by Lemma 2.23, $\bigwedge^r \mathcal{E}$ is locally free of rank one. □

**Definition 2.27.** Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. An *extension of $\mathcal{F}$ by $\mathcal{G}$* is an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We will occasionally refer to $\mathcal{E}$ as being an extension of $\mathcal{F}$ by $\mathcal{G}$.

**Example 2.28.** The simplest extensions are just direct sums of sheaves—for any two sheaves $\mathcal{F}, \mathcal{G}$, we always have the exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0$$

and also

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow 0$$

so that the sheaf $\mathcal{F}\oplus\mathcal{G}$ is both an extension of $\mathcal{F}$ by $\mathcal{G}$ and an extension of $\mathcal{G}$ by $\mathcal{F}$. These are known as *split* or *trivial* extensions. We shall see some examples of non-trivial extensions via Proposition 2.42, Theorem 2.53, and Theorem 2.54. △

We have a nice correspondence between line bundles and divisors.

**Definition 2.29.** Let $D$ be a divisor. Then we can form the sheaf associated to $D$, denoted $\mathcal{O}(D)$, by defining[7]

$$\mathcal{O}(D)(U) := \{f \in \mathcal{O}_X(U) \mid \mathrm{div\ f} + D \geq 0\}.$$

**Proposition 2.30.** *Let $X$ be any projective curve.*

1. *The sheaf $\mathcal{O}(D)$ is a line bundle, and every line bundle is isomorphic to one of this form. If $D$ is of degree $d$, then $\mathcal{O}(D)$ is also of degree $d$ as a line bundle.*
2. *If $D_1 \sim D_2$, then $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$.*
3. *For any divisors $D_1, D_2$ we have $\mathcal{O}(D_1 - D_2) \cong \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)^*$.*

*Proof.* The first point follows from Propositions 6.13 and 6.15 in [7, Chapter 2], and the remaining just by Proposition 6.13. □

**Remark 2.31.** Given a line bundle $\mathcal{L}$, this shows that $\mathcal{L} \otimes \mathcal{L}^* \cong \mathcal{O}_X$. As a result, line bundles are referred to as *invertible sheaves.*

From this, we can obtain an additional invariant associated to any vector bundle.

**Definition 2.32.** Given a line bundle $\mathcal{L}$, the *degree of* $\mathcal{L}$ is the degree of the corresponding divisor class. For any vector bundle $\mathcal{E}$ of rank $r$, the *degree of* $\mathcal{E}$ is the degree of $\bigwedge^r \mathcal{E}$ (see Proposition 2.26).

With the notion of degree for higher rank vector bundles, we can state Riemann-Roch:

**Theorem 2.33** (Riemann-Roch)**.** *If $X$ is an smooth projective curve, then for any vector bundle $\mathcal{E}$ over $X$, we have*

$$\dim \Gamma(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) = \deg(\mathcal{E}) + r(1 - g),$$

*where $H^1(X, \mathcal{E})$ is the first cohomology group of $\mathcal{E}$ under the $\Gamma(X, \cdot)$ functor (see [7, Chapter 3]).*

*Proof.* This is proven in [1, Page 420]. □

[7] There is some ambiguity when using functional notation for both the sheaf associated to $D$ and the ring associated to $U$. If the ambiguity ever arises, we will be careful to remind the reader—though the distinction between open sets and divisors is always made clear.

Now we introduce the main tool used in the thesis, transition matrices.

**Proposition 2.34.** *Let $X$ be a projective variety together with an open cover $\{U_i\}_{i=0}^n$ and matrices $M_{ij} \in GL_r(\mathcal{O}_X(U_i \cap U_j))$ (called transition matrices) satisfying the compatibility conditions*

$$\begin{aligned} M_{ik} &= M_{ij} M_{jk} \text{ on } U_i \cap U_j \cap U_k \\ M_{ij} &= M_{ji}^{-1} \text{ on } U_i \cap U_j. \end{aligned} \tag{2.8}$$

*We can define a presheaf $\mathcal{E}$ via the matrices by setting $\mathcal{E}(U)$ to be the set of $n$-tuples $s_i \in \mathcal{O}_X(U \cap U_i)^r$ such that for any pair $i \neq j$,*

$$s_i|_{U \cap U_i \cap U_j} = M_{ij} s_j|_{U \cap U_i \cap U_j}. \tag{2.9}$$

*We have that $\mathcal{E}$ is a vector bundle of rank $r$ over $X$.*

*Proof.* The compatibility conditions of the matrices ensure that this is a sheaf. Furthermore, we claim that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^r$, since sections of $\mathcal{E}|_{U_i}(U)$ are completely determined by the $i$-th term in the $n$-tuple $s_i \in \mathcal{O}_X(U \cap U_i)^r$. The remaining terms are just restrictions of the $i$-th term to a smaller open set. Therefore $\mathcal{E}$ is locally free of rank $r$. □

**Example 2.35.** Let $X = \mathbb{P}^1$. We have the standard affine open cover of $X$ given by

$$X = D_+(x_0) \cup D_+(x_1) = U_0 \cup U_1,$$

and by setting $x := \frac{x_1}{x_0}$, we have

$$\begin{aligned} \mathcal{O}_X(U_0) &= \mathbb{K}[x] \\ \mathcal{O}_X(U_1) &= \mathbb{K}\left[x^{-1}\right] \\ \mathcal{O}_X(U_0 \cap U_1) &= \mathbb{K}\left[x^{\pm 1}\right]. \end{aligned}$$

Then we can define

$$M_{10} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x^{-2} \end{pmatrix},$$

together with $M_{01} = M_{10}^{-1}$. These are transition matrices, and global sections of the bundle associated to these matrices are pairs of the form

$$(s,t) \in \mathcal{O}_X(U_0)^2 \times \mathcal{O}_X(U_1)^2$$

satisfying $t = M_{10}s$ (or equivalently $s = M_{01}t$). For example, the pair

$$\left(\begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{-2} \end{pmatrix}\right)$$

represents a global section. $\triangle$

**Remark 2.36.** Every vector bundle is isomorphic to one arising from some set of transition matrices as in Proposition 2.34. Given a vector bundle $\mathcal{E}$, there are isomorphisms $\alpha_i : \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^r$. From these, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{E}|_{U_i}(U_i \cap U_j) \arrow[r, "\mathrm{id}"] & \mathcal{E}|_{U_j}(U_i \cap U_j) \arrow[d, "(\alpha_j)_{U_i \cap U_j}"] \\
\mathcal{O}_X(U_i \cap U_j)^r \arrow[u, "(\alpha_i^{-1})_{U_i \cap U_j}"] \arrow[r, "L_{ij}"] & \mathcal{O}_X(U_i \cap U_j)^r
\end{tikzcd}$$

where the $L_{ij}$ are defined by

$$L_{ij} = (\alpha_j)_{U_i \cap U_j} \circ (\alpha_i^{-1})_{U_i \cap U_j}.$$

We obtain transition matrices via matrix representations of the linear maps $L_{ij}$. In this way, finding transition matrices for a bundle is analogous to finding a matrix representation for a linear map between vector spaces.

The correspondence in Lemma 2.30 gives a construction to form transition matrices for the bundle associated to a Cartier divisor. In particular, if $D = \{(U_i, f_i)\}_{i=1}^n$, then we can set the transition matrices

$$M_{ij} = \begin{pmatrix} \frac{f_i}{f_j} \end{pmatrix}.$$

Since $D$ is a Cartier divisor, $\frac{f_i}{f_j} \in \mathcal{O}_X(U_i \cap U_j)^*$ so that $M_{ij}$ is invertible. Furthermore, the matrices satisfy the compatibility conditions in (2.8):

$$\begin{aligned}
M_{ij}^{-1} &= \begin{pmatrix} \frac{f_j}{f_i} \end{pmatrix} = M_{ji}, \\
M_{ij}M_{jk} &= \begin{pmatrix} \frac{f_i}{f_j} \end{pmatrix} \cdot \begin{pmatrix} \frac{f_j}{f_k} \end{pmatrix} = \begin{pmatrix} \frac{f_i}{f_k} \end{pmatrix} = M_{ik}.
\end{aligned}$$

Let $\mathcal{L}$ be the bundle associated to these matrices.

**Lemma 2.37.** *We have an isomorphism $\mathcal{L} \cong \mathcal{O}(D)$.*

*Proof.* This is proven in Theorem 6.0.18 of [3, Chapter 6]. $\square$

**Example 2.38.** Let $X$ be an elliptic curve. In Example 2.19, we computed a representation for the divisor $O$ as a Cartier divisor, and obtained

$$O = \{(U_0, 1), (U_2, \omega_1)\}.$$

Therefore we have the transition matrices

$$M_{20} = (\omega_1),\ \ M_{02} = M_{20}^{-1}.$$

Consequently, the line bundle $\mathcal{O}(O)$ (Lemma 2.30) corresponding to the divisor $O$ is isomorphic to the bundle arising from these transition matrices. This should be expected—since $\omega_1$ is a uniformizer for local ring $\mathcal{O}_{X,O}$, and we have chosen our open cover with its zeroes and poles in mind ($\omega_1$ is invertible on $U_0 \cap U_2$). More generally, we can always pick our open cover appropriately so that any uniformizer at a point $P$ can represent a transition matrix for the divisor $P$. △

More generally, whenever we produce an open cover consisting of only two open sets $U_1, U_2$ (such as (2.3) or later (3.11)), only one matrix $M_{21} \in \mathrm{GL}_r(\mathcal{O}_X(U_1 \cap U_2))$ is required to describe a vector bundle (though the cover may depend on the bundle). We will denote the vector bundle arising from $M_{21}$ as $\mathcal{B}_{U_1,U_2}(M_{21})$. When no ambiguity arises to the open cover we are using, we will simply write $\mathcal{B}(M_{21})$.

**Remark 2.39.** As part of our notation, we are insisting to choose $M_{21}$ instead of $M_{12}$ because how global sections are described. Namely, global sections of $\mathcal{B}(M_{21})$ are pairs

$$(s, t) \in \mathcal{O}_X(U_1)^r \times \mathcal{O}_X(U_2)^r$$

satisfying $t = M_{21}s$. In this way, they are completely determined by our choice of $s \in \mathcal{O}_X(U_1)$. On the other hand, if we were to use $M_{12}$, our compatibility condition would give us $s = M_{12}t$, so that a global section would be completely determined by our choice of $t$. This will be important to keep in mind when we are directly working with global sections, such as in Theorem 3.19 or Theorem 4.8.

We can translate some sheaf operations such as tensoring and dualizing into the language of transition matrices:

**Lemma 2.40.** *Suppose we have an open cover $U_1, U_2$ of $X$, and $f \in \mathcal{O}_X(U_1 \cap U_2)^*$. Then for any matrix $M \in GL_r(\mathcal{O}_X(U_1 \cap U_2))$, we have*

$$\mathcal{B}(M) \otimes \mathcal{B}((f)) \cong \mathcal{B}(f \cdot M).$$

*Proof.* Denote the isomorphisms $\mathcal{B}(M)|_{U_i} \cong \mathcal{O}_{U_i}^r$ and $\mathcal{B}(f)|_{U_i} \cong \mathcal{O}_{U_i}$ by

$$\alpha_1 : \mathcal{B}(M)|_{U_1} \to \mathcal{O}_{U_1},\ \alpha_2 : \mathcal{B}(M)|_{U_2} \to \mathcal{O}_{U_2},$$
$$\beta_1 : \mathcal{B}((f))|_{U_1} \to \mathcal{O}_{U_1},\ \beta_2 : \mathcal{B}((f))|_{U_2} \to \mathcal{O}_{U_2}.$$

Then $\alpha_i \otimes \beta_i$ give isomorphisms

$$\mathcal{B}(M)|_{U_1} \otimes \mathcal{B}((f))|_{U_1} \cong (\mathcal{B}(M) \otimes \mathcal{B}((f)))|_{U_1} \cong \mathcal{O}_{U_1}^r \otimes \mathcal{O}_{U_1} \cong \mathcal{O}_{U_1}^r,$$
$$\mathcal{B}(M)|_{U_2} \otimes \mathcal{B}((f))|_{U_2} \cong (\mathcal{B}(M) \otimes \mathcal{B}((f)))|_{U_2} \cong \mathcal{O}_{U_2}^r \otimes \mathcal{O}_{U_2} \cong \mathcal{O}_{U_2}^r.$$

Furthermore, this sheaf can be described by a transition matrix via the linear map (see Remark 2.36)

$$L_{12} = (\alpha_2)_{U_1 \cap U_2} \otimes (\beta_2)_{U_1 \cap U_2} \circ ((\alpha_1)_{U_1 \cap U_2} \otimes (\beta_1)_{U_1 \cap U_2})^{-1}$$

But tensors commute with composition, so

$$\begin{aligned} L_{12} &= (\alpha_2)_{U_1 \cap U_2} \circ (\alpha_1^{-1})_{U_1 \cap U_2} \otimes (\beta_2)_{U_1 \cap U_2} \circ (\beta_1^{-1})_{U_1 \cap U_2} \\ &= M \otimes f \\ &= f \cdot M \otimes 1. \end{aligned}$$

Therefore we have that a transition matrix for $\mathcal{B}(M) \otimes \mathcal{B}((f))$ is $f \cdot M$ and we are done. $\square$

**Lemma 2.41.** *Suppose we have an open cover $U_1, U_2$ of $X$, and a matrix $M \in GL_r(\mathcal{O}_X(U_1 \cap U_2))$. Then*

$$\mathcal{B}(M)^* \cong \mathcal{B}((M^{-1})^T).$$

*Proof.* Recall that $\mathcal{B}(M)^* = \mathcal{H}om(\mathcal{B}(M), \mathcal{O}_X)$. Similarly to our proof above, we have isomorphisms

$$\alpha_1 : \mathcal{B}(M)|_{U_1} \to \mathcal{O}_{U_1}^r,\ \alpha_2 : \mathcal{B}(M)|_{U_2} \to \mathcal{O}_{U_2}^r.$$

We can use the standard basis in $\mathcal{O}_{U_i}(U)^r$ to describe linear maps from $\mathcal{B}(M)|_{U_i}(U)$ to $\mathcal{O}_{U_i}(U)$ by passing through the isomorphism. More precisely, if $\phi \in \mathcal{H}om(\mathcal{B}(M)|_{U_i}, \mathcal{O}_{U_i})(U)$, then $\phi$ is of the form

$$\phi = \left[(\alpha_i)_U^{-1}(e_i) \mapsto s_i\right],$$

where the $e_i$ denote the standard basis vectors in $\mathcal{O}_{U_i}(U)^r$, and $s_i \in \mathcal{O}_{U_i}(U)$ depend on $\phi$. Therefore homomorphisms $\phi$ correspond with row vectors $(s_1, s_2, ..., s_r) \in \mathcal{O}_{U_i}(U)^r$. These then give isomorphisms

$$\alpha_1^* : \mathcal{H}om(\mathcal{B}(M), \mathcal{O}_{U_1})|_{U_1} \to \mathcal{O}_{U_1}^r,\ \ \alpha_2^* : \mathcal{H}om(\mathcal{B}(M), \mathcal{O}_{U_2})|_{U_2} \to \mathcal{O}_{U_2}^r,$$

where we have

$$\begin{aligned}(\alpha_i^*)_U : \mathcal{H}om(\mathcal{B}(M), \mathcal{O}_{U_0})(U) &\to \mathcal{O}_{U_0}^r(U)\\ \Big[(\alpha_i^{-1})_U(e_i) \mapsto s_i\Big] &\mapsto (s_1, s_2, ..., s_r).\end{aligned}$$

Furthermore we have that the linear map $(\alpha_2^*)_{U_1\cap U_2} \circ (\alpha_1^*)^{-1}_{U_1\cap U_2}$ describes a transition matrix for $\mathcal{B}(M)^*$, and in this case it is just

$$\begin{aligned}(\alpha_2^*)_{U_1\cap U_2} \circ (\alpha_1^*)^{-1}_{U_1\cap U_2} &= \Big[(\alpha_2^{-1})_{U_1\cap U_2} \circ (\alpha_1)_{U_1\cap U_2}(e_i) \mapsto s_i\Big]\\ &= \Big[M^{-1}(e_i) \mapsto s_i\Big]\\ &= (M^{-1})^T.\end{aligned}$$

□

**Lemma 2.42.** *Suppose we have an open cover $U_1, U_2$ of $X$, and a matrix $M \in GL_r(\mathcal{O}_X(U_1 \cap U_2))$ which is in block form*

$$M = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

*where $B_{11}$ and $B_{22}$ are square matrices. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{B}(B_{11}) \xrightarrow{\phi} \mathcal{B}(M) \xrightarrow{\psi} \mathcal{B}(B_{22}) \longrightarrow 0.$$

*Proof.* Let $B_{11}$ be of size $n \times n$, and $B_{22}$ of size $m \times m$. Then we represent global sections in $\mathcal{B}(B_{11})(U)$ by pairs

$$(s, t) \in \mathcal{O}_X(U \cap U_1)^n \times \mathcal{O}_X(U \cap U_2)^n,$$

sections in $\mathcal{B}(B_{22})(U)$ by pairs

$$(s, t) \in \mathcal{O}_X(U \cap U_1)^m \times \mathcal{O}_X(U \cap U_2)^m,$$

and sections in $\mathcal{B}(M)(U)$ by pairs

$$((s_1, s_2), (t_1, t_2)) \in (\mathcal{O}_X(U \cap U_0)^n \times \mathcal{O}_X(U \cap U_0)^m) \times (\mathcal{O}_X(U \cap U_2)^n \times \mathcal{O}_X(U \cap U_2)^m).$$

We can define the maps in the exact sequence explicitly. Define

$$\begin{aligned} \phi_U : \mathcal{B}(B_{11})(U) &\to \mathcal{B}(M)(U) \\ (s,t) &\mapsto \left( \begin{pmatrix} s \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right), \end{aligned}$$

and

$$\begin{aligned} \psi_U : \mathcal{B}(M)(U) &\to \mathcal{B}(B_{22})(U) \\ \left( \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right) &\mapsto (s_2, t_2). \end{aligned}$$

where each $\begin{pmatrix} a \\ b \end{pmatrix}$ is a pair of vectors of length $n$ and $m$ respectively. These are both well-defined morphisms: If $t = B_{11}s$, then

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} B_{11}s \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Similarly if

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

then we have $t_2 = B_{22}s_2$. Consequently, $\phi_U$ and $\psi_U$ both give rise to well-defined sheaf morphisms. To prove exactness, it suffices to do so on the stalks (Exercise 1.1 in [7, Chapter 2]). To this end, fix $x \in X$. Since each sheaf involved is locally free, we have the diagram

$$\mathcal{O}_{X,x}^n \xrightarrow{\phi_x} \mathcal{O}_{X,x}^{n+m} \xrightarrow{\psi_x} \mathcal{O}_{X,x}^m,$$

where the maps $\phi_x$ and $\psi_x$ are defined similarly to above. Since these are just the standard inclusion and projection maps on free modules, we can conclude that this sequence is exact.

□

**Remark 2.43.** We can apply row operations with coefficients in $\mathcal{O}_X(U_2)$ and column operations with coefficients in $\mathcal{O}_X(U_1)$ to reduce $M$ (since we are transitioning from $U_2$ to $U_1$). If $M$ can be block diagonalized (i.e. $B_{12} = 0$), then the resulting extension of bundles is split. On the other hand, if we cannot block diagonalize the matrix, then the extension of bundles is non-trivial.

Finally, we will review some of the necessary tools to understand global generation.

**Definition 2.44.** A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *globally generated* if for any point $x \in X$, there exist sections $s_1, ..., s_n \in \Gamma(X, \mathcal{F})$ such that

$$\langle (s_1)_x, (s_2)_x, ..., (s_n)_x \rangle_{\mathcal{O}_{X,x}} = \mathcal{F}_x.$$

In other words, images of $s_i$ in the stalk at $x$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module.

**Example 2.45.** If $X$ is a projective variety, then trivial bundle $\mathcal{I}_r$ is always globally generated. In particular, $\Gamma(X, \mathcal{O}_X^r) = \mathbb{K}^r$ since $X$ is projective, and so the standard basis vectors $e_i$ (i.e. 1 in the $i$-th component) will globally generate the bundle, since the image of 1 in any stalk is still 1. △

**Example 2.46.** On the other hand, if $\mathcal{E}$ is a bundle of rank $r$ such that $\dim \Gamma(X, \mathcal{E}) = k < r$, then it is not globally generated, since the generating the module $\mathcal{E}_x \cong \mathcal{O}_{X,x}^r$ requires at least $r$ independent elements. △

By Lemma 2.23 the stalk of a vector bundle of rank $r$ at a point $x \in X$ is isomorphic to $\mathcal{O}_{X,x}^r$. Hence it is useful to understand generating sets of free $\mathcal{O}_{X,x}$-modules.

**Definition 2.47.** Let $X$ be a smooth projective curve and $x \in X$. Then a set of elements

$$s_i = \begin{pmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{ir} \end{pmatrix} \in \mathcal{O}_{X,x}^r$$

for $1 \le i \le r$ form an *upper triangular generating set of* $\mathcal{O}_{X,x}^r$ if each $s_i$ satisfies

(i) The entry $s_{ii}$ has valuation $\nu_x(s_{ii}) = 0$.

(ii) For $i < j \le r$, the entry $s_{ij}$ has valuation $\nu_x(s_{ij}) > 0$.

**Example 2.48.** Let $X = \mathbb{P}^1$. We have that the stalk at the point $(1:0) \in X$ is $\mathbb{K}[x]_{\langle x \rangle}$, the localization of $\mathbb{K}[x]$ at the ideal $\langle x \rangle$ (where $x = \frac{x_1}{x_0}$). Let $\mathcal{F} = \mathcal{I}_3$ be the trivial bundle, and consider the following set of global sections of $\mathcal{F}$

$$s_1 = \begin{pmatrix} 1+x \\ 0 \\ x^2+x \end{pmatrix}, \; s_2 = \begin{pmatrix} 1 \\ 1 \\ x \end{pmatrix}, \; s_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The images of $s_i$ at the stalk of $(1:0) \in X$ form an upper triangular generating set, since the valuation of $x$ is one at $(1:0)$. In other words, it *vanishes* at $(1:0)$. In this way, if one imagines evaluating $x$ at zero, the resulting generating set would be upper triangular in the usual sense. △

To prove that these are actual generating sets of the stalks, we will need a standard tool from algebra.

**Lemma 2.49** (Local Nakayama)**.** *Let $R$ be a local ring with unique maximal ideal $\mathfrak{m}$, and let $M$ be an $R$-module. If $r_1, ..., r_k \in R$ are such that*

$$\langle \overline{r_1}, ..., \overline{r_k} \rangle_{R/\mathfrak{m}} = M/\mathfrak{m}M,$$

*then we have*

$$\langle r_1, ..., r_k \rangle_R = M$$

*Proof.* This is a special case of Proposition 2.6 in [2, Chapter 2]. □

**Lemma 2.50.** *An upper triangular generating set $s_1, s_2, ..., s_r$ of $\mathcal{O}^r_{X,x}$ indeed satisfies*

$$\langle (s_1)_x, (s_2)_x, \cdots, (s_r)_x \rangle_{\mathcal{O}_{X,x}} = \mathcal{O}^r_{X,x}.$$

*Proof.* We will apply Nakayama's lemma in the case of local rings above. In our situation, $R = \mathcal{O}_{X,x}$, and $M = \mathcal{O}^r_{X,x}$. In order to apply the lemma, we must show that

$$\left\langle \overline{(s_1)_x}, ..., \overline{(s_r)_x} \right\rangle_{\mathcal{O}_{X,x}/\mathfrak{m}_x} = \mathcal{O}^r_{X,x}/\mathfrak{m}_x\mathcal{O}^r_{X,x},$$

where $\overline{(s_i)_x}$ are the residues of $(s_i)_x$ modulo $\mathfrak{m}_x\mathcal{O}^r_{X,x}$. Since $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is a field, this is essentially just linear algebra. By property (i), the $i$-th entry of $\overline{(s_i)_x}$ is invertible (hence non-zero). Furthermore, property (ii) tells us that for $i < j \le r$, the $j$-th entry of $\overline{(s_i)_x}$ is a zero, since elements of valuation one or higher at $x$ lie in $\mathfrak{m}_x$. Consequently, the $\overline{(s_i)_x}$ form an upper triangular generating set in the usual sense, and hence generate $\mathcal{O}^r_{X,x}/\mathfrak{m}_x\mathcal{O}^r_{X,x}$. □

## 2.4 Atiyah's Results

We will review the necessary tools developed by Atiyah in his 1957 paper [1]. Throughout this discussion, $X$ will refer to a smooth irreducible projective curve of genus one over an algebraically closed field $\mathbb{K}$ (not necessarily with a base point). First, we present the tools needed to prove that the bundles we construct in Chapter 3 are indecomposable.

**Lemma 2.51.** *Let $\mathcal{E}$ be a vector bundle of rank $r$ and degree $d$ over $X$, and suppose we have an exact sequence*

$$0 \longrightarrow \mathcal{I}_d \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0.$$

*If either*

1. $d = \dim \Gamma(X, \mathcal{E}) = \dim \Gamma(X, \mathcal{E}') = 1$, *or*

2. $\dim \Gamma(X, \mathcal{E}^*) = 0$ *and* $\dim \Gamma(X, \mathcal{E}') = d$,

*then* $\mathcal{E}$ *is indecomposable.*

*Proof.* Recall that from any short exact sequence of sheaves, there is a long exact sequence of cohomology from the global sections functor ([7, Chapter 3]). So we have

$$0 \longrightarrow \Gamma(X, \mathcal{E}'^*) \xrightarrow{\phi_1} \Gamma(X, \mathcal{E}^*) \xrightarrow{\phi_2} \Gamma(X, \mathcal{I}_d^*) \xrightarrow{\delta} H^1(X, \mathcal{E}'^*) \longrightarrow \cdots .$$

First, we show that $\delta$ is injective. If (1) holds, then by exactness at $\Gamma(X, \mathcal{E}'^*)$, we know that $\phi_1$ is an inclusion, and since the dimensions of the vector spaces are equal, is surjective. Furthermore, by exactness at $\Gamma(X, \mathcal{E}^*)$, we must conclude that $\phi_2 = 0$ since $\ker \phi_2 = \mathrm{Im}\ \phi_1$. Then by exactness at $\Gamma(X, \mathcal{I}_d^*)$, we have

$$\ker \delta = \mathrm{Im}\ \phi_2 = 0$$

On the other hand if (2) holds, then $\Gamma(X, \mathcal{E}^*) = 0$, so obviously $\phi_2 = 0$, implying $\delta$ is injective.

Now Lemma 13* in [1] states that $\mathcal{E}$ is so-called $\mathcal{I}_d$-complete if and only if $\delta$ is injective. Furthermore, as part of the proof of Lemma 16 in [1], Atiyah shows that if $\dim \Gamma(X, \mathcal{E}') = d$ (satisfied in both cases above), then $\mathcal{E}$ is indecomposable if and only if it is $\mathcal{I}_d$-complete, so we are done. □

**Lemma 2.52.** *Let* $\mathcal{E}$ *be an indecomposable vector bundle over* $X$ *of rank* $r$ *and degree* $d \geq 0$. *Then*

$$\dim \Gamma(X, \mathcal{E}) = \begin{cases} d & \text{if } d > 0 \\ 0 \text{ or } 1 & \text{if } d = 0 \end{cases}.$$

*Proof.* This is part (i) of Lemma 15 in [1]. □

Atiyah's classfication of vector bundles over elliptic curves is summarized via the following three theorems.

**Theorem 2.53.** *If* $X$ *is an elliptic curve, then for any rank* $r \in \mathbb{N}$, *there is an indecomposable vector bundle* $\mathcal{F}_r$ *of degree zero over* $X$ *with* $\Gamma(X, \mathcal{F}_r) \neq 0$ *which is unique up to isomorphism. Furthermore, there is an exact sequence*

$$0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow 0.$$

*If* $\mathcal{E}$ *is any indecomposable vector bundle of rank* $r$ *and degree zero, then there exists a unique degree zero line bundle* $\mathcal{L}$ *such that* $\mathcal{E} \cong \mathcal{F}_r \otimes \mathcal{L}$.

*Proof.* This is Theorem 5 in [1]. □

Note that $\mathcal{F}_1$ is isomorphic to the trivial line bundle by uniqueness, since $\mathcal{I}_1$ has a one-dimensional space of global sections.

**Theorem 2.54.** *Let $\mathcal{A}$ be a fixed degree one line bundle over $X$. For every rank $r \in \mathbb{N}$ and degree $d \in \mathbb{Z}$, there is a unique indecomposable vector bundle $\mathcal{E}(r,d)$ of rank $r$ and degree $d$ characterized by the properties*

*(i)* $\mathcal{E}(r,0) = \mathcal{F}_r$.

*(ii)* $\mathcal{E}(r,d) = \mathcal{E}(r,d-r) \otimes \mathcal{A}$.

*(iii) If* $0 < d < r$ *then there is an exact sequence*

$$0 \longrightarrow \mathcal{I}_d \longrightarrow \mathcal{E}(r,d) \longrightarrow \mathcal{E}(r-d,d) \longrightarrow 0.$$

*Proof.* This is Theorem 6 in [1]. □

**Remark 2.55.** Fixing a degree one line bundle over $X$ is equivalent to fixing a base point by Lemmas 2.8 and 2.30—whichever point corresponds to this bundle becomes the marked point $O$. In Example 2.38, we had seen that the line bundle associated to the divisor $O$ was $\mathcal{B}((\omega_1))$ over the open cover $U_0, U_2$ (2.3). Consequently, we can see that representing $X$ in Legendre form will give an isomorphism $\mathcal{B}((\omega_1)) \cong \mathcal{A}$.

**Theorem 2.56.** *If $\mathcal{E}$ is an indecomposable vector bundle over an elliptic curve of rank $r$ and degree $d$, then there is a unique degree zero line bundle $\mathcal{L}$ such that*

$$\mathcal{E} \cong \mathcal{E}(r,d) \otimes \mathcal{L}.$$

*Proof.* This is Theorem 7 in [1]. □

These results show that any indecomposable bundle over an elliptic curve is completely determined by its rank, degree, and a point on the curve—since degree zero line bundles are in correspondence with the points of the curve (Lemmas 2.8 and 2.30).

## 2.5 Integer Partitions

In this section, we will present a short discussion on partitions, which are necessary for our main construction.

**Definition 2.57.** Given a positive integer $n$, an *integer partition $\lambda$ of $n$* is a chain $\lambda$ of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ such that

$$\sum_{i=1}^{m} \lambda_i = n.$$

We denote $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$. Each $\lambda_i$ is referred to as a *part*, and $m$ is the number of parts of $\lambda$.

**Remark 2.58.** There is a related notion of integer combinations, which do not impose an order on the $\lambda_i$. Removing the different possible rearrangements of the $\lambda_i$ is one of the main reasons to study partitions instead of combinations. Conventionally, the order is always chosen to be greatest to least. However, any ordering would suffice. In Chapter 3, we will instead choose to order the $\lambda_i$ from least to greatest, as it serves our purposes better.

There is a natural involution on partitions which is not immediately obvious from the definition. To understand this involution, we first recall a nice way to represent partitions via Ferrers diagrams:

**Definition 2.59.** Given an integer partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$, we form the *Ferrers diagram of* $\lambda$ as the combinatorial object

$$\begin{array}{llllllll}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & & \lambda_1 \text{ dots} \\
\bullet & \bullet & \cdots & \bullet & \bullet & & & \lambda_2 \text{ dots} \\
\vdots & & & & & & & \\
\bullet & \cdots & \bullet & \bullet & & & & \lambda_m \text{ dots}
\end{array}$$

**Example 2.60.** The Ferrers diagram for the partition $\lambda = (6, 3, 3, 2, 1)$ is

$$\begin{array}{llllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & \\
\bullet & & & & &
\end{array}$$

$\triangle$

**Definition 2.61.** Given a Ferrers diagram for a partition $\lambda$, the *conjugate diagram* is the reflection of the Ferrers diagram along the diagonal line $x = -y$ (the top-leftmost point is considered to be the origin (0,0)). The corresponding partition to the conjugate diagram is called the *conjugate partition of* $\lambda$.

**Example 2.62.** The conjugate diagram for $\lambda = (6, 3, 3, 2, 1)$ is

$$\begin{array}{lllll}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & & \\
\bullet & & & & \\
\bullet & & & & \\
\bullet & & & &
\end{array}$$

which corresponds to the partition $\mu = (5, 4, 3, 1, 1, 1)$. Note that the number of parts of $\mu$ is $\lambda_1$, and conversely the number of parts of $\lambda$ is $\mu_1$. $\triangle$

**Lemma 2.63.** *Let $\lambda$ be a partition, and $\mu$ its corresponding conjugate partition. Then the number of parts of $\lambda$ is $\mu_1$.*

*Proof.* The number of rows in the Ferrers diagram is equal to the number of parts. Furthermore, the leftmost column necessarily contains one dot for each row. Clearly by reflecting along the diagonal, the leftmost column becomes the first row, and hence $\mu_1$ is the number of parts of $\lambda$. $\square$

# Chapter 3

# Central Result

In this chapter, we present our main construction. Throughout this chapter, $X$ will denote an elliptic curve in Legendre form, with marked point $O$—which forces char $\mathbb{K} \neq 2$. In Section 3.1, we construct the distinguished bundles $\mathcal{E}(r,d)$ from Theorem 2.54, and then conclude by providing the general case in Section 3.2.

## 3.1 Central Construction

Throughout this section, we fix the open cover $U_0, U_2$ of $X$ in (2.3). We will produce a transition matrix $M(r,d)$ such that $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$. Recall from Remark 2.39 that we will be constructing the matrix $M_{20}$— so that global sections of $\mathcal{B}(M(r,d))$ will be pairs of the form $(s, M(r,d)s)$ for $s \in \mathcal{O}_X(U_0)$. To do so, we will make careful use of the elements $\omega_k$ from (2.6); their relation to the rings of regular functions on this open cover via Lemmas 2.11, 2.12, and 2.13; and the tools in Section 2.4.

In order to construct $M(r,d)$ we must first construct a special integer partition of $d$ using the Euclidean algorithm.

**Remark 3.1.** As a reminder, integer partitions are conventionally ordered from greatest to least. However, we will be ordering them from least to greatest in the interest of cleaner notation later on.

Starting with two integers $(r, d)$ such that $0 < d < r$, suppose that the Euclidean algorithm returns

$$\begin{aligned} r_{-1} = r &= q_1 d + r_1 \\ r_0 = d &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ r_2 &= q_4 r_3 + r_4 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k, \end{aligned} \qquad \text{and} \qquad \begin{aligned} & r_1 < d \\ & 0 < r_i < r_{i-1} \text{ for } i = 2, ..., k-1 \\ & r_k = 0 \end{aligned}$$

Depending on the parity of the number of steps $k$, we can define a partition of $d$. If $k$ is even, then define

$$\mu(r, d) = \overbrace{r_{k-1} + \cdots + r_{k-1}}^{q_k \text{ terms}} + \overbrace{r_{k-3} + \cdots + r_{k-3}}^{q_{k-2} \text{ terms}} + \cdots + \overbrace{r_1 + \cdots + r_1}^{q_2 \text{ terms}}.$$

If $k$ is odd, then define

$$\mu(r, d) = r_{k-1} + \overbrace{r_{k-2} + \cdots + r_{k-2}}^{q_{k-1} \text{ terms}} + \overbrace{r_{k-4} + \cdots + r_{k-4}}^{q_{k-3} \text{ terms}} + \cdots + \overbrace{r_1 + \cdots + r_1}^{q_2 \text{ terms}}.$$

In other words, the $r_i$ appear $q_{i+1}$ times, for all of the odd indices $i$. If $k$ is odd, then there is an additional appearance of a single $r_{k-1}$. Alternatively, this definition can be expressed recursively as

$$\begin{aligned} \mu(r, 1) &= 1 \\ \mu(1, d) &= \overbrace{1 + 1 + \cdots + 1}^{d \text{ terms}} \\ \mu(r, d) &= \mu(r_1, r_2) + \overbrace{r_1 + r_1 + \cdots + r_1}^{q_2 \text{ terms}}. \end{aligned} \tag{3.1}$$

**Example 3.2.** Let us carry out the construction for the pair $(8, 5)$. In this case, $k = 4$:

$$\begin{aligned} 8 &= 1 \cdot 5 + 3 \\ 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

The relevant $r_i$ are on the odd steps: $r_1 = 3,\ r_3 = 1$, and the relevant $q_i$ are on the even steps: $q_2 = 1,\ q_4 = 2$. Hence the partition is

$$r_3 + r_3 + r_1 = 1 + 1 + 3.$$

△

**Example 3.3.** Now consider the pair $(45, 17)$. In this case, $k = 5$:

$$\begin{aligned} 45 &= 2 \cdot 17 + 11 \\ 17 &= 1 \cdot 11 + 6 \\ 11 &= 1 \cdot 6 + 5 \\ 6 &= 1 \cdot 5 + 1 \\ 5 &= 5 \cdot 1 + 0. \end{aligned}$$

Again the relevant $r_i$ are the on the odd steps: $r_1 = 11$, $r_3 = 5$, and the relevant $q_i$ are on the even steps, $q_2 = 1$, $q_4 = 1$. Since $k$ is odd, we also include $r_{k-1}$, in this case $r_4 = 1$. Therefore the partition is

$$r_4 + r_3 + r_1 = 1 + 5 + 11.$$

△

**Lemma 3.4.** *As constructed above, $\mu(r, d)$ is a partition of $d$. Its largest part is $d$ if $d|r$, and $r_1 = r - q_1 d$ otherwise.*

*Proof.* We proceed by induction on $k$, the number of steps of the Euclidean algorithm. If $k = 1$, then $d \mid r$ and

$$r = q_1 d + r_1.$$

Therefore by definition $\mu(r, d) = d$, which is clearly a partition of $d$. Note that $k = 1$ if and only if $d \mid r$.

Now suppose that for any pair $(r', d')$ with $0 < d' < r'$ such that the Euclidean algorithm takes $k$ steps, $\mu(r', d')$ is a partition of $d'$ with largest part either $d'$ or $r'_1$, and suppose $(r, d)$ is a pair that takes $k + 1$ steps. Recall from (3.1),

$$\mu(r, d) = \mu(r_1, r_2) + \overbrace{r_1 + r_1 + \cdots + r_1}^{q_2 \text{ terms}}.$$

By induction, $\mu(r_1, r_2)$ is a partition of $r_2$. Then since $d = q_2 r_1 + r_2$, we can see that $\mu(r, d)$ is a partition of $d$. Furthermore, by induction we can see that $\mu(r_1, r_2)$ either has

$r_2$ or $r_3 = r_1 - q_3 r_2$ as its largest part. Since the Euclidean algorithm produces strictly decreasing $r_i$, we know that $r_3 < r_2 < r_1$, and hence $r_1$ is the largest part of $\mu(r,d)$. $\square$

**Definition 3.5.** The *GCD partition of $d$ with respect to $r$* is the conjugate (Definition 2.61) of $\mu(r,d)$, which we denote $\lambda(r,d) = (\lambda_1, \lambda_2, ..., \lambda_\ell)$.

**Corollary 3.6.** *The number of parts of $\lambda(r,d)$ is $\ell = d$ if and only if $d|r$, otherwise $\ell = r - q_1 d$.*

*Proof.* This follows from Lemmas 2.63 and 3.4. $\square$

**Example 3.7.** Listed below are various pairs $(r,d)$ and their corresponding GCD partition.

$$\begin{aligned} \lambda(20,7) &= 1+1+1+1+1+2, & \lambda(38,17) &= 4+4+4+5 \\ \lambda(8,5) &= 1+1+3, & \lambda(15,13) &= 6+7, \\ \lambda(16,9) &= 1+1+1+1+1+1+3, & \lambda(18,12) &= 2+2+2+2+2+2, \\ \lambda(24,19) &= 3+3+3+3+7, & \lambda(32,29) &= 9+9+11. \end{aligned}$$

We can also examine the Ferrers diagrams:

$$\lambda(9,6) = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array} \qquad \lambda(13,10) = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \end{array}$$

$\triangle$

**Remark 3.8.** We shall see in Lemma 3.16 that $\lambda(r,d)$ completely describes what the diagonal of the matrices $M(r,d)$ will be.

Now we will construct the upper right block of $M(r,d)$. First, define $\omega(1) = (\omega_{-1})$, and for $k \geq 2$

$$\omega(k) := \begin{pmatrix} \omega_{-1} \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{k-1} \end{pmatrix} \tag{3.2}$$

where the $\omega_i$ are the special elements of valuation $i$ at the base point $O$ (see (2.6)).

Now suppose $0 < d < r$. If $2d < r$, define the block matrix

$$A(r,d) := \begin{pmatrix} \omega_{-1} I_d & 0_{d,r-2d} \end{pmatrix} \tag{3.3}$$

and otherwise if $2d \geq r$, set

$$A(r,d) := \begin{pmatrix} \omega(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & \omega(\lambda_2) & 0 & & 0 \\ 0 & 0 & \omega(\lambda_3) & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega(\lambda_\ell) \end{pmatrix} \tag{3.4}$$

where $\lambda(r,d) = (\lambda_1, ..., \lambda_\ell)$ is the GCD partition of $d$ with respect to $r$.

**Lemma 3.9.** *When $0 < d < r$, the matrix $A(r,d)$ is a $d \times (r-d)$ matrix.*

*Proof.* If $2d < r$, we have constructed $A(r,d)$ so that it consists of a $d \times d$ identity matrix followed by $r - 2d$ columns of zeroes, so it has exactly $d$ rows and $d + (r - 2d) = r - d$ columns.

If $2d \geq r$, then $A(r,d)$ is constructed to have $d$ rows, since $\lambda(r,d)$ is a partition of $d$ by Lemma 3.4, and $\ell$ columns, where $\ell$ is the number of parts of $\lambda(r,d)$. Hence it suffices to see that $\ell = r - d$. By Corollary 3.6, there are two cases:

1. If $\ell = d$, then we have that $d|r$, and since $d < r$ and $2d \geq r$, we conclude that $2d = r$, so that $r - d = d = \ell$.

2. Otherwise $\ell = r_1 = r - q_1 d$. But since $2d \geq r$, we must have that $q_1 = 1$, since $r_1 > 0$. Then $r_1 = r - d = \ell$.

□

**Example 3.10.** Listed below are some examples of $A(r,d)$.

$$A(5,3) = \begin{pmatrix} \omega_{-1} & 0 \\ 0 & \omega_{-1} \\ 0 & \omega_1 \end{pmatrix} \; A(7,3) = \begin{pmatrix} \omega_{-1} & 0 & 0 & 0 \\ 0 & \omega_{-1} & 0 & 0 \\ 0 & 0 & \omega_{-1} & 0 \end{pmatrix} \; A(11,9) = \begin{pmatrix} \omega_{-1} & 0 \\ \omega_1 & 0 \\ \omega_2 & 0 \\ \omega_3 & 0 \\ 0 & \omega_{-1} \\ 0 & \omega_1 \\ 0 & \omega_2 \\ 0 & \omega_3 \\ 0 & \omega_4 \end{pmatrix} \tag{3.5}$$

The important observation to make in the construction of $A(r,d)$ is that the columns have strictly increasing valuations in the non-zero entries. Additionally, for our later explicit proof of global generation, note that there are exactly $d$ non-zero entries, one for each row. △

Finally, having constructed this block, we can form the matrix $M(r,d)$ inductively. We form "the initial condition", for any $r \in \mathbb{N}$ and $d = 0$, as the $r \times r$ matrix

$$M(r,0) = \begin{pmatrix} 1 & \omega_{-1} & 0 & \cdots & 0 \\ 0 & 1 & \omega_{-1} & & 0 \\ \vdots & & \ddots & & \\ & & & 1 & \omega_{-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_1 & A(r,1) \\ 0 & M(r-1,0) \end{pmatrix}. \tag{3.6}$$

For any $r \in \mathbb{N}$, $d \in \mathbb{Z}$, the "first operation," defines

$$M(r,d) = \omega_1 M(r, d-r). \tag{3.7}$$

For $0 < d < r$, the "second operation," defines

$$M(r,d) = \begin{pmatrix} I_d & A(r,d) \\ 0 & M(r-d,d) \end{pmatrix}. \tag{3.8}$$

**Remark 3.11.** Notice the resemblance between these conditions and those characterizing $\mathcal{E}(r,d)$ in Theorem 2.54.

**Example 3.12.** Let us construct $M(6,2)$. The goal here is to use the initial condition or the second operation to reduce to a smaller problem—and the first operation is used to ensure $0 \leq d < r$, so that we can do this.

Since $0 < d < r$, we start with the second operation, to see that

$$M(6,2) = \begin{pmatrix} I_2 & A(6,2) \\ 0 & M(4,2) \end{pmatrix}.$$

Since $2d < r$, we don't need $\lambda(6,2)$ to compute $A(6,2)$, and we can see that the first two rows of the matrix are

$$M(6,2)_{\text{1-2,1-6}} = \begin{pmatrix} 1 & 0 & \omega_{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{-1} & 0 & 0 \end{pmatrix}.$$

Next we need to compute $M(4,2)$, so we use the second operation again:

$$M(4,2) = \begin{pmatrix} I_2 & A(4,2) \\ 0 & M(2,2) \end{pmatrix},$$

which then requires us to compute $M(2,2)$. In this case, we must use the first operation, $M(2,2) = \omega_1 M(2,0)$, where from (3.6), we see

$$M(2,0) = \begin{pmatrix} 1 & \omega_{-1} \\ 0 & 1 \end{pmatrix}.$$

Putting all of the pieces together, we arrive at the final result:

$$M(6,2) = \begin{pmatrix} 1 & 0 & \omega_{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & \omega_{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & \omega_{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & \omega_{-1} \\ 0 & 0 & 0 & 0 & \omega_1 & \omega_1\omega_{-1} \\ 0 & 0 & 0 & 0 & 0 & \omega_1 \end{pmatrix}.$$

△

**Example 3.13.** We will do one more example to illustrate some of the structure of the matrices, as well as a useful example for our explicit proof of global generation. Let us construct $M(5,8)$. We start with the first operation, since $d \geq r$, which tells us that $M(5,8) = \omega_1 M(5,3)$. Then we use the second operation to see

$$M(5,3) = \begin{pmatrix} I_3 & A(5,3) \\ 0 & M(2,3) \end{pmatrix}.$$

Recall that we had given $A(5,3)$ in (3.5). One can compute $M(2,3)$ by using the first operation to reduce the problem to computing $M(2,1)$, the second operation to reduce the problem to $M(1,1)$, and the first operation to reduce to $M(1,0)$, which is just the initial condition. This gives

$$M(2,3) = \begin{pmatrix} \omega_1 & \omega_1\omega_{-1} \\ 0 & \omega_1^2 \end{pmatrix}.$$

Putting all of the pieces together yields

$$M(5,8) = \begin{pmatrix} \omega_1 & 0 & 0 & \omega_1\omega_{-1} & 0 \\ 0 & \omega_1 & 0 & 0 & \omega_1\omega_{-1} \\ 0 & 0 & \omega_1 & 0 & \omega_1^2 \\ 0 & 0 & 0 & \omega_1^2 & \omega_1^2\omega_{-1} \\ 0 & 0 & 0 & 0 & \omega_1^3 \end{pmatrix}. \tag{3.9}$$

Note that all entries in the block $\omega_1 A(5,3)$ have a lower valuation at $O$ than the diagonal entry in their corresponding column. We shall see in Lemma 3.17 that this is no coincidence, and the matrix $A(r,d)$ is chosen specifically to ensure this structure. $\triangle$

**Lemma 3.14.** *For any $r \in \mathbb{N},\ d \in \mathbb{Z}$, the matrix $M(r,d)$ is uniquely determined by (3.6), (3.7), and (3.8). Furthermore, $M(r,d)$ is upper triangular.*

*Proof.* We will induct on $r$. The base case is $r = 0$, which is trivial. For the inductive step, we first note that the only way to compute $M(r,d)$ is using either the initial condition (3.6), or if $0 < d < r$, we can use the second operation (3.8). Let $m$ be the greatest integer such that $mr \leq d$, and set $d' = d - mr$, so that $0 \leq d' < r$. In other words, $d'$ is the unique residue of $d$ modulo $r$.

We can apply the first operation (3.7) $m$ times, and use the initial condition (3.6) if $d' = 0$ or the second operation (3.8) otherwise. In the former, we are done and it is upper triangular by construction. In the latter, we can apply induction to $M(r - d', d')$ to obtain our results.

**Lemma 3.15.** *For any $r \in \mathbb{N},\ d \in \mathbb{Z}$ the diagonal of $M(r,d)$ consists only of powers of $\omega_1$, and* $\det M(r,d) = \omega_1^d$.

*Proof.* We proceed by induction on $r$. The base case is $r = 0$, which is trivial. For the inductive step, let $m$ be the greatest integer such that $mr \leq d$, and set $d' = d - mr$, so that $0 \leq d' < r$. If $d' = 0$, then use the first operation (3.7) $m$ times and the initial condition (3.6) to see that

$$M(r,d) = \begin{pmatrix} \omega_1^m & \omega_1^m\omega_{-1} & 0 & \cdots & 0 \\ 0 & \omega_1^m & \omega_1^m\omega_{-1} & & 0 \\ \vdots & & \ddots & & \\ & & & \omega_1^m & \omega_1^m\omega_{-1} \\ 0 & 0 & \cdots & 0 & \omega_1^m \end{pmatrix},$$

which clearly has $\omega_1^m$ on the diagonal, and $\det M(r,d) = \omega_1^{mr} = \omega_1^d$. On the other hand, if $0 < d' < r$, then we apply the first operation $m$ times and then the second operation (3.8) to get

$$M(r,d) = \begin{pmatrix} \omega_1^m I_{d'} & \omega_1^m A(r,d') \\ 0 & \omega_1^m M(r - d', d') \end{pmatrix}.$$

By induction the $M(r-d',d')$ diagonal consists only of powers of $\omega_1$, establishing the first part. Furthermore

$$\begin{aligned}\det M(r,d) &= \det \omega_1^m I_{d'} \cdot \det \omega_1^m M(r-d',d') \\ &= \omega_1^{md'} \cdot \omega_1^{m(r-d')} \omega_1^{d'} \\ &= \omega_1^{mr+d'} \\ &= \omega_1^d.\end{aligned}$$

□

**Lemma 3.16.** *If $0<d<r$, the diagonal of $M(r,d)$ is the sequence*

$$(\overbrace{1,1,\cdots,1}^{d},\omega_1^{\lambda_1},\omega_1^{\lambda_2},\cdots,\omega_1^{\lambda_\ell}),$$

*where $\lambda(r,d)=(\lambda_1,...,\lambda_\ell)$ is the GCD partition of $d$ with respect to $r$.*

*Proof.* By Lemma 3.15, the diagonal consists only of powers of $\omega_1$. Furthermore, note that if $i<j$, then the exponent of the $i$-th diagonal entry is less than or equal to that of the $j$-th diagonal entry. This follows from our construction, since the exponent of a diagonal entry either increases via the first operation (3.7) in which case all entries increase by the same amount; or it increases via the second operation (3.8) in which case only the bottom right block raises the exponents.

Since $M(r,d)$ is upper-triangular, the determinant is the product of the diagonal entries, and $\det M(r,d)=\omega_1^d$ by Lemma 3.15. This means that the (non-zero) exponents of $\omega_1$ in the diagonal sequence of $M(r,d)$ form a partition of $d$ (particularly with our convention of ordering). Therefore, it suffices to show that this partition, call it $m(r,d)$, is equal to $\lambda(r,d)$. We will instead show that the conjugate partition $m(r,d)^*=\mu(r,d)$ by showing it satisfies the same recursion (3.1), and therefore $m(r,d)=\lambda(r,d)$.

Since $M(1,d)=(\omega_1^d)$, we have $m(1,d)=d$, and so

$$m(1,d)^*=(\overbrace{1+1+\cdots+1}^{d\text{ ones}}).$$

To compute $m(r,1)^*$, we use the second operation (3.8) $r$ times on $M(r,1)$, and see that

$$M(r,1)=\begin{pmatrix}1 & \omega_{-1} & 0 & \cdots & 0\\ 0 & 1 & \omega_{-1} & & 0\\ \vdots & & \ddots & & \\ & & & 1 & \omega_{-1}\\ 0 & 0 & \cdots & 0 & \omega_1\end{pmatrix}.$$

so that $m(r,1) = 1$ and hence

$$m(r,1)^* = 1.$$

Finally, if

$$r = q_1 d + r_1,$$
$$d = q_2 r_1 + r_2,$$

then

$$m(r,d)^* = m(r_1,r_2)^* + \overbrace{r_1 + r_1 + \cdots + r_1}^{q_2 \text{ terms}}$$

since we apply the first operation to $M(r,d)$ $q_2$ times so that $0 \le d - q_2 r_1 < r_1$. Then we use the initial condition or second operation, which in either case reduces the problem to computing the partition for the matrix

$$M(r_1, d - q_2 r_1) = M(r_1, r_2).$$

Therefore we conclude that $m(r,d)^* = \mu(r,d)$, and hence the conjugate partitions are also the same. □

**Corollary 3.17.** *If $0 < d < r$ the valuations of all non-zero entries in the $i$-th column of $A(r,d)$ are strictly lower than the valuation of the $(d+i)$-th diagonal entry of $M(r,d)$.*

*Proof.* If $2d < r$, then the valuation of all non-zero entries in $A(r,d)$ is $-1$, and the diagonal of $M(r,d)$ has valuation 0 or higher by Lemma 3.16.

If $2d \ge r$, then the non-zero entries in the $i$-th column of $A(r,d)$ are determined by the vector $\omega(\lambda_i)$ (3.2). The $(d+i)$-th diagonal entry of $M(r,d)$ in this case is $\omega_1^{\lambda_i}$ by Lemma 3.16. By construction, this has valuation strictly higher than any entry in $\omega(\lambda_i)$. □

**Lemma 3.18.** *For any $r \in \mathbb{Z}^+$, we have*

$$\dim \Gamma(X, \mathcal{B}(M(r,0))) = 1.$$

*Proof.* Recall that global sections of $\mathcal{B}(M(r,0))$ are determined by pairs (Proposition 2.34, Remark 2.39)

$$(s, M(r,0)s) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

In particular, we have the section represented by the pair

$$s = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; M(r,0)s = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

using the structure of the initial condition ((3.6)). Now we need to show there cannot be any other sections. Suppose

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{pmatrix} \in \mathcal{O}_X(U_0)^r.$$

Then

$$M(r,0)s = \begin{pmatrix} s_1 + \omega_{-1}s_2 \\ s_2 + \omega_{-1}s_3 \\ \vdots \\ s_{r-1} + \omega_{-1}s_r \\ s_r \end{pmatrix}.$$

But elements in $\mathcal{O}_X(U_0)$ have valuation 0 or less at $O$, and elements in $\mathcal{O}_X(U_2)$ must have non-negative valuation, so this forces $s_2 = s_3 = \cdots = s_r = 0$. Furthermore, we are forced to have $s_1 \in \mathcal{O}_X(U_0) \cap \mathcal{O}_X(U_2)$, which is equal to $\mathbb{K}$ by Lemma 2.13. □

With these tools in mind, we can present our central result.

**Theorem 3.19.** *We have an isomorphism*

$$\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d).$$

*Proof.* Let $m$ be the greatest integer such that $mr \leq d$, and set $d' = mr - d$, so that $0 \leq d' < r$. In order to show that $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$ it suffices to show that $\mathcal{B}(M(r,d')) \cong \mathcal{E}(r,d')$, since $\mathcal{E}(r,d) \cong \mathcal{E}(r,d') \otimes \mathcal{A}^m$ (Theorem 2.54) and similarly

$$\mathcal{B}(M(r,d)) = \mathcal{B}(\omega_1^m M(r,d')) \cong \mathcal{B}((\omega_1^m)) \otimes \mathcal{B}(M(r,d')) \cong \mathcal{A}^m \otimes \mathcal{B}(M(r,d')),$$

where the first equality is from applying the first operation (3.7) $m$ times, and the subsequent isomorphisms follow from Lemma 2.40 and Remark 2.55. Therefore we have reduced to showing that $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$ for $0 \leq d < r$.

If $d = 0$, then by Lemma 2.42 and (3.6), we have the exact sequence

$$0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{B}(M(r,0)) \longrightarrow \mathcal{B}(M(r-1,0)) \longrightarrow 0.$$

and we may apply Lemma 3.18 to see that

$$\Gamma(X, \mathcal{B}(M(r,0)) = \Gamma(X, \mathcal{B}(M(r-1,0) = 1.$$

Therefore the first part of Lemma 2.51 implies that $\mathcal{B}(M(r,0))$ is indecomposable. We conclude by Theorem 2.53 that $\mathcal{B}(M(r,0)) \cong \mathcal{F}_r = \mathcal{E}(r,0)$ since it has a non-trivial global section (again by Lemma 3.18).

Now suppose $0 < d < r$. Let us induct on $r$. The base case $r = 1$ is trivial as there is no integer between 0 and 1. For the inductive step suppose $r > 1$ and for any $0 < d' < r' < r$, we have an isomorphism $\mathcal{B}(M(r', d')) \cong \mathcal{E}(r', d')$. The second operation yields

$$M(r,d) = \begin{pmatrix} I_d & A(r,d) \\ 0 & M(r-d,d) \end{pmatrix}$$

which then gives an exact sequence by Lemma 2.42

$$0 \longrightarrow \mathcal{I}_d \longrightarrow \mathcal{B}(M(r,d)) \longrightarrow \mathcal{B}(M(r-d,d)) \longrightarrow 0.$$

Therefore $\mathcal{B}(M(r,d))$ is an extension of $\mathcal{B}(M(r-d,d))$ by $\mathcal{I}_d$. We claim that to show $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$, it suffices to show that $\Gamma(X, \mathcal{B}(M(r,d))^*) = 0$. Indeed, suppose this was the case. First, we can apply induction to see that $\mathcal{B}(M(r-d,d)) \cong \mathcal{E}(r-d,d)$ after reducing $d$ just as we had done above. Then by Lemma 2.52, we have that

$$\dim \Gamma(X, \mathcal{B}(M(r-d,d))) = d.$$

Then by the second part of Lemma 2.51, the extension $\mathcal{B}(M(r,d))$ is indecomposable. Finally we apply Theorem 2.54 (iii) to conclude that $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$.

Thus we have reduced to the situation of proving $\mathcal{B}(M(r,d))^*$ has no non-trivial global sections, for $0 < d < r$, which we will do by induction on $r$. The base case is $r = 1$, which is trivial. Now let $r > 1$ and $0 < d < r$. Suppose that for any $0 < d' < r' < r$ we have

$$\Gamma(X, \mathcal{B}(M(r', d'))^*) = 0.$$

Recall that $\mathcal{B}(M(r,d))^* \cong \mathcal{B}((M(r,d)^{-1})^T)$ by Lemma 2.41, so our goal is to show that there are no non-zero pairs

$$(s,t) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

satisfying

$$s = M(r,d)^T t.$$

Note that we have taken the inverse to move the matrix onto the left instead of the right (cf. Remark 2.39). Let

$$t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_r \end{pmatrix} \in \mathcal{O}_X(U_2)^r$$

and suppose the pair

$$(M(r,d)^T t, t) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

is a global section of $\mathcal{B}(M(r,d))^*$. Let us consider the structure of $M(r,d)^T$:

$$M(r,d)^T = \begin{pmatrix} I_d & 0 \\ A(r,d)^T & M(r-d,d)^T \end{pmatrix}.$$

Therefore we have that the first $d$ entries of $M(r,d)^T t$ are just $t_1, t_2, ..., t_d$. Since

$$(M(r,d)^T t, t) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

represents a global section, we must therefore conclude that

$$t_1, t_2, ..., t_d \in \mathcal{O}_X(U_0) \cap \mathcal{O}_X(U_2) = \mathbb{K} \text{ (Lemma 2.13)} \tag{3.10}$$

Ultimately, our goal is to show that all of the $t_1, t_2, ..., t_r$ vanish—which shows that any global section of $\mathcal{B}(M(r,d))^*$ is trivial. We will split into two cases:

1. Suppose $2d < r$. In this case, we will instead just show that the first $t_1, t_2, ..., t_d$ vanish, and conclude the remaining by induction. Let $1 \leq i \leq d$ be fixed, and consider the $(d+i)$-th entry of $M(r,d)^T t$. It is the dot product of the $(d+i)$-th row of $M(r,d)^T$ with $t$. The $(d+i)$-th row of $M(r,d)^T$ is the concatenation of the $i$-th rows of $A(r,d)^T$ and $M(r-d,d)^T$. Since $2d < r$, we have that the $i$-th row of $A(r,d)^T$ consists of a single $\omega_{-1}$ in the $i$-th position ((3.3)). Similarly, since $2d < r$, we have that $0 < d < r-d$, so that the second operation is needed to compute $M(r-d,d)$. Consequently, the first $d$ rows of $M(r-d,d)^T$ form the identity matrix, so that the $i$-th row only has

one non-zero entry which is a 1. Therefore the $(d+i)$-th entry of $M(r,d)^T t$ is

$$(M(r,d)^T t)_i = t_i \omega_{-1} + t_{d+i}$$

where $t_i \in \mathbb{K}$ by (3.10) and $t_{d+i} \in \mathcal{O}_X(U_2)$. In order for this to represent a global section, we must then have that $t_i = 0$, since $\omega_{-1} \notin \mathcal{O}_X(U_0)$, and there are no valuation $-1$ elements in $\mathcal{O}_X(U_2)$, so $t_{d+i}$ could not cancel $\omega_{-1}$.

This argument works for any $1 \leq i \leq d$, so we have $t_1 = t_2 = \cdots = t_d = 0$. Now a global section of $\mathcal{B}(M(r,d))^*$ is completely determined by global sections of $\mathcal{B}(M(r-d,d))^*$, and we can apply induction to conclude that there are no non-trivial global sections.

2. Now suppose $2d \geq r$. Let $\lambda(r,d) = (\lambda_1, ..., \lambda_\ell)$ be the GCD partition of $d$ with respect to $r$. In this case we will not need the inductive hypothesis. We will instead prove that $t_{d+i} = 0$ for $1 \leq i \leq r-d$ by induction on $i$. Along the way, we will also see that every other $t_i$ for $1 \leq i \leq d$ vanishes, and therefore there are no non-trivial global sections.

   For the base case $i = 1$, let us examine the $(d+1)$-th entry of $M(r,d)^T t$. This is determined by the dot product of $t$ with the $(d+1)$-th row of $M(r,d)^T$. Recall that

$$M(r,d)^T = \begin{pmatrix} I_d^T & 0 \\ A(r,d)^T & M(r-d,d)^T \end{pmatrix},$$

   so that the $(d+1)$-th row is just the first row of $A(r,d)^T$ concatenated with the first row of $M(r-d,d)^T$. The first row of $A(r,d)^T$ is just $\omega(\lambda_1)^T$, and since $M(r-d,d)^T$ is lower triangular, its first row consists of a single non-zero entry on the diagonal. By Lemma 3.16, this is $\omega_1^{\lambda_1}$, and hence the $(d+1)$-th entry of $M(r,d)^T t$ is

$$(M(r,d)^T t)_{d+1} = t_1 \omega_{-1} + t_2 \omega_1 + t_3 \omega_2 + \cdots + t_{\lambda_1} \omega_{\lambda_1 - 1} + t_{d+1} \omega_1^{\lambda_1}.$$

   In order for this element to lie in $\mathcal{O}_X(U_0)$, it cannot have positive valuation at $O$ by Lemma 2.11, nor valuation $-1$ at $O$. However, since $t_1, t_2, ..., t_d \in \mathbb{K}$, then in particular we have that $t_1, ..., t_{\lambda_1} \in \mathbb{K}$, since $\lambda_1 \leq d$. Assuming $t_i \neq 0$ for $1 \leq i \leq \lambda_1$, we have

$$\begin{aligned} \nu_O(t_1 \omega_{-1}) &= -1, \\ \nu_O(t_j \omega_{j-1}) &= j-1, \ 2 \leq j \leq \lambda_1, \\ \nu_O(t_{d+1} \omega_1^{\lambda_1}) &\geq \lambda_1. \end{aligned}$$

The last inequality comes from the fact that $t_{d+1} \in \mathcal{O}_X(U_2)$ and hence $\nu_O(t_{d+1}) \geq 0$ (Lemma 2.12). Therefore all of the terms involved have distinct valuations at $O$, no matter the choice of $t_i$, and as a result, no linear combination of the terms can yield an element of lower valuation—or cancel out to get zero. Finally, all of these valuations are forbidden by Lemma 2.11, so we conclude that all of the $t_i$ involved vanish.

Now suppose $i > 1$, and set $s = \sum_{k=1}^{i-1} \lambda_i$. Our inductive hypothesis is that $t_j = 0$ for the ranges

$$\begin{aligned} 1 \leq j &\leq s, \\ d+1 \leq j &< d+i. \end{aligned}$$

Now consider the $(d+i)$-th entry of $M(r,d)^T t$. Just as above, this is the dot product of $t$ with the $(d+i)$-th row of $M(r,d)^T$. The $(d+i)$-th row is exactly the $i$-th row of $A(r,d)^T$ concatenated with the $i$-th row of $M(r-d,d)^T$. However, by the inductive hypothesis, we have assumed that $t_{d+j} = 0$ for $d < j < d+i$. Since $M(r-d,d)^T$ is lower triangular, this means that its only contribution is the $i$-th diagonal entry. Furthermore, Lemma 3.16 tells us that the $i$-th diagonal entry is $\omega_1^{\lambda_i}$, and again the $i$-th row of $A(r,d)^T$ is the vector $\omega(\lambda_i)^T$. Then the $(d+i)$-th entry of $M(r,d)^T t$ is

$$(M(r,d)^T t)_{d+i} = t_{s+1}\omega_{-1} + t_{s+2}\omega_1 + t_{s+3}\omega_2 + \cdots + t_{s+\lambda_i}\omega_{\lambda_i - 1} + t_{d+i}\omega_1^{\lambda_i}$$

However, just as above, since $t_1, ..., t_d \in \mathbb{K}$, and $s \leq d$ (since $\lambda(r,d)$ is a partition of $d$ and we are taking only the first $i$ terms), we also have that $t_{s+1}, t_{s+2}, ..., t_{s+\lambda_i} \in \mathbb{K}$. Therefore, assuming none of them are zero, we have

$$\begin{aligned} \nu(t_{s+1}\omega_{-1}) &= -1, \\ \nu(t_{s+j}\omega_{j-1}) &= j-1, \ 2 \leq j \leq \lambda_i, \\ \nu(t_{d+i}\omega_1^{\lambda_i}) &\geq \lambda_i. \end{aligned}$$

Once again, the valuations of each term are distinct, and by Lemma 2.11, no elements of $\mathcal{O}_X(U_0)$ can attain these valuations, so we must conclude that all terms involved vanish. This establishes the induction.

Furthermore, along the way we also showed that $t_i = 0$ for all $1 \leq i \leq d$, since $\sum_{i=1}^{\ell} \lambda_i = d$. Therefore all of the $t_i$ vanish for $1 \leq i \leq r$, and so we conclude there are no non-trivial global sections of $\mathcal{B}(M(r,d))^*$.

□

We will conclude with an additional property of the matrices which will be useful for our explicit proof of global generation. □

**Lemma 3.20.** *If $d \geq r$, then for any $i, j$, we have $\nu_O(M(r,d)_{i,j}) \geq 0$.*

*Proof.* We proceed by induction on $r$. The base case $r = 0$ is trivial. Now suppose that for any $r' < r$ and $d' \geq r'$ we have that the valuation of every non-zero entry of $M(r', d')$ is non-negative. Let $m$ be the greatest integer such that $mr \leq d$, and set $d' = d - mr$, so that $0 \leq d' < r$. First suppose $d' = 0$. Then to compute $M(r, d)$, we apply the first operation (3.7) $m$ times, and then the initial condition (3.6) to get

$$M(r,d) = \begin{pmatrix} \omega_1^m & \omega_1^m\omega_{-1} & 0 & \cdots & 0 \\ 0 & \omega_1^m & \omega_1^m\omega_{-1} & & 0 \\ \vdots & & \ddots & & \\ & & & \omega_1^m & \omega_1^m\omega_{-1} \\ 0 & 0 & \cdots & 0 & \omega_1^m \end{pmatrix},$$

which clearly by construction only has terms of $\omega_1^m$, $\omega_1^m\omega_{-1}$ or zero. The valuations of these elements are $m$, $m-1$ and $\infty$ respectively, but since $m \geq 1$, they are all non-negative. Now suppose $0 < d' < r$. Then we use the first operation $m$ times and instead use the second operation (3.8) to see that

$$M(r,d) = \begin{pmatrix} \omega_1^m I_{d'} & \omega_1^m A(r,d') \\ 0 & \omega_1^m M(r-d', d') \end{pmatrix}.$$

Clearly every entry in $\omega_1^m I_{d'}$ is of non-negative valuation since it is either $\omega_1^m$ or zero. Furthermore, by using the first operation in reverse, the bottom right block is the same as $M(r-d', m(r-d')+d')$. Since $m \geq 1$ and $d' > 0$, we can conclude that $m(r-d')+d' \geq r-d'$, so that we can apply induction on this block and conclude all of the entries have non-negative valuation at $O$. Finally, we also have that the valuation of all entries in $\omega_1^m A(r, d')$ are non-negative: By construction the minimum valuation of entries in $A(r, d)$ is $-1$ (see (3.4)), and since $m \geq 1$, we are increasing the valuation by at least one. Therefore all of the entries in that block are of valuation at least zero. $\square$

## 3.2 Twisted Case

To conclude our construction, we will provide a transition matrix for any degree zero line bundle, thereby giving us a complete description of indecomposable vector bundles over elliptic curves via Theorem 2.56, Theorem 3.19, and Lemma 2.40.

Recall that any degree zero divisor over an elliptic curve is linearly equivalent to $P - O$ for a unique $P \in X$ (Lemma 2.8). We will first represent this as a Cartier divisor. Let $L_P \in \mathcal{O}_X(U_0)$ be such that the variety $X(L_P)$ is a line which passes through $P$ transversely and does not pass through $O$. By transversely, we mean that $X(L_P)$ is not tangent to $P$, which is possible since $X$ is smooth at $P$. Denote

$$\text{div } L_P = P + Q_1 + Q_2 - 3 \cdot O,$$

where $Q_1, Q_2$ are the two other points (possibly equal) where $X(L_P)$ intersects with $X$ (since $X$ is a degree three curve, see Bezout's Theorem, Corollary 7.8 in [7, Chapter 1]). Now we will modify our open cover from (2.3). Set

$$\begin{aligned} V_0 &= U_0 \backslash \{Q_1, Q_2\}, \\ V_2 &= U_2 \backslash \{P\}, \end{aligned} \tag{3.11}$$

so that $V_0$ is the open set consisting of $X$ without $O$, $Q_1$, and $Q_2$; and $V_2$ is the open set where we have removed $P, (1:0:0), (1:1:0)$, and $(1:\lambda:0)$. Now form the Cartier divisor

$$D = \left\{ (V_0, L_P), \left( V_2, \frac{1}{\omega_1} \right) \right\}.$$

By construction, we have $(\text{div } L_P)|_{V_0} = P$, and $\left( \text{div } \frac{1}{\omega_1} \right)\Big|_{V_2} = -O$, so that this represents the Weil divisor $P - O$. We then obtain

$$\mathcal{O}(P - O) \cong \mathcal{B}_{V_0, V_2}\left( \left( \frac{1}{\omega_1 L_P} \right) \right).$$

**Theorem 3.21.** *Let $X$ be an elliptic curve in Legendre form. Let $\mathcal{E}$ be any indecomposable vector bundle of rank $r$ and degree $d$ over $X$. Then either $\mathcal{E} \cong \mathcal{B}_{U_0, U_2}(M(r, d))$ or*

$$\mathcal{E} \cong \mathcal{B}_{V_0, V_2}\left( \frac{1}{\omega_1 L_P} M(r, d) \right)$$

*for a unique $P \in X$.*

*Proof.* Therorem 2.56 gives us that $\mathcal{E} \cong \mathcal{E}(r, d) \otimes \mathcal{L}$ for some degree zero line bundle $\mathcal{L}$. By Theorem 3.19, we have that $\mathcal{E}(r, d) \cong \mathcal{B}_{U_0, U_2}(M(r, d))$. In fact, we further have that $\mathcal{E}(r, d) \cong \mathcal{B}_{V_0, V_2}(M(r, d))$, since refining the open cover will not change the bundle. By the above, we can represent $\mathcal{L}$ as $\mathcal{B}_{V_0, V_2}\left( \left( \frac{1}{\omega_1 L_P} \right) \right)$ for a unique $P \in X$. Then the result follows from Lemma 2.40. $\square$

**Remark 3.22.** We have insisted that $X$ be in Legendre form, which implies that the characteristic of $\mathbb{K}$ is not two. However, it may be possible to extend this result to characteristic two by finding a suitable open cover of the elliptic curve which doesn't depend on the Legendre form. The main properties needed from the open cover are summarized by Lemmas 2.11, 2.12, and 2.13. If another open cover can be found which does not depend on the Legendre form which satisfies these Lemmas, then it is possible to reproduce the proof for any elliptic curve.

# Chapter 4

# Applications

In this chapter we give a short proof exactly determining which indecomposable bundles over an elliptic curve are globally generated via their degree and rank. As an application of our constructed matrices, we give a constructive proof of this result for the distinguished bundles $\mathcal{E}(r,d)$. Finally, we conclude by giving a theoretical verification that our included software will successfully compute global sections of vector bundles over an elliptic curve.

## 4.1 Global Generation

**Theorem 4.1.** *Let $X$ be any elliptic curve over an algebraically closed field $\mathbb{K}$ (no restriction on characteristic), and $\mathcal{E}$ a non-trivial indecomposable vector bundle over $X$ of rank $r$ and degree $d$. Then $\mathcal{E}$ is globally generated if and only if*

$$d \geq r + 1.$$

*Proof.* First, if $d = 0$, since we assume $\mathcal{E}$ is non-trivial, then Lemma 2.52 tells us that $\mathcal{E}$ has non-trivial global sections. Similarly, if $0 < d < r$, Lemma 2.52 gives us that $\mathcal{E}$ has $d < r$ independent sections, and they hence cannot possibly span an $r$-dimensional module. If $d = r$ and $\mathcal{E}$ is globally generated, then it is isomorphic to the trivial bundle, which contradicts our assumption. This establishes the forward direction.

Now suppose $d \geq r + 1$. Let $\mathcal{E}$ be an indecomposable bundle over $X$ of rank $r$ and degree $d$. Recall from Definition 2.44 that for any point $P \in X$ we must produce sections $s_1, ..., s_k \in \Gamma(X, \mathcal{E})$ such that their images $(s_1)_P, ..., (s_k)_P$ span $\mathcal{E}_P$ as an $\mathcal{O}_{X,P}$-module. By Lemma 2.49, this is equivalent to showing that

$$\left\langle \overline{(s_1)_P}, ..., \overline{(s_k)_P} \right\rangle_{\mathcal{O}_{X,P}/\mathfrak{m}_P} = \mathcal{E}_P / \mathfrak{m}_P \mathcal{E}_P,$$

where $\mathfrak{m}_P$ is the unique maximal ideal of $\mathcal{O}_{X,P}$, and $\overline{(s_i)_P}$ denotes the image of $(s_1)_P$ modulo $\mathfrak{m}_x \mathcal{E}_P$. To this end, fix $P \in X$. Consider $\mathcal{E} \otimes \mathcal{O}_X(-P)$ (Lemma 2.30). Note that

$\mathcal{E} \otimes \mathcal{O}_X(-P)$ is still indecomposable—if it decomposed into $\mathcal{E}_1 \oplus \mathcal{E}_2$, then twisting back by $\mathcal{O}_X(P)$ would yield a decomposition of $\mathcal{E}$. Furthermore, we note that there is a natural inclusion of $\mathcal{E} \otimes \mathcal{O}_X(-P)$ into $\mathcal{E}$ given by the presheaf morphism defined by

$$\phi_U : (\mathcal{E} \otimes \mathcal{O}_X(-P))(U) \to \mathcal{E}(U)$$
$$s \otimes t \mapsto t \cdot s.$$

Note that since $\mathcal{O}_X(-P)(U) \subset \mathcal{O}_X(U)$, this is a well-defined map, hence defining a presheaf morphism (and consequently a sheaf morphism), and it is injective since $t \cdot s = 0$ if and only if $t$ or $s$ is zero, either of which would imply $s \otimes t = 0$.

Since $\mathcal{O}(-P)$ is of degree $-1$, we have that $\mathcal{E} \otimes \mathcal{O}_X(-P)$ has degree $d - r \geq 1$, and so by Lemma 2.52 we have

$$\dim \Gamma(X, \mathcal{E} \otimes \mathcal{O}(-P)) = d - r$$

and

$$\dim \Gamma(X, \mathcal{E}) = d.$$

Therefore

$$\dim \Gamma(X, \mathcal{E}) - \dim \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X(-P)) = r$$

so there are $r$ global sections $s_1, ..., s_r \in \Gamma(X, \mathcal{E})$ whose images in

$$\Gamma(X, \mathcal{E}) / \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X(-P))$$

are linearly independent. Now suppose

$$c_1 \overline{(s_1)_P} + \cdots + c_r \overline{(s_r)_P} = 0,$$

where $c_i \in \mathcal{O}_{X,x}/\mathfrak{m}_x$, and $\overline{(s_i)_P}$ are the residues of $(s_i)_P$ modulo $\mathfrak{m}_x \mathcal{E}_P$. Then we have

$$c_1(s_1)_P + \cdots + c_r(s_r)_P \in \mathfrak{m}_P \mathcal{E}_P,$$

so that $c_1 s_1 + \cdots + c_r s_r \in \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X(-P))$. But then this would imply that $c_1 = \cdots = c_r = 0$, so $\overline{(s_1)_P}, ..., \overline{(s_r)_P}$ are linearly independent in $\mathcal{E}_P / \mathfrak{m}_P \mathcal{E}_P$. Furthermore, $\mathcal{E}_P / \mathfrak{m}_P \mathcal{E}_P$ is $r$-dimensional, and hence this is a spanning set, so that we are done. $\square$

Now we present an explicit construction to produce global sections generating the distinguished bundles $\mathcal{E}(r, d)$. Our strategy is to find global sections which span every stalk except for at $O$, then we will deal with $O$ separately. The former is very straightforward:

**Proposition 4.2.** *If $d \geq r$, then for $1 \leq i \leq r$, the pairs*

$$(e_i, M(r,d)e_i) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

*represent global sections of $\mathcal{E}(r,d)$, where the $e_i$ are the standard basis vectors (i.e. the $i$-th component is a one, and all other entries are zeroes).*

*Proof.* Obviously we have $e_i \in \mathcal{O}_X(U_0)^r$. The vector $M(r,d)e_i$ is just the $i$-th column of $M(r,d)$, which by Lemma 3.20 consists of entries in $\mathcal{O}_X(U_0 \cap U_2)$ which are of non-negative valuation at $O$. Therefore by Lemma 2.12, they all lie in $\mathcal{O}_X(U_2)$, so that $M(r,d)e_i \in \mathcal{O}_X(U_2)^r$. Consequently, by Proposition 2.34, they form global sections of $\mathcal{B}(M(r,d)) \cong \mathcal{E}(r,d)$ (Theorem 3.19). $\square$

For the stalk at $O$, we will need a much more technical construction.

**Proposition 4.3.** *If $d \geq r+1$ there are $s_1, s_2, ..., s_r \in \mathcal{O}_X(U_0)^r$ such that if $t_i = M(r,d)s_i$, then the pairs*

$$(s_i, t_i) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$$

*represent global sections of $\mathcal{E}(r,d)$, and the images $(t_i)_O$ in the stalk at $O$ form an upper triangular generating set for $\mathcal{E}(r,d)_O$ (see Definition 2.47).*

*Proof.* As a rough proof overview, this will proceed in two steps. First, we view $\mathcal{E}(r,d)$ as an extension

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}(r,d) \longrightarrow \mathcal{E}(r',d') \longrightarrow 0,$$

and lift global sections of $\mathcal{E}(r',d')$ to $\mathcal{E}(r,d)$, so that they span the last $r-d'$ components of the stalks. Second, we span the remaining $d'$ components by manipulating the structure of the matrix $M(r,d)$—particularly coming from the block $A(r,d)$.

We proceed by induction on $r$. The base case is $r=0$, which is trivial. Now suppose that for $\mathcal{B}(M(r',d'))$ with $r' < r$ and $d' > r'$ such $s_i$ exist. Let $m$ be the greatest integer such that $mr \leq d$, and set $d' = d - mr$ so that $0 \leq d' < r$. Let $q = d'$ if $d' \neq 0$, and otherwise set $q = 1$. In either case, we get the exact sequence

$$0 \longrightarrow \mathcal{I}_q \longrightarrow \mathcal{E}(r,d') \longrightarrow \mathcal{E}(r-q,d') \longrightarrow 0,$$

using Theorem 2.53 if $d' = 0$, and otherwise applying Theorem 2.54 (iii). Tensoring by line bundles preserves exactness, so we may tensor by $\mathcal{A}^m$ to obtain

$$0 \longrightarrow \mathcal{A}^m \otimes \mathcal{I}_q \longrightarrow \mathcal{A}^m \otimes \mathcal{E}(r,d') \longrightarrow \mathcal{A}^m \otimes \mathcal{E}(r-q,d') \longrightarrow 0.$$

The middle term above is isomorphic to $\mathcal{E}(r,d)$ using Theorem 2.54 (ii), and similarly we have

$$\mathcal{A}^m \otimes \mathcal{E}(r-q,d') \cong \mathcal{E}(r-q, m(r-q)+d').$$

In either case, this bundle has degree at least one higher than its rank: If $d'=0$, then we must have $m \geq 2$ since we are assuming $d \geq r+1$, and hence $m(r-q) \geq r-q+1$. If $d' \neq 0$, then $d' \geq 1$, and so $m(r-q)+d' \geq r-q+1$. Therefore, in both cases we can apply the inductive hypothesis to $\mathcal{A}^m \otimes \mathcal{E}(r-q,d')$ to obtain pairs

$$(s'_i, \omega_1^m M(r-q,d')s'_i) \in \mathcal{O}_X(U_0)^{r-q} \times \mathcal{O}_X(U_2)^{r-q}$$

for $q+1 \leq i \leq r$. We will extend these to global sections on $\mathcal{E}(r,d)$. Denote

$$M(r,d) \cdot (0, s'_i) = (\mathbf{f}_i, t'_i),$$

where each pair $(a,b)$ consists of a vector $a$ of length $q$, and a vector $b$ of length $r-q$, so that their combination gives a vector of length $r$. Furthermore, we have

$$s'_i \in \mathcal{O}_X(U_0)^{r-q},\ t'_i \in \mathcal{O}_X(U_2)^{r-q}$$

by induction, and $\mathbf{f}_i \in \mathcal{O}_X(U_0 \cap U_2)^q$. Now we apply Lemma 2.16 to $\mathbf{f}_i$ to obtain $\mathbf{g}_i \in \mathcal{O}_X(U_0)^q$ such that (note that we have $m \geq 1$)

$$\mathbf{f}_i - \omega_1^m I_q \mathbf{g}_i \in \mathcal{O}_X(U_2)^q$$

and set

$$s_i = (\mathbf{g}_i, s'_i) \in \mathcal{O}_X(U_0)^q \times \mathcal{O}_X(U_0)^{r-q}.$$

To compute $M(r,d)$, we apply the first operation (3.7) $m$ times, then apply either the initial condition (3.6) or the second operation (3.8). In either case, the first $q$ rows and columns of $M(r,d)$ form the identity matrix multiplied by $\omega_1^m$. Therefore

$$t_i = M(r,d)s_i = (\mathbf{f}_i - \omega_1^m I_q \mathbf{g}_i, t'_i).$$

Thus $t_i \in \mathcal{O}_X(U_2)^r$, and by induction the $t_i$ satisfy the two properties (i) and (ii) for an upper triangular generating set at $O$ (Definition 2.47). Now it suffices to find $s_1, s_2, \cdots, s_q \in \mathcal{O}_X(U_0)^r$ satisfying that their corresponding $t_i = M(r,d)s_i$ are in $\mathcal{O}_X(U_2)^r$ and satisfy the properties defining an upper triangular generating set at $O$—in other words, they satisfy:

(i) For $1 \leq i \leq q$, $s_i \in \mathcal{O}_X(U_0)^r$.

(ii) For $1 \leq i \leq q$, and $1 \leq j < i$, the $j$-th component of $t_i$ has non-negative valuation at $O$.

(iii) For $1 \leq i \leq q$, the $i$-th component of $t_i$ is of valuation zero at $O$.

(iv) For $1 \leq i \leq q$, and $i < j \leq r$, the $j$-th component of $t_i$ is of valuation strictly greater than zero at $O$.

First, we tackle the easier case of $m \geq 2$. In this case, for $1 \leq i \leq q$, we can set

$$s_i = \omega_{-m} e_i.$$

By Lemma 2.11, these vectors lie in $\mathcal{O}_X(U_0)^r$, and furthermore we had just noticed that the first $q$ rows and columns of $M(r,d)$ are the identity matrix multiplied by $\omega_1^m$, so

$$t_i = M(r,d)s_i = \omega_{-m}\omega_1^m e_i.$$

Then each $t_i$ has valuation zero at $O$ in the $i$-th entry, and all other entries are zero, so that this is a diagonal generating set for the stalk at $O$.

**Remark 4.4.** This completely finishes the case where $d' = 0$, since we cannot have $m = 1$—otherwise $d = r$, failing the assumption.

From now on, we are only concerned with the case $m = 1$, and therefore $d' > 0$ so that $q = d'$. Once again we remind the reader that we are attempting to attain conditions (i)–(iv). As an overview of the proof method, we use the structure of $A(r,d)$ to choose a vector immediately satisfying (iii), then we apply Lemma 2.15 to attain (ii) and (iv) while still maintaining (i). Since this is technical, we will give an example to aid the reader. The proof will continue on page 54.

**Example 4.5.** Recall from (3.9) that

$$M(5,8) = \begin{pmatrix} \omega_1 & 0 & 0 & \omega_1\omega_{-1} & 0 \\ 0 & \omega_1 & 0 & 0 & \omega_1\omega_{-1} \\ 0 & 0 & \omega_1 & 0 & \omega_1^2 \\ 0 & 0 & 0 & \omega_1^2 & \omega_1^2\omega_{-1} \\ 0 & 0 & 0 & 0 & \omega_1^3 \end{pmatrix}.$$

Note that $m = 1$ in this case. Our goal here is to find three sections $s_1, s_2, s_3$ satisfying the four properties above. What we shall see is that the sections essentially arise from the non-zero entries in the block $\omega_1 A(r,q)$ which in this case is the block

$$\omega_1 A(5,3) = \begin{pmatrix} \omega_1\omega_{-1} & 0 \\ 0 & \omega_1\omega_{-1} \\ 0 & \omega_1^2 \end{pmatrix}.$$

First, set

$$s_1 = e_4,$$

the standard basis vector. Then this satisfies (i), and picks out the first row element of $\omega_1 A(5,3)$ and places it in the first entry of

$$t_1 = M(5,8)s_1 = \begin{pmatrix} \omega_1\omega_{-1} \\ 0 \\ 0 \\ \omega_1^2 \\ 0 \end{pmatrix},$$

so that it it satisfies (iii). Additionally, this happens to automatically satisfy (ii), and (iv) particularly since $\omega_1^2$ has valuation $2 \geq 1$ (and all other entries are zeroes). What happened here will essentially be what happens in Case 1 below—i.e. picking out one term of valuation zero and another of valuation greater than zero. However, we note that the pair (5,8) isn't in Case 1.

Now for the second component, we set

$$s_2 = e_5$$

so that it satisfies (i) and places the second row element of $\omega_1 A(5,3)$ in the second entry of

$$t_2 = M(5,8)s_2 = \begin{pmatrix} 0 \\ \omega_1\omega_{-1} \\ \omega_1^2 \\ \omega_1^2\omega_{-1} \\ \omega_1^3 \end{pmatrix},$$

making it satisfy (iii). Once again this automatically satisfies (ii) and (iv). The only case where cancelling is involved is the third component. First we try

$$s_3' = \omega_{-2}e_5,$$

and note that this is in $\mathcal{O}_X(U_0)^5$ by Lemma 2.11. Then this picks out the third row element of $\omega_1 A(5,3)$ and scales it so that its valuation is zero at $O$:

$$M(5,8)s_3' = \begin{pmatrix} 0 \\ \omega_1\omega_{-1}\omega_{-2} \\ \omega_1^2\omega_{-2} \\ \omega_1^2\omega_{-1}\omega_{-2} \\ \omega_1^3\omega_{-2} \end{pmatrix},$$

so that it satisfies (iii). However, it fails (ii) and (iv) in the second and fourth entries respectively. Furthermore, note that the fifth entry does *not* fail (iv)—this will always be the case by Lemma 3.16 and Corollary 3.17. To resolve the second and fourth entries, we must use the cancelling lemma, Lemma 2.15. This must be done from the bottom up, due to the upper triangular structure of $M(r,d)$. We apply the lemma to the fourth entry $\omega_1^2\omega_{-1}\omega_{-2}$ first, to obtain $g_1$ such that

$$f_1 := \omega_1^2\omega_{-1}\omega_{-2} - \omega_1^2 g_1 \in \mathcal{O}_X(U_2).$$

Furthermore, we can insist that $\nu_O(f_1) > 0$, as per the second part of the lemma. We can then set

$$s_3'' = \omega_{-2}e_5 - g_1 e_4$$

which will give us

$$M(5,8)s_3'' = \begin{pmatrix} -\omega_1\omega_{-1}g_1 \\ \omega_1\omega_{-1}\omega_{-2} \\ \omega_1^2\omega_{-2} \\ f_1 \\ \omega_1^3\omega_{-2} \end{pmatrix},$$

so that (iv) is satisfied—namely, when we take the image at the stalk at $O$, the fourth and fifth entries vanish, leaving us with the desired upper triangular form. Finally, we conclude by cancelling the first and second entry using Lemma 2.15 or Lemma 2.16 to achieve (ii) (we are using the former to be more explicit). To this end, let $g_2, g_3 \in \mathcal{O}_X(U_0)$ be such that

$$f_2 := \omega_1\omega_{-1}\omega_{-2} - \omega_1 g_2 \in \mathcal{O}_X(U_2),$$

and

$$f_3 := -\omega_1\omega_{-1}g_1 - \omega_1 g_3 \in \mathcal{O}_X(U_2).$$

We set

$$s_3 = \omega_{-2}e_5 - g_1e_4 - g_2e_2 - g_3e_1.$$

Then

$$t_3 = M(5,8)s_3 = \begin{pmatrix} f_3 \\ f_2 \\ \omega_1^2\omega_{-2} \\ f_1 \\ \omega_1^3\omega_{-2} \end{pmatrix},$$

where

$$\begin{aligned} \nu_O(f_2), \nu_O(f_3) \geq 0 \quad &((ii)), \\ \nu_O(\omega_1^2\omega_{-2}) = 0 \quad &((iii)), \\ \nu_O(f_1), \nu_O(\omega_1^3\omega_{-2}) > 0 \quad &((iv)) \end{aligned}$$

so that all of the conditions are successfully satisfied.

We summarize the approach as follows: The block $\omega_1 A(r,q)$ has precisely one entry in each row, and $q$ rows—these are exactly used to construct the sections. We pick a column of $\omega_1 A(r,q)$, and multiply it by an appropriate $\omega_{-k}$ such that the desired entry has valuation 0 at $O$, ensuring (iii). Then we apply Lemma 2.15 strategically to ensure that (ii) and (iv) hold, while still maintaining (i).

To motivate why we consider $m \geq 2$ first, note that when $m = 2$, the term $\omega_1^2\omega_{-1}$ appears in $\omega_1 A(r,q)$, which is of valuation one at $O$. However, there are no elements of valuation $-1$ in $\mathcal{O}_X(U_0)$ by Lemma 2.11, and so we cannot find $f \in \mathcal{O}_X(U_0)$ so that $f\omega_1^2\omega_{-1}$ has valuation zero at $O$, which is vital to our method. $\triangle$

We remind the reader that $q = d' > 0$, and $m = 1$, and we are constructing $s_1, ..., s_q$ to satisfy the conditions (i)–(iv). From here, we can compute $M(r,d)$ by using the first operation once, and then the second operation.

$$M(r,d) = \begin{pmatrix} \omega_1 I_q & \omega_1 A(r,q) \\ 0 & \omega_1 M(r-q,q) \end{pmatrix}.$$

We separate into two cases:

1. Suppose $2q < r$. Then take the standard basis vectors (which obviously satisfy (i))

$$s_i = e_{q+i}, \ 1 \leq i \leq q.$$

Then the corresponding $t_i = M(r,d)s_i$ is just the $(q+i)$-th column of $M(r,d)$, which is the concatentation of the $i$-th columns of $\omega_1 A(r,q)$ and $\omega_1 M(r-q,q)$. Since $2d < r$, the $i$-th column of $\omega_1 A(r,q)$ consists of a single non-zero entry $\omega_1\omega_{-1}$ in the $i$-th row ((3.3)). Furthermore, $2q < r$ implies that the second operation is needed to compute $M(r-q,q)$—so the $i$-th column of $\omega_1 M(r-q,q)$ is part of the identity matrix in the construction, and hence is also just a single entry of $\omega_1$ in the $i$-th row (hence the $(q+i)$-th row of $M(r,d)$). So

$$t_i = \omega_1\omega_{-1}e_i + \omega_1 e_{q+i}.$$

We have that $\nu_O(\omega_1\omega_{-1}) = 0$, and $\nu_O(\omega_1) = 1$, and all of the remaining entries are zeroes, with infinite valuation. Therefore $t_i$ satisfies (ii), (iii), and (iv) as desired.

2. Now suppose $2q \geq r$. First, observe that in $\omega_1 A(r,q)$, each row has exactly one non-zero entry by construction (3.4)), so let $\alpha_i$ denote the unique non-zero entry in the $i$-th row of $\omega_1 A(r,q)$, and set $c(i)$ to be the corresponding column in which it lies in $\omega_1 A(r,q)$ (so that the term appears in the $(q+c(i))$-th column of $M(r,d)$). Set $a_i = \nu_O(\alpha_i)$, and note that $a_i \geq 0$, $a_i \neq 1$ since the lowest valuation in $\omega_1 A(r,q)$ comes from the term(s) $\omega_1\omega_{-1}$, and $A(r,q)$ never contains $\omega_0$ by construction.

   **Example 4.6.** In Example 4.5, we can see that for $\omega_1 A(5,3)$

   $$\begin{aligned}\alpha_1 &= \omega_1\omega_{-1},\ c(1) = 1\\ \alpha_2 &= \omega_1\omega_{-1},\ c(2) = 2\\ \alpha_3 &= \omega_1^2,\ c(3) = 2.\end{aligned}$$

   We have that $\alpha_1$ appears in the fourth column of $M(5,8)$, and $\alpha_2, \alpha_3$ appear in the fifth column of $M(5,8)$. △

   So, for $1 \leq i \leq q$, we will first set

   $$s_i' = \omega_{-a_i} e_{q+c(i)}$$

   which is in $\mathcal{O}_X(U_0)^r$ since $-a_i \leq -2$ or $a_i = 0$ (Lemma 2.11), hence satisfying (i).

   Applying $M(r,d)$ to $s_i'$ just returns the multiplication of the $(q+c(i))$-th column of $M(r,d)$ by $\omega_{-a_i}$. In this case there are two terms which we are certain of—the entry in $\omega_1 A(r,q)$ which we just defined, and the diagonal entry (of $\omega_1 M(r,q)$) via Lemma 3.17. Hence

   $$M(r,d)s_i' = \omega_{-a_i}\alpha_i e_i + \omega_{-a_i}\omega_1\omega_1^{\lambda_{c(i)}} e_{q+c(i)} + \mathbf{f}_i, \tag{4.1}$$

where $\lambda = \lambda(r,q)$ is the GCD partition of $q$ with respect to $r$, and $\mathbf{f}_i$ is some vector in $\mathcal{O}_X(U_0 \cap U_2)^r$. In other words, the $i$-th entry is $\omega_{-a_i}\alpha_i$ which has valuation zero at $O$; the $(q+c(i))$-th entry is $\omega_{-a_i}\omega_1\omega_1^{\lambda_{c(i)}}$ which has strictly positive valuation at $O$ by Corollary 3.17; and $\mathbf{f}_i$ consists of the remaining entries, with only the first $q+c(i)-1$ terms possibly non-zero since $M(r,d)$ is upper triangular.

It remains to modify $s'_i$ in such a way that $M(r,d)s'_i$ satisfies (ii) and (iv). Due to the upper triangular structure of $M(r,d)$, it is necessary to start with (iv), which we attain via the following lemma:

**Lemma 4.7.** *Fix $1 \le i \le q$. Then there exists $\boldsymbol{g}_i \in \mathcal{O}_X(U_0)^r$ such that*

$$M(r,d)(s'_i - \boldsymbol{g}_i)$$

*satisfies that its $i$-th entry has valuation 0 at $O$, and for $i < j \le r$, the $j$-th entry has strictly positive valuation at $O$.*

*Proof.* Denote

$$M(r,d)s'_i = \begin{pmatrix} f_1 & f_2 & \cdots & f_{q+c(i)} & 0 & \cdots & 0 \end{pmatrix}^T. \tag{4.2}$$

Recall that $c(i)$ is the column containing the unique non-zero entry $\alpha_i$ in the $i$-th row of $\omega_1 A(r,q)$ (in other words, $\alpha_i$ appears in the $c(i)$-th non-zero block). As a consequence, the only non-zero entries of $\omega_1 A(r,q)$ appear in entries indexed by $(i, c(i))$.

First, we claim that if $i < j \le q$, then $\nu_O(f_j) > 0$ already. For $i < j \le q$, the entry $f_j$ is determined by the block $\omega_1 A(r,q)$: it picks out the $(j, c(i))$-th entry. But if $c(j) \ne c(i)$ then we can conclude $f_j = 0$ by our above observation (and hence has infinite valuation at $O$).

On the other hand, if $c(j) = c(i)$, then by the construction of $\omega(k)$ ((3.2)), since $j > i$, the valuation of $\alpha_j$ is strictly higher than that of $\alpha_i$. Furthermore, we have (by our choice of $s'_i$)

$$f_i = \alpha_i \omega_{-a_i}, \ f_j = \alpha_j \omega_{-a_i},$$

and hence $\nu_O(f_j) > \nu_O(f_i) = 0$. This proves the cases $i < j \le q$.

We will prove the remaining cases, $q < j \le r$, by reverse induction on $j$. We want to show there exists $\mathbf{g}_i$ such that the last $r-j+1$ entries of $M(r,d)(s'_i - \mathbf{g}_i)$ have strictly positive valuation at $O$, and all entries of $M(r,d)\mathbf{g}_i$ are zero in the range $i \le k \le q$ (so that they do not affect the range we proved above, nor the $i$-th entry).

The base case will be $j = q + c(i)$, since the last $r - q - c(i)$ entries already have infinite valuation at $O$ ((4.2)). We had already seen in (4.1) that the $(q + c(i))$-th entry of $M(r, d)s_i'$ has positive valuation at $O$, establishing the base case.

Now suppose $j < q + c(i)$, and that there exists a vector $\mathbf{g}_i'$ satisfying the conditions for $j+1$. Since $j < q+c(i)$, we have that $j-q > 0$, so that $\lambda_{j-q}$ is well-defined—where $\lambda(r, q)$ is the GCD partition of $q$ with respect to $r$. Furthermore, by definition, we have that $\lambda_{j-q} \geq 1$, so that we can apply the second part of Lemma 2.15 to find $g_{ij}$ such that

$$\nu_O((M(r,d)(s_i' - \mathbf{g}_i'))_j - \omega_1\omega_1^{\lambda_{j-q}} g_{ij}) > 0$$

In other words, $g_{ij}$ cancels the $j$-th entry of $M(r, d)(s_i' - \mathbf{g}_i')$. Now set

$$\mathbf{g}_i = g_{ij}e_j + \mathbf{g}_i'.$$

Then

$$M(r,d)\mathbf{g}_i = \omega_1\omega_1^{\lambda_{j-q}} g_{ij}e_j + \mathbf{g}_i'' + M(r,d)\mathbf{g}_i',$$

where $\mathbf{g}_i''$ is some vector in $\mathcal{O}_X(U_0 \cap U_2)^r$, only possibly non-zero in the first $j - 1$ components. Note that the exponent of $\omega_1$ in the first term above comes from Lemma 3.16; we are choosing the $j$-th diagonal entry, which in this case is the $(j - q)$-th term in the partition $\lambda(r, q)$. The additional $\omega_1$ comes from the fact that we applied the first operation once. By induction, the last $r - j + 2$ entries of $M(r, d)(s_i' - \mathbf{g}_i)$ have strictly positive valuation, and by our choice of $g_{ij}$, we also have that the $j$-th entry has strictly positive valuation.

It remains to show that for $i \leq k \leq q$, the $k$-th entry of $M(r, d)\mathbf{g}_i$ is a zero. By induction we already know that $M(r, d)\mathbf{g}_i'$ has a zero in all entries of this range, so we just need to check $\mathbf{g}_i''$. The $k$-th entry of $\mathbf{g}_i''$ is determined by the block $\omega_1 A(r, q)$: it picks out the $((k, j - q)$-th entry. But since $j < q + c(i)$, this is some column *before* the $c(i)$-th column of $\omega_1 A(r, d)$. Consequently, it lands in the lower triangle of blocks in $\omega_1 A(r, d)$, which are all zero blocks ((3.4)), so that the $k$-th entry of $\mathbf{g}_i''$ must be zero. □

To conclude the proof of Proposition 4.3, we can apply Lemma 4.7 to obtain a modification $s_i''$ such that $M(r, d)s_i''$ satisfies (iv), and we can finally apply Lemma 2.16 to $M(r, d)s_i''$ to obtain $s_i$ such that $M(r, d)s_i$ additionally satisfies (ii). The second part of Lemma 2.16 ensures that we do not violate (iii) or (iv), thereby fulfilling all of the four conditions.

□

**Theorem 4.8.** *If $d \geq r+1$, then the distinguished bundle $\mathcal{E}(r,d)$ is globally generated.*

*Proof.* By Proposition 4.2, every stalk in $U_0$ is spanned by a set of global sections. By Proposition 4.3, the stalk at $O$ is also spanned by a set of global sections. Then since (as a set) $X = U_0 \cup \{O\}$, every stalk is spanned by a set of global sections, and hence $\mathcal{E}(r,d)$ is globally generated. □

**Remark 4.9.** This proof can also be adapted for the general twisted case, but it is similarly long and technical. As a rough sketch, Theorem 3.21 shows that a general indecomposable bundle has the same transition matrix, but divided by $\omega_1 L_P$. Since $L_P$ lies in $\mathcal{O}_X(U_0)$, we can just multiply all of our sections $s_i$ by $L_P$ to clear the denominator. This leaves us with a division by $\omega_1$, thereby lowering the valuation at $O$ of everything by one. We can then construct a special element $f_P$ so that $\frac{f_P}{L_P}$ has valuation $-1$ at $O$, and can thereby cancel all of the valuation $-1$ elements that are introduced.

## 4.2 Computing Global Sections

In this section, we discuss a method for computing global sections of $\mathcal{B}_{U_0,U_2}(M)$, where $M$ is some upper triangular matrix and $U_0, U_2$ is the open cover given by (2.3). First, recall that sections of $\mathcal{B}_{U_0,U_2}(M)$ are of the form

$$(s, Ms) \in \mathcal{O}_X(U_0)^r \otimes \mathcal{O}_X(U_2)^r.$$

Therefore, to compute global sections, it suffices to compute the possible $s \in \mathcal{O}_X(U_0)^r$ that satisfy $Ms \in \mathcal{O}_X(U_2)^r$. First, let us provide a way to explicitly represent these sections:

**Lemma 4.10.** *Any element $\overline{f} \in \mathcal{O}_X(U_0)$ can be uniquely represented in the form*

$$\overline{f} = c_0\omega_0 + \sum_{i=2}^{N} c_i \omega_{-i}$$

*for some $N \in \mathbb{N}$.*

*Proof.* Fix the monomial ordering $x > y$. Recall that $\mathcal{O}_X(U_0) = \mathbb{K}[x,y]/\langle y^2 - x(x-1)(x-\lambda)\rangle$ (2.5). For any $\overline{f} \in \mathcal{O}_X(U_0)$, we can take its representative in $\mathbb{K}[x,y]$, and apply Gröbner basis reduction via the ideal $\langle y^2 - x(x-1)(x-\lambda)$. Denote this reduction as $\overline{f'}$. Note that $\overline{f} \equiv \overline{f'}$ modulo the ideal, so that they are equal in $\mathcal{O}_X(U_0)$.

Furthermore, since $y^2 - x(x-1)(x-\lambda)$ is degree three in $x$, all monomials of degree three or higher in $x$ in $f$ will be cancelled in $f'$. We had constructed $\omega_k$ (2.6) so that they do not exceed degree three in $x$, so these will be the only monomials which appear in $f'$. □

Now to compute global sections, we can set up a system of linear equations in the coefficients of the monomials:

**Example 4.11.** Suppose we have the matrix

$$M = \begin{pmatrix} \omega_1 & \omega_{-1} \\ 0 & \omega_2 \end{pmatrix},$$

and we wish to compute global sections of $\mathcal{B}_{U_0,U_2}(M)$. They will be of the form

$$\begin{aligned} s_1 &= c_{01} + \sum_{i=2}^{-\nu_O(s_1)} c_{i1}\omega_{-i} \\ s_2 &= c_{02} + \sum_{i=2}^{-\nu_O(s_1)} c_{i2}\omega_{-i}, \end{aligned}$$

and

$$M \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \omega_1 s_1 + \omega_{-1} s_2 \\ \omega_2 s_2 \end{pmatrix}.$$

In order for this to represent a global section, we need $\omega_1 s_1 + \omega_{-1} s_2$ and $\omega_2 s_2$ to lie in $\mathcal{O}_X(U_2)$. Consequently, this imposes a condition on $c_{2i}$ for $i \geq 3$: if $c_{2i} \neq 0$ for $i \geq 3$, then $\nu_O(s_2) \leq -3$, so that

$$\nu_O(\omega_2 s_2) \leq -1,$$

and hence does not lie in $\mathcal{O}_X(U_2)$. Similarly, the first entry also gives conditions on the coefficients, such as

$$c_{21} + c_{02} = 0,$$

since if this was not satisfied, then $\omega_1 s_1 + \omega_{-1} s_2$ would have a non-zero valuation $-1$ term—and hence would not lie in $\mathcal{O}_X(U_2)$. We know there are only finitely many $c_{ij}$ which are non-zero, but *a priori* we are unsure of how many coefficients to check for computation. This motivates the next lemma. △

Let $M \in \mathrm{GL}_r(\mathcal{O}_X(U_0 \cap U_2))$ be upper triangular. We form the preliminary bounds

$$\begin{aligned} B_1 &= -\max\{0, \nu_O(M_{11})\}, \\ B_j &= -\max\{0, \nu_O(M_{ii})\} + \min_{1 \leq i \leq r}\{0, \nu_O(M_{ij})\}, \; 2 \leq j \leq r. \end{aligned}$$

**Example 4.12.** In the above example with

$$M = \begin{pmatrix} \omega_1 & \omega_{-1} \\ 0 & \omega_2 \end{pmatrix},$$

we have

$$B_1 = -1,\ B_2 = 2 - (-1) = -3.$$

Note that by definition, we always have that the $B_i$ are non-positive. The preliminary bounds $B_j$ tell us what the lowest valuation each column could theoretically yield without breaking the condition $(Ms)_j \in \mathcal{O}_X(U_2)$.

For instance, in the second column, the lowest valuation we can pick without failing $(Ms)_2 \in \mathcal{O}_X(U_2)$ would be $-2$, but this introduces a valuation $-3$ term in $(Ms)_1$. Similarly, in the first column, the lowest valuation we can pick would be $-1$ (theoretically—Lemma 2.11 tells us this is impossible).

However, the bound for the second column influences the first. If the pair $(s, Ms)$ represent a global section and $(Ms)_1$ contains a term of valuation $-3$ coming from $s_2$, then we must conclude that $M_{11}s_1$ also has a valuation $-3$ term to cancel it out. Consequently, $s_1$ would have valuation $-4$, so that its valuation is determined by all columns proceeding it. This motivates the next lemma. △

**Lemma 4.13.** *Suppose $M \in GL_r(\mathcal{O}_X(U_0 \cap U_2))$ is upper triangular, and suppose the pair $(s, Ms) \in \mathcal{O}_X(U_0)^r \times \mathcal{O}_X(U_2)^r$ represents a global section of $\mathcal{B}_{U_0,U_2}(M)$, where $s = (s_1, s_2, \cdots, s_r)^T$. Then we have*

$$\nu_O(s_j) \geq \sum_{k=1}^{r-j} B_{r-k}$$

*Proof.* We will proceed by reverse induction on $j$. The base case is $j = r$. In this case, we can easily see that the last entry of $Ms$ is just $M_{rr}s_r$, and in order for this to lie in $\mathcal{O}_X(U_2)$, we must have that its valuation is non-negative. Therefore

$$\nu_O(M_{rr}s_r) = \nu_O(M_{rr}) + \nu_O(s_r) \geq 0,$$

so that

$$\nu_O(s_r) \geq -\nu_O(M_{rr}) \geq -\max\{0, \nu_O(M_{rr})\} + \min_{1\leq i\leq r}\{0, \nu_O(M_{ir})\} = B_r.$$

This establishes the base case. Now for the inductive step, we suppose that $j < r$, and for $j < j'$, we have that

$$\nu_O(s_{j'}) \geq \sum_{k=1}^{r-j'} B_{r-k}.$$

Consider the $j$-th entry of $M(r,d)s$. Explicitly, it is

$$(M(r,d)s)_j = M_{jj}s_j + M_{j(j+1)}s_{j+1} + \cdots + M_{jr}s_r.$$

Since we are assuming the pair $(s, M(r,d)s)$ represents a global section, we must have that this entry lies in $\mathcal{O}_X(U_2)$, so that in particular its valuation must be non-negative. Since all of the $B_j$ are non-positive, we have that $\sum_{k=1}^{r-j} B_k$ is decreasing as $j$ decreases. Therefore the valuation of $M_{j(j+1)}s_{j+1}$ is minimal, and hence

$$\begin{aligned}
\nu_O(M_{j(j+1)}s_{j+1} + \cdots + M_{jr}s_r) &= \nu_O(M_{j(j+1)}) + \nu_O(s_{j+1}) \\
&\geq \nu_O(M_{j(j+1)}) + \sum_{k=1}^{r-j-1} B_{r-k} \\
&\geq \min_{1\leq i\leq r}\{0, \nu_O(M_{i(j+1)})\} + \sum_{k=1}^{r-j-1} B_{r-k}.
\end{aligned}$$

Consequently, we conclude that $M_{jj}s_j$ must also be bounded by this valuation, since if it were lower, it could not be cancelled out, and we would conclude that the $i$-th entry did not lie in $\mathcal{O}_X(U_2)$. Therefore

$$\nu_O(M_{jj}s_j) \geq \min_{1\leq i\leq r}\{0, \nu_O(M_{i(j+1)})\} - \sum_{k=1}^{r-j-1} B_{r-k}.$$

Using valuation rules and rearranging, this gives

$$\begin{aligned}
\nu_O(s_i) &\geq -\nu_O(M_{ii}) + \min_{1\leq i\leq r}\{0, \nu_O(M_{i(j+1)})\} + \sum_{k=1}^{r-j-1} B_{r-k} \\
&\geq -\max\{0, \nu_O(M_{ii})\} + \min_{1\leq i\leq r}\{0, \nu_O(M_{i(j+1)})\} + \sum_{k=1}^{r-j-1} B_{r-k} \\
&= B_j + \sum_{k=1}^{r-j-1} B_{r-k} \\
&= \sum_{k=1}^{r-j} B_{r-k}
\end{aligned}$$

$\square$

With this lemma in hand, we only have to consider finitely many coefficients in order to set up our linear system of equations.

# Appendix A

# Software for Computing Global Sections

For our computations, we opt to manually work with Laurent polynomial rings. In the future, it would be prudent to make use of SageMath's QuotientRing construction. Furthermore, we note that in the interest of possibly implementing this method as a Class (perhaps EllipticVectorBundle or some other suitable name), some methods are prefixed with an underscore, so as to be private.

```
# Base polynomial ring. Here, xz = 1, yw = 1. a is the constant in the
equation:
# y^2 = x(x-1)(x-a).
R.<x,y,z,w,a> = QQ[];
S.<a> = QQ[]
```

The ring $R$ forms our Laurent polynomial ring, where $z = x^{-1}$ and $w = y^{-1}$. The variable $a$ denotes the coefficient $\lambda$ specified in the Legendre form of the elliptic curve. The ring $S$ is meant to act as our ground field $\mathbb{K}$.

```
def _Reduction(f):
    """

    Computes the Groebner basis reduction of the input under the
    relation y^2 = x(x-1)(x-a), with monomial ordering x > y.

    INPUT:
```

```
    f - a polynomial in R.

    OUTPUT: The Groebner basis reduction of f.
    """
    I = ideal(y**2 - x*(x-1)*(x-a),x*z-1,y*w-1);
    B = I.groebner_basis();
    newf = R(f).reduce(B);
    return newf;

def _wdegree(f,low=False):
    """
    Computes the weighted grading of the ring R (deg x = 2, deg y =
    3) on an element.

    INPUT:

    f - a polynomial in R.

    low - A boolean determining whether to find the highest degree
    term, or lowest degree term.

    OUTPUT: The weighted degree of f.
    """
    # Initial degree.
    d = (2*(f.monomials()[0].degrees()[0] -
    f.monomials()[0].degrees()[2]) + 3*(f.monomials()[0].degrees()[1]
        - f.monomials()[0].degrees()[3]));
    for m in f.monomials():
        if low:
            d = min(d, 2*(m.degrees()[0] - m.degrees()[2]) +
            3*(m.degrees()[1] - m.degrees()[3]));
        else:
            d = max(d, 2*(m.degrees()[0] - m.degrees()[2]) +
            3*(m.degrees()[1] - m.degrees()[3]));
    return d;
```

This method computes the valuation of a monomial in $R$. See (2.6) and Lemma 2.10. Note that in this code, we are taking the negative of the valuation, and we refer to it as the degree. This is because it forms a weighted degree on $\mathbb{K}[x, y]$, where the degree of $x$ is 2, and the degree of $y$ is 3, hence the abbreviation "wdegree."

```
def _basismon(n):
    """
    Computes the monomial omega_n.

    INPUT:

    n - an integer.

    OUTPUT: The monomial omega_n.
    """
    mon = 0;
    if (n % 3 == 0):
        if (n < 0):
            mon = R(w^(-n/3));
        else:
            mon = R(y^(n/3));
    if (n % 3 == 1):
        if (n-4 < 0):
            mon = R(x^2*w^(-(n-4)/3));
        else:
            mon = R(x^2*y^((n-4)/3));
    if (n % 3 == 2):
        if (n-2 < 0):
            mon = R(x^1*w^(-(n-2)/3));
        else:
            mon = R(x^1*y^((n-2)/3));
    return mon;

def _MonomialCleanup(f):
    """
    Clears redundant variables.

    INPUT:

    f - a polynomial in R.
```

```
    OUTPUT: Applies the relation xz = 1, yw = 1 to clear redundancies.
    """
    newf = 0;
    for m in f:
        xdeg = min(m[1].degrees()[0],m[1].degrees()[2]);
        ydeg = min(m[1].degrees()[1],m[1].degrees()[3]);
        newf += m[0]*(m[1]/(x**xdeg*y**ydeg*z**xdeg*w**ydeg));
    return newf;
```

As mentioned above, we opt to manually work over the Laurent polynomial ring, so this method is meant to apply relations such as $xx^{-1} = 1$.

```
def _intpart(r,d):
    """

    Computes a special integer partition related to the construction of
    M(r,d).

    INPUT:

    r - a positive integer.

    d - an integer.

    OUTPUT: The GCD partition of d with respect to r.
    """
    r1 = r;
    d1 = d;
    intpart = [];
    while(d1 != 0):
        if (d1 < r1):
            r1 -= d1;
        else:
            d1 -= r1;
            intpart += [r1];
    intpart = Partition(intpart).conjugate();
    intpart = list(intpart);
    intpart.reverse();
    return intpart;
```

```
def _omegavector(k):
    """
    Forms the vector omega(k).

    INPUT:

    k - a positive integer.

    OUTPUT:

    The vector omega(k).
    """
    if k < 1:
        raise ValueError("k must be positive");
    vectorlist = [_basismon(1)];
    for i in range(k-1):
        vectorlist += [_basismon(-i-1)];
    return Matrix(k,1,vectorlist);

def _matrixA(r,d):
    """
    Constructs the upper right block in M(r,d).

    INPUT:

    r - a positive integer.

    d - an integer.

    OUTPUT: The block A(r,d).
    """
    if (2*d <= r):
        return block_matrix(1,2,[x**2*w*matrix.identity(R,d),matrix.zer
        o(R,d,r-2*d)],subdivide=False);
    else:
        intpart = _intpart(r,d);
        blocklist = [];
        for i in intpart:
```

```
            blocklist += [_omegavector(i)];
        return block_diagonal_matrix(blocklist,subdivide=False);

def _matrixM(r,d):
    """
    Constructs the matrix M(r,d).

    INPUT:

    r - a positive integer.

    d - an integer.

    OUTPUT: The matrix M(r,d).
    """
    if (r <= 0):
        raise ValueError("Rank must be positive.");
    if (d % r == 0):
        M = [[0 for i in range(r)] for j in range(r)];
        for i in range(r-1):
            M[i][i+1] = x**2*w;
        M = matrix(M) + matrix.identity(R,r);
    else:
        newd = d % r;
        M = block_matrix([[matrix.identity(R,newd),_matrixA(r,newd)],[m
        atrix.zero(R,r-newd,newd),_matrixM(r-newd,newd)]],subdivide=Fal
        se);
    if (floor(d/r) < 0):
        M = (z*y)**(-floor(d/r))*M;
    else:
        M = (x*w)**(floor(d/r))*M;
    return M;


def In_Rn(f,n):
    """
    Determines whether an element is contained in a ring.

    INPUT:
```

```
f - a polynomial in R.

n - an integer, indicating the number of the ring we want.

OUTPUT: Returns [true, []] if f is in R_n.
        Else, returns [false, mon], where mon is the list of
        monomials not in R_n.
"""
flag = True;
newf = _Reduction(f);

# For each monomial, check if the degree satisfies that it is in R_n.
failedmons = [];
for m in newf:
    # We consider a as a coefficient, not a variable, so we need to
    do a proper unpacking.
    coeff = S(m[0]*S(a)**(R(m[1]).degrees()[4]));
    mon = R(x)**(m[1].degrees()[0])*R(y)**(m[1].degrees()[1])*R(z)*
    *(m[1].degrees()[2])*R(w)**(m[1].degrees()[3]);
    mdeg = _wdegree(mon);
    # Clean up the monomials, since they might have some x*z's and
    y*w's lying around.
    mon = _MonomialCleanup(mon);
    if (n == 1):
        # Check if it's not in R1. These are the degrees that work.
        if (mdeg == 1 or mdeg < 0):
            # If it's in not in the ring, we add it to the failed
            monomials
            failedmons += [[mon,coeff]];
    elif (n == 2):
        # Check if it's not in R2. These are the degrees that work.
        if (mdeg > 0):
            # If it's in not in the ring, we add it to the failed
            monomials
            failedmons += [[mon,coeff]];
    else:
        # Something went wrong.
        raise ValueError("Invalid value for n");
```

```
    # If the failed monomial list is empty, then return True. Else,
    False, along with the list of failed monomials.
    if not(failedmons == []):
        flag = False;
    return [flag, failedmons];
```

This method checks the valuation of a polynomial to determine whether it lies in $\mathcal{O}_X(U_0)$ or $\mathcal{O}_X(U_2)$. In this case, $n = 1$ checks if the polynomial lies in $\mathcal{O}_X(U_0)$, and $n = 2$ checks if it lies in $\mathcal{O}_X(U_2)$. The output is a dictionary of monomials which failed, so that we can use them to form our system of equations.

```
def GlobalSections(M,dual=False):
    """
    Method for computing global sections of a vector bundle over an
    elliptic curve defined by the zero locus of
        f = y^2 - x(x-1)(x-a)

    INPUT:

    M - an upper triangular n x n matrix with entries in R.

    OUTPUT: A list of vectors in k[x,y]/f forming a basis for the space
    of global sections of the bundle associated to M.

    NOTES: Perhaps make a class: EllipticVectorBundle?
           Include Quotient rings to include hyperelliptic curves,
           potentially.
    """
    ringnum = 2;
    if (dual == True):
        ringnum = 1;

    # Initializing variables.
    [n, n1] = M.dimensions();
    if not(n == n1):
        raise AttributeError("Matrix is not square.");
    monorder = [];
    equations = [];
```

```
# First, we compute a naive bound for which monomials to use.
m = [0 for i in range(n)];
m[n-1] = -min(_wdegree(M[n-1][n-1]), 0) + max(max([_wdegree
(M[n-i-1][n-1]) for i in range(n-1)]),0);
for j in range(n-1):
    m[n-j-2] += m[n-j-1];
    m[n-j-2] += -min(_wdegree(M[n-j-2][n-j-2]),0) + max(max
    ([_wdegree(M[n-i-1][n-j-2]) for i in range(n-j)]),0);

# This will be the number of columns in our system.
totcnum = sum(m[i]+1 for i in range(n));

# This will be our linear system in the end.
equations = [];
# An intermediate storage variable.
tempterms = [];
# Primary loop. Starting from the top row:
for i in range(n):
    # This tracks the monomial each row in our system corresponds to
    monorder = [];
    # These are all of the new rows we add to the system.
    newrows = [];
    # For each entry in the row, compute the eligible basis monomials.
    for j in range(n):
        # For every monomial under the magic bound,
        # check the monomials outside of R_n and compile
        # their coefficients.
        # Depending on which ring we're looking at, R_1
        # or R_2, we need a different range for the
        # degrees.
        basisrange = range(m[j]+1)
        if (dual==False):
            if (m[j] > 0):
                basisrange += [m[j]+1];
                basisrange.remove(1);
        else:
            for k in range(m[j]+1):
                basisrange[k] = -basisrange[k];
```

```
        # For each degree
        for k in range(m[j]+1):
            tempterms =
            In_Rn(R(M[i][j]*_basismon(basisrange[k])),ringnum);
            # Check if there were any failed monomials.
            if not(tempterms[0]):
                # If there were, iterate over those
                for t in tempterms[1]:
                    # As long as we haven't seen this
                    # monomial, add it to the system,
                    # and keep track of it.
                    if (monorder.count(t[0]) == 0):
                        monorder += [t[0]];
                        newrows += [[0 for a in range(totcnum)]];

            # Fill in the rows. For each row (monomial)
            for a in range(len(monorder)):
                # Index tracking variable
                p = sum(m[q]+1 for q in range(j));
                # Go through the terms we'd computed.
                # Check if there were any failed monomials.
                if not(tempterms[0]):
                    # Over each failed monomial
                    for t in tempterms[1]:
                        # Check if it's equal to the one
                        # we're looking for
                        if (t[0] == monorder[a]):
                            # If so, extract its
                            # coefficient, put it in the row.
                            newrows[a][p+k] += S(t[1]);
    # Add these new rows to the matrix.
    equations += newrows

# Form the matrix. We reverse the order of equations here
# because it makes computing the kernel easier. This just
# makes the computation a bit faster due to the structure
# of the systems constructed. The point here is that
# reversing the order of equations yields a system which is
# ''almost lower-triangular.''
```

```
    equations.reverse();
    equationsmatrix = Matrix(equations);
    # Find the right kernel.
    solutions = equationsmatrix.right_kernel();
    vectorlist = solutions.basis();

    # This just concerns translating the results of the
    # kernel. This is just a bunch of index juggling
    # and keeping track of which coefficient corresponds to
    # which basis monomial.
    gsections = [];
    for v in vectorlist:
        section = [0 for i in range(n)];
        p = 0;
        for j in range(n):
            basisrange = range(m[j]+1)
            if (dual==False):
                if (m[j] > 0):
                    basisrange += [m[j]+1];
                    basisrange.remove(1);
            else:
                for k in range(m[j]+1):
                    basisrange[k] = -basisrange[k];
            for k in range(m[j]+1):
                section[j] += v[p+k]*_basismon(basisrange[k]);
            p += (m[j]+1);
        gsections += [section];

    if gsections == []:
        gsections = [0];
    return gsections;
```

This is the primary method for computation. Roughly speaking, the goal is to create a matrix whose columns are in correspondence with the coefficients $c_{ij}$ from

$$s_j = c_{0j} + \sum_{i=2}^{-\nu_O(s_j)} c_{ij}\omega_{-i},$$

and the rows are in correspondence with "illegal" monomials.

**Example A.1.** We will carry out the procedure on the matrix

$$M = \begin{pmatrix} \omega_1 & \omega_{-1} \\ 0 & \omega_2 \end{pmatrix}.$$

In the interest of clarity, we will express this matrix explicitly as

$$M = \begin{pmatrix} xy^{-1} & x^2y^{-1} \\ 0 & x^2y^{-2} \end{pmatrix},$$

however we note that these entries are representatives in the ring $\mathbb{K}[x,y]/\langle y^2 - x(x-1)(x-\alpha)\rangle$ (i.e. we are excluding the overline for convenience). We begin by using the bound from Lemma 4.13 to see that $s_2$ has valuation at least $-2$ and $s_1$ has valuation at least $-3$, so that

$$\begin{aligned} s_1 &= c_{01} + c_{21}x + c_{31}y, \\ s_2 &= c_{02} + c_{22}x. \end{aligned}$$

Now we multiply this vector by the matrix, and examine which monomials lie outside of $\mathcal{O}_X(U_2)$. Namely, which monomials have strictly negative valuation. In this case, we have

$$M \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} c_{01} + c_{21}x + c_{31}y + x^2y^{-1}(c_{02} + c_{22}x) \\ x^2y^{-2}(c_{02} + c_{22}x) \end{pmatrix},$$

and we further apply Gröbner basis reduction via the method "_Reduction" to find

$$M \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} c_{01} + c_{21}x + (c_{31} + c_{22})y + (c_{02} + (1+\alpha)c_{22})x^2y^{-1} - \alpha c_{22}xy^{-1} \\ (c_{02} + (1+\alpha)c_{22})x^2y^{-2} + c_{22} - \alpha c_{22}xy^{-2} \end{pmatrix}.$$

In order to represent a global section, we must have that the valuations of each entry are non-negative, so we must set the coefficients of all negative valuation monomials to zero. In this case, these come from the monomials $x, y$ and $x^2y^{-1}$. Therefore we get three equations in four unknowns:

$$\begin{aligned} c_{21} &= 0, \\ c_{31} + c_{22} &= 0, \\ c_{02} + (1+\alpha)c_{22} &= 0. \end{aligned}$$

The remaining coefficients are free variables. Solving this system then provides a basis for the space of global sections. In general, the method performs these products and reductions, tracks which monomials are disallowed, and forms the corresponding linear system. We then use a standard kernel solver to obtain a solution.

We note that the method can also compute sections of the dual, without computing the inverse of the matrix.

```
def is_globally_generated(M):
    """
    Checks whether the bundle associated to an upper
    triangular matrix M is globally generated.

    INPUT:

    M - an upper triangular matrix with coefficients
    in R.

    OUTPUT: Whether or not B(M) is globally generated.
    """
    [n,n1] = M.dimensions();
    gsects = GlobalSections(M);
    r = len(gsects);
    # If we don't have enough sections, obviously it'll fail.
    if r < n:
        return False;
    sectM = Matrix(gsects);
    I = ideal(sectM.minors(n));
    B = I.groebner_basis();
    # If there is some non-trivial solution, then we can
    # make a minor vanish, and hence drop the rank.
    if B != [1]:
        return False;
    # Otherwise, we move on to the point at infinity.
    else:
        infgsects = [M*vector(s) for s in gsects];
        # We evaluate each entry at (0,0).
        for t in infgsects:
            for i in range(n):
                t[i] = _Reduction(_MonomialCleanup(t[i]))(0,0,0,0,a);
        infsectM = Matrix(infgsects);
        infI = ideal(infsectM.minors(n));
        infB = infI.groebner_basis();
        if infB != [1]:
```

```
            return False;
    return True;
```

This method checks whether $\mathcal{B}(M)$ is globally generated. This is done by placing the global sections into a matrix, and computing its rank $n$ minors, where $n$ is the size of $M$. If the ideal of minors generates the ring $R$, then we can conclude that the matrix is full rank no matter which point one evaluates the matrix at. Finally, we check the fibre at $(0:0:1)$ by directly evaluating at this point and computing the rank.

**Example A.2.** We include some examples of using the code:

```
load ("GlobalSections.sage")
M = _matrixM(3,2);
print M;

> [      1       0   x^2*w]
> [      0       1     x*w]
> [      0       0 x^2*w^2]

gsects = GlobalSections(M);
print "The global sections of B(M) are:",gsects;

> The global sections of B(M) are: [[1, 0, 0], [0, 1, 0]]

print is_globally_generated(M);

> False

M = _matrixM(4,5);
print M;

> [    x*w x^3*w^2       0       0]
> [      0     x*w x^3*w^2       0]
> [      0       0     x*w x^3*w^2]
> [      0       0       0 x^2*w^2]

gsects = GlobalSections(M);
print "The global sections of B(M) are:",gsects;
```

```
> The global sections of B(M) are: [[1, 0, 0, 0], [-x*y - y*a - y, x^2, -y,
x], [0, -1, 0, 0], [0, 0, -1, 0], [0, 0, 0, -1]]
```

```
print is_globally_generated(M);
```

```
> True
```

We can compute global sections for matrices other than $M(r, d)$ as well, such as for the matrix

$$M = \begin{pmatrix} 1 & \omega_{-1} & 0 & \omega_2 \\ 0 & \omega_1^2 & \omega_{-1} & 0 \\ 0 & 0 & 1 & \omega_1 \\ 0 & 0 & 0 & \omega_3 \end{pmatrix}.$$

```
M = Matrix([[1, x^2*w, 0, x^2*w^2],[0, x^2*w^2, x^2*w, 0],
[0, 0, 1, x*w],[0, 0, 0, w]]);
print M;
```

```
> [      1   x^2*w       0 x^2*w^2]
> [      0 x^2*w^2   x^2*w       0]
> [      0       0       1     x*w]
> [      0       0       0       w]
```

```
gsects = GlobalSections(M);
print "The global sections of B(M) are:",gsects;
```

```
> The global sections of B(M) are: [][1, 0, 0, 0], [-x^2*a
- x^2 - y^2 + x*a, x*y - 1, -x, y], [y, -x + a + 1, 0, 0],
[x^2, -y, 1, 0], [0, 0, 0, 1]]
```

```
print is_globally_generated(M);
```

```
> True
```

△

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