Nowhere-zero flows and structure theory for signed graphs

by

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B.Sc., Shahid Beheshti University 2021

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Spring 2024

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Abstract

This thesis explores two variations of signed graphs. For the first variation we study nowhere-zero flows. Here we develop algorithms for computing the circular flow number in cubic graphs and we establish some theorems giving bounds on the circular flow number. For the second variation we prove a theorem showing the existence of a decomposition of a signed graph into positive cycles and a related theorem implying the existence of a removable cycle. We also establish a new structure theorem characterizing when a 3-connected signed graph has a path between two distinguished vertices that is disjoint from a negative cycle.

Keywords: Signed graphs; Circular flow; Circular coloring; Nowhere-zero flow; Positive cycle decomposition; Removable positive cycle; Two disjoint negative cycles

Dedication

This thesis is dedicated to the memory of Pouneh Gorji and Arash Pourzarabi, the passengers of flight 752 who departed from us far too soon. They will forever remain in our hearts and memories, never to be forgotten.

Acknowledgements

I want to thank my family for always being there for me. Mom, Parvin Hashemian, you have been my inspiration. Dad, Masoud Yousefi, you have supported me whenever I needed you. My sister, Parnian Yousefi, your belief in me has meant everything, and I owe much of my success to you. Aunt, Behnaz Hashemian, whose love and support have been akin to that of a second mother.

To my supervisors, Bojan Mohar and Matt DeVos, your guidance and wisdom have been incredibly inspiring. Thank you for the countless meetings we have had. I appreciate our talks when I was stressed. Thank to Matt for your contagious enthusiasm and passion for mathematics. I have come to love Graph Theory because of you.

I would like to extend my gratitude to Zhouningxin for her valuable assistance in deepening my understanding of signed graphs during her visit. Additionally, I want to thank Katheryn for the enjoyable research collaboration we shared together.

Thanks to my friends at office - MacKenzie, Danielle, Alexander, Lyn, Reza, Jesse, and Peter - for the laughs and support.

And to my friends Fatemeh, Yalda, Amin, Eugene, and Atabak, your laughter and encouragement have brightened even the darkest days.

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Introduction

1.1 Basic Definitions

Throughout this thesis, we will use the standard notation and definitions from 'Introduction to Graph Theory' by West [21]. We review some of these essential definitions first. A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct), called its *endpoints*. We draw a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. When u and v are the endpoints of an edge, they are *adjacent* and are *neighbors*. To denote an edge with endpoints u and v, we generally use uv. A *directed graph*, or *digraph*, D, consists of a set of vertices $V(D)$, a set of edges $E(D)$, and a function which assigns each edge e an ordered pair of vertices (u, v). The *degree* of a given vertex v, denoted as $d(v)$, is the number of edges incident to v. A *cubic* graph is a graph in which all vertices have a degree of three. A *walk* W of a graph G is an alternating sequence $v_1, e_1, v_2, \ldots, e_{k-1}, v_k$ of vertices and edges (allowing repetition) such that for $1 \leq i \leq k-1$, the edge e_i has endpoints v_i and v_{i+1}. A walk of a graph is said to be *closed* if v_1 and v_k are identical. A *path* P of a graph is a walk where all the vertices v_i are distinct. A *cycle* of a graph is a closed walk with all the vertices v_i (except $v_1 = v_k$) distinct. The *length* of a walk is the number of its edges. An *odd* (or *even*) cycle is a cycle of odd (or even) length. If X, Y are disjoint subsets of $V(G)$ with $X \cup Y = V(G)$, then the set of edges with one end in X and the other in Y is called an *edge cut* (or *cut* for short) and is denoted $E(X,Y)$. A graph is *k-edge-connected* if every proper nonempty subset of vertices $\emptyset \neq X \subset V(G)$ has $|E(X, V(G) \setminus X)| \geq k$. If G is a directed graph, then we let $\overrightarrow{E}(X,Y)$ ($\overleftarrow{E}(X,Y)$) denote the set of edges directed from X to Y (Y to X). A graph G' is a *subgraph* of another graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. A *k-coloring* of a graph G is a labeling $f\colon V(G) \to S$, where $|S| = k$ (often we use $S = [k]$). The labels are *colors*; the vertices of one color form a *color class*. A k-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number*, denoted as $\chi(G)$, is the least k such that G is k-colorable.

Let G be a graph and let $G_1, G_2 \subseteq G$. We call (G_1, G_2) a *k-separation* (or *separation of order k*) if the following conditions are satisfied:

- $G_1 \cup G_2 = G$
- $E(G_1) \cap E(G_2) = \emptyset$
- $|V(G_1) \cap V(G_2)| = k$

The separation is *proper* if $V(G_1) \setminus V(G_2) \neq \emptyset \neq V(G_2) \setminus V(G_1)$.

If G is a graph, an *embedding* of G in the plane is a function ϕ which assigns each vertex of G a distinct point in the plane and assigns to each edge e with ends u, v a simple rectifiable curve with ends $\phi(u)$ and $\phi(v)$ so that this curve minus its ends is disjoint from the image of $V(G) \cup (E(G) \setminus \{e\})$. A *plane graph* is a graph G together with an embedding of G in the plane. A graph is *planar* if there exists an embedding of it in the plane. Let G be a plane graph, and construct a new plane graph G^* as follows. For each face $a \in F(G)$, add a vertex a^* in a. For each edge $e \in E(G)$ which lies in the boundary walk of the faces a, b, add an edge from a^* to b^* which crosses e but is otherwise disjoint from the image of G (if e appears twice in the boundary walk of a, then e^* is a loop at a^*). This can be done so that G^* is a plane graph, and we call any plane graph constructed in this manner a *dual* of G. Note: for convenience we sometimes treat a pair of dual planar graphs G and G^* as having the same edge set with an edge e in G corresponding to e in G^*.

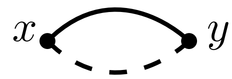

Figure 1.1: A digon in a signed graph

1.2 Signed Graphs and Switching

A *signed graph* (G, σ) consists of a graph G together with an assignment $\sigma\colon E(G) \to \{-1, 1\}$, referred to as a *signature*. Given a signed graph (G, σ), we denote by $E^+(G, \sigma)$ and $E^-(G, \sigma)$ the sets of *positive and negative edges* of (G, σ), respectively.

A signed graph with all edges being negative is denoted by $(G, -)$, while a signed graph with all edges positive is denoted by $(G, +)$. A signed multigraph on two vertices with two parallel edges of different signs is called a *digon*. We say a signed graph (H, π) is an *(induced) subgraph* of (G, σ) if H is an (induced) subgraph of G, and π is a signature on H such that for every $e \in E(H)$, $\pi(e) = \sigma(e)$. For simplicity, we may write (H, σ) as a subgraph of (G, σ) if H is a subgraph of G.

In this thesis, for drawing a signed graph, we use solid lines to represent positive edges, and dashed or dotted lines to represent negative edges.

A negative *theta* is constructed from a negative cycle C and a signed path P such that only the endpoints of the path P are on the cycle C. Let Q_1 and Q_2 be the unique paths of cycle C with same endpoints of the path P. By observation exactly one of cycles $Q_1 \cup P$ and $Q_2 \cup P$ is negative. We call this property *theta property*.

In this thesis we consider two different notions of equivalence for signed graphs. The first of these is called switching equivalence, and we introduce this next. Given a signed graph (G, σ) and a vertex v of (G, σ), a *switching at the vertex* v involves changing the sign of every edge incident to v. Note that if there is a loop at vertex v, the change will occur twice, thus nullifying the effect. Given a set $A \subseteq V(G)$, a *switching at the set* A involves switching the signs of all edges in the edge-cut $E(A, V \setminus A)$. This operation is equivalent to successively switching at each vertex in A.

Definition 1.2.1. *Let G be a graph and let σ, σ' be signatures of G. We say that (G, σ) is* switching equivalent *to (G, σ') if it may be obtained from (G, σ) through a series of switching* *at vertices to the opposite sign. We may also simply say that σ' is switching-equivalent to σ.*

If (G, σ) is a signed graph and $H \subseteq G$, the *sign* of H is defined to be $\sigma(H) = \prod_{e \in E(H)} \sigma(e)$. It is straightforward to verify that switching at a vertex has no effect on the sign of any cycle. So for every cycle C, any two switching equivalent signatures σ, σ' will satisfy $\sigma(C) = \sigma'(C)$. In fact, two signatures are switching equivalent if and only if they are either both positive or negative on all cycles.

Proposition 1.2.1. *[22] Two signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.*

Proof. We already observed the "only if" direction. For the "if" direction we assume that σ and σ' have the same set of negative cycles and we will show how to transform σ into σ' by a sequence of switches. We may assume without loss that G is connected and we choose a spanning tree $T \subseteq G$. For every edge $e \in E(T)$ the graph $T \setminus \{e\}$ has two components, say with vertex sets X and Y. We call $E(X, Y)$ the *fundamental cut* of T associated with e. Note that each fundamental cut contains exactly one edge of the tree T. Let $E(T) = \{e_1, \ldots, e_m\}$ and we modify σ in steps. On step i we do nothing if $\sigma(e_i) = \sigma'(e_i)$ but if $\sigma(e_i) \neq \sigma'(e_i)$ we modify σ by switching on the fundamental cut associated with e_i. At the end of these steps, the signatures σ and σ' agree on $E(T)$. For every edge $f \in E(G) \setminus E(T)$ there is a unique cycle C of G with $E(C) \subseteq E(T) \cup \{f\}$ called the *fundamental cycle* associated with f. By assumption $\sigma(C) = \sigma'(C)$ but since σ and σ' agree on $e_1, \ldots, e_m$ it follows that they must satisfy $\sigma(f) = \sigma'(f)$. Therefore, after our switching we have arranged that $\sigma = \sigma'$ as desired. □

So in the setting of signed graphs under switching equivalence, the signs of cycles carry all of the important information, not the signs of the edges. We call a signed graph *balanced* if every cycle is positive and note that in this case our signed graph (G, σ) is switching equivalent to $(G, +)$.

There is a vast body of structure theorems for minors and other containment relations for ordinary graphs, but there are comparatively few structure theorems for signed graphs.

One such theorem of interest is the following result first discovered by Lovász et. al. and proved by Slilaty. Before stating his theorem, we require one added bit of terminology. A graph G is *internally* k-connected if for every $X \subseteq V(G)$ with $|X| < k$ the graph $G - X$ is either connected, or has exactly two components, one of which is an isolated vertex. A graph G is a k-sum of the graphs G_1 and G_2 if for $i = 1, 2$ there exists a k-element subset of $V(G_i)$ called X_i so that G may be obtained from G_1 and G_2 by identifying the cliques X_1 and X_2.

Theorem 1.2.1. *[18] [Slilaty] Let (G, σ) be a 3-connected signed graph. If (G, σ) does not contain two vertex disjoint negative cycles, one of the following holds:*

1. *$G - v$ is balanced for some vertex $v \in V(G)$*
2. *$|V(G)| \leq 5$*
3. *G can be embedded in the projective plane so that every face is bounded by a positive cycle.*
4. *G can be expressed as a 3-sum of a graph with no two disjoint negative cycles and a balanced graph with at least 5 vertices.*

Our main theorem in this area is the following result characterizing when a signed graph has a path between a distinguished pair of vertices disjoint from a negative cycle. (In fact we prove a stronger form of this theorem that applies under the weaker assumption of 3-connectivity.)

Theorem 1.2.2. *Let (G, σ) be a signed 3-connected and internally 4-connected graph and let $u, v \in V(G)$. If G does not contain a u, v path P and a negative cycle C so that $V(P) \cap V(C) = \emptyset$ then one of the following is true*

- *$G \setminus \{u, v\}$ is balanced.*
- *G is planar and every negative face contains u or v.*

Another question of interest for signed graphs (under switching) is the existence of a decomposition into positive cycles. We prove an extension of the following well known theorem of Seymour on this problem.

Theorem 1.2.3 (Seymour)**.** *[17] Let* (G, σ) *be a signed planar 2-connected Eulerian graph. If* $|E^-(G, \sigma)|$ *is even, there exist positive cycles* $C_1, \ldots, C_k$ *so that* $\{E(C_1), \ldots, E(C_k)\}$ *is a partition of* $E(G)$*.*

Theorem 1.2.4. *Let* (G, σ) *be a signed 2-connected Eulerian planar graph with minimum degree* ≥ 4 *and let* $e \in E(G)$ *be distinguished. Then there exists either a positive removable cycle* C *with* $e \notin E(C)$ *or a removable balanced sausage* H *with* $e \notin E(H)$*.*

1.3 Nowhere-Zero Flows

The study of nowhere-zero flows has a prominent place within graph theory. Tutte introduced this subject and showed that for planar graphs nowhere-zero flows are dual to colourings. He then made three far-reaching conjectures concerning the exisence of these flows that remain open despite considerable work. In this section we introduce this subject which is central to our research.

Given a graph G, an *orientation* D of G is a directed graph obtained from G by assigning a direction to each edge of G. For each edge $e = uv$, if e is *oriented* from u to v in D, then we have $(u, v) \in E(D)$. Given a graph G and an Abelian group A, an *A-flow* of G is a pair (D, f) where D is an orientation of G and $f \colon E(G) \to A$ satisfies the following property called the *Kirchoff Rule* at every vertex v

$$\sum_{(v,w)\in E(D)} f(vw) - \sum_{(u,v)\in E(D)} f(uv) = 0. \tag{1.1}$$

An A-flow f is called *nowhere-zero* if $f \colon E(G) \to A \setminus \{0\}$. A $\mathbb{Z}$-flow f is called a *k-flow* if $|f(e)| \leq k - 1$ holds for every $e \in E(G)$. Tutte proved the following theorem showing that nowhere-zero k-flows are dual to k-colourings for planar graphs.

Theorem 1.3.1 (Tutte)**.** *If* G *and* G^* *are dual planar graphs, then* G *has a* k*-colouring if and only if* G^* *has a nowhere-zero* k*-flow.*

We will provide a proof of this theorem in Chapter 2. Surprisingly, Tutte proved the following theorem showing that the existence of a nowhere-zero A-flow depends only on the order of the group A.

Theorem 1.3.2 (Tutte)**.** *Let G be a graph and let A be an abelian group of order k. Then G has a nowhere-zero A-flow if and only G has a nowhere-zero k-flow.*

Let us pause to make a quick observation concerning A-flows. Suppose that (D, f) is an A-flow of the graph G and let $\{X, Y\}$ be a partition of $V(G)$. Then

$$\sum_{e \in \overrightarrow{E}(X,Y)} f(e) - \sum_{e \in \overleftarrow{E}(X,Y)} f(e) = \sum_{x \in X} \Big(\sum_{(x,w) \in E(D)} f(xw) - \sum_{(u,x) \in E(D)} f(ux) \Big) = 0$$

So in words, the net flow across every edge-cut must be zero. It follows immediately from this that every graph with a cut-edge cannot have a nowhere-zero A-flow for any abelian group A. Note: for planar graphs this is dual to the statement that a planar graph with a loop has no proper colouring. Tutte made the following conjecture that is central to this subject.

Conjecture 1.3.1 (Tutte)**.** *Every graph without a cut-edge has a nowhere-zero 5-flow.*

The Petersen graph does not have a nowhere-zero 4-flow, so the above conjecture is best possible if true. Seymour proved that every graph without a cut-edge has a nowhere-zero 6-flow, but Tutte's 5-flow conjecture remains open.

1.4 Circular Colouring and Flow

There is a natural refinement of graph colouring and nowhere-zero flows called circular colouring and circular flow that we introduce next. Let $r \geq 1$ be a real number. The standard circle of circumference r is defined as $C^r = \mathbb{R}/r\mathbb{Z}$. We can get C^r from the interval $[0, r]$ by identifying the two endpoints, 0 and r and will usually denote points in C^r using the corresponding real numbers in the interval $[0, r)$. However, the above quotient structure also equips C^r with the structure of an additive (abelian) group. For two points, x and y, on C^r, the *distance* between x and y on C^r, denoted by $d_{C^r}(x, y)$ or $d_{\pmod r}(x, y)$, is the length of the shorter arc of C^r connecting x and y. Given a graph G, a *circular r-coloring* of G is a mapping $f : V(G) \to C^r$ such that every edge $uv \in E(G)$ satisfies $d_{C^r}(f(u), f(v)) \geq 1$.

The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{r \mid G \text{ admits a circular } r\text{-coloring}\}.$$

The concept of circular coloring of graphs was introduced by Vince in 1988 in [20], where a different but equivalent definition was given, and the parameter was called the "star chromatic number." Later, the above definition was given in [24], and the term "circular chromatic number" was coined in [25]. The invariant $\chi_c(G)$ is a refinement of $\chi(G)$ and contains more information since $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.

If D is an orientation of G and C is a cycle of G, we let $\overrightarrow{C}$ denote the "forward" edges of C and $\overleftarrow{C}$ denote the "backward" edges. The *imbalance* of C is defined to be the maximum of $\left\{\frac{|E(C)|}{|\overleftarrow{C}|}, \frac{|E(C)|}{|\overrightarrow{C}|}\right\}$ (where we treat this maximum as ∞ if $C^+ = \emptyset$ or $C^- = \emptyset$). We define the *cycle imbalance* of the orientation D to be the maximum over all cycles C of the imbalance of C. Note that every acyclic orientations of graphs have infinite imbalance. Minty proved that a graph G is k-colourable if and only if it has an orientation of imbalance at most k. More generally we have the following.

Theorem 1.4.1. *(See [9]) For every graph G, the circular chromatic number $\chi_c(G)$ is equal to the minimum cycle imbalance of an orientation of G.*

Dualizing circular colouring gives rise to the notion of a circular modulo r-flow. For a real number $r \geq 0$ we define a *circular modulo r flow* to be a pair (D, f) where D is an orientation of G and $f : E(G) \to C^r$ is a flow satisfying the property $d_{C^r}(0, f(e)) \geq 1$ for every $e \in E(G)$. The *circular flow number* of G is defined as

$$\phi_c(G) = \inf\{r \mid G \text{ admits a circular modulo } r \text{ flow}\}.$$

Let us note that if G has a nowhere-zero k-flow, then it has a nowhere-zero $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$-flow which is also a circular modulo k flow so in this case we have $\phi_c(G) \leq k$. With this definition in place, Tutte's flow-colouring duality naturally extends to give us the following relationship for a pair of dual planar graphs G and G^*

$$\chi_c(G) = \phi_c(G^*)$$

There is also a similar result relating orientations to circular flow number. If D is a directed graph and $\{X, Y\}$ is a partition of $V(D)$ then we define the *imbalance* of the edge cut $E(X, Y)$ to be the maximum of $\left\{ \frac{|E(X,Y)|}{|\overrightarrow{E}(X,Y)|}, \frac{|E(X,Y)|}{|\overleftarrow{E}(X,Y)|} \right\}$ and we treat this as ∞ if one of $\overrightarrow{E}(X, Y)$ or $\overleftarrow{E}(X, Y)$ is empty. The *cut imbalance* of D is defined to be the maximum imbalance of an edge cut. Note that only strongly connected digraphs have finite cut imbalance.

Theorem 1.4.2. *(See [9]) For every graph G, the circular flow number $\phi_c(G)$ is equal to the minimum cut imbalance of an orientation of G.*

For the sake of completeness, we give a proof of this result in Chapter 2. Let us note here that Theorems 1.4.1 and 1.4.2 imply that $\chi_c(G)$ and $\phi_c(G)$ are always rational numbers. In fact whenever one of these parameters is equal to the fraction $\frac{p}{q}$ it is possible to realize the optimal colouring or flow using only p equally spaced points on the circle C^r.

1.5 Signed Graphs and Inversion

We have already introduced signed graphs and an equivalence based on switching (on edge-cuts). Here we introduce a new type of equivalence based on changing the signature on a cycle. Let (G, σ) be a signed graph and let C be a cycle of G, an *inversion* on the cycle C modifies the signature σ by changing the sign on all edges of $E(C)$.

Definition 1.5.1. *Let G be a graph and let σ, σ' be signatures of G. We say that (G, σ) is* inversion equivalent *to (G, σ') if it may be obtained from (G, σ) through a series of inversions on cycles. We may also simply say that σ' is inversion-equivalent to σ.*

Suppose that we wish to modify our signed graph (G, σ) into a new signed graph (G, σ') by a sequence of cycle inversions on the cycles $C_1, \ldots, C_k$. In this case, the set of edges on which σ and σ' differ is precisely

$$S = E(C_1) \oplus E(C_2) \oplus \ldots \oplus E(C_k)$$

where we use $A \oplus B$ to denote the symmetric difference of A and B. Observe that since $C_1, \ldots, C_k$ are cycles the set S has even degree at every vertex, so S can be expressed as a disjoint union of cycles. So in fact we can transform (G, σ) to (G, σ') by a sequence of inversions on edge-disjoint cycles.

Let (G, σ) be a signed graph embedded in the plane, let G^* be an embedded dual graph of G and assume that $E(G^*) = E(G)$ and that every edge e appears in our drawing as a pair of crossing edges one from G and the other from G^*. In this case (G^*, σ) is also a signed graph. By planar duality, If C is the edge set of a cycle in G^*, then C is a bond (a minimal nonempty edge-cut) in G. So inversion the signed graph (G^*, σ) on C changes the signature in exactly the same way as switching the signed graph (G, σ) on the edge-cut C. So switching and inversion may be viewed as dual operations for planar graphs.

We previously showed that two graphs are switching equivalent if they have the same negative cycles, next we show the corresponding result for inversions.

Lemma 1.5.1. *Two signed graphs (G, σ) and (G, σ') are inversion equivalent if and only if they have the same set of negative cuts.*

Proof. We have already observed that if (G, σ) and (G, σ') are inversion equivalent, then each cut has the same sign in these two signed graphs. For the converse, assume that the sign of each cut in (G, σ) is the same as its sign in (G, σ'). In particular, this holds for the cuts of the form $[\{v\}, V(G) \setminus \{v\}]$ for every $v \in V(G)$. Hence, if we take the symmetric difference S of the sets of negative edges in (G, σ) and (G, σ'), then the subgraph induced by these edges will have an even degree on each vertex. Therefore, (G, σ') is obtained from (G, σ) by inversion on the even-degree subgraph induced by S. $\square$

When investigating signed graphs it is frequently convenient to work with a signature that has relatively few negative edges. In our investigations, we considered the special family of signed cubic graphs and wished to determine when one could find an inversion equivalent signature for which no vertex was incident with 3 negative edges. It turns out that this is always possible when the graph is 3-connected, and in fact we prove the following more general result in Chapter 3.

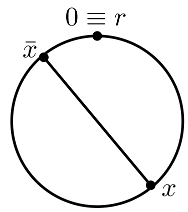

Figure 1.2: A point x and its antipodal point on a circle of circumference r create a pair of diametrically opposite points.

Theorem 1.5.1. *If (G, σ) is a simple k-edge-connected k-regular signed graph, then there is a signature σ' inversion-equivalent to σ so that every vertex is incident with at most 2 edges from the set $E^-(G, \sigma')$.*

1.6 Circular coloring and flow for signed graphs

We return to the setting of flows and our standard circle of circumference r, given by $C^r = \mathbb{R}/r\mathbb{Z}$. For a point $x \in C^r$, there is a unique point at a distance $r/2$ from x that is called the *antipodal point* of x and is denoted by $\overline{x}$. Next we introduce our main colouring definition.

Definition 1.6.1. *Given a signed graph (G, σ) and a real number r, a* circular r-coloring *of (G, σ) is a mapping $f: V(G) \to C^r$ satisfying the following conditions:*

- *For every positive edge uv of (G, σ), $d_{C^r}(f(u), f(v)) \geq 1$.*
- *For every negative edge uv of (G, σ), $d_{C^r}(f(u), \overline{f(v)}) \geq 1$.*

The circular chromatic number *of (G, σ) is defined as $\chi_c(G) = \inf\{r \mid (G, \sigma)$ admits a circular r-coloring$\}$.*

Note that for any negative edge uv, the condition $d_{C^r}(f(u), \overline{f(v)}) \geq 1$ is equivalent to $d_{C^r}(f(u), f(v)) \leq r/2 - 1$.

Proposition 1.6.1. *Let (G, σ) and (G, σ') be two switching-equivalent signed graphs. Then every circular r-coloring of (G, σ) corresponds to a circular r-coloring of (G, σ'). In particular, $\chi_c(G, \sigma) = \chi_c(G, \sigma')$.*

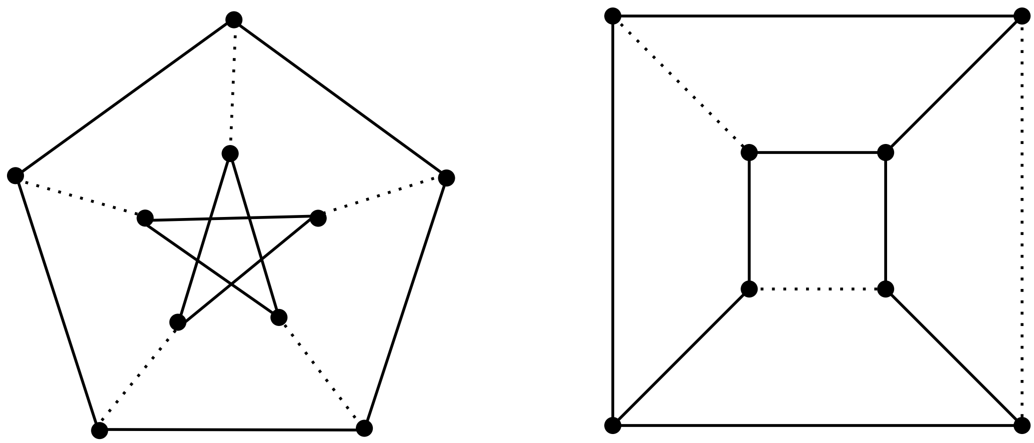

Figure 1.3: As a notable example, the circular chromatic number of these two signed graphs are both 10/3. (See [23])

Proof. As signed graphs (G, σ) and (G, σ') are switching equivalent, without loss of generality, we assume that (G, σ') is obtained from (G, σ) by switching at a vertex set A. Let f be a circular r-coloring of (G, σ). We define $g\colon V(G) \to C^r$ as follows:

$$g(v) = \begin{cases} f(v) & \text{if } v \in V(G) \setminus A \\ \overline{f(v)} & \text{if } v \in A \end{cases} \tag{1.2}$$

It is straightforward to verify that g is a circular r colouring of (G, σ') as desired. □

Observation 1.6.1. *If G is a graph with no loop, then $\chi_c(G, +) = \chi_c(G)$.*

Next we turn our attention to flows in signed graphs. Our main definition is next.

Definition 1.6.2. *Given a signed graph (G, σ) and a real number r, a* circular modulo r-flow *of (G, σ) is a pair (D, f) where D is an orientation of G and $f : E(G) \to C^r$ is a flow that satisfies the following conditions:*

- *For every positive edge e of (G, σ), $d_{C^r}(0, f(e)) \geq 1$*
- *For every negative edge e of (G, σ), $d_{C^r}(\frac{r}{2}, f(e)) \geq 1$.*

The circular flow number *of (G, σ) is defined as $\phi_c(G, \sigma) = \inf\{r \mid (G, \sigma)$ admits a circular modulo r-flow $\}$.*

As should be, the circular flow number of a signed graph is invariant under inversion.

Proposition 1.6.2. *Let (G, σ) and (G, σ') be two inversion-equivalent signed graphs. Then every circular modulo r-flow of (G, σ) corresponds to a circular modulo r-flow of (G, σ'). In particular, $\phi_c(G, \sigma) = \phi_c(G, \sigma')$.*

Proof. Let (D, f) be a circular modulo r flow of (G, σ) and assume that σ' is obtained from σ by switching on the even set of edges S. Define the function $g : E(G) \to C^r$ as follows:

$$g(e) = \begin{cases} f(e) & \text{if } e \in E(G) \setminus S \\ \overline{f(e)} & \text{if } e \in S \end{cases} \tag{1.3}$$

For every vertex $v \in V(G)$ there are an even number of edges of S incident with v and the flow value on each of these edges is modified by adding $\frac{r}{2}$. It follows that (D, g) is a flow. By construction it is a circular modulo r flow. □

As we will show in Chapter 2, flow-colouring duality extends naturally to this setting.

Theorem 1.6.1. *Let G and G^* be dual planar graphs with common edge set E and let $\sigma : E \to \{-1, 1\}$. Then $\chi_c(G, \sigma) = \phi_c(G^*, \sigma)$.*

1.7 Computing the circular flow number

For all $k \geq 3$ determining if a graph has chromatic number at most k is NP-complete. In contrast, by Seymour's 6-flow theorem [17], it is easy to test if a graph has flow number at most 6 (we need only check for cut edges). Despite this fact, it is still NP-complete to determine if a graph has flow number at most 4 (for cubic graphs this is equivalent to the existence of a 3-edge-colouring). For an integer $3 \leq k \leq 5$ it is straightforward to find an exponential time algorithm to determine if a graph G has a nowhere-zero $\mathbb{Z}_k$-flow. For instance, one may go over all orientations, choose a spanning tree T and for every edge $e \in E(G) \setminus E(T)$ pick a value for $\phi(e) \in \mathbb{Z}_k \setminus \{0\}$. Now ϕ can be completed to a flow by assigning every edge $f \in E(T)$ a value so that the net flow across the fundamental cut of T relative to f is zero. It can be shown that the resulting function ϕ is always a flow, and we can check if it is nowhere-zero by looking at the edges in $E(T)$.

In contrast, determining the circular flow number of a particular graph G is considerably more complicated. Lukot'ka found an interesting algorithm achieving the following result.

Theorem 1.7.1 (Lukot'ka)**.** *[14] There exists an algorithm that inputs a cubic graph G and either computes $\phi_c(G)$ in time $O(2^{0.6|V(G)|})$ or determines that G is a counterexample to Tutte's 5-flow conjecture.*

In Chapter 2 we consider the setting of circular flows on signed graphs and prove the following theorem.

Theorem 1.7.2. *There exists an algorithm that inputs a signed graph (G, σ) and computes the circular flow number of (G, σ) in time $O(2^{|V(G)|+|E(G)|})$.*

Chapter 2

Determining the circular flow number of signed graphs

In this chapter we explore circular flows for signed graphs and prove Theorem 1.7.2.

2.1 Flow-colouring duality

Our goal in this section is to prove two theorems that we restate from the introduction.

Theorem 1.3.1. *If G and G^* are dual planar graphs with common edge set E, then G has a proper k-colouring if and only if G^* has a nowhere-zero k-flow.*

Theorem 1.6.1. *Let G and G^* be dual planar graphs with common edge set E and let $\sigma : E \to \{-1, 1\}$. Then $\chi_c(G, \sigma) = \phi_c(G^*, \sigma)$.*

In fact, both of these theorems follow immediately from the following more general result.

Theorem 2.1.1. *Let G and G^* be dual planar graphs with a common edge set E. Let A be an Abelian group and let D be an orientation of G^*.*

1. *If $f : V(G) \to A$, there exists a function $g : E \to A$ so that (D, g) is a flow of G^* and every $e \in E$ incident with $u, v \in V(G)$ satisfies $g(e) = \pm(f(u) - f(v))$*

2. *If $g : E \to A$ and (D, g) is a flow of G^*, then there exists a function $f : V(G) \to A$ so every $e \in E$ incident with $u, v \in V(G)$ satisfies $g(e) = \pm(f(u) - f(v))$.*

Proof. For the first part, we may view f as assigning elements of A to the faces of G^*. Define the function $g : E \to A$ by the rule: For every directed edge $e \in D$ suppose that f assigns the face on the left of e the value a and the face on the right a b. Then we define $g(e) = a - b$. We claim first that g is a flow. First suppose that x is a vertex of G^* with all edges directed away from it. In this case it is easy to check that the sum of $g(e)$ over all edges incident with x is equal to 0. Switching the direction of an edge negates the value on it, so more generally, every vertex $x \in V(G^*)$ will satisfy Equation (1.1). Therefore g is a flow. It follows from the construction that whenever $e \in E$ is incident with $u, v \in V(G)$ we must have $g(e) = \pm(f(u) - f(v))$ as desired.

For the second part we assume that (D, g) is a flow of G^* and choose a spanning tree T of G. Choose a vertex $v \in V(G)$ and assign each vertex $u \in V(G)$ a value $f(u)$ by the following rule: Let P be the unique path in T from v to u. Observe that for every edge of P, the corresponding dual edge has an orientation given by D and as we move along the path this dual edge either crosses our path left-to-right or right-to-left. Define $f(u)$ to be the sum of g over all left-to-right edges minus the sum of g over all right-to-left edges. If e along P is an edge of T incident with the vertices $u, w \in V(G)$ then it is an immediate consequence of the definition that $g(e) = \pm(f(u) - f(w))$. Next suppose that $e \in E(G) \setminus E(T)$ is an edge incident with $v, w \in V(G)$. Consider the fundamental cycle of e and the tree T. In the dual graph this corresponds to an edge cut, and since g is a flow, the net flow of g across this edge cut must be 0. It follows from this that $g(e) = \pm(f(u) - f(v))$. □

The proofs of Theorems 1.3.1 and 1.6.1 are immediate consequences of the above result.

2.2 Circular flows

We have already defined circular modulo r-flows for both graphs and signed graphs, but we will require a stronger type of real valued flow for our investigations.

Definition 2.2.1. *Let G be a graph, let $f : E(G) \to \mathbb{R}$, let D be an orientation of G and assume that (D, f) is a flow of G.*

1. *f is a* circular *r-flow of G if $|f(e)| \in [1, r - 1]$ holds for every $e \in E(G)$.*

2. Let $\sigma : E(G) \to \{-1, 1\}$ be a signature. Then f is a circular r-flow *of (G, σ) if*

- *$|f(e)| \in [1, r-1]$ holds for every $e \in E^+(G, \sigma)$*
- *$|f(e)| \in [0, r/2-1] \cup [r/2+1, r]$ holds for every $e \in E^-(G, \sigma)$*

If G is a graph and (D, f) is a circular r-flow of G, then f can also be viewed as a circular modulo r-flow of G (more precisely, $f + r\mathbb{Z}$ is the modulo r-flow). The next lemma is the key result from this section, it shows that we can also go in the other direction. The proof of this well known property is based on Tutte's proof equating the existence of a nowhere-zero k-flow and a nowhere-zero $\mathbb{Z}_k$-flow.

Lemma 2.2.1. *Let G be a graph with orientation D, let $f : E(G) \to C^r$, and assume that (D, f) is a flow. Then there exists $g : E(G) \to (-r, r)$ so that (D, g) is a flow and $g(e) + r\mathbb{Z} = f(e)$ holds for every $e \in E(G)$.*

Before proving the lemma we require another definition and a simple observation. Let G be a graph, let D be an orientation of G, let A be an Abelian group, and let $f : E(G) \to A$. We define the *boundary* function of f denoted $\partial f : V(G) \to A$ by the following rule:

$$\partial f(v) = \sum_{(v,w) \in E(D)} f(vw) - \sum_{(u,v) \in E(D)} f(uv).$$

Note that for any such function f we must have $\sum_{v \in V(G)} \partial f(v) = 0$ since each edge contributes 0 to the sum.

Proof. For proving, choose a function $g : E(G) \to (-r, r)$ so that $g(e) + r\mathbb{Z} = f(e)$ holds for every $e \in E(G)$ and subject to this $\sum_{v \in V(G)} |\partial(g(v))|$ is minimum. If there is an edge e with $g(e) < 0$ then we reverse e and negate $g(e)$. By this operation we may assume $g(e) \geq 0$ holds for every $e \in E(G)$.

There are three distinct types of vertices in the graph G, namely V_1, V_2, and V_3, with the property that $\partial(g(v))$ assumes a positive value for vertices in V_1, a value of zero for vertices in V_2, and a negative value for vertices in V_3. For the sake of contradiction, let's assume that $\sum_{v \in V(G)} |\partial(g(v))| \neq 0$. This implies that both V_1 and V_3 are non-empty.

If D contains a directed path from a vertex in V_1 to a vertex in V_3, we can modify g by subtracting r from every edge in this path, and this gives a new function that contradicts the choice of g. Therefore, no such path exists, and it follows that there is a partition of $V(G)$ into $\{X, Y\}$ so that $V_3 \subseteq X$ and $V_1 \subseteq Y$ and all edges between X and Y are directed from X to Y. Now consider the following sum

$$\sum_{x \in X} \partial g(x)$$

On one hand, this quantity must be negative since $V_3 \subseteq X$ and $V_1 \cap X = \emptyset$. On the other hand, every edge with both ends in X contributes nothing, so this sum is also equal to $\sum_{e \in \overrightarrow{E}(X,Y)} g(e)$ and this must be non-negative. This contradiction implies that ∂g is the constant 0 function, and g is the desired flow. □

2.3 Hoffman's circulation theorem

In this section we introduce and prove an important theorem due to Hoffman that we require in the proof of our main result from this chapter. We will also use this theorem to deduce the following theorem from the introduction.

Theorem 1.4.2. *(see [9]) For every graph G, the circular flow number $\phi_c(G)$ is equal to the minimum cut imbalance of an orientation of G.*

Next we state and prove Hoffman's Theorem. Note that it gives a flow-based generalization of the standard Ford-Fulkerson max-flow/min-cut thoerem.

Theorem 2.3.1. *(Hoffman's Circulation Theorem [10]). Let D be a digraph and let $s, t : E(D) \to \mathbb{R}_{\geq \emptyset}$ satisfy $s(e) \leq t(e)$ for each $e \in E(D)$. Then there exists a function $f : E(D) \to \mathbb{R}_{\geq \emptyset}$ satisfying that for every $v \in V$, $\sum_{(v,u) \in E(D)} f(vu) = \sum_{(w,v) \in E(D)} f(wv)$ with $s(e) \leq f(e) \leq t(e)$ for each $e \in E(D)$ if and only if*

$$\sum_{(u,v) \in E(D), u \in U, v \notin U} s(uv) \leq \sum_{(x,y) \in E(D), x \notin U, y \in U} t(xy) \tag{2.1}$$

for every $U \subset V(D)$.

Proof. Define the slack of a set $U \subseteq V(D)$ to be

$$l(U) = \sum_{(u,v)\in A(G),u\in U,v\notin U} t(uv) - \sum_{(x,y)\in A(G),x\notin U,y\in U} s(xy). \tag{2.2}$$

It suffices to prove the existence of a flow ϕ under the assumption that every set has slack greater or equal than zero. If every edge e satisfies $s(e) = t(e)$, then we define $\phi = s = t$. Now for every $v \in V$ we have

$$\sum_{(x,y)\in E(D),x\in\{v\},y\in V\setminus\{v\}} \phi(xy) - \sum_{(x,y)\in E(D),x\in V\setminus\{v\},y\in\{v\}} \phi(xy) = l(v) \geq 0 \tag{2.3}$$

and

$$\sum_{(x,y)\in E(D),x\in V\setminus\{v\},y\in\{v\}} \phi(xy) - \sum_{(x,y)\in E(D),x\in\{v\},y\in V\setminus\{v\}} \phi(xy) = l(V-v) \geq 0. \tag{2.4}$$

Since the Left hand side of equation (2.4) equals negative the Left hand side of equation (2.3), they both equal to 0. So ϕ is a flow. We shall now modify s, t one edge at a time (maintaining nonnegative slack everywhere) until we achieve $s = t$. To do this, choose an edge f with $s(f) \neq t(f)$. Choose a set $X \subseteq V(D)$ with minimum slack so that $f \in (X, X^c)$ and choose a set $Y \subseteq V(D)$ with minimum slack so that $f \in (Y^c, Y)$. Set S be the set of edges with one end in $X \setminus Y$ and one end in $Y \setminus X$ and note that $f \in S$. Now we have

$$l(X) + l(Y) = l(X \cap Y) + l(X \cup Y) + \sum_{e\in S} (t(e) - s(e)) \geq t(f) - s(f). \tag{2.5}$$

So, we may choose $x, y \geq 0$ with $x \leq l(X)$ and $y \leq l(Y)$ and $x + y = t(f) - s(f)$. Now increase $s(f)$ by x and decrease $t(f)$ by y. Then $s(f) = t(f)$ and it follows from our choice of X, Y that the resulting functions s, t still have nonnegative slack for every set. □

2.4 Orientations and flows

Let G be a graph with orientation D, let $f : E(G) \to \mathbb{R}$ and assume that (D, f) is a flow of G. We call f *nonnegative* if $f(e) \geq 0$ holds for every $e \in E(G)$. If there exists an edge e

with $f(e) < 0$ then we may modify our orientation by reversing this edges and negate $f(e)$ to get another flow. By repeating this operation we can always turn a real valued flow into a nonnegative one. The resulting orientations can be very useful in working with flows. In particular, they feature in our proof of the following theorem restated from the introduction.

Theorem 1.4.2. *For every graph G, the circular flow number $\phi_c(G)$ is equal to the minimum cut imbalance of an orientation of G.*

Proof of Theorem 1.4.2. First suppose that G has an orientation D with cut imbalance r. Define the functions $\ell, u : E(G) \to \mathbb{R}$ by the rule that ℓ is constantly 1 and u is constantly $r - 1$. It now follows from Hoffman's theorem that we may choose a flow $f : E(G) \to \mathbb{R}$ with $1 \leq f(e) \leq r - 1$ for every $e \in E(G)$ and this is a circular r-flow.

Next suppose that G has a circular r-flow (D, f). By reversing edges, we may assume that f is nonnegative. We claim that D has cut imbalance at most r. To see this, consider a cut $E(X, Y)$ of the graph. Since f is a flow we must have

$$\sum_{e \in \overrightarrow{E}(X,Y)} f(e) = \sum_{e \in \overleftarrow{E}(X,Y)} f(e)$$

It now follows from $1 \leq f(e) \leq r - 1$ that the imbalance of this cut is at most r. Thus D is an orientation with imbalance at most r. □

Let (D, f) be a circular r-flow of the graph G, and assume that f is nonnegative. If (X, Y) is a partition of $V(G)$ then we call (X, Y) *tight* if every edge in $\overrightarrow{E}(X, Y)$ has $f(e) = 1$ and every edge in $\overleftarrow{E}(X, Y)$ has $f(e) = r - 1$. Observe that for our orientation, $E(X, Y)$ has cut imbalance exactly r. So for this orientation D there does not exist a nonnegative function $f' : E(G) \to \mathbb{R}$ for which (D, f') is a circular r'-flow for some $r' < r$. On the other hand, if there is no tight cut, then it is possible to apply Hoffman's Theorem to get a smaller circular flow. In particular, this gives rise to following observation.

Observation 2.4.1. *A graph G has $\phi_c(G) = r$ if and only if G has a circular r-flow, and every such flow has a tight cut.*

In fact, a similar tight cut property for circular flows in signed graphs. Let's begin with the definitions. Let (G, σ) be a signed graph and let (D, f) be a circular r-flow of (G, σ). By possibly reversing edges, we assume that f is nonnegative. If (X, Y) is a pair of nonempty disjoint sets with union $V(G)$ then we call (X, Y) *tight* if the following holds:

$$f(e) = \begin{cases} 1 & \text{if } e \in \overrightarrow{E}(X,Y) \cap E^+(G,\sigma) \\ r-1 & \text{if } e \in \overleftarrow{E}(X,Y) \cap E^+(G,\sigma) \\ \frac{r}{2}+1 & \text{if } e \in \overrightarrow{E}(X,Y) \cap E^-(G,\sigma) \\ \frac{r}{2}-1 \text{ or } r & \text{if } e \in \overleftarrow{E}(X,Y) \cap E^-(G,\sigma) \end{cases}$$

Theorem 2.4.1. *A signed graph (G, σ) has $\phi_c(G, \sigma) = r$ if and only if (G, σ) has a circular r-flow, but every such flow has a tight cut.*

Proof sketch. First suppose that (G, σ) has a circular s-flow (D, f) for some $s < r$. In this case $(D, \frac{r}{s}f)$ is a circular r-flow with no tight pairs.

Next suppose that (D, f) is a nonnegative circular r-flow of (G, σ) with no tight pair. Let $\epsilon > 0$ be a real number to be chosen later. Now we replace our flow f with an upper and lower bound function on every edge so we can apply Theorem 2.3.1. We let $m = |E(G)|$ and proceed as follows for every edge $e \in E(G)$.

- If $f(e) = 1$ and $\sigma(e) = 1$ then let $\ell(e) = 1 + \epsilon$ and $u(e) = 1 + m\epsilon$
- If $f(e) = r - 1$ and $\sigma(e) = 1$ then let $\ell(e) = r - 1 - m\epsilon$ and $u(e) = r - 1 - \epsilon$.
- If $f(e) = r$ and $\sigma(e) = -1$ then let $\ell(e) = r - m\epsilon$ and let $u(e) = r - \epsilon$
- If $f(e) = \frac{r}{2} + 1$ and $\sigma(e) = -1$ then let $\ell(e) = \frac{r}{2} + 1 + \epsilon$ and $u(e) = \frac{r}{2} + 1 + m\epsilon$
- If $f(e) = \frac{r}{2} - 1$ and $\sigma(e) = -1$ then let $\ell(e) = \frac{r}{2} - 1 - m\epsilon$ and let $u(e) = \frac{r}{2} - 1 - \epsilon$.
- Otherwise we let $\ell(e) = f(e) - m\epsilon$ and let $u(e) = f(e) + m\epsilon$.

We may choose ϵ suitably small so that every positive edge e has $\ell(e) > 1$ and $u(e) < r - 1$ and so that every negative edge e has either $\ell(e) > -1$ and $u(e) < \frac{r}{2} - 1$ or has $\ell(e) > \frac{r}{2} + 1$ and $u(e) < r$.

It follows from the assumption that no cut is tight that for every cut, all suitably small choices for ϵ will not violate the condition in Hoffman's Theorem. Therefore, for some $\epsilon > 0$ we satisfy the conditions for every cut to apply Hoffman's Theorem, and the resulting output is an r' circular colouring for some $r' < r$. □

Based on the above theorem, if a signed graph (G, σ) has $\phi_c(G) = r$ and (D, f) is a nonnegative circular r-flow of G, then there must be a tight cut. We can then express the value r based on the number of tight edges of different types. In particular this means that r must always be a rational number.

2.5 Balanced valuations

Let D be an orientation of the graph G. For every $X \subseteq V(G)$ we define $d^+(X) = |\overrightarrow{E}(X, V(G) \setminus X)|$ and $d^-(X) = |\overleftarrow{E}(X, V(G) \setminus X)|$. For a single vertex v we simplify the notation by defining $d^+(v) = d^+(\{v\})$ and $d^-(v) = d^-(\{v\})$. Define the function $b : V(G) \to \mathbb{Z}$ by the rule that $b(v) = d^+(v) - d^-(v)$. For every $X \subseteq V(G)$ we have the following equation (this follows from the fact that every edge with both ends in X makes no contribution).

$$\sum_{x \in X} b(x) = \sum_{x \in X} \left(d^+(x) - d^-(x) \right) = d^+(X) - d^-(X) \tag{2.6}$$

It follows from this equation and elementary considerations that the function b satisfies all of the following properties:

(i) $\sum_{v \in V(G)} b(v) = 0$

(ii) $b(v) \equiv_2 \deg(v)$ for every $v \in V(G)$.

(iii) $|E(X, V(G) \setminus X)| \geq \sum_{x \in X} b(x)$

This brings us to the key definition for this section.

Definition 2.5.1. *If G is a graph a function $b : V(G) \to \mathbb{Z}$ is called a* balanced valuation *if it satisfies properties (i), (ii), and (iii) from the above list.*

We have already seen that for every orientation D of G, the function $b : V(G) \to \mathbb{Z}$ given by outdegree minus indegree gives rise to a balanced valuation. In this case we say that the orientation D is *associated* with b. The next lemma shows that in fact, every such balanced valuation can be constructed in this manner.

Lemma 2.5.1. *Let G be a graph and let $b : V(G) \to \mathbb{Z}$ be a balanced valuation. Then there exists an associated orientation of G (i.e. an orientation of G for which $b(v) = d^+(v) - d^-(v)$ holds for every $v \in V(G)$).*

Proof. We construct an auxiliary digraph D^* from G together with functions $\ell : E(D^*) \to \mathbb{R}$ and $u : E(D^*) \to \mathbb{R}$ according to the following rules: The vertex set of D^* consists of $V(G)$ together with a single new vertex w. For every edge $xy \in E(G)$ we add two directed edges (x, y) and (y, x) to $E(D^*)$ and we set $\ell(x, y) = \ell(y, x) = 0$ and $u(x, y) = u(y, x) = 1$. Finally, for every vertex $z \in V(G)$ with $b(z) > 0$ we add the edge (w, z) to $E(D^*)$ and we assign $u(w, z) = \ell(w, z) = b(z)$ and for every $z \in V(G)$ with $b(z) < 0$ we add the edge (z, w) to $E(D^*)$ and we assign $u(w, z) = \ell(w, z) = -b(z)$. It follows from parts 1 and 3 in the definition of balanced valuation that the digraph D^* together with the functions ℓ and u satisfy the conditions for Hoffman's theorem. Therefore we may choose a real valued flow $f : E(D^*) \to \mathbb{R}$ so that every edge $e \in E(D^*)$ satisfies $\ell(e) \leq f(e) \leq u(e)$. Call an edge $xy \in E(G)$ *tight* if one of $f(x, y), f(y, x)$ is equal to 1 and the other is 0. Over all possible real valued flows, let us assume that f has been chosen as follows:

1. The number of tight edges is maximum
2. Subject to 1, the number of edges $(x, y) \in E(D^*)$ with $f(x, y) = 0$ is maximum.

We claim that our flow f chosen as above has every edge tight. If there is an edge $xy \in E(G)$ with $f(x, y), f(y, x) > 0$ then we may modify f by decreasing the value on each of these edges by $\min\{f(x, y), f(y, x)\}$ while maintaining our flow and this contradicts the choice of f. It follows that for every edge $xy \in E(G)$ at least one of $f(x, y)$ or $f(y, x)$ is equal to 0. In particular, if xy is not tight, then $f(x, y) + f(y, x) < 1$. Suppose that not all edges in G are tight. In this case our construction together with property (ii) imply that there cannot be just one non-tight edge incident with a vertex in $V(G)$. It follows from this that we may choose a cycle C of non-tight edges of G. Let C_1 and C_2 be the two directed cycles of D^* corresponding to C. Assume that either C_1 contains an edge (x, y) with $f(x, y) > 0$ or that neither C_1 nor C_2 contains such an edge. Let $M = \max\{f(x,y) \mid e \in E(C_1)\}$ and modify f by adding $1 - M$ to every edge of $E(C_1)$. It follows from our assumptions that the resulting flow has more tight edges thus contradicting our choice. Therefore, our chosen function f must have every edge tight. Now modify D^* to form D by deleting every $(x,y) \in E(D)$ with $f(x,y) = 0$ and deleting the vertex w. It follows from this construction that D is an orientation of G and $d^+(v) - d^-(v) = b(v)$ holds for every $v \in V(G)$. □

We have seen that finding a circular r-flow of a graph is equivalent to finding an orientation of the graph with cut imbalance at most r. However, if b is a balanced valuation of G and D is an associated orientation, then it follows from equation 2.6 that the imbalance of every edge cut $E(X,Y)$ can be computed based on the size of the cut in the underlying graph $|E(X,Y)|$ and the function b. More precisely, this edge cut will have imbalance given by $\frac{|E(X,Y)|}{t}$ where $t = \frac{1}{2}\left(|E(X,Y)| - |\sum_{x \in X} b(x)|\right)$. For every balanced valuation b of G let us define the *imbalance* of b to be the maximum imbalance of an edge cut for some (and thus every) orientation of G associated to b. We now have the following equivalence.

Theorem 2.5.1. *[12] Let G be a graph and let $r \geq 2$ be a real number. Then the following statements are equivalent:*

- *G admits a circular r-flow.*
- *G has an orientation with imbalance at most r.*
- *G has a balanced valuation with imbalance at most r.*

2.6 Computing ϕ_c for graphs

In this section we introduce Lukot'ka's algorithm for computing circular flow numbers in cubic graphs. We begin with the key observation. Let G be a cubic graph and let (D, f) be a circular r flow of G for some $r < 5$ and assume that f is nonnegative. Now define the balanced valuation $b : V(G) \to \mathbb{Z}$ by the rule $b(v) = d^+(v) - d^-(v)$ (where d^+ and d^- indicate indegree and outdegree in D). Since every edge has positive flow value and the graph is cubic, every vertex v has $b(v) = \pm 1$. For $i = 1, 2$ let V_i be the set of vertices v with $b(v) = (-1)^i$. It follows from the fact that every edge has flow value at least one but less than 4 that every component of the graph induced by V_i has size at most 2. Let us define this new concept.

Definition 2.6.1. *For a positive integer k a k-*bisection *of a graph G is a partition of $V(G)$ into $\{V_1, V_2\}$ so that the graph induced by V_i has all components of size at most k for $i = 1, 2$.*

Based on the above discussion, for every cubic graph G with $\phi_c(G) < 5$ we can compute $\phi_c(G)$ by first finding every 2-bisection. Then for every 2-bisection $\{V_1, V_2\}$ define the function $b : V(G) \to \mathbb{Z}$ by $b(v) = 1$ if $v \in V_1$ and $b(v) = -1$ if $v \in V_2$ and check if b is a balanced valuation. If so, then we find an associated orientation D and compute the imbalance of D. The smallest imbalance found in this exhaustive search will be the circular chromatic number of G. Lukot'ka constructed a series of algorithms to perform these steps effectively.

Theorem 2.6.1. *[14] For a cubic graph G there exists an algorithm with running time $O(2^{0.6|V(G)|})$ that finds all 2-bisections of G. (see appendix A).*

Theorem 2.6.2. *[14] Let G be a graph and let b be a mapping from $V(G)$ to $\mathbb{Z}$. There exists an algorithm with running time $O(|E(G)|^{3/2})$ that takes G and b as the input and either shows that b is not a balanced valuation, or constructs an associated orientation.*

Theorem 2.6.3. *[14] Let G be a graph and let D be an orientation of G. There exists an algorithm that computes the cut imbalance of G, say r, and if $r < \infty$ also returns a nonnegative circular r-flow (D, f). This algorithm has running time $O(|E(G)|^{1.5} \log^2 (|E(G)|))$.*

In fact, using Lukot'ka's algorithms and some earlier work one may obtain an algorithm that inputs a cubic graph G and either determines that G is a counterexample to Tutte's 5-flow Conjecture or computes $\phi_c(G)$ and runs in time $2^{0.6|V(G)|}$, as claimed in Theorem 1.7.1. An algorithm of Fomin and Høie [8] uses dynamic programming over path decompositions to find all integer 5-flows of G in time $O(2^{0.34|V(G)|})$. If no such 5-flow exists, then G is a counterexample to the 5-flow Conjecture and there is nothing left to show. If every 5-flow has a tight cut, then we must have $\phi_c(G) = 5$ by Theorem 2.4.1. Otherwise we have $\phi_c(G) < 5$ and we proceed with Lukot'ka's algorithms.

2.7 Determining ϕ_c for signed graphs

Unfortunately, balanced valuations do not seem helpful in computing circular flow numbers of signed graphs. This is because negative edges can have arbitrarily small flow values in a nonnegative flow. Nevertheless, we will provide an exponential time algorithm that can compute ϕ_c for a signed graph by reducing this to a certain restricted flow problem on unsigned graphs.

Let (G, σ) be a signed graph and let T be the set of all $v \in V(G)$ so that v is incident with an odd number of negative edges. We form a new graph $\hat{G}$ by adding a new vertex w and edges joining w to every vertex in T. For every orientation D of G we let $\hat{D}$ be the corresponding orientation of $\hat{G}$ in which all edges incident to w are directed toward it.

Here is our lemma proved jointly with Devos and Mohar.

Lemma 2.7.1. *Let (G, σ) be a signed graph with orientation D and let $\hat{G}$ and $\hat{D}$ be as above. Let $f : E(G) \to C^r$ and define $\hat{f} : E(\hat{G}) \to C^r$ as follows:*

$$\hat{f}(e) = \begin{cases} f(e) & \text{if } e \in E(G) \text{ and } \sigma(e) = 1 \\ f(e) + \frac{r}{2} & \text{if } e \in E(G) \text{ and } \sigma(e) = -1 \\ \frac{r}{2} & \text{if } e \notin E(G). \end{cases}$$

Then (D, f) is a circular modulo r flow of the signed graph (G, σ) if and only if $(\hat{D}, \hat{f})$ is a circular modulo r flow of the (not signed) graph $\hat{G}$.

Proof. For a vertex $v \in V(G)$ we consider the boundary of the function f relative to D and the boundary of $\hat{f}$ relative to $\hat{D}$. If $v \notin T$ then at the vertex v the only difference between $\hat{f}$ and f is that we have added $\frac{r}{2}$ on an even number of edges incident with v. Since an even multiple of $\frac{r}{2}$ is 0 in C^r this means that the Kirchoff rule holds for f at v if and only if it holds for $\hat{f}$ at v. Next suppose that $v \in T$. In this case when moving from G to $\hat{G}$ we have added $\frac{r}{2}$ on an odd number of edges of G incident with v but in $\hat{G}$ the vertex v is adjacent to the new vertex w and $\hat{f}(vw) = \frac{r}{2}$. So again the Kirchoff rule holds for v and the function f if and only if it holds for $\hat{f}$. Finally, by parity we must have $|T|$ even and it follows that the Kirchoff rule must hold at the new vertex w of $\hat{G}$.

We have determined that (D, f) is a flow of (G, σ) if and only if $(\hat{D}, \hat{f})$ is a flow of $\hat{G}$. In order for f to be a circular modulo r-flow, every positive edge e must have $d_{C^r}(0, f(e)) \geq 1$ and every negative edge e must have $d_{C^r}(\frac{r}{2}, f(e)) \geq 1$. Thanks to the construction this translates into the condition $d_{C^r}(0, \hat{f}(e)) \geq 1$ in $\hat{G}$. It follows that (D, f) is a circular modulo r-flow of (G, σ) if and only if $(\hat{D}, \hat{f})$ is a circular modulo r-flow of $\hat{G}$ as desired. $\square$

We have now shown the following chain of equivalences

$$\begin{aligned}
&(G, \sigma) \text{ has a modulo circular } r\text{-flow} \\
\Leftrightarrow\ &\hat{G} \text{ has a modulo circular } r\text{-flow with all new edges having flow } \tfrac{r}{2} \\
\Leftrightarrow\ &\hat{G} \text{ has a circular } r\text{-flow with all new edges having flow } \pm\tfrac{r}{2}
\end{aligned}$$

For any orientation of $\hat{G}$ a straightforward modification of Theorem 2.6.3 allows us to determine the minimum r for which there exists a nonnegative circular r-flow of $\hat{G}$ with the added restriction that all new edges have flow value $\frac{r}{2}$. This algorithm still runs in time $O(|E(\hat{G})|^{1.5} \log^2(|E(\hat{G})|))$. So we can compute $\phi_c(G, \sigma)$ by considering all $2^{|E(G)|+|V(G)|}$ orientations of $\hat{G}$ and running this algorithm on each one. This gives us Theorem 1.7.2 and completes this chapter.

Chapter 3

On the maximum negative degree of signed graphs

This thesis aims to enhance Lukot́ka's theorems and algorithms for cubic signed graphs. To achieve this, our initial objective was to demonstrate that every 3-connected cubic signed graph could be transformed using a sequence of cycle-inversions, resulting in a new signed graph where no vertex is adjacent to three negative edges. Although, we later discovered that this assumption was not needed, it is still interesting that it is possible to get such an outcome. Existence of series of cycle-inversions with this property is the central goal of this chapter. To lay the foundation for our approach, we will begin by clarifying essential definitions, theorems, and lemmas, as outlined below.

3.1 T-joins and cycle-inversion on signed graphs

Let $G = (V, E)$ be a connected graph, and let $T \subseteq V$ have $|T|$ even. A subset of edges $J \subseteq E$ is called a *T-join* if the graph $H = (V, J)$ has the property that $d_H(v)$ is odd for every $v \in T$ and even for every $v \in V(G) \setminus T$. The following lemma shows that T-joins always exist.

Lemma 3.1.1. *Let $G = (V, E)$ be a connected graph and let $T \subseteq V$ have $|T|$ even. If $H \subseteq G$ is a spanning tree, then there exists a T-join J with $J \subseteq E(H)$.*

Proof. We proceed by induction on $|V(H)|$. For the base case $|V(H)| = 1$ so $T = \emptyset$ and $J = \emptyset$ satisfies the lemma. For the inductive step we may choose a leaf vertex v with unique neighbour u. If $v \notin T$ the result follows by applying induction to $H - v$ and T. If $v \in T$ then we apply the theorem inductively to $H - v$ and the set $T' = (T \setminus \{v\}) \oplus \{u\}$ to choose a T'-join J'. Now $J' \cup \{uv\}$ is the desired T-join in H. □

For a pair of sets A, B we denote the symmetric difference by $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Observe that whenever J is a T-join and C is the edge set of a cycle, then $J \oplus C$ is also a T-join. It follows immediately from this that every minimal T-join is the edge set of a forest. Call a set of edges $S \subseteq E$ *even* if the graph (V, S) has all vertices of even degree, and note that every even set of edges can be expressed as a disjoint union of edge sets of cycles. If J and J' are T-joins, then the set $J \oplus J'$ must be even, and it follows that we can transform J into J' by cycle inversions using edge-disjoint cycles. The following observation is an immediate consequence.

Observation 3.1.1. *Let $G = (V, E)$ be connected and let $T \subseteq V$ have $|T|$ even. If $J \subseteq E$ is a T-join, then it is minimal if and only if (V, J) is a forest.*

If $\{X, Y\}$ is a partition of V and C is the edge cut consisting of the edges with one end in X and one in Y, then we call C a *T-cut* if $|X \cap T|$ is odd. Observe that a T-cut is minimal if and only if it is a bond (a minimal nonempty edge cut).

Observation 3.1.2. *Let $G = (V, E)$ be connected and let $T \subseteq V$ have $|T|$ even. If $C \subseteq E$ is a T-cut and $J \subseteq E$ is a T-join, then $C \cap J \neq \emptyset$.*

Proof. Suppose for a contradiction that $C \cap J = \emptyset$ and let $\{X, Y\}$ be the partition of V associated with the edge cut C. Now, consider the subgraph H of (V, J) induced by X. The vertices of odd degree in H are precisely $X \cap T$, but this set has odd size, which is contradictory. □

In fact, the minimal sets hitting every T-cut are the minimal T-joins and the minimal sets hitting every T-join are the minimal T-cuts as we show next.

Theorem 3.1.1. *Let $G = (V, E)$ be connected and let $T \subseteq V$ have $|T|$ even.*

1. *If $S \subseteq E$ has nonempty intersection with every T-cut, then S contains a T-join.*

2. *If $S \subseteq E$ has nonempty intersection with every T-join, then S contains a T-cut.*

Proof. For the first part, let $H_1, \ldots, H_k$ be the components of the graph (V, S). For every $1 \leq i \leq k$ the edge cut of G separating $V(H_i)$ from the remaining vertices cannot be a T-cut since every T-cut intersects S. It follows that $T_i := T \cap V(H_i)$ has $|T_i|$ even. Lemma 3.1.1 implies that there is a T_i-join, say $J_i \subseteq E(H_i)$. Now $J = \cup_{i=1}^{k} J_i$ is a T-join contained in S, as desired.

For the second part, consider the components of the graph $G \setminus S$, denoted $H_1, \ldots, H_k$. If $T_i := T \cap V(H_i)$ is even for every $1 \leq i \leq k$, then every H_i has a T_i-join, J_i and $J = \cup_{i=1}^{k} J_i$ is a T-join disjoint from S, and this is a contradiction. It follows that there exists $1 \leq j \leq k$ for which $|T_j|$ is odd. But then the edge-cut separating $V(H_j)$ from the remaining vertices is a T-cut contained in S as desired. □

In fact, T-joins provide an equivalent way of working with signatures under cycle inversions. Let σ be a signature for the graph $G = (V, E)$ and define T_σ to be the set of vertices $v \in V$ incident to an odd number of edges in $E^-(G, \sigma)$. Note that T_σ is the set of odd degree vertices in the graph $(V, E^-(G, \sigma))$ so $|T_\sigma|$ is necessarily even, and moreover $E^-(G, \sigma)$ is a T_σ-join. The following lemma characterizes inverse equivalence by way of this connection.

Lemma 3.1.2. *Let $G = (V, E)$ be a connected graph with signatures $\sigma, \sigma' : E \to \{-1, 1\}$. Then σ is inversing equivalent to σ' if and only if $T_\sigma = T_{\sigma'}$*

Proof. To establish the "only if" part, observe that applying a cycle inversion to σ has no effect on the set T_σ. So whenever σ and σ' are invering equivalent we must have $T_\sigma = T_{\sigma'}$.

For the "if" part, suppose $T = T_\sigma = T_{\sigma'}$. Consider the set $S = E^-(G, \sigma) \oplus E^-(G, \sigma')$. It follows from our assumptions that S is even, so there exist edge disjoint cycles $C_1, \ldots, C_k$ with $S = \bigcup_{i=1}^{k} E(C_i)$. Starting with the signed graph (G, σ) and performing cycle inversions on $C_1, C_2, \ldots, C_k$ enables us to transform it into the signed graph (G, σ'). □

3.2 An extension of Tutte's theorem

In this section we bootstrap Tutte's famous theorem on perfect matchings to a more general result on subgraphs with both parity and upper bound constraints on the vertex degrees. Let us begin by stating Tutte's classical theorem.

Theorem 3.2.1 (Tutte's 1-Factor Theorem, see [2]). *A graph $G = (V, E)$ has a perfect matching if and only if, for every $U \subseteq V$, the subgraph $G \setminus U$ has at most $|U|$ components of odd size.*

For the purpose of our investigations into resigning, we need the following generalization of Theorem 3.2.1 where higher vertex degrees are permitted. The proof bootstraps from the original.

Here is our theorem, proved jointly with DeVos and Mohar, but we actually found that it was already proved by Lovász.

Theorem 3.2.2. *Let G be a graph and let $T \subseteq V(G)$ have $|T|$ even. Let $f \colon V(G) \to \mathbb{Z}$ and assume that $f(v)$ is odd if and only if $v \in T$. There exists a subgraph $H \subseteq G$ such that*

1. *for every $v \in V(G)$, $deg_H(v)$ is odd if and only if $v \in T$, and*
2. *$deg_H(v) \leq f(v)$ holds for every $v \in V(G)$*

if and only if there does not exist $X \subseteq V(G)$ such that number of components of $G \setminus X$ with an odd number of vertices in T is greater than $\sum_{v \in X} f(v)$.

Proof. The "only if" direction is fairly straightforward, so we prove it first. Suppose that there exists $X \subseteq V(G)$ so that $H_1, \ldots, H_t$ are distinct components of $G \setminus X$ with $|V(H_i) \cap T|$ odd and assume that $t > \sum_{v \in X} f(v)$. Every subgraph of G whose edge set is a T-join must contain at least one edge between X and $V(H_i)$ for every $1 \leq i \leq t$. It follows that any subgraph $H \subseteq G$ satisfying the first constraint must have at least t distinct edges incident with vertices of X. However, no such subgraph can satisfy the second constraint. It follows that under this assumption, no subgraph can satisfy both constraints.

For the "if" direction, we create a graph G' from the original graph G as follows. Create the new graph G' by replacing each vertex $v \in V(G)$ with new vertices $v_1, \ldots, v_{f(v)}$. We say that $v_1, \ldots, v_{f(v)}$ are *associated* with v. For new vertices v_i and u_j ($1 \leq i \leq f(v)$ and $1 \leq j \leq f(u)$) in $V(G')$, add the edge v_iu_j if the corresponding vertices v and u are neighbors in G. For every vertex $v \in V(G)$ add all edges of the form v_iv_j where $1 \leq i < j \leq f(v)$ to G' (so the subgraph induced by $\{v_1, \ldots, v_{f(v)}\}$ is a complete graph). Next is the key claim

Claim: G' has a perfect matching if and only if G has the desired subgraph.

To prove this claim, first suppose that G has the desired subgraph H we are interested in. Consider following algorithm:

1. Let G^+ and H^+ be copies of G' and H respectively.
2. Let S be the empty set.
3. Let $uv \in E(H^+)$, choose $u_iv_j \in E(G^+)$ such that u_i and v_j (in G^+) are corresponding vertices of u and v (in H^+) respectively.
4. Delete uv from $E(H^+)$ and u_i, v_j from $V(G^+)$.
5. Add edge u_iv_j to S.
6. Go to step 3 if $E(H^+) \neq \emptyset$ (note that since $deg_{H^+}(v) \leq f(v)$ you can go back to step 3 while $E(H^+) \neq \emptyset$).
7. Output G^+ and S

Since $deg_{H^+}(v)$ is odd if and only if $v \in T$, the final output graph $V(G^+)$ has a vertex partition (corresponding to vertices of G) into sets $\{V_1, \ldots, V_n\}$ so that each V_i induces a clique on an even number of vertices. Choosing a perfect matching from each of these induced subgraphs extends S to a perfect matching in G'.

Next suppose that G' has a perfect matching. Over all perfect matchings M choose one to minimize the number of edges of the form u_iv_j where $u, v \in V(G)$ are distinct. Suppose (for a contradiction) that M contains distinct edges u_iv_j and $u_{i'}v_{j'}$ where $u, v \in V(G)$ are distinct. In this case we can construct a new perfect matching by removing these two edges and adding the edges $u_iu_{i'}$ and $v_jv_{j'}$. This new perfect matching contradicts the choice of

M and it follows that for every $uv \in E(G)$ there is at most one edge of the form $u_i v_j \in M$. Let H be the spanning subgraph of G with edge set

$$E(H) = \{uv \in E(G) \mid u_i v_j \in M \text{ for some } 1 \leq i \leq f(u) \text{ and } 1 \leq j \leq f(v)\}$$

For every $v \in V(G)$, we have $deg_H(v) \leq f(v)$ since there are $f(v)$ vertices in G' associated to this v. By the choice of our M, every $v \in G$ has the property that $deg_H(v)$ has the same parity as $f(v)$ and this completes the proof of the claim.

Having proved the claim, we can complete the proof of the theorem by showing that whenever G' does not have a perfect matching, the graph G has the obstruction (i.e. a set $X \subseteq V(G)$ such that number of components of $G \setminus X$ with an odd number of vertices in T is greater than $\sum_{v \in X} f(v)$.) Assume now that G' does not have a perfect matching and apply Tutte's 1-factor theorem 3.2.1 to choose a minimal set X' with the property that the number of components of $G' \setminus X'$ with an odd number of vertices is greater than $|X'|$.

Let $1 \leq i, j \leq f(v)$ with $i \neq j$ and consider the vertices $v_i, v_j \in V(G')$. Note that since v_i and v_j are adjacent, they cannot be in distinct components of $G' \setminus X'$. If $v_i \in X'$ and $v_j \notin X'$ then the only vertices of $G' \setminus X'$ adjacent to v_i lie in the component of this graph containing v_j. It follows from this that $X'' := X' \setminus \{v_i\}$ contradicts our choice of the set X'. Therefore, for every $v \in V(G)$ all of the vertices of G' associated to v either lie in X' or they are all in the same component of $G' \setminus X'$.

Define the set $X = \{v \in V(G) \mid v_i \in X' \text{ for some } 1 \leq i \leq f(v)\}$. Let $H'_1, \ldots, H'_t$ be the components of $G' \setminus X'$ of odd size. For every $1 \leq i \leq t$ let H_i be the subgraph of G induced by those vertices (of G) associated with vertices in $V(H'_i)$. Since every H'_i has odd size, it follows that $|V(H_i) \cap T|$ is odd for every $1 \leq i \leq t$. So $H_1, \ldots, H_t$ are all components of $G \setminus X$ containing an odd number of vertices in T. Now we have found our obstruction set X since $\sum_{v \in X} f(v) = |X'| < t$. □

Corollary 3.2.1. *Given a signed graph (G, σ) and a positive integer k, we can decide in polynomial time if G has an inversing equivalent signature such that the maximum degree of vertices adjacent to negative edges is less than or equal to k.*

Proof. We may assume that G is simple since there is never a need to use two negative edges in parallel (or a loop). We may now assume $k < |V(G)|$ as otherwise the signature σ satisfies the corollary. Following the proof of the above theorem, we form the graph G'. By this theorem, our graph G has the desired signature if and only if G' has a perfect matching. The perfect matching problem for an n vertex m edge graph can be solved efficiently by Edmond's Blossom Algorithm in time $O(n^2m)$ (see [6]) and this gives us the desired polynomial algorithm for G. □

3.3 k-regular signed graphs

In this section we consider k-regular signed graphs and we are interested in determining if there is an inversing equivalent signature for which no vertex is incident with more than two negative edges. We will prove that this property holds true under the assumption that the graph is k-edge-connected, but may fail under the weaker assumption of $(k-1)$-edge-connectivity.

Corollary 3.3.1. *For every simple k-edge-connected k-regular signed graph (G, σ), there exists an inversing equivalent signature to σ, where each vertex is incident to at most two negative edges.*

Proof. Suppose for contradiction no such signature exists. Let $T = \{v \in V(G) \mid v$ is incident to odd number of negative edges$\}$. Define function $f \colon V(G) \to \mathbb{Z}$ such that for every $v \in T$, $f(v) = 1$ else $f(v) = 2$. Now by applying Theorem 3.2.2, either G has a subgraph $H = (V, E')$ such that each vertex in T is adjacent to one negative edge, and each vertex in $V \setminus T$ is adjacent to either 0 or 2 negative edges, or there exists a set $X \subseteq V(G)$ such that $|\sum_{v \in X} f(v)|$ is less than number of components of $G \setminus X$ with an odd number of vertices in T. Let $H_1, \ldots, H_t$ be the components of $G \setminus X$ containing an odd number of vertices in T. Let S be set of edges between the set X and the $\cup_{i=1}^{t} V(H_i)$. It follows from the k-edge-connectivity of G that there are at least k edges between X and $V(H_i)$ for every i, thus $|S| \geq kt$. On the other hand every vertex in X has degree k so $|S| \leq k \cdot |X|$. Thus $kt \leq |S| \leq k|X| \leq k \cdot (|X \cap T| + 2 \cdot |X \setminus T|) = k \cdot (\sum_{v \in X} f(v))$, a contradiction. □

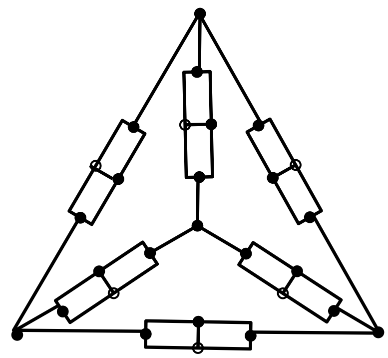

Figure 3.1: Examples for Theorem 3.3.2 where $k = 3$

Next we show a family of graphs demonstrating that the above corollary does not hold under the weaker assumption of $(k-1)$-edge-connectivity. To verify $(k-1)$-edge-connectivity of our graphs, the following classical theorem is useful.

Theorem 3.3.1. *(Menger's Theorem, see[15]) For every graph, the size of a minimum nonempty edge cut is equal to the maximum number k so that for every pair of vertices, there exist k pairwise edge-disjoint paths between them.*

Now we are ready to give our example graphs.

Here is our theorem proved jointly with Devos and Mohar.

Theorem 3.3.2. *For every odd $k \geq 3$ there exists a simple $(k-1)$-edge-connected k-regular signed graph (G, σ) that has no equivalent signature for which every vertex is incident with at most two negative edges.*

Proof. Define G_1 to be a graph with vertex set $\{1, 2\} \times \mathbb{Z}_{2(k-1)}$ and an edge between $(1, a)$ and $(2, b)$ if $b - a \in \{0, 1, \ldots, k-2\} \pmod{2(k-1)}$. It is straightforward to verify that G_1 is a $(k-1)$-regular $(k-1)$-edge-connected bipartite graph with bipartition $(\{1\} \times \mathbb{Z}_{2(k-1)}, \{2\} \times \mathbb{Z}_{2(k-1)})$. Define G_2 to be the graph obtained from G_1 by adding two new vertices u, v where u is adjacent to all vertices of the form $(1, a)$ where a is odd and v is adjacent to all vertices of the form $(1, b)$ where b is even. Note that G_2 is a $(k-1)$-edge-connected graph with bipartiton (V_1, V_2) where $V_1 = \{1\} \times \mathbb{Z}_{2(k-1)}$ and $V_2 = \{2\} \times \mathbb{Z}_{2(k-1)} \cup \{u, v\}$. Furthermore every vertex in V_1 has degree k and every vertex in V_2 has degree $k-1$. Now modify G_2 to form a new graph G_3 as follows: For every vertex $w \in V_2$ let H_w be a graph obtained from a complete graph on $k+1$ vertices by deleting a matching of size $\frac{k-1}{2}$ and modify G_2 by adding the graph H_w, deleting the vertex w and then adding edges between every neighbour of w in G_2 and every vertex of degree $k-1$ in H_w. The resulting graph G_3 is now k-regular and $(k-1)$-edge-connected. Let T be a set consisting of V_1 together with exactly one vertex from each subgraph of the form H_w. Now choose a signature σ so that $T = T_\sigma$. The signed graph (G_3, σ) cannot have an inversing equivalent signature so that every vertex is incident with at most two negative edges, since every vertex in V_1 would have to be incident with exactly one negative edge, and after deleting V_1 we are left with $|V_1| + 2$ components each of which has an odd number of vertices in T. $\square$

Chapter 4

The path and negative cycle property

In this chapter we will be interested in signed graphs under switching equivalence. For brevity we will frequently refer a signed graph G without explicitly naming the signature. In this case it is understood that G is equipped with a signature σ.

Our interest is in studying a property of a signed graph G with a pair of distinguished vertices u, v. We say that G has the *Path-Negative-Cycle Property* or *PNC property* (for short) if there exists a path P from u to v and a negative cycle C for which $V(P) \cap V(C) = \emptyset$. Our goal in this chapter is to provide a structural characterization of all 3-connected graphs G with distinguished vertices u, v that do not satisfy the PNC property.

Observation 4.0.1. *Let G be a signed planar graph with distinguished vertices u, v. If G has the PNC property, then there exists a u, v path P and vertex disjoint negative cycle C where C is the boundary of a face.*

Proof. Let P be a u, v path and C a negative cycle with $V(P) \cap V(C) = \emptyset$. We may assume without loss of generality that in our embedding the path P lies outside the disc bounded by C. Now the edge set of C can be expressed as the symmetric difference of the edge sets of all faces inside C and it follows that there is a face inside C bounded by a negative cycle C' and C' together with P yield the observation. □

It follows from the above observation that if G is a signed planar graph with a pair of distinguished vertices u, v that G does not have the PNC property if every negative face of

G is incident with either u or v. In fact this is the main obstruction to the PNC property as shown in the following theorem, the main result in this chapter.

Theorem 4.0.1. *Let G be a 3-connected internally 4-connected signed graph with distinguished vertices u, v. If G does not have the PNC property then one of the following holds:*

1. *$G - \{u, v\}$ is balanced*
2. *G can be embedded in the plane with all negative faces incident to either u or v.*

Before diving into the proof of this theorem, we begin by establishing some motivation and background.

4.1 Motivation

The motivation for this problem is an attempt to generalize two famous structure theorems for graphs and signed graphs. One of these we have already mentioned, but we restate it for convenience.

Theorem 1.2.1 (Slilaty)**.** *[18] Let (G, σ) be a 3-connected signed graph. If (G, σ) does not contain two vertex disjoint negative cycles, one of the following holds:*

1. *$G - v$ is balanced for some vertex $v \in V(G)$*
2. *$|V(G)| \leq 5$*
3. *G can be embedded in the projective plane so that every face is bounded by a positive cycle.*
4. *G can be expressed as a 3-sum of a graph with no two disjoint negative cycles and a balanced graph with at least 5 vertices.*

The most interesting case in this theorem is the third. In such an embedding of a signed graph in the projective plane all contractible cycles must be balanced, and any two non-contractible cycles intersect are not balanced (thanks to the topology of the projective plane). The other classical theorem of interest is as follows.

Let G be a graph and let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a collection of pairwise disjoint two element subsets of $V(G)$. We say that G has a *$\mathcal{T}$-linkage* if there exist k pairwise vertex disjoint paths $P_1, \ldots, P_k$ so that P_i has ends T_i for $1 \leq i \leq k$.

Theorem 4.1.1 (Thomassen's 2-Linkage Theorem [19])**.** *Let graph G be 3-connected and internally 4-connected, with distinct vertices $s_1, s_2, t_1, t_2 \in V(G)$. Then, either there exist vertex-disjoint paths P_1 and P_2 such that P_i has ends s_i and t_i for $i = 1, 2$, or G has an embedding in the plane with s_1, s_2, t_1, t_2 on the outer face occurring in that cyclic order.*

Let us note that in the latter case the plane embedding of G forbids the presence of the desired paths. Also note that there are (stronger) versions of this theorem that weaken the connectivity assumption on G. In this section we consider the following problem in an attempt to generalize both of the aforementioned theorems.

Problem 4.1.1 (The two signatures two cycles problem)**.** *Let G be a 3-connected and internally 4-connected graph with signatures σ_1 and σ_2. If G does not contain a pair of vertex disjoint cycles C_1, C_2 with $\sigma_i(C_i) = -1$, what is the structure of G?*

The special case for the above problem when $\sigma_1 = \sigma_2$ is solved by Slilaty's structure theorem. Another case of interest is when σ_i is a signature with a single negative edge $s_i t_i$ for $i = 1, 2$. in this case the disjoint negative cycles will exist if and only if the graph $G - \{s_1t_1, s_2t_2\}$ contains vertex disjoint paths P_1, P_2 where P_i has ends s_i, t_i for $i = 1, 2$. So this case of the problem is solved by the 2-linkage theorem. Indeed, our main result from this chapter provides a partial solution to this problem in the special case when σ_1 has just one negative edge as we highlight below.

Observation 4.1.1. *Let G be a graph with signatures σ_1 and σ_2. If uv is the unique edge of G with $\sigma_1(uv) = -1$ then the signed graph $(G - uv, \sigma_2)$ has the PNC property if and only if G contains a pair of vertex disjoint cycles C_1, C_2 with $\sigma_i(C_i) = -1$.*

Examples

In this section we introduce some families of graphs with a pair of signatures that do not have two vertex disjoint cycles one negative in each signature.

Note that in our illustrations, blue edges represent negative edges in one signature, red edges indicate negative edges in the second signature, and purple edges signify negative edges present in both signatures.

Example 4.1.1. *Let G be a graph with signatures σ_1, σ_2 and assume that uv is the unique negative edge in σ_1. Suppose that $G - uv$ is planar and moreover, in a planar embedding of $G - \{u, v\}$, there are exactly two negative faces, F_1 and F_2 with F_1 containing u and F_2 containing v (see Figure 4.1).*

Observe that in a signed planar graph with exactly two negative faces, say F_1 and F_2, a cycle C is negative if and only if exactly one of F_1, F_2 is inside C. So in the above example every cycle in $G - \{u, v\}$ that is negative in σ_2 separates F_1 and F_2 and therefore intersects every path in $G - uv$ from u to v.

Example 4.1.2. *Let G be a graph with vertex disjoint 3-cycles C_1, C_2, let $u \in V(C_1)$, $v \in V(C_2)$ be adjacent, and assume that $\deg(u) = 3 = \deg(v)$. Let σ_1 be the signature with uv as the unique negative edge and let σ_2 be the signature with negative edge set $E(C_1) \cup E(C_2)$ (see Figure 4.2).*

In this example, every cycle C that is negative in σ_1 contains a path from u to v and therefore uses at least two vertices from $V(C_1)$ ($V(C_2)$). It follows that there cannot exist a cycle disjoint from C that is negative in σ_2.

Example 4.1.3. *Let $G_1, \ldots G_{2k}$ be vertex disjoint graphs and for $1 \leq i \leq 2k$ let $u_i, u'_i, v_i, v'_i \in V(G_i)$ be distinct. Assume that for every odd $1 \leq i \leq 2k$ the graph G_i has a planar embedding with u_i, v_i, v'_i, u'_i on the outer face in this cyclic order. Let G be constructed from the union of $G_1, \ldots, G_{2k}$ by adding the edges $u_i v_{i+1}$ and $u'_i v'_{i+1}$ for all $1 \leq i \leq 2k$ (indices modulo $2k$). Define σ_1 to be the signature with negative edge set $\{u_1 v_2, u'_1 v'_2\}$ and σ_2 to be the signature with negative edge set $\{u_i v_{i+1} \mid 1 \leq i \leq 2k\}$ (see Figure 4.3)*

.

In a graph from the above example consider a cycle C that is negative in σ_1 and observe that C must contain exactly one of the edges $u_i v_{i+1}, u'_i v'_{i+1}$ for every $1 \leq i \leq 2k$. For every

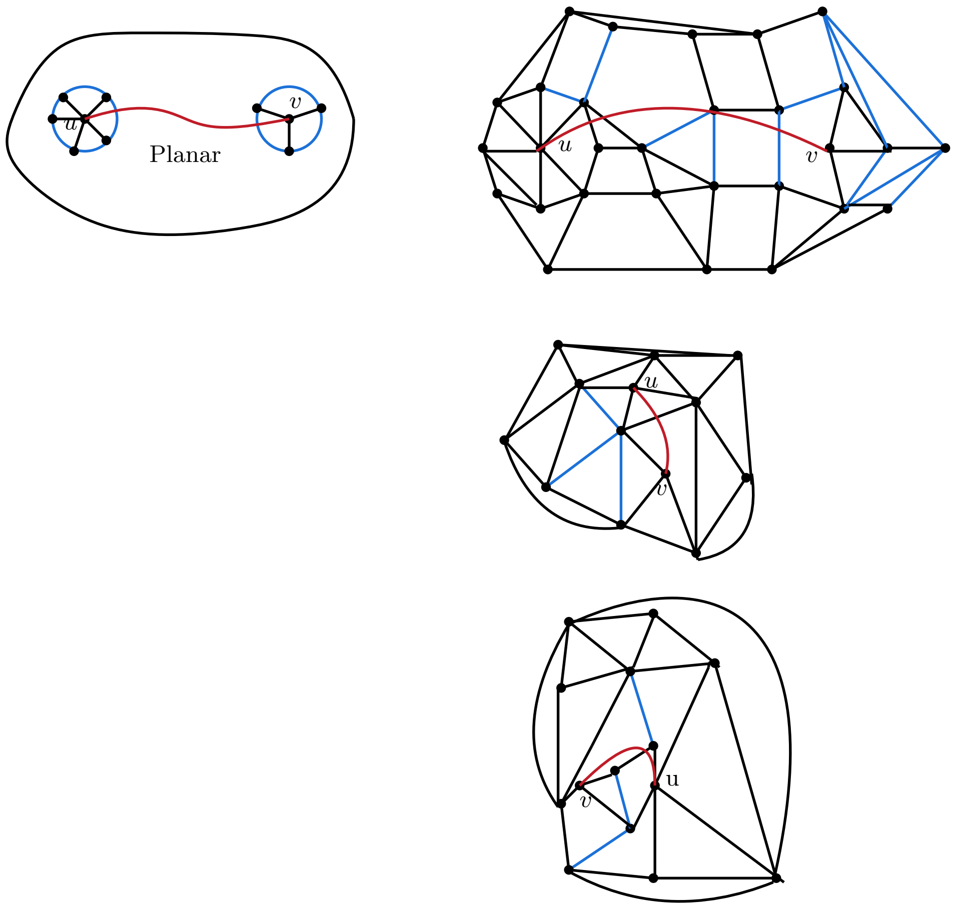

Figure 4.1: Illustrations of examples 4.1.1

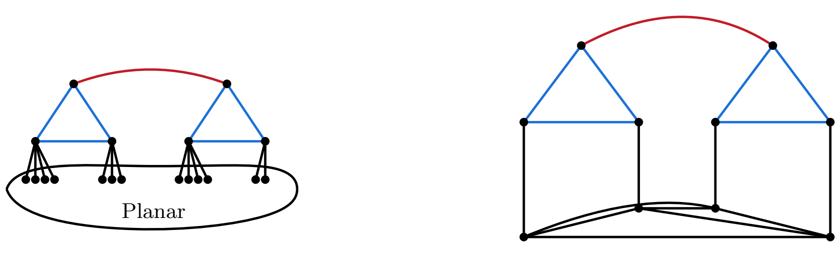

Figure 4.2: Illustrations of examples 4.1.2

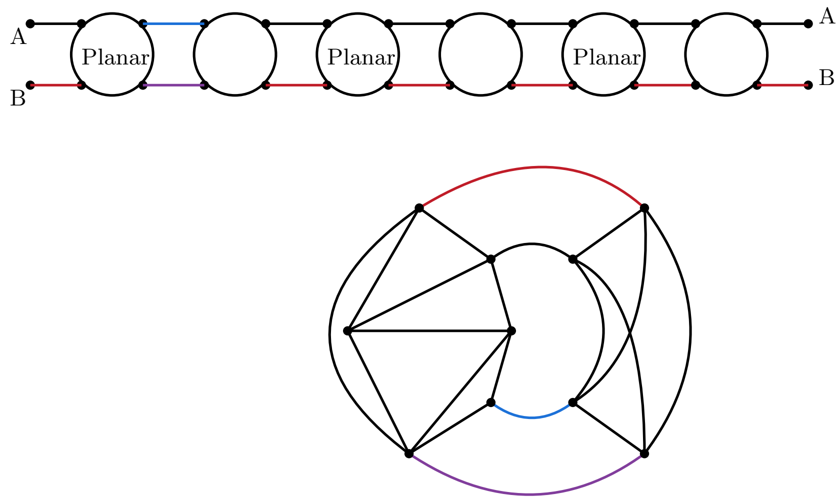

Figure 4.3: Illustrations of examples 4.1.3

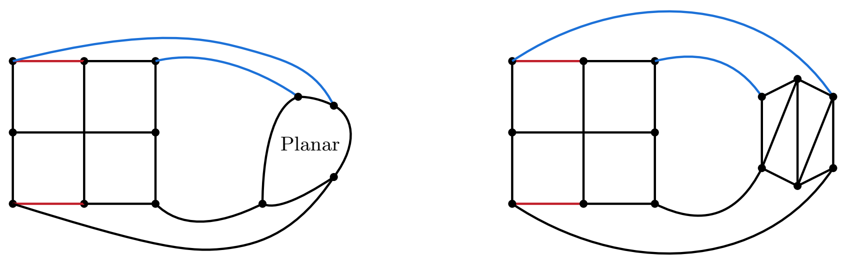

Figure 4.4: Illustrations of examples 4.1.4

odd $1 \leq i \leq 2k$, it follows from the planarity of G_i that every cycle vertex disjoint from C must contain both of $u_{i-1}v_i, u_iv_{i+1}$ or neither. It follows that there cannot exist a cycle disjoint from C that is negative in C_2.

Example 4.1.4. *Let G_1 be an 8 vertex cycle with vertices in cyclic order $v_1, \ldots, v_8$ together with the additional edges v_2v_6 and v_4v_8. Let G_2 be a graph embedded in the plane with four distinct vertices $u_1, \ldots, u_4$ arranged in this cyclic order along the outer face. Assume that G_1 and G_2 are vertex disjoint and then let G be the graph obtained from $G_1 \cup G_2$ by adding the edges v_1u_1, v_3u_2, v_5u_3, and v_7u_4. Let σ_1 be a signature with negative edge set $\{v_1v_2, v_6v_7\}$ and let σ_2 be a signature with negative edge set $\{v_1u_1, v_3u_2\}$ (see Figure 4.4).*

A small case analysis reveals that graphs constructed as in the above example do not contain the desired two vertex disjoint cycles.

Let G be a graph with a signature σ for which all negative edges are incident with the vertex w. Define S^+ (S^-) as the set of positive (negative) edges incident to w. We say w is

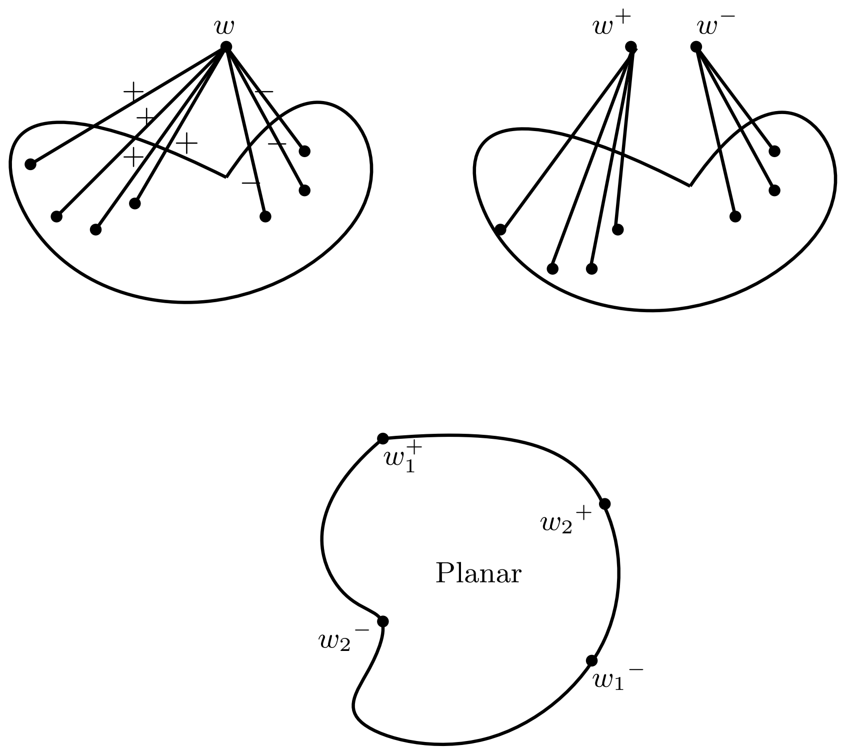

Figure 4.5: Illustrations of examples 4.1.5

a *balanced vertex*. We modify G to form a new signed graph G' as follows: We introduce two new vertices, w^+ and w^-, and modify each edge in S^+ (S^-) by replacing the endpoint w with w^+ (w^-). Subsequently, we remove the vertex w (which becomes isolated at then end of the process). We call G' a *σ-vertex-split* of G. Note that a negative cycle in G corresponds to a path in G' from w^- to w^+.

Example 4.1.5. *Let G have signatures σ_1 and σ_2. Let w_1, w_2 be distinct vertices and assume that all negative edges of σ_i are incident with w_i. Let G' be obtained from G by doing a σ_1-vertex-split (at w_1) and then let G'' be obtained from G' by doing a σ_2-vertex-split (at w_2). Assume that G'' can be embedded in the plane such that ${w_1}^+$, ${w_2}^+$, ${w_1}^-$, and ${w_2}^-$ appear consecutively in this order along the outer face (see Figure 4.5).*

In this case G'' does not have two vertex disjoint paths one from w_1^- to w_1^+ and the other from w_2^- to w_2^+ and it follows that the original graph G does not have the desired disjoint negative cycles.

4.2 Using a negative K_4

Now we begin toward the proof of our theorem characterizing when a signed graph has the PNC property. In this section we will show how to use a certain subgraph to deduce this property.

First some definitions. If G is a graph and $H \subseteq G$, a *bridge* of H is one of the following:

- A subgraph $F \subseteq G$ isomorphic to K_2 with $V(F) \subseteq V(H)$ and $E(F) \cap E(H) = \emptyset$.
- A subgraph of G obtained by taking a component of $G - V(H)$ together with every edge joining a vertex of this component to a vertex in $V(H)$ (with its end vertices).

Note that $E(G)$ is the disjoint union of $E(H)$ and the edge sets of its bridges. If F is a bridge of H we call $V(H) \cap V(F)$ *attachment* vertices. For the next lemma there is a particular signed graph of interest. If G is a signed graph a subgraph $H \subseteq G$ is called a *negative* K_4 if H is a subdivision of K_4 and every cycle of H that corresponds to a 3-cycle in K_4 is negative.

In the figures below, blue edges represent a negative cycle in one signature, while red edges indicate disjoint uv-paths.

Lemma 4.2.1. *Let G be a signed 3-connected graph with distinguished vertices u, v. If G has a negative K_4 vertex disjoint from $\{u, v\}$, then G has the PNC property with respect to u, v.*

Proof. Begin by choosing a negative K_4 subgraph called H, vertex disjoint from $\{u, v\}$ so as to maximize the size of the bridge containing u. Let B_u (B_v) be the bridge of H containing u (v). If $B_u = B_v$ then this bridge contains a u, v path that is vertex disjoint from a negative cycle in H and the proof is complete, so we may assume $B_u \neq B_v$.

Let $v_1, \ldots, v_4$ be the degree 3 vertices in H and for $1 \leq i \leq 4$ let C_i be the unique cycle in H not containing v_i and note that C_i is negative. If there exists $1 \leq i \leq 4$ so that B_u and B_v both have attachments not contained in $V(C_i)$, then we have the desired path from u to v disjoint from C_i. Accordingly we may assume no such i exists.

For $1 \leq i < j \leq 4$ let P_{ij} be the unique path in H from v_i to v_j with $V(P_{ij}) \cap \{v_1, \ldots, v_4\} = \{v_i, v_j\}$. First suppose that B_u has an attachment vertex in the interior of

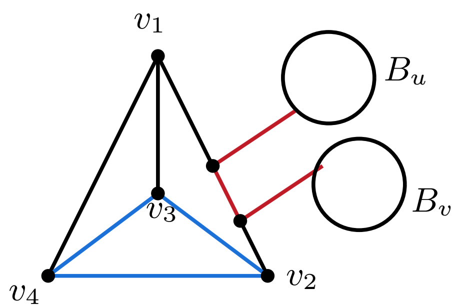

Figure 4.6: B_u and B_v both have attachments not contained in $V(C_1)$.

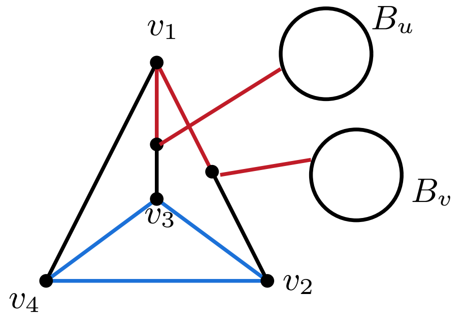

Figure 4.7: B_u and B_v both have attachments not contained in $V(C_1)$.

the path P_{13}. In this case, all attachments of B_v must lie in the path P_{24} (or there would be a cycle C_i not containing an attachment vertex of B_u and not containing an attachment vertex of B_v). Since B_v has at least three attachment vertices, this means that B_v must contain an attachment vertex in the interior of P_{24}. Observe that a similar argument works with B_u and B_v interchanged. In particular, if one of B_u or B_v has an "interior" attachment vertex, then the other one does too.

If the only attachment vertices of both B_u and B_v lie in $\{v_1, \ldots, v_4\}$, then it follows from the fact that B_u and B_v each have at least 3 attachment vertices that there exists $1 \leq i \leq 4$ so that v_i is an attachment of both B_u and B_v. But in this case there exists a path from u to v vertex disjoint from C_i and this completes the proof. By the above argument, we may now assume (by possibly relabelling) that all attachment vertices of B_u lie in P_{13} and all attachment vertices of B_v lie in P_{24}.

Let Q_{13} (Q_{24}) be the unique minimal subpath of P_{13} (P_{24}) containing all attachment vertices of B_u (B_v). Deleting the ends of the path Q_{13} from the graph cannot disconnect $B_u \cup Q_{13}$ with the rest of the graph and it follows that there exists another bridge B of H with

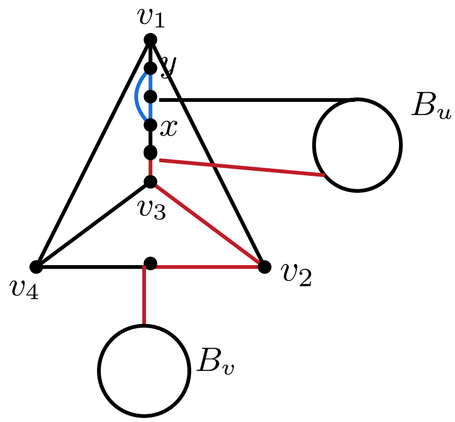

Figure 4.8: Case 1: y is in P_{13}.

an attachment x in the interior of Q_{13} and another attachment vertex $y \in V(H) \setminus V(Q_{13})$. Choose a path $R \subseteq B$ from x to y internally disjoint from $V(H)$. Now we consider cases.

Case 1: $y \in V(P_{13})$

If the unique cycle in $P_{13} \cup R$ is negative, then this negative cycle is disjoint from a path from u to v completing the proof. Otherwise, we can modify H to form another negative K_4, call it H', by replacing the subpath of P_{13} from x to y with the path R. Now H' contradicts the choice of H as the bridge containing u has increased in size.

Case 2: y is in the interior of P_{24}.

Since B_v has at least 3 attachment vertices, we may choose a subpath R' of P_{24} from y to v_3 or v_4 so that B_v has an attachment vertex not in $V(R')$. By possibly relabelling, we may assume R' has ends v_4, y. Observe that the graph $(H \cup R) \setminus \{v_2\}$ has exactly two cycles that contain the path R, say C and C'. Since $E(C) \bigoplus E(C') = E(C_2)$ exactly one of C or C' is negative and in either case we have a negative cycle disjoint from a u, v path as desired.

Case 3: y is not in P_{13} or the interior of P_{24}.

In this case the ends of the path R are either both in $V(C_2)$ or both in $V(C_4)$ and by relabelling we may assume the former case. Now $C_2 \cup R$ has two cycles containing R exactly

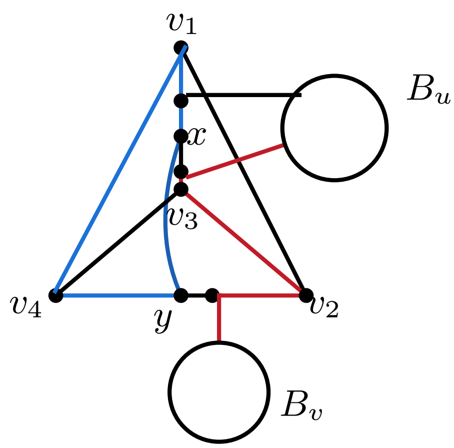

Figure 4.9: Case 2: y is in the interior of P_{24}.

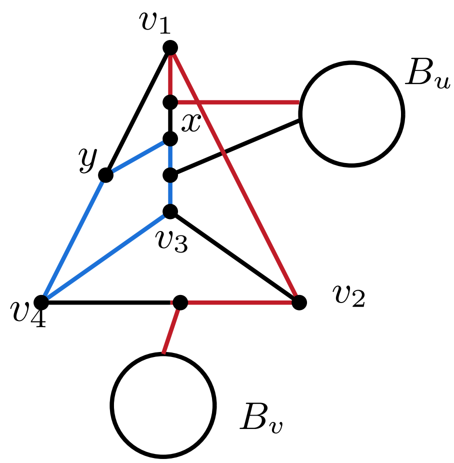

Figure 4.10: Case 3: y is not in P_{13} or the interior of P_{24}.

one of which is negative and this cycle is disjoint from a u, v path thus completing the proof.

□

4.3 Preparations

In this section we prove some lemmas that will be required for characterizing the PNC property.

Lemma 4.3.1. *Let G be a 3-connected signed graph and let (G_1, G_2) be a 3-separation of G with $|V(G_2) \setminus V(G_1)| \geq 1$. If G_2 is not balanced, then there exists a negative cycle $C \subseteq G_2$ so that $|V(G_1) \cap V(C)| \leq 2$.*

Proof. Let $X = V(G_1) \cap V(G_2)$ and choose a negative cycle $C \subseteq G_2$. We may assume $X \subseteq V(C)$ as otherwise we are done. If $V(C) = X$ [21] we may choose a vertex $v \in V(G_2) \backslash X$ and three internally disjoint paths starting at v and ending at distinct vertices in X and the subgraph consisting of C and these paths contains a negative cycle not containing all of X. So we may assume that $V(C)$ contains a vertex $z \notin X$. Now let $Q \subseteq C$ be the unique maximal path of C containing z and not containing a vertex of X in the interior. Let Z be the set of ends of Q and note that $Z \subseteq X$. By 3-connectivity $G \setminus Z$ contains a path between the two components of $C - Z$ and it follows from applying the theta property to C and this path that G_2 contains the desired negative cycle. □

Our next lemma will handle a special case in the proof when $G - \{u, v\}$ is planar.

Lemma 4.3.2. *Let G be a 3-connected signed graph with distinguished vertices u, v. Assume that $G' = G \setminus \{u, v\}$ is a subdivision of a 3-connected planar graph. Then either G has the PNC property or one of the following holds:*

1. *$G - \{u, v\}$ is balanced, or*

2. *G is planar and every negative face is incident with u or v.*

Proof. Suppose first that C is a negative cycle bounding a face in our planar embedding of G'. If there exists a vertex u' adjacent to u and v' adjacent to v with $u', v' \notin V(C)$ then our connectivity implies that $G' \setminus V(C)$ is connected and it follows that G has the PNC property. So it must be that every negative cycle bounding a face contains either all neighbours of u or all neighbours of v. Note that by parity, the total number of negative faces is even. If this number is zero, then $G - \{u, v\}$ is balanced and we are done. By the above analysis, the only other possibility is that G' has exactly two negative faces, one of which contains all neighbours of u and the other contains all neighbours of v. In this case the original graph G is planar with every negative face incident to u or v. □

Before we prove the next lemma we state a strong form of the 2-linkage theorem that we require.

Theorem 4.3.1 (Thomassen's 2-Linkage Theorem (strong form)). *[19] Let G be a graph with distinct vertices s_1, s_2, t_1, t_2 and assume that G does not contain a 3-separation (G_1, G_2) with $s_1, s_2, t_1, t_2 \in V(G_1)$ and $|V(G_2) \backslash V(G_1)| \geq 2$. If G does not contain a $\{\{s_1, t_1\}, \{s_2, t_2\}\}$-linkage, then G can be embedded in the plane with the vertices s_1, s_2, t_1, t_2 appearing on the outer face in this order.*

Our next lemma handles another key case in the proof of our main theorem.

Lemma 4.3.3. *Let G be a 3-connected signed graph with distinguished vertices u, v and assume the following:*

- *$G \setminus \{u, v\}$ is an unbalanced graph with a balancing vertex w,*
- *There does not exist a 3-separation (G_1, G_2) of G with $u, v, w \in V(G_1)$ and $|V(G_2) \setminus V(G_1)| \geq 2$.*

Then either G has the PNC property, or G is planar and every negative face is incident with u or v.

Proof. By possibly resigning the graph G we may assume that our signature σ has the property that all negative edges are incident with at least one of u, v, w. Further modify σ by changing the sign of every edge incident with u or v to be positive (and note that this does not affect the PNC property). Let G' be obtained from G by doing a σ split that splits the vertex w into w^+ and w^-. If the graph G' has a $\{\{u, v\}, \{w^+, w^-\}\}$-linkage then G has the PNC property. Otherwise it follows from Theorem 4.3.1 that G' can be embedded in the plane with u, w^+, v, w^- appearing in this order on the boundary of the infinite face. In this case the graph G is also planar with exactly two negative faces, one containing u and the other v. □

4.4 Reductions

Our main theorem from this section requires the introduction of numerous reductions that move from a 3-connected signed graph G to a smaller 3-connected graph G'. In all cases these reductions will not effect the PNC property.

For the purpose of our reductions we will assume that G is a signed 3-connected graph with two distinguished vertices u and v and we will modify preserving these features. We begin by defining the first reduction.

Definition 4.4.1 (Reduction R1)**.** *Let G be a signed 3-connected graph with distinguished vertices u, v and let (G_1, G_2) be a 3-separation of G satisfying the following properties:*

- $u, v \in V(G_1)$
- $|V(G_2) \setminus V(G_1)| \geq 2$
- *All edges in $E(G_2 \setminus \{u, v\})$ are positive*

Let G' be the signed graph obtained from G_1 by adding a new vertex w and positive edges joining w to all three vertices in $V(G_1) \cap V(G_2)$. We call G' an R1 reduction *of G.*

Next we prove that this reduction preserves the PNC property.

Lemma 4.4.1 (First reduction)**.** *Let (G, σ) be a signed 3-connected graph with distinguished vertices u, v and let (G_1, G_2) be a 3-separation of G with $\{u, v\} \subseteq V(G_1)$ and $|V(G_2) \setminus V(G_1)| \geq 2$. Then*

1. *If $G_2 \setminus \{u, v\}$ is unbalanced, then G has the PNC property.*
2. *If $G_2 \setminus \{u, v\}$ is balanced, then by possibly resigning G we may assume that all edges in this subgraph are positive and perform an R1 reduction to get the graph (G', σ'). Then G has the PNC property if and only if G' has the PNC property.*

Proof. First assume that $G_2' = G_2 \setminus \{u, v\}$ contains a negative cycle C. Note that by Lemma 4.3.1 we may assume that C contains at most two vertices from the set $X = V(G_1) \cap V(G_2)$. It now follows from the 3-connectivity of G that there exists a path P from u to v vertex disjoint from C thus giving the PNC property.

For the second part of the proof we assume that G_2' is balanced and carry out the reduction defined above to form G'. It is immediate from our assumptions that if G satisfies the PNC property with path P and cycle C then one of $E(P) \cap E(G_2)$ or $E(C) \cap E(G_2)$ is empty. It follows immediately from this that G has the PNC property if and only if G' does. □

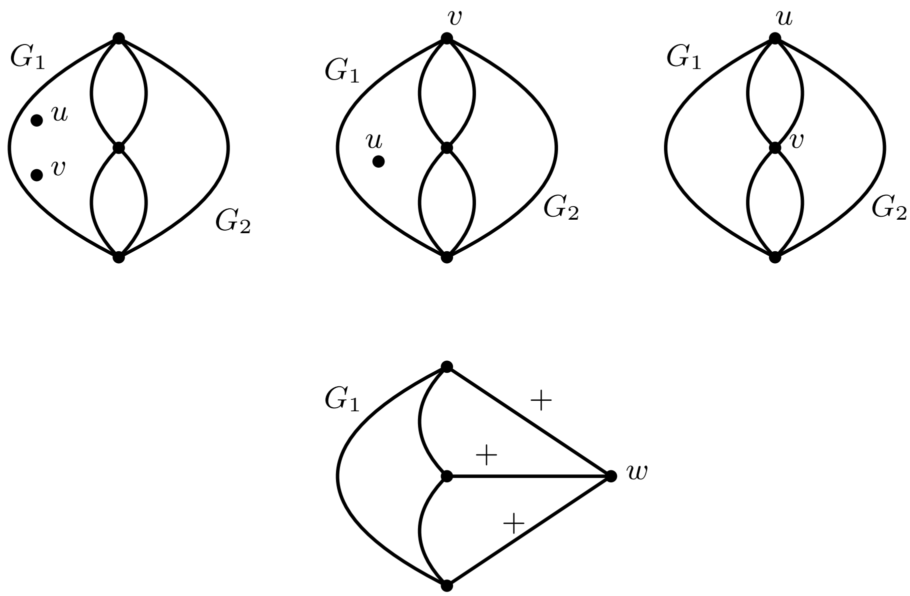

Figure 4.11: R1 - First Reduction

Definition 4.4.2 (Reduction R2)**.** *Let G be a signed 3-connected graph with distinguished vertices u, v and let (G_1, G_2) be a 3-separation of G satisfying the following properties:*

- $u \in V(G_1) \setminus V(G_2)$ *and* $v \in V(G_2) \setminus V(G_1)$
- $|V(G_2) \setminus V(G_1)| \geq 2$
- *All edges in $E(G_2 \setminus v)$ are positive.*

Let G' be the signed graph obtained from G_1 by adding a new vertex v and positive edges joining v to all three vertices in $X = V(G_1) \cap V(G_2)$. In addition, for every $x \in X$, if G_2 contains a $\{\{x, v\}, X \setminus \{x\}\}$ linkage, then we add to G' a positive edge with ends $X \setminus \{x\}$. We call G' an R2 reduction of G.

Let us comment that the strong form of Thomassen's 2-linkage theorem gives a complete characterization of when any linkage of size 2 does not exist and thanks to this we can give a precise structural description of the behaviour of the R2 reduction.

Lemma 4.4.2 (Second reduction)**.** *Let G be a signed 3-connected graph with distinguished vertices u, v. If G' is an R2 reduction of G, then G has the PNC property if and only if G' has the PNC property.*

Proof. Let (G_1, G_2) be the 3-separation on which the R2 reduction is performed and let $X = V(G_1) \cap V(G_2)$. First suppose that G' has the PNC property with path P and cycle C. If C does not contain any edge with both ends in X, then C is a negative cycle in G and it follows from 3-connectivity that C is disjoint from a u, v path in G giving the PNC property. If C contains an edge xx' with $x, x' \in X$, then P must contain the unique vertex in $X \setminus \{x, x'\}$ say x''. It now follows from our construction that G_2 contains the linkage $\{\{v, x''\}, \{x, x'\}\}$ and it follows that G has the PNC property.

Next suppose that G has the PNC property with path P and cycle C. If $C \subseteq G_1$ then it is immediate that G' will also have the PNC property using the cycle C. Otherwise C must contain exactly two vertices, say $x, x' \in X$ and P must contain the unique remaining vertex in $X \setminus \{x, x'\}$, say x''. It follows from this that G_2 contains the linkage $\{\{x, x'\}, \{v, x''\}\}$ so G' contains the positive edge xx'. Since all edges of $G' \setminus v$ are positive, we can reroute the negative cycle $C \subseteq G$ to a new negative cycle using xx' and it follows that G' has the PNC property. □

Definition 4.4.3 (Reduction R3)**.** *Let G be a signed 3-connected graph with distinguished vertices u, v, let (G_1, G_2) be a 4-separation of G with $X = V(G_1) \cap V(G_2)$ satisfying the following properties:*

- $u \in V(G_1) \setminus V(G_2)$ *and* $v \in V(G_1) \cap V(G_2)$
- $|V(G_2) \setminus V(G_1)| \geq 2$
- *At least one component H of $G_2 \setminus V(G_1)$ has all of X as neighbours of $V(H)$.*
- *All edges in $E(G_2 \setminus v)$ are positive.*

Let G' be the signed graph obtained from G_1 by adding a new vertex w and positive edges joining w to all four vertices in $X = V(G_1) \cap V(G_2)$. In addition, for every $x \in X$, if G_2 contains a $\{\{x, v\}, X \setminus \{v, x\}\}$ linkage, then we add to G' a positive edges with ends $X \setminus \{v, x\}$ and a positive edge with ends $\{v, x\}$. We call G' an R3 reduction of G.

Next we show that this reduction works as desired.

Lemma 4.4.3 (Third reduction). *Let G be a signed 3-connected graph with distinguished vertices u, v. If G' is an R3 reduction of G, then G has the PNC property if and only if G' has the PNC property.*

Proof. We omit this argument since it is nearly identical to that for the R2 reduction. The only cases of interest concern the PNC property with a path P and cycle C where $C \nsubseteq G_1$. However in all such cases we can move between G and G' by interchanging edges with both ends in X with suitable paths in G_2. □

4.5 Characterizing the PNC property

Our main goal for this subsection is to complete our characterization of the PNC property for 3-connected graphs.

Here is our theorem proved jointly with Devos, Nurse, and Mohar.

Theorem 4.5.1. *Let G be a 3-connected signed graph with distinguished vertices u, v. Either G has the PNC property or one of the following holds:*

1. *$G - \{u, v\}$ is balanced, or*
2. *After applying reductions (R1) and (R2) the graph G is planar and all negative faces are incident with either u or v.*

The proof of our main theorem relies upon another structure theorem for signed graphs due to Lov'asz, Seymour, Schrijver, and Truemper and written by Gerards.

Theorem 4.5.2. *Let G be a simple signed 3-connected graph with no negative K_4. Then G is either balanced, has a balancing vertex, or G can be reduced to a planar graph with exactly two negative faces by the following operation:*

- *If (G_1, G_2) is a 3-separation and G_2 is a balanced graph with $|V(G_2) \setminus V(G_1)| \geq 2$ then resign G so that G_2 is positive and replace G by the graph obtained from G_1 by adding a new vertex w and positive edges joining w to every vertex in $V(G_1) \cap V(G_2)$.*

Proof of Theorem 4.5.1. Suppose (for a contradiction) that G is a counterexample to the theorem with $|V(G)|$ minimum. We proceed to establish properties of G in steps.

(1) G does not have a 3-separation (G_1, G_2) with $u, v \in V(G_1)$ and $|V(G_2) \setminus V(G_1)| \geq 2$.

In this case it follows from Lemma 4.4.1 that either G has the PNC property and we are done, or $G_2 \setminus \{u, v\}$ is balanced and we can perform reduction (R1) transforming G into the smaller graph G'. By the minimality of our counterexample, G' must satisfy the theorem. If G' has the PNC property, then so does G (by Lemma 4.4.1) which is contradictory. Otherwise G' reduces by (R1) and (R2) to a planar graph with the structure in the theorem statement, but then so does G. This proves (1).

(2) $G - \{u, v\}$ does not have a balancing vertex.

Suppose (for a contradiction) that w is a balancing vertex in $G - \{u, v\}$. In this case it follows from Lemma 4.3.3 that the theorem holds for G.

(3) G does not have a vertex w adjacent to both u and v

In this case (2) implies that $G - \{u, v, w\}$ contains a negative cycle C, but then we the three vertex path given by u, w, v shows that G has the PNC property.

(4) G does not have a 4-separation (G_1, G_2) with $u, v \in V(G_1) \cap V(G_2)$ so that $|V(G_1) \setminus V(G_2)|, |V(G_2) \setminus V(G_1)| \geq 2$.

Suppose (for a contradiction) that (G_1, G_2) is a 4-separation violating (4). Let $w \in (V(G_1) \cap V(G_2)) \setminus \{u, v\}$ and note that by (2) the graph $G - \{u, v, w\}$ must contain a negative cycle C. Without loss of generality we may assume $C \subseteq G_1$. It now follows from our connectivity that G_2 contains a path P from u to v and thus G has the PNC property. This proves (4).

(5) The graph G is not planar

Suppose (for a contradiction) that G is planar. Then it follows from (1) (3), and (4) that $G - \{u, v\}$ is a subdivision of a 3-connected graph. But now Lemma 4.3.2 implies that G satisfies the theorem.

(6) G does not have an internal 3-separation (G_1, G_2) with $u \in V(G_1) \setminus V(G_2)$ and $v \in V(G_2) \setminus V(G_1)$.

Suppose (for a contradiction) that (G_1, G_2) is such a 3-separation and let $X = V(G_1) \cap V(G_2)$. Form G_1^* from G_1 by adding the vertex v and positive edges between v and all vertices in X. Similarly, form G_2^* from G_2 by adding the vertex u and positive edges between u and all vertices in X. If G_i^* has the PNC property for some $i = 1, 2$ then it follows from our connectivity that G also satisfies the PNC property, which is a contradiction. Next suppose $G_2^* \setminus \{u, v\} = G_2 \setminus \{v\}$ is balanced. In this case we may resign G so that all edges in this subgraph are positive and then apply reduction R2 to G to obtain the new graph G'. It follows from the minimality of our counterexample that the theorem holds for G'. If G' has the PNC property, then so does G by Lemma 4.4.2. Otherwise G' can be reduced by R1 and R2 to a planar graph with all negative faces incident with u or v, but then this conclusion also holds for G. This proves (6).

(7) G does not have a 4-separation (G_1, G_2) with $u \in V(G_1)\setminus V(G_2)$ and $v \in V(G_1) \cap V(G_2)$ and $G_2 \setminus V(G_1)$ a connected graph of size ≥ 2.

Suppose (for a contradiction) that such a 4-separation exists. If $G_2 \setminus \{v\}$ contains a negative cycle, then it follows from our connectivity and Lemma 4.3.1 that $G_2 \setminus \{v\}$ contains such negative cycle C using at most two vertices in $V(G_1) \cap V(G_2)$. But then we can find a path in $G_1 \setminus V(C)$ from u to v thus giving G the PNC property. Therefore $G_2 \setminus \{v\}$ is balanced, and by resigning G we may assume these edges are positive. Now apply an R3 reduction to G forming the graph G'. By the minimality of our counterexample, the theorem holds for G'. If G' has the PNC property, then Lemma 4.4.3 implies that G has it too. Otherwise G' can be reduced by R1 and R2 reductions to a planar graph with the stated properties. Now consider this R3 reduction that was performed. Let $X = V(G_1) \cap V(G_2)$ and let x be the newly added vertex when doing R3. Since G' reduces by R1 and R2 reductions to a planar graph, the subgraph induced by $X \cup \{x\}$ is not K_5. It follows from this that at least one of the 2-linkages problems for G_2 given by partitioning X into two sets of size two cannot be solved. It follows from this that G_2 is planar, and moreover the embedding of G' can be extended to give a planar embedding of G. So the R3 reduction was not actually necessary to get the desired structure.

(8) The graph $G - \{u, v\}$ is a subdivision of a 3-connected and internally 4-connected graph.

It follows from (1), (3), and (4) that $G - \{u, v\}$ is a subdivision of a 3-connected graph. If this graph is not internally 4-connected, consider a violating 3-separation, and extend this to a 5-separation in the original graph G of the form (G_1, G_2) with $u, v \in V(G_1) \cap V(G_2)$. Note that both u and v have neighbours in both $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ by (7). First suppose that $G_2 \setminus \{u, v\}$ contains a negative cycle C. Then by Lemma 4.3.1 we may choose such a negative cycle containing at most two vertices from $V(G_1) \cap V(G_2)$. Now our assumptions imply the existence of a path $P \subseteq V(G_1) \setminus V(C)$ from u to v and thus G has the PNC property. A similar argument shows that $G_1 \setminus \{u, v\}$ is balanced. First resign G so that all edges in $G_1 \setminus \{u, v\}$ are positive and then resign G so that all edges in $G_2 \setminus \{u, v\}$ are positive. This second resigning only uses vertices in $V(G_2)$ moreover, we can do this resigning using at most one vertex in $(V(G_1) \cap V(G_2)) \setminus \{u, v\}$. To see this, note that if more vertices were used, we could achieve the same result by switching at the complementary set of vertices. It now follows that G has a balancing vertex, but this contradicts (2).

With the last property in place we are ready to complete the proof. Let G' be the graph obtained from $G - \{u, v\}$ by suppressing degree 2 vertices. If G' has a pair of parallel edges forming a negative cycle, then this negative cycle is disjoint from a u, v path and G has the PNC property. So we may assume that if G' has any parallel edges, they have the same sign, and then we modify G' to form G'' by removing any such extra edges.

Now apply Theorem 4.5.2 to G''. If this graph is balanced or has a balancing vertex then we are done. If G'' contains a negative K_4 then Lemma 4.2.1 implies that G satisfies the PNC property. In the remaining case $G - \{u, v\}$ is planar and Lemma 4.3.2 implies the result. $\square$

Chapter 5

Decomposition of signed graphs

A *decomposition* of a graph G is a list of subgraphs $H_1, \ldots, H_k$ so that $\{E(H_1), \ldots, E(H_k)\}$ is a partition of $E(G)$. It is elementary that a graph G has a decomposition into cycles if and only if every vertex of G has even degree. In this chapter we will explore the following more complicated notion.

Definition 5.0.1. *A signed graph (G, σ) has a* positive cycle decomposition *if there is a decomposition of G given by $C_1, \ldots, C_k$ where C_i is a positive cycle of G for every $1 \leq i \leq k$.*

Our main result from this section is an extension of a well-known theorem of Seymour on positive cycle decompositions of planar graphs.

5.1 Positive-cycle decomposition of graphs

For a graph G and its associated negative signed graph $(G, -)$, we see that a cycle $C \subseteq G$ is odd if and only if it is negative in $(G, -)$. So a positive cycle decomposition of $(G, -)$ corresponds to a decomposition of G into even cycles. On the other hand, if (H, σ) is a signed graph, we can modify the graph H to form H' by subdividing every positive edge of H. Now H' is an (unsigned) graph and a decomposition of H' into even cycles corresponds to a positive cycle decomposition of (H, σ). So positive cycle decomposition problems are equivalent to even cycle decomposition problems. Indeed, this is a common in the world of signed graphs. We prefer the setting of signed graphs since in many cases the connectivity of the signed graph is a key parameter that is more natural to work with in this setting.

If (G, σ) is a signed graph that has a positive cycle decomposition, then every block of G must contain an even number of negative edges. Seymour proved that this necessary condition is also sufficient for planar graphs.

Theorem 5.1.1. *(Seymour's Theorem, see [17]) Let (G, σ) be a signed planar graph. Then (G, σ) has a positive cycle decomposition if and only if all vertices of G have even degree and every block of G has an even number of negative edges.*

The following theorem shows that this result does not hold true for all graphs.

Theorem 5.1.2. *[11] There exists a 2-connected Eulerian negative-K_5-minor-free signed graph with an even number of negative edges which does not have a positive cycle decomposition.*

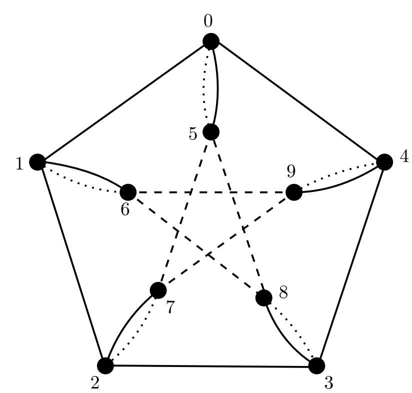

Figure 5.1: 2-connected Eulerian loopless negative-K_5-minor-free signed graph with an even number of negative edges which is not balanced-cycle decomposable.

Proof of Theorem 5.1.2. We claim that the signed graph (G, σ) in Figure 5.1 is such a signed graph. Evidently, G is 2-connected, Eulerian and loopless. We claim that (G, σ) does not have a positive cycle decomposition.

Suppose (for a contradiction) that a positive cycle decomposition of (G, σ) exists and let C be an arbitrary cycle in this decomposition. Note that G only contains cycles of length 2, 5, 6, 8, or 9. Since all 2-cycles are unbalanced, C must have a length of 5, 6, 8, or 9.

It is easy to check that if C has a length of 5 or 6, then some block of $(G - E(C), \sigma - E(C))$ is a negative 2-cycle, leading to a contradiction. Thus, C must have a length of 8 or 9.

However, since $|E(G)| = 20$, we can observe that $8a + 9b \neq 20$ for all non-negative integers a and b.

Now, we need to show that (G, σ) does not contain an negative-K_5-minor. By degree considerations, the only way to obtain a K_5-minor from the Petersen graph is to contract a perfect matching. Therefore, to obtain an negative-K_5 from (G, σ), we must delete exactly one edge from each 2-cycle and then contract a perfect matching M.

Let C be the balanced 5-cycle 04321. It is easy to see that $|M \cap E(C)| \in \{0, 2\}$. Therefore, our negative-K_5-minor either contains a balanced 5-cycle or a balanced 3-cycle, which leads to a contradiction. □

Theorem 5.1.3. *[11] Every signed loopless 2-connected Eulerian negative-K_4-minor-free graph with an even number of negative edges has a positive cycle decomposition.*

5.2 Removable positive cycles

Next we introduce some concepts that we will use to find positive cycle decompositions. A cycle C in a graph G is called *removable* if $G - E(C)$ is 2-connected. Now assume that (G, σ) is a signed graph. We say that a subgraph $H \subseteq G$ is a *balanced sausage* if the following properties hold:

- H can be obtained from a path of length ≥ 2 by adding a second copy of every edge.
- All cycles of H are positive
- Every $v \in V(H)$ with $deg_H(v) = 4$ satisfies $deg_G(v) = 4$

If H is a balanced sausage, then we call it *removable* if the graph obtained from G by deleting $\{v \in V(G) \mid \deg_H(v) = 4\}$ is 2-connected. Our main result in this chapter is the following theorem showing the existence of removable positive cycles and balanced sausages.

Here is our theorem proved jointly with Devos, Mohar, Wang, and Nurse.

Theorem 5.2.1. *Let (G, σ) be a signed 2-connected Eulerian planar graph with minimum degree ≥ 4 and let $e \in E(G)$ be distinguished. Then there exists either a positive removable cycle C with $e \notin E(C)$ or a removable balanced sausage H with $e \notin E(H)$.*

In fact, the above theorem implies Seymour's Theorem. Suppose we are given a 2-connected Eulerian planar graph (G, σ) with an even number of negative edges. If G has a vertex of degree 2, then by possibly resigning, we may assume this vertex is incident with a positive edge and contract it. Repeat until all vertices have degree ≥ 4. Now apply the above theorem to choose either a removable cycle or sausage. After removing it, we repeat the above process to get rid of degree 2 vertices, then apply the theorem again and so on.

Proof. Suppose that it is false and consider a minimum counterexample (G, σ). We establish properties of G in steps.

(1) (G, σ) does not contain a balanced sausage H with $e \notin E(H)$.

Suppose (for a contradiction) that G contains a balanced sausage H. Let u_1, u_2 be the two vertices in H with $deg_H(u_i) = 2$ and let $S = V(H) \setminus \{u_1, u_2\}$ (note that $S \neq \emptyset$ by definition). By possibly resigning G we may assume that all edges of H are positive. Now modify G to form the signed graph G' by deleting S and then adding two positive edges in parallel with ends u_1, u_2. By the minimality of our counterexample G' contains a removable positive cycle or removable balanced sausage and in either case this immediately implies that G has the same property.

(2) G does not contain a parallel class of size ≥ 3

If (2) were violated, this parallel class would contain a balanced removable cycle by theta property.

(3) G does not have a proper 2-separation (G_1, G_2) so that G_1, G_2 have odd degree vertices.

Suppose (for a contradiction) that such a separation exists and choose one (G_1, G_2) so that $e \in E(G_1)$ and so that G_2 is minimal. Letting $\{x, y\} = V(G_1) \cap V(G_2)$ it then follows that x and y both have odd degree in both G_1 and G_2, and by the minimality of G_2 (and (2)) it must be that $deg_{G_2}(x), deg_{G_2}(y) \geq 3$. The theorem holds for the graph obtained from G_2 by adding the distinguished edge xy and it follows that the result holds for G.

(4) G does not have a proper 2-separation (G_1, G_2) so that G_1 and G_2 are Eulerian.

Suppose (for a contradiction) that such a separation (G_1, G_2) exists and let $\{x, y\} = V(G_1) \cap V(G_2)$. Note that since x and y have even degree in both graphs, they must have

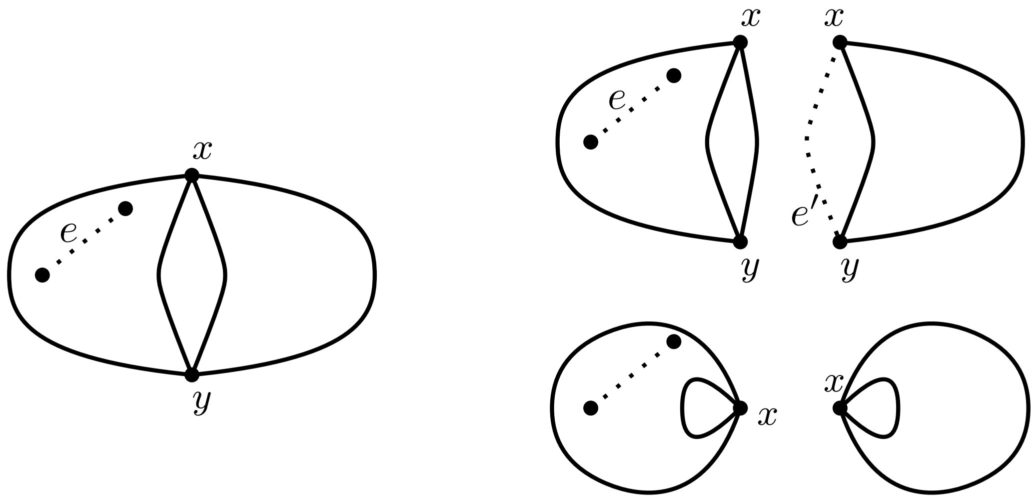

Figure 5.2: Operations on part (3) and (4)

degree at least two in both G_1 and G_2. For $i = 1, 2$ let G'_i be the graph obtained from G_i by identifying the vertices x and y. By induction we may apply the theorem to both G'_1 and G'_2 using the distinguished edge e or an arbitrary edge. Let C_1, C_2 be the two removable cycles from G'_1, G'_2. If one of C_1 or C_2 does not contain the vertex obtained by identifying x and y, then this is a removable cycle in G and we are done. Otherwise, in the original graph G the cycle C_i corresponds to an x, y path $P_i \subseteq G_i$. In this case the cycle $P_1 \cup P_2$ is a positive removable cycle.

Based on (3) and (4) it must be that G is 3-connected. By possibly changing the sign of the edge e we may assume that G has an even number of negative edges. Now apply Seymour's theorem to choose a decomposition of (G, σ) into positive cycles $C_0, C_1, \ldots, C_k$ and assume that $e \in E(C_0)$. Note that C_0 may be positive or negative in the original signature, but every C_i with $1 \leq i \leq k$ is positive. We will show that one of these cycles is removable. Choose $1 \leq j \leq k$ as follows:

- Maximize the block B of $G - E(C_j)$ containing C_0.
- Maximize the component of $G - E(C_j)$ containing C_0.

We claim that according to this choice the graph $G - E(C_j)$ is 2-connected. Suppose this is false. If $G - E(C_j)$ is not connected and H is a component not containing C_0 then choose any $1 \leq i \leq k$ with $C_i \subseteq H$. Now C_i contradicts the choice of j. So it must be that

$G - E(C_j)$ is connected. Suppose this graph is not 2-connected and let $x_1, \ldots, x_t$ be the cut vertices of $G - E(C_j)$ contained in $V(B)$. For every $1 \leq i \leq t$ choose a 1-separation (H_i, H_i') of $G - E(C_j)$ so that $V(H_i) \cap V(H_i') = \{x_i\}$ and $B \subseteq H_i'$ and subject to this H_i is maximal. It follows from these choices that G is the edge disjoint union of the block B and the vertex disjoint subgraphs $H_1, \ldots, H_t$.

First consider the case that $t \geq 2$. Suppose there exists an edge $e \in E(C_j)$ incident with a vertex in $V(H_s) \setminus \{x_s\}$ and a vertex in $V(B) \setminus \{x_s\}$. In this case we get a contradiction to the choice of C_j by selecting any cycle C_i from a subgraph H_r with $r \neq s$. So we may assume no such edge exists. In this case the 3-connectivity of G implies that $t \geq 3$. Furthermore, there must exist $1 \leq s < s' \leq t$ so that $E(C_j)$ contains an edge joining a vertex in $V(H_s)$ and a vertex in $V(H_{s'})$. Now choose $1 \leq r \leq t$ with $r \neq s, s'$ and a cycle $C_i \subseteq H_r$. We see that C_i contradicts the choice of C_j and this is a contradiction.

In the remaining case G can be decomposed into B, H_1, and C_j. It follows from the 3-connectivity of G that $|V(C_j) \cap V(B)| \geq 2$ and thus $B \cup C_j$ is 2-connected. Therefore we get a contradiction to our choice by choosing any C_i with $C_i \subseteq H_1$. This final contradiction completes the proof.

□

5.3 Path and cycle decompositions

A graph with a vertex of odd degree cannot have a decomposition into cycles. Here is a definition offering a more general type of decomposition.

Definition 5.3.1. *Let G be a graph with $2k$ vertices of odd degree. A* good decomposition *of G is a decomposition into $P_1, \ldots, P_k, C_1, \ldots, C_h$ where $P_1, \ldots, P_k$ are paths and $C_1, \ldots, C_h$ are cycles.*

Note that in a good decomposition, every odd degree vertex must appear as the end of exactly one P_i path.

Let G and G' be graphs, and let $e = uv$ be an edge of G, and $e' = u'v'$ be an edge of G'. A *2-sum* of G and G' is obtained by deleting edge e from G, deleting edge e' from G', identifying vertex u with vertex u', and identifying vertex v with vertex v'. If G and G' are

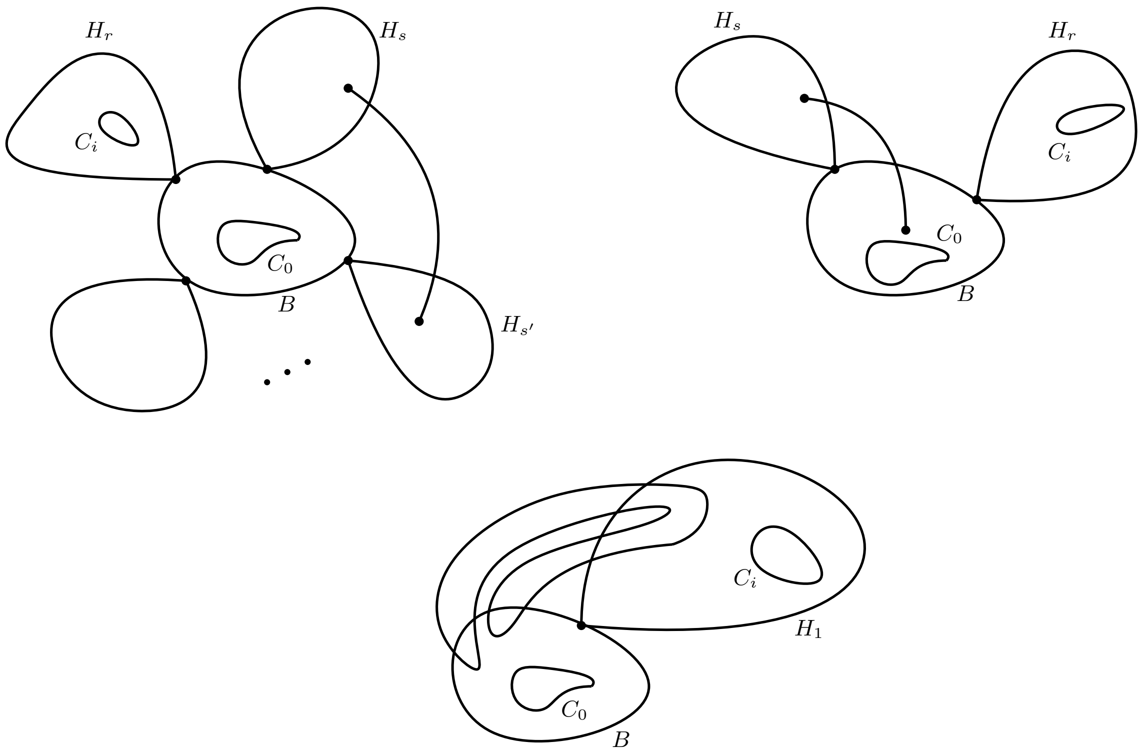

Figure 5.3: Three possible cases if $G - E(C_j)$ is not 2-connected

signed graphs, a 2-sum is only permitted when edges e and e' have the same sign (though you can always reassign signs to make this true).

We say that a 2-connected signed graph G with a distinguished edge e is *tame* if the following conditions hold: G is Eulerian, has an even number of negative edges, and every edge appears in a 2-edge-cut.

Observation 5.3.1. *If G with distinguished edge e is tame, and G' has no good decomposition, then a 2-sum of G' and G over the edge e will not have a good decomposition (to see why, note that by assumption there is an edge f of G so that $\{e, f\}$ is an edge-cut. If our 2-sum were to have a decomposition, any path or cycle not using the edge f must be completely contained in either G or G'. Since G has only vertices of even degree, this means that apart from the path or cycle using f, all other edges of G are contained in a positive cycles from the decomposition. But this would give a good decomposition of G, thus a contradiction. Based on this we have the following conjecture).*

Conjecture 5.3.1. *If G is a 2-connected signed planar graph that is not Eulerian, then G has a good decomposition unless G may be obtained from a negative K_5 by doing 2-sums with tame planar graphs.*

We believe to approach this conjecture by considering a minimum counterexample, graph G, and then attempting to establish certain properties of G.

Chapter 6

An Upper bound on circular chromatic number of signed graphs

A classic theorem due to Brooks on graph colouring is as follows.

Theorem 6.0.1 (Brooks)**.** *If G is a connected graph with maximum degree Δ then G is Δ-colourable, unless G is either a complete graph or an odd cycle.*

In our investigations of signed graphs we looked to consider colouring properties. We had hoped to show an analogue of Brooks' Theorem in this setting, but discovered that this result was already implied by an existing theorem on a different type of colouring. In this chapter we briefly recount some colouring properties and mention this implication. We begin with a definition for colouring of signed graphs due to Zaslavsky.

Definition 6.0.1. *Let (G, σ) be a signed graph and let k be a positive integer. A $2k$-colouring of (G, σ) is a function $f : V(G) \to \{\pm 1, \ldots, \pm k\}$ with the following property:*

- *If $uv \in E(G)$ has $\sigma(uv) = 1$ then $f(u) \neq f(v)$.*
- *If $uv \in E(G)$ has $\sigma(uv) = -1$ then $f(u) \neq -f(v)$.*

We say that (G, σ) is $2k$-colourable *if such a colouring exists.*

Let us note that (G, σ) is $2k$-colourable if and only if (G, σ) has a circular $2k$-colouring where we only use a subset of C^{2k} consisting of 2k points forming the vertices of a regular $2k$-gon. In particular, by the same proof as Proposition 1.6.1 we have that a $2k$-colouring of a signed graph gives a corresponding $2k$-colouring for any switching equivalent signed graph.

An important variation of graph colouring is the notion of list colouring which was first considered by Erdős, Rubin, and Taylor [16].

Definition 6.0.2. *Let G be a graph and for every $v \in V(G)$ let $L(v) \subseteq \mathbb{N}$. We say that G is L-*colourable *if there exists a function $f : V(G) \to \mathbb{N}$ so that $f(v) \in L(v)$ holds for every $v \in V(G)$ and $f(u) \neq f(v)$ whenever $uv \in E(G)$. If G is L-colourable for every L with $|L(v)| \geq k$ then we say that G is k-*choosable.

In fact, Brooks' Theorem can be generalized to the setting of list colourings as follows.

Theorem 6.0.2. *([4, 3, 7]; a simple proof in [13]) If G is a simple connected graph with maximum degree Δ then G is Δ-choosable unless G is either a complete graph or an odd cycle.*

There is another even stronger form of colouring recently introduced by Dvorak and Postle [5]. They called this correspondence coloring, but we will refer to it as DP-coloring (as is now standard).

Definition 6.0.3. *Let G be a graph. A* cover *of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\cup_{v \in V(G)} L(v)$, satisfying the following conditions.*

- *For every $v \in V(G)$, $H[L(v)]$ is a complete graph.*
- *For every $uv \in E(G)$, the edges of H between $L(u)$ and $L(v)$ form a matching.*
- *For every distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $L(u)$ and $L(v)$.*

*An (L, H)-*coloring *of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$. If an (L, H)-colouring exists whenever $|L(v)| \geq k$ holds for every $v \in V(G)$ then we say that G is DP k-colourable.*

In fact, DP colouring is stronger that both list colouring and signed graph colouring.

Here is our theorem proved jointly with Devos, Bradshaw, and Mohar.

Theorem 6.0.3. *Let G be a graph.*

1. *If G is DP k-colourable, then is k-choosable*

2. *If G is DP $2k$-colourable, then for every signature σ the signed grpah (G, σ) is $2k$-colourable.*

Proof. For the first part, let L be an assignment of lists to the vertices of G with $|L(v)| \geq k$ for every $v \in V(G)$. Form a cover of G, called (L', H), by the rule that $L'(v) = \{v\} \times L(v)$ (so the union of these sets forms $V(H)$ and for every edge $uv \in E(G)$ we add to H all edges of the form $(v, c)(u, c)$ where $c \in L(v) \cap L(u)$. Since G is DP k-colourable, H has an independent set of size $|V(G)|$ and this gives an L-colouring of G.

For the second part let $\sigma : E(G) \to \{-1, 1\}$ be a signature of G. Form a cover of G, denoted (L, H) by the rule that $L(v) = \{v\} \times \{\pm 1, \ldots, \pm k\}$ (again the union of these sets form $V(H)$). For every positive edge $uv \in E(G)$ we add all edges of the form $(u, i)(v, i)$ to H and for every negative edge $uv \in E(G)$ we add all edges of the form $(u, i)(v, -i)$ to H. Since G is DP k-colourable, H has an independent set of size $|V(G)|$ and this gives a colouring of the signed graph (G, σ). □

In fact, there is a generalization of Brooks's Theorem to DP colouring as follows.

Theorem 6.0.4. *[1] If G is a simple connected graph with maximum degree Δ then either G is Δ-DP-colourable, or G is a complete graph or cycle.*

By the above discussion this immediately yields the following Brooks type theorem for signed graph colouring which was our original aim.

Corollary 6.0.1. *Let (G, σ) be a connected signed graph with maximum degree Δ and assume that Δ is even. Then (G, σ) is Δ-colourable unless G is either a complete graph or a cycle.*

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Appendix A

Code

Let G be a simple graph. Assume $G \neq K_4$. This is Lukot'ka's algorithm for finding all 2-bisection cubic graphs.

1. Find a proper 3-vertex-colouring of G.
2. Extend one colour class into a maximal (with respect to inclusion) independent set. The graph induced by the remaining vertices is an union of paths (possibly containing only one vertex) and even circuits.
3. Recursively generate all possible (not necessarily proper) 2-vertex-colourings of the paths and cycles such that there are no three consecutive edges of the same colour.
4. For each such colouring we do the following.
   (a) We extend the colouring of the paths and circuits to the independent set. Consider a vertex v in the independent set. As all three vertices neighbouring v are already coloured, two of them must have the same colour and thus we have to colour v with the other colour. We check if this colour does not introduce a monochromatic connected subgraph on three vertices.
   (b) We check if the colouring is a bipartition (we keep track of the number of vertices of each colour during step 3 process so that this check is done in constant time).