Hamiltonicity of Covering Graphs of Trees

by

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Abstract

In this thesis we investigate sufficient conditions for the Hamiltonicity of highly symmetric graphs. The subject of Hamiltonicity in such graphs, in particular, vertex-transitive graphs and specifically Cayley graphs has attracted significant attention from researchers over the years. The thesis focuses on a class of highly symmetric graphs known as regular coverings of graphs, or lifts of voltage graphs. We restrict our attention to lifts of voltage trees over cyclic groups. This class of graphs had been investigated for Hamiltonicity already by Batagelj and Pisanki in [11] and later by Hell et al. in [26]. Our results in Chapter 3 generalize the results in [11] and are independent of the results in [26]. Informally, we derive sufficient conditions for Hamiltonicity of covering graphs of trees based on special decompositions of such trees. In Chapter 4, we first derive two sufficient conditions for the existence of Hamiltonian cycle in covering graphs of trees over a cyclic group of large prime order. Then we use these conditions to prove a probabilistic result that given a tree T and a random voltage assignment on T with a fixed probability distribution, the limit of the probability that the covering graph over T is Hamiltonian for large enough cyclic group of prime order tends to 1 as the order of T tends to infinity. Finally, we prove that the covering graph of a tree T of a fixed maximum degree contains a cycle through almost all vertices provided the voltage assignment assigns elements to every loop of T that are coprime to the order of a large enough cyclic group. Most of the results of this thesis are included in the submitted manuscript [12].

Keywords: Hamiltonicity; Voltage graph; Covering graph; Cayley graph; Vertex-transitive graph; Cyclic group

Dedication

I am dedicating this thesis to my brother, Yilin Michael Zhang, a brilliant mind and a passionate soul in the area of discrete mathematics. Though he is no longer with us, his spirit and love for mathematics continue to guide and motivate me. His dream to further his academic journey in mathematics has been a profound inspiration to me. As I started on my master's journey, I felt his absence deeply, but his enduring passion for the subject has been my guiding light. This work is a tribute to his memory, his dreams, and the unending quest for knowledge that he so loved.

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I would like to express my sincere gratitude to my supervisor, Professor Ladislav Stacho, for his invaluable guidance throughout my Master's journey. From my undergraduate studies to this point, he has been a mentor, introducing me to the field of graph theory. His wisdom and consistent support have been crucial in my academic journey. I am deeply thankful for his support throughout the development of this thesis.

I would also like to extend my thanks to Dr. Peter Bradshaw for his great help and collaboration on the manuscript [12] on which this thesis is based. Working with Dr. Bradshaw has been both an enlightening and enriching experience. His expertise, attention to detail, and innovative ideas have greatly enhanced the quality of our work.

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Chapter 1

Basic Notation and Overview

1.1 Definitions and Notation

We follow notation and terminology of graph theory as in the book *Introduction to Graph Theory* by West [37]. We say a *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an unordered pair of vertices. With the same notation, a *directed graph* G is a triple consisting of a vertex set $V(G)$, an arc set $A(G)$, and a function assigning each arc (oriented edge) an ordered pair of vertices. Sometimes, if no confusion can arise, in directed graphs, these arcs are referred to as edges. In this thesis, we often will interchange a graph with a special directed graph where each edge of the graph gives rise to two oppositely directed arcs (including loops) without explicitly speaking about this graph as directed. Two vertices $v, w \in V(G)$ are called *adjacent* and are *neighbours* if there exists an edge $e \in E(G)$ such that v and w are both endpoints of e. We write $N_G(v)$ to denote the set of all neighbours of v in G. A *loop* is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints. If a graph has no loops and multiple edges we call the graph *simple graph*. We say that a graph is *reflexive* if there exists exactly[1] one loop at every vertex. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *decomposition* of a graph G is a list of subgraphs such that each edge of G appears in exactly one subgraph in the list. A *spanning subgraph* of G is a subgraph containing each vertex of G.

A *walk* is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i, has endpoints v_{i-1} and v. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. When we say a path P is in a graph G, it means that P is a path as a subgraph of G. We denote P, a u, v*-path* if the endpoints of P are vertex u and v. We say G is *connected* if there exist a u, v-path for all $u, v \in V(G)$, otherwise G is *disconnected*. The *connectivity* of a graph G is the minimum size of a vertex set $S \subseteq V(G)$ such

[1] The common definition of a reflexive graph requires at least one loop at every vertex of the graph. Our definition in the thesis is slightly stricter by restricting the number of loops to be exactly one

that $G - S$ is disconnected or has one vertex. G is *k-connected* if its connectivity is at least k. An *independent set* in a graph is a set of pairwise nonadjacent vertices and the *independence number* of a graph is the maximum size of an independent set of vertices. The *components* of G are G's maximal connected subgraphs. A *cycle* is a simple graph with the same number of vertices and edges whose vertices can be placed in a cyclic order so that two vertices are adjacent if and only if they are consecutive in the order. A *Hamiltonian path* in a graph G is a spanning path in G, and a *Hamiltonian cycle* in G is a spanning cycle in G. We call a graph G *Hamiltonian* if G has a Hamiltonian cycle. The *circumference* of a graph is the length of its longest cycle.

An *isomorphism* from a graph G to a H is a bijection f that maps $V(G)$ to $V(H)$ and $E(G)$ to $E(H)$ such that each edge of G with endpoints u and v is mapped to an edge with endpoints $f(u)$ and $f(v)$, respectively. An *automorphism* of a graph G is an isomorphism from G to G. We follow notation and definitions related to vertex-transitive graphs found in Godsil and Royle's *Algebraic Graph Theory* [22]. It is customary to denote a group by the letter G. Therefore when groups and graphs will be involved in our discussion, we will reserve G to denote the group and Γ to denote the graph. A group G is called a *cyclic group* if there is an element $a \in G$ such that every element of G is some integral power of a. The *order of an element* a in a group G is the least positive integer n such that $a^n = id$, where $id \in G$ is the identity of the group. A non-empty subset H of a group G is said to be a *subgroup* of G if H itself is a group under the operations of G. If H is a subgroup of G and $a \in G$, then the set $aH = \{ah : h \in H\}$ is called the *left coset of* H in G and the set $Ha = \{ha : h \in H\}$ is called the *right coset of* H in G. A *homomorphism* is a map from one group to another that preserves the group operation and it is *isomorphism* if the map is one-to-one and onto. An isomorphism from a group G onto itself is called an *automorphism* of G. A graph Γ is *vertex-transitive* if its automorphism group acts transitively on $V(\Gamma)$ and is *edge-transitive* if its automorphism group acts transitively on $E(\Gamma)$. Let G be a group and let S be a subset of G that is closed under taking inverses and does not contain the identity. Then the Cayley graph $Cay(G, S)$ is the graph with the vertex set $V(Cay(G, S)) = G$ and the edge set $E(Cay(G, S)) = \{gh : hg^{-1} \in S\}$.

1.2 Thesis Overview

The thesis has two parts divided into four chapters. The first part, which includes the first two chapters, starts with an introduction to the topic. We will first provide necessary definitions and notation which will be used throughout the thesis. Then we will review several fundamental results in the area of existence of Hamiltonian cycles, including Dirac's theorem and Ore's theorem as the fundamental results in the study of Hamiltonicity. After this overview, the thesis will re-focus on the review of results on highly symmetric graphs and their Hamiltonicity. Chapter 2 concludes with previous known results on which this thesis will build.

In the second part of the thesis, we will focus on the Hamiltonicity of covering graphs of voltage graphs. In Chapter 3, we will introduce a useful tool called "Billiard Strategy" and will improve it to “Extended Billiard Strategy”. These strategies are designed for studying the Hamiltonicity of covering graphs of trees. We will prove new sufficient conditions for the Hamiltonicity of the lifts of trees based on a decomposition of these trees into paths similar to the one described in [26]. In Chapter 4, we start by deriving two sufficient conditions for the existence of the Hamiltonian cycle in covering graphs of trees over a cyclic group of large prime order. Following this, we use these conditions to prove a probabilistic result that given a tree T and a random voltage assignment on T with a fixed probability distribution, the limit of the probability that the covering graph over T is Hamiltonian for some large enough cyclic group of prime order tends to 1 as the order of T tends to infinity. The last part of Chapter 4 is devoted to proving that the covering graph of a tree T of a fixed maximum degree contains a cycle through almost all vertices provided the voltage assignment assigns elements to every loop of T that are coprime to the order of a large enough cyclic group.

Chapter 2

Background and Known Results

The concept of Hamiltonian cycle is due to Sir William Rowan Hamilton. In 1857, he invented the mathematical game the “Icosian Game”. The aim of this game is to find a walk along the vertices (in the game named by major cities) and the edges of a dodecahedron, which visits every vertex exactly once and then returns to the starting vertex. The game did not gain much popularity, but it turned out that it had much more fundamental theoretical value. Instead of a dodecahedron, one can ask about the existence of such a walk (cycle) on any graph. Such a walk (cycle) is today referred to as the Hamiltonian cycle, and this concept has secured a central place within graph theory. Until the 1970s, interest in Hamiltonian cycles was more focused on their relationship to the Four Color Problem [37] and a few closely related problems. Since the late 1970s, there has been a rapid increase in research concerning Hamiltonian cycles and paths. The area of Hamiltonicity stands out for its breadth and complexity, addressing and linking together a variety of topics which are not limited only to the coloring problem. These subtopics include but are not limited to the existence or construction of Hamiltonian cycles and paths, Hamiltonian problems in special graph classes, relationship to various graph invariants, travelling salesman problems, related complexity issues etc. Determining whether a graph contains a Hamiltonian cycle is an NP-complete problem [21]. One of the main lines of research on the topic of Hamiltonian cycles is sufficient conditions for their existence [14, 17, 25, 28, 34]. Further research on the topic includes graphs with some structure, such as planar graphs, graphs with high symmetry, chordal graphs, sparse graphs, etc. [3, 4, 15, 32]

In this thesis, we will concentrate on graphs with some degree of symmetry. In 1969, Lovász [32] asked whether every connected vertex-transitive graph has a Hamiltonian path, and since then, this question motivated considerable interest in questions about the existence of Hamiltonian paths and cycles in graphs with a high degree of symmetry. While Lovász’s question has not yet been answered affirmatively, a vertex-transitive graph with no Hamiltonian path has not yet been found either. Furthermore, there are only four known vertex-transitive graphs (on at least three vertices): the Petersen graph, the Coxeter graph, and a modification of each of the two which are not Hamiltonian (but they all have a Hamiltonian path).

One particular type of vertex-transitive graph are Cayley graph. A simple fact that none of the four known vertex-transitive non-Hamiltonian graphs is a Cayley graph has led to a folklore conjecture that every Cayley graph is Hamiltonian. This conjecture appears, for example, in [31]. A partial result of this conjecture is the classical result which states that Cayley graphs of finite abelian groups are guaranteed to be Hamiltonian [33]. In contrast to the original conjecture by Lovász, Babai [9] conjectured that there exists a positive constant $\epsilon > 0$ so that there exist infinitely many connected vertex-transitive graphs G with the length of a longest cycle at most $(1-\epsilon)V(G)$. Babai in [8] also proved that every connected vertex-transitive graph on $n \geq 4$ vertices has circumference greater than $\sqrt{3n}$. Further research has focused on proving special cases of Lovász's conjecture [2, 3, 4, 5, 6, 7, 13, 30].

Voltage graphs were introduced by Gross [23] to provide an efficient procedure for constructing a larger graph from a smaller one, the larger being called a lift or covering graph. In this thesis, we will focus our attention on voltage graphs and their lifts/coverings in regard to their Hamiltonicity.

Below, we will survey in more detail some of the ideas and concepts sketched above. In particular, we will focus on the relationship of covering graphs of voltage graphs to Hamiltonian cycles.

2.1 Hamiltonicity of General Graphs

There exist many sufficient conditions guaranteeing Hamiltonicity of a given graph. Many of these conditions are based on edge density requirements. Many of these conditions are derived from the fundamental idea described in the following theorem proved by Dirac.

Theorem 2.1.1. *[17] If G is a simple graph on at least three vertices and the minimum degree $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.*

Proof. We first claim that G is connected, for otherwise we can suppose G has more than one component, and then the degree of any vertex in the smallest component will be less than $\frac{n}{2}$, a contradiction.

Let $|V(G)| = n$ and $P = v_1 v_2 \dots v_k$ be a longest path in G. We can observe that both v_1 and v_k cannot be adjacent to any vertex not in P, otherwise P can be extended to a longer path. Therefore $N_G(v_1) \subseteq V(P)$ and $N_G(v_k) \subseteq V(P)$ since G is simple. Because $deg(v_1) \geq \delta(G) \geq \frac{n}{2}$, we can get that $|N_G(v_1)| \geq \frac{n}{2}$, and we can argue similarly about v_k. Therefore v_1 has at least $\frac{n}{2}$ neighbours in $\{v_2 \dots v_k\}$ and v_k has at least $\frac{n}{2}$ neighbours in $\{v_1 \dots v_{k-1}\}$. We know $k \leq n$ so by pigeonhole principle, there exist a vertex $v_i \in \{v_2 \dots v_{k-1}\}$ such that v_i is adjacent to v_k and v_{i+1} is adjacent to v_1. Now we can see that there exists a cycle $C = v_1 v_{i+1} v_{i+2} \dots v_k v_i v_{i-1} \dots v_2 v_1$. We claim that C is a Hamiltonian cycle of G. Suppose not, then $V(G) \setminus V(C) \neq \emptyset$ so there must exist a vertex u in G such that $u \in V(G) \setminus V(C)$. Because G is connected u must be adjacent to a vertex in P, say v_j. Then we can break C into a path P' with endpoints v_j and v_{j+1} and the path uP' is a longer path than P, contradicting with P being the longest. $\square$

While Dirac's theorem sets a foundational basis for Hamiltonicity, by careful examination of vertices in Dirac's argument, Ore in 1960 established the following theorem as a generalization of Dirac's theorem. He took into account the degree restriction on pairs of non-adjacent vertices, rather than individual vertices.

Theorem 2.1.2. *[34] Let G be a simple graph on at least three vertices. If for each pair of distinct non-adjacent vertices $u, v \in V(G)$, $d(u) + d(v) \geq |V(G)|$, then G is Hamiltonian.*

The correlation between Hamiltonicity and the degree of vertices of the graphs is an example of edge density conditions. Also, note that both Dirac's theorem and Ore's theorem are the best possible as there exist non-Hamiltonian graphs having just one vertex of degree less than $\frac{|V(G)|}{2}$ or having pairs of vertices with degree sum just one less than Ore's bound requires. The degree conditions can be relaxed in the presence of other graph theoretic properties such as regularity, planarity, high connectivity, high toughness, or high symmetry.

Both theorems above are foundational in Hamiltonian graph theory. Since their introduction, many significant results have been build upon them and the concept of degree conditions has come to form a major discussion point in Hamiltonicity problems. In what follows, we offer a short survey of such results.

By imposing some regularity condition on the graph, Jackson in [28] established that the degree condition in Dirac's theorem can be notably lowered.

Theorem 2.1.3. *[28] Every 2-connected d-regular graph G with $d \geq |V(G)|/3$ is Hamiltonian.*

Later, in 1986, the degree condition was slightly refined to $|V(G)|/3 - 1$ by Hilbig [27], with the exception of the Petersen graph and a graph obtained by replacing one vertex of the Petersen graph with a triangle.

Besides degree conditions, connectivity is also a major factor in Hamiltonicity problems. Haggkvist and Nicoghossian et. al. [25] were able to greatly decrease the minimum degree requirement for Hamiltonicity when the graph has sufficient connectivity, as presented in the following theorem.

Let $\kappa(G)$ denote the vertex connectivity of a graph G.

Theorem 2.1.4. *[25] Let G be a graph on n vertices and $\kappa(G) = k \geq 2$. If the minimum degree $\delta(G) \geq \frac{1}{3}(n + k)$, then G is Hamiltonian.*

Let $\alpha(G)$ denote the independence number of a graph G. In contrast to many previously known degree conditions for Hamiltonicity, Chvátal and Erdős [14] obtained an important sufficient condition for a graph to be Hamiltonian in terms of its connectivity and independence number.

Theorem 2.1.5. *[14] If G is a simple graph with at least three vertices and $\kappa(G) \geq \alpha(G)$ then G is Hamiltonian.*

Proof. If $\alpha(G) = 1$, G is a complete graph and thus Hamiltonian. Suppose that $\kappa(G) \geq \alpha(G) > 1$. Let $k = \kappa(G)$ and C be a longest cycle in G. Since every graph with $\delta(G) \geq 2$ has a cycle of length at least $\delta(G) + 1$, and since $\delta(G) \geq \kappa(G) = k$, C has at least $k + 1$ vertices.

Let H be a connected component of $G - V(C)$. The cycle C has at least k vertices with edges to H; otherwise, deleting the vertices of C with edges to H contradicts $\kappa(G) = k$. Let $u_1, \ldots, u_k$ be the k vertices of C with edges to H.

For $i = 1, \ldots, k$, let a_i be the vertex immediately following u_i on C. If any two of these vertices are adjacent to each other, say $a_i a_j$ then we construct a longer cycle by using $a_i a_j$, and the portions of C from a_i to u_j, a_j to u_i, and a u_i, u_j-path through H. Since H is a connected component, the last path exists. This longer cycle contradicts the choice of C.

If a_i has a neighbour in H, then we can detour to H between u_i and a_i along C, again resulting in a longer cycle that C. Thus, we also conclude that no a_i has a neighbour in H. Therefore $\{a_1, \ldots, a_k\}$ plus any vertex of H forms an independent set of size $k + 1$, a final contradiction. $\square$

The inequality $\kappa(G) \geq \alpha(G)$ is called "Chvátal-Erdős" condition. The condition can be considerably improved to $\kappa(G) \geq \alpha(G^2)$ if G is claw-free, as proved by Ainouche et al. [1]. In 1990, Jackson and Ordaz conjectured in [29] that "Chvátal-Erdős" condition also implies pancyclicity, having cycles of all possible lengths. A recent result in [19] proves asymptotically this conjecture by showing that a graph G with $\kappa(G) > (1 + o(1))\alpha(G)$ is pancyclic.

Beyond the theorems that we have outlined above, Hamiltonian graph theory is rich with other theorems and conjectures which make the subject very fruitful for research, and our list of results above is just a brief glimpse on the subject.

2.2 Highly Symmetric Graphs

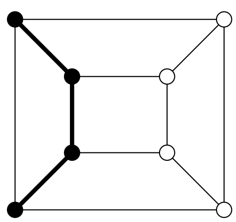
An important class of highly symmetric graphs is the class of transitive graphs which are graphs whose automorphism group acts vertex, edge, or arc transitively on the corresponding sets. In particular, a transitive graph G is *vertex-transitive* if its automorphism group acts transitively on the vertex set $V(G)$, and G is *edge-transitive*, if its automorphism group acts transitively on the edge set $E(G)$.

A graph is *symmetric (or arc-transitive)* if its automorphism group acts transitively on ordered pairs of adjacent vertices, that is, the set of arcs. By definition, symmetric is a stronger property than vertex-transitive.

An *s-arc* is a sequence of $s + 1$ vertices $(v_0, \ldots, v_s)$ such that consecutive vertices are adjacent and $v_{i-1} \neq v_{i+1}$ for all $0 < i < s$. A graph is called *s-arc-transitive*, if it has an s-arc and if there is always a graph automorphism of G sending each s-arc onto any other s-arc. When $s = 0$, G is vertex-transitive. A *3-arc graph* of G, denoted by $X(G)$, is defined to have the vertex set $A(G)$ such that two vertices corresponding to two arcs (u, v) and (x, y) are adjacent if and only if (v, u, x, y) is a 3-arc of G.

Example 2.2.1. A few examples of s-arc transitive graphs for small values of s are as follows.

- The cubical graph is 2-arc-transitive but is not 3-arc-transitive. Figure 2.1 shows that any 3-arc that is contained in a 4-cycle is not equivalent to any 3-arc that is not contained in a 4-cycle, we cannot find a graph automorphism sending each 3-arc onto any other 3-arc.
- The complete bipartite graph $K_{n,n}$ is 3-arc but not 4-arc-transitive, for all $n \geq 2$.
- The Heawood graph is 4-arc-transitive, but not 5-arc-transitive.



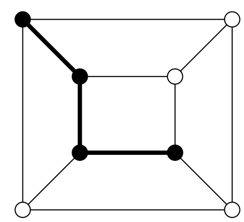


Figure 2.1: An example of two 3-arcs that fail cubical graph to be a 3-arc-transitive graph.

Theorem 2.2.2. *[39] A 3-arc graph is Hamiltonian if and only if it is connected.*

Corollary 2.2.3. *[39] If a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is Hamiltonian.*

We next introduce a class of graphs which will be important for this thesis.

In our thesis the definition of a covering graph will be described in terms of voltage graph, informally speaking, a *covering graph* is constructed from a *base graph* Γ, a group G, and a mapping $\sigma : E(\Gamma) \to G$. In the construction, each vertex of Γ is replicated for every element of G, and its edges are added based on specific criteria influenced by the labels given by σ. Gross initially proposed covering graphs as a way to describe graph embeddings on surfaces [23]. Their importance has grown since then, for example being used in the construction of graphs that possess large girth while maintaining a small number of vertices, providing a counterexample to Greenwell and Kronk's conjecture on an edge coloring and Hamiltonicity of cubic graphs [36]. Despite not being vertex-transitive, covering graphs still exhibit high symmetry. This makes them have many properties in common with vertex-transitive graphs. In this thesis, we focus on the covering graphs of voltage graphs, and we provide a formal definition of the covering graph in terms of a voltage graph in Definition 2.2.6. Note that the covering graph is a special case of a more general notion in topology called covering projections. We refer an interested reader to [24].

Before we formally define a covering graph we need to define voltage assignment and voltage graph. Voltage graphs are obtained from ordinary graphs (we think of them as directed graphs where every edge is thought of as two oppositely directed arcs) by assigning a group element to each arc over a fixed group, in this thesis, usually a cyclic group.

Definition 2.2.4. Let Γ be a graph with a loop at every vertex. For each edge $[u, v] \in E(\Gamma)$, we say that Γ has corresponding *arcs* (u, v) and (v, u), which correspond to the two opposite directions in which the edge $[u, v]$ can be traversed. Accordingly, we define the *arc set* of Γ as $A(\Gamma) = \{(u, v), (v, u) : [u, v] \in E(\Gamma)\}$. If $[v, v]$ is a loop of Γ, then we let $A(\Gamma)$ contain two elements (v, v) corresponding to $[v, v]$.

Definition 2.2.5. Given a group G, we say that a *voltage assignment* on Γ is a function $\sigma : A(\Gamma) \to G$ that satisfies $\sigma(u, v) = \sigma(v, u)^{-1}$ for every edge $[u, v] \in E(\Gamma)$, and such that σ also assigns inverse elements to each pair of loops (v, v). The triple (Γ, G, σ) is called the *voltage graph*. We will often refer to the values assigned by σ as *labels* or *voltages*.

Definition 2.2.6. Given a voltage graph (Γ, G, σ), where $\sigma : A(\Gamma) \to G$ is a voltage assignment to a group G, we define the *covering graph* of (Γ, G, σ), written Γ^σ, to be the graph defined as follows:

- $V(\Gamma^\sigma) = V(\Gamma) \times G$. Thus, for each vertex $v \in V(\Gamma)$ and element $a \in G$, there is a vertex (v, a) in Γ^σ, and we will often write v_a instead of (v, a).
- For any vertices $u, v \in V(\Gamma)$ and elements $a, b \in G$, $(u, a) \sim (v, b)$ if and only if there exists an arc $e = uv \in A(\Gamma)$ satisfying $\sigma(e) = a^{-1}b$.

Definition 2.2.7. For a vertex $v \in V(\Gamma)$, we write v^σ for the subgraph of Γ^σ induced by the vertex set $\{v_g : g \in G\}$. We say that v^σ is the *fibre* of Γ over v.

Let us observe a simple example that familiarize ourselves with the concept.

In all voltage graphs that we will present in the thesis, we always assume G is a cyclic group with an additive operator. Therefore $-a$ denotes the inverse of a for all $a \in G$.

Example 2.2.8. Figure 2.2 shows an example of constructing a covering graph (on the right) from a voltage graph (on the right) with voltage assignment $\sigma : E(H) \to \mathbb{Z}_3$ such that $\sigma(v, v) = 1$, $\sigma(v, u) = 0$, and $\sigma(u, u) = 2$.

We will next list some properties of covering graphs. We will particularly explore how a walk or a component in the voltage graph lifts into the covering graph. We aim to provide a clearer picture of the underlying mechanics of the process.

Theorem 2.2.9. *[24] Let W be a walk in a voltage graph (Γ, G, σ) with initial vertex u. Then for each vertex u_a in the fibre over u, there is a unique lift of W that starts at u_a.*

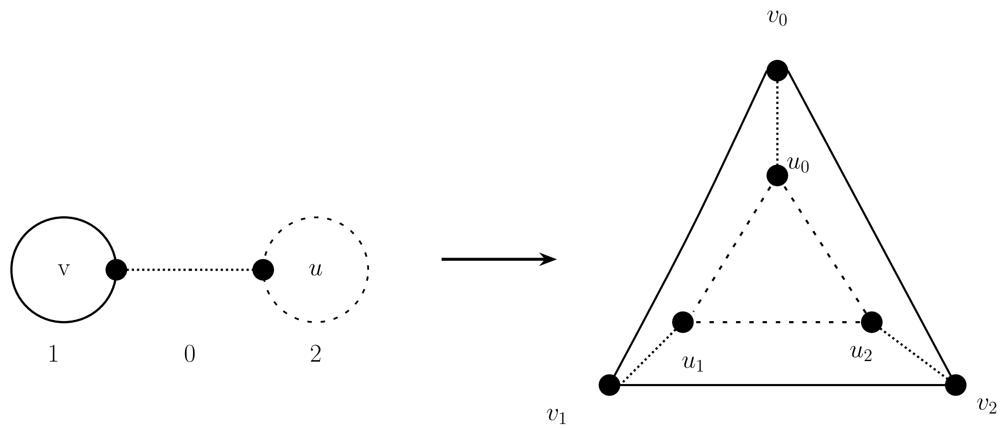


Figure 2.2: Covering graph (on the right) of the voltage graph (on the left). The corresponding group is $\mathbb{Z}_3$.

Proof. Consider the first oriented edge of W, say e^+ or e^-. [1] If it is e^+ then since only one plus-directed edge of fibre over e starts at the vertex u_a, i.e, the edge e_a^+, that edge must be the first edge in the lift of W starting at u_a. If it is e^-, then since only one minus-directed edge of the fibre over e starts at u_a, it follows that the edge must be the first edge in the lift of W starting at u_a.

Similarly, there is only one possible choice for a second edge of the lift of W, since the initial point of that second edge must be the terminal point of the first edge, and since that second edge of the lift must lie in the fibre over the second edge of the base walk W. This uniqueness holds, by the same arguments, for all the remaining edges as well. □

Definition 2.2.10. Given a voltage graph (Γ, G, σ), the *net voltage* on a walk $W = e_1^{\psi_1}, \ldots, e_n^{\psi_n}$ where each $\psi_i = -$ or $+$ is defined to be the sum $\sigma(e_1^{\psi_1}) + \cdots + \sigma(e_n^{\psi_n})$ of the voltages on the directed edges of W in the order and their direction along W. Note that the voltage on a minus-directed edge e^- is understood to be the group inverse of the voltage on e^+, in our case, $\sigma(e^-) = -\sigma(e^+)$.

Theorem 2.2.11. *[24] Let W be a walk from u to v in a voltage graph (Γ, G, σ), and let b be the net voltage on W. Then the lift W_a starting at u_a terminates at the vertex v_{a+b}.*

Proof. Let $b_1, \ldots, b_n$ be the successive voltages encountered on the walk W such that $\sum_{i=1}^{n} b_i = b$. Then the subscripts of the vertices of W_a are

$$a, a + b_1, a + b_1 + b_2, \ldots, a + b_1 \cdots + b_n = a + b$$

Since W_a terminates in the fibre over v, its final vertex is v_{a+b}. □

[1]Note that in directed voltage graph whereas e_l^+ runs from u_l to v_{l+k} the reverse edge e_l^- runs from v_{l+k} to u_l, for some k.

Theorem 2.2.12. *[24] Consider a voltage graph* (Γ, G, σ). *Let* C *be a* k*-cycle in the base graph* Γ *such that the net voltage on* C *has order* m *in* G. *Then each component of the preimage* $p^{-1}(C)$ *is a* km*-cycle and there are* $|G|/m$ *such components.*

Proof. Let the cycle C be represented by a closed $u-$base walk W, let b be the net voltage on W, and let u_a be a vertex in the fibre over u. Then the component of $p^{-1}(C)$ containing u_a is formed by the edges in the walks $W_a, W_{a+b}, \ldots, W_{a+b^{m-1}}$, which attach end-to-end to form a $km-$cycle. For each of the $|G|/m$ left cosets of the cyclic group generated by the net voltage b, there is a unique component of $p^{-1}(C)$. $\square$

Although covering graphs are not vertex-transitive graphs in general, they are still highly symmetric and share many properties with vertex-transitive graphs. As we will show in the following examples, several classes of vertex-transitive graphs can be thought of as covering graphs of very small graphs. In particular, complete graphs, Cayley graphs, and n-cube graphs are covering graphs of a graph with only one or two vertices with a possible edge and some loops.

Example 2.2.13. Any complete graph K_n can be constructed from a single vertex base graph $\Gamma = v$ with loops. Figure 2.3 shows an example of construction of K_4 by $\sigma : A(\Gamma) \to \mathbb{Z}_4$ where Γ is a single vertex with two loops with $\sigma(e) = 1$, $\sigma(f) = 2$.

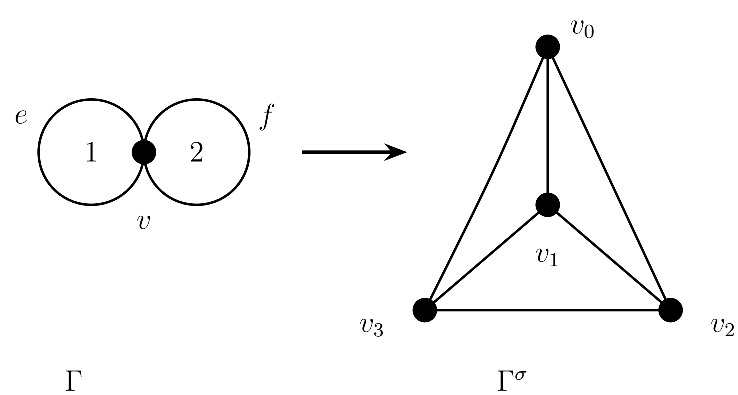


Figure 2.3: A base graph Γ of single vertex with a loop and the covering graph $\Gamma^\sigma = K_4$

Example 2.2.14. Observe that any generalized Petersen graph can be constructed from a voltage graph with two vertices, two self-loops, and one other edge, that is, a dumbbell graph. Figure 2.4 shows an example of lifting a dumbbell graph to $GP(5, 2)$, a Petersen Graph by $\mathbb{Z}_5$.

Example 2.2.15. All Cayley graphs are lifts of a one-vertex bouquet of loops. In particular, a Cayley graph $Cay(G, S)$ is a covering graph of the voltage graph with one vertex and a loop for each element of S.

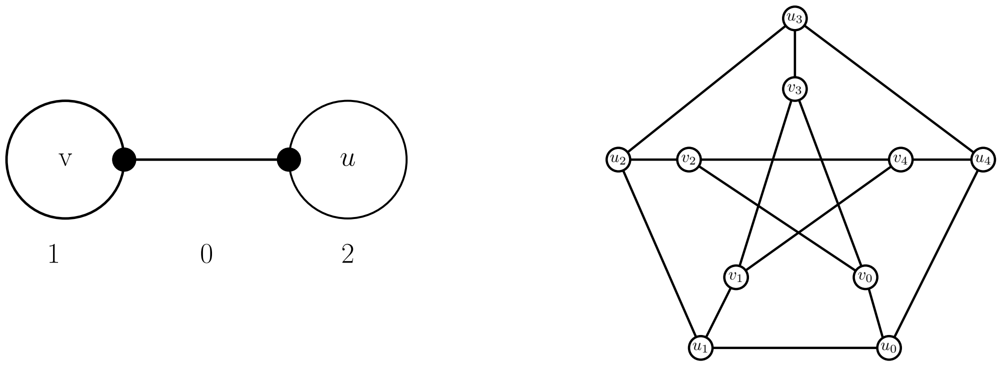


Figure 2.4: Petersen graph as a lift of a dumbbell graph by $\mathbb{Z}_5$

2.3 Hamiltonicity of Highly Symmetric Graphs

Lovász posed the question of whether every connected vertex-transitive graph has a Hamiltonian path in 1969. There is a great interest in solving this long-standing problem and still, it remains widely open.

Conjecture 2.3.1. *[32] Every finite connected vertex-transitive graph contains a Hamiltonian path.*

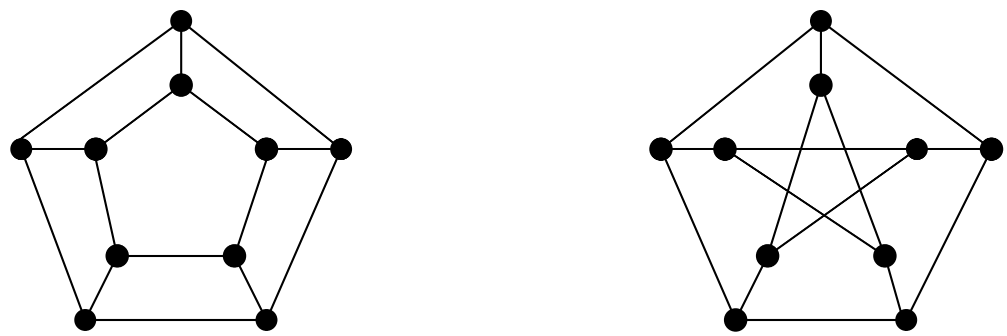
It's known that just four connected vertex-transitive graphs with a minimum of three vertices do not possess a Hamiltonian cycle: the Petersen graph, the Coxeter graph, and two variations formed from the Petersen graph and the Coxeter graph by substituting every vertex with a triangle in each. Given that all of these are cubic graphs, hinting that a comprehensive study of the Hamiltonicity of cubic vertex-transitive graphs is essential to address the stated conjecture.

Another important class of highly symmetric graphs is the class of generalized Petersen graphs. We have seen one example of such graph above. In general, let n and k be two integers such that $1 \leq k \leq n-1$. The *generalized Petersen graph* $GP(n,k)$ is defined to have vertex-set $\{u_i, v_i \mid i = 0, 1, ..., n-1\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 0 \leq i \leq n-1\}$ where subscripts are reduced modulo n.

A generalized Petersen graph can also be defined as a covering graph as follows. We let Γ be a K_2 with a loop at each vertex, and we let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ assign the values 1 and -1 to the pair of loops at one vertex of Γ, the values k and $-k$, for some $1 \leq k \leq n-1$, to the pair of loops at the other vertex of Γ, and finally a value of 0 to the arcs corresponding to the cut-edge of Γ. Then, the covering graph Γ^σ is isomorphic to the generalized Petersen graph $P(n,k)$. For examples, see Figure 2.5.

Theorem 2.3.2. *[3] The generalized Petersen graph $GP(n,k)$ is Hamiltonian if and only if it is neither*

- $GP(n,2) \cong GP(n,n-2) \cong GP(n,(n-1)/2) \cong GP(n,(n+1)/2)$ *when* $n \equiv 5 \pmod 6$*, nor*



$GP(5,1)$ $GP(5,2)$

Figure 2.5: Examples of generalized Petersen graph $GP(5,1)$, the 5-prism graph and $GP(5,2)$, the Petersen Graph

- $GP(n, n/2)$ *when* $n \equiv 0 \pmod 4$ *and* $n \geq 8$

The proof of the above theorem in the original paper [3] is long, and we offer an outline of the original proof.

Proof. To describe a Hamiltonian cycle for a given generalized Petersen graph, an important tool for proving the theorem is the definition and the construction of a *lattice diagram* for a $GP(n,k)$. A lattice diagram corresponds to a Hamiltonian cycle in $GP(n,k)$. In a plane for integers x and y we call (x,y) a *lattice point*. We label lattice points in a plane with elements $0,\ldots,n-1$ in $\mathbb{Z}_n$ such that whenever point (x,y) is labelled with $i \in \mathbb{Z}_n$, then point $(x+1,y)$ is labelled with $i+1$ and $(x,y-1)$ is labelled with $i+k$. By the labelling, we can construct a labelled graph $H(n,k)$ with vertices (x,y) and (x',y') adjacent if and only if $|x-x'|+|y-y'|=1$. Then one considers a subgraph $L(n,k)$ of $H(n,k)$ containing a closed or an opened Eulerian trail which satisfies the following three conditions:

- If a vertex of degree four is entered vertically (horizontally), then it must be departed vertically (horizontally);
- Each label $0,1,\ldots,n-1$ is encountered once in the horizontal direction and once in the vertical direction;
- If $L(n,k)$ has an open Eulerian trail, then the two vertices of odd degree must have the same label and either both have degree one or one has degree one and the other has degree three.

Using these properties, one can show that such a trail in $L(n,k)$ represents a Hamiltonian cycle in $GP(n,k)$, and such $L(n,k)$ can be defined as a lattice diagram for $GP(n,k)$. Since each lattice diagram corresponds to a Hamiltonian cycle in $GP(n,k)$, to prove the Hamiltonicity of $GP(n,k)$

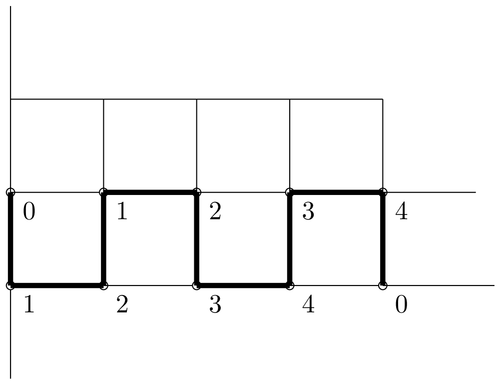


Figure 2.6: Example of a lattice diagram $L(5,1)$ (in bold) and it can represent a Hamiltonian cycle in $GP(5,1)$.

it is sufficient to find certain labelled Eulerian graphs. Figure 2.6 is an example of labelled lattice points and lattice diagram $L(5,1)$.

If $k = n/2$, for $GP(n, n/2)$ and n even, it is not hard to see that $GP(n, n/2)$ is Hamiltonian if and only if $n = 4$ or $n \equiv 2 \mod 4$ by observing that any Hamiltonian cycle must contain paths $u_i v_i v_{i+n/2} u_{i+n/2}$ for $i = 0, \ldots, (n-2)/2$.

If $k \neq n/2$, it is necessary to have $k \leq \lfloor n/2 \rfloor$ since $GP(n,k) \cong GP(n, n-k)$. Based on Bannai's result in [10], we only need to show that $GP(n,k)$ is Hamiltonian when n and k are not relative primes. Alspach [3] proved the following two lemmas in order to prove this case.

- The cubic generalized Petersen graph $GP(n,k)$ is Hamiltonian whenever $gcd(n,k)$ is even.
- Let n and k be relatively prime with $1 \leq k < n/2$. If there exists an extendible lattice diagram $L(3n, 3k)$, then $GP(dn, dk)$ is Hamiltonian for all odd $d > 1$.

The first statement leaves us with the case of showing that $GP(dn, dk)$ is Hamiltonian for all odd $d > 1$ when $gcd(n,k) = 1$ which leads to the second statement. Proofs of both statements use the lattice diagram technique. Due to its extensive nature, incorporating the complete proof here would detract from the conciseness and focused narrative we aim to maintain. Therefore, we encourage interested readers to refer to the original paper for a detailed examination of the proof. □

Theorem 2.3.3. *[2] With the exception of the Petersen graph, every connected vertex-transitive graph of order* $2p$*, where* p *is prime, is Hamiltonian.*

Moreover, it is known that connected vertex-transitive graphs of order pq, where p and q are primes, admitting an imprimitive subgroup [2] of automorphisms contains a Hamiltonian cycle. Also, a Hamiltonian path is known to exist in connected vertex-transitive graphs of order $5p$ and $6p$.

We observed above that every Cayley graph of a group G is a covering graph of a graph on a single vertex, with one loop for each generator of G. Among the four non-Hamiltonian vertex-transitive graphs, none of them is a Cayley graph. In view of the Lovász conjecture, the attention has focused more toward the Hamiltonicity of Cayley graphs.

Theorem 2.3.4. *[33] Let G be a finite abelian group with at least three elements. Then any Cayley graph of G has a Hamiltonian cycle.*

Before proving the theorem, Marušič introduced several notations that will be used specifically for the proof: Let G be a group with identity element id. If M is a subset of G then we denote $M^{-1} = \{x^{-1} : x \in M\}$, $M_0 = M - \{id\}$ and $M^* = M_0 \cup M_0^{-1}$. Let $S = [s_1, s_2, \dots, s_r]$ and $T = [t_1, t_2 \dots, t_q]$ be sequences on G. We say S is Hamiltonian if $r = |G|$, $\prod_{i=1}^{r} s_i = id$ and the partial products $\prod_{i=1}^{j} s_i = g_j$ where g_j are all distinct non-empty elements of G for different j. If $s_i \in M$, for $i = 1, 2, \dots, r$ then S is called an *M-sequence* on G. We use $H(M, G)$ to denote the set of all Hamiltonian M^*-sequence on G. The inverse sequence S^{-1} of S is the sequence $[s_r^{-1}, s_{r-1}^{-1}, \dots, s_1^{-1}]$. $\bar{S}$ denotes the sequence $[s_1, s_2, \dots, s_{r-1}]$ and $\hat{S}$ denotes the sequence $[s_2, s_3, \dots, s_{r-1}]$ for $r \geq 2$. Furthermore, the product ST is defined to be the sequence $[s_1, s_2, \dots, s_r, t_1, t_2, \dots, t_q]$.

In his proof, Marušič used the following lemma.

Lemma 2.3.5. *Suppose G is a finite abelian group with a generating set M. Let M' be a non-empty subset of M_0. If we have a Hamiltonian $(M')^*$-sequence on $\langle M' \rangle$ denoted S, then there exist some sequence Q in G such that $\bar{S}Q$ is a Hamiltonian M^*-sequence on G.*

Proof. The proof will use the induction on the cardinality of $M_0 \setminus M'$. For the base case, if $M_0 \setminus M' = \emptyset$, then $\langle M' \rangle = \langle M \rangle = G$ and take $Q = s_n$.

Now suppose $M_0 \setminus M' \neq \emptyset$ so there is some g in $M_0 \setminus M'$. Then let $H = M_0 \setminus \{g\}$, and by inductive hypothesis, there exist some Q in H such $\bar{S}Q$ is a Hamiltonian $(M_0 \setminus g)^*$-sequence on H. Let $W = \bar{S}Q$. Let j be the smallest integer such that g^j is in H. Then if j is odd, let $T = \bar{Q}(W, [g]^j) l_w [g^{-1}]$ Then $\bar{S}T$ is a Hamiltonian M^*-sequence on G. We can see this by noting that the removal of a non-redundant generator g yields disjoint isomorphic subgraphs. We move through copies of $H, gH, \dots, g^{j-1}H$ and then backtrack by way of g^{-1}. If j is even, let $T = \bar{Q}(W, [g]^j)[g](\bar{W})^{-1}[g^{-1}]^{j-1}$. Then $\bar{S}T$ is a Hamiltonian M^*-sequence on G. □

Now we can briefly explain the proof of Theorem 2.3.4.

[2] In contrast to primitive permutation group, if a group is transitive and does preserve a nontrivial partition, then it is called imprimitive.

Proof. Using the previous lemma, when G is finite and abelian with generating set M, if for some nonempty subset M' of M_0, where M_0 denote $M - \{id\}$, we have $S \in H(M', \langle M' \rangle)$, then there exists some sequence Q in G such that $\bar{S}Q$ is a Hamiltonian M^*-sequence on G.

By claiming that $Cay(G, M)$ is Hamiltonian if and only if $H(M, G) \neq \emptyset$, it suffices to show that if M is a generating set of an abelian group G of order at least 3, there exists $S \in H(M, G)$ such that $S \neq \emptyset$. If exists an element $c \in M$ of order ≥ 3, then let $M' = \{c\}$ and $S = [c]^n$. If there is no element in M that has order ≥ 3, then M contains two distinct elements a, b so that each have order 2. Then let $M' = \{a, b\}$ and $S = [abab]$ is a Hamiltonian M'-cycle on $\langle M' \rangle$, as follows by above lemma. □

Theorem 2.3.6. *[20] Every connected Cayley graph of a group with prime order commutator group has a Hamiltonian cycle.*

The condition above has later been generalized to that with the exception of the Petersen graph, every connected vertex-transitive graph whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime power order is Hamiltonian [18].

While Cayley graphs have received significant attention in the literature, most established results rely on sets of restrictions, either to the group's class and order or to the generating sets used in the Cayley graph.

Theorem 2.3.7. *[35] Any finite group G of size $|G| \geq 3$ has a generating set S of size $|S| \leq \log_2 |G|$, such that the corresponding Cayley graph $Cay(G, S)$ is Hamiltonian.*

Theorem 2.3.8. *[38] Every connected Cayley digraph on a group of prime-power order greater than 2 is Hamiltonian.*

Theorem 2.3.9. *[15] Let S be a set of generators for a finite group that gives a Coxeter presentation for G. Then the Cayley graph $Cay(G, S)$ is Hamiltonian.*

Inspired by Example 2.2.14, we can ask a motivational question

Question 2.3.10. *Given a reflexive tree Γ, a cyclic group G, and a corresponding voltage graph (Γ, G, σ) where $\sigma : E(\Gamma) \to G$. Under what conditions does the covering graph Γ^σ contain a Hamiltonian cycle?*

Addressing this question proves to be a nontrivial task even when dealing with small trees. For instance, when the base tree Γ consists of a single vertex, there exists a variety of potential covering graphs, among which Cayley graphs constitute a specific category. It is noteworthy that the conjecture stating "Every finite connected Cayley graph contains a Hamiltonian cycle" remains unproven. In the case where Γ is a 2-vertex reflexive tree, the class of all corresponding covering graphs contains generalized Petersen graphs.

Next, we will focus on finding sufficient conditions for Hamiltonicity of covering graphs of trees. The aim is to extend the discussion from specific cases involving a tree on one or two vertices to larger trees. Our inspiration is the following result.

Theorem 2.3.11. *[11] Let $\Gamma = T \times C_n$ be the Cartesian product of an n-cycle C_n and a tree T with maximum degree $\Delta(T) \geq 2$. Then G has a Hamiltonian cycle if and only if $\Delta(T) \leq n$.*

Proof. For any vertex $u \in V(T)$ with $deg(u) = \Delta(T)$, let S be a set such that $S = \{u^i \in V(G) \mid i \in \mathbb{Z}_n\}$. Let H be the graph obtained from G by removing the n vertices in S. We can observe that H is disconnected and has at least $\Delta(T)$ components. If $n < \Delta(T)$ then G has no Hamiltonian cycle. In the case $n \geq \Delta(T)$, we construct a Hamiltonian cycle: Take an arbitrary proper edge-coloring of T with n colors. For notational convenience, we choose the color set to be $\mathbb{Z}_n$. The n-edge-coloring of T exists as $n \geq \Delta(T)$ and the tree T is a bipartite graph which is $\Delta(T)$-edge-colorable.

Let $V(T) = V^- \cup V^+$ be the bipartition of T. Define a permutation $h : V(G) \to V(G)$ by following rules:

- If an edge (s,t) of T with $s \in V^-$ and $t \in V^+$ is colored with $i \in \mathbb{Z}_n$, then let $h(s^i) = t^i$ and $h(t^{i+1}) = s^{i+1}$;
- If $s \in V^-$ and there is no edge (s,t) colored with $i \in \mathbb{Z}_n$ then $h(s^i) = s^{i+1}$;
- If $t \in V^+$ and there is no edge (s,t) colored with $i \in \mathbb{Z}_n$ then $h(t^{i+1}) = t^i$

Observe that h is a cyclic permutation which defines a Hamiltonian cycle in G.

□

We can reformulate Theorem 2.3.11 in terms of lifts (covering graphs) of special voltage graphs. We observe that the graph $T \times C_n$ can be seen as the lift of a reflexive tree Γ, isomorphic to T (after deleting all loops from Γ), with a voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$, where each arc of Γ corresponding to a cut-edge has voltage 0, and each loop of Γ has voltage 1. Theorem 2.3.11 gives a necessary and sufficient condition for the lift Γ^σ to be Hamiltonian. Hell et al. [26] considered lifts of reflexive trees Γ with more general voltage assignments σ on $\mathbb{Z}_n$. They allowed every loop of Γ be assigned a value coprime to n, the order of the group, and gave a necessary and sufficient condition for the lift Γ^σ to be Hamiltonian. They proved two results (stated in the following theorem as part (a) and (b), respectively).

Theorem 2.3.12. *[26] Let Γ be a reflexive tree and let L be the set of self-loops. Let $\sigma : A(\Gamma) \to \mathbb{Z}_p$. Suppose the voltage graph $(\Gamma, \mathbb{Z}_p, \sigma)$ satisfies the following conditions:*

1. *There exists a system of paths $P_1, P_2, \ldots, P_k$ of Γ such that $E(P_1), E(P_2), \ldots E(P_k)$ is a partition of $E(\Gamma) \setminus L$, the paths P_i and P_j are internally vertex disjoint for any $i \neq j$, and for all $i, (1 \leq i \leq k)$, P_i satisfies either of the following:*

(a) the two ends of P_i have the same label, or

(b) there are two adjacent vertices u_i, v_i in P_i which have the same label, both u_i and v_i have degree at most two in $\Gamma - L$

2. $\sigma(w, w)$ is coprime to p for every $w \in V(\Gamma)$.

Then the covering graph Γ^σ is Hamiltonian if and only if $p \geq \Delta$, where Δ is the maximum degree of $\Gamma - L$.

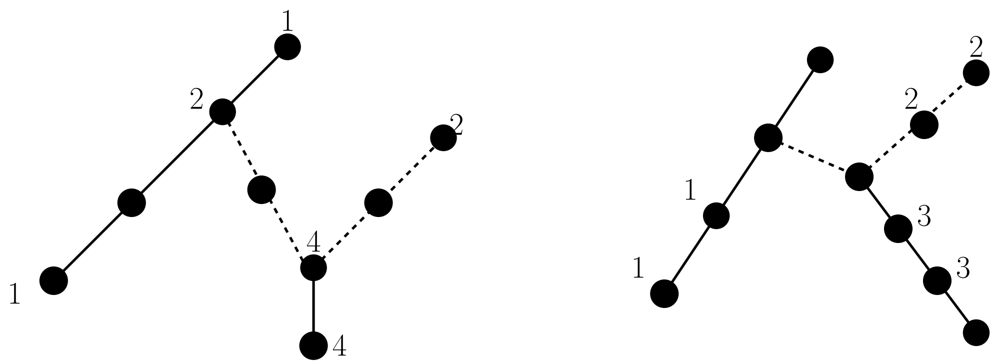


Figure 2.7: Two examples that satisfy either condition 1(a) or 1(b) in Theorem 2.3.12. The left figure is an example satisfying the condition 1(a). The right figure is an example satisfying the condition 1(b). The number at a vertex denotes the label of its self-loop. If a vertex has no number next to it, it means its self-loop can have any label coprime to p. Then, the paths with different types of lines form a system of paths satisfying the conditions in Theorem 2.3.12.

For an example of trees that satisfy the conditions of Theorem 2.3.12, see Figure 2.7.

This thesis aims to find further conditions under which the lift of a reflexive tree with a voltage assignment on a cyclic group is Hamiltonian.

2.4 Long Paths and Cycles in Highly Symmetric Graphs

Recall that in the previous section, we mentioned that there are only four known connected vertex-transitive graphs that do not have a Hamiltonian cycle. These four graphs include the Petersen graph, the Coxeter graph and two graphs derived from them. Although they are not Hamiltonian, we can observe that the Petersen graph and the Coxeter graph both have cycles of length $n-1$. The two graphs derived from them have a longest cycle of length $n-3$. Thus it is natural to consider the circumference of a graph as an approximation to a Hamiltonian cycle length. For example for vertex-transitive graphs, we can also approach Lovàsz's conjecture by finding a lower bound on path or cycle lengths in connected vertex-transitive graphs. The best-known result in this direction is that of Babai, who has shown the following.

Theorem 2.4.1. *[8] Every connected vertex-transitive graph with $n \geq 3$ vertices contains a cycle of length at least $\sqrt{3n}$.*

Proof. Let Γ be a graph and let G be the automorphism group of Γ. It is shown and proved in [8] that a connected vertex-transitive graph with minimum degree at least three is at least 3-connected. Let C be a maximum length cycle of Γ. It follows from [8] that any two cycles of maximum length in a 3-connected graph have at least three vertices in common. Therefore, $|C \cup C^g| \geq 3$ for any automorphism $g \in G$. Consider the number of pairs (g, x) where $x \in C \cup C^g$. For each element of G there are at least 3 such vertices in C, and therefore there are at least $3|G|$ such pairs. On the other hand, the elements of G that map x to y form a coset of G_x, and so there are exactly $|C||G_x|$ elements g^{-1} of G such that $x^{g^{-1}} \in C$, that is, $x \in C^g$. Therefore, $3|G| \leq |C|^2|G_x|$, and since G is transitive, $|G|/|G_x| = |V| = n$ by orbit stabilizer theorem. That is, $|C| \geq \sqrt{3n}$. $\square$

DeVos recently modified Babai's approach and gave a better bound for long cycles in vertex-transitive graph.

Theorem 2.4.2. *[16] Every connected vertex-transitive graph with $n \geq 3$ vertices contains a cycle of length at least $(1 - o(1))n^{3/5}$.*

Chapter 3

Decomposition Theorems for Hamiltonicity of Coverings of Trees

3.1 Billiard Strategy and Extended Billiard Strategy

The *billiard strategy* was first introduced in [26] as a method to construct a Hamiltonian cycle of a covering graph Γ^σ of a voltage graph (Γ, G, σ). Recall that G is a cyclic group of order n, and let us assume that Γ is a path with ends u and v. Start by considering the $u_0 - u_1$ Hamiltonian path in the fibre over u, leaving its two ends u_0 and u_1 open. Extend the path to the next fibre on the right from these ends to their corresponding vertices in this new fibre. Next include all remaining vertices of this fibre onto the constructed path by adding them in clockwise (or counter-clockwise) order from these starting vertices. This process will create new ends in this fibre, which are extended to the next fibre to the right. Repeat this process until we get to the fibre over v.

An important fact here is that the two new ends have the difference of their labels equal to 1, the same as the difference of u_0 and u_1. This difference is preserved in the fibre over v as well. If certain conditions are satisfied, using this process we can build a Hamiltonian cycle in Γ^σ.

The billiard strategy is at the heart of constructing a Hamiltonian cycle in a covering graph of a tree as presented in [26]. The focus of the remainder of this section is a refinement of the billiard strategy. We will describe and prove the refinement in the following lemmas. We call the new strategy the "extended billiard strategy".

Roughly speaking, given a covering graph of a reflexive path Γ over a cyclic group, we will be able to guarantee the existence of a family $\mathcal{P}$ of paths in the covering graph such that the paths in $\mathcal{P}$ include all vertices in the fibres over internal vertices of Γ, and such that the endpoints of paths in $\mathcal{P}$ appear in the fibre over each endpoint of Γ at voltages form an arithmetic progression in $\mathbb{Z}_n$. The original billiard strategy in [26] can be obtained from this by requiring that every voltage assignment on the path Γ be coprime to our group size n and then setting $|\mathcal{P}| = 2$.

For a path P with endpoints u and v, we will often give a direction to P and say that P *begins* at u and *ends* at v, or that P begins at v and ends at u. For a path P that begins at u and ends at v, if S is a vertex set satisfying $S \cap V(P) \neq \emptyset$, then we say that P *arrives* in S at w if $w \in V(P) \cap S$ and w is at the minimum distance from u along P among all vertices in $V(P) \cap S$.

Before stating and proving our lemma, we will state the following lemma from [26], by which we can always assume that $\sigma(e) = 0$ for each non-loop edge of Γ.

Lemma 3.1.1. *Let Γ be a reflexive graph, and let $(u, v) \in A(\Gamma)$ be an arc corresponding to a cut-edge $[u, v] \in E(\Gamma)$. Let G be a group, and for a pair $g, h \in G$, let $\sigma_g : A(\Gamma) \to G$ and $\sigma_h : A(\Gamma) \to G$ be voltage assignments. Suppose that σ_g and σ_h satisfy the following properties:*

- *$\sigma_g(u, v) = g$, and $\sigma_h(u, v) = h$;*
- *For each arc $e \in A(\Gamma)$ satisfying $e \notin \{(u, v), (v, u)\}$, $\sigma_g(e) = \sigma_h(e)$.*

Then $\Gamma^{\sigma_g} \cong \Gamma^{\sigma_h}$.

The following is the extended billiard strategy lemma.

Lemma 3.1.2. *For an integer $m \geq 1$, let $\Gamma = (v_1, \ldots, v_m)$ be a reflexive path, and let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ be a voltage assignment. Let $l, r \in [0, n-1]$ and $d \in [1, \gcd(r, n)]$ be constants. Then, there exists a family of d vertex-disjoint paths $\mathcal{P} = \{P_0, \ldots, P_{d-1}\}$ in Γ^σ satisfying the following properties:*

- *The paths $P_0, \ldots, P_{d-1}$ begin at the vertices $(v_1, l), (v_1, l+r), \ldots, (v_1, l+(d-1)r)$, respectively.*
- *For each $2 \leq t \leq m$, the paths of $\mathcal{P}$ arrive in the fibre v_t^σ at a set of d vertices $\{(v_t, i_t), (v_t, i_t + r), \ldots, (v_t, i_t + (d-1)r)\}$, for some value $i_t \in \mathbb{Z}_n$, where addition is calculated in $\mathbb{Z}_n$.*
- *For each $2 \leq t \leq m-1$, if a path $P \in \mathcal{P}$ visits a component K of a fibre v_t^σ, then every vertex of K is visited by a path from $\mathcal{P}$.*
- *For each $1 \leq t \leq m-1$, after a path $P \in \mathcal{P}$ leaves a fibre v_t^σ, P never returns to v_t^σ.*

Proof. By Lemma 3.1.1, we may assume that $\sigma(e) = 0$ for each arc $e \in A(\Gamma)$ that is not a loop. By applying an appropriate automorphism to Γ^σ, we may also assume without loss of generality that $l = 0$.

We induct on m, the number of vertices of Γ. When $m = 1$, for $j \in \{0, \ldots, d-1\}$, we let P_j be a path of length 0 containing the single vertex (v_1, jr). Since $d \leq \gcd(r, n)$, the first statement of the lemma holds, and the other three statements hold vacuously.

Now, suppose $m \geq 2$. We construct our family of paths as follows. By the induction hypothesis, there exists a vertex-disjoint family of paths $P_0, \ldots, P_{d-1}$ starting at $(v_1, 0), (v_1, r), \ldots, (v_1, (d-1)r)$ ending at a vertex in the set $\{(v_{m-1}, i), (v_{m-1}, i + r), \ldots, (v_{m-1}, i + (d-1)r)\}$, respectively, and satisfying the last three conditions of the lemma after replacing m with $m - 1$.

Let $a = \sigma(v_{m-1})$ be the voltage of the primary loop at v_{m-1}. Consider each path P_j, which arrives at the fibre v^σ_{m-1} at the vertex $(v_{m-1}, i_{m-1} + jr) \in V(v^\sigma_{m-1})$, where $i_{m-1} \in \mathbb{Z}_n$. If $a \neq 0$, then we perform the following steps for each P_j. We extend P_j by adding the vertices

$$(v_{m-1}, i_{m-1} + jr + a), (v_{m-1}, i_{m-1} + jr + 2a), (v_{m-1}, i_{m-1} + jr + 3a), \ldots$$

until we reach a vertex $(v_{m-1}, i_{m-1}+jr+sa)$ such that $(v_{m-1}, i_{m-1}+jr+(s+1)a)$ already belongs to a (not necessarily distinct) path P_k. This extension is depicted in Figure 3.1. As $(v_{m-1}, i_{m-1} + jr + (s+1)a)$ is the first vertex encountered by P_j that already belongs to a path P_k, it follows from the way that we have extended P_j that $(v_{m-1}, i_{m-1} + jr + (s+1)a)$ must be the vertex at which P_k arrived at the fibre v^σ_{m-1}; that is, $jr + (s+1)a = i_k$ At this point, we stop adding vertices from v^σ_{m-1} to P_j, with $(v_{m-1}, i_{m-1} + jr + sa)$ being the last vertex from v^σ_{m-1} added.

We claim that after applying this technique at v^σ_{m-1}, the endpoints of the paths $P_0, \ldots, P_{d-1}$ form the set

$$S_{m-1} := \{(v_{m-1}, i_{m-1} - a), (v_{m-1}, i_{m-1} - a + r), \ldots, (v_{m-1}, i_{m-1} - a + (d-1)r)\}.$$

(Note that we do not make any claims about the order in which these vertices appear as the endpoints of paths $P_0, \ldots, P_{k-1}$.) Indeed, if $a = 0$, then this claim clearly holds. If $a \neq 0$, then we recall that the paths $P_0, \ldots, P_{d-1}$ arrive at v^σ_{m-1} at vertices of the set

$$\{(v_{m-1}, i_{m-1}), (v_{m-1}, i_{m-1} + r), \ldots, (v_{m-1}, i_{m-1} + (d-1)r)\}.$$

Then, in the process of extending our paths, each path P_j is extended by adding vertices of the fibre v^σ_{m-1} to P_j until P_j reaches a vertex $(v_{m-1}, i_{m-1} + tr - a)$, where $t \in \{0, \ldots, d-1\}$. Furthermore, after extending each path P_j, the endpoints of the paths $P_0, \ldots, P_{d-1}$ must still be distinct. Therefore, it must follow that after extending each path P_j, the endpoints of $P_0, \ldots, P_{d-1}$ make up the set S_{m-1}. Thus, the claim holds.

Finally, for each path P_j, we add an edge $[(v_{m-1}, i_{m-1}+jr-a), (v_m, i_{m-1}+jr-a)]$ to extend P_j to v^σ_m. This completes our construction of paths $P_0, \ldots, P_{m-1}$. Observe that our family of paths is still vertex-disjoint.

We check that the four properties of the lemma hold. The first property holds by the induction hypothesis. The second property holds for $2 \leq t \leq m-1$ by the induction hypothesis and holds for $t = m$ by the construction. The third property holds for $2 \leq t \leq m-2$, by the induction hypothesis. For $t = m-1$, the statement must hold, as each component is a cycle and each path P_j does not exit a component of v^σ_{m-1} until P_j cannot visit any more vertices in that component. The fourth statement also holds by the induction hypothesis and by construction. Thus, induction is complete, and the theorem is proven. $\square$

Recall we will use the name *extended billiard strategy* to refer to the method used in Lemma 3.1.2 to generate our family $\mathcal{P}$ of paths. Lemma 3.1.2 tells us that given a path Γ, a voltage assignment σ, and a value d as outlined in the lemma statement, if we follow the extended billiard strategy as outlined to produce paths $P_0, \ldots, P_{d-1}$, then for each value $2 \leq t \leq m-1$, the paths arrive at the fibre v_t^σ at a set of d vertices $\{(v_t, i_t), (v_t, i_t + r), \ldots, (v_t, i_t + (d-1)r)\}$, for some value $i_t \in \mathbb{Z}_n$, where addition is calculated modulo n. By following the proof of Lemma 3.1.2, we see that this value i_t is in fact $-(\sigma(v_2) + \sigma(v_3) + \cdots + \sigma(v_{t-1}))$. Furthermore, after applying our method at v_t^σ so that the paths in $\mathcal{P}$ contain all vertices of v_t^σ, we see that the endpoints of the paths occupy the vertex set $\{(v_t, \alpha_t), (v_t, \alpha_t + r), \ldots, (v_t, \alpha_t + (d-1)r)\}$, where $\alpha_t = i_t - \sigma(v_t, v_t)$. Using this fact, we define the *order* of the paths $P_0, \ldots, P_{d-1}$ at v_t^σ as follows. After applying our method at v_t^σ, the paths with endpoints at $(v_t, \alpha_t), (v_t, \alpha_t + r), \ldots, (v_t, \alpha_t + (d-1)r)$, respectively, are $P_{a_1}, \ldots, P_{a_d}$, where $(a_1, \ldots, a_d)$ is some permutation of the set $\{0, \ldots, d-1\}$. We write $\pi(v_t) = (a_1, \ldots, a_d)$, and we say that the permutation $\pi(v_t)$ gives the *order* of the paths $P_0, \ldots, P_{d-1}$ at v_t. Note that $\pi(v_t)$ depends on t, d, n, σ, and r. It will be convenient to define $\pi(v_1) = id$.

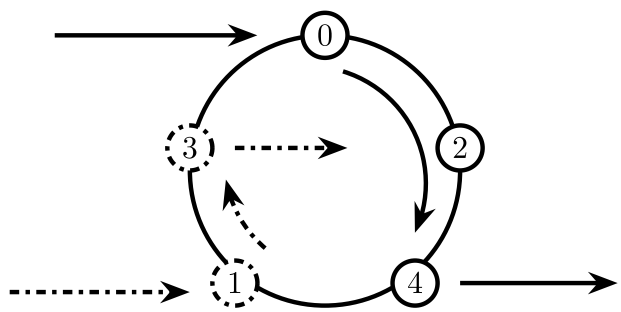


Figure 3.1: The figure shows an example of the path extension in Lemma 3.1.2. The underlying group is $\mathbb{Z}_5$. The cycle is a fibre over some vertex v_i with the voltage assignment $\sigma(v_i) = 2$. We have depicted two paths P_0 (the solid path) and P_1 (the dashed path). The path P_0 arrives at the fibre at 0, and P_1 arrives at the fibre at 1. We then extend P_0 to $0 + 2 = 2 \pmod 5$, and then to $2 + 2 = 4 \pmod 5$, and then stop because $4 + 2 = 1 \pmod 5$, and this vertex is already visited by P_1.

Definition 3.1.3. Let $\Gamma = (v_1, \ldots, v_m)$ be a reflexive path with a voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$. When $m \geq 3$, we define

$$c(\Gamma) = \left\lceil \frac{1}{2} \max\{\gcd(n, \sigma(v_t, v_t)) : 2 \leq t \leq m-1\} \right\rceil .$$

When $m = 2$, we say $c(\Gamma) = 1$.

The following corollary gives a simple condition for when a system of paths constructed in Lemma 3.1.2 includes all vertices in fibres over the internal vertices of Γ.

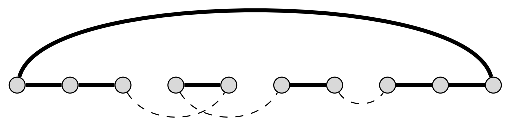


Figure 3.2: The figure shows a fibre v^σ in the lift Γ^σ of a voltage graph Γ. The edges in bold show the intersection of a subgraph $H \subseteq \Gamma^\sigma$ with v^σ, and the dashed edges represent the edges of H outside of v^σ. The subgraph H intersects v^σ at all edges except for three edges, and these three missing edges form an alternating consecutive edge set in v^σ.

Corollary 3.1.4. *For an integer $m \geq 2$, let $\Gamma = (v_1, \ldots, v_m)$ be a reflexive path, and let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ be a voltage assignment. Let $\mathcal{P} = \{P_0, \ldots, P_{d-1}\}$ be a family of paths on Γ^σ constructed according to Lemma 3.1.2 with a value r coprime to n. If $2c(\Gamma) \leq d \leq n$, then the paths of $\mathcal{P}$ visit all vertices in each fibre v_i^σ, for $2 \leq i \leq m-1$.*

Proof. Consider the fibre v_t^σ for a value $2 \leq t \leq m-1$. This fibre contains a component C_a for each coset $a + \langle \sigma(v_t) \rangle$ of $\mathbb{Z}_n$, and the number of such cosets is $\gcd(n, \sigma(v_t))$. By Lemma 3.1.2, the paths $P_0, \ldots, P_{d-1}$ arrive at v_t^σ at a vertex set of the form $\{(v_i, t), (v_i, t + r), \ldots, (v_i, t + r(d-1))\}$, so as $(r, n) = 1$, by the assumption $d \geq 2c(\Gamma) \geq \gcd(n, \sigma(v_t))$, the vertices of the paths $P_0, \ldots, P_{d-1}$ meet every component of v_t^σ. Therefore, by the third property of Lemma 3.1.2, $V(v_t^\sigma) \subseteq V(P_0) \cup \ldots \cup V(P_{k-1})$. $\square$

3.2 2-Factors in Covering Graphs of a Path

Given a path Γ satisfying the conditions of Corollary 3.1.4, we would like to find conditions for when the paths $P_0, ..., P_{d-1}$ can be joined together to form a Hamiltonian cycle.

Our first step in locating a Hamiltonian cycle involves establishing a necessary definition and a key lemma. The lemma provides the conditions for the presence of a 2-factor with special properties in the covering graph. We will apply this lemma later to find a Hamiltonian cycle in the lift of a path.

Definition 3.2.1. Given a vertex v whose loop has a nonzero voltage assigned by σ, we say that a set of edges $E \subseteq E(v^\sigma)$ is in *alternating consecutive order* if E forms a color class of a proper 2-coloring of the edges of some path in v^σ. We sometimes call E an *alternating consecutive edge set*.

For a graph Γ with a voltage assignment σ, we will often consider subgraphs of Γ^σ that intersect some fibre v^σ of Γ^σ in all edges except for some alternating consecutive edge set. In other words, we may consider a subgraph $H \subseteq \Gamma^\sigma$ for which $E(H) \cap E(v^\sigma) = E(v^\sigma) \setminus E$, where E is an alternating consecutive edge set in v^σ. We show an example of such a subgraph H in Figure 3.2.

In the following lemma and in later lemmas, we consider a path whose endpoints both have the label r which is coprime to n . However, we note that given a path Γ with such a voltage r at its endpoints, we can give each vertex $v \in V(\Gamma)$ a new label $\varphi(v) = \sigma(v)r^{-1} \in \mathbb{Z}_n$, and then Γ^φ is isomorphic to Γ^σ. Therefore, by relabelling group elements appropriately, we may assume in the proofs of this lemma and later lemmas that the endpoints of our path Γ have a voltage of 1.

Lemma 3.2.2. *Let $\Gamma = (v_1, \dots, v_m)$ be a reflexive path with a voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$. Suppose that $\sigma(v_1) = \sigma(v_m) = r$, where r is coprime with n. Then, for each even integer d satisfying*

$$2c(\Gamma) \leq d \leq n,$$

and for each integer l satisfying $0 \leq l \leq n-1$, there exists a 2-factor F of Γ^σ, as well as two edge subsets $E^L \subseteq E(v_1^\sigma)$ and $E^R \subseteq E(v_m^\sigma)$, such that

- $|E^L| = |E^R| = d/2$;
- $E^L = \{[(v_1, l), (v_1, l+r)], [(v_1, l+2r), (v_1, l+3r)], \dots, [(v_1, l+(d-2)r), (v_1, l+(d-1)r)]\}$;
- *E^L and E^R are alternating consecutive edge sets in v_1^σ and v_m^σ, respectively;*
- *F contains all edges of v_1^σ except E^L and all edges of v_m^σ except E^R.*

Proof. Let d and l be as in the statement of the lemma. By our previous discussion, we may assume that $r = 1$ by relabelling our group elements. We construct a family of d paths $\mathcal{P} = \{P_0, ..., P_{d-1}\}$ by the process of Lemma 3.1.2 (using our values l and $r = 1$), each with one endpoint in the fibre v_1^σ and other endpoint in the fibre v_m^σ. By Lemma 3.1.2, we may assume that the endpoints of each path P_i are $(v_1, l+i)$ and $(v_m, l+i+g)$, for a single value $g \in \mathbb{Z}_n$. Furthermore, by Corollary 3.1.4, we may assume that the paths of $\mathcal{P}$ contain all vertices of the fibres $v_2^\sigma, v_3^\sigma, \dots, v_{m-1}^\sigma$. We construct the 2-factor F from the union $P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ by adding the following edges:

- all edges in the unique perfect matching on the vertex set $\{(v_1, l+i) : 1 \leq i \leq d-2\}$ in v_1^σ;
- all edges in the unique perfect matching on the vertex set $\{(v_m, l+i+g) : 1 \leq i \leq d-2\}$ in v_m^σ;
- all edges of the path $(v_1, l+d-1), (v_1, l+d), (v_1, l+d+1), \dots, (v_1, l-1), (v_1, l)$;
- all edges of the path $(v_m, l+d+g-1), (v_m, l+d+g), (v_m, l+d+g+1), \dots, (v_m, l+g-1), (v_m, l+g)$.

It is straightforward to check that F is a 2-factor of Γ^σ. We let E^L to consist of all edges in the unique perfect matching on $\{(v_1, l+i) : 0 \leq i \leq d-1\}$ in v_1^σ. Similarly, we let the set E^R to consist of all edges in the unique perfect matching on $\{(v_m, l+i+g) : 0 \leq i \leq d-1\}$ in v_m^σ. By construction, all four properties of the lemma are satisfied for sets E^L and E^R. $\square$

Example 3.2.3. In Figure 3.3 an example illustrating application of Lemma 3.2.2 is presented. The underlying group is $\mathbb{Z}_n$, and the voltage of each vertex from $\mathbb{Z}_n$ is shown. Here, we suppose that $d = 4 \geq 2c(\Gamma)$ and $g = 3$. The two different shades of vertices show the two components of the 2-factor F. Moreover, the component of F containing P_1 and P_2 is highlighted in bold. The set E^L is depicted by the two dashed edges in v_1^σ, and the set E^R is depicted by the two dashed edges in v_m^σ.

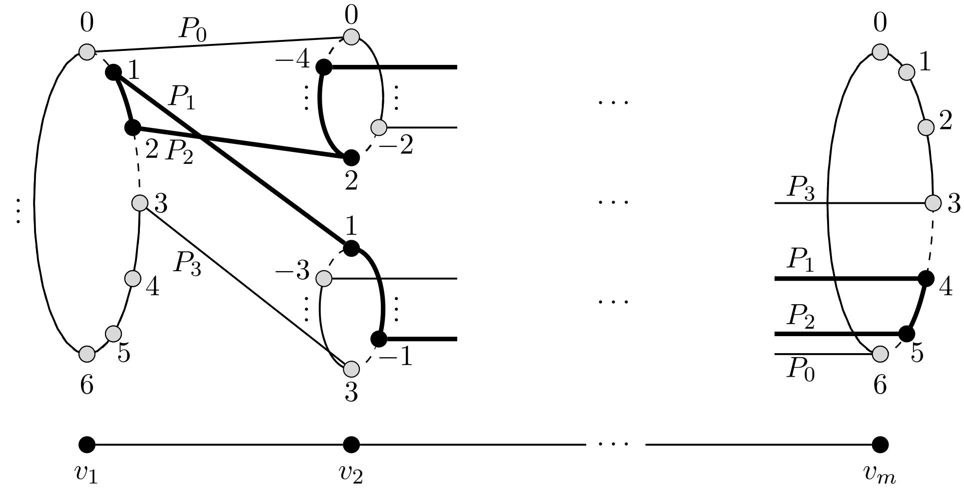


Figure 3.3: An example of construction of the 2-factor F described in Lemma 3.2.2 constructed using a family $\mathcal{P} = \{P_0, P_1, P_2, P_3\}$ of paths. The underlying group is $\mathbb{Z}_n$, and voltage of each vertex from $\mathbb{Z}_n$ is shown. Here, we suppose that $d = 4 \geq 2c(\Gamma)$ and $g = 3$. The two different shades of vertices show the two components of the 2-factor F. Moreover, the component of F containing P_1 and P_2 is highlighted in bold. The set E^L is depicted by the two dashed edges in v_1^σ, and the set E^R is depicted by the two dashed edges in v_m^σ.

The Lemma 3.2.2 will be the key ingredient in proving the Hamiltonicity of coverings of trees in the later sections.

3.3 Spider Decomposition

In the previous section, we have proved a sufficient condition when a lift of a path has a 2-factor. Using this result, in this section and the following one, we give sufficient conditions that guarantee that a lift of a tree is Hamiltonian. Our sufficient conditions will resemble the condition of [26].

Proposition 3.3.1. *Let Γ be a reflexive spider with m leaves $v_1, ..., v_m$, and a center v_0. Let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ be a voltage assignment. For each leaf v_i, let P_i be the path in Γ with endpoints v_0 and v_i. Suppose that for some value p, coprime to n, and for all $0 \leq i \leq m$, $\sigma(v_i, v_i) = p$, and that $\sum_{i=1}^m 2c(P_i) \leq n$. Then there exists a 2-factor F of Γ^σ and $m+1$ edge sets $E_i \subseteq E(v_i^\sigma)$, $0 \leq i \leq m$, satisfying the following properties:*

- *$|E_i| = c(P_i)$, $1 \leq i \leq m$, and $|E_0| = \sum_{i=1}^{m} c(P_i)$;*

- *E_i is alternating consecutive edge set, $0 \leq i \leq m$;*

- *F contains all edges of v_i^σ except E_i, $0 \leq i \leq m$.*

Proof. We write Γ as the union of m paths $P_1, \ldots, P_m$, where for $1 \leq i \leq m$, P_i has endpoints v_0 and v_i. Then, by Lemma 3.2.2, for each $1 \leq i \leq m$, each P_i^σ contains a 2-factor F_i along with an alternating consecutive edge sets $E_i^L \subseteq v_0^\sigma$ and $E_i^R \subseteq v_i^\sigma$ such that $E(F_i) \cap E(v_0^\sigma) = E(v_0^\sigma) \setminus E_i^L$ and $E(F_i) \cap E(v_i^\sigma) = E(v_i^\sigma) \setminus E_i^R$. As $\sum_{i=0}^{m-1} 2c(P_i) \leq n$, by applying an appropriate automorphism to the lift P_i^σ of each path P_i, we may assume that

$$E_0 := E_1^L \cup \cdots \cup E_m^L$$

is an alternating consecutive set, and $|E_0| = \sum_{i=1}^{m} c(P_i)$. Since, $E_0 \subseteq E(v_0^\sigma)$, E_0 satisfies both conditions of the proposition. Similarly, we set $E_i = E_i^R$ for $i = 1, .., m$. We obtain F as the graph on $V(\Gamma^\sigma)$ with the following edge set:

$$\left(\bigcup_{i=1}^{m} E(F_i) \setminus E(v_0^\sigma) \right) \cup \bigcap_{i=1}^{m} \left(E(F_i) \cap E(v_0^\sigma) \right).$$

By construction of F, every vertex outside of v_0^σ has degree 2. Furthermore, $E(F) \cap E(v_0^\sigma) = E(v_0^\sigma) \setminus E_0$, and since E_0 an alternating consecutive set, each vertex of v_0^σ also has degree 2 in F. Therefore, F is a 2-factor on Γ^σ. Finally, since $E_i = E_i^R$ for $i = 1, .., m$, and each E_i^R is an alternating consecutive edge set, so is E_i. □

Proposition 3.3.2. *Let Γ be a reflexive spider with m leaves $v_1, ..., v_m$, a center v_0, and a voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$. For $1 \leq i \leq m$, let P_i be the path in Γ with endpoints v_0 and v_i. Suppose that for some value p coprime to n, for $0 \leq i \leq m$, $\sigma(v_i, v_i) = p$, and $\sum_{i=1}^{m} 2c(P_i) \leq n$. Further suppose that the length of P_1 is one, i.e., v_0 and v_1 are adjacent in Γ. Then Γ^σ contains a Hamiltonian cycle C, as well as edge sets $E_i \subseteq E(v_i^\sigma)$ $0 \leq i \leq m$ satisfying the following properties:*

- *E_0 is a consecutive edge set, $|E_0| = \sum_{i=1}^{m} 2c(P_i)$, and C contains all edges of v_0^σ except those of E_0;*

- *E_1 is an alternating consecutive edge set, $|E_1| = \sum_{i=1}^{m} c(P_i)$, and C contains all edges of v_1^σ except those of E_1;*

- *For $2 \leq i \leq m$, E_i is alternating consecutive edge sets, $|E_i| = c(P_i)$, and C contains all edges of v_i^σ except those of E_i.*

Proof. We first write Γ as the union of the reflexive 2-path P (v_0, v_1) and the reflexive spider S with center v_0 and leaves $v_2, \ldots, v_m$. For $i \in \{2, 3, , \ldots, m\}$, let P_i be the path in Γ with endpoints

v_0 and v_i. By Proposition 3.3.1, we may find a 2-factor F on S^σ and an alternating consecutive edge set $F_0 \subseteq E(v_0^\sigma)$ of size $\sum_{i=2}^m c(P_i)$ such that $E(F) \cap E(v_0^\sigma) = E(v_0^\sigma) \setminus F_0$. Furthermore, as $\sum_{i=2}^m 2c(P_i) \leq n$, there exists an alternating consecutive edge set $F \subseteq E(v_0^\sigma)$ of size $|F_0|$ such that F contains at least one edge of each component of F. Now for each edge $v_0^j v_0^{j+1} \in F$ where $0 \leq j \leq n$, we remove the edge from v_0^σ and edge $v_1^j v_1^{j+1}$ from $E(v_1^\sigma)$. By connecting each endpoint v_0^j to the corresponding vertex v_1^j in v_1^σ we can get a Hamiltonian cycle C where $E(C) \cap E(F) = E(F) \setminus F$. Let $E_0 = F_0 \cup F$ and we can see that E_0 is a consecutive edge set on v_0^σ of size $2\sum_{i=2}^m c(P_i)$. Let E_1 be the set of edges that are removed from v_1^σ and we can see that $|E_1| = |F| = |F_0| = \sum_{i=2}^m c(P_i)$ and $E(C) \cap E(v_1^\sigma) = E(v_1^\sigma) - E_1$, that is, C contains all edges of v_1^σ except E_1. By Proposition 3.3.1 F contains all edges of v_i^σ except E_i which is an alternating consecutive edge set of size $|c(P_i)|$ for all $2 \leq i \leq m$. We can see that $E(C) = (E(F) \setminus E_0) \cup (E(v_1^\sigma \setminus E_1)$, so C contains all edges of v_i^σ except E_i as well. $\square$

We define a *base spider* as a triple (T, G, σ), where T is a subdivision of a star, and T has a central vertex v with $\sigma(v, v) = 1$. We also require each leaf l of T to have voltage $\sigma(l, l) = 1$. Finally, we require that $T \setminus v$ has at least one single-vertex component, and we choose one such single-vertex component and name this vertex w. We give every leaf of T a weight of 1, except for w. We give w the weight of $\sum c(C_i)$, where the sum runs over each component C_i of $T \setminus v$ apart from the vertex w. The central vertex v, and all leaves of T are called *joint* vertices.

We say two base spiders S_1 and S_2 are *spider internally vertex disjoint* if they satisfy one of the following:

1. S_1 and S_2 are vertex disjoint;

2. $V(S_1) \cap V(S_2) = \{v\}$, where v is a joint of both S_1 and S_2;

Example 3.3.3. Figure 3.4 gives an example of a spider with that satisfies the Proposition 3.3.2.

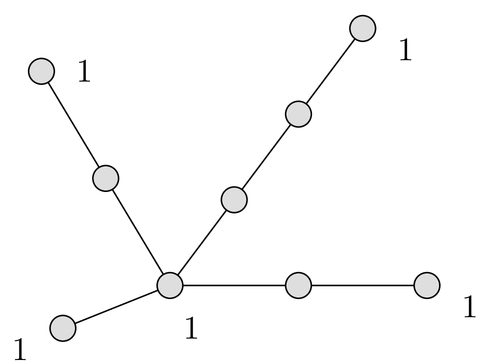


Figure 3.4: A spider graph Γ with voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$ such that each leaf and the center have a voltage that coprime to n, i.e 1. The spider graph also contains a "short leg" that meets the condition in Proposition 3.3.2.

Now, we have the following main result of this section.

Theorem 3.3.4. *Let T be a reflexive tree, and let L be the set of loops and $\sigma : E(T) \to \mathbb{Z}_n$ be a voltage assignment on T. If the voltage graph $(T, \mathbb{Z}_n, \sigma)$ satisfies the following conditions:*

- *There exists a system of base spiders $S_1, S_2, \dots S_k$ such that $\{E(S_1), E(S_2), \dots E(S_k)\}$ is a partition of $E(T) \setminus L$ and any pair of base spiders are spider internally vertex disjoint.*
- *For each vertex $v \in V(T)$ that is a joint of some base spider, the total weight that v receives from all of the base spiders to which it belongs is at most n.*

Then T^σ is Hamiltonian.

Proof. The proof is by induction on k, the number of base spiders in T, and we that there exists a Hamiltonian cycle C in T^σ that satisfies the following stronger property:

(1) For each leaf or branching vertex $v \in V(T)$, there exist a consecutive edge set $E_v \subseteq E(C) \cap E(v^\sigma)$ such that,

$$|E_v| \geq n - \sum_{i: v \in V(S_i)} \omega_{S_i}(v).$$

For a vertex $v \in V(T)$, we use $\deg(v)$ to denote the number of edges in T incident to v, not counting loops. In a base spider S with center v_0, we let P_i be the path with endpoints v_0 and v_i for $1 \leq i \leq \deg(v_0)$. Note that the path P_1 is the short leg of S, i.e., have the length one. For the base case, when $k = 1$, Proposition 3.3.2 implies that there exists a Hamiltonian cycle C in T. Furthermore there exist a set of consecutive edges of size $2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$ on v_0^σ that are not in $E(C)$. Therefore Hamiltonian cycle in a base spider contains

$$|E(v_0^\sigma)| - 2\sum_{i=2}^{\deg(v_0)} c(P_i) = n - 2\sum_{i=2}^{\deg(v_0)} c(P_i)$$

consecutive edges of $E(v_0^\sigma)$. For the short leg P_1 and its leaf v_1, Proposition 3.3.2 implies that there exists a set of alternating consecutive edges of size $\sum_{i=2}^{\deg(v_0)} c(P_i)$ on v_1^σ that are not in $E(C)$, so C contains

$$|E(v_1^\sigma)| - \sum_{i=2}^{\deg(v_0)} c(P_i) = n - \sum_{i=2}^{\deg(v_0)} c(P_i)$$

edges of $E(v_1^\sigma)$. It follows that $n - 2\sum_{i=2}^{\deg(v_0)} c(P_i)$ of these edges are consecutive in the fibre v_1^σ. For remaining leaves $v_2, \dots, v_{\deg(v_0)}$, by Proposition 3.3.2, C contains $n - 2c(P_i)$ consecutive edges in v_i^σ for $i = 2, \dots, \deg(v_0)$. Therefore for any leaf or branching vertex in the base spider S, statement (1) holds.

For the inductive step, suppose the proposition holds for all values up to $k - 1$. Let T be a tree with a corresponding system $S_1, \dots, S_k$ of base spiders that satisfies both conditions of the proposition. There exists at least one base spider, say S_k, such that

$$T_0 = \bigcup_{i=1}^{k-1} S_i$$

is connected.

We can see that $|V(S_k) \cap V(T)| = 1$, as otherwise, T_0 is disconnected. Let $\{u\} = V(S_k) \cap V(T)$, and let

$$\omega_{T_0}(u) = \sum_{i=1, u \in V(S_i)}^{k-1} \omega_{S_i}(u).$$

By the induction hypothesis, T_0^σ has a Hamiltonian cycle C_0 and there is a consecutive edge set

$$F = \{e_1, e_2, \ldots, e_x\} \in E(C_0) \cap E(u^\sigma)$$

of x edges, where $x \geq n - \omega_{T_0}(u)$ and e_i, e_{i+1} are consecutive for all $e_i \in F$.

Recall that $\omega_{S_k}(u)$ is the weight that u received in S_k. By the second condition of the proposition,

$$\omega_{S_k}(u) \leq n - \omega_{T_0}(u).$$

Now we show that we can extend C_0 to a Hamiltonian cycle in T^σ, say C, and C also satisfies statement (1). Following the notation of Proposition 3.3.1, we let v_0 be the central vertex of S_k, and we write $\deg_{S_k}(v_0) = m$. Then, we let v_1 be the leaf of S_k that is adjacent to v_0, and we let $v_2, \ldots, v_m$ denote the other leaves of S_k, for $2 \leq i \leq m$. By *strictly internally vertex disjoint rule*, u belongs to one of following three cases:

1. $u \in \{v_i : 2 \leq i \leq \deg_{S_k}(v_0)\}$;

2. u is v_1;

3. u is v_0 .

In Case 1, $\omega_{S_k}(u) = 2c(P_i)$. As S_k is base spider, we know that S_k^σ is Hamiltonian and contains a Hamiltonian cycle C_{S_k} such that there is a alternating consecutive edge set E_{leaf} of size $c(P_i)$ and $E(C_{S_k}) \cap E(u^\sigma) = E(u^\sigma) \setminus E_{leaf}$. Since $x \geq n - \omega_{T_0}(u) \geq \omega_{S_k}(u) = 2c(P_i)$, we can remove $c(P_i)$ alternating consecutive edges $e_1, e_3, \ldots, e_{2c(P_i)-1}$ from F and obtain $E_{leaf} = F \setminus \{e_1, e_3, \ldots, e_{2c(P_i)-1}\}$ satisfying the requirement above.

Now we can join C_{S_k} and C_0 as one cycle C that is a Hamiltonian cycle of T. We can find a consecutive edge set $E = F \setminus \{e_1, e_2, \ldots, e_{2c(P_i)-1}\} = \{e_{2c(P_i)}, e_{2c(P_i)+1} \ldots, e_x\} \in E(C) \cap E(u^\sigma)$ and

$$|E| = x - 2c(P_i) + 1 \geq n - \omega_{T_0}(u) - 2c(P_i) + 1 > n - (\omega_{T_0}(u) + \omega_{S_k}(u)) = n - \omega_T(u)$$

For Case 2, $\omega_{S_k}(u) = 2\sum_{i=1}^{deg(v_0)-1} c(P_i)$. Proposition 3.3.2 implies that there is a Hamiltonian cycle C_{S_k} in S_k such that there exist a alternating consecutive edge set $E_1 \in E(u^\sigma)$ of size $\sum_{i=1}^{deg(v_0)-1} c(P_i)$ such that

$$E(C_{S_k}) \cap E(u^\sigma) = E(u^\sigma) \setminus E_1$$

Since $x \geq n - \omega_{T_0}(u) \geq \omega_{S_k}(u) = 2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$, we can remove $\sum_{i=1}^{\deg(v_0)-1} c(P_i)$ alternating consecutive edges $e_1, e_3, \ldots, e_y$ from F and let it be E_1, where $y = 2\sum_{i=1}^{\deg(v_0)-1} c(P_i) - 1$. Now we can join C_{S_k} and C_0 as one cycle C that is a Hamiltonian cycle of T. We can find a consecutive edge set $E = F \setminus \{e_1, e_2, \ldots, e_y\} = \{e_{y+1}, e_{y+2} \ldots, e_x\} \in E(C) \cap E(u^\sigma)$ and

$$|E| = x - y = x - 2\sum_{i=1}^{\deg(v_0)-1} c(P_i) + 1 > n - (\omega_{T_0}(u) + \omega_{S_k}(u)) = n - \omega_T(u)$$

For case 3, $\omega_{S_k}(u) = 2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$. Proposition 3.3.2 implies that there is a Hamiltonian cycle C_{S_k} in S_k such that there exist a consecutive edge set $E_0 \in E(u^\sigma)$ of size $2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$ such that

$$E(C_{S_k}) \cap E(u^\sigma) = E(u^\sigma) \setminus E_0$$

Since $x \geq n - \omega_{T_0}(u) \geq \omega_{S_k}(u) = 2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$ so we can remove $2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$ consecutive edges $e_1, e_2, \ldots, e_y$ from F and let it be E_1, where $y = 2\sum_{i=1}^{\deg(v_0)-1} c(P_i)$. Now we can join C_{S_k} and C_0 as one cycle C that is a Hamiltonian cycle of T. We can find a consecutive edge set $E = F \setminus \{e_1, e_2, \ldots, e_y\} = \{e_{y+1}, e_{y+2} \ldots, e_x\} \in E(C) \cap E(u^\sigma)$ and

$$|E| = x - y = x - 2\sum_{i=1}^{\deg(v_0)-1} c(P_i) \geq n - (\omega_{T_0}(u) + \omega_{S_k}(u)) = n - \omega_T(u)$$

In all three cases, we have enough edges to combine C_0 and C_{S_k} to a Hamiltonian cycle C that satisfies (1). $\square$

3.4 Odd Shifting Decomposition

We now further generalize the labelling on trees which will give Hamiltonian lifts. For this, we need more definitions and terminologies that allow us to define such labellings.

Definition 3.4.1. Let $\Gamma = (v_1, \ldots, v_m)$ be a reflexive path, and let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ be a voltage assignment satisfying $\sigma(v_1, v_1) = \sigma(v_m, v_m)$. Let $\mathcal{P} = \{P_0, P_1, ..., P_{d-1}\}$ be a family of $d = 2c(\Gamma)$ paths in Γ^σ as defined in Lemma 3.1.2. For $2 \leq t \leq m-1$, recall that $\pi(v_t)$ denotes the permutation on $\{0, \ldots, d-1\}$ describing the order of the paths $P_0, \ldots, P_{d-1}$ as they leave v_t^σ. Thus, the permutation $\pi(v_{m-1})$ describes also the order of the paths in $\mathcal{P}$ as they enter v_m^σ. We say that (Γ, G, σ) is an *odd shifting path* if the permutation $\pi(v_{m-1})$ is of the form $(d-s, d-s+1, \ldots, d-1, 0, 1, \ldots, d-s-1)$

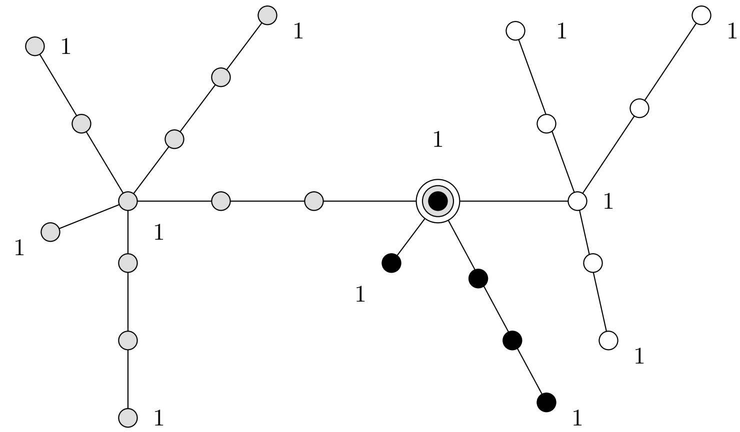


Figure 3.5: A voltage graph that is a tree that satisfies the conditions in Theorem 3.3.4 and can be decomposed into three base spiders (in white, black and gray). The number beside each vertex denotes the voltage assignment of its self-loop. If a vertex has no number beside it, it means that its self-loop can be any voltage within $\mathbb{Z}_n$.

for some odd number $0 < s < d$. In other words, if (Γ, G, σ) is an odd shifting path, then an odd number of the d paths in $\mathcal{P}$ that were at the beginning of order $\pi(v_1)$ are shifted to the order $\pi(v_{m-1})$ at the end. We also say that the value $r = \sigma(v_1, v_1) = \sigma(v_m, v_m)$ is the *endpoint voltage* of (Γ, G, σ).

In the following example, we will demonstrate what an odd shifting path may look like.

Example 3.4.2. For an integer $m \geq 1$, let $\Gamma = (v_1, \ldots, v_m)$ be a reflexive path, and let $\sigma : A(\Gamma) \to \mathbb{Z}_n$ be a voltage assignment. Let $\{P_0, P_1, ..., P_d\}$ be the paths that are used to perform billiard strategy on Γ^σ. Consider a pair of consecutive vertices on Γ, v_i and v_{i+1}. If $\pi(v_{i-1}) = \pi(v_{i+1})$ then we call v_i and v_{i+1} as *order preserving pair*. One can easily check that an *inverse pair*, where $\sigma(v_i, v_i)$ is the inverse of $\sigma(v_{i+1}, v_{i+1})$ in $\mathbb{Z}_n$ is an order preserving pair.

Let v be an internal vertex on a path Γ and u,w are the two vertices adjacent to v. We remove v and both edges adjacent to v and then connect u and w with an edge. The operation is called *smoothing out* v from Γ.

For a path Γ with an even number of vertices, if we can recursively find and smooth out pairs of order preserving pairs among internal vertices until there are only two endpoints left, we say the path Γ is *order preserving path*.

If we form a single path Γ by joining an even number of order preserving paths $\Gamma_1, \ldots, \Gamma_{2k}$ all with a common voltage r coprime to n at their endpoints, then Γ is an odd shifting path. See Figure 3.6.

We are now ready to state our result which is a sufficient condition when an odd shifting path lifts to a Hamiltonian cycle. Because we will be using this result later for general trees, we prove a stronger result where we put several conditions on the resulting Hamiltonian cycle, in particular, how the cycle traverses through the fibres of the first and the last vertex of the path.

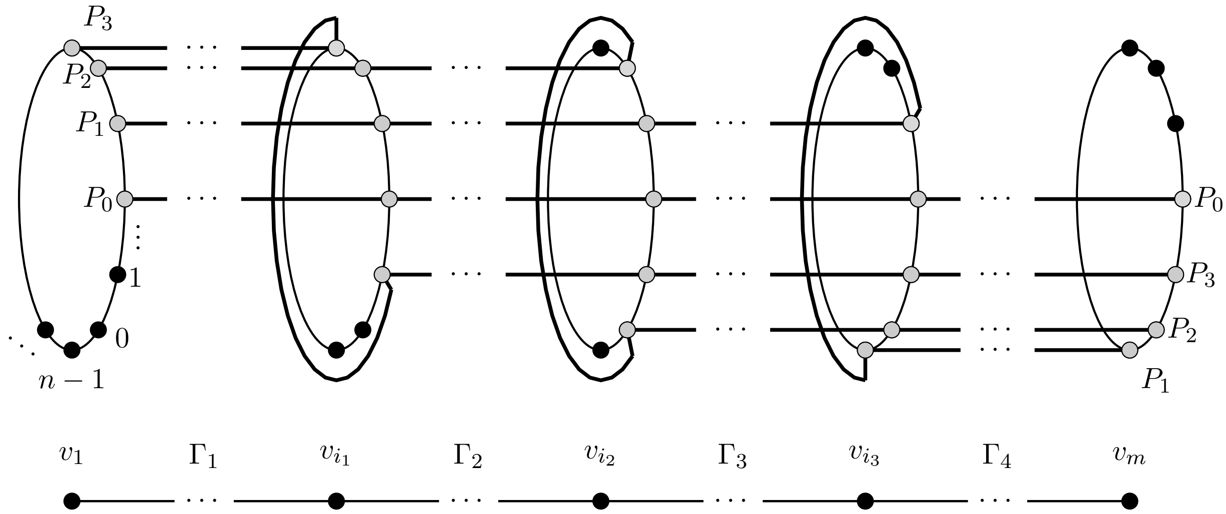


Figure 3.6: The figure shows an example of an odd shifting path consisting of four order-preserving paths $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ with endpoints $v_0, v_{i_1}, v_{i_2}, v_{i_3}, v_m$. In the figure, we show four paths P_0, P_1, P_2, P_3 in the lift that are constructed by the process of Lemma 3.1.2. Since each path Γ_i is order preserving, the final order of the paths P_0, P_1, P_2, P_3 arriving at $v_{i_4}^{\sigma}$ is ultimately only affected by three single shifts that occur in fibres $v_{i_1}^{\sigma}$, $v_{i_2}^{\sigma}$, and $v_{i_3}^{\sigma}$, and therefore the final order of these paths in v_m^{σ} is P_1, P_2, P_3, P_0. Also note that while $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 1$ in for simplicity, the vertices $v_{i_1}, v_{i_2}, v_{i_3}$ can have any common voltage r coprime to n in our construction.

Theorem 3.4.3. *For an integer $m \geq 2$, let $\Gamma = (v_1, \ldots, v_m)$ be an odd shifting path with a voltage assignment $\sigma : A(\Gamma) \to \mathbb{Z}_n$ and an endpoint voltage r coprime to n. Let $\mathcal{P} = \{P_0, \ldots, P_{d-1}\}$ be the family of paths constructed by the process of Lemma 3.1.2 using $d = 2c(\Gamma)$ and an element $l \in \mathbb{Z}_n$. Then, Γ^{σ} contains a Hamiltonian cycle C and two edge subsets $E^L \subseteq E(v_1^{\sigma})$ and $E^R \subseteq E(v_m^{\sigma})$ such that*

- *C contains all edges of the paths $P_0, \ldots, P_{d-1}$;*
- *$|E^L| = |E^R| = d/2$;*
- *$E^L = \{[(v_1, l), (v_1, l + r)], [(v_1, l + 2r), (v_1, l + 3r)], \ldots, [(v_1, l + (d-2)r), (v_1, l + (d-1)r)]\}$;*
- *E^L and E^R are alternating consecutive edge sets;*
- *C contains all edges of v_1^{σ} except E^L and C contains all edges of v_m^{σ} except E^R.*

Proof. As discussed before, we may assume without loss of generality that $r = 1$ and $l = 0$. Let $P_0, \ldots, P_{d-1}$ be the d paths produced by the extended billiard strategy described in Lemma 3.1.2.

Since (Γ, G, σ) is an odd shifting path, the paths $P_0, \ldots, P_{d-1}$ arrive in v_m^σ respectively at the voltages

$$g + (d - s), g + (d - s + 1), \ldots, g + (d - 1), g, g + 1, \ldots, g + (d - s - 1)$$

for some element $g \in \mathbb{Z}_n$ and some odd number $0 < s < d$. We now obtain a Hamiltonian cycle C on Γ^σ from $P_0 \cup P_1 \cup \cdots \cup P_{d-1}$ by adding the following edges:

- all edges in the unique perfect matching on the vertex set $\{(v_1, i) : 1 \leq i \leq d - 2\}$ in v_1^σ;
- all edges in the unique perfect matching on the vertex set $\{(v_m, g + i) : 1 \leq i \leq d - 2\}$;
- all edges of the path $(v_1, d - 1), (v_1, d), \ldots, (v_1, n - 1), (v_1, 0)$;
- all edges of the path $(v_m, g + d - 1), (v_m, g + d), \ldots, (v_m, g + n - 1), (v_m, g)$.

We note that for each odd value $1 \leq i \leq d - 1$, the endpoints of P_i and P_{i+1} in v_1^σ are joined by a path in $C \cap E(v_1^\sigma)$ (with d and 0 identified). Furthermore, since (Γ, G, σ) is an odd shifting path, for each even value $0 \leq i \leq d - 2$, the endpoints of P_i and P_{i+1} in v_m^σ are joined by a path in $C \cap E(v_m^\sigma)$. Hence, C is a Hamiltonian cycle that, starting in v_1^σ, visits $P_0, P_1, \ldots, P_{d-1}$ in order.

Finally, we let E^L be the unique perfect matching on $\{(v_1, i) : 0 \leq i \leq d - 1\}$ in v_1^σ which is not in C. Similarly we let E^R be the unique perfect matching on $\{(v_m, g + i) : 0 \leq i \leq d - 1\}$ in v_m^σ. We observe that both E^L and E^R are alternating consecutive edge sets. This completes the proof. $\square$

Now we will show that if a tree can be decomposed into multiple odd shifting paths, then the lift of this tree is often Hamiltonian. Beforehand, we need a bit more definitions.

Definition 3.4.4. Let T be a reflexive tree with a voltage assignment $\sigma : E(T) \to \mathbb{Z}_n$, and L be its set of loops. If there exists a system of odd shifting paths $\{Q_1, \ldots, Q_k\}$ such that $\{E(Q_1) \setminus L, \ldots, E(Q_k) \setminus L\}$ is a partition of $E(T) \setminus L$, and if the paths Q_i and Q_j are strictly internally vertex disjoint[1] for any $i \neq j$, then we say that T can be *odd shifting decomposed*, and we say that $\{Q_1, \ldots, Q_k\}$ is the *odd shifting decomposition* of T. Observe that in an odd shifting decomposition $\{Q_1, \ldots, Q_k\}$, since the odd shifting paths only intersect at their endpoints which we will call *joints*, the paths $Q_1, \ldots, Q_k$ all have the same endpoint voltage.

Definition 3.4.5. Let Q_i be a path from an odd shifting decomposition $\{Q_1, \ldots, Q_k\}$ of T, with endpoints u and v. We assign *weight* ω_{Q_i} to u and v, as follows $\omega_{Q_i}(u) = \omega_{Q_i}(v) = 2c(Q_i)$. We define the *tree weight* of v in T as $\Omega_T(v) = \sum \omega_{Q_i}(v)$, where the sum is over all paths in $\{Q_1, \ldots, Q_k\}$ for which v is an endpoint. We say (T, Z_n, σ) is *properly weighted* if $\Omega_T(v) \leq n$ (the order of the group $\mathbb{Z}_n$) holds for each joint v of T.

[1]Two paths P and Q are strictly internally vertex disjoint if each vertex in $V(P) \cap V(Q)$ is an endpoint of both P and Q.

Finally, we have the following sufficient condition for the existence of a Hamiltonian cycle in the lift of a tree.

Theorem 3.4.6. *Let T be a reflexive tree and let $\sigma : E(T) \to \mathbb{Z}_n$ be a voltage assignment on T. Suppose the voltage graph (T, Z_n, σ) can be odd shifting decomposed into paths $\{Q_1, \ldots, Q_k\}$ whose common endpoint voltage is coprime to n. If (T, Z_n, σ) is properly weighted, then T^σ is Hamiltonian.*

Proof. As discussed before, we may relabel the elements of our group G so that our paths Q_i all have endpoint voltage 1. Our proof is by induction on k, the number of odd shifting paths in the decomposition of T. We will prove the stronger statement that there exists a Hamiltonian cycle C in T^σ that satisfies the following property:

(1) For each joint vertex $v \in V(T)$, there exists a path P with $E(P) \subseteq E(C) \cap E(v^\sigma)$ such that

$$|E(P)| \geq n - \Omega_T(v).$$

For the base case, when $k = 1$, T is a path with two joints u and v. Theorem 3.4.3 implies that there exists a Hamiltonian cycle C in T along with two alternating consecutive edge sets $E^R \subseteq E(v^\sigma)$ and $E^L \subseteq E(u^\sigma)$ of size at most $c(\Gamma)$ such that C contains all edges of u^σ and v^σ except E^R and E^L. Therefore, $E(C) \cap E(v^\sigma)$ contains a path of at least $n - 2c(T) = n - \Omega_T(v)$ edges; that is, (1) holds.

For the inductive step, suppose that (1) holds for all values up to $k - 1$. Let T be a tree with a corresponding system $\{Q_1, \ldots, Q_k\}$ of odd shifting paths that satisfies the conditions of the theorem. Since $\{Q_1, \ldots, Q_k\}$ is an edge-partition of the tree T, there exists at least one odd shifting path, without loss of generality Q_k, that intersects $\bigcup_{i=1}^{k-1} V(Q_i)$ at only one of its endpoints. Observe that

$$T_0 := \bigcup_{i=1}^{k-1} Q_i$$

is connected and hence a tree. Moreover, $\{Q_1, \ldots, Q_{k-1}\}$ is an odd shifting decomposition of T_0, and T_0 is properly weighted. Let u be the vertex where Q_k is joined to T_0—that is, $\{u\} = V(Q_k) \cap V(T_0)$. Furthermore, let

$$\Omega_{T_0}(u) = \sum_{\substack{\{i \in [1,k-1]: \\ u \in V(Q_i)\}}} \omega_{Q_i}(u).$$

By the induction hypothesis, T_0^σ has a Hamiltonian cycle C_0, and for each joint vertex of T_0, there exists a path which satisfies (1). In particular, we can find a path

$$P = (e_1, e_2, \ldots, e_s) \subseteq E(C_0) \cap E(u^\sigma),$$

where $s \geq n - \Omega_{T_0}(u)$.

Now, recall that u is the endpoint of Q_k, and u is in T_0, and let $v \neq u$ be the other endpoint of Q_k. According to Theorem 3.4.3, Q_k^σ contains a Hamiltonian cycle C_k along with two alternating consecutive edge sets $E_k^L \subseteq E(u^\sigma)$ and $E_k^R \subseteq E(v^\sigma)$ of size $c(Q_k)$, for which

$$E(C_k) \cap u^\sigma = u^\sigma \setminus E_k^L \text{ and } E(C_k) \cap v^\sigma = u^\sigma \setminus E_k^R.$$

Now, we show that we can combine C_0 and C_k to a Hamiltonian cycle C in T^σ, such that C will also satisfy the statement (1). By our assumption, $\omega_{Q_k}(u) = 2c(Q_k)$, and $n - \Omega_{T_0}(u) \geq \omega_{Q_k}(u)$. Since

$$s \geq n - \Omega_{T_0}(u) \geq 2c(Q_k) = |E_k^L|,$$

by choosing an appropriate value for l in Theorem 3.4.3 such that (u, l) is an endpoint of e_1 but not of e_2, we observe that $E_k^L = \{e_1, e_3, \dots, e_{2c(Q_k)-1}\}$ in C_k. Without loss of generality, we assume that $l = 0$, so that the endpoints of each edge $e_i \in P$ are $(u, i-1)$ and (u, i).

We define $A_0 = C_0 \setminus P$, and we see that A_0 is a Hamiltonian path on $T_0^\sigma \setminus P$ with endpoints $(u, 0)$ and (u, s). Similarly, we define

$$A_k = C_k \setminus (E(u^\sigma) \setminus P)$$

and we see that A_k is a Hamiltonian path on the graph $Q_k^\sigma \setminus (E(u^\sigma) \setminus P)$ with endpoints $(u, 0)$ and (u, s). Therefore, $V(A_0) \cup V(A_k) = V(T^\sigma)$, and since A_0 and A_k only intersect at their endpoints, it follows that $C = A_0 \cup A_k$ is a Hamiltonian cycle on T^σ. See Figure 3.7 for an example of this construction.

Now, we show that C satisfies (1). We observe that $E(C) \cap v^\sigma = v^\sigma \setminus E_k^R$. Since E_k^R is an alternating consecutive edge set of size $c(Q_k)$, $E(v^\sigma) \cap E(C)$ contains a path of at least $n - 2c(Q_k) = n - \Omega_T(v)$ edges. Next, we consider $E(u^\sigma) \cap E(C)$. When constructing C from C_k, only the edge set $\{e_1, e_3, \dots, e_{2c(Q_k)-1}\}$ was deleted from C_0, so by the induction hypothesis, $u^\sigma \cap E(C)$ still has a path $(e_{2c(Q_k)}, \dots, e_s)$ with

$$r - 2c(Q_k) + 1 > (n - \Omega_{T_0}(u)) - \omega_{Q_k}(u) = n - \Omega_T(u)$$

edges. Hence, both joint vertices satisfy the condition (1), and the condition (1) has not changed for the remaining joint vertices of T. Therefore, (1) holds for all vertices of T, and induction is complete. $\square$

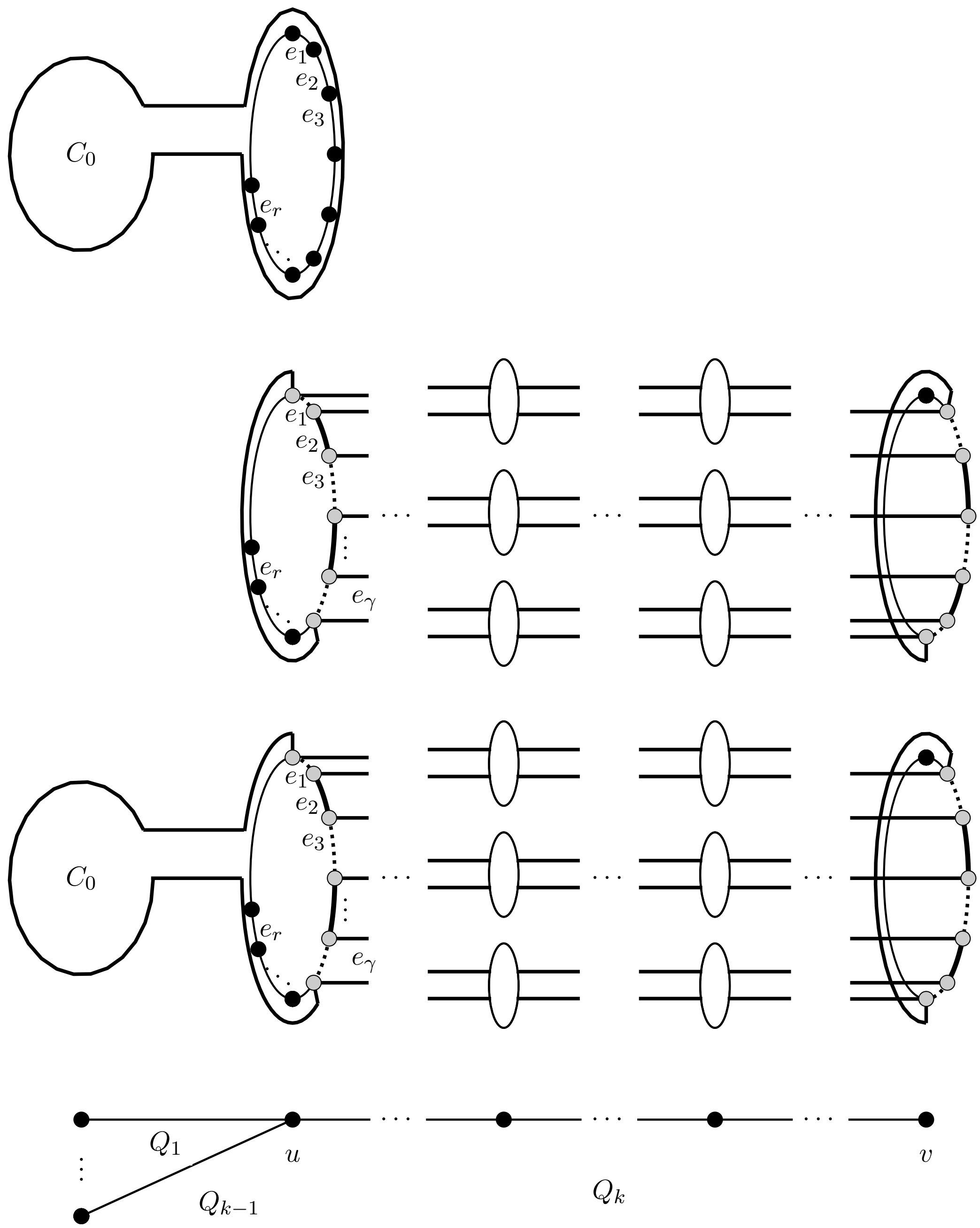


Figure 3.7: The figure shows four graphs. The first (top) graph represents T_0^σ, and a Hamiltonian cycle C_0 is shown. The second graph shows the Hamiltonian cycle C_k constructed on Q_k^σ using Theorem 3.4.3. The third graph shows the combination of C_0 and C_k that gives a Hamiltonian cycle C on T^σ as in the proof of Theorem 3.4.6. (In the second and third graphs, we write $\gamma = 2c(Q_k) - 1$.) Finally, the last figure shows the base graph T.

Chapter 4

Cycles in Coverings of Trees Over a Large Prime Order Cyclic Groups

We want to acknowledge that the main ideas of the work presented in this chapter are of Peter Bradshaw.

4.1 Hamiltonicity of coverings of trees over a large prime order cyclic groups

Results so far have been for special voltage assignments and worked even for smaller order groups. In this chapter, we want to consider general voltage assignments when the group order is large prime. Before starting our discussion on this topic we will establish some terminologies and notations for convenience.

Given a reflexive tree T, let $\varphi : E(T) \to \mathbb{Z}$ be an assignment of integers to the edges of T, and we will always let φ be positive at each loop of T. We define a homomorphism $\psi : \mathbb{Z} \to \mathbb{Z}_p$ such that $x \mapsto x + \langle p \rangle$ where p will always be prime and typically large. Then, for each edge $e \in E(T)$ with corresponding arcs $e^+, e^- \in A(T)$, we let $\sigma(e)$ be $\psi \circ \varphi(e)$ and $-\psi \circ \varphi(e)$ for e^+ and e^-, respectively. When interpreting the integers given by φ as group elements in $\mathbb{Z}_p$, we can observe that given T and φ, σ gives a voltage assignment to $A(T)$ over $\mathbb{Z}_p$. For a vertex $v \in V(T)$, we will often say that $\sigma(e)$ is the voltage of v, where e is the loop at v, and we often abuse the notation to $\sigma(v)$. Note that since p is prime and φ is nonzero, for each vertex $v \in V(T)$, $\sigma(v)$ is a generator for $\mathbb{Z}_p$, and therefore every fibre of T^σ contains a single cycle. Now we can consider the following specific question:

Question 4.1.1. *Let T be a reflexive tree, and let φ be an assignment of integers to $E(T)$ as described above. Does there exist a number $N \in \mathbb{N}$ such that T^σ is Hamiltonian whenever $p \geq N$?*

We do not have an answer to Question 4.1.1, but we are able to show the following results.

Theorem 4.1.2. *Let T be a reflexive tree, and let φ be an assignment of integers to $E(T)$ as described above. Suppose that there exists a vertex $v \in V(T)$ such that for every neighbour u of v, $\varphi(u) = \varphi(v)$. Then, when p is a sufficiently large prime, T^σ is Hamiltonian.*

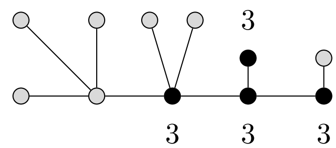


Figure 4.1: This figure shows a reflexive voltage tree (with loops omitted) that satisfies the conditions in Theorem 4.1.2. The number beside each vertex denotes the voltage assignment of its loop. If a vertex has no number beside it, then its loop can have any nonzero voltage within $\mathbb{Z}_p$.

Theorem 4.1.3. *Let T be a reflexive tree, and let φ be an assignment of integers to $E(T)$ as described above. Suppose that there exist two adjacent vertices $u, v \in V(T)$ for which $\varphi(u) = \varphi(v) = 1$. Then, when p is a sufficiently large prime, T^σ is Hamiltonian.*

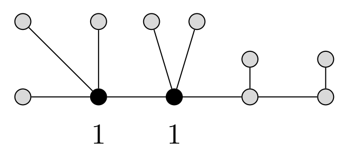


Figure 4.2: A reflexive voltage tree (with loops omitted) is shown that satisfies the conditions of Theorem 4.1.3. The number next to each vertex denotes the voltage assignment on its loop. If a vertex has no number next to it, its loop can have any nonzero voltage from $\mathbb{Z}_p$ assigned.

Trees that satisfy the conditions of Theorem 4.1.2 and Theorem 4.1.3 are depicted in Figures 4.1 and 4.2.

Before presenting our proofs, we first make some assumptions, state some definitions and prove lemmas that will help us prove the two results. As before, we will assume without loss of generality that $\varphi(e) = 0$ for every cut-edge $e \in E(T)$, which can be done according to Lemma 3.1.1. Moreover, we will assume that the order of the group p is a sufficiently large prime number, and finally, we will write $\varphi(v)$ instead of $\varphi(v, v)$ for a vertex $v \in V(T)$.

For a vertex $v \in V(T)$, and for two integers $0 \leq a, b \leq p - 1$, we define $v^\sigma[a, b]$ to be the graph induced by the vertex set $\{(v, i) : a \leq i \leq b\}$. We also have the following definition.

Definition 4.1.4. Let $v \in V(T)$. Then for each positive multiple N of $2\varphi(v)$, we let $M_v(N)$ denote the matching in v^σ containing all edges of the form

$$\{(v, 2a\sigma(v) + i), (v, (2a + 1)\sigma(v) + i)\}$$

for $a \in \{0, 1, \ldots, \frac{N}{2\varphi(v)} - 1\}$, $i \in \{0, 1, \ldots, \varphi(v) - 1\}$. Furthermore, for an element $g \in G$, we say that $M_v(N) + g$ denotes the matching obtained from $M_v(N)$ by applying the automorphism $(v, t) \mapsto (v, t + g)$.

Examples of $M_v(12)$ and $M_v(12) + 1$ where $\varphi(v) = 3$ are shown in Figure 4.3. The matchings described in Definition 4.1.4 will be useful in proofs of our theorems when we build a 2-factor in T^σ under certain conditions and then connect all components of the 2-factor into a Hamiltonian cycle. Informally, for two adjacent vertices $w, x \in V(T)$, if we have a 2-factor F of the lift of the component of $T - x$ containing w that contains many edges of w^σ and a second 2-factor F_0 of the lift of the component of $T - w$ containing x that contains many edges of x^σ, then we may "join" F and F_0 by removing two matchings of this form of equal size from w^σ and x^σ and replacing them with a matching consisting of edges of the form $[(w, g), (x, g)]$. Figure 4.4 shows two 2-factors being joined in a similar way to what we have described here. The removed edges are depicted in gray, and the added edges are shown as vertical edges. The following observations will be useful for us when we use these matchings.

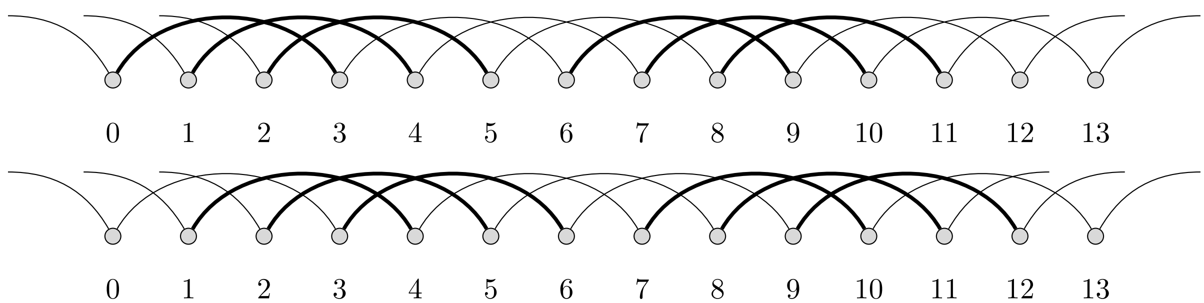


Figure 4.3: We show here a part of a fibre v^σ of a vertex v with $\varphi(v) = 3$ which belongs to the lift of a voltage graph over a large cyclic group. On the top row, the bolded edges make up the edges in the matching $M_v(12)$. In the bottom row, the bolded edges make up the edges in the matching $M_v(12) + 1$.

Observation 4.1.5. *For each positive integer N, when p is sufficiently large, the matchings $M_v(N)$ and $M_v(N) + \sigma(v)$ are edge-disjoint.*

Observation 4.1.6. $M_v(2N) = M_v(N) \cup (M_v(N) + N)$.

In next lemma, we show that for any vertex $v \in V(T)$, T^σ has a 2-factor H each of whose components contains at least one edge in a certain local structure of the fibre v^σ. This property will be later used to "attach" components of a 2-factor together into a Hamiltonian cycle. We have this property that an edge of every component of the 2-factor can be found in a specific part of T^σ.

Lemma 4.1.7. *Suppose that p is a sufficiently large prime in terms of T and φ. Then, there exists a positive integer $N = N(\varphi, T)$ such that for each vertex $v \in V(T)$, T^σ contains a 2-factor H*

satisfying $M_v(N) \subseteq E(H)$, and such that each component of H has an edge in the matching $M_v(N)$. Furthermore, H may be chosen such that H contains every edge of $v^\sigma[N, p-1]$.

Proof. The proof is based on induction on the number of vertices of T. For the base case, $|V(T)| = 1$, we can observe that T^σ is a cycle. Therefore we can let $H = T^\sigma$ and $N = 2\varphi(v)$, and the statement holds.

For the inductive steps, we suppose $|V(T)| > 1$. Let $v \in V(T)$ with neighbours $u_0, \dots, u_{r-1}$. Let $T \setminus \{v\}$ have components $T_0, \dots, T_{r-1}$, so that each neighbour of v is contained in each component, i.e. $u_i \in V(T_i)$ for each $i \in \{0, \dots, r-1\}$. By the induction hypothesis, for each $i \in \{0, \dots, r-1\}$, we may choose an integer $N_i = N_i(\varphi_{|T_i}, T_i)$ and a 2-factor F_i on T_i such that the matching $M_{u_i}(N_i)$ is a subset of $E(F_i)$ and contains an edge of each component of F_i, and such that F_i contains every edge of $u_i^\sigma[N_i, p-1]$. Then, we define

$$N' = 2\varphi(v) N_1 N_2 \dots N_r,$$

and by applying automorphisms to each F_i, it follows that for each $i \in \{0, \dots, r-1\}$, we may find a 2-factor H_i on T_i such that the matching $M_{u_i}(N_i) + iN' + \sigma(v)$ is a subset of $E(H_i)$ and contains an edge of each component of H_i. Furthermore, since F_i contains every edge of $u_i^\sigma[N_i, p-1]$, and since p is sufficiently large, we can also assume that $M_{u_i}(N') + iN' + \sigma(v) \subseteq E(H_i)$ is also a subset of $E(H_i)$.

Next, we consider the graph $H \subseteq G$ containing the edges of the cycle v^σ as well as $E(H_0) \cup \dots \cup E(H_{r-1})$. As H is a union of the 2-factors $H_0, \dots, H_{r-1}$ as well as the cycle v^σ, H is 2-regular. Let us modify the graph H as follows. For each $i \in \{0, \dots, r-1\}$, add the edges

$$[(v, \sigma(v) + iN' + a), (u_i, \sigma(v) + iN' + a)]$$

for each value $a \in \{0, \dots, N'-1\}$ into H, and for each $i \in \{0, \dots, r-1\}$, remove the matching $M_{u_i}(N') + iN' + \sigma(v)$ from H. Furthermore, remove the matching $M_v(rN') + \sigma(v)$ from H. This leaves us with a new 2-factor H in T^σ. An example of a construction of H is shown in Figure 4.4. In the figure, $r = 2$, $\varphi(u_0) = 1$, $\varphi(u_1) = \varphi(v) = 2$, and $N' = 8$.

After the set-ups, we will prove the lemma by demonstrating the existence of an integer $N = rN' + 2\varphi(v)$ that satisfies the conditions of Lemma 4.1.7. We will first observe two claims that will help to complete the proof.

Claim 4.1.8. *$M_v(N) \subseteq E(H)$. Furthermore, H contains every edge of $v^\sigma[N, p-1]$.*

Proof of claim: It follows from the construction, that $E(H)$ contains every edge of v^σ except for those contained in $M_v(rN') + \sigma(v)$, which gives that H contains every edge of $v^\sigma[N, p-1]$. To prove

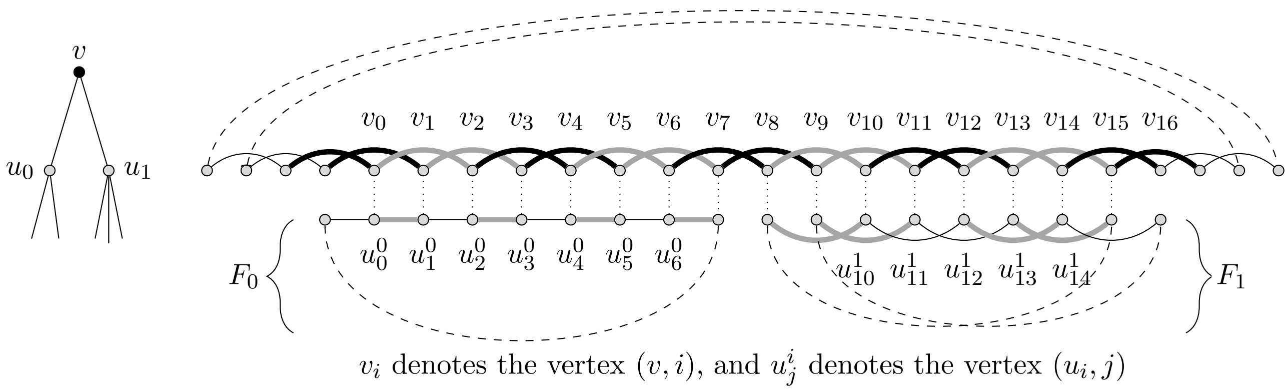


Figure 4.4: The drawing on the left-hand side shows part of a reflexive tree T with a voltage assignment σ and a specified vertex v which has two neighbours u_0 and u_1. The drawing on the right-hand side shows parts of the fibres v^σ, u_0^σ, and u_1^σ, with the dashed lines representing the undrawn parts of the fibres. We may seek a 2-factor F each of whose components has an edge in a certain matching in v^σ as follows. We find 2-factors F_0 and F_1 in the two components of $T \setminus \{v\}$, which are shown as cycles. We then may remove a matching from each of F_0 and F_1, as well as from v^σ and add the dotted edges between the fibre v^σ and the other two fibres. The removed edges are shown in gray. Finally, we obtain a larger 2-factor H, each of whose components must contain at least one of the edges in the bolded matching.

that $M_v(N) \subseteq E(H)$, it will suffice to show that $M_v(N)$ is disjoint from $M_v(rN') + \sigma(v)$. As

$$M_v(rN') + \sigma(v) \subseteq M_v(rN' + 2\varphi(v)) + \sigma(v) = M_v(N) + \sigma(v)$$

it suffices to show that $M_v(N)$ is disjoint from $M_v(N) + \sigma(v)$. It follows directly from Observation 4.1.5 that $M_v(N)$ is disjoint from $M_v(N) + \sigma(v)$. Hence, we see that when we remove $M_v(rN') + \sigma(v)$ from v^σ, we do not remove any edges from $M_v(N)$. Therefore, each edge of $M_v(N)$ belongs to $E(H)$ and the claim is proved. $\square$

Claim 4.1.9. *Each component of H has an edge in $M_v(N)$.*

Proof of claim: It follows from our construction, that $E(H) \cap E(v^\sigma) = E(v^\sigma) \setminus (M_v(rN') + \sigma(v))$. Let $S = \{\sigma(v), \ldots, \sigma(v) + rN' - 1\}$ be a subset of $\mathbb{Z}_p$. One can see that elements of S are the group values of the endpoints of $M_v(rN') + \sigma(v)$. Furthermore, S is a subset of the group values of the vertex set $V(M_v(N))$. Therefore, as H is 2-regular, any component in H that contains a vertex (v, b) for $b \in S$ must also have an edge in $M_v(N)$.

Let H_i be 2-factors obtained from the induction hypothesis. Consider $H \cap H_i$ for each 2-factor H_i. We note that $H \cap v^\sigma$ is a forest of paths. Therefore, since H is 2-regular, every component C of H must contain both a vertex from v^σ and a vertex from a 2-factor H_i. Hence, it follows that C must contain an edge of the form

$$[(v, \sigma(v) + iN' + a), (u_i, \sigma(v) + iN' + a)]$$

for some value $i \in \{0, \ldots, r-1\}$ and some value $a \in \{0, \ldots, N'-1\}$. Since $b = \sigma(v) + iN' + a \in S$, it then follows from the argument above that C must contain an edge in $M_v(N)$, that is, each component of H has at least one edge in the matching $M_v(N)$. The claim is proved now. $\square$

Claims 4.1.8 and 4.1.9 directly imply Lemma 4.1.7. $\square$

Using Lemma 4.1.7 we are now ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Let T be a reflexive tree, and let v be a vertex of T with r neighbours $u_0, \ldots, u_{r-1}$. Furthermore, for every neighbour u_i of v, let $\varphi(u_i) = \varphi(v)$. For each $i \in \{0, \ldots, r-1\}$, let T_i be the component of $T \backslash \{v\}$ containing u_i and F_i be a 2-factor of T_i^σ. It follows by Lemma 4.1.7, there exist 2-factors F_i, along with a corresponding integer $N_i = N_i(\varphi_{|T_i}, T_i)$, such that each component of F_i has at least one edge in $M_{u_i}(N_i)$ and such that each edge induced by $u_i^\sigma[N_i, p-1]$ belongs to $E(F_i)$. We let

$$N = 2\varphi(v)N_1N_2 \ldots N_r.$$

By an appropriate application of automorphisms on $T_0, \ldots, T_{r-1}$, we may obtain that for each $i \in \{0, \ldots, r-1\}$ a 2-factor H_i, each of whose components contains at least one edge in the matching $M_{u_i}(N) + iN$.

Now we will describe the process of constructing a Hamiltonian cycle by connecting each component of each 2-factor together. For each $i \in \{0, \ldots, r-1\}$ and each component C of F_i, we choose an edge

$$[(u_i, a), (u_i, a + \sigma(v))] \in M_{u_i}(N) + iN$$

of C, by choosing an appropriate a. (Note that this is possible, since $\sigma(u_i) = \sigma(v)$.) Then, we remove the edges $[(u_i, a), (u_i, a + \sigma(v))]$ and $[(v, a), (v, a + \sigma(v))]$ from C and v^σ, respectively, and add the edges $[(u_i, a), (v_i, a)]$ and $[(u_i, a + \sigma(v)), (v, a + \sigma(v))]$. We can perform the above steps for each i, by beginning with a cycle $A = v^\sigma$, and each time we replace edges for a component C as described above, we extend A to include every vertex of C. After repeating the process for every component of every 2-factor H_i, A will be extended to a cycle that visits every vertex of T^σ. Therefore, T^σ is Hamiltonian. Finally, note that a sufficient bound on p is $p > 2^{|V(T)|} \prod_{v \in V(T)} \varphi(v)$. $\square$

Next, we will prove Theorem 4.1.3 with the similar technique using Lemma 4.1.7.

Proof of Theorem 4.1.3. Let T be a reflexive tree, and let $u, v \in V(T)$ be an adjacent pair of vertices for which $\varphi(u) = \varphi(v) = 1$. We define $T_u \subseteq T$ and $T_v \subseteq T$ as the subtrees obtained by removing the edge uv and taking the component with u and the component containing v,

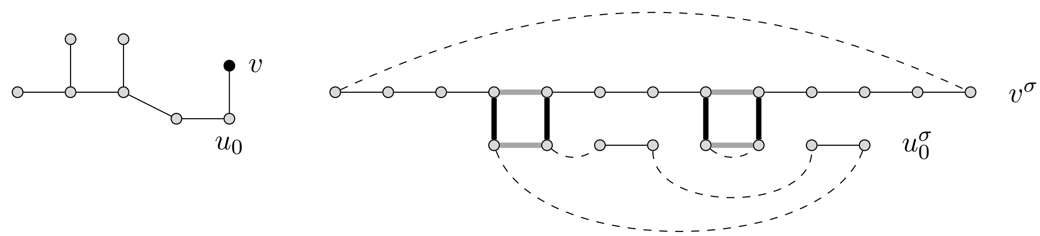


Figure 4.5: The figure on the left shows a tree T with a voltage assignment σ and a vertex v, all of whose neighbours (here, just u_0) have the same voltage as v. The figure on the right shows parts of the fibres v^σ and u^σ (with their vertices depicted horizontally) and the graph T_0^σ, with the dashed lines representing the undrawn parts of the graphs. T_0^σ has a 2-factor F_0 each of whose components has an edge in a certain local part of the fibre u_0^σ. Here, we may create a Hamiltonian cycle on T^σ by removing one edge in u_0^σ from each cycle of F_0 and then replacing these edges with a matching between u_0^σ and v^σ. Here, the removed edges are shown in gray, and the added edges are shown vertically in bold.

respectively. By Lemma 4.1.7, there exists a 2-factor F_u in T_u^σ in which each component of F_u has an edge in the matching $M_u(N_u)$, for some integer $N_u = N_u(\varphi_{|T_u}, T_u)$. Similarly, there exists a 2-factor F_v in T_v^σ in which each component has an edge in the matching $M_v(N_v) + N_v$, for some integer $N_v = N_u(\varphi_{|T_v}, T_v)$. Note that as $\varphi(u) = \varphi(v) = 1$, the edges induced by $u^\sigma[N_u, p-1]$ all belong to a single component A_u of F_u, and the edges induced by $v^\sigma[N_v + N_u, p-1]$ all belong to a single component A_v of F_v.

Let $H = F_u \cup F_v$, we will show the process of modifying H into a Hamiltonian cycle on T^σ. For each component C in F_u, we remove from H two edges, $[(u,i),(u,i+1)] \in E(C) \cap M_u(N_u)$ and $[(v,i),(v,i+1)] \in E(C) \cap M_v(N_v)$. Then, we add to H the edges $[(u,i),(v,i)]$ and $[(u,i+1),(v,i+1)]$. Through the above steps, we can extend A_v by connecting component C to it. Figure 4.6 shows how we perform the "remove and attach" process described above. We repeat this process for every component C of F_u, and we attach every component of F_u, apart from A_u, to A_v. Similarly, through this process, we attach every component of F_v, apart from A_v, to A_u. This leaves us with two cycles, which we may attach by choosing some value i satisfying $N_u + N_v < i < p-1$, removing the edges $[(u,i),(u,i+1)]$ from H and $[(v,i),(v,i+1)]$, and adding the edges $[(u,i),(v,i)]$ and $[(u,i+1),(v,i+1)]$ to H. This is possible because we can choose $p > N_u + N_v + 1$. Through this process, we obtain a single Hamiltonian cycle H on T^σ. This completes the proof. $\square$

A notable implication of Theorem 4.1.2 is that although the answer to Question 4.1.1 remains open, we are able to answer Question 4.1.1 for "almost every" labelling in case of large enough trees by following theorem.

Theorem 4.1.10. *Let $P : \mathbb{Z} \to [0,1]$ be a fixed probability distribution on the positive integers. For a positive integer n, let T be an arbitrarily chosen reflexive tree on n vertices, and let $\varphi : E(T) \to \mathbb{Z}$*

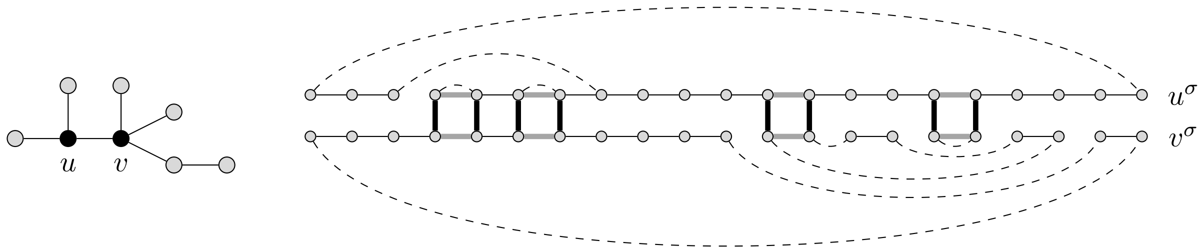


Figure 4.6: The figure on the left shows a tree T with a voltage assignment σ and two vertices u, v, both of which have voltage 1. The figure on the right shows parts of the fibres u^σ, v^σ.We may seek a 2-factor F each of whose components has an edge in a certain matching in v^σ as follows. We find 2-factors F_u and F_v in the two components of $T \setminus \{uv\}$, which are depicted using cycles in the figure on the right, with the dashed lines representing the undrawn parts of the graph. We then may remove a matching from each of F_u and F_v, as well as from v^σ, and add edges between u^σ and v^σ. The edges that we remove are shown in gray, and the edges that we add are shown in bold. We then obtain a Hamiltonian cycle on T^σ.

be a random assignment of positive integers to the edges of T, where each edge $e \in E(T)$ is given a positive integer randomly according to the distribution P. If A is the event that for some sufficiently large prime p, T^σ is Hamiltonian, then $\lim_{n\to\infty} \Pr(A) = 1$.

Proof. Without loss of generality, by Lemma 3.1.1, we will assume $\varphi(e) = 0$ for every cut-edge $e \in E(T)$. Let a be some positive integer that has a positive probability $\epsilon > 0$ under P. We show that for large n, there almost surely exists a vertex $v \in V(T)$ such that, for each neighbour u of v, $\varphi(u) = \varphi(v) = a$.

Since $\sum_{v\in V(T)} \deg(v) = 2n-2$, and T is a connected graph, we can see that at least $\frac{1}{2}n$ vertices of T must have degree at most 2. Therefore there must exist a set $U \subseteq V(T)$ of $\frac{1}{6}n$ vertices of degree at most 2, and the closed neighbourhoods of the vertices in U are disjoint. For a vertex $v \in U$, let X be the event that v and all neighbours of v are assigned the value a to their loops and Y be the event that no vertex in U has the same assignment as all its neighbours. Then we can see that $P(X) = \epsilon^3$ so $P(Y) = (1-\epsilon^3)^{\frac{1}{6}n}$ and $P(Y)$ approaches 0 as n approaches infinity. Therefore, as n increases, the probability of a Hamiltonian cycle existing in T^σ for a large prime p approaches 1. □

4.2 Coverings of trees having large circumference

We've previously explored the Hamiltonian nature of covering graphs derived from base trees with distinct voltages over a prime cyclic group. This relationship hinges on certain voltage conditions. But what if these conditions falter? Our current focus shifts to this scenario. Interestingly, regardless of how voltages are assigned, almost every vertex is included in a cycle of the covering graph when any base tree is lifted over such a large prime cyclic group. A noteworthy observation is that even with a non-prime order n for our group, this holds true if each T label is mutually prime with n.

Theorem 4.2.1. *Let $\Delta \geq 0$ be an integer, let $0 < \epsilon \leq \frac{1}{2}$, and let $n \geq \frac{5\Delta}{\epsilon^2}$ be an integer. If T is a reflexive tree of maximum degree at most Δ, and if $\sigma : \mathbb{Z}_n \to E(T)$ is a mapping that assigns group elements coprime to n to each loop of T, then there exists a cycle C on T^σ that contains at least $(1-\epsilon)|V(T^\sigma)|$ vertices.*

To use induction to prove Theorem 4.2.1, we will prove the following stronger theorem and will also show that the stronger theorem implies Theorem 4.2.1.

Theorem 4.2.2. *Let n and ω be positive integers satisfying $n\left(1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}\right) \geq \lceil\sqrt{\omega}\rceil$. If T is a reflexive tree of maximum degree at most $\frac{n}{\omega}$, and if $\sigma : \mathbb{Z}_n \to E(T)$ is a mapping that assigns group elements coprime to n to each loop of T, then there exists a cycle C on T^σ that contains at least $\left(1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}\right)|V(T^\sigma)|$ vertices.*

Claim 4.2.3. *Theorem 4.2.2 implies Theorem 4.2.1.*

Proof. We assume that Theorem 4.2.2 holds and we will prove the theorem by induction on the number of vertices of T. For the base case, $|V(T)| = 0$ so $\Delta = 0$ in Theorem 4.2.1, then the theorem is trivial. Therefore we assume that $\Delta \geq 1$. We will show that the hypotheses of Theorem 4.2.1 satisfy the hypotheses of Theorem 4.2.2, and we will also show that the conclusion of Theorem 4.2.2 implies the conclusion of Theorem 4.2.1.

Let n, Δ, and ϵ be chosen as in Theorem 4.2.1. We choose ω to be the smallest integer so that $\frac{1}{\lceil\sqrt{\omega}\rceil} + \frac{\lceil\sqrt{\omega}\rceil}{\omega} < \epsilon$, and we observe that $\epsilon \leq \frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1}$ because ω is the smallest. We want to show that when $n \geq \frac{5\Delta}{\epsilon^2}$ then $n\left(1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}\right) \geq \lceil\sqrt{\omega}\rceil$ and $\Delta \leq \frac{n}{\omega}$. We note that since $\frac{1}{\lceil\sqrt{\omega}\rceil} + \frac{\lceil\sqrt{\omega}\rceil}{\omega} \geq \frac{1}{2}$ for $1 \leq \omega \leq 16$, it must hold that $\omega \geq 17$.

We start with proving the first inequality, $n\left(1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}\right) \geq \lceil\sqrt{\omega}\rceil$. Since $n \geq \frac{5}{\epsilon^2}$, and $\epsilon \leq \frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1}$, it is enough to show that

$$\frac{5}{\left(\frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1}\right)^2} \geq \frac{\lceil\omega\rceil}{1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}}$$

holds for $\omega \geq 17$. When ω approaches infinity, we have

$$\frac{\lceil\omega\rceil}{1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}}\left(\frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1}\right)^2 \to 4$$

and it is easy to check for small values of ω that the inequality holds. Therefore, it holds that

$$n\left(1 - \frac{1}{\lceil\sqrt{\omega}\rceil} - \frac{\lceil\sqrt{\omega}\rceil}{\omega}\right) \geq \lceil\sqrt{\omega}\rceil$$

.

Next, we show that $\Delta \leq \frac{n}{\omega}$ holds. Since $n \geq \frac{5\Delta}{\epsilon^2}$, and since $\epsilon \leq \frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1}$, it is sufficient to show that

$$5 \geq \omega \left(\frac{1}{\lceil\sqrt{\omega-1}\rceil} + \frac{\lceil\sqrt{\omega-1}\rceil}{\omega-1} \right)^2 .$$

When ω approaches infinity, the right-hand side expression has a limit of 4, and it is easy to check the inequality for small ω. Therefore, it holds that $\Delta \leq \frac{n}{\omega}$. Hence, we have shown that the hypotheses of Theorem 4.2.2 hold, so by assumption, the conclusions of Theorem 4.2.2 hold as well.

Now, since $\frac{1}{\lceil\sqrt{\omega}\rceil} + \frac{\lceil\sqrt{\omega}\rceil}{\omega} < \epsilon$, it is clear that the conclusion of Theorem 4.2.2 implies that the conclusion of Theorem 4.2.1 also holds. This completes the proof of the claim. $\square$

The above claim shows that it is sufficient to prove Theorem 4.2.2 to prove Theorem 4.2.1. For the proof of Theorem 4.2.2, we will need the following definition and lemma.

Definition 4.2.4. For an integer $n \geq 1$ and a pair $g, h \in \mathbb{Z}_n$, we say that the *distance* between g and h is the minimum number of terms 1 or -1 that must be added to g to obtain h.

Lemma 4.2.5. *Let $1 \leq m \leq n$ be integers. For each generator $g \in \mathbb{Z}_n$, there exists an integer $1 \leq k \leq m$ and an element $h \in \mathbb{Z}_n$ at a distance of at most $\lfloor \frac{n}{m} \rfloor$ from 0 for which $kg = h$.*

Proof. Consider the set $K = \{kg : 1 \leq k \leq m\} \subseteq \mathbb{Z}_n$. For each element $a \in K$, we define the set $R_a \subseteq \mathbb{Z}_n$ as the set $\{a, a+1, \ldots, a + \lfloor \frac{n}{m} \rfloor\}$. For two distinct elements $a, b \in K$, if $R_a \cap R_b \neq \emptyset$, then a and b are at a distance of at most $\lfloor \frac{n}{m} \rfloor$. Now, since g is a generator of $\mathbb{Z}_n$, all elements of K are distinct. Therefore, since $m(\lfloor \frac{n}{m} \rfloor + 1) > n$, there must be two elements $k_1 g, k_2 g \in K$ for which $R_{k_1 g}$ and $R_{k_2 g}$ intersect. Without loss of generality, we assume that $k_2 g \in R_{k_1 g}$, which implies that modulo n, $k_2 g - k_1 g$ is at most $\lfloor \frac{n}{m} \rfloor$. However, this implies that $|k_2 - k_1| g$ is at a distance of at most $\lfloor \frac{n}{m} \rfloor$ from 0, so letting $k = |k_2 - k_1|$ and letting $h = |k_2 - k_1| g$ gives us our result. $\square$

With Lemma 4.2.5 we can now prove Theorem 4.2.2 and then Theorem 4.2.1 will follow directly by Claim 4.2.3.

Proof of Theorem 4.2.2. By Lemma 3.1.1, we may assume that every cut-edge $e \in E(T)$ satisfies $\varphi(e) = 0$. We give an orientation to the cut-edges of T so that each vertex of T has out-degree at most 1. We prove the stronger statement by choosing C such that C contains all but at most

$$\deg^+(v) \frac{n}{\lceil\sqrt{\omega}\rceil} + \deg^-(v) \lceil\sqrt{\omega}\rceil$$

edges from each fibre v^σ in T^σ. Furthermore, $E(C) \cap E(v^\sigma)$ forms a path.

We prove the statement by induction on $|V(T)|$. For the base case, when $|V(T)| = 1$, T^σ is a single cycle, so the statement holds. For inductive steps, suppose $|V(T)| > 1$. Let ℓ be a leaf of T with out-degree 1 and a neighbour $v \in V(T)$. By the induction hypothesis, $(T - \ell)^\sigma$ contains a cycle C' that satisfies our stronger condition in Theorem 4.2.2, and in particular, C' contains all but at most

$$\deg^+(v)\frac{n}{\lceil\sqrt{\omega}\rceil} + \deg^-(v)\lceil\sqrt{\omega}\rceil$$

edges of v^σ and such that $E(C') \cap E(v^\sigma)$ is a path.

Since $\sigma(v)$ is coprime to n, we can assume that $\sigma(v) = 1$ by applying an appropriate automorphism. Now, we extend C' to ℓ^σ as follows. Using Lemma 4.2.5, we choose an integer $1 \leq k \leq \frac{n}{\lceil\sqrt{\omega}\rceil}$ and an element $h \in \mathbb{Z}_n$ at a distance $d \leq \lceil\sqrt{\omega}\rceil$ from 0 for which $k\sigma(\ell) = h$. Now, since C' intersects v^σ in at least

$$n - \frac{n}{\lceil\sqrt{\omega}\rceil} - \frac{n}{\omega}\cdot\lceil\sqrt{\omega}\rceil \geq \lceil\sqrt{\omega}\rceil$$

edges, and since these edges form a path in v^σ, we may find some path $P \subseteq v^\sigma \cap C'$ of length at least $\lceil\sqrt{\omega}\rceil$. By applying an appropriate automorphism to T^σ, we may assume that P is of the form $(v_0, v_1, \ldots, v_{|E(P)|})$ and that v_0 is an endpoint of the path $E(C') \cap E(v^\sigma)$. Let $P^* = (v_0, v_1, \ldots, v_d)$. Note that P^* is a subpath of C'. We remove all edges of P^* from C' and add edges $[v_0, \ell_0]$ and $[v_d, \ell_d]$ to C'. Now, since $d \in \{h, -h\}$ modulo n, and $k\sigma(\ell) = h$, there exists a path of length k from ℓ_0 to ℓ_d in ℓ^σ. Then, since ℓ^σ is a cycle, there also exists a path P' of length $n-k$ in ℓ^σ from ℓ_0 to ℓ_d. We add this path P' to C', which gives us our final cycle C.

Now we need to show that the induction hypothesis holds for C. We note that C contains all but $k \leq \frac{n}{\lceil\sqrt{\omega}\rceil}$ edges of ℓ^σ. Additionally, when $T - \ell$ was extended to T and C' was extended to C, the in-degree of v increased by one, and C lost $d \leq \lceil\sqrt{\omega}\rceil$ edges from v^σ compared to C'. Therefore, C contains all but at most

$$\frac{n}{\lceil\sqrt{\omega}\rceil}\deg^+(w) + \lceil\sqrt{\omega}\rceil\deg^-(w)$$

edges from each fibre w^σ in T^σ. Since $E(v^\sigma) \cap E(C)$ was obtained from the path $E(v^\sigma) \cap E(C')$ by removing a subpath containing an endpoint, we can see that $E(C) \cap E(\ell^\sigma)$ and $E(v^\sigma) \cap E(C)$ are two paths. Therefore, the induction hypothesis holds for T and C, and the proof is complete. □

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