An Investigation into Historic and Contemporary Presentations of Logarithms in Textbooks: Can Historic Presentations Enhance the Contemporary Presentations of Logarithms?

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Abstract

This dissertation explores the presentation of logarithms in current textbooks and historic texts. The aim of the research is to discover any past textual presentations of logarithms that could enhance modern presentations. Based on the theory of Multiple Perspectives, the scope of the historical text analysis includes not only foundational mathematics texts around logarithms, but also textbooks, dictionaries, and encyclopaedias. The theoretical construct of a Concept Image is discussed, focusing on a concept's need to connect to other mathematics for students to have a full view of the concept.

To understand modern presentations of logarithms, a survey of modern textbooks is completed including an analysis of connections, or pathways, created between logarithms and other mathematical topics. This analysis shows that logarithms today are solely related to exponents, with all aspects of the concept coming through the students' understanding of exponents and exponential functions.

To explore the historical presentation of logarithms, the foundations and early history of the logarithm is traced, gathered both from primary and secondary sources. Then the presentation of logarithms in historical texts between 1614 and 1750 is explored, focusing on how these texts introduce logarithms, define logarithms, justify the properties of logarithms, and use them in applications. Finally, the term *logarithm*, and how it is translated and then defined in texts from 1614 to 1850, is surveyed. This analysis demonstrates the connections between logarithms and mathematical concepts other than exponents, focusing mostly on those connections, such as sequences, ratios, and proportions, which were not apparent in modern textbooks.

The result of the analysis produced in this dissertation is a set of recommendations for modern textbook authors or mathematics instructors. These recommendations do not argue against the modern presentation of logarithms and their overwhelming connection to exponents, but instead suggest bringing additional connections to other mathematical concepts into multiple levels of lessons around logarithms. These additional connections could strengthen any student's concept image of logarithms, but perhaps more importantly, they could help students who do not have a strong understanding of exponents still have a pathway to understanding logarithms.

Keywords: logarithms; history of logarithms; textbook analysis; historic text; concept image; multiple perspectives

Dedication

This dissertation is dedicated to my family, parents Sam and Eileen along with my sister, Karin's family, her husband David and daughters Sierra and Keri, for supporting, if not completely understanding, my midlife crisis that led me back to school and this program. I know it was hard to have one of us move across the continent to another country, but I am so happy that you accepted my choice to embark on this journey. Six years in, and I do think this was the right move for me and have been so happy that you have been able to visit as often as you have. Even though none of you will ever read this dissertation, it felt important to have you all in here as you all have always been a primary part of any stage of my life.

I would also like to dedicate this to Sarah Felkar. I met you partway through this journey, but in the last three years you have been my biggest cheerleader, greatest friend, and favourite person. I could not ask for a better partner during this phase of my life or the next. I only hope to support you in your future goals in anywhere close to the manner that you have supported me. I am so excited to explore our life together after this chapter is complete.

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I need to acknowledge everyone that attended the 2019 International Conference for the History of Mathematics Education in France. Just listening to their research helped guide this dissertation but specifically to those that gave feedback to my research, I thank for showing me a path forward. That was my first foray into this type of research and while I went in a different direction I would have no been able to complete dissertation without that experience.

I would like to thank my friends that kept me sane along the way. Those in Vancouver, in Baltimore, and elsewhere, for checking in, especially during the Covid lockdowns to make sure I'm talking to more than just my cat(s).

Basically, to all the faculty, colleagues, family, and friends that have helped me in this process. It was a long, and sometimes arduous, journey and I would have not have made it without the support of so many people. Thank you.

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Glossary

As most of my dissertation is concerned with how logarithms were discussed in the 17th and 18th century, when I am specifically referring to those texts, some of the terms are different than how we mean them now. I use those terms so that when I am referring to figures or quotations from those texts, the language stays the same.

Series	Denotes a sequence of numbers. If it is an arithmetic series then the sequence will have a common difference, if geometric then a common ratio. Series was the word used in most of the historical texts I am reviewed. In the dissertation, I most often use sequence or progression for the ease of the readers but in some of the images from text, the word used will be series.
Exponent	Late 1500s and again in the late 1600s it was another word for the individual terms in an arithmetic sequence when compared to a geometric sequence. In the 1700s it became used in its current meaning: where a base is raised to a number, the exponent would be that number to which the base is raised. For example in 3^2, 2 is the exponent.
Index	A synonym for exponent, used more frequently for most of the 1600s into the early 1700s. It was used before *exponent* as the word for the numbers in an arithmetic sequence when compared to a geometric sequence. It is still used today in some countries instead of exponent, so where a base is raised to a number, the index would be that number to which the base is raised. For example in 3^2, 2 is the index.
Mean Proportional	A commonly used mathematical expression in 1600s and 1700s. If it is singular it is comparable to the contemporary expression ‘geometric mean’. Given two numbers a and b, to find the geometric mean between them, one could use proportions: $\frac{a}{x} = \frac{x}{b}$, the mean would be the square root of the product of the two numbers. If it is pluralised, ‘means proportional’, then the set up would be $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$ and can be solved through a series of equations. This expression is explored more in Section 4.5.
Power	Often used in relation with an exponent, but refers to the entire expression and can be used to signify the operation of exponentiation: where a base is raised to a number, it could be written as a base raised to the power

	of the number. The power signifies the effect of the exponent upon the base.
Progression	As I mainly looked at geometric and arithmetic sequences (called series in many of the texts I reviewed due to naming conventions of the time), a geometric and arithmetic progression would be a synonym and is used mainly when discussing historic texts as they often used this phrasing.

Chapter 1. The ‘Sequence’ of Events Leading to these Studies

My experience with logarithms could be considered common. They were something that I encountered in mathematics, found a bit abstract, but useful, and that I did not think about much until I started teaching. I taught College Algebra for 15 years at the University of Maryland, Baltimore County (UMBC), in the United States of America (USA), and logarithms had always presented a challenge for my students. We used a few different textbooks over the time-frame that I taught, both in print and on-line, but the presentation of logarithms was consistent throughout each one – logarithms were related solely to exponents. My experience with logarithms did not allow me to reach a different conclusion; the way that I had learned and used logarithms also relied on their association with exponents. So, I researched various ways to present them, looking through all manner of textbooks and any internet research that seemed relevant. I did find a few different presentations, and developed one that consistently improved the students’ manipulation of logarithms.

This presentation focused on the position of terms in relation to the exponent. In teaching exponent rules, I had reinforced that if a base had a negative exponent, then it meant the base should be a reciprocal with a now-positive exponent. If the base with the negative exponent was in the numerator, then the reciprocal would put it in the denominator. If the base with the negative exponent was in the denominator, then the reciprocal would put it in the numerator. So, in practice, the negative exponent would effectively flip that base over a fraction bar. I had first gone through the mathematical reasoning with the students, but then focused on this idea going forward. When it came time for logarithms, I introduced logarithms as an exponent, so the same idea would apply: positive logarithms would not need to be flipped over a fraction bar, and negative logarithms would need to flip to the denominator. After introducing the properties of logarithms through this method, the students’ scores on quizzes and tests improved by a significant level, but their understanding of the topic did not. They could manipulate logarithms, but if I asked the students about the manipulations, they could not explain them.

At the time, I reasoned that their understanding would come later, the more they worked with the topic. Today, I do not totally disagree with that assessment, as I believe students add to their understanding of concepts each time they work with that concept, each time they are reintroduced to that concept, and each time that concept relates to another part of mathematics. While the students in my past may have come to an understanding about logarithms, that would have relied on how their subsequent courses introduced and worked with them. But, in my experience, since logarithms are consistently taught through their relationship with exponents, there would not be too many new connections to other topics in mathematics to help aid in their understanding of the logarithmic concept.

This dissertation starts from this frustration, I reviewed modern textbooks to see if there were other conceptions of logarithms. Then I searched through historical texts, specifically the time period from 1614, when logarithms were first introduced by John Napier to 1750, the end of the decade when they first became defined through the exponent, to find presentations in those texts that would give today's students a new way of envisioning this concept. I also exampined the word 'logarithm', both the translation and related definitions, and combined this study with the others to suggest new ways of presenting logarithms in contemporary textbooks.

1.1. Path Taken to this Dissertation

When I started my Ph.D. journey at Simon Fraser University (SFU) in 2017, I did not have a firm topic. I had previously done research in mathematics education based around hybrid learning and the flipped classroom, and had created on-line training for mathematics tutors through my Masters program, but I wanted to research a topic not based around technology. I enjoyed my past research, but within a few years it was too dated to have much relevance, and for a project as big as a dissertation, I wanted my research to have a chance to make a more lasting impact. It was through the direction of my senior supervisor, Dr. Rina Zazkis, that I went back to my frustrations with logarithms. I reviewed a past study that she had supervised, started reading other literature around the topic and found myself being drawn into the history of logarithms and how they were historically presented.

1.1.1. A Few False Starts

In the beginning, the sources that I reviewed were more focused on the teaching of logarithms, so I went through the archives of *The Mathematics Teacher* (from 1908) to see different presentation methods that teachers brought to the concept. While these articles were interesting, they either did not feel overly relevant to today's classroom, as they would often focus on using tables, or they were too similar to modern teaching to create connections to mathematical concepts other than exponents.

Next, I spent time reading other papers, books, and dissertations based around teaching logarithms using their history. I found influence in Clark's (2006) dissertation, a case-study on teachers' use of history in teaching logarithms. I read Toumasis (1993) paper about a three-year study in a Grade 11 classroom in Greece, where he introduced logarithms through the relationship between arithmetic and geometric sequences, and built that toward logarithms being an exponent. These studies felt closer to a topic I would want to explore, so I continued with papers that theorised ways to introduce history in the teaching of logarithms (Katz, 1986; Panagiotou, 2011), where the authors discussed the different ways that teaching the history could assist in the current teaching of the topic. I realised two things at this time: first, that some of the studies provided a useful, and interesting, way to introduce logarithms, but they often did not have much effect on the concept past the initial introduction. Logarithms may be introduced as a relationship between sequences, but once logarithms became associated with exponents, the relationship to sequences disappeared. Secondly, I realised that much of this material was very advanced. I had trouble following John Napier's (1550 – 1716) initial construction of logarithms, and trying to explore it with a Grade 11 classroom felt overwhelming. Some of the authors even mentioned that these ideas were only for their top students or should be significantly altered when teaching, so all modern students could relate (Katz, 1986; Panagiotou, 2011). This led me to wonder if there were a way to use history in teaching logarithms that would not confound the students and could influence their way of thinking about logarithms even past the initial introduction.

Before I went any further, I did a few pilot studies on the relationship between history and textbooks. The first study looked at the reasons history was included in textbooks (Riley, 2018). I had been a Teaching Assistant for a Linear Algebra course at SFU, so I used the textbook for the course, *Contemporary Linear Algebra* (Anton &

Busby, 2003), to get an idea on what place history of mathematics has in a modern textbook. My results showed that, in general, history was employed to add some colour, or commentary, usually to the side of the more technical writing used in the main text. The history was mainly incidental to the mathematics. The second study looked at how logarithms were taught in one textbook starting in 1800 and then a single textbook every 50 years there-after. This study was my first attempt in reviewing past textbooks and taught me a lot about the use of theory and establishing appropriate parameters early in the process. While I did present that research at the International Conference on the History of Mathematics Education (2019) conference, it was a very different research paper that was published (Riley, 2020a).

1.1.2. On My Way

After these initial attempts, I stepped back from writing and began devouring books, papers and journals around the history of mathematics education. I was unsure how to narrow my focus, so I browsed texts about the history of mathematics and mathematics education (Fauvel & Gray, 1987; Karp & Schubring, 2016) and also bought the entire set of issues from the *International Journal of the History of Mathematics Education*, which was published from 2006 to 2015. These resources gave me a better idea of how to conduct research in this field, as well as what has been done in this field in the recent past. While most of this reading did not make it explicitly into this dissertation, there were studies that I strongly connected, therefore this process definitely influenced the final direction.

I also dedicated time to understand fully the early timeline of logarithms. I began, as most researchers in this area begin, by reading through Florian Cajori's (1913) series on the history of logarithms and exponents. I then moved onto any other study that I could find that cited Cajori. With these studies, I tried to comprehend, and repeat, the mathematics that were discussed. It took too long, but I was finally able to move on to looking at the primary sources and restarted trying to understand and repeat the mathematics. While I eventually did get a decent understanding, it took a lot of practice and rewriting to explain any of the mathematics I was learning. This process sparked the idea that perhaps focusing on the texts of the 'official' history was not the only way to explore historical ideas of logarithms with students: perhaps other texts written around that time, such as textbooks, could be a good intermediary.

Exploring Some Side Paths

At this point, I had found some interesting asides when reviewing the history of logarithms and thought that my dissertation could consist of three smaller studies exploring these episodes. I attempted some small studies during this time to see what research would sustain my interest enough to continue it further.

An opportunity arose to work for a few hours with high school students at a mathematics camp one summer. Together, we explored the relationship between geometric and arithmetic sequences, concluding with students creating their own slide rules and using them for calculations. While it was a fun experience, and the group seemed to enjoy it, they had trouble connecting it with logarithms. I thought that manipulatives like slide rules may be too removed from the current idea of logarithms to be understood without an instructor dedicating extra time and energy which they may not have. I am sure it can work; it just was not something that I felt drawn to pursuing.

I found it really interesting that Cajori's (1913a) write-up on the history of logarithms credited William Jones, via William Gardiner, with first presenting logarithms as exponents and yet did not comment further on either of these researchers. I also could not find much other research on William Jones (1675–1749), so I did a deep dive into his life and texts to understand better how he made this connection. While I did find evidence that at least one other person presented logarithms as exponents before Jones (Nicholas Saunderson (1682–1739), as discussed throughout this dissertation), Jones generalised the idea and seemed to be able to derive the properties through this idea. I was able to present this exploration as a second conference paper (Riley, 2020b), but it did not seem enough to me to make up a chapter of a dissertation, therefore much of that paper is included in Chapter 4, the history of logarithms.

I spent quite a bit of time looking through texts trying to find any place where John Napier's formulation of logarithms was actually used in a presentation on logarithms or any other mathematics. Many authors included snippets, or some sort of description, but just did as an aside to their main presentation. They would talk about the invention, that Napier envisioned logarithms as the relationship between two lines, and then move on to other ways of understanding them. I then found Colin MacLaurin's (1698 – 1746) *Treatise of Fluxions*, published in 1742, a defense of Newtonian Calculus where he actually used the ideas that Napier developed. I did not go into detail in how he used these ideas, as it would have meant learning more about Newtonian Calculus than I desired, and, given that it was considered a very dense mathematical tome in its time, there is a chance that even spending years, I would still not have a full understanding. But I did get a much greater understanding of Napier's vision of logarithms through this text, so, once again, parts of this research are presented in Chapter 4.

1.1.3. The Questions That Started It All

During this time, I was participating in a monthly seminar with some peers where we shared bits of our research. I submitted a piece about William Jones' different ways of defining logarithms (discussed more in Chapters 4 and 6) before the transition to exponents. I remember one of the faculty involved in the seminar, David Pimm, made a statement about how those ideas of logarithms were lost today and then he wondered what else was missing from today's understanding of the concept. This statement is what started this dissertation. By this time, I had already decided to write this dissertation around the history of logarithms and their intersection with mathematics education, but I still thought it would be a series of studies. But now, I realised that this was what I wanted to focus on; I wanted to go through texts from the past and see what presentations of logarithms existed, how they were used to establish the properties of logarithms and the definitions that influenced their use. I wanted to catalogue all the meanings, and subsequent uses, of logarithms that were missing from today's presentation of the topic.

It was at this point that I went back to MacLaurin's (1742) treatise, as I appreciated his explanation of Napier's original vision of logarithms and knew I would need to include some of his ideas in my dissertation, no matter the exact focus. I wrote a short paper summarising his explanation for the next seminar. Again, a comment from David Pimm guided this dissertation. He mentioned that, in order to understand if this idea would be useful today, there would have to be a study about the current presentation of logarithms in textbooks. This study could frame the dissertation to give an idea of the modern understanding of this concept. While I did not use this idea to frame the dissertation, it is prevalent in Chapters 5 and 8.

The last major influence on this dissertation came at a departmental conference when I was presenting my study on William Jones (Riley, 2020b). My supervisor, Rina

Zazkis, had asked why logarithms were called 'logarithms'. At this time, I could answer that it was a translation from Greek for the number of ratios, but I knew enough to be uncomfortable with that being the full answer. I realised that I had read that information in various modern and past texts, but I had never actually researched it deeply. This led to months of trying to find definite proof on the translation that Napier used, the reason that he chose this portmanteau, which eventually became Chapter 7.

1.2. The Path of the Dissertation

As I have presented the pathway arriving at this dissertation, I now present the pathway of the dissertation itself. Directly to come in this chapter is a discussion about the use of history in mathematics education and the research questions that directed this study.

In Chapter 2, I explore how the theory of Multiple Perspectives (Kjeldsen, 2010) guides the way that I view the historic sources, and the modern sources, as well as their interactions. I trace the trail that the theoretical construct of a Concept Image (Tall & Vinner, 1981) has taken over the past decades as I wrestle with what meaning of that construct will influence the methods used when deciding upon the data, and the analysis of that data.

Chapter 3 is a review of the relevant literature around logarithms with a focus on how history has been employed to assist in student comprehension. Chapter 4 does a deep dive into the history of logarithms. As the goal of the dissertation is how history can influence teaching and learning today, a thorough explanation of the 'official' history is presented.

The next few chapters are the thrust of the dissertation. The presentation of logarithms in modern textbooks makes up Chapter 5. The focus of the study is on English language texts, so textbooks used in the United States, Canada and Great Britain are considered. In the study, I review texts where logarithms are introduced and subsequent texts where logarithms are used in further mathematics up through integration.

In Chapter 6, I review English language texts published between 1614 and 1750 to explore their presentation of logarithms. I look at the introduction of logarithms, the derivation or justification of the properties of logarithms, and where appropriate, the solving of exponential equations.

In Chapter 7, I take a little aside to look at the word *logarithm* itself and the translations and definitions that were assigned to it between 1614 and 1850. While this may be viewed as supplementary, many studies have shown there is confusion about the word for students, so this small study aims to provide a few descriptions that could help provide understanding.

Chapter 8 ties together the previous three chapters, suggesting how historical ideas of logarithms could help today's students create new connections from various mathematics to logarithms. While the suggestions do focus on creating new connections, most of the connections do also reinforce the contemporary relationship between logarithms and exponents. And even though the focus is on how the historical conception could help with a contemporary conception, there are small inclusions on the etymology of the term *logarithm* and how the suggested inclusions could help make meaning of the concepts name.

Chapter 9 is the conclusion of the dissertation. It holds my responses to the research questions, limitations of this study, the contributions of this study, and ideas for further research. There are two appendices, one that explores a way of calculating logarithms that did not fit naturally into this study and the other that lists the texts used in this study for anyone that would want to review them.

1.3. History in Mathematics Education

Here is where I should put up a disclaimer that I am foremost a mathematics educator, not a historian. While my academic background is both in Mathematics and Ancient Studies (a Bachelor's degree that is a combination of Ancient History, Ancient Greek, Archaeology and Classics), my work has primarily been related to the teaching of Mathematics. I taught Mathematics at Community Colleges and Universities in the United States, as well as at Colleges in Canada. I have immersed myself in how adults learn and succeed in mathematics. I have stayed interested in history, and in mathematics history in particular, but that was as a hobby, not as something I trained in, or worked in, until pursuing my PhD.

As the history of mathematics education is a very diverse field, before entering into the main body of the dissertation, there needs to be a discussion about history and its place in mathematics education. This discussion will lead to an understanding of the view toward history and mathematics that are contained within.

1.3.1. Why Include History in Mathematics Education

There are many books, journals and papers dedicated to history in mathematics education and they all list arguments for the reasons it should be included and suggestions on how to achieve the goals desired by the inclusion of history. The most well-known among these is Fauvel (1991) who listed 15 different reasons, and Tzanakis and Arcavi (2000) who came up with 17 reasons in the *International Commission on Mathematical Instruction's* (ICMI) study on History in Mathematics Education. As many of these reasons can fall under a theme, I group them as other researchers have done (Fried, 2001; Furinghetti, 2004; Jankvist, 2009):

- to aid the student in the learning of mathematics and the teacher in the teaching of mathematics;
- to connect mathematics to other school subjects;
- to place mathematics as a part of culture;
- to humanise mathematics.

Jankvist (2009) simplifies any groupings further by listing two overarching reasons for having history in mathematics education: the goal can be the learning of history or history can be a tool for the learning of mathematics. For example, the first group above, the learning of mathematics, is a use under 'history as a tool' with the main goal being the mathematics. The second group is mixed; it would be a tool if the focus is on how other subjects influenced the direction of mathematics, or a goal if it involves a cross-curricular project with another department. The third group, appreciating mathematics as being a part of culture, is also to be mixed: the goal would be situating mathematical innovations within the historical and, therefore, current culture. Of course, doing so could also turn history into a tool if it advances 'new' ways of thinking or doing mathematics. The last is more personal: while mathematics is a part of culture, it is individuals who struggle, and invent, and innovate mathematics. History in this case is a goal – students learn about the humans that created these concepts and can then view themselves in relation to these people.

Outside of the reasons above, there is the oft-mentioned idea of the evolutionary argument, “ontogenesis recapitulates phylogenesis”,[1] the idea that, in order to learn a concept, one must go through the evolutionary stages of the concept in order to discover it for oneself the way it was initially discovered in general (Furinghetti, 2004; Jankvist, 2009; Katz, 1997). This argument is not one that is commonly held today, as since the history of discovery is seldom straightforward instead of providing a nice pathway for the student, it would instead create confusion. Though there are researchers (Luis et al., 1991) who argue that an artificial path through a concept development, one designed to cover the primary stages involved in discovery, could mirror the students’ own discovery of a topic.

This dissertation is focused on using history as a tool in learning about logarithms. I looked at historical sources to find what has disappeared from the way that we present logarithms, and which parts of history could help students as they work with the modern presentations. The focus of these studies is on the mathematics, not the history, though there will be suggestions in Chapter 8 that refer to the historical background of logarithms meant to humanise logarithms and to place them into historical-cultural context. I am also not looking to recreate the evolutionary development of mathematics in the classroom. The suggestions made in Chapter 8 may not be in any particular evolutionary order, they are just ideas of logarithms, from history, that could help students conceive of logarithms within a particular topic. While the focus is on history as a tool, history as a goal is not completely abandoned. Jankvist (2009) actually separates course materials in his study, as they could use inspiration from history which would have the double focus.

1.3.2. How to Include History in Mathematics Education

Now that the reasons to incorporate history into mathematics teaching have been discussed, the topic naturally turns to ways to include history into a mathematics

[1] Credited to Ernst Haeckel as his Biogenetic Law in 1866 (Barnes, 2014).

classroom or textbook. Researchers in this area listed a few different ways to achieve this goal. Tzanakis and Arcavi (2000), in the ICMI study, listed three things that could be the focus when including history in mathematics education: the history, the mathematics through following the historical pathway of a concept in the teaching of that concept, and the combination of history and the nature of mathematics by understanding the cultural context of its discovery and evolution. Furinghetti (1997) interviewed teachers and found they searched history to find new problems or activities, to find different constructions of a concept, and as a way to affect students' feelings about mathematics. Karp and Wasserman (2015) also suggested using history to bring in other interests of students to mathematics. These additional problems or projects combined history in mathematics in a way that provides a new context for the mathematics while not altering the current conception of the mathematical concept. This addition would support an idea of Fried's (2001) that currently history in mathematics education is "not something studied but used" (p. 395).

As the focus of this dissertation is the 'use' of history to aid modern students, I feel I must further Fried's (2001) comments and respond to them. He continues

> Now, just because a body of knowledge is used does not in itself prove it has been misused, and the one who uses it should not necessarily be condemned. However, when history is used to justify, enhance, explain, and encourage distinctly modern subjects and practices, it inevitably becomes what is called "anachronical" (Kragh 1987) or "Whig" history (Butterfield 1931/1951, p. 395).

'Anachronical' or 'whig' history is where the determination of the focus in research, what is important, is determined by the present day. This study is an example of 'whig' history; I am directly looking to the past with the idea that logarithms are exponents, and relating the history, and the path taken, through that lens. For a historian of mathematics, this could be a problem as their research into logarithms should not automatically build to exponents; but as I am foremost a mathematics educator, looking to use history to broaden our modern conception of logarithms, I am comfortable embracing this approach to combining history and mathematics.

While Fried (2001) eventually argued for completely separate, but complementary, classes around mathematics and history of mathematics, Jankvist (2009) took a less severe approach. In his review of history in mathematics education studies, he noted three main ways to combine the two subjects: as an aside, as an entire section or module, or in a way that is based around history. The first is what is often seen in textbooks now, namely the history is included in a side-box, usually as a way to humanise mathematics. Having an entire section or module approach would be dedicating time in class to teaching the history of the mathematics. A history-based approach would incorporate the different techniques and strategies found in history into modern classrooms, sometimes touching upon the history, but with the focus on the mathematics. This last approach is the idea that I use throughout this study: history can be researched in order to find ways that it can lend itself to modern mathematics.

1.3.3. Last notes on History in Mathematics Education

I end this section by summarizing Eric Jensen's (2003) classification system for those that use history in their work, but, as he writes in Dutch, I will trust Kjeldsen's (2012) translation and use of his work, and apply it to how I plan to incorporate history. Jensen classifies the study of history as either "(1) lay history and professional history; (2) pragmatic history and scholarly history; (3) action history and observer history" (p. 4).

This dissertation is not the work of a professional historian; it is the work of a doctoral candidate in mathematics education who is interested in history and has some familiarity with studying the subject. It is not a scholarly historical paper, while I hope that Chapter 4 does contribute to the understanding of the invention and evolution of logarithms, overall the purpose of the dissertation is to explore conceptual understandings of logarithms in the past with an eye on how that understanding could assist students today. The last pairing takes a little more explanation: action history looks at history with the purpose of using it to effect today, observer looks at history to understand it through the lens of the time with no expectation of modern applications. In general, this study falls on the side of action history, though Chapters 4 and 7 could fall under observer history. These categories help set the frame for this study and lead us into the research questions.

1.4. Research Questions

Within this dissertation, the research questions I aim to respond to are organised in four sets:

1. How are logarithms currently presented in textbooks? What connections do they have to a student's past work in mathematics? What conceptions of logarithms are continued as a student continues further into mathematics?
2. How are logarithms presented in historic texts and textbooks? How does their presentation affect their use in those texts?
3. What is the etymology of the term *logarithm*? How did that term interact with the meaning and use of logarithms? How can that information be useful to students and teachers today?
4. How could the historic conceptions of logarithms tie into students' understanding and use of logarithms today? Do they bring in new connections to students past mathematics?

Chapter 2. 'Multiple' Theories

In studies I completed early in my PhD career, I had used some methods known in the history of mathematics education community, such as Gert Schubring's (1987) method of analysing texts which laid a framework to follow a topic through a selection of historical texts. But, as I continued my studies, I realised that the dissertation I wanted to write was a blend of the history of mathematics education, modern textbook analysis, and a look at how logarithms are currently presented. Due to this, I am using a combination of two theories, Multiple Perspectives, which comes from History, and the theoretical construct of a concept image, found in Mathematics Education.

2.1. Multiple Perspectives

To discuss ideas from history, and specifically from the history of mathematics, I look to the theory of Multiple Perspectives. Multiple Perspectives is a newer theory developed and used by Tinne Hoff Kjeldsen, who borrowed it from a Danish historian Eric Jensen. As Jensen wrote exclusively in Danish, I stick to Kjeldsen's (2012) explanation of his work in which the main theme is that in order to research history, one needs to pick a focus and then study the perspectives that matter to that focus. These perspectives can be very different from each other, one can view a part of mathematics history by looking at the inventors and innovators, the scientific disciplines that effected it, the cultural components of the time, or really anything that would build to create a complete picture of the topic being studied. For this study of how readers could have encountered logarithms in the reviewed time-span, that will mean: looking at the 'official' texts that are commonly thought of as foundational in the history of logarithms; researching how logarithms were presented in mathematics, and other, texts used by students; and seeing how the word *logarithm* was translated by both mathematicians and non-mathematicians. I aim to find any instance of logarithms in text that could have given meaning of the concept to a learner of mathematics.

Another major component of the Multiple Perspectives theory is that, "people are understood as being shaped by history and being shapers of history" (Kjeldsen, 2012, p. 3). This statement places humans as the protagonist in history, looking at their actions and the consequences, while realising that these consequences create the next humans.

Kjeldsen then applies this view to history of mathematics by declaring that mathematics is created by human beings, at a certain point in history, due to the cultural and social abilities of the time. Mathematical advancements do not generally appear fully formed due to theoretical proofs, but come to be because of a concrete need that may or may not be related to mathematics. This approach does not just look at the influence of prior mathematics on new mathematical developments, but looks at how the wider world would effect these new developments. She ties this into mathematics education by stating that, “studying history of mathematics from practices of mathematics brings history close to mathematical activities, to processes of knowledge production in mathematics, and hence to mathematics education” (Kjeldsen & Petersen, 2014, p. 31).

Those who invented logarithms did so as a way to simplify complex calculations. Those who innovated logarithms 50 or 100 years after their invention had many other reasons for these innovations, only some related to mathematics. They did not first read of them from Napier’s texts, they first learned of them through their schooling or other texts written about logarithms. There were some who would have read the primary texts discussed in Chapter 4, a chapter reviewing the history of logarithms, but many would have continued with just reviewing other contemporaneous books. Looking at only the texts that are known to have contributed to the history of logarithms does not give a complete picture of their development. Studying these textbooks around mathematics and other disciplines that use logarithms, learning how these past mathematicians learned, can shed light to the ‘knowledge production’ at that time.

While Kjeldsen focuses her Multiple Perspectives theory on how to write about the history of mathematics, she does combine it with other theories to try and introduce historical episodes to modern classrooms (Kjeldsen, 2010, 2011, 2012). She argues that having students question mathematical concepts from a historical perspective (why was this important? what problem did this help solve?) can help them unlock new conceptions of the topic (Kjeldsen, 2010). I would take this a step further and view the introduction of historical topics also under a Multiple Perspectives lens as there are multiple views to consider, mainly the historical conception of the topic and today’s students' conception of the topic, which could lead to the students' future conception of the topic.

I do not expect current students to immerse themselves in centuries-old mathematics in the hope of understanding the little changes that logarithms went through in their first 150 years. I also do not believe that the students of the past had a full understanding of these changes, and I relate today's students more to these past students than I do to the mathematicians. Even with that, I am fully cognisant that past students had a much different background and ways of performing mathematics than students today. Throughout this dissertation, I work to conceptualise logarithms in a way that is true to the time it was written, though I do at times put it in contemporary notation or summarise it through a contemporary lens. In this study, I am looking for the examples and explanations that can transcend that border. In any analysis of historical texts, my eye is equally on how past students would understand the material given the historical circumstances, as well as on how it could help today's students broaden their concept image of logarithms.

2.1.1. Personal Perspective

Importantly, the Multiple Perspectives theory also takes into account those using it by introducing two overarching filters. The first focuses on the interplay between the research questions, the contemporary sources, and the historical sources. The second looks for the researcher needs to position themselves in reference to the history. They should acknowledge their choice of research, their perspectives, and their goals. In short, researchers have to consider the perspectives of the actors in the history they are studying, as well as their own perspectives on why they are doing the study.

Kjeldsen (2019) uses the image below to describe a historian's perspective on their research (Figure 1).

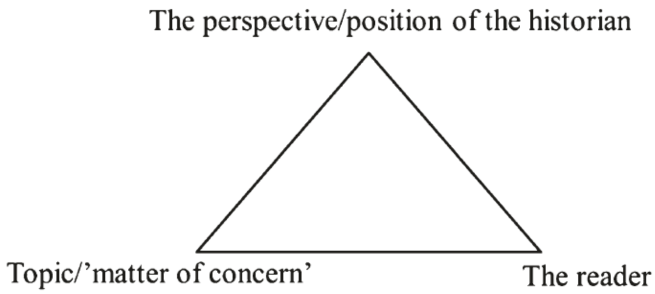


Figure 2.1: The relationship between the historian, the topic, and the reader (Kjeldsen, 2019, p. 146)

Kjeldsen writes that the research cannot be consumed outside of the perspective of the researcher and the interest of the reader. The reader and the historian both speak to the validity of the topic and to the current culture around the topic. Kjeldsen takes this part as an opportunity to position the researcher in relation to the topic and to the reader. I now follow her footsteps and try to position myself in relation to both.

But first, I would like to go a little further than what the Figure 2.1 implies and update it by adding just two words, 'other researcher', to 'reader' to show better the relationship as I see it. This addition would make this a cyclical relationship, like the relationship described in multiple perspectives; culture influences people which influences future culture which influences future people. All nodes of this triangle lead to another. As this is the filter through which this research is written and consumed, all nodes need to be addressed.

Starting with the historian, as discussed in Chapter 1, I found this topic both through my personal experience, and because my personal experience is common. I read the work of other researchers concerning logarithms and found a place where I wanted to contribute. My culture – past frustrations, current interests, the large amount of other scholarly works and historical texts – directly influenced my choice of topic and the direction of this dissertation.

The topic itself, the presentation of logarithms in current and historical textbooks, most likely has a small audience: those who are required to read this dissertation, those who teach logarithms, those who are interested in history of teaching mathematics, those who are interested in history of teaching, those who are interested in history of textbooks, and those who are interested in logarithms. So for those readers/researchers, the topic chosen will maybe point them in new directions for ideas in teaching or research, or expand upon what they know as it has done for me.

The last node, the reader/researcher, directly influences the topic as they represent what already exists, what is available, and what would be of interest. They set the parameters of the topic, and their interest guides where the topic can expand to next. The historian/writer sees them as the same researchers that are cited in this dissertation. They built the background of this topic and the thought of them guides this research.

I want my research to be useful both for historians of mathematics and for mathematics education, but also for current teachers. I am sure there are many who would like to revisit different ways to present logarithms. I have tried to keep both audiences in mind during my research and writing.

This is the overlying filter of this research. Any choices made in choosing the data, the analysis, and the conclusions are with these things in mind; that I come into this from a place of frustration and genuine interest, that I build upon the scholarship done before me, and that I write for historians, for the mathematics education field, and for those who teach logarithms; my dissertation should support all three of these nodes.

2.2. Concept Image

The idea of *Concept Image* came from Shlomo Vinner, who began exploring this topic in 1975, though at the time he called it a ‘mental image’. He connected the mental image to the person; there was an object, and then there was the individual’s conception of the object. The person’s conception of the object was the ‘mental image’. He did not expand upon this idea more in this paper, but obviously spent the next five years pondering this idea as much of his research in the 1980s derived from it.

His collaboration with Rina Herschowitz (Herschowitz & Vinner, 1980) expanded upon the idea of ‘mental images’ by stating that, besides the picture of the object, there could be properties associated with the concept in a person’s mind that would be included in this ‘image’. It also gave us the terms ‘concept image’ and ‘concept definition’. Concept definitions at this stage were the words or symbols that relate to a concept, and it was presumed that, while they may help give meaning to a concept, they were not as strong as the image that eventually forms in a person’s mind, and that they would largely be forgotten overtime while that image would remain. The concept image at this point was the ‘mental image’ along with the concept definition; it was the words, properties, and visualisations that made up a concept.

It is no surprise that the collaboration of Tall and Vinner (1981) is the one that most researchers cited when discussing concept image, as they defined it beautifully in a short paragraph. Most subsequent researchers only cited the first sentence when using this theory, but I choose to include the whole thing.

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (p. 152)

The main idea is that the way that people learn and understand concepts is often through experience directly with the concepts, not through their definitions. The process of using the concepts, making mistakes and revising the ideas associated with the concept, a ‘concept image’ is born. It is then refined over time, as familiar concepts are introduced to new procedures, the concept image that a student has is tested, and hopefully expanded. The student then creates a ‘personal concept definition’ which is not a technical definition, but is the words and symbols that the student uses to specify their concept image.

Over the next few decades, the two scholars differed a bit over how they meant their theory to be understood and employed. Vinner did research into how concept images are formed (Herschowitz & Vinner, 1983; Vinner, 1991), but expended more of his energy on how a student’s concept image could lead to misunderstandings when confronted with a concept definition (Rasslan & Vinner, 1998; Vinner, 1983; Vinner & Dreyfus, 1989). In recent years, he has looked to ideas outside of mathematics education to come up with ways to deal with these confrontations such as ‘conflict resolution’ (Vinner, 2018) and inhibiting students from using any intuition that would arise from their concept image (Vinner, 2008). He has been very careful to keep the ideas of concept image and concept definition apart; to him, differences between the two mean that there was something out of alignment and the student needed to work to fix the problem. Once the two are in accordance, then the student has a true understanding of the concept.

David Tall took a different look at these two ideas, which he even noted in a footnote in a 2006 article:

> he [Vinner] and I have quite different meanings, with his original meaning being a distinction between mental pictures and concept definitions which are separate cells while mine has a cognitive biological meaning constructed in the brain where the concept definition (if it exists) is part of the concept image. This significant difference in meaning has had no effect in the shared used of the term in the mathematical education community who are largely unaware of it. (p. 206)

To me, the difference between the two readings of the authors' ideas is that, for Vinner, there was an end-point to the process of creating a concept image; eventually your personal image of the concept aligns perfectly with all mathematical definitions of the concept. For Tall, one never stopped adding to the concept image; it was more personal and was created from every interaction that one has with that concept, including a definition. Like Vinner, he pushed conflict in the work with students' concept images by introducing uncommon situations, but not to rectify misunderstandings, but to expand a student's idea of a concept so they have a fuller idea of how the concept could work in all circumstances (Tall, 1987). He also discussed intuition, but not to inhibit it, rather to encourage it in students as they transitioned to more formal mathematics. He believed that intuition comes from a student's concept image, and if that could be broadened and strengthened then students could use it successfully (Tall, 1989).

2.2.1. A Focus on Tall's Later Work around Concept Image

In 1996, Tall and DeMarois discussed how concept image could be viewed as a combination of depth, and breadth, of understanding, where breadth was different representations of a concept. And in 1999, there were a series of papers he co-authored that revisited this idea. He started to look into how students went from elementary mathematics to formal mathematics, defining two paths, one where they built up to the formal definition from the concept image, specifically the properties; and one where the properties were proven from the formal definition (Gray et al., 1999). In this transitional phase, as students were learning how to deal with the formalism of mathematics, the student who could work from concept image to formal definition would be able to be “constantly working on various images, reconstructing ideas so that they support the formal theory” (Pinto & Tall, 1999, p. 7). He also wrote a paper about using a computer and computer imaging to support the development of concept images, saying that, “it could be inappropriate to always insist that a mathematical process must be presented first so that it may then be encapsulated as a mathematical concept. Other cognitive processes can be used to build up useful parts of the concept image as foundations for later formal or symbolic development” (Tall et al., 1999, p. 233). Further Tall and colleagues argued that, in order to build up a student’s concept image, the topic needed to be presented in multiple ways (Akkoç & Tall, 2002; Giraldo et al., 2003). A 2022 study (Gülbağci Dede et al.) argued that the pre-service mathematics teachers needed a wide breadth of ideas on functions, and that visualising the relationship between these ideas helped to create the comprehensive understanding that would aid in their future teaching. These papers lend support for breadth being important in building a student’s concept image. In order to advance in mathematics, and to make sense of formal definitions, they would have to have a wide variety of representations to use as reference.

More recently, Tall has revisited his idea in looking at historical mathematics and textbooks. In discussing the nineteenth-century mathematician Augustin-Louis Cauchy’s role in the development of calculus, he noted that, “we should reflect more carefully on the way that we, as ‘experts’, view the conceptions of students as they too go through a developmental process” (Tall & Katz, 2011, p. 20), realising that the mathematics that Cauchy used was not formalized, and was something that could resonate more with students and broaden their understanding of the topics.

I have gone through Vinner and Tall’s work since the notion of concept image was first introduced, as I do feel that their original meaning has changed for both authors. While I do appreciate and respect Vinner’s work, it is Tall’s that speaks to me for this dissertation. In my experience with teaching logarithms, and in reviewing the literature, it seems that modern students have a very limited view of the concept. They are mainly taught that logarithms are one thing, the inverse of an exponential function, and I believe that, if the concept were expanded, they will be able to draw these relationships (the breadth) that will help build and expand their concept images. In reading mathematics theory, I keep coming back to how Tall used concept image in the most recent past; as something that is built up through connections made to the concept and as something that is needed if students are going to formalize the concept. I think logarithms are a topic that is taught with some depth, but less breadth, and, like Tall, I feel both are equally important in building up the student's concept image. I do think that looking to the past is a way to build breadth, as those mathematicians and students are often not working with the same tools, so the mathematics is often presented differently. I also believe that how things are presented in textbooks often guide both teachers and students, and so the breadth needs to be apparent there.

I do want to discuss one last paper dealing with concept image that did not fit previously in this section, *Concept Image Revisited* (Bingolbali & Monaghan, 2008). In this study, the authors brought the idea of concept image from an individual construct to a social one by comparing two different tracks–mathematics and mechanical engineering understanding of a derivative. They were not able to speak for what the classes' concept image was, but could talk toward trends and what influenced those trends. This study has been cited in many of the later papers, as researchers have gone from research questions around what and how a student is thinking to moulding that student's thinking. The idea that concept image is both individual and social, and draws influences from the class, the teacher, and the text, is one that speaks to this study.

2.3. Last Notes on the Theory

My research into historical textbooks revolves around the ideas that modern textbooks do not show the breadth needed to build a strong concept image of logarithms, and that there are ideas from the past that if incorporated can help expand the breadth and depth of logarithms. The idea from the theoretical construct of a concept image, that in order to understand a concept, one needs to connect it to different mathematical topics, is mirrored in the theory of Multiple Perspectives. Both theories focus on the need to approach a topic from various points of view to gain a better understanding of the topic. While Multiple Perspectives can guide the choice of data and the initial preparation of analysis, the concept image construct can guide how to analyse the data and to connect them across the studies.

Chapter 3. An 'Index' of Past Studies

The studies surrounding logarithms tended to focus on the breadth of understanding (is logarithm a number? a function? a graph?) or the depth of understanding (what does it mean for a logarithm to be a number?). As this dissertation hopes to inform both ways of viewing logarithms, I filtered the reviewed studies through this lens. This chapter reviews the literature around logarithms, with a special focus on the use of history in teaching logarithms and on logarithms in text analysis.

3.1. Studies focused on the students' conceptions of logarithms

Logarithms are a topic that can be confusing for students. There are many studies that discuss students' misconceptions around logarithms and which try to single out common errors. While the general finding was that students were competent in completing mechanical tasks with logarithms, they did not actually comprehend the mathematics behind the tasks (Berezovski, 2006; DePierro et al, 2008; Ganesan & Dindyal, 2014; Hoon et al., 2010; Liang & Wood, 2005). Therefore, most studies that reported on errors found less in the operational aspect (Aziz et al., 2017; de Gracia, 2016; DePierro et al., 2008) than those who did a more comprehensive study where they discovered that students could not explain why operations happened, or why logarithms simplified, they could just repeat what they had seen in class (Kenney, 2005; Kusuma & Masduki, 2016). Interestingly, de Gracia (2016) conducted an experiment designed to teach to the common errors students make with logarithms. After assessing the common errors among his students, he focused on four: using 'log' as a variable, properties of logarithms, changing the base, and the concept of a logarithm being the inverse of an exponential. He found that teaching to these errors reduced them by over 60% for all but the properties of logarithms, which were only reduced by 10%. So even though the properties of logarithms seemed like a less common error overall, it was one that was not just mechanical, students' conception of the logarithmic properties were constructed incorrectly and would take a lot of work to tear down and rebuild.

It was commonly found that students focused too much on logarithms as the inverse of exponential functions, therefore, they did not see a logarithm as its own mathematical concept. (DePierro et al., 2008; Ganesan & Dindyal, 2014; Hoon, et al., 2010; Liang & Wood, 2005; Kenney & Kastberg, 2013) This idea feeds into the breadth focus of this dissertation, while going into more depth would be helpful for those students that already have a grasp on that foundation, providing another way to access logarithms can help the students that are struggling with the relationship between logarithms and exponents.

3.1.1. Logarithms Through Concept Image

The idea of concept images of logarithms has been explored quite a bit, most often in lamenting students' depth or breadth of understanding. Siyepu (2015) discussed the lack of understanding of a natural logarithm, while Lappa and Nikolantonakis (2019) noticed that students had a poor understanding of how exponentiation tied into the logarithmic object. Valtoribio et al. (2018) studied Grade 11 students in the Phillipines and pointed to logarithmic functions as one of the least understood and noticed that students did not often refer back to definitions, so having a strong concept image was important. When looking at those teaching logarithms, Boz-Yaman and Yigit (2019) were not optimistic; pre-service mathematics teachers were found lacking both in concept images and in concept definitions around logarithms, which the researchers believed kept the individuals in an early phase of understanding.

So while there was much agreement in recent studies that students, and teachers, were lacking in their concept image of logarithms, there was less agreement on how to rectify this situation. Technology was offered as a tool to help students experience the relationship between these more difficult concepts and their subsequent graphs (Lappa, 2020; Lee, 2012; Perrotta & Rogora, 2021). More communication between scholars and students was also listed as a way to understand different conceptions of logarithms. Direct communication from teachers to students in the form of questions and discussions to understand a class' concept image of logarithms was suggested before moving forward with any new material (Valtoribio et al., 2018). The focus here was on correcting some students' erroneous concept images before building new ones.

Hamdam (2008) expanded on the idea of building new logarithmic images by looking at connections. She discussed how logarithms are presented throughout the mathematics curriculum and that the instructor has to juggle the idea of logarithm as the inverse and anti-derivative of two different functions $\left(e^x \text{ and} \frac{1}{x}\right)$ without any background work that ties them together.

3.1.2. The Breadth of Logarithms

Other studies did not use the theory of concept image but did argue for expanding the breadth of what a logarithm could be, while still primarily defining a logarithm as the inverse of an exponential equation. Berezovski and Zazkis (2006) created a framework for logarithms consisted of viewing logarithms as numbers, the operational meaning of logarithms (properties of logarithms), and understanding logarithms as functions. This framework was subsequently adopted by other scholars to help categorize students' understanding of logarithms where students were found to be deficient in all three areas of this framework (Berezovski, 2006; Ganesan & Dindyal, 2014; Kenney & Kastberg, 2013).

Building off students' lack of conception around logarithms, some researchers have looked at how well pre-service mathematics teachers understand logarithms (Berezovski, 2009; Okoye-Ogbalu & Mthethwa, 2019). In both cases, these were people that would soon be teaching mathematics at a secondary school and were self-acknowledged to be good at mathematics. Okoye-Ogbalu and Mthethwa came from a constructivist perspective, conducting interviews around logarithmic problems to determine if the pre-service teachers could list the pre-requisite knowledge needed to successfully complete the exercises. Berezovski (2009) tested a research methodology where pre-service teachers wrote out both sides of a 'practice interview' for a job involving teaching chapter around logarithms. In both cases, the pre-service teachers showed little understanding of what concepts built into logarithms nor could they logarithms within any of the three levels of the Berezovski and Zazkis framework, though there was a good understanding of difficulties that students experienced when working with the topic.

Students did not have a clear concept of a logarithm as a number (Berezovksi, 2006; Kenney & Kastberg, 2013; Liang & Wood, 2005). Two of the studies (Kenney & Kastberg, 2013; Liang & Wood, 2005) argued that the material should be retaught, focusing on the logarithm as a number and its own function separate from the exponential function, but there were no follow-up studies by the researchers focused on helping students see a logarithm this manner. The studies that did attempt to focus on the logarithm as a number go back to the history, which is discussed later in this chapter, but, for example, Toumasis (1993) included rewriting the relationship as ratios, so tying it back to one of the nomenclatures of *logarithm*, while also giving background context on how these logarithms would have been used in calculations in the past. Another paper went through the lens of variation theory. The study had students come up with multiple equivalent logarithmic expressions. While those students were found to have a solid view of logarithms as a number, they were not able to build that in to understanding operations with logarithms (O'Neil & Doerr, 2015).

Hoping to increase operational understanding, a few studies turned to technology. Keith Weber, in 2002, taught a lesson on exponentiation to two classes, then had students in one class write out a computer program performing the operation. He followed this by introducing logarithms as repeated division and asked a series of computation, rule, and conceptual questions about logarithms. The students who worked through the coding of exponentiation had a better understanding of the idea behind logarithmic operations and were able to answer some of the conceptual problems compared with his control group. While he did not actually use a computer in the classroom, he did have his students translate the lesson to a computer-based language seemingly to positive results. Later, Christof Weber (2019a) had students compare division with logarithms, creating an algorithm to calculate logarithms through repeated division. This new algorithm was built on the idea of long division, so reinforced that concept while also expanding it in a new way. While it is an interesting premise and would be a different way to view logarithms, I could not find any other place it had been put into practice outside his classroom.

Two other studies looked at logarithms in different ways through computing. Sand et al. (2022) had students create the logarithmic function from a Taylor polynomial using a computer to do much of the calculations, while Toumasis (2006) had students use a computer and a graphical program to answer the question whether a logarithm ever equals itself. Students in both of these studies explored aspects of logarithms that are not commonly discussed and could lead to a wider understanding of the underlying concept.

3.1.3. The Depth of Logarithms

A few studies did not try to look at logarithms in a new way, but tried to make new connections between exponential equations/functions and logarithms. Makgakga and Sepeng (2013) used their arms and the rest of their bodies to demonstrate graphical transformations of exponential functions to interpret logarithms, to the success of their students. Using higher technology, a common approach was to have the students work with a computer graphing system to try and determine what the changes to the graphs could tell them about the logarithmic function (Getenet & Beswick, 2014; Koştur & Yilmaz, 2017). While experiments using this technique did have students understand the boundaries of logarithms, those who relied on graphical techniques did not always understand where the graphs came from, or what exactly was a logarithm (Getenet & Beswick, 2014). Budinski and Takači (2013) also used graphing technology, but tied it into modeling as a way to introduce logarithms and to give students a real-life example of this abstract concept.

In a similar manner, two papers looked at *Realistic Mathematics Education (RME)*, the idea of using a concrete context to build to the more abstract (Engbersen, 2009; Webb et al., 2011). Logarithms were introduced as a way to work with common, exponential, real-world problems, such as compounding interest, and then expanded upon them from there. RME, and programs like it, have been shown to help contextualize logarithms, while also helping students communicate about the nature of logarithms. (Engbersen, 2009; Lelu & Julie, 2008). The students in these programs have shown to be more engaged, so as long as the material highlights the connection between logarithms and exponentials, they could grasp the concept more easily (Budinski & Takači, 2013; Webb et al., 2011; Wood, 2005).

In a study that showed the difficulty of presenting logarithms in new ways, Ural (2017) tried to have students construct the function and operations of logarithms by themselves through working with the inverse of exponential functions. The students had trouble with this premise and were found to have a poorer understanding of logarithms than expected. This study highlighted the need for breadth in teaching logarithms; students needed more past experiences to grasp onto when building to new concepts.

3.1.4. The Symbology of Logarithms

Studies have shown that students were confused around the symbols used for logarithms. Students saw ‘log’ not as an operation but as its own term or value and would apply algebraic rules to the ‘log’ such as reducing it in a fraction or factoring it in a polynomial (Aziz et al., 2017; de Gracia, 2016; Hoon et al., 2010; Kenney & Kastberg, 2013; Liang & Wood, 2005; Rafi & Retnawati, 2018). Some researchers suggested that teachers should emphasise the symbol through oral or written communication to correct this misunderstanding of a logarithm (Hoon, Singh & Ayop, 2010; Kenney & Kastberg, 2013; Liang & Wood, 2005; Wood, 2005). Others suggested changing the symbol altogether and using something new to indicate logarithms (Brennan, 2007; Hammack & Lyons, 1995; Hurwitz, 1999). I touch on the word *logarithm* and its meaning later in Chapter 7, as it contributes to the confusion found among students.

3.2. Studies Focused on Using Historical Ideas Around Logarithms

The incorporation of the history of mathematics into mathematics education has been frequently studied, though the studies have a diversity of purpose (e.g. Fauvel, 1991; Fauvel & van Maanen, 2006; Jankvist, 2009). Jankvist (2009) described the purpose of history of mathematics as being two-fold: it could be a goal upon itself or it could be a tool to strengthen the understanding of mathematics. As a goal, he did not mean that history of mathematics should be taught as only its own discipline, but rather that the purpose of including historical lessons in mathematics was to show that it was a product of human endeavour with all the fits and starts that implies (Jankvist & Kjeldsen, 2011). As this dissertation is focused mainly on using the ideas in the past to help the students make sense of mathematics today, namely to build up their concept image, I focus on the use of history as a tool, though history as a goal comes up in a few areas.

Some studies that shared this purpose used the theoretical construct of concept image to focus on expanding a student’s concept image by relating it to the past (Barahmand, 2020; Mosvold et al., 2014; Mota et al., 2013). The Mota et al. conducted their research over two school terms, where they introduced historical definitions of tangents to students multiple times. They found that students who had been in both studies had a stronger concept image and a correct concept definition, which could be an argument for continually revisiting the history of a concept as it is introduced. Two studies by Vagliardo (2004, 2009) seemed to support this. He did a concept mapping of logarithms through their history, their conceptual meaning to mathematicians, to teachers, and to students through various interviews. In his 2009 article, he found that there is an overfocus in education to logarithm as an exponent, which left students with a limited view of the function.

The idea of touching on logarithms' history has been studied quite a bit, it was even the example used in the introductory chapter to one of the foundational books about history in mathematics education (Fauvel & van Maanen, 2006). Most of these studies, though, focused on how instructors could incorporate the history of logarithms into their lesson plans. Much of it was theoretical, going through the history of logarithms and writing out suggestions for their readers (Confrey & Smith, 1995; Dennis & Confrey, 1997; Katz, 1995; Mendes & da Silva, 2018; Ostler, 2013; Panagiotou, 2011; Smith & Confrey, 1994; Vagliardo, 2004; Weber, 2016). While most of the papers did focus on the idea of logarithms as a relationship between arithmetic and geometric sequences, not many report having tested their ideas. Ferrari-Escolá et al. (2016), as well as Fermsjö (2014), did use the relationship between the sequences to have students discover the operations with logarithms. While both authors were successful in their initial endeavour, Fermsjö did find that students had difficulty seeing the logarithm as the inverse of exponentiation, so it created some confusion in students when they were expected to use that relationship in later mathematics.

There are a few studies that looked at other aspects of their history. Some branched out to focus on logarithmic relationship to division/splitting (Confrey & Smith, 1995; Ferrari-Escolá et al., 2016; Smith & Confrey, 1994); another examined the idea of a logarithmic curve being related to geometric means (Dennis & Confrey, 1997); another looked at the tools created due to logarithms, specifically slide rules (Ostler, 2013); and finally, Panagiotou (2011) went through a detailed history starting with the sequences and up through the inverse of exponents. This last article included suggestions for introducing each of these ideas in the classroom but did not follow-up with any implementation study. All of these studies argued for a more breadth to logarithms, logarithms were to remain the inverse of an exponential function, but they had other relationships that could be explored.

There are some studies that used episodes from the history of logarithms in the classroom. Toumasis (1993) created a unit that looked at the historical context leading to logarithms, the foundation of e, tying logarithms to exponents, and finally the logarithm as a function. He conducted this study for three years and found it advantageous for many reasons, one of them being that it tied logarithms to more prior knowledge, a way to create a better concept image. A similar study was run by Tsang-Yi (2012), but this time he looked at Napier and Brigg's logarithms and their ties to astronomy.

Textbooks are discussed for not including more than a superficial history in their pages (Vagliardo, 2009; Vural, 2021). This absence was listed as a reason that history was not more integrated into the classroom, the text often did not include the mathematics in its historical blurbs, but instead included notes about figures or colourful anecdotes (Ferreira & Rich, 2001; Martins Moura & de Jesus Brito, 2019; Riley, 2018). Tsang-Yi (2012) went further, noting that history would not be used in a mathematics classroom until it was presented in a way that teachers could incorporate into their lesson plans and be accessible in a place that teachers could easily find. While he does not mention textbooks, including historical notes that were useful for the teaching of mathematics in them would seem to fulfill his requirements.

A different tack was to use primary sources in the classroom such as Leonhard Euler's explanation of logarithms in his text *Introductio in analysin infinitorum* (ca. 1748) and having students work through that material (Demattè, 2021; Lappa, 2020). In both cases, the researchers were concerned about how the students interacted with the material. Lappa noted that reading the material was difficult for the students and they needed teacher intervention, but that the material did increase their interest. In a case-study done around a pre-service mathematics teacher, an early 20th-century textbook was used to get a feel of how logarithms were explained through arithmetic and geometric progressions. This student teacher also had trouble translating the words and symbols in the old text to today's language and to today's meaning of logarithms. His previous experience with logarithms did not help much in understanding them in this different way (Martins Moura & de Jesus Brito, 2019).

Relatedly, a few studies looked at current student's conception of a mathematical idea in relation to the mathematicians of the past (Keiser, 2004; Kjeldsen & Petersen, 2014), with the thought that if their concept image was more in line with the past, then maybe revisiting the history could strengthen it. Bråting and Pejlare (2015) argued against this, stating, “one cannot assume that students are at the same cognitive level as famous mathematicians of the past” (p. 259). Karp and Wasserman (2015) suggested introducing an aspect of logarithms used in the past, their ability to simplify calculations with trigonometric entities, to start a discussion with students on why that aspect is no longer used today.

I think it is interesting to present famous mathematical arguments and famous mathematicians to students, but understanding their work could be too difficult for some students just being introduced to the material. I believe that those students could still find conceptual understanding from historical ideas around logarithms, but perhaps they should be introduced to those ideas via appropriate elaborations and adaptations rather than via original sources. Including past presentations of logarithms in contemporary textbooks could give students a new representations of this concept which would hopefully strengthen their mathematical understanding, but also, as they become familiar with historical ideas in mathematics, it could serve as a pathway toward using primary sources.

3.3. Studies on Textbook Analysis with a Focus on Logarithms

Jeremy Kilpatrick (2014) stated that, “throughout history, the principal function of mathematics textbooks has been to serve as repositories of authorized knowledge” (p. 4). While there is a place in mathematics education to analyse the style and writing of a textbook, I will be looking back at this quotation, what idea of mathematics are they trying to pass on to students. In using concept image with textbook analysis, researchers tended to go two ways: they either kept it purely in the theoretical background of their research, or they used it as part of their methodology. In using it the first way, it was there as a justification for the study. For example, Kim (2012) researched the non-text part of a textbook as these “representations […] play important roles in one’s creation of “concept image”” (p. 184). In this same vein, Alyami (2020) looked at radian measurements, Avcu (2019) studied representations of special quadrilaterals, and Pieronkiewicz (2019) examined tangent lines, all arguing that the representations of the above could add to a reader’s concept image.

Other researchers used concept image as a part of their methods when analysing the texts. There were again two ways that they did this. One way was to go through the text and summarise either the personal concept definitions that could be created (Mai et al., 2017) or the concept images (Lo et al., 2006). The other prominent way was to use the ideas behind concept image to create themes that can be used in classifying the text. Vollstedt et al. (2014) started with concept image to create their themes of inner-mathematical and extra-mathematical. Extra-mathematical themes would relate the mathematics to concepts outside of school mathematics, while inner ones did the opposite. To them, this was especially important in the introduction of a new topic, as that was how students built new ideas onto an existing concept image. Bayda and Sutliff (2020) looked at the definitions in an Algebra 1 text and classified the texts as building to stipulated (purely mathematical) or extracted (from external sources) definitions. These ideas came from concept images, as extracted definitions helped students make connections to other concepts and building up a concept image, while stipulated definitions was how students become proficient at mathematics; the study looked at these instances in the text, but also at the interplay between the two. Similar to above, Czocher and Baker (2010) used the ideas behind concept image to classify definitions and exercises in the text as low-level and high-level, with the high-level ones building connections to other mathematics, or the external world, supporting concept image formation.

Building to studies of textbooks around logarithms, Engbersen (2009) compared textbooks from three different countries and how their presentation added breadth and depth to logarithms, though it was termed horizontal and vertical mathematization in his study. Along with concept image, he used the theories of Concept Rich Mathematics combined with Realistic Mathematics Education, which were normally based around instruction. I do appreciate the work that Engbersen did in translating the theories to textbooks, as they helped me see how the depth and breadth could be used in a textbook analysis with logarithms, so I took these ideas with me as I worked through this dissertation.

There were not many more studies to be found that look at logarithms in textbooks. Three of them looked at examples of proofs in texts (Bergwall, 2021; Johnson et al., 2010; Thompson et al., 2012) and used logarithms as one of the topics that they reviewed. The three studies reviewed textbooks used in American schools and schools in Finland and Sweden respectively, and found that justifications for logarithmic content were there over half the time. There was also expectations that the students would be able to prove some properties in the exercises. Bergwall (2021) joined other researchers (Vagliardo, 2009; Vural, 2021) in noting that logarithms always appeared together with exponential equations.

Senk and Thompson (2006) conducted a project where some classes were assigned readings from the text and worked with them in their class. The students in the project did better analysing non-standard mathematics, including logarithms, than those that were taught in a more traditional way. While this study did not speak for the presentation of logarithms in textbook, it did speak for the importance of students using textbooks as a resource.

3.4. Studies not about Logarithms but Related to this Dissertation

Eisenberg (2003) looked at pre-service mathematics teachers and their ability to evaluate square roots without a calculator. He noted that, while definitions and current usage are the most important, wading into the technicalities helped build up a rich concept image. Revisiting how to find a square root and deducing different methods was one such method to achieve that. I do not get into the calculating of logarithms much in this study, but it was something that I kept in mind and included it when it was presented in a particularly interesting way that could still tie into methods that they learn today.

And lastly, I want to bring to attention a study that looked at how mathematics anxiety affected students' learning of logarithms, and, perhaps not surprisingly, found that students with high anxiety had a low-level ability to solve logarithmic problems (Hayati et al., 2019). Perhaps, introducing new ways to conceive of logarithms could help with this problem.

3.5. Last Notes on the Studies in this Chapter

The breadth of the studies in this chapter show how concerned researchers are on students' conception of logarithms. Studies have shown that students are confused about almost all aspects of the concept, the name, the operations associated with it, and its relationship to other operations. Meanwhile, teachers and researchers have looked beyond mathematics to help introduce this concept, including coming to logarithms through their history and through technology. These ideas included introducing logarithms through modeling, through progressions, and through repeated division. While some of the ideas have been tried in the classroom, the majority are in the theoretical stage.

This dissertation is an addition to the theoretical ideas put forth to introduce logarithms, though it attempts to go past their introduction by establishing connections to related mathematics in most of the topics where logarithms are seen, up through integration. The suggestions put forth in Chapter 8 also work to clarify some of the students' main misconceptions around logarithms by reinforcing the idea of the logarithm as a number, introducing new ideas of graphing logarithms, and new ways to think of a logarithm as a function.

Chapter 4. A Lo(n)g History

In this chapter, I provide an overview of the history of logarithms, first coming through the relationship between arithmetic and geometric sequences, followed by the immediate after-effects, such as the creation of physical tools for logarithmic calculations, logarithms in geometric space, and the creation of logarithmic sequences. Next, I explore the history of exponents and their eventual connection to logarithms. I detail the mathematical foundations and explanations of logarithms in this section, as I explore how these ideas were presented to students, past and present, in Chapters 5 and 6.

In writing about the history of logarithms, there are quite a few strands to follow. The first book on logarithms was published in 1614 and, over their first 150 years, logarithms were presented as a way to simplify calculations, as a comparison between two progressions, as an area, as a single sequence, and, finally, as an exponent. While these strands were interwoven at many points, there were definite times that one came into the forefront, and so I shall try focus on them individually, but tie them together when needed.

As a note, I have some experience with the Latin and Ancient Greek languages, but am unfamiliar with other languages. I use the work of several history of mathematics scholars and their translations for most of the texts that are not in English. Due to this, some of my work on the early history of logarithms refers to other scholars, as well as to primary sources.

I will also use the terms ‘discovery’, ‘invention’, and ‘creation’ in reference to logarithms. In historic texts, logarithms were often called ‘invented’ numbers (Briggs, 1631, Leadbetter, 1728), while modern researchers more often refer to the ‘discovery’ of logarithms (Clark & Montelle, 2012; Waldvogel, 2014). I feel stressing that logarithms were something invented or created by humans speaks to mathematics as a human endeavour, an important idea that is sometimes missing from mathematics classrooms. I also appreciate that mathematics is often considered universal, something that exists outside of humans, but there for humans to discover (Fine, 2012). The history of logarithms falls in both categories to me, the idea was there, the relationship between these sequences had existed since the existence of numbers and was waiting to be discovered. But the ability to make use of this relationship in a meaningful way was a powerful invention, or creation.

4.1. Following the Progressions

Given that both the discoverers of logarithms, John Napier and Jost Bürgi (1552 – 1632), took their cue from the relationship between an arithmetic and geometric sequence, I spend time exploring that relationship, namely that the two sequences and their special property between addition and multiplication has been known since at least the time of the Ancient Greeks. While there were tables of powers of perfect squares in ancient Babylon (Boyer & Merzbach, 2011), they did not focus on the relationship between the arithmetic and geometric sequences; that multiplication in the geometric translated to addition in the arithmetic. It was not until Ancient Greece that more solid evidence of these properties were found. Euclid (c. 300BCE) wrote of geometric sequence of integer terms, though not using that language (Smith & Confrey, 1994), while Archimedes' work (c. 250BCE) was an early example of noting the connection between multiplication and placement in a geometric sequence.

In *The Sand Reckoner*, Archimedes (ca. 220 BCE./2009) was determined to name a number that would be higher than the amount of sand found on Earth. In order to accomplish this, he had to create the language to discuss very large numbers and then also maneuver his new large numbers in a way that gave his upper bound a number-name. For the first part, he started with the highest single named number, a myriad, or 10,000. He then labeled the numbers between a unit and a myriad-myriads (100,000,000) the first order. The second order would start with the 10^8 and included every multiple of that number by the numbers in the first order, so it continued to 200,000,000 then 300,000,000 and ended at $10^{8\cdot 2}$ or 10^{16}. The third would start with 10^{16} and included every multiple of that number and those in the first order, so it would end at $10^{8\cdot 3}$ or 10^{24}. He would continue this pattern until he had myriad orders, ending at $10^{8\cdot 10,000}$ or $10^{80,000}$. He went further to classify even larger numbers, but for the purposes of this explanation, I will stop here.

The basic outline of Archimedes' argument was that if he could estimate the grains of sand that would fit into a tiny sphere, then he could choose a number higher (making it a power of 10) and use proportionality to find how many grains of sand would fit inside a sphere larger than the Earth. Therefore, the number of grains of sand on the earth would be fewer than that number. Archimedes split the orders he created into 8 parts, all multiples of 10 so all the numbers would be in proportion. His first order would be, in contemporary notation, $1, 10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7$; similarly, his second octad would be $10^8, 10^9, 10^{10}, 10^{11}, 10^{12}, 10^{13}, 10^{14}, 10^{15}$; and so forth. He then started with a small sphere that would fit 10 units of the second order, or 10^9, grains of sand.

Archimedes then did calculations with his new numbers, so established that if any two terms in any of the octads are multiplied, then the distance of the product will be at least as much from the larger one as the smaller one was from the unit. So he slowly started to increase his sphere holding his 10^9 grains of sand; first by 100 (and since it was a volume, he was careful to increase it by 100^3 or 1,000,000) and stated that he was multiplying the 7th term of the sequence (1,000,000) by the 10th term of the sequence (10^9) and since the 7th term is 6 away from the unit, then the product will be 6 away from the 10th term so the result was the 16th term or 10^{15}. Therefore, to multiply the two numbers, Archimedes just needed to add their space in the sequence and subtract 1, his starting value. He continued in this manner to build up his sphere and subsequently the number of grains of sand, being able to put a name to the number of grains until it was larger than the Earth.

In writing about numbers in proportion, in a sequence, this way, Archimedes was examining the relationship between multiplication and addition that was the backbone of logarithms and a major reason for their invention. It was an idea that had been known since ancient times, but how to harness its power was not understood until the early 1600s.

Another work of note for logarithms in Ancient Greek mathematics was by Diophantus of Alexandria (c. 250CE) over 500 years later, who publicized writing mathematics with a syncopated algebra notation, using more symbols than the rhetorical algebra of the past (Stallings, 2000). His was one of the many works that was translated into Arabic and expanded upon during the Islamic Golden Age (Rashed, 2013). While Islamic scholars, and Diophantus, did much work with algebra and the rules of exponents, as will be seen in Section 4.8, their contribution to these two sequences appears to be passing along the Hindu–Arabic number system, the numerals used for our current base-ten system, and the work of the Ancient Greeks. Of course, there could be texts found in the future that expand on their contributions, but as they are currently not known, I will move on to Europe.

Thomas Bradwardine (1295 – 1339), in England, followed in the footsteps of Aristotle, with his work around the movement of objects and the relationship among velocity, resistance, and force. He saw that velocity varied in proportion to the ratio of force to resistance. He did more than any scholar to build out the mathematics of those ratios and so developed their multiplicative structure, but kept it contained to that one instance (Smith & Confrey, 1994). In France, Nicholas Oresme (ca. 1320 – 1382) separated Bradwardine's ideas from physics, while also expanding upon them. Bradwardine had referred to fractional exponents, but it was Oresme who built up the density of geometric sequences. To understand his writing, we have to give meaning to some of the more common words: part – a root (in his words, an aliquot) of a number (2 is a part of 8, as $2^3 = 8, \text{or } 2 = 8^{\frac{1}{3}}.$); parts – when you have more than one part of the same number (4 is parts of 8, as $4 = 8^{\frac{2}{3}}$); and commensurable – two ratios are commensurable if they have a common measure that is a part of both of them (4:1 and 8:1 are commensurable as 2:1 is a common measure of both, 4:1 and 5:1 are incommensurable). Oresme saw that roots and parts of a ratio were commensurable with that ratio, creating a much larger and denser sequence.

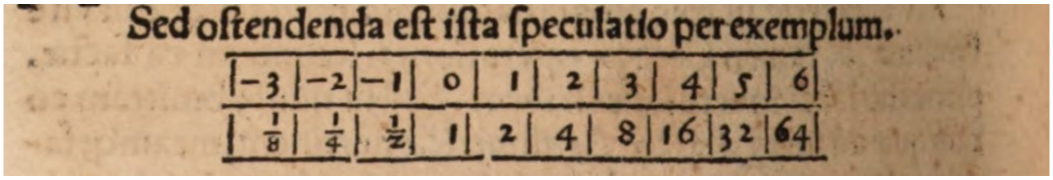
Using the ideas above, Oresme looked at a sequence of ratios (he used the word *proportio* which today would translate just as proportion, but in reading the text it is used both as ratio and proportion, depending on context) and showed that, if you have a base ratio of A:1, then part and parts of that ratio could be commensurable with each other and with the base ratio. He further built it out that everything in this sequence would be commensurable to everything else in this sequence, but incommensurable to any other ratio (Grant, 1960). To expand upon Oresme's idea, the ratio 2:3 could lie in a sequence with $\sqrt{2}:\sqrt{3}$ as well as $8:27$, and they would be commensurable with each other as well as with every other instance of 2 and 3 raised to the same rational exponent. This supposition contributed to the density of geometric sequences, as in today's terms his idea would be expressed as $f(x) = \left(\frac{2}{3}\right)^x$, with $f(x)$ a continuous function. Oresme's world would not have allowed for this function, but by building up the density of both sequences he could spark that idea in others.

The relationship between a geometric sequence and arithmetic sequence appeared in many texts in the 15th century, but none were as influential as *Triparty en la Science des nombres* by Nicolas Chuquet (1445 – 1488). In it, he listed out two tables of numbers, and noted the multiplication/addition relationship as seen in Figure 4.1. Next to this list he wrote how multiplying 2^1 and 4^2 gives 8^3 (it would be written today as $2x^1 \times 4x^2 = 8x^3$), which was a manner of exploring the relationship between the geometric and arithmetic sequences. He did not go further than that; he was mainly using this idea to further his work on multiplication and division of variables.

Nombres	Denoĩacions
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14

Figure 4.1: Representation of the geometric and arithmetic sequence around 2 (Chuquet, 1484/1881, p. 151)

Chuquet's work perhaps reached Michael Stifel (1487 – 1567), a German mathematician, who included the same sequence in his text *Arithmetica Integra.* Unlike others before him, Stifel continued the sequences both to the right and to the left, relating the negative numbers in the arithmetic sequence with the fractions in the geometric one (Figure 4.2). He then followed with specifying the three main operations that this relationship simplifies – multiplication, division, and working with powers. His text was the most clear-cut example of the relationship between these two sequences before the time of Napier and Bürgi. Like Chuquet, Stifel was using this relationship to work with monomials, so did not focus on what this relationship could mean for mathematics (Panagiotou, 2011).


Sed oſtendenda eſt iſta ſpeculatio per exemplum.

−3	−2	−1	0	1	2	3	4	5	6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64

Figure 4.2: Expanding the arithmetic/geometric sequence around 2 to the left (Stifel, 1544, p. 249)

This prehistory all had to happen before logarithms could be discovered. There needed to be a fully formed relationship between the geometric and arithmetic sequences. There had to be the idea that a geometric sequence could include a dense-enough collection of numbers that the operations that consisted of this relationship would be useful. And there had to be a reason even to search for a dense sequence. The late 1500s was perfect timing as there was a strong emphasis on astronomy, and the computations were tedious. Prosthaphaeresis, processes to change multiplication and division into addition and subtraction, often by use of trigonometric identities, was mainly employed to do the calculations (Panagiotou, 2011), but scientists were desperate for something that could simplify the process even further.

4.2. Discovering Logarithms: A Tale of Two Mathematicians

While Jost Bürgi and John Napier both saw the potential in relationships between the arithmetic and geometric sequence, they approached this same calculating tool in very different ways. As it is hard to actually name a ‘true’ creator of logarithms, since both these scholars appeared to have worked independently, I discuss them separately and then try to specify how their creations built into what we now know of logarithms.

4.2.1. Jost Bürgi (1552–1632)

Jost Bürgi was born in Switzerland, but lived most of his adult life in modern-day Germany and Czechia. He was an astronomer and well-known clock–maker, devising the first clock precise to the second and later becoming an Imperial Clock–Maker in Prague, as well as the colleague and aide to the Imperial Astronomer, Johannes Kepler (1571 – 1630) (Clark, 2015). He was also a prolific mathematician, devising a new algorithm to create more accurate tables of sines than those that existed at the time Unlike previous ways of calculating sines, which were mainly through inscribing polygons into a circle, he managed to create an arithmetic method that was even more accurate (Folkerts et al., 2016). He took a similar tack in creating logarithms, finding a way to simplify the calculations while maintaining their accuracy. He took a brilliantly simple tack; if he could get a dense enough arithmetic sequence, then his geometric sequence had the potential to include, or estimate, the sequence of natural numbers. To do this, in today's parlance, he chose a number close to 1, in this case $1\frac{1}{10{,}000}$ or 1.0001 as the common ratio in a geometric progression (Smith & Confrey, 1994). Clark (2015) translated Bürgi's work from German to English and includes his handwritten foreward, where he gave clear reasonings for his creation

> Dear friendly reader: though many excellent and various tables have been invented to remove the difficulties involved in calculating multiplications, divisions, and extractions of roots, these have always been only for particular [calculations]. So multiplication and division have their own tables, e.g., the Pythagorean table, the extraction of square roots has its table of squares, the cubical extraction has its table of cubes, and thus continuing, every quantity needs its special tables; the multitude of tables is not only annoying but also cumbersome and difficult. I therefore searched for all time and worked to invent general tables with which you would like to do all of the above things. Consider therefore the property and correspondence of two progressions. One is arithmetic, the other geometric; what is multiplication is only addition, and what is division is subtraction in that, and what is in the extraction of a square root is only halving in that, extraction of a cube root is only dividing in 3, extraction of a fourth root to divide in 4, fifth root in 5, and so on in other quantities. I have considered nothing more useful than to create these tables so it may happen that all the numbers may be found in the same way. (Bürgi, 1620/2015, p. 121)

Going from the same geometric progression based around 2 that was used in earlier texts,[2] he showed the ease of calculations using these progressions and that, with the creation of tables, users could just use this one tool instead of flipping to many different calculating aides (Clark, 2015).

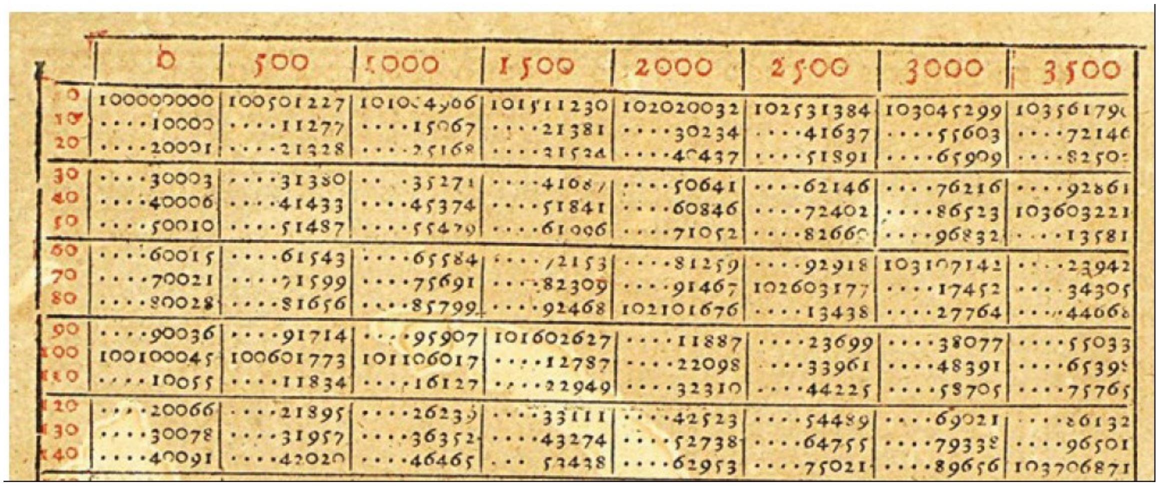
[2] He does not reference Stifel, but references Simon Jacob's *Rechenbuch auf den Linien und mit Ziffern* published in 1557 on Page 21b which has the same geometric progression based around 2 and the relationship presented. He follows this with the name Moritius Zons, whose work with a geometric progression based around 3 can be found in his 1616 text (Clark, 2015).

Bürgi did not leave any details on how he constructed his tables, but a few scholars (Roegel, 2010c; Waldvogel, 2014) have derived a path that seems reasonable. To build up a dense geometric progression, his baseline started with having 0 correspond to 10^8 and then he would multiply that number by the ratio for his geometric sequence, $1 + \frac{1}{10{,}000}$, and continue. In today's terms the sequence would be the terms associated with $10^8\left(1 + \frac{1}{10^4}\right)^n$. To make his tables, though, he used a linear extrapolation to go to his next number. I have written out the first few terms:

$$a_0 = 100{,}000{,}000$$
$$a_1 = 100{,}000{,}000\left(1 + \frac{1}{10{,}000}\right) \text{ or } a_0\left(1 + \frac{1}{10{,}000}\right)$$
$$a_2 = 100{,}000{,}000\left(1 + \frac{1}{10{,}000}\right)\left(1 + \frac{1}{10{,}000}\right) \text{ or } a_1\left(1 + \frac{1}{10{,}000}\right)$$

It is pretty easy to see now how this can be rewritten to $a_0 = 10^8, a_{n+1} = a_n + \frac{a_n}{10{,}000}$, with n starting at 0 and continuing until $a_n = 10^9$, which was at $n = 23{,}027$. The larger values, such as 10^8 and 10^9, were used both by Bürgi and Napier, so that all logarithms, even if written out to 8 digits, would be well above zero. While fractions were common at this time, decimal notation had not yet been standardised, so both mathematicians were avoiding the need for fractions by using such large values. Napier actually introduced the decimal notation that we use to this day during in his work around logarithms (Jourdain, 1914).

Bürgi's tables (Figure 4.3) were more properly described as a table of anti-logarithms, since the tables are double-entry where they were organized by logarithm. He did not use these terms, instead, he colour-coded his tables. The logarithms, or the arithmetic sequence, were called 'red numbers' in his text and were in red; the anti-logarithms, the geometric sequence, were in black. For example, to multiply $100521328 \times 100090036$, search for them among the black, 10052138 is associated with the 500 column and 20 row, equalling 520; 100090036 would be 090. Adding the two would give 610, which should now be found among the red, it would be the 500 column and the 110 row, so a result of 100611834 which then would be appended an appropriate number of 0s (a single digit off of today's calculator computation). He included directions on how to use his tables to do straightforward calculations, as well as linear interpolations for numbers in between and higher than those in the table. (Clark, 2015).



	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	10356179
10	····10000	····11277	····15067	····21381	····30234	····41637	····55603	····72146
20	····20001	····21328	····25168	····31524	····40437	····51891	····65909	····8250
30	····30003	····31380	····35271	····41687	····50641	····62146	····76216	····92861
40	····40006	····41433	····45374	····51841	····60846	····72402	····86523	103603221
50	····50010	····51487	····55479	····61996	····71052	····82660	····96832	····13581
60	····60015	····61543	····65584	····72153	····81259	····92918	103107142	····23942
70	····70021	····71599	····75691	····82309	····91467	102603177	····17452	····34305
80	····80028	····81656	····85799	····92468	102101676	····13438	····27764	····4466
90	····90036	····91714	····95907	101602627	····11887	····23699	····38077	····55033
100	100100045	100601773	101106017	····12787	····22098	····33961	····48391	····6539
110	····10055	····11834	····16127	····22949	····32310	····44225	····58705	····75765
120	····20066	····21895	····26239	····33111	····42523	····54489	····69021	····86132
130	····30078	····31957	····36352	····43274	····52738	····64755	····79338	····96501
140	····40091	····42020	····46465	····53438	····62953	····75021	····89656	103706871

Figure 4.3: Part of the first page of Bürgi's Table (Bürgi, 1620/2015, p. 190)

Bürgi's work in logarithms did not go much beyond these tables, but that should not minimize what he did. He looked at a long-known mathematical property between two progressions, realized that he could condense it, and used that to make a mechanical tool for calculations. But, from this description, I am not sure if that should be considered as the creation of a new operation, or if what he created was a new, and very important, tool. In the long run, it does not really matter. While the creation of Bürgi's tables is easier to grasp than Napier's and, with the colour-coding, arguably easier to use, they did not have much influence at the time. Napier published his text, after being rewritten with the help of Henry Briggs (1561 – 1630), it had been translated into multiple languages (Clark & Montelle, 2012). He would be the one to popularise these tables and, through his work on creating a structure around logarithms, would set the foundation for them being more than just a calculating tool. I turn to a more detailed description of Napier and his work in the next section.

4.2.2. John Napier (1550 – 1617)

Napier was born near Edinburgh, Scotland and was a well-known theologian, inventor, and mathematician. He was educated at St Andrews University, followed that by a few years of studying on the continent, before returning to Scotland where he wrote texts around religion and invented wartime instruments (Jourdain, 1914). His scholarship put him in touch with astronomers, and the use of prosthaphaeresis which piqued his interest in creating a calculating tool (Pierce, 1977).

Like Bürgi, Napier realised that, if a small enough ratio were chosen for the geometric progression, then it could be near enough to every integer between $10^8 \text{ and } 10^9$ (today this would be 1 and 10, but he was avoiding using fractions or the nascent decimal notation), so that estimation or interpolation could be used to find all the integers in that span. And, while he was very much focused on the application of logarithms, he also worked to apply theoretically a continuity to the relationship between the arithmetic and geometric progressions; the tables are discrete values and Napier knew that numbers had to be skipped in his geometric progression, but he was able to envision it as a line where all values are included. He saw the importance of building mathematical structure to logarithms, which became apparent when looking through his two texts around the subject. *Mirifici Logarithmorum Canonis Descriptio* ('Description of the Wonderful Canon of Logarithms', 1614), which was published first but written second, discussed his idea of the logarithm function (though not called a function yet), introduced the tables and explained their use. His need for continuity of logarithms may have led Napier to define them in a way that few have done since.[3] In order for his logarithms to be more than just a dense progression, he viewed them in a kinematic manner which I explain in some detail below.

But first, the situation in which Napier was using logarithms was different from today. He planned them for use in astronomy, so he based his logarithms around the sines of angles. Sines at that time were not considered as the ratio between sides of a triangle, but were visualised as a line, specifically the half the chord that corresponds to the angle at the centre of a circle of a given radius (Hobson, 1914). They were not beholden to a specific circle, but would alter depending upon the radius. In Figure 4.4, half of BD would be the sine of the angle θ (left), while "the whole sine", a term used by Napier and others at that time to refer to where the angle is 90º, would be the radius BA (right) as BD would just be the entire diameter.

[3] One of the few examples I have come across is Colin MacLaurin's (1742) *Treatise of Fluxions*, which was written to justify Newtonian calculus from the ground up. In this case, logarithms based around motion makes a certain sort of sense.

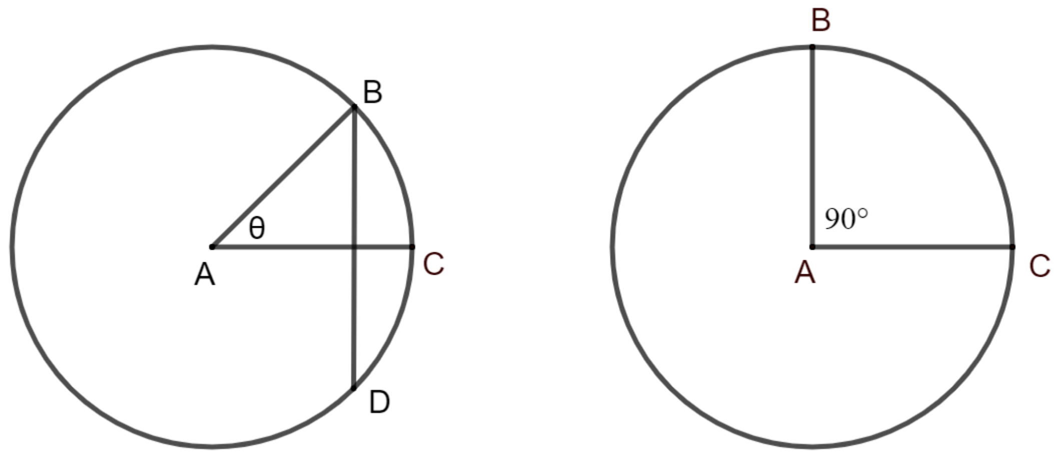


Figure 4.4: Left) illustration of sin θ being half of BD, Right) illustration of sin 90º as BA

All the following mathematics based around Napier's construction of logarithms came from two translations of *Descriptio* into English – Edward Wright (1616) and Ian Bruce (2012) – as well as two translations of Napier's second work, *Mirifici Logarithmorum Canonis Constructio* (1619, 'The Construction of the Wonderful Canon of Logarithms') – William Rae MacDonald (1888) and Ian Bruce (2012). The nicknames of *Descriptio* and *Constructio* are often used with these texts, so I will follow suit in this dissertation. Napier wrote in Latin, so while I did refer back to the original texts, I used these other sources to make sense of Napier's work for my own knowledge and then to explain it for the reader.

In order to make logarithms useful for his primary audience of astronomers, Napier chose a large circle and began with the logarithm of the whole sine (sin 90º) equal the radius, 10^7. He did this for the same reason that Bürgi did above, namely to side-step decimals and the notation that was still in flux.

As Napier was working within the confines of sines, he imagined a line that would hold all values of the sine of an angle, with the length of 10^7. As his whole sine (90º) is the radius (10^7), and the sine of no angle would be 0, it is reasonable to conclude that this line would hold all other sine values, as they would be between these two extremes. He declared that this line could be made of various segments that diminish in proportion to the distance remaining between them and the end of the line.

I will follow the example that Napier put in *Constructio,* starting with Figure 4.5 and using the proportion of $\frac{1}{10}$. The entire line segment is TS, a tenth of the way down that line, is point A. With the new line segment AS, a tenth of the way down that line there is the point B. Then with BS, a tenth of the way down that segment there is a point C, and so on. So the line segment TA equals $\frac{1}{10}$TS, AB is $\frac{1}{10}$AS, and BC is $\frac{1}{10}$BS, and so forth.

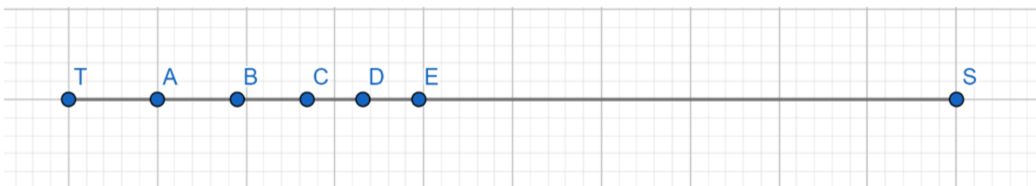


Figure 4.5: A line segment in diminishing proportion

These line segments represent a geometric progression when looking at the space remaining on the original line segment. Since each point is at a tenth of the way down the line, the remaining segment is nine-tenths of the original line.

To write it out in a more modern notation: $AS = \frac{9}{10}TS$; $BS = \frac{9}{10}AS$, so, by substitution, $\left(\frac{9}{10}\right)^2 TS$; $CS = \frac{9}{10}BS$, so $\left(\frac{9}{10}\right)^3 TS$. This is a diminishing geometric proportion represented in line segments.

Napier called these remaining line segments the sines of the angles from 0 to 90º of a circle with the radius 10^7. He then put this line segment into a system with a different, continuous line, which has points of equal spacing (Figure 4.6).

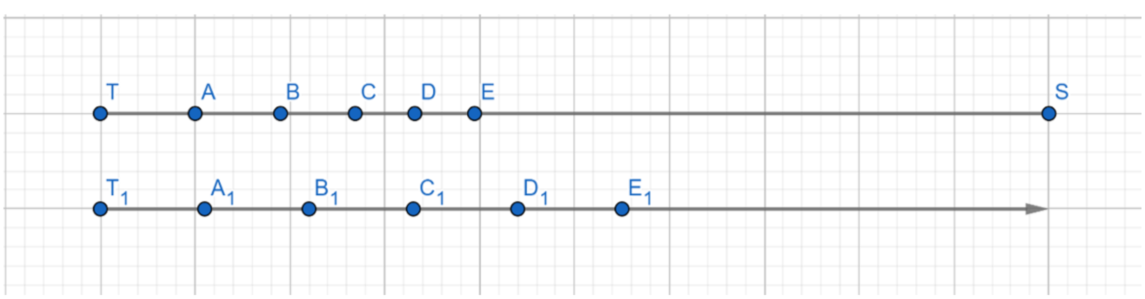


Figure 4.6: The line segment TS with segments of diminishing proportion and the continuous line T_1 with evenly spaced segments

A line beginning with point T_1 has points equally spaced, it is consistent with an arithmetic sequence, since the distances build equally along the continuous line. For more modern notation, each distance is 5.5 units, therefore, T_1A_1 is 5.5, T_1B_1 is 2×5.5 and T_1C_1 is 3×5.5.

Napier at this point was able to illustrate what he meant by the term 'logarithm'. He imagined a point running along the two lines with the same starting velocity (Figure 4.7). The point on the line segment TS slowed in proportion to the distance left to the end of the line. The point on the line starting with T_1 stayed at an even velocity. He then corresponded the points at various times, and called displacement of the point on the continuous line the logarithm.

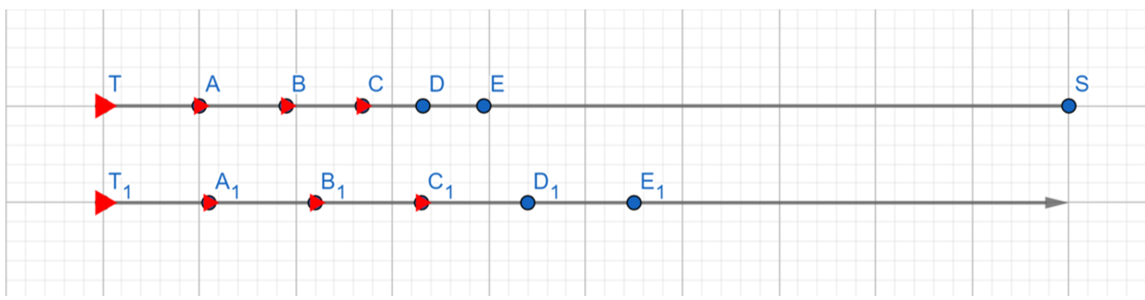


Figure 4.7: Two points with the same starting velocity running on parallel lines. The velocity of the bottom stays consistent while the velocity of the top slows in proportion to the distance remaining in the line segment.

In the example presented, both points start at T and T_1 with a velocity of 50 units/time, but by the time they get to A, the top one has slowed to 45 units/time $\left(\frac{9}{10} \times 50\right)$, by B it has slowed to around 40.5 units/time ($\frac{9}{10} \times 45$ or $\left(\frac{9}{10}\right)^2 \times 50$), and by C the velocity is 36.45 units/time ($\frac{9}{10} \times 40.5$ or $\left(\frac{9}{10}\right)^3 \times 50$). It will continue to reduce velocity and, by design, will never reach S. In the bottom line starting with T_1, the point travels at a consistent velocity so remains at 50 units/time.

Napier correlated the lengths of segments remaining on the top line (the sines) with the space travelled on the bottom line (the logarithms), so, at A, the sine would be AS, and the logarithm would be T_1A_1 (Figure 4.8). At B, the sine would be BS, and the logarithm would be T_1B_1, and so on.

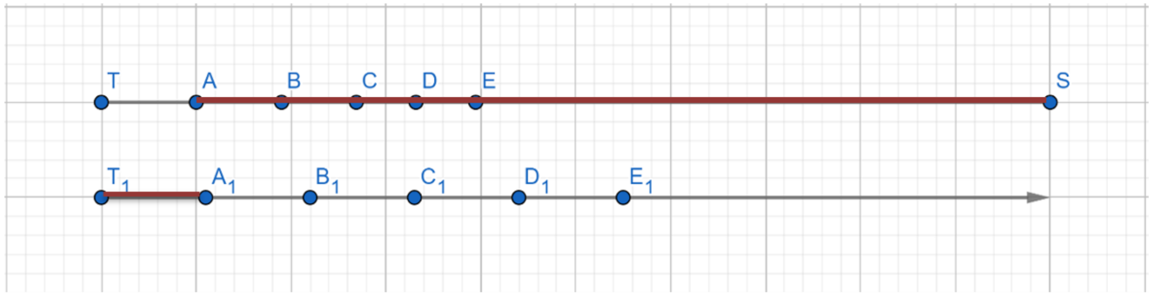


Figure 4.8: The relationship between the sine (top line) and logarithm (bottom line) at a given point

To bring this back to the geometric and arithmetic comparison, it can be rewritten in modern notation as discussed earlier.

AS is $\left(\frac{9}{10}\right) \times 1\mathrm{TS}$ and the logarithm is $\mathrm{T_1A_1} = 1 \times (5.5)$

BS is $\left(\frac{9}{10}\right)^2 \times \mathrm{TS}$ and the logarithm is $\mathrm{T_1B_1} = 2 \times (5.5)$

CS is $\left(\frac{9}{10}\right)^3 \times \mathrm{TS}$ and the logarithm is $\mathrm{T_1C_1} = 3 \times (5.5)$

So even though this is an unusual way to visualise logarithms, the connection between the two sequences becomes clear. Napier managed to use a kinetic/dynamic geometrical approach to imagine and illustrate his logarithms. To make this a more generalized definition, he went away from the points on the segment being placed at a tenth of the length to the end, and instead used an undefined ratio (Figure 4.9).

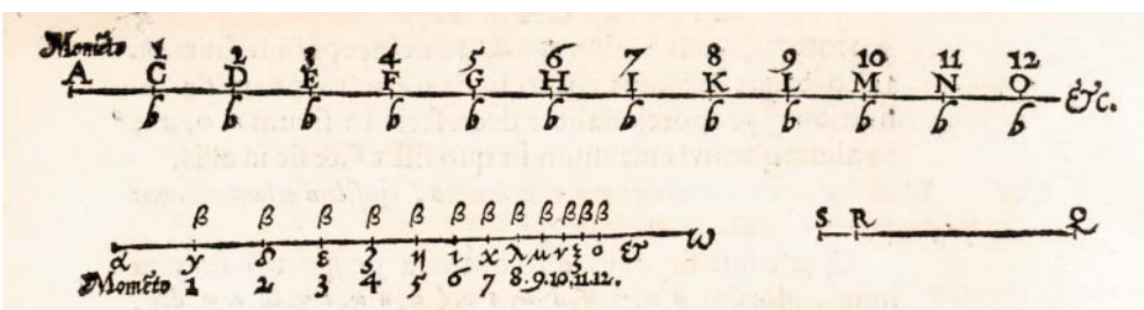


Figure 4.9: Image from *Descriptio*, explaining logarithms using a generalized proportion. (Napier, 1614, p. 4)

The bottom line segment $\alpha\omega$ was segmented according to the proportion $\frac{SR}{SQ}$, so the first point γ would be $\left(\frac{SR}{SQ}\right) \times \alpha\omega$, the second point δ would be $\left(\frac{SR}{SQ}\right) \times \gamma\omega$, and so on.

The top line was Napier's continuous line, where the points were spaced evenly apart. To relate this back previous example:

$\gamma\omega$ is $\left(\frac{SR}{SQ}\right)^1 \times \alpha\omega$ and the logarithm is $\mathrm{AC} = 1 \times \mathrm{AC}$

$\delta\omega$ is $\left(\frac{SR}{SQ}\right)^2 \times \alpha\omega$ and the logarithm is $\mathrm{AD} = 2 \times \mathrm{AC}$

$\varepsilon\omega$ is $\left(\frac{SR}{SQ}\right)^3 \times \alpha\omega$ and the logarithm is $\mathrm{AE} = 3 \times \mathrm{AC}$

In *Constructio* (1619), the text Napier wrote first but was published second, he went into detail on how to calculate his logarithms. He first set out his geometric sequence. As his radius was 10^7, he began by choosing a ratio $\left(\frac{1}{10}\right)^7$, so there was high density and relatively easy calculations. As he was working with a decreasing geometric sequence, he subtracted instead of adding, unlike Bürgi. His sequence, in modern terminology was $a_0 = 10^7; a_n = a_{n-1}\left(1 - \frac{1}{10^7}\right)$

To begin calculating his logarithms, Napier again used the idea of motion. He added a small section on his line with the decreasing velocity (the top line in Figure 4.10) where he could imagine that his point started at an even greater velocity.

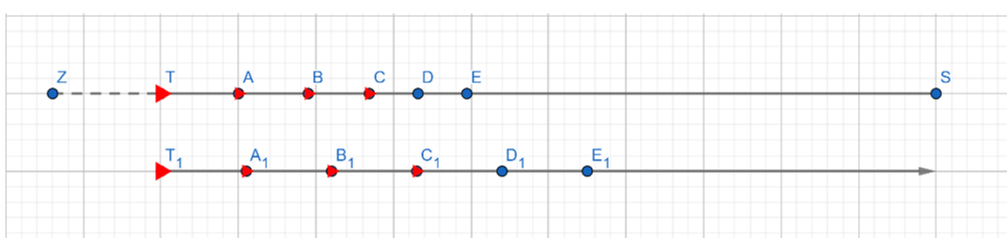


Figure 4.10: Napier's extension of the line segment representing a geometric sequence

Given that the point on the line segment TS is slowing down, there were some natural boundaries for the distances travelled by points on both segments, which meant there were some boundaries for the logarithms. Looking at point A (which as discussed above would be Napier's first sine AS at 9,999,999) the sine would be AS and the logarithm of AS would be T_1A_1. At that point, a few things were true:

1) the point takes the same time to travel from ZT to TA, so $\frac{TS}{AS} = \frac{ZS}{TS}$;

2) $TA < T1A1$; and

3) $ZT > T1A1$.

Using those ideas logarithm T_1A_1 could be estimated by computing the boundaries of TA and ZT.

Napier now knew that $TA = TS - AS$ so $10^7 - AS$ and that $ZT = ZS - TS$. But since Z was a point not initially on his line, he would want to use the ratio in 1) to rewrite it in known terms.

Napier could start by rewriting it as $TS\left(\frac{ZS}{TS} - 1\right)$ and then substitute so $TS\left(\frac{TS}{AS} - 1\right)$ or $TS\left(\frac{TS-AS}{AS}\right)$ or $10^7\left(\frac{10^7 - AS}{AS}\right)$ so that his logarithm T_1A_1 has the boundaries, $10^7 - AS < T_1A_1 < 10^7\left(\frac{10^7-AS}{AS}\right)$.

Now given that AS is 9,999,999 then: $1 < T_1A_1 < 10^7\left(\frac{1}{9{,}999{,}999}\right)$ or $1 < T_1A_1 < 1.0000001$. And at this point Napier averaged to find his estimated logarithm for this sine.

Napier did this for the first 100 terms, setting them along an arithmetic sequence starting at 1.00000005 and then began the process again for a second table at 10^7, but this time with a ratio of $\left(\frac{1}{10}\right)^5$ which let him start where the first 100 terms ended. His arithmetic sequence now increased by 100 for each term to account for the jumps. He did this for the next 50 terms. For his third table, he again started at 10^7, but this time divided by 2000, and his arithmetic sequence jumped by 5,000 each time. He did these jumps to lessen the work and the errors. He had 20 sequences in the third table with 69 subsequences; which gave him a dense enough progression to create his logarithms for the sines.

Napier's method of logarithms did not last past this text, even at the end of *Constructio,* he wrote an appendix which began: "On the construction of another and better kind of logarithms, namely one in which the logarithm of unity is 0" (Napier, 1619/1888, p. 48). At this time, he had already been working with Henry Briggs on the logarithms that would become popularised throughout the mathematics world and beyond.

While there are arguments over who began working on logarithmic tables first, Napier or Bürgi, Napier did publish a few years before Bürgi. It is because of that, and the immediate fanfare that followed, along with his rigorous construction of logarithms, that John Napier had been considered the discoverer, or inventor, of logarithms. By stretching his logarithms to infinity, and placing the geometric sequence on a line implying a density that could lead to a continuous function, he gave them a form that others could use to continue his work. In his hands, they were more than just a wonderful new calculating tool; they were a proper new mathematical entity.

4.3. Immediate Improvements to Napier's Work

As discussed, Napier's work made an immediate impact. Edward Wright (1561 – 1615), an English mathematician, translated *Descriptio* into English, which was published through the work of his friend, another English mathematician, Henry Briggs, in 1616, as Wright had died shortly before completing the translation (Napier/Wright, 1616). Briggs was so taken by this new creation that he immediately wrote to Napier expressing both his admiration and suggestions for improving Napier's tables. He then travelled to Scotland in 1615 to meet with Napier and they decided to base logarithms on the powers of ten, with log 1 = 0. While they started the work together, Briggs finished the calculations as Napier was too ill to continue. Briggs met with Napier once more the following year to review his work on the tables, and planned to meet again the next year, but John Napier had died by that time (Hobson, 1914).

Briggs' way of expressing logarithms was different from Napier's and did influence his choice in values to start his logarithmic tables. He built off a foundational idea between geometric sequences and an arithmetic sequences: that if any pair of numbers in the geometric sequence have the same ratio, then their corresponding numbers in the arithmetic sequence will have an equal difference. For example, in Figure 4.11 (left), the first column is the geometric sequence, so the pairs 128 & 32 and 16 & 4 have the same ratio, 4:1. The next four columns are various corresponding arithmetic sequences and their corresponding numbers will have an equal difference: 8 – 6 = 5 – 3; 12 – 10 = 9 – 7; 26 – 20 = 17 – 11; and 14 – 20 = 23 – 29.

	A	B	C	D
1	1	5	5	35
2	2	6	8	32
4	3	7	11	29
8	4	8	14	26
16	5	9	17	23
32	6	10	20	20
64	7	11	23	17
128	8	12	26	14
numeri Proportionales.	Logar	Logar	Logar	Logar

A	B	C	A	B	C	A	B	C
1	0	0000	1	0	0	1	0	0
10	1	1000	3	1	047712	2	1	03010299
100	2	2000	9	2	095424	4	2	06020599
1000	3	3000	27	3	143136	8	3	09030899
10000	4	4000	81	4	190848	16	4	12041199
100000	5	5000	243	5	238561	32	5	15051499
1000000	6	6000	729	6	286272	64	6	18061799

Figure 4.11: (Left) Logarithms without log 1 = 0, and (Right) logarithms with that requirement (Briggs, 1624, pp. 1–2)

As he and Napier had discussed, Briggs started with the idea that the log of unity should be zero. A main reason for this was that it let you choose any common difference in your arithmetic sequence and your formulations would remain the same (Figure 4.11). To explain, I return to the ideas discussed earlier by Archimedes that if two numbers are multiplied that are in a geometric sequence, then their result was the same distance from the higher of the numbers as the distance between the lower number and the start of the sequence. With Briggs' idea to set the first term at 0, then the distance between the first term (0) and the lower number would just be the lower number. So the product would be at the place occupied by the higher number added to the lower number. If the starting term is not 0, then that distance would play a factor. Looking at Figure 4.11 (left), multiplying 2 & 8 using the numbers in column B (representing the spaces that the geometric sequence occupies) would mean that the difference between 6 (the logarithm of 2) and 5 (the starting value) to be 1, and would add that to 8 (the logarithm of 8) to get 9, which then would correspond back to 16 from the first column. But if the starting point was 0, then going to Figure 4.11 (right), any time logarithms are used (columns B & C for the different geometric sequences in Column A), the starting value does not have to be taken into account as the difference is just the lower number, so the logarithms could just be added.

As Briggs had set values for the logarithms of the power of 10, he used a much different process than Napier or Bürgi in creating logarithmic tables. But his initial work with logarithms was done in concert with Napier, calling back to his idea of logarithms lying along a line. Napier and Bürgi had individually focused on having a very small ratio in their geometric progressions, but Briggs and Napier started anew from sets of known logarithms: the logarithm of 1 being 0 and the logarithm of 10 being 1,000,000,000,000,000. They chose such a large number so their logarithms would not be decimal fractions, and so they would have up to 15 significant digits.

Figure 4.12 is a visualisation of their ideas. There are three lines, the bottom one is the set of values, 1, 3, 7, and 10. The middle line has each value increased exponentially by a factor of 100,000, as I will not go as far as they did in the need for 15 significant digits, but will settle for 5. The top line has the count of the number of 10s that need to be multiplied by themselves to get that number (essentially, how many times 10 divides the number on the middle line which will make it the distance from 0). So the bottom line will be the base numbers, the middle line will be the alteration of the base numbers to the correct amount of significant digits, and will count as our geometric progression, and the top line will be the arithmetic progression, or the logarithms.

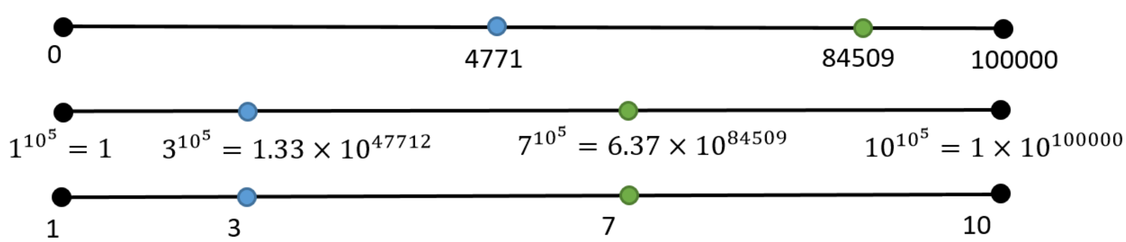


Figure 4.12: Calculating Logarithms – bottom: normal scale, middle: all increased by a power of 100,000, top: logarithmic scale

While Briggs and Napier explained their reasoning using the ideas of lines, they did not draw them out, they instead set up tables showing how they were calculating these many digit numbers. The tables were in tetrads of consisting of the tens-multiples of 2, 4, 8, and 10, which is shown in Table 1 with the number 2. Briggs and Napier would multiply the two by itself to get the index of 2, then that by itself to get the index of 4, then that by itself to get the index of 8. That number is now multiplied by the value that corresponds to the index of 2 in order to get an index of 10. They then restarted the process continuing till the index was a multiple of 10 to the amount of significant digits that they chose.

To exemplify the method, I will choose four significant digits as that will be enough to show the method, and I will also do what they did and not write out all the digits, but copy the first few and then keep track of the number of digits that were produced, as one less than the amount of digits would be the amount of repeated 10s needed to make that number. This number would have been placed on the top line in Figure 4.12, and is the logarithm.

Table 4.1: Raising 2 to subsequently higher powers

Original Value: 2	Index	Digits		Index	Digits
4	2	1	1,606,938…	200	61
16	4	2	2,582,249,878…	400	121
256	8	3	6,668,014,433…	800	241
$256 \times 4 = 1024$	10	4	1,071,508,607…	1000	302
1,048,576	20	7	1,148,130,695…	2000	602
1,099,511,627…	40	13	1,318,204,093…	4000	1204
12,089,258,196…	80	25	1,737,662,032…	8000	2409
12,676,506…	100	31	1,995,063,117…	10,000	3011

From Table 4.1, it is apparent that at an index of 10, we have one significant digit, 3, while at 100, 30 is two significant digits, and at 1000, 301, finally at 10,000, 3010 would be the correct value of the logarithm of 2 for four significant digits. If we go out to five significant digits as we did in Figure 4.12 then we will find the logarithm of 2 is 30102. Now combining this with Figure 4.12, the rest of the whole number logarithms from 1 to 10 can be found, 4 would just be double the logarithm of 2; 5 would be the logarithm of 10 minus the logarithm of 2; 6 would be the logarithm of 3 plus the logarithm of 2; 8 would be the logarithm of 4 plus the logarithm of 2; and 9 would be double the logarithm of 3. Napier and Briggs had realised this early, and so only went through these extreme measures for the prime numbers.

While this way of calculating logarithms fit well with Napier's original creation, it did involve a lot of difficult calculations, so Briggs soon worked out a different way to calculate the logarithms of prime numbers. His idea seems to still take in Napier's vision of two lines, but viewing it a bit like we today view continuous functions. He saw that if you could find two values, that were infinitesimally close together, and they were in proportion to their outputs (in this case logarithms), then a value in between them would also be in proportion. The function he was working with was not the logarithmic function though, he began by taking repeated square roots of 10, and then only looking at the decimal portion; $10^{1/(2^m)} - 1$.

To understand his thinking today, I will rewrite it using limits. As Briggs started at 10, and approached 1 from the right side, as he continued to take square roots, if he just concentrates on the decimal section, it will have a limit approaching $\frac{1}{2}$.

$$\lim_{x \to 1^+} \frac{\sqrt{x}-1}{x-1} = \frac{1}{2};$$

$$\text{factor the denominator: } \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} \text{ to get } \lim_{x \to 1^+} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

This result means that once his square roots are small enough, they will be proportional to the logarithms of those square roots, since the logarithm of square root over its original value will be $\frac{1}{2}$. Briggs did not need to know limits to discover this, it had been something that came from his and Napier's work with numbers.

Briggs did this to 15 significant digits, which took 54 square roots of 10. As before, I will illustrate this to 5 significant digits, we can stop once there is a pair of 5 leading zeroes (Table 4.2, first two columns).

Table 4.2: Calculating the Logarithm of Root 2 to five significant digits

Root 10	Log of Root 10	Root 2	Row Number
10	1	2	0
3.1622776601…	0.5	1.4142135623…	1
1.7782794100…	0.25	1.1892071150…	2
1.3335214321…	0.125	1.0905077326…	3
1.1547819846…	0.0625	1.0442737824…	4
1.0746078283…	0.03125	1.0218971486…	5
1.0366329284…	0.015625	1.0108892860…	6
1.0181517217…	0.0078125	1.0054299011…	7
1.0090350448…	0.00390625	1.0027112750…	8
1.0045073642…	0.001953125	1.0013547198…	9
1.0022511482…	0.0009765625	1.0006771306…	10
1.0011249413…	0.00048828125	1.0003385080…	11
1.0005623126…	0.000244140625	1.0001692397…	12
1.0002811167…	0.0001220703125	1.0000846162…	13
1.0001405485…	0.00006103515625	1.0000423072…	14
1.0000702717…	0.000030517578125	1.0000211533…	15
1.0000351352…	0.0000152587890625	1.0000105766…	16
1.0000175674…	0.00000762939453125	1.00000528830729176…	17
1.00000878370363461…	0.000003814697265625		18
1.00000439184217316…	0.0000019073486328125		19

From Table 4.2, it is apparent that the significant portion of the square roots are relatively in proportion to the significant portion of their logarithms: $\frac{439184217316}{878370363461} \approx \frac{19073486328125}{3814697265625}$. Briggs now would take the last root (the 19th) and raised it to different powers, to show that using proportions would give reasonably accurate logarithms up to his desired number of significant digits (Table 4.3). If he was going for five significant digits, then he would use $\frac{439184217316}{1907348633}$ as the one side of the proportion in doing all of his calculations.

Table 4.3: Verifying Logarithms using Brigg's method.

Index	19th square root of 10 raised to subsequent powers	Actual Logarithm using today's technology (the logarithm of the 19th square root times the index)	Estimated logarithm through proportions to 5 significant digits
1	1.00000439184217316…	0.0000019073486328125	
2	1.00000878370363461…	0.000003814697265625	0.00000381470
3	1.0000131755843844…	0.0000057220458984375	0.00000572207
4	1.0000175674844226…	0.00000762939453125	0.00000762944
5	1.0000219594037494…	0.0000095367431640625	0.00000953682

Briggs used this idea in finding the logarithms of primes. If he could take enough roots of 2, so that the result lay between or beyond his lowest roots of 10, then it could also be put into a proportion to solve for the logarithm. Turning back to Table 4.2: $\frac{878370363461}{3814697265625} = \frac{528830729176}{x}$. Solving this equation and returning it behind the 5 zeroes would give the logarithm of the seventeenth square root of 2 as 0.0000022966725887. To find the actual logarithm of 2, they would just need to double this number 17 times, so the logarithm of 2, to 5 significant digits was 0.30102.

Briggs continued this with the other primes, though he also found many shortcuts along the way. An easier one to understand is that he would first rewrite the prime number to different powers so he could use the known logarithms to simplify the work. For example, to find the logarithm of 2 he rewrote it as 1024, or 2^{10}, which then could be expressed as $\log 1024 = \log 1000 \times 1.024 = 3 + \log 1.024$. Now, he was starting off with a number closer to 1 so he did not have to take as many square roots. In this case, he hit his selected amount of significant digits with his 46th and 47th square root. (Briggs, 1624/2006; Roegel, 2010a).

Briggs published the logarithms of the numbers 1 – 1,000 in 1617 in a pamphlet to his friends and colleagues and up to 30,000 additional ones in 1624, in his text *Arithmetica Logarithmica* ('Logarithmic Arithmetic'), along with the construction methods. Briggs planned to continue work on the tables, and hired quite a few mathematicians to help in the work, however, before he could complete it, a Dutch mathematician named Adrian Vlacq (1600 – 1667) published the complete tables from 0 to 100,000 in 1628. While he did call it a second edition, and it used 30,000 of Brigg's logarithms, Vlacq's *Arithmetica Logarithmica* added an additional 70,000 which had been calculated by him and Ezechiel De Decker (ca. 1603 – ca. 1647) using construction methods learned from Briggs and De Decker (Miller, 1979).

These tables, started by Briggs, and finished by Vlacq and De Decker, used methods that were further refined by other mathematicians up until the use of series in creating logarithms was popularized by Halley and Wallis in 1695. Section 4.6 will expand upon that idea. But as logarithmic tables proved to be so popular and useful, it feels almost natural that the next big step would be creating tools to ease the use of these tables.

4.4. Tools for Logarithms

Only six years on from *Descriptio* (1614), and before Briggs' *Arithmetica Logarithmica*, the English mathematician Edmund Gunter (1581 – 1626) published his table of logarithms. Gunter was a colleague of Briggs and was familiar with the latter's first pamphlet of logarithms published in 1617 (van Poelje, 2004). Gunter took these tables and reduced the significant digits from fifteen digits to eight and, while he included the work that Briggs had already done on logarithms with a base 10, he went back to the ideas behind Napier's tables and included tables that were the logarithms of the sine and the tangent of angles (Gunter, 1620). He followed this up three years later with a description of how to make a 'line of numbers', which included Briggs' logarithms along with the logarithms of sines and tangents of angles (Gunter, 1623). His lines are shown below (Figure 4.13), and include a single line that recorded the numbers represented by the distances of their logarithms. A divider, similar to a compass, could be placed upon these distances and then moved to calculate the multiplication/division or doubled/halved to find the exponent/root of numbers. The Gunter Rule, which was used in navigation up to throughout the 19th century, was 2ft long and 2in wide, and included his logarithmic scales, plus others scales useful to seamen (van Poelje, 2004). It is a good representation of a static logarithmic tool.

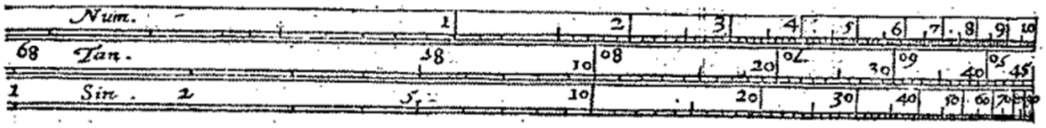


Figure 4.13: Drawing of original Gunter scale [I have altered this image by rotating it 90 degrees]. (Gunter, 1623, p. 168)

Both the invention of the slide-rule and the circular slide-rule are credited to William Oughtred (1574 – 1660), a mathematician from London. His first description of his slide rules (Figure 4.14) was in *The Circles of Proportion and the Horizontal Instrument,* published in Latin in 1632, but the introduction of the English translation published a year later talked about how, for years, he had been sliding two Gunter scales along each other. Though the body of the text mainly discussed the circular slide-rule, it is worth noting that Bürgi's 1620 cover also included a concentric circle illustrating his logarithms (Figure 4.14), suggesting that he had imagined this type of tool could exist someday, though there is no indication that Oughtred had seen Bürgi's text.

After the inventions of these tools, other slide-rules abounded, but all were based on these initial concepts.

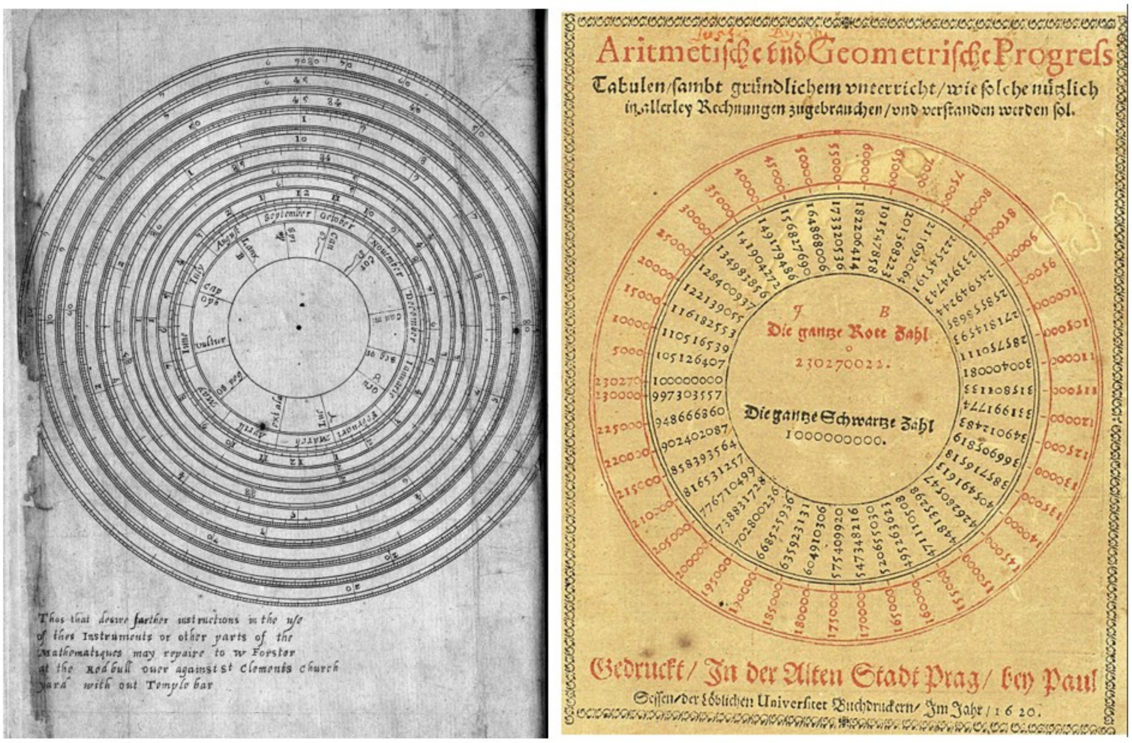


Figure 4.14: Left: Image of Oughtred's Circular Slide Rule. (Oughtred, 1632/1633, cover); Right: Cover of Bürgi's book (Bürgi, 1620/2015)

4.5. Logarithms in Geometric Space

The next big advancement in logarithms came in the visualisation of this new concept, or more specifically how this new concept related to geometric figures that were often studied in the 17th century. Conic sections were one such area of study and Gregory of Saint-Vincent (1584 – 1667), a Jesuit priest and mathematics teacher from Belgium, proved that the area under a hyperbola could be understood as a sequence of strips, all with the same area, in geometric progression. While, at the time, this could have been viewed as the logarithm, he never called it that. He published this in his text *Opus Geometricum Quadraturae Circuli et Sectionum Coni*, published in 1647, whose main component was around squaring a circle (Burn, 2001).

In order to understand what Gregory of Saint-Vincent did, we need to remember that one of the main ideas of logarithms was that numbers with an equal difference were matching numbers that were in same ratio (Figure 4.11left). He worked with line segments that were in ratio, and then proved that the corresponding areas under the hyperbola were equal using the same method of exhaustion that had been used since before Euclid, though he was the first to call it that (Burn, 2001). If the line segments lay in a geometric sequence, and the area of each corresponding rectangle lay in an arithmetic sequence, then that pointed to the relationship between the area under a hyperbola and logarithms. He had two proofs for the area under the hyperbola, one looking at triangular segments, and one looking at rectangles. As this dissertation is focused on ways the presentation of logarithms in the past can help currently, I chose to focus on the rectangles, as it is not all that dissimilar to Riemann Sums.

Gregory of Saint-Vincent first drew two asymptotes, AB and AC, and a Hyperbola, DEF (Figure 4.15 original on the left, and reproduced on the right adding points so it accords to our modern notation of geometric figures). He then drew lines parallel to the AB asymptote DH, LI, EG, MK, and FC so they had a common ratio, in this case a decreasing one. Earlier in the text, he had discussed how the nature of hyperbolas transforms a ratio to the reverse from one axis to the other, so in this case AH, AI, AG, AK, and AC would all be increasing by the opposite ratio (Dhombres, 2015).

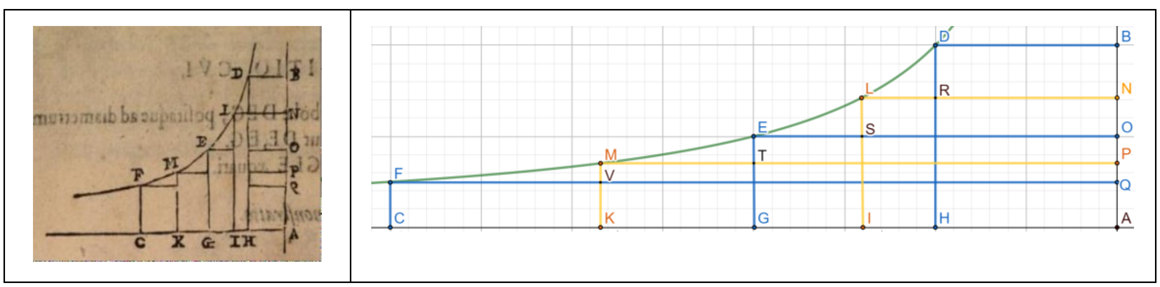


Figure 4.15: Left – Hyperbola DEF, with the axes having lines in ratio (Saint-Vincent, 1647, p. 586) and Right – a clearer representation

Gregory of Saint-Vincent next looked at the rectangles embedded in the hyperbola and proved that each rectangle had the same area. As his proof used ideas of ratios that are not commonly seen today, I will go through the steps of his proof, while adding in line lengths and flipping the graph so it is a bit more familiar to the modern reader (Figure 4.16). The proof starts with the premise that the lines DH:LI, LI:EG, EG:MK, MK:FC are in proportion, which in Figure 4.16 we can see they have a common ratio of r as DH = $1/a$, LI = $1/ar$, and so forth. Then AH:AI = AI:AG = AG:AK = AK:AC due to the nature of logarithms as stated above. The common ratio would be $1/r$, as given our lengths AH = a and AI = ar.

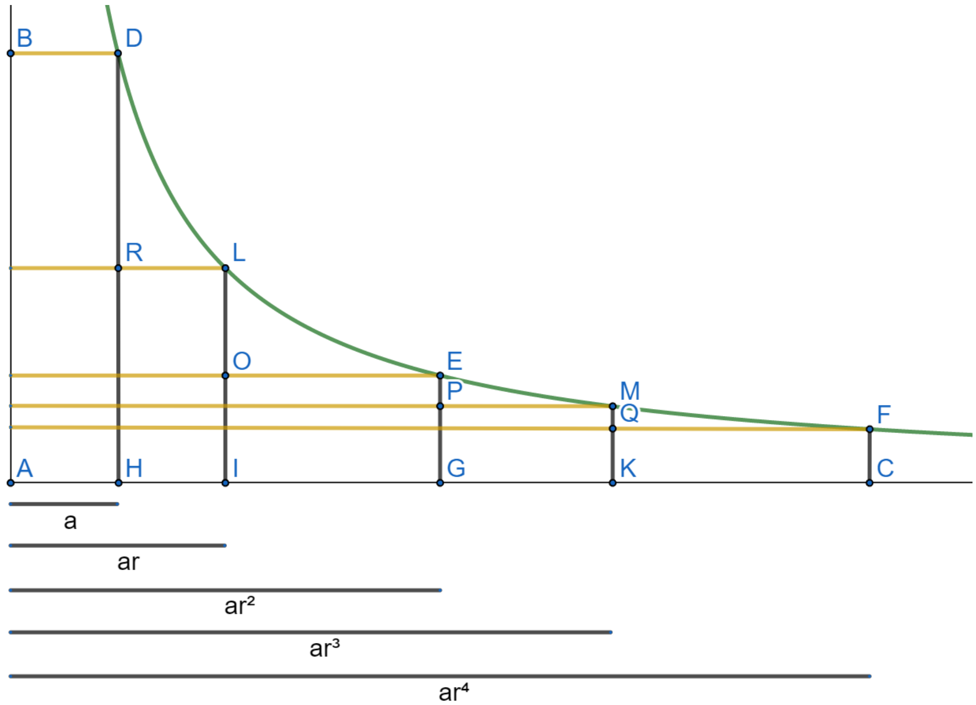


Figure 4.16: Areas under a hyperbola being equal if y-values are in proportion

To prove the areas of the rectangles are equal, we start with just proving two of them are equal, LRHI and EOIG. This equality can be written as $LI \times HI = EG \times IG$ which is equivalent to proving that $\frac{HI}{IG} = \frac{EG}{LI}$. Gregory of Saint-Vincent did this using common ratio principles of his time (Dhombres, 1993), but for a more modern notation, one can incorporate their lengths. As the ratio EG:LI is 1/*r*, it is HI:IG that needs to be found.

$$HI = AI - AH = ar - a = a(r - 1)$$
$$IG = AG - AI = ar^2 - ar = ar(r - 1)$$
$$\frac{HI}{IG} = \frac{a(r - 1)}{ar(r - 1)} = \frac{1}{r}$$

Therefore, the proportion can be EG:LI = HI:IG, so the two rectangles are equal. We can do this again for the other visible rectangles: MTGK and FVKC; or we could insert a sequence of mean proportional numbers to get continuously denser geometric progressions so the rectangles created come closer to filling that hyperbolic area.

The idea of two things having a mean proportional was common at the time, especially in geometry (Newton 1658; Rathborne, 1616). But the concept of a mean proportional became separated from the specific geometric drawings as tools were invented to simplify the geometry (Gunter, 1623), and eventually, in part due to logarithms driving interest in geometric and arithmetic progressions, became something itself to study (Wingate, 1645). Today, we would call it a geometric mean, and it is very important to geometric progressions where it can be used to fill in the space between the known numbers of the progression. Wingate (1645) gave the examples of 16 being the mean proportional between 8 and 32 since 8:16 as 16:32, while Gunter (1623) explained that the mean proportional can be found by multiplying the two numbers and taking the square root. Wingate (1645) went a bit further and found two means proportional between 8 and 27, 12 and 18 given that 8:12 as 12:18 and as 18:27. Today, we could solve this in two ways, we could try to find the corresponding geometric sequence that 8, 12, 18, 27 belongs to, $8r^0, 8r^1, 8r^2, 8r^3$, where $r = \frac{3}{2}$ or we could write his work algebraically as a system of equations and solve:

$$\frac{27}{x} = \frac{x}{y} = \frac{y}{8} \begin{cases} \frac{27}{x} = \frac{x}{y} \\ \frac{27}{x} = \frac{y}{8} \\ \frac{x}{y} = \frac{y}{8} \end{cases}$$

$$x = 18, y = 12$$

Going back to Figure 4.16, even if there was a denser set of rectangles, as long as the sides remain in geometric progression, using the idea of means proportional, then the lengths of the sides of the rectangles would have a common ratio (1/r and r), while the areas have a common difference (in this case 1). Therefore, the areas would correspond to logarithms.

Even though Gregory of Saint-Vincent developed this proof, he did not directly associate this area with logarithms. That was done a few years later by his colleague Alphonse Antonio de Sarasa (1617 – 1667), who was also a Jesuit priest and mathematician from Belgium (Burn, 2001).

Years later, John Keill (1671 – 1721) described a different logarithmic curve that was appended to his translation of *Euclid's Elements of Geometry* (1723). He drew a horizontal line created of many smaller lines of equal length: AC, CE, EG, GI, IL to the right side, and Ar, rπ, to the left side (Figure 4.16). All of the endpoints had perpendiculars, AB being 1 and the rest in continual proportion to AB, so if CD was a then EF was a^2, while Ar would be $\frac{1}{a}$ and rπ = $\frac{1}{a^2}$.

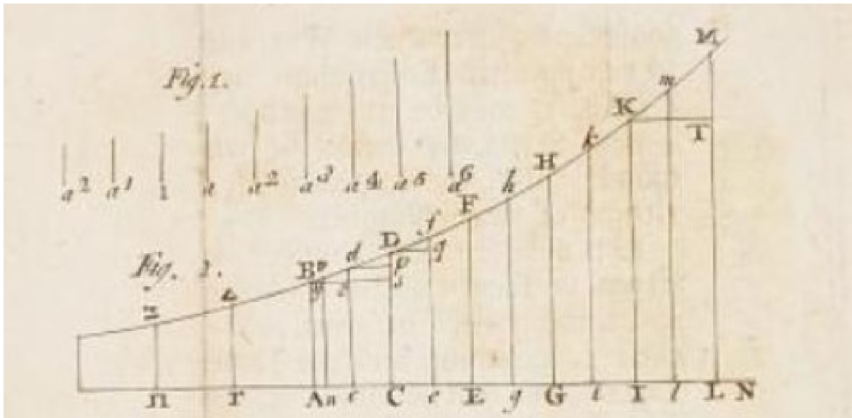


Figure 4.17: Fig 1- geometric sequence as lines, Fig 2. Keill's logarithmic curve (Keill, 1723, p. 143)

If the horizontal lines are bisected and new vertical lines were raised (*cd, ef, gh, ik, lm*), keeping in geometric proportion, then one could create density. Keill used this curve to explain the doctrine of logarithms, the horizontal being the logarithm of the vertical starting from A, AC was the logarithm of DC; AG the logarithm of GH; and AR the logarithm of rΔ. While a few authors kept the name of this curve as logarithmic (Dodson, 1742; Leslie, 1821), today it would be called exponential. Another writer (Coolidge, 1950) credited this curve to Christiann Huygens (1621 – 1695) in 1661, but I was not able to find the text, so it may have been known before Keill. Sufficient to say that this curve was being used to express the idea of logarithms at this time.

4.6. Logarithmic Series

The idea of logarithms being represented as the area under a hyperbola led many to try and more easily calculate logarithms. Nicholas Mercator (1620 - 1687), was a German mathematician who published *Logarithmo-technia* in 1668 (Panagiotou, 2011). In this text, he was creative with his hyperbola and planned so the calculations could start at 0. In today's parlance, he looked at the hyperbola $\frac{1}{x+1}$ between 0 and 1, and represented it with the series: $\frac{1}{x+1} = 1 - x + x^2 - x^3 + \ldots$. He then explained how this could be used this to find the area under the hyperbola, but in words not mathematical notations (Coolidge, 1950).

It was John Wallis (1616 – 1703), an English mathematician, who put it into symbols in his account of the text in a Royal London Philosophical Transaction also in 1668. The information on this section comes from Wallis and from Coolidge (1950). Working from Mercator's drawings and results, he started with the series for $\frac{1}{x+1}$ between 0 and A, but then subdivided that into *n* subdivisions using the labels *a*, 2*a*, 3*a,* and so on, to get a more accurate result. In Figure 4.18 I chose a larger *a* so it was easier to see, but in his case *a* would have been representing a number significantly close to 1, so that the shapes under the curve are primarily filled by one large rectangle.

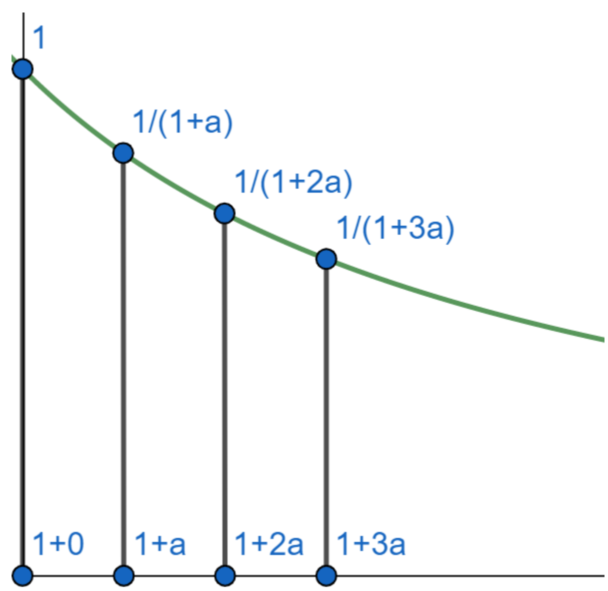


Figure 4.18: The breakdown of the hyperbola into smaller rectangles

Wallis then found the area of each rectangle and added them to find the area of the hyperbola from 1 to n.

$$a\left(\frac{1}{1+a}+\frac{1}{1+2a}+\frac{1}{1+3a}+\frac{1}{1+4a}+\cdots\right)$$

Using the series found by Mercator, this could be expanded

$$a[(1-a+a^2-a^3\ldots)+(1-2a+4a^2-8a^3\ldots)+(1-3a+9a^2-27a^3\ldots)+\cdots]$$

And then rearranged.

$$a[(1+1+1+\cdots)-a(1+2+3+\cdots)+a^2(1^2+2^2+3^2+\cdots)-a^3\ldots]$$

Modern mathematics has their summations for these series, as did the mathematics in Wallis' time, but I will use the modern notation as it is more familiar, though I do rearrange the terms on the right side of the equation in a way that is true to Wallis' thinking.

$$1 + 1 + 1 + \cdots + 1 = n$$

$$1 + 2 + 3 + \cdots + n - 1 = \frac{(n-1)n}{2} = n^2\left(\frac{1}{2} - \frac{1}{2n}\right)$$
$$1^2 + 2^2 + 3^2 + \cdots + n - 1 = \frac{(n-1)(n)(2n-1)}{6} = n^3\left(\frac{2}{6} - \frac{1}{6n} - \frac{2}{6n^2} + \frac{1}{6n^3}\right)$$
$$1^3 + 2^3 + 3^3 + \cdots + n - 1 = \frac{(n-1)^2 n^2}{4} = n^4\left(\frac{1}{4} - \frac{2}{4n} + \frac{1}{4n^2}\right)$$

Wallis argued that as density increased (the number of n rectangles) many of the fractions would vanish so subbing back in would result in

$$an - \frac{a^2n^2}{2} + \frac{a^3n^3}{3} - \frac{a^4n^4}{4} \ldots$$

Wallis realised that the product an is equivalent to the entire line which ran from 0 to A, therefore $an = A$. And as he was calculating the area under the hyperbola, he called it a logarithm:

$$\log(1 + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots$$

Wallis noted that this result was only applicable when *A* was less than one (Hofmann, 1939). Isaac Newton (1643 – 1727), working around the same time, came to the same conclusion as Mercator through integration (Panagiotou, 2011).

Edmond Halley (1656 – 1742) in a Philosophical Transaction printed in 1696 managed to remove this series from its geometric construct, which resulted in a series that could incorporate all numbers, not just those close to 1. He first redefined the idea of logarithms to be the number of equal ratios between two numbers; if there is a defined minimum ratio between two numbers, which he called ratiuncula, then the logarithm of the ratio of those two numbers is the number of ratiunculae are between the two numbers. Halley did two examples, one that speaks to modern day natural logarithms where the ratiuncula is the smallest increase from one. For example, if he chose

1.00000001, then he would find the natural logs up to 7 correct digits: $1.0000001^x = 2 \Rightarrow x = 6{,}931{,}472$, so there would be 6,931,472 ratiuncula between 1 and 2, in this system.

If starting from the standard base-10 logarithms, then the ratiunculae would be different as Halley had to ascertain that the logarithm of 10 was 1. To be correct to seven digits he would change the 1 to 10,000,000 and then would want the logarithm, the exponent to be that 10,000,000. A modern equivalent is $x^{10{,}000{,}000} = 10 \Rightarrow x = 1.0000002302585$. Now to find the base-10 logarithm of 2, Halley could write a new equation $1.0000002302585^y = 2 \Rightarrow y = 3010300$, therefore there were that many ratiuncula between 1 and 2 in this system. The system could change as necessary following this model.

For Halley, solving $1.0000002302585^y = 2 \Rightarrow y = 3010300$ was not an easy feat, so he took advantage of the fact that he was dealing with numbers very close to 1 and proposed that to find the logarithm of 2, one just had to take the 2302585th root of 2. Basically he was saying that the following two powers both equal 2:

$$1.0000002302585^{3010300} = 2$$
$$1.0000003010300^{2302585} = 2$$

He does not explain why this is true, but we can surmise his thinking by using some simple algebra and knowledge of Pascal's triangle. If we generalise the equations above, and separate the unit from the decimal fraction, then we can get the equation below, making sure that x and y are numbers a few orders less than the denominator:

$$\left(1 + \frac{x}{1{,}000{,}000{,}000{,}000}\right)^y = \left(1 + \frac{y}{1{,}000{,}000{,}000{,}000}\right)^x$$

If we follow Pascal's triangle in expanding this binomial, the first two terms on each side will be $1 + \frac{xy}{1{,}000{,}000{,}000{,}000}$ while the remainder of the terms will have the denominator of one trillion squared, and then cubed, and so forth; therefore, those terms are far removed from our significant digits, while the first two terms are still relevant. This means that for the purposes of finding the logarithm of a value, say 2, solving $1.0000002302585^x = 2$ is equivalent to solving $\left(1 + \frac{x}{1{,}000{,}000{,}000{,}000}\right)^{2302585} = 2$. But solving the second equation means taking a very large root of 2, which thanks to a new method, proposed by Isaac Newton, could now just be written as series.

After a few more simplifications, Halley found the same series as Mercator and Wallis, but was also others which eased the time and energy needed to calculate logarithms. More work was done surrounding series expansion and natural logarithms, but the baseline had been set.

4.7. Natural Logarithms

The logarithms discussed in Sections 4.6 and 4.7 were at this time, the 17th century, often called hyperbolic logarithms, but today we would call them natural logarithms. At this time in history, there was not much consideration of a 'base' of a logarithm; as long as there was a sequence with a common ratio set to a sequence with a common difference, then the second sequence would be considered logarithms. Early on Napier and Briggs decided to set the sequences so that the multiples of 10 lined up with integers (1 corresponding to 0), but it was not thought of as a logarithm with a base of 10. Before that, Napier had created logarithms by looking at an appropriately small rate close to the number 1, so the geometric sequence could represent approximations of all integers. His geometric sequence was decreasing, but two authors, Oughtred and John Speidell (1600 – 1634), in the early-1600s, computed tables of logarithms using his method but with an increasing rate, and they are almost identical to current tables of natural logarithms (Mitchell & Strain, 1936). The work done by Halley above, in Section 4.6, also explores this idea of natural logarithms, where a number close to one was chosen as the starting point for his idea of logarithms as a 'number of ratios'

This same idea came forth when logarithms began to be associated with hyperbolas, having a denser geometric sequence so the rectangles created in Figure 4.15 would occupy the vast majority of the area under the hyperbolic curve. Having dense geometric sequences naturally fit in with the idea of natural logarithms which have a base of e. One of the ways to understand e is through the idea that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (Coolidge, 1950). This representation models Napier's idea of a miniscule rate between a number and 1 as well as the geometers attempting to create a dense sequence of rectangle under a hyperbola. Therefore without knowing about e, or understanding the base of a logarithm, natural logarithms appeared naturally.

4.8. Exponents

It may appear surprising to us today that logarithms were not associated with exponents until after a century had passed from Napier's texts. The history of exponents is long, but I will do a condensed version that will help establish the connection of exponents and logarithms.

Throughout the history of exponents, there has been disagreement on whether their writing followed an additive or multiplicative property. In ancient India, it was mainly multiplicative, meaning that if there was a word for 'square' and a word for 'cube' then 'square-cube' would mean 'the value of something raised to the 6^{th} power' and there would be separate designations for 5^{th} or 7^{th}. The way of writing powers (the calculated result of terms raised to exponents, while exponents or index are term that something is raised to) varied in Greece and Rome, and in early Europe it started with additive but then switched to multiplicative representation. Diophantus's text *Arithmetics* followed an arithmetic argument, and the Islamic scholars that translated it, and later studied from it, agreed with that standard (Rosenfeld & Černova, 1967). Diophantus did not just set a standard that would be passed on, he also created a new way of writing it, called syncopated algebra. In Ancient Greek, the word δύναμις meant 'power' but, in mathematics, was commonly used for the square and κύβος meant cubed. Since Diophantus was working with higher powers, he immediately set about creating symbols for them. In Line 15 of the first book in *Arithmetica*, he labeled an unknown squared as Δ^Y and an unknown cube as K^Y, thereby using the first letters of their names, discussed above. As he was working with equations to the 6^{th} degree, he went further and used the terminology of a square-square $\Delta^Y\Delta$ to mean the 4^{th}, a square-cube ΔK^Y for the 5^{th} and a cube-cube K^YK for the 6^{th}. (Diophantus, ca 250 C.E./1893; Heath, 1910). This representation is the additive property of exponents and something was passed to the scholars of the Islamic Golden Age and then to Europe through them.

With the use of syncopated algebra, and the Hindu-Arabic number system, mathematics flourished in the Middle East. Al-Kwharizmi (ca. 780 – ca. 850) proposed algebra, the mathematics of solving equations with unknown values. Though many of the equations he was solving were tackled in the past, he did it with the systematic rigour necessary to create a new area of mathematics. In his text, he introduced six types of problems, proved their solution geometrically, but then dispensed with the geometry to focus on the algorithm (Katz, 2009). It was written in rhetorical form, as it was before the translation of Diophantus, but his text was used as a teaching tool throughout Europe (Stallings, 2000). Later, Al-Karaji (953 – 1029) working from Al-Khwarizmi and other Arabic scholars, as well as the newly translated *Arithmetics*, furthered exponential operations outside of geometry, including operations with higher powers (Rashed, 2013). Al-Samaw'al (ca. 1130 – ca. 1180) took Al-Karaji's work and writing it in symbols was able to extend it both to the left and right of one, in modern terminology, with negative and positive exponents. While it is still in a syncopated style, he did use tables and the already shortened symbols that Al-Karaji used for the different powers to symbolise the operations taking place (Figure 4.19) (Berggren, 2016).

I pccc	H pmcc	G pmmc	F pcc	:E pmc	D pmm	C pc	B pm	A pt	0 unit	A t	B m	C c	D mm	E mc	F cc	G mmc	H mcc	I ccc
$\frac{1}{8}\frac{1}{8}\frac{1}{8}$	$\frac{1}{4}\frac{1}{8}\frac{1}{8}$	$\frac{1}{4}\frac{1}{4}\frac{1}{8}$	$\frac{1}{8}\frac{1}{8}$	$\frac{1}{4}\frac{1}{8}$	$\frac{1}{4}\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512
$\frac{1}{27}\frac{1}{27}\frac{1}{27}$	$\frac{1}{9}\frac{1}{27}\frac{1}{27}$	$\frac{1}{9}\frac{1}{9}\frac{1}{27}$	$\frac{1}{27}\frac{1}{27}$	$\frac{1}{9}\frac{1}{27}$	$\frac{1}{9}\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243	729	2,187	6,561	19,683

Figure 4.19: Chart of powers, 'm' stood for square, 'c' stood for cubed, and 'p' in front was 'part of' (Berggren, 2016, p. 136)

Al-Khwarizmi's text is considered foundational to algebra and Al-Karaji as very influential to western thinkers, while there is less knowledge of Al-Samaw'al, his ideas continued to influence the Arab world, and indirectly the west. The Islamic Golden Age inherited Greek mathematics, expanded upon them, and used them to create completely new branches, before passing both the old and new knowledge to Western Europe. By the 1300s, Europe had been introduced to the Hindu-Arabic numeral system, as well as to algebra, and had started rediscovering some of the Greek texts (Stallings, 2000).

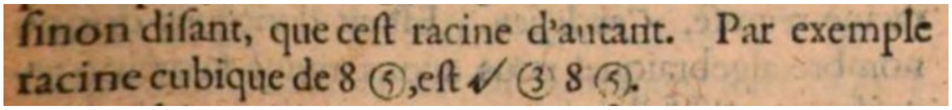
4.8.1. Exponential Notation

For most of this history, powers were referred to rhetorically as above, or as repeated characters even as the understanding of them branched out. Bradwardine in the early 1300s made specific reference to half the proportion when referring to square roots, and Oresme in building up his multiplicative world realized that part of something could be represented by a fraction, which in his description meant fractional exponent (Grant, 1960). As we saw in Section 4.1, it was Nicolas Chuquet in the late 1400s who is credited for putting a new notation to exponents. In his 1484 text, *Triparty en la science des nombres,* he writes out an exponent that could be used for a single value (Figure 4.20) and included negative numbers a possibility by writing an m after the exponent, explaining that numbers less than one could be expressed in the manner of $12.^{1.\tilde{m}.}$, while negative 12 would still be expressed as $.\tilde{m}.12.^{1}$ (Chuquet, 1484/1881). This notation did not immediately catch on though; René Descartes (1596 – 1650) is credited with popularizing the superscript notation for the exponent in 1637, after the creation of logarithms (Boyer, 1943).

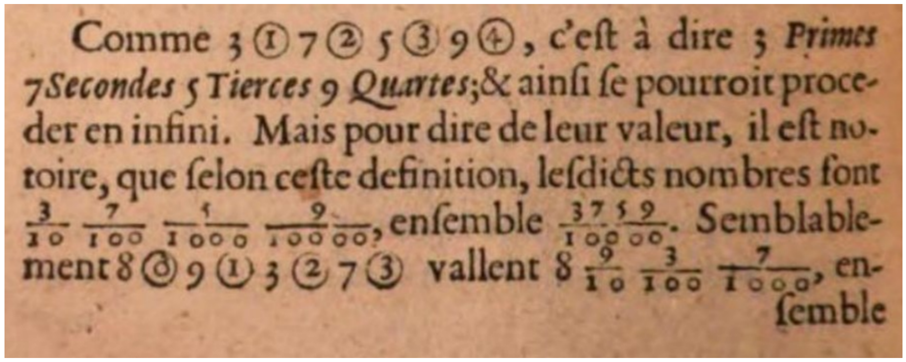
Àulcunesfoiz lung peult estre plus et laultre mōins vel e9.ª comme. 12. premiers moins que lon peult ainsi noter .12.$^{1.\tilde{m}.}$ ou moins 12. p̃miers que lon peult ainsi noter .m̃. 12.^{1} Et aulcunesfoiz lung et laultre est moins comme

Figure 4.20: Early modern notation for exponents (Chuquet, 1484/1881, p. 153)

A hundred years after Chuquet, Simon Stevin (1548 – 1620) started the process of writing roots and exponents with the same symbol. He used circles to note the type of radical and also the exponent, his symbology in Figure 4.21-top loosely translates in modern notation to "For example, the cube root of $8x^5$, is $\sqrt[3]{8x^5}$" (Stevin, 1585, p. 257). He used the exact same symbols for his attempt at writing decimals (Figure 4.21-bottom), but had no clear indication of when the circled number would indicate a positive or negative exponent, other than the context of the problem. This uncertainty was maybe a reason his notation never became popular, or that it was a bit unwieldy to use. (Boyer, 1943; Stevin, 1585).



ſinon diſant, que ceſt racine d'autant. Par exemple racine cubique de 8 ⑤, eſt √ ③ 8 ⑤.



Comme 3 ① 7 ② 5 ③ 9 ④, c'eſt à dire 3 *Primes* 7 *Secondes* 5 *Tierces* 9 *Quartes*; & ainſi ſe pourroit proceder en infini. Mais pour dire de leur valeur, il eſt notoire, que ſelon ceſte definition, leſdicts nombres ſont $\frac{3}{10}$ $\frac{7}{100}$ $\frac{5}{1000}$ $\frac{9}{10000}$, enſemble $\frac{3759}{10000}$. Semblablement 8 ⓪ 9 ① 3 ② 7 ③ vallent 8 $\frac{9}{10}$ $\frac{3}{100}$ $\frac{7}{1000}$, enſemble

Figure 4.21: Top- The same notation used for the root and exponent (Stevin, 1585, p. 257). Bottom- The circle notation is also used to rewrite fractions (Stevin, 1585, p. 140)

John Wallis in *Arithmetica Infinitorum* (1656) was thought to be the main force behind combining the root and the exponent into a fractional exponent, and then expanding into irrational exponents. Though he never actually used any notation for fractional exponents in his text, he did discuss how a negative 'index' (exponent) will lead to repeated reciprocals, and that an 'index' of -1/2 would be a reciprocal square root. This lead into the possibility of an irrational index. So while he did not work much with these ideas, he did start building toward them as objects that exist in mathematics (Wallis, 1656).

4.8.2. Exponential Language

Michael Stifel put the name to this symbol in mid-1500s, though in the form of geometric and arithmetic sequences. He called his arithmetic sequence the 'exposed' numbers and used that terminology when talking about the relationship between the two sequences. Figure 4.22 is an example using the word 'exponens' for 'exposed' while explaining that 64 divides 1/8 resulting in 1/512, and that this is the same as 6 subtracted from -3 which is -9. This -9 is the exponent of the fraction 1/512 (Figure 4.22).

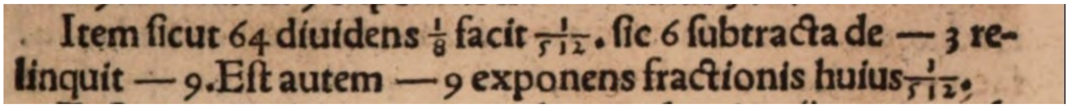
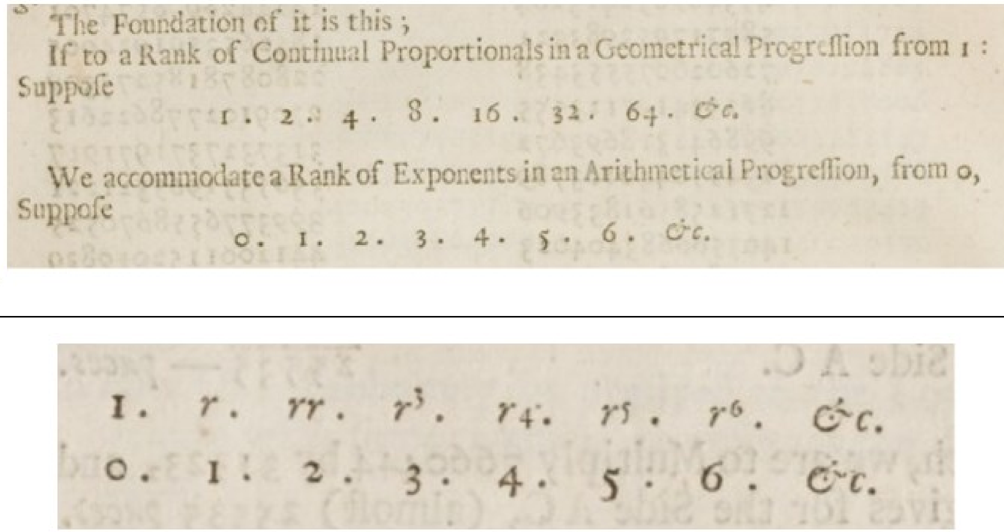

Item ſicut 64 diuidens $\frac{1}{8}$ facit $\frac{1}{512}$. ſic 6 ſubtracta de — 3 relinquit — 9. Eſt autem — 9 exponens fractionis huius $\frac{1}{512}$.

Figure 4.22: Use of the word 'exponens' in practice (Stifel, 1544, p. 250)

This terminology did not immediately catch on, and for most of the first 100 years of logarithms, the arithmetic sequence was referred to as the index. Wallis (1685) was a main contributor of 'exponent' coming into fashion in the mathematics world when he labeled his arithmetic sequence a 'Rank of Exponents' (Figure 4.23). Like the background with the arithmetic and geometric progression, this work had to be done before logarithms and exponents could officially be related. The operations with exponents had to be established, and as logarithms were seen as continuous, exponents had to encompass all numbers. Wallis, and others, could use this previous work make that last connection.

4.9. Connecting Progressions and Exponents

In Section 4.8, I detailed how Wallis, and others, had started the work of tying the idea of the 'index' in a sequence to the 'index' that is the exponent. In 1685, Wallis went further as he put together the older term 'exponent' for an arithmetic progression written in relation to a geometric progression. In Figure 4.23, Wallis had created an arithmetic and geometric progression, but his arithmetic progression used both the term 'exponent', while his next set uses the symbol for the exponent.



The Foundation of it is this ;
If to a Rank of Continual Proportionals in a Geometrical Progreſſion from 1 :
Suppoſe
1. 2. 4. 8. 16. 32. 64. &c.

We accommodate a Rank of Exponents in an Arithmetical Progreſſion, from 0,
Suppoſe
0. 1. 2. 3. 4. 5. 6. &c.

$1.\ r.\ rr.\ r^3.\ r4.\ r5.\ r^6.$ &c.
$0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.$ &c.

Figure 4.23: Top- Logarithms with the term and Bottom- notation ‘exponent’ (Wallis, 1685, p. 56)

After Wallis, mathematicians begin to use the term ‘exponent’ along with index to mean both the arithmetic sequence, and our current usage of the word. This included Edmund Halley, who, in working on a new way to construct logarithms, ended up also redefining them.

> They may much more properly be said to be *Numeri Rationum Exponentes*: Wherein we consider *ratio* as a *Quantitas sui generis*, beginning from the *ratio* of *equality* or 1 to 1 = 0; being affirmative when the *ratio* is increasing, as of Unity and a greater Number, but Negative when decreasing (Halley, 1696, pp. 58-59)

This redefinition brought the idea of logarithms to be the number or ratios it took to get to the desired value in the geometric progression, which connects logarithms with the multiplicative idea of exponents. In Briggs’ initial text around logarithms, they were defined as numbers with a common difference in relation to numbers with a common ratio as in Figure 4.10 where arithmetic sequences were shown, starting with a variety of numbers, but all would be considered logarithms of the given geometric sequence. This new definition is based on place, and while new, does call to mind one of the methods used by Briggs and Napier in calculating logarithms illustrated in Figure 4.11. The geometric sequence could no longer be tied to ‘any’ arithmetic sequence, but now was anchored at 1, where there are so far 0 ratios. From here it could go up (positive number of ratios) or down (negative number of ratios) any amount of ratios, and those ratios would be the logarithm of that value (Figure 4.24)

New value	1/4	½	1	2	4	8
Number of ratios to new value	-2	-1	0	1	2	3

Figure 4.24: Logarithms are the number of ratios from 1.

After Isaac Newton published his *Method of Fluxions* in 1636 Charles Hayes (1678 – 1760) wrote an explainer in English. In it, he made a note that cannot be found in Newton's work or the official translation from Latin to English by Colson. He set out how to solve an exponential equation directly using a logarithm, which previously exponential equations had mainly been accomplished through viewing them as a sequence and using proportionality. He instead made use of the properties of logarithms to justify his work (Figure 4.25). Looking at the power property: if a number was squared, then the logarithm of that squared number would be twice the logarithm of the number, so $2^2 = 4 \Rightarrow 2\log 2 = \log 4$, and then he generalised the result.

401. For the Logarithm of the Square, Cube or Biquadrate, &c. of any Number, is equal to twice, thrice or fourtimes, &c. the Logarithm of the Root. Therefore Univerfally, the Logarithm of x^v is $= v \times lx$; but $x^v = y$, therefore the Logarithm of x^v is equal to the Logarithm of y, that is $v \times lx = ly$.

Figure 4.25: Solving an exponential equation directly through logarithms (Hayes, 1704, p. 307)

The ground had now been set to tie these two concepts. In the next section, this connection is discussed from perhaps an unexpected source, Tables of Logarithms.

4.10. Logarithms and Exponents

In reading through the literature on the history of logarithms, it is noted the William Gardiner wrote the first text to base the definition of logarithms on exponents in his *Tables of Logarithms*, published in 1742, with Panagiotou (2011) citing Cajori (1913b) who cites Johannes Tropfke's *History of Mathematics* (1902). But Gardiner is not credited with this definition: these researchers looked to the note in his introduction where he credited William Jones, a well-known Welsh mathematician, with his knowledge of logarithms (Cajori, 1913b; Panagiotou, 2011).

Jones had a history of working on ships, so his first text in 1702 was a text on navigation. In it, he appended a chapter on logarithms from Samuel Heynes (1701) which used the language of exponents, and tied the operations in the sequence to the operations with exponents (Figure 4.26).

If to a Rank of continual Proportionals in a Geometric Progreſſion from Unity. Suppoſe,

1, 2, 4, 8, 16, 32, 64, &c.

We accommodate a Rank of Exponents in an Arithmetick Progreſſion, from o. Suppoſe,

o, 1, 2, 3, 4, 5, &c.

The Terms.	1.	r.	rr.	rrr.	r^4.	r^5.	r^6.	&c.
Exponents.	o.	e.	2e.	3e.	4e.	5e.	6e.	&c.

Then as $rr \times r^3 = r^5$ & $\sqrt{rr} \times r^3 = r^5 \sqrt{r}$

So $2e \times 3e = 5e$ & $2\frac{1}{2} e \times 3e = 5\frac{1}{2} e$.

Figure 4.26: Comingling logarithms and exponents (Jones, 1702, p. 1)

After settling into London, Jones became a tutor and began a library that was well-known to the mathematical elite, including correspondence with Wallis and Halley. He was also well respected and trusted by Isaac Newton, and was actually the one to publish the paper discussed in sub-section 4.8.1 that first used the rational exponent symbols (Cannon, 2006). In 1706, he published his most mathematical work and this time his presentation of logarithms followed Halley's method and definition (Jones, 1706). He was well-read and was eager to build his library, so after this text, he did less on his own writing and focused more on curating texts and teaching (Cannon, 2006).

In 1740, another mathematician, Nicholas Saunderson, published his ten books of algebra, and in the section on logarithms, he defined them both as the relationship between sequences, and a geometric progression, but also by what he called: "Another idea of logarithms"

> In the system here described, every natural number is, or may be considered as some power of 10, and it's logarithm as the index of that

power: for let a be the logarithm of any natural number as A; then since Brigg's logarithms of 10 is 1, his logarithm of 10^a will be a; this evident from art. 390 concept 6 [note: that if $\log A = a$ then $\log A^2 = 2a$ and $\log A^m = ma$]; therefore A must be equal to 10^a, since they both have the same logarithms; that is, the natural number A is such a power of 10 as is expressed by it's logarithm a. This consideration gives us a new idea of logarithms, and to one acquainted with the nature of powers and their indexes, it will be no wonder that the addition, subtraction, multiplication and division of these logarithms answer to the multiplication, division, involution and evolution of their natural numbers. (Saunderson, 1740, pp. 622–623).

The idea of logarithms being exponents was out there, and since Jones and Saunderson had correspondence with the same people, it is likely that they knew of each other's work. While Saunderson made it explicit that the logarithm was an exponent, it is not until two years later that Jones' work expanded it to an exponent of any base, not just the Briggsian base of 10.

In 1742, two books, William Gardiner's *Tables of Logarithms* and James Dodson's (ca. 1705 – 1757) *The Anti-Logarithmic Canon,* came out which credited Jones with their ideas around logarithms. Gardiner's text is referenced more often as noted above, but both texts help show how Jones thought about logarithms (Figure 4.27). Both of these explanations removed all instances of progressions or sequences, and define a logarithm strictly from a single instance of an exponential expression or equation.

XXIX. BUT *William Jones*, Efq; (to whom I am extremely obliged, not only for his general rule of proportion, but alfo for the cafes of compound intereft, with other curious problems relating to that Subject, and fome rules concerning Menfuration, as alfo the liberty of his Study for collecting the materials of this Introduction) fhewed me his Method of finding the Logarithms of Numbers, which he does from this principle only.

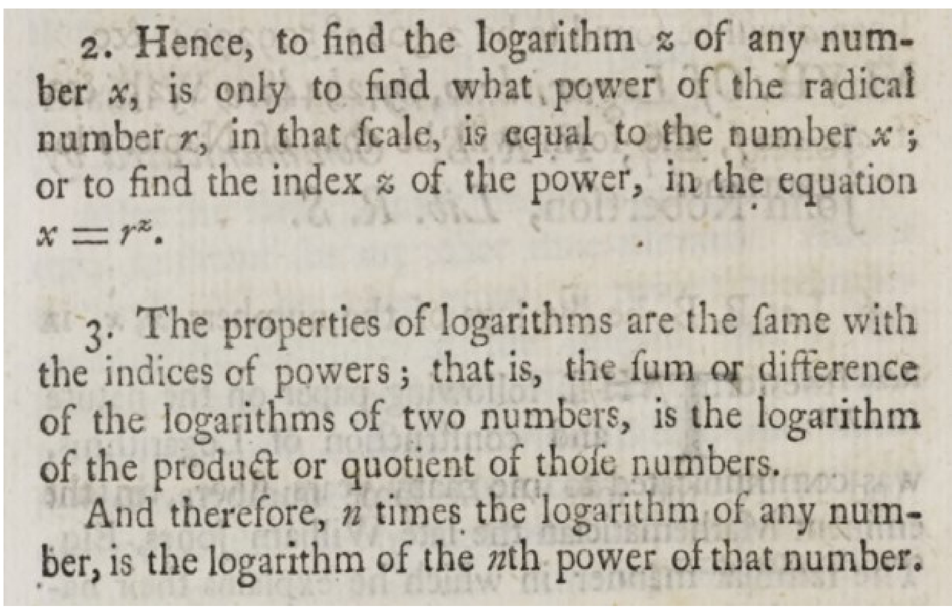
That any Number (x), *whatever may be expreffed by fome fingle power* (r^z) *of the fame radical Number* (r), *where the Index* (z), *of the power of* r, *is every where the Logarithm of the number* x; and by putting 10 for r, he

THE common Logarithm of a number is the Index of that power of 10, which is equal to the number: That is, The Logarithm of any number $a = \overline{10}|^{+x}$, or $\overline{10}|^{-x}$, is $+x$, or $-x$.

Figure 4.27: Top – Dodson (1742, p. viii); Bottom – Gardiner (1742, p. 1)

Dodson's text (Figure 4.27) included an introduction tracing the history of logarithms, which was purportedly written by William Jones (Wilkinson, 1853a). Another source (Jones & Robertson, 1771) suggested that Jones also authored the chapter on logarithms in Gardiner's text (Figure 4.27), so he may have been more involved with the writing of these texts, specifically the parts relating to logarithms, than just being the teacher and inspiration.

It was well-known that William Jones was working on another mathematics text and had left it with a peer, John Robertson. He later published quite a few of his papers including the excerpt in Figure 4.28, *Of Logarithms* (Wilkinson, 1835b). Here, Jones exclusively used an exponential equation to define logarithms, in fact calling the logarithm, the power. Using this new definition he was able to use exponents to derive the main properties, which is similar to how logarithms and their operations are presented in the modern day.



2. Hence, to find the logarithm z of any number x, is only to find what power of the radical number r, in that ſcale, is equal to the number x; or to find the index z of the power, in the equation $x = r^z$.

3. The properties of logarithms are the ſame with the indices of powers; that is, the ſum or difference of the logarithms of two numbers, is the logarithm of the product or quotient of thoſe numbers.

And therefore, n times the logarithm of any number, is the logarithm of the nth power of that number.

Figure 4.28: Jones' definition of logarithms and their operations, using the idea of exponents, called index in the text (Jones & Robertson, 1771, p. 456)

Of course, we cannot know that these words were strictly Jones' or if they were altered a bit to show the change in thinking since his death. And while this definition is very similar to Leonhard Euler's (1707 – 1783), there is no concrete connection between him and Euler. Jones was well known and prolific at sharing his ideas, so perhaps Euler did learn of this connection through Jones or one of his students, or Euler could have just as easily come to the same conclusion by reading the same material that Jones had read.

It was Leonhard Euler though, the famous Swiss mathematician, who put that notion into his *Introductio in Analysin Infinitorom* in 1748 (Figure 4.29), and, even more importantly, in his *Complete Guide to Algebra* in 1770, who paved the way for this definition of logarithms being so wide-spread. His text was used as a foundation for many schools and so many generations of students learned this definition as primary (Panagiotou, 2011).


Quod si ergo fuerit $a^z = y$, erit $z = ly$:

Figure 4.29: Euler's formulation of the relationship between logarithms and exponents (Euler, 1748, p. 73)

4.11. Last Notes on the History of Logarithms

This chapter on the history of logarithms attempts to follow the course of discovery of this mathematical creation and then to explore further the new understandings and redefinitions of this concept, as well as how the notation evolved, over 150 years. As the research in the rest of this dissertation looks at how logarithms were presented to students from 1616 to 1750 and then how they are presented today, it is important to have this background on what the mathematicians at different times knew and were discovering about this concept. To conclude this chapter, Table 4.4 (overleaf) lists the important dates and events in the history of logarithms from Archimedes to Euler.

Table 4.4: Timeline of Events in the History of Logarithms

ca. 220BCE	Archimedes explores operations of terms in a geometric progression in 'The Sand Reckoner'
ca. 250CE	Diophantus uses syncopated algebra for exponential expressions
ca. 1000	Al-Karaji using a translated text of Diophantus furthers work with higher powers
ca. 1150	Al-Samaw'al uses symbols for exponential expressions and extends them to positive and negative exponents
1328	Bradwardine builds up the multiplicative structure of ratios
ca. 1360	Oresme builds up the density of geometric sequences and includes irrational numbers
1484	Chuquet explores the relationship between the arithmetic and geometric progression and creates a notation for exponents
1544	Stifel extends progressions to include numbers between 0 and 1 and uses a latin variant of 'expose' for the arithmetic sequence
1585	Stevin uses the same notation for the root as he does the exponent
ca. 1588 – ca. 1614	Napier and Bürgi independently develop logarithms
1614	**Napier's first book on logarithms is published**
1617	Briggs publishes the first table of base 10 logarithms, constructed with Napier
1623	Gunter creates a precurser to the slide rule
1637	Descartes invents the modern notation for exponents
1647	Gregory of St-Vincent discovers the connection between logarithms and the area under a hyperbola
1656	Wallis combines the root and exponent into one symbol
1668	Mercator uses the connection between logarithms hyperbolas to represent a logarithm as a series
1685	Wallis uses the word 'exponent' for the arithmetic sequence and writes the geometric sequence with exponential notation
1695	Halley redefines logarithms as a number of ratios
1740	Saunderson expresses that the logarithm is an exponent, but focuses on the base of 10
1742	Gardiner and Dodson publish tables and credit William Jones for his work expressing the logarithm as an exponent
1748	Euler's formulation of logarithms is printed for the first time and will be the most popular definition of logarithms for the future

Chapter 5. An ‘Exposé’ of Current Textbooks

As discussed in Chapter 4, logarithms can be expressed in many different ways. Before I delve into past views of logarithms, I want to set the picture for how logarithms are presented in today’s textbooks. The focus of this chapter is to look at the presentation of logarithms in current textbooks to see which conceptions of them are explored, as well as the connections that exist between logarithms and a student’s past mathematics.

In the discussion around concept image in Chapter 2, it was argued that, in order to build a strong concept image, there needed to be connections made between concepts. These connections could consist of scaffolding, using past concepts to build to a newer one, and also of backfilling new information into a previous concept. While looking through the textbooks in this chapter, I am focusing on what connections are made between logarithms and other aspects of mathematics. As new topics are often only introduced using one pathway, I do not just look at the texts where logarithms are introduced, but also look at the textbooks for possible subsequent courses, to see if different connections are built around logarithms as readers get deeper into their studies.

Therefore, in this study, I focus on the first set of research questions presented in Section 1.4: How are logarithms currently presented in textbooks? What connections do they have to a student’s past work in mathematics? What conceptions of logarithms are continued as a student continues further into mathematics?

5.1. Background on Logarithms in Secondary School Education in the chosen areas in Canada, Great Britain and the United States

Secondary School Education means different things to each country discussed in this study. In the US, it generally means high school, encompassing grades 9-12, and the school systems themselves are run through the states, half of which currently follow the national Common Core Curriculum (Edgate, 2023). In Canada, secondary education can switch meanings depending on the province. In Ontario, it more often means grades 9 to 12, but an increasing number of secondary schools are beginning in Grade 7 or 8 (Ontario Association of School Districts International, 2023), while in Alberta it means high school encompassing grades 10 to 12 (Government of Alberta, 2023a). There is no nationalised curriculum for the country, but each province has a Ministry of Education which determines the provincial standards. In Great Britain, all three countries[4] follow a national curriculum and logarithms are first brought in during A-level, the last two years of school. England and Wales share the same content on the Mathematics A-levels (Qualifications Wales, 2023), while Scotland has its own curriculum (Scottish Qualifications Authority, 2023).

As the curriculums and textbooks in the US and Canada often differed by state/province, I chose the three most populous states/provinces. In the US, this was California, Texas and Florida, their combined population is over a quarter of the country's total (United States Census Bureau, 2021). California does use the Common Core Curriculum, while the other two use their own state's guidelines. In all three of these states, Algebra I and Geometry is required, but students can choose to take Algebra II, which is the course where logarithms are introduced (California Department of Education, 2015; Florida State University, 2019; Office of the Secretary of State, 2012). While students are not required to take Algebra II to graduate in each state, in 2009, 75.8% of students in the US took Algebra II in high school, and data showed an upward trend from 1990, so it is likely that the same amount or more take Algebra II now (National Center for Education Statistics, 2022). While most students will first see logarithms in high school in the US, there is still a portion who may not encounter them until a post-secondary College Algebra course.

For Canada, the three most populous provinces are Ontario, Quebec and British Columbia (BC), but as this dissertation is focused on English-only texts, I chose not to review Quebec and instead added the fourth most populous province, Alberta, instead. The three provinces I review make up over 60% of Canada's population, so I am reviewing textbooks in use by the majority of the students in the country (Statistics Canada, 2022). In Ontario, logarithms are first seen in Grade 12, a University preparatory course, Advanced Functions (Ontario Ministry of Education, 2023). In Alberta, the mathematics program is split into sequences, one for University needing Calculus, another for University not needing Calculus, and a last one for students not

[4] There is a larger conversation about what to call England, Scotland, and Wales, but for the purpose of this dissertation I am calling them separate countries, in terms of education, within the larger country of the United Kingdom of Great Britain and Northern Ireland.

planning on entering into University. Logarithms appear in the last mathematics course in both of the sequences bound for University. BC is similar, logarithms first appear in the Grade 12 Foundations of Mathematics or in Pre-Calculus (Province of British Columbia, 2022a/b). Grade 12 mathematics is not required in either province to graduate with a diploma, but is required to enter most post-secondary institutions (Government of Alberta, 2023b). In 2013, less than half of Canadian students took Grade 12 mathematics (CTVNews, 2013), so, like US institutions, it does follow that some students would see logarithms for the first time in their undergraduate education.

5.2. Method

There is a total of 26 texts included in this study, thirteen from a student's first text introducing logarithms, and thirteen from subsequent texts (these texts can be found in Appendix B). There is at least one text included from each level of the curriculum in each of the chosen areas along with one International Baccalaureat textbook. As there are now texts that exist fully on-line being used in all levels of schooling, I included an equal number of both on-line and traditional printed texts. A few of the texts included are used in multiple countries.

5.2.1. Vertical and Horizontal Notions

While all my studies included in this dissertation are around logarithms and texts, overall they are not that similar, therefore I use slightly different methods for each one. But, as there is an overarching thought process behind the methods used, I introduce the background to the methods in this chapter and a version of this method is then seen in the next two chapters.

When starting my program and looking into how to do a text analysis, I ran across the ideas of Thematic Analysis as laid out by Braun & Clarke (2006, 2019). Thematic analysis is a reflexive way of looking at data, the researcher usually has some themes for the data to fit into and/or more likely discovers themes after familiarising themselves with the data. Braun & Clark (2019) described it as, "themes are creative and interpretive stories about the data, produced at the intersection of the researcher's theoretical assumptions, their analytic resources and skill, and the data themselves" (p. 594). It was only after looking at the historical data that I started to see what conceptions of logarithms were missing for modern students. This led to my choice of theory with the overarching idea of breadth and depth. I am not doing a true Thematic Analysis, but it is my starting point in how I view and come to textual analysis.

In going more specifically toward textbook analysis, I turned to a framework developed by Charalambous et al. (2010), who tried to bridge the space between horizontal and vertical analysis of a textbook. In the original formulation of the framework, the horizontal analysis looks at the audience and layout of the text, while the vertical delves into the mathematics presented. I follow this framework closely in this study, the horizontal analysis is done first, and then influences the vertical analysis which focuses on the various conceptions of logarithms and what connections are made to previous mathematics.

Horizontal Analysis ⇨	Audience of textbook	Layout of textbook
Vertical Analysis ⇩	Connections to past mathematics	Introduction of new material around logarithms
		Conceptions of logarithms

Initially, the chosen texts were grouped by whether they were the first time a student interacted with logarithms or whether they were a subsequent text. The texts were then coded around how they introduced logarithms, and then again for each ensuing mention of logarithms paying special attention to whether they were presented as a number or value, function, or operation following some of the difficulties with logarithms that arose while doing the background reading in Chapter 3 (Berevoski & Zazkis, 2006). While that paper looked at the operational aspect of logarithms, referring to the Product, Quotient and Power Properties, my focus changed through the study to whether the text viewed the logarithm as an operation, so I used the word differently from my initial source. In coding, I also added the views of a logarithm as a graph, an exponent, a button on a calculator, as the inverse of an exponential, and as Euler's relationship that if $a^z = y,$ then $z = \log_a y$ (Euler, 1748).

The following coding was used initially:

- Inverse of an exponential: is the word *inverse* used; is the logarithm described in relation to the exponential; is the logarithm graphed using the ideas of inverse; is the logarithm used to solve an equation using the ideas of inverse?

- Euler's formulation: is it stated specifically; is it used to evaluate or solve problems with logarithms?

- Number: does the text evaluate logarithms; is there is space for the reader to understand that $\log_a b$ can be a value?

- Button on a calculator: does the text direct the reader to their calculator to evaluate a logarithm or to graph a logarithm or to check their work around logarithms?

- Function: is the word *function* used; does the text discuss how/why it is a function; are familiar designations of a function employed such as f(x), domain, and range?

- Graph: is there a graphical representation of logarithms; is the graph of a logarithm discussed?

- Exponent: does the text call the logarithm an exponent; is there words or work that would make it clear to the student that the logarithm is an exponent?

- Operation: is the word *operation* used; does the text evaluate logarithms from that format without rewriting in exponential form; is the text clear that when using logarithms in an equation, the logarithm is an operation?

Once the initial coding was completed, I redid the exercise, this time also looking for any other connections to past mathematics that occurred in the narrative, examples or exercises. I took special care to see what ideas of logarithms continued throughout the mathematics pathway, which ones drop off and which ones are added later. Returning to the theoretical construct of a concept image, there should be multiple connections made between logarithms and other elements of mathematics, and between logarithms and real-life examples, so that was at the forefront of my mind when doing this study.

Below I present the data by level of textbook from introduction to a student through their use in integration.

5.3. Analysis of First Textbooks with Logarithms

The thirteen textbooks I review in this section includes those from Algebra II in the US, Grade 12 Mathematics in Canada, and the Higher Maths or A-Level Maths courses in Great Britain, along with five textbooks where logarithms are introduced in College or University, and the first part of an International Baccalaureate (IB) text.

Reviewing the placement of logarithms in these textbooks show that they are primarily contained to their own sections. In only four out of the thirteen texts do logarithms appear in other sections of the book, three times of which is in sections related specifically to functions, generally around the composition of functions or transformations of graphs, once in the section around geometric sequences, and once in an exercise in the section around the binomial theorem. In the rest of the texts, logarithms are introduced in a separate section and only appear there; they are not included in any examples or exercises in the rest of the textbook.

In twelve out of thirteen of the textbooks, logarithms appear as the second part of a single chapter or unit that also includes exponents, most often called some variation of 'Exponential and Logarithmic Functions'. In only one of the texts do logarithms rate their own chapter, though it comes directly after the chapter on exponential functions.

The location of the chapter does vary, but not widely: three times it appears in the first quarter of the textbook, but on average the chapter on logarithms appears two-thirds of the way through the text, and twice it is actually the last chapter. It most often located near the end of chapters that look at other functions, such as polynomial functions and rational functions, so, by the time logarithms are introduced, readers are familiar with function notation and the idea of inverses.

The chapters themselves all cover the same main sections with a few different add-ons. They all have an introduction to logarithms which usually includes a definition and how to evaluate a logarithm, a section on the properties or laws of logarithms and a section on solving exponential and logarithmic equations. All but one included graphs of logarithmic functions in either their own section or in the introductory section. Slightly more than half of the texts have extra sections on modeling or other applications, but it is not consistent in terms of which applications are included. One has a section on Geometric Series and Sequences, though it never ties back to logarithms (even though it could). As the primary focus of all the texts are the sections listed directly above, the vertical analysis will go through that in order: introduction and definition of a logarithm, evaluating logarithms, properties of logarithms, solving equations and graphing.

5.3.1. Introduction to Logarithms

Logarithms are introduced in three ways in the texts chosen for analysis:

- as the inverse of an exponential function (8 out of 13);
- as a way to solve an exponential equation; $b^x = c$ (3 out of 13);
- as an exponent (2 out of 13).

A third of the texts that introduced the logarithm as the inverse of an exponential function began through a graphical means – either starting with a graph of an exponential function and then reflecting it across the line $y = x$, or using the table of values from an exponential function and flipping the y- and x-values, then plotting. Two more texts used a graphical means to introduce logarithms, one used it as a way to solve an exponential equation, while the other just introduced the parent graph of a logarithm in an earlier section on translations of graphs. A little over a third of the total texts employed graphs to introduce logarithms, giving a visual component to this new concept.

Those that did not use a graph would state their main idea of logarithms, one of the three listed above, and then showed examples to justify their claim. If logarithms were introduced as a way to solve an exponential equation, then the logarithm would next be an exponent too, or the operation that undoes exponentiation. If they started with logarithms being an exponent, then the text would lead into Euler's formulation of a logarithm.

5.3.2. Definition of Logarithms

Most of the texts took Euler's formulation as the definition of a logarithm: $\log_b y = x \Leftrightarrow b^x = y$. At times, this formulation would have restrictions on the variables such as $b \neq 1 \text{ or } x > 0$, but other times these restrictions would come in later and not be considered under the official 'definition'.

While 85% of the texts did define a logarithm using this formulation, a third of them wrote it with the exponential expression first, and one of the texts had the definition only leading one way from logarithm to exponential. The remaining texts defined a logarithm as the unique solution of an exponential equation and/or as the inverse of an exponential function. No matter the definition, though, in every text the next step was to signify that a logarithm could be rewritten in exponential form and vice versa.

5.3.3. Evaluation of Logarithms

All manner of introduction and definitions led to evaluating a logarithm of which there were again three ways that were initially presented;

- from the logarithm, find the power the base would be raised to give you the argument (10 out of 13);
- rewriting into exponential form (2 out of 13);
- calculator (1 out of 13).

While evaluation of a logarithm was most often initially presented from logarithmic form, in exactly half of these instances the text immediately moved to rewriting the expression into exponential form. So, in only half of those ten texts where logarithms were evaluated from the logarithms, where readers would find the power the base would be raised to give the argument, were students spending any time working with the logarithm.

In all the texts, the introduction of logarithms, the definition and evaluation were closely aligned. As these concepts were the first ideas any reader had around logarithms, these are the instances that would need the greatest connections to students' previous experiences with mathematics. In these cases, students were bringing quite a few previous ideas with them. Firstly, they would need to understand exponentiation and exponential equations, and in most cases they needed an understanding of functions, inverse functions and graphs of inverse functions.

5.3.4. Properties of Logarithms

The three properties/laws of logarithms were the same in all the texts:

- the product property: $\log_a xy = \log_a x + \log_a y$;
- the quotient property: $\log_a \frac{x}{y} = \log_a x - \log_a y$;
- and the power property: $\log_a x^b = b \log_a x$.

For the majority of the texts (9 out of 13), the properties were shown, sometimes with a sentence saying how they were related to the properties of exponents, and then they were proved by setting the individual logarithms equal to a different variable, converting from logarithmic form to exponential form and using the laws of exponents similar Figure 5.1. Two-thirds of the time not all three properties were proven, but one or two were left for the readers to work out in the exercises.

Proof of Product Property of logarithms: $\log_a(M \times N) = \log_a(M) + \log_a(N)$

Let $x = \log_a(M)$ and $y = \log_a(N)$ then $a^x = M$ and $a^y = N$

From the laws of exponents: $a^x \times a^y = a^{x+y} = M \times N$

Rewriting it in logarithmic form: $x + y = \log_a(M \times N)$

And substitute back in for x and y: $\log_a(M) + \log_a(N) = \log_a(M \times N)$

Figure 5.1: Example of a Properties of Logarithms Proof

Of the remaining four texts, three had the student calculate assigned logarithms and then try to make a conjecture for operations involving the logarithms and their corresponding arguments. While the examples in Figure 5.2 let readers derive the quotient property for themselves, in every instance where this happened, it was immediately followed by proof of at least one of the laws by again converting the logarithms to exponentials. As the exercises in Figure 5.2b were purely calculations, the concept of a logarithm is not explored. The logarithm is a button on a calculator and the students are not exploring the relationship between the conception of a logarithm and the operations that it simplifies. There is no context in this exercise for why some of these expressions are equivalent, it is just presented as a something to find, not understand.

2. a) Show that $\log \frac{1000}{100} \neq \frac{\log 1000}{\log 100}$.

b) Use a calculator to find the approximate value of each expression, to four decimal places.

i) $\log 12$ **ii)** $\log 35 - \log 5$

iii) $\log 36$ **iv)** $\log 72 - \log 2$

v) $\log 48 - \log 4$ **vi)** $\log 7$

c) Based on the results in part b), suggest a possible law for $\log M - \log N$, where M and N are positive real numbers.

Figure 5.2: Exercise to encourage readers to surmise the Quotient Law of Logarithms (McAskill et al., 2012, p. 393)

The only outlier in this group introduced logarithms as a way to solve an exponential equation using the idea that a logarithm is an operation opposite of exponentiation. So, in this instance, they did state the properties and then they proved them by exponentiating each side of the equation by the base of the logarithm.

5.3.5. Graphing Logarithmic Functions

There were only two ways that the textbooks presented graphing a logarithmic function, both taking advantage of the logarithmic function being the inverse function to an exponential function. Exactly half of the texts that included this topic wrote it in exponential form, graphed it and then reflected it over the line $y = x$. The other half wrote it in exponential form, interchanged the x- and y-values and then graphed those points. One text did not cover graphing, so is excluded from this set of data. The logarithm itself was not evaluated or manipulated to create a graph, instead, any logarithm was immediately rewritten to an exponential function and graphed from that form. Once the reader was comfortable with the intercepts, shape, domain, range and asymptote of the graph of a basic logarithmic function, then it would be expressed that this graph is of the parent function and the reader would be expected to apply transformations taught previously to this graph. At this time, the logarithmic graph is a static shape that is moved along the rectangular co-ordinate system according to the ideas apparent in transformation of functions, but there is not an active relationship between the logarithmic function and the movements that are happening.

In both of these instances, properties of logarithms and graphing logarithmic functions, the authors leaned on the relationship they had established between the logarithmic function and the exponential function. Though the reader may have understood the logarithm to be an exponent, and even understood a bit about evaluating logarithms from their logarithm notation, in these sections the logarithmic notation is immediately eschewed for the familiarity of the exponential form. Some of the authors did highlight how the properties of logarithms tied into the numerical values of logarithms but that is quickly overtaken by the proofs that relied on first rewriting the logarithmic expression in exponential form. The material in these sections tied heavily into the students' past experience with exponents, exponent rules, graphing exponential functions and inverse functions, and their more current experience with Euler's formulation of a logarithm and the common introduction of a logarithm as being the inverse to an exponential function.

5.3.6. Solving Logarithmic Equations or Solving Exponential Equations Using Logarithms

For equations that included logarithms, there were two types that were shown in the texts: those where every term is a logarithm, and those where there is a mix of terms, both logarithmic and constant. The ones which were both logarithmic and constant were generally solved by the vast majority (10 out of 13 texts) by rewriting the problem as an exponential equation and then solving the result (Figure 5.3 top). Two of the texts did not have the reader do the conversion from logarithmic to exponential, but instead they had them exponentiate each side of the equation, which eventually has the same effect (Figure 5.3 bottom). The last text has the student rewrite the constant as a logarithm with a common base so then an equality principle (Figure 5.4) could be invoked and the logarithmic part of the equation can be ignored.

$$R = \log\left(\frac{a}{T}\right) + B$$

$$5.5 = \log\left(\frac{a}{1.8}\right) + 3.2$$ Substitute the given values into the equation.

$$2.3 = \log\left(\frac{a}{1.8}\right)$$ Isolate the term with the unknown, *a*, by subtracting 3.2 from both sides.

$$10^{2.3} = \frac{a}{1.8}$$ Rewrite the equation in exponential form.

$$10^{2.3} \times 1.8 = a$$ Multiply both sides by 1.8 to solve for *a*.

$$359.1\ \mu = a$$

EXPONENTIATING TO SOLVE EQUATIONS The property of equality for exponential equations on page 515 implies that if you are given an equation $x = y$, then you can *exponentiate* each side to obtain an equation of the form $b^x = b^y$. This technique is useful for solving some logarithmic equations.

EXAMPLE 5 **Exponentiate each side of an equation**

Solve $\log_4(5x - 1) = 3$.

$\log_4(5x - 1) = 3$	**Write original equation.**
$4^{\log_4(5x-1)} = 4^3$	**Exponentiate each side using base 4.**
$5x - 1 = 64$	$b^{\log_b x} = x$
$5x = 65$	**Add 1 to each side.**
$x = 13$	**Divide each side by 5.**

Figure 5.3: top - (Kirkpatrick et al., 2012, p. 487); bottom - (Larson et al., 2007, p. 517)

For equations where every term was a logarithm, the ten out of thirteen texts that included this type of problem invoked an Equality Principle of Logarithms, even if they did not all call it that (Figure 5.4). Of those that used this principle, four texts had no reasoning on why this principle was true; three referred back to the one-to-one property of logarithms; two exponentiated both sides of the logarithmic equation as a proof; and a last one leaned into the idea behind logarithms: “It seems reasonable that if the exponent we raise b to in order to get x is the same exponent that we raise b to in order to get y, then x and y are the same thing” (Beveridge, 2018, p. 175).

CONCEPT Property of Equality for Logarithmic Equations

Symbols If $x > 0$, then $\log_b x = \log_b y$ if and only if $x = y$.

Words If two logarithms (exponents) of the same base are equal, then the quantities are equal; if two quantities are equal, and the bases are the same, then the logarithms (exponents) are equal.

Figure 5.4: Example of the equality principle for logarithms (Kennedy et al., 2020, p. 333)

This principle of equality was also followed for solving exponential equations. All thirteen texts did include this concept, and twelve out of the thirteen had the reader 'take the logarithm of both sides'. Of those, exactly half cited the equality principle, and like above, only half of those gave any justification for the equality principle, this time all citing the one-to-one property, while the other half gave no justification. Following that, five out of these twelve texts just presented 'taking a logarithm of both sides' as something that is possible with no reasoning for why it can be done, and two texts gave the reason that logarithms are an operation, just like squaring or adding something to both sides. The text that did not fit into this pattern at all presented the problems so the reader could rewrite into logarithmic form and after introducing the change of base formula (Change of base formula: $\log_a b = \frac{\log b}{\log a}$) directed the reader to use their calculator to evaluate the answer, three other texts also had this option as an alternative method (Figure 5.5).

Examples. Solve the equations.

(1) $5^{2x-1} = 3$

(2) $3^{2x} - 5 \times 3^x + 6 = 0$

Solutions.

(1) The idea is substitution. Let $y = 2x - 1$; then the equation becomes $5^y = 3$, so $2x - 1 = y = \log_5 3$ (we can find logarithms of the two sides with base 5 to get $2x - 1 = \log_5 3$ as well); then, solving for x we get $x = \dfrac{1}{2}(\log_5 3 + 1)$.

Figure 5.5: Solving exponential equations (Ji & Ge, 2015, p. 165).

Solving logarithmic and exponential equations had students work both with logarithms, with functions, and with exponentials. In order to succeed, students needed to understand the properties/laws of logarithms, so they could simplify the equations before solving, may have needed to understand that the logarithm is an exponent, and that the logarithm is an operation that can be applied. Most students would need some background in exponentiation as an operation and basic exponent rules, while some students would need to understand one-to-one functions in order to follow the justifications for the equality principle of logarithms.

5.3.7. Are logarithms a number?, a function?, an operation?

Logarithms had many conceptions throughout the chapters reviewed (Figure 5.6). As stated earlier, they were most often introduced as the inverse of an exponential function. Euler's formulation was usually present and led into evaluating a logarithm, and either that or graphing led to the log or ln button on a calculator. While there were some discrepancies in the texts, this was a common presentation.

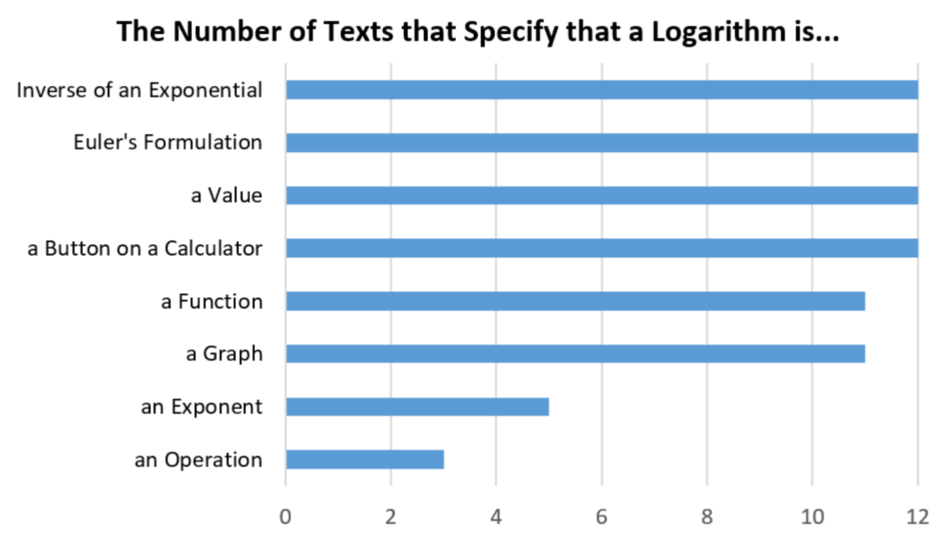


Figure 5.6: Number of textbooks that present logarithms in these different forms

No text specified that they were all the things listed in Figure 5.6, but they all presented at least four ways of looking at logarithms with the most two common words used around logarithms being 'function' and 'graph'. The authors in these texts focused on the idea of a logarithm as a function, though only 38% spent any time justifying that claim, the rest labelled it a function and then moved on, calling it that many times throughout the text. As discussed in the horizontal analysis though, the logarithm section usually came in the latter half of the textbook, so perhaps most authors were assuming that the no longer needed to justify the name ‘function’? Either way, readers of these texts would come away with the idea that the logarithm is a function.

The next most common phrases associated with logarithms were equally split over implying that a logarithm was an operation or whether a logarithm was a value. These instances were far less common than the wording around function, and in both cases were not always clear. The word ‘exponent’ was used several times: sometimes it was obvious it was referring to the logarithm as a number, while other times it would be part of an operation. Specific words were used more often with the idea of a logarithm as a number such as ‘evaluate’, while the words around logarithm as an operation were more abstract. Phrases such as “inverse of exponentiation” and “take the logarithm” did imply that there is an action associated with logarithms, but only in very few cases did they call the logarithm an operation. Readers of these texts could come away confused over whether a logarithm was its own operation or if it was just another way of writing an exponential operation.

5.3.8. Summary of the Textbooks where Students Encounter Logarithms for the First Time

Only three of the texts reviewed stated pre-requisites for the sections on logarithms: they were graphing functions, inverse functions and properties of exponents, and after analysis I would agree that those are the prerequisites most needed to understand these presentations. In the majority of the texts, logarithms were strictly tied to exponents and the exponential function. To understand a logarithm was to understand it is the inverse of an exponential function; to evaluate a logarithm was to convert it into exponential form; to graph a logarithm was to go from the inverse of a graph of an exponential function; the laws of logarithms were derived from rules of exponents; and if not taking the logarithm of each side, then to solve logarithmic equations meant to exponentiate each side. For the most part, logarithms were only associated with exponents; every action with logarithms ran through the idea of exponents.

There were a few exceptions to the above, three texts worked a bit more with logarithms as numbers: having students estimate a logarithm and using basic table of logarithms to justify some of the properties. This presentation tied logarithms more strongly to calculations, logarithms were values where multiplication becomes addition and connected them to other mathematical ideas that students would have familiarity with: numbers, number lines, and calculations with numbers.

5.4. Subsequent Textbooks with Logarithms

The thirteen textbooks in this section are meant for students who have already had at least one class that introduced logarithms, so they are labeled either Pre-calculus, Calculus, or further texts for A-levels, Advanced Higher Maths and IB.

Very few of the texts had a specific section for logarithms, mainly the five Pre-calculus texts. These sections were located in the first few chapters of the book and were fundamentally the same as the discussion in Section 5.3. Two of the texts were continuations of series by the same authors and in both of those, the Exponential and Logarithmic Function chapter was the exact same.

Of the remaining eight texts, half had a short section specifically on logarithms, usually in a review section or an appendix, which gave the main definition (Euler's formulation), explained that a logarithmic function is the inverse of an exponential function, stated the properties/laws of logarithms, and showed to solve an exponential equation by taking the logarithm of each side. The only thing that was new in these sections was the emphasis on the natural logarithm, so the inverse relationship between the natural logarithm and e was highlighted along with a change of base formula so more logarithms could be rewritten as a natural logarithm.

The main difference in all of these texts compared to those discussed in the previous section were that logarithms were no longer confined to their specific chapter, they were found throughout the book. So while I did not redo the analysis on the logarithmic sections, as it would be the same as above, I did analyse all the other instances of logarithms in these texts.

5.4.1. Incidental Usage of Logarithms

In the texts that immediately followed a reader's introduction to logarithms, logarithms often showed up in a more incidental manner. They were there remind students of logarithms, not because it advanced the ideas of logarithms or because logarithms were needed for the new concept. For instance, the logarithm of a number may have been included as the radius of a circle to be graphed, or elements in a matrix of which a determinant needs to be found. In these cases, the logarithm was seen as a number, having examples like this throughout the texts could help students become used to the nomenclature of logarithms and used to the idea that logarithms can have a value but did not expand upon the idea of logarithms.

One instance which could have gone a bit further appeared in two texts. There was a question in the exercises around arithmetic and geometric sequences where students were asked to prove that the logarithms of the terms of a given geometric sequence result in an arithmetic sequence. The reader most likely would use a calculator to find the value of the logarithms of those terms and then see if they have a common difference, so it could just be an example of using logarithms in lieu of numbers. But given the history of logarithms, it could also lead a reader to investigate further, especially if they questioned if that was the only geometric sequence where this prompt was true?

Properties, or laws, of logarithms also often appeared throughout these texts, such as with trigonometric identities or in exercises around mathematical induction. Logarithms did not have to be used in these examples, but including them made them more ubiquitous in mathematics and reinforced the students' understanding and use of the properties of logarithms.

Ideas around functions were expanded upon in many of these texts with specific units on inverse functions, operations with functions, among others. Logarithms were present throughout all these chapters, and while it did not introduce much new about the logarithmic function, it reinforced the idea that it was a function.

Similar to above, solving exponential equations using logarithms was seen frequently throughout these texts. Logarithms appeared in trigonometric equations, work around sequences and in quite a few applications. Interestingly enough, solving logarithmic equations all but disappeared, though students did get more work with ‘taking the log’ of both sides, usually to finish a problem.

The use of logarithms described in this sub-section were sprinkled into the texts to remind students of this concept and to give them more experience with logarithms. The use of logarithms in these sections reinforced previous conceptions of the function and worked to expand the idea of a logarithm. For most of these instances, though, it was not necessary to include logarithms, so, in the following sub-section, I review the parts of the text where logarithms were an essential part of the discussed concepts.

5.4.2. Parametric and Polar Equations

I am putting these two textbook sections, parametric equations and polar equations, together since, while technically logarithms do not need to be part of these sections, their absence would be notable. Both of these sections used similar ideas of logarithms, they expected the student to know and understand the graph of a logarithm, that logarithms and exponentials were inverse functions, and to be able to work with exponential and logarithmic equations. The unit on parametric equations did introduce something new to the student, the idea that an object could move in a manner affected by logarithms. And, in graphing a logarithmic equation on a polar axis, students gained a new view of the logarithmic graph. While the ideas that build up to logarithms might not be strengthened in these units, they may have introduced new ideas around logarithms.

5.4.3. Limits and Continuity

Logarithms appeared in all the units around limits and continuity, and while students needed to understand the graph of a logarithmic function in order to understand this new material, the idea of limits and continuity is also enhanced by the students understanding of a graph. While before the graph of a logarithm was the inverse of an exponential function, these new ideas can give students more insight into what was happening in all parts of the graph. The idea of limits let students envision, in a more mathematically sound way, the behaviour of logarithms at the extreme edges of the graph, and the idea of continuity could help students realise that there was a logarithm of every number within the domain. The presentation of both of these ideas did call back to the logarithm as the inverse of an exponential function at least once, but students now spent more time working with the actual logarithmic function and with the actual graph of a logarithm. This sub-section did not give students any new foundation upon which to build out their idea of logarithms, but did work to further their understanding of the logarithmic graph and the logarithm as a function.

5.4.4. Hyperbolic Trigonometric Functions

Several books that included hyperbolic trigonometric functions also included the inverse of these functions. Since hyperbolic trigonometric functions can be written using exponentials, those texts solved related equations to write the inverse in terms of logarithms. The operations were not new, it mainly referred to solving exponential equations, but it did connect logarithms to a new idea that students would maybe be able to build on in the future.

5.4.5. Derivatives

Logarithms were present throughout the units on differentiation in any of the texts that included that topic. They usually appeared pretty early, as the textbook author would work through deriving logarithmic and exponential functions.

Only one text tried to use the limit definition to find the derivative of the natural logarithm function.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} \Rightarrow \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} \Rightarrow \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

From here they multiplied the expression by $\frac{x}{x}$ so they could get closer to having the limit equal e.

$$\lim_{h \to 0} \frac{1}{x} \times \frac{x}{h} \ln\left(1 + \frac{h}{x}\right) \Rightarrow \frac{1}{x} \times \lim_{h \to 0} \ln\left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \Rightarrow \frac{1}{x} \times \ln e \Rightarrow \frac{1}{x}$$

Most though had not previously defined *e* in this way in the text, so they did not go through this method.

Two of the texts drew upon the idea of graphs and inverse graphs, and stated that an unusual feature about the derivative of $y = e^x$ is that it equals the y-value at every given point: $\frac{dy}{dx} = y$. And since the derivative is a slope, it may be written as $\frac{dy}{dx} = \frac{y}{1}$. Since the logarithmic function is the inverse, then they argue that the derivative of a logarithm should correspond to the x-value, but in the form of $\frac{dy}{dx} = \frac{1}{x}$ to keep in the same manner as the derivative of the exponential function. Both the texts admitted that this was not a valid proof, but a way for the readers to visualize the derivative and to conceptualise it using their previous understanding of logarithms.

The remaining texts worked from the idea of the logarithm being the inverse of the exponential to rewrite the expression as $x = e^y$ and then used implicit differentiation.

$$\ln x = y \Rightarrow x = e^y \Rightarrow 1 = e^y \frac{dy}{dx} \Rightarrow \frac{1}{e^y} = \frac{dy}{dx} \Rightarrow \frac{1}{x} = \frac{dy}{dx}$$

The last two methods made strong use of the inverse properties of logarithmic and exponential functions, while the first used properties of logarithms and a solid understanding of *e.* All three methods did show a different way of looking at logarithms, and a different way of viewing derivation. I argue that any of these add to a student's conception of a logarithm as it is now related to a new function; but those that work more strongly with the logarithm, not immediately transforming it to an exponential function, could be a greater help to students' understanding of the logarithmic function.

In the two texts that justified the derivation of $\log_a(x)$, it was accomplished by rewriting it using the change of base formula as $\frac{\ln x}{\ln a}$ and then deriving that function. And while the derivative of $f(x) = e^x$ did not involve logarithms, the derivative of $f(x) = a^x$ did and was shown in one of two ways.

The first way rewrote it as $f(x) = e^{\ln a^x}$ and used the power property of logarithms and the chain rule to find the derivative:

$$f(x) = e^{x \ln a} \Rightarrow f'(x) = e^{x \ln a} \times \ln(a) \Rightarrow a^x \times \ln a$$

The second way let $y = a^x$ and then took the natural logarithm of both sides and used implicit differentiation.

$$\ln y = x \ln a \Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y \times \ln a \Rightarrow \frac{dy}{dx} = a^x \times \ln a$$

Both ways incorporated advanced ideas in differentiation, but they primarily used the inverse relationship between an exponential and a logarithm in setting up their derivation. And in both of these, the reader needed to be comfortable with the idea that the logarithm of a constant is a constant. While neither of these exercises added new ideas to the concept of a logarithm, the rewriting of a value or variable as the composition of a logarithm and exponential, as well as the inserting an equals sign so the student can 'taking the logarithm of both sides' is something that the reader has not often seen previously. These sections did add new tools to incorporate logarithms.

Those tools were apparent in the way that logarithms come in use during all units around differentiation, primarily the sections around logarithmic differentiation and L'Hopital's rule. Logarithmic differentiation involved letting an expression that had many products, quotients, or powers equal a random variable then taking the logarithm of both sides so the properties of logarithms could be used to write the more complex expression as separate, simpler terms. These terms were then much easier to derive. The way logarithms were used in L'Hopital's rule is similar, but this time working with limits. While these were not new ideas of logarithms they did use logarithms and specifically the properties of logarithms in new ways.

Logarithms were also used in the solutions of various applications. They were present in the work around Taylor and MacLaurin series, and one text even did a linear approximation of ln 2 using differentials. While logarithms here were more similar to the way logarithms were used in sub-section 5.4.1, having them so present could increase students' familiarity and comfort with the concept. Combine that with the newer ways to use logarithms created through differentiation, and the increased importance of the natural logarithm, and students should have a feel for the diversity of this function.

5.4.6. Integration

In the reviewed texts, the integral of $\frac{1}{x}$ was related strictly to the derivative of the natural logarithm. None of the books drew any connection to this integral and the area under the curve of a hyperbola in the main text, nor of any other way to develop this integral. The integral of the logarithm came after students had learned to do integration by parts and was shown either as an example or given in an exercise. While the act of finding the integral of these functions did not build any new connections, this is an important connection that students will expand upon of going forward.

An Interesting Aside

James Stewart (2006) did define the logarithm as an integral in an Appendix. As it is not in the main text, I will not spend much time on it, but it is an interesting idea. He started off by defining the natural logarithm as the integral from $1 \text{ to } x \text{ of } \frac{1}{t} \text{ given } x > 1$, but did not give any justification other than "If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = \frac{1}{t}$ from $t = 1$ to $t = x$" (p. A50). This definition of logarithms had been discussed by May Hamdan (2006) and here Stewart showed how it could be used to prove the properties of logarithms and then used to define exponential functions. I think it is a fruitful area that could be explored in future texts, once the relationship between logarithms and the hyperbolic curve leading to the integral have been better established outside of exponents. Even if it does come much later than the current introduction of logarithms via exponents, if this connection is explored it could give students another pathway to conceiving logarithms.

5.5. Last Notes on Current Textbooks

This review focused on the presentation of logarithms in the textbooks where they would first be seen as well as subsequent textbooks. It included a look at the pre-requisite mathematics that contributed to the new concept of logarithms and what ideas of logarithms continued throughout their use in mathematics up through integration. For the most part, logarithms strongly tied into any idea relating to exponential functions. Logarithms also built upon basic foundations covered in units on functions, and inverse functions. And less often, but still present were logarithms connections with numerical values and the operations between those values. This list did not grow through the review of subsequent texts, logarithms were thought of in different ways, but they did not get any new roots, they still grow from exponents and their primary associations were exponential functions.

What did come out in the review of the later textbooks was that the way that logarithms were utilized were not always front and centre in the earlier textbooks. One thing that I noted is that the idea of logarithms as operations is not always apparent. In the textbooks where logarithms were introduced, the idea of logarithms being an operation was there, but it was not often written plainly or justified. But later, expressions are turned into equations just so the logarithm can be taken of both sides; the use of logarithms as an operation is prevalent and necessary. Likewise, the properties of logarithms are not kept to a single section, they present a major component of logarithms in later mathematics. The use of the operations of logarithms, along with using a logarithm as an operation, is so common in these later texts that the reader's conception of a logarithm will most likely have the operational understanding as primary. Therefore, it does seem that this understanding should be stressed in multiple ways in the earlier textbooks.

In this dissertation, I again emphasise that I am not arguing against presenting logarithms as the inverse of exponential functions, as that is a necessary way to understand logarithms and something that follows them throughout mathematics. I am arguing that there are other foundations of logarithms that could serve students well in later mathematics. That logarithms came to us primarily as a way to simplify calculations would lead straight into the properties of logarithms which become fundamental to advanced mathematics students. That the logarithms are the area under a hyperbola could help students understand integration in a different way and could be an entry point into the rest of the ideas around the function, as shown by Stewart (2006). Having different ideas to refer to when learning or expanding upon a new concept could help the reader get a grasp of that concept, and therefore will give them more space and resources when trying to move forward.

Chapter 6. The 'Progression' of Logarithms through Historical Texts

As discussed in Chapter 4, Napier's envisioning of logarithms was kinematic, but that was not often the way that they were presented in subsequent texts. Edward Wright's (1618) translation of Napier, John Keill's (1723) examination of logarithms through a curve (for him logarithmic, for us exponential) and Colin MacLaurin's (1742) defense of Newtonian Calculus are the only places that I have found where the kinetic nature of logarithms was discussed. Between the time when logarithms were kinetic, to the time when they became entwined with exponents and exponential equations, they had been presented in quite a few different ways, with all of those ways focused on the connection between the operations with numbers and the operations with their logarithms.

This connection was so primary, as it was the main use of logarithms for the first part of their history. Many of the textbooks in the early 1600s focused on using tables to make calculations easier using logarithms. But even in these early texts, there were spaces where authors were trying to broaden the concept of a logarithm by introducing how they were calculated or starting to relate them more with solving problems involving proportions. By the late 17th century and into the 18th century, logarithms became more associated geometric curves and as a quicker way to solve various problems.

In this chapter, I focus on the ways that logarithms are introduced and presented to the reader in historical texts and how that introduction paved the way for the operations with logarithms. As my focus is on what parts of the past could help current students increase their concept image of logarithms, I am only looking at areas that are still relevant today. Therefore, the large use of logarithms in trigonometry, where they eased the calculations involved in using the Pythagorean Theorem, for example, will be ignored. The goal of this chapter is to answer the second set of questions found in Section 1.4: How are logarithms presented in historic texts and textbooks? How does their presentation affect their use in those texts?

6.1. Method

To start this project, I searched Google Books, Early English Books Online, Haithitrust, and the SFU website for any instance of the word 'logarithm', including alternate spellings (e.g. logarithme, logarythm), in between the years of 1614 (Napier's first text) and 1750, the end of the decade that saw logarithms defined as exponents for the first time. I then discounted any texts that were not in English, any text that did not include any mathematics, and any texts that had a limited use of logarithms, such as it appeared only once in the text to simplify one calculation. Overall, I reviewed 80 texts, almost all from the United Kingdom or Northern Ireland. These texts can be found in Appendix B.

I once again returned to the framework of Charalambous et al. (2010) and redefined what is meant by the horizontal and vertical analysis. In the initial framework, the horizontal analysis reviewed the audience of the text and layout, while vertical analysis focused more on the mathematics and the connections created within the mathematics.

Horizontal Analysis	Type of text	Common Definitions
Vertical Analysis ⇩	Connections to other mathematics	Introduction and presentation of logarithms
		Derivation of properties

In this chapter, the horizontal analysis was used to guide the choices of texts and presentations of the data around logarithms made in this section. Logarithms were introduced at many levels of mathematics during the span of time studied so there needed to be a limit on the type of text included in the study. The 80 texts were organised by subject and level of mathematics then perused to see purpose of logarithms for those texts. From this analysis, common threads were also found within the types of textbooks or across types of textbooks. This categorising lead to the coding of every definition of a logarithm that was found in the text. I reasoned that, if a common definition or usage of logarithms was found across types of textbooks, then it had a larger reach than if it was just isolated to a single type.

For the vertical analysis, I selected those texts that introduced the concept of logarithms, not just how to use the tables with no explanation. These were then coded with respect to how the logarithmic operations were derived. If logarithms were used in solving equations that could be classified as exponential, then that was also noted and coded it. The coding here is more straightforward so will be easy to follow throughout the analysis.

6.2. Horizontal Analysis of Past Textbooks

I start with the horizontal analysis, reviewing the types of textbooks and definitions that were found in the textbooks. Once that is complete, I probe further into the texts that work more deeply with logarithms.

6.2.1. Types of Textbooks

The textbooks were first divided by their most prominent features which helped narrow down the audience for the texts. Since many texts were written for multiple audiences, these added to more than the number of texts (80):

- Applications (39 texts): including astronomy (3), dyalling/gauging (6), finance (2), military (3) navigation (19), surveying (5), and veterinary (1) texts;
- Arithmetic texts (19);
- Algebra texts (7);
- Geometry texts (9);
- Trigonometry texts (10);
- Tables of logarithms (9);
- Texts on the study of logarithms (5);
- Calculus (2).

Within each type of texts, there were levels to how much they explored the concept of logarithms. Some texts went into detailed definitions and used those definitions to derive the properties of logarithms, while others assumed the reader was already familiar with the concept and proceeded to use them without explanation, or just explained how to use the tables without any discussion of logarithms themselves (Figure 6.1).

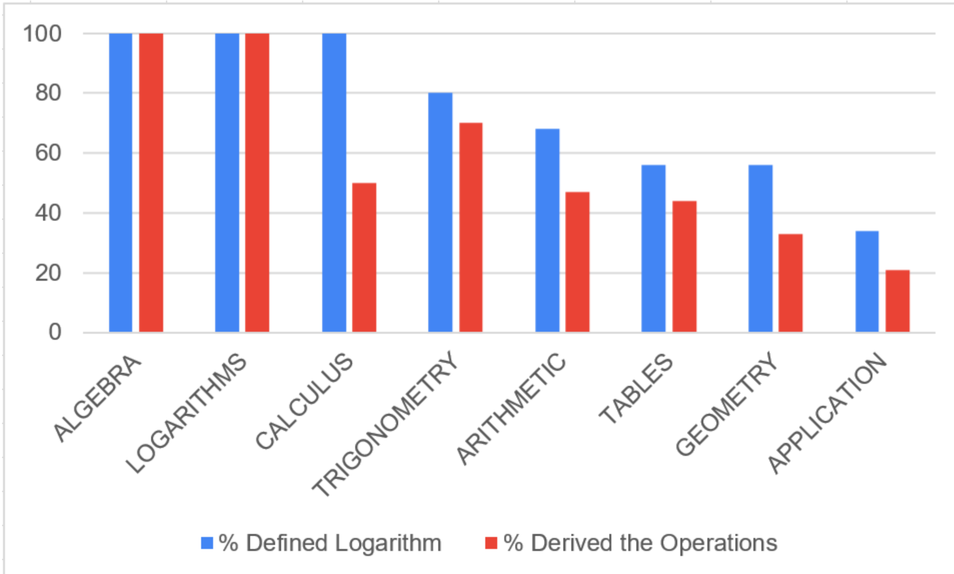


Figure 6.1: Percent of the types of texts that defined logarithms and derived the properties

It is perhaps not surprising that every text that was focused on examining logarithms did have at least one definition for them and used that definition to derive their properties. Every text that was primarily an algebra text did the same.

The other types of texts all had examples where they would use logarithms in problems, but would not examine the concept of a logarithm. I only included ones where there was a modicum of explanation, so, for example, "but if you have a Table of Logarithms, which are now very common, (though not known in the time of this Author) the work is most easie. For half the number of any Logarithm is the Logarithm of the Square Root" (Phillipe, 1669, p. 161). While there was a bit of explanation on how to use a logarithm, there was no discussion on definition of a logarithm, or the reasons they could be used in such a way. I expect that some students at the time already were familiar with the concept, though for a few this may have been the first introduction and they could have explored the concept later.

Application-based texts, where the reader is not primarily learning mathematics but rather using mathematics to gain expertise in another area, were the majority of the texts where there was less of an explanation of the concept of a logarithm. The authors of those texts were maybe less interested in exploring mathematical concepts, focusing instead on explaining the calculations needed for the reader to succeed in their chosen field. For these texts, logarithms were most often strictly a calculating tool. Students would need to understand the tables and how to use them, but would not need any comprehension of the concept beyond their use.

A similar idea was apparent in Arithmetic texts; authors explained how to use logarithms to simplify the operations but did not expand on that idea, though some of the authors did explore logarithms in arithmetic texts meant for higher mathematics students (Moore, 1681; Martin, 1740). In the Trigonometry and Geometry texts, it appeared to be the opposite; students at this level were expected already to be familiar with the concept, so the authors went straight into using logarithms to ease calculations.

While there were texts where the concept of logarithms was not explored, as discussed above, 50 of the 80 texts did have some aspect of definition for the concept of a logarithm. Those are the texts I focus on for the remainder of the study.

6.2.2. Types of Definitions

There were multiple introductions to logarithms in the examined texts, which are detailed below in Section 6.3. But, for the horizontal analysis, I considered whether these definitions ran across types of textbooks or were instead contained within one type. Most texts used more than one definition of logarithms, a short explanation of each along with how I labeled them is presented next.

- Artificial Number (AN): “a sort of Artificial Numbers, so adapted to correspond with Natural Numbers, that the Addition and Subtraction of them, do exactly answer to the Multiplication and Division of those Natural Numbers they are adapted to” (Ward, 1710, p. 10).
    - while I do not consider this one a true definition, it is present in enough of the texts that I deem it important to have its own code.
- Operations (O): “They perform that by *Addition* and *Subtraction*, that Natural Numbers do by *Multiplication* and *Division*” (Leadbetter, 1739, p. 19).
- Comparison of Progressions (CP): "There may be several kinds of Logarithms contrived; for any Series of Numbers in Arithmetical Progression are the Logarithms of those right against them in Geometrical Proportion" (Hatton, 1721, p. 249).
- Equal Differences given proportional numbers (ED): "The logarithms of proportional quantities are equally differing" (Napier/Wright, 1616, p. 7).

- Number of ratios (NR): “The Logarithm of any Number is the Logarithm of the Ratio of Unity to that Number; or It is the Distance between Unity and that Number” (Keill, 1723, p. 339).

- Exponents (E): “The common Logarithm of a number is the Index of that power of 10, which is equal to the number: That is, The Logarithm of any number $a = 10^{+x}$, or 10^{-x} is $+x$, or $-x$" (Gardiner, 1742, p. 1).

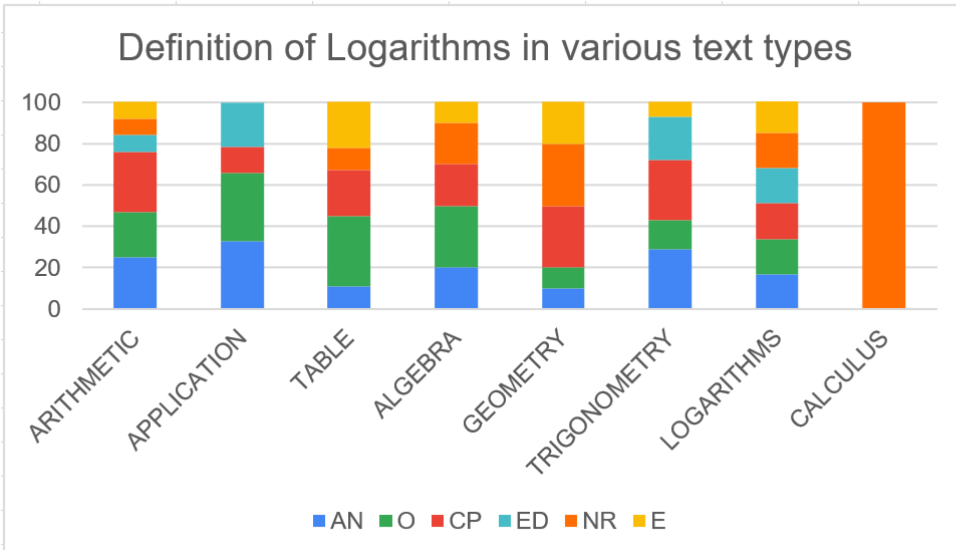


Figure 6.2: The distribution of the definitions of logarithms across different types of texts

Figure 6.2 shows how often, by percent, each type of definition appears in each type of text. The chart is organised by level of text as it was expected that the ones based around arithmetic or application-based would be less focused on higher concepts of mathematics, while Algebra, Geometry and Trigonometry will have gone into more detail. The texts that were mainly tables of logarithms, with some explanatory material fit between these two levels, and the ones focused on the study of logarithms at the end as they had the more detailed explanations, followed by the Calculus texts.

From Figure 6.2, we can see that the first three definitions, AN, O and CP, were present in almost all levels of text, though AN and O were more common in the earlier levels of mathematics. Presenting logarithms as artificial numbers that ease difficult operations was a common and easy way to explain them to students who mainly be using them to perform calculations. This definition remained even into higher mathematics texts but it was joined by the more detailed definitions of CP and NR.

While CP existed at almost all levels, it was more apparent in the trigonometry and geometry texts. Over time it combines with NR in the geometry texts as the idea of an arithmetic progression following a geometric progression evolved to become the distance of a term in that geometric progression from the starting value of 1. Both presentations were a very visual way of viewing logarithms which fit with the visual aspect of geometry. These both led into the idea of logarithms as an exponent, which was seen in most of the types of texts, but unusually it was most prevalent in tables of logarithms. In Chapter 4, it was shown that the first definitions that introduced logarithms as exponents were found in tables of logarithms, so it was perhaps to be expected that the data in this section agreed with the previous findings.

While ED was present in four of the types of texts, it was not a common way of defining logarithms after the early 1700s. I present further detail on the timeline of each definition in Section 6.3, but this definition was present in the original translation of Napier's text (Napier/Wright, 1616) and had some popularity for early writers. It was seen most often in application-based texts, where authors would give the definition, an example, and use the example to show the properties of logarithms before doing exercises.

It is interesting to note that the texts purely exploring the logarithmic concept presented all of these definitions equally in the time that they were active. There were five of these texts, ranging from 1616 to 1740, and yet not one definition was prioritised. As the authors of these texts wrote about many aspects of logarithms, they defined them in multiple ways, using all the definitions that had come before.

Lastly, with the two Calculus texts, the NR definition was the only one presented to the readers. I presume that the authors of these texts expected readers to already be familiar with logarithms so did a cursory introduction with one definition and then went straight into using logarithms in their problems. As there was not much time spent on introducing or defining logarithms, and I had such a limited number of texts of this type to review, I cannot draw a conclusion on why NR was the definition chosen in the calculus texts. I could surmise though that it had something to do with the geometric indications of the NR definition.

Using a combination of definitions was prevalent throughout most of the texts, there were only 14 out of the 50 texts that had a single definition: 0 had AN, 2 had O, 2 had CP, 5 had ED, 3 had NR, and 2 had E. The other 36 texts combined the definitions above to build up their concept of a logarithm. The definitions were not treated equally, though, as seen in Figure 6.3, only two combinations used ED while CP was the most popular definition being employed in 12 different combinations and in 28 different texts.

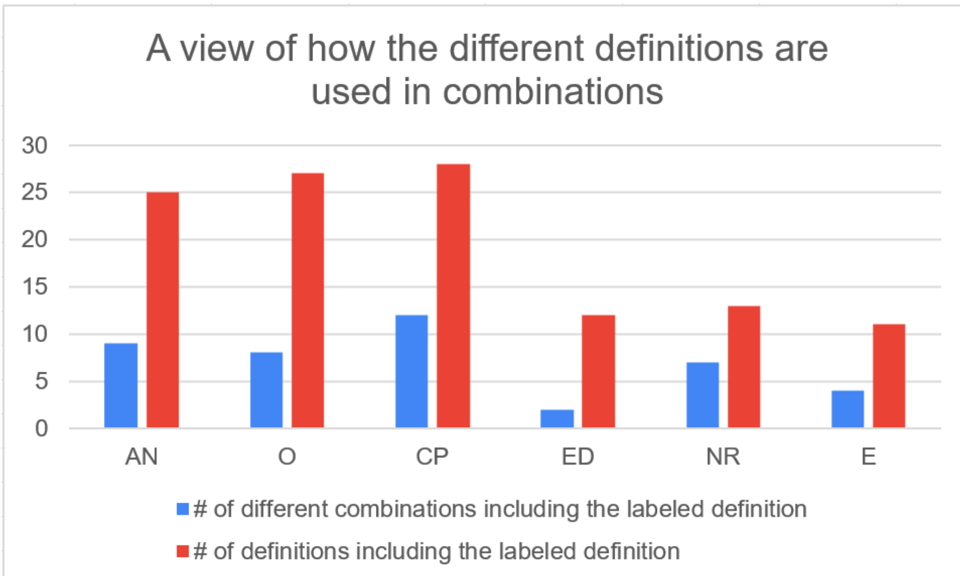


Figure 6.3: How the definitions are used in combination with others

AN, O, and CP were the most common definitions during the timespan reviewed. ED was a definition that fell out of favour during this time, while NR and E only appeared closer to the end of the span. Next I look at each definition in detail and how they were used to give meaning to the logarithmic concept.

6.3. Analysis of the Definition of Logarithms

In the vertical analysis, I first review the different definitions of logarithms used in these texts along with how they interact with each other to create a full idea of a logarithm as presented by the authors. I then focus on the justification for the properties of logarithms and how the different definitions build into solving exponential equations using logarithms. As logarithms have been used in the past, and are still presently used to solve problems around compound interest, I focused on this type of equation, though at times I did branch out as warranted. Before logarithms, compound interest was mainly managed by the use of tables, but with logarithms bringing its own set of tables that could greatly simplify calculations, solving compound interest problems changed rapidly to reflect the introduction of logarithms (Lewin, 2019).

6.3.1. Artificial Numbers

The idea of logarithms as Artificial Numbers appeared as early as in Henry Briggs' *Logarithmicall Arithmetike* in 1631, though there they were called 'invented'. This idea continued being a prominent part of the logarithmic concept, showing up 25 times in definitions through the 1740s (Saunderson, 1740). While they were most often called 'artificial numbers', they were also called 'borrowed', 'contrived', and 'substituted' numbers.

As stated in sub-section 6.2.2, this idea of logarithms never made a complete definition, but was used often in concert with the other definitions to round out the concept (Figure 6.4). It was most often used with the idea of operations as a way to simplify calculations:

> Iogarithmetique is a logicall kinde of Arithmetique, or artificiall use of numbers invented for the ease of calculation wherein each number is fitted with an Artificiall, and these artificiall numbers so ordered, that what is produced by multiplication of naturall numbers, the same may be by the addition of these their artificiall numbers; what they performe by division, the same is here done by subtraction. (Gunter, 1636, p. 1)

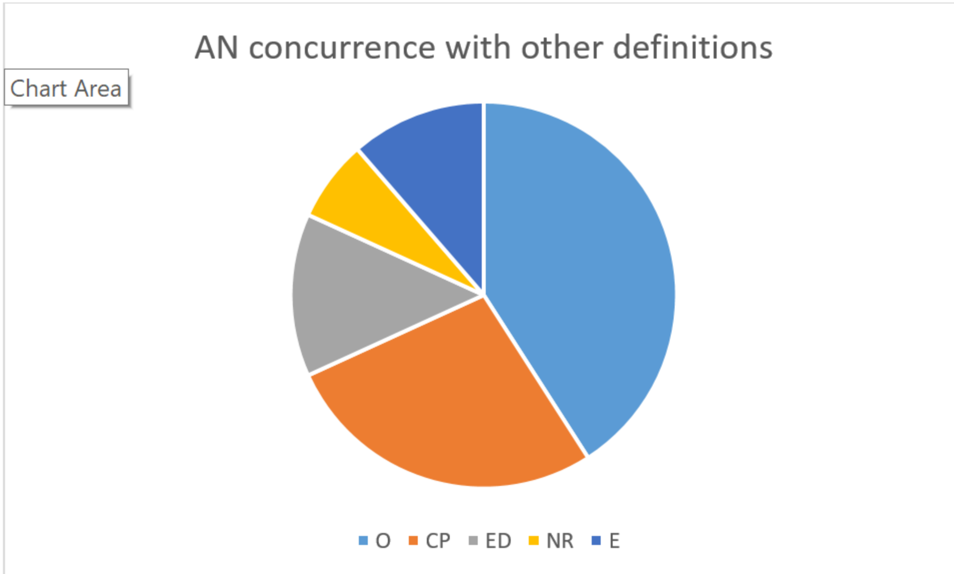


Figure 6.4: AN concurrence with the definitions of logarithms

While AN did not comprise a full definition of logarithms it was present in the majority of the definitions (52%) and continued to appear in the entire span of the time that was reviewed. Authors of this time period seemed to believe it was important for the readers to know that logarithms were ‘artificial’ numbers, compared with the ‘natural’ numbers that they were used to operating. This distinction could be a way of humanising mathematics. Logarithms were numbers invented to make mathematics easier for future generations. A few authors did seem to take this tack, stating that logarithms “were invented by the Lord Nepeir, Baron of Merchiston in Scotland; who ingeniously contrived how to perform Multiplication or Division of Natural Numbers; by adding or subtracting certain Artificial Numbers (fitted to correspond with the Natural,) called Logarithms” (Ward, 1695, p. 91). Mathematics in this case was created by humans to help other humans in understanding the world.

6.3.2. Operations

Logarithms as a way to simplify operations was a main definition through the entire time-span reviewed, first seen in Briggs’ 1631 text and continuing until Francis Holliday’s *Syntagma Mathesios* in 1745. In this definition, the reason for logarithms was to simplify calculation, and when it was the sole definition, no other ideas around logarithms were explored. It was only the sole definition twice in the reviewed texts, all other 25 appearances were in combination with other definitions. Authors of these texts seemed to appreciate the primary function of logarithms at this time and wanted to highlight it for their readers, but the majority included one of the other definitions of logarithms so they could more easily explore the mathematics of the logarithmic concept.

6.3.3. Equal Differences Given Proportional Numbers

The definition of logarithms as being numbers having equal differences when paired with proportional numbers came early but did not last far into the 18th century. It first appeared in Edward Wright's 1616 translation of Napier's work and was last seen in John Hill's 1713 Arithmetic text. In the review of texts during that century, it only appeared 12 times.

The ED definition was commonly paired with AN, and mainly stated that "Logarithmes are borrowed numbers, which differ amongst themselves by Arithmetical proportion, as the numbers that borrow them differ by Geometrical proportion" (Wingate, 1635, p. 1). Figure 6.5 continues Wingate's text, he first gave an example of a sequence of proportional numbers (a geometric sequence though he does not call it that) and then next to it, he placed columns of numbers that have equal differences (so arithmetic sequences).

Prop. nu.	A	B	C	D
1	1	5	5	35
2	2	6	8	32
4	3	7	11	29
8	4	8	14	26
16	5	9	17	23
32	6	10	20	20
64	7	11	23	17
128	8	12	26	14
256	9	13	29	11
512	10	14	32	8
1024	11	15	35	5
	Log.	*Log.*	*Log.*	*Log.*

Figure 6.5: A sequence of proportional numbers set alongside four sequences of numbers that have equal differences (Wingate, 1635, p. 2)

In Figure 6.5, he showed four different sets of logarithms, and at this point, all of these were seen equally as logarithms since all four columns fit the definition, though with different starting points and differences: A started at 1 and had a difference of 1, B started at 5 and also had a difference of 1, C also started at 5 but had a difference of 3, and D is the opposite of C, started at 35 and had a difference of -3. When each of these sequences were aligned with the sequence of proportional numbers in the first column, then they were the logarithms of those numbers.

For the authors who used this definition, the idea that logarithms could be any sequence that had equal difference was an important idea to convey. Briggs tied logarithms to the power of 10 using an arithmetic sequence starting at 0 and increasing (and decreasing) by 1, but that was not the only option that existed. At this time, when logarithms were invented numbers (whether the authors label them as that or not) mathematicians and students were able to pick and choose the set of sequences that they related. ED started to disappear after the CP definition gained prominence, which was mainly just a different way of stating the same idea and had completely fallen out of favour by the time NR and E became more popular, so was not ever seen in concert with those definitions. CP had come in and built the bridge from the earlier work of ED to the later definitions.

6.3.4. Comparison of Progressions

The idea of logarithms as being a comparison between a geometric progression and an arithmetic progression was apparent in their conception, but it was primarily expressed as the ED definition until John Moore in 1681. Early presentations of the CP definition even used similar tables to those seen with the ED definition (Figure 6.6). And like the ED definition, any arithmetic progression, set aside a geometric progression would be a logarithm, though now the arithmetic progression was more likely to start at zero.

Definition. Logarithms are borrowed Numbers in Arithmetical Progreſſion, fitted or aſſigned to a rank of Numbers in Geometrical progreſſion. Therefore, any rank of Numbers being given in Geometrical progreſſion to them may be annexed for Logarithms, any rank of Numbers in Arithmetical progreſſion at pleaſure.

As in this Table, in the firſt column thereof, is a rank of numbers in Geometrical progreſſion from 1. Now to thoſe or to any other rank of proportionals may be adjoined for Logarithms either of thoſe ranks in the 2d. and 3 d.col. or any other rank of equi-different numbers, but thoſe are moſt commodious for uſe which have 0 aſſigned for the Logarithms of Unity.

Num.	Log.	Log.
1	0	0
2	1	3
4	2	6
8	3	9
16	4	12
32	5	15
64	6	18
128	7	21
256	8	24
512	9	27
&c.	&c.	&c.

Theſe Logarithms or Artificial numbers being thus contrived and aſſigned to proportional numbers and reſpectively ſubſtituted inſtead thereof; thoſe concluſions which in the proportional numbers are wrought by Multiplication and Diviſion, may be performed by the Addition and Subſtraction of their correſpondent Logarithms; and the extraction of roots in the proportionals may be ef-

N fected

Figure 6.6: CP definition of logarithms containing tables commonly seen in ED definitions (Forster, 1690, p. 177)

Once it was introduced, the CP definition became immensely common, appearing 28 times in texts overall, and it continued to be used throughout the end of this study with the latest appearance being Dodson's *The Anti-Logarithmic Canon* from 1742. Figure 6.7 shows that CP was used equally in conjunction with all the other definitions aside from ED, and we saw in sub-section 6.3.3 that ED could be thought of as just a slightly different way to discuss how the progressions relate to logarithms. CP was the bridge between the earlier ideas of logarithms, it was the main idea that drove Napier and Bürgi, and then built into the later conceptions.

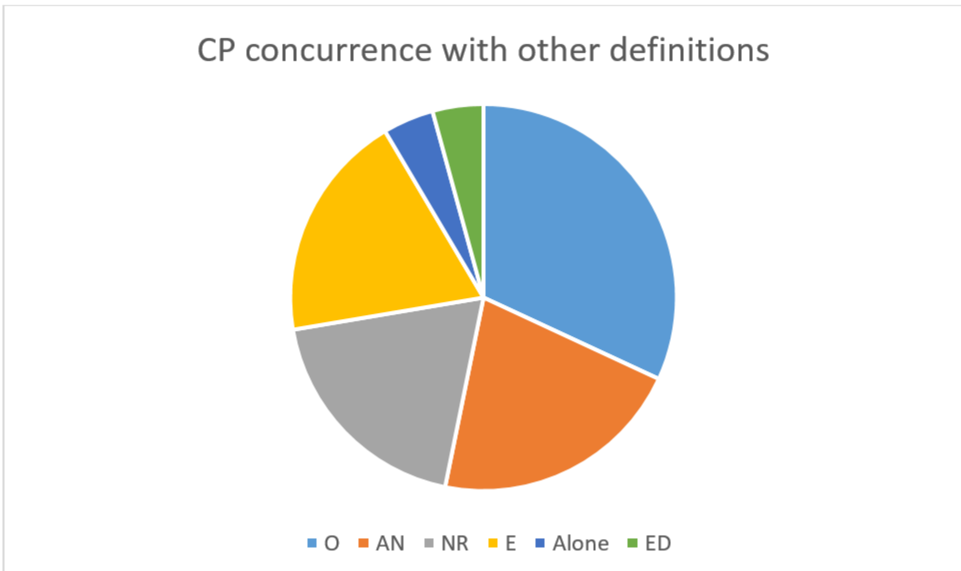
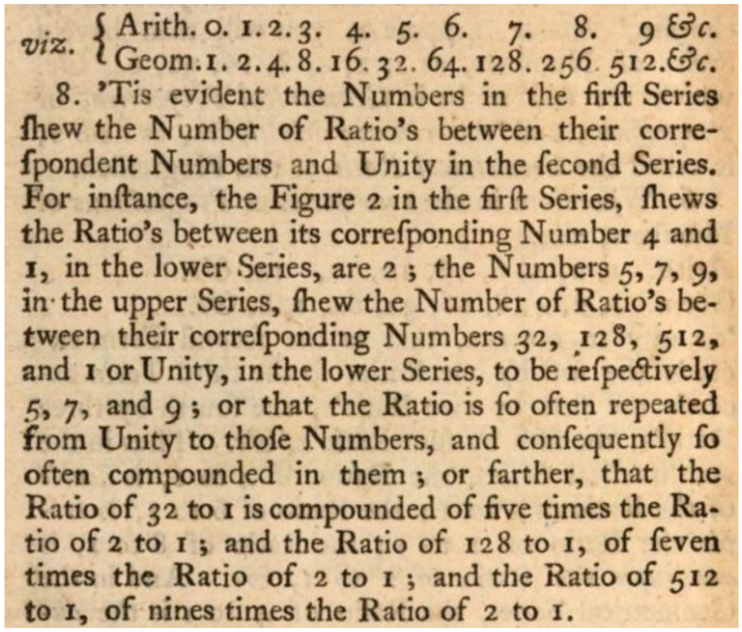


Figure 6.7: Concurrence of CP with other definitions of logarithms

Benjamin Martin's (1740) text is a good example of how CP leads to NR, so I feel it is worth exploring. He presumed that the reader is familiar with the idea of a logarithm in the sense of progressions, and attempted to redefine it as the number of ratios. In the first few pages he spent time defining 'ratio' and explored a few geometric sequences showing that, if a sequence began at 1 and the next term was *a,* then the following term would be a duplicate of the ratio *a*:1, the next a triplicate. He used 'duplicate' and 'triplicate' to show that they refer to multiplication.

Martin then placed an arithmetic progression beginning at 0 with a difference of 1 over a geometric progression beginning at 1 with a ratio of 2. Using these sequences Martin explored how the number in the arithmetic sequence could be defined as the number of ratios needed to get from 2:1 to the ratio of the corresponding number to 1 in the geometric sequence (Figure 6.8).



viz. { Arith. 0. 1. 2. 3. 4. 5. 6. 7. 8. 9 &c.
Geom. 1. 2. 4. 8. 16. 32. 64. 128. 256. 512. &c.

8. 'Tis evident the Numbers in the firſt Series ſhew the Number of Ratio's between their correſpondent Numbers and Unity in the ſecond Series. For inſtance, the Figure 2 in the firſt Series, ſhews the Ratio's between its correſponding Number 4 and 1, in the lower Series, are 2; the Numbers 5, 7, 9, in the upper Series, ſhew the Number of Ratio's between their correſponding Numbers 32, 128, 512, and 1 or Unity, in the lower Series, to be reſpectively 5, 7, and 9; or that the Ratio is ſo often repeated from Unity to thoſe Numbers, and conſequently ſo often compounded in them; or farther, that the Ratio of 32 to 1 is compounded of five times the Ratio of 2 to 1; and the Ratio of 128 to 1, of ſeven times the Ratio of 2 to 1; and the Ratio of 512 to 1, of nines times the Ratio of 2 to 1.

Figure 6.8: Logarithms going from progressions to number of ratios (Martin, 1740, p. 4)

Logarithms as a comparison between these two progressions has been built in from their invention, so it is maybe not surprising that it is such a huge part of the next steps of this concept. Logarithms as a number of ratios, and logarithms as an exponent come directly from this idea and are explored in further detail in the next two sections.

6.3.5. Number of Ratios

The NR definition of a logarithm, that the logarithm is the number of places in a geometric progression, was explored a few ways in the 13 texts in which it appears. The logarithm could be the rank of each term in a geometric progression (4 out of 13); it could be the distance from 1 to the term in the geometric progression (4 out of 13); or it could be the number of ratios between the term and unity (5 out of 13). While these three definitions are slightly different, I have combined them in this category as they were presented in very similar ways and all have the aspect where the arithmetic sequence could no longer be random as it could in previous definitions, but is now tied to the position of the terms in the geometric sequence. They also follow each other in chronological order. Logarithm being the place of the number in a geometric progression started with John Wallis in 1685 and was mainly found before the 1700s. The other two definitions quickly followed with the distance definition being more popular first and then dying out in favour of the definition pointing to the number of ratios.

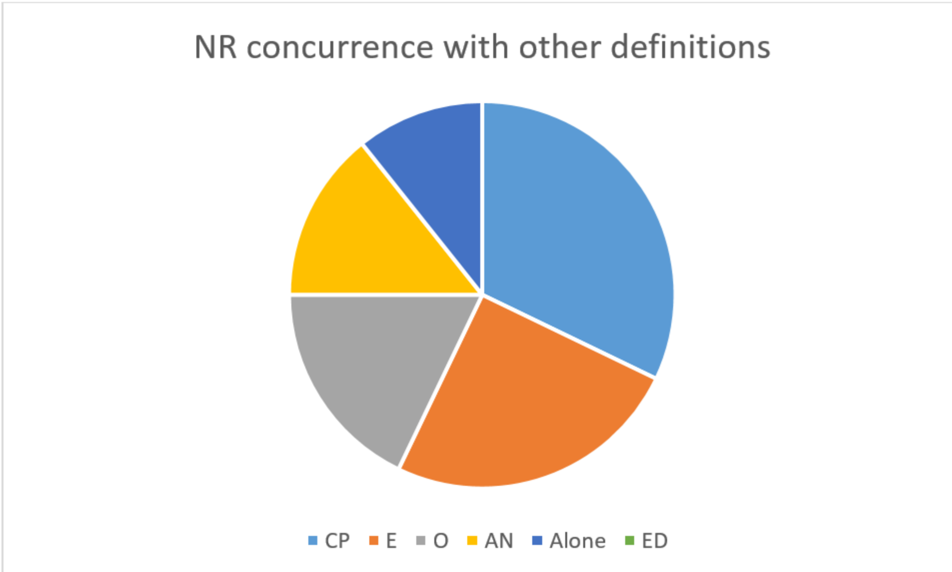


Figure 6.9: Concurrence of NR with other definitions of logarithms

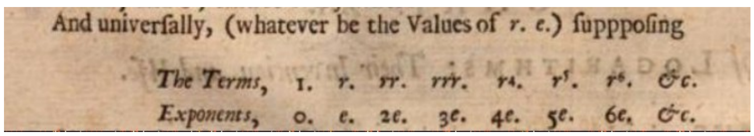
As seen in Figure 6.9, the NR was almost equally combined with CP and E, while ED had dropped off completely. The chronology above could perhaps justify the coming transformation from logarithms being related to progressions to logarithms being related to exponents as the 'number of ratios' speaks directly to the idea of an exponent and is explored in sub-section 6.3.6 through the work by Saunderson (1740).

As logarithms started to be related to just a single geometric progression, whether they were the ordinal number in the progression, the distance from 1, or the number of ratios, they were well on their way to being associated with exponents. By this time, exponents as a word had started to take root, thanks in a large part to John Wallis (1685) and exponents as a notation had a firm hold. Once the connection was made, the mathematics world would not turn back.

6.3.6. Exponents

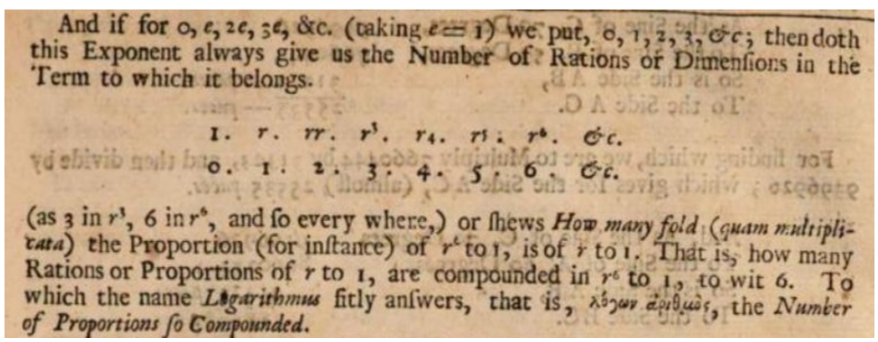
While logarithms were not fully defined through exponential equations until 1742 with Dodson and Gardner, they were associated with exponents before that. In this study, there were 11 texts that included some idea of exponents with the definition of logarithms. John Wallis (1685) was the first to call logarithms exponents, but at that time he meant that the arithmetic sequence was called exponents, an idea that was continued by other authors (Jeake, 1696; Ozanam, 1712).

Wallis does go a bit further than the others though as he also defines logarithms as the number of ratios (NR) and, in doing so, relates the exponent symbol with the logarithm (Figure 6.10). Other authors continued this idea, using exponent notation in their discussion of a logarithm as the distance from one (Keill, 1723), or the number of ratios (Jones, 1706; Martin 1740).



And univerſally, (whatever be the Values of *r. e.*) ſuppofing

The Terms,	1.	*r.*	*rr.*	*rrr.*	r^4.	r^5.	r^6.	&c.
Exponents,	0.	*e.*	2*e.*	3*e.*	4*e.*	5*e.*	6*e.*	&c.



And if for 0, *e*, 2*e*, 3*e*, &c. (taking *e* = 1) we put, 0, 1, 2, 3, &c; then doth this Exponent always give us the Number of Rations or Dimenſions in the Term to which it belongs.

1.	*r.*	*rr.*	r^3.	r^4.	r^5:	r^6.	&c.
0.	1.	2.	3.	4.	5.	6.	&c.

(as 3 in r^3, 6 in r^6, and ſo every where,) or ſhews *How many fold* (*quam multiplicata*) the Proportion (for inſtance) of r^6 to 1, is of *r* to 1. That is, how many Rations or Proportions of *r* to 1, are compounded in r^6 to 1, to wit 6. To which the name *Logarithmus* fitly anſwers, that is, λόγων ἀριθμός, the *Number of Proportions ſo Compounded.*

Figure 6.10: Logarithms as number of ratios using exponent symbols (Wallis, 1685, pp. 56-57)

The 1740s were when logarithms as exponents begin to take precedence. Benjamin Martin (1740) defined logarithms two ways in one chapter, through their progressions and as the number of ratios. Then, in a separate chapter, he set up a geometric progression starting with 1 and with a common ratio of $a{:}\,1$, using exponent notation and then states “observe, that the *Exponents* of the *Powers* of the Terms in the Series $1, a^1, a^2, a^3, a^4, a^5, a^6, a^7,$ &c. are the *Logarithms* of those Terms respectively” (Martin, 1740, p. 18). While this definition was still associated with the relationship between the progressions, and with the idea of logarithms as the number of ratios, it was setting a new way of expressing logarithms.

Nicholas Saunderson (1740) went a bit further. After defining logarithms through their operations, and as artificial numbers, he defined them as the number of ratios as discussed in sub-section 6.3.5. He then began to explore Briggs' logarithms, logarithms with a base of 10, explaining "*in the system here described, every natural number is, or may be considered as some power of 10, and its logarithm as the index of that power*: for let *a* be the logarithm of any natural number as *A*; then since Briggs' logarithm of 10 is 1, his logarithm of 10^a will be *a*; [...] therefore *A* must be equal to 10^a, since they have both the same logarithm; that is, the natural number *A* is such a power of 10 as is expressed by it's logarithm *a*" (Saunderson, 1740, pp. 622–623).

Saunderson was harking back to Napier's and Bürgi's initial ideas for logarithms, that 'every' number could be represented in a geometric sequence. Napier and Bürgi succeeded by choosing a ratio very close to one, Saunderson instead was looking at the most popular version of logarithms of his day, base 10, and stating that every number was equal to some power of 10. While not writing it with equation notation, he was basically saying that $A = 10^a \Rightarrow \log A = a$.

In 1742, Gardner and Dodson solely defined logarithms through exponents. Dodson's definition was very close to Saunderson's while Gardner's defined the logarithm through an exponential equation (Figure 4.26). Until these two authors, exponents were just another way to define logarithms, in concurrence with other definitions, but never as the primary definition.

6.4. Analysis of the Properties of Logarithms

The properties of logarithms were derived, or justified, in multiple ways, often depending on what definition was primary in that text. As many of the texts used multiple definitions, some texts tried to combine the definitions in their derivations of the properties of logarithms.

6.4.1. Through Operations Definition

For those texts, whose primary definition was based around Operations (O), the properties of logarithms were generally easily derived as it came from the definition. Logarithms were numbers that were invented to ease calculations by making multiplication addition and making division subtraction. Often, though, the power property of a logarithm was not explained, just shown in examples. As the power property was not often expressly stated as part of the definition, it is perhaps not surprising that it just appeared. The Operational definition itself justified the Product and Quotient properties, but the Power property would have to be derived, and that was maybe too advanced mathematics for those texts that focused on the O definition.

6.4.2. Through the ED Definition

Those that defined logarithms through the ED definition, tended to derive the properties by looking at a set of four numbers in proportion in the geometric sequence and comparing the extremes to the numbers in the middle. They would then relate this to the corresponding numbers in the arithmetic sequence. Wingate's (1635) text discussed in sub-section 6.3.3 is a good example of this process.

First Wingate noted a known fact about a sequence of four number that are in proportion, the product of the extremes will be equal to the product of the middle. Using algebraic notation, if $a{:}\,b :: c{:}\,d \Rightarrow \frac{a}{b} = \frac{c}{d} \Rightarrow a \times d = b \times c$.

Then Wingate noted the relation to his sequences of logarithms, "when of foure numbers given, the second exceedes the first as much as the fourth exceedes the third; the summe of the first and the fourth is equall to the summe of the second and the third" (Wingate, 1635, pp. 2–3). To use more contemporary algebraic notation, there is an arithmetic sequence with a common difference of g, the first number then would be la (for $\log a$), the second $la + g$, the third lc, and the fourth $lc + g$. If we apply these values to Wingate's words then it is quickly seen he is correct, $la + lc + g = la + g + lc$. Given that $la + g$ would be the second term, it could be represented by lb, and same for the last term $lc + g$, it could be ld.

	Extremes	Middle
Product of Extremes = Product of Middle	$a \times d$	$b \times c$
Sum of Extremes = Sum of Middle	$la + ld$	$lb + lc$

In this way, Wingate had justified the addition property and continued by using the table in Figure 6.5 to provide some examples. Starting with the Proportional

Numbers (the left column) of 1, 4, 8, and 32, he set up the proportion and subsequent products of $1\!:\!4 :: 8\!:\!32 \Rightarrow 1 \times 32 = 4 \times 8$. Given that the products of the extremes were equal, the addition of the related logarithms in all four tables would also be equal.

- $1\&3\!: 4\&6 \Rightarrow 1 + 6 = 3 + 4$
- $5\&7\!: 8\&10 \Rightarrow 5 + 10 = 7 + 8$
- $5\&11\!: 14\&20 \Rightarrow 5 + 20 = 11 + 14$
- $35\&29\!: 26\&20 \Rightarrow 35 + 20 = 29 + 26$

After presenting this idea, he next visited how Briggs set up his tables so the proportional numbers would be based around a common ratio of 10 and that the logarithmic numbers would start at 0 (Figure 6.11). He noted that due to this particular setup, as long as the set of proportions started at 1, then the corresponding logarithm would start at 0. Given that, the addition of the extremes equalling the addition of the middle would really just be the addition of the middle equalling the far end extreme; this would correspond to the logarithms of the two things being multiplied adding to equal the logarithm of the product.

I

A	B
1	0,000000
10	1,000000
100	2,000000
1000	3,000000
10000	4,000000
100000	5,000000
1000000	6,000000
10000000	7,000000

&c.

II

C	D
1	0,000000
2	0,301029
4	0,602059
8	0,903089
16	1,204119
32	1,505149
64	1,806179
128	2,107209
256	2,408239
512	2,709269
1024	3,010299

&c.

Figure 6.11: A set of two tables showing Briggs' logarithms. (Wingate, 1635, p. 5)

As an example, he worked through the proportion, $1\!:\!16 :: 4\!:\!64 \Rightarrow 1 \times 64 = 16 \times 4$ which would correspond to the logarithms: $0, 1.204119, 0.602059,$ and $1{,}806179 \Rightarrow 1{,}204119 + 0602059 = 0 + 1{,}806179$ (he does note that the sum is off by a unit, but stated that “the want of an unit or two in the last figure of the Logarithme either in this or any other operation whatsoever begetteth no error in the worke (Wingate, 1635, pp. 9–10). He used the same idea with examples from the tables to explore the division property of logarithms including generalising the property. But for the power rules he relied just on the examples from the table without trying to explain why it would be true in a general sense.

In seven of the twelve texts that used the ED definition for logarithms, the justification for the multiplication and division properties followed these ideas presented by Wingate. Two of those texts worked to also justify the power property tying them into the multiplication and division properties. Richard Norwood (1656) is a good example as he put Corollaries at the end of his discussion on the properties stating that if multiplying numbers meant adding the corresponding logarithms, then doubling the logarithm would mean squaring the associated number and opposite for the square root. He then went through a few examples with his given tables. The other five texts had no justification for the properties of logarithms, just stated them as fact and then moved on to examples.

6.4.3. Through the CP Definition

Many of the texts that used the CP definition derived or justified the properties of logarithms through one of the ways already discussed as CP was often used in concordance with one of the previous definitions. But the slim majority (15 out of 28) of the texts drew the multiplication and division properties directly from the comparison of the progressions.

Continuing from Ward’s (1695) quotation in sub-section 6.3.2, two definitions were expressed: logarithms are artificial numbers to assist in calculations and that logarithms are found in the relationship between two progressions. Ward continued this by showing two examples of geometric and arithmetic progressions and how that related to the common properties of logarithms (Figure 6.12). While Ward did not generalise the operations, that had already been done in 1685 by Wallis.

Viz. $\begin{cases} 1 . 2 . 4 . 8 . 16 . 32 . 64 . 128 \text{ \&c. Geometrical} \\ 0 . 1 . 2 . 3 . 4 . 5 . 6 . 7 \text{ \&c. Arithmetical} \end{cases}$

It is very perceptible, That as the Numbers in the Geometrical Progreſſion are produced by *Multiplication* or *Diviſion*, thoſe in the Arithmetical Progreſſion are produced by *Addition*, or *Subſtraction*; as doth appear by this Example.

Viz. $\begin{cases} 4 \times 32 = 128 \\ 2 + 5 = 7 \end{cases}$ or $\begin{cases} 128 \div 32 = 4 \text{ Geometr.} \\ 7 - 5 = 2 \text{ Arithmet.} \end{cases}$

Again, $\begin{cases} 1 . 10 . 100 . 1000 . 10000 . 100000 \text{ \&c. Geometric.} \\ 0 . 1 . 2 . 3 . 4 . 5 \text{ \&c. Arithmetic.} \end{cases}$

The ſame Agreement is betwixt theſe latter, as was between the Two Firſt Ranks.

Viz. $\begin{cases} 1000 \times 10 = 10000 \\ 3 + 1 = 4 \end{cases}$ or $\begin{cases} 100000 \div 1000 = 100 \text{ Geom.} \\ 5 - 3 = 2 \text{ Arith.} \end{cases}$

Either of theſe Examples do ſufficiently ſhew the Reaſon, and very Ground of Logarithms.

Figure 6.12: Exploring the properties of logarithms from a comparison between two progressions (Ward, 1695, p. 91)

Of the remaining texts that include CP among their definitions, seven out of 28 write out two proportions and then look at the extremes vs middle terms as discussed in sub-section 6.4.2. Five texts did not give any reasoning, just presented a definition for logarithms and then stated the properties without tying the two together.

The remaining text used the properties of exponents in their justification, though still strongly tied to the idea of the progressions. This is discussed in sub-section 6.4.6.

6.4.4. Through the NR Definition

How the authors defined logarithms under the NR umbrella influenced how they presented the properties of logarithms. Those who stated that logarithms were “the number of places of such Geometrical Progression” (Wallis, 1685, p. 16) and those that defined logarithms as being “the distance of the proportional Numbers from Unity” (Wolfius, 1739, p. 28) basically replaced the arithmetic progression which was able to fluctuate with one that was static, but treated logarithms still as the relationship between the two progressions which included how the properties were justified.

The only definition in this grouping that led to new ways of presenting the properties was the idea that “logarithm will be the numeral exponent of the ratio of its natural number to unity” (Saunderson, 1740, p. 619). Saunderson first introduced logarithms through their operational definition, and explored them through progressions, but then he started a new section with the above definition. While Saunderson officially justified the properties in the previous sections, his work in this section created a new way of thinking about logarithms and their properties.

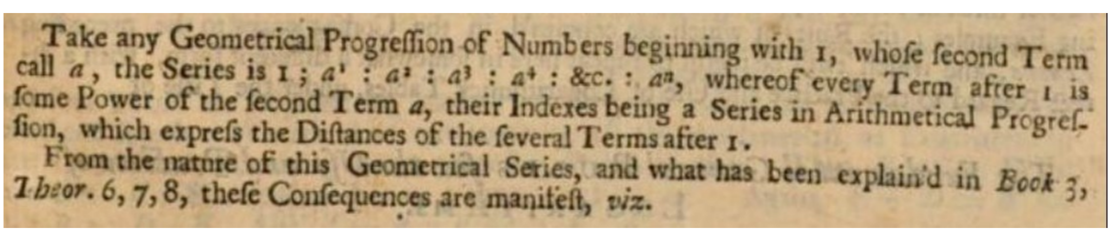
Saunderson began by giving an example of a ratio 81:1 and stated that it can be broken into four ratios: 81:27, 27:9, 9:3, 3:1. This breakdown of 81:1 referred to a section earlier in the text where he notes that “in a series of quantities of any kind whatsoever increasing or decreasing from the first to the last, the ratio of the extremes is said to be compounded of all the intermediate ratios” (Saunderson, 1740, p. 469). The numbers in his ratios consisted of a geometric sequence (series in his text), 1, 3, 9, 27, 81, so more naturally fit the idea of the relationship between sequences and logarithms, perhaps even implying a base of three, though any number could be the base. He noted that all these ratios are the same as 3:1, so “the ratio of 81 to 1 is said to be four times as big as the ratio of 3 to 1 […], and hence it is that the logarithm of 81 is four times as big as the logarithm of 3” (Saunderson, 1740, p. 619). While he did not go further into this idea, a reader could affirm that the power of a ratio (such as four times as big) is equivalent to multiplication in logarithms (as times in ratio would imply raising to a power, and times for numbers implies multiplication).

Saunderson followed this with a different example, 24:1, which he broke down to 24:12, 12:4, 4:1. Again following his text from page 469 in the last paragraph, the sequence would be 1, 4, 12, 24, which was not an obvious geometric sequence and as such did not immediately imply logarithms and building from one term to the next lacked a clear connection. Saunderson then noted that these ratios are equivalent to 2:1, 3:1, and 4:1 and stated as “the ratio of 24 to 1 is equal to the ratios of 2 to 1, 3 to 1, and 4 to 1 put together; and hence it is that the logarithm of 24 is equal to the logarithms of 2, 3, and 4 put together” (Saunderson, 1740, p. 619). He did generalise this outcome a bit but was reliant on the reader understanding that ‘put together’ for ratios implied multiplication while for numbers it would imply addition. He ends with saying he could have presented his entire section on logarithms just using the NR definition but to do so the reader would have to have a strong understanding on the mathematics of ratios, and as he did not want to assume, he had also presented them the other ways. In 1745, Francis Holliday also worked through logarithms in this manner, but in a book based around calculus so he could more likely expect the readers to have this firm understanding.

6.4.5. Through the E Definition

Only one text derived the properties of logarithms through exponents, though first Alexander Malcolm (1730), in Book III of his *New System of Arithmetick, Theorical and Practical,* derived some of the properties of exponents. To explain the multiplication property of exponents, he explained, “$A^n \text{ and } A^m$ being each a Product of A continually multiplied by itself, their Product must be a Product of A continually by itself; *i.e.* a Power of A. Also it’s plain, that in $A^n \times A^m$, the Root *A* is applied as a Factor as oft as the Sum $n + m$, so that $A^n \times A^m = A^{n+m}$ (Malcolm, 1730, pp. 145–146). He called this ‘Theorem 6’ and continued with ‘Theorems 7 & 8’ the division of common bases and raising the power to an exponent using the same ideas.

In Book V, Malcolm introduced logarithms. Using the CP definition of arithmetic progression relating to geometric progression, he presented a geometric progression starting at 1 and increasing by a common ratio *a*. Referring back to the theorems established in Book III, Malcolm justified the properties of logarithms (Figure 6.13).


Take any Geometrical Progreſſion of Numbers beginning with 1, whoſe ſecond Term call a, the Series is 1 ; a^1 : a^2 : a^3 : a^4 : &c. : a^n, whereof every Term after 1 is ſome Power of the ſecond Term a, their Indexes being a Series in Arithmetical Progreſſion, which expreſs the Diſtances of the ſeveral Terms after 1.

From the nature of this Geometrical Series, and what has been explain'd in *Book* 3, *Theor.* 6, 7, 8, theſe Conſequences are manifeſt, *viz.*

Figure 6.13: A definition of a logarithm that ties together progressions and exponents (Malcolm, 1730, p. 485)

Christian Wolfius (1739) does almost the exact opposite; he first worked with logarithms as the correlation of the progressions and justified the properties of logarithms through those means. Later, he built to the idea of exponents. Once he had an example similar to Figure 6.13 where there is a geometric progression and the exponents were seen to be in an arithmetic progression, he used the properties of logarithms as justification for the properties of exponents. Interestingly, he did not actually tie together logarithms and exponents, his definition was a combination of CP and NR.

6.5. Analysis of Solving Exponential Equations

Logarithms were often used to aid in solving equations, most often in their capacity to ease calculations. For almost the first hundred years, though, of their existence, they were not overly present in solving exponential equations. In this analysis, I focused on problems related to compound interest, as that is an exponential equation that is still in use today, but did expand some to look at other comparable equations.

6.5.1. Through the O, ED & CP Definitions

When logarithms are defined primarily through the operations that they ease, solving problems with logarithms is not readily apparent. John Ward (1710), in his finance text, provides a common example of applying logarithms to a compound interest equation. He first writes out the equation and defines the variables (Figure 6.14, top), but, since logarithms are just a way to simplify calculations, the word does not exist to show how they could be used to solve this equation. Instead, he writes out the calculations needed to be completed to calculate each of the four variables as separate rules (Figure 6.14, bottom).

Viz. $\begin{cases} P. & \text{Denotes any Principal put to Intereſt.} \\ t. & \text{The Time of its Continuance at Intereſt.} \\ A. & \text{The Amount or Principal and its Intereſt.} \end{cases}$

And *R.* Denotes the Amount of 1 *l.* for One Year. That is, the Ratio of the given Rate more 1.

Then it will be, $PR^t = A.$

Caſe 4. Having *P*, *R*, and *A*, given; To find *t*.

Rule. *From the Log. of the propoſed Amount, Subſtract the Log. of the propoſed Principal* (as before) *and Divide their Difference by the Log. of R the Amount of* 1 l. *at the given Rate, and the Quotient will ſhew the Time required.*

Figure 6.14: Solving compound interest problem using logarithims when they are defined strictly through their operations (Ward, 1710, p. 53 (top) & p. 55 (bottom))

Defining logarithms through their operations may have been enough for those readers who would just be using logarithms to simplify calculations that were already presented. But, as the method in Figure 6.14 was in solving problems was common (Hodgson, 1723; Leadbetter, 1739; Ronayne, 1717; Ward, 1695), defining logarithms through their operations did not seem to give the reader the tool to use or view logarithms in any other manner. This limitation is most likely the reason it was so often presented alongside other definitions. Though combining the Operation definition with the ED or the CP definitions would not have changed the way that exponential equations were solved as all but one text who defined logarithms primarily through the ED or CP definitions solved these problems similar to Figure 6.14. There are a few moments in these texts though that are worth exploring.

John Ward (1695) solved equations in the same manner presented in Figure 6.14, except he actually showed the work to get his equation and he used repeated division to arrive at a solution (Figure 6.15).

Queſt. III. *Suppoſe* 250 l. *hath been at Intereſt, and the Amount is* 375 l. 18 *s.* 2 *d. at* 6 per Cent. *Compound Intereſt, How long hath it been foreborn.*

Here is given $P = 250$ $z = 375{,}9075$ And $a = 1{,}06$ for one Year. Thence to find $t =$ the Index Power of a.

General Theorem $Pa^t = z$. Therefore $z \div p = a^t$

Conſequently if a^t be continually Divided by a, until it become $a \div a = 1$ the Number of ſuch Diviſions will be $= t$. *For ſuch Number of Diviſions diſcovers how oft* (a) was involved.

But $375{,}9075 \div 250 = 1{,}50363 = a^t$.

And $1{,}50363 \div 1{,}06 = 1{,}418518$

Again, $1{,}418518 \div 1{,}06 = 1{,}338225$

And ſo on, until it become $1{,}06 \div 1{,}06$ which will be at the Seventh Operation.

Then will $t = 7$ the Number of Years required.

Figure 6.15: A demonstration of solving an exponential equation with repeated division (Ward, 1695, p. 108)

While Ward did present the thought process and calculations involved in solving this compound interest problem, he did not tie it to logarithms. He suggested that the reader use logarithms to ease the calculations, so a clever student could maybe make the connection to solving this problem using logarithms. The student would already have found the logarithm of 1.50363 from repeatedly subtracting the logarithm of 1.06 until reaching 0; so perhaps the student would have connected the repeated subtraction with division. Even if this process was not explicitly related to logarithms, this solution did touch on another idea of solving an exponential equation.

John Collins (1685) used logarithms in his finance text to convert yearly interest to months and then to days. He began by presenting compound interest as a sequence of proportions. If there was a principle of \$1, at a rate of 6%, then the amount after a year would be \$1.06. If one was looking for the amount after 2 years (x), or 3 years (y), then proportions could be set up comparing each subsequent year. Using more modern notation:

$$1{:}\,1.06 :: 1.06{:}\,x \Rightarrow x = 1.06^2$$
$$1.06{:}\,1.06^2 :: 1.06^2{:}\,y \Rightarrow y = 1.06^3$$

Collins then directed readers to the compound interest tables that were common at this time where the amount could be found for many different years. In order to find the amount in months or days, he instructed the reader on the use of a mean proportional. To determine how much compound interest goes into one month, one would start with the idea that there is some number, multiplied by itself 12 times, to get to the 1.06 (the amount after a year). Today, this would be solved by taking the twelfth root, and Collins was also doing that operation, but through the lens of finding the mean proportional. Mean proportional numbers were discussed in more detail in Section 4.5, but to explain them in the way that Collins was using them, they are the geometric mean or multiplicative halfway point between two numbers; to find the mean proportional between 2 and 5, the two numbers are multiplied together and the square root is taken of the product: $\sqrt{10}$.

For the instance above, as Collins was looking for the interest monthly between 0 months and 12 months, the corresponding amounts would be 1 and 1.06. These values could be viewed as the endpoints of arithmetic and geometric progressions, the arithmetic one starting at 0 and ending at 12, and the geometric on starting at 1 and ending at 1.06. The first mean proportional would be $\sqrt{1 \times 1.06} = 1.02956$ which would correspond to the average of the months: $\frac{0+12}{2} = 6$; so, at 6 months, the amount was the number above. Collins continued this pattern, finding the mean proportional again (1.01467) corresponding it with 3 months, and again (1.00731) which would be 1.5 months. The arithmetic sequence then would be: 0, 1.5, 3, 6, 12, while the geometric sequence would correspond with: 1, 1.00731, 1.01467, 1.02956, 1.06. More of each progression could be filled in if one chose to, and Collins suggested that his readers use their knowledge of logarithms to simplify matters. While he did not specify exactly what he meant, we can guess that he meant to use logarithms to simplify the repeated square roots, or perhaps even to use the logarithms to take the twelfth root of 1.06, saving the intermediate steps.

None of these methods specifically used logarithms to solved exponential equations, they mainly used logarithms to simplify the calculations. But by presenting different methods that exponential equations could be solved, and different ways to use logarithms in that process, an astute reader could have expanded their understanding of exponents and logarithms.

6.5.2. Through the NR & E Definitions

The authors that presented logarithms using the ‘number of ratios’ definition, started solving exponential equations by taking the logarithm of both sides. Charles Hayes (1704), in his calculus text, introduced logarithms through two curves, the hyperbolic and the exponential (which he called ‘logarithmic’, similar to Keill as discussed in Section 4.5). As this was a book of calculus, he stated that the logarithm of a number was the sum of the ratio of the derivative (fluxion) at that number and of the number itself; basically a number of ratios. He also justified taking the logarithm of both sides of an equation in a general sense though this justification came from the operations, not any definition (Figure 4.24 repeated here as Figure 6.16).

401. For the Logarithm of the Square, Cube or Biquadrate, *&c.* of any Number, is equal to twice, thrice or fourtimes, *&c.* the Logarithm of the Root. Therefore Univerſally, the Logarithm of x^v is $= v \times l x$; but $x^v = y$, therefore the Logarithm of x^v is equal to the Logarithm of y, that is $v \times l x = l y$.

Figure 6.16: Solving an exponential equation directly through logarithms (Hayes, 1704, p. 307)

Two years later, William Jones (1706) referenced Edmund Halley’s method of creating logarithms (Section 4.6) when writing his *Synopsis Palmariorum Matheseos,* therefore his definition also simplified logarithms to a ‘number of ratios’. He also presented the idea of taking the logarithm of both sides (Figure 6.17), and similar to Hayes, he justified it using the properties of logarithms as did Christian Wolfius in his 1739 text.

2. The *Logarithms* of the *Powers* of any *Number* are obtain’d by *Multiplying the Logarithm of that Number by the Index of the Power*; for theſe *Indices* are *Proportional* to thoſe *Logarithms*: Thus the *Logarithms* of 4, 8, 16, 32, 64, &c. are found by *Multiplying* the *Logarithm* of the *Root* 2, by the *Indices* 2, 3, 4, 5, 6, &c. of thoſe *Powers* reſpectively. Theref. if $x^n = a$, then n L., x = L, a.

Figure 6.17: Jones’ justification for taking a logarithm of both sides. (Jones, 1706, p. 186)

Solving exponential equations using logarithms was established before logarithms were strictly defined as an exponent, which perhaps was a catalyst for the redefinition of a logarithm in terms of the exponent. While none of the texts I reviewed that defined logarithms as an exponent included solving exponential equations, it seems likely that once logarithms were redefined in relation to exponential equations, those ways of solving would come to the forefront.

6.6. Last Notes on the Analysis of Past Textbooks

Logarithms were heralded since their invention and immediately put to use in a variety of fields, which could be seen in the variety of application based texts that appear in this study. While a third of these texts did not delve into the concept of logarithms, focusing instead on how to use them to simplify calculations, the remaining two-thirds of the authors did feel that some explanation of the concept was important for their readers. Most of these application-based texts still tied their definition to the operations, while the more mathematical texts tended toward a more even split among the definitions.

The definitions of logarithms seemed to follow a path, they started with the description of them as artificial numbers and defined them related either to their operations or as numbers with an equal difference attached to numbers in proportion. While the idea of logarithms as artificial numbers, and being defined through their operations, continued being used through the time span reviewed (1614–1750), the equal difference definition gave way to a definition built around an arithmetic progression in relation to a geometric progression. This definition based around progressions stood the test of time and is still used today as an alternative definition of logarithms as seen in Chapter 5. Eventually, the progression definition evolved to one that counted the number of ratios, which led to logarithms being defined as an exponent. In going through the historic texts, there is an order to when these definitions first appeared, though with the exception of the equally differing definition, once introduced, they all continued to appear during this span.

Solving exponential equations also had a pathway, in the earlier texts they were solved by using proportions, later each variable would be solved so they could be evaluated by substitution, and even later the authors would take the logarithms of both sides. There were some interesting asides though, which could expand the concept of an exponent or a logarithm, as authors explored how to use logarithms to simplify the process of solving the equation.

The properties of logarithms did not fall into a chronological timeline. For the most part, authors either used the definition, if the definition was based around the operations, to justify the properties of logarithms, or they used examples from the two progressions to explain them. Rarely were other ways presented, though a few authors leaned into their chosen definition and explored a different way to derive the properties of logarithms. Most noteworthy were those that were working with the NR definition and those that worked with the logarithm as an exponent.

Given the major changes that happened in this short span (1614–1750), it is surprising that the definition of a logarithm being used in modern texts is recognisable as one used in the 1740s. The way the authors after the 1700s solved exponential equations is also recognisable to the modern reader. Most of the other definitions discussed in this chapter would be unfamiliar to the present classroom, as well as the derivations of the properties of logarithms, and the other ways of solving exponential equations. As most of these presentations of logarithms do not connect to exponents and exponential functions, using any of the ideas presented would create new pathways to logarithms for today's students. As shown in Chapter 8, there is are many ideas from these texts that can be pulled into present-day textbooks.

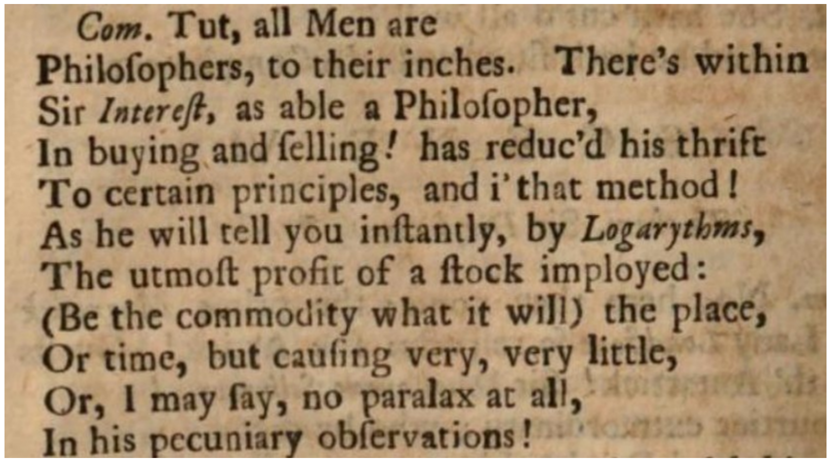
Chapter 7. A Rithmos by any other Logos

The term ‘logarithm’ is one that vexes students and teachers today (Hoon et al., 2010; Kenney & Kastberg, 2013; Liang & Wood, 2005; Wood, 2005). Some have even suggested changing the name of the function (Brennan, 2007; Hammack & Lyons, 1995; Hurwitz, 1999). I can understand that students, having trouble with a somewhat abstract concept, also struggle with an unusual name the concept. I do not believe the term ‘logarithm’ needs to be changed, the struggle is with the concept, whatever the name chosen for it, but I do think that giving a reason for the naming of logarithms could help students find a hold when trying to grasp onto this new mathematical idea. In order for the name to have meaning to students though, it would need to connect to an idea of logarithms today. While the name ‘logarithm’ is a very small issue in the research around teaching logarithms, it was one that drew my interest as I delved deeper into their history and realised that students of the past were getting translations of the word that then tied into the concept, something that today’s students are missing.

It may be of interest to today’s teachers and students that the word ‘logarithm’ did not obviously imply the operations or actions associated with logarithms, even at the time of their creation. This chapter looks at the etymology of *logarithm* and its relationship to the mathematical concept ‘logarithm’ throughout the first 250 years of its history. In doing so, this chapter attempts to respond to the third set of questions in Section 1.4: What is the etymology of the term *logarithm*? How did that name interact with the meaning and use of logarithms? How can that information be useful to students and teachers today?

7.1. The Historic Commonality of the Term ‘logarithm’

Ever since they were introduced in 1614 through John Napier’s book, *Mirifici Logarithmorum Canonis Descriptio*, logarithms have entranced both the mathematics, and the non-mathematics world. While mathematicians and astronomers were quick to embrace them (J. Napier/M. Napier, 1839), they were almost as quick to enter the public consciousness. A few years after the idea was first published logarithms had entered the world of the stage, with Ben Johnson including them (Figure 7.1) in poetic stanzas first shown in 1632 (Johnson, 2000).



Com. Tut, all Men are
Philoſophers, to their inches. There's within
Sir *Intereſt*, as able a Philoſopher,
In buying and ſelling! has reduc'd his thrift
To certain principles, and i'that method!
As he will tell you inſtantly, by *Logarythms*,
The utmoſt profit of a ſtock imployed:
(Be the commodity what it will) the place,
Or time, but cauſing very, very little,
Or, I may ſay, no paralax at all,
In his pecuniary obſervations!

Figure 7.1: Logarithms in a play (Johnson, 1716, p. 382)

Six years later, in 1638, Sir Thomas Herbert included them as one of his reasons for the superiority of British navigation in comparison to Chinese navigation in a book describing his travels, though by 1653 a treatise of logarithms had been published in China (Maor, 1994). And, by the 1800s, logarithms were such a ubiquitous part of the world, that they could be used as a metaphor without much explanation, as Häusser (1868/1873) does in a text discussing the religious strife around Charles V, the Holy Roman Emperor. Häusser used logarithms as a metaphor, writing:

> It was in this that the fundamental error of Charles's policy lay with regard to the great question of the age. He made his calculations in a wonderful manner; in the long labour of a lifetime he cast up everything figure by figure; but one thing he could not discover, the logarithm for the religious commotions of his time. (p. 40)

As explored in Chapter 4, logarithms were discovered to simplify complicated calculations with large numbers and, since Charles V could not discover that process for this religious strife, he was stuck trying other, harder and more laborious options.

7.2. The Portmanteau of *Logos* and *Arithmos*

For how common and well-known logarithms were, there has not always been agreement on the meaning of the term. In the very first writings around logarithms, they did not actually have a name. Napier's initial text, *Mirifici Logarithmorum Canonis Constructio* (1619) introduced them as 'artificial numbers' (in latin: *numeris artificialis*). It was not until his second book, *Mirifici Logarithmorum Canonis Descriptio* (1614) which was published first, that he had decided to call them 'logarithms' (Napier, 1839). He did not give a reason for choosing this term, though common thought took it as a portmanteau of the Greek words *logos* and *arithmos* (λογος and αριθμος) and then translated them as 'ratio' and 'numbers.' (Havil, 2014; Panagiotou, 2011; Pierce, 1977; Weber, 2016).

The scholars who provide this portmanteau do not always have the same reasoning. Havil (2014) said, "they are abstract numbers which have been contrived to assist with the manipulation of ratios of real-world numbers" (p.67). Pierce (1977) claimed they were named that way as, "Napier constructed the entire Table of Radicals by multiplication and subtraction, using ratios" (p.26). Panagiotou (2011) decided that, "The term logarithm means precisely: the number that measures the ratios (the Greek logos). [to show] 'how many ratios (logos)' are required in the continuous proportion" (p. 6). And Weber (2016) went a completely different way by stating that Napier meant them to count the number of divisions. It seems that, even with the same starting translations, applying those words to the idea of logarithms offers up different ideas.

7.3. The Problem with 'Logos'

A significant reason that there is such confusion over the name is the Greek word *logos*. While *arithmos* is consistently translated into 'number', starting from the time of Homer (Lidell, Scott & Jones, 1940), *logos* has many translations. The Lidell–Scott–Jones lexicon of Greek words, a dictionary that last printed in 1940 yet is still considered the pre-eminent Greek lexicon (Stray, Clarke & Katz, 2019) has ten main translations for *logos* along with many sub-definitions. The most common of these have to do with a version of 'word' or 'narrative' as *logos* is the noun form of λεγω, meaning 'to speak.' (Lidell, Scott & Jones, 1940). This meaning was true in the earliest written versions of the word, as Herodotus (ca. 424 – ca. 425 BCE) used it to describe his histories; they were stories that were told to him from a personal account, in comparison with mythology, stories that were passed down through generations. This meaning of *logos* then turned into 'reason', as it was something that can be understood empirically (Fowler, 2011). Later *logos* became associated with the word of the biblical god (Lidell, Scott & Jones, 1940), credited to the Gospel of John, "In the beginning was the Word, and the Word was with God, and the Word was God" (King James Bible, 2022, John 1:1). *Logos* also has some mathematical meanings. It could mean 'account' such as an accounting of spending perhaps. And 'reason' could turn into a 'relation' between things, which could mean a ‘proportion’ or ‘ratio,’ which it meant in writings by Euclid (ca. 300 BCE) (Lidell, Scott & Jones, 1940). All of these notions could derive from the initial meaning of ‘word’.

To complicate this story further is that, while Napier did appear to have coined the word ‘logarithm’, there was a similar word used a few decades earlier. Florian Cajori (1930) added this to his detailed history of logarithms when a reader noted that the word *logarithmentai* had been in use since at least 1553. The word can be found in a book on Divination written by Caspar Peucer (1525 – 1602), a German mathematician and astronomer, to refer to ‘triangular numbers’, objects that corresponds to words and back to numbers to make meaning of biblical passages. While I could not find the 1553 text, I did find it in the 1560 edition. The translation of the first sentence (Figure 7.2) reads roughly [Greek language translation in brackets]: ‘‘Among [Arithmetic] there is a new number which we name a new thing ‘logarithm’[5].

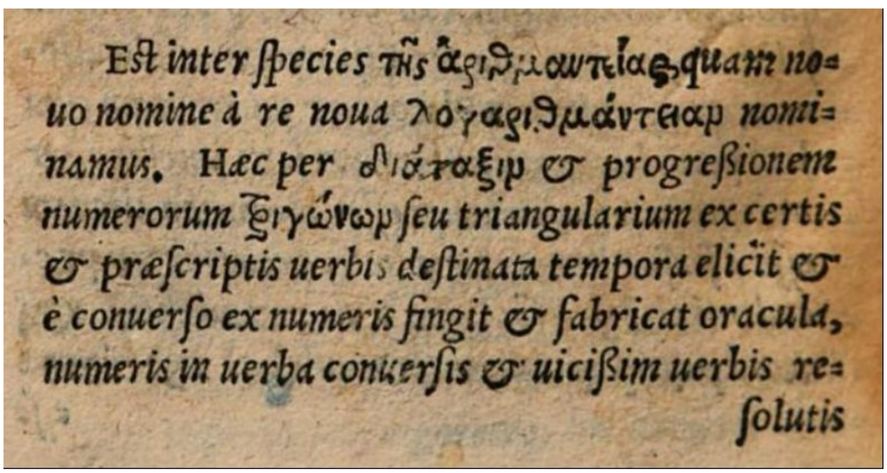
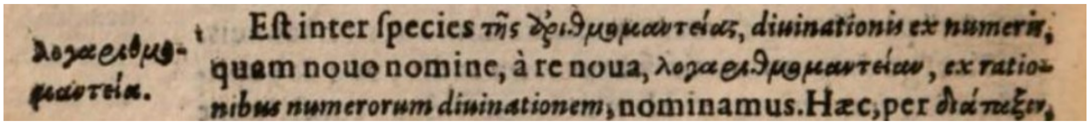

Est inter species τῆς ἀριθμαντίας, quam no=
uo nomine à re noua λογαριθμάντειαν nomi=
namus. Hæc per διάταξιν & progreßionem
numerorum τριγώνων seu triangularium ex certis
& præscriptis uerbis destinata tempora elicit &
è conuerso ex numeris fingit & fabricat oracula,
numeris in uerba conuersis & uicißim uerbis re=
solutis

Figure 7.2: Usage of the word Logarithm before Napier (Peucer, 1560, p. 239)

By 1593, the text had changed, adding a more specific meaning to the type of numbers (Figure 7.3). Loosely translated, it reads: ‘Among [Arithmetic] there is a kind of divining numbers, it has a new name as it is a new thing, it is called [logarithm] *an account of divine numbers*.”

[5] Translations of Figure 7.2 & 7.3 of the first sentence done by me, with help from Florian Cajori’s translation, Wheelock’s Dictionary, and the Perseus Project website.



λογαριθμο-μαντεία. Est inter species τῆς ἀριθμομαντείας, diuinationis ex numeris, quam nouo nomine, à re noua, λογαριθμομαντείαν, ex rationibus numerorum diuinationem, nominamus. Hæc, per διάπεξιν,

Figure 7.3: The use of the word 'Logarithm' in Peucer's 1593 text (p. 414)

It is unknown if Napier had familiarity with this use of the word, but the book was popular enough to be republished multiple times and at least one modern author felt "Napier had derived the word "logarithm" from "logarithmanteia", meaning something like "prediction of word by number and vice versa" (Osterhage, 2020, pp. 78–79). Napier had a well-known interest in theology, his most widely read book in his lifetime was *A Plaine Discovery of the Whole Revelation of St. John*, printed in 1593, which was a mathematical analysis of the end of the world (Rice et al., 2017). It would not be hard to reason that he could have read this work and been interested in the biblical mathematics, though it should be noted that he did not use any version of this word in *A Plaine Discovery*.

It would be hard to ascertain that Napier definitely took the name of his invention from Peucer, but there are enough similarities between their new numbers at least to take it into consideration. Peucer used 'artificial numbers' to create a new kind of operation in 'arithmetic' to make sense of the divine words, while Napier created 'artificial numbers' to create a new kind of arithmetic to make sense of higher calculations. They were both trying to create reason from chaos and, in doing so, each created new numbers to use in their mathematics. It would also bring in the faith that Napier had behind his invention. Both of his books started with *Mirifici Logarithmorum Canonis,* or 'Wonderful/Marvelous Canon of Logarithms', and both his texts were aimed to ease the calculations of astronomers, those making sense of the cosmos (Rice et al., 2017). It is not out of the realm of possibility for Napier to believe that he had made a discovery inspired by the divine. He could have then reached back to his days of researching *A Plaine Discovery* and remembered the word *logarithmanteia,* which at first Peucer seemed to have meant to be word-numbers though, given his subject matter, he may have meant numbers that make sense of the 'Word of God'. Napier could have taken this and altered it to different meanings: proportion-numbers, ratio-numbers, numbers that make reason, and maybe in that same vein, numbers that are given by God, that make sense of God's creation, are the word of God. In this way, Napier's choice of name would fit with the excitement that he had for his invention.

7.4. Method

To look at how logarithms got their name, I reviewed the texts by the mathematicians who were excited by their creation. I also looked at the textbooks that taught logarithms, the dictionaries that defined them and the encyclopaedias that expanded upon that definition. My review of texts begins in 1616, the year that Napier's text was published, and ends in 1850, approximately 100 years after Euler published his reformulation of logarithms as I wanted to see if the translations had changed after this new definition of logarithms was presented.

To choose dictionaries, I started with the list of major English Dictionaries presented in John Longmuir's (1864) combination of Walker and Webster's dictionaries. I then searched through the SFU library database, Google Books and the website Early English Books Online to expand upon the dictionaries, and to find encyclopedias, mathematics texts and mathematics textbooks that gave an etymology of 'logarithm'. There are a total of 27 dictionaries, 10 encyclopedias and 24 mathematics texts or textbooks included in this study.

7.4.1. Horizontal and Vertical Analysis

I began with a horizontal analysis which organised the data around the type of text and the mathematical level of the work around logarithms. For the vertical analysis, I looked at whether the etymology calls to other mathematical ideas, or to ideas outside mathematics that could help the reader make sense of the term; and if the etymology of the word was used to give the reader insight into concept of a logarithm. For this study, the breadth was looking at the different uses of the word *logos* while the depth was looking at how that translation expands or deepens the concept of a logarithm.

7.5. Data and Analysis

The horizontal analysis led to the bundling of the dictionaries and the encyclopaedias together, as both were written for the wider public. Even though encyclopaedias went into much more detail than dictionaries, their ideas were still an outline compared to any mathematical texts which attempted to derive meaning from the concept.

I originally thought that I would have more groupings in the mathematical texts, but, after an initial review, I realised that they were very similar in the way defined and used logarithms in in the text. They would begin by defining logarithms in one section of the book, but then continue to interact with them, and maybe expand upon their meaning throughout the rest of the text. And even though the texts were written for various groups: mathematicians, mathematics students, navigation students and mathematics teachers, all the texts had the reader working with their specific iteration of logarithms. The audience were people that needed to understand a definition of logarithms and how to use them to further their own studies.

For the data analysis, I organise the chosen artifacts from the least technically detailed to the most, so I start with dictionaries, then encyclopedias, then mathematical textbooks and texts. I coded the texts by the translations used for ‘logos’ and then by if and how that translation built into the concept of a logarithm.

7.5.1. Dictionaries and Encyclopedias

Dictionaries translated *logos* three ways, “a word”, “ratio” and “reason”. One dictionary used multiple translations for *logos,* and eight more had more than one definition for the word ‘logarithm’. As these were dictionaries, none went into detailed explanations, they limited their definitions to those found in the below Table 7.1.

Table 7.1: The translation(s) of ‘logos’ against the stated definition(s) of logarithms in dictionaries.

Logos Means:	Logarithms are:			
	Artificial numbers to aid calculations	Rank of numbers in arithmetic progression fitted to a sequence of numbers in geometric progression.	Index of Ratios of Numbers to one another	The exponents of the powers to which a constant must be raised to equal a given number.
a word	4	2	1	0
ratio	5	2	2	2
reason	3	2	0	0
no translation	4	8	3	0

Dictionaries were written to be succinct descriptions of a concept, so it was not surprising that many of them did not use the given translation of *logos* to build the definition, but it was notable that many of the translations did not seem to have any relation to the definition. Only two definitions included the translation of *logos* directly in the definition, though if the reader already had a base understanding of mathematics and logarithms, then ‘ratio’ could enhance the progression and exponent definitions as a geometric progression has a common ratio and exponent could be thought of as implying repeated multiplication by a common ratio. ‘Reason’ also can stretch to fit as the reason for logarithms was to aid calculations, or again if the reader has a mathematical background then the contemporary mathematical definition of ‘reason’ which was akin to ‘ratio’ (Moxon, 1679) could fit. Even for readers who had a previous understanding of mathematics, it was still a little under half the occurrences where the given translation would work itself into the definition.

There were also many dictionaries that did include the etymology, saying that the word ‘logarithm’ came from the two Greek words *logos* and *arithmos*, but did not translate those words. If the reader had a background in the Greek language, then the etymology without the translation, could perhaps still be helpful for the given explanation. One example to highlight is Nathan Bailey’s multiple versions of *A Universal Etymological English Dictionary.* In his first edition (1721), he did break down the etymology to the two words in Greek, but provided no translation. His definition of logarithm at this time was the relationship between geometric and arithmetic progressions which ease calculations. In his second edition (1724), he had translated ‘logos’ to ‘word’ though he did not change his definition to account for this translation. Did he feel that having the translation helped readers make sense of this topic? I cannot see how, but he kept this translation and definition through the next 20 editions. Dictionaries, and one encyclopedia, were the only texts that did not translate the two words; the other more detailed texts that were building up to and onto the definition of logarithms wanted their chosen translation to create connections to the concept.

Encyclopedias, for the most part, did go into some detail, most including multiple definitions of logarithms, as well as different uses and characteristics of logarithms. Of the encyclopaedias that used the translation of ‘ratio’, over 80% of them included that word in creating their definition. The translation of *logos* to ‘proportion’ only came after the logarithm is defined through the relationship to exponents, and those are the only translations that have one definition (Table 7.2). This definition though did not include any stated connection to the translation of ‘proportion’.

Table 7.2: The translation of logos against the primary definition of logarithms in encyclopedias.

Logos Means:	Logarithms are:			
	Artificial numbers to aid calculations	Rank of numbers in arithmetic progression fitted to a sequence of numbers in geometric progression.	Index of Ratios of Numbers to one another	The exponents of the powers to which a constant must be raised to equal a given number.
ratio	5	5	5	3
proportion	0	0	0	2
no translation	1	0	2	1

For about half of the dictionaries, if they translated *logos* and if the public had some understanding of mathematics, then the translation of the term could help explain the concept, though it was not usually apparent in the text. The encyclopedias had more space to expand on the definitions, so they could explain why they chose a particular translation and how it connected to the concept of logarithms. And, in the beginning they did, specifically when it is defined through the number of ratios, though once the logarithm began to be defined through an exponent, their explanation of that connection disappeared. While the word ‘ratio’ continued being associated with logarithm, the authors no longer connected the translation of *logos* with the given definition.

7.5.2. Mathematical Texts and Textbooks

All but one of the mathematics texts that delved into the etymology of *logarithm* tried to use their chosen translation of *logos* in their explanation of the concept. The one exception is the book authored by David Morrice (1801) who, after translating *logos* as ‘a description or account’ admitted that, “I do not recollect, in the whole course of my reading, to have seen the derivation of the word explained; and I do not believe that many pupils could tell me what the word logarithm means, or what the science is” (p. 410). As the other texts tied the etymology to the definitions, I focus on those, looking at each translation individually.

Translations around ‘Ratio’

Twenty out of 24 mathematics texts reviewed use the translation of ‘reason’, ‘ratio’, or ‘proportion’ for *logos*. The first text to mention the etymology is by Henry Briggs’ (1631) *Logarithmicall arithmetike*, which was an English translation of his previous work which was printed in Latin. Unfortunately, Briggs passed before finishing this text, so it was completed by Henry Gellibrand (Wallis, 1685), and as it included many changes from Briggs' previous work, it is unknown whether this meaning of *logos* and *logarithm* was something discussed with Napier, or if it is a later addition by Gellibrand. In the text, *logos* was translated as 'reason', stating that "the whole word signifies rationall or proportinall numbers" (p. 1). The primary definition of logarithm in this text followed the idea of artificial numbers to ease calculations. Other authors (Jeake, 1696; Leadbetter, 1728) followed this train of thought with the translation of 'reason' for *logos* and a definition around easing calculations, though Jeake did add in the translation of 'proportion'. Edmund Wingate (1635) used the same translation of 'reason', but again added in 'proportion' when discussing constructing logarithmic tables, that a table of logarithms around a different base could be "calculated according to the *reason* and proportion of the Logarithmes [already known]" (p. 5).

There are two ways that one could read this, in all of the texts, logarithms were primarily things that were created for the 'reason' of easing calculations, so perhaps the authors wanted the name to indicate this use as 'reason' did have a primary meaning of 'cause' (Bailey, 1724). But, in three of the texts, the idea of logarithms as related to proportions took centre stage, and as at this time 'reason' also had a meaning in mathematics that is akin to proportion (Moxon, 1679). Therefore, in mathematics texts, it seemed correct to replace 'reason' with 'proportion'. Given that Briggs created his tables by first defining all the base 10 logarithms, and then by finding other logarithms through a series of proportions, this meaning of the word 'logarithm' followed his work. And given that he worked closely with Napier, it is an indication of Napier's thought process in choosing this name, that logarithms arose from numbers in proportion. In Edward Wright's translation of Napier's work from Latin to English, he had said that, "The logarithms of proportional quantities are equally differing" (1616, p. 7) and Briggs' defined logarithms to be numbers in companion to proportional numbers which have equal differences (Briggs/Bruce, 1624/2006). Logarithms being in relation to proportional numbers seems a reasonable understanding of the etymology.

In 1685, Wallis published a text where he translated *logos* as 'proportion', and used that translation in his definition of logarithms to be "number of proportions so compounded" (p. 57). This idea of logarithms was new and would be popularized by Edmund Halley ten years later, as he used this idea to calculate logarithms in a novel manner. Out of the 24 mathematics texts, thirteen used the translation of 'ratio' and, after

Halley, they all defined logarithms in some manner that includes 'the number of ratios'. This definition speaks directly to the etymology, so it is no surprise the texts using this definition were most often the ones to include the etymology. After Euler introduced his new depiction of logarithms in 1748, this translation was still used in texts that had retained some manner of the older definition even as they started to include the idea of logarithms being related to exponential equations.

It is likely prudent to note here that the last known works of Napier were published in 1839, by his distant relative Mark Napier, and used this formulation in the introduction "Αριθμοι signifies numbers; λογαριθμοι, the ratios of numbers, or rather the number of ratios, λογων αριθμος" (p. x). As this text was a collection of his writings which are all from before his invention of logarithms, so before he had chosen a name, the above etymology was his descendant's terminology, not necessarily Napier's.

Translations around 'Word'

There are two other main translations of *logos* that made it into mathematics texts, *logos* as 'description' or 'account', and *logos* as 'speech', or 'voice' or 'language'. As discussed at the start of this sub-section, Morrice could not find a way to use his translation of 'account' in any of his work around logarithms, and the other text using that translation is similar. They translated *logos* as 'description', and then defined it through the relationship between two sequences, without ever utilising the translation.

Those that defined it as a variant of 'speech' did a bit more in their connections. In the texts that used this idea, logarithms were at least partly defined as artificial numbers to assist in calculations, furthermore, logarithms were a language given to us by numbers to ease tedious work. A text that used the translation 'divine' implied the same thing, except this time they were numbers given by a higher power. This translation of *logos* seamlessly fit into the definitions that were based on logarithms being artificial numbers created to ease calculations, but once the definitions became more advanced, multiple translations of *logos* were needed. Mudie (1836) began with a translation of 'voice', but as he was writing after Euler's formulation of logarithms as exponents had been popularised, he transitioned into the translation of 'ratios':

> exponents are called **LOGARITHMS**, or "the voices of numbers;" and there is something in the word *logos*, of which the first part of this name is composed, which always worthy of our attention when we meet with the

word alone or in a compound. *Logos* is not the mere sound—the noise in the ear made by an uttered word; for the Greek expression for that is *phone*, which means “a noise," and is equally applicable to all noises, whether there is any sense in them or not. *Logos*, on the other hand, is the information which the thing alluded to is capable of giving in answer to our inquiry or our observation; and therefore, as there may be a sound or *phone* where there is no *logos*, so there may be a *logos* without any *phone*, in which the information may be communicated to the eye or any other of the senses, or to the mind, without any instrumentality of the senses, which is in fact the case in our general perception of the meaning of ratios. *Logos* is nearly synonymous with ratio, and with relation; for as it is impossible for the mind to judge of anything without a standard of judgment, mentioned or implied, there is reference to such a standard in every case where truth or correct information of any kind is acquired.

Hence, in the plainest language of mathematics, Logarithms are the ratios of numbers; the expressions for Logarithms, whether given in the common figures of arithmetic, or in any other way, are the exponents of those ratios; and the operations which we are enabled to shorten, or perform in cases where common arithmetic is unequal to them, constitute the Arithmetic of Exponents (Mudie, 1836, pp. 350–351).

This long quotation illustrates some of the difficulties involved in trying to tie the etymology of a word with a changing definition. This quotation also demonstrates the idea of Etymological Fallacy: that the original meaning of the word should still have bearing on the modern definition (Kolb, 2018). The earliest understanding of the word ‘logarithm’ comes from either Briggs or Gellibrand, depending on the author of the relevant section of *Logarithmicall arithmetike* (1631) and they would have had long conversations with either Napier or Briggs (again depending), but it is uncertain that their understanding of the word would have matched Napier’s. The true etymology of ‘logarithm’ may never actually be known. But also, does it matter? Is Napier’s chosen translation of *logos* important for our modern understanding of logarithms in mathematics?

7.6. Why care about a name?

Perhaps Edmund Stone (1743) has the answer:

Dr. Wallis, in his History of Algebra, calls Logarithms the Indexes of the Ratio's of Numbers to one another.— Dr. Halley, in the Philosophical Transactions, No 216. says, they are the Exponents of the Ratio's of Unity to Numbers.— So also Mr. Cotes, in his Harmonia Mensurarum, says, they are the Numerical Measures of Ratio's; but all these convey but a very confused Notion of Logarithms. Nay, if what the great Dr. Barrow says, in

> one of his Mathematical Lectures, be admitted for Truth, (where he treats of the Nature of a Ratio, and denies it to be any manner of Quantity.) those Gentlemen's Definitions must be either Nonsense, or very near it. (Logarithm Section).

Trying to tie the name of the function directly to the operation leads to nonsense. But while the name does not have to lead into a definition, learning where the name came from, and all the meanings connected to it can help us make sense of the word. Logarithms can be the ‘Ratio’ of ‘Numbers’. They can also be the nature of a ‘Ratio’. They can make ‘reason’ out of complex operations. They can be an operation, and a quantity, and a relationship. The name can even refer back to the god-like power of this creation. Or it can go back to the root word of ‘logos’ and just be a collection of words, an account, a story.

As I get to the end of this dissertation, that is what I choose to get across, that logarithms are a story, a story around and about and of numbers, and one that can be useful for students to learn. Currently, the meaning of the name ‘logarithm’ does not imply the definition that is in use, the one based upon the inverse of exponentiation. But if the definition coincided with a translation of *logos,* then the etymology of the word could enhance readability and understanding of the unusual term. An understanding of *logos* as ‘ratio’, or ‘reason’, and *arithmos* as ‘numbers’ could reinforce the concept of ‘Number of ratios’, or ‘artificial number for the reason of easing calculations’. If students’ concept image of logarithms increased to include these new definitions, perhaps with some of the presentations of the past, then the etymology would again be of service in explaining upon the concept.

Chapter 8. Some 'Reason'able Suggestions

This dissertation was written to consider whether and how ideas about logarithms in the past could increase today's students' concept image about logarithms. This chapter seeks to tie together the last three chapters, by looking to the historic sources and conceptions of logarithms to see if they could influence students' understanding and use of logarithms today. In this chapter, I respond to the last set of questions in Section 1.4: How could the historic conceptions of logarithms tie into students' understanding and use of logarithms today? Do they bring in new connections to students past mathematics?

While there have been studies on incorporating the history of logarithms into classrooms, they focus on the mathematics of logarithms as written by past mathematicians and scientists, not the mathematics that was presented to past students. Along with that, a major focus of this study is the new connections that could be made between logarithms and other mathematics, so for both these reasons recommendations in this section may differ from those previous studies. This chapter also discusses the created nature of logarithms by discussing the circumstances of their invention and the innovations to logarithms over time. As explained in Chapter 2, mathematics is a human endeavour, and that is something that should be highlighted to students. Mathematics is invented and then changed depending on the needs of the time, and logarithms provide an excellent example of that process.

8.1. Should Original Sources Be Used?

In this dissertation, I reviewed over a hundred original sources, from tomes by famous mathematicians to textbooks for arithmetic students. There is a discussion among researchers in the history of mathematics education field over the use of original sources in mathematics classrooms. It is almost impossible to use a 100% original source in the classroom. They will often need to be altered in some way, such as translating or using modern symbology, in order for contemporary students to attempt to understand them without having to learn entire new subjects (Schubring, 2008). The context around the original sources could also have to be explained to make sense of the language employed, which means the context would be filtered through the instructor or modern texts. As this dissertation is focused on providing suggestions for textbooks, including original sources and the context in that format would take up an inordinate amount of space that textbook authors would want to use for other mathematical material.

In this dissertation, while I did review these original sources, I am choosing not to use them in my suggestions in this chapter. I did gain a lot of understanding from the original sources, but that understanding only came after hours of work with the sources, with multiple translations and with reading texts about the original works. It is not something that I expect to be able to fit into the space allotted in a modern mathematics textbook. I do think original sources have a space in mathematics education. They could be included as supplementary material for the student to work on with the guidance of an instructor, or a project for a group of students. There is also space for completely new textbooks to be written for mathematics classes around original sources, but as that is not the focus of this dissertation, I am looking where to include new representations of logarithms in today's mathematics textbooks, I do not attempt to do that here.

8.2. Incorporating Historical Ideas of Logarithms into Today's Lessons

This chapter is organised with headings similar to those of Chapter 5, the chapter on the presentation of logarithms in modern textbooks. Included are suggestions on how historic conceptions of logarithms could be incorporated into contemporary textbooks or lesson plans in a way that offer a new pathway toward the logarithmic concept.

8.2.1. Introducing Logarithms

As discussed in sub-section 5.3.1, logarithms in contemporary textbooks were primarily introduced as the inverse of an exponential equation, function or graph. Euler's formulation of logarithms was included as was is a rewriting of an exponential equation into a logarithmic equation. The idea of a logarithm as an exponent, or inverse of exponentiation, followed their trajectory through modern mathematics, so it made sense to be the foundation of the concept. Some historical ideas of logarithms could also be incorporated at this time to connect logarithms to other aspects of mathematics.

The definition of a logarithm as an arithmetic sequence in sync with a geometric sequence, specifically using Wallis' (1685) nomenclature of 'exponent' for the arithmetic sequence could tie logarithms to their initial conception. Wallis' work could be followed further by next writing the geometric sequence with exponents, while explaining that the logarithm could also be defined as the number of ratios needed to get to a certain number (Figure 8.1).

<table>
<tr>
<td>Introduction as an inverse</td>
<td>Recall that the exponential function: $y = f(x) = a^x$ is one-to-one therefore it has an inverse function: $x = f(y) = a^y$. That inverse is called the logarithmic function and is written as: $f(x) = \log_a x$.

The logarithmic function of base a is defined by the relationship with the exponential function. If $a > 0$ and $a \neq 1$, then $\log_a x = y \Leftrightarrow a^y = x$</td>
</tr>
<tr>
<td>Introduction as a relationship between sequences</td>
<td>Logarithms did not actually arise from their relationship with exponents, but instead from the relationship between an arithmetic and a geometric sequence. At the time, the arithmetic sequence was often called the index or exponent, and it is also the logarithm of the term in the geometric sequence.

An arithmetic sequence has a common difference, or a common number that is added to each term to create the next term. In the sequence below, that common difference would be 1.

A geometric sequence has a common ratio, or a common number that is multiplied by each term to create the next term. In the sequence below, that common number would be 2.

<table>
<tr><td>Arithmetic Sequence (exponent/logarithm)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>Geometric Sequence</td><td>1</td><td>2</td><td>4</td><td>8</td><td>16</td><td>32</td></tr>
</table>

For this pairing, the base of geometric sequence is 2, so the base of the logarithm (the arithmetic sequence) would be 2.</td>
</tr>
<tr>
<td>Introduction as the number of ratios</td>
<td>If we rewrote the geometric sequence above with a common ratio of a, then we could see that the arithmetic sequence is represented in the exponent.

<table>
<tr><td>Geometric Sequence</td><td>a^0 or 1</td><td>a^1</td><td>a^2</td><td>a^3</td><td>a^4</td><td>a^5</td></tr>
</table>

In this case, the base is a, so logarithm has a base of a and it is the exponent of the given term. It can also be thought of as the number of ratios from the starting value of 1. That is one of the translations of the word logarithm, it is a portmanteau made up of logos which can translate to 'ratio' and arithmos which translates to 'number'.</td>
</tr>
</table>

Figure 8.1: A look at logarithms expressed in the way common in modern textbooks with some of the historical introductions included

The bottom two ways of introducing logarithms shown in Figure 8.1 offer two different views of the concept, but both views still show that logarithms can be expressed by their relationship to exponents. This view of logarithms also leads to an explanation of the word *logarithm* as it can translate from the Greek to mean the 'number of ratios'.

While expanding introductory text would result in a much longer section than the introduction through an inverse, it would also expand the world of logarithms presented to students. Students would connect the concept of a logarithm to sequences and to ratios, building onto their concept image of this function. Both the suggested extensions also keep logarithms closely tied with exponents, and can even help further the meaning of exponents, which means it could be weaved into current texts without needing too much explanatory material.

8.2.2. Calculating Logarithms and Properties of Logarithms

As discussed in sub-section 5.3.3, current methods of evaluating logarithms rely on viewing the logarithm as an exponent, either by deciding the power that the base must be raised to in order to equal the argument, or by rewriting the logarithm into exponential form. Both of these methods further the relationship between logarithms and exponents, and start the investigation of the notion of a logarithm as a number.

After reviewing past textbooks, I am not convinced that any of the presentations around calculating logarithms should be brought into the present and chose not to focus on that topic in Chapter 6. Calculating logarithms was a big focus of these textbooks, but the strategy was either very convoluted as many of the texts traced the work that Briggs did, or it involved topics that the modern student, learning logarithms for the first time, has yet to encounter, such as series and areas under a curve.

The only method that could be useful here – though probably not be in the main text, but perhaps in an activity in the exercises, or side-section of the text, draws from a presentation discussed in Appendix A that used the mean proportional, or in modern terms the geometric mean. This presentation could help students understand the nature of ratios and their connection to logarithms. If presented visually (Figure 8.2) it can also lead students into understanding the continuous nature of logarithms as well as some of their boundaries.

While $\log 100$ and $\log 1000$ can be pretty easily calculated by discovering what power 10 must be raised to in order to get either 100 or 1000, it is not as easy to calculate $\log 9$. Today, we have calculators, so we can just punch in the expression, but before calculators mathematicians created a variety of ways to evaluate this expression. One of the ways uses the idea discussed above in Figure 8.8.1 where logarithms began as a relationship between two sequences. If the geometric sequence had a ratio of 10, and the arithmetic a difference of 1, then the relationship would look like:

Arithmetic sequence (exponent/logarithm)	0	1	2	3	4
Geometric sequence (argument)	1	10	100	1000	10000

We can work with these two sequences to find an approximation of $\log 9$. First let us understand arithmetic and geometric means. To find the mean of two numbers in an arithmetic sequence, you would add the two numbers and divide it by 2. To find the mean in a geometric one, you would multiply the two numbers and take its square root. So an arithmetic mean between 0 and 1 would be $\frac{0+1}{2} = \frac{1}{2} = 0.5$ and a geometric mean between 1 and 10 would be $\sqrt{1 \times 10} = \sqrt{10} \approx 3.162$.
We can put these numbers into our sequences below. These new numbers would still fit into the sequences and these sequences would still include all the numbers of the original.

Arithmetic sequence (exponent/logarithm)	0	0.5	1	2	3
Geometric sequence (argument)	1	3.162	10	100	1000

We would like to calculate the $\log 9$, so we could keep finding means until the argument is close to 9, then we will have an approximation of the logarithm. 9 is between $\sqrt{10}$ and 10, so we should find the geometric mean between those numbers and the arithmetic mean between the corresponding ones in the arithmetic sequence. $\sqrt{\sqrt{10} \times 10} \approx 5.623$ and $\frac{\frac{1}{2}+1}{2} = \frac{3}{4}$.
Here again, we have sequences which also contain all the values of the original. We could also think of it as a new sequence, just with a smaller ratio (much smaller than 10) in the geometric sequence and a smaller difference in the arithmetic. This small ratio was an original idea of logarithms, if the ratio is small enough then the geometric sequence would not skip over whole numbers (as our sequence currently does) and could be useful in calculations (as seen in the section on ‘properties of logarithms’).

Arithmetic sequence (exponent/logarithm)	0	0.5	0.75	1	2
Geometric sequence (argument)	1	3.162…	5.623…	10	100

If we found the geometric mean to the two numbers immediately surrounding 9 eleven more times, our geometric sequence keeps getting closer and closer to 9, while the corresponding number in the arithmetic sequence gets closer to 0.9542, which is a good approximation of $\log 9$ to four digits as shown below. Please notice, that as we are trying to reach 9, in the geometric means, our first number in the square root is the newest number that would be less than 9 and our second number in the square root is the newest number that is greater than 9. In our arithmetic means, the lowest number and the highest numbers match with their geometric counterparts.

Arithmetic means (exponent/logarithm)	Geometric means (argument)
$\frac{0+1}{2} = 0.5$	$\sqrt{1 \times 10} \approx 3.162277\ldots$
$\frac{0.5+1}{2} = 0.75$	$\sqrt{3.162277 \ldots \times 10} \approx 5.623413\ldots$
$\frac{0.75+1}{2} = 0.875$	$\sqrt{5.623413 \ldots \times 10} \approx 7.498942\ldots$
$\frac{0.875+1}{2} = 0.9375$	$\sqrt{7.498942 \ldots \times 10} \approx 8.659643\ldots$
$\frac{0.9375+1}{2} = 0.96875$	$\sqrt{8.659643 \ldots \times 10} \approx 9.305720\ldots$
$\frac{0.9375+0.96875}{2} = 0.953125$	$\sqrt{8.659643 \ldots \times 9.305720 \ldots} \approx 8.976871\ldots$
$\frac{0.953125+0.96875}{2} = 0.9609375$	$\sqrt{8.976871 \ldots \times 9.305720 \ldots} \approx 9.139816\ldots$
$\frac{0.953125+0.9609375}{2} = 0.95703125$	$\sqrt{8.976871 \ldots \times 9.139816 \ldots} \approx 9.057977\ldots$
$\frac{0.953125+0.95703125}{2} = 0.955078125$	$\sqrt{8.976871 \ldots \times 9.057977 \ldots} \approx 9.017333\ldots$
$\frac{0.953125+0.955078125}{2} = 0.954101563$	$\sqrt{8.976871 \ldots \times 9.017333 \ldots} \approx 8.997079\ldots$
$\frac{0.954101563+0.955078125}{2} = 0.954589844$	$\sqrt{8.997079 \ldots \times 9.017333 \ldots} \approx 9.007200\ldots$
$\frac{0.954101563+0.954589844}{2} = 0.954345703$	$\sqrt{8.997079 \ldots \times 9.007200 \ldots} \approx 9.002138\ldots$
$\frac{0.954101563+0.954345703}{2} = 0.954223633$	$\sqrt{8.997079 \ldots \times 9.002138 \ldots} \approx 8.999608\ldots$

We could keep going to find even closer approximations, but this is enough for our purposes. We could repeat this process to find $\log 2$ or $\log \frac{1}{2}$ or the logarithm of any positive number, but we would not be able to calculate the logarithm of any negative number. Try to reason why that is by taking a look back at the geometric mean. Does a square root ever result in a negative number?

This is one way to think through calculating logarithms, past mathematicians invented many ways to do so, but today we are lucky enough today to use our calculators to evaluate logarithms.

Figure 8.2: Calculating a logarithm using geometric and arithmetic means

The only other note from past textbooks that should be included today is around language, the idea from the introduction of logarithms by Napier that a logarithm is an artificial or an invented number. This idea could be the link that carries calculating logarithms into properties of logarithms.

8.2.3. Properties of Logarithms

While every current textbook that included the properties of logarithms derived them through exponent rules, a few did include other ways to work out the properties in the exercises, though never directly through the relationship between the two sequences. Doing so would be an easy connection for students to make between the idea of a logarithm and basic numerical operations, which would reinforce the idea of a logarithm as a number (Figure 8.3). It also supports the idea that logarithms are something invented for a specific purpose, and that has been altered to meet modern needs, introducing the fluidity of mathematics.

When logarithms were first invented by John Napier in 1614, they were often called 'artificial numbers' or 'invented numbers'. Calculators were not yet in existence, so calculations, especially multiplication, division, raising numbers to an exponent, and taking the roots of numbers, were very difficult. Napier noticed that these operations could be transformed to simpler operations using the relationship between an arithmetic sequence and a geometric sequence. Two of the translations of the word *logarithm* is 'the speech of numbers' and 'numbers that make reason', and some think that this is why the word was used, these new numbers are talking to us, showing us a way to make mathematics easier.

Arithmetic sequence (exponent/logarithm)	0	1	2	3	4	5	6	7
Geometric sequence (argument)	1	2	4	8	16	32	64	128

1) Look at the above geometric sequence, multiply 4 and 16. Now compare to the corresponding values in the arithmetic sequence. Try again with 2 and 32.

4 in the geometric matches with 2 in the arithmetic. 16 in the geometric matches with 4 in the arithmetic:
$4 \times 16 = 64$ in the geometric, which matches with 6 in the arithmetic and is the sum of 2 and 4.
$2 \times 32 = 64$ in the geometric which corresponds with $1 + 5 = 6$ in the arithmetic.
Multiplication in the geometric corresponds with addition in the arithmetic (the logarithms).

2) Again looking at the geometric sequence, divide 128 by 32. Now compare with the corresponding values in the arithmetic sequence. Try again with 64 and 4.

128 in the geometric is 7 in the arithmetic. 32 in the geometric is 5 in the arithmetic.
$128 \div 32 = 4$ in the geometric, which is 2 in the arithmetic and also the difference of 7 and 5.
$64 \div 4 = 16$ in the geometric which corresponds with $6 - 2 = 4$ in the arithmetic.
Division in the geometric corresponds with subtraction in the arithmetic (the logarithms).

3) Still starting from the geometric sequence, take the square-root of 64. And the cube root of 64. Try to find a pattern in the matching numbers from the arithmetic sequence.

64 in the geometric matches to 6 in the arithmetic.
$\sqrt{64} = 8$, which is 3 in the arithmetic.
$\sqrt[3]{64} = 4$, which is 2 in the arithmetic.
From the square root, $6 \div 2 = 3$, and from the cube root, $6 \div 3 = 2$.
The root in the geometric corresponds with division by the root in the arithmetic (the logarithms).

4) Lastly, from the geometric sequence, square 4 and cube 4. Compare to their matching numbers in the arithmetic sequence and try to find a pattern.

4 in the geometric matches to 2 in the arithmetic.
$4^2 = 16$ which is 4 in the arithmetic.
$4^3 = 64$ which is 6 in the arithmetic.
From squaring, $2 \times 2 = 4$, from the cubing $2 \times 3 = 6$.
Raising a number in the geometric sequence to an exponent corresponds with multiplying by that exponent in the arithmetic (the logarithms).

These were the four main properties of logarithms at the time of their founding and they were the primary reason for the invention of logarithms. Being able to transform operations, from multiplication to addition, division to subtraction, roots to division, and powers to multiplication, saved mathematicians, and scientists, time and energy. These calculations were often called logarithm arithmetic and were done with the 'invented numbers' that are logarithms. Today we can also derive these properties through working with exponents.

Figure 8.3: Illustrating the properties of logarithms by working through the relationship between the geometric and arithmetic sequence

The presentation in Figure 8.3 could be expanded to incorporate sequences where the geometric sequence starts at 1 and has a common ratio of a (Figure 8.4). This presentation would bridge the space between logarithms coming from the comparison of two progressions to logarithms being an exponent.

Remember from the introduction of logarithms that they can be seen as the number of ratios needed to make a term in a geometric sequence. If we look at the below geometric progression, the common ratio is a, the exponent is the arithmetic sequence, which is also the number of ratios to any given term, so it is the logarithm of that term given a base of a.

Geometric sequence	a^0 or 1	a^1	a^2	a^3	a^4	a^5	a^6	a^7

Following the discoveries from Figure 8.3, if we instead work with this sequence, we will derive the same properties of logarithms.

Multiply two of the terms, for example: $a^2 \times a^4 = a^6$, then you add the exponents (the logarithms)
Divide two of the terms: $\frac{a^6}{a^2} = a^4$, then you subtract their exponents (the logarithms)
Take a root of one of the terms: $\sqrt[3]{a^6} = a^2$, then you divide their exponents (the logarithms)

And raise one of the terms to a new exponent: $(a^2)^3 = a^6$, then you multiply their exponents (the logarithms)

These properties of logarithms follow directly from the properties of exponents.

Figure 8.4: Exploring the properties of logarithms through a geometric progression

Both Figures 8.4 and 8.5 explore the properties of logarithms through the geometric series and that relationship to an arithmetic series, and both start with the fixed relationship of 1 in the geometric corresponding to 0 in the arithmetic. Removing that fixed relationship could lead to a discussion about the development of logarithms, that even though they initially could be 'any' geometric sequence paired with 'any' arithmetic sequence, eventually it was decided that there needed to be that one fixed correlation. While the reasoning for that fixed correlation was to simplify calculations even more, it also helped further the evolution of logarithms to exponents. This idea was the middle/extremes method explored in sub-section 6.4.2 is pretty advanced and very removed from our current understanding of logarithms, so would most likely be an extra activity. The presentation of this idea in Figure 8.4, though, would continue building the concept of logarithms as 'invented' numbers, logarithms as the relationship between sequences, and as the extreme/middle method is visual, could further the idea of logarithms as the number of ratios, where the logarithm is the space between the start of a sequence and the term of interest.

Initially the word 'logarithm' applied to ANY arithmetic sequence attached to ANY geometric sequence. A few examples are below:

Geometric Sequence (argument)	Arithmetic Sequence A (exponent/logarithm)	Arithmetic Sequence B (exponent/logarithm)	Arithmetic Sequence C (exponent/logarithm)
1	5	0	33
2	7	1	30
4	9	2	27
8	11	3	24
16	13	4	21
32	15	5	18
64	17	6	15

As you know the properties of logarithms, see if they still apply given these sequences. Multiply 4×8 in the geometric sequence and then add the corresponding logarithms.

Geometric Sequence: $4 \times 8 = 32$
Arithmetic Sequence A: $9 + 11 = 20 \neq 15$

Arithmetic Sequence B: $2 + 3 = 5$
Arithmetic Sequence C: $27 + 24 = 51 \neq 18$

The only one above that seems to follow the properties that we know is Sequence B! Let's explore why that is.

When we are working with the geometric sequence, we are working with ratios, which we know because a geometric sequence means that there is a common ratio between all the terms, the ratio in this case is 2 or $\frac{2}{1}$.

When we multiply numbers in the geometric sequence, we are still working with ratios, which can also be thought of as a series of terms in proportion. If we multiply 4×8, it is the same thing as imagining starting at a point (in this case 1 is the starting point) and quickly getting to 4 by moving two spaces down the geometric sequence. Next you would start at 8 and also move 2 spaces down the geometric sequence, doing so would keep the terms in proportion. Starting at 1 and going to 4 could be written as a ratio of $\frac{4}{1}$, while starting at 8 and moving those two spaces would end at 32, so a ratio of: $\frac{32}{8}$, which is simplified is also $\frac{4}{1}$. Due to that simplification, we can see these two ratios are equal: $\frac{4}{1} = \frac{32}{8}$, which means if we cross multiply: $32 \times 1 = 4 \times 8$.

This cross multiplication leads us to the idea that when working in a geometric sequence, the product of the middle numbers, the numbers you want to multiply, will equal the product of the extremes, the starting number and your ending number. So, from the above chart, if I want to multiply 2×32 then they would be the middle numbers and would equal the extremes 1×64.

This idea in a geometric sequence maybe seems pretty self-explanatory, but let us now compare with the matching numbers in the arithmetic sequence.

Geometric Sequence	Arithmetic Sequence A	Arithmetic Sequence B	Arithmetic Sequence C
$4 \times 8 = 1 \times 32$	$9 + 11 = 5 + 15$	$2 + 3 = 0 + 5$	$27 + 24 = 33 + 18$
$2 \times 32 = 1 \times 64$	$7 + 15 = 5 + 17$	$1 + 5 = 0 + 6$	$30 + 18 = 33 + 15$

Now all of these work as expected, the properties of logarithms still apply, multiplication does convert to addition in logarithms! Why did Sequence B work both times? What is special about it?

That Sequence B starts with a 0 means that when adding it to the 'ending extreme number', the product, nothing changes as adding 0 to any number does not change that number. So the addition of the two 'middles' the two numbers that correspond to the numbers you are multiplying, directly add to the ending number, the product. Mathematicians realized this in the early history of logarithms and quickly decided to fix the correspondence between 1 in the geometric sequence and 0 in the arithmetic sequence so calculations would be even easier.

There is another reason to choose to have the fixed correspondence between 0 and 1, it means that we can imagine logarithms as the number of ratios in the geometric sequence, or the distance that the term in the geometric ratio is from the value of 1. This correspondence lets us evaluate and estimate logarithms quickly. Since I know the common ratio in the geometric sequence above is 2, I could quickly know that the $\log_2 8 = 3$ as it would be 3 ratios of 2 to create 8, or in the sequence 8 would be 3 spaces from the start. I could also know that the $\log_2 15 \approx 4$ as it would take 4 ratios of 2 to create 16, so would need slightly less than 15, or again 15 would be slightly less than 4 spaces from the start. This idea was a major step in logarithms becoming exponents and will come up again when logarithmic scales are introduced.

Figure 8.5: Reasons for there to be a fixed correspondence of 1 on the geometric sequence to 0 on the arithmetic sequence

The multiplication property in Figures 8.4 and 8.5 could also be derived by looking at the distance from 1 in the geometric progression, similar to Archimedes (ca. 220BCE) in *The Sand Reckoner,* but as every instance of logarithms in modern textbooks came after the introduction of exponents and their properties, it feels more natural to use those mathematical concepts. As discussed in Chapter 5, not much space is devoted to deriving or proving the properties of logarithms, and as they are such a large part of the mathematics around logarithms going forward, introducing them through multiple methods could strengthen students' ability to connect these properties to their understanding of ratios, proportions, sequences, and basic operations learned in previous mathematics.

8.2.4. Solving Logarithmic and Exponential Equations

'Taking the log of each side' did not appear in the historic texts until the 1700s and the justification for this action is different than those used in modern times. In sub-section 5.3.6, the equality principle, or the one-to-one nature of functions was justification for this operation. In sub-section 6.3.5, solving these exponential equations was justified through the properties of logarithms. While this justification would most likely keep logarithms associated with exponents, it connects the action to the properties of logarithms, and so can further reinforce those concepts (Figure 8.6).

We can also come to the idea of solving exponential equations purely through the properties of logarithms! Let's start out with an equation that we want to solve:

$$a^x = y$$

This is a hard equation to solve without logarithms, but now that we know of them, we can recall that one of the properties states that if a term in the logarithm is raised to an exponent, then it is the same as that exponent times the logarithm. That would mean that if we had a logarithm in there, then our x would no longer be in the exponent space, but it would be multiplied by the logarithm! That would be a much easier to solve equation. Let's write out this Power Property of Logarithms:

$$\log_b a^x = x \log_b a$$

Then by substitution (since a^x and y are the exact same thing) we can say

$$\log_b y = x \log_b a$$

You will in the future read this as 'taking the logarithm' of both sides and it will be justified later using an 'equality property', but before that was a common idea, this process of substitution was used to explore this concept.

Figure 8.6: Using properties of logarithms to justify using logarithms to solve exponential equations

Even though in the historical texts, logarithms were not commonly used to solve exponential equations, there are a few other concepts that were used and could be informative for today's students. Before logarithms, most problems that would now be expressed through exponential equations were instead viewed as a set, or sets, of proportional equations. This presentation could be an interesting aside to readers; just a quick explanation of how equations were solved before the exponent was more common, which would also further the connection between the exponent, the logarithm, and ratios.

The more relevant idea from the past textbooks was the process of solving an exponential equation by repeated division. This presentation introduced a new way of looking at logarithms, while further connecting them to their definition as the inverse of an exponential equation. Given that exponents are first taught as repeated multiplication, and that logarithms are introduced as their inverse, the view of logarithms as repeated division can strengthen this connection. Since it is not a process that leads into 'taking the logarithm of each side' it most likely does not belong in the main text, but it is something that readers can presumably grasp without much extra instruction so it could be a set of exercises or an extra activity (Figure 8.7).

Before logarithms, scholars used many different methods to solve exponential equations. One way was to use an operation that is the opposite of an exponent. Since an exponent can mean repeated multiplication, the opposite would be repeated division.

Given the equation: $3^x = 81$, use repeated division to solve for x.

Solution: This problem can be rewritten to ask: "How many times must 3 multiply by itself to get 81?" And one way to solve this would be by division. Instead of seeing how many times 3 must be multiplied by itself, see how many times 3 divides 81 until the multiplicative identity of 1 is achieved.

	81
Divide 1 time	27
Divide 2 times	9
Divide 3 times	3
Divide 4 times	1

3 divides 81 four times, so the answer must be 4. We can check that $3^4 = 81$, which is does.

Try this again: $2^x = 128$. How many times will 2 divide 128 until left with 1?

	128
Divide 1 time	64
Divide 2 times	32
Divide 3 times	16
Divide 4 times	8
Divide 5 times	4
Divide 6 times	2
Divide 7 times	1

It took 7 divisions, so the solution is 7. Check: $2^7 = 128$.

If the solution was not a whole number, this process is good for estimating a solution, but not adequate to find the actual value.

For example: Given $5^x = 1215$

	1215
Divide 1 time	243
Divide 2 times	48.6
Divide 3 times	9.72
Divide 4 times	1.944
Divide 5 times	0.3888

From the work, we can estimate that the solution to the problem will lie between 4 and 5. If we solve this problem using logarithms:

$5^x = 1215$	Take the logarithm of both sides
$\log 5^x = \log 1215$	Bring down the x, using the power property of logarithms, so it is multiplied by $\log 5$
$x \log 5 = \log 1215$	Divide by $\log 5$ to isolate the x
$x = \dfrac{\log 1215}{\log 5} \approx 4.413$	

Figure 8.7: Solving or estimating exponential equations by repeated division

There have been some contemporary researchers who theorise that students should be introduced to logarithms through this method of repeated division (Vos & Bwrge, 2016; Weber, 2019b). That idea does have some basis in past understandings of exponential equations, as seen in sub-section 6.5.1, but did not show up in any of the work surrounding logarithms. It is an intriguing newer idea though and perhaps one that should be studied more.

Logarithmic equations did not appear in the historic texts, so this topic is the last of those found in the introductory chapters on logarithms. There is a lot from past texts that could be incorporated into modern books which could help students make new connections between logarithms and their prior mathematics. Expanding the idea of logarithms to be more than just their relationship with exponents gives another entryway into this difficult concept. It also opens up an understanding of logarithms that could influence how students view their use going forward.

8.2.5. Parametric Equations, Continuity, Limits and Derivation

Given that Napier's formulation of logarithms did not much influence past textbooks, it is perhaps surprising that this formulation could be relevant in modern times. Napier's original construction was based around the kinetic/geometric model of two lines with a point traveling at different velocities as discussed in Section 4.2. While this model would have to be adjusted to fit with the modern understanding of a logarithmic graph, it could be a good inclusion or introduction into parametric equations. Napier was working before the establishment of the rectangular co-ordinate system and his work quite naturally fit into a parametric system. Students could work through a revised version of Napier's work which would re-establish the link between logarithms and the relationship between arithmetic and geometric sequences while also establishing how that relationship could also correspond to the already-learned logarithmic graph (Figure 8.8)

John Napier invented logarithms before the establishment of exponents. He imagined them as two lines, one arithmetic and one geometric, where points would travel along the lines at the same starting rate. On the arithmetic line (the logarithm), the rate would remain constant. On the geometric line, it would increase by a ratio that is in relation to the starting place of the point, for the purposes of this exercise the distance of the point would increase by a constant rate. (Napier actually looked at a decreasing geometric sequence, but we will use an increasing one.)

As he had two lines, both depending on the same factor, *time*, this idea of logarithms can be expressed as a parametric system of equations using t as the parameter. We will say that the geometric sequence, the $x(t)$, starts at 5 and is increasing at a constant ratio of 1/10 with each jump in t. The arithmetic sequence (the logarithm) will be the $y(t)$ and each jump in that line will be a space of 0.5. After, we can graph these equations.

Find the $y(t)$ first. This line has a point that increases 0.5 units for each increase of t. The placement (the y) of the point will be the distance of the point from the start.

Time (t)	0	1	2	3	4
Distance (y)	0	0.5	1	1.5	2

This table shows that the distance is an arithmetic sequence with a common difference of 0.5. We can write that in an equation dependent on time as: $y(t) = 0.5t$

Now we need to find the $x(t)$. This is the line where the point is increasing its space by a factor of $\frac{1}{10}$, or 0.1 at each t. We'll do another table to see if we can figure out an equation.

Time (t)	0	1	2	3
Distance (x)	5	$5 + 5 \times 0.1 = 5.5$	$5.5 + 5.5 \times 0.1 = 5.55$	$5.55 + 5.55 \times 0.1 = 5.555$

This table is not leading to as obvious a pattern as the last one, so maybe we should write it out a different way.

At $t = 1$:
The point would be at: $5 + 5 \times 0.1 \Rightarrow 5(1 + .1) \Rightarrow 5(1.1)$

At $t = 2$:
The point would be at: $5(1.1) + 5(1.1) \times 0.1 \Rightarrow 5(1.1)(1 + 0.1) \Rightarrow 5(1.1)^2$

At $t = 3$
The point would be at: $5(1.1)^2 + 5(1.1)^2 \times 0.1 \Rightarrow 5(1.1)^2(1 + 0.1) \Rightarrow 5(1.1)^3$

It is now a bit easier to see the pattern, it is a geometric sequence starting at 5 and with a common ratio of 1.1. It can be written as $x(t) = 5(1.1)^t$

We now have the following parametric equations: $\begin{cases} x(t) = 5(1.1)^t \\ y(t) = 0.5t \end{cases}$

Writing it this way establishes that this equation is a relationship between a geometric sequence, $x(t)$, and an arithmetic sequence (the logarithm), $y(t)$. Graphing this will help us further 'see' this relationship as a logarithm.

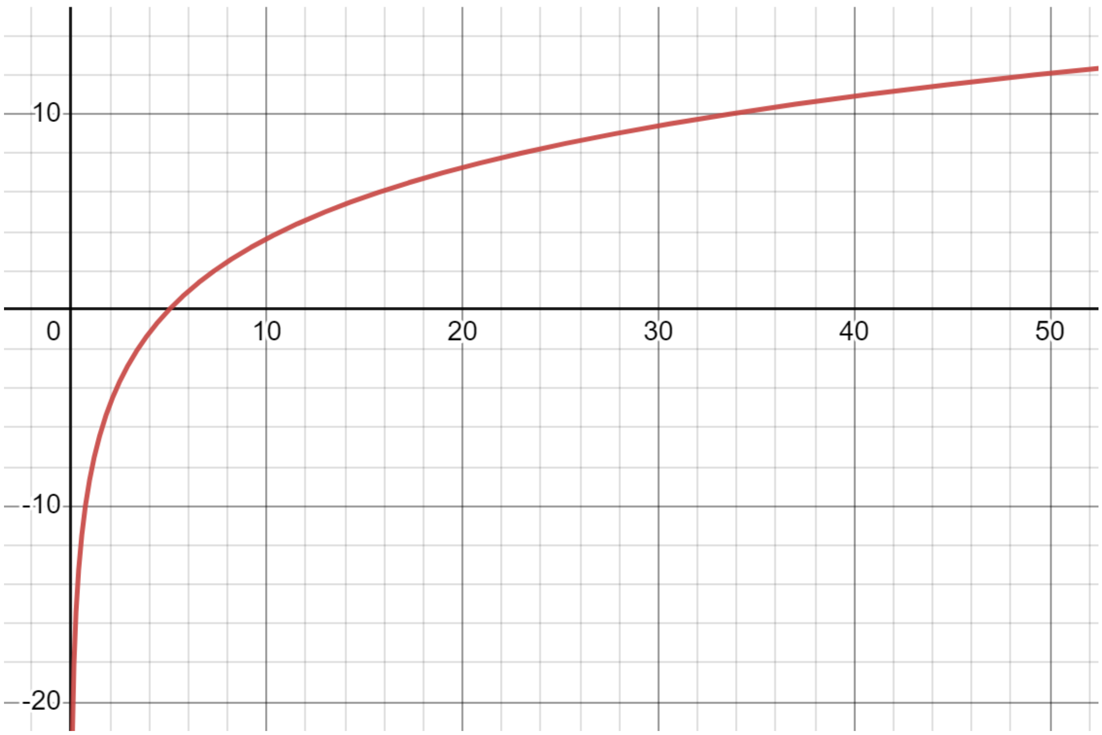


Figure 8.8: Re-imagining Napier's logarithms through parametric equations

Though none of the texts that were reviewed presented parametric equations, the Napier's envisioning of logarithms fits with this topic. Using a modified version of Napier's model brings a little history to the explanation and hopefully further bundles the idea of logarithms and these two sequences. Though the lines that Napier used were mentioned in this exercise, they were not included in lieu of tables as it was easier to present the information. But that is something that could be explored in the future.

This presentation can also reconfirm the ideas that the reader has about continuity and limits in logarithmic functions, coming through the parametric equation. The $y(t)$ has no limits, while the $x(t)$ has a limit approaching 0, and both are continuous. By the time that students study parametric equations, they should have covered the topics of limits and continuity topics, but exploring these ideas presented another way, through another idea of logarithms, should help verify their information.

Lastly, this presentation could be touched upon when starting derivatives. During the time period reviewed in this dissertation (1614–1750), in English language texts, calculus was mainly done in the way of Newton, a method that is not currently used in English language classrooms. Therefore, not much in the study leant itself to the derivation of logarithms, but derivation itself can thank Napier and others like him for furthering the idea of looking at mathematics in a kinetic way, drawing out shapes and lines through the movement of points over time. That idea was one of many that lead to the development of calculus. While this problem (Figure 8.8) does not directly lead to calculus, discussing a modified form of how Napier envisioned logarithms can create new connections between algebra and calculus.

8.2.6. Integration

In his textbook, James Stewart (2006) explored the logarithm being defined as the area under a hyperbola from 1 to x, and others have written about it in research papers (Hamdan, 2008). These authors begin their definition with already knowing that $\int \frac{1}{x} dx = \ln x$ and then working through the properties of logarithms, calculating logarithms and other uses from that starting point. This suggestion by Stewart and Hamdan does have the student making new connections between their current understanding of integration and past algebra, but it does not build the initial connection between the logarithm and the hyperbola.

Following from the idea of logarithms as a relationship between the arithmetic sequence and geometric sequence, and then working through Gregory of Saint Vincent's (1647) work, could create these new connections. Presuming that the readers already have an understanding of the logarithm as a relationship between the two sequences, Figure 8.9 explores how the logarithm is the area under a hyperbola given bounds.

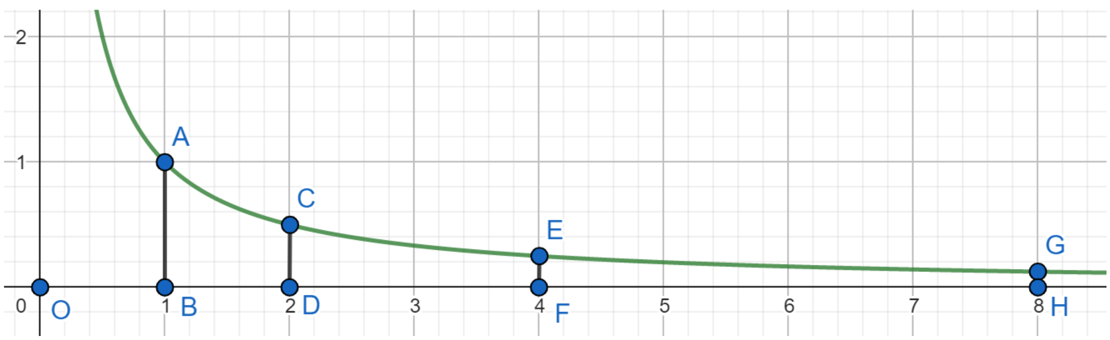
To review, remember that the logarithm is equivalent to an arithmetic sequence that is in concert to a geometric sequence such as with the example below, the common ratio in the geometric sequence is 10, and the common difference in the arithmetic sequence is 1.

Arithmetic Sequence (Logarithm)	0	1	2	3	4	5
Geometric Sequence	1	10	100	1,000	10,000	100,000

The common ratio does not have to be 10, it can also be e, or any other number greater than one, though 10 and e are the most common bases for logarithms. As long as there is a geometric sequence connected to an arithmetic sequence, then that arithmetic sequence can be considered a logarithm.

This idea was used in the past to find the area under a hyperbolic curve, using a process that was somewhat similar to today's Riemann sums.

Given $f(x) = \frac{1}{x}$ and the corresponding graph below, finish drawing in the marked rectangles and then label the width, height, and area in the provided table. As with Riemann sums, one of the right-hand, left-hand or middle must be chosen, use the right-hand part of the rectangle for the height.



	Length	Height	Area
ABDC	1	$\frac{1}{2}$	$\frac{1}{2}$
CDFE	2	$\frac{1}{4}$	$\frac{1}{2}$
EFHG	4	$\frac{1}{8}$	$\frac{1}{2}$

From here we can see that the length is in a geometric sequence with a common ratio of 2, the height is in a geometric sequence with a common ratio of ½, and the area is the same in all rectangles.

Let's go one step further, label the length and the area (using the areas found above) in the below table.

Length	Area (using areas above)
OD = 2	ABDC = $\frac{1}{2} = 0.5$
OF = 4	ABFE = $\frac{1}{2} + \frac{1}{2} = 1$
OH = 8	ABHG = $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$

From this point, it is hopefully apparent that the lengths being in geometric sequence, with a common ratio of 2, leads to the areas being in arithmetic sequence, with a common difference of 0.5.

The rectangles we chose though did not fill much of the area of each segment. You could try this again by choosing to use more rectangles in each area. Try it with 2 in each area. And then 5. And then 10.

Using 2 Rectangles	Area Rectangle 1	Area Rectangle 2	Total Area
ABDC	$0.5\left(\frac{1}{1.5}\right) = 0.\overline{333}$	$0.5\left(\frac{1}{2}\right) = 0.25$	$0.58\overline{333}$
CDFE	$1\left(\frac{1}{3}\right) = 0.\overline{333}$	$1\left(\frac{1}{4}\right) = 0.25$	$0.58\overline{333}$
EFHG	$2\left(\frac{1}{6}\right) = 0.\overline{333}$	$2\left(\frac{1}{8}\right) = 0.25$	$0.58\overline{333}$

Using 5 Rectangles	Area Rectangle 1	Area Rectangle 2	Area Rectangle 3	Area Rectangle 4	Area Rectangle 5	Total Area
ABDC	$0.2\left(\frac{1}{1.2}\right)$	$0.2\left(\frac{1}{1.4}\right)$	$0.2\left(\frac{1}{1.6}\right)$	$0.2\left(\frac{1}{1.8}\right)$	$0.2\left(\frac{1}{2}\right)$	0.6456349…
CDFE	$0.4\left(\frac{1}{2.4}\right)$	$0.4\left(\frac{1}{2.8}\right)$	$0.4\left(\frac{1}{3.2}\right)$	$0.4\left(\frac{1}{3.6}\right)$	$0.4\left(\frac{1}{4}\right)$	0.6456349…
EFHG	$0.8\left(\frac{1}{4.8}\right)$	$0.8\left(\frac{1}{5.6}\right)$	$0.8\left(\frac{1}{6.4}\right)$	$0.8\left(\frac{1}{7.2}\right)$	$0.8\left(\frac{1}{8}\right)$	0.6456349…

We'll not list the areas of each rectangle if you use 10 of them, but your total area for each segment should come to 0.6687714…

In fact, if you use 100 rectangles, your total area for each segment will come to 0.6906534… and if you use 1000 rectangles, the total area for each segment will be 0.6928972… and finally if you do 10,000 rectangles (so getting very close to covering the entire area of each segment!) you get 0.6931221…

Go ahead and use your calculator to find the natural logarithm of 2. You should get 0.6931471…, that is the true area under each of these segments. We got close to there, but even with a 1000 rectangles, we were still a bit off.

This idea of an area under the hyperbola will be true no matter how small you make the common ratio. Therefore, like Riemann sums, you could choose a very small number so the rectangles occupy most of the space under the hyperbola and as long as the lengths are in geometric sequence then the areas will be in arithmetic sequence, therefore the areas are a logarithm. Choosing a small ratio is actually one of the foundations of e.

Figure 8.9: The area under a hyperbola as a logarithm

This idea of logarithms is of course very important with integrals, but it can also lead to deriving a series for logarithms, and could perhaps fit into sections on the MacLaurin series. It also brings a new geometric idea to e. Most of the other topics in derivation and integration, such as L'Hopital's rule, depend on the properties of logarithms, so while these topics were a large part of Chapter 5, there were not concepts from historical texts that specifically speak to these ideas. Though the previous suggestions based around properties of logarithms should strengthen that concept which will hopefully lead to an easier time with the other topics, such as L'Hopital's Rule.

8.3. Last Notes on Incorporating Historic Conceptions of Logarithms into Modern Lessons

There were other topics that were discussed in the study of contemporary textbooks that were skipped in this chapter, mainly Polar Equations and Hyperbolic Trigonometric Equations. In the review of past texts, I did not find any new ways to address them, but that does not mean that new ways do not exist, just that they are still to be discovered. But all other topics were covered in this section: the historic definitions of logarithms as 'artificial numbers', 'based on operations', 'common differences relating to common ratios' and 'a relationship between two sequences' could all be utilised to create new connections between logarithms and other aspects of mathematics.

While logarithms are currently overwhelmingly associated with exponents, exponential functions and exponential equations in modern times, broadening their understanding can invite students who have a weak understanding around exponents to still engage with logarithms. These new connections include relating logarithms to sequences, to ratios and proportions, to numerical operations, to linear and exponential functions, to areas of rectangles, and to repeated division. These new ideas about logarithms can perhaps even lead to new ways of visualising or using this concept. Logarithms will most likely still be taught primarily through their relationship with exponents, but adding in these other ideas could enhance a student's concept image so they have a fuller understanding of this tricky topic.

Chapter 9. A 'Power'ful Ending

In this last chapter, I return to the research questions listed in Chapter 1 and present a summary of the analysis and discussion that took place in the individual chapters relating to each of those questions. I also look at the limitations around this study, as well as discuss the contributions this study makes to the discipline of mathematics education. I also suggest some ideas for future studies.

9.1. Response to Research Question 1

How are logarithms currently presented in textbooks? What connections do they have to a student's past work in mathematics? What conceptions of logarithms are continued as a student continues further into mathematics?

The focus of Chapter 5 was on the presentation of logarithms in modern-day textbooks. They are most often introduced as an exponent, inverse of an exponential function or a way to solve an exponential equation. Logarithms are primarily defined through a formulation similar to Euler's definition, where $a^x = b \Leftrightarrow x = \log_a b$, so again related to solving exponential equations. Their properties are derived through the properties of exponents, and they are generally graphed in a way that presents them being the inverse of the exponential function. Solving logarithmic and exponential equations steps away a little from their association with exponents and connects with their being labeled a function. Overall, the introduction of logarithms to new students relies heavily on both their connection to exponents and to their status as a function.

After their initial intro, logarithms continued to be seen in subsequent textbooks. These textbooks primarily take advantage of the properties of logarithms to simplify expressions, and do not go back to introduce the properties in any other way but do reinforce them by their continued use. However, when students begin topics related to differentiation and integration, logarithms do get another connection in hyperbolas. This connection is not often explained geometrically, but again through logarithms connections to exponents which students are expected to make use of going forward.

The primary connections that logarithms have to students' past mathematics is to exponents, exponential functions and equations. All subsequent work with logarithms relate back to this connection. Logarithms are also firmly established as a function, and aspects of functions are later used to help conceptualise the logarithmic function as a graph and how to use logarithms in an equation. Lastly, logarithms are related to hyperbolas, but this connection is usually brought to life through exponents, though the tenuous thread tying these together is usually cut and the logarithm-hyperbola connection is expected to stand on its own.

9.2. Response to Research Question 2

How are logarithms presented in historic texts and textbooks? How does their presentation affect their use in those texts?

Chapter 6 showed that logarithms were presented in myriad ways in historic texts spanning the years 1614 to 1750. These sources could be grouped into the description of ‘artificial number’, or into definitions including: they are numbers that simplify operations, they are an arithmetic progression connected to a geometric progression, they are a sequence of numbers that equally differ in relation to a sequence of numbers in proportion, they are the number of ratios and they are an exponent. It was infrequent for logarithms to have a single definition, for the most part they were used in combination to explore the concept better. These definitions appeared in all sorts of texts that included logarithms, including those that could be classified as mathematics texts, scientific texts and economics texts.

Each definition led to a different way of justifying the properties of logarithms, which included their definition if the definition was based around the operations, or by using the idea of middles and extremes or by showing examples based around the progressions. There was one text that explored the properties through breaking down numbers into various proportions, but it only appeared once. There were two attempts to justify the properties through their connection with exponents, but neither was explored in much depth. I presume the work connecting logarithms as exponents and the properties of logarithms emerged in texts that are outside the scope of this study.

Logarithms were not much used in solving exponential equations, and when they did appear, the process was not explained outside the relationship to the properties of logarithms. For the most part, though, proportions were the main mechanism to solve these types of equations, while logarithms were used as a way to simplify calculations. There were a few examples in which equations were solved by taking the logarithms of each side, though this was explained through their operational definition. And there was example in which exponential equations were solved by repeated division, but that was not necessarily tied to logarithms.

Logarithms were discussed much more in these texts than appear in this study; they were primarily a tool to simplify calculations, but there were also sections about how to use the tables of logarithms, on how to calculate the logarithms that appear in the tables and a small bit of Newtonian Calculus that focused on logarithms. As none of these topics are currently included in modern mathematics, I therefore chose not to focus on them.

There were also a few topics missing from these texts that I found surprising.. There were no logarithmic equations, so there were no processes to solve these equations. The relationship between logarithms and hyperbolas did not appear much in these texts, and when it did it was simply to state the relationship as a fact. There was no attempt to graph the modern logarithmic graph, instead what was called the 'logarithm graph' is what modern mathematicians would recognise as a graph of an exponential function.

9.3. Response to Research Question 3

What is the etymology of the term 'logarithm'? How did that name interact with the meaning and use of logarithms? How can that information be useful to students and teachers today?

Chapter 7 explored the etymology of the word *logarithm* and discovered that it is not an easy word to translate today or in the past. It comes from the two root words, *logos* and *arithmos,* but there is not agreement on the translation of *logos,* while *arithmos* is universally translated as 'number'. Those closest to Napier translated *logos* as 'reason', 'proportion', or 'ratio', but others have different translations based around 'word'. Some of these translations have been utilised when discussing logarithms in the historical texts included in this study [1614–1850] to various degrees of success. Some of the translations worked well with the given definition of logarithms, while for others, the translation was provided, but not fully incorporated into the description of the concept. There were times when the translation was provided, and the author specified that they did not understand why this word *logarithm* was chosen for this concept. In that respect, these authors mirror contemporary students.

There are ways that the translation of *logarithm* could be explored by teachers and students. For example, the ‘reason’ for them is to ease ‘numerical’ calculations; or they give ‘voice’ to ‘numerical’ operations; or they are the ‘number of ratios’; or they are related to a sequence of ‘numbers in proportion’. I am sure there are more, but these are the translations that would most easily fit into today’s classroom. While ‘number of ratios’ could make sense with the exponent relationship to logarithms, which is the main connection for contemporary students, in order for the other translations to have meaning, new connections between other mathematics and logarithms need to be forged. Looking at the relationship between sequences would give meaning to ‘numbers in proportion’ and ‘number of ratios’, while exploring the properties of logarithms could explain the other two translations.

As stated in Section 7.4, I am unsure that the term *logarithm* must have meaning derived from its translation. Students and teachers do not seem to have difficulty with the word *exponent,* whose etymology was explored in sub-section 4.8.2: they accept the modern mathematical meaning of the word. However, as this does seem to be a point of contention in various studies around logarithms, so perhaps these translations and the expressed connections to other mathematics could give the word more meaning to today’s students and teachers.

9.4. Response to Research Question 4

How could the historic conceptions of logarithms tie into students understanding and use of logarithms today? Do they bring in new connections to students’ past mathematics?

Chapter 8 explored a few ideas about how historical conceptions of logarithms could better fit into today’s classrooms and textbooks. Returning to the definition of logarithms through the relationship between the arithmetic and geometric sequences could give students a new connection to logarithms, while still being able to draw the line to exponents. This connection to sequences could build into properties of logarithms and even to the properties of exponents. This connection could even allow students to explore some of the constants of logarithms, such as $\log 1 = 0$, from a perspective removed from exponents. And importantly, exploring logarithms through the relationship between sequences can proovide some clarity to the relationship between the logarithm and the area under a hyperbolic curve.

While the relationship between sequences is perhaps the easiest definition to incorporate into a contemporary classroom or textbook, there are other aspects of logarithms that were used in the past that could enable new mathematical connections for today's students. The relationship between logarithms and repeated division in solving exponential equations gives a new operation to logarithms, while actually tying it closer to the inverse of exponentiation. Viewing logarithms through Napier's construction, via a set of parametric equations can give a visual/kinectic look at this function. And exploring how historical figures calculated logarithms can provide students with a new appreciation and understanding of a logarithm as a number, beyond just a button on the calculator.

Sprinkling these connections into a contemporary classroom or textbook could help students come to logarithms through a new pathway allowing those that are having trouble understanding the logarithmic function as the inverse of an exponential function to still have a relationship with the concept of a logarithm. Quite a few of the suggestions in Chapter 8 focus on the idea of the logarithm as a number, which is something that Chapter 3 showed was problematic for many students. By talking about the invented nature of these artificial numbers, and coming to the properties of logarithms by manipulating numbers, or by calculating the logarithm, students could have a better grasp of a logarithm as a number. However, most of the suggestions made in Chapter 8 still connect to the idea of the logarithm as an exponent, so incorporating any of these suggestions would help build up the primary relationship that logarithms have today.

9.5. Limitations of this Study

Although there are several limitations to this dissertation, there were also a few surprises. Many of the limitations were related to the design of the study itself, but they should be examined as these limitations could lead to interesting future studies.

9.5.1. Language and Geography

The most obvious limitation from this set of studies is language. As I am primarily an English language speaker, I did not include any texts written in other languages in the studies described in Chapters 6 and 7. This exclusion was very limiting in the early decades of logarithms, as Latin was a primary language for scholarly texts and, again, as the study approached Euler's time since he wrote in French. Future studies should be done on textbooks in those languages, as well as in German, as that was the language of Bürgi and Leibniz, as well as any other languages that may be of interest to the researcher.

The limitation of language also showed during the study of modern textbooks in Chapter 5, as I immediately had to remove Quebec textbooks from the study as they are written in French. Geography was also a big limitation in Chapter 5, as I only focused on countries where I was familiar with their school systems. I am from the United States, studying in Canada, and it felt necessary to include Great Britain due to it being the place that founded and established logarithms. Luckily, when I was 10 years old, I attended school in Birmingham, England, and also had a supervisor throughout this process who is British.

Even within the countries that were included, I needed to find a way to limit the texts. I chose to review textbooks from the most populous areas in each country (if the textbooks were not nationalised), but there are many other ways the restrictions could have been drawn. For example, Future scholars could look at the regions with the highest or lowest mathematics scores and review their associated textbooks. There remains a lot of space for research into logarithms in modern textbooks, both in the countries I chose and, obviously, in those that I could not, or did not, include here.

9.5.2. Time-span

The time-span for the historic textbooks in Chapter 6 was narrowed to cover the founding of logarithms (1614) up through the decade where logarithms were defined as an exponent (1750). As seen in Chapter 7's study that went to 1850, logarithms were not exclusively viewed as being related to exponents for quite some time, at least in the larger community of dictionaries and encyclopaedias. A future study exists for textbooks after this range. There could also be a study done on how problems similar to those today were done prior to the invention of logarithms. There is room on both sides of the span, before 1614 and after 1750, to expand upon this study.

9.5.3. Tools and Tables

While the use of logarithmic tools, such as slide rules, was touched upon in Chapter 4, these were not included in any of the studies that followed. Their absence was actually surprising, as I expected them to appear in the historic textbook review in Chapter 6, but not a single text discussed how to use slide rules or use Gunter's Line. A few textbooks mentioned them in relation to an exercise, but the reader was expected already to be familiar with these tools.

Many of these same textbooks, though, did go into detail on how to use the logarithmic tables that were present in the book, or would recommend their own favourite set of tables. Though I did begin to code the presentation on how to use the tables, I soon realised that, while their use may help students better understand that a logarithm can be a number, I could not envision how this lesson would build into any other modern topic around logarithms. Therefore, I decided to focus on the introduction and definition of logarithms, as well as how their properties were established, and how they were used in solving equations.

Both of these topics, the use of tools and tables, are still ripe for further study. Using the tables does go hand-in-hand with the properties of logarithms, as well as calculating logarithms, and therefore is something that should be considered. And while the use of tools did not appear in the texts that I reviewed, they may have appeared in others, and also deserve further study on how including manipulatives into the teaching of logarithms could affect student understanding.

9.5.4. Level of Mathematics

In reviewing modern textbooks, I began the study at the first introduction of logarithms and then worked to follow the concept through subsequent texts that still explored logarithms, up to their use in integration. I stopped at integration because that topic was covered in at least one text from each country that was reviewed. However, there were topics that were included in some textbooks from a geographic area that were not included in others. These topics should be included in a future study, as they include the topic of 'Sequences in Series'. Going into more detail on how logarithms are presented in a single progression of texts, or expanding the scope to include all the topics that could have been covered, would be an interesting in-depth future study.

Similarly, with the historic texts in Chapter 6, there were many more topics included in the texts than were chosen to review. Again, I looked for those topics that appeared consistently, and this time I also looked for topics that still had a place in modern mathematics. But there are still topics about logarithms, such as using the tables discussed above, that should be studied further.

Lastly, I avoided most calculus texts from the 1600s and 1700s in this dissertation. The main impetus for this absence is that, at the time, calculus in the English language texts primarily meant Newtonian Calculus. While these texts were interesting, and I did review quite a few of them, the span from Newtonian Calculus to the calculus we use now, when paired with also trying to understand historic views of logarithms, felt too great. This area is ripe for further study though. I did a deep dive into Colin MacLaurin's *A Treatise of Fluxions* where he used Napier's two-line model of logarithms in his defense of Newtonian Calculus, and it should be explored. While I felt the bridge was too great, I am sure that there are other scholars out there that could cross it.

9.6. Contributions and Further Study

While there were some necessary and some unexpected limitations to this study, there is still much in this dissertation that could be of interest to other scholars. As I reflect back upon each chapter, I want to highlight the work that I feel could help other researchers in this area.

9.6.1. Theory

In Chapter 2, the theoretical framework of Multiple Perspectives was explored, though mainly as it related to historical research. I think it also fits in with using history to enhance modern mathematics. As teachers, we should look to many sources to improve our explanations to our students, which includes our textbooks, trusted internet sources, colleagues, peers and texts in applicable areas of employment and, of course, the history of how the topic was discovered and developed. Looking for these explanations through history may also humanise logarithms, and mathematics as a whole, by slowly giving a story of their invention and evolution. But we should also look to the texts that would never appear in any 'official' history. In this dissertation, I have attempted to show that, historic textbooks can add a lot of information that is missed when one focuses on the better known texts. This information maybe did not lead to the development of the topic but it did round out the concept in a way that makes it easier to understand.

9.6.2. History of Logarithms

While, in Chapter 4, I used many contemporary sources to make sense of the mathematics of the past, most of the sources skipped over much of the detail. I found myself drawn back to the original sources, but also to other contemporaneous texts to try and fill in the missing pieces. Sub-section 4.2.2 explains Napier's work with logarithms in a way that I had not seen before. I worked through his text, through every English translation I could find of his text, as well as Colin MacLaurin's treatise to find methods, both equations and visuals, that hopefully helped the reader comprehend Napier's vision of logarithms.

Section 4.6 discussed Edmond Halley and his method for calculating logarithms. His method was included in many tables of logarithms in the past (all of Sherwin's tables for example) and yet I did not find a contemporary discussion of Halley's work. I pored over his text in an effort to understand the mathematical leaps that were most likely common in his time, but foreign to me now. There were manipulations with numbers that were second-hand to Halley, that having an understanding of today could open up new avenues of study. His initial set-up of a logarithm as a specific root is an example of one such avenue that may be worth exploring moving forward.

The other items in this section that felt 'new' when researching and writing this chapter included adding Saunderson and Dodson to the list of texts that first defined logarithms as an exponent. William Jones is credited with writing, or helping to write, Dodson's definition of logarithms, which if added to Gardiner's text, strongly implies he was a primary communicator of this new way of thinking, as was often shown in my research. But Dodson's text had not been included in any of the history of logarithm timelines that I encountered, and Saunderson's text, published two years earlier was missing completely. Now that so many of these historic texts are becoming more accessible, I would expect others to appear in the future that will further fill in the story of logarithms.

9.6.3. Current Textbook Analysis

Chapter 5 looked at the state of presenting logarithms in textbooks that are currently in use in a classroom, for a specified geographic area and language. Textbook analysis for logarithms have been done, even recently, but usually confined to one textbook or one country. This widespread analysis, going from the introduction of logarithms to their use in integration, in three distinct geographic areas is something I hope scholars can draw from in their research. It is also something that I hope other scholars will repeat, as times and ways of presenting logarithms change.

9.6.4. Historic Text Analysis

Chapter 6 is a newer style of study around the history of logarithms, where it does not solely look at how logarithms were understood by the mathematicians that wrote the texts, but also by those that authored texts meant for students of all calibres. Doing a widespread study on the presentation of logarithms in textbooks introduced some new ways of viewing and working with logarithms. Foremost among these was the realisation that logarithms retained multiple definitions throughout this span (1614–1750), even when a new definition was introduced, the authors still found reason to keep older ones, with the exception of the 'equal differences' definition. I had expected one definition to supersede the others, so this was unexpected and interesting.

There were two ways of working with logarithms that I did not see in my research into the history of logarithms. One was repeated division, which contemporary scholars have discussed, as mentioned in Chapter 3, but I was not expecting to see it in one of the historic textbooks. The other method was the work around the properties of logarithms done with ratios in sub-section 6.4.4. While a logarithm as the number of ratios was familiar to me as a result of my research, the method used in this section was new and could be another pathway into working with logarithms for a future scholar.

9.6.5. The ‘Logarithm’ term study

The study in Chapter 7 came from attempting to find the ‘true’ etymology of the word *logarithm* in response to a supervisor’s query, and so that it could be included in my introduction. After hours of researching both contemporary and historic sources, I decided to include my struggle as a chapter in this dissertation. The known translations are not new to any researcher of logarithms, but the combination of the translations, the authors’ thoughts, and the authors’ subsequent definition and use of logarithms is unique in this field. Similar to sub-section 9.6.4, I expect that, as more historic texts become readily accessible, this study may need to be updated.

9.6.6. Combining Historic and Contemporary Ideas of Logarithms

Chapter 8 is the main output of the combination of studies that make up this dissertation. The suggestions pull from Chapters 4, 6 and 7 hopefully to create new connections to the mathematics surrounding logarithms that were discussed in Chapter 5. Some of these suggestions have been made before, such as using the relationship between sequences to explore the properties of logarithms, while some of the others are unique. Specifically, any work around calculating logarithms in sub-section 8.2.2, the work around justifying $\log 1 = 0$ in sub-section 8.2.3, and the idea of Napier’s construction as a set of parametric equations in sub-section 8.2.5, are suggestions I had not previously seen, that should be explored further.

9.7. Last Notes at the End of this Dissertation

As I reflect back on the process of researching and writing this dissertation, I want to address a few lessons I have learned and where I hope to go with it in the future. As discussed in Chapter 1, it took awhile to find a focus for this dissertation. While I knew relatively early that I was interested in the history of logarithms, combining it with a textbook analysis came later. Through this process, I have been reintroduced to the importance of textbooks, both for students and instructors, especially new instructors. I am interested in the future in looking at other aspects of modern textbooks, specifically the connections that the authors make among the mathematical content in their texts.

In completing this process, I have learned that I love much of the research required in writing a dissertation, I can immerse myself for years in a specific subject, following any trails to which the research leads. I also enjoy synthesising the information for a reader and using that information to promote new ideas. The suggestions provided in Chapter 8 lead to the fact that this is a dissertation of ideas and arguments, not of action. These are a collection of studies based around texts, which lead to some ideas for new ways of presenting logarithms, but none of these suggestions have yet been tested. This study does not feel like it is yet complete. The next step would be to try these suggestions, either in written form as shown in this chapter (though most likely redone with a small committee), or presented in a classroom by a teacher. I hope to pick this study up in the future, though I also welcome any other interested researchers to do so, and how that they send along results.

As I plan to teach Adult Basic Education, I feel I will use this study as a blueprint for future studies based around my students' struggles. Researching the history, both the 'official' and from alternative sources can bring new conceptions of a difficult topic. I hope to be in a space where I can facilitate pilot studies around my research with students and also, perhaps, expand my research into a series of textbooks.

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Van Poelje, O. (2004). Gunter rules in navigation. *Journal of the Oughtred Society, 13*(1), 11-22.

Vinner, S. (1975). The naive platonic approach as a teaching strategy in arithmetics. *Educational Studies in Mathematics*, *6*(3), 339-350.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, *14*(3), 293-305. https://doi.org/10.1080/0020739830140305

Vinner, S. (1991). The Role of Definitions in the Teaching and Learning of Mathematics. In D. Tall (Ed.). *Advanced Mathematical Thinking* (pp. 65-81). Springer Netherlands. https://doi.org/10.1007/0-306-47203-1_5

Vinner, S. (2008). Beyond Curriculum and Evaluation-Mathematics Education for Example. In J. Casey & R. Upton (Eds.), *Educational Curricula: Development and Evaluation (*pp. 183-199). Nova Science Publishers, Inc.

Vinner, S. (2018). The Role of Examples in the Learning of Mathematics and in Everyday Thought Processes. In M. Danesi (Ed). *Mathematics, Education, and Other Endangered Species. Mathematics in Mind*. Springer, Cham. https://doi.org/10.1007/978-3-319-90035-3_9

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for research in mathematics education*, *20*(4), 356-366. https://doi.org/10.5951/jresematheduc.20.4.0356

Vollstedt, M., Heinze, A., Gojdka, K., & Rach, S. (2014). Framework for examining the transformation of mathematics and mathematics learning in the transition from school to university. In *Transformation-a fundamental idea of mathematics education* (pp. 29-50). Springer, New York, NY.

Vos, P., & Espedal, B. (2016). Logarithms - a meaningful approach with repeated division. *Mathematics Teaching, 251*, 30-33.

Vural, D. (2021). Investigating History of Mathematics for Teaching Mathematics: The Case of Logarithm. *Erciyes Journal of Education, 5*(2), 208-220. https://doi.org/10.32433/eje.1004600

Waldvogel, J. (2014). Jost Bürgi and the discovery of the logarithms. *Elemente der Mathematik, 69*(3), 89-117.

Wallis, J. (1656). *Arithmetica Infinitorum [Arithmetic of the Infinite]*. Printed for Tho. Robinson

Wallis, J. (1668). An account of two books. […] II. Logarithmotechnia Nicolai Mercatoris. Concerning which we shall here deliver the account of the Judicious Dr. I. Wallis, given in a letter to the Lord Vis-count Brouncker, as follows. Some illustration of the Logarithmotechnia of M. Mercator, who communicated it to the Publisher, as follows. *Philosophical Transactions Royal Society, 38*(3), 750-764. https://doi.org/10.1098/rstl.1668.0034

Wallis, J. (1685). *A treatise of algebra, both historical and practical […]*. Printed by John Playford for Richard Davis.

Ward, J. (1695). *A Compendium of Algebra. Consisting of plain, easie and concise rules for the Speedy Attaining to that Art […]*. Printed by John Ward for John Ward.

Ward, J. (1710). *Clavis Usuræ: Or, A Key to Interest both Simple and Compound […]*. Printed for J. Cecill.

Webb, D., van der Kooij, H., & Geist, M. (2011). Design Research in the Netherlands: Introducing Logarithms Using Realistic Mathematics Education. *Journal of Mathematics Education at Teachers College, 2*(1), 47-52. https://doi.org/10.7916/jmetc.v2i1.708

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Weber, C. (2019a). Comparing the structure of algorithms: the case of long division and log division. In U. Jankvist, M. Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 698-705). Freudenthal Group; Freudenthal Institute.

Weber, C. (2019b). Making sense of logarithms as counting divisions. *The Mathematics Teacher*, *112*(5), 374-380. https://doi.org/10.5951/mathteacher.112.5.0374

Weber, K. (2002). Developing Students' Understanding of Exponents and Logarithms. In D. Mewborn, P. Sztain, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.) *Proceedings of the 24th Annual Meeting [of the] North American Chapter of the International Group for the Psychology of Mathematics Education*, 2, 1019-1027. Athens, United States.

Wilkinson, T. T. (1853a). XXIX. Mathematics and mathematicians. The journal of the late Reuben Burrow. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 5*(31), 185-193. https://doi.org/10.1080/14786445308647223

Wilkinson, T. T. (1853b). LXXX. Mathematics and mathematicians. The journal of the late Reuben Burrow. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 5*(35), 514-522. https://doi.org/10.1080/14786445308647299

Wilson, J. (1714). *Trigonometry: With an Introduction to the Use of Both Globes, and Projection of the Sphere in Plano. […]*. Printed by James Watson.

Wingate, E. (1635). *Λογαριθμοτεχνία, or The construction and use of the logarithmeticall tables […]*. Printed for Miles Flesher.

Wingate, E. (1645). *The Use of the Rule of Proportion: In Arithmetique and Geometrie […].* Printed by M.F. for P. Stephens.

Wolfius, C. (1739). *Elementa matheseos universæ [Universal Mathematical Elements]. A Treatise of Algebra […]*. Printed for A. Bettesworth and C. Hitch.

Wood, E. (2005). Understanding logarithms. *Teaching Mathematics and its Applications, 24*(4), 167-178. https://doi.org/10.1093/teamat/hrh023

Appendix A:

Alternative Method of Creating Logarithms

In Sections 4.2, 4.3, 4.6, and Appendix A, ideas about calculating logarithms and creating the tables were discussed. In my review of historic textbooks, 25% would include a section on calculating logarithms and creating the tables. The most common methods used in these texts were Briggs' repeated root of 2 (Section 4.3), and Halley's series (Section 4.6), but there was a third way that appeared a little under a quarter of the time. I first noticed this method in the text by Johann Christoff Sturm (1700), who attempted to tie together the traditional math leading back to the Ancient Greeks with the new mathematics which included Isaac Newton.

I will call this method 'repeated means', and in the text Sturm credits Vlacq, saying that it is a way he created his tables. More recent scholarship, though, has come to the conclusion that Ezechiel De Decker invented many of the ways that logarithms were being determined, though it was in tables credited to Vlacq (Miller, 1979). While it is not spelled out in the Vlacq's text, scholars have uncovered some of the methods by looking through personal notes and texts by De Decker. One of the methods to find the logarithms of large primes uses the idea of means between two known logarithms. This method will not be correct to as many significant figures as Briggs' method without either more work or taking into account the differential, which is the way that De Decker approached it (Roegel, 2010b).

For students though, this method is perhaps easier to understand than Briggs' method. Sturm (1700) wrote his explanation for the use of students, so they would have an idea on how logarithms were calculated, therefore, he focused on the logarithms of smaller numbers, specifically 9 (which is the same in every other text which this appears). The student would start with two known logarithms, 1 and 10, and take the geometric and arithmetic means of them. To find the geometric mean, the two numbers are multiplied and then the square root taken of the product. The arithmetic mean adds the two numbers and then takes half of the sum. If the geometric mean was less than 9, than they would repeat the procedure with the new number and 10. If it was ever above

9, then they would repeat with the new number and the one preceding it, and then continue with the appropriate upper and lower bounds (Table A.1).

Table A.1: Method to calculate the logarithm of 9 using repeated means

Lower Bound	Geometric Mean	Upper Bound	Lower bound Logarithm	Arithmetic Mean (Logarithm)	Upper Bound Logarithm
1	3.16227766…	10	0	0.5	1
3.16227766…	5.623413252…	10	0.5	0.75	1
5.623413252…	7.498942093…	10	0.75	0.875	1
7.498942093…	8.659643234…	10	0.875	0.9375	1
8.659643234…	9.305720409…	10	0.9375	0.96875	1
8.659643234…	8.976871324…	9.305720409…	0.9375	0.953125	0.96875
8.976871324…	9.139816995…	9.305720409…	0.953125	0.9609375	0.96875
8.976871324…	9.057977759…	9.139816995…	0.953125	0.95703125	0.9609375
8.976871324…	9.017333353…	9.057977759…	0.953125	0.955078125	0.95703125
8.976871324…	8.997079593…	9.017333353…	0.953125	0.954101563	0.955078125
8.997079593…	9.00720078…	9.017333353…	0.954101563	0.954589844	0.955078125
8.997079593…	9.002138764…	9.00720078…	0.954101563	0.954345703	0.954589844
8.997079593…	8.999608823…	9.002138764…	0.954101563	0.954223633	0.954345703
8.999608823…	9.000873705…	9.002138764…	0.954101563	0.954284668	0.954345703
8.999608823…	9.000241242…	9.000873705…	0.954223633	0.95425415	0.954284668
8.999608823…	8.999925027…	9.000241242…	0.954223633	0.954238892	0.95455415

With fifteen rows, the logarithm of approximately 9 is approximately 0.95424, which is correct to that many digits. More rows would have to be done in order to have a more accurate logarithm. From here, students could now find the logarithm of 3 by halving 0.95424, and subsequently the logarithm of any power relating to 3. This method could then be repeated to find the logarithms of the other primes.

While it does not seem that this was an often-used method, it did come up in multiple texts (Sturm, 1700; Ozanam, 1712; Wilson, 1714; Malcolm, 1730; Kirkby, 1735; Martin, 1740). The method fits nicely with the idea of logarithms being an arithmetic progression attached to numbers in a geometric progression. Personally, I find it easier to follow and explain than some of the more intense methods discussed in Chapter 4 and if modern-day students were to broaden their understanding of a logarithms to include these progressions, it would be a way that they could discover how the logarithms of numbers are constructed using just a simple calculator and the operations of logarithms.

Appendix B:

List of Texts Analysed in the Chapters 5, 6, and 7

Below is the list of all the texts analysed in the above chapters. I did not include these in the references as most of them were not cited in the dissertation, but I wanted to include them in the Appendix for any future scholars who would like to attempt a similar study. As those listed for the Chapter 5 study are modern textbooks, I have followed the APA reference style, for the other two studies I organised the texts by year and listed the attributes that would be most helpful in searching for the text.

Chapter 5 Study: Textbooks in Use at the Time of this Dissertation

Thirteen textbooks used in the analysis for the first course where students encounter logarithms

1. Abramson, J. (2021). *College Algebra 2e.* Rice University, OpenStax.
2. Attwood, G., Barraclough, J., Bettison, I., Macpherson, A., Moran, B., Nicholson, S., Oliver, D., Petran, J., Pledger, K., Smith, H., Staley, G., Ward-Penny, R., Wilkins, D. (2017). *EdExcel AS and A level Mathematics: Pure Mathematics Year 1* (ed. H. Smith). Pearson Education Limited.
3. Barclay, R., Logan, B., & Smith, M. (2021). *Higher Maths, Second Edition: Boost Ebook.* Hodder Gibson
4. Beveridge, R. (2018). *College Algebra & Trigonometry*. University of Minnesota, Open Textbook Library.
5. Buckle, N., Dunbar, I. (2004). *International Baccalaureate Mathematics Higher Level (Core) 3rd Ed.* (ed. F. Cirrito). IBID Press.
6. Ji, X., Ge, M. (2015). *College Algebra: Special Equations, Functions, and Logarithms.* Cognella Academic Publishing.
7. Kennedy, D., Milou, E., Thomas, C., Zbiek, R. (2020). *Envision Florida Algebra 2.* Pearson Education.
8. Kirkpatrick, C., Aldred, B., Chilvers, C., Farahani, B., Farentino, K., Lillo, A., Macpherson, I., Rodger, J., Trew, S. (2012). *Advanced Functions*. Nelson Publishing.
9. Larson, R., Boswell, L., Kanold, T., Stiff, L. (2007). *Algebra 2 Texas Edition.* Mcdougal Littell, a division of the Houghton Mifflin Company.
10. Larson, R., Boswell, L. (2014). *Big Ideas Math Algebra 2 A Common Core Curriculum.* Big Ideas Learning.

11. McAskill, B., Watt, W., Balzarini, E., Johnson, B., Kennedy, R., Melnyk, T., Zarski, C. (2012). *PreCalculus 12.* McGraw-Hill Ryerson.

12. Stitz, C. & Zeager, J. (2013). *College Algebra, 3rd Edition*. Kirtland, OH: Stitz Zeager Textbooks.

13. Sullivan, M. (2012). *College Algebra, 9th Edition*. Prentice Hall.

Textbooks used in subsequent courses where students encounter logarithms

1. Abramson, J. (2014). *Precalculus.* Houston, TX: Rice University, OpenStax.

2. Attwood, G., Barraclough, J., Bettison, I., Gallick, K., Goldberg, D., Mcateer, A., Macpherson, A., Moran, B., Petran, J., Pledger, K., San, C., Smith, H., Staley, G., Wilkins, D. (2017). *EdExcel AS and A level Mathematics: Pure Mathematics Year 2* (ed. H. Smith). London: Pearson Education Limited.

3. Buckle, N., Dunbar, I. (2004). *International Baccalaureate Mathematics Higher Level (Core) 3rd Ed.* (ed. F. Cirrito). Victoria, Aus: IBID Press.

4. Collingwood, D., Prince, K.D., & Conroy, M. (2011). *PreCalculus*. Minneapolis, MN: University of Minnesota, Open Textbook Library.

5. Feldman, J., Rechnitzer, A., & Yeager, E. (2016). *Differential Calculus.* Vancouver: University of British Columbia.

6. Feldman, J., Rechnitzer, A., & Yeager, E. (2017). *Integral Calculus.* Vancouver: University of British Columbia.

7. Guichard, D., Bailey, J., Blenkinsop, M., Cavers, M., Hartman, G., & Ling, J. (2020). *Calculus: Early Transcendentals, 2021A version (Lyryx)*. Lyryx Learning.

8. Larson, R. & Hostetler, R. (2007). *Precalculus 7th edition*. Boston, MA: Houghton Mifflin.

9. Lippman, D. & Rasmussen, M. (2021). *Precalculus An Investigation of Functions.* Washington Open Course Library.

10. Saint Andrew's Academy Department of Mathematics (2020). *Advanced Higher Course Textbook Volume 1 & 2*. Paisley.

11. Stewart, J. (2006). *Calculus: Early Transcendentals, 6E*. Belmont, CA: Thompson Brooks/Cole

12. Stewart, J., Redlin, L., Watson, S. (2009). *Precalculus: Mathematics for Calculus, Fifth Edition.* Belmont: Brooks Cole/Cengage Learning.

13. Stitz, C. & Zeager, J. (2013). *Precalculus, 3rd Edition*. Kirtland, OH: Stitz Zeager Textbooks.

Chapter 6 Study: Historic Texts and Textbooks

Listed are the 80 texts analysed in Chapter 6 listed by year. I have also notated if they were primarily for academics ([a]) of that time, for the use of students ([s]), or texts that were mainly application based, such as tables of logarithms ([t]). The 50 texts that were examined more closely are notated by a star (*).

YEAR	AUTHOR	TRUNCATED TITLE
1618[a*]	Edward Wright	A Description of the Admirable Table of Logarithmes: With a Declaration of the Most Plentifull, Easie, and Speedy Use Thereof in Both Kinds of Trigonometry, as Also in All Mathematicall Calculations
1631[a*]	Henry Briggs	Logarithmicall arithmetike
1633[t]	Nathaniel Roe	*Tabulæ logarithmicæ, or Tvvo tables of logarithmes*
1635[a*]	Edmund Wingate	Λογαριθμοτεχνία, or The construction and use of the logarithmeticall tables (second edition)
1635[s]	Henry Gellibrand	An institution trigonometricall
1636[t*]	Edmund Gunter	The Works of Edmund Gunter
1637[t]	John Wells	The compleat art of dyalling.
1645[t*]	Urquhart, Thomas	*The trissotetras: Or, a most exquisite table for resolving all manner of triangles*
1652[s*]	Edm. Wingate	Arithmetique made easie the second book
1653[t*]	William Leybourn	The Compleat Surveyor
1654[t]	John Newton	Institutio mathematica, or, A mathematical institution shewing the construction and use of the naturall and artificiall sines, tangents, and secants
1654[a*]	Edmund Wingate	Ludus mathematicus: or the mathematical game:
1654[s*]	John Newton	Tabulæ mathematicæ
1656[t*]	Richard Norwood	Trigonometrie, or, The doctrine of triangles, etc. Third Edition
1658[t*]	John Newton	Doctrine of triangles, in two books, the one composed, the other translated
1659[t]	John Collins	The Mariners Plain Scale New Plain'd: Volume 2
1669[t]	Henry Phillipe	The Sea-man's Kalender
1669[t]	Samuel Sturmy	The Mariner's Magazine: Or, Sturmy's Mathematical and Practical Arts
1669[s]	William Leybourn	Nine geometricall exercises, for young sea-men, and others that are studious
1675[s]	John Mayne	Arithmetick: ... in a most ... facile method for common capacities.
1676[t]	John Seller	Practical Navigation, Or, An Introduction to the Whole Art
1678[t]	Henry Phillipe	A Mathematical Manual; containing tables of logarithms for numbers ... Volume 1
1679[t]	John Newton	Cosmographia, or, A view of the terrestrial and cœlestial globes in a brief explanation of the principles of plain and solid geometry applied to surveying and gauging of cask
1681[s*]	John Moore	A New Systeme of the Mathematicks

1683 [t*]	William Hunt	Practical gauging epitomized, with the use and construction Volume 1
1685 [a*]	John Wallis	a Treatise of Algebra, Historical and Practical
1685 [s*]	E. Cocker	Cocker's Decimal Arithmetick
1685 [s*]	John Collins	The Doctrine of Decimal Arithmetick, Simple Interest, etc. as Also of Compound Interest and Annuities
1687 [t]	Peter Blackborow	Navigation Rectified
1687 [t]	John Taylor	Thesaurium Mathematicae: Or, The Treasury of the Mathematicks:
1688 [s]	John Moore	Moore's Arithmetick
1690 [s*]	Mark Forster	Arithmetical trigonometry being the solution of all the usual cases in plain trigonometry by common arithmetick without any tables whatsoever
1691 [t]	John Seller	The Sea-Gunner: Shewing the Practical Part of Gunnery, as it is Used at Sea
1692 [t*]	P. Perkins	The Seaman's Tutor: Explaining Geometry, Cosmography, and Trigonometry
1692 [t]	Matthew Norwood	Norwood's System of Navigation
1694 [s]	William Oughtred	Mr William Oughtred's Key of the Mathematicks newly translated [by E. Halley]
1695 [s*]	John Ward	A Compendium of Algebra. Consisting of plain, easie and concise rules
1695 [t]	Jonas Moore	A Mathematical Compendium: Or, Useful Practices in Arithmetick, Geometry, and Astronomy, Geography and Navigation, Embattelling, and Quartering of Armies; Fortification and Gunnery; Gauging and Dialling
1696 [a*]	Samuel Jeake	Λογιστολογία, or Arithmetick surveighed and reviewed: in four books
1700 [a*]	Johann Christophorus Sturm	Mathesis enucleata, or, The elements of the mathematicks by J. Christ. Sturmius ; made English by J.R. and R.S.S.
1701 [a*]	Samuel Heynes	A Treatise of Trigonometry: Plane and Spherical, Theoretical and Practical
1701 [t]	James Atkinson	Epitome of Navigation
1701 [t]	Nathaniel Colson	The Mariners New Kalendar
1702 [t*]	William Jones	A New Compendium of the Whole Art of Practical Navigation
1702 [s*]	Ignace Gaston Pardies	Short, but yet plain Elements of Geometry and Plain Trigometry
1704 [a*]	Charles Hayes	A Treatise of Fluxions: Or, an Introduction to Mathematical Philosophy.
1706 [a*]	William Jones	Synopsis Palmariorum Matheseos
1706 [t*]	Henry Sherwin	Mathematical Tables: Contrived After a Most Comprehensive Method
1708 [t]	James Keill	An Account of Animal Secretion, the quantity of blood in the humane body
1710 [t*]	John Ward	Clavis Usuræ: Or, A Key to Interest
1712 [s*]	Jacques Ozanam	Cursus mathematicus: or, a compleat course of the mathematicks. In five volumes. Volume 2
1713 [s*]	Edward Wells	The Young Gentleman's Arithmetick, and Geometry
1713 [s*]	John Hill	Arithmetick both in the theory and practice made plain

1714 [s*]	John Wilson	Trigonometry: With an Introduction to the Use of Both Globes
1714 [s]	Edward Wells	The Young Gentleman's Trigonometry, Mechanicks, and Opticks
1715 [t]	Henry Wilson	Navigation New Modell'd
1717 [a*]	Philip Ronayne	A Treatise of Algebra in Two Books
1717 [s]	Abraham Sharp	Geometry Improv'd
1721 [a*]	Edward Hatton	An Intire System of Arithmetic: or arithmetic in all it's parts
1723 [s*]	John Keill	Euclid's Elements of Geometry: From the Latin Translation of Commandine.
1723 [s*]	James Hodgson	A System of the Mathematics volume 1
1728 [t*]	Charles Leadbetter	A Compleat System of Astronomy, Volume 1
1729 [t*]	Henry Coggeshall	The Art of Practical Measuring
1730 [s*]	Alexander Malcolm	A New System of Arithmetick Theorical and Practical
1731 [s*]	John Ward	The Young Mathematician's Guide ... The sixth edition
1733 [s]	Henry Boad	Artium Principia: or, the Knowledge of the first principles
1735 [s*]	John Kirkby	Arithmetical Institutions: Containing a Compleat System of Arithmetic, Natural, Logarithmic, in all their Branches.
1735 [s]	Benjamin Martin	A New Compleat and Universal System Or Body of Decimal Arithmetick
1738 [t*]	James Hodgson	The Theory of Navigation Demonstrated: And Its Rudiments Clearly and Plainly… Third edition
1739 [a*]	Christian Wolfius	Elementa matheseos universæ. A Treatise of Algebra
1739 [t*]	Archibald Patoun	A Compleat Treatise of Practical Navigation Demonstrated from It's First… second edition
1739 [s*]	Charles Leadbetter	The Young Mathematician's Companion, Being a Compleat Tutor to the Mathematicks
1739 [s]	Benjamin Martin	Πανγεωμετρια; or the Elements of Geometry
1740 [a*]	Nicholas Saunderson	The Elements of Algebra, in Ten Books volume the second containing the last 5 books
1740 [s*]	Benjamin Martin	Logarithmologia: Of The Whole Doctrine of Logarithms, Common and Logistical
1742 [t*]	James Dodson	The Anti-logarithmic Canon.
1742 [t*]	William Gardiner	Tables of logarithms
1742 [t]	William Gardiner	Sherwin's Mathematical Tables: Contriv'd After a Most Comprehensive Method Third edition
1745 [a*]	Francis Holliday	Syntagma Mathesios: Containing the Resolution of Equations
1748 [s]	James Dodson	The Mathematical Repository: Containing Analytical Solutions of Five Hundred Questions

Chapter 7 Study: Texts that gave the two words ‘logos’ and ‘arithmos’ when introducing logarithms.

As this study covered three distinct styles of texts, I have categorised them as Dictionaries, Encyclopaedias, or Mathematical Texts.

Thirty Dictionaries, organised by year

Year	Author	Truncated Title
1692	John Moxon	Mathematicks made easie: or, a mathematical dictionary,
1721	Nathan Bailey	A Universal Etymological English Dictionary
1724	Nathan Bailey	A Universal Etymological English Dictionary
1736	John Harris	Lexicon Technicum, Volume 2
1741	Daniel Fenning	The Royal English Dictionary 2nd edition
1742	Charles Leadbetter	A compleat system of astronomy in two volumes. 2nd edition
1749	Benjamin Martin	Lingua Britannica Reformata: or, A New English Dictionary
1752	Theodor Arnold	A Compleat English Dictionary:
1755	Samuel Johnson	A dictionary of the English language; in which the words are deduced from their originals and illustrated in their different significations by examples from the best writers
1759	William Rider	A New Universal English Dictionary: or A Compleat Treasure of the English Language
1765	Samuel Clark	The Complete Dictionary of Arts and Sciences. In which the Whole Circle of Human Learning is Explained, Volume 2
1768	Samuel Johnson	A Dictionary of the English Language
1772	Frederick Barlow	The complete English dictionary: or, general repository of the English language
1772	John Draper	The young student's pocket companion, or Arithmetic, geometry, trigonometry, and Mensuration
1773	William Kenrick	A new Dictionary of the English Language
1775	John Ash	The New and Complete Dictionary of the English Language
1777	Samuel Johnson	A Dictionary of the English Language 4th edition
1802	Abala Kanta Sen	The Student's Comprehensive Anglo-Bengali Dictionary
1805	William Perry	The Synonymous, Etymological, and Pronouncing English Dictionary
1812	James Barclay	A Complete and Universal Dictionary of the English Language
1815	William Burney	A New Universal Dictionary of the Marine
1832	D. J. Browne	The Etymological Encyclopaedia of Technical Words and Phrases
1837	John Wood	Etymological Guide to the English Language: Being a Collection
1844	Alexander Reid	A dictionary of the English language
1846	J. E. Worcester	A universal and critical dictionary of the English language: to which are added Walker's Key to the pronunciation of classical and Scripture proper names
1848	William Grimshaw	An Etymological Dictionary of the English Language 3rd edition
1861	John Ogilvie	The Imperial Dictionary: English, Technological, and Scientific

Ten Encyclopaedias organised by year

Year	Author	Truncated Title
1738	Ephraim Chambers	Cyclopaedia: or, an Universal Dictionary of Arts and Sciences
1788	William Henry Hall	The New Royal Encyclopaedia
1797	various authors, anonymous	Encyclopædia Britannica, Volume 10, Part 1, 3rd edition
1797	William Henry Hall	The new encyclopædia; or, modern universal dictionary of arts and sciences, on a new and improved plan. ... In three volumes
1800	W.M. Johnson and Thomas Exley	The new imperial encyclopaedia, Volume 3
1807	various authors, anonymous	The New Encyclopedia: or, Universal Dictionary of Arts and Sciences
1819	Abraham Rees	Cyclopaedia: or, an Universal Dictionary of Arts and Sciences, Volume 21
1831	edited by Francis Lieber	Encyclopædia Americana
1834	Society for the Diffusion of Useful Knowledge	Library of Useful Knowledge: Natural Philosophy
1836	D.K. Sandford, Thomas Thompson, and Allan Cunningham	The Popular Encyclopedia, pt. 1

Twenty-Four Mathematical Texts and Textbooks, organised by year

Year	Author	Truncated Title
1624	Henry Briggs	Arithmetica logarithmica
1631	Henry Briggs	Logarithmicall arithmetike
1635	Edmund Wingate	Λογαριθμοτεχνία, or The construction and use of the logarithmeticall tables
1685	John Wallis	A Proposal about Printing a Treatise of Algebra, Historical and Practical
1687	John Taylor	Thesaurarium Mathematicæ, or the Treasury of the Mathematicks
1692	P. Perkins	The Seaman's Tutor
1696	Samuel Jeake	Λογιστολογία, or Arithmetick surveighed and reviewed: in four books
1700	Edmund Halley	Philosophical Transactions to the Year - Abridged and Disposed, Volume 1
1701	Samuel Heynes	A Treatise of Trigonometry: Plane and Spherical, Theoretical and Practical
1706	William Jones	Synopsis Palmariorum Matheseos
1728	Charles Leadbetter	A Compleat System of Astronomy
1730	Alexander Malcolm	A New System of Arithmetick Theorical and Practical
1740	Nicholas Saunderson	The Elements of Algebra, in Ten Books
1740	Benjamin Martin	Logarithmologia: Of The Whole Doctrine of Logarithms, Common and Logistical

1750	John Barrow	Navigatio Britannica
1760	Francis Maseres	Elements of Plane Trigonometry: In which is Introduced, a Dissertation on the Nature and Use of Logarithms
1785	Benjamin Donne	The British Mariner's Assistant
1801	David Morrice	The Art of Teaching Examined
1818	Elijah Hinsdale Burritt	Logarithmick Arithmetick, containing a new and correct table of logarithms
1827	George Darley	A system of popular trigonometry
1834	Baden Powell	An Historical View of the Progress of the Physical and Mathematical Sciences
1836	Robert Mudie	Popular Mathematics
1843	M. F. Maury	An Elementary, Practical and Theoretical Treatise on Navigation
1846	James W. Kavanagh	Arithmetic, its principles and practice