Union of Sums of Completely Simple Matrix Semigroups

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Abstract

The set $M_n(\boldsymbol{k})$ of $n \times n$ matrices over a commutative field $\boldsymbol{k}$ is a semigroup with respect to matrix multiplication. This thesis focuses on subsemigroups of $M_n(\boldsymbol{k})$.

Let $\mathbf{0}$ be the $n \times n$ zero matrix. A collection $\{S_i : i \in \Lambda\}$ of sets in $M_n(\boldsymbol{k})$ is a **0**-*meet collection* if $S_i S_j = \mathbf{0}$ whenever $i \neq j$. Since a $\mathbf{0}$-meet collection $F = \{S_i : i \in \Lambda\}$ of completely simple semigroups is finite, it is unambiguous to define

$$\boldsymbol{S}_I^F = \begin{cases} \{\mathbf{0}\} & \text{if } I = \varnothing \\ \sum_{i \in I} S_i & \text{if } I \neq \varnothing \end{cases}$$

for any subset I of Λ. It turns out that $\bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$ (the union of sums of S_i) is a semigroup, which is called a Σ-*semigroup*.

For convenience, the symbol $\mathcal{V}$ is used to represent certain varieties of normal orthogroups. The objective of this thesis is to show that each semigroup in $\mathcal{V}$ is contained in a smallest Σ-semigroup which is in $\mathcal{V} \vee \mathcal{SL}$, where $\mathcal{SL}$ is the variety of semilattices. Consequently, each maximal semigroup in $\mathcal{V} \vee \mathcal{SL}$ is a Σ-semigroup.

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Semigroups

This chapter contains background material, most of which can be found in Howie. Sections 2 to 6 contain definitions and properties of several types of semigroups which we will encounter in Chapter II. Section 7 describes how these semigroups relate in the context of varieties, while Section 8 discusses properties of matrix semigroups.

I.1 Notation

Throughout this thesis, let $\mathbf{N}$ denote the set of positive integers and let $\mathbf{N}_m$ denote the set of the first m positive integers. If A, B are subsets of a ring R, then:

- $A + B = \{a + b : a \in A, b \in B\}$;
- $A \setminus B = \{a \in A : a \notin B\}$;
- $AB = \{ab : a \in A, b \in B\}$;
- $A^m = \{a_1a_2\cdots a_m : a_i \in A, 1 \leq i \leq m\}$;
- $\langle A \rangle = \{a_1a_2\cdots a_r : a_i \in A, 1 \leq i \leq r, r \in \mathbf{N}\}$.

In addition, we abbreviate $A\{b\}$ to Ab and $\{a\}B$ to aB. If $\{A_i : i \in I\}$ is a collection of subsets of R, then we denote their cartesian product by $\prod_{i \in I} A_i$ and write a typical element as $(a_i : i \in I)$, where a_i is in A_i. If $I = \{1, 2, \ldots, m\}$, then we simply write $\prod_{i \in I} A_i$ as $A_1 \times A_2 \times \cdots \times A_m$ and $(a_i : i \in I)$ as $(a_1, a_2, \ldots, a_m)$.

I.2 Introduction

A *semigroup* is a non-empty set S with an associative binary operation $\cdot$, which we often call *multiplication.* We abbreviate $a{\cdot}b$ to ab for any elements a, b of S. Associativity allows products of any finite length in a semigroup to be written unambiguously without the need for parentheses. Hence if A, B, C are subsets of a semigroup, then $(AB)C = A(BC)$.

A semigroup with only one element is called a *trivial* semigroup. A semigroup with an identity element is called a *monoid.* If a semigroup S has no identity element, an extra element 1 can be adjoined to S to form the monoid $S \cup \{1\}$ with the obvious multiplication ($1a = a1 = a$ for all a in S). Hence it is convenient to define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

A *zero* of a semigroup S is an element z of S such that $S \neq \{z\}$ and $zS = Sz = \{z\}$. The condition $S \neq \{z\}$ is added to the usual definition of a zero so that the only element of a trivial semigroup is an identity element and not a zero.

Let S be a semigroup. A non-empty subset T of S is called a *subsemigroup* of S if it is closed under multiplication, i.e. if $T^2 \subseteq T$. A subsemigroup which is also a group is called a *subgroup.* An element e of S is an *idempotent* if $e^2 = e$, and the set of idempotents of S is denoted by $E(S)$. A semigroup consisting entirely of idempotents is called a *band*, and a commutative band is called a *semilattice*. For any idempotents e, f of S, we shall write $e \geq f$ if $ef = fe = f$, and write $e > f$ if $e \geq f$ and $e \neq f$. It is straightforward to show that $\geq$ is a partial order on $E(S)$. Note that if S contains an identity element 1_S and a zero 0_S, then $1_S \geq e \geq 0_S$ for all e in $E(S)$.

Let S, T be any semigroups. A map $\phi : S \to T$ is a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b)$$

for all a, b in S. A bijective homomorphism is called an *isomorphism.* If $\phi : S \to T$ is an isomorphism, then we say S and T are *isomorphic* and write $S \cong T$. The *direct product* of semigroups S_i ($i \in I$) is the semigroup with the cartesian product $\prod_{i \in I} S_i$ as its underlying set and componentwise multiplication: $(s_i : i \in I)(t_i : i \in I) = (s_i t_i : i \in I)$ for all s_i, t_i in S_i.

I.3 Semilattices of Semigroups

Recall that a semilattice Y is a commutative semigroup of idempotents, i.e. $\alpha^2 = \alpha$ and $\alpha\beta = \beta\alpha$ for all α, β in Y.

Lemma I.3.1 *If ϕ is a homomorphism from a semigroup S onto a semilattice Y, then:*
(1) *$\phi^{-1}(\alpha)$ is a subsemigroup of S for each α in Y;*
(2) *$\{\phi^{-1}(\alpha) : \alpha \in Y\}$ is a partition of S;*
(3) *$\phi^{-1}(\alpha)\phi^{-1}(\beta) \subseteq \phi^{-1}(\alpha\beta)$ for all α, β in Y.*

Proof (1) If $a, b \in \phi^{-1}(\alpha)$, then $\phi(a) = \alpha = \phi(b)$. Hence $\phi(ab) = \phi(a)\phi(b) = \alpha^2 = \alpha$, which implies that $ab \in \phi^{-1}(\alpha)$.

(2) It is trivial that each element of S belongs to some $\phi^{-1}(\alpha)$. If $a \in \phi^{-1}(\alpha) \cap \phi^{-1}(\beta)$, then $\alpha = \phi(a) = \beta$, and so $\phi^{-1}(\alpha) = \phi^{-1}(\beta)$. Consequently, $\{\phi^{-1}(\alpha) : \alpha \in Y\}$ is a disjoint collection of subsets of S.

(3) Suppose $a \in \phi^{-1}(\alpha)$ and $b \in \phi^{-1}(\beta)$. Then $\phi(a) = \alpha$ and $\phi(b) = \beta$. Hence $\phi(ab) = \phi(a)\phi(b) = \alpha\beta$, from which we deduce that $ab \in \phi^{-1}(\alpha\beta)$. ■

In Lemma I.3.1, if we denote $\phi^{-1}(\alpha)$ by S_α, then S is the disjoint union of the subsemigroups S_α ($\alpha \in Y$) and

$$S_\alpha S_\beta \subseteq S_{\alpha\beta},$$

where $\alpha\beta$ is the product of α and β in the semilattice Y. We call S a *semilattice of semi-*

groups and write $S = (Y; S_\alpha)$. Knowledge that a semigroup S is a semilattice of its subsemigroups S_α may give us a useful decomposition of S, but it provides no information on how elements from different S_α multiply. We now introduce a construction where multiplication is specified by a collection of homomorphisms.

Let Y be a semilattice and let $\{S_\alpha : \alpha \in Y\}$ be a collection of disjoint semigroups. For each pair of elements α, β of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism and suppose that:

(1) $\phi_{\alpha,\alpha}$ is the identity map of S_α for each α in Y;

(2) $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ for every α, β, γ in Y such that $\alpha \geq \beta \geq \gamma$.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ and define a multiplication on S by the rule that if $a_\alpha \in S_\alpha$ and $b_\beta \in S_\beta$, then

$$a_\alpha b_\beta = (\phi_{\alpha,\alpha\beta} a_\alpha)(\phi_{\beta,\alpha\beta} b_\beta).$$

It is not difficult to show that S is a semigroup. Furthermore, each S_α is a subsemigroup of S such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Hence S is certainly a semilattice of semigroups. We shall write $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ and call it a *strong semilattice of semigroups.*

I.4 Green's Equivalences

Definition I.4.1 Let a, b be elements of a semigroup S. Then:

(1) $a\mathcal{L}b$ if there exist x, y in S^1 such that $xa = b, yb = a$;

(2) $a\mathcal{R}b$ if there exist x, y in S^1 such that $ax = b, by = a$;

(3) $a\mathcal{H}b$ if $a\mathcal{L}b$ and $a\mathcal{R}b$;

(4) $a\mathcal{J}b$ if there exist u, v, x, y in S^1 such that $uav = b, xby = a$.

It is straightforward to show that $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ are equivalence relations on a semigroup, which are called *Green's equivalences*. Note that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{L} \cup \mathcal{R} \subseteq \mathcal{J}$.

Denote the $\mathcal{L}$-class containing a by L_a, and define the sets R_a, H_a and J_a similarly. The following result is Theorem 2.2.5 of Howie.

Theorem I.4.2 *If H is an $\mathcal{H}$-class of a semigroup S, then either $H^2 \cap H = \varnothing$, or $H^2 = H$ and H is a subgroup of S.* ■

Consequently, each $\mathcal{H}$-class of a semigroup S can contain at most one idempotent, and if e is an idempotent of S, then H_e is a subgroup with identity element e. We shall end this section with a lemma, the proof of which is routine.

Lemma I.4.3 *Let S be a semigroup. Then:*

(1) $(\forall a, b, c \in S)\ a\mathcal{L}b \Rightarrow ac\mathcal{L}bc$;

(2) $(\forall a, b, c \in S)\ a\mathcal{R}b \Rightarrow ca\mathcal{R}cb$. ■

I.5 Completely Simple Semigroups

An element a of a semigroup S is *regular* if $axa = a$ for some x in S, and S is *regular* if all its elements are regular. Bands and groups are examples of regular semigroups. Note that if $axa = a$, then ax and xa are idempotents. Hence a regular semigroup contains at least one idempotent.

Let e, f be idempotents of a semigroup S. Recall that $e \geq f$ if and only if $ef = fe = f$. A nonzero idempotent e of S is said to be *primitive* if it is minimal in the set of nonzero idempotents of S with respect to the partial order $\geq$, i.e. if f is nonzero in $E(S)$ and $e \geq f$, then $e = f$.

A semigroup S is called *simple* if $\mathcal{J} = S \times S$, or equivalently, if S is the only $\mathcal{J}$-class of itself. A simple semigroup S does not have a zero; for if 0_S is a zero of S, then $\{0_S\}$ is a $\mathcal{J}$-class different from S. A simple semigroup is *completely simple* if it contains a primitive idempotent. The following result is Theorem 3.3.1 of Howie.

Theorem I.5.1 *Let G be a group, let I, Λ be non-empty sets and let $P = [p_{\lambda i}]$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define a multiplication on S by*

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

Then S is a completely simple semigroup.

Conversely, each completely simple semigroup is isomorphic to a semigroup constructed in this way. ■

The semigroup in Theorem I.5.1 is called the *Rees matrix semigroup* and is denoted by $\mathcal{M}[G; I, \Lambda; P]$.

Lemma I.5.2 *If S is a completely simple semigroup with a primitive idempotent e, then:*
(1) *S is regular;*
(2) *$L_e = Se$ and $R_e = eS$;*
(3) *every $\mathcal{H}$-class of S is a subgroup.*

Proof (1) For any $a \in S$, we have $a\mathcal{J}e$ since S is simple. Hence there exist $u, v, x, y \in S^1$ such that $a = uev$ and $e = xay$. Let $f = evyexue$. Then

$$f^2 = evyex(ueev)yexue = evye(xay)exue = evyexue = f.$$

Thus $f \in E(S)$, and it is obvious that $e \geq f$. Consequently, $e = f$ for e is primitive. Now

$$a = uev = ufv = (uev)yex(uev) = a(yex)a,$$

which shows that a is regular.

(2) It suffices to show $L_e = Se$. If $x \in L_e$, then $x = ye$ for some $y \in S^1$. So $x = ye \in S^1e = Se \cup \{e\} = (S \cup \{e\})e = Se$.

Conversely, if $xe \in Se$, then $e\mathcal{J}xe$ because S is simple. Hence there exist $u, v \in S^1$ such that $e = uxev$. Now let $f = eveuxe$. Then

$$f^2 = eve(uxeev)euxe = eveuxe = f \quad \text{and} \quad ef = fe = f.$$

Therefore f is an idempotent such that $e \geq f$. It follows that $e = f$ because e is primitive.

Thus $e = (eveu)xe$, which implies that $xe \in L_e$.

(3) If $a \in S$, then there exist $u, v \in S^1$ such that $a = uev$ for S is simple. Hence $a^2 = auev$. Since $ue, aue \in Se$, we deduce that $ue\mathcal{L}aue$ by (2). By Lemma I.4.3(1), we have $a = (ue)v\mathcal{L}(aue)v = a^2$. The proof of $a^2\mathcal{R}a$ is similar. It follows that $a^2\mathcal{H}a$.

Now if H is an $\mathcal{H}$-class of S, then we have $a^2 \in H^2 \cap H$ for any $a \in H$. Therefore H is a subgroup (Theorem I.4.2). ■

A consequence of Lemma I.5.2 is

Corollary I.5.3 *Each completely simple semigroup is a disjoint union of groups.* ■

We now present some alternate characterizations of completely simple semigroups.

Theorem I.5.4 *The following conditions on a semigroup S without zero are equivalent.*

(1) *S is completely simple;*

(2) *S is regular and weakly cancellative, i.e. for all a, b, c, d in S,*

$$(ac = bc \ \&\ da = db) \Rightarrow a = b;$$

(3) *S is regular and every idempotent of it is primitive.*

Proof (1) $\Rightarrow$ (2). Let S be a completely simple semigroup. By Lemma I.5.2, S is regular. We may assume that $S = \mathcal{M}[G; I, \Lambda; P]$ (Theorem I.5.1). Let $a = (i, x, \lambda)$, $b = (j, y, \mu)$, $c = (k, z, \gamma)$, $d = (l, w, \eta)$, and suppose that $ac = bc$ and $da = db$. Then

$$(i, xp_{\lambda k}z, \gamma) = (j, yp_{\mu k}z, \gamma) \text{ and } (l, wp_{\eta i}x, \lambda) = (l, wp_{\eta j}y, \mu),$$

from which we deduce that $i = j$, $\lambda = \mu$ and $x = y$. Hence $a = b$ as required.

(2) $\Rightarrow$ (3). Let e, f be idempotents of S such that $e \geq f$. Then $ef = ff$ and $fe = ff$. Therefore we obtain $e = f$ by the weakly cancellative property of S.

(3) $\Rightarrow$ (1). For any $a \in S$, there exists $b \in S$ such that $a = aba$ because S is regular. Then ab is an idempotent such that $a\mathcal{J}ab$. Hence each $\mathcal{J}$-class of S contains an idempotent. Now suppose J_e, J_f are any $\mathcal{J}$-classes of S, where $e, f \in E(S)$. Choose any idempo-

tent $g \in J_{ef}$. Then $g = xefy$ for some $x, y \in S^1$. Let $u = efygxe$. Then

$$u^2 = efyg(xeefy)gxe = efygxe = u \quad \text{and} \quad eu = ue = u.$$

Hence $e \geq u$. But since every idempotent is primitive, it follows that $e = u$. Consequently,

$$g = xefy \quad \text{and} \quad e = efygxe,$$

which implies $g\mathcal{J}e$. Therefore $J_g = J_e$. Similarly, if we consider the idempotent $v = fygxef$, we can deduce that $J_g = J_f$. Hence $J_e = J_g = J_f$. In conclusion, S contains only one $\mathcal{J}$-class, and so it is simple. ■

The remainder of this section introduces all the completely simple semigroups which we will encounter in Chapter II.

Recall that a trivial semigroup contains only one element and so is obviously completely simple. A semigroup S is called *left zero* if $ab = a$ for all a, b in S. A *right zero semigroup* is analogously defined. Note that a left (right) zero semigroup is a band, therefore we may also call it a *left (right) zero band.* A semigroup S is called a *rectangular band* if $aba = a$ for all a, b in S. The following justifies why 'band' is being used in naming rectangular bands.

Proposition I.5.5 *The following conditions on a semigroup S are equivalent.*

(1) *S is a rectangular band;*

(2) $(\forall a, b, c \in S)\ a^2 = a \ \&\ abc = ac$;

(3) *S is a direct product of a left zero band and a right zero band;*

(4) *S is a completely simple band.*

Proof (1) $\Rightarrow$ (2). Suppose $aba = a$ for all $a, b \in S$. Then certainly $a^3 = a$ for all $a \in S$. Hence $a^4 = a^2$ and so $a^2 = a^4 = aa^2a = a$.

Next, for any $a, b, c \in S$, we have

$$abc = ab(cac) = (abca)c = ac.$$

(2) $\Rightarrow$ (3). Choose any element c from S. Let $L = Sc$ and $R = cS$. Then for any $x = uc$

and $y = vc$ in L,

$$xy = u(cvc) = uc^2 = uc = x,$$

and so L is a left zero band. Similarly, R is a right zero band. Define a map $\phi : S \to L \times R$ by $\phi(x) = (xc, cx)$. Then ϕ is one-one; for if $(xc, cx) = (yc, cy)$, then

$$x = x^2 = xcx = ycx = ycy = y^2 = y.$$

Also, ϕ is onto, since for all $(ac, cb) \in L \times R$, we have $\phi(ab) = (abc, cab) = (ac, cb)$. Finally, ϕ is a homomorphism because for all $x, y \in S$,

$$\phi(xy) = (xyc, cxy) = (xcyc, cxcy) = (xc, cx)(yc, cy) = \phi(x)\phi(y).$$

Hence ϕ is an isomorphism and $S \cong L \times R$.

(3) $\Rightarrow$ (4). Suppose $S \cong L \times R$ for some left zero band L and right zero band R. Then S is trivially a band and so is regular. Let $e, f \in S$. We may assume $e = (x, y)$ and $f = (z, w)$ for some $x, z \in L$ and $y, w \in R$. If $e \geq f$, then

$$e = (x, y) = (xzx, ywy) = (x, y)(z, w)(x, y) = e(fe) = ef = f.$$

Therefore every idempotent of S is primitive. It follows by Theorem I.5.4 that S is completely simple.

(4) $\Rightarrow$ (1). Let S be a completely simple band. If $a, b \in S$, then a, aba are idempotents such that $a \geq aba$. Consequently, $a = aba$ because every idempotent of S is primitive (Theorem I.5.4). ■

It is not difficult to show that left zero and right zero bands are special cases of rectangular bands, so they must also be completely simple.

A regular semigroup is called *orthodox* if its set of idempotents is a subsemigroup. A band is orthodox, but the converse is not generally true. For example, a non-trivial group is an orthodox semigroup which is not a band.

Definition I.5.6 Let S be a completely simple semigroup. Then:

(1) S is a *rectangular group* if it is orthodox;

(2) S is a *left group* if $E(S)$ is a left zero band;

(3) S is a *right group* if $E(S)$ is a right zero band.

Note that a group satisfies all requirements of being a left, right and rectangular group because it is completely simple and contains only one idempotent. We shall characterize these orthodox semigroups in the next section.

I.6 Completely Regular Semigroups

Let a, x be elements of a semigroup S. Then x is an *inverse* of a if $axa = a$ and $xax = x$, and the set of inverses of a is denoted by $V(a)$. Note that an element with an inverse must be regular. On the other hand, every regular element has an inverse; for if $axa = a$, then xax is an inverse of a.

Let S be a semigroup. An element a of S is *completely regular* if there exists x in $V(a)$ such that $ax = xa$, and S is *completely regular* if all its elements are completely regular. Clearly, a completely regular element is regular. Therefore a completely regular semigroup is also regular.

Lemma I.6.1 *Let a be an element of a completely regular semigroup. If x is an inverse of a such that* $ax = xa$, *then*:

(1) $e = ax$ *is an idempotent*;

(2) $a\mathcal{H}x\mathcal{H}e$;

(3) x *is unique.*

In particular, x *is the inverse of* a *in the group* H_a.

Proof To verify (1) and (2) is straightforward. To prove (3), suppose y is an inverse of a such that $ay = ya$. Then by (1) and (2), ay is an idempotent of $H_a = H_e$, and so $ay = e$ because H_e is a group (Theorem I.4.2). Therefore

$$y = yay = ye = yax = ex = xax = x. \qquad \blacksquare$$

In the light of Lemma I.6.1, it is reasonable to adopt the following convention: if a is a completely regular element of a semigroup and x is the inverse of a such that $ax = xa$, then we write $a^{-1} = x$ and $a^0 = ax$. Hence if a is an element of a completely regular semigroup S, then we have:

- $aa^{-1} = a^{-1}a = a^0$;
- $aa^0 = a^0a = a$;
- $a^{-1}a^0 = a^0a^{-1} = a^{-1}$;
- $(a^{-1})^{-1} = a$;
- $(a^{-1})^0 = (a^0)^{-1} = a^0$;
- $(a^0)^0 = a^0 = (a^0)^2$;
- $a \in E(S) \Leftrightarrow a^0 = a$.

Since any element a of a completely regular semigroup S is contained in the subgroup H_a, S is a disjoint union of its subgroups. Conversely, any disjoint union of groups forming a semigroup can easily be seen to be completely regular. Therefore we have proved

Theorem I.6.2 *A semigroup is completely regular if and only if it is a disjoint union of groups.* ■

It follows from Corollary I.5.3 that a completely simple semigroup is a disjoint union of groups, and so must be completely regular. In fact we have

Theorem I.6.3 *A semigroup is completely simple if and only if it is completely regular and simple.*

Proof It suffices to prove the converse of the theorem. Suppose S is a completely regular and simple semigroup. We shall show that every idempotent of S is primitive. If $e, f \in E(S)$ and $e \geq f$, then there exist $u, v \in S^1$ such that $e = ufv$ because S is simple. Therefore we have

$$\begin{aligned}(euf)^0 &= (euf)^{-1}euf = [(euf)^{-1}euf]f \\ &= (euf)^0f = (euf)^0e^2f = (euf)^0eufvf \\ &= e(ufv)f = e^2f = f.\end{aligned}$$

Hence,

$$f = fe^2 = (euf)^0 e^2 = (euf)^0 eufv = e(ufv) = e^2 = e.$$

By Theorem I.5.4 the proof is complete. ■

A semilattice of completely simple semigroups is a disjoint union of groups, and therefore is a completely regular semigroup. But the converse is also true by Howie (Theorem 4.1.3).

Theorem I.6.4 *Each completely regular semigroup is a semilattice of completely simple semigroups.* ■

Whenever we have a completely regular semigroup $(Y; S_\alpha)$, it is understood that S_α is a completely simple semigroup for each α in the semilattice Y. Furthermore, note that each S_α is a $\mathcal{J}$-class of S.

We shall prove the following lemma before characterizing the rectangular groups introduced in the previous section.

Lemma I.6.5 *Let S, T be completely simple semigroups and let $\phi : S \to T$ be a homomorphism. Then $\phi(x^0) = \phi(x)^0$ for all x in S.*

Proof If $x \in S$, then $\phi(x^0)$ is an idempotent of T because $\phi(x^0)\phi(x^0) = \phi(x^0x^0) = \phi(x^0)$. Next, we have

$$\phi(x^0)\phi(x)^0 = [\phi(x^0)\phi(x)]\phi(x)^{-1} = \phi(x^0x)\phi(x)^{-1} = \phi(x)\phi(x)^{-1} = \phi(x)^0.$$

Similarly, we can show $\phi(x)^0\phi(x^0) = \phi(x)^0$. Hence we deduce $\phi(x^0) \geq \phi(x)^0$. But since every idempotent of the completely simple semigroup T is primitive (Theorem I.5.4), we also have $\phi(x^0) = \phi(x)^0$. ■

Proposition I.6.6 *The following conditions on a completely regular semigroup S are equivalent.*

(1) *S is a rectangular group*;

(2) $(\forall a, b, c \in S)\ a^0b^0c^0 = a^0c^0$;

(3) $(\forall a, b \in S)\ a^0b^0a^0 = a^0$.

Proof (1) $\Rightarrow$ (2). Let a, b, c be elements of a rectangular group S. Since $E(S)$ is regular and all its elements are primitive, $E(S)$ is completely simple (Theorem I.5.4). It follows that $E(S)$ is a rectangular band (Proposition I.5.5). Hence $a^0b^0c^0 = a^0c^0$ as required.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose $a^0 = a^0b^0a^0$ for all $a, b \in S$. Let $e, f \in E(S)$. Then

$$(ef)^2 = (e^0f^0e^0)f^0 = e^0f^0 = ef,$$

from which we deduce that $ef \in E(S)$. Hence S is orthodox. Furthermore, if $e \geq f$, then

$$e = e^0 = e^0f^0e^0 = e(fe) = ef = f.$$

So each idempotent of S is primitive. By Theorem I.5.4, S is completely simple. ∎

Corollary I.6.7 *Let S be a completely regular semigroup. Then*:
(1) *S is a left group if and only if $ab^0 = a$ for all a, b in S*;
(2) *S is a right group if and only if $a^0b = b$ for all a, b in S*;
(3) *S is a group if and only if $a^0 = b^0$ for all a, b in S.*

Proof (1) If S is a left group, then $E(S)$ is a left zero band. Thus

$$ab^0 = aa^0b^0 = aa^0 = a.$$

Conversely, suppose $ab^0 = a$ for all $a, b \in S$. Then we certainly have $a^0b^0a^0 = a^0$. Hence S is a rectangular group (Proposition I.6.6). Now if $e, f \in E(S)$, then $ef = ef^0 = e$. So $E(S)$ is a left zero band, which implies that S is a left group.

(2) This is symmetrical to the proof of (1).

(3) The identity element of a group is the only idempotent. Therefore every group satisfies the identity $a^0 = b^0$.

Conversely, suppose $a^0 = b^0$ for all $a, b \in S$. Then S contains only one idempotent, say $E(S) = \{e\}$. To show that S is group, it suffices to show that the inverse of each element $a \in S$ is unique. Let $x, y \in V(a)$. Then since ax, xa, ay, ya are idempotents, we must have $ax = ay = e = xa = ya$. Hence

$$x = xax = xay = yay = y. \qquad \blacksquare$$

Lemma I.6.8 *Define a binary relation* $\geq$ *on a semigroup* S *by*

$$a \geq b \quad \text{if} \quad ae = fa = b \text{ for some } e, f \text{ in } E(S).$$

If S *is completely regular, then the binary relation* $\geq$ *is a partial order on* S.

Proof For any $a \in S$, we have $aa^0 = a^0a = a$. Thus $a \geq a$, which implies that $\geq$ is reflexive. To show that $\geq$ is antisymmetric, suppose $a \geq b$ and $b \geq a$. Then $ae = b$ and $a = fb$ for some $e, f \in E(S)$. It follows that $fa = ffb = fb = a$ and so

$$a = fb = (fa)e = ae = b.$$

Finally, to show that $\geq$ is transitive, suppose $a \geq b$ and $b \geq c$. Then $ae = fa = b$ and $bg = hb = c$ for some $e, f, g, h \in E(S)$. Note that

$$\begin{aligned} c &= hb = h(hb) = hc = hbg \\ &= hfag = hf(fa)g = hf(bg) = hfc \end{aligned}$$

and that $c = bg = aeg$. Hence

$$\begin{aligned} a^{-1}ca^{-1}c &= a^{-1}hba^{-1}c = a^{-1}hf(aa^{-1}a)eg \\ &= a^{-1}hf(aeg) = a^{-1}(hfc) = a^{-1}c, \end{aligned}$$

which implies that $a^{-1}c \in E(S)$. Now

$$a(a^{-1}c) = (aa^{-1}a)eg = aeg = c.$$

By a symmetrical argument, we can show that $(ca^{-1})a = c$ where $ca^{-1} \in E(S)$. Therefore we may conclude that $a \geq c$. $\blacksquare$

Note that the partial order in Lemma I.6.8 generalizes the partial order defined on $E(S)$; for if a, b, e, f are elements of $E(S)$ such that $ae = fa = b$ (i.e. $a \geq b$), then

$$ab = a(ae) = ae = b \quad \text{and} \quad ba = (fa)a = fa = b.$$

In Chapter II, our primary interest lies in completely regular semigroups that are strong semilattices of completely simple semigroups introduced in the previous section. Before we introduce these semigroups, first note that a band is completely regular and hence is a semilattice of its completely simple subsemigroups. But these subsemigroups are also bands, so by Proposition I.5.5 they must be rectangular bands. Hence we have established the following, the converse of which is obviously true.

Theorem I.6.9 *Every band is a semilattice of rectangular bands.* ■

Consequently, if we say $(Y; S_\alpha)$ is a band, then it is understood that S_α is a rectangular band for every α in the semilattice Y.

Definition I.6.10 Let S be a semigroup. Then:

(1) S is a *normal band* if it is a strong semilattice of rectangular bands;

(2) S is a *left normal band* if it is a strong semilattice of left zero bands;

(3) S is a *right normal band* if it is a strong semilattice of right zero bands.

There is no special name given to a strong semilattice of trivial semigroups because it is necessarily a semilattice. If we say $(Y; S_\alpha; \phi_{\alpha,\beta})$ is a normal (left normal, right normal) band, then each S_α is a rectangular (left zero, right zero) band.

Proposition I.6.11 *The following conditions on a band $S = (Y; S_\alpha)$ are equivalent.*

(1) *S is a normal band;*

(2) $(\forall a, b, c \in S)$ $a^2 = a$ & $abca = acba$;

(3) $(\forall a, b, c, d \in S)$ $a^2 = a$ & $abcd = acbd$.

Proof (1) $\Rightarrow$ (2). Let $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ be a normal band. For any $a, b, c \in S$, we may assume that $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. Certainly we have $a^2 = a$ for S is a band. Now let $\eta = \alpha\beta\gamma$. Note that since $\phi_{\alpha,\eta}a$, $\phi_{\beta,\eta}b$ and $\phi_{\gamma,\eta}c$ are elements of the rectangular band S_η, we have

$$abca = (\phi_{\alpha,\eta}a)(\phi_{\beta,\eta}b)(\phi_{\gamma,\eta}c)(\phi_{\alpha,\eta}a) = (\phi_{\alpha,\eta}a)$$
$$= (\phi_{\alpha,\eta}a)(\phi_{\gamma,\eta}c)(\phi_{\beta,\eta}b)(\phi_{\alpha,\eta}a) = acba.$$

(2) $\Rightarrow$ (3). Assume (2). Then

$$abcd = a(bcdabc)d = (abca)(dbcd)$$
$$= a(cbadcb)d = (acbd)(acbd) = acbd.$$

(3) $\Rightarrow$ (1). Suppose $\alpha \geq \beta$ in Y and let $a \in S_\alpha$. Then for any $x, y \in S_\beta$, we have $xyx = x$ and $yxy = y$, and so

$$axa = ax(yx)a = ayxxa = ay(yx)a = a(yxy)a = aya.$$

Hence the element axa is independent of the choice of x in S_β. It follows that whenever $\alpha \geq \beta$ in Y, the map $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ defined by

$$a \mapsto axa,$$

where x is any element of S_β, is well-defined. It is straightforward to show that $\phi_{\alpha,\alpha}$ is the identity map on S_α and that $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ whenever $\alpha \geq \beta \geq \gamma$. To show $\phi_{\alpha,\beta}$ is a homomorphism, let $a, b \in S_\alpha$ and choose any x from S_β. Then

$$\phi_{\alpha,\beta}(ab) = ab(xa)b = axabb = ax(xab)b$$
$$= (axa)(bxb) = (\phi_{\alpha,\beta}a)(\phi_{\alpha,\beta}b).$$

Finally, for any $\alpha, \beta \in Y$, let $a \in S_\alpha$, $b \in S_\beta$. Choose any x from $S_{\alpha\beta}$. Then

$$ab = ab(xa)b = axabb = ax(xab)b$$
$$= (axa)(bxb) = (\phi_{\alpha,\alpha\beta}a)(\phi_{\beta,\alpha\beta}b),$$

where the first equality holds because ab, x belong to the same rectangular band $S_{\alpha\beta}$. We may thus conclude that $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ is a strong semilattice of rectangular bands. ■

Corollary I.6.12 *Let S be a band. Then*:

(1) *S is a left normal band if and only if $abc = acb$ for all $a, b, c \in S$;*

(2) *S is a right normal band if and only if $abc = bac$ for all $a, b, c \in S$.*

Proof It suffices to prove (1), for (1) and (2) are symmetrical. Let $S = (Y; S_\alpha ; \phi_{\alpha,\beta})$ be a left normal band. If $a, b, c \in S$, then $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. Let $\eta = \alpha\beta\gamma$. Since $\phi_{\alpha,\eta}a$, $\phi_{\beta,\eta}b$, $\phi_{\gamma,\eta}c$ belong to the same left zero band S_η, we have

$$abc = (\phi_{\alpha,\eta}a)(\phi_{\beta,\eta}b)(\phi_{\gamma,\eta}c) = \phi_{\alpha,\eta}a = (\phi_{\alpha,\eta}a)(\phi_{\gamma,\eta}c)(\phi_{\beta,\eta}b) = acb.$$

Conversely, if $abc = acb$ for all $a, b, c \in S$, then certainly we have $abca = acba$. Hence S is a normal band (Proposition I.6.11). We may thus assume $S = (Y; S_\alpha ; \phi_{\alpha,\beta})$ where each S_α is a rectangular band. Now if $a, b \in S_\alpha$, then $aba = a$. Hence

$$ab = a(ab) = aba = a.$$

Therefore each S_α is a left zero band, which implies that S is a left normal band. ■

Definition I.6.13 Let S be a semigroup. Then:

(1) S is a *normal orthogroup* if it is a strong semilattice of rectangular groups;

(2) S is a *left normal orthogroup* if it is a strong semilattice of left groups;

(3) S is a *right normal orthogroup* if it is a strong semilattice of right groups;

(4) S is a *Clifford semigroup* if it is a strong semilattice of groups.

Analogous to normal bands, if we say $(Y; S_\alpha ; \phi_{\alpha,\beta})$ is a normal (left normal, right normal) orthogroup, then it is understood that each S_α is a rectangular (left, right) group. Similarly, if $(Y; S_\alpha ; \phi_{\alpha,\beta})$ is a Clifford semigroup, then each S_α is a group.

Proposition I.6.14 *The following conditions on a completely regular semigroup S are equivalent.*

(1) *S is a normal orthogroup*;

(2) $(\forall a, b, c \in S)\ ab^0ca = acb^0a$;

(3) $(\forall a, b, c, d \in S)\ ab^0cd = acb^0d$;

(4) *$E(S)$ is a normal band.*

Proof By Theorem IV.2.7 of Petrich and Reilly, (1), (2) and (4) are equivalent. Note that in Petrich and Reilly a normal orthogroup S is defined to be a completely regular semigroup satisfying the condition in (4). Now assume (2), then

$$\begin{aligned} ab^0cd &= (ab^0ca)b^0c(ab^0c)^{-1}d = acb^0(ab^0c)(ab^0c)^{-1}d \\ &= acb^0(ab^0c)^{-1}(ab^0c)d = acb^0(ab^0c)^{-1}ab^0cd(b^0cd)^0 \\ &= ac[b^0(ab^0c)^0db^0]cd(b^0cd)^{-1} = acb^0d(ab^0c)^0b^0cd(b^0cd)^{-1} \\ &= acb^0d[(ab^0c)^0(b^0cd)^0]. \end{aligned}$$

Similarly, we have that $acb^0d = ab^0cd[(acb^0)^0(cb^0d)^0]$. So $ab^0cd\mathcal{R}acb^0d$. By duality we deduce $ab^0cd\mathcal{L}acb^0d$. It follows that $ab^0cd\mathcal{H}acb^0d$, which implies $(ab^0cd)^0 = (acb^0d)^0$. Hence

$$\begin{aligned} ab^0cd &= (acb^0d)^0(ab^0cd)(acb^0d)^0 \\ &= (acb^0d)^{-1}acb^0(dab^0cda)cb^0d(acb^0d)^{-1} \\ &= (acb^0d)^{-1}(acb^0d)acb^0d(acb^0d)(acb^0d)^{-1} = acb^0d. \end{aligned}$$

We thus have (2) $\Rightarrow$ (3). It is trivial that (3) $\Rightarrow$ (2). ■

Corollary I.6.15 *Let S be a completely regular semigroup. Then*:

(1) *S is a left normal orthogroup if and only if $ab^0c = acb^0$ for all a, b, c in S;*

(2) *S is a right normal orthogroup if and only if $ab^0c = b^0ac$ for all a, b, c in S;*

(3) *S is a Clifford semigroup if and only if $ab^0 = b^0a$ for all a, b in S.*

Proof (1) Let $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ be a left normal orthogroup. If $a, b, c \in S$, then $a \in S_\alpha$, $b \in S_\beta$, $c \in S_\gamma$ for some $\alpha, \beta, \gamma \in Y$. Letting $\eta = \alpha\beta\gamma$, we have

$$\begin{aligned} ab^0c &= (\phi_{\alpha,\eta}a)(\phi_{\beta,\eta}b^0)(\phi_{\gamma,\eta}c) \overset{(a)}{=} (\phi_{\alpha,\eta}a)(\phi_{\beta,\eta}b)^0(\phi_{\gamma,\eta}c) \\ &\overset{(b)}{=} (\phi_{\alpha,\eta}a)(\phi_{\gamma,\eta}c) \overset{(b)}{=} (\phi_{\alpha,\eta}a)(\phi_{\gamma,\eta}c)(\phi_{\beta,\eta}b)^0 \\ &\overset{(a)}{=} (\phi_{\alpha,\eta}a)(\phi_{\gamma,\eta}c)(\phi_{\beta,\eta}b^0) = acb^0, \end{aligned}$$

where (a) holds by Lemma I.6.5 and (b) holds because $\phi_{\alpha,\eta}a$, $(\phi_{\beta,\eta}b)^0$, $\phi_{\gamma,\eta}c$ belong to the same left group S_η.

Conversely, if $ab^0c = acb^0$ for all $a, b, c \in S$, then we also have $ab^0ca = acb^0a$. Hence S is a normal orthogroup (Proposition I.6.14). Assume $S = (Y; S_\alpha; \phi_{\alpha,\beta})$. If $a, b \in S_\alpha$, then $a^0b^0a^0 = a^0$ for $E(S_\alpha)$ is a rectangular band. Therefore

$$ab^0 = aa^0(a^0b^0) = a(a^0b^0a^0) = aa^0 = a,$$

which implies that S_α is a left group.

(2) This is symmetrical to the proof of (1).

(3) Suppose $S = (Y; S_\alpha; \phi_{\alpha,\beta})$ is a Clifford semigroup. Let $a, b \in S$. Then $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$, and

$$\begin{aligned} ab^0 &= (\phi_{\alpha,\alpha\beta}a)(\phi_{\beta,\alpha\beta}b^0) \overset{(c)}{=} (\phi_{\alpha,\alpha\beta}a)(\phi_{\beta,\alpha\beta}b)^0 \\ &\overset{(d)}{=} (\phi_{\beta,\alpha\beta}b)^0(\phi_{\alpha,\alpha\beta}a) \overset{(c)}{=} (\phi_{\beta,\alpha\beta}b^0)(\phi_{\alpha,\alpha\beta}a) = b^0a, \end{aligned}$$

where (c) holds by Lemma I.6.5 and (d) holds because $\phi_{\alpha,\alpha\beta}a$, $(\phi_{\beta,\alpha\beta}b)^0$ belong to the same group $S_{\alpha\beta}$.

Conversely, if $ab^0 = b^0a$ for all $a, b \in S$, then we also have $ab^0c = b^0ac$. Hence S is a right normal orthogroup. Assume $S = (Y; S_\alpha; \phi_{\alpha,\beta})$. If $a, b \in S_\alpha$, then since $E(S_\alpha)$ is a right zero band, we have

$$a^0 = b^0a^0 = a^0b^0 = b^0.$$

Thus S_α is a group by Corollary I.6.7(3). ■

Having introduced the required semigroups, we shall end this section by showing that the binary relation (partial order) $\geq$ defined on a completely regular semigroup S in Lemma I.6.8 has an easier characterization if S is a strong semilattice of completely simple semigroups.

Lemma I.6.16 *If S is a strong semilattice of completely simple semigroups, then*

$$(\forall a, b \in S)\ a \geq b \Leftrightarrow ab^0 = b^0a = b.$$

Proof Let $S = (Y; S_\alpha; \phi_{\alpha,\beta})$, and suppose $a, b \in S$ and $a \geq b$. Then there exist $e, f \in E(S)$ such that $ae = fa = b$. We may assume that $a \in S_\alpha$, $b \in S_\beta$, $e \in S_\gamma$, $f \in S_\eta$ for some α, β, γ, $\eta \in Y$. Since $b = ae = (\phi_{\alpha,\alpha\gamma}a)(\phi_{\gamma,\alpha\gamma}e)$, we have $\alpha\gamma = \beta$ (i.e. $\alpha \geq \beta$) and that

$$b(\phi_{\gamma,\beta}e) = (\phi_{\alpha,\beta}a)(\phi_{\gamma,\beta}e)(\phi_{\gamma,\beta}e) = (\phi_{\alpha,\beta}a)(\phi_{\gamma,\beta}e).$$

Similarly, we can deduce that $(\phi_{\eta,\beta}f)b = (\phi_{\eta,\beta}f)(\phi_{\alpha,\beta}a)$ by considering $b = fa$. Now $\phi_{\alpha,\beta}a$, b, $\phi_{\gamma,\beta}e$, $\phi_{\eta,\beta}f$ belong to the same completely simple semigroup S_β. Therefore $b = \phi_{\alpha,\beta}a$ because S_β is weakly cancellative (Theorem I.5.4). Hence

$$ab^0 = (\phi_{\alpha,\beta}a)(\phi_{\beta,\beta}b^0) = bb^0 = b = b^0b = (\phi_{\beta,\beta}b^0)(\phi_{\alpha,\beta}a) = b^0a.$$

The converse is obvious. ■

I.7 Varieties of Normal Orthogroups

An *algebra* is an ordered pair $(A, \{f_i : i \in I\})$, where A is a non-empty set and f_i is an n_i-ary operation defined on A for all i in I. We may write $(A, f_1, f_2, \dots, f_m)$ if $I = \mathbf{N}_m$. When the n_i-ary operations f_i are understood we shall write A rather than $(A, \{f_i : i \in I\})$. In this section, we are only concerned with the special case when A is a (2, 1)-*algebra*, i.e. a triplet $(A, \cdot, ')$ where $(x, y) \mapsto x{\cdot}y$ is a binary operation and $x \mapsto x'$ is a unary operation. The reader is referred to Burris and Sankappanavar for the general case.

If $(A, \cdot, ')$ is a (2, 1)-algebra, then a *subalgebra* of A is a (2, 1)-algebra $(B, \cdot, ')$ such that:

(1) B is a non-empty subset of A;

(2) $(\forall x, y \in B)\ x{\cdot}y \in B$;

(3) $(\forall x \in B)\ x' \in B$.

If $(A, \cdot, ')$, $(B, \bullet, \hat{})$ are (2, 1)-algebras, then a map $\phi : A \to B$ is a *homomorphism* if

$$\phi(x\cdot y) = \phi(x)\bullet\phi(y) \quad \text{and} \quad \phi(x') = \phi(x)\hat{}.$$

In addition, if ϕ is onto, then B is said to be a *homomorphic image* of A.

Let $(A, \cdot, ')$ be a (2, 1)-algebra and let X be a set of symbols (an alphabet). Together with the symbols $\cdot$ and $'$, we define a *word* recursively as follows:

(1) every element in X is a word;

(2) if p, q are words, then $p{\cdot}q$ is a word;

(3) if p is a word, then p' is a word.

The expression $p = q$ is called an *identity* if p, q are words. If the identity $p = q$ holds for any substitution of elements from A, then we say A *satisfies* the identity $p = q$.

For any set of identities **R**, define [**R**] to be the class of all (2, 1)-algebras satisfying the identities in **R**. A class of (2, 1)-algebras **K** is an *equational class* if **K** = [**R**] for some set of identities **R**.

Definition I.7.1 Let **K** be a non-empty class of (2, 1)-algebras with the following properties:

(1) if $A \in \mathbf{K}$ and B is a subalgebra of A, then $B \in \mathbf{K}$;

(2) if $A \in \mathbf{K}$ and B is a homomorphic image of A, then $B \in \mathbf{K}$;

(3) if $A_\alpha \in \mathbf{K}$ $(\alpha \in \Lambda)$, then $\prod_{\alpha\in\Lambda} A_\alpha \in \mathbf{K}$.

Then we say **K** is a *variety*. If **H** and **K** are varieties such that $\mathbf{H} \subseteq \mathbf{K}$, then we say **H** is a *subvariety* of **K**.

The following is a specialization of a result (due to Birkoff) which can be found in Burris and Sankappanavar (Theorem 11.9).

Theorem I.7.2 A class of (2, 1)-algebras is a variety if and only if it is an equational class. ■

In the light of Theorem I.7.2, each variety of (2, 1)-algebras is of the form [**R**], where

R is a set of identities. If $\mathbf{R} = \{p_1 = q_1, p_2 = q_2, \ldots, p_m = q_m\}$, then we may write [**R**] simply as $[p_1 = q_1, p_2 = q_2, \ldots, p_m = q_m]$.

Recall that the unary operation $a \mapsto a^{-1}$ is well-defined in a completely regular semigroup. Thus together with its multiplication, a completely regular semigroup S is a (2, 1)-algebra $(S, \cdot\,, {}^{-1})$. It is a not difficult to see that a (2, 1)-algebra is a completely regular semigroup if and only if it satisfies the identities $a(bc) = (ab)c$, $a = aa^{-1}a$, $(a^{-1})^{-1} = a$ and $aa^{-1} = a^{-1}a$. Hence the variety of completely regular semigroups is

$$\mathcal{CR} = [a(bc) = (ab)c,\ a = aa^{-1}a,\ (a^{-1})^{-1} = a,\ aa^{-1} = a^{-1}a].$$

We often omit the identity $a(bc) = (ab)c$ in varieties of (completely regular) semigroups for convenience.

The following lists contain varieties of semigroups introduced in Sections 5 and 6. Since these varieties are subvarieties of $\mathcal{CR}$, we shall assume the identities for $\mathcal{CR}$ without further comment. We also write $a^0 = aa^{-1}$.

List A (see Proposition I.5.5):

- trivial semigroups $\mathcal{T} = [a = b]$;
- left zero bands $\mathcal{LZ} = [ab = a]$;
- right zero bands $\mathcal{RZ} = [ab = b]$;
- rectangular bands $\mathcal{RB} = [a^2 = a,\ abc = ac]$ $= [aba = a]$.

List B (see Proposition I.6.11 and Corollary I.6.12):

- semilattices $\mathcal{SL} = [a^2 = a,\ ab = ba]$;
- left normal bands $\mathcal{LN} = [a^2 = a,\ abc = acb]$;
- right normal bands $\mathcal{RN} = [a^2 = a,\ abc = bac]$;
- normal bands $\mathcal{N} = [a^2 = a,\ abcd = acbd]$ $= [a^2 = a,\ abca = acba]$.

List C (see Proposition I.6.6 and Corollary I.6.7):

- groups $\mathcal{G} = [a^0 = b^0]$;
- left groups $\mathcal{LG} = [ab^0 = a]$;
- right groups $\mathcal{RG} = [a^0b = b]$;
- rectangular groups $\mathcal{ReG} = [a^0b^0c^0 = a^0c^0]$
  $= [a^0b^0a^0 = a^0]$.

List D (see Proposition I.6.14 and Corollary I.6.15):

- Clifford semigroups $\mathcal{SG} = [ab^0 = b^0a]$;
- left normal orthogroups $\mathcal{LNO} = [ab^0c = acb^0]$;
- right normal orthogroups $\mathcal{RNO} = [ab^0c = b^0ac]$;
- normal orthogroups $\mathcal{NO} = [ab^0cd = acb^0d]$
  $= [ab^0ca = acb^0a]$.

Following the practice of Petrich and Reilly, we denote the variety of Clifford semigroups by $\mathcal{SG}$ (strong semilattices of groups), where in Howie, it is denoted by $\mathcal{CL}$. Note that Lists A and C contain varieties of completely simple semigroups.

The intersection of a non-empty collection $\{[\mathbf{R}_i] : i \in I\}$ of varieties is a variety; for a semigroup S belongs to $\bigcap_{i\in I}[\mathbf{R}_i]$ if and only if the collection $\bigcup_{i\in I}\mathbf{R}_i$ of identities is satisfied by S, i.e. $\bigcap_{i\in I}[\mathbf{R}_i] = \left[\bigcup_{i\in I}\mathbf{R}_i\right]$. For example, the intersection $\mathcal{B} \cap \mathcal{C}$ of the variety $\mathcal{B} = [a^2 = a]$ of bands and the variety $\mathcal{C} = [ab = ba]$ of commutative semigroups is the variety $\mathcal{SL} = [a^2 = a, ab = ba]$ of semilattices.

Define the *join* $[\mathbf{R}_1] \vee [\mathbf{R}_2]$ of the varieties $[\mathbf{R}_1]$, $[\mathbf{R}_2]$ to be the intersection of the collection of all varieties containing $[\mathbf{R}_1]$ and $[\mathbf{R}_2]$. The join of any arbitrary collection of varieties is similarly defined. Since every variety is contained in the variety $[a = a]$ of all semigroups, the join of any collection of varieties exists.

With respect to the operations $\cap$ and $\vee$, the set of varieties of completely regular semigroups becomes a lattice with greatest element the variety $[a = a]$ of all completely regular semigroups and least element the variety $[a = b]$ of trivial semigroups.

The following are lattice diagrams of varieties in Lists A, B, C and D. The justification of these diagrams can be found in Chapter 4.6 of Howie and Diagram V.5.6 of Petrich and Reilly.

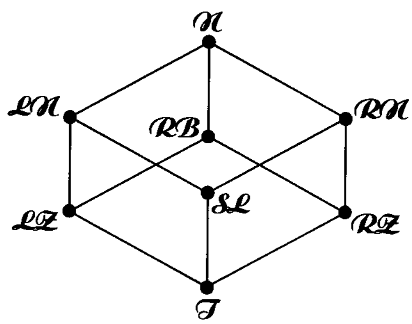

Figure I.1
The lattice of varieties in Lists A and B.

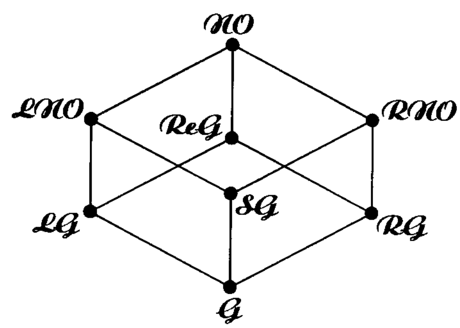

Figure I.2
The lattice of varieties in Lists C and D.

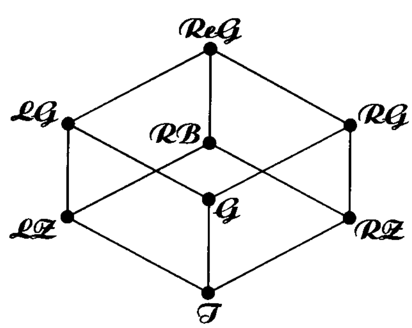

Figure I.3
The lattice of varieties in Lists A and C.

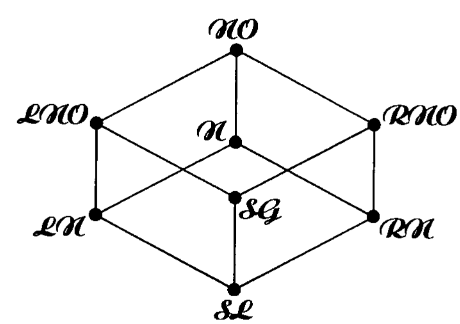

Figure I.4
The lattice of varieties in Lists B and D.

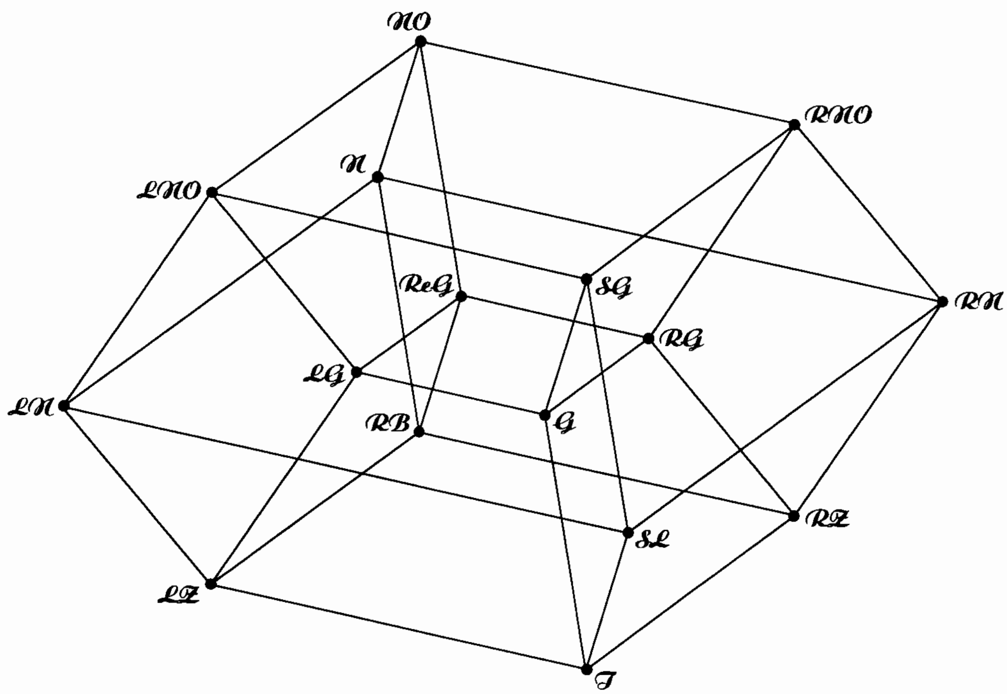

Figure I.5
The lattice of varieties in Lists A, B, C and D.

I.8 Matrix Semigroups

Let $\boldsymbol{k}$ be a commutative field. We denote the vector space of $n \times 1$ matrices over $\boldsymbol{k}$ by $\boldsymbol{k}^n$ and denote the ring of $n \times n$ matrices over $\boldsymbol{k}$ by $M_n(\boldsymbol{k})$. If a is a matrix of any size, then let a^{T}, $\mathrm{Im}(a)$ and $\mathrm{rank}(a)$ denote the transpose of a, image of a and rank of a respectively. The reader is referred to Lancaster and Tismenetsky for other concepts in linear algebra. In this section, we consider subsemigroups of $(M_n(\boldsymbol{k}), \cdot\,)$ where $\cdot$ is the usual matrix multiplication. Our goal is to show that the number of $\mathcal{J}$-classes in a completely regular subsemigroup of $M_n(\boldsymbol{k})$ is finite.

Lemma I.8.1 *If* a, b *are any elements of* $M_n(\boldsymbol{k})$, *then* $\mathrm{rank}(ab) \leq \mathrm{rank}(a)$ *and* $\mathrm{rank}(ab) \leq \mathrm{rank}(b)$.

Proof Since $\mathrm{Im}(ab) \subseteq \mathrm{Im}(a)$, it follows that $\mathrm{rank}(ab) \leq \mathrm{rank}(a)$. Similarly, we have $\mathrm{Im}(b^{\mathrm{T}}a^{\mathrm{T}}) \subseteq \mathrm{Im}(b^{\mathrm{T}})$, which implies that $\mathrm{rank}(b^{\mathrm{T}}a^{\mathrm{T}}) \leq \mathrm{rank}(b^{\mathrm{T}})$. Therefore

$$\mathrm{rank}(ab) = \mathrm{rank}[(ab)^{\mathrm{T}}] = \mathrm{rank}(b^{\mathrm{T}}a^{\mathrm{T}}) \leq \mathrm{rank}(b^{\mathrm{T}}) = \mathrm{rank}(b). \quad \blacksquare$$

Corollary I.8.2 *If* a, b *are any elements of a subsemigroup* S *of* $M_n(\boldsymbol{k})$ *such that* $a\mathcal{J}b$, *then* $\mathrm{rank}(a) = \mathrm{rank}(b)$.

Proof If $a, b \in S$ and $a\mathcal{J}b$, then there exist $u, v, x, y \in S^1$ such that $a = ubv$, $b = xay$. By Lemma I.8.1, we have $\mathrm{rank}(a) = \mathrm{rank}(ubv) \leq \mathrm{rank}(bv) \leq \mathrm{rank}(b)$. To show $\mathrm{rank}(b) \leq \mathrm{rank}(a)$ is a similar task. ■

Corollary I.8.3 *Every element of a simple subsemigroup of* $M_n(\boldsymbol{k})$ *has the same rank.* ■

We need the following result from Putcha (Lemma 1.6) to prove the main theorem of this section (which is actually a special case of Theorem 1.7 of Putcha).

Lemma I.8.4 *If* E *is an infinite set of idempotents in* $M_n(\boldsymbol{k})$ *of rank* r, *then there exist distinct* e, f *in* E *such that* $\mathrm{rank}(ef) = \mathrm{rank}(fe) = r$. ■

Theorem I.8.5 *If* $S = (Y; S_\alpha)$ *is a completely regular subsemigroup of* $M_n(\boldsymbol{k})$, *then* Y *is finite. Consequently, the number of* $\mathcal{J}$*-classes of* S *is finite.*

Proof Seeking a contradiction, suppose that Y is infinite. Then we can choose an idempotent from each S_α to form the infinite subset E of $E(S)$. Hence we have

$$(\forall e, f \in E)\ e\mathcal{J}f \Rightarrow e = f.$$

The set $E_r = \{e \in E : \mathrm{rank}(e) = r\}$ is infinite for some $r \in \mathbf{N}_n$. By Lemma I.8.4, there exist $e, f \in E_r$ such that $e \neq f$ and $\mathrm{rank}(ef) = \mathrm{rank}(fe) = r$. Since $ef\mathcal{H}(ef)^0$, we have $\mathrm{rank}(ef)$

$= \mathrm{rank}[(ef)^0]$ by Corollary I.8.2. Note that $(ef)^0\boldsymbol{k}^n$ and $e\boldsymbol{k}^n$ are subspaces of $\boldsymbol{k}^n$ such that $(ef)^0\boldsymbol{k}^n = ef(ef)^{-1}\boldsymbol{k}^n \subseteq e\boldsymbol{k}^n$. Therefore together with

$$\dim[(ef)^0\boldsymbol{k}^n] = \mathrm{rank}[(ef)^0] = \mathrm{rank}(ef) = r = \mathrm{rank}(e) = \dim(e\boldsymbol{k}^n),$$

we have $(ef)^0\boldsymbol{k}^n = e\boldsymbol{k}^n$. Now for any $x \in \boldsymbol{k}^n$, there exists $y \in \boldsymbol{k}^n$ such that $ex = (ef)^0y$. Hence $(ef)^0ex = (ef)^0(ef)^0y = (ef)^0y = ex$, from which we deduce that $e = (ef)^0e$ because x is arbitrary in $\boldsymbol{k}^n$. Thus $e = (ef)^{-1}efe$. Similarly, if we consider the dual $fe\mathcal{H}(fe)^0$, we obtain $f = (fe)^{-1}fef$. It follows then that $e\mathcal{J}f$, which implies the contradiction $e = f$. ■

Corollary I.8.6 *Every subsemilattice of $M_n(\boldsymbol{k})$ is finite.* ■

Chapter II

Σ-Semigroups

This chapter introduces the definition and properties of a Σ-semigroup. Since we consider only subsemigroups of $M_n(\boldsymbol{k})$ with respect to matrix multiplication $\cdot$, every semigroup in this chapter is a subsemigroup of $(M_n(\boldsymbol{k}), \cdot\,)$. We reserve the symbol $\mathcal{V}$ to represent any variety in Lists A, B, C or D, and we reserve the symbol $\mathcal{W}$ to represent any variety of completely regular semigroups in general. For convenience, if we say S is a $\mathcal{V}$-semigroup ($\mathcal{W}$-semigroup), we mean that S is a semigroup in the variety $\mathcal{V}$ ($\mathcal{W}$). Also, let $\mathbf{0}$ denote the $n \times n$ zero matrix and let $\mathbf{1}$ denote the $n \times n$ identity matrix.

II.1 Introduction

Let $F = \{S_i : i \in \Lambda\}$ be a finite collection of sets in $M_n(\boldsymbol{k})$ such that $S_i \neq \{\mathbf{0}\}$ for all i in Λ. We say F is a $\mathbf{0}$-*meet collection* if $S_iS_j = \{\mathbf{0}\}$ whenever $i \neq j$. For any finite subset I of Λ, define

$$\boldsymbol{S}_I^F = \begin{cases} \{\mathbf{0}\} & \text{if } I = \varnothing \\ \sum_{i \in I} S_i & \text{if } I \neq \varnothing. \end{cases}$$

In this section, we will construct a semigroup using a $\mathbf{0}$-meet collection F of completely simple semigroups. But first note that if $F = \{S_i : i \in \Lambda\}$ is such a collection in $M_n(\boldsymbol{k})$, then Λ must be finite; for if Λ is infinite, then choosing an idempotent from each S_i will generate an infinite subsemilattice of $M_n(\boldsymbol{k})$, which contradicts Corollary I.8.6.

Hence whenever we consider a **0**-meet collection of completely simple semigroups, it is understood that it is a finite collection. Next, note that $S_iS_j = \{\mathbf{0}\}$ whenever $i \neq j$. So if $\sum_{i\in I} x_i$, $\sum_{i\in J} y_i$ exist in $\boldsymbol{S}_I^F$, $\boldsymbol{S}_J^F$ respectively, then

$$\sum_{i\in I} x_i \sum_{i\in J} y_i = \sum_{i\in I\cap J} x_i y_i = \sum_{i\in I\cap J} x_i \sum_{i\in I\cap J} y_i .$$

For the following three lemmas, let $F = \{S_i : i \in \Lambda\}$ be a **0**-meet collection of completely simple $\mathcal{W}$-semigroups.

Lemma II.1.1 *For any subset I of Λ, each element of $\boldsymbol{S}_I^F$ is uniquely represented,* i.e. *if there exist x_i, y_i in S_i and $I, J \subseteq \Lambda$ such that $\sum_{i\in I} x_i = \sum_{i\in J} y_i$, then $I = J$ and $x_i = y_i$ for all i in I. Consequently, if $e = \sum_{i\in I} e_i$ is an idempotent of $\boldsymbol{S}_I^F$, then e_i is an idempotent of S_i for all i in I.*

Proof Suppose $x_i, y_i \in S_i$ and $I, J \subseteq \Lambda$ such that $\sum_{i\in I} x_i = \sum_{i\in J} y_i$. For any $j \in I$, choose any element a_j from S_j. Then

$$a_j x_j = a_j \sum_{i\in I} x_i = a_j \sum_{i\in J} y_i = \begin{cases} a_j y_j & \text{if } j \in J \\ \mathbf{0} & \text{if } j \notin J. \end{cases}$$

Since $a_jx_j \neq \mathbf{0}$, we must have $a_jx_j = a_jy_j$ and $I \subseteq J$. By symmetry, we obtain $x_ja_j = y_ja_j$ and $J \subseteq I$. Hence $x_j = y_j$ (for S_j is weakly cancellative by Theorem I.5.4) and $I = J$. ■

Lemma II.1.2 *For each subset I of Λ, define a map $\sigma_I : \boldsymbol{S}_I^F \to \prod_{i\in I} S_i$ by*

$$\sum_{i\in I} x_i \mapsto (x_i : i \in I)$$

(for example, if $I = \{1, 2, 3\}$, then $\sigma_I(x_1 + x_2 + x_3) = (x_1, x_2, x_3)$).

Then σ_I is an isomorphism and $\boldsymbol{S}_I^F$ is a completely simple $\mathcal{W}$-semigroup for each subset I of Λ.

Proof Let $I \subseteq \Lambda$ and let $x = \sum_{i \in I} x_i$, $y = \sum_{i \in I} y_i$ be elements of $\boldsymbol{S}_I^F$. To show σ_I is well-defined, suppose $x = y$. Then by Lemma II.1.1, $x_i = y_i$ for all $i \in I$. Hence

$$\sigma_I(x) = (x_i : i \in I) = (y_i : i \in I) = \sigma_I(y).$$

It is not difficult to see that σ_I is a bijection. To show that σ_I is a homomorphism, first note that $xy = \sum_{i \in I} x_i y_i$. Therefore

$$\sigma_I(xy) = (x_i y_i : i \in I) = (x_i : i \in I)(y_i : i \in I) = \sigma_I(x)\sigma_I(y).$$

Hence σ_I is an isomorphism, i.e. $\boldsymbol{S}_I^F \cong \prod_{i \in I} S_i$. It suffices now to show that $\prod_{i \in I} S_i$ is a completely simple $\mathcal{W}$-semigroup. But a direct product of completely simple semigroups is certainly completely simple, and a direct product of $\mathcal{W}$-semigroups is a $\mathcal{W}$-semigroup (see Definition I.7.1). Hence $\prod_{i \in I} S_i$ is a completely simple $\mathcal{W}$-semigroup as required. ∎

Lemma II.1.3 *For any subsets I, J of Λ such that $I \supseteq J$, define a map $\phi_{I,J} : \boldsymbol{S}_I^F \to \boldsymbol{S}_J^F$ by*

$$\sum_{i \in I} x_i \mapsto \sum_{i \in J} x_i$$

(for example, if $I = \{1, 2, 3\}$ and $J = \{1, 2\}$, then $\phi_{I,J}(x_1 + x_2 + x_3) = x_1 + x_2$).
Then:

(1) *$\phi_{I,I}$ is the identity map of $\boldsymbol{S}_I^F$;*

(2) *$\phi_{I,J}$ is a homomorphism;*

(3) *if I, J, K are subsets of Λ such that $I \supseteq J \supseteq K$, then $\phi_{J,K} \circ \phi_{I,J} = \phi_{I,K}$.*

Proof Let I, J be subsets of Λ such that $I \supseteq J$. To show that $\phi_{I,J}$ is well-defined is a routine application of Lemma II.1.1. To prove (1) and (3) is straightforward. As for the proof of (2), note that $\phi_{I,J} = \sigma_J^{-1} \circ \pi_{I,J} \circ \sigma_I$, where σ_I, σ_J are isomorphisms defined in Lemma II.1.2, and $\pi_{I,J}$ is the projection homomorphism from $\prod_{i \in I} S_i$ onto $\prod_{i \in J} S_i$. Hence $\phi_{I,J}$ is a homomorphism. ∎

We are now ready for the following semigroup construction.

Theorem II.1.4 *If $F = \{S_i : i \in \Lambda\}$ is a* **0***-meet collection of completely simple $\mathcal{W}$-semigroups, then $S = \bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$ is a strong semilattice of completely simple $\mathcal{W}$-semigroups.*

Proof We may write $S = \bigcup_{I \in Y} \boldsymbol{S}_I^F$, where $Y = 2^\Lambda$ (the set of all subsets of Λ) is the semilattice with $\cap$ as binary operation. Hence for any $I, J \in Y$,

$$I \geq J \;\Leftrightarrow\; I \supseteq J \;\Leftrightarrow\; I \cap J = J.$$

Let $x, y \in S$. We may assume $x \in \boldsymbol{S}_I^F$ and $y \in \boldsymbol{S}_J^F$. Then there exist $x_i, y_i \in S_i$ such that $x = \sum_{i \in I} x_i$ and $y = \sum_{i \in J} y_i$. Now

$$xy = \sum_{i \in I} x_i \sum_{i \in J} y_i = \sum_{i \in I \cap J} x_i \sum_{i \in I \cap J} y_i = \phi_{I, I \cap J}(x)\, \phi_{J, I \cap J}(y).$$

Hence together with Lemma II.1.3, we deduce that $S = (Y; \boldsymbol{S}_I^F; \phi_{I,J})$. But by Lemma II.1.2, $\boldsymbol{S}_I^F$ is a completely simple $\mathcal{W}$-semigroup for all $I \in Y$. Therefore S is a strong semilattice of completely simple $\mathcal{W}$-semigroups. ■

Definition II.1.5 We call the semigroup S introduced in Theorem II.1.4 the Σ-*semigroup with foundation F*. More generally, by a Σ-*semigroup* S we mean a semigroup for which there exists a **0**-meet collection $F = \{S_i : i \in \Lambda\}$ of completely simple semigroups such that S is the Σ-semigroup with foundation F.

In Definition II.1.5, since F is a **0**-meet collection and each S_i cannot contain **0**, we must have $S_i \cap S_j = \varnothing$ whenever $i \neq j$. Also, we allow F to be empty, which gives the Σ-semigroup $S = \{\mathbf{0}\}$. In addition, we have

Corollary II.1.6 *Each Σ-semigroup is a strong semilattice of completely simple semigroups. Consequently, each Σ-semigroup is completely regular.* ■

If S is a semigroup, then the smallest Σ-semigroup containing S, if it exists, is called the *Σ-closure of S* and is denoted by S^{Σ}. Not every semigroup has a Σ-closure (Example II.3.3), but every $\mathcal{V}$-semigroup has a Σ-closure and there is a procedure to obtain it. Section 3 introduces the concepts (the +closure and −closure of a semigroup) which are crucial in calculating the Σ-closure of a $\mathcal{V}$-semigroup. Furthermore, we can show that if S is a $\mathcal{V}$-semigroup, then S^{Σ} is a semigroup in the variety $\mathcal{V} \vee \mathcal{SL}$.

II.2 Characterization of Σ-Semigroups

Let α, β be elements of a semilattice Y. We say *α covers β* and write $\alpha \succ \beta$ if $\alpha > \beta$ and there is no γ in Y such that $\alpha > \gamma > \beta$.

Let S be a semigroup. We define S to be *closed under* + if it contains $\mathbf{0}$ and

$$(\forall a, b \in S) \quad ab = ba = \mathbf{0} \;\Rightarrow\; a + b \in S.$$

In addition, we define S to be *closed under* − if

$$(\forall a, b \in S) \quad a \geq b \;\Rightarrow\; a - b \in S,$$

where $\geq$ is defined in Lemma I.6.8 and − is the usual matrix subtraction. Note that a completely regular semigroup S closed under − necessarily contains $\mathbf{0}$ because $a \geq a$ for all a in S. Hence whenever we say that a completely regular semigroup $S = (Y; S_\alpha)$ is closed under − (or contains $\mathbf{0}$), we may assume that Y contains 0 and $S_0 = \{\mathbf{0}\}$.

In this section we will show that necessary and sufficient conditions for a completely regular semigroup to be a Σ-semigroup are precisely being closed under + and −. We need the following three lemmas to prove the main theorem of this section.

Lemma II.2.1 *If $S = (Y; S_\alpha)$ is a completely regular semigroup and $\alpha \geq \beta$ in Y, then*

$$(\forall x \in S_\alpha)(\forall y \in S_\beta) \quad x \geq (xyx)^0 x.$$

Proof Let $e = (xyx)^0$ and $f = x^{-1}(xyx)^0x$. Then e, f are idempotents of S such that $ex = (xyx)^0x$ and

$$xf = xx^{-1}(xyx)^0x = x^0xyx(xyx)^{-1}x = xyx(xyx)^{-1}x = (xyx)^0x.$$

Hence $x \geq (xyx)^0x$. ■

Lemma II.2.2 *Let $S = (Y; S_\alpha)$ be a completely regular semigroup closed under $-$ and let $F = \{S_\alpha : \alpha \in Y, \alpha \succ 0\}$. Then S is contained in the Σ-semigroup with foundation F.*

Proof First, it is not difficult to see that F is a **0**-meet collection. By Theorem I.8.5, Y is finite. So if $Y = \{0\}$, then $F = \varnothing$ and we are done. Otherwise we may assume $\{\alpha \in Y : \alpha \succ 0\} = \mathbf{N}_m$. Suppose $x \in S_\alpha$ where $\alpha \in Y\backslash\{0\}$. Since i covers 0 for each $i \in \mathbf{N}_m$, exactly one of $\alpha \geq i$, $\alpha i = 0$ is true. We may therefore assume, without loss of generality, that

$$\alpha \geq i \ \text{ if } \ 1 \leq i \leq r \ \text{ and } \ \alpha i = 0 \ \text{ if } \ r+1 \leq i \leq m.$$

For each $i \in \mathbf{N}_m$, choose a representative y_i from S_i. By Lemma II.2.1, $x \geq (xy_ix)^0x$ whenever $1 \leq i \leq r$. By hypothesis and since $x \geq (xy_1x)^0x$, we have $x - (xy_1x)^0x \in S$. But $e = (xy_2x)^0$ and $f = x^{-1}(xy_2x)^0x$ are idempotents of S such that

$$e[x - (xy_1x)^0x] = (xy_2x)^0x - (xy_2x)^0(xy_1x)^0x = (xy_2x)^0x - \mathbf{0} = (xy_2x)^0x$$

and

$$[x - (xy_1x)^0x]f = xx^{-1}(xy_2x)^0x - (xy_1x)^0xx^{-1}(xy_2x)^0x = (xy_2x)^0x - \mathbf{0} = (xy_2x)^0x.$$

Therefore $x - (xy_1x)^0x \geq (xy_2x)^0x$, which implies that $x - (xy_1x)^0x - (xy_2x)^0x \in S$. Continuing, we have $x - \sum_{i=1}^r (xy_ix)^0x \in S$. Assume $x - \sum_{i=1}^r (xy_ix)^0x \in S_\beta$ for some $\beta \in Y$. For each $i \in \mathbf{N}_m$, define

$$z_i = \begin{cases} (xy_ix)^0 & \text{if } 1 \leq i \leq r \\ y_i & \text{if } r+1 \leq i \leq m. \end{cases}$$

Note that each z_i belongs to S_i, and it is not difficult to show $z_j\left(x - \sum_{i=1}^r (xy_ix)^0x\right) = \mathbf{0}$

for all $j \in \mathbf{N}_m$. Hence $\beta \mathbf{N}_m = 0$, from which we deduce that $\beta = 0$ because $\mathbf{N}_m$ contains all the covers of 0. Consequently, $x - \sum_{i=1}^{r} (xy_i x)^0 x = \mathbf{0}$, i.e. $x = \sum_{i=1}^{r} (xy_i x)^0 x \in \boldsymbol{S}_{\mathbf{N}_r}^F$. We may thus conclude that $S \subseteq \bigcup_{I \subseteq \mathbf{N}_m} \boldsymbol{S}_I^F$. ∎

Lemma II.2.3 *Let S be a Σ-semigroup with foundation $F = \{S_i : i \in \Lambda\}$. Let $x = \sum_{i \in I} x_i$ and $y = \sum_{i \in J} y_i$, where x_i, y_i exist in S_i and $I, J \subseteq \Lambda$. If $x \geq y$, then $I \supseteq J$ and $x_i = y_i$ for all i in J.*

Proof If $x \geq y$, then there exist idempotents e, f of S such that $ex = xf = y$. We may assume $e = \sum_{i \in K} e_i \in \boldsymbol{S}_K^F$ for some $K \subseteq \Lambda$. Now

$$\sum_{i \in K \cap I} e_i x_i = \sum_{i \in K} e_i \sum_{i \in I} x_i = ex = y = \sum_{i \in J} y_i .$$

Thus by Lemma II.1.1, $K \cap I = J$ $(I \supseteq J)$ and $e_i x_i = y_i$ for all $i \in J$. By the same lemma, since $e = \sum_{i \in K} e_i$ is an idempotent of $\boldsymbol{S}_K^F$, e_i is also an idempotent of S_i for all $i \in K$. Hence

$$e_i x_i = e_i(e_i x_i) = e_i y_i .$$

Similarly, we may suppose $f = \sum_{i \in L} f_i$ for some $L \subseteq \Lambda$ and deduce that $x_i f_i = y_i f_i$ for all $i \in J$. Therefore, by Theorem I.5.4, $x_i = y_i$ for all $i \in J$. ∎

Having the required lemmas, we are ready to characterize Σ-semigroups.

Theorem II.2.4 *A completely regular semigroup is a Σ-semigroup if and only if it is closed under + and −.*

Proof Let $S = (Y; S_\alpha)$ be a completely regular semigroup.

Suppose $S = \bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$ where $F = \{S_i : i \in \Lambda\}$ is its foundation. Let $x, y \in S$. We may assume $x = \sum_{i\in I} x_i \in \boldsymbol{S}_I^F$ and $y = \sum_{i\in J} y_i \in \boldsymbol{S}_J^F$ for some $I, J \subseteq \Lambda$. If $xy = \mathbf{0}$, then $\mathbf{0} = xy = \sum_{i\in I\cap J} x_i y_i \in \boldsymbol{S}_{I\cap J}^F$, which implies $I \cap J = \varnothing$. Hence $x + y \in \boldsymbol{S}_I^F + \boldsymbol{S}_J^F = \boldsymbol{S}_{I\cup J}^F \subseteq S$. If $x \geq y$, then by Lemma II.2.3, $I \supseteq J$ and $x_i = y_i$ for all $i \in J$. Thus $x - y = \sum_{i\in I\setminus J} x_i \in \boldsymbol{S}_{I\setminus J}^F \subseteq S$. Therefore S is closed under under + and –.

Conversely, suppose S is closed under + and –. Let $F = \{S_\alpha : \alpha \in Y, \alpha \succ 0\}$, which we may assume to be $\{S_i : i \in \mathbf{N}_m\}$. By Lemma II.2.2, $S \subseteq \bigcup_{I\subseteq \mathbf{N}_m} \boldsymbol{S}_I^F$. To show the reverse containment, let $x \in \bigcup_{I\subseteq \mathbf{N}_m} \boldsymbol{S}_I^F$. Then $x = \sum_{i\in I} x_i$ for some $x_i \in S_i$ and $I \subseteq \mathbf{N}_m$. Note that $x_i x_j = \mathbf{0}$ whenever $i \neq j$. Without lost of generality assume $I = \mathbf{N}_r$. Now $x_1 x_2 = \mathbf{0}$, so $x_1 + x_2 \in S$. But $(x_1 + x_2)x_3 = \mathbf{0}$, so $x_1 + x_2 + x_3 \in S$. Continuing, we deduce that $x = \sum_{i\in I} x_i \in S$. Hence $\bigcup_{I\subseteq \mathbf{N}_m} \boldsymbol{S}_I^F \subseteq S$. In conclusion, S is a Σ-semigroup with foundation F. ∎

II.3 The +Closure and –Closure of a Semigroup

The following are some consequences of Theorem II.2.4.

Theorem II.3.1 *If $\mathcal{A}$ is a collection of Σ-semigroups, then $\bigcap\mathcal{A}$ is a Σ-semigroup. Furthermore, if each semigroup in $\mathcal{A}$ is a $\mathcal{O}$-semigroup, then $\bigcap\mathcal{A}$ is also a $\mathcal{O}$-semigroup.*

Proof If $\mathcal{A}$ is a collection of Σ-semigroups, then $\bigcap\mathcal{A}$ is non-empty because every $S \in \mathcal{A}$ contains $\mathbf{0}$. Since every $S \in \mathcal{A}$ is closed under + and – (Theorem II.2.4), $\bigcap\mathcal{A}$ is also closed under + and –. Hence $\bigcap\mathcal{A}$ is a Σ-semigroup. It is obvious that if every $S \in \mathcal{A}$ is a $\mathcal{O}$-semigroup, then $\bigcap\mathcal{A}$ is a $\mathcal{O}$-semigroup. ∎

Corollary II.3.2 *If a completely regular semigroup S is closed under –, then S^Σ exists.*

Proof If S is a completely regular semigroup closed under $-$, then S is contained in some Σ-semigroup (Lemma II.2.2). Hence the collection $\mathcal{A}$ of all Σ-semigroups containing S is non-empty. But any intersection of Σ-semigroups is a Σ-semigroup (Theorem II.3.1). Therefore $\bigcap\mathcal{A}$ is a Σ-semigroup. It follows that $S^{\Sigma} = \bigcap\mathcal{A}$. ■

By Corollary II.1.6, every Σ-semigroup is completely regular. But an arbitrarily given completely regular semigroup may not be contained in any Σ-semigroup, as we shall see in the following example.

Example II.3.3 *Consider the subband* $B = \{\mathbf{1}, x, y\}$ *of* $M_2(\boldsymbol{k})$ *where*

$$x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad y = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then B is a completely regular semigroup that is not contained in any Σ-semigroup.

Proof Suppose $S = \bigcup_{I\subseteq\Lambda} \boldsymbol{S}_I^F$ is a Σ-semigroup containing B. Since $\mathbf{1} \geq x$, $\mathbf{1} \geq y$ and S is closed under $-$ (Theorem II.2.4), we have $\mathbf{1} - x$, $\mathbf{1} - y \in S$. It is straightforward to show that $\{x, y\}$ is a left zero band and that $\{\mathbf{1} - x, \mathbf{1} - y\}$ is a right zero band. Therefore we have $x\mathcal{J}y$ and $(\mathbf{1} - x)\mathcal{J}(\mathbf{1} - y)$, which implies that $x, y \in \boldsymbol{S}_I^F$ and $\mathbf{1} - x$, $\mathbf{1} - y \in \boldsymbol{S}_J^F$ for some $I, J \subseteq \Lambda$. Now $\mathbf{0} = x(\mathbf{1} - x) \in \boldsymbol{S}_I^F \boldsymbol{S}_J^F = \boldsymbol{S}_{I\cap J}^F$. Thus $\boldsymbol{S}_{I\cap J}^F = \{\mathbf{0}\}$. But $(\mathbf{1} - x)y = y - x \neq \mathbf{0}$ is a contradiction. ■

We now present the main concept of this section.

Let S be a semigroup. We define the *+closure of S* to be the smallest semigroup closed under $+$ containing S and denote it by S^+. Note that S^+ is just the intersection of all semigroups closed under $+$ containing S. We will show how to generate S^+ by the following construction.

For $i = 0, 1, \ldots$, define

$$S^+(i) = \begin{cases} S \cup \{\mathbf{0}\} & \text{if } i = 0 \\ \langle a+b : a, b \in S^+(i-1), ab = ba = \mathbf{0} \rangle & \text{if } i \geq 1. \end{cases} \tag{II.3.1}$$

It is straightforward to show that $S^+(0) \subseteq S^+(1) \subseteq \cdots$ is a chain of semigroups, and that $\bigcup_{i\geq 0} S^+(i)$ is a semigroup closed under +. Indeed, $\bigcup_{i\geq 0} S^+(i)$ is the smallest semigroup closed under + containing S; for if T is any semigroup closed under + containing S, then T must contain every $S^+(i)$. Hence $S^+ = \bigcup_{i\geq 0} S^+(i)$.

Similarly, for any semigroup S, we define the *−closure of S* to be the smallest semigroup closed under − containing S and denote it by S^-. We can also show that S^- = $\bigcup_{i\geq 0} S^-(i)$, where

$$S^-(i) = \begin{cases} S \cup \{\mathbf{0}\} & \text{if } i = 0 \\ \langle a-b : a, b \in S^-(i-1), a \geq b \rangle & \text{if } i \geq 1. \end{cases} \tag{II.3.2}$$

Hence we have

Theorem II.3.4 *The +closure and −closure of a semigroup S are given by*

$$S^+ = \bigcup_{i\geq 0} S^+(i) \quad \textit{and} \quad S^- = \bigcup_{i\geq 0} S^-(i),$$

where $S^+(i)$ *and* $S^-(i)$ *are defined in* (II.3.1) *and* (II.3.2) *respectively.* ■

Lemma II.3.5 *If S is a completely simple semigroup, then*

$$S^+ = S^- = S \cup \{\mathbf{0}\}.$$

Consequently, if S is a completely simple $\mathcal{O}$*-semigroup, then both* S^+ *and* S^- *belong to the variety* $\mathcal{O} \vee \mathcal{SL}$.

Proof Since a completely simple semigroup S different from $\{\mathbf{0}\}$ does not contain $\mathbf{0}$, we can easily deduce that $S^+ = S \cup \{\mathbf{0}\}$ because $S \cup \{\mathbf{0}\} = S^+(0) = S^+(1) = \cdots$. Now if $x \geq y$ in S, then $xe = fx = y$ for some idempotents e, f of S. Thus we have $xe = ye$ and $fx = fy$. It follows by Theorem I.5.4 that $x = y$. Hence we again have $S \cup \{\mathbf{0}\} = S^-(0) = S^-(1) = \cdots$, which implies that $S^- = S \cup \{\mathbf{0}\}$. ■

For the next two sections, we will prove the analogy of Lemma II.3.5 for non-completely simple $\mathcal{V}$-semigroups.

II.4 The +Closure of a $\mathcal{V}$-Semigroup

For the following lemmas, let S be a non-completely simple $\mathcal{V}$-semigroup (i.e. semigroups in the varieties of Figure I.4) and let

$$C = \{a + b : a, b \in S, ab = ba = \mathbf{0}\}.$$

Lemma II.4.1 *If $x = a + b$ is an element of C (where $a, b \in S$ such that $ab = ba = \mathbf{0}$), then x is a completely regular element of $M_n(\mathbb{k})$ such that:*

(1) $x^{-1} = a^{-1} + b^{-1}$;

(2) $x^0 = a^0 + b^0$.

Furthermore, the elements x^{-1}, x^0 belong to C.

Proof Let $S = (Y; S_\alpha)$. Then $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Since $ab = \mathbf{0}$, we have $\alpha\beta = 0$, i.e. $S_\alpha S_\beta \cup S_\beta S_\alpha = \{\mathbf{0}\}$. Because $a^{-1}, a^0 \in S_\alpha$ and $b^{-1}, b^0 \in S_\beta$, we have $a^{-1}b^{-1} = b^{-1}a^{-1} = \mathbf{0} = a^0b^0 = b^0a^0$. Therefore $a^{-1} + b^{-1}, a^0 + b^0 \in C$. It is not difficult to show that $a^{-1} + b^{-1}$ is an inverse of x that commutes with it, and that $(a + b)(a^{-1} + b^{-1}) = a^0 + b^0$. For example, to show the latter result:

$$\begin{aligned}(a + b)(a^{-1} + b^{-1}) &= aa^{-1} + ab^{-1} + ba^{-1} + bb^{-1} \\ &= a^0 + \mathbf{0} + \mathbf{0} + b^0 = a^0 + b^0.\end{aligned}$$

Consequently, we have $x^{-1} = a^{-1} + b^{-1}$ and $x^0 = a^0 + b^0$ (see Lemma I.6.1 and the remark following it). ■

Lemma II.4.2 *For any x, y, z, w in C:*

(1) $x^2 = x$ *and* $xy = yx$ *if* $\mathcal{V} = \mathcal{SL}$; (2) $x^2 = x$ *and* $xyz = xzy$ *if* $\mathcal{V} = \mathcal{LN}$;

(3) $x^2 = x$ *and* $xyz = yxz$ *if* $\mathcal{V} = \mathcal{RN}$; (4) $x^2 = x$ *and* $xyzw = xzyw$ *if* $\mathcal{V} = \mathcal{N}$;

(5) $xy^0 = y^0x$ *if* $\mathcal{V} = \mathcal{SG}$; (6) $xy^0z = xzy^0$ *if* $\mathcal{V} = \mathcal{LNO}$;

(7) $xy^0z = y^0xz$ *if* $\mathcal{V} = \mathcal{RNO}$; (8) $xy^0zw = xzy^0w$ *if* $\mathcal{V} = \mathcal{NO}$.

Proof (8) Let $\mathcal{V} = \mathcal{NO}$ and let $x, y, z, w \in C$. Then $x = a + b$, $y = c + d$, $z = e + f$, $w = g + h$ for some $a, b, c, d, e, f, g, h \in S$. For any $u, v \in S$, we have

$$uy^0zv \overset{(a)}{=} u(c^0 + d^0)(e + f)v = uc^0ev + uc^0fv + ud^0ev + ud^0fv$$
$$\overset{(b)}{=} uec^0v + ufc^0v + ued^0v + ufd^0v = u(e + f)(c^0 + d^0)v \overset{(a)}{=} uzy^0v,$$

where (a) holds by Lemma II.4.1 and (b) holds by Proposition I.6.14. Hence

$$xy^0zw = (a + b)y^0z(g + h) = ay^0zg + ay^0zh + by^0zg + by^0zh$$
$$= azy^0g + azy^0h + bzy^0g + bzy^0h = (a + b)zy^0(g + h) = xzy^0w.$$

(5) Replace '$\mathcal{NO}$' by '$\mathcal{SG}$', 'Proposition I.6.14' by 'Corollary I.6.15(3)', and remove 'x', 'w', 'u', 'v' in the proof of (8).

(6) Replace '$\mathcal{NO}$' by '$\mathcal{LNO}$', 'Proposition I.6.14' by 'Corollary I.6.15(1)', and remove 'w', 'v' in the proof of (8).

(7) Replace '$\mathcal{NO}$' by '$\mathcal{RNO}$', 'Proposition I.6.14' by 'Corollary I.6.15(2)', and remove 'x', 'u' in the proof of (8).

(4) Let $\mathcal{V} = \mathcal{N}$ and let $x = a + b \in C$. Since S is a band, we have, by Lemma II.4.1, that $x^0 = a^0 + b^0 = a + b = x$. Hence every element of C is an idempotent, and so $x^2 = x$ for all $x \in C$. Now S is also a normal orthogroup. Therefore by (8), $xy^0zw = xzy^0w$ for all $x, y, z, w \in C$, which is equivalent to $xyzw = xzyw$ because $y^0 = y$ in C.

(1) Replace '$\mathcal{N}$' by '$\mathcal{SL}$', 'normal orthogroup' by 'Clifford semigroup', '(8)' by '(5)', and remove 'x', 'w' in the proof of (4).

(2) Replace '$\mathcal{N}$' by '$\mathcal{LN}$', 'normal orthogroup' by 'left normal orthogroup', '(8)' by

'(6)', and remove 'w' in the proof of (4).

(3) Replace '$\mathcal{N}$' by '$\mathcal{RN}$', 'normal orthogroup' by 'right normal orthogroup', '(8)' by '(7)', and remove 'x' in the proof of (4). ■

Lemma II.4.3 *$\langle C\rangle$ is a completely regular semigroup.*

Proof Let $x = \prod_{i=1}^{m} x_i \in \langle C\rangle$ where $x_i \in C$. Our objective is to show that x has an inverse in $\langle C\rangle$ with which it commutes. Note that since S is a normal orthogroup, we may invoke Lemma II.4.2(8). Also, x_i^{-1}, x_i^0 exist in C by Lemma II.4.1.

First we use induction on m to prove that $x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0 \in V(x)$. The case $m = 1$ is trivial. Suppose $m \geq 2$ and $x_1^0\left(\prod_{i=r}^{1} x_i^{-1}\right)x_r^0 \in V(x)$ whenever $r < m$. So when $r = m$,

$$\begin{aligned}
x\left[x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0\right]x &= \left(\prod_{i=1}^{m-1} x_i\right)x_m x_1^0 x_m^{-1}\left(\prod_{i=m-1}^{1} x_i^{-1}\right)x_m^0\left(\prod_{i=1}^{m-1} x_i\right)x_m \\
&\overset{(a)}{=}\left(\prod_{i=1}^{m-1} x_i\right)x_1^0(x_m x_m^{-1})\left(\prod_{i=m-1}^{1} x_i^{-1}\right)\left(\prod_{i=1}^{m-1} x_i\right)x_m^0 x_m \\
&= \left(\prod_{i=1}^{m-1} x_i\right)x_1^0 x_m^0\left[x_{m-1}^0\left(\prod_{i=m-1}^{1} x_i^{-1}\right)\right]\left(\prod_{i=1}^{m-1} x_i\right)x_m \\
&\overset{(a)}{=}\left[\left(\prod_{i=1}^{m-1} x_i\right)x_1^0\left(\prod_{i=m-1}^{1} x_i^{-1}\right)x_{m-1}^0\left(\prod_{i=1}^{m-1} x_i\right)\right]x_m^0 x_m \\
&\overset{(b)}{=}\left(\prod_{i=1}^{m-1} x_i\right)x_m = x,
\end{aligned}$$

where (a) indicates repeated use of Lemma II.4.2(8) and (b) holds by induction hypothesis. To show

$$\left[x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0\right]x\left[x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0\right] = x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0$$

is similar. Therefore we have $x_1^0\left(\prod_{i=m}^{1} x_i^{-1}\right)x_m^0 \in V(x)$.

We now use induction on m to prove that

$$x\left[x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0\right] = \prod_{i=1}^{m}x_i^0 = \left[x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0\right]x.$$

It suffices to show $x\left[x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0\right] = \prod_{i=1}^{m}x_i^0$ because proving the other equality is a similar task. The case $m = 1$ is trivial. Suppose $m \geq 2$ and

$$\left(\prod_{i=1}^{r}x_i\right)x_1^0\left(\prod_{i=r}^{1}x_i^{-1}\right)x_r^0 = \prod_{i=1}^{r}x_i^0$$

whenever $r < m$. Then when $r = m$,

$$\begin{aligned}
x\left[x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0\right] &= \left(\prod_{i=1}^{m-1}x_i\right)x_m x_1^0 x_m^{-1}\left(\prod_{i=m-1}^{1}x_i^{-1}\right)x_m^0\\
&\overset{(c)}{=}\left(\prod_{i=1}^{m-1}x_i\right)(x_m x_m^{-1})x_1^0\left(\prod_{i=m-1}^{1}x_i^{-1}\right)x_m^0\\
&=\left[\left(\prod_{i=1}^{m-1}x_i\right)x_{m-1}^0\right]x_m^0 x_1^0\left(\prod_{i=m-1}^{1}x_i^{-1}\right)x_m^0\\
&\overset{(c)}{=}\left[\left(\prod_{i=1}^{m-1}x_i\right)x_1^0\left(\prod_{i=m-1}^{1}x_i^{-1}\right)x_{m-1}^0\right]x_m^0 x_m^0\\
&\overset{(d)}{=}\left(\prod_{i=1}^{m-1}x_i^0\right)x_m^0 = \prod_{i=1}^{m}x_i^0,
\end{aligned}$$

where (c) holds by Lemma II.4.2(8) and (d) holds by induction hypothesis.

We may conclude that $x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0$ is an inverse of x that commutes with it. Hence x is completely regular. ■

It is clear from the proof of Lemma II.4.3 that if $x = \prod_{i=1}^{m}x_i$ is in $\langle C\rangle$, then

$$x^{-1} = x_1^0\left(\prod_{i=m}^{1}x_i^{-1}\right)x_m^0 \quad\text{and}\quad x^0 = \prod_{i=1}^{m}x_i^0.$$

We are now ready to prove the following.

Theorem II.4.4 *The +closure of a semigroup in $\mathcal{O}$ is a semigroup in $\mathcal{O} \vee \mathcal{SL}$.*

Proof Let S be a $\mathcal{V}$-semigroup. If S is completely simple, then we are done by Lemma II.3.5. So suppose S is non-completely simple. But we only need to prove the theorem for $\mathcal{V} = \mathcal{NO}$, because the proofs of the other cases are very similar.

Recall that the +closure of S is $S^+ = \bigcup_{i\geq 0} S^+(i)$, where $S^+(i)$ is defined in (II.3.1). We use induction to prove that every $S^+(i)$ is a normal orthogroup. The case $i = 0$ is obvious. Suppose $S^+(m-1)$ is a normal orthogroup. Let $C = \{a + b : a, b \in S^+(m-1), ab = ba = \mathbf{0}\}$. By Lemma II.4.3, $\langle C\rangle$ is a completely regular semigroup. Now let $x, y, z \in \langle C\rangle$. Then $x = \prod_{i=1}^{m_1} x_i$, $y^0 = \prod_{i=1}^{m_2} y_i^0$, $z = \prod_{i=1}^{m_3} z_i$ for some $x_i, y_i, z_i \in C$. Therefore

$$\begin{aligned} xy^0zx &= \left(\prod_{i=1}^{m_1} x_i\right)\left(\prod_{i=1}^{m_2} y_i^0\right)\left(\prod_{i=1}^{m_3} z_i\right)\left(\prod_{i=1}^{m_1} x_i\right) \\ &= \left(\prod_{i=1}^{m_1} x_i\right)\left(\prod_{i=1}^{m_3} z_i\right)\left(\prod_{i=1}^{m_2} y_i^0\right)\left(\prod_{i=1}^{m_1} x_i\right) = xzy^0x, \end{aligned}$$

where the second equality holds by repeated use of Lemma II.4.2(8). Hence $S^+(m) = \langle C\rangle$ is a normal orthogroup (Proposition I.6.14) and induction is completed. Since $S^+(0) \subseteq S^+(1) \subseteq \cdots$, it follows that $S^+ = \bigcup_{i\geq 0} S^+(i)$ is a normal orthogroup. ■

II.5 The –Closure of a $\mathcal{V}$-Semigroup

Recall that a $\mathcal{V}$-semigroup is a strong semilattice of completely simple semigroups. Hence by Lemma I.6.16, the partial order $\geq$ defined on a completely regular semigroup can be simplified to

$$a \geq b \iff ab^0 = b^0a = b.$$

For the following lemmas, let S be a non-completely simple $\mathcal{V}$-semigroup and let

$$D = \{a - b : a, b \in S, a \geq b\}.$$

Lemma II.5.1 *Let a, b be elements of S. If $a \geq b$, then*:

(1) $ab^0 = b^0a = b$;
(2) $ab^{-1} = b^{-1}a = b^0$;
(3) $a^0 \geq b^0$, *i.e.* $a^0b^0 = b^0a^0 = b^0$;
(4) $a^0b = ba^0 = b$;
(5) $a^0b^{-1} = b^{-1}a^0 = b^{-1}$;
(6) $a^{-1}b = ba^{-1} = b^0$;
(7) $a^{-1} \geq b^{-1}$, *i.e.* $a^{-1}b^0 = b^0a^{-1} = b^{-1}$.

Proof Since $a \geq b$ is equivalent to (1), it suffices to show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) $\Rightarrow$ (7). Moreover, only one-sided proofs are necessary.

(1) $\Rightarrow$ (2). $ab^{-1} = (ab^0)b^{-1} = bb^{-1} = b^0$.

(2) $\Rightarrow$ (3). $a^0b^0 = (a^0a)b^{-1} = ab^{-1} = b^0$.

(3) $\Rightarrow$ (4). $a^0b = (a^0b^0)b = b^0b = b$.

(4) $\Rightarrow$ (5). $a^0b^{-1} = (a^0b)b^{-1}b^{-1} = bb^{-1}b^{-1} = b^{-1}$.

(1) $\Rightarrow$ (6). If $ab^0 = b^0a = b$, then we also have (3). Hence $a^{-1}b = (a^{-1}a)b^0 = a^0b^0 = b^0$.

(6) $\Rightarrow$ (7). $a^{-1}b^0 = (a^{-1}b)b^{-1} = b^0b^{-1} = b^{-1}$. ■

Lemma II.5.2 *If $x = a - b$ is an element of D (where $a, b \in S$ such that $a \geq b$), then x is a completely regular element of $M_n(\boldsymbol{k})$ such that*:

(1) $x^{-1} = a^{-1} - b^{-1}$;

(2) $x^0 = a^0 - b^0$.

Furthermore, the elements x^{-1}, x^0 belong to D.

Proof If $a, b \in S$ such that $a \geq b$, then $a^{-1}, a^0, b^{-1}, b^0 \in S$, and by Lemma II.5.1, we have $a^{-1} \geq b^{-1}$ and $a^0 \geq b^0$. Therefore $a^{-1} - b^{-1}, a^0 - b^0 \in D$. Also, we have

$$\begin{aligned}(a-b)(a^{-1}-b^{-1}) &= aa^{-1} - ab^{-1} - ba^{-1} + bb^{-1} \\ &= a^0 - b^0 - b^0 + b^0 = a^0 - b^0,\end{aligned}$$

and similarly, $(a^{-1} - b^{-1})(a - b) = a^0 - b^0$. Thus $a^{-1} - b^{-1}$ and x commute. It remains to show that $a^{-1} - b^{-1}$ is an inverse of x, which is just a routine application of Lemma II.5.1. Hence by Lemma I.6.1 and the remark following it, we may conclude that $x^{-1} = a^{-1} - b^{-1}$

and $x^0 = a^0 - b^0$. ■

Lemma II.5.3 *For any x, y, z, w in D:*

(1) $x^2 = x$ and $xy = yx$, if $\mathcal{V} = \mathcal{SL}$; (2) $x^2 = x$ and $xyz = xzy$, if $\mathcal{V} = \mathcal{LN}$;

(3) $x^2 = x$ and $xyz = yxz$, if $\mathcal{V} = \mathcal{RN}$; (4) $x^2 = x$ and $xyzw = xzyw$, if $\mathcal{V} = \mathcal{N}$;

(5) $xy^0 = y^0x$ if $\mathcal{V} = \mathcal{SG}$; (6) $xy^0z = xzy^0$ if $\mathcal{V} = \mathcal{LNO}$;

(7) $xy^0z = y^0xz$ if $\mathcal{V} = \mathcal{RNO}$; (8) $xy^0zw = xzy^0w$ if $\mathcal{V} = \mathcal{NO}$.

Proof This is very similar to the proof of Lemma II.4.2. For example, to show (5), let $x, y \in D$. Then $x = a - b$ and $y = c - d$ for some $a, b, c, d \in S$ such that $a \geq b$ and $c \geq d$. Using Lemma II.5.2 and the assumption that S is a Clifford semigroup, we have

$$\begin{aligned} xy^0 &= (a-b)(c^0 - d^0) = ac^0 - ad^0 - bc^0 + bd^0 \\ &= c^0a - d^0a - c^0b + d^0b = (c^0 - d^0)(a-b) = y^0x. \end{aligned}$$ ■

Lemma II.5.4 *$\langle D \rangle$ is a completely regular semigroup.*

Proof This is almost identical to the proof of Lemma II.4.3. Just replace 'C' by 'D', use Lemma II.5.2 instead of Lemma II.4.1, and use Lemma II.5.3(8) instead of Lemma II.4.2(8). ■

Apply Lemma II.5.3 and II.5.4 in the same way as we have applied Lemma II.4.2 and II.4.3 in Theorem II.4.4, we obtain

Theorem II.5.5 *The –closure of a semigroup in $\mathcal{V}$ is a semigroup in $\mathcal{V} \vee \mathcal{SL}$.* ■

II.6 The Σ-Closure of a $\mathcal{V}$-Semigroup

We can now combine the main theorems of the last two sections to prove the main result of this thesis.

Theorem II.6.1 *The* Σ*-closure of a semigroup in* $\mathcal{V}$ *is a semigroup in* $\mathcal{V} \vee \mathcal{SL}$. *In particular, if* S *is a* $\mathcal{V}$*-semigroup, then* $S^{\Sigma} = (S^{-})^{+}$.

Proof Let S be a $\mathcal{V}$-semigroup. By Theorem II.5.5, S^{-} is a semigroup in $\mathcal{V} \vee \mathcal{SL}$. We may assume $S^{-} = (Y; S_\alpha; \phi_{\alpha,\beta})$ and $F = \{S_\alpha : \alpha \in \Lambda\}$, where $\Lambda = \{\alpha \in Y : \alpha \succ 0\}$. Since S^{-} is closed under $-$, we have $S^{-} \subseteq \bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$ by Lemma II.2.2. On the other hand, $\bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$ is a Σ-semigroup and so it must be closed under + (Theorem II.2.4). Thus $(S^{-})^{+} \subseteq \bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$. Note that

$$
\begin{aligned}
x \in \bigcup\nolimits_{I \subseteq \Lambda} \boldsymbol{S}_I^F \quad &\Rightarrow \quad x \in \boldsymbol{S}_I^F \text{ for some } I \subseteq \Lambda \\
&\Rightarrow \quad x = \sum\nolimits_{\alpha \in I} x_\alpha \text{ for some } x_\alpha \in S_\alpha \subseteq S^{-} \\
&\Rightarrow \quad x \in (S^{-})^{+}.
\end{aligned}
$$

Therefore $(S^{-})^{+} = \bigcup_{I \subseteq \Lambda} \boldsymbol{S}_I^F$, which implies that $(S^{-})^{+}$ is a Σ-semigroup. To show that $(S^{-})^{+}$ is the smallest Σ-semigroup containing S, let T be any Σ-semigroup containing S. Then $S^{-} \subseteq T$ because T is closed under $-$, and $(S^{-})^{+} \subseteq T$ because T is closed under +. Thus $S^{\Sigma} = (S^{-})^{+}$. Finally, since S^{-} is a semigroup in $\mathcal{V} \vee \mathcal{SL}$, it follows by Theorem II.4.4 that $(S^{-})^{+}$ is a semigroup in $(\mathcal{V} \vee \mathcal{SL}) \vee \mathcal{SL} = \mathcal{V} \vee \mathcal{SL}$. ■

Note that if S is a $\mathcal{V}$-semigroup which is not completely simple, then $\mathcal{V} \geq \mathcal{SL}$ in the lattice of Figure I.5. Hence $\mathcal{V} \vee \mathcal{SL} = \mathcal{V}$. Consequently, the Σ-closure of S is in $\mathcal{V}$. We may thus conclude that

Corollary II.6.2 *A maximal, non-completely simple* $\mathcal{V}$*-semigroup is a* Σ*-semigroup.* ■

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List of Symbols

$\mathbf{0}$	the $n \times n$ zero matrix	
$\mathbf{1}$	the $n \times n$ identity matrix	
a^{-1}	the inverse of a in the group H_a	11
a^0	the identity element of the group H_a	11
a^{T}	the transpose of a matrix a	
$a \geq b$	the binary relation defined on a semigroup	14
$\alpha \succ \beta$	α covers β	32
$(A, \{f_i : i \in I\})$	an algebra with underlying set A and n_i-ary operations f_i	20
$\mathcal{CR}$	the variety of completely regular semigroups	22
$e \geq f$	the partial order defined on the set of idempotents of a semigroup	2, 5
$e > f$	$e \geq f$ and $e \neq f$	2
$E(S)$	the set of idempotents of a semigroup S	2
$\mathcal{G}$	the variety of groups	23
$\mathcal{H}$	Green's equivalence	4
H_a	the $\mathcal{H}$-class containing a	5
$\mathrm{Im}(a)$	the image of a matrix a	
$\mathcal{J}$	Green's equivalence	4
J_a	the $\mathcal{J}$-class containing a	5
$\boldsymbol{k}$	a commutative field	
$\boldsymbol{k}^n$	the vector space of $n \times 1$ matrices over $\boldsymbol{k}$	
$\mathcal{L}$	Green's equivalence	4

L_a	the $\mathcal{L}$-class containing a	5
$\mathcal{LG}$	the variety of left groups	23
$\mathcal{LN}$	the variety of left normal bands	22
$\mathcal{LNO}$	the variety of left normal orthogroups	23
$\mathcal{LZ}$	the variety of left zero bands	22
$\mathcal{M}[G; I, \Lambda; P]$	the Rees matrix semigroup	6
$M_n(\boldsymbol{k})$	the ring of $n \times n$ matrices over $\boldsymbol{k}$	
$\mathbf{N}$	the set of positive integers	
$\mathcal{N}$	the variety of normal bands	22
$\mathbf{N}_m$	the set of the first m positive integers	
$\mathcal{NO}$	the variety of normal orthogroups	23
$p = q$	an identity	21
$\mathcal{R}$	Green's equivalence	4
R_a	the $\mathcal{R}$-class containing a	5
$[\mathbf{R}]$	an equational class (variety) of (2, 1)-algebras determined by a set of identities $\mathbf{R}$	21
$[\mathbf{R}_1] \vee [\mathbf{R}_2]$	the join of varieties $[\mathbf{R}_1]$, $[\mathbf{R}_2]$	23
$\text{rank}(a)$	the rank of a matrix a	
$\mathcal{RB}$	the variety of rectangular bands	22
$\mathcal{ReG}$	the variety of rectangular groups	23
$\mathcal{RG}$	the variety of right groups	23
$\mathcal{RN}$	the variety of right normal bands	22
$\mathcal{RNO}$	the variety of right normal orthogroups	23
$\mathcal{RZ}$	the variety of right zero bands	22
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