On the Lattice of Rees-Sushkevich Varieties

by

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Abstract

A solution to the problem of computing a finite semigroup generating the intersection of two finitely generated exact Rees-Sushkevich varieties is presented. This solution uses the generalization of the concept of critical groups to the context of completely simple semigroups and the characterization of critical semigroups that are central completely simple. Next, the lattice of subvarieties of a non-exact variety generated by a semigroup of order four is completely characterized. This lattice is infinite and semi-modular, and it contains only finitely based varieties.

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Dedication

To my parents, on their 30th anniversary.

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Chapter 1

Introduction

A semigroup with zero (respectively, without zero) is said to be *completely* 0*-simple* (respectively, *completely simple*) if it contains a nonzero primitive idempotent and has no nonzero proper ideals. Completely (0-)simple semigroups were one of the very first semigroups to be studied and they remain one of the most interesting semigroups due to their frequent occurrence in many areas. For example, it is well known that minimal nonzero ideals and non-null principal factors of finite semigroups are completely 0-simple semigroups. The theorem of Rees [16], representing each completely (0-)simple semigroup as a semigroup of matrices over a group with a zero possibly adjoined, has played a dominant role in the developement of the theory.

Let $\mathbf{CS}_n^0$ denote the class of all completely 0-simple semigroups with subgroups of exponent dividing n, and let $\mathbf{CS}_\mathbf{f}^0$ denote the class of all finite completely 0-simple semigroups. A subvariety of the variety generated by $\mathbf{CS}_n^0$ is called a *Rees-Sushkevich variety*, and a *Rees-Sushkevich pseudovariety* is similarly defined. A Rees-Sushkevich variety or pseudovariety is *exact* if it is generated by completely (0-)simple semigroups. Exact (pseudo)varieties have recently been investigated by Kublanovsky [7, 9], and Reilly [17].

Most of the background material that is required in this thesis is contained in Chapter 2, while the main results are contained in Chapters 3 and 4. The contents of Chapters 3 and 4 are related since they both concern aspects of the lattice of Rees-Sushkevich varieties, but they can almost be read independently.

In [7], Kublanovsky showed that the set of all exact pseudovarieties, each of which is generated by a single finite semigroup, forms a lattice; this result led him to pose the problem of finding a finite completely (0-)simple semigroup that will generate the intersection of two such pseudovarieties. This problem shall be called Kublanovsky's Intersection Problem.

Chapter 3 is a join work by Reilly and the author, and it contains a solution to Kublanovsky's Intersection Problem within the context of $\mathbf{CCS} \vee \mathbf{NB_2}$, where $\mathbf{CCS}$ is the variety of central completely simple semigroups, and $\mathbf{NB_2}$ is the variety generated by the semigroups L, R and B_2. The concept of critical groups is essential in the proof of the famous theorem of Oates and Powell [12], which states that the variety generated by a finite group is a Cross variety and therefore finitely based. Although most properties of groups do not hold for completely simple semigroups in general, the correspondence between congruences with admissible triples of the form $(\varepsilon : N : \varepsilon)$ on a completely simple semigroup $\mathcal{M}(I, G, \Lambda; P)$ and the normal subgroups N of G enables certain results concerning critical groups to be generalized to the context of completely simple semigroups (Theorems 3.1.11, 3.1.12 and Corollary 3.1.13). These generalizations permit the characterization of central completely simple semigroups that are critical (Theorem 3.2.5). By using an isomorphism from the lattice $\mathcal{LE}(\mathbf{CCS}_n \vee \mathbf{NB_2})$ onto a sublattice of $\mathcal{L}(\mathbf{RB}) \times \mathcal{L}(\mathbf{A}_n) \times \mathcal{L}(\mathbf{G}_n) \times \{\mathbf{T}, \mathbf{Y}, \mathbf{B_2}\}$, a solution of Kublanovsky's Intersection Problem within the context of $\mathbf{CCS} \vee \mathbf{NB_2}$ can be constructed from semigroups generating intersections of varieties from $\mathcal{L}(\mathbf{RB}), \mathcal{L}(\mathbf{A}_n), \mathcal{L}(\mathbf{G}_n)$ and $\{\mathbf{T}, \mathbf{Y}, \mathbf{B_2}\}$ individually (see Section 5). Examples of critical semigroups are presented in the last section of the chapter. These examples show that there is no immediate relationship between the criticality of a completely simple semigroup and the criticality of its subgroups. It will also be shown that the monoid obtained from adjoining an identity element to a critical central completely simple semigroup is also critical (Proposition 3.6.9).

The variety $\mathbf{A_0}$ generated by a well-known semigroup A_0 of order four is aperiodic and non-exact, and it figures prominently in the study of exact and non-exact varieties (see [9], [17]). Chapter 4 contains a complete characterization of the lattice $\mathcal{L}(\mathbf{A_0})$ of subvarieties of $\mathbf{A_0}$. The non-exact variety $\mathbf{A_0}$ contains several infinite families of varieties. (By way of contrast, Reilly [17] showed that there are only 13 exact aperiodic varieties). For each introduced semigroup S in $\mathbf{A_0}$, a set of identities for S and a canonical form for words in S will be presented. It will be shown that each word can be reduced to one in canonical form using these identities, and that two words form an identity of S if and only if their canonical forms are identical. These results effectively imply that the set of identities initially presented constitutes a basis for S, that two words have the same interpretation in S if and only if their canonical forms are identical, and that the elements of the relatively free semigroup of S can be taken to be exactly those words in canonical form with an appropriate law of composition.

It turns out that $\mathcal{L}(\mathbf{A_0})$ contains a complete sublattice $\mathfrak{LNR}^*$ (Corollary 4.8.8), and each subvariety of $\mathbf{A_0}$ that is not in $\mathfrak{LNR}^*$ is the intersection of a variety in $\mathfrak{LNR}^*$ and a permutation variety (Propositions 4.9.2 and 4.9.5). Important properties, such as joins, intersections, and coverings of varieties in $\mathcal{L}(\mathbf{A_0})$ will be fully described. It will be shown that all varieties in $\mathcal{L}(\mathbf{A_0})$ are finitely based (Corollary 4.9.6). But $\mathcal{L}(\mathbf{A_0})$ contains an infinite interval all members of which are non-finitely generated (Proposition 4.12.1); the least variety from this interval is thus minimal with respect to being locally finite and non-finitely generated.

Chapter 2

Preliminaries

2.1 Semigroups

A *semigroup* is a nonempty set S with an associative binary operation $\cdot$, which we often call *multiplication*. For each pair of elements a, b from S, the product $a \cdot b$ will be abbreviated to ab. Associativity allows products of any finite length in a semigroup to be written unambiguously without the need for parentheses. In this thesis, S stands for an arbitrary semigroup unless otherwise specified.

An element e of S is an *identity* if $ea = ae = a$ for all $a \in S$. An element z of S is a *zero* if $S \neq \{z\}$ and $za = az = z$ for all $a \in S$. If a semigroup contains an identity or zero, then they are unique and are usually denoted by 1 and 0 respectively. A semigroup with an identity is called a *monoid*. If S has no identity, then an extra element 1 can be adjoined to S to form the monoid $S \cup \{1\}$ with obvious multiplication. For convenience, define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

A subset T of S is a *subsemigroup* of S (written as $T \leq S$) if T is closed under multiplication. A subsemigroup T of S is *proper* (written as $T < S$) if $T \neq S$. If $A \subseteq S$ then $\langle A \rangle$ denotes the subsemigroup of S *generated by* elements of A. A subsemigroup which is also a group is called a *subgroup*. As usual, if G is a group, then we write $H \leq G$ (respectively, $H \trianglelefteq G$) when H is a subgroup (respectively, normal subgroup) of G.

An element e of S is an *idempotent* if $e^2 = e$, and the set of idempotents of S is denoted by $E(S)$. For any $e, f \in E(S)$, we write $e \geq f$ if $ef = fe = f$. It is straightforward to show that $\geq$ is a partial order on $E(S)$. A nonzero idempotent that is minimal with respect to $\geq$ is called *primitive*. The subsemigroup of S generated by $E(S)$ is called the *core* of S and is denoted by $C(S)$. A semigroup in which all its elements are idempotents is called a *band*, and a commutative band is called a *semilattice*. The set $\{0,1\}$ under usual multiplication is a semilattice and is denoted by Y. A semigroup the idempotents of which form a subsemigroup is called *orthodox*.

Note that a group G is just a semigroup with an identity and in which every element of G has an inverse in G. Therefore group theoretic definitions (which do not involve the identity) such as direct products, subdirect products, homomorphisms etc., can be carried over to the context of semigroups. Let T be a semigroup. We say S is a *factor* of T if S is a homomorphic image of a subsemigroup of T. A direct product P of semigroups S_i $(i \in I)$ is written as $\prod_{i \in I} S_i$. If $I = \{1, \ldots, n\}$ then we may write $P = S_1 \times \cdots \times S_n$. The projective homomorphism from P onto S_i is denoted by π_i. If $a \in P$, then a_i denotes the i^{th} component of a, that is, $a_i = \pi_i a$.

2.2 Congruences and Lattice Homomorphisms

Let $\mathcal{L}$ and $\mathcal{L}'$ be lattices. A mapping $\alpha : \mathcal{L} \longrightarrow \mathcal{L}'$ is a (*lattice*) *homomorphism* if it preserves meets and joins, that is,

$$\alpha(x \wedge y) = \alpha x \wedge \alpha y, \qquad \alpha(x \vee y) = \alpha x \vee \alpha y$$

for all $x, y \in \mathcal{L}$. If α is also a bijection, then it is a (*lattice*) *isomorphism*. If $\mathcal{L}$ and $\mathcal{L}'$ are complete lattices and α preserves arbitrary meets and joins, then α is a *complete homomorphism*. It is straightforward to show that if α is a homomorphism, then it also preserves ordering in the sense that $x \leq y$ implies $\alpha x \leq \alpha y$. For any $x, y \in \mathcal{L}$, we say x *covers* y (and write $x \succ y$) if $x > y$ and there is no $z \in \mathcal{L}$ such that $x > z > y$.

Let ρ be an equivalence relation on S and let $a, b \in S$. We often consider relations on S as subsets of $S \times S$. The ρ-class of a will be denoted by $a\rho$. If a and b are ρ-related, then we write $a\rho b$ or $(a,b) \in \rho$. We say that ρ is a *congruence* if $a\rho b$ implies $ca\rho cb$ and $ac\rho bc$ for all $a, b, c \in S$. Equivalently, ρ is a congruence if and only if $a\rho b$ and $c\rho d$ imply $ac\rho bd$. The trivial congruence $\{(a,a) \mid a \in S\}$ is denoted by ε.

Let ρ be a congruence on S. It is easy to show that $(a\rho)(b\rho) = (ab)\rho$ for all $a, b \in S$ so that the set S/ρ of all ρ-classes forms a semigroup called the *quotient semigroup* induced by ρ. If T is a subsemigroup of S, then ρ induces the congruence $[\rho]_T = \rho \cap (T \times T)$ on T.

Let $\mathrm{Con}\,(S)$ denote the collection of all congruences on S. Note that $\mathrm{Con}\,(S)$ is closed under arbitrary intersection. Hence for each subset A of $S \times S$, there exists a least congruence ρ_A on S containing A. Consequently, $\mathrm{Con}\,(S)$ is a complete lattice under $\cap$ and $\vee$, where

$$\bigvee \{\rho_i \mid i \in I\} = \bigcap \{\rho \in \mathrm{Con}\,(S) \mid \rho_i \subseteq \rho \text{ for all } i \in I\}.$$

Define a relation $\mathcal{H}$ on S as follows: $(a,b) \in \mathcal{H}$ if $S^1 a = S^1 b$ and $aS^1 = bS^1$. It is easy to see that $\mathcal{H}$ is an equivalence relation. Furthermore, it is well-known that if H is a $\mathcal{H}$-class, then either $H^2 \cap H = \emptyset$ or H is a subgroup of S.

Lemma 2.2.1 ([14], Corollary I.7.10) *The maximal subgroups of a semigroup coincide with the $\mathcal{H}$-classes containing idempotents. Any two distinct maximal subgroups are disjoint and any $\mathcal{H}$-class can contain at most one idempotent.*

A semigroup is *completely regular* if all its $\mathcal{H}$-classes are subgroups.

Lemma 2.2.2 (Lallement [10]) *Let S, T be completely regular semigroups and $\varphi : S \longrightarrow T$ be a surjective homomorphism. If $f \in E\,(T)$, then there exists $e \in E\,(S)$ with $\varphi e = f$.*

Lallement's original result is more general than the preceding, but it is sufficient for our purposes. We refer the reader to [14] for more information on completely regular semigroups.

2.3 Rees Matrix Semigroups

A semigroup S with zero is 0-*simple* if $SaS = S$ for all nonzero $a \in S$. A 0-simple semigroup is *completely* 0-*simple* if it has a primitive idempotent. Let I, Λ be nonempty sets, G be a group and $P = [p_{\lambda i}]$ be a $\Lambda \times I$ matrix with entries from $G^0 = G \cup \{0\}$ such that it has no row or column consisting entirely of zeros. Define an operation on $(I \times G \times \Lambda) \cup \{0\}$ by

$$(i,a,\lambda)(j,b,\mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$
$$(i,a,\lambda)\,0 = 0\,(i,a,\lambda) = 00 = 0.$$

Then $(I \times G \times \Lambda) \cup \{0\}$ is a completely 0-simple semigroup called a *Rees matrix semigroup* (*over G^0 with sandwich matrix P*) and is denoted by $\mathcal{M}^0\,(I, G, \Lambda; P)$. The fundamental Rees

Theorem establishes that a semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup of the form $\mathcal{M}^0(I,G,\Lambda;P)$. The following Rees matrix semigroups of order five are required later in this thesis:

$$B_2=\mathcal{M}^0\left(\{1,2\},\{1\},\{1,2\};\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),$$
$$A_2=\mathcal{M}^0\left(\{1,2\},\{1\},\{1,2\};\begin{bmatrix}1&1\\1&0\end{bmatrix}\right).$$

A semigroup S without zero is *simple* if $SaS=S$ for all $a\in S$, and a simple semigroup is *completely simple* if it has a primitive idempotent. A Rees matix semigroup construction also exists for completely simple semigroups. Let I,Λ be nonempty sets and $P=[p_{\lambda i}]$ be a $\Lambda\times I$ matrix with entries from a group G. Then the set $I\times G\times\Lambda$ with operation defined by

$$(i,a,\lambda)(j,b,\mu)=(i,ap_{\lambda j}b,\mu)$$

is a completely simple semigroup. This semigroup is called a *Rees matrix semigroup* (*over* G *with sandwich matrix* P) and is denoted by $\mathcal{M}(I,G,\Lambda;P)$. A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup of the form $\mathcal{M}(I,G,\Lambda;P)$. Furthermore, for any Rees matrix semigroup of the form $\mathcal{M}(I,G,\Lambda;P)$, we may assume the matrix P to be *normalized* in the sense that entries in the first row and column are all equal to the identity 1 of G (see [14], Theorem III.2.6).

Note that groups are examples of completely simple semigroups. A *left zero band* is a semigroup that obeys the multiplication rule $ab=a$. *Right zero bands* are defined dually. A direct product of a left zero and a right zero band is a *rectangular band*, and a direct product of a rectangular band and a group is a *rectangular group*. Left zero, right zero and rectangular bands are the only bands that are completely simple. Denote by L (respectively, R) the left zero (respectively, right zero) band with exactly two elements.

Let $S=\mathcal{M}(I,G,\Lambda;P)$ be a Rees matrix semigroup. It is straightforward to show that $(i,a,\lambda)\,\mathcal{H}\,(j,b,\mu)$ if and only if $i=j$ and $\lambda=\mu$. Therefore S is partitioned into the $\mathcal{H}$-classes $H_{i\lambda}=\{i\}\times G\times\{\lambda\}$, each containing the idempotent $\left(i,p_{\lambda i}^{-1},\lambda\right)$. Consequently each completely simple semigroup is completely regular. But a completely 0-simple semigroup may contain a non-group $\mathcal{H}$-class and so may not be completely regular.

Let P be the matrix of a Rees matrix semigroup over G or G^0. Then $\langle P\rangle$ denotes the subgroup of G generated by the nonzero entries of P. The core of a completely simple Rees matrix semigroup then has a very simple form:

Lemma 2.3.1 ([14], Lemma III.2.10) *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Then $C(S) = \mathcal{M}(I, \langle P \rangle, \Lambda; P)$.*

Lemma 2.3.2 ([14], Theorem III.5.2) *Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Then the following statements are equivalent.*

(1) *S is orthodox;*

(2) *S is a rectangular group;*

(3) *All entries of P are* 1.

In view of Lemma 2.3.2, rectangular groups are the only orthodox completely simple semigroups. Other classes of completely simple semigroups will be introduced later when they are required.

2.4 Congruences on a Rees Matrix Semigroup

Let $S = \mathcal{M}(I, G, \Lambda; P)$ with P normalized. Let $\mathcal{E}(I)$ (respectively, $\mathcal{E}(\Lambda)$) denote the set of all equivalence relations on I (respectively, Λ), and let $\mathcal{N}(G)$ denote the set of all normal subgroups of G. A triple $(r, N, \pi) \in \mathcal{E}(I) \times \mathcal{N}(G) \times \mathcal{E}(\Lambda)$ is *admissible* for S if

$$(i, j) \in r \Longrightarrow p_{\lambda i} p_{\lambda j}^{-1} \in N \text{ for all } \lambda \in \Lambda,$$
$$(\lambda, \mu) \in \pi \Longrightarrow p_{\lambda i} p_{\mu i}^{-1} \in N \text{ for all } i \in I.$$

Theorem 2.4.1 ([14], Theorem III.4.6) *If (r, N, π) is an admissible triple for S, then the relation $\rho_{(r,N,\pi)}$ on S defined by*

$$(i, a, \lambda)\, \rho_{(r,N,\pi)}\, (j, b, \mu) \quad \text{if} \quad (i, j) \in r, \;\; ab^{-1} \in N, \;\; (\lambda, \mu) \in \pi$$

is a congruence on S. Conversely, if ρ is a congruence on S, then $\rho = \rho_{(r,N,\pi)}$ for a unique admissible triple (r, N, π) for S. Moreover,

$$S/\rho_{(r,N,\pi)} \cong \mathcal{M}(I/r, G/N, \Lambda/\pi; P/N)$$

where P/N is the $(\Lambda/\pi) \times (I/r)$ matrix with $(\lambda\pi, ir)$-entry equal to $p_{\lambda i} N$.

We write $\rho \longleftrightarrow (r, N, \pi)$ to mean that the congruence ρ is induced by the admissible triple (r, N, π), that is, $\rho = \rho_{(r,N,\pi)}$. For convenience, ρ_N denotes the congruence induced by the admissible triple $(\varepsilon, N, \varepsilon)$. Note that $\mathcal{H}$ is the congruence ρ_G on S. Recall that $\mathrm{Con}\,(S)$ is a complete lattice. If congruences on a completely simple semigroup are expressed as admissible triples, then joins and meets can be found as follows:

Lemma 2.4.2 ([14], Lemma III.4.10) *The set of admissible triples for a completely simple semigroup is a complete lattice under the operations*

$$\bigwedge\nolimits_{\alpha\in A} (r_\alpha, N_\alpha, \pi_\alpha) = \left(\bigcap\nolimits_{\alpha\in A} r_\alpha, \bigcap\nolimits_{\alpha\in A} N_\alpha, \bigcap\nolimits_{\alpha\in A} \pi_\alpha\right),$$
$$\bigvee\nolimits_{\alpha\in A} (r_\alpha, N_\alpha, \pi_\alpha) = \left(\bigvee\nolimits_{\alpha\in A} r_\alpha, \bigvee\nolimits_{\alpha\in A} N_\alpha, \bigvee\nolimits_{\alpha\in A} \pi_\alpha\right).$$

For more information on congruences on Rees matrix semigroups, see [14].

2.5 Semigroup Varieties

This section summarizes some important facts concerning varieties of semigroups. All the statements here generalize to algebras and we refer the reader to [2] for a general treatment. A *variety* is a class of semigroups that is closed under the formation of homomorphic images, subsemigroups, and direct products. A subclass of a variety $\mathbf{V}$ which is also a variety is a *subvariety* of $\mathbf{V}$. Let $\mathfrak{C}$ be a class of semigroups throughout this section. The variety *generated by* $\mathfrak{C}$ (written as $V(\mathfrak{C})$) is the least variety containing $\mathfrak{C}$ and is just the intersection of all varieties containing $\mathfrak{C}$. When $\mathfrak{C}$ is $\{S_1, \ldots, S_n\}$ or $\{S \mid S \text{ has property X}\}$, then we also write $V(S_1, \ldots, S_n)$ or $V(S \mid S \text{ has property X})$ for $V(\mathfrak{C})$ respectively. Furthermore, if it is clear from the context that $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$ are classes of semigroups and that $S_1, \ldots, S_n$ are semigroups, then we may write $V(\mathfrak{C}_1, \ldots, \mathfrak{C}_m, S_1, \ldots, S_n)$ for the variety generated by all semigroups in $\mathfrak{C}_1 \cup \cdots \cup \mathfrak{C}_m \cup \{S_1, \ldots, S_n\}$.

If $\mathbf{V} = V(\mathfrak{C})$ then the semigroups in $\mathfrak{C}$ are called *generators* of $\mathbf{V}$. For any variety $\mathbf{V}$, the lattice $\mathcal{L}(\mathbf{V})$ of subvarieties of $\mathbf{V}$ is a complete lattice with operations given by

$$\bigwedge\nolimits_{i\in I} \mathbf{V}_i = \bigcap\nolimits_{i\in I} \mathbf{V}_i,$$
$$\bigvee\nolimits_{i\in I} \mathbf{V}_i = \bigcap \{\mathbf{W} \in \mathcal{L}(\mathbf{V}) \mid \mathbf{V}_i \subseteq \mathbf{W} \text{ for all } i \in I\}.$$

Note that if $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are classes of semigroups, then

$$V(\mathfrak{C}_1) \vee V(\mathfrak{C}_2) = V(\mathfrak{C}_1 \cup \mathfrak{C}_2).$$

But in general, it is very difficult to find a generating set of semigroups for the intersection $V(\mathfrak{C}_1) \cap V(\mathfrak{C}_2)$.

Let $\mathbf{U}$ and $\mathbf{V}$ be varieties such that $\mathbf{U} \subseteq \mathbf{V}$. The *interval* $[\mathbf{U}, \mathbf{V}]$ is defined to be the set

$$\{\mathbf{W} \in \mathcal{L}(\mathbf{V}) \mid \mathbf{U} \subseteq \mathbf{W} \subseteq \mathbf{V}\}.$$

Note that $\mathcal{L}(\mathbf{V}) = [\mathbf{T}, \mathbf{V}]$ where $\mathbf{T}$ is the variety generated by the trivial semigroup. For convenience, let

$$\begin{aligned}[\mathbf{U}, \mathbf{V}) &= \{\mathbf{W} \in \mathcal{L}(\mathbf{V}) \mid \mathbf{U} \subseteq \mathbf{W} \subset \mathbf{V}\} \\ &= [\mathbf{U}, \mathbf{V}] \setminus \{\mathbf{V}\}.\end{aligned}$$

Let $\mathrm{H}(\mathfrak{C}), \mathrm{S}(\mathfrak{C})$ and $\mathrm{P}(\mathfrak{C})$ represent, respectively, the classes of all semigroups that may be obtained as homomorphic images, subsemigroups and direct products from semigroups in $\mathfrak{C}$. Then

Theorem 2.5.1 (Tarski) $V(\mathfrak{C}) = \mathrm{HSP}(\mathfrak{C})$.

In view of Theorem 2.5.1, if S is in $V(\mathfrak{C})$, then it is a factor of some product of semigroups $S_i \in \mathfrak{C}$ $(i \in I)$. Equivalently, there exist a subsemigroup T of $\prod_{i \in I} S_i$ and a homomorphism φ from T onto S:

$$S \xleftarrow{\varphi} T \leq \prod_{i \in I} S_i.$$

Let X be a countable alphabet whose elements we refer to as variables and let X^+ denote the set of all words formed from the alphabet X with concatenation as operation. Then X^+ is the *free semigroup* on X in the sense that for any semigroup S and any mapping $\varphi : X \longrightarrow S$, there exists a unique homomorphism $\overline{\varphi} : X^+ \longrightarrow S$ such that $\overline{\varphi}|_X = \varphi$, where $\overline{\varphi}|_X$ is the restriction of $\overline{\varphi}$ to X. If $\mathbf{u} = \mathbf{u}(x_1, \ldots, x_n)$ is a word in X^+ and S denotes the substitution $x_i \longrightarrow a_i$ into a semigroup A, then $\mathbf{u}(\mathrm{S})$ denotes the element $\mathbf{u}(a_1, \ldots, a_n)$ in A.

An *identity* is a pair $(\mathbf{u}, \mathbf{v})$ in $X^+ \times X^+$, usually written as $\mathbf{u} = \mathbf{v}$. A semigroup S *satisfies* an identity $\mathbf{u} = \mathbf{v}$ and we write $S \vDash \mathbf{u} = \mathbf{v}$ if $\varphi\mathbf{u} = \varphi\mathbf{v}$ for all homomorphism $\varphi : X^+ \longrightarrow S$. Equivalently, if $x_1, \ldots, x_n$ are the variables in $\mathbf{u}$ and $\mathbf{v}$, then $S \vDash \mathbf{u} = \mathbf{v}$ if and only if $\mathbf{u}(a_1, \ldots, a_n) = \mathbf{v}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in S$. For a set Σ of identities, we write $\mathfrak{C} \vDash \Sigma$ if $S \vDash \mathbf{u} = \mathbf{v}$ for all $S \in \mathfrak{C}$ and $\mathbf{u} = \mathbf{v} \in \Sigma$.

The set of all identities on X satisfied by all semigroups of a class $\mathfrak{C}$ is denoted by $\mathrm{Id}_{\mathfrak{C}}(X)$. It is not difficult to show that $\mathrm{Id}_{\mathfrak{C}}(X)$ is a *fully invariant congruence* on X^+, that is, a congruence on X^+ such that $(\mathbf{u}, \mathbf{v}) \in \mathrm{Id}_{\mathfrak{C}}(X)$ implies $(\varphi\mathbf{u}, \varphi\mathbf{v}) \in \mathrm{Id}_{\mathfrak{C}}(X)$ for all endomorphism φ of X^+. Hence for simplicity, we also write $\equiv_{\mathfrak{C}}$ for $\mathrm{Id}_{\mathfrak{C}}(X)$ and call it the *fully invariant congruence over* $\mathfrak{C}$. The mapping $\mathbf{V} \longmapsto \equiv_{\mathbf{V}}$ is an anti-isomorphism between the lattice of semigroup varieties and the lattice of fully invariant congruences ([14], Theorem I.8.14). The semigroup $F_X(\mathfrak{C}) = X^+/\mathrm{Id}_{\mathfrak{C}}(X)$ is called the *$\mathfrak{C}$-free semigroup on X*. This semigroup has the *universal mapping property for $\mathfrak{C}$ over X*, that is, for any $S \in \mathfrak{C}$ and function $\varphi : X \longrightarrow S$, there exists a unique homomorphism $\overline{\varphi} : F_X(\mathfrak{C}) \longrightarrow S$ such that $\overline{\varphi}|_X = \varphi$. Note that $X^+ = F_X(\mathbf{S})$ where $\mathbf{S}$ is the variety of all semigroups. If $|X| = n$ then we write $F_n(\mathfrak{C})$ instead of $F_X(\mathfrak{C})$. For any set X and variety $\mathbf{V}$, the semigroup $F_X(\mathbf{V})$ is subdirectly embeddable in a direct product of semigroups in $\mathbf{V}$. Therefore $\langle F_X(\mathbf{V})\rangle = \mathbf{V}$.

Lemma 2.5.2 *Let S be an m-generated semigroup and let $S_1, \ldots, S_n$ be finite semigroups such that $S \in \mathbf{V} = V(S_1, \ldots, S_n)$. Then S divides a finite direct product of semigroups from $\{S_1, \ldots, S_n\}$ and is finite. In particular, $F_m(\mathbf{V})$ is finite.*

PROOF. Let $P = S_1 \times \cdots \times S_n$ and $|P| = k$ (whence $\mathbf{V} = V(P)$). Then S is a homomorphic image of $F_m(\mathbf{V})$, and $F_m(\mathbf{V})$ can be embedded in a product of k^m copies of P since that is the number of possible mappings of the m generators into P. Therefore S divides a direct product of nk^m semigroups from $\{S_1, \ldots, S_n\}$. ■

A semigroup (respectively, group) is *locally finite* if each of its finitely generated subsemigroups (respectively, subgroups) is finite, and a variety is *locally finite* if all its members are locally finite. In view of Lemma 2.5.2, we have

Corollary 2.5.3 *The variety generated by a finite semigroup or a finite group is locally finite.*

Given a set Σ of identities, let $[\Sigma]$ denote the class of all semigroups that satisfy all identities in Σ. A class of semigroups of the form $[\Sigma]$ for some set Σ of identities is said to be an *equational class.* The following is a very special case of a Theorem by Birkhoff.

Theorem 2.5.4 (Birkhoff) *A class of semigroups is equational if and only if it is a variety.*

Let $\mathbf{V} = [\Sigma]$. Then we say $\mathbf{V}$ is *defined by* Σ (or Σ *defines* $\mathbf{V}$), and Σ is called a *basis* of $\mathbf{V}$. If Σ is finite then $\mathbf{V}$ is said to be *finitely based.* A semigroup is also said to be *finitely based* if the variety it generates is finitely based. A semigroup S (respectively, variety $\mathbf{V}$) is *non-finitely based* if $V(S)$ (respectively, $\mathbf{V}$) cannot be defined by any finite set of identities. For any sets Σ, Π of identities, it is clear that

$$[\Sigma] \cap [\Pi] = [\Sigma \cup \Pi].$$

But it is very difficult in general to find a set of identities that defines $[\Sigma] \vee [\Pi]$.

Let $\Sigma \cup \{\mathbf{u} = \mathbf{v}\}$ be a set of identities on X. We say $\mathbf{u} = \mathbf{v}$ is a *consequence* of Σ (or Σ *implies* $\mathbf{u} = \mathbf{v}$) and write $\Sigma \vdash \mathbf{u} = \mathbf{v}$ if $\mathbf{u} = \mathbf{v}$ can be deduced from identities in Σ. If Π is a set of identities on X, then we write $\Sigma \vdash \Pi$ if $\Sigma \vdash \mathbf{u} = \mathbf{v}$ for all $\mathbf{u} = \mathbf{v} \in \Pi$. It is easy to show that $S \models \Sigma \vdash \mathbf{u} = \mathbf{v}$ implies $S \models \mathbf{u} = \mathbf{v}$.

A *pseudovariety* is a class of finite semigroups that is closed under the formation of homomorphic images, subsemigroups, and finite direct products. The pseudovariety *generated by* $\mathfrak{C}$ is the least pseudovariety containing $\mathfrak{C}$, namely the intersection of all pseudovarieties containing $\mathfrak{C}$. The pseudovariety generated by $\mathfrak{C}$ is denoted by $P_S(\mathfrak{C})$. Further information regarding varieties and pseudovarieties can be found in [1].

The following list contains notation for specific varieties that will be required in later chapters. In some cases, well known characterizations by bases of identities and semigroup generators are provided (see [1] or [14]). Since we are concerned only with semigroups in this thesis, it is more convenient to assume the identity $(xy)z = x(yz)$ without including it in the basis of each variety.

$$\begin{aligned}
\mathbf{T} &= [x = y] : \text{trivial semigroups} \\
\mathbf{L} &= [xy = x] = V(L) : \text{left zero bands} \\
\mathbf{R} &= [xy = y] = V(R) : \text{right zero bands} \\
\mathbf{RB} &= \left[x^2 = x, xyz = xz\right] = V(L, R) : \text{rectangular bands} \\
\mathbf{Y} &= \left[x^2 = x, xy = yx\right] = V(Y) : \text{semilattices} \\
\mathbf{B_2} &= V(B_2) \\
\mathbf{A_2} &= V(A_2)
\end{aligned}$$

The variety of groups (respectively, abelian groups) is denoted by $\mathbf{G}$ (respectively, $\mathbf{A}$).

2.6 Rees-Sushkevich Varieties

Let $\mathbf{CS}$ denote the class of all completely simple semigroups, and $\mathbf{CS^0}$ the class of all completely 0-simple semigroups. For each $n \in \mathbb{N}$ and subclass $\mathbf{V}$ of completely (0-)simple semigroups, let $\mathbf{V}_n$ denote the class of all semigroups from $\mathbf{V}$ with subgroups of exponent dividing n. It is straightforward to show that $\mathbf{CS}$ and $\mathbf{CS}_n$ are varieties contained in $\mathbf{CS^0}$ and $\mathbf{CS}^0_n$ respectively. But $\mathbf{CS^0}$ and $\mathbf{CS}^0_n$ are not varieties since the direct product of two completely 0-simple semigroups is not completely 0-simple in general.

Let $\mathbf{RS}_n$ be the variety generated by $\mathbf{CS}^0_n$. Any variety in $\mathcal{L}(\mathbf{RS}_n)$ is called a *Rees-Sushkevich variety.* A Rees-Sushkevich variety is *exact* if it is generated by completely 0-simple semigroups. If $\mathbf{V}$ is exact, then $\mathbf{V}$ is clearly generated by a single semigroup in $\mathbf{CS}^0_n$.

Proposition 2.6.1 (Kublanovsky [9], Corollary 2) *The exact subvarieties of* $\mathcal{L}(\mathbf{RS}_n)$ *constitute a sublattice.*

In the light of Proposition 2.6.1, let $\mathcal{LE}(\mathbf{RS}_n)$ denote the sublattice of $\mathcal{L}(\mathbf{RS}_n)$ consisting of the exact subvarieties of $\mathbf{RS}_n$. For any exact Rees-Sushkevich variety $\mathbf{V}$, let $\mathcal{LE}(\mathbf{V})$ denote the sublattice of $\mathcal{L}(\mathbf{V})$ consisting of the exact subvarieties of $\mathbf{V}$. Examples of exact varieties include $\mathbf{RS}_n$ and $\mathbf{B_2}$.

Proposition 2.6.2 (Hall et al., [5]) *The identities*

$$x^{n+2} = x^2,$$
$$(xy)^{n+1} x = xyx,$$
$$(xyz)^n xwz = xwz (xyz)^n$$

constitute a basis for $\mathbf{RS}_n$.

Proposition 2.6.3 (Trakhtman, [18]) *The identities*

$$x^3 = x^2,$$
$$(xy)^2 x = xyx,$$
$$x^2y^2 = y^2x^2$$

constitute a basis for $\mathbf{B_2}$.

Proposition 2.6.4 ([17], Corollary 5.10) $\mathbf{A_2} = \mathbf{RS}_1$.

Let $\mathbf{CS}_\mathbf{f}^\mathbf{0}$ denote the class of all finite completely 0-simple semigroups. Then a pseudovariety in $\mathcal{L}\left(P_S\left(\mathbf{CS}_\mathbf{f}^\mathbf{0}\right)\right)$ is called a *Rees-Sushkevich pseudovariety.* A Rees-Sushkevich pseudovariety is *exact* if it is generated by completely 0-simple semigroups.

Let $\mathrm{FB}\,(n)$ (respectively, $\mathrm{NFB}\,(n)$) denote the number of finitely based (respectively, non-finitely based) semigroups with n elements. It has been shown that

$$\lim_{n\to\infty} \left(\mathrm{NFB}\,(n)\,/\mathrm{FB}\,(n)\right) = 0$$

(see [20]). Therefore a randomly chosen semigroup is more likely to be finitely based than non-finitely based. But Volkov [21] provided a recipe for constructing non-finitely based completely 0-simple semigroups.

Proposition 2.6.5 ([21], Proposition 7) *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be a Rees matrix semigroup with matrix $P = [p_{\lambda i}]$. If G has finite exponent and does not belong to $V(\langle P\rangle)$, and if there exist $j, k \in I$, $\sigma, \tau \in \Lambda$ such that $p_{\sigma j}, p_{\sigma k}, p_{\tau k} \neq 0$ but $p_{\tau j} = 0$, then S is non-finitely based.*

Therefore the varieties generated by semigroups described in Proposition 2.6.5 are all exact and non-finitely based. A simple example of such a semigroup is

$$\mathcal{M}^0\left(\{1,2\}, G, \{1,2\}; \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}\right)$$

where G is the cyclic group of order two with identity e; this semigroup has nine elements.

A semigroup S is *periodic* if there exists $n \geq 1$ such that a^n is an idempotent of S for all $a \in S$. Note that $x^3 = x^2$ implies $\left(x^2\right)^2 = x^2$, and that $x^{n+2} = x^2$ implies

$$\left(x^n\right)^2 = x^{n-2}x^{n+2} = x^{n-2}x^2 = x^n$$

for $n \geq 2$. Therefore in view of Proposition 2.6.2, all semigroups in $\mathbf{RS}_n$ are periodic. The semigroups given by the following presentations play an important role in determining the exactness of a variety:

$$N_1 = \left\langle a \mid a^2 = 0 \right\rangle;$$
$$A_0 = \left\langle \mathfrak{e}, \mathfrak{f} \mid \mathfrak{e}^2 = \mathfrak{e}, \mathfrak{f}^2 = \mathfrak{f}, \mathfrak{f}\mathfrak{e} = 0 \right\rangle.$$

Let $\mathbf{N}_1$ and $\mathbf{A_0}$ be the varieties generated by N_1 and A_0 respectively.

Theorem 2.6.6 ([9], Theorem 2) *A variety* $\mathbf{V}$ *of periodic semigroups is exact if and only if* $\mathbf{V}$ *is a Rees-Sushkevich variety and one of the following conditions holds*:

(1) $N_1 \notin \mathbf{V}$;

(2) $B_2 \in \mathbf{V}$ *and* $A_0 \notin \mathbf{V}$;

(3) $A_2 \in \mathbf{V}$.

A semigroup S is *uniformly periodic* if it satisfies an identity $x^{n+k} = x^n$ for some $n, k \geq 1$; if $k = 1$ then S is said to be *aperiodic*. Reilly [17] proved that there are precisely 13 aperiodic exact varieties, and these varieties form a lattice as shown in Figure 2.1.

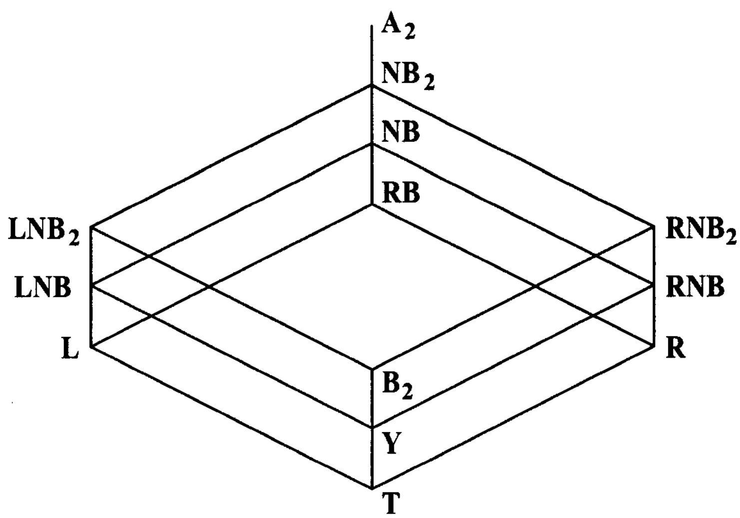

Figure 2.1: The lattice of aperiodic exact varieties.

Generators of previously undefined varieties in Figure 2.1 are given below. For their bases of identities, see [17].

$$\mathbf{LNB} = V(L, Y) = \mathbf{L} \vee \mathbf{Y};$$
$$\mathbf{RNB} = V(R, Y) = \mathbf{R} \vee \mathbf{Y};$$
$$\mathbf{NB} = V(L, R, Y) = \mathbf{RB} \vee \mathbf{Y};$$
$$\mathbf{LNB_2} = V(L, B_2) = \mathbf{L} \vee \mathbf{B_2};$$
$$\mathbf{RNB_2} = V(R, B_2) = \mathbf{R} \vee \mathbf{B_2};$$
$$\mathbf{NB_2} = V(L, R, B_2) = \mathbf{RB} \vee \mathbf{B_2}.$$

In contrast to the 13 aperiodic exact varieties, Reilly [17] showed that the aperiodic non-exact varieties are precisely contained in the two intervals

$$[\mathbf{N}_1, \mathbf{ANCB_2}], \qquad [\mathbf{A_0} \vee \mathbf{B_2}, \mathbf{ANCA_2}]$$

where $\mathbf{ANCB_2}$ (respectively, $\mathbf{ANCA_2}$) is the largest aperiodic non-exact variety not containing $\mathbf{B_2}$ (respectively, $\mathbf{A_2}$). Furthermore, both intervals are infinite. In Chapter 4, we present a complete description of the lattice $\mathcal{L}(\mathbf{A_0})$.

2.7 Permutation Varieties

For $p, q \in \mathbb{N}$ such that $p \leq q$, let $\mathbb{I}_p^q = \{p, \ldots, q\}$. Let S_k denote the group of permutations on $\mathbb{I}_1^k$. It is well known that each permutation can be written as a product of disjoint cycles. Let $(a_1 \cdots a_m)$ be an m-cycle with $a_i \in \mathbb{I}_1^k$. Note that $(a_1 \cdots a_m)$ can be interpreted as an element in S_n for any $n \geq k$. To avoid this ambiguity, we write $(a_1 \cdots a_m)_n$ if we want to specify that the permutation $(a_1 \cdots a_m)$ is an element of S_n. For example, the cycles $(13)_3, (13)_4$ denote the permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

respectively. When necessary, commas will be used to seperate the entries in a cycle for clarity. For example, we write $(n-1, n, n+1)$ instead of $(n-1 \ \ n \ \ n+1)$.

A permutation $\pi \in S_k$ corresponds uniquely to the identity

$$x_1 \cdots x_k = x_{\pi(1)} \cdots x_{\pi(k)}. \tag{2.1}$$

Let $\mathrm{Id}\pi$ denote the identity in (2.1) and call it the *permutation identity* (*associated with* π). Whenever we write $\mathrm{Id}\pi$, it is understood that π is a permutation in S_k for some k, and that $\mathrm{Id}\pi$ is the permutation identity associated with π. A variety defined only by permutation identities is called a *permutation variety.* For any set Π of permutations, let $\mathrm{Id}\Pi = \{\mathrm{Id}\pi \mid \pi \in \Pi\}$.

For each $\pi \in S_k$ and $l, r \geq 0$, define the permutation $(l : \pi : r) \in S_{l+k+r}$ by

$$(l : \pi : r)(i) = \begin{cases} l + \pi(i - l) & \text{if } l + 1 \leq i \leq l + k, \\ i & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Id}(l:\pi:r)$ denotes the identity

$$\begin{aligned} & x_1 \cdots x_l x_{l+1} \cdots x_{l+k} x_{l+k+1} \cdots x_{l+k+r} \\ & = x_1 \cdots x_l x_{l+\pi(1)} \cdots x_{l+\pi(k)} x_{l+k+1} \cdots x_{l+k+r}. \end{aligned}$$

For example,

$$\begin{aligned} \mathrm{Id}(0:(123)_3:0) &: x_1x_2x_3 = x_2x_3x_1, \\ \mathrm{Id}(2:(12)_2:1) &: x_1x_2x_3x_4x_5 = x_1x_2x_4x_3x_5, \\ \mathrm{Id}(2:(12)_3:1) &: x_1x_2x_3x_4x_5x_6 = x_1x_2x_4x_3x_5x_6. \end{aligned}$$

Whenever we write $(l:\pi:r)$, it is understood that π is a permutation and $l, r \geq 0$. Note that if $\pi(1) \neq 1$ and $\pi(k) \neq k$ then $(l:\pi:r)$ is a permutation that fixes exactly the first l and the last r variables.

We write $\mathrm{Id}(l_1:\alpha:r_1) = \mathrm{Id}(l_2:\beta:r_2)$ when $\mathrm{Id}(l_1:\alpha:r_1)$ and $\mathrm{Id}(l_2:\beta:r_2)$ are, letter for letter, the same identities. For example, $\mathrm{Id}(2:(23)_4:1) = \mathrm{Id}(3:(12)_2:2)$. If Π is a subset of permutations, then let

$$\begin{aligned} (l:\Pi:r) &= \{(l:\pi:r) \mid \pi \in \Pi\}, \\ \mathrm{Id}(l:\Pi:r) &= \{\mathrm{Id}(l:\pi:r) \mid \pi \in \Pi\}. \end{aligned}$$

The following are summarized results of Pollák [15].

Proposition 2.7.1 ([19], Corollary 2.3) *Let Π be a subgroup of S_n that fixes neither 1 nor n. Then $\mathrm{Id}\Pi \vdash \mathrm{Id}S_{2n}$.*

Corollary 2.7.2 ([19], Corollary 2.4) *Each permutation variety is finitely based.*

In this section, we present a bound on the number of variables required to define an arbitrary permutation variety.

Lemma 2.7.3 *Let $\alpha, \beta \in S_k$. Then $\{\mathrm{Id}\alpha, \mathrm{Id}\beta\} \vdash \mathrm{Id}(\alpha \circ \beta)$.*

PROOF. Substituting $x_i \longrightarrow x_{\alpha(i)}$ for the variables in $\mathrm{Id}\beta$, we obtain

$$x_{\alpha(1)} \cdots x_{\alpha(k)} = x_{\alpha(\beta(1))} \cdots x_{\alpha(\beta(k))}.$$

Hence

$$\begin{aligned} \{\mathrm{Id}\alpha, \mathrm{Id}\beta\} &\vdash \left\{ \begin{array}{l} x_1 \cdots x_k = x_{\alpha(1)} \cdots x_{\alpha(k)}, \\ x_{\alpha(1)} \cdots x_{\alpha(k)} = x_{\alpha(\beta(1))} \cdots x_{\alpha(\beta(k))} \end{array} \right\} \\ &\vdash \left\{ x_1 \cdots x_k = x_{\alpha(\beta(1))} \cdots x_{\alpha(\beta(k))} \right\} \\ &= \{\mathrm{Id}\,(\alpha \circ \beta)\}. \end{aligned}$$

■

Lemma 2.7.4 *Let $\alpha \in S_m$, $\beta \in S_n$, and $l, r, l_0, r_0 \geq 0$ be such that*

(1) $l_0 \leq l$;

(2) $r_0 \leq r$;

(3) $\alpha(1) \neq 1$ *and* $\alpha(m) \neq m$;

(4) $\beta(1) \neq 1$ *and* $\beta(n) \neq n$.

Then

$$\{\mathrm{Id}\,(l_0 : \alpha : r), \mathrm{Id}\,(l : \beta : r_0)\} \vdash \mathrm{Id}\,(l_0 : \gamma : r_0)$$

for some $\gamma \in S_t$ with $t = m + n + l + r$ such that $\gamma(1) \neq 1$ and $\gamma(t) \neq t$.

PROOF. Let $p = l_0 + m + r$ and $q = l + n + r_0$. It is easy to show that

$$\{\mathrm{Id}\,(l_0 : \alpha : r), \mathrm{Id}\,(l : \beta : r_0)\} \vdash \{\mathrm{Id}\,(l_0 : \alpha : r + q), \mathrm{Id}\,(l + p : \beta : r_0)\}$$

where $(l_0 : \alpha : r + q)$ and $(l + p : \beta : r_0)$ are both in S_{p+q}. By Lemma 2.7.3, the two identities associated with these permutations imply

$$\mathrm{Id}\,[(l_0 : \alpha : r + q) \circ (l + p : \beta : r_0)],$$

and it is straightforward to show that $(l_0 : \alpha : r + q) \circ (l + p : \beta : r_0) = (l_0 : \gamma : r_0)$ where $\gamma \in S_t$ with $t = m + n + l + r$, and that $\gamma(1) \neq 1$ and $\gamma(t) \neq t$. ■

Now let Π be a set of permutations. Without loss of generality express each permutation in Π in the form $(l : \pi : r)$ where π does not fix the first and last symbol in its domain. Then there exist $(l_0 : \alpha : r), (l : \beta : r_0) \in \Pi$ with least possible l_0 and r_0, that is,

$$(l : \pi : r) \in \Pi \implies l_0 \leq l \text{ and } r_0 \leq r.$$

Assume $\alpha \in S_m$ and $\beta \in S_n$. By Lemma 2.7.4, there exists $\gamma \in S_t$ $(t = m + n + l + r)$ with $\gamma(1) \neq 1$ and $\gamma(t) \neq t$ such that $\mathrm{Id}\,(l_0 : \gamma : r_0)$ is a consequence of $\mathrm{Id}\Pi$. Hence by Proposition 2.7.1, $\mathrm{Id}\Pi$ implies $\mathrm{Id}\,(l_0 : S_{2t} : r_0)$. But for any $(l : \pi : r) \in \Pi$,

$$\mathrm{Id}\,(l : \pi : r) = \mathrm{Id}\,(l_0 : (l - l_0 : \pi : r - r_0) : r_0)$$

so that all permutation consequences of $\mathrm{Id}\Pi$ can be expressed in the form $\mathrm{Id}\,(l_0 : \pi : r_0)$. Thus

$$[\mathrm{Id}\Pi] = [\mathrm{Id}\,(l_0 : \pi : r_0) \in \mathrm{Id}\Pi \mid \pi \in S_2 \cup \cdots \cup S_{2t}]\,,$$

and the variety $[\mathrm{Id}\Pi]$ has a basis that requires at most $l_0 + 2t + r_0$ variables.

Chapter 3

Intersections of Exact Pseudovarieties

Kublanovsky's Intersection Problem concerns the computation of a finite completely (0-)simple semigroup that will generate the intersection of two given singly generated exact pseudovarieties. This chapter contains a solution to this problem within the class $\mathbf{CCS} \vee \mathbf{NB_2}$. Section 1 generalizes the concept of critical groups to completely simple semigroups. This generalization enables the characterization of critical central completely simple semigroups in Section 2 (Theorem 3.2.5). Section 3 solves Kublanovsky's Intersection Problem to within the class $\mathbf{G}$ and Section 4 extends this solution to the class $\mathbf{CCS}$ (Corollary 3.4.4). This solution is then further extended in Section 5 to a solution in the class $\mathbf{CCS} \vee \mathbf{NB_2}$. Section 6 presents several examples of critical semigroups, including an infinite class of critical monoids (Proposition 3.6.9).

3.1 Critical Semigroups

Let S be a finite nontrivial semigroup. Recall that a quotient (or homomorphic image) of a subsemigroup of S is called a *factor* of S. A factor of S is *proper* if it is not isomorphic to S. If S is not contained in the variety generated by its proper factors then it is *critical*. Critical semigroups play an important role in the varieties that are generated by finite semigroups. This section contains generalizations of some results of critical groups in [12] to completely simple semigroups.

Lemma 3.1.1 *A variety generated by a finite semigroup is generated by its critical semigroups.*

PROOF. Let A be a finite semigroup and $\mathbf{V} = V(S \in V(A) \mid S \text{ is critical})$. Suppose $\mathbf{V} \neq V(A)$. Since $A \notin \mathbf{V}$ there exists a semigroup $T \in V(A) \setminus \mathbf{V}$ of minimal order. Then T is not critical so that it is contained in the variety generated by its proper factors. But by the minimality of $|T|$, all proper factors of T are in $\mathbf{V}$, so that T is contradictorily in $\mathbf{V}$. ■

Lemma 3.1.2 *A finite nontrivial subdirectly irreducible rectangular band is either L or R. Consequently a finite rectangular band is critical if and only if it is subdirectly irreducible.*

PROOF. Let S be a nontrivial rectangular band. Then $S \cong L' \times R'$ for some left zero band L' and right zero band R'. In order for S to be subdirectly irreducible one of L', R' must be trivial. Without loss of generality assume R' is trivial so that $S \cong L'$. For each pair of distinct elements $x, y \in L'$, let $\pi_{x,y}$ be the equivalence relation on L' that only identifies x and y:

$$\pi_{x,y} = (\{x,y\} \times \{x,y\}) \cup \{(z,z) \mid z \in L'\}.$$

It is straightforward to show that every equivalence relation on L' is a congruence, whence $\pi_{x,y}$ is also a congruence. If L' contains three distinct elements a, b and c, then $\pi_{a,b} \cap \pi_{b,c} = \varepsilon$ so that L' has no minimal nontrivial congruence and is subdirectly reducible. Therefore L' has exactly two elements. ■

Unless otherwise stated, let $S = \mathcal{M}(I, G, \Lambda; P)$ be a finite nontrivial completely simple semigroup with normalized m by n matrix P throughout this section.

Lemma 3.1.2 characterized finite completely simple bands that are critical. Therefore we will assume that S is not a band, whence $|G| > 1$. For convenience, if ρ is the minimum nontrivial congruence on S, then it is called the *monolith congruence* on S. Similarly, if M is the minimum nontrivial normal subgroup of G then it is called the *monolith subgroup* of G. For example, the subgroup $\mathbb{Z}_p$ of a cyclic p-group is monolithic, and clearly each simple group is a monolith subgroup of itself. A proof of the following well-known result can be found in [14].

Lemma 3.1.3 ([14], Lemma I.5.8) *A semigroup is subdirectly irreducible if and only if it has a monolith congruence. In particular, a group is subdirectly irreducible if and only if it has a monolith subgroup.*

A subdirectly reducible semigroup is embeddable in a direct product of its homomorphic images. Therefore a critical semigroup is subdirectly irreducible and has a monolith congruence. The following lemma describes how S being subdirectly irreducible affects its monolith congruence and the group G.

Lemma 3.1.4 *If S is subdirectly irreducible, then*

(1) *G has a monolith subgroup M;*

(2) *ρ_M is the monolith congruence on S.*

Proof. (1) If A and B are nontrivial normal subgroups of G such that $A \cap B = \{1\}$, then S can be embedded into $S/\rho_A \times S/\rho_B$ because $\rho_A \cap \rho_B = \varepsilon$. So for S to be subdirectly irreducible, G must contain a monolith subgroup M.

(2) Let σ be a nontrivial congruence on S with $\sigma \longleftrightarrow (r, N, \pi)$. If $N = \{1\}$, then S contradictorily has no monolith congruence since

$$\begin{aligned}\sigma \cap \rho_M &\longleftrightarrow (r, \{1\}, \pi) \cap (\varepsilon, M, \varepsilon) \\ &= (\varepsilon, \{1\}, \varepsilon) \longleftrightarrow \varepsilon.\end{aligned}$$

If $N \neq \{1\}$ then $M \subseteq N$ since M is monolithic. Therefore

$$\rho_M \longleftrightarrow (\varepsilon, M, \varepsilon) \subseteq (r, N, \pi) \longleftrightarrow \sigma,$$

whence $\rho_M \subseteq \sigma$. Since σ is arbitrary and nontrivial, ρ_M is monolithic. ■

In order that S be critical, it is clearly necessary that S be subdirectly irreducible. But that is by no means sufficient. Examples of subdirectly irreducible semigroups that are non-critical will be given in Section 6. Furthermore it will be shown that there is no direct relationship between the criticality of S and G.

An m by n matrix P is *diverse* if any one of the following conditions holds.

(D1) $m = n = 1$;

(D2) $m, n \geq 2$, no two rows of P are identical, and no two columns of P are identical.

Lemma 3.1.5 *Let M be a normal subgroup of G. Then:*

(1) *The congruence ρ_M on S is monolithic if and only if M is monolithic and P is diverse.*

(2) *S is subdirectly irreducible if and only if G is subdirectly irreducible and P is diverse.*

(3) *If S is critical and not a group then it is not orthodox.*

PROOF. (1) Suppose that ρ_M is the monolith congruence on S, whence M is nontrivial. Then clearly M is monolithic. If P is not diverse, then one of the following holds:

(a) $1 = m < n$;

(b) $1 = n < m$;

(c) P has two identical rows;

(d) P has two identical columns.

Suppose (a). Then P is a row matrix. But since P is normalized, all entries of P are 1. The congruence induced by the admissible triple $(\varepsilon, \{1\}, \Lambda \times \Lambda)$ is nontrivial but intersects ρ_M trivially, contradicting ρ_M being monolithic. Therefore (a) is impossible. By symmetry, (b) is also impossible.

Now suppose (c). Let the λ^{th} and μ^{th} rows of P be identical, and let $\pi_{\lambda,\mu}$ be the equivalence relation on Λ which identifies λ and μ. Then the congruence induced by the admissible triple $(\varepsilon, \{1\}, \pi_{\lambda,\mu})$ is nontrivial but intersects ρ_M trivially, contradicting ρ_M being monolithic. Therefore (c) is impossible. By symmetry, (d) is also impossible.

Conversely, assume that M is monolithic and P is diverse. Let σ be a congruence on S with $\sigma \longleftrightarrow (r, N, \pi)$. If $N \neq \{1\}$, then clearly $\rho_M \subseteq \sigma$. Therefore it suffices to assume that $N = \{1\}$. By the definition of an admissible triple,

$$
\begin{aligned}
(i,j) \in r &\Longrightarrow p_{\lambda i} p_{\lambda j}^{-1} = 1 \text{ for all } \lambda \in \Lambda \\
&\Longrightarrow p_{\lambda i} = p_{\lambda j} \text{ for all } \lambda \in \Lambda \\
&\Longrightarrow i^{\text{th}} \text{ and } j^{\text{th}} \text{ columns of } P \text{ are identical.}
\end{aligned}
$$

Therefore $r = \varepsilon$ since P is diverse; by a similar argument, $\pi = \varepsilon$. Consequently, σ is trivial.

(2) By Lemmas 3.1.3, 3.1.4 and (1),

$$
\begin{aligned}
S \text{ is subdirectly irreducible} &\Longleftrightarrow S \text{ has a monolith congruence } \rho_M \\
&\Longleftrightarrow G \text{ has a monolith subgroup and } P \text{ is diverse} \\
&\Longleftrightarrow G \text{ is subdirectly irreducible and } P \text{ is diverse.}
\end{aligned}
$$

(3) Suppose $S = \mathcal{M}(I, G, \Lambda; P)$ is critical and not a group. Then either $m \geq 2$ or $n \geq 2$. But if S is orthodox, then by Theorem 2.3.2, all entries of P are 1 so that $S \cong I \times G \times \Lambda$ where G and either I or Λ are nontrivial. Therefore S is contradictorily non-critical. ■

Let $\mathfrak{D}$ be a finite set of finite completely simple semigroups which is *factor closed* in the sense that each factor of a semigroup in $\mathfrak{D}$ is isomorphic to a semigroup in $\mathfrak{D}$. If S is in $V(\mathfrak{D})$ then by Theorem 2.5.1 and Lemma 2.5.2, it can be represented as a factor of a finite direct product of semigroups $D_1, \ldots, D_n$ in $\mathfrak{D}$. In general the choice of the semigroups $D_1, \ldots, D_n$ is not unique. But every such choice determines a non-increasing sequence of integers consisting of their orders. Ordered lexicographically – one sequence is smaller than another if its entry in the first place where they differ is the smaller of the two – the set of these sequences has a minimum. The representations of S corresponding to this minimum sequence are the *minimal representations.*

All subsequent statements in this section refer to a fixed minimal representation of the finite semigroup $S = \mathcal{M}(I, G, \Lambda; P)$ (with P normalized) in $V(\mathfrak{D})$, that is,

$$S \cong T/\kappa \text{ such that } T \leq \prod_{i=1}^{n} D_i \text{ and } \kappa \in \mathrm{Con}(T)$$

where $(|D_1|, \ldots, |D_n|)$ is minimal. Since each D_i is finite completely simple, T is also finite completely simple, whence we can adopt the following notation:

$$\begin{aligned} T &= \mathcal{M}(*, H, *; *), \\ D_i &= \mathcal{M}(*, G_i, *; *), \\ \kappa &\longleftrightarrow (r, K, \pi) \text{ where } K \trianglelefteq H \leq \prod_{i=1}^{n} G_i. \end{aligned}$$

A subgroup U_j of G_j is isomorphic to a subgroup $\overline{U_j}$ of $\prod_{i=1}^{n} G_i$:

$$\overline{U_j} = \left\{ g \in \prod_{i=1}^{n} G_i \middle| g_j \in U_j \text{ and } (i \neq j \Rightarrow g_i = 1) \right\}.$$

Similarly, a congruence ρ_j on D_j corresponds to a congruence $\overline{\rho_j}$ on $\prod_{i=1}^{n} D_i$:

$$\overline{\rho_j} = \left\{ (x, y) \in \prod_{i=1}^{n} D_i \times \prod_{i=1}^{n} D_i \middle| x_j \rho_j y_j \text{ and } (i \neq j \Rightarrow x_i = y_i) \right\}.$$

Lemma 3.1.6 *Each semigroup D_i is critical, and if any D_i is replaced by one of its proper factors, then the resulting direct product has no factor isomorphic to S.*

PROOF. If D_i is not critical then it is a factor of $\prod_j D_{i,j}$ where each $D_{i,j}$ is a proper factor of D_i. Hence each $D_{i,j}$ belongs to $\mathfrak{D}$ and has order smaller than D_i. But then S is a factor of $\prod_{k\neq i} D_k \times \prod_j D_{i,j}$ and this representation is contradictorily smaller than the original minimal representation. If D_i is replaced by one of its proper factors then the sequence of integers is replaced by a smaller one which again by minimality cannot correspond to a representation of S. ■

Lemma 3.1.7 *The semigroup T is a subdirect product of $\prod_{i=1}^{n} D_i$. Consequently H is a subdirect product of $\prod_{i=1}^{n} G_i$.*

PROOF. If $\pi_i T$ were a proper subsemigroup of D_i then $\pi_i T$ could have been chosen in place of D_i to represent S in $V(\mathfrak{D})$, contradicting the minimality assumption. ■

Lemma 3.1.8 *A subgroup of $\overline{G_i}$ is normal in $\overline{G_i}$ if and only if it is normalized by H. If $\rho_i \in \mathrm{Con}(D_i) \setminus \{\varepsilon\}$ with $\rho_i \longleftrightarrow (*, N_i, *)$, then $\overline{N_i} \cap H \neq \{1\}$.*

PROOF. The first part is immediate from Lemma 3.1.7. Suppose $\overline{N_i} \cap H = \{1\}$. Let $\sigma \in \mathrm{Con}(D_i)$ with $\sigma \longleftrightarrow (\varepsilon, N_i, \varepsilon)$ and define a mapping $\Psi : T \longrightarrow \prod_{j\neq i} D_j \times D_i/\sigma$ by

$$(\Psi x)_j = \begin{cases} x_j & \text{if } j \neq i, \\ x_i\sigma & \text{otherwise.} \end{cases}$$

Clearly Ψ is a homomorphism. Let $x = (s, a, \alpha)$ and $y = (t, b, \beta)$ be in T. Since

$$\begin{aligned} \Psi x = \Psi y &\Longrightarrow \begin{cases} x_j = y_j & \text{if } j \neq i \\ (x_i, y_i) \in \sigma & \text{otherwise} \end{cases} \\ &\Longrightarrow s = t, \alpha = \beta \text{ and } ab^{-1} \in \overline{N_i} \cap H = \{1\} \\ &\Longrightarrow x = y, \end{aligned}$$

Ψ is injective. But now D_j $(j \neq i)$ and D_i/σ form a smaller representation of S in $V(\mathfrak{D})$. ■

Lemma 3.1.9 *The normal subgroup K of H intersects each $\overline{G_i}$ trivially.*

PROOF. By Lemma 3.1.8, $K \cap \overline{G_i}$ is normal in both $\overline{G_i}$ and H. Identifying $K \cap \overline{G_i}$ as a normal subgroup in G_i, let $\sigma_i \in \mathrm{Con}(D_i)$ with $\sigma_i \longleftrightarrow (\varepsilon, K \cap \overline{G_i}, \varepsilon)$. Then define a mapping $\Omega : T \longrightarrow \prod_{j \neq i} D_j \times D_i/\sigma_i$ by

$$(\Omega x)_j = \begin{cases} x_j & \text{if } j \neq i \\ x_i\sigma_i & \text{otherwise.} \end{cases}$$

Clearly Ω is a homomorphism. Since, for $x, y \in T$,

$$\begin{aligned} \Omega x = \Omega y &\Longrightarrow \begin{cases} x_j = y_j & \text{if } j \neq i \\ (x_i, y_i) \in \sigma_i & \text{otherwise} \end{cases} \\ &\Longrightarrow (x, y) \in [\overline{\sigma_i}]_T, \end{aligned}$$

$T/[\overline{\sigma_i}]_T$ is embeddable into $\prod_{j \neq i} D_j \times D_i/\sigma_i$, and since

$$\begin{aligned} [\overline{\sigma_i}]_T &\longleftrightarrow (\varepsilon, K \cap \overline{G_i}, \varepsilon) \\ &\subseteq (r, K, \pi) \longleftrightarrow \kappa, \end{aligned}$$

$S \cong T/\kappa$ is a homomorphic image of $T/[\overline{\sigma_i}]_T$. If $\sigma_i \neq \varepsilon$ then D_j $(j \neq i)$ and D_i/σ_i will form a smaller representation of S in $V(\mathfrak{D})$. Hence $\sigma_i = \varepsilon$ and $K \cap \overline{G_i} = \{1\}$ as required. ■

Lemma 3.1.10 *If S is subdirectly irreducible then*

$$S/\mu \in V(D_i/\mu_i \mid 1 \leq i \leq n)$$

where μ and μ_i $(1 \leq i \leq n)$ are the monolith congruences of S and D_i $(1 \leq i \leq n)$ respectively.

PROOF. Since each D_i is critical (Lemma 3.1.6) it can be assumed non-band; for if D_i is a band then it is either L or R by Lemma 3.1.2, whence D_i/μ_i is trivial. Furthermore, each D_i is subdirectly irreducible and so by Lemma 3.1.3 has a monolith congruence μ_i. By Lemma 3.1.4,

$$\mu_i \longleftrightarrow (\varepsilon, M_i, \varepsilon)$$

where M_i is the monolith subgroup of G_i. Since μ is the monolith congruence of $S \cong T/\kappa$, there exists μ' in $\mathrm{Con}(T)$ such that μ'/κ is the monolith congruence of T/κ. Note that μ' is necessarily the smallest congruence on T strictly containing κ, say $\mu' \longleftrightarrow (r', M', \pi')$

with $r \subseteq r', K \subseteq M'$ and $\pi \subseteq \pi'$. If $K = M'$ then $\mu'/\kappa \longleftrightarrow (*, \{1\}, *)$ is the monolith congruence of $T/\kappa \cong S$, contradicting Lemma 3.1.4(2). Therefore $K \neq M'$ and

$$\kappa \longleftrightarrow (r, K, \pi) \subset (r, M', \pi) \subseteq (r', M', \pi').$$

Hence $r' = r$, $\pi' = \pi$ and

$$\mu' \longleftrightarrow (r, M', \pi),$$

where $K \lhd M' \unlhd H$ and any normal subgroup of H that contains K as a proper subgroup also contains M'.

Now consider the monolith M_i of G_i. Then $\overline{M_i}$ is normal in $\overline{G_i}$ and, by Lemma 3.1.8, $\overline{M_i} \cap H$ is nontrivial in $\prod_{j=1}^{n} G_j$ so that $\pi_i\left(\overline{M_i} \cap H\right)$ is nontrivial in G_i. Again by Lemma 3.1.8, $\overline{M_i}$ is normalized by H so that $\overline{M_i} \cap H \unlhd H$. Consequently $\pi_i\left(\overline{M_i} \cap H\right) \unlhd \pi_i(H) = G_i$. Hence $M_i \subseteq \pi_i\left(\overline{M_i} \cap H\right)$. Since

$$\begin{aligned} |M_i| &\leq \left|\pi_i\left(\overline{M_i} \cap H\right)\right| \leq \left|\overline{M_i} \cap H\right| \\ &\leq \left|\overline{M_i}\right| = |M_i|, \end{aligned}$$

we have $\overline{M_i} \cap H = \overline{M_i}$, whence $\overline{M_i} \unlhd H$.

Suppose that A is a nontrivial normal subgroup of H contained in $\overline{M_i}$. Then $\pi_i(A) \leq M_i$ and $\pi_i(A) \unlhd \pi_i(H) = G_i$. Since $A \subseteq \overline{M_i}$, $\pi_i(A)$ is nontrivial. Consequently $\pi_i(A) = M_i$ by the minimality of M_i, whence $A = \overline{M_i}$. Thus we have shown $\overline{M_i}$ is a minimal normal subgroup of H.

By Lemma 3.1.9, K intersects $\overline{M_i}$ trivially so that $K\overline{M_i}/K \cong \overline{M_i}$. Since the normal subgroup $\overline{M_i}$ of H is minimal, the normal subgroup $K\overline{M_i}/K$ of H/K must also be minimal, whence $K\overline{M_i}/K$ must coincide with the monolith subgroup M'/K of H/K. Therefore $M' = K\overline{M_i}$ and

$$\begin{aligned} M' &= K\overline{M_1} \cdot K\overline{M_2} \cdots K\overline{M_n} \\ &= K\left(\overline{M_1} \cdot \overline{M_2} \cdots \overline{M_n}\right) = K\textstyle\prod_{i=1}^{n} M_i, \end{aligned}$$

which implies that

$$\begin{aligned} \mu' &\longleftrightarrow (r, M', \pi) \\ &= (r, K\textstyle\prod_{i=1}^{n} M_i, \pi) \\ &= (r, K, \pi) \vee (\varepsilon, \textstyle\prod_{i=1}^{n} M_i, \varepsilon) \\ &\longleftrightarrow \kappa \vee [\textstyle\prod_{i=1}^{n} \mu_i]_T. \end{aligned}$$

Now

$$\begin{aligned} S/\mu &\cong (T/\kappa)\,/\,(\mu'/\kappa) \\ &\cong T/\mu' = T/\left(\kappa \vee [\prod_{i=1}^{n}\mu_i]_T\right), \end{aligned}$$

where the last quotient is a homomorphic image of $T/\,[\prod_{i=1}^{n}\mu_i]_T$, which in turn is a subsemigroup of $\prod_{i=1}^{n}D_i/\prod_{i=1}^{n}\mu_i \cong \prod_{i=1}^{n}(D_i/\mu_i)$. ■

For a semigroup S, let

$$\mathfrak{F}_S = \{T/\rho \mid T \leq S, \rho \in \mathrm{Con}\,(T)\,, (T,\rho) \neq (S,\varepsilon)\}$$

be the set of proper factors of S.

Theorem 3.1.11 *A finite completely simple semigroup is subdirectly irreducible if and only if it is not contained in the variety generated by its proper quotients.*

PROOF. Let S be a finite completely simple semigroup that is subdirectly irreducible. If S is a band then the result holds by Lemma 3.1.2. So assume that S is not a band.

Let μ be the monolith congruence on S. The result is immediate if S is critical. So suppose that S is non-critical, that is, $S \in V\,(\mathfrak{F}_S)$. Let $\mathfrak{D}$ be a finite subset of $\mathfrak{F}_S$ which is minimal with respect to being factor closed and generating S. Since $\mathfrak{F}_S$ satisfies these conditions except for minimality, the existence of $\mathfrak{D}$ is guaranteed. Let D have maximal order in $\mathfrak{D}$ and let $\mathfrak{D}_0 = \mathfrak{D}\backslash\{D\}$. Then $S \notin V\,(\mathfrak{D}_0)$. But $\mathfrak{D}_0$ is still factor closed and every proper factor of a semigroup in $\mathfrak{D}$ is isomorphic to some semigroup in $\mathfrak{D}_0$. Therefore $S/\mu \in V\,(\mathfrak{D}_0)$ by Lemma 3.1.10. Since each proper quotient of S is a homomorphic image of S/μ, the variety generated by all proper quotients of S is contained in $V\,(\mathfrak{D}_0)$ and so cannot contain S.

Conversely, if S is subdirectly reducible then it belongs to the variety generated by its proper quotients. ■

Theorem 3.1.12 *A finite subdirectly irreducible completely simple semigroup that is not critical is contained in the variety generated by its proper subsemigroups.*

PROOF. Let S be a finite subdirectly irreducible completely simple semigroup that is not critical, and let μ be its monolith congruence. Suppose S is not contained in the variety $\mathbf{V}$ generated by its proper subsemigroups. Let

$$\mathfrak{H}_S = \{T/\rho \mid T < S, \rho \in \mathrm{Con}\,(T)\}.$$

Then

$$\begin{aligned}\mathfrak{F}_S &= \{T/\rho \mid T \leq S, \rho \in \mathrm{Con}\,(T)\,, (T,\rho) \neq (S,\varepsilon)\} \\ &= \{T/\rho \mid T < S, \rho \in \mathrm{Con}\,(T)\} \cup \{S/\rho \mid \rho \in \mathrm{Con}\,(S) \setminus \{\varepsilon\}\} \\ &= \mathfrak{H}_S \cup \{S/\rho \mid \rho \in \mathrm{Con}\,(S) \setminus \{\varepsilon\}\}.\end{aligned}$$

By assumption $S \in V(\mathfrak{F}_S)$ and $S \notin \mathbf{V} = V(\mathfrak{H}_S)$. Clearly $\mathfrak{H}_S \subset \mathfrak{F}_S$, so let $\mathfrak{D}$ be a subset of $\mathfrak{F}_S$ which is minimal with respect to being factor closed, generating S and strictly containing $\mathfrak{H}_S$. Since $\mathfrak{F}_S$ satisfies these conditions except for minimality, the existence of $\mathfrak{D}$ is guaranteed. Since $\mathfrak{D}\backslash\mathfrak{H}_S \neq \emptyset$ there exists a semigroup D of maximal order in $\mathfrak{D}\backslash\mathfrak{H}_S$; let $\mathfrak{D}_0 = \mathfrak{D}\backslash\{D\}$. Then $S \notin V(\mathfrak{D}_0)$ by the minimality of $\mathfrak{D}$. Furthermore every proper factor of every semigroup in $\mathfrak{D}$ is isomorphic to some semigroup in $\mathfrak{D}_0$ so that $S/\mu \in V(\mathfrak{D}_0)$ by Lemma 3.1.10. Consequently every proper quotient of S is in $V(\mathfrak{D}_0)$, that is, $\mathfrak{F}_S\backslash\mathfrak{H}_S \subseteq V(\mathfrak{D}_0)$. Now the inclusions $\mathfrak{H}_S \subseteq \mathfrak{D}_0 \subseteq \mathfrak{F}_S$ imply that

$$\mathfrak{F}_S = (\mathfrak{F}_S\backslash\mathfrak{H}_S) \cup \mathfrak{H}_S \subseteq V(\mathfrak{D}_0)\,,$$

whence $S \in V(\mathfrak{F}_S) \subseteq V(\mathfrak{D}_0)$, a contradiction. ■

Corollary 3.1.13 *If a finite completely simple semigroup is contained neither in the variety generated by its proper subsemigroups nor in the variety generated by its proper quotients, then it is critical.*

PROOF. If S is not contained in the variety generated by its proper quotients, then it is subdirectly irreducible by Theorem 3.1.11. Furthermore if S is not contained in the variety generated by its proper subsemigroups, then it is critical by Theorem 3.1.12. ■

3.2 Critical Semigroups in CCS

The material from this section to Section 5 inclusively are join work by Reilly and the author. Let $\mathbb{N}$ (respectively, $\mathbb{P}$) denote the set of all positive (respectively, prime) integers. A completely simple semigroup S is *central* if the product of any two idempotents of S lies in the center of the maximal subgroup containing it. The class **CCS** of all central completely simple semigroups is a variety (see [14], Proposition III.6.7(ii)). This section characterizes central completely simple semigroups that are critical.

Proposition 3.2.1 ([14], Proposition III.6.2) *Let $S = \mathcal{M}(I, G, \Lambda; P) \in \mathbf{CS}_n$ with P normalized. Then the following are equivalent.*

(1) *S is central;*

(2) *$\langle P \rangle$ is contained in the centre $Z(G)$ of G;*

(3) *S satisfies the identity $x^n y^n x = x y^n x^n$.*

From Proposition 3.2.1, the class $\mathbf{CCS}_n$ of all central completely simple semigroups with subgroups of exponent dividing n is a variety. Let $\mathfrak{I}$ denote the class of all idempotent generated completely simple semigroups:

$$\mathfrak{I} = \{S \in \mathbf{CS} \mid S = C(S)\}.$$

Lemma 3.2.2 *Let $\mathbf{V} = V(\mathfrak{C})$ be such that $\mathfrak{C} = \{S_\alpha \mid \alpha \in \Gamma\}$ is a collection of finite central completely simple semigroups $S_\alpha = \mathcal{M}(I_\alpha, G_\alpha, \Lambda_\alpha; P_\alpha)$ with P_α normalized. Then*

(1) $\mathbf{V} \cap \mathbf{RB} = V(I_\alpha \times \Lambda_\alpha \mid \alpha \in \Gamma)$;

(2) $\mathbf{V} \cap \mathbf{G} = V(G_\alpha \mid \alpha \in \Gamma)$;

(3) $V(\mathfrak{I} \cap \mathbf{V}) \cap \mathbf{A} = V(\langle P_\alpha \rangle \mid \alpha \in \Gamma)$.

PROOF. (1) Suppose $S \in \mathbf{V} \cap \mathbf{RB}$. Then there exist $S_\sigma \in \mathfrak{C}$ ($\sigma \in \Sigma$), a subsemigroup T of $\prod_{\sigma \in \Sigma} S_\sigma$ and a homomorphism ψ from T onto S. It is straightforward to show that $a\mathcal{H}b$ implies $\psi a \mathcal{H} \psi b$ for all $a, b \in T$. But since S is a band, the congruence $\mathcal{H}$ is trivial on S. Therefore $a\mathcal{H}b$ implies $\psi a = \psi b$ for all $a, b \in T$. Hence S is a homomorphic image of $T/\mathcal{H} \leq \prod_{\sigma \in \Sigma}(S_\sigma/\mathcal{H})$, and

$$\begin{aligned} S \in V(S_\sigma/\mathcal{H} \mid \sigma \in \Sigma) &\subseteq V(S_\alpha/\mathcal{H} \mid \alpha \in \Gamma) \\ &= V(I_\alpha \times \Lambda_\alpha \mid \alpha \in \Gamma). \end{aligned}$$

Consequently, $\mathbf{V} \cap \mathbf{RB} \subseteq V(I_\alpha \times \Lambda_\alpha \mid \alpha \in \Gamma)$, and the reverse inclusion is obvious.

(2) This is ([14], Lemma VIII.1.2).

(3) Suppose $S \in \mathfrak{J} \cap \mathbf{V}$. Then there exist $S_\sigma \in \mathfrak{C}$ $(\sigma \in \Sigma)$, a subsemigroup T of $\prod_{\sigma\in\Sigma} S_\sigma$ and a homomorphism ψ from T onto S. Since $S \in \mathfrak{J}$, S is generated by $E(S)$ so that by Lemma 2.2.2, there exists a subset U of $E(T)$ such that $\psi\langle U\rangle = S$. It is easy to see that $\langle U\rangle$ is a subsemigroup of $\prod_{\sigma\in\Sigma} C(S_\sigma)$, whence

$$S \in V(C(S_\sigma) \mid \sigma \in \Sigma) \subseteq V(C(S_\alpha) \mid \alpha \in \Gamma).$$

Hence $V(\mathfrak{J} \cap \mathbf{V}) \subseteq V(C(S_\alpha) \mid \alpha \in \Gamma)$. By (2) and Lemma 2.3.1,

$$\begin{aligned} V(\mathfrak{J} \cap \mathbf{V}) \cap \mathbf{A} &\subseteq V(C(S_\alpha) \mid \alpha \in \Gamma) \cap \mathbf{A} \\ &= V(\mathcal{M}(I_\alpha, \langle P_\alpha\rangle, \Lambda_\alpha; P_\alpha) \mid \alpha \in \Gamma) \cap \mathbf{A} \\ &= V(\langle P_\alpha\rangle \mid \alpha \in \Gamma). \end{aligned}$$

The inclusion $V(\langle P_\alpha\rangle \mid \alpha \in \Gamma) \subseteq V(\mathfrak{J} \cap \mathbf{V}) \cap \mathbf{A}$ is obvious. ■

The following is a special case of ([14], Theorem VIII.8.4).

Theorem 3.2.3 *The mapping*

$$\mathbf{V} \overset{\varkappa}{\longmapsto} (\mathbf{V} \cap \mathbf{RB}, V(\mathfrak{J} \cap \mathbf{V}) \cap \mathbf{A}, \mathbf{V} \cap \mathbf{G})$$

is an isomorphism of $\mathcal{L}(\mathbf{CCS}_n)$ onto the sublattice $\varkappa(\mathcal{L}(\mathbf{CCS}_n))$ of $\mathcal{L}(\mathbf{RB}) \times \mathcal{L}(\mathbf{A}_n) \times \mathcal{L}(\mathbf{G}_n)$.

For any $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $\mathbb{Z}_{p^n}$ be the cyclic group of order p^n generated by a. Define the Rees matrix semigroup

$$C_{p,n} = \mathcal{M}\left(\{1,2\}, \mathbb{Z}_{p^n}, \{1,2\}; \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}\right).$$

Note that the sandwich matrix of $C_{p,n}$ is normalized.

Lemma 3.2.4 *The completely simple semigroup $C_{p,n}$ is central and generated by its idempotents.*

PROOF. By Proposition 3.2.1, $C_{p,n}$ is central. By Lemma 2.3.1,

$$\begin{aligned} C\left(C_{p,n}\right) &= \mathcal{M}\left(\{1,2\}, \langle a\rangle, \{1,2\}; \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}\right) \\ &= \mathcal{M}\left(\{1,2\}, \mathbb{Z}_{p^n}, \{1,2\}; \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}\right) = C_{p,n}. \end{aligned}$$

Therefore $C_{p,n}$ is generated by its idempotents. ■

We are now ready to characterize critical central completely simple semigroups.

Theorem 3.2.5 *A finite central completely simple semigroup*

$$S = \mathcal{M}(I, G, \Lambda; P)$$

with normalized matrix P is critical if and only if exactly one of the following holds:

(1) *S is isomorphic to L or R;*

(2) *S is a critical group;*

(3) *$S = C_{p,n}$ for some $(p,n) \in \mathbb{P} \times \mathbb{N}$.*

PROOF. Let S be critical. Since S is central and P is normalized, we have $\langle P\rangle \subseteq Z(G)$. If $|G| = 1$ then S is a band so that (1) holds by Lemma 3.1.2. Thus suppose that $|G| > 1$. Since P is diverse by Lemma 3.1.5(2), either $|I| = |\Lambda| = 1$ or $|I|, |\Lambda| \geq 2$. The former implies (2), so assume that $|I|, |\Lambda| \geq 2$, whence $\langle P\rangle \neq \{1\}$ because P has distinct rows and distinct columns, and therefore $Z(G) \neq \{1\}$. Since G is subdirectly irreducible, $Z(G)$ must be a cyclic p-group so that $\langle P\rangle$ is a cyclic p-subgroup of $Z(G)$, say $\langle P\rangle \cong \mathbb{Z}_{p^n}$ for some $n \in \mathbb{N}$. Therefore $\langle P\rangle = \langle p_{\lambda i}\rangle$ for some $(i,\lambda) \in I \times \Lambda$. Now

$$T = \mathcal{M}\left(\{1,i\}, \langle P\rangle, \{1,\lambda\}; \begin{bmatrix} 1 & 1 \\ 1 & p_{\lambda i} \end{bmatrix}\right)$$

is a subsemigroup of S. By Lemma 3.2.2 and Theorem 3.2.3,

$$\begin{aligned} \varkappa V(T,G) &= \left(V\left(\{1,i\} \times \{1,\lambda\}\right), V\left(\langle p_{\lambda i}\rangle\right), V\left(\langle P\rangle, G\right)\right) \\ &= \left(V\left(I \times \Lambda\right), V\left(\langle P\rangle\right), V\left(G\right)\right) \\ &= \varkappa V(S). \end{aligned}$$

Since $\varkappa$ is injective, $V(S) = V(T, G)$. But S is critical and not a group (since $|I|, |\Lambda| \geq 2$). Consequently $S = T$ and (3) follows.

Conversely, if either (1) or (2) prevails then S is obviously critical. So it suffices to assume $S = C_{p,n}$ for some $(p, n) \in \mathbb{P} \times \mathbb{N}$. By Lemma 3.1.5(2), S is subdirectly irreducible and by Lemma 2.3.2, S is not orthodox. Seeking a contradiction, suppose S is not critical. Then by Theorem 3.1.12, S is contained in the variety $\mathbf{V}$ generated by its proper subsemigroups. But it is straightforward to show that each proper subsemigroup of S is orthodox and thus a rectangular group by Lemma 2.3.2. Since the class **ReG** of all rectangular groups is a variety ([14], Corollary III.5.3), $S \in \mathbf{V} \subseteq \mathbf{ReG}$ is contradictorily orthodox. ■

3.3 Intersection of Singly Generated Subvarieties of Groups

A (pseudo)variety is *singly generated* if it is generated by a single finite semigroup. In [7], Kublanovsky announced the following interesting result:

Theorem 3.3.1 *For any pair of finite completely 0-simple semigroups A and B, there exists a finite completely 0-simple semigroup C such that*

$$P_S(A) \cap P_S(B) = P_S(C).$$

Hence the set of all pseudovarieties singly generated by finite completely 0-simple semigroups forms a sublattice of the lattice of semigroup pseudovarieties.

This result led Kublanovsky to pose the following problem in [8].

Problem 3.3.2 *Find (in terms of Rees matrix representation) an operation $\circ$ which given two finite completely 0-simple semigroups A and B returns a finite completely 0-simple semigroup $A \circ B$ with*

$$P_S(A) \cap P_S(B) = P_S(A \circ B).$$

The remainder of this chapter is devoted to a partial solution to this problem.

Lemma 3.3.3 *Let A, B and C be finite semigroups. Then the following statements are equivalent:*

(1) $P_S(A) \cap P_S(B) = P_S(C)$;

(2) $V(A) \cap V(B) = V(C)$.

PROOF. Assume that (1) holds and let $\mathbf{W} = V(A) \cap V(B)$. By Corollary 2.5.3, $V(A)$ is locally finite since A is finite. Thus $F_n(V(A))$ is finite and belongs to $P_S(A)$ for all $n \in \mathbb{N}$. Since $F_n(\mathbf{W})$ is a homomorphic image of $F_n(V(A))$, it also belongs to $P_S(A)$. Similarly, $F_n(\mathbf{W}) \in P_S(B)$, whence $F_n(\mathbf{W}) \in P_S(A) \cap P_S(B) = P_S(C)$. Therefore $\mathbf{W} \subseteq V(C)$. Now since $C \in P_S(C) = P_S(A) \cap P_S(B) \subseteq \mathbf{W}$ it follows that $V(C) \subseteq \mathbf{W}$ and equality prevails.

Now assume that (2) holds. Let $S \in P_S(A) \cap P_S(B)$. Then S is finite and $S \in V(A) \cap V(B) = V(C)$. Hence S is a homomorphic image of $F_m(V(C))$ with $m = |S|$. But since C is finite, we have $F_m(V(C)) \in P_S(C)$ and $S \in P_S(C)$. Consequently $P_S(A) \cap P_S(B) \subseteq P_S(C)$.

Conversely, consider any $S \in P_S(C)$ and let $m = |S|$. Then

$$S \in P_S(C) \subseteq V(C) \subseteq V(A) \cap V(B)$$

so that S is a homomorphic image of $F_m(V(A))$ and of $F_m(V(B))$. As before $F_m(V(A)) \in P_S(A)$ and $F_m(V(B)) \in P_S(B)$. Consequently, $S \in P_S(A) \cap P_S(B)$ and so $P_S(C) \subseteq P_S(A) \cap P_S(B)$. ■

In the light of Lemma 3.3.3, it is immaterial whether we deal with varieties or pseudovarieties in solving Problem 3.3.2.

In this section we solve Problem 3.3.2 within the context of groups. Fortunately, in this case it is a matter of piecing together known results about varieties generated by finite groups that can all be found in [12].

A variety of groups is a *Cross variety* if it is locally finite, finitely based, and contains finitely many non-isomorphic critical groups.

Lemma 3.3.4 ([12], 51.52) *Each subvariety of a Cross variety is a Cross variety.*

The key to this whole discussion is the following famous result due to Oates and Powell.

Theorem 3.3.5 ([12], 52.11) *The variety generated by a finite group is a Cross variety.*

Now by Lemma 3.1.1, any locally finite variety of groups is generated by its critical members so that, in particular, any Cross variety is so generated. In addition, every subvariety of a Cross variety is again a Cross variety.

So suppose that G and H are finite groups. Then $\mathbf{U} = V(G)$ and $\mathbf{V} = V(H)$ are Cross varieties whence $\mathbf{W} = \mathbf{U} \cap \mathbf{V}$ is also a Cross variety and therefore generated by its critical groups. The next step is to find a bound on the size of the critical groups in $\mathbf{W}$.

It so happens that there is a computable function $f(x, y)$ such that if A is a group of exponent e_A and if c_A is the maximal class of any of its nilpotent factors then the order of any critical group in $V(A)$ is bounded by $f(e_A, c_A)$. For details see ([12], Chapter 5).

Thus the order of any critical group in $\mathbf{W}$ is bounded by

$$m = \min\left(f(e_G, c_G), f(e_H, c_H)\right).$$

Now every group in $\mathbf{W}$ that has a set of k generators (with $k \leq m$) is a homomorphic image of the $\mathbf{W}$-free group $F_m(\mathbf{W})$ in $\mathbf{W}$ on m generators. In particular, the critical groups in $\mathbf{W}$ are all homomorphic images of $F_m(\mathbf{W})$. Consequently $F_m(\mathbf{W})$ generates $\mathbf{W}$ and, since $\mathbf{W}$ is a Cross variety and so locally finite, $F_m(\mathbf{W})$ is finite. Thus $F_m(\mathbf{W})$ is a finite group that generates $\mathbf{W}$. The next question is how to compute $F_m(\mathbf{W})$.

Since $F_m(\mathbf{W}) \in \mathbf{W} = \mathbf{U} \cap \mathbf{V}$, it follows that $F_m(\mathbf{W})$ is a homomorphic image of both $F_m(\mathbf{U})$ and $F_m(\mathbf{V})$ where $F_m(\mathbf{U})$ and $F_m(\mathbf{V})$ can be computed as subgroups of the direct product of $|G|^m$ copies of G and $|H|^m$ copies of H, respectively. Now any group that is a homomorphic image of both $F_m(\mathbf{U})$ and $F_m(\mathbf{V})$ must be m-generated and lie in $\mathbf{U} \cap \mathbf{V} = \mathbf{W}$ and therefore be a homomorphic image of $F_m(\mathbf{W})$. Hence $F_m(\mathbf{W})$ is (isomorphic to) $F_m(\mathbf{U})/N$ where N is the smallest normal subgroup of $F_m(\mathbf{U})$ such that $F_m(\mathbf{U})/N$ is a homomorphic image of $F_m(\mathbf{V})$. Equivalently, N is the smallest normal subgroup of $F_m(\mathbf{U})$ such that the mapping $x_i \longmapsto Nx_i$ $(1 \leq i \leq m)$ extends to a homomorphism of $F_m(\mathbf{V})$ to $F_m(\mathbf{U})/N$.

First note that since $\mathbf{W} \subseteq \mathbf{U}$, every critical group in $\mathbf{W}$ also belongs to $\mathbf{U}$ so that we may assume that $n = f(e_G, c_G) \geq m$. Next, we know that $\mathbf{U}$ is generated by the $\mathbf{U}$-free semigroup on generators $x_1, \ldots, x_n$. Moreover $F_n(\mathbf{U})$ and $F_n(\mathbf{V})$ are computable as subgroups of the direct product of $|G|^n$ copies of G and $|H|^n$ copies of H, respectively. It is then possible to compute the normal subgroups of $F_n(\mathbf{U})$. There must be a smallest normal subgroup N of $F_n(\mathbf{U})$ such that $F_n(\mathbf{U})/N \in \mathbf{V}$. It is possible to identify N as it is the smallest normal subgroup such that the identity mapping on $\{x_1, \ldots, x_n\}$ extends to a homomorphism of $F_n(\mathbf{V})$ onto $F_n(\mathbf{U})/N$. We then have $F_n(\mathbf{U})/N \in \mathbf{U} \cap \mathbf{V}$ and so $F_n(\mathbf{U})/N$ must, in fact, be isomorphic to the free group in $\mathbf{U} \cap \mathbf{V}$ on n generators. Since $n \geq m$, $F_n(\mathbf{U})/N$ generates $\mathbf{U} \cap \mathbf{V}$ and is computable.

3.4 Intersection of Singly Generated Subvarieties of CCS

This section extends the solution from the previous section to the larger variety **CCS** of central completely simple semigroups. Recall from Theorem 3.2.5 that the semigroup

$$C_{p,n} = \mathcal{M}\left(\{1,2\}, \mathbb{Z}_{p^n}, \{1,2\}; \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}\right)$$

(where $\mathbb{Z}_{p^n} = \langle a \rangle$) are, up to isomorphism, precisely all the critical central completely simple semigroups that are not orthodox. Let $\mathcal{C} = \{C_{p,n} \mid (p,n) \in \mathbb{P} \times \mathbb{N}\}$. If A is a semigroup then let

$$\mathcal{B}_A = V(A) \cap \{L, R\}, \qquad \mathcal{C}_A = V(A) \cap \mathcal{C}.$$

Clearly if $A \in$ **CCS** is finite then $\mathcal{B}_A$ is computable. But $\mathcal{C}_A$ is also computable by the following.

Lemma 3.4.1 *Let $A = \mathcal{M}(I, G, \Lambda; P)$ be a finite central completely simple semigroup with P normalized. Then*

$$\mathcal{C}_A = \{C_{p,n} \mid \mathbb{Z}_{p^n} \textit{ is a factor of } \langle P \rangle\}.$$

Consequently $\mathcal{C}_A$ is a finite computable set of semigroups.

Proof. Let $\Xi = \{C_{p,n} \mid \mathbb{Z}_{p^n}$ is a factor of $\langle P \rangle\}$. Note that since P is normalized, Ξ is empty if $|I|$ or $|\Lambda|$ is 1. So assume $|I|, |\Lambda| \geq 2$. Next note that $V(\Xi) \cap \mathbf{G} = V(\langle P \rangle) \subseteq V(G)$. By Lemma 3.2.2 and Theorem 3.2.3,

$$\begin{aligned} \varkappa V(I \times \Lambda, G, \Xi) &= (V(I \times \Lambda), V(\mathbb{Z}_{p^n} | \mathbb{Z}_{p^n} \text{ is a factor of } \langle P \rangle), V(G)) \\ &= (V(I \times \Lambda), V(\langle P \rangle), V(G)) \\ &= \varkappa V(A) \end{aligned}$$

so that $V(I \times \Lambda, G, \Xi) = V(A)$. Hence

$$\Xi \subseteq V(A) \cap \mathcal{C} = \mathcal{C}_A.$$

Conversely, if $C_{p,n} \in V(A) \cap \mathcal{C}$, then by Theorem 3.2.3, $\mathbb{Z}_{p^n} \in V(\langle P \rangle)$ so that $\mathbb{Z}_{p^n}$ is a factor of $\langle P \rangle$. Hence $C_{p,n} \in \Xi$ so that $\mathcal{C}_A \subseteq \Xi$ and equality prevails. ■

Lemma 3.4.2 *If $A = \mathcal{M}(*, G, *; *)$ is a finite central completely simple semigroup with normalized sandwich matrix, then*

$$V(A) = V(\mathcal{B}_A, \mathcal{C}_A, G).$$

PROOF. Let S be a critical semigroup in $V(A)$. By Theorem 3.2.5, either $S \in \{L, R\}, S \in \mathcal{C}$ or $S \in \mathbf{G}$. The first and second imply $S \in \mathcal{B}_A$ and $S \in \mathcal{C}_A$, respectively. If $S \in \mathbf{G}$ then by Lemma 3.2.1, $S \in V(A) \cap \mathbf{G} = V(G)$. So $V(\mathcal{B}_A, \mathcal{C}_A, G)$ contains all the critical semigroups of $V(A)$. By Lemma 3.1.1, $V(A) \subseteq V(\mathcal{B}_A, \mathcal{C}_A, G)$. The inclusion $V(\mathcal{B}_A, \mathcal{C}_A, G) \subseteq V(A)$ is obvious. ■

Theorem 3.4.3 *If $A = \mathcal{M}(*, G, *; *)$ and $B = \mathcal{M}(*, H, *; *)$ are finite central completely simple semigroups with normalized sandwich matrices, then*

$$V(A) \cap V(B) = V(\mathcal{B}_A \cap \mathcal{B}_B, \mathcal{C}_A \cap \mathcal{C}_B, G \circ H).$$

PROOF. Let $\mathbf{V} = V(\mathcal{B}_A \cap \mathcal{B}_B, \mathcal{C}_A \cap \mathcal{C}_B, G \circ H)$. Then by invoking Theorem 3.3.1 and Lemma 3.3.3, $V(A) \cap V(B) = V(S)$ for some finite semigroup S, which clearly must be central completely simple. Suppose that $S = \mathcal{M}(*, K, *; *)$. By Lemma 3.2.2,

$$\begin{aligned} V(K) &= V(S) \cap \mathbf{G} = V(A) \cap V(B) \cap \mathbf{G} \\ &= V(G) \cap V(H) = V(G \circ H). \end{aligned}$$

Note that

$$\mathcal{B}_S = V(S) \cap \{L, R\} = V(A) \cap V(B) \cap \{L, R\} = \mathcal{B}_A \cap \mathcal{B}_B$$

and similarly, $\mathcal{C}_S = \mathcal{C}_A \cap \mathcal{C}_B$. Therefore by Lemma 3.4.2,

$$\begin{aligned} V(S) &= V(\mathcal{B}_S, \mathcal{C}_S, K) \\ &= V(\mathcal{B}_S, \mathcal{C}_S, G \circ H) \\ &= V(\mathcal{B}_A \cap \mathcal{B}_B, \mathcal{C}_A \cap \mathcal{C}_B, G \circ H). \end{aligned}$$

■

Corollary 3.4.4 *Let $A = \mathcal{M}(*, G, *; *)$ and $B = \mathcal{M}(*, H, *; *)$ be finite central completely simple semigroups with normalized sandwich matrices. Then*

$$V(A) \cap V(B) = V(A \circ B)$$

where $A \circ B = G \circ H \times \prod [(\mathcal{B}_A \cap \mathcal{B}_B) \cup (\mathcal{C}_A \cap \mathcal{C}_B)]$ is a computable finite central completely simple semigroup.

3.5 Intersection of Singly Generated Subvarieties of $\mathbf{CCS} \vee \mathbf{NB_2}$

Recall that $\mathbf{NB_2}$ is the variety generated by L, R and the completely 0-simple semigroup B_2 of order five. This section presents a solution to Problem 3.3.2 within the context of completely (0-)simple semigroups from $\mathbf{CCS} \vee \mathbf{NB_2}$. It is an extension of the solution in the previous section which concerned semigroups from the variety $\mathbf{CCS}$ of all central completely simple semigroups.

The following two results, the justifications of which are very lengthy, are dependent on results from a paper of Kublanovsky waiting for publication. Therefore their proofs have been omitted.

Proposition 3.5.1 ([11], Theorem 6.3) *Let $S = \mathcal{M}^0(I, G, \Lambda; P) \in \mathbf{CS}_n^0$. Then the following statements are equivalent.*

(1) $S \in \mathbf{CCS}_n \vee \mathbf{NB_2}$;

(2) *S satisfies the identities*

$$x^{2n}y^{2n} = \left(x^{2n}y^{2n}\right)^{n+1}, \qquad x^{2n}y^{2n}x = xy^{2n}x^{2n}.$$

Recall that for any exact Rees-Sushkevich variety $\mathbf{V}$, the set $\mathcal{LE}(\mathbf{V})$ denote the sublattice of $\mathcal{L}(\mathbf{V})$ consisting of the exact subvarieties of $\mathbf{V}$.

Theorem 3.5.2 ([11], Theorem 7.3) *The mappings*

$$\mathbf{U} \overset{\varphi}{\longmapsto} (\mathbf{U} \cap \mathbf{CS}_n, \mathbf{U} \cap \mathbf{B_2})$$

and

$$(\mathbf{V}, \mathbf{W}) \overset{\chi}{\longmapsto} \mathbf{V} \vee \mathbf{W}$$

are inverse isomorphisms between $\mathcal{LE}(\mathbf{CCS}_n \vee \mathbf{NB_2})$ and $\mathcal{L}(\mathbf{CCS}_n) \times \{\mathbf{T}, \mathbf{Y}, \mathbf{B_2}\}$.

Lemma 3.5.3 ([17], Lemma 4.2(i)) *If S is a finite semigroup in $\mathbf{CS}_n^0$, then*

$$V(S) \cap \mathbf{CS}_n = V(T \leq S \mid T \in \mathbf{CS}).$$

Consequently, there exists a finite computable semigroup S' in $V(S)$ such that

$$V(S) \cap \mathbf{CS}_n = V(S').$$

Let S be a finite Rees matrix semigroup with sandwich matrix P. It is straightforward to find the completely simple subsemigroups of S; the direct product S' of these subsemigroups of S, by Lemma 3.5.3, is a completely simple semigroup such that $V(S) \cap \mathbf{CS}_n = V(S')$.

Next we find a generator for the variety $V(S) \cap \mathbf{B_2}$. Since S is completely (0-)simple, $V(S)$ is exact. Therefore $V(S) \cap \mathbf{B_2}$ is exact and must be one of $\mathbf{T}, \mathbf{Y}$ or $\mathbf{B_2}$ by Figure 2.1. It follows from ([17], Corollary 6.3) that $V(S) \cap \mathbf{RS}_1 = V(S/\mathcal{H})$. Therefore, since $\mathbf{B_2} \subseteq \mathbf{RS}_1$,

$$V(S) \cap \mathbf{B_2} = (V(S) \cap \mathbf{RS}_1) \cap \mathbf{B_2} = V(S/\mathcal{H}) \cap \mathbf{B_2}.$$

If S is completely simple then $V(S/\mathcal{H}) \cap \mathbf{B_2} \subseteq \mathbf{RB} \cap \mathbf{B_2} = \mathbf{T}$. If S is completely 0-simple but P has no zero entries, then $V(S/\mathcal{H}) \cap \mathbf{B_2} = \mathbf{Y}$. Hence it remains to consider the case when S is completely 0-simple with P having a zero entry. Note that by Theorem 2.4.1,

$$S/\mathcal{H} = S/\rho_G \cong \mathcal{M}^0(I, \{1\}, \Lambda; P/G).$$

Since P has no row or column consisting entirely of zeros, P/G must have a submatrix of one of the following forms

$$\begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$$

where $a \in \{0,1\}$. It is obvious that rearranging rows and columns of a Rees matrix semigroup does not change the semigroup isomorphically. Therefore rearranging the rows and columns appropriately, $S/\mathcal{H}$ is guaranteed to contain either A_2 or B_2 as a subsemigroup, whence $V(S/\mathcal{H}) \cap \mathbf{B_2} = \mathbf{B_2}$. For convenience, let S_1 be the semigroup in $\{\{1\}, Y, B_2\}$ such that $V(S) \cap \mathbf{B_2} = V(S_1)$.

We are now ready to present a solution to Problem 3.3.2 for semigroups in $\mathbf{CCS} \vee \mathbf{NB_2}$ by using the operation $\circ$ from the previous section. Suppose A, B are finite Rees matrix semigroups in $\mathbf{CCS} \vee \mathbf{NB_2}$. Then there exists $n \in \mathbb{N}$ such that $A, B \in \mathbf{CCS}_n \vee \mathbf{NB_2}$. Let A', A_1, B', B_1 be semigroups such that

$$V(A) \cap \mathbf{CS}_n = V(A'), \qquad V(A) \cap \mathbf{B_2} = V(A_1),$$
$$V(B) \cap \mathbf{CS}_n = V(B'), \qquad V(B) \cap \mathbf{B_2} = V(B_1).$$

Since A' and B' are central completely simple semigroups, $A' \circ B'$ is computable by Corollary 3.4.4. Clearly, $A_1 \circ B_1$ is computable by Figure 2.1 since $A_1, B_1 \in \{\{1\}, Y, B_2\}$. Therefore, with φ and χ as in Theorem 3.5.2,

$$\begin{aligned}
\varphi(V(A) \cap V(B)) &= \varphi V(A) \cap \varphi V(B) \\
&= (V(A) \cap \mathbf{CS}_n, V(A) \cap \mathbf{B_2}) \cap (V(B) \cap \mathbf{CS}_n, V(B) \cap \mathbf{B_2}) \\
&= (V(A'), V(A_1)) \cap (V(B'), V(B_1)) \\
&= (V(A') \cap V(B'), V(A_1) \cap V(B_1)) \\
&= (V(A' \circ B'), V(A_1 \circ B_1))
\end{aligned}$$

and

$$\begin{aligned}
V(A) \cap V(B) &= \chi\left(V(A' \circ B'), V(A_1 \circ B_1)\right) \\
&= V\left((A' \circ B') \times (A_1 \circ B_1)\right).
\end{aligned}$$

3.6 Examples of Critical Semigroups

Examples of subdirectly irreducible semigroups that are not critical will be presented first. Then examples will be given to demonstrate that there is no direct relation between the criticality of a finite completely simple semigroup $\mathcal{M}(I, G, \Lambda; P)$ and its underlying structure group G. The section will end by showing all monoids $C^1_{p,n}$ are critical.

Example 3.6.1 ([12], 51.33) *Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the Quaternion group and let*

$$G_Q = \frac{Q \times Q}{N},$$

where $N = \{(1,1), (-1,-1)\}$ is a normal subgroup of $Q \times Q$. Then G_Q is subdirectly irreducible but not critical.

PROOF. Note that $M = \{N, (1,-1)N\}$ is a normal subgroup of G_Q. It is easy to show that if $(a,b)N \neq N$ then $((a,b)N)^2 = (1,-1)N$. Hence each nontrivial normal subgroup of G_Q contains M so that G_Q is subdirectly irreducible with monolith subgroup M. Now the mapping $a \longmapsto (a,1)N$ is an embedding of Q into G_Q so that Q is a proper factor of G_Q. But by definition, $G_Q \in V(Q)$ so that G_Q is not critical. ■

Example 3.6.2 *Let*

$$T_1 = \mathcal{M}\left(\{1,2,3\}, \mathbb{Z}_p, \{1,2\}; \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \end{bmatrix}\right)$$

where $\mathbb{Z}_p = \langle a \rangle$, $p \in \mathbb{P}$ *and* $p \geq 3$. *Then* T_1 *is subdirectly irreducible and not critical.*

PROOF. Since T_1 possesses the monolith congruence $\rho_{\mathbb{Z}_p}$ it is subdirectly irreducible. But $V(T_1) = V(C_{p,1})$ by Lemma 3.2.2 and Theorem 3.2.3, where $C_{p,1}$ is clearly a proper subsemigroup of T_1. Therefore T_1 is not critical. ■

A completely simple semigroup $S = \mathcal{M}(I, G, \Lambda; P)$ is *overabelian* if G is abelian. By the following lemma the class of all completely simple semigroups in $\mathbf{CS}_n$ the cores of which are overabelian form a semigroup variety.

Lemma 3.6.3 ([14], Proposition III.6.1) *Let* $S = \mathcal{M}(I, G, \Lambda; P) \in \mathbf{CS}_n$ *with* P *normalized. Then the following statements are equivalent.*

(1) $C(S)$ *is overabelian;*

(2) *The entries of* P *commute;*

(3) S *satisfies the identity* $xy^nx^nz^nx = xz^nx^ny^nx$.

Example 3.6.4 *Let* $T_2 = \mathcal{M}(\{1,2,3\}, G_Q, \{1,2,3\}; P)$ *where*

$$P = \begin{bmatrix} N & N & N \\ N & (i,1)N & (1,i)N \\ N & (1,j)N & (j,1)N \end{bmatrix}.$$

Then T_2 *is critical.*

PROOF. Suppose T_2 is not critical. Then since T_2 is subdirectly irreducible (with monolith congruence ρ_M, where M is defined as in Example 3.6.1) it belongs to the variety generated by its proper subsemigroups (Theorem 3.1.12). Now $\langle P\rangle = G_Q$ implies that each proper subsemigroup of T_2 must have the form $\mathcal{M}(I, H, \Lambda; R)$ where $H \leq G_Q$ and R is a proper submatrix of P. It is then routine to show that the core of each proper subsemigroup of T_2 is overabelian. This would imply that $C(T_2)$ is also overabelian by Lemma 3.6.3, which is impossible because P is normailized and its entries do not commute. ■

Corollary 3.6.5 (1) *G_Q is a subdirectly irreducible group that is non-critical;*

(2) *T_1 is a subdirectly irreducible semigroup that is non-critical;*

(3) *$C_{p,n}$ is critical with critical structure group $\mathbb{Z}_{p^n}$;*

(4) *T_2 is critical with non-critical structure group;*

(5) *$L \times G$ is non-critical for any structure group G.*

For any congruence ρ on a semigroup S, let ρ^1 be the congruence $\rho \cup \{(1,1)\}$ on the monoid S^1. For the remainder of this section, fix a prime integer p and let

$$C_n = \begin{cases} C_{p,n} & \text{if } n \geq 1, \\ L \times R & \text{if } n = 0. \end{cases}$$

For $i, \lambda \in \{1,2\}$, let $e_{i\lambda}$ denote the idempotent of the $\mathcal{H}$-class $\{i\} \times \mathbb{Z}_{p^n} \times \{\lambda\}$ of C_n. Note that C_n^1 is completely regular since each of its five $\mathcal{H}$-classes is a subgroup. Since C_n^1 satisfies the identity $x^{p^n+1} = x$, each semigroup S in $V(C_n^1)$ also satisfies this identity so that a^{p^n} is an idempotent of S for all $a \in S$. For convenience, let Z denote $p^{n-1}\mathbb{Z}_{p^n}$, the unique subgroup of $\mathbb{Z}_{p^n}$ of order p.

Lemma 3.6.6 (1) *If ρ is a congruence on C_n^1 then either $1\rho = \{1\}$ or $\rho = C_n^1 \times C_n^1$.*

(2) *If $n \geq 1$ then the monolith congruence on C_n^1 is ρ_Z^1.*

PROOF. (1) Let $i, j, \lambda, \mu \in \{1,2\}$ be such that $i \neq j$ and $\lambda \neq \mu$. Suppose $1\rho \neq \{1\}$. Then $a\rho 1$ for some $a \in C_n$ so that $a^{p^n}\rho 1$ where a^{p^n} is an idempotent. Without loss of generality assume $a^{p^n} = e_{i\lambda}$. Then

$$1\rho e_{i\lambda} = (e_{i\lambda}e_{j\lambda})^{p^n}\,\rho\,(1 \cdot e_{j\lambda})^{p^n} = e_{j\lambda}\,,$$
$$1\rho e_{i\lambda} = (e_{i\mu}e_{i\lambda})^{p^n}\,\rho\,(e_{i\mu} \cdot 1)^{p^n} = e_{i\mu}\,,$$
$$1\rho e_{i\lambda}\rho\,(e_{j\lambda}e_{i\mu})^{p^n} = e_{j\mu}\,.$$

Hence $e\rho 1$ for all $e \in E(C_n)$, whence $x\rho 1$ for all $x \in C(C_n)$. The result now follows since $C_n = C(C_n)$ by Lemma 3.2.4.

(2) This follows from (1) and the fact that ρ_Z is the monolith congruence on C_n. ■

Lemma 3.6.7 *For $n \geq 1$, the variety generated by the proper factors of C_n^1 is $V\left(C_n, C_{n-1}^1\right)$.*

Proof. Let $\mathbf{V}$ be the variety generated by the proper factors of C_n^1. Since $C_n^1/\rho_Z^1 \cong C_{n-1}^1$, it follows from Lemma 3.6.6 that C_{n-1}^1 is the maximum proper quotient of C_n^1. It is easy to show that each maximal proper subsemigroup of C_n^1 is isomorphic to either C_n, $(\mathbb{Z}_{p^n} \times L)^1$ or $(\mathbb{Z}_{p^n} \times R)^1$. Hence

$$\begin{aligned}\mathbf{V} &= V\left(C_{n-1}^1, C_n, (\mathbb{Z}_{p^n} \times L)^1, (\mathbb{Z}_{p^n} \times R)^1\right)\\ &= V\left(C_{n-1}^1, C_n, \mathbb{Z}_{p^n}^1, L^1, R^1\right)\\ &= V\left(C_{n-1}^1, C_n\right),\end{aligned}$$

where the last equality holds because letting $L = \{e, f\}$, we have

$$\mathbb{Z}_{p^n}^1 \cong \left\{\left(1_{\mathbb{Z}_{p^n}}, 1\right)\right\} \cup \left(\mathbb{Z}_{p^n} \times \{e\}\right) \leq C_n \times L^1.$$

■

Lemma 3.6.8 *Let S be a completely regular semigroup. Then*

$$V(S) \cap \mathbf{CS} = V(T \leq S \mid T \in \mathbf{CS}).$$

Proof. This follows from ([14], Theorem IX.9.1). ■

Proposition 3.6.9 *The monoid C_n^1 is critical for $n \geq 1$.*

Proof. Suppose that C_n^1 is not critical. Then $C_n^1 \in V\left(C_n, C_{n-1}^1\right)$ by Lemma 3.6.7. Hence there exist a completely simple semigroup A, a monoid B and a homomorphism ψ from a subsemigroup T of $A \times B$ onto C_n^1. By Lemma 2.2.2 there exists $e \in E(T)$ such that $\psi e = 1$, whence $M = eTe$ is a monoid with identity e and $\psi M = C_n^1$. Suppose that $e = (u, *)$. Then $e \geq (a, b)$ for any other idempotent (a, b) of M, whence $u \geq a$. But u and a

are idempotents of A so that $u = a$ since all idempotents in a completely simple semigroup are primitive. Hence

$$E(M) \subseteq \{(u, b) \mid b \in E(B)\},$$

from which it follows that $\pi_1 M$ is a subgroup of $A \in V(C_n)$. Therefore C_n^1 divides $\pi_1 M \times B$ so that $C_n^1 \in V(\mathbb{Z}_{p^n}, C_{n-1}^1)$. Now by Lemma 3.6.8,

$$\begin{aligned} V(C_n) &= V(C_n^1) \cap \mathbf{CS} \\ &= V(\mathbb{Z}_{p^n}, C_{n-1}^1) \cap \mathbf{CS} \\ &= V(\mathbb{Z}_{p^n}, C_{n-1}), \end{aligned}$$

which is a contradiction because by Lemma 3.2.2 and Theorem 3.2.3,

$$\begin{aligned} \varkappa V(C_n) &= (*, V(\mathbb{Z}_{p^n}), *) \\ &\neq (*, V(\mathbb{Z}_{p^{n-1}}), *) \\ &= \varkappa V(\mathbb{Z}_{p^n}, C_{n-1}). \end{aligned}$$

■

Chapter 4

The Lattice of Subvarieties of $\mathbf{A_0}$

This chapter characterizes the lattice $\mathcal{L}(\mathbf{A_0})$ of subvarieties of $\mathbf{A_0}$. Sections 2 to 8 introduce some infinite classes of semigroups in $\mathbf{A_0}$; the varieties generated by these semigroups form the complete sublattice $\mathfrak{LNR}^*$ of $\mathcal{L}(\mathbf{A_0})$ (Corollary 4.8.8). Subvarieties of $\mathbf{A_0}$ that are not in $\mathfrak{LNR}^*$ are investigated in Sections 9 to 11; each of these subvarieties are the intersection of a variety in $\mathfrak{LNR}^*$ and a permutation variety (Propositions 4.9.2 and 4.9.5). It will also be shown that the subvarieties of $\mathbf{A_0}$ are all finitely based (Corollary 4.9.6). Sections 10 to 12 describe how intersections and joins of varieties in $\mathcal{L}(\mathbf{A_0})$ can be found. The chapter ends by showing that the non-finitely generated subvarieties of $\mathbf{A_0}$ constitute a subinterval in $\mathcal{L}(\mathbf{A_0})$ (Proposition 4.12.1).

4.1 Notation

Let X be a countably infinite alphabet. Our description of canonical forms will depend on an ordering of the elements of X; when such an ordering is required then we let $X = \{z_1, z_2, \ldots\}$. Throughout this chapter let small letters $a, b, \ldots, x, y, z$ be variables and bold letters $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{x}, \mathbf{y}, \mathbf{z}$ be words over X. Let $|\mathbf{w}|$ be the number of variables (counting multiplicity) in $\mathbf{w}$. If $\mathbf{u}, \mathbf{v} \in X^+$, then we write $\mathbf{u} \equiv \mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are identical words in X^+, and write $\mathbf{u} = \mathbf{v}$ to stand for a semigroup identity. The *head* (respectively, *tail*) of $\mathbf{u}$ is the first (respectively, last) variable to appear in $\mathbf{u}$ and is denoted by $\mathfrak{h}(\mathbf{u})$ (respectively, $\mathfrak{t}(\mathbf{u})$). The *content* of $\mathbf{u}$ is the set of variables that appear in $\mathbf{u}$ and is denoted by $\mathfrak{c}(\mathbf{u})$. An identity $\mathbf{u} = \mathbf{v}$ is *balanced* if $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. Recall that the variety defined by a set Σ of identities is denoted by $[\Sigma]$. For the two element semilattice Y, it is well known that

$Y \models \mathbf{u} = \mathbf{v}$ if and only if $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$.

If $\mathbf{U}, \mathbf{V}$ are varieties and Σ is a set of identities, then we write $\mathbf{U} \cap [\Sigma] = \mathbf{U}^{\Sigma} = [\Sigma]^{\mathbf{U}}$. If $\mathbf{U}$ is nontrivial and it has no proper nontrivial subvarieties, then $\mathbf{U}$ is an *atom.* If $\mathbf{V} \subset \mathbf{U}$ and there is no subvariety $\mathbf{W}$ such that $\mathbf{V} \subset \mathbf{W} \subset \mathbf{U}$, then $\mathbf{V}$ is an *anti-atom* of $\mathcal{L}(\mathbf{U})$.

The main semigroup under investigation is A_0. Recall that

$$\begin{aligned} A_0 &= \langle \mathfrak{e}, \mathfrak{f} \mid \mathfrak{e}^2 = \mathfrak{e}, \mathfrak{f}^2 = \mathfrak{f}, \mathfrak{f}\mathfrak{e} = 0 \rangle \\ &= \{0, \mathfrak{e}, \mathfrak{f}, \mathfrak{e}\mathfrak{f}\}. \end{aligned}$$

This chapter examines semigroups in the variety $\mathbf{A_0} = V(A_0)$ and the subvarieties they generate. The investigation of each semigroup in each section follows a similar approach. For each semigroup $S \in \mathbf{A_0}$ targeted for investigation, canonical forms for words in $V(S)$ will be defined; words that are in these canonical forms are called $V(S)$-*words.* We show that each word $\mathbf{u} \in X^+$ is equivalent in $V(S)$ to a (necessary unique) $V(S)$-word, which will be denoted by $\mathbf{u}^{V(S)}$.

4.2 The Variety $\mathbf{A_0}$

In [3], Edmunds established a finite basis for $\mathbf{A_0}$:

$$x^3 = x^2, \quad xyx = x^2yx = xy^2x = xyx^2 = xyxy = yxy. \tag{4.1}$$

He accomplished the task by showing each word is equivalant to one in canonical form, and that no two distinct canonical words constitute an identity of $\mathbf{A_0}$. By examining these canonical words, we are able to show (in the next section) that all balanced identities not satisfied by $\mathbf{A_0}$ have a common consequence which is also not satisfied by $\mathbf{A_0}$. This common consequence hence defines an anti-atom of $[\mathbf{Y}, \mathbf{A_0}]$, and it will be shown to be the only anti-atom of $\mathcal{L}(\mathbf{A_0})$ (Proposition 4.6.13), a result which is extremely important in the characterization of the lattice $\mathcal{L}(\mathbf{A_0})$.

We could begin immediately by working with the canonical words defined by Edmunds to find the anti-atom of $\mathcal{L}(\mathbf{A_0})$. But for the sake of completeness and clarity for the reader who refers to Edmunds's work [3], we present and elaborate completely on Edmunds's method to establish a basis for $\mathbf{A_0}$, although the basis we find will be slightly different than the one in (4.1).

Theorem 4.2.1 *The following identities are satisfied by A_0.*

$$\mathrm{I}: x^3 = x^2,$$
$$\mathrm{II}: xyx = yxy,$$
$$\mathrm{III}: x^2yx = xy^2x = xyx^2 = xyx.$$

PROOF. Verification of this theorem is straightforward. ■

Define

$$\mathcal{M} = \{z_{\sigma_1} \cdots z_{\sigma_k} \mid \sigma_1 < \cdots < \sigma_k, k \in \mathbb{N}\}.$$

Note that because of the ordering of the indices, if two words in $\mathcal{M}$ have the same content then they must be identical. The following simple observations will be useful in the sequel.

Corollary 4.2.2 *Let $a, x, y \in X$ and $b, c \in X \cup \{\emptyset\}$. Then:*

(1) $\{\mathrm{II}, \mathrm{III}\} \vdash abx^2ca = abxca$;

(2) $\{\mathrm{II}, \mathrm{III}\} \vdash abxyca = abyxca$;

(3) $\{\mathrm{II}, \mathrm{III}\} \vdash xbycx = ybxcy$;

(4) *Let $\mathbf{a} = \mathbf{b}$ be a balanced identity involving at least two distinct variables. If $\mathfrak{h}(\mathbf{a}) = \mathfrak{t}(\mathbf{a})$ and $\mathfrak{h}(\mathbf{b}) = \mathfrak{t}(\mathbf{b})$, then $A_0 \models \mathbf{a} = \mathbf{b}$.*

PROOF. (1) The following argument holds for $b, c \in X \cup \{\emptyset\}$:

$$\begin{aligned} a\left(bx^2c\right)a &= bx^2cabx^2c && \text{by II} \\ &= (bxc)\,a\,(bxc) && \text{by III} \\ &= abxca && \text{by II.} \end{aligned}$$

(2) The following argument holds for $b, c \in X \cup \{\emptyset\}$:

$$\begin{aligned} abxyca &= ab\,(xyx)\,yca && \text{by (1)} \\ &= ab\,(yxy)\,yca && \text{by II} \\ &= aby\,(yxy)\,ca && \text{by III} \\ &= abyxyxca && \text{by II} \\ &= abyxca && \text{by (1).} \end{aligned}$$

(3) Clearly (3) holds if $b, c = \emptyset$.

Case (i) Suppose $b, c \neq \emptyset$. Then

$$\begin{aligned} x\,(byc)\,x &= b\,(ycxby)\,c && \text{by II} \\ &= (byb)\,x\,(cyc) && \text{by (2)} \\ &= ybyxycy && \text{by II} \\ &= ybxcy && \text{by (2) and II.} \end{aligned}$$

Case (ii) Suppose exactly one of b, c is empty. By symmetry, it suffices to consider $b = \emptyset \neq c$. Then

$$\begin{aligned} xycx &= x\,(xyc)\,x && \text{by III} \\ &= y\,(xxc)\,y && \text{by Case (i)} \\ &= yxcy && \text{by III} \end{aligned}$$

(4) There exist $x, y \in X$ and $\mathbf{u}, \mathbf{v} \in X^+$ such that

$$\mathbf{a} \equiv x\mathbf{u}x, \qquad \mathbf{b} \equiv y\mathbf{v}y.$$

By (1), (2), (3) and III we may assume that $\mathbf{u}, \mathbf{v} \in \mathcal{M}$ with $x \notin \mathfrak{c}(\mathbf{u})$ and $y \notin \mathfrak{c}(\mathbf{v})$. There are two cases to consider: $x = y$ and $x \neq y$.

Case (i) If $x = y$ then $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ so that $\mathbf{u} \equiv \mathbf{v}$. Hence

$$\mathbf{a} = x\mathbf{u}x \equiv x\mathbf{v}x = \mathbf{b}.$$

Case (ii) If $x \neq y$ then we have $y \in \mathfrak{c}(\mathbf{u})$. Hence $\mathbf{u} = \mathbf{u}_0 y \mathbf{u}_1$ for some $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{M} \cup \{\emptyset\}$. Note that $y \notin \mathfrak{c}(\mathbf{u}_0\mathbf{u}_1)$. Now by (3),

$$\mathbf{a} \equiv x\mathbf{u}_0 y \mathbf{u}_1 x = y\mathbf{u}_0 x \mathbf{u}_1 y$$

and $\mathfrak{c}(\mathbf{u}_0 x \mathbf{u}_1) = \mathfrak{c}(\mathbf{a}) \setminus \{y\} = \mathfrak{c}(\mathbf{b}) \setminus \{y\} = \mathfrak{c}(\mathbf{v})$. We may then use (1), (2) and III to reduce $\mathbf{u}_0 x \mathbf{u}_1$ to a word in $\mathcal{M}$, which reduces this case to (i) and thus the result follows. ■

The following subsets of X^+ are required for the definition of canonical forms in $\mathbf{A_0}$:

$$\begin{aligned} \mathcal{P} &= \left\{ z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k} \mid \sigma_1, \ldots, \sigma_k \text{ are distinct, } \alpha_i \in \{1,2\},\ k \in \mathbb{N} \right\} \cup \{\emptyset\}, \\ \mathcal{A} &= \{ z_{\sigma_1} \cdots z_{\sigma_k} z_{\sigma_1} \mid \sigma_1 < \cdots < \sigma_k,\ k \geq 2 \}. \end{aligned}$$

A word $\mathbf{u} \in X^+$ will be called an $\mathbf{A_0}$-*word* if either $\mathbf{u} \in \mathcal{P}$ or

$$\mathbf{u} \equiv \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1 \cdots \mathbf{a}_k\mathbf{p}_k \tag{4.2}$$

where $\mathbf{p}_i \in \mathcal{P}$, $\mathbf{a}_i \in \mathcal{A}$, and $\mathfrak{c}(\mathbf{p}_0), \ldots, \mathfrak{c}(\mathbf{p}_k), \mathfrak{c}(\mathbf{a}_1), \ldots, \mathfrak{c}(\mathbf{a}_k)$ are pairwise disjoint. By definition each word in $\mathcal{P}$ or $\mathcal{A}$ is an $\mathbf{A_0}$-word, as is any word in $\mathcal{M}$ since $\mathcal{M} \subset \mathcal{P}$. For convenience, whenever we say an expression written in the form (4.2) is an $\mathbf{A_0}$-word, then it is understood that those conditions on $\mathbf{p}_i, \mathbf{a}_i$ will be assumed. Furthermore, $\mathbf{p}$ and $\mathbf{q}$ (possibly with subscripts) always denote elements from $\mathcal{P}$, while $\mathbf{a}$ and $\mathbf{b}$ (possibly with subscripts) denote elements from $\mathcal{A}$. Note that $\mathcal{P}$ contains $\emptyset$ but $\mathcal{A}$ does not.

Let $\mathbf{D} = [\mathrm{I}, \mathrm{II}, \mathrm{III}]$ in this section. Recall that for a variety $\mathbf{V}$, the fully invariant congruence on X^+ over $\mathbf{V}$ is denoted by $\equiv_{\mathbf{V}}$. By Theorem 4.2.1, it is clear that $\mathbf{A_0} \subseteq \mathbf{D}$, or equivalently, that $\equiv_{\mathbf{D}} \subseteq \equiv_{\mathbf{A_0}}$. We now proceed to show that each word in X^+ is $\equiv_{\mathbf{D}}$-related to a unique $\mathbf{A_0}$-word. It will then be shown that no two distinct $\mathbf{A_0}$-words are $\equiv_{\mathbf{A_0}}$-related, thus establishing that $\equiv_{\mathbf{D}} = \equiv_{\mathbf{A_0}}$ and $\mathbf{D} = \mathbf{A_0}$.

Let $\mathbf{u} \equiv x_1^{\alpha_1} \cdots x_k^{\alpha_k} \in X^+$ be such that $x_i \neq x_{i+1}$ and let $x \in X$. We say x *appears* n *times* in $\mathbf{u}$ if it shows up n times in the list $x_1, \ldots, x_k$. The *multiplicity* $\mathfrak{m}_{\mathbf{u}}(x)$ of x is the number of occurences of x in $\mathbf{u}$. For example, x appears three times in $\mathbf{v} \equiv x^2yxzx^3$ while $\mathfrak{m}_{\mathbf{v}}(x) = 6$.

Proposition 4.2.3 *Each word in X^+ is $\equiv_{\mathbf{D}}$-related to an $\mathbf{A_0}$-word.*

Proof. Let $\mathbf{u} \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_k} \in X^+$ be such that $x_i \neq x_{i+1}$ $(1 \leq i \leq n-1)$. We may assume by I that $\alpha_i \in \{1, 2\}$ for all i. If all x_i are distinct then $\mathbf{u} \in \mathcal{P}$. Otherwise there exists a variable $y \in \mathfrak{c}(\mathbf{u})$ that is the first (from the left) to appear twice in $\mathbf{u}$. Then

$$\mathbf{u} \equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \mathbf{v}_1 y^{\delta_2} \mathbf{w}_1 \tag{4.3}$$

for some $\mathbf{p}_0 \in \mathcal{P}$, $\mathbf{v}_1 \in X^+$, $\mathbf{w}_1 \in (X \backslash \{y\})^+ \cup \{\emptyset\}$ and $1 \leq \delta_1, \delta_2 \leq 2$ such that

$$\mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{v}_1) = \mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{w}_1) = \emptyset.$$

Note that $y \notin \mathfrak{c}(\mathbf{p}_0)$. We show that $\mathbf{u}$ is $\equiv_{\mathbf{D}}$-related to an expression of the same form as in (4.3) with the stronger condition that $\mathfrak{c}(\mathbf{p}_0)$, $\mathfrak{c}(\mathbf{v}_1)$ and $\mathfrak{c}(\mathbf{w}_1)$ are pairwise disjoint. We proceed by defining $\mathbf{v}_i, \mathbf{w}_i \in X^+$ inductively such that

$$\mathbf{u} \equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \mathbf{v}_i y^{\delta_2} \mathbf{w}_i$$

and

$$\mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{v}_i) = \mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{w}_i) = \emptyset.$$

If $\mathfrak{c}(\mathbf{v}_i) \cap \mathfrak{c}(\mathbf{w}_i) = \emptyset$ then $\mathbf{v}_{i+1}$ and $\mathbf{w}_{i+1}$ are not defined. Otherwise, there exists a variable z common to both $\mathbf{v}_i$ and $\mathbf{w}_i$. Moreover, this z can be chosen to be the last variable in $\mathbf{w}_i$ that is also in $\mathfrak{c}(\mathbf{v}_i)$, whence there exist $\mathbf{v}_i^{\mathrm{h}}, \mathbf{v}_i^{\mathrm{t}}, \mathbf{w}_i^{\mathrm{h}}, \mathbf{w}_i^{\mathrm{t}} \in X^+ \cup \{\emptyset\}$ such that

$$\mathbf{v}_i \equiv \mathbf{v}_i^{\mathrm{h}} z^{\epsilon_1} \mathbf{v}_i^{\mathrm{t}}, \qquad \mathbf{w}_i \equiv \mathbf{w}_i^{\mathrm{h}} z^{\epsilon_2} \mathbf{w}_i^{\mathrm{t}},$$

where $\epsilon_1, \epsilon_2 \in \{1, 2\}$ and $\mathfrak{c}(\mathbf{v}_i) \cap \mathfrak{c}(\mathbf{w}_i^{\mathrm{t}}) = \emptyset$. Then

$$\begin{aligned}
\mathbf{u} &\equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \mathbf{v}_i^{\mathrm{h}} \left(z^{\epsilon_1} \mathbf{v}_i^{\mathrm{t}} y^{\delta_2} \mathbf{w}_i^{\mathrm{h}} z^{\epsilon_2} \right) \mathbf{w}_i^{\mathrm{t}} \\
&\equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \mathbf{v}_i^{\mathrm{h}} \left(z \mathbf{v}_i^{\mathrm{t}} y^{\delta_2} \mathbf{w}_i^{\mathrm{h}} z \right) \mathbf{w}_i^{\mathrm{t}} && \text{by III} \\
&\equiv_{\mathbf{D}} \mathbf{p}_0 \left(y^{\delta_1} \mathbf{v}_i^{\mathrm{h}} y^{\delta_2} \mathbf{v}_i^{\mathrm{t}} z \mathbf{w}_i^{\mathrm{h}} y^{\delta_2} \right) \mathbf{w}_i^{\mathrm{t}} && \text{by Corollary 4.2.2(3)} \\
&\equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \left(\mathbf{v}_i^{\mathrm{h}} z^{\epsilon_1} \mathbf{v}_i^{\mathrm{t}} \right) \mathbf{w}_i^{\mathrm{h}} y^{\delta_2} \mathbf{w}_i^{\mathrm{t}} && \text{by Corollary 4.2.2(4)} \\
&\equiv \mathbf{p}_0 y^{\delta_1} \left(\mathbf{v}_i \mathbf{w}_i^{\mathrm{h}} \right) y^{\delta_2} \mathbf{w}_i^{\mathrm{t}},
\end{aligned}$$

Define

$$\mathbf{v}_{i+1} \equiv \mathbf{v}_i \mathbf{w}_i^{\mathrm{h}}, \qquad \mathbf{w}_{i+1} \equiv \mathbf{w}_i^{\mathrm{t}}.$$

Since $\mathbf{v}_{i+1}$ and $\mathbf{w}_{i+1}$ are subwords of $\mathbf{v}_i \mathbf{w}_i$ and that $\mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{v}_i \mathbf{w}_i) = \emptyset$, we have

$$\mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{v}_{i+1}) = \mathfrak{c}(\mathbf{p}_0) \cap \mathfrak{c}(\mathbf{w}_{i+1}) = \emptyset.$$

Now whenever $\mathbf{w}_i$ and $\mathbf{w}_{i+1}$ are defined, then $|\mathbf{w}_{i+1}| < |\mathbf{w}_i|$ because $\mathbf{w}_{i+1}$ is a proper subword of $\mathbf{w}_i$. Hence there must exists k such that $\mathbf{w}_k$ is defined but $\mathbf{w}_{k+1}$ is not. Therefore we have

$$\mathbf{u} \equiv_{\mathbf{D}} \mathbf{p}_0 y^{\delta_1} \mathbf{v}_k y^{\delta_2} \mathbf{w}_k$$

such that $\mathfrak{c}(\mathbf{p}_0), \mathfrak{c}(\mathbf{v}_k)$ and $\mathfrak{c}(\mathbf{w}_k)$ are pairwise disjoint. If $\mathfrak{c}(y\mathbf{v}_k) = \{z_{i_1}, \ldots, z_{i_m}\}$ with $i_1 < \cdots < i_m$, then by Corollary 4.2.2(4), $y^{\delta_1} \mathbf{v}_k y^{\delta_2} \equiv_{\mathbf{D}} z_{i_1} \cdots z_{i_m} z_{i_1} \in \mathcal{A}$. Let $\mathbf{a}_1 \equiv z_{i_1} \cdots z_{i_m} z_{i_1}$ and $\mathbf{u}_1 \equiv \mathbf{w}_k$. Then

$$\mathbf{u} \equiv_{\mathbf{D}} \mathbf{p}_0 \mathbf{a}_1 \mathbf{u}_1$$

with $\mathfrak{c}(\mathbf{p}_0), \mathfrak{c}(\mathbf{a}_1)$ and $\mathfrak{c}(\mathbf{u}_1)$ pairwise disjoint. If $\mathbf{u}_1 \neq \emptyset$ then repeat the procedure from the beginning of this proof on the word $\mathbf{u}_1$ and we will have the desired result. ■

Note that the proof of Proposition 4.2.3 is an algorithm that outputs an $\mathbf{A_0}$-word for each word in X^+; let $\mathbf{u}^{\mathbf{A_0}}$ denote the $\mathbf{A_0}$-word that is obtained from $\mathbf{u}$ by this algorithm. Some immediate consequences of Proposition 4.2.3 are included in the following corollary.

Corollary 4.2.4 *Let* $\mathbf{u}$ *be an* $\mathbf{A_0}$*-word and* $\mathbf{h},\mathbf{t} \in X^+ \cup \{\emptyset\}$. *Then:*

(1) *If* $x \in \mathfrak{c}(\mathbf{u})$ *then* $\mathfrak{m}_{\mathbf{u}}(x) \leq 2$;

(2) *If* $\mathbf{u} \equiv \mathbf{h}x\mathbf{v}x\mathbf{t}$ *then* $x\mathbf{v} \in \mathcal{M}$;

(3) *If* $\mathbf{u} \equiv \mathbf{h}x\mathbf{v}x\mathbf{t}$ *then* $\mathfrak{c}(\mathbf{h})$, $\mathfrak{c}(\mathbf{v})$ *and* $\mathfrak{c}(\mathbf{t})$ *are pairwise disjoint.*

The next goal (achieved in Theorem 4.2.10) is to show that no two distinct $\mathbf{A_0}$-words are $\equiv_{\mathbf{A_0}}$-related. Before proceeding, note that by II and I respectively,

$$x^2\left(y^2x^2y^2\right) = \left(x^2x^2\right)y^2x^2 = x^2y^2x^2.$$

Hence $\{\mathrm{I},\mathrm{II}\}$ implies (and A_0 satisfies) the identity

$$\mathrm{IV}: x^2y^2x^2y^2 = x^2y^2x^2.$$

But since $\mathfrak{e}^2\mathfrak{f}^2\mathfrak{e}^2 \neq \mathfrak{e}^2\mathfrak{f}^2 \neq \mathfrak{e}^2\mathfrak{f}^2\mathfrak{e}^2\mathfrak{f}^2$, we have

$$A_0 \nvDash x^2y^2x^2 = x^2y^2,\ x^2y^2x^2y^2 = x^2y^2. \tag{4.4}$$

In the next few results, we investigate what properties two $\mathbf{A_0}$-words $\mathbf{u},\mathbf{v}$ must possess if they are to form an identity of $\mathbf{A_0}$. Of course, we will eventually show that they must be identical. There are many ways in which $\mathbf{u},\mathbf{v}$ can be distinct: $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$, $\mathfrak{m}_{\mathbf{u}}(x) \neq \mathfrak{m}_{\mathbf{v}}(x)$, etc. But note that in each of these cases when $\mathbf{u},\mathbf{v}$ are distinct, we will show that $\{\mathrm{I},\mathrm{II},\mathrm{III},\mathbf{u}=\mathbf{v}\}$ implies one or both identities in (4.4). This fact is crucial in the following sections for finding the anti-atom of $\mathcal{L}(\mathbf{A_0})$.

Lemma 4.2.5 *If* $\mathbf{u},\mathbf{v}$ *are* $\mathbf{A_0}$*-words such that* $A_0 \models \mathbf{u}=\mathbf{v}$, *then* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v}), \mathfrak{h}(\mathbf{u}) = \mathfrak{h}(\mathbf{v})$ *and* $\mathfrak{t}(\mathbf{u}) = \mathfrak{t}(\mathbf{v})$.

Proof. Since $Y \in \mathbf{A_0} \models \mathbf{u}=\mathbf{v}$, the identity $\mathbf{u}=\mathbf{v}$ is satisfied by Y so that $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. If $\mathfrak{c}(\mathbf{u}) = \{x\}$ then $\mathbf{u},\mathbf{v} \in \{x,x^2\}$. Since A_0 is not a band it does not satisfy $x^2=x$, whence $\mathbf{u}$ and $\mathbf{v}$ are identical. Therefore we may assume $|\mathfrak{c}(\mathbf{u})| \geq 2$. It suffices just to prove $\mathfrak{h}(\mathbf{u}) = \mathfrak{h}(\mathbf{v})$ because to prove $\mathfrak{t}(\mathbf{u}) = \mathfrak{t}(\mathbf{v})$ is symmetrical. Seeking a contradiction, suppose $\mathfrak{h}(\mathbf{u}) \neq \mathfrak{h}(\mathbf{v})$, whence

$$\mathbf{u} \in \{\mathbf{p}_0, \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1\cdots\mathbf{a}_k\mathbf{p}_k\}, \qquad \mathbf{v} \in \{\mathbf{q}_0, \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1\cdots\mathbf{b}_l\mathbf{q}_l\},$$

with $\mathfrak{h}(\mathbf{u}) = z_r$, $\mathfrak{h}(\mathbf{v}) = z_s$. Without loss of generality assume $r > s$. Let S denote the following substitution into X^+:

$$w \longrightarrow \begin{cases} x^2 & \text{if } w = z_r, \\ x^2 & \text{if } \mathbf{p}_0 = \emptyset \text{ and } w \in \mathfrak{c}(\mathbf{a}_1), \\ y^2 & \text{otherwise.} \end{cases}$$

Note that if $\mathbf{p}_0 = \emptyset$, then $z_r = \mathfrak{h}(\mathbf{u}) = \mathfrak{h}(\mathbf{a}_1)$. Since $s < r$, this means that $z_s \notin \mathfrak{c}(\mathbf{a}_1)$. In any case, $\mathbf{u}(\mathrm{S}) \equiv_{\mathbf{D}} x^2y^2$ by I. Note that $z_s = y^2$ under S, whence $\mathbf{v}(\mathrm{S})\, y^2 \equiv_{\mathbf{D}} y^2x^2y^2$ by I. Therefore A_0 satisfies

$$\begin{aligned} x^2y^2x^2 = y^2x^2y^2 & \qquad \text{by II} \\ = \mathbf{v}(\mathrm{S})\, y^2 = \mathbf{u}(\mathrm{S})\, y^2 & \\ = x^2y^2 & \qquad \text{by I,} \end{aligned}$$

which contradicts (4.4). ■

Lemma 4.2.6 *Let*

$$\mathbf{u} \equiv \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1 \cdots \mathbf{a}_k\mathbf{p}_k, \qquad \mathbf{v} \equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1 \cdots \mathbf{b}_l\mathbf{q}_l$$

be $\mathbf{A_0}$*-words. If* $A_0 \models \mathbf{u} = \mathbf{v}$*, then:*

(1) $\mathbf{p}_0 = \emptyset$ *if and only if* $\mathbf{q}_0 = \emptyset$;

(2) $\mathbf{p}_k = \emptyset$ *if and only if* $\mathbf{q}_l = \emptyset$.

Proof. By symmetry it suffices just to prove (1). Suppose $\mathbf{p}_0 = \emptyset$ and $\mathbf{q}_0 \neq \emptyset$. Then

$$\mathbf{u} \equiv \mathbf{a}_1\mathbf{p}_1 \cdots \mathbf{a}_k\mathbf{p}_k, \qquad \mathbf{v} \equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1 \cdots \mathbf{b}_l\mathbf{q}_l.$$

By Lemma 4.2.5, we have $\mathfrak{h}(\mathbf{a}_1) = \mathfrak{h}(\mathbf{q}_0)$. Let S denote the the following substitution into X^+:

$$w \longrightarrow \begin{cases} x^2 & \text{if } w = \mathfrak{h}(\mathbf{a}_1), \\ y^2 & \text{otherwise.} \end{cases}$$

Then $\mathbf{u}(\mathrm{S}) \equiv_{\mathbf{D}} x^2y^2x^2$ by I and IV, and $\mathbf{v}(\mathrm{S}) \equiv_{\mathbf{D}} x^2y^2$ by I. Now

$$A_0 \models \mathbf{u}(\mathrm{S}) = \mathbf{v}(\mathrm{S}) \vdash x^2y^2x^2 = x^2y^2$$

contradicts (4.4). ■

Lemma 4.2.7 *Let*

$$\mathbf{u} \equiv \mathbf{hpat}_1, \qquad \mathbf{v} \equiv \mathbf{hqbt}_2,$$

where

(1) $\mathbf{h}, \mathbf{t}_1, \mathbf{t}_2 \in X^+ \cup \{\emptyset\}$;

(2) $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, $\mathbf{a}, \mathbf{b} \in \mathcal{A}$;

(3) $\mathfrak{c}(\mathbf{h}), \mathfrak{c}(\mathbf{p}), \mathfrak{c}(\mathbf{a}), \mathfrak{c}(\mathbf{t}_1)$ *are pairwise disjoint;*

(4) $\mathfrak{c}(\mathbf{h}), \mathfrak{c}(\mathbf{q}), \mathfrak{c}(\mathbf{b}), \mathfrak{c}(\mathbf{t}_2)$ *are pairwise disjoint.*

If $A_0 \models \mathbf{u} = \mathbf{v}$ *then* $\mathbf{p} \equiv \mathbf{q}$.

PROOF. By assumption

$$\mathbf{p} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}, \qquad \mathbf{q} \equiv z_{\tau_1}^{\beta_1} \cdots z_{\tau_l}^{\beta_l},$$

where $i \neq j$ implies $\sigma_i \neq \sigma_j$ and $\tau_i \neq \tau_j$. Without loss of generality assume $k \leq l$. For convenience let

$$\mathbf{p}_{i,j} \equiv z_{\sigma_i}^{\alpha_i} \cdots z_{\sigma_j}^{\alpha_j}, \qquad \mathbf{q}_{i,j} \equiv z_{\tau_i}^{\beta_i} \cdots z_{\tau_j}^{\beta_j}$$

for $i \leq j$. Suppose $\mathbf{p}_{1,r} \equiv \mathbf{q}_{1,r}$ for some $r < k$. If $\sigma_{r+1} \neq \tau_{r+1}$, then let S be the following substitution into X^+:

$$w \longrightarrow \begin{cases} x^2 & \text{if } w \in \mathfrak{c}\left(\mathbf{hp}_{1,r+1}\right), \\ y^2 & \text{otherwise.} \end{cases}$$

Note that we may assume $\mathbf{h} \neq \emptyset$ because otherwise we can always premultiply both $\mathbf{u}$ and $\mathbf{v}$ by any word $\mathbf{h}$ such that $\mathfrak{c}(\mathbf{h}) \cap \mathfrak{c}(\mathbf{uv}) = \emptyset$. Furthermore, since $z_{\tau_{r+1}} \notin \mathfrak{c}\left(\mathbf{hq}_{1,r}\right) = \mathfrak{c}\left(\mathbf{hp}_{1,r}\right)$ and $z_{\tau_{r+1}} \neq z_{\sigma_{r+1}}$, we have $z_{\tau_{r+1}} \notin \mathfrak{c}\left(\mathbf{hp}_{1,r+1}\right)$ so that $z_{\tau_{r+1}} \longrightarrow y^2$ under S. Since $z_{\sigma_{r+1}} \in \mathfrak{c}\left(\mathbf{q}_{r+2,l}\mathbf{bt}_2\right)$, the word $\left(\mathbf{q}_{r+2,l}\mathbf{bt}_2\right)(\mathrm{S})$ contains x^2 as a subword. Hence A_0 satisfies

$$\begin{aligned} x^2y^2 &= \mathbf{u}(\mathrm{S}) && \text{by I} \\ &= \mathbf{v}(\mathrm{S}) \\ &= \left(\mathbf{hq}_{1,r}\right)(\mathrm{S})\, z_{\tau_{r+1}}(\mathrm{S})\left(\mathbf{q}_{r+2,l}\mathbf{bt}_2\right)(\mathrm{S}) \\ &= x^2 \cdots y^2 \cdots x^2 \cdots \\ &= x^2y^2x^2 && \text{by I and IV,} \end{aligned}$$

which contradicts (4.4). Therefore $\sigma_{r+1} = \tau_{r+1}$. Next, letting T be the substitution

$$w \longrightarrow \begin{cases} x^2 & \text{if } w \in \mathfrak{c}\left(\mathbf{hp}_{1,r}\right), \\ xy & \text{if } w = z_{\sigma_{r+1}}, \\ y^2 & \text{otherwise.} \end{cases}$$

into X^+, we have

$$x^2 (xy)^{\alpha_{r+1}} y^2 = \mathbf{u}(\mathrm{T}) = \mathbf{v}(\mathrm{T}) = x^2 (xy)^{\beta_{r+1}} y^2$$

by I. If $\alpha_{r+1} \neq \beta_{r+1}$ then the above identity implies $x^2xyy^2 = x^2 (xy)^2 y^2$ so that A_0 satisfies

$$\begin{aligned} x^2y^2 &= x^2xyy^2 && \text{by I} \\ &= x^2xyxyy^2 \\ &= x^2yxy^2 && \text{by I} \\ &= x^2y^2x^2y^2 && \text{by III,} \end{aligned}$$

contradicting (4.4). Therefore $\alpha_{r+1} = \beta_{r+1}$, whence $z_{\sigma_{r+1}}^{\alpha_{r+1}} \equiv z_{\tau_{r+1}}^{\beta_{r+1}}$ and $\mathbf{p}_{1,r+1} \equiv \mathbf{q}_{1,r+1}$. By induction, $\mathbf{p}_{1,k} \equiv \mathbf{q}_{1,k}$.

It remains to show that $k = l$. Suppose $k < l$. Then since $\mathbf{a} \equiv z\mathbf{a}_1 z$ for some $z \in X$ and $\mathbf{a}_1 \in X^+$, we have

$$\mathbf{u} \equiv \mathbf{hp}z\mathbf{a}_1 z\mathbf{t}_1, \qquad \mathbf{v} \equiv \mathbf{hpq}_{k+1,l}\mathbf{bt}_2.$$

Let U denote the following substitution in A_0:

$$w \longrightarrow \begin{cases} x^2 & \text{if } w \in \mathfrak{c}\left(\mathbf{hp}z_{\tau_{k+1}}\right), \\ y^2 & \text{otherwise.} \end{cases}$$

Then $z \longrightarrow y^2$ under U. Also, since $z_{\tau_{k+1}} \in \mathfrak{c}(\mathbf{a}_1\mathbf{t}_1)$, one of $\mathbf{a}_1(\mathrm{U}), \mathbf{t}_1(\mathrm{U})$ contains x^2 as a subword. Hence A_0 satisfies

$$\begin{aligned} x^2y^2 &= \mathbf{v}(\mathrm{U}) && \text{by I} \\ &= \mathbf{u}(\mathrm{U}) \\ &= (\mathbf{hp}z\mathbf{a}_1 z\mathbf{t}_1)(\mathrm{U}) \\ &= x^2 \cdots y^2\mathbf{a}_1(\mathrm{U})\, y^2\mathbf{t}_1(\mathrm{U}) \\ &= x^2y^2x^2 && \text{by I and IV,} \end{aligned}$$

contradicting (4.4). Consequently $k \not< l$ as required. ■

Lemma 4.2.8 *Let*

$$\mathbf{u} \equiv \mathbf{hat}_1, \qquad \mathbf{v} \equiv \mathbf{hbt}_2,$$

where

(1) $\mathbf{h}, \mathbf{t}_1, \mathbf{t}_2 \in X^+ \cup \{\emptyset\}$;

(2) $\mathbf{a}, \mathbf{b} \in \mathcal{A}$;

(3) $\mathfrak{c}(\mathbf{h}), \mathfrak{c}(\mathbf{a}), \mathfrak{c}(\mathbf{t}_1)$ *are pairwise disjoint;*

(4) $\mathfrak{c}(\mathbf{h}), \mathfrak{c}(\mathbf{b}), \mathfrak{c}(\mathbf{t}_2)$ *are pairwise disjoint.*

If $A_0 \models \mathbf{u} = \mathbf{v}$ *then* $\mathbf{a} \equiv \mathbf{b}$.

PROOF. By assumption

$$\mathbf{a} \equiv z_k \mathbf{a}_1 z_k, \qquad \mathbf{b} \equiv z_l \mathbf{b}_1 z_l$$

for some $z_k, z_l \in X$. Since $\mathbf{a}, \mathbf{b}$ are $\mathbf{A}_0$-words, $z_k \mathbf{a}_1, z_l \mathbf{b}_1 \in \mathcal{M}$ by Corollary 4.2.4(2). Without loss of generality we may assume $k \leq l$. Then

$$\mathbf{u} \equiv \mathbf{h} z_k \mathbf{a}_1 z_k \mathbf{t}_1, \qquad \mathbf{v} \equiv \mathbf{h} z_l \mathbf{b}_1 z_l \mathbf{t}_2.$$

Using a similar argument as in the proof of Lemma 4.2.7, we may assume $\mathbf{h} \neq \emptyset$. Suppose that $k < l$. Let $\mathfrak{c}(\mathbf{b}_1) = \{z_{i_1}, \ldots, z_{i_r}\}$. Then by Corollary 4.2.4(2) we have $l < \min\{i_1, \ldots, i_r\}$. Hence $z_k \notin \mathfrak{c}(\mathbf{b}_1)$. Since $z_k \notin \mathfrak{c}(\mathbf{h})$ by assumption, it follows that $z_k \notin \mathfrak{c}(\mathbf{h} z_l \mathbf{b}_1)$. However $z_l \in \mathfrak{c}(\mathbf{a}_1 \mathbf{t}_1)$. Let S denote the following substitution in X^+:

$$w \longrightarrow \begin{cases} x^2 & \text{if } w \in \mathfrak{c}(\mathbf{h} z_l \mathbf{b}_1), \\ y^2 & \text{otherwise.} \end{cases}$$

Note that since $z_l \in \mathfrak{c}(\mathbf{a}_1 \mathbf{t}_1)$, at least one of $\mathbf{a}_1(\mathrm{S}), \mathbf{t}_1(\mathrm{S})$ must contain x^2 as subword. Therefore $\mathbf{u}(\mathrm{S}) = x^2 y^2 \mathbf{a}_1(\mathrm{S}) y^2 \mathbf{t}_1(\mathrm{S})$ is a product of x^2 and y^2 which must contain the subword $x^2 y^2 x^2$, whence A_0 satisfies

$$\begin{aligned} x^2 y^2 &= \mathbf{v}(\mathrm{S}) && \text{by I} \\ &= \mathbf{u}(\mathrm{S}) \\ &= \cdots x^2 y^2 x^2 \cdots \\ &= x^2 y^2 x^2 && \text{by I and IV,} \end{aligned}$$

contradicting (4.4). Therefore $k = l$ and

$$\mathbf{u} \equiv \mathbf{h} z_k \mathbf{a}_1 z_k \mathbf{t}_1, \qquad \mathbf{v} \equiv \mathbf{h} z_k \mathbf{b}_1 z_k \mathbf{t}_2.$$

Now let $w \in \mathfrak{c}(\mathbf{a}_1)$ and suppose $w \notin \mathfrak{c}(\mathbf{b}_1)$. Then $w \in \mathfrak{c}(\mathbf{t}_2)$ and so $w \longrightarrow y^2$ under S, whence A_0 satisfies

$$\begin{aligned} x^2y^2 &= \mathbf{v}\,(\mathrm{S}) && \text{by I} \\ &= \mathbf{u}\,(\mathrm{S}) \\ &= \cdots x^2 \cdots y^2 \cdots x^2 \cdots \\ &= x^2y^2x^2 && \text{by I and IV,} \end{aligned}$$

again a contradiction. Therefore $w \in \mathfrak{c}\,(\mathbf{b}_1)$ and $\mathfrak{c}\,(\mathbf{a}_1) \subseteq \mathfrak{c}\,(\mathbf{b}_1)$. To show that $\mathfrak{c}\,(\mathbf{b}_1) \subseteq \mathfrak{c}\,(\mathbf{a}_1)$ is symmetrical so that $\mathfrak{c}\,(\mathbf{a}_1) = \mathfrak{c}\,(\mathbf{b}_1)$. Consequently $z_k = z_l$ and

$$\mathfrak{c}\,(\mathbf{a}) = \mathfrak{c}\,(z_k\mathbf{a}_1z_k) = \mathfrak{c}\,(z_l\mathbf{b}_1z_l) = \mathfrak{c}\,(\mathbf{b})\,,$$

which implies $\mathbf{a} \equiv \mathbf{b}$ because $\mathbf{a}, \mathbf{b} \in \mathcal{A}$. ■

Corollary 4.2.9 *Let* $\mathbf{u}, \mathbf{v}$ *be* $\mathbf{A_0}$*-words such that* $A_0 \models \mathbf{u} = \mathbf{v}$. *Then*

(1) $\mathbf{u} \in \mathcal{P}$ *if and only if* $\mathbf{v} \in \mathcal{P}$;

(2) $\mathbf{u} \in \mathcal{A}$ *if and only if* $\mathbf{v} \in \mathcal{A}$.

Furthermore, $\mathbf{u} \equiv \mathbf{v}$ *in both cases.*

PROOF. (1) Suppose $\mathbf{u} \in \mathcal{P}$ and $\mathbf{v} \notin \mathcal{P}$. Then

$$\mathbf{u} \equiv \mathbf{p}, \qquad \mathbf{v} \equiv \mathbf{qbt}$$

for some $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, $\mathbf{b} \in \mathcal{A}$ and $\mathbf{t} \in X^+ \cup \{\emptyset\}$ such that $\mathfrak{c}\,(\mathbf{q}), \mathfrak{c}\,(\mathbf{b})$ and $\mathfrak{c}\,(\mathbf{t})$ are pairwise disjoint. Choose $\mathbf{a} \in \mathcal{A}$ such that $\mathfrak{c}\,(\mathbf{a}) \cap \mathfrak{c}\,(\mathbf{uv}) = \emptyset$. Then $A_0 \models \mathbf{pa} = \mathbf{qbta}$, which by Lemma 4.2.7 implies $\mathbf{p} \equiv \mathbf{q}$. But now the contents of $\mathbf{u}$ and $\mathbf{v}$ are distinct, contrary to Lemma 4.2.5. Consequently $\mathbf{u} \in \mathcal{P}$ implies $\mathbf{v} \in \mathcal{P}$.

By symmetry $\mathbf{v} \in \mathcal{P}$ also implies $\mathbf{u} \in \mathcal{P}$. Now choose $\mathbf{a} \in \mathcal{A}$ such that $\mathfrak{c}\,(\mathbf{a}) \cap \mathfrak{c}\,(\mathbf{uv}) = \emptyset$. Then $A_0 \models \mathbf{ua} = \mathbf{va}$ implies $\mathbf{u} \equiv \mathbf{v}$ by Lemma 4.2.7.

(2) Suppose $\mathbf{u} \in \mathcal{A}$ and $\mathbf{v} \notin \mathcal{A}$. Then by Lemma 4.2.6, $\mathbf{v}$ begins and ends with a word from $\mathcal{A}$. Hence

$$\mathbf{u} \equiv \mathbf{a}, \qquad \mathbf{v} \equiv \mathbf{b}_0\mathbf{w}\mathbf{b}_1$$

for some $\mathbf{a}, \mathbf{b}_0, \mathbf{b}_1 \in \mathcal{A}$ and $\mathbf{w} \in X^+ \cup \{\emptyset\}$ such that $\mathfrak{c}\,(\mathbf{b}_0), \mathfrak{c}\,(\mathbf{w})$ and $\mathfrak{c}\,(\mathbf{b}_1)$ are pairwise disjoint. Then $\mathbf{a} \equiv \mathbf{b}_0$ by Lemma 4.2.8. Since $\mathfrak{c}\,(\mathbf{u}) = \mathfrak{c}\,(\mathbf{v})$ by Lemma 4.2.5, $\mathbf{b}_1$ is contradictorily empty. Therefore $\mathbf{u} \in \mathcal{A}$ implies $\mathbf{v} \in \mathcal{A}$.

By symmetry $\mathbf{v} \in \mathcal{A}$ also implies $\mathbf{u} \in \mathcal{A}$. Now $\mathbf{u} \equiv \mathbf{v}$ by Lemma 4.2.8. ■

Theorem 4.2.10 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{A_0}$*-words such that* $A_0 \models \mathbf{u} = \mathbf{v}$*, then* $\mathbf{u} \equiv \mathbf{v}$.

PROOF. The cases when $\mathbf{u}, \mathbf{v} \in \mathcal{P}$ or $\mathbf{u}, \mathbf{v} \in \mathcal{A}$ are covered in Corollary 4.2.9. So suppose that

$$\mathbf{u} \equiv \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1 \cdots \mathbf{a}_k\mathbf{p}_k, \qquad \mathbf{v} \equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1 \cdots \mathbf{b}_l\mathbf{q}_l$$

are $\mathbf{A_0}$-words that are not in $\mathcal{P} \cup \mathcal{A}$. Without loss of generality assume $k \leq l$. Choose $\mathbf{a} \in \mathcal{A}$ such that $\mathfrak{c}(\mathbf{a}) \cap \mathfrak{c}(\mathbf{uv}) = \emptyset$. Then $A_0 \models \mathbf{ua} = \mathbf{va}$. By Lemma 4.2.7 and Lemma 4.2.8 alternately, we deduce successively that

$$\begin{aligned} \mathbf{p}_0 &\equiv \mathbf{q}_0, \\ \mathbf{p}_0\mathbf{a}_1 &\equiv \mathbf{q}_0\mathbf{b}_1, \\ \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1 &\equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1, \\ &\vdots \\ \mathbf{p}_0\mathbf{a}_1\mathbf{p}_1 \cdots \mathbf{a}_k\mathbf{p}_k &\equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1 \cdots \mathbf{b}_k\mathbf{q}_k, \end{aligned}$$

that is, $\mathbf{u} \equiv \mathbf{q}_0\mathbf{b}_1\mathbf{q}_1 \cdots \mathbf{b}_k\mathbf{q}_k$. Since $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ by Lemma 4.2.5, we must have $k = l$, whence $\mathbf{u} \equiv \mathbf{v}$. ■

Theorem 4.2.11 *Let* $\mathbf{u}, \mathbf{v} \in X^+$.

(1) $\mathbf{A_0} \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{A_0}} \equiv \mathbf{v}^{\mathbf{A_0}}$;

(2) *The set of all* $\mathbf{A_0}$*-words constitutes the* $\mathbf{A_0}$*-free semigroup with the operation* $\cdot$ *given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{A_0}};$$

(3) $\mathbf{A}_0 = [\mathrm{I}, \mathrm{II}, \mathrm{III}]$.

PROOF. It suffices to prove that $\equiv_{\mathbf{D}} = \equiv_{\mathbf{A_0}}$. Suppose that $\mathbf{u} \equiv_{\mathbf{A_0}} \mathbf{v}$. Since $\mathbf{u} \equiv_{\mathbf{D}} \mathbf{u}^{\mathbf{A_0}}$ and $\mathbf{v} \equiv_{\mathbf{D}} \mathbf{v}^{\mathbf{A_0}}$ by Proposition 4.2.3, and since $\equiv_{\mathbf{D}} \subseteq \equiv_{\mathbf{A_0}}$, we have $\mathbf{u}^{\mathbf{A_0}} \equiv_{\mathbf{A_0}} \mathbf{v}^{\mathbf{A_0}}$. Hence $\mathbf{u}^{\mathbf{A_0}} \equiv \mathbf{v}^{\mathbf{A_0}}$ by Theorem 4.2.10, whence $\mathbf{u} \equiv_{\mathbf{D}} \mathbf{u}^{\mathbf{A_0}} \equiv \mathbf{v}^{\mathbf{A_0}} \equiv_{\mathbf{D}} \mathbf{v}$, that is, $\mathbf{u} \equiv_{\mathbf{D}} \mathbf{v}$. Therefore $\equiv_{\mathbf{A_0}} \subseteq \equiv_{\mathbf{D}}$ and by Proposition 4.2.3, we have equality. ■

4.3 The Variety $\mathbf{B_2^-}$

Recall that

$$B_2 = \mathcal{M}^0\left(\{1,2\}, \{1\}, \{1,2\}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

is a completely 0-simple semigroup with five elements. Letting $\mathfrak{a} = (1,1,2)$ and $\mathfrak{b} = (2,1,1)$ in B_2, this semigroup has the following presentation:

$$\begin{aligned} B_2 &= \langle \mathfrak{a}, \mathfrak{b} \mid \mathfrak{a}^2 = \mathfrak{b}^2 = 0, \mathfrak{aba} = \mathfrak{a}, \mathfrak{bab} = \mathfrak{b} \rangle \\ &= \{0, \mathfrak{a}, \mathfrak{b}, \mathfrak{ab}, \mathfrak{ba}\}. \end{aligned}$$

It is easy to show that $B_2^- = \{0, \mathfrak{a}, \mathfrak{ab}, \mathfrak{ba}\}$ is a subsemigroup of B_2. It is helpful to note that $\mathfrak{ab}, \mathfrak{ba}$ and 0 are the idempotents in B_2. Following the same approach employed to find a basis for $\mathbf{A_0}$, Edmunds [3] established a finite basis for the variety $\mathbf{B_2^-} = V(B_2^-)$:

$$x^3 = x^2, \quad xyx = x^2yx = xy^2x = xyx^2 = xyxy = x^2y^2 = yxy.$$

In this section, we show that all balanced identities not satisfied by $\mathbf{A_0}$ have a common consequence which is also not satisfied by $\mathbf{A_0}$ (Lemma 4.3.9). By investigating Edmunds's canonical words for $\mathbf{B_2^-}$, we also show that all balanced identities not satisfied by $\mathbf{B_2^-}$ have a common consequence which is not satisfied by $\mathbf{B_2^-}$ (Lemma 4.3.10). Similar to the case for $\mathbf{A_0}$, this common consequence not satisfied by $\mathbf{B_2^-}$ defines an anti-atom of $[\mathbf{Y}, \mathbf{B_2^-}]$, and it will eventually be shown to be the unique anti-atom of $\mathcal{L}(\mathbf{B_2^-})$ (Proposition 4.8.11).

Following the approach taken in the previous section, we first present and elaborate on Edmunds's method to establish a basis for $\mathbf{B_2^-}$.

Theorem 4.3.1 *The semigroup* B_2^- *satisfies the identities* I, II, III *and*

$$\mathrm{V} : x^2y^2 = y^2x^2.$$

Consequently, B_2^- *belongs to* $\mathbf{A_0}$.

PROOF. Verification of this theorem is straightforward. ■

Recall that

$$\mathcal{P} = \left\{z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k} \mid \sigma_1, \ldots, \sigma_k \text{ are distinct}, \ \alpha_i \in \{1,2\}, \ k \in \mathbb{N}\right\} \cup \{\emptyset\}.$$

A word $\mathbf{u} \in X^+$ will be called a $\mathbf{B_2^-}$*-word* if $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k} \in \mathcal{P}$ such that

$$\alpha_i = \alpha_{i+1} = 2 \implies \sigma_i < \sigma_{i+1}. \tag{4.5}$$

Whenever we say $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is a $\mathbf{B_2^-}$-word, then it will be understood that $\sigma_1, \ldots, \sigma_k$ are all distinct, $\alpha_i \in \{1, 2\}$, and the condition in (4.5) is satisfied.

If $\mathbf{a} = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ is a word such that $x_i \neq x_{i+1}$ and $\alpha_i \in \mathbb{N}$, then define $\overline{\mathbf{a}} = x_1^2 \cdots x_k^2$. For each $m \in \mathbb{N}$, define the identities

$$\mathrm{E}_m : (x_1^{\alpha_1} \cdots x_m^{\alpha_m})\, x_1^{\beta} = x_m^{\beta}\, (x_1^{\alpha_1} \cdots x_m^{\alpha_m}) = x_1^2 \cdots x_m^2$$

where $\alpha_1, \ldots, \alpha_m, \beta \in \mathbb{N}$. Throughout this chapter, let

$$\nabla = \{\mathrm{I}, \mathrm{II}, \mathrm{III}\ , \mathrm{V}\}\,. \tag{4.6}$$

and let $\equiv_\nabla$ be the fully invariant congruence over $[\nabla]$.

Lemma 4.3.2 *Let* $\mathbf{u}, \mathbf{v} \in X^+$ *and* $m \in \mathbb{N}$. *Then:*

(1) $\nabla \vdash \mathbf{u}^2 = \overline{\mathbf{u}}$;

(2) $\nabla \vdash \mathrm{E}_m$;

(3) $\nabla \vdash \mathbf{u}^2 = x_1^2 \cdots x_m^2$ *if* $\mathfrak{c}(\mathbf{u}) = \{x_1, \ldots, x_m\}$;

(4) $\nabla \vdash \mathbf{u}^2 = \mathbf{v}^2$ *if* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$.

PROOF. (1) We may assume $\mathbf{u} \equiv x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ where $\alpha_i \in \mathbb{N}$ and $x_i \in X$ with $x_i \neq x_{i+1}$. Then

$$\begin{aligned}
\mathbf{u}^2 &\equiv \left(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} x_1^{\alpha_1}\right) x_2^{\alpha_2} \cdots x_k^{\alpha_k} \\
&= x_1^2 x_2^2 \cdots \left(x_k^2 x_1^2 x_2^{\alpha_2} \cdots x_k^{\alpha_k}\right) && \text{by I, III and Corollary 4.2.2(1)} \\
&= x_1^2 x_2^2 \cdots x_k^2 x_1^2 x_2^2 \cdots x_k^2 && \text{by I, III and Corollary 4.2.2(1)} \\
&= \left(x_1^2 x_1^2\right)\left(x_2^2 x_2^2\right) \cdots \left(x_k^2 x_k^2\right) && \text{by V} \\
&= x_1^2 x_2^2 \cdots x_k^2 && \text{by I} \\
&\equiv \overline{\mathbf{u}}.
\end{aligned}$$

(2) Clearly $\nabla \vdash \mathrm{E}_1$, so suppose that $m \geq 2$. Then

$$\begin{aligned}
x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} x_1^{\beta} &= \left(x_1^2 x_2^2 \cdots x_m^2\right) x_1^2 && \text{by I, III and Corollary 4.2.2(1)} \\
&= \left(x_1^2 x_1^2\right) x_2^2 \cdots x_m^2 && \text{by V} \\
&= x_1^2 x_2^2 \cdots x_m^2 && \text{by I.}
\end{aligned}$$

To show $x_m^{\beta}(x_1^{\alpha_1}\cdots x_m^{\alpha_m}) = x_1^2\cdots x_m^2$ is similar. Therefore $\nabla \vdash \mathrm{E}_m$.

(3) This is a consequence of (1) and V.

(4) This is a consequence of (3). ■

For the rest of this thesis, all other varieties introduced will be subvarieties of $\mathbf{B}_2^-$. Lemma 4.3.2 will be used very frequently in their investigation.

Proposition 4.3.3 *Each word in X^+ is $\equiv_\nabla$-related to a $\mathbf{B}_2^-$-word.*

PROOF. By Lemma 4.3.2(2), it is easy to show that each word $\mathbf{u} \in X^+$ is $\equiv_\nabla$-related to a word $z_{\sigma_1}^{\alpha_1}\cdots z_{\sigma_k}^{\alpha_k}$ in $\mathcal{P}$. If $\alpha_i = \alpha_{i+1} = 2$ and $\sigma_i > \sigma_{i+1}$, then V can be invoked to replace the subword $z_{\sigma_i}^2 z_{\sigma_{i+1}}^2$ by $z_{\sigma_{i+1}}^2 z_{\sigma_i}^2$. Hence the condition in (4.5) can be achieved by V so that $\mathbf{u}$ is $\equiv_\nabla$-related to a $\mathbf{B}_2^-$-word. ■

The $\mathbf{B}_2^-$-word that is obtained from $\mathbf{u}$ by the method in the above proof is denoted by $\mathbf{u}^{\mathbf{B}_2^-}$. The following two lemmas are required to prove the uniqueness of each $\mathbf{B}_2^-$-word in $\mathbf{B}_2^-$.

Lemma 4.3.4 *Let $\mathbf{u}, \mathbf{v}$ be $\mathbf{B}_2^-$-words such that $\mathbf{B}_2^- \models \mathbf{u} = \mathbf{v}$. Then $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ and $\mathfrak{m}_{\mathbf{u}}(x) = \mathfrak{m}_{\mathbf{v}}(x)$ for all $x \in X$.*

PROOF. Clearly $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ since $Y \in \mathbf{B}_2^-$. By the definition of a $\mathbf{B}_2^-$-word $\mathfrak{m}_{\mathbf{u}}(x) = \mathfrak{m}_{\mathbf{v}}(x) \in \{0,1,2\}$. Clearly $\mathfrak{m}_{\mathbf{u}}(x) = 0$ if and only if $\mathfrak{m}_{\mathbf{v}}(x) = 0$. So suppose that $\mathfrak{m}_{\mathbf{u}}(x) = 1$ and $\mathfrak{m}_{\mathbf{v}}(x) = 2$. Then

$$\mathbf{u} \equiv \mathbf{u}_1 x \mathbf{u}_2, \qquad \mathbf{v} \equiv \mathbf{v}_1 x^2 \mathbf{v}_2$$

for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in X^+ \cup \{\emptyset\}$ such that $x \notin \mathfrak{c}(\mathbf{u}_1\mathbf{u}_2\mathbf{v}_1\mathbf{v}_2)$ and

$$\mathfrak{c}(\mathbf{u}_1) \cap \mathfrak{c}(\mathbf{u}_2) = \mathfrak{c}(\mathbf{v}_1) \cap \mathfrak{c}(\mathbf{v}_2) = \emptyset.$$

Letting S denote the substitution

$$w \longrightarrow \begin{cases} \mathfrak{a} & \text{if } w = x, \\ \mathfrak{ab} & \text{if } w \in \mathfrak{c}(\mathbf{u}_1), \\ \mathfrak{ba} & \text{if } w \in \mathfrak{c}(\mathbf{u}_2), \end{cases}$$

into B_2^-, we contradictorily have $\mathbf{u}(\mathrm{S}) = \mathfrak{a} \neq 0 = \mathbf{v}(\mathrm{S})$. Therefore $\mathfrak{m}_{\mathbf{u}}(x) = \mathfrak{m}_{\mathbf{v}}(x)$. ■

Lemma 4.3.5 *Let*

$$\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_n}^{\alpha_n}, \qquad \mathbf{v} \equiv z_{\tau_1}^{\beta_1} \cdots z_{\tau_n}^{\beta_n}$$

be $\mathbf{B_2^-}$*-words. If* $\mathbf{B_2^-} \models \mathbf{u} = \mathbf{v}$*, then* $\sigma_i = \tau_i$ *for all* i*.*

PROOF. Seeking a contradiction, suppose that k is the least integer such that $\sigma_k \neq \tau_k$. Then

$$\mathbf{u} \equiv \mathbf{w} z_{\sigma_k}^{\alpha_k} \mathbf{u}_1 z_{\tau_k}^{\alpha_s} \mathbf{u}_2, \qquad \mathbf{v} \equiv \mathbf{w} z_{\tau_k}^{\beta_k} \mathbf{v}_1 z_{\sigma_k}^{\beta_t} \mathbf{v}_2$$

for some words $\mathbf{w}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in (X \backslash \{z_{\sigma_k}, z_{\tau_k}\})^+ \cup \{\emptyset\}$ with pairwise disjoint contents. Without loss of generality assume $\sigma_k > \tau_k$. If the multiplicity of each variable in $z_{\sigma_k}^{\alpha_k} \mathbf{u}_1 z_{\tau_k}^{\alpha_s}$ is two, then by condition (4.5), we contradictorily have $\sigma_k < \tau_k$. Therefore there must be a variable z_{σ_m}, the multiplicity of which in $z_{\sigma_k}^{\alpha_k} \mathbf{u}_1 z_{\tau_k}^{\alpha_s}$ (and hence $\mathbf{u}$) is one. Let S denote the following substitution into B_2^-:

$$z_{\sigma_i} \longrightarrow \begin{cases} \mathfrak{a} & \text{if } i = m, \\ \mathfrak{ab} & \text{if } i < m, \\ \mathfrak{ba} & \text{if } i > m. \end{cases}$$

Then $\mathbf{u}(\mathrm{S}) = \mathfrak{a}$. It is straightforward to see that $\mathfrak{a} B_2^- \mathfrak{a} = \{0\}$. Therefore, since

$$z_{\tau_k}(\mathrm{S}) \in \{\mathfrak{a}, \mathfrak{ba}\}, \qquad z_{\sigma_k}(\mathrm{S}) \in \{\mathfrak{a}, \mathfrak{ab}\},$$

we contradictorily have $\mathbf{v}(\mathrm{S}) \in B_2^- \mathfrak{a} B_2^- \mathfrak{a} B_2^- = \{0\}$. ■

Theorem 4.3.6 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{B_2^-}$*-words such that* $\mathbf{B_2^-} \models \mathbf{u} = \mathbf{v}$*, then* $\mathbf{u} \equiv \mathbf{v}$*.*

PROOF. The result follows easily from Lemma 4.3.4 and Lemma 4.3.5. ■

Theorem 4.3.7 (1) $\mathbf{B_2^-} \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{B_2^-}} \equiv \mathbf{v}^{\mathbf{B_2^-}}$;

(2) *The set of all* $\mathbf{B_2^-}$*-words constitutes the* $\mathbf{B_2^-}$*-free semigroup with the operation* $\cdot$ *given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{B_2^-}};$$

(3) $\mathbf{B_2^-} = [\nabla] = [\mathrm{V}]^{\mathbf{A_0}} = \mathbf{A_0} \cap \mathbf{B_2}$.

PROOF. Proving (1), (2) and $\mathbf{B_2^-} = [\nabla]$ is similar to proving Theorem 4.2.11. By definition,

$$[\nabla] = [\mathrm{V}] \cap [\mathrm{I}, \mathrm{II}, \mathrm{III}] = [\mathrm{V}] \cap \mathbf{A_0} = [\mathrm{V}]^{\mathbf{A_0}}.$$

Thus it remains to prove the last equality in (3). Now by Lemma 4.3.2(3) and III respectively,

$$(xy)^2 x = x^2 y^2 x = xyx$$

so that $\nabla \vdash (xy)^2 x = xyx$. Hence, by Proposition 2.6.3,

$$\begin{aligned} \mathbf{A_0} \cap \mathbf{B_2} &= [\mathrm{I}, \mathrm{II}, \mathrm{III}] \cap \left[\mathrm{I}, \mathrm{V}, (xy)^2 x = xyx\right] \\ &= [\nabla] \cap \left[(xy)^2 x = xyx\right] = [\nabla]. \end{aligned}$$

■

In view of Theorem 4.3.7(3), a semigroup $S \in \mathbf{A_0}$ belongs to $\mathbf{B_2^-}$ if and only if it satisfies V. But in order to show that S satisfies V, it suffices (by the following lemma) just to verify that its generators satisfy V.

Lemma 4.3.8 *Let $S = \langle s_i \mid i \in I \rangle \in \mathbf{A_0}$. If $s_i^2 s_j^2 = s_j^2 s_i^2$ for all $i, j \in I$, then $S \models \nabla$.*

PROOF. Since $\mathbf{A_0} = [\mathrm{I}, \mathrm{II}, \mathrm{III}]$ it only remains to show that $S \models \mathrm{V}$. Let $a = s_{i_1} \cdots s_{i_m}$ and $b = s_{j_1} \cdots s_{j_n}$. By assumption and since $S \in \mathbf{A_0}$, the generators $\{s_i \mid i \in I\}$ satisfy the identities I, II, III and V. Hence they also satisfy the identity in Lemma 4.3.2(3), and

$$\begin{aligned} a^2 b^2 &= s_{i_1}^2 \cdots s_{i_m}^2 \cdot s_{j_1}^2 \cdots s_{j_n}^2 \\ &= s_{j_1}^2 \cdots s_{j_n}^2 \cdot s_{i_1}^2 \cdots s_{i_m}^2 \\ &= b^2 a^2. \end{aligned}$$

■

We shall end this section by presenting the identity that is a common consequence within $\mathbf{A_0}$ (respectively, $\mathbf{B_2^-}$) of each balanced identity not satisfied by $\mathbf{A_0}$ (respectively, $\mathbf{B_2^-}$). These results play important roles in determining anti-atoms of $\mathcal{L}(\mathbf{A_0})$ and $\mathcal{L}(\mathbf{B_2^-})$ in later sections.

Lemma 4.3.9 *If π is a balanced identity not satisfied by $\mathbf{A_0}$, then*

$$\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \pi\} \vdash \mathrm{V}.$$

PROOF. Since $\mathbf{A_0}$ does not satisfy π, there must be two distinct $\mathbf{A_0}$-words $\mathbf{u}, \mathbf{v}$ with $\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \pi\} \vdash \mathbf{u} = \mathbf{v}$. From the proofs of Lemmas 4.2.5, 4.2.6, 4.2.7, 4.2.8, and Corollary 4.2.9, it is straightforward to deduce that $\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathbf{u} = \mathbf{v}\}$ implies one or both of

$$x^2y^2x^2 = x^2y^2, \qquad x^2y^2x^2y^2 = x^2y^2;$$

let (A), (B) denote these identities respectively. Now $\{(\mathrm{A}), \mathrm{II}\} \vdash \mathrm{V}$ since

$$\begin{aligned} x^2y^2 &= x^2y^2x^2 && \text{by (A)} \\ &= y^2x^2y^2 && \text{by II} \\ &= y^2x^2 && \text{by (A)}, \end{aligned}$$

and $\{(\mathrm{B}), \mathrm{I}, \mathrm{II}\} \vdash \mathrm{V}$ since

$$\begin{aligned} x^2y^2 &= \left(x^2y^2x^2\right) y^2 && \text{by (B)} \\ &= y^2x^2 \left(y^2y^2\right) && \text{by II} \\ &= y^2x^2y^2 && \text{by I} \\ &= x^2y^2x^2 && \text{by II} \\ &= \left(x^2y^2x^2\right) x^2 && \text{by I} \\ &= y^2x^2y^2x^2 && \text{by II} \\ &= y^2x^2 && \text{by (B)}. \end{aligned}$$

Therefore $\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \pi\} \vdash \mathrm{V}$. ■

Lemma 4.3.10 *If π is a balanced identity not satisfied by $\mathbf{B_2^-}$, then*

$$\nabla \cup \{\pi\} \vdash x^2y^2w^2 = x^2yw^2.$$

PROOF. Note that the pair of words forming π are each $\equiv_{[\nabla\cup\{\pi\}]}$-related to a $\mathbf{B_2^-}$-word, say $\mathbf{u}, \mathbf{v}$, whence the variety defined by $\nabla \cup \{\mathbf{u} = \mathbf{v}\}$ is identical to that defined by $\nabla \cup \{\pi\}$. Therefore we may assume π to be $\mathbf{u} = \mathbf{v}$, where

$$\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}, \qquad \mathbf{v} \equiv z_{\tau_1}^{\beta_1} \cdots z_{\tau_k}^{\beta_k}$$

with $\{\sigma_1, \dots, \sigma_k\} = \{\tau_1, \dots, \tau_k\}$. Suppose $\sigma_i = \tau_i$ for all i. Then $\alpha_i \neq \beta_i$ for some i, say $(\alpha_j, \beta_j) = (1, 2)$. Letting S be the following substitution

$$z_{\sigma_i} \longrightarrow \begin{cases} x^2 & \text{if } i < j, \\ y & \text{if } i = j, \\ w^2 & \text{if } i > j, \end{cases}$$

into X^+, we have $x^2\mathbf{u}(\mathrm{S})\,w^2 = x^2yw^2$ and $x^2\mathbf{v}(\mathrm{S})\,w^2 = x^2y^2w^2$ by I, so that

$$\begin{aligned}\{\mathrm{I}, \mathbf{u} = \mathbf{v}\} &\vdash \{\mathrm{I}, x^2\mathbf{u}(\mathrm{S})\,w^2 = x^2\mathbf{v}(\mathrm{S})\,w^2\} \\ &\vdash x^2yw^2 = x^2y^2w^2.\end{aligned}$$

Therefore it remains to consider when $\sigma_i \neq \tau_i$ for some i. Let j be the least such that $\sigma_j \neq \tau_j$. Then

$$\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_{j-1}}^{\alpha_{j-1}} z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^{\alpha} \mathbf{u}_2, \qquad \mathbf{v} \equiv z_{\sigma_1}^{\beta_1} \cdots z_{\sigma_{j-1}}^{\beta_{j-1}} z_{\tau_j}^{\beta_j} \mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2$$

for some $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \left(X \backslash \{z_{\sigma_j}, z_{\tau_j}\}\right)^+ \cup \{\emptyset\}$, $\alpha, \beta \in \{1, 2\}$, and

$$\mathfrak{c}(\mathbf{u}_1) \cap \mathfrak{c}(\mathbf{u}_2) = \mathfrak{c}(\mathbf{v}_1) \cap \mathfrak{c}(\mathbf{v}_2) = \emptyset.$$

Now

$$\mathbf{u}' \equiv h^2 z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^{\alpha} \mathbf{u}_2, \qquad \mathbf{v}' \equiv h^2 z_{\tau_j}^{\beta_j} \mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2$$

are obtained from $\mathbf{u}$ and $\mathbf{v}$, respectively, by either making the substitutions $z_{\sigma_1} = \cdots = z_{\sigma_{j-1}} = h^2$ into X^+ if $j > 1$, or premultiplication by h^2 if $j = 1$. Without any loss of generality assume $\sigma_j > \tau_j$. If the multiplicity of each variable in $z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^{\alpha}$ is two, then by condition (4.5), we contradictorily have $\sigma_j < \tau_j$. Therefore there must be a variable t in $z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^{\alpha}$ the multiplicity of which is one. There are two cases: $t \in \mathfrak{c}\left(z_{\sigma_j}^{\alpha_j} \mathbf{u}_1\right)$ or $t = z_{\tau_j}$.

Case (i) Suppose $t \in \mathfrak{c}\left(z_{\sigma_j}^{\alpha_j} \mathbf{u}_1\right)$. Then postmultiplying $\mathbf{u}'$ and $\mathbf{v}'$ each by z_{τ_j}, we have

$$\begin{aligned}\mathbf{u}' z_{\tau_j} &\equiv h^2 z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 \left(z_{\tau_j}^{\alpha} \mathbf{u}_2 z_{\tau_j}\right) \\ &\equiv_{\nabla} h^2 z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^2 \mathbf{u}_2^2 && \text{by Lemma 4.3.2(2)} \\ &\equiv_{\nabla} h^2 z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^2 \overline{\mathbf{u}_2} && \text{by Lemma 4.3.2(1),}\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}' z_{\tau_j} &\equiv h^2 \left(z_{\tau_j}^{\beta_j} \mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2 z_{\tau_j}\right) \\ &\equiv_{\nabla} h^2 z_{\tau_j}^2 \left(\mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2\right)^2 && \text{by Lemma 4.3.2(2)} \\ &\equiv_{\nabla} h^2 z_{\tau_j}^2 \overline{\mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2} && \text{by Lemma 4.3.2(1).}\end{aligned}$$

Let $\mathbf{u}'' \equiv h^2 z_{\sigma_j}^{\alpha_j} \mathbf{u}_1 z_{\tau_j}^2 \overline{\mathbf{u}_2}$ and $\mathbf{v}'' \equiv h^2 z_{\tau_j}^2 \overline{\mathbf{v}_1 z_{\sigma_j}^{\beta} \mathbf{v}_2}$. Consider the following substitution T into X^+:

$$s \longrightarrow \begin{cases} x^2 & \text{if } s \text{ preceeds } t \text{ in } \mathbf{u}'', \\ y & \text{if } s = t, \\ w^2 & \text{if } s \text{ follows } t \text{ in } \mathbf{u}''. \end{cases}$$

Since the multiplicity of t in $z_{\sigma_j}^{\alpha_j} \mathbf{u}_1$ is one, its multiplicity in $\mathbf{u}''$ is also one, whence $\mathbf{u}''(\mathrm{T}) = x^2 y w^2$ by invoking I if necessary. Note that $\mathbf{v}''$ is a product of squares so that $\mathbf{v}''(\mathrm{T}) = x^2 y^2 w^2$ by I and V. Hence

$$\begin{aligned} \nabla \cup \{\mathbf{u} = \mathbf{v}\} &\vdash \nabla \cup \{\mathbf{u}'' = \mathbf{v}''\} \\ &\vdash x^2 y^2 w^2 = x^2 y w^2. \end{aligned}$$

Case (ii) Suppose that $t = z_{\tau_j}$ (whence $\alpha = 1$). Then premultiplying $\mathbf{u}'$ and $\mathbf{v}'$ by z_{σ_j}, we have

$$\begin{aligned} z_{\sigma_j} \mathbf{u}' &\equiv \left(z_{\sigma_j} h^2 z_{\sigma_j}^{\alpha_j}\right) \mathbf{u}_1 t \mathbf{u}_2 \\ &\equiv_\nabla h^2 z_{\sigma_j}^2 \mathbf{u}_1 t \mathbf{u}_2 \qquad \text{by Lemma 4.3.2(2)} \end{aligned}$$

and

$$\begin{aligned} z_{\sigma_j} \mathbf{v}' &\equiv \left(z_{\sigma_j} h^2 t^{\beta_j} \mathbf{v}_1 z_{\sigma_j}^{\beta}\right) \mathbf{v}_2 \\ &\equiv_\nabla h^2 t^2 \mathbf{v}_1^2 z_{\sigma_j}^2 \mathbf{v}_2 \qquad \text{by Lemma 4.3.2(2)} \\ &\equiv_\nabla h^2 t^2 \overline{\mathbf{v}_1} z_{\sigma_j}^2 \mathbf{v}_2 \qquad \text{by Lemma 4.3.2(1)}. \end{aligned}$$

Let $\mathbf{u}''' \equiv h^2 z_{\sigma_j}^2 \mathbf{u}_1 t \mathbf{u}_2$ and $\mathbf{v}''' \equiv h^2 t^2 \overline{\mathbf{v}_1} z_{\sigma_j}^2 \mathbf{v}_2$. Consider the following substitution U into X^+:

$$s \longrightarrow \begin{cases} x^2 & \text{if } s \in \mathfrak{c}\left(h^2 z_{\sigma_j}^2 \mathbf{u}_1\right), \\ y & \text{if } s = t, \\ w^2 & \text{if } s \in \mathfrak{c}(\mathbf{u}_2). \end{cases}$$

Then $\mathbf{u}'''(\mathrm{U})\, w^2 = x^2 y w^2$ by I and V. But note that $\mathbf{v}'''(\mathrm{U})\, w^2$ is a product of words from $\{x^2, y^2, w^2\}$ so that $\mathbf{v}'''(\mathrm{U})\, w^2 = x^2 y^2 w^2$ by I and V. Therefore

$$\begin{aligned} \nabla \cup \{\mathbf{u} = \mathbf{v}\} &\vdash \nabla \cup \{\mathbf{u}''' = \mathbf{v}'''\} \\ &\vdash x^2 y^2 w^2 = x^2 y w^2. \end{aligned}$$

■

Corollary 4.3.11 (1) *If* $\mathbf{U} \in [\mathbf{Y}, \mathbf{A_0})$ *then* $\mathbf{U} \subseteq \mathbf{B_2^-}$;
(2) *If* $\mathbf{U} \in [\mathbf{Y}, \mathbf{B_2^-})$ *then* $\mathbf{U} \subseteq [x^2y^2w^2 = x^2yw^2]^{\mathbf{B_2^-}}$.

PROOF. (1) Suppose $\mathbf{U} \in [\mathbf{Y}, \mathbf{A_0})$. Then there exists some balanced identity π such that $\mathbf{U} \models \pi$ and $\mathbf{A_0} \not\models \pi$. By Lemma 4.3.9, $\mathbf{U} \in [\mathrm{I}, \mathrm{II}, \mathrm{III}, \pi] \subseteq [\nabla] = \mathbf{B_2^-}$.

(2) This is similar to (1) but by invoking Lemma 4.3.10 instead of Lemma 4.3.9. ■

4.4 The Semigroup U_∞ in $\mathbf{A_0}$

In this short section, we present an infinite subsemigroup U_∞ of A_0^∞, the cartesian product of countably infinite copies of A_0. It will be shown that U_∞ actually generates $\mathbf{A_0}$ (Proposition 4.6.15). Therefore we record only those observations that are required in later sections. Being infinite, U_∞ is obviously not a very convenient generator of $\mathbf{A_0}$. But its size has other advantages, one of which is the ability to possess infinitely many proper factors generating infinitely many subvarieties of $\mathbf{A_0}$ (which will be introduced later in the chapter). Furthermore, the proper factors of U_∞ serve as infinite models for identities involving arbitrarily many variables to test against, a task for which the four elements of A_0 are less well suited.

Let the components of A_0^∞ be indexed by $\mathbb{N}$, that is, $A_0^\infty = S_1 \times S_2 \times \cdots$ where $S_1 = S_2 = \cdots = A_0$. For each $i \in \mathbb{N}$, define the element $\mathfrak{u}_i$ of A_0^∞ componentwise by

$$(\mathfrak{u}_i)_m = \begin{cases} \mathfrak{f} & \text{if } m < i, \\ \mathfrak{ef} & \text{if } m = i, \\ \mathfrak{e} & \text{if } m > i, \end{cases}$$

and define $\mathbf{0} = (0, 0, \ldots)$. Then $U_\infty = \langle \mathfrak{u}_i \mid i \in \mathbb{N} \rangle \cup \{\mathbf{0}\}$ is a subsemigroup of A_0^∞. Note that $\mathfrak{u}_i\mathfrak{u}_j\mathfrak{u}_k = \mathfrak{u}_i\mathfrak{u}_k$ if $i < j < k$, and that

$$\mathfrak{u}_i^\alpha = (\underbrace{\mathfrak{f}, \ldots, \mathfrak{f}}_{i-1}, (\mathfrak{ef})^\alpha, \mathfrak{e}, \mathfrak{e}, \ldots) = \begin{cases} (\mathfrak{f}, \ldots, \mathfrak{f}, \mathfrak{ef}, \mathfrak{e}, \mathfrak{e}, \ldots) & \text{if } \alpha = 1, \\ (\mathfrak{f}, \ldots, \mathfrak{f}, 0, \mathfrak{e}, \mathfrak{e}, \ldots) & \text{if } \alpha \geq 2. \end{cases}$$

Lemma 4.4.1 *A nonzero element of* A_0^∞ *is in* U_∞ *if and only if it is of the form*

$$(\underbrace{\mathfrak{f}, \ldots, \mathfrak{f}}_{m}, x_1, \ldots, x_n, \mathfrak{e}, \mathfrak{e}, \ldots) \tag{4.7}$$

where $x_1, \ldots, x_n \in \{0, \mathfrak{ef}\}$, $m \geq 0$ *and* $n \geq 1$.

PROOF. It is straightforward to show that nonzero elements of U_∞ are of the form (4.7). Conversely, note that

$$\mathfrak{u}_{m+1}^{\alpha_1}\cdots\mathfrak{u}_{m+n}^{\alpha_n} = (\underbrace{\mathfrak{f},\ldots,\mathfrak{f}}_{m}, (\mathfrak{ef})^{\alpha_1}, \ldots, (\mathfrak{ef})^{\alpha_n}, \mathfrak{e}, \mathfrak{e}, \ldots)$$

where

$$(\mathfrak{ef})^{\alpha_i} = \begin{cases} \mathfrak{ef} & \text{if } \alpha_i = 1, \\ 0 & \text{if } \alpha_i = 2. \end{cases}$$

Hence elements of the form in (4.7) are also in U_∞. ■

The nonempty sequence $(x_1, \ldots, x_n)$ in (4.7) shall be called the *nilpotent sequence* of x. The nilpotent sequence of $\mathbf{0}$ is defined to be itself. The following lemma, the verification of which is straightforward, will be useful in later sections.

Lemma 4.4.2 *Let $x \in U_\infty \cup \{\emptyset\}$. Then:*

(1) *If $i < j$ then $(\mathfrak{u}_i\mathfrak{u}_j)_k = \mathfrak{ef}$ for $i \le k \le j$;*

(2) *If $i \ge j$ then $(\mathfrak{u}_i x \mathfrak{u}_j)_k = 0$ for $i \le k \le j$.*

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_0^\infty = \mathbb{N} \cup \{0, \infty\}$. For convenience, a sequence $s_1, \ldots, s_m$ (respectively, word $s_1 \cdots s_m$) is associated with the empty sequence (respectively, empty word) whenever $m = 0$. For any $l, n, r \in \mathbb{N}_0^\infty$, define the following identities:

$$(l : n : r) : \begin{cases} \mathbf{x}_l \mathbf{y}_{n+1}^2 \mathbf{w}_r = \mathbf{x}_l \mathbf{y}_{n+1} \mathbf{w}_r & \text{if } n \in \mathbb{N}_0, \\ \mathbf{x}_l y^2 z \mathbf{w}_r = \mathbf{x}_l y z^2 \mathbf{w}_r & \text{if } n = \infty, \end{cases}$$

where

$$\mathbf{x}_l = \begin{cases} x_1 \cdots x_l & \text{if } l \in \mathbb{N}_0, \\ x^2 & \text{if } l = \infty, \end{cases}$$

and $\mathbf{y}_{n+1}$ and $\mathbf{w}_r$ are defined analogously. These identities will be used extensively in later sections to define the subvarieties generated by the proper factors of U_∞. The variety defined by $(l : n : r)$ shall simply be written as $[l : n : r]$.

Lemma 4.4.3 *For all $l, r \in \mathbb{N}_0^\infty$ and $n \in \mathbb{N}_0$,*

$$[l : n : r]^{\mathbf{B}_2^-} = \left[\mathbf{x}_l y_1^{\alpha_1} \cdots y_{n+1}^{\alpha_{n+1}} \mathbf{w}_r = \mathbf{x}_l y_1 \cdots y_{n+1} \mathbf{w}_r \mid 1 \le \alpha_i \le 2\right]^{\mathbf{B}_2^-}.$$

PROOF. Let $\mathbf{U}$ (respectively, $\mathbf{V}$) denote the variety on the left (respectively, right) of the above equality. Since

$$\begin{aligned}\mathbf{x}_l y_1^{\alpha_1}\cdots y_{n+1}^{\alpha_{n+1}}\mathbf{w}_r &\equiv_{\mathbf{U}} \mathbf{x}_l\left(y_1^{\alpha_1}\cdots y_{n+1}^{\alpha_{n+1}}\right)^2\mathbf{w}_r && \text{by } (l:n:r)\\ &\equiv_{\mathbf{U}} \mathbf{x}_l\left(y_1\cdots y_{n+1}\right)^2\mathbf{w}_r && \text{by Lemma 4.3.2(4)}\\ &\equiv_{\mathbf{U}} \mathbf{x}_l y_1\cdots y_{n+1}\mathbf{w}_r && \text{by } (l:n:r),\end{aligned}$$

we have $\mathbf{U}\subseteq\mathbf{V}$. Conversely, assuming the identities in $\mathbf{U}$,

$$\begin{aligned}\mathbf{x}_l\mathbf{y}_{n+1}^2\mathbf{w}_r &\equiv_{\mathbf{V}} \mathbf{x}_l y_1^2\cdots y_{n+1}^2\mathbf{w}_r && \text{by Lemma 4.3.2(3)}\\ &\equiv_{\mathbf{V}} \mathbf{x}_l y_1\cdots y_{n+1}\mathbf{w}_r \equiv \mathbf{x}_l\mathbf{y}_{n+1}\mathbf{w}_r.\end{aligned}$$

Therefore $\mathbf{V}\subseteq\mathbf{U}$. ■

In view of Lemma 4.4.3, the identities

$$\mathbf{x}_l y_1^{\alpha_1}\cdots y_{n+1}^{\alpha_{n+1}}\mathbf{w}_r = \mathbf{x}_l y_1\cdots y_{n+1}\mathbf{w}_r \qquad (1\le\alpha_i\le 2)$$

and $\mathbf{x}_l\mathbf{y}_{n+1}^2\mathbf{w}_r = \mathbf{x}_l\mathbf{y}_{n+1}\mathbf{w}_r$ define the same subvariety of $\mathbf{B_2^-}$ whenever $n\in\mathbb{N}_0$. We shall use both forms interchangeably as the former has wider applications while the latter has a simpler expression.

4.5 The Varieties $\mathbf{L}_l$ and $\mathbf{R}_r$

In this section, we introduce the semigroups L_l and R_r indexed by $l,r\in\mathbb{N}_0^\infty$. As suggested by the choice of notation, these semigroups are symmetrical duals of each other. Thus it suffices (and we choose) to investigate L_l, the approach for which follows very similarly to that in Sections 2 and 3. It will also be shown that the varieties $\mathbf{L}_l$ (respectively, $\mathbf{R}_r$) generated by L_l (respectively, R_r) form a strictly increasing complete chain (Corollary 4.5.8 and Proposition 4.5.12).

We begin by presenting the foundations needed for the introduction of L_l. Let $x,y\in U_\infty$. We write $x\sim y$ if the nilpotent sequences of x and y occur at the same coordinates, and write $x\lambda y$ if the first 0 entry (if any) of x and y occur at the same coordinate. Let $\overset{\lambda}{\sim}$ denote the intersection of the relations $\sim$ and λ. Note that by definition, $x\not\sim\mathbf{0}$ for all $x\in U_\infty\backslash\{\mathbf{0}\}$.

Proposition 4.5.1 *The relations* $\sim$ *and* $\overset{\lambda}{\sim}$ *are congruences on* U_∞.

PROOF. It is easy to see that $\sim$ and λ are equivalence relations and so is $\overset{\lambda}{\sim}$. It is routine to verify that if $x \sim y$ (respectively, $x \overset{\lambda}{\sim} y$), then $\mathfrak{u}_i x \sim \mathfrak{u}_i y$ and $x\mathfrak{u}_i \sim y\mathfrak{u}_i$ (respectively, $\mathfrak{u}_i x \overset{\lambda}{\sim} \mathfrak{u}_i y$ and $x\mathfrak{u}_i \overset{\lambda}{\sim} y\mathfrak{u}_i$) for all i. Therefore both $\sim$ and $\overset{\lambda}{\sim}$ are congruences on U_∞. ■

Note that for fixed $i, j, k \geq 0$ and $x_1, \dots, x_k \in \{0, \mathfrak{ef}\}$,

$$(\underbrace{\mathfrak{f}, \dots, \mathfrak{f}}_{i}, \underbrace{\mathfrak{ef}, \dots, \mathfrak{ef}}_{j}, 0, x_1, \dots, x_k, \mathfrak{e}, \mathfrak{e}, \dots) \overset{\lambda}{\sim} (\underbrace{\mathfrak{f}, \dots, \mathfrak{f}}_{i}, \underbrace{\mathfrak{ef}, \dots, \mathfrak{ef}}_{j}, \underbrace{0, 0, \dots, 0}_{k+1}, \mathfrak{e}, \mathfrak{e}, \dots)$$

Note also that the $\overset{\lambda}{\sim}$-class of each $\mathfrak{u}_i$ is singleton. Let $\mathfrak{l}_i$ denote the $\overset{\lambda}{\sim}$-class of each $\mathfrak{u}_i$. Then for each $l \in \mathbb{N}_0^\infty$,

$$L_l = \begin{cases} \{\mathfrak{l}_1^2, \mathfrak{l}_1^2\mathfrak{l}_2^2\} & \text{if } l = 0, \\ \langle \mathfrak{l}_1, \dots, \mathfrak{l}_l, \mathfrak{l}_{l+1}^2 \rangle & \text{if } l \in \mathbb{N}, \\ \langle \mathfrak{l}_i \mid i \in \mathbb{N} \rangle & \text{if } l = \infty, \end{cases}$$

is a subsemigroup of $U_\infty / \overset{\lambda}{\sim}$. Note that L_0 is isomorphic to the semilattice Y of order two. If $l \in \mathbb{N}$, then $\mathfrak{l}_1^2 \cdots \mathfrak{l}_{l+1}^2$ is the zero element of L_l, while L_∞ has no zero. It is easy to show that $\mathfrak{u}_i^2\mathfrak{u}_j^2 \overset{\lambda}{\sim} \mathfrak{u}_j^2\mathfrak{u}_i^2$ for all $i, j \in \mathbb{N}$ so that $\mathfrak{l}_i^2\mathfrak{l}_j^2 = \mathfrak{l}_j^2\mathfrak{l}_i^2$.

Let $\mathbf{L}_l$ denote the variety generated by L_l, and let $\mathfrak{L} = \{\mathbf{L}_l \mid l \in \mathbb{N}_0^\infty\}$. It is easy to see that $L_l \subset L_m$ whenever $l, m \in \mathbb{N}_0^\infty$ and $l < m$. Therefore the varieties in $\mathfrak{L}$ form the chain

$$\mathbf{Y} = \mathbf{L}_0 \subsetneq \mathbf{L}_1 \subsetneq \cdots \subsetneq \mathbf{L}_\infty. \tag{4.8}$$

We now proceed to show that $\mathbf{L}_l = [l : 0 : 0]^{\mathbf{B}_2^-}$. In this section, l always denote an element in $\mathbb{N}_0^\infty$ unless otherwise specified.

Theorem 4.5.2 *The semigroup L_l satisfies the identities in $\nabla \cup \{(l : 0 : 0)\}$. Consequently, L_l belongs to $[l : 0 : 0]^{\mathbf{B}_2^-}$.*

PROOF. Clearly L_l satisfies I, II, and III since it belongs to $\mathbf{A_0}$. By the discussion following the definition of L_l, $\mathfrak{l}_i^2\mathfrak{l}_j^2 = \mathfrak{l}_j^2\mathfrak{l}_i^2$ for all $i, j \in \mathbb{N}$. Thus L_l satisfies V by Lemma 4.3.8, whence $L_l \models \nabla$. It remains to show that $L_l \models (l : 0 : 0)$. There are two cases: $l \in \mathbb{N}_0$ and $l = \infty$.

Case (i) Suppose $l \in \mathbb{N}_0$. Then $(l : 0 : 0)$ is the identity $\mathbf{x}_l y^2 = \mathbf{x}_l y$. Since L_0 is isomorphic to Y which satisfies the identity $y^2 = y$, we assume that $l \geq 1$. We first show that $\mathbf{x}_l y^2 \overset{\lambda}{\sim} \mathbf{x}_l y$ for all $x_1, \dots, x_l, y \in \{\mathfrak{u}_1, \dots, \mathfrak{u}_l, \mathfrak{u}_{l+1}^2\}$. We may assume $y \neq \mathfrak{u}_{l+1}^2$ because

$\mathfrak{u}_{l+1}^2$ is an idempotent. Since $y^2 \sim y$ and $\sim$ is a congruence on S, we have $\mathbf{x}_l y^2 \sim \mathbf{x}_l y$. Therefore it remains to show $\mathbf{x}_l y^2 \lambda \mathbf{x}_l y$.

Note that the nilpotent sequences of $\mathbf{x}_l y^2$ and $\mathbf{x}_l y$ occur between the first and $l+1^{\text{st}}$ coordinates inclusively. Seeking a contradiction, suppose $\mathbf{x}_l y^2$ and $\mathbf{x}_l y$ are not λ-related. Then there exists a least integer $m \leq l+1$ such that $\left(\mathbf{x}_l y^2\right)_m \neq (\mathbf{x}_l y)_m$, whence one of $\left(\mathbf{x}_l y^2\right)_m, (\mathbf{x}_l y)_m$ is 0 while the other is $\mathfrak{ef}$. Since $\mathbf{x}_l y^2 \sim \mathbf{x}_l y$, the m^{th} coordinate is also the first coordinate in which either $\mathbf{x}_l y^2$ or $\mathbf{x}_l y$ is 0. If $(\mathbf{x}_l y)_m = 0$, then we contradictorily have $\left(\mathbf{x}_l y^2\right)_m = (\mathbf{x}_l y)_m (y)_m = 0$. Hence we must have

$$(\mathbf{x}_l y)_m = \mathfrak{ef}, \qquad \left(\mathbf{x}_l y^2\right)_m = 0. \tag{4.9}$$

Now

$$0 = \left(\mathbf{x}_l y^2\right)_m = (\mathbf{x}_l y)_m (y)_m = \mathfrak{ef}\,(y)_m$$

so that $(y)_m \in \{0, \mathfrak{e}, \mathfrak{ef}\}$. If $(y)_m = 0$ then $y = \mathfrak{u}_{l+1}^2$ in violation of our assumption. If $(y)_m = \mathfrak{e}$ then $y = \mathfrak{u}_j$ for some $j < m$ so that $\left(\mathbf{x}_l y^2\right)_j = (\mathbf{x}_l)_j (\mathfrak{ef})^2 = 0$, contradicting the minimality of m. Therefore $(y)_m = \mathfrak{ef}$ and $y = \mathfrak{u}_m$ with $m \leq l$ necessarily.

Having shown that y can only be $\mathfrak{u}_m$, we proceed to show that any choices of x_i from $\left\{\mathfrak{u}_1, \ldots, \mathfrak{u}_l, \mathfrak{u}_{l+1}^2\right\}$ will result in a contradiction. If $x_i = \mathfrak{u}_{l+1}^2$ for some i then

$$(\mathbf{x}_l y)_m = \left(\cdots \mathfrak{u}_{l+1}^2 \cdots \mathfrak{u}_m\right)_m = 0$$

by Lemma 4.4.2(2), which contradicts (4.9). Therefore $x_i \neq \mathfrak{u}_{l+1}^2$ for all i. If $x_1, \ldots, x_l \in \{\mathfrak{u}_1, \ldots, \mathfrak{u}_{m-1}\}$, then $\mathfrak{u}_i$ will appear twice in $\mathbf{x}_l$ for some $i \in \{1, \ldots, m-1\}$ so that $(\mathbf{x}_l y)_i = 0$ by Lemma 4.9(2), contradicting the minimality of m. Hence $x_i \in \{\mathfrak{u}_m, \ldots, \mathfrak{u}_l\}$ for some i; but

$$(\mathbf{x}_l y)_m = (\cdots x_i \cdots \mathfrak{u}_m)_m = 0$$

by Lemma 4.4.2(2), contradicting (4.9). But that exhausts the possibilities and in all cases we have a contradiction. Consequently $\mathbf{x}_l y^2 \overset{\lambda}{\sim} \mathbf{x}_l y$, or equivalently, the identity $\mathbf{x}_l y^2 = \mathbf{x}_l y$ is satisfied by $\mathfrak{l}_1, \ldots, \mathfrak{l}_l, \mathfrak{l}_{l+1}^2$.

Now for any elements $a_1, \ldots, a_l, b \in L_l$, we have $b = b_1 \cdots b_k$ where $b_i \in \left\{\mathfrak{l}_1, \ldots, \mathfrak{l}_l, \mathfrak{l}_{l+1}^2\right\}$.

Then

$$\begin{aligned} a_1 \cdots a_l b^2 &= a_1 \cdots a_l (b_1 \cdots b_k)^2 \\ &= a_1 \cdots a_l b_1^2 \cdots b_k^2 && \text{by Corollary 4.3.2(3)} \\ &= a_1 \cdots a_l b_1 \cdots b_k && \text{by first part of the proof} \\ &= a_1 \cdots a_l b \end{aligned}$$

Hence $L_l \models \mathbf{x}_l y^2 = \mathbf{x}_l y$.

Case (ii) Suppose $l = \infty$. Then $(\infty : 0 : 0)$ is the identity $x^2y^2 = x^2y$. It is straightforward to show that $x^2y^2 = x^2y$ holds for all $x, y \in \{\mathfrak{l}_i \mid i \in \mathbb{N}\}$. Therefore if $a, b \in L_\infty$, say $a = a_1 \cdots a_p$ and $b = b_1 \cdots b_q$ with $a_i, b_j \in \{\mathfrak{l}_i \mid i \in \mathbb{N}\}$, then

$$\begin{aligned} a^2b^2 &= (a_1 \cdots a_p)^2 (b_1 \cdots b_q)^2 \\ &= a_1^2 \cdots a_p^2 b_1^2 \cdots b_q^2 && \text{by Corollary 4.3.2(3)} \\ &= a_1^2 \cdots a_p^2 b_1 \cdots b_q \\ &= (a_1 \cdots a_p)^2 b_1 \cdots b_q && \text{by Corollary 4.3.2(3)} \\ &= a^2 b. \end{aligned}$$

Hence $L_\infty \models x^2y^2 = x^2y$. ■

Lemma 4.5.3 $[0:0:0]^{\mathbf{B}_2^-} \subseteq [1:0:0]^{\mathbf{B}_2^-} \subseteq \cdots \subseteq [\infty:0:0]^{\mathbf{B}_2^-}$.

PROOF. Clearly $(0:0:0) \vdash (1:0:0)$. Substituting $x_l \longrightarrow x_l x_{l+1}$ in the identity $\mathbf{x}_l y^2 = \mathbf{x}_l y$ yields the identity $\mathbf{x}_{l+1} y^2 = \mathbf{x}_{l+1} y$. Hence $(l:0:0) \vdash (l+1:0:0)$ for all $l \in \mathbb{N}$. If all x_i in $\mathbf{x}_l y^2 = \mathbf{x}_l y$ are replaced by x^2, then we have $\mathbf{x}_\infty y^2 = \mathbf{x}_\infty y$ by I. Therefore $[l:0:0]^{\mathbf{B}_2^-} \subseteq [\infty:0:0]^{\mathbf{B}_2^-}$ for all $l \in \mathbb{N}$. ■

A $\mathbf{B}_2^-$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ will be called an $\mathbf{L}_l$-*word* if

(L1) $k \geq l+1 \Longrightarrow \alpha_{l+1} = 2$;

(L2) $\alpha_i = 2 \Longrightarrow \alpha_i = \cdots = \alpha_k = 2$.

Note that condition (L1) is vacuous when $l = \infty$. The following lemma contains a more convenient description of $\mathbf{L}_l$-words.

Lemma 4.5.4 *A word* $\mathbf{u}$ *is an* $\mathbf{L}_l$-*word if and only if one of the following statements hold:*

(1) $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$ *where* $\sigma_1, \ldots, \sigma_k$ *are distinct and* $1 \leq k \leq l$;

(2) $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_k}^2$ *where* $\sigma_1, \ldots, \sigma_k$ *are distinct,* $\sigma_p < \cdots < \sigma_k$ *and* $1 \leq p \leq l+1$.

PROOF. Suppose $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is an $\mathbf{L}_l$-word. If $\alpha_i = 2$ for some i, then there exists a least p such that $\alpha_p = 2$. Note that $p \leq l+1$ by (L1). Hence

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 z_{\sigma_{p+1}}^{\alpha_{p+1}} \cdots z_{\sigma_k}^{\alpha_k}.$$

Now $\alpha_p = \cdots = \alpha_k = 2$ by (L2). Since $\mathbf{u}$ is also a $\mathbf{B_2^-}$-word, it satisfies the condition in (2). If $\alpha_i = 1$ for all i, then $k \leq l$ by (L1), whence $\mathbf{u}$ satisfies the condition in (1).

Conversely, it is easy to see that the words in (1) and (2) are $\mathbf{L}_l$-words. ■

For this section, let $\equiv_{(l:0:0)}$ denote the fully invariant congruence over $[\nabla \cup \{(l:0:0)\}]$. By Lemma 4.5.3,

$$\nabla \cup \{(0:0:0)\} \vdash \nabla \cup \{(1:0:0)\} \vdash \cdots \vdash \nabla \cup \{(\infty:0:0)\} \vdash \nabla.$$

Proposition 4.5.5 *Each word in* X^+ *is* $\equiv_{(l:0:0)}$*-related to an* $\mathbf{L}_l$*-word.*

PROOF. Let $\mathbf{u} \in X^+$. Since $\nabla_{(l:0:0)} \vdash \nabla_{(\infty:0:0)}$, any word deduced from $\mathbf{u}$ by invoking identities in $\nabla \cup \{(l:0:0), (\infty:0:0)\}$ is $\equiv_{(l:0:0)}$-related to $\mathbf{u}$. In particular, $\mathbf{u} \equiv_{(l:0:0)} \mathbf{u}^{\mathbf{B_2^-}}$, say $\mathbf{u}^{\mathbf{B_2^-}} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$.

Case (i) Suppose $\alpha_i = 2$ for some i. It is easy to see that $(l:0:0)$ and $(\infty:0:0)$ can be used to deduce from $\mathbf{u}^{\mathbf{B_2^-}}$ a word of the form $\mathbf{u}_1 \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_k}^2$ with $1 \leq p \leq l+1$. The ordering $\sigma_p < \cdots < \sigma_k$ can be achieved by invoking V, so that by Lemma 4.5.4(2), $\mathbf{u}_1$ (and hence $\mathbf{u}^{\mathbf{B_2^-}}$) is $\equiv_{(l:0:0)}$-related to an $\mathbf{L}_l$-word.

Case (ii) Suppose $\alpha_i = 1$ for all i. If $k \geq l+1$ then $\alpha_{l+1} = 2$ by $(l:0:0)$ so the result follows from Case (i). If $k \leq l$ then $\mathbf{u}^{\mathbf{B_2^-}}$ is already an $\mathbf{L}_l$-word by Lemma 4.5.4(1). ■

The $\mathbf{L}_l$-word that is obtained from $\mathbf{u}$ by the method in the above proof is denoted by $\mathbf{u}^{\mathbf{L}_l}$. We need to introduce another concept on $\mathbf{B_2^-}$-words to help show that $\equiv_{(l:0:0)}$ distinguishes distinct $\mathbf{L}_l$-words. Let $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ be a $\mathbf{B_2^-}$-word. Define the *l-left segment* $\overleftarrow{\mathbf{u}}^l$ of $\mathbf{u}$ as follows. If $l = 0$ or $\alpha_1 = 2$, then $\overleftarrow{\mathbf{u}}^l = \emptyset$; if $l \geq 1$ and $\alpha_1 = 1$, then $\overleftarrow{\mathbf{u}}^l$ is the subword $z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_p}^{\alpha_p}$ of $\mathbf{u}$ where p is maximal in $\mathbb{I}_1^l$ such that $\alpha_1 = \cdots = \alpha_p = 1$. If we say $z_{\sigma_1} \cdots z_{\sigma_p}$ is an l-left segment for some word, then it is understood that the subscripts $\sigma_1, \ldots, \sigma_p$ are all distinct and that $p \leq l$. Recall (by Lemma 4.5.4) that an $\mathbf{L}_l$-word has one of the following forms (with $\sigma_1, \ldots, \sigma_k$ distinct):

$$\begin{aligned} \mathbf{u}_1 &\equiv z_{\sigma_1} \cdots z_{\sigma_k} && 1 \leq k \leq l, \\ \mathbf{u}_2 &\equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_k}^2 && \sigma_p < \cdots < \sigma_k \text{ and } 1 \leq p \leq l+1 \end{aligned}$$

(note: if $p = 1$ then $z_{\sigma_1} \cdots z_{\sigma_{p-1}} = \emptyset$). In this case, $\overleftarrow{\mathbf{u}_1}^l \equiv z_{\sigma_1} \cdots z_{\sigma_m}$ where $m = \min\{k, l\}$, and $\overleftarrow{\mathbf{u}_2}^l \equiv z_{\sigma_1} \cdots z_{\sigma_n}$ where $n = \min\{p-1, l\}$.

Lemma 4.5.6 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be* $\mathbf{B}_2^-$*-words such that* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. *If* $\overleftarrow{\mathbf{u}}^l \not\equiv \overleftarrow{\mathbf{v}}^l$ *then*

$$\nabla \cup \{\mathbf{u} = \mathbf{v}\} \vdash (s : 0 : \infty)$$

for some $s < l$. *Furthermore,* $\overleftarrow{\mathbf{u}}^s \equiv \overleftarrow{\mathbf{v}}^s$ *and* $\overleftarrow{\mathbf{u}}^{s+1} \not\equiv \overleftarrow{\mathbf{v}}^{s+1}$.

Proof. Let $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ and $\mathbf{v} \equiv z_{\tau_1}^{\beta_1} \cdots z_{\tau_k}^{\beta_k}$ be $\mathbf{B}_2^-$-words. Then there exist $p, q \leq l$ such that

$$\overleftarrow{\mathbf{u}}^l \equiv z_{\sigma_1} \cdots z_{\sigma_p}, \qquad \overleftarrow{\mathbf{v}}^l \equiv z_{\tau_1} \cdots z_{\tau_q}.$$

Without loss of generality assume that $p \leq q$. Note that $(s : 0 : \infty)$ is the identity $\mathbf{x}_s y w^2 = \mathbf{x}_s y^2 w^2$.

Suppose $\overleftarrow{\mathbf{u}}^l \not\equiv \overleftarrow{\mathbf{v}}^l$. Then there are two cases: $p = q$ and $p < q$.

Case (i) Suppose $p = q$. Then $(\sigma_1, \ldots, \sigma_p) \neq (\tau_1, \ldots, \tau_p)$, whence there exists a smallest possible $s < p$ such that $(\sigma_1, \ldots, \sigma_s) = (\tau_1, \ldots, \tau_s)$ and $\sigma_{s+1} \neq \tau_{s+1}$. Let S denote the following substitution into X^+:

$$z_{\sigma_i} \longrightarrow \begin{cases} x_i & \text{if } 1 \leq i \leq s, \\ y & \text{if } i = s+1, \\ w^2 & \text{if } i > s+1. \end{cases}$$

Then $\mathbf{u}\,(\mathrm{S})\, w^2 \equiv_\nabla \mathbf{x}_s y w^2$ by I, and

$$\begin{aligned} \mathbf{v}\,(\mathrm{S})\, w^2 &\equiv \mathbf{x}_s w^2 \cdots y \cdots w^2 \\ &\equiv_\nabla \mathbf{x}_s y^2 w^2 \qquad \text{by Lemma 4.3.2(2).} \end{aligned}$$

Hence $\nabla \cup \{\mathbf{u} = \mathbf{v}\}$ implies $\mathbf{x}_s y w^2 = \mathbf{x}_s y^2 w^2$.

Case (ii) Suppose $p < q$. If $(\sigma_1, \ldots, \sigma_p) \neq (\tau_1, \ldots, \tau_p)$ then the result holds by an identical argument to that in Case (i). Hence it remains to consider the case when

$(\sigma_1, \ldots, \sigma_p) = (\tau_1, \ldots, \tau_p)$. Note that since $p < q$, we must have $\beta_{p+1} = 1$. But if $\alpha_{p+1} = 1$, then $\overleftarrow{\mathbf{u}}^l$ would be $z_{\sigma_1} \cdots z_{\sigma_{p+1}} \cdots$ instead of $z_{\sigma_1} \cdots z_{\sigma_p}$, a contradiction. Therefore $\alpha_{p+1} = 2$. Now let T denote the following substitution in X^+:

$$z_{\tau_i} \longrightarrow \begin{cases} x_i & \text{if } 1 \leq i \leq p, \\ y & \text{if } i = p+1, \\ w^2 & \text{if } i > p+1. \end{cases}$$

Then $\mathbf{v}\,(\mathrm{T})\, w^2 \equiv_\nabla \mathbf{x}_p y w^2$ by I, and

$$\begin{aligned} \mathbf{u}\,(\mathrm{S})\, w^2 &\equiv \begin{cases} \mathbf{x}_p y^2 w^2 \cdots w^2 & \text{if } \sigma_{p+1} = \tau_{p+1}, \\ \mathbf{x}_p w^2 \cdots y \cdots w^2 & \text{otherwise,} \end{cases} \\ &\equiv_\nabla \begin{cases} \mathbf{x}_p y^2 w^2 & \text{by I,} \\ \mathbf{x}_p y^2 w^2 & \text{by Lemma 4.3.2(2).} \end{cases} \end{aligned}$$

Hence $\nabla \cup \{\mathbf{u} = \mathbf{v}\} \vdash \mathbf{x}_p y w^2 = \mathbf{x}_p y^2 w^2$ and the claim holds with $p = s$. ■

Proposition 4.5.7 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be* $\mathbf{B}_2^-$*-words. Then* $L_l \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ *and* $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$.

PROOF. Suppose $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ and $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$, say $\overleftarrow{\mathbf{u}}^l \equiv z_{\sigma_1} \cdots z_{\sigma_p}$. Then there exist $\mathbf{u}_1, \mathbf{v}_1 \in X^+ \cup \{\emptyset\}$ such that

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_p} \mathbf{u}_1, \qquad \mathbf{v} \equiv z_{\sigma_1} \cdots z_{\sigma_p} \mathbf{v}_1.$$

Clearly $\mathbf{u}_1 = \emptyset$ if and only if $\mathbf{v}_1 = \emptyset$, whence we may assume $\mathbf{u}_1, \mathbf{v}_1 \neq \emptyset$. There are two cases: $p < l$ and $p = l$.

Case (i) Suppose $p < l$. If the first variable a of $\mathbf{u}_1$ has multiplicity one, then $\overleftarrow{\mathbf{u}}^l$ would be $z_{\sigma_1} \cdots z_{\sigma_p} a \cdots$ instead of $z_{\sigma_1} \cdots z_{\sigma_p}$. Hence $\mathbf{u}_1 \equiv a^2 \mathbf{u}_2$ for some $\mathbf{u}_2 \in X^+ \cup \{\emptyset\}$. Similarly, $\mathbf{v}_1 \equiv b^2 \mathbf{v}_2$ for some $\mathbf{v}_2 \in X^+ \cup \{\emptyset\}$ and $b = \mathfrak{h}(\mathbf{v}_2)$. Note that $\mathfrak{c}(a\mathbf{u}_2) = \mathfrak{c}(b\mathbf{v}_2)$, therefore

$$\begin{aligned}
\mathbf{u}_1 &\equiv a^2\mathbf{u}_2 \\
&\equiv_{(\infty:0:0)} a^2\mathbf{u}_2^2 && \text{by } (\infty:0:0) \\
&\equiv_{\nabla} (a\mathbf{u}_2)^2 && \text{by Lemma 4.3.2(3)} \\
&\equiv_{\nabla} (b\mathbf{v}_2)^2 && \text{by Lemma 4.3.2(4)} \\
&\equiv_{\nabla} b^2\mathbf{v}_2^2 && \text{by Lemma 4.3.2(3)} \\
&\equiv_{(\infty:0:0)} b^2\mathbf{v}_2 && \text{by } (\infty:0:0) \\
&\equiv \mathbf{v}_1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
L_l &\models \nabla \cup \{(l:0:0)\} && \text{by Theorem 4.5.2} \\
&\vdash \nabla \cup \{(\infty:0:0)\} && \text{by Lemma 4.5.3} \\
&\vdash \mathbf{u}_1 = \mathbf{v}_1 \vdash \mathbf{u} = \mathbf{v}.
\end{aligned}$$

Case (ii) Suppose $p = l$. Then

$$\begin{aligned}
\mathbf{u} &\equiv z_{\sigma_1}\cdots z_{\sigma_l}\mathbf{u}_1 \\
&\equiv_{(l:0:0)} z_{\sigma_1}\cdots z_{\sigma_l}\mathbf{u}_1^2 && \text{by } (l:0:0) \\
&\equiv_{\nabla} z_{\sigma_1}\cdots z_{\sigma_p}\mathbf{v}_1^2 && \text{by Lemma 4.3.2(4)} \\
&\equiv_{(l:0:0)} z_{\sigma_1}\cdots z_{\sigma_p}\mathbf{v}_1 && \text{by } (l:0:0) \\
&\equiv \mathbf{v}.
\end{aligned}$$

Therefore $L_l \models \nabla \cup \{(l:0:0)\} \vdash \mathbf{u} = \mathbf{v}$.

Conversely, suppose $L_l \models \mathbf{u} = \mathbf{v}$. Then clearly $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ since L_l contains $L_0 \cong Y$. Seeking a contradiction, suppose $\overleftarrow{\mathbf{u}}^l \not\equiv \overleftarrow{\mathbf{v}}^l$. Then by Lemma 4.5.6, $\nabla \cup \{\mathbf{u} = \mathbf{v}\} \vdash (s:0:\infty)$ for some $s < l$. Thus L_l satisfies

$$(s:0:\infty): x_1\cdots x_s y^2 w^2 = x_1 \cdots x_s y w^2.$$

But $\mathfrak{l}_1\cdots\mathfrak{l}_s\mathfrak{l}_{s+1}^2\mathfrak{l}_{s+2}^2 \neq \mathfrak{l}_1\cdots\mathfrak{l}_s\mathfrak{l}_{s+1}\mathfrak{l}_{s+2}^2$ because the first 0 entry of the element on the left occurs at the $s+1^{\text{st}}$ coordinate, while that on the right at the $s+2^{\text{nd}}$. Hence $L_l \not\models (s:0:\infty)$ contradictorily. ■

Corollary 4.5.8 *The varieties in $\mathfrak{L}$ form the strictly increasing chain*

$$\mathbf{L}_0 \subset \mathbf{L}_1 \subset \cdots \subset \mathbf{L}_\infty.$$

PROOF. Suppose $l < m$. By Proposition 4.5.7, L_m is a semigroup in $\mathbf{L}_m$ that does not satisfy $(l : 0 : 0)$. Therefore $L_m \notin \mathbf{L}_l$ and $\mathbf{L}_l \neq \mathbf{L}_m$. The result now follows from (4.8). ■

Theorem 4.5.9 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{L}_l$*-words such that* $\mathbf{L}_l \models \mathbf{u} = \mathbf{v}$, *then* $\mathbf{u} \equiv \mathbf{v}$.

PROOF. By Lemma 4.5.4, an $\mathbf{L}_l$-word has one of the following forms

$$\begin{array}{ll} z_{\sigma_1} \cdots z_{\sigma_k} & 1 \leq k \leq l, \\ z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_k}^2 & 1 \leq p \leq l+1. \end{array}$$

It is easy to see that if two such words form an identity of $\mathbf{L}_l$, they must be identical by Proposition 4.5.7. ■

Theorem 4.5.10 (1) $\mathbf{L}_l \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{L}_l} \equiv \mathbf{v}^{\mathbf{L}_l}$;

(2) *The set of all* $\mathbf{L}_l$*-words constitutes the* $\mathbf{L}_l$*-free semigroup with the operation* $\cdot$ *given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{L}_l};$$

(3) $\mathbf{L}_l = [\nabla \cup \{(l : 0 : 0)\}] = [l : 0 : 0]^{\mathbf{B}_2^-}$.

PROOF. The proof of this theorem follows closely to that of Theorem 4.2.11. ■

If l is finite then $\mathbf{L}_l$ is finitely based and generated by the finite semigroup L_l. Although $\mathbf{L}_\infty$ is also finitely based, it is not generated by any finite semigroups.

Proposition 4.5.11 *The variety* $\mathbf{L}_\infty$ *is not generated by any finite semigroup. Consequently,* $\mathbf{L}_l$ *is generated by a finite semigroup if and only if* l *is finite.*

PROOF. Suppose $S = \{s_1, \ldots, s_k\}$ is a nontrivial semigroup in $\mathbf{L}_\infty$. Let $x_1, \ldots, x_{k+1} \in S$. Then $x_i = x_j$ in S for some i, j with $i \leq j$. If $j = k+1$, then

$$x_1 \cdots x_{i-1} (x_i \cdots x_k x_{k+1}) = x_1 \cdots x_{i-1} x_i \cdots x_k x_{k+1}^2$$

by III. If $j < k+1$ then

$$\begin{aligned}
& x_1 \cdots x_{i-1} (x_i \cdots x_{j-1} x_j) x_{j+1} \cdots x_{k+1} \\
&= x_1 \cdots x_i \cdots x_{j-1} x_j^2 x_{j+1} \cdots x_{k+1} && \text{by III} \\
&= x_1 \cdots x_i \cdots x_{j-1} x_j^2 (x_{j+1} \cdots x_{k+1})^2 && \text{by } (\infty : 0 : 0) \\
&= x_1 \cdots x_i \cdots x_{j-1} x_j^2 \left(x_{j+1} \cdots x_k x_{k+1}^2\right)^2 && \text{by Corollary 4.3.2(4)} \\
&= x_1 \cdots \left(x_i \cdots x_{j-1} x_j^2\right) x_{j+1} \cdots x_k x_{k+1}^2 && \text{by } (\infty : 0 : 0) \\
&= x_1 \cdots x_i \cdots x_{j-1} x_j x_{j+1} \cdots x_k x_{k+1}^2 && \text{by III.}
\end{aligned}$$

Therefore, S satisfies $(k : 0 : 0)$ and is contained in $\mathbf{L}_k$, whence by Corollary 4.5.8, it cannot generate $\mathbf{L}_\infty$. ■

For a subset C of $\mathbb{N}_0^\infty$, let $\sup C$ denote the least upper bound (*supremum*) of C in $\mathbb{N}_0^\infty$, that is,

$$\sup C = \begin{cases} \max C & \text{if } |C| < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 4.5.12 *The chain $\mathfrak{L}$ is complete. In particular, if $C \subseteq \mathbb{N}_0^\infty$ then*

$$\bigcap\nolimits_{l \in C} \mathbf{L}_l = \mathbf{L}_{\min C}, \qquad \bigvee\nolimits_{l \in C} \mathbf{L}_l = \mathbf{L}_{\sup C}.$$

PROOF. Clearly $\bigcap_{l \in C} \mathbf{L}_l = \mathbf{L}_{\min C}$ and that $\bigvee_{l \in C} \mathbf{L}_l = \bigcup_{l \in C} \mathbf{L}_l = \mathbf{L}_{\max C}$ if C is finite. Hence suppose C is infinite. Seeking a contradiction, assume $\bigvee_{l \in C} \mathbf{L}_l = \mathbf{V} \neq \mathbf{L}_\infty$. Then there exists an identity $\mathbf{u} = \mathbf{v}$ such that $\mathbf{V} \models \mathbf{u} = \mathbf{v}$ and $\mathbf{L}_\infty \not\models \mathbf{u} = \mathbf{v}$. Note that $\mathbf{u}^{\mathbf{L}_\infty} = \mathbf{v}^{\mathbf{L}_\infty}$ will still be an identity such that $\mathbf{V} \models \mathbf{u}^{\mathbf{L}_\infty} = \mathbf{v}^{\mathbf{L}_\infty}$ and $\mathbf{L}_\infty \not\models \mathbf{u}^{\mathbf{L}_\infty} = \mathbf{v}^{\mathbf{L}_\infty}$. Since $\mathbf{u}^{\mathbf{L}_\infty}$ and $\mathbf{v}^{\mathbf{L}_\infty}$ have the same finite content, say with m variables, they are indeed $\mathbf{L}_m$-words. Now there exists some $k \in C$ such that $k \geq m$. Since

$$\mathbf{L}_m \subseteq \mathbf{L}_k \subset \mathbf{V} \models \mathbf{u}^{\mathbf{L}_\infty} = \mathbf{v}^{\mathbf{L}_\infty},$$

we have $\mathbf{u}^{\mathbf{L}_\infty} \equiv \mathbf{v}^{\mathbf{L}_\infty}$ by Theorem 4.5.9, whence $\mathbf{u}^{\mathbf{L}_\infty} = \mathbf{v}^{\mathbf{L}_\infty}$ is contradictorily satisfied by $\mathbf{L}_\infty$. Consequently $\bigvee_{l \in C} \mathbf{L}_l = \mathbf{L}_\infty$ and $\mathfrak{L}$ is complete. ■

We now define the semigroup R_n that is the dual of L_n for $n \in \mathbb{N}_0^\infty$. Let $R_n = (L_n, *)$ be the semigroup with L_n as its underlying set of elements and $*$ as binary operation defined by

$$x * y = yx,$$

where the product on the right of this equality occurs within L_n. Then the variety $\mathbf{R}_n$, generated by R_n is defined by the dual basis of $\mathbf{L}_n$:

$$\mathbf{R}_n = [\nabla \cup \{(0:0:n)\}] = [0:0:n]^{\mathbf{B}_2^-} .$$

It is more suggestive to use r as the subscript for R_r, just as l is dually used in L_l. Define

$$\mathfrak{R} = \{\mathbf{R}_r \mid r \in \mathbb{N}_0^\infty\} .$$

It is clear that all the concepts and properties of L_l and $\mathfrak{L}$ carry over symmetrically to R_r and $\mathfrak{R}$; we state some of these for the convenience of the reader.

A $\mathbf{B_2^-}$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ will be called an $\mathbf{R}_r$*-word* if

(R1) $k \geq r+1 \Longrightarrow \alpha_{k-r} = 2$;

(R2) $\alpha_i = 2 \Longrightarrow \alpha_1 = \cdots = \alpha_i = 2$.

Note that condition (R1) is vacuous when $r = \infty$. The $\mathbf{R}_r$-word that is $\equiv_{\mathbf{R}_r}$-related to a word $\mathbf{v}$ is denoted by $\mathbf{v}^{\mathbf{R}_r}$.

Let $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ be a $\mathbf{B_2^-}$-word. The *r-right segment* $\overrightarrow{\mathbf{u}}^r$ of $\mathbf{u}$ is defined as follows. If $r = 0$ or $\alpha_k = 2$ then $\overrightarrow{\mathbf{u}}^r = \emptyset$; if $r \geq 1$ and $\alpha_k = 1$, then $\overrightarrow{\mathbf{u}}^r$ is the subword $z_{\sigma_q}^{\alpha_q} \cdots z_{\sigma_k}^{\alpha_k}$ of $\mathbf{u}$ where q is minimal in $\mathbb{I}_{k-r+1}^k$ such that $\alpha_q = \cdots = \alpha_k = 1$. If we say $z_{\sigma_q} \cdots z_{\sigma_k}$ is an r-right segment for some word, then it is understood that the subscripts $\sigma_q, \ldots, \sigma_k$ are all distinct and that $k - r + 1 \leq q$ necessarily.

As a dual to Lemma 4.5.4, an $\mathbf{R}_r$-word has one of the following forms (with $\sigma_1, \ldots, \sigma_k$ distinct):

$$\begin{aligned} &\mathbf{u}_1 \equiv z_{\sigma_1} \cdots z_{\sigma_k} && 1 \leq k \leq l, \\ &\mathbf{u}_2 \equiv z_{\sigma_1}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k} && \sigma_1 < \cdots < \sigma_q \text{ and } k - r \leq q \leq k, \end{aligned}$$

(note: if $q = k$ then $z_{\sigma_{q+1}} \cdots z_{\sigma_k} = \emptyset$). In this case, $\overrightarrow{\mathbf{u}_1}^r \equiv z_{\sigma_m} \cdots z_{\sigma_k}$ where $m = \max\{1, k-r+1\}$, and $\overrightarrow{\mathbf{u}_2}^r \equiv z_{\sigma_n} \cdots z_{\sigma_k}$ where $n = \max\{q+1, k-r+1\}$.

Proposition 4.5.13 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be* $\mathbf{B_2^-}$*-words. Then* $R_r \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ *and* $\overrightarrow{\mathbf{u}}^r \equiv \overrightarrow{\mathbf{v}}^r$.

Theorem 4.5.14 (1) $\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{R}_r} \equiv \mathbf{v}^{\mathbf{R}_r}$;

(2) *The set of all* $\mathbf{R}_r$*-words constitutes the* $\mathbf{R}_r$*-free semigroup with the operation* $\cdot$ *given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{R}_r} ;$$

(3) $\mathbf{R}_r = [0:0:r]^{\mathbf{B}_2^-}$.

Proposition 4.5.15 *The variety* $\mathbf{R}_r$ *is generated by a finite semigroup if and only if* r *is finite.*

Proposition 4.5.16 *The chain* $\mathfrak{R}$ *is complete. In particular, if* $C \subseteq \mathbb{N}_0^\infty$ *then*

$$\bigcap_{r\in C} \mathbf{R}_r = \mathbf{R}_{\min C}, \qquad \bigvee_{r\in C} \mathbf{R}_r = \mathbf{R}_{\sup C}.$$

4.6 The Varieties $\mathbf{N}_n$

In this section, we introduce the semigroups N_n, indexed by $n \in \mathbb{N}_0^\infty$, and investigate them with a similar approach to what we did for L_l and R_r in Section 5. For each $n \in \mathbb{N}_0^\infty$, the variety $\mathbf{N}_n$ generated by N_n is contained in both $\mathbf{L}_n$ and $\mathbf{R}_n$ (Corollary 4.6.2(1)). But the semigroups N_n do resemble L_l and R_r as we will show that the varieties $\mathbf{N}_n$ also form a strictly increasing complete chain (see Corollary 4.5.8 and Proposition 4.5.12), and that $\mathbf{N}_\infty$ cannot be generated by any finite semigroup.

Let ψ be the relation on U_∞ that only identifies elements that have at least one zero coordinate. Equivalently,

$$\psi = (Z \times Z) \cup \{(x, x) \mid x \in U_\infty\}$$

where Z is the set of all elements of U_∞ with some zero coordinates. It is straightforward to show that ψ is a congruence on U_∞. For $n \in \mathbb{N}_0^\infty$, define the subsemigroup N_n of U_∞/ψ by

$$N_n = \langle \mathfrak{n}_i \mid 1 \leq i \leq n \rangle \cup \{\mathbf{0}\}$$

where $\mathfrak{n}_i = \mathfrak{u}_i/\psi$. Note that if $i < j < k$, then the element $\mathfrak{u}_i\mathfrak{u}_j\mathfrak{u}_k = \mathfrak{u}_i\mathfrak{u}_k$ has no 0 coordinates, so that $\mathfrak{n}_i\mathfrak{n}_j\mathfrak{n}_k = \mathfrak{n}_i\mathfrak{n}_k$. But if $i \geq j$, then $\mathfrak{n}_i a \mathfrak{n}_j = \mathbf{0}$ for all $a \in N_n$. Therefore all nonzero elements of N_n are of the form $\mathfrak{n}_i\mathfrak{n}_j$ with $i < j$. If we define $\binom{0}{2} = \binom{1}{2} = \binom{\infty}{2} = 0$, then N_n is a semigroup of order $\binom{n}{2} + n + 1$ for all $n \in \mathbb{N}_0^\infty$. In this section, n always denote an element in $\mathbb{N}_0^\infty$ unless otherwise stated.

Let $\mathbf{N}_n$ denote the variety generated by N_n, and let $\mathfrak{N} = \{\mathbf{N}_n \mid n \in \mathbb{N}_0^\infty\}$. Clearly $N_m \subset N_n$ whenever $m, n \in \mathbb{N}_0^\infty$ and $m < n$. Therefore the varieties in $\mathfrak{N}$ form the chain

$$\mathbf{T} = \mathbf{N}_0 \subseteq \mathbf{N}_1 \subseteq \cdots \subseteq \mathbf{N}_\infty. \tag{4.10}$$

Let $\mathbf{u}$ be a word that involves the variables $x_1, \ldots, x_n$, and suppose $x \neq x_i$ for all i. If S is a semigroup satisfying the identities $\mathbf{u}x = x\mathbf{u} = \mathbf{u}$, then $\mathbf{u}(a_1, \ldots, a_n)$ is the zero in S

for all $a_1, \ldots, a_n \in S$, since

$$\mathbf{u}(a_1, \ldots, a_n)\, b = b\mathbf{u}(a_1, \ldots, a_n) = \mathbf{u}(a_1, \ldots, a_n)$$

for all $b \in S$. Hence it is unambiguous to write $\mathbf{u} = \mathbf{0}$ in place of $\mathbf{u}x = x\mathbf{u} = \mathbf{u}$. Recall from Section 4 that

$$\mathbf{x}_n = \begin{cases} x_1 \cdots x_n & \text{if } n \in \mathbb{N}_0, \\ x^2 & \text{if } n = \infty. \end{cases}$$

For this section, let $\nabla_n = \nabla \cup \{\mathbf{x}_{n+1} = \mathbf{0}\}$. In particular, $\nabla_\infty = \nabla \cup \{x^2 = \mathbf{0}\}$.

Theorem 4.6.1 *The semigroup N_n satisfies the identities in ∇_n. Consequently, N_n belongs to $[\mathbf{x}_{n+1} = \mathbf{0}]^{\mathbf{B}_2^-}$.*

PROOF. It is easy to show that $N_n \models x^2 = \mathbf{0} = xyx \vdash \nabla$. Also it is easy to show that $N_n \models \mathbf{x}_{n+1} = \mathbf{0}$ if $n \in \{0, \infty\}$. Therefore we may assume $n \in \mathbb{N}$. Now if $a_1, \ldots, a_{n+1} \in N_n$, then some generator from $\{\mathfrak{n}_1, \ldots, \mathfrak{n}_n\}$ must appear twice in the factorization of $a_1 \cdots a_{n+1}$. Since N_n satisfies $x^2 = \mathbf{0} = xyx$, this product must be $\mathbf{0}$, whence $N_n \models \mathbf{x}_{n+1} = \mathbf{0}$. ■

Corollary 4.6.2 (1) *$\mathbf{N}_n \subseteq \mathbf{L}_n \cap \mathbf{R}_n$ for all $n \in \mathbb{N}_0^\infty$;*

(2) *$\mathbf{Y} \not\subseteq \mathbf{N}_n$ for all $n \in \mathbb{N}_0^\infty$.*

PROOF. (1) Since

$$\begin{aligned} \mathbf{x}_{n+1} = \mathbf{0} &\vdash \begin{cases} x_1 \cdots x_n x_{n+1} = \mathbf{0} & \text{if } n \in \mathbb{N}_0 \\ x^2 = \mathbf{0} & \text{if } n = \infty \end{cases} \\ &\vdash \begin{cases} x_1 \cdots x_n y^2 = \mathbf{0} = x_1 \cdots x_n y & \text{if } n \in \mathbb{N}_0 \\ x^2 y^2 = \mathbf{0} = x^2 y & \text{if } n = \infty \end{cases} \\ &\vdash (n : 0 : 0), \end{aligned}$$

$\mathbf{N}_n \subseteq \mathbf{L}_n$ by Theorems 4.5.10(3) and 4.6.1. To show $\mathbf{N}_n \subseteq \mathbf{R}_n$ is symmetrical.

(2) Clearly Y does not satisfy $\mathbf{x}_{n+1} = \mathbf{0}$. Therefore $Y \notin \mathbf{N}_n$ by Theorem 4.6.1. ■

Lemma 4.6.3 $\nabla_0 \vdash \nabla_1 \vdash \cdots \vdash \nabla_\infty \vdash xyx = \mathbf{0}$.

PROOF. Clearly $\mathbf{x}_{n+1} = \mathbf{0} \vdash \mathbf{x}_{n+2} = \mathbf{0}$ for each $n \in \mathbb{N}_0$ so that $\nabla_n \vdash \nabla_{n+1}$. For $n \geq 1$, $\mathbf{x}_{n+1} = \mathbf{0} \vdash x^2 = x^{n+1} = \mathbf{0}$ by I. Hence $\nabla_n \vdash \nabla_\infty$. Finally, $x^2 = \mathbf{0} \vdash xyx = x^2yx = \mathbf{0}$ by III. Hence $\nabla_\infty \vdash xyx = \mathbf{0}$. ■

A word $\mathbf{u} \in X^+$ will be called an $\mathbf{N}_n$-*word* if either $\mathbf{u} \equiv z_1^2$, or

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$$

for some distinct $\sigma_1, \ldots, \sigma_k \in \mathbb{N}$ and $k \leq n$. Let $\equiv_n$ denote the fully invariant congruence over $[\nabla_n]$ in this section. As in previous sections, it will be shown that each word in X^+ is $\equiv_n$-related to an $\mathbf{N}_n$-word. Note that $\nabla_n \vdash x^2 = \mathbf{0}$ by Lemma 4.6.3. Therefore the $\equiv_n$-class of z_1^2 is actually the zero of $X^+/\equiv_n$. Thus it is unambiguous and more convenient to write $\mathbf{0}$ instead of the $\mathbf{N}_n$-word z_1^2. If we say $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$ is a (nonzero) $\mathbf{N}_n$-word, then it is understood that $\sigma_1, \ldots, \sigma_k$ are all distinct with $k \leq n$.

Proposition 4.6.4 *Each word in* X^+ *is* $\equiv_n$*-related to an* $\mathbf{N}_n$*-word.*

PROOF. If $\mathbf{u}$ is linear with at most n variables, then it is already an $\mathbf{N}_n$-word by definition. Otherwise $\mathbf{u}$ contains a variable with multiplicity two, or it is a product of at least $n+1$ variables. In the former case, $\mathbf{u} \equiv_n \mathbf{0}$ since $\nabla_n \vdash x^2 = xyx = \mathbf{0}$ by Lemma 4.6.3; in the latter, $\mathbf{u} \equiv_n \mathbf{0}$ by $\mathbf{x}_{n+1} = \mathbf{0}$. ■

The $\mathbf{N}_n$-word that is obtained from $\mathbf{u}$ by the method in the above proof is denoted by $\mathbf{u}^{\mathbf{N}_n}$. Among all canonical words defined for different varieties in this thesis, $\mathbf{N}_n$-words are the simplest and easiest to describe:

$$\mathbf{u}^{\mathbf{N}_n} \equiv \begin{cases} \mathbf{u} & \text{if } \mathbf{u} \text{ is linear and } |\mathbf{u}| \leq n, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Lemma 4.6.5 *Let* $\mathbf{u}, \mathbf{v}$ *be nonzero* $\mathbf{N}_n$*-words. Then* $\mathbf{N}_n \models \mathbf{u} = \mathbf{v}$ *implies* $\mathbf{u} \equiv \mathbf{v}$.

PROOF. We may assume that

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_p}, \qquad \mathbf{v} \equiv z_{\tau_1} \cdots z_{\tau_q}$$

where $p, q \leq n$. If $\sigma_j \notin \{\tau_1, \ldots, \tau_q\}$, then letting S be the following substitution

$$z_\varphi \longrightarrow \begin{cases} \mathrm{n}_i & \text{if } \varphi = \tau_i, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

into $\mathbf{N}_n$, we contradictorily have $\mathbf{u}(\mathrm{S}) = \mathbf{0} \neq \mathfrak{n}_1\mathfrak{n}_q = \mathbf{v}(\mathrm{S})$. Therefore $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ and $p = q$. Now the sequences $(\sigma_1, \ldots, \sigma_p)$ and $(\tau_1, \ldots, \tau_q)$ must be the same because otherwise, for some $i > j$, we have

$$\begin{aligned} \mathbf{u}(\mathrm{S}) &= \cdots \mathfrak{n}_i \cdots \mathfrak{n}_j \cdots \\ &= \mathbf{0} \\ &\neq \mathfrak{n}_1\mathfrak{n}_q = \mathbf{v}(\mathrm{S}). \end{aligned}$$

■

Theorem 4.6.6 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{N}_n$*-words such that* $\mathbf{N}_n \models \mathbf{u} = \mathbf{v}$, *then* $\mathbf{u} \equiv \mathbf{v}$.

PROOF. It is obvious that if one of $\mathbf{u}, \mathbf{v}$ is $\mathbf{0}$ then the other must also be $\mathbf{0}$. Therefore assume both $\mathbf{u}, \mathbf{v}$ to be nonzero, whence the result follows from Lemma 4.6.5. ■

Theorem 4.6.7 (1) $\mathbf{N}_n \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n}$;

(2) *The set of all* $\mathbf{N}_n$*-words constitutes the* $\mathbf{N}_n$*-free semigroup with the operation* $\cdot$ *given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{N}_n};$$

(3) $\mathbf{N}_n = [\nabla \cup \{\mathbf{x}_{n+1} = \mathbf{0}\}] = [\mathbf{x}_{n+1} = \mathbf{0}]^{\mathbf{B}_2^-} = \left[\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}\right]$.

PROOF. To prove (1), (2) and the first two equalities of (3) is similar to proving Theorem 4.2.11. It remains to verify the last equality of (3). By Lemma 4.6.3,

$$[\mathbf{x}_{n+1} = \mathbf{0}]^{\mathbf{B}_2^-} \subseteq \left[\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}\right],$$

and the reverse inclusion follows from the easy observation that $x^2 = xyx = \mathbf{0} \vdash \nabla$. ■

For the remainder of this section, we will establish some results for N_n that are common to L_l and R_r, namely, that $\mathfrak{N}$ is a complete chain and $\mathbf{N}_\infty$ is not generated by any finite semigroup. We will also characterize, with the help of $\mathbf{N}_\infty$, all subvarieties of $\mathbf{A_0}$ that do not contain $\mathbf{Y}$. Some consequences of this characterization include a representation of $\mathcal{L}(\mathbf{A_0})$ as a disjoint union of two intervals, and that $\mathbf{B}_2^-$ is the unique anti-atom of $\mathbf{A_0}$.

Corollary 4.6.8 *The varieties in* $\mathfrak{N}$ *form the strictly increasing chain*

$$\mathbf{N}_0 \subset \mathbf{N}_1 \subset \cdots \subset \mathbf{N}_\infty$$

PROOF. Suppose $m < n$. By Theorem 4.6.7(3), N_n is a semigroup in $\mathbf{N}_n$ that does not satisfy $\mathbf{x}_{m+1} = \mathbf{0}$. Therefore $N_n \notin \mathbf{N}_m$ and $\mathbf{N}_m \neq \mathbf{N}_n$. The result now follows from (4.10). ■

Proposition 4.6.9 *The variety* $\mathbf{N}_\infty$ *is not generated by any finite semigroup. Consequently,* $\mathbf{N}_n$ *is generated by a finite semigroup if and only if* n *is finite.*

PROOF. Suppose that S is a nontrivial semigroup in $\mathbf{N}_\infty$ with k elements and let $s_1, \ldots, s_{k+1} \in S$. Then two of $s_1, \ldots, s_{k+1}$ must coincide so that since S satisfies $x^2 = xyx = \mathbf{0}$, we must have $s_1 \cdots s_{k+1} = \mathbf{0}$. Consequently S satisfies $\mathbf{x}_{k+1} = \mathbf{0}$ and is in $\mathbf{N}_k$. Therefore by Corollary 4.6.8, S cannot generate $\mathbf{N}_\infty$ ■

Proposition 4.6.10 *The chain* $\mathfrak{N}$ *is complete. In particular, if* $C \subseteq \mathbb{N}_0^\infty$ *then*

$$\bigcap\nolimits_{n \in C} \mathbf{N}_n = \mathbf{N}_{\min C}, \qquad \bigvee\nolimits_{n \in C} \mathbf{N}_n = \mathbf{N}_{\sup C}.$$

PROOF. Clearly $\bigcap_{n \in C} \mathbf{N}_n = \mathbf{N}_{\min C}$ and that $\bigvee_{n \in C} \mathbf{N}_n = \bigcup_{n \in C} \mathbf{N}_n = \mathbf{N}_{\max C}$ if C is finite. Hence suppose C is infinite. Seeking a contradiction, assume $\bigvee_{n \in C} \mathbf{N}_n = \mathbf{V} \neq \mathbf{N}_\infty$. Then there exists an identity $\mathbf{u} = \mathbf{v}$ such that $\mathbf{V} \models \mathbf{u} = \mathbf{v}$ and $\mathbf{N}_\infty \not\models \mathbf{u} = \mathbf{v}$. Note that $\mathbf{u}^{\mathbf{N}_\infty} = \mathbf{v}^{\mathbf{N}_\infty}$ will still be an identity such that $\mathbf{V} \models \mathbf{u}^{\mathbf{N}_\infty} = \mathbf{v}^{\mathbf{N}_\infty}$ and $\mathbf{N}_\infty \not\models \mathbf{u}^{\mathbf{N}_\infty} = \mathbf{v}^{\mathbf{N}_\infty}$.

Suppose $\mathbf{u}^{\mathbf{N}_\infty}$ is $\mathbf{0}$. Then $\mathbf{v}^{\mathbf{N}_\infty}$ cannot be $\mathbf{0}$ and so must be linear. Hence $\mathbf{V} \models \mathbf{x}_k = \mathbf{0}$ where $k = \left|\mathbf{v}^{\mathbf{N}_\infty}\right|$. But there exists $s \in C$ such that $s \geq k$, whence we have $\mathbf{N}_s \subset \mathbf{V} \models \mathbf{x}_k = \mathbf{0}$, contradicting Corollary 4.6.8. Hence both $\mathbf{u}$ and $\mathbf{v}$ are different from $\mathbf{0}$; let $n = \max\left\{\left|\mathbf{u}^{\mathbf{N}_\infty}\right|, \left|\mathbf{v}^{\mathbf{N}_\infty}\right|\right\}$. Then $\mathbf{u}^{\mathbf{N}_\infty}$ and $\mathbf{v}^{\mathbf{N}_\infty}$ are both $\mathbf{N}_n$-words. Now there exists some $t \in C$ such that $t \geq n$. Since $\mathbf{N}_n \subseteq \mathbf{N}_t \subset \mathbf{V} \models \mathbf{u}^{\mathbf{N}_\infty} = \mathbf{v}^{\mathbf{N}_\infty}$, we have $\mathbf{u}^{\mathbf{N}_\infty} \equiv \mathbf{v}^{\mathbf{N}_\infty}$ by Theorem 4.6.6, whence $\mathbf{u} = \mathbf{v}$ is contradictorily satisfied by $\mathbf{N}_\infty$. ■

Lemma 4.6.11 *Let* $\mathbf{V}$ *be a subvariety of* $\mathbf{A_0}$. *Then* $\mathbf{Y} \nsubseteq \mathbf{V}$ *if and only if* $\mathbf{V} \subseteq \mathbf{N}_\infty$.

PROOF. Suppose $\mathbf{Y} \nsubseteq \mathbf{V}$. Then there exists an identity $\mathbf{u} = \mathbf{v}$ of $\mathbf{V}$ such that $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$, say $z \in \mathfrak{c}(\mathbf{u}) \setminus \mathfrak{c}(\mathbf{v})$. Let S denote the following substitution into X^+:

$$w = \begin{cases} y^2 & \text{if } w = z, \\ x^2 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} x^2y^2x^2 &\equiv_{\mathbf{A_0}} x^2\mathbf{u}\,(\mathrm{S})\,x^2 && \text{by I} \\ &\equiv_{\mathbf{V}} x^2\mathbf{v}\,(\mathrm{S})\,x^2 \\ &\equiv_{\mathbf{A_0}} x^2 && \text{by I} \end{aligned}$$

so that $\mathbf{V} \models x^2y^2x^2 = x^2$. But

$$\begin{aligned} x^2y &\equiv_{\mathbf{V}} x^2\left(y^2x^2y\right) \\ &\equiv_{\mathbf{A_0}} \left(x^2x^2\right)y^2x^2 && \text{by Corollary 4.2.2(4)} \\ &\equiv_{\mathbf{A_0}} x^2y^2x^2 && \text{by I} \\ &\equiv_{\mathbf{V}} x^2 \end{aligned}$$

and, by symmetry, $yx^2 \equiv_{\mathbf{V}} x^2$. Therefore $\mathbf{V}$ satisfies $x^2y = yx^2 = x^2$, or equivalently, $\mathbf{V} \models x^2 = \mathbf{0}$. Furthermore,

$$\begin{aligned} xyx &\equiv_{\mathbf{A_0}} x^2y^2x^2 && \text{by III} \\ &\equiv_{\mathbf{V}} \mathbf{0}. \end{aligned}$$

Consequently, $\mathbf{V} \subseteq \left[x^2 = \mathbf{0} = xyx\right] = \mathbf{N}_\infty$ by Theorem 4.6.7(3).

The converse clearly holds since $\mathbf{Y} \nsubseteq \mathbf{N}_\infty$. ■

Corollary 4.6.12 *The lattice $\mathcal{L}(\mathbf{A_0})$ is the disjoint union of the intervals $[\mathbf{Y}, \mathbf{A_0}]$ and $[\mathbf{T}, \mathbf{N}_\infty]$.*

Proposition 4.6.13 *The subvariety $\mathbf{B_2^-}$ is the unique anti-atom of $\mathbf{A_0}$.*

PROOF. Suppose $\mathbf{V} \in \mathcal{L}(\mathbf{A_0})$. If $\mathbf{Y} \subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{B_2^-}$ by Lemma 4.3.11(1). If $\mathbf{Y} \nsubseteq \mathbf{V}$ then $\mathbf{V} \in [\mathbf{T}, \mathbf{N}_\infty]$ by Lemma 4.6.11 so that $\mathbf{V} \subseteq \mathbf{N}_\infty \subseteq \mathbf{B_2^-}$. ■

Corollary 4.6.14 *If* $\mathbf{A_0} \nvDash \mathbf{u} = \mathbf{v}$ *then* $[\mathbf{u} = \mathbf{v}]^{\mathbf{A_0}} = [\mathbf{u} = \mathbf{v}]^{\mathbf{B_2^-}}$.

PROOF. By Proposition 4.6.13, all proper subvarieties of $\mathbf{A_0}$ satisfies the identity V. Therefore if $\mathbf{A_0} \nvDash \mathbf{u} = \mathbf{v}$, then $[\mathbf{u} = \mathbf{v}]^{\mathbf{A_0}} \subseteq \mathbf{B_2^-}$ and

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{A_0}} = [\mathbf{u} = \mathbf{v}]^{\mathbf{A_0}} \cap \mathbf{B_2^-} = [\mathbf{u} = \mathbf{v}]^{\mathbf{B_2^-}} .$$

■

Except for $\mathbf{B_2^-}$, all subvarieties of $\mathbf{A_0}$ that we have encountered so far and those that will be investigated later are of the form $[\pi]^{\mathbf{B_2^-}}$ for some identity π not satisfies by $\mathbf{A_0}$. Therefore in view of Corollary 4.6.14, all these subvarieties may actually be written as $[\pi]^{\mathbf{A_0}}$. For example,

$$\mathbf{R}_r = [0 : 0 : r]^{\mathbf{A_0}} , \qquad \mathbf{N}_n = [\mathbf{x}_{n+1} = \mathbf{0}]^{\mathbf{A_0}} .$$

But it is more suggestive to write $[\pi]^{\mathbf{B_2^-}}$ since the important identity V can be "read off" from $\mathbf{B_2^-}$.

Corollary 4.6.15 *The semigroup* U_∞ *generates* $\mathbf{A_0}$.

PROOF. If $V(U_\infty) \neq \mathbf{A_0}$ then $U_\infty \in \mathbf{B_2^-}$ by Proposition 4.6.13. But it is easy to show that the idempotents of U_∞ do not commute so that $U_\infty \nvDash$ V, a contradiction. ■

4.7 The Varieties $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$

Having described the varieties $\mathbf{L}_l, \mathbf{R}_r$ and $\mathbf{N}_n$ in previous sections, the next natural step is to investigate the intersections and joins formed by these varieties. For simplicity, we write $\mathbf{UV} = \mathbf{U} \vee \mathbf{V}$ for any varieties $\mathbf{U}$ and $\mathbf{V}$. In particular, we write

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{L}_l \vee \mathbf{N}_n \vee \mathbf{R}_r .$$

Note the following basic cases of joins and bases of identities:

$$\mathbf{L}_l\mathbf{N}_0\mathbf{R}_0 = \mathbf{L}_l \vee \mathbf{T} \vee \mathbf{Y} = \mathbf{L}_l = [l : 0 : 0]^{\mathbf{B_2^-}} ,$$
$$\mathbf{L}_0\mathbf{N}_0\mathbf{R}_r = \mathbf{Y} \vee \mathbf{T} \vee \mathbf{R}_r = \mathbf{R}_r = [0 : 0 : r]^{\mathbf{B_2^-}} .$$

It will be shown that the subscripts of the varieties $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ correspond closely (but not exactly) to the triples $(l:n:r)$ in the defining identities.

Let

$$\mathfrak{LRN} = \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid l,n,r \in \mathbb{N}_0^\infty\}.$$

Similar to the $\mathfrak{L}, \mathfrak{R}$ and $\mathfrak{N}$, the sets $\mathfrak{LRN}$ and $\mathfrak{LRN} \cup \mathfrak{N}$ are complete sublattices of $\mathcal{L}(\mathbf{A_0})$; these results will be verified in the next section.

In this section, we investigate the varieties $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ for general $l,n,r \in \mathbb{N}_0^\infty$ (with certain necessary constraints), the approach follows closely to those taken in previous sections on $\mathbf{L}_l, \mathbf{R}_r$ and $\mathbf{N}_n$. But we first consider $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ with $n = l+m+r$, where $(l,m,r) \in \mathbb{U}$ and

$$\mathbb{U} = \{(l,m,r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \mid (l=\infty \text{ or } r=\infty) \Rightarrow m=0\}.$$

Note that if l or r is infinite, then $l+m+r = l+r = \infty$.

Theorem 4.7.1 *If* $(l,m,r) \in \mathbb{U}$, *then the semigroup* $L_l \times N_{l+m+r} \times R_r$ *satisfies the identities* I, II, III, V *and* $(l:m:r)$. *Consequently,* $L_l \times N_{l+m+r} \times R_r$ *belongs to* $[l:m:r]^{\mathbf{B}_2^-}$.

PROOF. It is easy to show that $(l:0:0)$ and $(0:0:r)$ each implies $(l:m:r)$. Hence $L_l, R_r \models (l:m:r)$. It is also easy to show that $N_{l+m+r} \models (l:m:r)$ for $l,r \neq \infty$. If $l=\infty$, then $m=0$ and $l+m+r = l$, whence

$$\begin{aligned} N_{l+r+m} &\in \mathbf{L}_l \qquad \text{by Corollary 4.6.2(1)} \\ &\models (l:0:0) \vdash (l:m:r). \end{aligned}$$

Similarly, if $r=\infty$ then $N_{l+r+m} \models (l:m:r)$. Therefore $N_{l+r+m} \models (l:m:r)$ in all cases. Consequently, $L_l \times N_{l+r+m} \times R_r \models (l:m:r)$. ■

Let $n = l+m+r$ with $(l,m,r) \in \mathbb{U}$. Then a $\mathbf{B}_2^-$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ will be called an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$*-word* if all of the following statements hold.

(J1) If $k > n$, then $\alpha_{l+1} = \alpha_{k-r} = 2$;

(J2) If $\alpha_i = 2$ with $i > l+1$, then $\alpha_{l+1} = 2$;

(J3) If $\alpha_i = 2$ with $i < k-r$, then $\alpha_{k-r} = 2$;

(J4) If $\alpha_i = \alpha_j = 2$ with $i < j$, then $\alpha_i = \cdots = \alpha_j = 2$.

Note that conditions (J1) and (J2) (respectively, (J1) and (J3)) are vacuous when $l = \infty$ (respectively, $r = \infty$).

Lemma 4.7.2 *Let $n = l + m + r$ and $(l, m, r) \in \mathbb{U}$. A word $\mathbf{u}$ is an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word if and only if one of the following statements hold:*

(1) $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$ *where $\sigma_1, \ldots, \sigma_k$ are distinct and $1 \leq k \leq n$;*

(2) $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{i-1}} z_{\sigma_i}^2 \cdots z_{\sigma_j}^2 z_{\sigma_{j+1}} \cdots z_{\sigma_k}$ *where $\sigma_1, \ldots, \sigma_k$ are distinct, $\sigma_i < \cdots < \sigma_j$, $1 \leq i \leq l+1$ and $0 \leq k - j \leq r$.*

PROOF. Suppose $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ -word. Then $\sigma_1, \ldots, \sigma_k$ are distinct since $\mathbf{u}$ is a $\mathbf{B}_2^-$-word. There are two cases: $\alpha_t = 2$ for some t, and $\alpha_t = 1$ for all t.

Case (i) Suppose $\alpha_t = 2$ for some t. Then there exists a least i and greatest j such that $\alpha_i = \alpha_j = 2$. Note that if $i > l + 1$, then $\alpha_{l+1} = 2$ by (J2), contradicting the minimality of i. Hence $i \leq l + 1$. By a symmetrical argument using (J3), we have $j \geq k - r$. Hence $\alpha_i = \cdots = \alpha_j = 2$ by (J4), and $\sigma_i < \cdots < \sigma_j$ because $\mathbf{u}$ is a $\mathbf{B}_2^-$-word. Therefore

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{i-1}} z_{\sigma_i}^2 \cdots z_{\sigma_j}^2 z_{\sigma_{j+1}} \cdots z_{\sigma_k}$$

with $1 \leq i \leq l + 1$ and $0 \leq k - j \leq r$. Thus $\mathbf{u}$ satisfies the conditions in (2).

Case (ii) Suppose $\alpha_t = 1$ for all t. Then $k \leq n$ by (J1) and $\mathbf{u}$ satisfies the conditions in (1).

Conversely, it is easy to show that each word satisfying the conditions in (1) or (2) is a $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word. ■

Note the following conventions in Lemma 4.7.2(2):

$$i = 1 \Longrightarrow z_{\sigma_1} \cdots z_{\sigma_{i-1}} = \emptyset, \qquad j = k \Longrightarrow z_{\sigma_{j+1}} \cdots z_{\sigma_k} = \emptyset.$$

Whenever we say $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$ (respectively, $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{i-1}} z_{\sigma_i}^2 \cdots z_{\sigma_j}^2 z_{\sigma_{j+1}} \cdots z_{\sigma_k}$) is an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word, it is understood that the conditions in Lemma 4.7.2(1) (respectively, Lemma 4.7.2(2)) hold.

In the light of Lemma 4.7.2, an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word reduces to a $\mathbf{L}_l$-word (respectively, $\mathbf{R}_r$-word) when $m = r = 0$ (respectively, $l = m = 0$).

Let $\equiv_{(l:m:r)}$ denote the fully invariant congruence over $[\nabla \cup \{(l : m : r)\}]$. As in previous sections, we need to show that each word is $\equiv_{(l:m:r)}$-related to some $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word, a result essential to the proof of the main theorem of this section.

Lemma 4.7.3 *If $(l, m, r) \in \mathbb{U}$, then:*

(1) $\nabla \cup \{(l : m : r)\} \vdash \{(l : 0 : \infty), (\infty : 0 : r)\}$;

(2) $\nabla \cup \{(l : m : r)\} \vdash (\infty : 0 : \infty)$.

PROOF. (1) If $r = m = 0$, then postmultiplying both sides of $(l:0:0)$ by w^2 yields $(l:0:\infty)$. Moreover, if $r = \infty$, then $m = 0$ and $(l:m:r)$ is exactly $(l:0:\infty)$. Therefore assume $r < \infty$ and $(r,m) \neq (0,0)$. Recall that

$$(l:m:r): \begin{cases} \mathbf{x}_l \mathbf{y}_{m+1}^2 \mathbf{w}_r = \mathbf{x}_l \mathbf{y}_{m+1} \mathbf{w}_r & \text{if } m < \infty, \\ \mathbf{x}_l y^2 z \mathbf{w}_r = \mathbf{x}_l y z^2 \mathbf{w}_r & \text{if } m = \infty. \end{cases}$$

If $m = \infty$, then letting S be the substitution $z, w_1, \ldots, w_r \longrightarrow w^2$ into X^+, we have

$$\begin{aligned} \mathbf{x}_l y^2 w^2 &\equiv_\nabla \mathbf{x}_l y^2 w^2 \left(w^2\right)^r \qquad \text{by I} \\ &\equiv \left(\mathbf{x}_l y^2 z \mathbf{w}_r\right)(\mathrm{S}) \equiv_{(l:m:r)} \left(\mathbf{x}_l y z^2 \mathbf{w}_r\right)(\mathrm{S}) \\ &\equiv \mathbf{x}_l y \left(w^2\right)^{r+2} \equiv_\nabla \mathbf{x}_l y w^2 \qquad \text{by I.} \end{aligned}$$

If $m < \infty$, then letting T be the substitution $y_2, \ldots, y_{m+1}, w_1, \ldots, w_r \longrightarrow w^2$ into X^+, we have

$$\begin{aligned} \mathbf{x}_l y_1^2 w^2 &\equiv_\nabla \mathbf{x}_l \left(y_1^2 w^2\right) w^{2r} \qquad \text{by I} \\ &\equiv_\nabla \mathbf{x}_l \left(y_1 w^{2m}\right)^2 \left(w^2\right)^r \qquad \text{by Lemma 4.3.2(3)} \\ &\equiv \left(\mathbf{x}_l \mathbf{y}_{m+1}^2 \mathbf{w}_r\right)(\mathrm{T}) \equiv_{(l:m:r)} \left(\mathbf{x}_l \mathbf{y}_{m+1} \mathbf{w}_r\right)(\mathrm{T}) \\ &\equiv \mathbf{x}_l y_1 w^{2m} w^{2r} \equiv_\nabla \mathbf{x}_l y_1 w^2 \qquad \text{by I.} \end{aligned}$$

Hence $\nabla \cup \{(l:m:r)\} \vdash (l:0:\infty)$ in all cases. To prove $\nabla \cup \{(l:m:r)\} \vdash (\infty:0:r)$ is symmetrical.

(2) This is a consequence of (1). ■

Proposition 4.7.4 *If $n = l + m + r$ and $(l,m,r) \in \mathbb{U}$, then each word in X^+ is $\equiv_{(l,m,r)}$-related to an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word.*

PROOF. Let $\mathbf{u} \in X^+$. Since

$$\nabla \cup \{(l:m:r)\} \vdash \{(l:0:\infty), (\infty:0:r), (\infty:0:\infty)\}$$

by Lemma 4.7.3, any word deduced from $\mathbf{u}$ by invoking identities in

$$\nabla \cup \{(l:m:r), (l:0:\infty), (\infty:0:r), (\infty:0:\infty)\}$$

is $\equiv_{(l:m:r)}$-related to $\mathbf{u}$. In particular, $\mathbf{u} \equiv_{(l:m:r)} \mathbf{u}^{\mathbf{B}_2^-}$, say $\mathbf{u}^{\mathbf{B}_2^-} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$. Therefore it suffices to show $\mathbf{u}^{\mathbf{B}_2^-}$ is $\equiv_{(l:m:r)}$-related to some $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word. There are two cases: $\alpha_i = 2$ for some i, or $\alpha_i = 1$ for all i.

Case (i) Suppose $\alpha_i = 2$ for some i. Let p (respectively, q) be the least (respectively, greatest) such that $\alpha_p = 2$ (respectively, $\alpha_q = 2$). If $p > l+1$, then

$$\begin{aligned} z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 &\equiv_{(l:m:r)} z_{\sigma_1} \cdots z_{\sigma_l} \left(z_{\sigma_{l+1}} \cdots z_{\sigma_{p-1}}\right)^2 z_{\sigma_p}^2 && \text{by } (l:0:\infty) \\ &\equiv_\nabla z_{\sigma_1} \cdots z_{\sigma_l} z_{\sigma_{l+1}}^2 \cdots z_{\sigma_{p-1}}^2 z_{\sigma_p}^2 && \text{by Lemma 4.3.2(3).} \end{aligned}$$

If $q < k - r$, then

$$\begin{aligned} z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k} &\equiv_{(l:m:r)} z_{\sigma_q}^2 \left(z_{\sigma_{q+1}} \cdots z_{\sigma_{k-r}}\right)^2 z_{\sigma_{k-r+1}} \cdots z_{\sigma_k} && \text{by } (\infty:0:r) \\ &\equiv_\nabla z_{\sigma_q}^2 z_{\sigma_{q+1}}^2 \cdots z_{\sigma_{k-r}}^2 z_{\sigma_{k-r+1}} \cdots z_{\sigma_k} && \text{by Lemma 4.3.2(3).} \end{aligned}$$

Therefore $\mathbf{u}^{\mathbf{B}_2^-}$ is $\equiv_{(l:m:r)}$-related to a word of the form

$$\mathbf{v} \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 z_{\sigma_{p+1}}^{\alpha_{p+1}} \cdots z_{\sigma_{q-1}}^{\alpha_{q-1}} z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k}$$

with $1 \le p \le l+1$ and $0 \le k - q \le r$. Now

$$\begin{aligned} \mathbf{v} &\equiv_{(l:m:r)} z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \left(z_{\sigma_{p+1}}^{\alpha_{p+1}} \cdots z_{\sigma_{q-1}}^{\alpha_{q-1}}\right)^2 z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k} && \text{by } (\infty:0:\infty) \\ &\equiv_\nabla z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 z_{\sigma_{p+1}}^2 \cdots z_{\sigma_{q-1}}^2 z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k} && \text{by Lemma 4.3.2(3),} \end{aligned}$$

and V can be invoked to arrange the sequence $(\sigma_p, \ldots, \sigma_q)$ into increasing order to obtain an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word (see Lemma 4.7.2(2)).

Case (ii) Suppose $\alpha_i = 1$ for all i. If $k > n$, then by Lemma 4.4.3, $\mathbf{u}^{\mathbf{B}_2^-}$ is $\equiv_{(l:m:r)}$-related to a word $z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ with $\alpha_{l+1} = \cdots = \alpha_{k-r+1} = 2$ and the result follows from Case (i). Therefore assume $k \le n$, whence $\mathbf{u}^{\mathbf{B}_2^-}$ is an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word by Lemma 4.7.2(1). ■

Let $\mathbf{u}^{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r}$ denote the $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word that is obtained from $\mathbf{u}$ by the method in the above proof. Compared to canonical forms of words in other varieties introduced in previous sections, the definition of a $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word is more complicated relatively. However, it turns out that we already have the required results to distinguish $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-words that are not $\equiv_{(l:m:r)}$-related.

Lemma 4.7.5 *Let $n = l + m + r$ and $(l, m, r) \in \mathbb{U}$. If*

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k},$$
$$\mathbf{v} \equiv z_{\tau_1} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_t}^2 z_{\tau_{t+1}} \cdots z_{\tau_n}$$

are $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-words such that $\mathbf{L}_l, \mathbf{R}_r \models \mathbf{u} = \mathbf{v}$, then $\mathbf{u} \equiv \mathbf{v}$.

Proof. Note that $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ because $Y \in \mathbf{L}_l, \mathbf{R}_r$. Also, note that $p-1, s-1 \leq l$ and $k-q, k-t \leq r$. Since $\mathbf{L}_l \models \mathbf{u} = \mathbf{v}$ by assumption, $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$ by Proposition 4.5.7, whence $(\sigma_1, \ldots, \sigma_{p-1}) = (\tau_1, \ldots, \tau_{s-1})$. By a dual argument and Proposition 4.5.13, we also have $(\sigma_{q+1}, \ldots, \sigma_k) = (\tau_{t+1}, \ldots, \tau_n)$. Now $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ so that $\{\sigma_p, \ldots, \sigma_q\} = \{\tau_s, \ldots, \tau_t\}$. Hence $(\sigma_p, \ldots, \sigma_q)$ and $(\tau_s, \ldots, \tau_t)$ must be identical sequences since they are increasing with identical contents. Consequently $\mathbf{u} \equiv \mathbf{v}$. ■

Theorem 4.7.6 *Let $n = l + m + r$ and $(l, m, r) \in \mathbb{U}$. If $\mathbf{u}, \mathbf{v}$ are $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-words such that $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$, then $\mathbf{u} \equiv \mathbf{v}$.*

Proof. Suppose that $\mathbf{u}$ is linear. Then by Lemma 4.7.2, $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$ where $\sigma_1, \ldots, \sigma_k$ are distinct and $1 \leq k \leq n$. Note then that $\mathbf{u}$ is actually a $\mathbf{N}_n$-word and $\mathbf{u} \equiv \mathbf{u}^{\mathbf{N}_n}$. Also, since $\mathbf{N}_n$ is a subvariety of $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ we have $\mathbf{N}_n \models \mathbf{u} = \mathbf{v}$. Hence $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n}$ by Theorem 4.6.7(1). Now

$$\mathbf{u} \equiv \mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n} \equiv \begin{cases} \mathbf{v} & \text{if } \mathbf{v} \text{ is linear,} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Since $\mathbf{u} \not\equiv \mathbf{0}$, $\mathbf{v}$ must also be linear and $\mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{v}$. Therefore $\mathbf{u} \equiv \mathbf{v}$. By symmetry, if $\mathbf{v}$ is linear then $\mathbf{u} \equiv \mathbf{v}$.

Therefore it remains to consider the case when both $\mathbf{u}$ and $\mathbf{v}$ are not linear, whence the result follows from Lemma 4.7.5. ■

Theorem 4.7.7 *Let $n = l + m + r$ and $(l : m : r) \in \mathbb{U}$. Then:*

(1) $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathbf{u}^{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r} \equiv \mathbf{v}^{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r}$;

(2) *The set of all $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-words constitutes the $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-free semigroup with the operation $\cdot$ given by*

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{uv})^{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r};$$

(3) $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = [\nabla \cup \{(l : m : r)\}] = [l : m : r]^{\mathbf{B}_2^-}$.

PROOF. The proof of this theorem follows closely to that of Theorem 4.2.11. ■

Note that Theorem 4.7.7 is only concerned with varieties of the form $\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r$ where $(l : m : r) \in \mathbb{U}$; we will proceed to show that this is sufficient as all varieties in $\mathfrak{LNR}$ can be expressed in this fashion. First consider $\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r$-words when $m = 0$.

Lemma 4.7.8 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r$*-words such that* $\mathbf{L}_l\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$, *then* $\mathbf{u} \equiv \mathbf{v}$.

PROOF. First note that $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ because $Y \in \mathbf{L}_l\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$. There are two cases: at least one of $\mathbf{u}, \mathbf{v}$ is linear, or both $\mathbf{u}, \mathbf{v}$ are not linear.

Case (i) By symmetry it suffices just to assume that $\mathbf{u}$ is linear. Then by Lemma 4.7.2,

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$$

where $\sigma_1, \ldots, \sigma_k$ are distinct and $1 \leq k \leq l + r$. Since $\mathbf{L}_l \models \mathbf{u} = \mathbf{v}$ and $\mathbf{R}_r \models \mathbf{u} = \mathbf{v}$ by assumption, we must have $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$ and $\overrightarrow{\mathbf{u}}^r \equiv \overrightarrow{\mathbf{v}}^r$ by Propositions 4.5.7 and 4.5.13 respectively. But note that

$$\overleftarrow{\mathbf{u}}^l \equiv z_{\sigma_1} \cdots z_{\sigma_p}, \qquad \overrightarrow{\mathbf{u}}^r \equiv z_{\sigma_q} \cdots z_{\sigma_k}$$

where $p = \min\{k, l\}$ and $q = \max\{1, k - r + 1\}$.

Suppose $p = k$. Then $\mathbf{u} \equiv \overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$ where $\overleftarrow{\mathbf{v}}^l$ is a subword of $\mathbf{v}$. If $\mathbf{v}$ is not linear, then $\overleftarrow{\mathbf{v}}^l$ is a proper subword of $\mathbf{v}$ with fewer variables involved, whence we contradictorily have

$$\mathfrak{c}(\mathbf{v}) \neq \mathfrak{c}\left(\overleftarrow{\mathbf{v}}^l\right) = \mathfrak{c}\left(\overleftarrow{\mathbf{u}}^l\right) = \mathfrak{c}(\mathbf{u}).$$

Therefore $\mathbf{v}$ is linear and $\mathbf{u} \equiv \overleftarrow{\mathbf{v}}^l \equiv \mathbf{v}$. Similarly, if $q = 1$ then $\mathbf{u} \equiv \mathbf{v}$.

Hence it remains to assume $p = l$ and $q = k - r + 1$, whence $\left|\overleftarrow{\mathbf{u}}^l\right| = l$ and $\left|\overrightarrow{\mathbf{u}}^r\right| = k - (q-1) = r$. Since $|\mathbf{u}| = k \leq l + r$, the segments $\overleftarrow{\mathbf{u}}^l$, $\overrightarrow{\mathbf{u}}^r$ are either adjacent or overlapping in $\mathbf{u}$ as shown in the following respectively:

$$\mathbf{u} \equiv \begin{cases} z_{\sigma_1} \cdots z_{\sigma_l} z_{\sigma_{k-r+1}} \cdots z_{\sigma_k} & \text{if } k = l + r, \\ (z_{\sigma_1} \cdots [z_{\sigma_{k-r+1}} \cdots z_{\sigma_l}) \cdots z_{\sigma_k}] & \text{if } k < l + r. \end{cases}$$

If $\mathbf{v}$ is not linear, then

$$\mathfrak{c}(\mathbf{v}) \neq \mathfrak{c}\left(\overleftarrow{\mathbf{v}}^l \overrightarrow{\mathbf{v}}^r\right) = \mathfrak{c}\left(\overleftarrow{\mathbf{u}}^l \overrightarrow{\mathbf{u}}^r\right) = \mathfrak{c}(\mathbf{u})$$

is a contradiction. Thus $\mathbf{v}$ is linear. Now since $\mathfrak{c}(\mathbf{v}) = \mathfrak{c}(\mathbf{u})$, $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$ and $\overrightarrow{\mathbf{u}}^r \equiv \overrightarrow{\mathbf{v}}^r$, the segments $\overleftarrow{\mathbf{v}}^l, \overrightarrow{\mathbf{v}}^r$ are adjacent or overlapping in $\mathbf{v}$ in the same way as $\overleftarrow{\mathbf{u}}^l, \overrightarrow{\mathbf{u}}^r$ do so in $\mathbf{u}$. Consequently $\mathbf{u} \equiv \mathbf{v}$.

Case (ii) Suppose both $\mathbf{u}, \mathbf{v}$ are not linear. Then the result holds by Lemma 4.7.5 with $m = 0$. ■

Corollary 4.7.9 *Let $l, n, r \in \mathbb{N}_0^\infty$. Then:*

(1) $\mathbf{L}_l\mathbf{R}_r = \mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r$;

(2) *If $n \le l + r$ then $\mathbf{N}_n \subset \mathbf{L}_l\mathbf{R}_r$.*

Proof. (1) Note that $\equiv_{(l:0:r)} \subseteq \equiv_{\mathbf{L}_l\mathbf{R}_r}$ since $\mathbf{L}_l\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r$. Let $(\mathbf{u}, \mathbf{v}) \in \equiv_{\mathbf{L}_l\mathbf{R}_r}$. Then $\left(\mathbf{u}, \mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}\right), \left(\mathbf{v}, \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}\right) \in \equiv_{(l:0:r)} \subseteq \equiv_{\mathbf{L}_l\mathbf{R}_r}$, whence

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) \in \equiv_{\mathbf{L}_l\mathbf{R}_r} &\Longrightarrow \left(\mathbf{u}, \mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}\right), \left(\mathbf{v}, \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}\right) \in \equiv_{\mathbf{L}_l\mathbf{R}_r} \\
&\Longrightarrow \left(\mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}, \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r}\right) \in \equiv_{\mathbf{L}_l\mathbf{R}_r} \\
&\Longrightarrow \mathbf{L}_l\mathbf{R}_r \models \mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} = \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} \\
&\Longrightarrow \mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} \equiv \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} \qquad \text{by Lemma 4.7.8} \\
&\Longrightarrow \mathbf{u} \equiv_{(l:0:r)} \mathbf{u}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} \equiv \mathbf{v}^{\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r} \equiv_{(l:0:r)} \mathbf{v} \\
&\Longrightarrow (\mathbf{u}, \mathbf{v}) \in \equiv_{(l:0:r)} .
\end{aligned}$$

Thus $\equiv_{\mathbf{L}_l\mathbf{R}_r} = \equiv_{(l:0:r)}$ and $\mathbf{L}_l\mathbf{R}_r = \mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r$.

(2) If $n \le l + r$, then $\mathbf{N}_n \subset \mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r = \mathbf{L}_l\mathbf{R}_r$ by (1). ■

Recall that

$$\mathfrak{LNR} = \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid l, n, r \in \mathbb{N}_0^\infty\}.$$

Let $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \in \mathfrak{LNR}$. In view of Corollary 4.7.9, if $n < l + r$, then

$$\begin{aligned}
\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r &= \mathbf{N}_n(\mathbf{L}_l\mathbf{R}_r) = \mathbf{N}_n(\mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r) \\
&= \mathbf{L}_l(\mathbf{N}_n\mathbf{N}_{l+r})\mathbf{R}_r = \mathbf{L}_l\mathbf{N}_{l+r}\mathbf{R}_r.
\end{aligned}$$

Therefore we have shown:

Proposition 4.7.10

$$\begin{aligned}\mathfrak{LNR} &= \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid l,n,r \in \mathbb{N}_0^\infty, n \geq l+r\} \\ &= \{\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r \mid (l,m,r) \in \mathbb{U}\}\,.\end{aligned}$$

Recall that if $(l,m,r) \in \mathbb{U}$ where either l or r is infinite, then $m = 0$ by the definition of $\mathbb{U}$. The following result shows that no new subvarieties of $\mathbf{A_0}$ are defined by $(l:m:r)$ if either l or r is infinite with $m \geq 1$.

Lemma 4.7.11 *If $l = \infty$ or $r = \infty$, then*

$$[l:0:r]^{\mathbf{B}_2^-} = [l:1:r]^{\mathbf{B}_2^-} = \cdots = [l:\infty:r]^{\mathbf{B}_2^-} = \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r.$$

Proof. By symmetry, it suffices to consider the case when $r = \infty$. Then

$$(l:m:\infty): \begin{cases} \mathbf{x}_l\,(y_1\cdots y_{m+1})^2\,w^2 = \mathbf{x}_l y_1\cdots y_{m+1}w^2 & \text{if } m \in \mathbb{N}_0, \\ \mathbf{x}_l y^2 z w^2 = \mathbf{x}_l y z^2 w^2 & \text{if } m = \infty. \end{cases}$$

Let $m \in \mathbb{N}_0$. Then it is obvious that $(l:m:\infty) \vdash (l:m+1:\infty)$ when y_{m+1} is replaced by $y_{m+1}y_{m+2}$. For $m \geq 1$,

$$\begin{aligned}\nabla \cup \{(l:m:\infty)\} &\vdash \nabla \cup \left\{\mathbf{x}_l y_1^{\alpha_1}\cdots y_{m+1}^{\alpha_{n+1}} w^2 = \mathbf{x}_l y_1\cdots y_{m+1}w^2 \mid 1 \leq \alpha_i \leq 2\right\} \\ &\vdash \left\{\mathbf{x}_l y_1^{\alpha_1} y_2^{\alpha_2} w^2 = \mathbf{x}_l y_1 y_2 w^2 \mid 1 \leq \alpha_i \leq 2\right\} \\ &\vdash \mathbf{x}_l y_1^2 y_2 w^2 = \mathbf{x}_l y_1 y_2 w^2 = \mathbf{x}_l y_1 y_2^2 w^2 \\ &\vdash \mathbf{x}_l y_1^2 y_2 w^2 = \mathbf{x}_l y_1 y_2^2 w^2 \\ &\vdash (l:\infty:\infty)\end{aligned}$$

where the first deduction holds by Lemma 4.4.3, and the second by the substitution $y_3,\ldots,y_{m+1} \longrightarrow w^2$ into X^+ and then reducing excess powers of w by I. Hence

$$\nabla \cup \{(l:0:\infty)\} \vdash \nabla \cup \{(l:1:\infty)\} \vdash \cdots \vdash \nabla \cup \{(l:\infty:\infty)\}\,.$$

Now we have $\mathbf{x}_l y^2 z w^2 = \mathbf{x}_l y z^2 w^2 \vdash \mathbf{x}_l y^2 w^2 = \mathbf{x}_l y w^2$ by the substitution $z \longrightarrow w$ into X^+ and reducing excess powers of w by I, so that $\nabla \cup \{(l:\infty:\infty)\} \vdash \nabla \cup \{(l:0:\infty)\}$. ■

Since none of $\mathbf{L}_\infty$, $\mathbf{R}_\infty$ and $\mathbf{N}_\infty$ are generated by finite semigroups, it is natural that $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ also has this property whenever one of l,n or r is infinite.

Proposition 4.7.12 *If one of l, n, or r is infinite, then $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ is not generated by any finite semigroup.*

PROOF. Suppose $S \in \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ is finite. Then S is a factor of $W = L_l^a \times N_n^b \times R_r^c$ for some $a, b, c \in \mathbb{N}$, whence there exists a surjecive homomorphism ψ from a subsemigroup T of W onto S. Since S is finite, there exists a finitely generated subsemigroup T_0 of T such that $\psi T_0 = S$. Since $\mathbf{A_0}$ is locally finite, T_0 is finite. Hence the projective images P_1, P_2 and P_3 of T_0 onto L_l^a, N_n^b and R_r^c respectively are finite semigroups. Now if $l = \infty$, then $P_1 \in \mathbf{L}_{l_0}$ for some $l_0 < \infty$ (Proposition 4.5.11). Hence $S \in \langle P_1, P_2, P_3 \rangle \subseteq \mathbf{L}_{l_0}\mathbf{N}_n\mathbf{R}_r$ and S cannot generate $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$, since $L_\infty \in \mathbf{L}_\infty\mathbf{N}_n\mathbf{R}_r$ and $L_\infty \notin \mathbf{L}_{l_0}\mathbf{N}_n\mathbf{R}_r$. Similar arguments apply by Proposition 4.6.9 if $n = \infty$, and by Proposition 4.5.11 if $r = \infty$. ■

4.8 The Lattice $\mathfrak{LNR}^*$

In this section we show that $\mathfrak{LNR}^* = \mathfrak{LNR} \cup \mathfrak{N}$ forms a complete sublattice of $\mathcal{L}(\mathbf{A_0})$. Note that the sets $\mathfrak{LNR}$ and $\mathfrak{N}$ are contained in $[\mathbf{Y}, \mathbf{A_0}]$ and $[\mathbf{T}, \mathbf{N}_\infty]$ respectively and hence must be disjoint in view of Corollary 4.6.12. More specifically, whether a variety $\mathbf{V}$ from $\mathfrak{LNR}^*$ belongs to $\mathfrak{LNR}$ or $\mathfrak{N}$ is dependent on whether or not it contains $\mathbf{Y}$.

Let $\mathbf{N}_k \in \mathfrak{N}$ and $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \in \mathfrak{LNR}$. Then clearly $\mathbf{N}_k \vee \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{L}_l\mathbf{N}_s\mathbf{R}_r$ with $s = \max\{k, n\}$. If $k > n = l + m + r$ then

$$\begin{aligned}
\mathbf{N}_n &\subseteq \mathbf{N}_k \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \\
&= \left[\mathbf{x}_{k+1} = x^2 = xyx = \mathbf{0},\ \mathbf{x}_l\mathbf{y}_{m+1}^2\mathbf{w}_r = \mathbf{x}_l\mathbf{y}_{m+1}\mathbf{w}_r\right]^{\mathbf{B}_2^-} \\
&\subseteq \left[\mathbf{x}_{k+1} = x^2 = xyx = \mathbf{0},\ \mathbf{0} = \mathbf{x}_l\mathbf{y}_{m+1}\mathbf{w}_r\right] \\
&= \left[\mathbf{x}_{k+1} = x^2 = xyx = \mathbf{0},\ \mathbf{x}_{n+1} = \mathbf{0}\right] \subseteq \mathbf{N}_n,
\end{aligned}$$

that is, $\mathbf{N}_k \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{N}_n$. If $k \leq n$ then $\mathbf{N}_k \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{N}_k$ because $\mathbf{N}_k \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$. Hence $\mathbf{N}_k \vee \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ and $\mathbf{N}_k \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ both belong to $\mathfrak{LNR}^*$. By Proposition 4.6.10, $\mathfrak{N}$ is a complete chain. Therefore to show that $\mathfrak{LNR}^*$ is a complete lattice, it suffices to show that $\mathfrak{LNR}$ is a complete lattice.

It was shown in Proposition 4.7.10 that

$$\mathfrak{LNR} = \{\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r \mid (l, m, r) \in \mathbb{U}\} \tag{4.11}$$

and

$$\mathfrak{LNR} = \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid (l,n,r) \in \mathbb{V}\} \tag{4.12}$$

where

$$\mathbb{V} = \{(l,n,r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \mid n \geq l + r\}.$$

The advantage of expressing varieties of $\mathfrak{LNR}$ in the form in (4.11) is that identity bases can be read off easily from subscripts:

$$\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r = [l : m : r]^{\mathbf{B}_2^-}.$$

This is less convenient but still possible with expressions in (4.12):

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = [l : n - (l+r) : r]^{\mathbf{B}_2^-}$$

with the conventions $\infty + \infty = \infty$ and $\infty - \infty = 0$. But it turns out that it is easier and more convenient when calculating intersections and joins of varieties expressed in the form (4.12).

Proposition 4.8.1 *Suppose $W \subseteq \mathbb{V}$. Then*

$$\bigvee_{(l,n,r)\in W} \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{L}_{l_0}\mathbf{N}_{n_0}\mathbf{R}_{r_0}$$

where

$$l_0 = \sup\{l \mid (l,n,r) \in W\}, \qquad r_0 = \sup\{r \mid (l,n,r) \in W\},$$
$$n_0 = \max\{l_0 + r_0, n_1\}, \qquad n_1 = \sup\{n \mid (l,n,r) \in W\}.$$

Consequently $(l_0, n_0, r_0) \in \mathbb{V}$ and $\mathfrak{LNR}$ is closed under arbitrary joins.

PROOF. Since $k_1 \leq k_2$ implies $\mathbf{L}_{k_1} \subseteq \mathbf{L}_{k_2}$, $\mathbf{N}_{k_1} \subseteq \mathbf{N}_{k_2}$ and $\mathbf{R}_{k_1} \subseteq \mathbf{R}_{k_2}$, we have

$$\bigvee_{(l,n,r)\in W} \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{L}_{l_0}\mathbf{N}_{n_1}\mathbf{R}_{l_0}.$$

If $n_1 \geq l_0 + r_0$ then $n_1 = n_0$ and we are done. If $n_1 < l_0 + r_0$, then by the remarks after Corollary 4.7.9,

$$\mathbf{L}_{l_0}\mathbf{N}_{n_1}\mathbf{R}_{l_0} = \mathbf{L}_{l_0}\mathbf{N}_{l_0+r_0}\mathbf{R}_{l_0} = \mathbf{L}_{l_0}\mathbf{N}_{n_0}\mathbf{R}_{l_0}.$$

■

In Proposition 4.8.1, n_1 could be used instead of n_0 in the join $\mathbf{L}_{l_0}\mathbf{N}_{n_0}\mathbf{R}_{r_0}$. The reason for using $n_0 = \max\{l_0 + r_0, n_1\}$ is to ensure a unique representation.

It remains to investigate arbitrary intersections of varieties from $\mathfrak{LNR}$. Since by definition, elements in $\mathfrak{LNR}$ are joins (of varieties $\mathbf{L}_l, \mathbf{N}_n$ and $\mathbf{R}_r$), it requires more effort to find intersections because in general, intersection is not always distributive over joins, that is,

$$\mathbf{U} \cap (\mathbf{V} \vee \mathbf{W}) \neq (\mathbf{U} \cap \mathbf{V}) \vee (\mathbf{U} \cap \mathbf{W}).$$

Several lemmas will be proved before we tackle this problem.

Lemma 4.8.2 *Let* $l, r \in \mathbb{N}_0^\infty$. *Then:*

(1) $\{(l:0:\infty), (\infty:0:r)\} \vdash (l:\infty:r)$;

(2) $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r$.

Proof. (1) Recall that $(l:0:\infty), (\infty:0:r)$ are

$$\mathbf{x}_l y^2 z^2 = \mathbf{x}_l y z^2, \qquad y^2 z^2 \mathbf{w}_r = y^2 z \mathbf{w}_r$$

respectively. Therefore

$$\mathbf{x}_l \left(y^2 z \mathbf{w}_r\right) = \left(\mathbf{x}_l y^2 z^2\right) \mathbf{w}_r = \mathbf{x}_l y z^2 \mathbf{w}_r,$$

which is $(l:\infty:r)$.

(2) This result is a consequence of (2). ■

Corollary 4.8.3 *Suppose* $(l_1, n_1, r_1), (l_2, n_2, r_2) \in \mathbb{V}$. *Then*

$$\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r$$

where $l = \min\{l_1, l_2\}$ *and* $r = \min\{r_1, r_2\}$.

Proof. The last of the following inclusions holds by Lemma 4.8.2(2):

$$\begin{aligned}\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} &\subseteq (\mathbf{L}_{l_1}\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_{r_1}) \cap (\mathbf{L}_{l_2}\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_{r_2}) \\ &= (\mathbf{L}_{l_1}\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_{l_2}\mathbf{N}_\infty\mathbf{R}_\infty) \cap (\mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_{r_1} \cap \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_{r_2}) \\ &= \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r.\end{aligned}$$

■

Lemma 4.8.4 *If* $(l,m,r) \in \mathbb{U}$, *then*

$$[l:m:r]^{\mathbf{B_2^-}} = [(l+m:0:r),(l:0:r+m)]^{\mathbf{B_2^-}}.$$

Equivalently, $\mathbf{L}_l\mathbf{N}_{l+m+r}\mathbf{R}_r = \mathbf{L}_{l+m}\mathbf{R}_r \cap \mathbf{L}_l\mathbf{R}_{r+m}$.

PROOF. Let $\mathbf{V}_m = [(l+m:0:r),(l:0:r+m)]^{\mathbf{B_2^-}}$. If either $l=\infty$ or $r=\infty$, then $m=0$ and the result is immediate. Therefore assume that $l,r<\infty$ and $m>0$. We first show that $[l:m:r]^{\mathbf{B_2^-}} \subseteq \mathbf{V}_m$. There are two cases: $m<\infty$ and $m=\infty$.

Case (i) Suppose $m<\infty$. Then by Lemma 4.4.3,

$$\begin{aligned}
[l:m:r]^{\mathbf{B_2^-}} &= \left[\mathbf{x}_l y_1^{\alpha_1}\cdots y_{m+1}^{\alpha_{m+1}}\mathbf{w}_r = \mathbf{x}_l y_1\cdots y_{m+1}\mathbf{w}_r \mid 1\le\alpha_i\le 2\right]^{\mathbf{B_2^-}}\\
&\subseteq \left[\begin{array}{l} \mathbf{x}_l y_1\cdots y_m y_{m+1}^2\mathbf{w}_r = \mathbf{x}_l y_1\cdots y_{m+1}\mathbf{w}_r,\\ \mathbf{x}_l y_1^2 y_2\cdots y_{m+1}\mathbf{w}_r = \mathbf{x}_l y_1\cdots y_{m+1}\mathbf{w}_r \end{array}\right]^{\mathbf{B_2^-}}\\
&= \mathbf{V}_m.
\end{aligned}$$

Case (ii) Suppose $m=\infty$. Then $(l:\infty:r)$ is $\mathbf{x}_l y^2 z\mathbf{w}_r = \mathbf{x}_l y z^2\mathbf{w}_r$. Making the substitutions $x_1,\ldots,x_l,y \longrightarrow x^2$ into X^+ and then reducing excessive powers of x by I, we have $x^2 z\mathbf{w}_r = x^2z^2\mathbf{w}_r$. Hence

$$[l:\infty:r]^{\mathbf{B_2^-}} \subseteq \left[x^2z\mathbf{w}_r = x^2z^2\mathbf{w}_r\right]^{\mathbf{B_2^-}} = [\infty:0:r]^{\mathbf{B_2^-}}.$$

By a symmetrical argument, $[l:\infty:r]^{\mathbf{B_2^-}} \subseteq [l:0:\infty]^{\mathbf{B_2^-}}$. Hence,

$$[l:\infty:r]^{\mathbf{B_2^-}} \subseteq [(\infty:0:r),(l:0:\infty)]^{\mathbf{B_2^-}} = \mathbf{V}_\infty.$$

We now show that $\mathbf{V}_m \subseteq [l:m:r]^{\mathbf{B_2^-}}$. Since

$$\begin{aligned}
\mathbf{V}_\infty &= [(\infty:0:r),(l:0:\infty)]^{\mathbf{B_2^-}}\\
&\subseteq [l:\infty:r]^{\mathbf{B_2^-}}
\end{aligned}$$

by Lemma 4.8.2, it suffices to assume $m<\infty$. Note that we have $\mathbf{V}_m \models (\infty:0:\infty)$ by Lemma 4.7.3(2). Now

$$\begin{aligned}
\mathbf{x}_l\mathbf{y}_{m+1}^2\mathbf{w}_r &\equiv_\nabla \mathbf{x}_l y_1^2\,(y_2\cdots y_m)^2\,y_{m+1}^2\mathbf{w}_r && \text{by Lemma 4.3.2(3)}\\
&\equiv_{\mathbf{V}_m} \mathbf{x}_l y_1^2 y_2\cdots y_m y_{m+1}^2\mathbf{w}_r && \text{by } (\infty:0:\infty)\\
&\equiv_{\mathbf{V}_m} \mathbf{x}_l y_1 y_2\cdots y_m y_{m+1}\mathbf{w}_r && \text{by } (l+m:0:r) \text{ and } (l:0:r+m)\\
&\equiv \mathbf{x}_l\mathbf{y}_{m+1}\mathbf{w}_r.
\end{aligned}$$

Therefore $\mathbf{V}_m \models (l : m : r)$ as required. ■

Lemma 4.8.5 *If* $l, r, m, l_1, r_1, m_1 \in \mathbb{N}_0$ *are such that* $l_1 \geq l, r_1 \geq r$ *and* $l_1 + r_1 + m_1 = l + r + m$*, then*

$$[(l_1 : m_1 : r_1), (l : \infty : r)]^{\mathbf{B}_2^-} \subseteq [l : m : r]^{\mathbf{B}_2^-}.$$

Proof. If $l_1 + r_1 = l + r$ then $l_1 = l$ and $r_1 = r$, whence $(l_1 : m_1 : r_1)$ and $(l : m : r)$ are the same identity. Therefore suppose $l_1 + r_1 > l + r$. Consider the identities:

$$(l_1 : 0 : r_1 + m_1) : x_1 \cdots x_{l_1} y^2 w_1 \cdots w_{r_1+m_1} = x_1 \cdots x_{l_1} y w_1 \cdots w_{r_1+m_1},$$
$$(l : \infty : r) : x_1 \cdots x_l y^2 z w_1 \cdots w_r = x_1 \cdots x_l y z^2 w_1 \cdots w_r.$$

First we show that

$$\{(l_1 : 0 : r_1 + m_1), (l : \infty : r)\} \vdash (l : 0 : r + m).$$

If $l_1 = l$ then $r_1 + m_1 = r + m$ so that the identities $(l_1 : 0 : r_1 + m_1)$ and $(l : 0 : r + m)$ are identical. Therefore assume $l_1 > l$. Since

$$\begin{aligned} |x_1 \cdots x_{l_1} y w_1 \cdots w_{r_1+m_1}| &= l_1 + r_1 + m_1 + 1 \\ &\geq (l + 1) + r + 0 + 1 \\ &= |x_1 \cdots x_l y z w_1 \cdots w_r|, \end{aligned}$$

the identity $(l : \infty : r)$ can be used to move the exponent "2" of the variable y in the identity $(l_1 : 0 : r_1 + m_1)$ forward to the variable x_{l+1} to obtain

$$x_1 \cdots x_l x_{l+1}^2 x_{l+2} \cdots x_{l_1} y w_1 \cdots w_{r_1+m_1} = x_1 \cdots x_{l_1} y w_1 \cdots w_{r_1+m_1}.$$

This identity is exactly $(l : 0 : r + m)$, since the number of variables that follow x_{l+1} is

$$l_1 - (l + 1) + 1 + r_1 + m_1 = l_1 + r_1 + m_1 - l = r + m.$$

By a symmetrical argument we have

$$\{(l_1 + m_1 : 0 : r_1), (l : \infty : r)\} \vdash (l + m : 0 : r).$$

Therefore,

$$\begin{aligned}[(l_1 : m_1 : r_1), (l : \infty : r)]^{\mathbf{B}_2^-} &= [(l_1 + m_1 : 0 : r_1), (l_1 : 0 : r_1 + m_1), (l : \infty : r)]^{\mathbf{B}_2^-} \\ &\subseteq [(l + m : 0 : r), (l : 0 : r + m)]^{\mathbf{B}_2^-} \\ &= [l : m : r]^{\mathbf{B}_2^-} \qquad \text{by Lemma 4.8.4.}\end{aligned}$$

■

Proposition 4.8.6 *Suppose* $(l_1, n_1, r_1), (l_2, n_2, r_2) \in \mathbb{V}$. *Then*

$$\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$$

where $l = \min\{l_1, l_2\}, r = \min\{r_1, r_2\}$ *and* $n = \min\{n_1, n_2\}$. *Consequently* $(l, n, r) \in \mathbb{V}$ *and* $\mathfrak{LNR}$ *is closed under* $\cap$.

PROOF. It is obvious that $\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} \supseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$. By Corollary 4.8.3, we have

$$\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r. \tag{4.13}$$

Thus equality holds if $n = \infty$. Therefore we may assume $n = n_1 < \infty$ and $n_1 \leq n_2$. Let m_1 and m be such that $n_1 = l_1 + r_1 + m_1$ and $n = l + r + m$. Since $l_1 \geq l$, $r_1 \geq r$ and $l_1 + r_1 + m_1 = n_1 = n = l + r + m$, we have

$$[(l_1 : m_1 : r_1), (l : \infty : r)]^{\mathbf{B}_2^-} \subseteq [l : m : r]^{\mathbf{B}_2^-}$$

by Lemma 4.8.5. That is, $\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$. Consequently,

$$\begin{aligned}\mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_{l_2}\mathbf{N}_{n_2}\mathbf{R}_{r_2} &\subseteq \mathbf{L}_{l_1}\mathbf{N}_{n_1}\mathbf{R}_{r_1} \cap \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \qquad \text{by (4.13)} \\ &\subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r.\end{aligned}$$

■

Proposition 4.8.7 *Suppose* $W \subseteq \mathbb{V}$. *Then*

$$\bigcap\nolimits_{(l,n,r)\in W} \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = \mathbf{L}_{l_0}\mathbf{N}_{n_0}\mathbf{R}_{r_0}$$

where

$$l_0 = \min\{l \mid (l,n,r) \in W\}, \quad r_0 = \min\{r \mid (l,n,r) \in W\},$$
$$n_0 = \min\{n \mid (l,n,r) \in W\}.$$

Consequently $(l_0, n_0, r_0) \in \mathbb{V}$ *and* $\mathfrak{LNR}$ *is closed under arbitrary meets.*

PROOF. This is a consequence of Proposition 4.8.6. ■

Corollary 4.8.8 *The set* $\mathfrak{LNR}^*$ *is a complete sublattice of* $\mathcal{L}(\mathbf{A_0})$.

PROOF. By Propositions 4.8.1 and 4.8.7, $\mathfrak{LNR}$ forms a complete sublattice of $\mathcal{L}(\mathbf{A_0})$. Hence by the remarks at the beginning of this section, $\mathfrak{LNR}^*$ is also a complete sublattice of $\mathcal{L}(\mathbf{A_0})$. ■

As mentioned in the beginning of this section, expressing varieties of $\mathfrak{LNR}$ in the forms of (4.12) allows us to compute intersections and joins in $\mathfrak{LNR}$. It is easy to see that the mapping $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \longmapsto (l,n,r)$ is an isomorphism of $\mathfrak{LNR}$ onto the sublattice $\mathbb{V}$ of $\mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty$, where the operations are obvious from Propositions 4.8.1 and 4.8.7. Apart from these contexts, we may sometimes choose not to use expressions in the forms of (4.11) or (4.12) for the sake of simplicity and convenience. For example we may write

$$\mathbf{L}_2\mathbf{N}_3 \vee \mathbf{L}_1\mathbf{R}_4 = \mathbf{L}_2\mathbf{R}_4,$$
$$\mathbf{L}_2\mathbf{N}_3 \cap \mathbf{L}_1\mathbf{R}_4 = \mathbf{L}_1\mathbf{N}_3$$

instead of

$$\mathbf{L}_2\mathbf{N}_3\mathbf{R}_0 \vee \mathbf{L}_1\mathbf{N}_5\mathbf{R}_4 = \mathbf{L}_2\mathbf{N}_6\mathbf{R}_4,$$
$$\mathbf{L}_2\mathbf{N}_3\mathbf{R}_0 \cap \mathbf{L}_1\mathbf{N}_5\mathbf{R}_4 = \mathbf{L}_1\mathbf{N}_3\mathbf{R}_0$$

respectively. In particular, $\mathbf{L}_0\mathbf{N}_n\mathbf{R}_0 = \mathbf{N}_n\mathbf{Y}$ and $\mathbf{L}_\infty\mathbf{R}_\infty = \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_\infty$.

The greatest element in $\mathfrak{LNR}^*$ is $\mathbf{L}_\infty\mathbf{R}_\infty$ so that $\mathfrak{LNR}^*$ is a sublattice of $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$. Due to the completeness and large (infinite) order of $\mathfrak{LNR}^*$, it was once conjectured by the author that $\mathfrak{LNR}^* = \mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$. But this turns out to be far from the truth, as it will be shown in the next section that there are still infinitely many more subvarieties of $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$

not captured by $\mathfrak{LNR}^*$. However, for the remainder of this section, we will investigate the covering situation in $\mathfrak{LNR}^*$ in the sense that for any given $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \in \mathfrak{LNR}^*$, which varieties in $\mathfrak{LNR}^*$ (if any) are anti-atoms of $\mathcal{L}(\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r)$.

For each fixed $n \in \mathbb{N}_0^\infty$, define

$$\mathfrak{H}_n = \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid (l,r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty, l+r \leq n\} \cup \{\mathbf{N}_n\}.$$

It is straightforward to show that $\mathfrak{H}_0 = \{\mathbf{T}, \mathbf{Y}\}$ and $\mathfrak{H}_\infty$ are complete sublattices of $\mathfrak{LNR}^*$. But if $n \in \mathbb{N}$ then $\mathfrak{H}_n$ is only a complete semilattice since it is not closed under taking joins. For example, $\mathbf{L}_n\mathbf{N}_n\mathbf{R}_0, \mathbf{L}_0\mathbf{N}_n\mathbf{R}_n \in \mathfrak{H}_n$ but $\mathbf{L}_n\mathbf{N}_n\mathbf{R}_0 \vee \mathbf{L}_0\mathbf{N}_n\mathbf{R}_n = \mathbf{L}_n\mathbf{N}_{2n}\mathbf{R}_n \notin \mathfrak{H}_n$. The following result describes the covering situation in $\mathfrak{H}_n$.

Lemma 4.8.9 *If $\mathbf{V} \in [\mathbf{N}_n, \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r)$ with $1 \leq l+r \leq n$, then either $\mathbf{V} \subseteq \mathbf{L}_s\mathbf{N}_n\mathbf{R}_r$ for some $s < l$, or $\mathbf{V} \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_t$ for some $t < r$.*

PROOF. If $\mathbf{V} = \mathbf{N}_n$ then the result follows immediately. Hence we may assume $\mathbf{N}_n\mathbf{Y} \subseteq \mathbf{V}$ by Corollary 4.6.12. By assumption, either $\mathbf{L}_l \nsubseteq \mathbf{V}$ or $\mathbf{R}_r \nsubseteq \mathbf{V}$. Suppose that $\mathbf{L}_l \nsubseteq \mathbf{V}$. Then there exists an identity $\mathbf{u} = \mathbf{v}$ such that $\mathbf{V} \models \mathbf{u} = \mathbf{v}$ and $\mathbf{L}_l \nvDash \mathbf{u} = \mathbf{v}$. Since $\mathbf{V} \subseteq \mathbf{B}_2^-$, the identity $\mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}$ will still have the properties $\mathbf{V} \models \mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}$ and $\mathbf{L}_l \nvDash \mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}$. Since $\mathbf{Y} \subseteq \mathbf{V}$, the identity $\mathbf{u} = \mathbf{v}$ is balanced. Hence $\overleftarrow{\mathbf{u}}^l \not\equiv \overleftarrow{\mathbf{v}}^l$ by Proposition 4.5.7. Therefore $\left[\mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}\right]^{\mathbf{B}_2^-} \models (s:0:\infty)$ for some $s < l$ by Lemma 4.5.6, whence

$$\begin{aligned} \mathbf{V} &\subseteq \left[\mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}\right]^{\mathbf{B}_2^-} \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \\ &\subseteq \mathbf{L}_s\mathbf{N}_\infty\mathbf{R}_\infty \cap \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \\ &= \mathbf{L}_s\mathbf{N}_n\mathbf{R}_r. \end{aligned}$$

A symmetrical argument shows that $\mathbf{R}_r \nsubseteq \mathbf{V}$ implies $\mathbf{V} \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_t$ for some $t < r$. ■

Corollary 4.8.10 *Let $l, r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$.*

(1) *The only anti-atoms of $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ in $\mathfrak{H}_n$ are $\mathbf{L}_{l-1}\mathbf{N}_n\mathbf{R}_r$ and $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_{r-1}$;*

(2) *The only anti-atom of $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty$ in $\mathfrak{H}_\infty$ is $\mathbf{L}_{l-1}\mathbf{N}_\infty\mathbf{R}_\infty$;*

(3) *The only anti-atom of $\mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_r$ in $\mathfrak{H}_\infty$ is $\mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_{r-1}$;*

(4) *The only anti-atom of $\mathbf{L}_0\mathbf{N}_n\mathbf{R}_0$ in $\mathfrak{H}_n$ is $\mathbf{N}_n$;*

(5) *$\mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_\infty$ has no anti-atoms in $\mathfrak{H}_\infty$.*

Consequently, there are at most two anti-atoms for each variety in $\mathfrak{H}_n$.

PROOF. (1) This is a consequence of Lemma 4.8.9.

(2) If $\mathbf{L}_s\mathbf{N}_\infty\mathbf{R}_r$ is an anti-atom of $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty$, then since

$$\mathbf{L}_s\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_s\mathbf{N}_\infty\mathbf{R}_{r+1} \subseteq \mathbf{L}_s\mathbf{N}_\infty\mathbf{R}_\infty,$$

we must have $r = \infty$. By Lemma 4.8.9, it is then straightforward to show that the only anti-atom of $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty$ in $\mathfrak{H}_\infty$ is $\mathbf{L}_{l-1}\mathbf{N}_\infty\mathbf{R}_\infty$.

(3) This is symmetrical to (2).

(4) Suppose $\mathbf{V} \in \mathfrak{H}_n$ is such that $\mathbf{V} \subsetneqq \mathbf{L}_0\mathbf{N}_n\mathbf{R}_0 = \mathbf{N}_n\mathbf{Y}$. If $\mathbf{Y} \subseteq \mathbf{V}$ then $\mathbf{V}$ must be of the form $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ so that it contradictorily contains $\mathbf{L}_0\mathbf{N}_n\mathbf{R}_0$. Therefore we must have $\mathbf{Y} \nsubseteq \mathbf{V}$, whence $\mathbf{V} \subseteq \mathbf{N}_\infty$ by Lemma 4.6.11. Consequently, $\mathbf{V} \in \mathcal{L}(\mathbf{N}_\infty) \cap \mathfrak{H}_n = \{\mathbf{N}_n\}$.

(5) If $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r$ is an anti-atom of $\mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_\infty$, then since

$$\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_{l+1}\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_\infty,$$
$$\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_{r+1} \subseteq \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_\infty,$$

we must have $l = r = \infty$, a contradiction. ■

Having found all the anti-atoms (if any) of each variety in $\mathfrak{H}_n$, we present in Figure 4.1 the diagrams of $\mathfrak{H}_n$ for several values of n. Our convention for (semi)lattice diagrams are as follows. A line joining a lower positioned variety to a higher positioned variety indicates containment in the same order, and a bolded line indicates containment with covering.

It is easy to see from Figure 4.1 that $\mathfrak{H}_n$ is the union of the intervals $[\mathbf{N}_n, \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r]$ with $l + r = n$. We shall establish two more results before we present the diagram of the sublattice $\mathfrak{LNR}^* \cup \{\mathbf{A_0}, \mathbf{A_0} \cap \mathbf{B_2}\}$ of $\mathcal{L}(\mathbf{A_0})$.

Proposition 4.8.11 *The subvariety* $\mathbf{L}_\infty\mathbf{R}_\infty$ *is the unique anti-atom of* $\mathbf{B}_2^-$.

PROOF. Recall that $\mathbf{L}_\infty\mathbf{R}_\infty = [\infty : 0 : \infty]^{\mathbf{B}_2^-}$ where

$$(\infty : 0 : \infty) : x^2y^2w^2 = x^2yw^2.$$

Suppose $\mathbf{V} \in \mathcal{L}(\mathbf{B}_2^-)$. If $\mathbf{Y} \subseteq \mathbf{V}$ then $\mathbf{V} \subseteq \mathbf{L}_\infty\mathbf{R}_\infty$ by Lemma 4.3.11(2). If $\mathbf{Y} \nsubseteq \mathbf{V}$ then

$\mathbf{V} \in [\mathbf{T}, \mathbf{N}_\infty]$ by Lemma 4.6.11. Hence $\mathbf{V} \subsetneq \mathbf{N}_\infty \subseteq \mathbf{L}_\infty\mathbf{R}_\infty$ by Corollary 4.7.9(2). ■

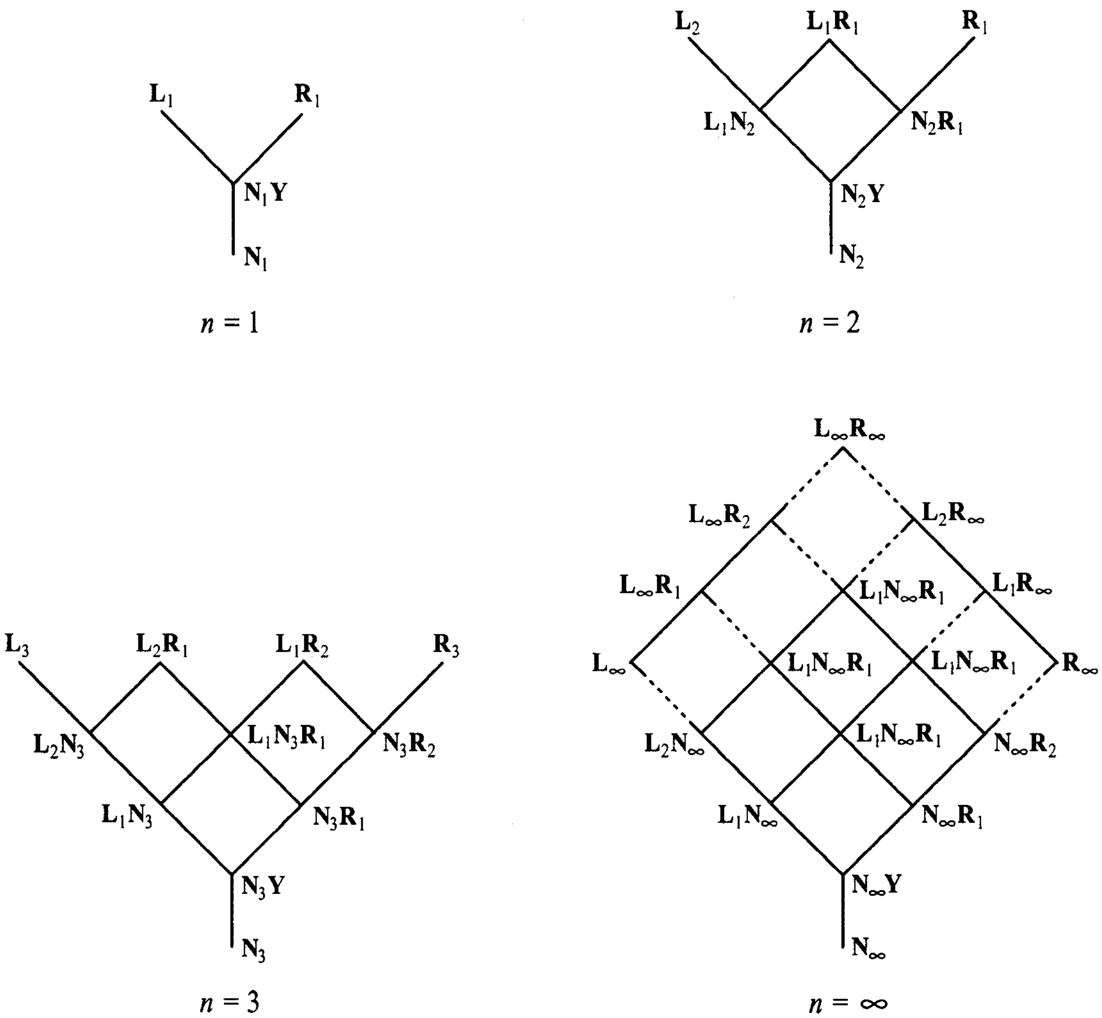

Figure 4.1: Semilattices $\mathfrak{H}_n$ for $n = 1, 2, 3, \infty$.

Corollary 4.8.12 *If* $\mathbf{B}_2^- \nvDash \mathbf{u} = \mathbf{v}$ *then* $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = [\mathbf{u} = \mathbf{v}]^{\mathbf{L}_\infty\mathbf{R}_\infty}$.

Proof. By Proposition 4.8.11, all proper subvarieties of $\mathbf{B}_2^-$ are contained in $\mathbf{L}_\infty\mathbf{R}_\infty$.

Therefore $[\mathbf{u}=\mathbf{v}]^{\mathbf{B}_2^-} = [\mathbf{u}=\mathbf{v}]^{\mathbf{B}_2^-} \cap \mathbf{L}_\infty\mathbf{R}_\infty = [\mathbf{u}=\mathbf{v}]^{\mathbf{L}_\infty\mathbf{R}_\infty}$. ■

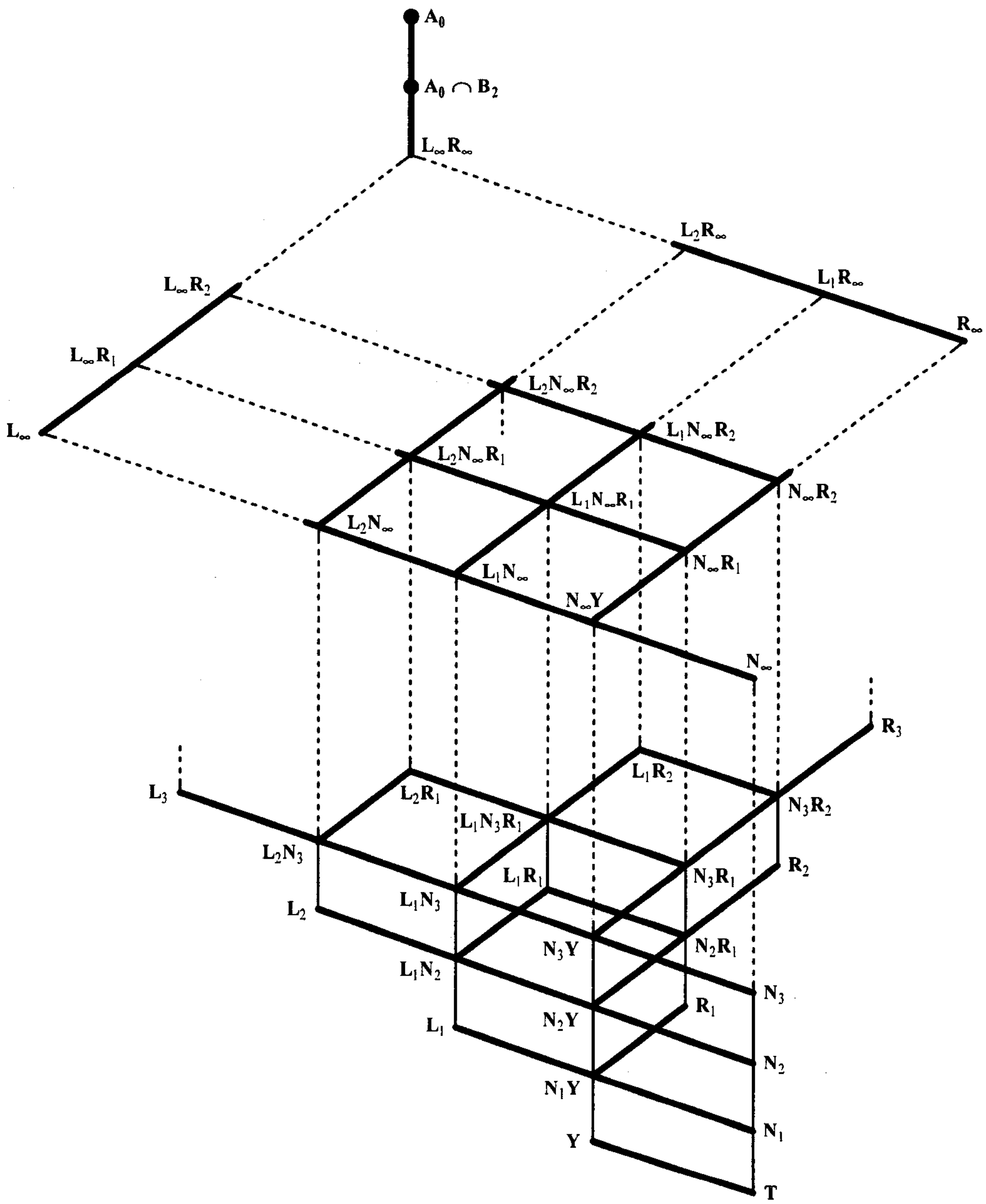

Figure 4.2: The sublattice $\mathfrak{LNR}^* \cup \{\mathbf{A_0}, \mathbf{A_0} \cap \mathbf{B_2}\}$ of $\mathcal{L}(\mathbf{A_0})$.

For the verification of coverings in Figure 4.2, see Propositions 4.6.13, 4.8.11, and Corollary 4.8.10. From this diagram, some basic lattice properties can be deduced easily.

A lattice $\mathcal{L}$ is *modular* if for all $a, b, c \in \mathcal{L}$,

$$a \leq c \implies (a \vee b) \wedge c = a \vee (b \wedge c),$$

and it is *semimodular* if for all $a, b \in \mathcal{L}$,

$$a \succ a \wedge b \text{ and } b \succ a \wedge b \implies a \vee b \succ a \text{ and } a \vee b \succ b.$$

The lattice of congruences on a completely 0-simple semigroup is semimodular ([6], Theorem 3.6.2), while the lattice of normal subgroups of a group [6] and the lattice of varieties of completely regular semigroups [13] are modular. It is well known that each modular lattice is semimodular ([6], Proposition 1.8.5).

Proposition 4.8.13 *The lattice* $\mathcal{L}(\mathbf{A_0})$ *is non-semimodular.*

PROOF. Note that $\mathbf{L_1}$ and $\mathbf{R_1}$ each covers $\mathbf{N_1Y} = \mathbf{L_1} \cap \mathbf{R_1}$ by Corollary 4.8.10. But $\mathbf{L_1R_1}$ covers neither $\mathbf{L_1}$ nor $\mathbf{R_1}$. Therefore the sublattice of $\mathcal{L}(\mathbf{A_0})$ shown in Figure 4.3 is non-semimodular. ■

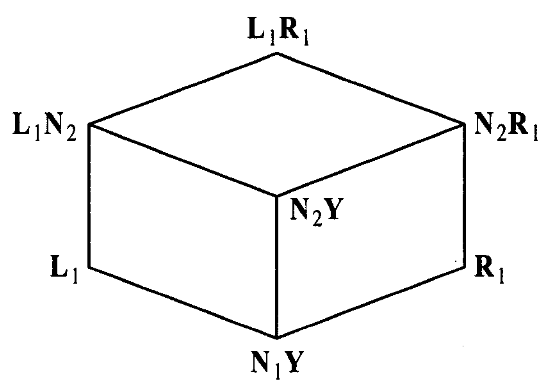

Figure 4.3: A non-semimodular sublattice of $\mathcal{L}(\mathbf{A_0})$.

4.9 Varieties in $\mathcal{L}(\mathbf{L_\infty R_\infty}) \setminus \mathfrak{LNR}^*$

Since $\mathfrak{LNR}^*$ is the union of all the semilattices $\mathfrak{H}_n$ and the covering aspect within each of these semilattices has been fully described by Proposition 4.8.10, we shall investigate what varieties in $\mathcal{L}(\mathbf{A_0})$ possibly lie outside of $\mathfrak{LNR}^*$; since

$$\mathcal{L}(\mathbf{A_0}) = \{\mathbf{A_0}, \mathbf{B_2^-}\} \cup \mathcal{L}(\mathbf{L_\infty R_\infty}),$$

these varieties can only exist in $\mathcal{L}(\mathbf{L_\infty R_\infty})$. Hence it suffices to investigate $\mathcal{L}(\mathbf{L_\infty R_\infty})$ instead of $\mathcal{L}(\mathbf{A_0})$. Furthermore, since by Proposition 4.6.12, $\mathcal{L}(\mathbf{L_\infty R_\infty})$ is the disjoint union of $[\mathbf{T}, \mathbf{N_\infty}]$ and $[\mathbf{Y}, \mathbf{L_\infty R_\infty}]$, it suffices to investigate these intervals seperately. With the results of $\mathfrak{N} \subset [\mathbf{T}, \mathbf{N_\infty}]$ and $\mathfrak{LNR} \subset [\mathbf{Y}, \mathbf{L_\infty R_\infty}]$ found in previous sections, we show that all varieties in $\mathcal{L}(\mathbf{L_\infty R_\infty})$ are of the form $\mathbf{V} \cap \mathbf{P}$ with $\mathbf{V} \in \mathfrak{LNR}^*$ and $\mathbf{P}$ a permutation variety (Propositions 4.9.2 and 4.9.5).

Lemma 4.9.1 *If* $\mathbf{u}, \mathbf{v}$ *are* $\mathbf{N_\infty}$*-words and* $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$, *then*

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{N_\infty}} = [\mathbf{x}_n = \mathbf{0}]^{\mathbf{N_\infty}}$$

with $n = \min\{|\mathfrak{c}(\mathbf{w})| \mid \mathbf{w} \in \{\mathbf{u}, \mathbf{v}\} \setminus \{\mathbf{0}\}\}$.

PROOF. The result clearly holds if one of $\mathbf{u}, \mathbf{v}$ is $\mathbf{0}$. So suppose $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_p}$ and $\mathbf{v} \equiv z_{\tau_1} \cdots z_{\tau_q}$ with $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$ and $p \leq q$. Then there exists some $\tau_i \notin \{\sigma_1, \ldots, \sigma_p\}$. Substituting z_{τ_i} by $\mathbf{0}$ we have $\mathbf{u} = \mathbf{0}$. Therefore $[\mathbf{u} = \mathbf{v}]^{\mathbf{N_\infty}} \subseteq [\mathbf{u} = \mathbf{0}]^{\mathbf{N_\infty}} = [\mathbf{x}_p = \mathbf{0}]^{\mathbf{N_\infty}}$. But

$$\mathbf{x}_p = \mathbf{0} \vdash \{\mathbf{u} = \mathbf{0}, \mathbf{v} = \mathbf{0}\} \vdash \mathbf{u} = \mathbf{v}.$$

Hence $[\mathbf{u} = \mathbf{v}]^{\mathbf{N_\infty}} = [\mathbf{x}_p = \mathbf{0}]^{\mathbf{N_\infty}}$. ■

For each identity $\mathbf{u} = \mathbf{v}$, the words $\mathbf{u}, \mathbf{v}$ are $\equiv_{\mathbf{N_\infty}}$-related to $\mathbf{N_\infty}$-words. In view of Lemma 4.9.1, if $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$, then

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{N_\infty}} = [\mathbf{u}^{\mathbf{N_\infty}} = \mathbf{v}^{\mathbf{N_\infty}}]^{\mathbf{N_\infty}} = [\mathbf{x}_n = \mathbf{0}]^{\mathbf{N_\infty}} = \mathbf{N}_n.$$

Otherwise, if $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$, then $[\mathbf{u} = \mathbf{v}]^{\mathbf{N_\infty}} = \mathbf{N_\infty} \cap [\mathbf{u}^{\mathbf{N_\infty}} = \mathbf{v}^{\mathbf{N_\infty}}]$ where $\mathbf{u}^{\mathbf{N_\infty}} = \mathbf{v}^{\mathbf{N_\infty}}$ is a permutation identity. More generally, we have

Proposition 4.9.2 *Each variety in* $[\mathbf{T}, \mathbf{N_\infty}]$ *is of the form*

$$\mathbf{N}_n \cap \mathbf{P}$$

where $n \in \mathbb{N}_0^\infty$ *and* $\mathbf{P}$ *is a permutation variety.*

PROOF. Let $\mathbf{V} \in [\mathbf{T}, \mathbf{N}_\infty]$. Then $\mathbf{V} = [\Sigma_1 \cup \Sigma_2]^{\mathbf{N}_\infty}$ for some (possibly empty) sets Σ_1, Σ_2 of identities formed by $\mathbf{N}_\infty$-words such that those in Σ_1 are balanced while those in Σ_2 are not. We may omit the identity $\mathbf{0} = \mathbf{0}$ from $\Sigma_1 \cup \Sigma_2$ since it is trivial in $\mathbf{N}_\infty$. Let $\mathbf{P} = [\Sigma_1]$. Since

$$\mathbf{P} = \begin{cases} [\Sigma_1] & \text{if } \Sigma_1 \neq \emptyset, \\ [x = x] & \text{if } \Sigma_1 = \emptyset, \end{cases}$$

$\mathbf{P}$ is always a permutation variety. If $\Sigma_2 = \emptyset$ then $[\Sigma_2]^{\mathbf{N}_\infty} = \mathbf{N}_\infty$; otherwise $[\Sigma_2]^{\mathbf{N}_\infty} = \mathbf{N}_n$ for some n (Lemma 4.9.1). Therefore

$$\mathbf{V} = [\Sigma_2]^{\mathbf{N}_\infty} \cap [\Sigma_1] = \mathbf{N}_n \cap \mathbf{P}$$

for some $n \in \mathbb{N}_0^\infty$. ■

We now extend the result in Proposition 4.9.2 to the context of $[\mathbf{Y}, \mathbf{L}_\infty \mathbf{R}_\infty]$. Consider a word of the form

$$\mathbf{u} \equiv x_1 \cdots x_{p-1} x_p^\alpha \cdots x_q^\alpha x_{q+1} \cdots x_k$$

where $x_1, \ldots, x_k$ are distinct variables, $1 \leq p \leq q \leq k$ and $\alpha \in \{1, 2\}$. Then $\mathbf{u}$ is *linear* if $\alpha = 1$, and it is *quadratic* otherwise. If $\alpha = 2$ then the subscripts $p, \ldots, q$ of variables of exponent two form the integer interval $\mathbb{I}_p^q$ which we call the *interval* of $\mathbf{u}$. Note that each $\mathbf{L}_\infty \mathbf{R}_\infty$-word is either linear or quadratic, and conversely, a linear word is an $\mathbf{L}_\infty \mathbf{R}_\infty$-word. But a quadratic word of the form

$$z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k}$$

may not be a $\mathbf{B}_2^-$-word (and hence not an $\mathbf{L}_\infty \mathbf{R}_\infty$-word) since the sequence $(\sigma_p, \ldots, \sigma_q)$ need not be increasing (see (4.5)).

Lemma 4.9.3 *Let $\mathbf{u}, \mathbf{v}$ be distinct quadratic $\mathbf{L}_\infty \mathbf{R}_\infty$-words with $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. Then*

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = \mathbf{L}_l \mathbf{N}_n \mathbf{R}_r$$

for some $(l, n, r) \in \mathbb{V}$ such that at least one of l, n or r is infinite.

PROOF. Since $\mathbf{u}, \mathbf{v}$ are quadratic $\mathbf{L}_\infty \mathbf{R}_\infty$-words with $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$, we may assume by Lemma 4.7.2 that

$$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_k},$$
$$\mathbf{v} \equiv z_{\tau_1} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_t}^2 z_{\tau_{t+1}} \cdots z_{\tau_k},$$

where $\sigma_1, \dots, \sigma_k$ are distinct with $\sigma_p < \cdots < \sigma_q$, and $\tau_1, \dots, \tau_k$ are distinct with $\tau_s < \cdots < \tau_t$. Moreover, $\{\sigma_1, \dots, \sigma_k\} = \{\tau_1, \dots, \tau_k\}$ since $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. Let $m = \min\{p-1, s-1\}$ and $n = \min\{k-q, k-t\}$. If $\overleftarrow{\mathbf{u}}^{\infty} \equiv \overleftarrow{\mathbf{v}}^{\infty}$ and $\overrightarrow{\mathbf{u}}^{\infty} \equiv \overrightarrow{\mathbf{v}}^{\infty}$ then clearly $\mathbf{u} \equiv \mathbf{v}$. Therefore assume that either $\overleftarrow{\mathbf{u}}^{\infty} \not\equiv \overleftarrow{\mathbf{v}}^{\infty}$ or $\overrightarrow{\mathbf{u}}^{\infty} \not\equiv \overrightarrow{\mathbf{v}}^{\infty}$.

Case (i) Suppose that $\overleftarrow{\mathbf{u}}^{\infty} \not\equiv \overleftarrow{\mathbf{v}}^{\infty}$ and $\overrightarrow{\mathbf{u}}^{\infty} \equiv \overrightarrow{\mathbf{v}}^{\infty}$. By Lemma 4.5.6, $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \subseteq [l : 0 : \infty]$ for some $l < \infty$ with $\overleftarrow{\mathbf{u}}^{l} \equiv \overleftarrow{\mathbf{v}}^{l}$ and $\overleftarrow{\mathbf{u}}^{l+1} \not\equiv \overleftarrow{\mathbf{v}}^{l+1}$ (note: $l \le m$ necessarily). Thus $(\sigma_1, \dots, \sigma_l) = (\tau_1, \dots, \tau_l)$ and $\{\sigma_{l+1}, \dots, \sigma_q\} = \{\tau_{l+1}, \dots, \tau_t\}$, whence

$$
\begin{aligned}
& z_{\sigma_1} \cdots z_{\sigma_l} z_{\sigma_{l+1}} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_{q-1}}^2 z_{\sigma_q}^2 \\
& \equiv_{(l:0:\infty)} \overleftarrow{\mathbf{u}}^{l} \left(z_{\sigma_{l+1}} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_{q-1}}^2 \right)^2 z_{\sigma_q}^2 \qquad \text{by } (l:0:\infty) \\
& \equiv_{\mathbf{V}} \overleftarrow{\mathbf{u}}^{l} \left[\left(z_{\sigma_{l+1}} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_{q-1}}^2 \right) z_{\sigma_q} \right]^2 \qquad \text{by Lemma 4.3.2(3)} \\
& \equiv_{\mathbf{V}} \overleftarrow{\mathbf{v}}^{l} \left[\left(z_{\tau_{l+1}} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_{t-1}}^2 \right) z_{\tau_t} \right]^2 \qquad \text{by Lemma 4.3.2(4)} \\
& \equiv_{\mathbf{V}} \overleftarrow{\mathbf{v}}^{l} \left(z_{\tau_{l+1}} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_{t-1}}^2 \right)^2 z_{\tau_t}^2 \qquad \text{by Lemma 4.3.2(3)} \\
& \equiv_{(l:0:\infty)} z_{\tau_1} \cdots z_{\tau_l} z_{\tau_{l+1}} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_{t-1}}^2 z_{\tau_t}^2 \qquad \text{by } (l:0:\infty).
\end{aligned}
$$

Thus $[l : 0 : \infty]^{\mathbf{B}_2^-} \subseteq [\mathbf{u} = \mathbf{v}]$, whence $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = [l : 0 : \infty]^{\mathbf{B}_2^-} = \mathbf{L}_l \mathbf{N}_\infty \mathbf{R}_\infty$. By symmetry, if $\overleftarrow{\mathbf{u}}^{\infty} \equiv \overleftarrow{\mathbf{v}}^{\infty}$ and $\overrightarrow{\mathbf{u}}^{\infty} \not\equiv \overrightarrow{\mathbf{v}}^{\infty}$ then $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = \mathbf{L}_\infty \mathbf{N}_\infty \mathbf{R}_r$ for some $r \le n$.

Case (ii) Suppose that $\overleftarrow{\mathbf{u}}^{\infty} \not\equiv \overleftarrow{\mathbf{v}}^{\infty}$ and $\overrightarrow{\mathbf{u}}^{\infty} \not\equiv \overrightarrow{\mathbf{v}}^{\infty}$. By Lemma 4.5.6 and its dual result, we have

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \subseteq [(l:0:\infty), (\infty:0:r)]^{\mathbf{B}_2^-} \subseteq [l : \infty : r]$$

for some $l < \infty$ and $r < \infty$ with $\overleftarrow{\mathbf{u}}^{l} \equiv \overleftarrow{\mathbf{v}}^{l}$ and $\overrightarrow{\mathbf{u}}^{r} \equiv \overrightarrow{\mathbf{v}}^{r}$. Note that $l \le m$, $r \le n$ and $\{\sigma_{l+1}, \dots, \sigma_{k-r}\} = \{\tau_{l+1}, \dots, \tau_{k-r}\}$. Therefore

$$
\begin{aligned}
\mathbf{u} & \equiv z_{\sigma_1} \cdots z_{\sigma_l} z_{\sigma_{l+1}} \cdots z_{\sigma_{p-1}} z_{\sigma_p}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}} \cdots z_{\sigma_{k-r}} z_{\sigma_{k-r+1}} \cdots z_{\sigma_k} \\
& \equiv_{(l:\infty:r)} \overleftarrow{\mathbf{u}}^{l} z_{\sigma_{l+1}}^2 \cdots z_{\sigma_{p-1}}^2 z_{\sigma_p}^2 \cdots z_{\sigma_q}^2 z_{\sigma_{q+1}}^2 \cdots z_{\sigma_{k-r}}^2 \overrightarrow{\mathbf{u}}^{r} \qquad \text{by } (l:\infty:r) \\
& \equiv_{\mathbf{V}} \overleftarrow{\mathbf{v}}^{l} z_{\tau_{l+1}}^2 \cdots z_{\tau_{s-1}}^2 z_{\tau_s}^2 \cdots z_{\tau_t}^2 z_{\tau_{t+1}}^2 \cdots z_{\tau_{k-r}}^2 \overrightarrow{\mathbf{v}}^{r} \qquad \text{by } \mathbf{V} \\
& \equiv_{(l:\infty:r)} z_{\tau_1} \cdots z_{\tau_l} z_{\tau_{l+1}} \cdots z_{\tau_{s-1}} z_{\tau_s}^2 \cdots z_{\tau_t}^2 z_{\tau_{t+1}} \cdots z_{\tau_{k-r}} z_{\tau_{k-r+1}} \cdots z_{\tau_k} \qquad \text{by } (l:\infty:r) \\
& \equiv \mathbf{v}
\end{aligned}
$$

and $[l : \infty : r]^{\mathbf{B}_2^-} \subseteq [\mathbf{u} = \mathbf{v}]$. Consequently, $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = [l : \infty : r]^{\mathbf{B}_2^-} = \mathbf{L}_l \mathbf{N}_\infty \mathbf{R}_r$. ■

Lemma 4.9.4 *Let* $\mathbf{u}, \mathbf{v}$ *be* $\mathbf{L}_\infty\mathbf{R}_\infty$*-words with* $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ *such that* $\mathbf{u}$ *is linear and* $\mathbf{v}$ *is quadratic. If* $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \notin \mathfrak{LNR}^* \cup \{\mathbf{B}_2^-\}$, *then*

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} = [l : m : r]^{\mathbf{B}_2^-} \cap [\mathrm{Id}\pi]$$

for some $l, m, r \in \mathbb{N}_0$ *and nontrivial permutation* $\pi \in S_k$ *with* $k = l + m + r + 1$.

Proof. Although $\mathbf{u}, \mathbf{v}$ are $\mathbf{L}_\infty\mathbf{R}_\infty$-words by assumption, the ordering on their subscripts is unnecessary in this proof. Therefore we assume that $\mathbf{u} \equiv z_1 \cdots z_k$ (instead of $\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$) for simplicity. Now since $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$ and $\mathbf{v}$ is quadratic,

$$\mathbf{v} \equiv z_{\pi(1)} \cdots z_{\pi(p-1)} z_{\pi(p)}^2 \cdots z_{\pi(q)}^2 z_{\pi(q+1)} \cdots z_{\pi(k)}$$

for some permutation $\pi \in S_k$ and $1 \le p \le q \le k$. Let $\mathbf{U} = [\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-}$. Suppose $\pi(i) = i$ for all $i \in \mathbb{I}_1^{p-1} \cup \mathbb{I}_{q+1}^k$. Then $\pi(\mathbb{I}_p^q) = \mathbb{I}_p^q$. Now

$$\begin{aligned} z_1 \cdots z_k &\equiv_{\mathbf{U}} z_1 \cdots z_{p-1} z_{\pi(p)}^2 \cdots z_{\pi(q)}^2 z_{q+1} \cdots z_k \\ &\equiv_{\mathbf{V}} z_1 \cdots z_{p-1} z_p^2 \cdots z_q^2 z_{q+1} \cdots z_k \qquad \text{by V} \\ &\equiv_{\mathbf{V}} z_1 \cdots z_{p-1} (z_p \cdots z_q)^2 z_{q+1} \cdots z_k \qquad \text{by Lemma 4.3.2(3)} \end{aligned}$$

implies that $\mathbf{U} \subseteq [p-1 : q-p : k-q]^{\mathbf{B}_2^-}$, and

$$\begin{aligned} z_1 \cdots z_k &\equiv_{(p-1:q-p:k-q)} z_1 \cdots z_{p-1} (z_p \cdots z_q)^2 z_{q+1} \cdots z_k \\ &\equiv_{\mathbf{V}} z_1 \cdots z_{p-1} z_p^2 \cdots z_q^2 z_{q+1} \cdots z_k \qquad \text{by Lemma 4.3.2(3)} \\ &\equiv_{\mathbf{V}} z_1 \cdots z_{p-1} z_{\pi(p)}^2 \cdots z_{\pi(q)}^2 z_{q+1} \cdots z_k \qquad \text{by V} \end{aligned}$$

implies that $[p-1 : q-p : k-q]^{\mathbf{B}_2^-} \subseteq \mathbf{U}$. Therefore we have the contradiction

$$\mathbf{U} = [p-1 : q-p : k-q]^{\mathbf{B}_2^-} \in \mathfrak{LNR}^*.$$

Hence $\pi(i) \ne i$ for some $i \in \mathbb{I}_1^{p-1} \cup \mathbb{I}_{q+1}^k$.

Note that if $\mathbf{B}_2^- \models \mathbf{u} = \mathbf{v}$, then $\mathbf{B}_2^- \models \mathbf{u}^{\mathbf{B}_2^-} = \mathbf{v}^{\mathbf{B}_2^-}$ so that by Theorem 4.3.6 we must have $\mathbf{u}^{\mathbf{B}_2^-} \equiv \mathbf{v}^{\mathbf{B}_2^-}$. But this is impossible since $\mathbf{u}^{\mathbf{B}_2^-} \equiv \mathbf{u}$ is linear but $\mathbf{v}^{\mathbf{B}_2^-}$ is clearly not. Thus $\mathbf{B}_2^- \not\models \mathbf{u} = \mathbf{v}$ and $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \models (\infty : 0 : \infty)$ by Lemma 4.8.11. Now consider a word $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ with $x_1, \ldots, x_k$ distinct and $\alpha_i \in \{1, 2\}$. Invoking $\mathbf{u} = \mathbf{v}$ on $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ yields

$$x_1^{\alpha_1} \cdots x_k^{\alpha_k} \equiv_{\mathbf{U}} x_{\pi(1)}^{\alpha_{\pi(1)}} \cdots x_{\pi(p-1)}^{\alpha_{\pi(p-1)}} x_{\pi(p)}^2 \cdots x_{\pi(q)}^2 x_{\pi(q+1)}^{\alpha_{\pi(q+1)}} \cdots x_{\pi(k)}^{\alpha_{\pi(k)}}.$$

Furthermore, by invoking $(\infty : 0 : \infty)$ and Lemma 4.3.2(3), the word on the right of this equation is $\equiv_{\mathbf{U}}$-related to a quadratic word, that is, a word such that

$$\left(\alpha_{\pi(i)} = 2 \text{ for some } i \in \mathbb{I}_1^{p-1}\right) \Longrightarrow \alpha_{\pi(i)} = \cdots = \alpha_{\pi(p-1)} = 2,$$
$$\left(\alpha_{\pi(j)} = 2 \text{ for some } j \in \mathbb{I}_{q+1}^{k}\right) \Longrightarrow \alpha_{\pi(q+1)} = \cdots = \alpha_{\pi(j)} = 2.$$

Hence the identity $\mathbf{u} = \mathbf{v}$ induces a mapping Ψ as follows:

$$x_1^{\alpha_1} \cdots x_k^{\alpha_k} \overset{\Psi}{\longmapsto} x_{\pi(1)}^{\alpha_{\pi(1)}} \cdots x_{\pi(p-1)}^{\alpha_{\pi(p-1)}} x_{\pi(p)}^2 \cdots x_{\pi(q)}^2 x_{\pi(q+1)}^{\alpha_{\pi(q+1)}} \cdots x_{\pi(k)}^{\alpha_{\pi(k)}}.$$

Note that Ψ maps both linear and quadratic words to quadratic words. In particular, $\Psi(\mathbf{u}) = \mathbf{v}$.

Now for each $i \geq 1$, let $\mathbf{I}_i$ be the interval of $\Psi^i(\mathbf{u})$. It is clear that

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \vDash \mathbf{u} = \Psi(\mathbf{u}) = \Psi^2(\mathbf{u}) = \cdots,$$

and that $\mathbf{I}_1 \subseteq \mathbf{I}_2 \subseteq \cdots$. Since all $\mathbf{I}_i$ are contained in $\mathbb{I}_1^k$, there exists some sufficiently large number h such that $\mathbf{I}_h = \mathbf{I}_{h+1} = \cdots = \mathbb{I}_s^t$ with $1 \leq s \leq t \leq k$. Now $\pi^{k!h}$ is just the trivial permutation and $\mathbf{I}_{k!h} = \mathbb{I}_s^t$. Therefore

$$\begin{aligned} z_1 \cdots z_k &\equiv_{\mathbf{U}} \Psi^{k!h}(\mathbf{u}) \\ &\equiv z_1 \cdots z_{s-1} z_s^2 \cdots z_t^2 z_{t+1} \cdots z_k \\ &\equiv_{\nabla} z_1 \cdots z_{s-1} (z_s \cdots z_t)^2 z_{t+1} \cdots z_k \qquad \text{by Lemma 4.3.2(3),} \end{aligned}$$

which is exactly the identity $(l : m : r)$ with $l = s-1$, $m = t-s$ and $r = k-t$. Thus $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \subseteq [l : m : r]$. Furthermore,

$$\begin{aligned} z_1 \cdots z_k &\equiv_{\mathbf{U}} \Psi^{k!h+1}(\mathbf{u}) \\ &\equiv z_{\pi(1)} \cdots z_{\pi(s-1)} z_{\pi(s)}^2 \cdots z_{\pi(t)}^2 z_{\pi(t+1)} \cdots z_{\pi(k)} \\ &\equiv z_{\pi(1)} \cdots z_{\pi(l)} z_{\pi(l+1)}^2 \cdots z_{\pi(k-r)}^2 z_{\pi(k-r+1)} \cdots z_{\pi(k)} \\ &\equiv_{\mathbf{U}} z_{\pi(1)} \cdots z_{\pi(k)} \qquad \text{by } (l : m : r). \end{aligned}$$

Hence $[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \subseteq [\mathrm{Id}\pi]$. Consequently,

$$[\mathbf{u} = \mathbf{v}]^{\mathbf{B}_2^-} \subseteq [l : m : r]^{\mathbf{B}_2^-} \cap [\mathrm{Id}\pi], \tag{4.14}$$

where $\pi \in S_k$ with $k = l + m + r + 1$.

Now note that $\mathbb{I}_p^q = \mathbf{I}_1 \subseteq \mathbb{I}_s^t$ implies

$$l+1 = s \leq p, \qquad q \leq t = k-r.$$

Therefore by Lemma 4.4.3,

$$\begin{aligned}[l:m:r]^{\mathbf{B}_2^-} &\models \left\{ z_1 \cdots z_l z_{l+1}^{\beta_{l+1}} \cdots z_{k-r}^{\beta_{k-r}} z_{k-r+1} \cdots z_k = z_1 \cdots z_k \middle| 1 \leq \beta_i \leq 2 \right\} \\ &\vdash z_1 \cdots z_{p-1} z_p^2 \cdots z_q^2 z_{q+1} \cdots z_k = z_1 \cdots z_k,\end{aligned}$$

and $\nabla \cup \{(l:m:r), \mathrm{Id}\pi\} \vdash \mathbf{u} = \mathbf{v}$ because

$$\begin{aligned}\mathbf{u} &\equiv z_1 \cdots z_k \\ &\equiv_{[\mathrm{Id}\pi]} z_{\pi(1)} \cdots z_{\pi(k)} \\ &\equiv_{(l:m:r)} z_{\pi(1)} \cdots z_{\pi(p-1)} z_{\pi(p)}^2 \cdots z_{\pi(q)}^2 z_{\pi(q+1)} \cdots z_{\pi(k)} \\ &\equiv \mathbf{v}.\end{aligned}$$

Consequently equality holds in (4.14). ∎

The result of Proposition 4.9.2 can now be extented to $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$.

Proposition 4.9.5 *Each variety in* $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$ *is of the form*

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}$$

where $(l,n,r) \in \mathbb{V}$ *and* $\mathbf{P}$ *is a permutation variety.*

PROOF. Let $\mathbf{V} \in [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$. Suppose $\mathbf{V} \models \mathbf{u} = \mathbf{v}$, and let $\mathbf{U} = [\mathbf{u} = \mathbf{v}]^{\mathbf{L}_\infty\mathbf{R}_\infty}$. Since

$$\mathbf{U} = [\mathbf{u} = \mathbf{v}]^{\mathbf{L}_\infty\mathbf{R}_\infty} = \left[\mathbf{u}^{\mathbf{L}_\infty\mathbf{R}_\infty} = \mathbf{v}^{\mathbf{L}_\infty\mathbf{R}_\infty}\right]^{\mathbf{L}_\infty\mathbf{R}_\infty},$$

we may assume $\mathbf{u}, \mathbf{v}$ to be $\mathbf{L}_\infty\mathbf{R}_\infty$-words to begin with. If one of $\mathbf{u}, \mathbf{v}$ is quadratic, then by Lemmas 4.9.3 and 4.9.4, $\mathbf{U} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}$ for some $(l,n,r) \in \mathbb{V}$ and permutation variety $\mathbf{P}$, possibly defined by trivial permutations. If both $\mathbf{u}, \mathbf{v}$ are linear, then $\mathbf{U} = \mathbf{L}_\infty\mathbf{R}_\infty \cap [\mathbf{u} = \mathbf{v}]$. Hence each identity of $\mathbf{V}$ defines a variety of the form $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}$ so that $\mathbf{V}$ itself also has the same form. ∎

Corollary 4.9.6 *Each variety in* $\mathcal{L}(\mathbf{A}_0)$ *is finitely based.*

PROOF. Since

$$\mathcal{L}(\mathbf{A_0}) = \{\mathbf{A_0}, \mathbf{B_2^-}\} \cup [\mathbf{T}, \mathbf{N}_\infty] \cup [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty],$$

the corollary is implied by Propositions 4.9.2, 4.9.5 and Corollary 2.7.2. ■

Having identified all varieties in $[\mathbf{T}, \mathbf{N}_\infty] \cup [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] = \mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$, we will continue the investigation with $[\mathbf{T}, \mathbf{N}_\infty]$ and $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$ respectively in the next three sections.

4.10 The Interval $[\mathbf{T}, \mathbf{N}_\infty]$

In this section, we examine varieties in $[\mathbf{T}, \mathbf{N}_\infty]$, their generating semigroups, and the intersections and joins they form. Let $\mathbf{V} \in [\mathbf{T}, \mathbf{N}_\infty]$. Then by Proposition 4.9.2 and Corollary 4.9.6,

$$\mathbf{V} = \mathbf{N}_n \cap [\mathrm{Id}\Pi] = \mathbf{N}_n^{\mathrm{Id}\Pi} \tag{4.15}$$

for some $n \in \mathbb{N}_0^\infty$ and finite set Π of (possibly trivial) permutations. If n is finite, then the defining identity $\mathbf{x}_{n+1} = \mathbf{0}$ of $\mathbf{N}_n$ implies all permutations with at least $n+1$ variables. Therefore we may assume throughout this section that each permutation in Π involves at most n variables.

Lemma 4.10.1 *Suppose* $\mathbf{N}_n^{\mathrm{Id}\Pi} \models \mathbf{u} = \mathbf{v}$. *Then* $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{0}$ *if and only if* $\mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{0}$.

PROOF. By Theorem 4.6.7(3), the variety $\mathbf{N}_n^{\mathrm{Id}\Pi}$ is defined by the identities in $\{\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}\}$ and $\mathrm{Id}\Pi$. Suppose $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{0}$ and $\mathbf{v}^{\mathbf{N}_n} \not\equiv \mathbf{0}$. Then $\mathbf{u}$ either contains a variable with multiplicity at least two, or it is a word of length at least $n+1$. But $\mathbf{v}$ is a linear word with at most n variables so that it is impossible to deduce $\mathbf{u}$ from $\mathbf{v}$ by the identities in $\{\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}\} \cup \mathrm{Id}\Pi$. Therefore $\mathbf{u}^{\mathbf{N}_n}$ and $\mathbf{v}^{\mathbf{N}_n}$ are both zero or nonzero simultaneously. ■

Proposition 4.10.2 $\mathbf{N}_n^{\mathrm{Id}\Pi} \models \mathbf{u} = \mathbf{v}$ *if and only if* $\mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$.

PROOF. Suppose $\mathbf{N}_n^{\mathrm{Id}\Pi} \models \mathbf{u} = \mathbf{v}$. Then since $\mathbf{N}_n^{\mathrm{Id}\Pi} \subseteq \mathbf{N}_n$, we have $\mathbf{N}_n^{\mathrm{Id}\Pi} \models \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$ and

$$\{\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}\} \cup \mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}.$$

If $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{0}$ then $\mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{0}$ by Lemma 4.10.1 so that $\mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$ vacuously. Otherwise, both $\mathbf{u}^{\mathbf{N}_n}$ and $\mathbf{v}^{\mathbf{N}_n}$ are not $\mathbf{0}$, whence they are linear with at most n variables. The identities $\mathbf{x}_{n+1} = x^2 = xyx = \mathbf{0}$ clearly cannot be used to deduce new words from $\mathbf{u}^{\mathbf{N}_n}$ and $\mathbf{v}^{\mathbf{N}_n}$. Consequently they are unnecessary in the duduction of $\mathbf{v}^{\mathbf{N}_n}$ from $\mathbf{u}^{\mathbf{N}_n}$ (and vice versa). Hence $\mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$.

Conversely, suppose $\mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$. Then $\mathbf{N}_n^{\mathrm{Id}\Pi} \vDash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}$. By Lemma 4.10.1, either $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{0}$ or $\mathbf{u}^{\mathbf{N}_n} \not\equiv \mathbf{0} \not\equiv \mathbf{v}^{\mathbf{N}_n}$. If $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{0}$ then

$$\mathbf{N}_n^{\mathrm{Id}\Pi} \vDash \mathbf{u} = \mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{v}^{\mathbf{N}_n} = \mathbf{v}.$$

If $\mathbf{u}^{\mathbf{N}_n} \not\equiv \mathbf{0} \not\equiv \mathbf{v}^{\mathbf{N}_n}$, then $\mathbf{u} \equiv \mathbf{u}^{\mathbf{N}_n}$ and $\mathbf{v} \equiv \mathbf{v}^{\mathbf{N}_n}$ are both linear with at most n variables, whence we again have $\mathbf{N}_n^{\mathrm{Id}\Pi} \vDash \mathbf{u} = \mathbf{v}$. ■

Since it is extremely straightforward to reduce words into $\mathbf{N}_n$-words, Proposition 4.10.2 provides an easy way to recognize identities of $\mathbf{N}_n^{\Pi}$:

$$\begin{aligned}\mathrm{Id}_{\mathbf{N}_n^{\mathrm{Id}\Pi}}(X) &= \left\{(\mathbf{u},\mathbf{v}) \in X^+ \times X^+ \mid \mathbf{N}_n^{\mathrm{Id}\Pi} \vDash \mathbf{u} = \mathbf{v}\right\} \\ &= \left\{(\mathbf{u},\mathbf{v}) \in X^+ \times X^+ \mid \mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}\right\}.\end{aligned}$$

In fact, it is easy to see that the congruence

$$\mathrm{Id}_{\mathbf{N}_n^{\mathrm{Id}\Pi}}(X_n) = \left\{(\mathbf{u},\mathbf{v}) \in X_n^+ \times X_n^+ \mid \mathrm{Id}\Pi \vdash \mathbf{u}^{\mathbf{N}_n} = \mathbf{v}^{\mathbf{N}_n}\right\}$$

on $X_n^+ \times X_n^+$ with $X_n = \{x_1, \dots, x_n\}$, is the restriction of $\mathrm{Id}_{\mathbf{N}_n^{\mathrm{Id}\Pi}}(X)$ to X_n^+ and so is also fully invariant.

Proposition 4.10.3 $\mathbf{N}_n^{\mathrm{Id}\Pi} = \left\langle F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right)\right\rangle$.

PROOF. Let $\mathbf{V} = \mathbf{N}_n^{\mathrm{Id}\Pi}$. Clearly $F_n(\mathbf{V})$ embeds into $F(\mathbf{V})$ in the obvious way so that $F_n(\mathbf{V}) \in \langle F(\mathbf{V})\rangle = \mathbf{V}$. Therefore it remains to show $F(\mathbf{V}) \in \langle F_n(\mathbf{V})\rangle$. Suppose $F_n(\mathbf{V}) \vDash \mathbf{u} = \mathbf{v}$. We may assume $\mathbf{u}$ and $\mathbf{v}$ are distinct $\mathbf{N}_n$-words. If $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$ then by Lemma 4.9.1, $[\mathbf{u} = \mathbf{v}]^{\mathbf{N}_\infty} = [\mathbf{x}_t = \mathbf{0}]^{\mathbf{N}_\infty}$ for some $t \leq n$ so that $F_n(\mathbf{V}) \models \mathbf{x}_t = \mathbf{0}$. But this is impossible since $\mathrm{Id}\Pi \nvdash \mathbf{x}_t = \mathbf{0}$. Therefore $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$, whence $\mathbf{u} = \mathbf{v}$ is a permutation involving at most n variables. By renaming variables we may assume $\mathbf{u}, \mathbf{v} \in X_n^+$ so that $(\mathbf{u},\mathbf{v}) \in \mathrm{Id}_{\mathbf{V}}(X_n)$. Consequently $(\mathbf{u},\mathbf{v}) \in \mathrm{Id}_{\mathbf{V}}(X)$ and $F(\mathbf{V}) \vDash \mathbf{u} = \mathbf{v}$. ■

Proposition 4.10.4 *The semigroup* $\mathbf{N}_n^{\mathrm{Id}\Pi}$ *is finitely generated if and only if* n *is finite.*

PROOF. If n is finite then by Proposition 4.10.3, $\mathbf{N}_n^{\mathrm{Id}\Pi}$ is generated by the semigroup $F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right)$. Since $\mathbf{A_0}$ is locally finite (Corollary 2.5.3) and $F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right) = X_n^+/\mathrm{Id}_{\mathbf{N}_n^{\mathrm{Id}\Pi}}(X_n)$ is n-generated, $F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right)$ is also finite. If n is infinite, then an argument similar to the one in the proof of Proposition 4.6.9 shows that $\mathbf{N}_n^{\mathrm{Id}\Pi}$ cannot be generated by any finite semigroup. ■

We now turn our attention to finding intersections and joins of varieties in $[\mathbf{T}, \mathbf{N}_\infty]$. For each $i \in I$, let $n_i \in \mathbb{N}_0^\infty$ and Π_i be a nonempty finite set of (possibly trivial) permutations so that $\left\{\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} \middle| i \in I\right\}$ is an arbitrary collection of varieties in $[\mathbf{T}, \mathbf{N}_\infty]$. Then

$$\bigcap\nolimits_{i\in I} \mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} = \mathbf{N}_n^{\mathrm{Id}\Pi}$$

where $n = \min\{n_i \mid i \in I\}$ and $\Pi = \bigcup_{i \in I} \Pi_i$. Clearly if I is finite then so is Π. If I is infinite, then Π can be replaced by a subset of Π involving at most k variables, where $k = \min\{n, \Gamma(\Pi)\}$ and $\Gamma(\Pi)$ is a bound found in Section 7 of Chapter 2 on the number of variables required to define $[\mathrm{Id}\Pi]$.

Finding joins of varieties in $[\mathbf{T}, \mathbf{N}_\infty]$ requires a little more preparation. For each $k \in \mathbb{N}$, let $\Pi_i^{(k)}$ be the set of all permutations involving exactly k variables the associated identities of which are satisfied by $\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}$, that is, $\Pi_i^{(k)} = \left\{\pi \in S_k \middle| \mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} \models \mathrm{Id}\pi\right\}$. Then

$$\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} = \mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)} \middle| k \in \mathbb{N}\right]. \tag{4.16}$$

Note that each $\Pi_i^{(k)}$ is (necessarily) a subgroup of S_k. In particular, if Π_i contains only trivial permutations then $\Pi_i^{(k)}$ is the trivial subgroup of S_k. Since the subgroups $\Pi_i^{(k)}$ ($k \in \mathbb{N}$) constitute all permutations the associated identities of which are satisfied by $\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}$, the expression in (4.16) is maximal and thus unique; we shall called it the *complete representation* of $\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}$.

Proposition 4.10.5 *For each* $i \in I$, *let* $n_i \in \mathbb{N}_0^\infty$ *and* Π_i *be a finite set of permutations. Let*

$$\mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)} \middle| k \in \mathbb{N}\right]$$

be the complete representation of $\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}$. *Then*

$$\bigvee\nolimits_{i\in I}\left(\mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)} \middle| k \in \mathbb{N}\right]\right) = \mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right] \tag{4.17}$$

where $n = \sup\{n_i \mid i \in I\}$ *and* $\Pi^{(k)} = \bigcap_{i\in I} \Pi_i^{(k)}$. *In particular, if* $n < \infty$ *then*

$$\bigvee_{i\in I} \left(\mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)} \middle| k \in \mathbb{N}\right]\right) = \mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \leq n\right].$$

PROOF. Let $\mathbf{U}$ (respectively, $\mathbf{V}$) denote the variety on the left (respectively, right) of the equation in (4.17). The containment $\mathbf{U} \subseteq \mathbf{V}$ can be verified easily. Thus it remains to show $\mathbf{V} \subseteq \mathbf{U}$.

Suppose $\mathbf{U} \models \mathbf{u} = \mathbf{v}$. Since $\mathbf{U} \subseteq \mathbf{N}_n$, we may assume $\mathbf{u}$ and $\mathbf{v}$ to be distinct $\mathbf{N}_n$-words. First suppose $\mathfrak{c}(\mathbf{u}) \neq \mathfrak{c}(\mathbf{v})$. Then by Lemma 4.9.1, $[\mathbf{u} = \mathbf{v}]^{\mathbf{N}_\infty} = [\mathbf{x}_s = \mathbf{0}]^{\mathbf{N}_\infty}$ for some finite $s \leq n$ so that $\mathbf{U} \models \mathbf{x}_s = \mathbf{0}$. If $s < n$, then there exist $j \in I$ with $s < n_j$, whence $F\left(\mathbf{N}_{n_j}^{\mathrm{Id}\Pi_j}\right)$ is contradictorily a semigroup in $\mathbf{U}$ that does not satisfy $\mathbf{x}_s = \mathbf{0}$. Therefore $s = n$ and $[\mathbf{u} = \mathbf{v}]^{\mathbf{N}_\infty} = [\mathbf{x}_n = \mathbf{0}]^{\mathbf{N}_\infty} = \mathbf{N}_n \supseteq \mathbf{V}$, whence $\mathbf{V} \models \mathbf{u} = \mathbf{v}$. Next, suppose $\mathfrak{c}(\mathbf{u}) = \mathfrak{c}(\mathbf{v})$. Then $\mathbf{u} = \mathbf{v}$ is a permutation identity, say in t variables. Since $\mathbf{u} = \mathbf{v}$ is satisfied by $\mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)} \middle| k \in \mathbb{N}\right]$ for all $i \in I$, it belongs to $\mathrm{Id}\left(\bigcap_{i\in I} \Pi_{n_i}^{(t)}\right)$, whence $\mathbf{V} \models \mathbf{u} = \mathbf{v}$. ■

Proposition 4.10.5 presented a basis of the arbitrary join of varieties from $[\mathbf{T}, \mathbf{N}_\infty]$. If I and $n = \sup\{n_i \mid i \in I\}$ are finite, then the basis of the join of varieties in Proposition 4.10.5 is computable. But this is possible even if n is infinite, as we shall demonstrate in the following result. Note that it suffices to consider the join of just two varieties as the general finite case is covered by induction.

Proposition 4.10.6 *Let* $\mathbf{N}_{n_1}^{\mathrm{Id}\Pi_1}, \mathbf{N}_{n_2}^{\mathrm{Id}\Pi_2} \in [\mathbf{T}, \mathbf{N}_\infty]$ *where* Π_1 *and* Π_2 *are finite collections of permutations, and let* $n = \max\{n_1, n_2\}$. *Then there exists* $m \in \mathbb{N}$ *such that*

$$\mathbf{N}_{n_1}^{\mathrm{Id}\Pi_1} \vee \mathbf{N}_{n_2}^{\mathrm{Id}\Pi_2} = \mathbf{N}_n \cap \left[\mathrm{Id}\Pi_1^{(k)} \cap \mathrm{Id}\Pi_2^{(k)} \middle| k \leq m\right].$$

PROOF. For $i \in \{1, 2\}$, there exist l_i, r_i, h_i such that all nontrivial identities in $\mathrm{Id}\Pi_i$ are of the form $\mathrm{Id}(l_i : \pi : r_i)$, and that $\mathrm{Id}\Pi_i \vdash \mathrm{Id}(l_i : S_{h_i} : r_i)$ (see Section 7 of Chapter 2). Therefore $\Pi_i^{(k)} = (l_i : S_{k-l_i-r_i} : r_i)$ for all $k \geq l_i + h_i + r_i$. Let $m = \max\{l_1 + h_1 + r_1, l_2 + h_2 + r_2\}$. By Proposition 4.10.5 it suffices to show

$$\mathbf{N}_n \cap \left[\mathrm{Id}\Pi_1^{(k)} \cap \mathrm{Id}\Pi_2^{(k)} \middle| k \in \mathbb{N}\right] = \mathbf{N}_n \cap \left[\mathrm{Id}\Pi_1^{(k)} \cap \mathrm{Id}\Pi_2^{(k)} \middle| k \leq m\right].$$

Let $\mathbf{U}$ (respectively, $\mathbf{V}$) denote the variety on the left (respectively, right) of the above equation. It is easy to show $\mathbf{U} \subseteq \mathbf{V}$.

Suppose $\mathbf{u} = \mathbf{v}$ is an identity satisfied by $\mathbf{U}$. We may assume the pair of words forming $\mathbf{u} = \mathbf{v}$ to be $\mathbf{N}_n$-words. Clearly, $\mathbf{U}$ satisfies an identity of the form $\mathbf{x}_s = \mathbf{0}$ if and only if $\mathbf{V}$ does. Therefore if $\mathbf{u} = \mathbf{v}$ is not balanced, then $\mathbf{V} \models \mathbf{u} = \mathbf{v}$ by Lemma 4.9.1. It remains to consider when $\mathbf{u} = \mathbf{v}$ is balanced, whence it is a permutation identity involving, say, q variables. If $q \leq m$ then clearly $\mathbf{V} \models \mathbf{u} = \mathbf{v}$. Otherwise,

$$\begin{aligned} q > m &\Longrightarrow \mathbf{u} = \mathbf{v} \in \mathrm{Id}\Pi_1^{(q)} \cap \mathrm{Id}\Pi_2^{(q)} \\ &\Longrightarrow \mathbf{u} = \mathbf{v} \in \mathrm{Id}\,(l_1 : S_{q-l_1-r_1} : r_1) \cap \mathrm{Id}\,(l_2 : S_{q-l_2-r_2} : r_2) \\ &\Longrightarrow \mathrm{Id}\,(l_1 : S_{m-l_1-r_1} : r_1) \vdash \mathbf{u} = \mathbf{v} \;\text{ and } \mathrm{Id}\,(l_2 : S_{m-l_2-r_2} : r_2) \vdash \mathbf{u} = \mathbf{v} \\ &\Longrightarrow \mathbf{u} = \mathbf{v} \in \mathrm{Id}\Pi_1^{(m)} \cap \mathrm{Id}\Pi_2^{(m)} \\ &\Longrightarrow \mathbf{V} \models \mathbf{u} = \mathbf{v}. \end{aligned}$$

Hence in all cases, $\mathbf{V} \models \mathbf{u} = \mathbf{v}$. Consequently, $\mathbf{V} \subseteq \mathbf{U}$. ■

Having investigated the intersections and joins of varieties in $[\mathbf{T}, \mathbf{N}_\infty]$, we describe the locations of varieties in $[\mathbf{T}, \mathbf{N}_\infty]$ with respect to one another by presenting a diagram.

Let α and β be identities. We say α is *shorter* than β if α involves fewer variables than β. For each $k \in \mathbb{N}$, let $\mathfrak{P}_k$ be the class of all nontrivial permutation varieties of the form $[\mathrm{Id}\Pi]$ such that the shortest nontrivial identities in $\mathrm{Id}\Pi$ involve k variables. Equivalently,

$$\mathfrak{P}_k = \{[\mathrm{Id}\Pi]\,|\Pi \cap S_k \nsubseteq \{1\}\,, \Pi \cap S_i \subseteq \{1\} \;\text{ if } i < k\}\,.$$

In particular, we have $\mathfrak{P}_2 = \{[xy = yx]\}$ since $xy = yx$ implies all nontrivial permutation identities. Letting $\mathsf{P}\,(n;k) = \{\mathbf{N}_n \cap \mathbf{P}|\, \mathbf{P} \in \mathfrak{P}_k\}$ and $\mathbf{C} = [xy = yx]$, a diagram of the interval $[\mathbf{T}, \mathbf{N}_\infty]$ can be seen in Figure 4.4. This diagram displays the location of each variety $\mathbf{N}_n \cap \mathbf{P}$ in $[\mathbf{T}, \mathbf{N}_\infty]$ to within the respective class $\mathsf{P}\,(n;k)$ it belongs to. It is possible but too difficult to include each variety in each $\mathsf{P}\,(n;k)$ individually in the diagram. If n and k are fixed, then it is easy to show that $\mathsf{P}\,(n;k)$ is closed under taking intersections but not closed under taking joins. To illustrate these properties, consider the interval $[\mathbf{N}_3 \cap \mathbf{C}, \mathbf{N}_3] = \mathsf{P}\,(3;3) \cup \{\mathbf{N}_3 \cap \mathbf{C}, \mathbf{N}_3\}$ as shown in Figure 4.5. For brevity, the varieties in $\mathsf{P}\,(3;3)$ are represented by the permutations associated with their defining identities. The justification of the intersections and joins of varieties in Figure 4.5 can be found in Lemma 4.10.7.

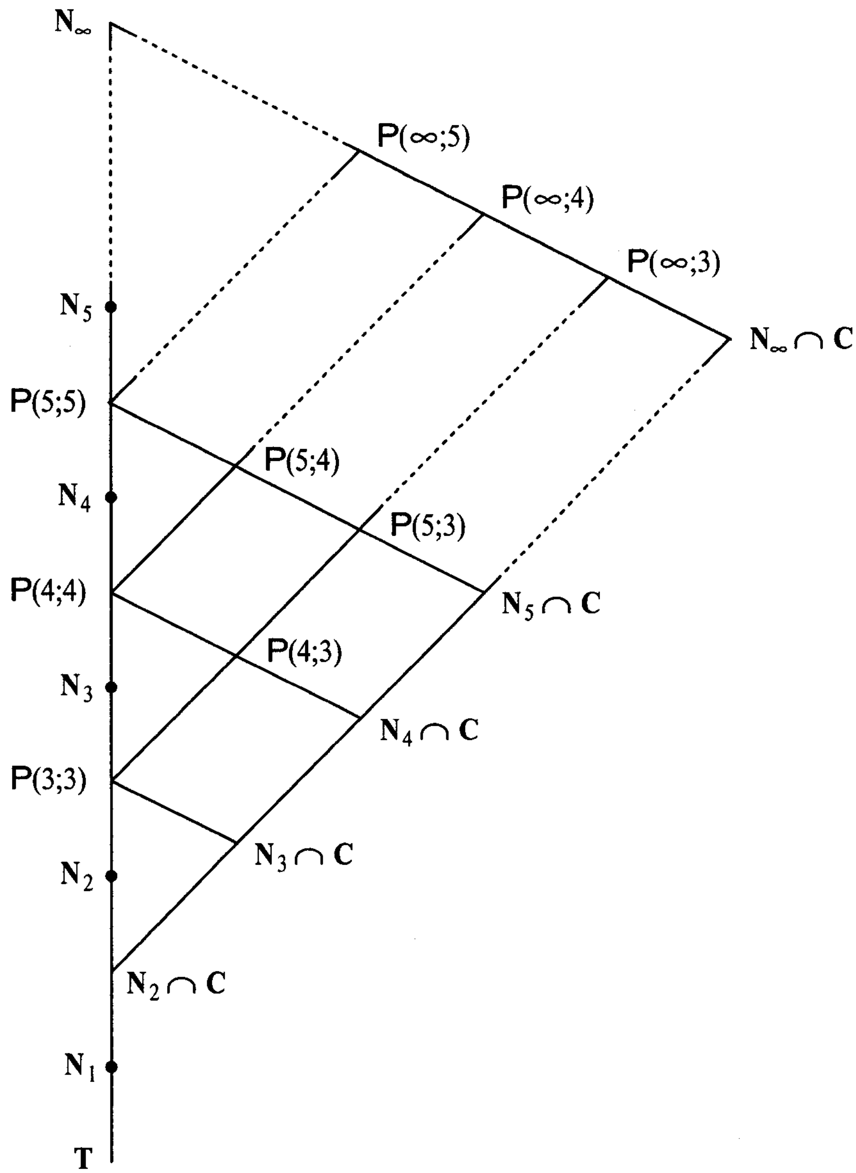

Figure 4.4: The interval $[\mathbf{T}, \mathbf{N}_\infty]$.

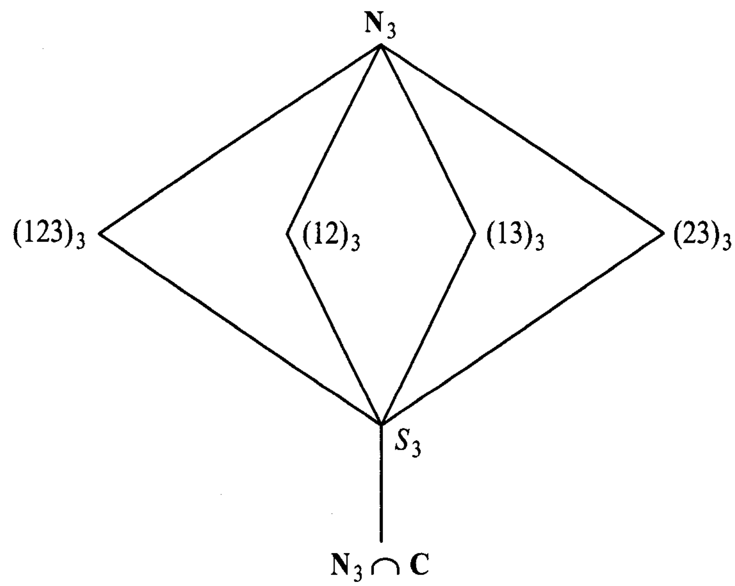

Figure 4.5: The interval $[\mathbf{N}_3 \cap \mathbf{C}, \mathbf{N}_3]$.

Lemma 4.10.7 *Let $a, b, c \in \{1,2,3\}$ be distinct. Then:*

(1) $\mathbf{N}_3^{\mathrm{Id}(ab)_3} \vee \mathbf{N}_3^{\mathrm{Id}(ac)_3} = \mathbf{N}_3^{\mathrm{Id}(ab)_3} \vee \mathbf{N}_3^{\mathrm{Id}(123)_3} = \mathbf{N}_3$;

(2) $\mathbf{N}_3^{\mathrm{Id}(ab)_3} \cap \mathbf{N}_3^{\mathrm{Id}(ac)_3} = \mathbf{N}_3^{\mathrm{Id}(ab)_3} \cap \mathbf{N}_3^{\mathrm{Id}(123)_3} = \mathbf{N}_3^{\mathrm{Id}S_3}$.

PROOF. (1) It suffice to assume $a = 1$, $b = 2$ and $c = 3$. We first show that $\mathbf{N}_3^{\mathrm{Id}(12)_3} \vee \mathbf{N}_3^{\mathrm{Id}(13)_3} = \mathbf{N}_3$. Since

$$\mathbf{N}_3^{\mathrm{Id}(12)_3} = \mathbf{N}_3 \cap [\{\mathbf{x}_2 = \mathbf{x}_2\}, \{\mathbf{x}_3 = \mathbf{x}_3, \mathrm{Id}\,(12)_3\}, \mathrm{Id}S_4, \mathrm{Id}S_5, \ldots],$$
$$\mathbf{N}_3^{\mathrm{Id}(13)_3} = \mathbf{N}_3 \cap [\{\mathbf{x}_2 = \mathbf{x}_2\}, \{\mathbf{x}_3 = \mathbf{x}_3, \mathrm{Id}\,(13)_3\}, \mathrm{Id}S_4, \mathrm{Id}S_5, \ldots]$$

are complete representations, we have, by Proposition 4.10.5,

$$\begin{aligned}\mathbf{N}_3^{\mathrm{Id}(12)_3} \vee \mathbf{N}_3^{\mathrm{Id}(13)_3} &= \mathbf{N}_3 \cap [\{\mathbf{x}_2 = \mathbf{x}_2\}, \{\mathbf{x}_3 = \mathbf{x}_3, \mathrm{Id}\,(12)_3\} \cap \{\mathbf{x}_3 = \mathbf{x}_3, \mathrm{Id}\,(13)_3\}] \\ &= \mathbf{N}_3 \cap [\{\mathbf{x}_2 = \mathbf{x}_2\}, \{\mathbf{x}_3 = \mathbf{x}_3\}] = \mathbf{N}_3.\end{aligned}$$

To show $\mathbf{N}_3^{\mathrm{Id}(12)_3} \vee \mathbf{N}_3^{\mathrm{Id}(123)_3} = \mathbf{N}_3$ is similar.

(2) This follows from Lemma 2.7.3 and the fact that

$$S_3 = \langle (ab)_3, (bc)_3 \rangle = \langle (ab)_3, (123)_3 \rangle$$

for any distinct $a, b, c \in \{1, 2, 3\}$. ■

4.11 The Interval $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$

This section extends the results in the previous section from $[\mathbf{T}, \mathbf{N}_\infty]$ to $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$. Varieties in $\mathfrak{LNR}$ have already been completely described in Section 7. Therefore our emphasis here is on varieties in $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] \setminus \mathfrak{LNR}$; we begin with a few observations to help narrow down the properties of such varieties.

Lemma 4.11.1 *Let $\pi \in S_k$ and $l, r \in \mathbb{N}_0$.*

(1) *If $\pi(1) \neq 1$ then $[\mathrm{Id}(l : \pi : r)]^{\mathbf{B}_2^-} \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty$;*

(2) *If $\pi(k) \neq k$ then $[\mathrm{Id}(l : \pi : r)]^{\mathbf{B}_2^-} \subseteq \mathbf{L}_\infty\mathbf{N}_\infty\mathbf{R}_r$.*

PROOF. (1) Let $\mathbf{U} = [\mathrm{Id}(l : \pi : r)]^{\mathbf{B}_2^-}$. By assumption, $\pi(i) = 1$ for some $i > 1$. Then

$$\begin{aligned} \mathbf{x}_l a_1 \cdots a_k \mathbf{w}_r &\equiv_{\mathbf{U}} \mathbf{x}_l a_{\pi(1)} \cdots a_{\pi(k)} \mathbf{w}_r && \text{by } \mathrm{Id}(l : \pi : r) \\ &\equiv \mathbf{x}_l a_{\pi(1)} \cdots a_{\pi(i-1)} a_1 a_{\pi(i+1)} \cdots a_{\pi(k)} \mathbf{w}_r. \end{aligned}$$

Letting S be the substitution

$$t \longrightarrow \begin{cases} y & \text{if } t = a_1, \\ w & \text{if } t \in \{a_2, \ldots . a_k, w_1, \ldots . w_k\}, \end{cases}$$

into X^+, we have

$$\begin{aligned} \mathbf{x}_l y w^2 &\equiv_{\mathbf{U}} (\mathbf{x}_l a_1 \cdots a_k \mathbf{w}_r)(\mathrm{S}) && \text{by I} \\ &\equiv_{\mathbf{U}} \left(\mathbf{x}_l a_{\pi(1)} \cdots a_{\pi(i-1)} a_1 a_{\pi(i+1)} \cdots a_{\pi(k)} \mathbf{w}_r\right)(\mathrm{S}) \\ &\equiv \mathbf{x}_l w^\alpha y w^\beta && \text{for some } \alpha, \beta \geq 1 \\ &\equiv_{\nabla} \mathbf{x}_l y^2 w^2 && \text{by Lemma 4.3.2(2)}. \end{aligned}$$

Hence $[\mathrm{Id}(l : \pi : r)]^{\mathbf{B}_2^-} \subseteq [l : 0 : \infty]^{\mathbf{B}_2^-} = \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_\infty$.

(2) This is symmetrical to (1). ■

Corollary 4.11.2 *Let $\pi \in S_k, (l_1, n, r_1) \in \mathbb{V}$ and $l_2, r_2 \in \mathbb{N}_0$.*

(1) *If $\pi(1) \neq 1$ and $l = \min\{l_1, l_2\}$, then*

$$\mathbf{L}_{l_1}\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l_2 : \pi : r)] = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l_2 : \pi : r)].$$

(2) *If $\pi(k) \neq k$ and $r = \min\{r_1, r_2\}$, then*

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_{r_1} \cap [\mathrm{Id}(l : \pi : r_2)] = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l : \pi : r_2)].$$

PROOF. (1) By Lemma 4.11.1 and Proposition 4.8.7,

$$\begin{aligned}\mathbf{L}_{l_1}\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l_2 : \pi : r)] &= \mathbf{L}_{l_1}\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l_2 : \pi : r)]^{\mathbf{B}_2^-} \\ &= \mathbf{L}_{l_1}\mathbf{N}_n\mathbf{R}_r \cap \left(\mathbf{L}_{l_2}\mathbf{N}_\infty\mathbf{R}_\infty \cap [\mathrm{Id}(l_2 : \pi : r)]^{\mathbf{B}_2^-}\right) \\ &= (\mathbf{L}_{l_1}\mathbf{N}_n\mathbf{R}_r \cap \mathbf{L}_{l_2}\mathbf{N}_\infty\mathbf{R}_\infty) \cap [\mathrm{Id}(l_2 : \pi : r)]^{\mathbf{B}_2^-} \\ &= \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}(l_2 : \pi : r)].\end{aligned}$$

(2) This is symmetrical to (1). ■

Proposition 4.11.3 *Let $\mathbf{V} \in [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] \backslash \mathfrak{LNR}$. Then*

$$\mathbf{V} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$$

for some $(l, n, r) \in \mathbb{V}$ and a nonempty finite set Π of permutations (not all trivial). Furthermore, l and r can be chosen to be finite such that each permutation in Π has the form $(l : \pi : r)$ with at most n variables.

PROOF. By Proposition 4.9.5, $\mathbf{V} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ where $(l, n, r) \in \mathbb{V}$ and Π is a nonempty finite set of permutations. Since $\mathbf{V} \in [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] \backslash \mathfrak{LNR}$ these permutations cannot all be trivial. By Corollary 4.11.2, we may assume that l and r are minimal in the sense that for all $(l' : \pi : r') \in \Pi$ with $\pi \in S_k$,

$$\pi(1) \neq 1 \Longrightarrow l \leq l', \qquad \pi(k) \neq k \Longrightarrow r \leq r'.$$

In particular, l, r can always be chosen to be finite. Therefore each permutation in Π can be expressed as $(l : \pi : r)$ (not necessarily having the properties $\pi(1) \neq 1$ or $\pi(k) \neq k$). Let

$m = n - (l + r)$. Consider a permutation $(l : \alpha : r)$ that involve at least $n+1$ variables, say $\alpha \in S_k$ so that $l + k + r \geq n + 1$. Since $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r = [l : m : r]^{\mathbf{B}_2^-}$ and $k \geq m + 1$,

$$\begin{aligned}
\mathbf{x}_l y_1 \cdots y_k \mathbf{w}_r &\equiv_{(l:m:r)} \mathbf{x}_l \left(y_1 \cdots y_k\right)^2 \mathbf{w}_r && \text{by } (l:m:r) \\
&\equiv_{\mathbf{V}} \mathbf{x}_l y_1^2 \cdots y_k^2 \mathbf{w}_r && \text{by Lemma 4.3.2(3)} \\
&\equiv_{\mathbf{V}} \mathbf{x}_l y_{\alpha(1)}^2 \cdots y_{\alpha(k)}^2 \mathbf{w}_r && \text{by V} \\
&\equiv_{\mathbf{V}} \mathbf{x}_l \left(y_{\alpha(1)} \cdots y_{\alpha(k)}\right)^2 \mathbf{w}_r && \text{by Lemma 4.3.2(3)} \\
&\equiv_{(l:m:r)} \mathbf{x}_l y_{\alpha(1)} \cdots y_{\alpha(k)} \mathbf{w}_r && \text{by } (l:m:r)\,.
\end{aligned}$$

Hence the identity $\mathrm{Id}\,(l : \alpha : r)$ is implied by identities the of $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$. Therefore the variety $\mathbf{V}$ will be unchanged if all permutations from Π involving at least $n+1$ variables are omitted. ■

The following result is crucial in finding generating semigroups, intersections and joins of varieties in $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$.

Proposition 4.11.4 *Let $(l, n, r) \in \mathbb{V}$ and Π be a nonempty finite set of permutations. If each permutation in Π has the form $(l : \pi : r)$ with at most n variables, then*

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi] = \mathbf{L}_l\mathbf{N}_n^{\mathrm{Id}\Pi}\mathbf{R}_r.$$

PROOF. Let $\mathbf{U}$ (respectively, $\mathbf{V}$) denote the variety on the left (respectively, right) of the equation above. We first prove that $\mathbf{U} \subseteq \mathbf{V}$. Suppose $\mathbf{u} = \mathbf{v}$ is an identity of $\mathbf{V}$. Since $\mathbf{V} \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$, we may assume $\mathbf{u}, \mathbf{v}$ to be $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-words. Note that $c(\mathbf{u}) = c(\mathbf{v})$ since $\mathbf{Y} \subseteq \mathbf{V}$. There are two cases: $\mathbf{u}$ is linear and $\mathbf{u}$ is quadratic.

Case (i) Suppose $\mathbf{u}$ is linear. Then $|\mathbf{u}| \leq n$ by (J1), whence $\mathbf{u}^{\mathbf{N}_n} \equiv \mathbf{u}$ is nonzero. By assumption $\mathbf{N}_n^{\mathrm{Id}\Pi} \vdash \mathbf{u} = \mathbf{v}$ so that $\{\mathbf{x}_{n+1} = \mathbf{0}\} \cup \mathrm{Id}\Pi \vdash \mathbf{u} = \mathbf{v}$. Therefore by Lemma 4.10.1, $\mathbf{v}^{\mathbf{N}_n} \equiv \mathbf{v}$ is also linear. Since $\mathrm{Id}\Pi \vdash \mathbf{u} = \mathbf{v}$ by Proposition 4.10.2, we have $\mathbf{U} \models \mathbf{u} = \mathbf{v}$.

Case (ii) Suppose $\mathbf{u}$ is quadratic. If $\mathbf{v}$ is linear then by an argument symmetrical to Case (i), we deduce that $\mathbf{u}$ is contradictorily linear. Therefore $\mathbf{v}$ is also quadratic. Now since $\mathbf{L}_l, \mathbf{R}_r \models \mathbf{u} = \mathbf{v}$, we have $\mathbf{u} \equiv \mathbf{v}$ by Lemma 4.7.5 so that $\mathbf{U} \models \mathbf{u} = \mathbf{v}$.

It remains to prove $\mathbf{V} \subseteq \mathbf{U}$. If $\mathbf{u} = \mathbf{v}$ is the permutation identity $\mathrm{Id}\,(l : \pi : r) \in \mathrm{Id}\Pi$, then $\overleftarrow{\mathbf{u}}^l \equiv \overleftarrow{\mathbf{v}}^l$ and $\overrightarrow{\mathbf{u}}^r \equiv \overrightarrow{\mathbf{v}}^r$. Thus $\mathbf{L}_l, \mathbf{R}_r \models \mathrm{Id}\,(l : \pi : r)$ by Propositions 4.5.7 and 4.5.13, whence $\mathbf{L}_l, \mathbf{R}_r \subseteq \mathbf{U}$. Clearly, $\mathbf{N}_n^{\mathrm{Id}\Pi} = \mathbf{N}_n \cap [\mathrm{Id}\Pi] \subseteq \mathbf{U}$. Consequently, $\mathbf{V} \subseteq \mathbf{U}$. ■

In view of Proposition 4.11.4, the variety $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ is generated by $L_l \times F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right) \times R_r$. If Π contains a nontrivial permutation, then $l, r < \infty$ by Proposition 4.11.3, so that $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ is finitely generated whenever n is finite. But $\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ cannot be generated by any finite semigroup because by the argument in the proof of Proposition 4.7.12, any finite semigroup $S \in \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ is contained in $\mathbf{L}_l\mathbf{N}_{n_0}\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ for some $n_0 < \infty$. Therefore we have shown:

Proposition 4.11.5 *Suppose $l, r < \infty$ and let Π be a nonempty finite set of permutations containing some nontrivial permutation. Then $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ is finitely generated if and only if n is finite.*

We now investigate the intersection and join of varieties from $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$. Consider a collection $\mathfrak{M}$ of varieties from $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$, which can be partitioned into $\mathfrak{M} \cap \mathfrak{LNR}$ and $\mathfrak{M} \backslash \mathfrak{LNR}$. We may express varieties from these two sets respectively as follow:

$$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [x = x], \qquad \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi],$$

for some $(l, n, r) \in \mathbb{V}$ and a nonempty finite set Π of permutations (not all trivial). By Proposition 4.11.3, each permutation in Π has the form $(l : \pi : r)$ with at most n variables. Therefore l and r in $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$ are finite. The following is a method of indexing these two types of varieties simultaneously.

For each $i \in I$, let $(l_i, n_i, r_i) \in \mathbb{V}$ and Π_i be a nonempty finite set of permutations of the form $(l_i : \pi : r_i)$ with at most n_i variables. Some Π_i can be trivial so that varieties in $\mathfrak{LNR}$ will be captured by $\mathfrak{M}$ for generality. If l_i or r_i is infinite, then obviously no permutation can be of the form $(l_i : \pi : r_i)$, thus we also include the requirement that $\mathrm{Id}\Pi_i = \{x = x\}$ if one of l_i or r_i is infinite. It is straightforward to check that

$$\mathfrak{M} = \{\mathbf{L}_{l_i}\mathbf{N}_{n_i}\mathbf{R}_{r_i} \cap [\mathrm{Id}\Pi_i] \mid i \in I\}$$

described above is an arbitrary collection of varieties from $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$. By Proposition 4.8.7,

$$\begin{aligned}\bigcap \mathfrak{M} &= \bigcap\nolimits_{i \in I} (\mathbf{L}_{l_i}\mathbf{N}_{n_i}\mathbf{R}_{r_i} \cap [\mathrm{Id}\Pi_i]) \\ &= \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]\end{aligned}$$

where

$$l = \min\{l_i \mid i \in I\}, \qquad n = \min\{n_i \mid i \in I\},$$
$$r = \min\{r_i \mid i \in I\}, \qquad \Pi = \bigcup_{i\in I} \Pi_i.$$

By an argument similar to the one following the proof of Corollary 4.10.4, the set Π can be chosen to be finite. Note that

$$\begin{aligned} l = \infty &\Longrightarrow l_i = \infty \text{ for all } i \in I \\ &\Longrightarrow \mathrm{Id}\Pi_i = \{x = x\} \text{ for all } i \in I \\ &\Longrightarrow \mathrm{Id}\Pi = \{x = x\}, \end{aligned}$$

and similarly, $r = \infty$ implies $\mathrm{Id}\Pi = \{x = x\}$. Therefore if one of l, r is infinite then $\bigcap \mathfrak{M} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ is generated by the semigroups L_l, R_r and N_n. But if l, r are finite, then Π can be a nonempty set of permutations, each of which is of the form $(l : \pi : r)$ so that by Proposition 4.11.4, the intersection

$$\bigcap \mathfrak{M} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi] = \mathbf{L}_l\mathbf{N}_n^{\mathrm{Id}\Pi}\mathbf{R}_r$$

is generated by the semigroups L_l, R_r and $F_n\left(\mathbf{N}_n^{\mathrm{Id}\Pi}\right)$.

It remains to determine a basis for $\bigvee \mathfrak{M}$. For each $i \in I$, consider the complete representation

$$\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} = \mathbf{N}_{n_i} \cap \left[\mathrm{Id}\Pi_i^{(k)}\middle|\, k \in \mathbb{N}\right].$$

Note that if one of l_i, r_i is infinite, then by assumption $\Pi_i^{(k)}$ is the trivial subgroup of S_k for each k. So assume all l_i, r_i are finite; in this case, invoking Proposition 4.10.5 yields:

$$\bigvee_{i\in I} \mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i} = \mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)}\middle|\, k \in \mathbb{N}\right]$$

where $n = \sup\{n_i \mid i \in I\}$ and $\Pi^{(k)} = \bigcap_{i\in I} \Pi_i^{(k)}$. Therefore by Proposition 4.11.4,

$$\begin{aligned} \bigvee \mathfrak{M} &= \bigvee_{i\in I} \left(\mathbf{L}_{l_i}\mathbf{N}_{n_i}\mathbf{R}_{r_i} \cap [\mathrm{Id}\Pi_i]\right) \\ &= \bigvee_{i\in I} \left(\mathbf{L}_{l_i}\mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}\mathbf{R}_{r_i}\right) \\ &= \left(\bigvee_{i\in I} \mathbf{L}_{l_i}\right)\left(\bigvee_{i\in I} \mathbf{N}_{n_i}^{\mathrm{Id}\Pi_i}\right)\left(\bigvee_{i\in I} \mathbf{R}_{r_i}\right) \\ &= \mathbf{L}_l\left(\mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)}\middle|\, k \in \mathbb{N}\right]\right)\mathbf{R}_r \end{aligned}$$

where $l = \sup\{l_i \mid i \in I\}$ and $r = \sup\{r_i \mid i \in I\}$. Then we have:

Proposition 4.11.6

$$\bigvee \mathfrak{M} = \begin{cases} \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right] & \text{if } n > l + r, \\ \mathbf{L}_l\mathbf{R}_r & \text{if } n \leq l + r. \end{cases}$$

PROOF. Note that if $n \leq l + r$, then $\mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right] \subseteq \mathbf{L}_l\mathbf{R}_r$ so that $\bigvee \mathfrak{M} = \mathbf{L}_l\mathbf{R}_r$. Therefore we may assume $n > l + r$, whence $l, r < \infty$. Now consider any $\alpha \in \Pi^{(k)}$. Then $\alpha \in \Pi_i^{(k)}$ for each $i \in I$ so that α can be expressed as $(l_i : * : r_i)$ for all $i \in I$. Consequently α can also be expressed as $(l : * : r)$. Hence all permutations in $\Pi^{(k)}$ can be expressed as $(l : * : r)$. By the remarks above and Proposition 4.11.4,

$$\begin{aligned} \bigvee \mathfrak{M} &= \mathbf{L}_l \left(\mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right]\right) \mathbf{R}_r \\ &= \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right]. \end{aligned}$$

■

Hence finding a basis for the join of varieties from $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] \setminus \mathfrak{LNR}$ is very similar to finding one for the join of varieties from $[\mathbf{T}, \mathbf{N}_\infty] \setminus \mathfrak{N}$ (see Proposition 4.10.5). Theoretically, a finite subset of Π can be chosen from $\bigcup_{k \in \mathbb{N}} \Pi^{(k)}$ so that $\bigvee \mathfrak{M} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$. But if $|I| < \infty$, then the basis for $\bigvee \mathfrak{M}$ is computable and finite since that of $\mathbf{N}_n \cap \left[\mathrm{Id}\Pi^{(k)} \middle| k \in \mathbb{N}\right]$ is also computable and finite by Proposition 4.10.6.

Following the direction of the previous section, we now describe the relative positions of each varieties in $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$. Due to the "larger" size of $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$ the task is necessarily more complicated than but is similar to that of $[\mathbf{T}, \mathbf{N}_\infty]$. Note that by Proposition 4.11.3, a typical variety in $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty] \setminus \mathfrak{LNR}$ is of the form

$$\mathbf{V} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]$$

where $(l, n, r) \in \mathbb{V}$, Π is a nonempty finite set of permutations (not all trivial), and each nontrivial permutation in Π has the form $(l : \pi : r)$ with at most n variables. From the proof of the same proposition, it has been shown that the identities of $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$ imply all identities of the form $\mathrm{Id}\,(l : \pi : r)$ involving at least $n + 1$ variables. Therefore a similar argument shows that the identities of $\mathbf{L}_l\mathbf{R}_r$ implies all identities in $\mathrm{Id}\Pi$. Hence

$$\mathbf{L}_l\mathbf{R}_r \subseteq \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi] \subseteq \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r$$

and $\mathbf{V} \in [\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$. Clearly, if we began with $\mathbf{V} = \mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \in \mathfrak{LNR}$ (and l, r are allowed to be infinite), then we also have $\mathbf{V} \in [\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$. Consequently,

Proposition 4.11.7 *The interval* $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$ *is the disjoint union of the intervals* $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$, *where* $l, r \in \mathbb{N}_0^\infty$.

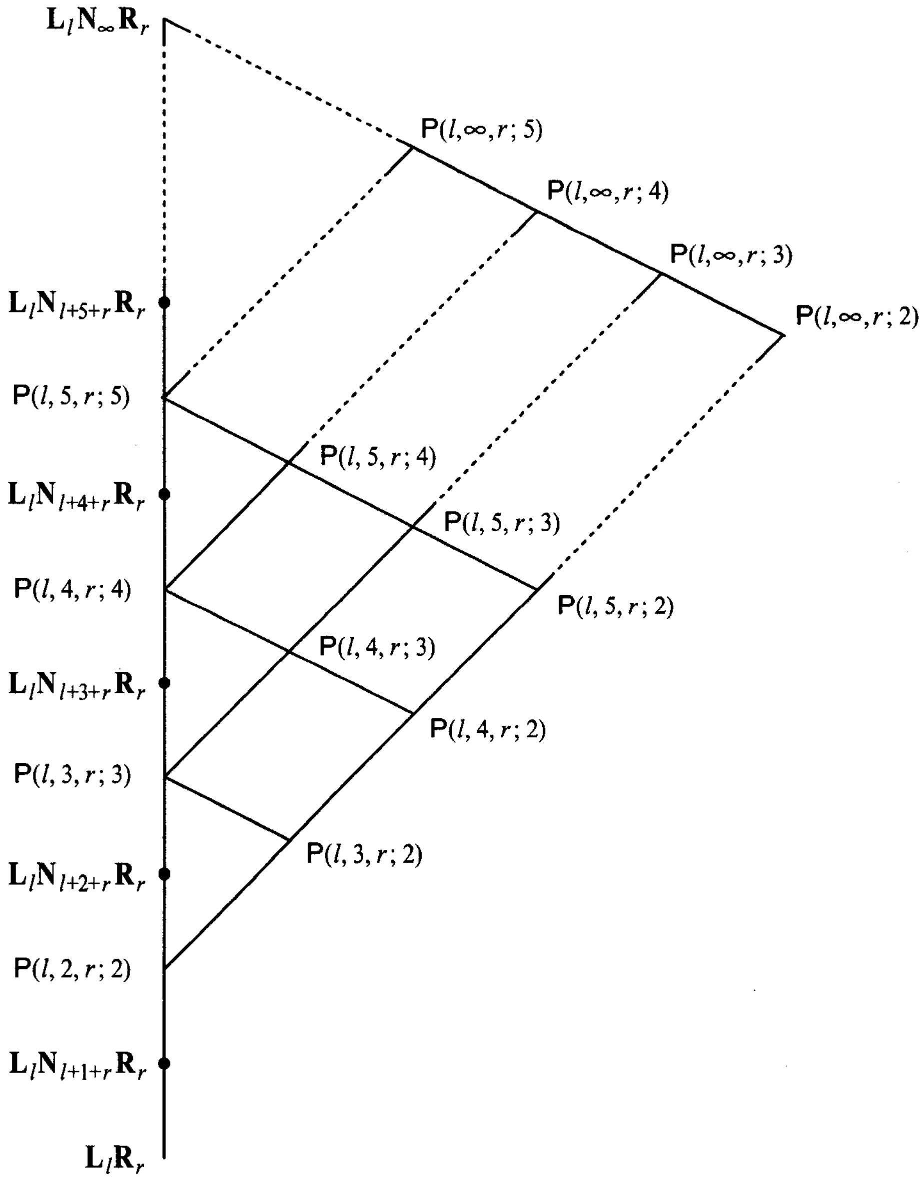

Figure 4.6: The interval $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$.

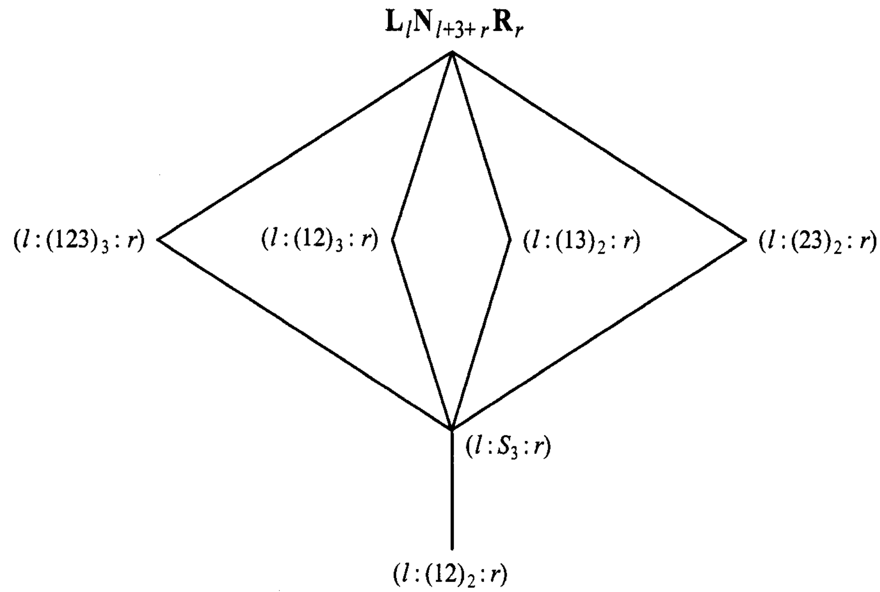

Figure 4.7: The interval $[\mathsf{P}(l,3,r;3), \mathbf{L}_l\mathbf{N}_{l+3+r}\mathbf{R}_r]$.

In view of Proposition 4.11.7, it suffices to divide the investigation of $[\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$ by investigating each interval $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$ individually. Note that if one of l, r is infinite, then $\mathbf{L}_l\mathbf{R}_r = \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r$ so that $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r] = \{\mathbf{L}_l\mathbf{R}_r\}$. Thus for the rest of this section, we fix l and r and assume them to be finite.

For $k \in \mathbb{N}$, let $\mathfrak{P}_{(l,r;k)}$ be the class of all nontrivial permutation varieties of the form $[\mathrm{Id}\,(l:\Pi:r)]$ such that the shortest nontrivial identity in $\mathrm{Id}\Pi$ involves k variables. Equivalently,

$$\mathfrak{P}_{(l,r;k)} = \{[\mathrm{Id}\,(l:\Pi:r)]\,|\Pi \cap S_k \nsubseteq \{1\}, \Pi \cap S_i \subseteq \{1\} \text{ if } i < k\}.$$

In particular, $\mathfrak{P}_{(l,r;2)} = \{[\mathrm{Id}\,(l:(12)_2:r)]\}$ since $\mathrm{Id}\,(l:(12)_2:r)$ implies all nontrivial permutation identities of the form $\mathrm{Id}\,(l:\pi:r)$. Letting $\mathsf{P}(l,n,r;k) = \{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P} \mid \mathbf{P} \in \mathfrak{P}_{(l,r;k)}\}$, a diagram of the interval $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$ can be seen in Figure 4.6. This diagram displays the location of each variety $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}$ in $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$ to within the respective class $\mathsf{P}(l,n,r;k)$ it belongs to. Similar to the interval $[\mathbf{T}, \mathbf{N}_\infty]$, it is possible but too difficult to include each variety in each $\mathsf{P}(l,n,r;k)$ individually in the diagram. If n and k are fixed, then it is easy to show that $\mathsf{P}(l,n,r;k)$ is closed under taking intersections but not closed under taking joins. By Proposition 4.11.4 and a result similar to Lemma 4.10.7, the interval

$[\mathsf{P}(l,3,r;3), \mathbf{L}_l\mathbf{N}_{l+3+r}\mathbf{R}_r]$ is described in Figure 4.7. For brevity, the varieties in $\mathsf{P}(l,3,r;2)$ and $\mathsf{P}(l,3,r;3)$ are represented by the permutations associated with their defining identities.

4.12 The Lattice $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$

In this short final section we investigate intersections and joins of varieties in $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$. Fortunately, it is a matter of combining some observations with results from previous sections. Furthermore, we investigate the varieties in $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$ that are not finitely generated.

Recall that $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty) = [\mathbf{T}, \mathbf{N}_\infty] \cup [\mathbf{Y}, \mathbf{L}_\infty\mathbf{R}_\infty]$, and a variety in this lattice is of the form $\mathbf{V} \cap \mathbf{P}$ where $\mathbf{V} \in \mathfrak{LNR}^* = \mathfrak{LNR} \cup \mathfrak{N}$ and $\mathbf{P}$ a permutation variety (possibly defined by trivial permutations). Consider a collection $\mathfrak{M} = \{\mathbf{V}_i \cap \mathbf{P}_i \mid i \in I\}$ of varieties in $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$. The previous two sections dealt with the cases when either all $\mathbf{V}_i$ belong to $\mathfrak{N}$, or all $\mathbf{V}_i$ belong to $\mathfrak{LNR}$ respectively. Therefore it remains to assume some $\mathbf{V}_i$ belong to $\mathfrak{LNR}$ and some to $\mathfrak{N}$. Suppose $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi], \mathbf{N}_k \cap [\mathrm{Id}\Sigma] \in \mathfrak{M}$. Then

$$\begin{aligned}(\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]) \cap (\mathbf{N}_k \cap [\mathrm{Id}\Sigma]) &= ((\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{N}_\infty) \cap [\mathrm{Id}\Pi]) \cap (\mathbf{N}_k \cap [\mathrm{Id}\Sigma]) \\ &= (\mathbf{N}_n \cap [\mathrm{Id}\Pi]) \cap (\mathbf{N}_k \cap [\mathrm{Id}\Sigma]).\end{aligned}$$

Therefore if just one $\mathbf{V}_i$ is in $\mathfrak{N}$, the problem of finding $\bigcap \mathfrak{M}$ reduces to a problem in $\mathfrak{N}$.

Now there exist $l_1, r_1 \in \mathbb{N}_0$ such that all nontrivial permutations in Σ are of the form $(l_1 : \pi : r_1)$. Since all nontrivial permutations in Σ can also be expressed in form $(l_0 : \pi : r_0)$ whenever $l_0 \le l_1$ and $r_0 \le r_1$, we may assume $l_1 \le l$ and $r_1 \le r$. Thus by Proposition 4.11.4,

$$\begin{aligned}(\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]) \vee (\mathbf{N}_k \cap [\mathrm{Id}\Sigma]) &= \mathbf{L}_l\mathbf{N}_n^{\mathrm{Id}\Pi}\mathbf{R}_r \vee \mathbf{N}_k^{\mathrm{Id}\Sigma} \\ &= \mathbf{L}_l\mathbf{N}_n^{\mathrm{Id}\Pi}\mathbf{R}_r \vee \mathbf{L}_{l_1}\mathbf{N}_k^{\mathrm{Id}\Sigma}\mathbf{R}_{r_1} \\ &= (\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap [\mathrm{Id}\Pi]) \vee (\mathbf{L}_{l_1}\mathbf{N}_k\mathbf{R}_{r_1} \cap [\mathrm{Id}\Sigma]).\end{aligned}$$

Hence if just one $\mathbf{V}_i$ is in $\mathfrak{LNR}$, the problem of finding $\bigvee \mathfrak{M}$ reduces to a problem in $\mathfrak{LNR}$.

We now consider the collection $\mathfrak{X}$ of varieties in $\mathcal{L}(\mathbf{L}_\infty\mathbf{R}_\infty)$ that are not finitely generated. By Propositions 4.6.9, 4.7.12, 4.10.4 and 4.11.5, all these varieties are precisely those of the forms

$$\begin{aligned}&\mathbf{N}_\infty, \qquad \mathbf{N}_\infty \cap [\mathrm{Id}\Pi], \\ &\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \quad (l, r \in \mathbb{N}_0^\infty), \\ &\mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \cap [\mathrm{Id}(l : \Pi : r)] \quad (l, r \in \mathbb{N}_0).\end{aligned}$$

Clearly each of these varieties contains $\mathbf{N}_\infty \cap \mathbf{C}$. Hence $\mathfrak{X} \subseteq [\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. Conversely, let $\mathbf{V} \in [\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. If $\mathbf{V}$ is of the form $\mathbf{N}_n \cap \mathbf{P}$ with permutation variety $\mathbf{P}$ (possibly defined by trivial permutations), then by Proposition 4.10.6 and since $\mathbf{N}_\infty \cap \mathbf{C} \subseteq \mathbf{V}$, we have

$$\begin{aligned} \mathbf{V} &= (\mathbf{N}_n \cap \mathbf{P}) \vee (\mathbf{N}_\infty \cap \mathbf{C}) \\ &= \mathbf{N}_\infty \cap \mathbf{Q} \end{aligned}$$

for some (possibly trivial) permutation variety $\mathbf{Q}$. Therefore $\mathbf{V}$ is not finitely generated by Proposition 4.10.4. If $\mathbf{V}$ is of the form $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}$ then by the discussion at the end of the previous section and the beginning of this section, we have

$$\begin{aligned} \mathbf{V} &= (\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}) \vee (\mathbf{N}_\infty \cap \mathbf{C}) \\ &= (\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \cap \mathbf{P}) \vee (\mathbf{L}_0\mathbf{N}_\infty\mathbf{R}_0 \cap \mathbf{C}) \\ &= \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r \cap \mathbf{Q} \end{aligned}$$

for some permutation variety $\mathbf{Q}$. Thus $\mathbf{V}$ is not finitely generated by Proposition 4.11.5. Therefore $\mathbf{V} \in \mathfrak{X}$ in all cases, whence $\mathfrak{X} = [\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. We have thus shown:

Proposition 4.12.1 *The subvarieties of* $\mathbf{A}_0$ *that are not finitely generated are precisely those varieties in* $[\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. *Consequently, the variety* $\mathbf{N}_\infty \cap \mathbf{C}$ *is minimal with respect to being locally finite and non-finitely generated.*

Proposition 4.12.2 *The interval* $[\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$ *is the disjoint union of the intervals* $[\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{N}_\infty]$ *and* $[\mathbf{N}_\infty\mathbf{Y} \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$.

PROOF. Let $\mathbf{V} \in [\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. If $\mathbf{Y} \subseteq \mathbf{V}$, then $(\mathbf{N}_\infty \cap \mathbf{C})\mathbf{Y} \subseteq \mathbf{V}$. But by Proposition 4.11.4,

$$\begin{aligned} (\mathbf{N}_\infty \cap \mathbf{C})\mathbf{Y} &= \mathbf{L}_0\mathbf{N}_\infty^{\mathbf{C}}\mathbf{R}_0 \\ &= \mathbf{L}_0\mathbf{N}_\infty\mathbf{R}_0 \cap \mathbf{C} = \mathbf{N}_\infty\mathbf{Y} \cap \mathbf{C} \end{aligned}$$

so that $\mathbf{V} \in [\mathbf{N}_\infty\mathbf{Y} \cap \mathbf{C}, \mathbf{L}_\infty\mathbf{R}_\infty]$. If $\mathbf{Y} \not\subseteq \mathbf{V}$, then $\mathbf{V} \subseteq \mathbf{N}_\infty$ by Lemma 4.6.11. Thus $\mathbf{V} \in [\mathbf{N}_\infty \cap \mathbf{C}, \mathbf{N}_\infty]$. ■

Appendix: Canonical Words

For a countably infinite alphabet $X = \{z_1, z_2, \ldots\}$ define

$$\begin{aligned}\mathcal{P} &= \left\{z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k} \mid \sigma_1, \ldots, \sigma_k \text{ are distinct, } \alpha_i \in \{1,2\},\ k \in \mathbb{N}\right\} \cup \{\emptyset\}, \\ \mathcal{A} &= \{z_{\sigma_1} \cdots z_{\sigma_k} z_{\sigma_1} \mid \sigma_1 < \cdots < \sigma_k,\ k \geq 2\}.\end{aligned}$$

Let $l, m, r \in \mathbb{N}_0^\infty$ and let $(l, m, r) \in \mathbb{U}$ where

$$\mathbb{U} = \{(l, m, r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \mid (l = \infty \text{ or } r = \infty) \Rightarrow m = 0\}.$$

1. A word $\mathbf{u} \in X^+$ is an $\mathbf{A_0}$-*word* if either $\mathbf{u} \in \mathcal{P}$ or

   $$\mathbf{u} \equiv \mathbf{p}_0 \mathbf{a}_1 \mathbf{p}_1 \cdots \mathbf{a}_k \mathbf{p}_k$$

   where $\mathbf{p}_i \in \mathcal{P}$, $\mathbf{a}_i \in \mathcal{A}$, and $\mathfrak{c}(\mathbf{p}_0), \ldots, \mathfrak{c}(\mathbf{p}_k), \mathfrak{c}(\mathbf{a}_1), \ldots, \mathfrak{c}(\mathbf{a}_k)$ are pairwise disjoint.

2. A word $\mathbf{u} \in X^+$ is a $\mathbf{B_2^-}$-*word* if $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k} \in \mathcal{P}$ such that

   $$\alpha_i = \alpha_{i+1} = 2 \implies \sigma_i < \sigma_{i+1}.$$

3. A $\mathbf{B_2^-}$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is an $\mathbf{L}_l$-*word* if
   (L1) $k \geq l + 1 \implies \alpha_{l+1} = 2$;
   (L2) $\alpha_i = 2 \implies \alpha_i = \cdots = \alpha_k = 2$.

4. A $\mathbf{B_2^-}$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is an $\mathbf{R}_r$-*word* if
   (R1) $k \geq r + 1 \implies \alpha_{k-r} = 2$;
   (R2) $\alpha_i = 2 \implies \alpha_1 = \cdots = \alpha_i = 2$.

5. A word $\mathbf{u} \in X^+$ is an $\mathbf{N}_n$-*word* if either $\mathbf{u} \equiv z_1^2$, or

   $$\mathbf{u} \equiv z_{\sigma_1} \cdots z_{\sigma_k}$$

   for some distinct $\sigma_1, \ldots, \sigma_k \in \mathbb{N}$ and $k \leq n$.

6. Let $n = l+m+r$. A $\mathbf{B}_2^-$-word $\mathbf{u} \equiv z_{\sigma_1}^{\alpha_1} \cdots z_{\sigma_k}^{\alpha_k}$ is an $\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-*word* if all of the following statements hold.

   (J1) If $k > n$, then $\alpha_{l+1} = \alpha_{k-r} = 2$;

   (J2) If $\alpha_i = 2$ with $i > l+1$, then $\alpha_{l+1} = 2$;

   (J3) If $\alpha_i = 2$ with $i < k-r$, then $\alpha_{k-r} = 2$;

   (J4) If $\alpha_i = \alpha_j = 2$ with $i < j$, then $\alpha_i = \cdots = \alpha_j = 2$.

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List of Symbols

$\mathfrak{h}(\mathbf{u})$	head of $\mathbf{u}$	45
I, II, III	identities of A_0	47
$\mathrm{Id}_{\mathfrak{C}}(X)$	set of identities on X satisfied by semigroups in $\mathfrak{C}$	11
$\mathrm{Id}\pi$	a permutation identity	16
$\mathrm{Id}\Pi$	$\{\mathrm{Id}\pi \mid \pi \in \Pi\}$	16
$\mathrm{Id}(l:\pi:r)$	a permutation identity	17
$\mathrm{Id}(l:\Pi:r)$	$\{\mathrm{Id}(l:\pi:r) \mid \pi \in \Pi\}$	17
IV	an identity of A_0	51
$\mathfrak{I}$	the class of all idempotent generated completely simple semigroups	30
$\mathbb{I}_p^q$	$\{p,\ldots,q\}$	16
$\varkappa$	a monomorphism of $\mathcal{L}(\mathbf{CCS}_n)$	31
L	the left zero band of order two	7
L_l	a semigroup in $\mathbf{A_0}$	69
$(l:n:r)$	an identity	67
$[l:n:r]$	the variety defined by $(l:n:r)$	67
$(l:\pi:r)$	a permutation identity	16
$(l:\Pi:r)$	$\{(l:\pi:r) \mid \pi \in \Pi\}$	17
$\mathbf{L}_l$	the variety generated by L_l	69
$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$	$\mathbf{L}_l \vee \mathbf{N}_n \vee \mathbf{R}_r$	85
$\mathfrak{L}$	$\{\mathbf{L}_l \mid l \in \mathbb{N}_0^\infty\}$	69
$\mathfrak{L}\mathfrak{R}\mathfrak{N}$	$\{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r \mid l,n,r \in \mathbb{N}_0^\infty\}$	86
$\mathfrak{L}\mathfrak{N}\mathfrak{R}^*$	$\mathfrak{L}\mathfrak{N}\mathfrak{R} \cup \mathfrak{N}$	94
$\mathfrak{l}_i$	element of L_l	69
$\mathcal{L}\mathcal{E}(\mathbf{RS}_n)$	the sublattice of exact subvarieties of $\mathbf{RS}_n$	13
$\mathcal{L}\mathcal{E}(\mathbf{V})$	the sublattice of exact subvarieties of $\mathbf{V}$	13
$\mathcal{L}(\mathbf{V})$	the lattice of subvarieties of $\mathbf{V}$	9
$\mathfrak{m}_{\mathbf{u}}(x)$	multiplicity of x in $\mathbf{u}$	49
$\mathcal{M}$	a set of words in X^+	47
$\mathcal{M}^0(I,G,\Lambda;P)$	Rees matrix semigroup	6
$\mathcal{M}(I,G,\Lambda;P)$	Rees matrix semigroup	7
N_1	a semigroup of order two	14

N_n	a semigroup in $\mathbf{A_0}$	79
$\mathbf{N}_1$	the variety generated by N_1	14
$\mathbf{NB_2}$	the variety generated by the semigroups L, R and B_2	15
$\mathbf{N}_n$	the variety generated by N_n	79
$\mathbf{N}_n^{\mathrm{Id}\Pi}$	$\mathbf{N}_n \cap [\mathrm{Id}\Pi]$	112
$\mathfrak{N}$	$\{\mathbf{N}_n \mid n \in \mathbb{N}_0^\infty\}$	79
$\mathfrak{n}_i$	element of N_n	79
$\mathbb{N}$	$\{1, 2, \ldots\}$	30
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	67
$\mathbb{N}_0^\infty$	$\mathbb{N} \cup \{0, \infty\}$	67
∇	$\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{V}\}$	59
∇_n	$\nabla \cup \{\mathbf{x}_{n+1} = \mathbf{0}\}$	80
$P_S(\mathfrak{C})$	pseudovariety generated by semigroups in $\mathfrak{C}$	12
P/N	$(\Lambda/\pi) \times (I/r)$ matrix	8
$\langle P \rangle$	subgroup generated by the entries of P	7
$\mathcal{P}$	a set of words in X^+	48
$\mathsf{P}(l, n, r; k)$	a class of subvarieties in $[\mathbf{L}_l\mathbf{R}_r, \mathbf{L}_l\mathbf{N}_\infty\mathbf{R}_r]$	126
$\mathsf{P}(n; k)$	a class of subvarieties in $[\mathbf{T}, \mathbf{N}_\infty]$	116
$\mathbb{P}$	the set of prime integers	30
$\Pi_i^{(k)}$	a set of permutations	114
$\prod_{i \in I} S_i$	direct product of semigroups S_i $(i \in I)$	5
π_i	projection homomorphism	5
R	the right zero band of order two	7
R_r	a semigroup in $\mathbf{A_0}$	77
$\mathbf{RB}$	the variety of rectangular bands	12
$\mathbf{R}_r$	the variety generated by R_r	78
$\mathbf{RS}_n$	the variety generated by $\mathbf{CS}_n^{\mathbf{0}}$	13
(r, N, π)	admissible triple	8
$\mathfrak{R}$	$\{\mathbf{R}_r \mid r \in \mathbb{N}_0^\infty\}$	78
ρ_N	the congruence induced by $\rho_{(\varepsilon, N, \varepsilon)}$	9
$\rho \longleftrightarrow (r, N, \pi)$	ρ is induced by (r, N, π)	9
$[\rho]_T$	$\rho \cap (T \times T)$	6
$\rho_{(r,N,\pi)}$	congruence induced by (r, N, π)	8

S^1	the semigroup S with adjoined identity element	4
$S_1 \times \cdots \times S_n$	direct product of semigroups $S_1, \ldots, S_n$	5
S_k	the group of permutations on $\mathbb{I}_1^k$	16
$S \models \mathbf{u} = \mathbf{v}$	S satisfies $\mathbf{u} = \mathbf{v}$	10
$\sup C$	supremum of C in $\mathbb{N}_0^\infty$	77
$\Sigma \vdash \mathbf{u} = \mathbf{v}$	Σ implies $\mathbf{u} = \mathbf{v}$	12
$\Sigma \vdash \Pi$	$\Sigma \vdash \mathbf{u} = \mathbf{v}$ for all $\mathbf{u} = \mathbf{v} \in \Pi$	12
$[\Sigma]$	variety defined by Σ	11
$\mathsf{t}(\mathbf{u})$	tail of $\mathbf{u}$	45
U_∞	a subsemigroup of A_0^∞	66
$[\mathbf{U}, \mathbf{V}]$	interval	10
$[\mathbf{U}, \mathbf{V})$	$[\mathbf{U}, \mathbf{V}] \setminus \{\mathbf{V}\}$	10
$\mathbf{U}^\Sigma$ or $[\Sigma]^\mathbf{U}$	$\mathbf{U} \cap [\Sigma]$	46
$\lvert\mathbf{u}\rvert$	the number of variables in $\mathbf{u}$	45
$\mathbf{u}^{\mathbf{A_0}}$	$\mathbf{A_0}$-word	51
$\mathbf{u}^{\mathbf{B_2^-}}$	$\mathbf{B_2^-}$-word	60
$\mathbf{u}^{\mathbf{L}_l}$	$\mathbf{L}_l$-word	72
$\mathbf{u}^{\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r}$	$\mathbf{L}_l\mathbf{N}_n\mathbf{R}_r$-word	89
$\mathbf{u}^{\mathbf{N}_n}$	$\mathbf{N}_n$-word	81
$\mathbf{u}^{\mathbf{R}_r}$	$\mathbf{R}_r$-word	78
$\mathbf{u}(\mathrm{S})$	the word $\mathbf{u}$ under substitution S	10
$\overleftarrow{\mathbf{u}}^l$	l-left segment of $\mathbf{u}$	72
$\overrightarrow{\mathbf{u}}^r$	r-right segment of $\mathbf{u}$	78
$\mathbf{u} = \mathbf{0}$	the identities $\mathbf{u}x = x\mathbf{u} = \mathbf{u}$	80
$\mathbf{u} = \mathbf{v}$	an identity	10
$\mathbf{u} \equiv \mathbf{v}$	$\mathbf{u}$ and $\mathbf{v}$ are identical words in X^+	45
u_i	element of U_∞	66
$\mathbb{U}$	$\{(l, m, r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \mid (l = \infty \text{ or } r = \infty) \Rightarrow m = 0\}$	86
V	an identity of B_2^-	58
$V(\mathfrak{C})$	the variety generated by $\mathfrak{C}$	9
$\mathbb{V}$	$\{(l, n, r) \in \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \times \mathbb{N}_0^\infty \mid n \geq l + r\}$	95
X	a countably infinite alphabet	10

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