Semistable Reduction of Hyperelliptic Curves Over Finite Extensions of the 2-adic Numbers

by

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Abstract

Constructing semistable models for hyperelliptic curves serves as an important ingredient in many interesting problems in mathematics such as solving generalized Fermat equations and generalizing the famous Tate's algorithm for hyperelliptic curves. In recent years, explicit methods for constructing semistable models for hyperelliptic curves defined over local field having characteristics not equal to 2 has been examined thoroughly by Dokchitser-Dokchitser-Maistret-Morgan (2017) (see [4]). Their method, however, relies heavily on the fact that the residue characteristics of the local fields are not 2 and does not apply for the characteristic 2 case. In this thesis, we take a different approach to construct semistable models for a specific class (double root clusters) of hyperelliptic curves defined over finite extensions of the 2-adic numbers. We then demonstrate our methods by constructing an explicit semistable model for a given hyperelliptic curve as a proof of concept. Our result serves as a small step towards a general method for computing semistable models of hyperelliptic curves defined over local fields with residue characteristic 2 for the specific class of curves that we are interested in.

Keywords: Arithmetic Geometry; Hyperelliptic Curves; Semistable Reduction; Characteristic 2

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Chapter 1

Introduction

1.1 Overview

A hyperelliptic curve C over a field K is an equation of the form

$$C : y^2 = f(x) \tag{1.1.1}$$

for some polynomial f in the variable x where $\deg f = 2g + 1$ or $2g + 2$ for an integer $g \geq 2$ called the genus. When K is algebraically closed, C as a geometric object is just the set of point (x, y) satisfying (1.1.1). For us, the field K is usually not algebraically closed however, and so we ought to use the scheme language, and take C as the affine variety

$$C = \operatorname{Spec} K[x, y]/(y^2 - f(x)). \tag{1.1.2}$$

We call this formulation the “affine model” of the hyperelliptic curve C and it has a singular point at infinity in the projective plane. What we really need is the so called the “non-singular completion” of C which will be defined later in this thesis. For the sake of introduction, we may think of hyperelliptic curves as affine varieties given by (1.1.2) for now. These curves arise naturally in many algebraic geometry and number theory problems. In particular, they show up in the classification of curves by genus as well as generalized Fermat equations.

Let C be a hyperelliptic curve defined over a local field K and let $\mathcal{O}_K$ be the ring of integers of K. If we can find a scheme $\mathcal{C}$ over $\mathcal{O}_K$ such that by viewing $\mathcal{C}$ as a scheme over K, $\mathcal{C}$ and C are isomorphic over K, then we call $\mathcal{C}$ a ‘model’ of C (this will be properly defined later in the thesis). The model $\mathcal{C}$ gives us a way of ‘representing’ C as a scheme over the ring $\mathcal{O}_K$ which in turn, allows us to study our curve in new ways. We note that the notion of a model fundamentally considers a base ring that is not a field, so we really need that schemes generalize the notion of a variety to objects defined over rings. In the particular case where K is a finite extension of the ℓ-adic numbers $\mathbb{Q}_\ell$ where ℓ is prime, we can take the reduction of $\mathcal{C}$ modulo ℓ (we will later define this as the special fiber of $\mathcal{C}$

and denote it as $\mathcal{C}_\ell$). We then study $\mathcal{C}_\ell$ to see whether it is smooth or singular which are properties analogous to smoothness in usual analysis. It turns out, two different models for the same hyperelliptic curve can have different smoothness behaviours. In particular, we say a model $\mathcal{C}$ is semistable if the singular points of $\mathcal{C}_\ell$ (non-smooth points) are all isolated nodal singularities which we will properly define later in the thesis.

The Frey-Ribet-Wiles approach to Fermat's Last Theorem can be adapted to apply to various generalized Fermat equations. In this approach one attaches an algebraic curve to a putative solution and then prove this curve would have such special properties that it cannot exist. In the original proof, this curve is an elliptic curve, but some variants use a hyperelliptic curve. The approach derives a Galois representation from the putative curve and requires detailed information on its conductor (for a complete description of this, see [2]). To compute this conductor, one determines explicit semistable models of the curve over finite extensions of $\mathbb{Q}_\ell$, for all primes ℓ; including $\ell = 2$ (cf. [3]).

Our question now boils down to how to compute semistable models of hyperelliptic curves explicitly? Using tools from algebraic geometry, we are at least guaranteed that semistable models always exist theoretically, over some base field extension. Explicitly constructing them however is very difficult in most cases. Being able to explicitly compute semistable models of arbitrary hyperelliptic curves over local fields is equivalent to generalizing Tate's algorithm for elliptic curves to the hyperelliptic case e.g. computing conductor exponents for hyperelliptic curves.

Over the recent years, the explicit construction of semistable models has been studied systematically in [4] in the case where ℓ is an odd prime. They introduced the notion of *cluster pictures* to classify the reduction behaviour of hyperelliptic curves, and then tackling each cluster picture class accordingly by gluing together models of hyperelliptic curves parametrized by discs in the projective line over K. The $\ell = 2$ case becomes much more difficult to deal with and the method provided in [4] does not apply. A numerical algorithm to compute the conductor of a specific genus 2 curve at $\ell = 2$ can also be found in [5]. In applications to generalized Fermat equations, we need to find the conductor in a parameterized family of hyperelliptic curves however, for which their algorithm is not guaranteed to work. This serves as the main motivation of our method.

In this thesis, we will demonstrate a explicit construction of semistable models for hyperelliptic curves when $\ell = 2$ for a specific "cluster picture" class described in [4]. Our method boils down to gluing together models that are locally elliptic curves, blowing them up at their non-reduced components to obtain locally semistable models, and then finally patching them together to a potential semistable model for the original hyperelliptic curve.

1.2 Thesis Outline

All the notations and terminologies used in this thesis is listed in Chapter 2. In order to define models of curves and semistable reduction type, we will need to review some basic background knowledge in algebraic geometry. In Chapter 3, we will go over these preliminaries. In Section 3.2 and 3.3, we will recall the definitions of proper morphisms and flat morphisms respectively. These two concepts are used in the definition of models. In Section 3.1 and 3.5, We will review the gluing and blow-up constructions of schemes respectively. We will later use these constructions to construct our explicit semistable model for the class of hyperelliptic curves we are interested in. The definition of semistable reduction is in Section 3.4 along with the definition for models of curves and hyperelliptic curves. In Chapter 4, we will state our main strategy and the required assumptions (Condition 1). In Section 4.1 and Section 4.1.1 we describe our construction in detail and in Section 4.1.2 we present an explicit example.

Chapter 2

Notations and Terminologies

$\{S_\alpha\}_I$	A indexed family of sets S_α with $\alpha \in I$ where I is an indexing set.
$\overline{F}$	F is a field and $\overline{F}$ is an algebraic closure of F.
$\mathrm{GF}(p^k)$	Finite field with p^k elements where p is a prime and $k \geq 1$ is a positive integer.
Ring	Every ring in this thesis is going to be commutative with unity.
UFD	Unique factorization domian.
PID	Principal ideal domain.
DVR	Discrete valuation ring.
$(r_1, ..., r_k) \subseteq R$	The ideal in R generated by $r_1, ..., r_k \in R$.
$R[S^{-1}], \operatorname{Spec} R[S^{-1}]$	R is a ring and $S \subseteq R$ is a subset. We denote by $R[S^{-1}]$ the localization of R away from the multiplicative subset generated by S. We extend this terminology to affine schemes by calling $\operatorname{Spec} R[S^{-1}]$ the localization of $\operatorname{Spec} R$ away from S.
R_p	R is a ring and $p \in \operatorname{Spec} R$ is a prime ideal of R. R_p denotes the localization of R at the multiplicative set $R \setminus p$.
$\operatorname{Frac} R$	The field of fractions of an integral domain R.
$\dim R$	The Krull dimension of a ring R.
$\widehat{R}$	For a local ring R with maximal ideal $\mathfrak{m}$, we denote $\widehat{R}$ as the completion of R i.e. $\widehat{R} = \varprojlim_n R/\mathfrak{m}^n$.
$R[\![x_1, ..., x_n]\!]$	Formal power series ring in the variables $x_1, ..., x_n$ over a ring R.
$K, \mathcal{O}_K, \pi$	K is a local field, $\mathcal{O}_K = \operatorname{Frac} K$ is a discrete valuation ring (DVR) and π is a uniformizer of $\mathcal{O}_K$ i.e. $\pi \in \mathcal{O}_K$ generates the unique maximal ideal of $\mathcal{O}_K$.
K^{sep}	Separable closure of the local field K in $\overline{K}$.
$(X, \mathcal{O}_X), k(x)$	X is a scheme with structure sheaf $\mathcal{O}_X$. For $x \in X$, we note $k(x)$ as the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ where $\mathcal{O}_{X,x}$ is the stalk of x and $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is its unique maximal ideal.
X/R	A scheme X over a ring R i.e. there is a structure morphism $X \to \operatorname{Spec} R$.
$D_X(f_1, ..., f_m)$	If $X = \operatorname{Spec} R$ is an affine scheme, we denote $D_X(f_1, ..., f_m) \subseteq X$ as the distinguished open set generated by $f_1, ..., f_m \in R$. When the coordinate ring R is clear from context, we will simply write $D(f_1, ..., f_m)$ instead.
$V(f_1, ..., f_m)$	Vanishing locus of the elements $f_1, ..., f_m \in R$ i.e. $(f_1, ..., f_m) = \operatorname{Spec}(R/(f_1, ..., f_m))$.
$\mathrm{cl}_U(X)$	The closure of $U \subseteq X$ in the topological space X.

Chapter 3

Preliminaries

3.1 Gluing and Local Properties

One of our main ingredients for constructing the desired semistable models of hyperelliptic curves in Chapter 4 is the gluing construction for schemes. We will first recall how to glue together a scheme and then introduce the notion of local properties for schemes.

Theorem 3.1.1 (Gluing Schemes, [6], Section I.2.4. and [13], Lemma 26.14.1)**.** *Let $\{X_\alpha\}_I$ be a family of schemes, and an open set $X_{\alpha,\beta} \subseteq X_\alpha$ for each $\alpha, \beta \in I$. Suppose we are also given a family of isomorphisms of schemes*

$$\psi_{\alpha,\beta} : X_{\alpha,\beta} \to X_{\beta,\alpha} \quad \textit{for each } \alpha, \beta \in I,$$

satisfying the conditions:

1. *$X_{\alpha,\alpha} = X_\alpha$ and $\psi_{\alpha,\alpha} : X_\alpha \to X_\alpha$ is the identity map for all $\alpha \in I$.*
2. *$\psi_{\beta,\alpha} = \psi_{\alpha,\beta}^{-1}$ for all $\alpha, \beta \in I$,*
3. *$\psi_{\alpha,\beta}(X_{\alpha,\beta} \cap X_{\alpha,\gamma}) = X_{\beta,\alpha} \cap X_{\beta,\gamma}$ for all $\alpha, \beta, \gamma \in I$.*
4. *$\psi_{\beta,\gamma} \circ \psi_{\alpha,\beta}\big|_{X_{\alpha,\beta} \cap X_{\alpha,\gamma}} = \psi_{\alpha,\gamma}\big|_{X_{\alpha,\beta} \cap X_{\alpha,\gamma}}$ for all $\alpha, \beta, \gamma \in I$ called the compatibility condition.*

Then there exists a unique scheme X with open cover $\{U_\alpha\}_I$ such that $U_\alpha \cong X_\alpha$ and the identity maps on the intersections $U_\alpha \cap U_\beta \subseteq X$ corresponds to the isomorphisms $\psi_{\alpha,\beta}$.

Furthermore, if Y is any scheme and for each $\alpha \in I$, there exists a morphism $f_\alpha : X_\alpha \to Y$ such that $f_\beta \circ \psi_{\alpha,\beta} = f_\alpha\big|_{X_{\alpha,\beta}}$ then there exist a unique morphism $f : X \to Y$ such that $f\big|_{U_\alpha} \circ \iota_\alpha = f_\alpha$ where $\iota_\alpha : X_\alpha \to U_\alpha$ is the natural isomorphism induced by the gluing.

Proof. The proof may be found in [13, Lemma 26.14.1 and Lemma 26.14.2]. □

Remark 1. As a set, X is the coproduct $\coprod_{\alpha \in I} X_\alpha / \sim$ where $x_i \sim x_j$ for $x_i \in X_i$ and $x_j \in X_j$ if and only if $\psi_{i,j}(x_i) = x_j$.

Remark 2. We see that if in particular, the X_α's are affine, then $\{U_\alpha\}_I$ is an affine open cover of the glued scheme X.

This remark leads us to the notion of affine-local properties and stalk-local properties of schemes.

Definition 3.1.2 (Affine-local Property)**.** Let P be a property of schemes. We say P is an affine-local property if for any affine open cover $\{U_\alpha\}_I$ of a scheme X, we have X has property P if and only if each U_α has property P.

Definition 3.1.3 (Stalk-local Property)**.** Let P be a property of schemes. We say P is a stalk-local property if for any scheme X, we have X has property P if and only if each stalk $\mathcal{O}_{X,x}$ has property Q where Q is some property for local rings.

Proposition 3.1.4. *P is stalk-local implies it is affine-local.*

Proof. Let X be any scheme with property P. By definition, for any $x \in X$, we have $\mathcal{O}_{X,x}$ satisfies property Q. Let $\{U_\alpha\}_I$ be any affine open cover of X. If we take any $\alpha \in I$ and consider U_α, then for every $x \in U_\alpha$ the stalk

$$\mathcal{O}_{U_\alpha,x} \cong \mathcal{O}_{X,x}$$

has property Q. This implies U_α has property P for all $\alpha \in I$ since P is stalk-local. Since the scheme X with property P and $\{U_\alpha\}_I$ are all arbitary, this implies P is affine-local. $\square$

We also give the notion of a property for morphisms of schemes being local on the target.

Definition 3.1.5 (Local On The Target)**.** Let P be a property defined for morphisms of schemes. We say P is *local on the target* if the following conditions are satisfied:

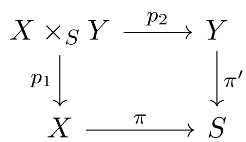
1. If $\pi : X \to Y$ is a morphism of schemes with property P, then for any open subset $V \subseteq Y$, the restricted morphism $\pi^{-1}(V) \to V$ has property P,

2. For any morphism of schemes $\pi : X \to Y$, if there exist an open cover $\{V_i\}$ of Y for which each restricted morphism $\pi^{-1}(V_i) \to V_i$ has property P, then π has property P.

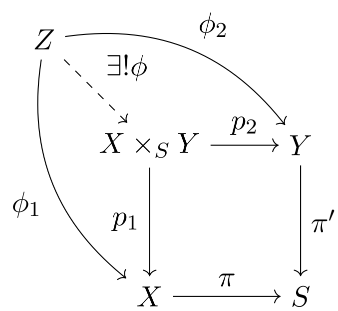
3.2 Fiber Products and Proper Morphisms

To give the definition of a curve being semistable, we will first recall what it means for a morphism between schemes to be proper. Before doing that, we will review some general definitions for schemes and their morphisms such as the fiber product construction and separatedness following the treatment in Hartshorne [8].

Definition 3.2.1 (Fiber Products of Schemes)**.** Let S be a scheme, and let X and Y be schemes over S i.e. we have morphisms $\pi : X \to S$ and $\pi' : Y \to S$. We define the *fiber product* of X and Y over S, denoted $X \times_S Y$ to be a scheme, together with two projection morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ which makes the following diagram commute:



as well as a universal property that for any scheme Z with morphisms $\phi_1 : Z \to X$ and $\phi_2 : Z \to Y$, there exists a unique morphism $\phi : Z \to X \times_S Y$ such that the following diagram commutes:



i.e. $\phi_1 = p_1 \circ \phi$ and $\phi_2 = p_2 \circ \phi$.

The next theorem guarantees that fiber products of schemes given in Definition 3.2.1 are well-defined and they always exists.

Theorem 3.2.2 ([8], Theorem II.3.3)**.** *For any two schemes X and Y over a scheme S, the fiber product $X \times_S Y$ exists, and it is unique up to unique ismorphism.*

We can now give another desirable property which morphisms of schemes can satisfy using the definition of fiber products.

Definition 3.2.3 (Preserved Under Pullback)**.** Let P be a property defined for morphisms of schemes. We say P is *preserved under pullback* if the following condition is satisfied: Let $\pi : X \to Y$ be any morphism of schemes with property P and let $\alpha : Y' \to Y$ be any morphism of schemes. If we take the fiber product $X \times_Y Y'$ induced by π and α then the projection map $X \times_Y Y' \to Y'$ has property P.

One particular important case of fiber products for us is base changing.

Definition 3.2.4 (Base Changing)**.** Let $f : X \to Y$ be a morphism of schemes. For any $y \in Y$, we set

$$X_y := X \times_Y \operatorname{Spec} k(y).$$

This is the *fiber of f over y*. The second projection map $X_y \to \operatorname{Spec} k(y)$ makes X_y into a scheme over $k(y)$.

We can now define what we mean by saying 'generic/special fibers of a morphism.

Definition 3.2.5 (Generic Fibers)**.** Let $f : X \to Y$ be a morphism of schemes with Y irreducible with unique generic point ξ. We call X_ξ in the sense of Definition 3.2.4 the *generic fiber* of f.

Proposition 3.2.6 ([9], Proposition III.1.16)**.** *Let $f : X \to Y$ be a morphism of schemes. Then for any $y \in Y$, the first projection map $p : X_y = X \times_Y \operatorname{Spec} k(y) \to X$ induces a homeomorphism from X_y onto $f^{-1}(y)$*

We recall the spectrum of a DVR only has two points: the generic point and the special point. Since we will study schemes that are defined over DVRs, the special fiber of such a scheme is defined using the special point.

Definition 3.2.7 (Special Fibers)**.** Let X be a scheme over a DVR R with uniformizer π. The unique maximal ideal (π) corresponds to the unique closed point $s \in \operatorname{Spec} R$. We denote X_s or X_π the *special fiber* of X (under the natural structure map $X \to \operatorname{Spec} R$).

Proposition 3.2.8 ([9], Example. III.1.18)**.** *Following the notations in Definition 3.2.7, the underlying topological space structure of X is the disjoint union of the generic X_η, which is a scheme over $K = \operatorname{Frac} R$, and of the special fiber X_π, which is a scheme over the residue field $R/(\pi)$. Moreover, X_η is open in X because $\{\eta\}$ is open in $\operatorname{Spec} R$. The special fiber X_π is closed in X because the special point s is closed.*

Remark 3 (Computing Base Changes For Affine Schemes)**.** In the case where $X = \operatorname{Spec} R[x_1, ..., x_n]/(f_1, ..., f_m) \to \operatorname{Spec} R$ for some ring R and $p \in \operatorname{Spec} R$. We have $X_p \cong \operatorname{Spec}(R/p)[x_1, ..., x_n]/(\tilde{f}_1, ..., \tilde{f}_m)$ where $\tilde{f}_1$ is the image of f in $(R/p)[x_1, ..., x_n]$. This is because

$$X_p = \operatorname{Spec} R[x_1, ..., x_n]/I \times_{\operatorname{Spec} R} \operatorname{Spec} R/p \cong \operatorname{Spec}\left(R[x_1, ..., x_n]/I \otimes_R (R/p)\right)$$

where $I = (f_1, ..., f_m)$. See [15] Section 10.1 for the fiber product construction of affine schemes.

We will now proceed to define separatedness for morphisms of schemes. We first need a few more definitions.

Definition 3.2.9 (Closed Immersions)**.** A morphism $\pi : X \to Y$ is called a *closed immersion* if it is affine (i.e. for each affine open $U \subseteq Y$ we have $\pi^{-1}(U) \subseteq X$ is affine), and for every affine open $\operatorname{Spec} B \subseteq Y$, with $\pi^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, the induced ring map $B \to A$ is surjective.

Definition 3.2.10 (Diagonal Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$.

We recall from general topology that a topological space X is Hausdorff if and only if the diagonal set

$$\{(x, x) : x \in X\} \subseteq X \times X$$

is closed in the product topology $X \times X$. Although the Zariski topology is almost never Hausdorff, we will see that the definition of separatedness mimics the Hausdorff condition.

Definition 3.2.11 (Separatedness)**.** We say a morphism $\pi : X \to Y$ of schemes is *separated* if the diagonal morphism Δ is a closed immersion. In that case we also say X is *separated* over Y. A scheme X is *separated* if it is separated over $\operatorname{Spec} \mathbb{Z}$, the final object in the category of schemes.

We will soon see that separatedness (which mimics Hausdorffness of topological spaces) is required to give the definition for properness of schemes which is the algebraic geometry equivalent for compactness.

We recall from general topology that a map $f : X \to Y$ between topological spaces X and Y is called proper (in the topological sense) if the preimage of every compact set in Y is compact in X. Some authors will also define proper maps as a continuous closed map such that the preimage of every point is compact. All of these definitions of properness give us problems in the category of schemes however. We first do not have the notion of compactness because our schemes are not Hausdorff. Secondly, a map being closed is not a strong enough condition for us as closedness is not preserved under pullback. To fix the second problem, we can simply define a stronger notion where we force closedness to be preserved under pullback.

Definition 3.2.12 (Universally Closed Morphisms)**.** We say a morphism $\pi : X \to Y$ of schemes is said to be *universally closed* if it is a closed map and for any $Y' \to Y$, the corresponding projection morphism $\pi' : X \times_Y Y' \to Y'$ coming from taking the fiber product $X \times_Y Y'$ is closed i.e. closedness is preserved under pullback.

Definition 3.2.13 (Finite Type Morphisms)**.** A morphism $\pi : X \to Y$ of schemes is said to be of *finite type* at $x \in X$ if there exist an affine open neighborhood $\operatorname{Spec} A = U \subset X$ of x and an affine open $\operatorname{Spec} B = V \subset Y$ with $\pi(U) \subset V$ such that the induced ring map $B \to A$ turns A into a finitely generated B-algebra. We say π is of *finite type* if π is of finite type at every point and π is quasi-compact i.e., the preimage of any affine open subset of Y is quasi-compact in X.

To address the problem of the Zariski topology not being Hausdorff, we can simply add the separatedness condition into our definition for properness. Hence we can finally define what a proper morphism between schemes is, in a similar fashion to the properness condition for continuous maps in the topological sense.

Definition 3.2.14 (Properness)**.** A morphism $\pi : X \to Y$ of schemes is called *proper* if it is separated, of finite type and universally closed. We often say a scheme X (over a field K) is *proper* if the morphism $X \to \operatorname{Spec} K$ is proper.

It is hard to check whether a given morphism is proper or not. However there is a very nice criterion to check properness when the morphism $\pi : X \to Y$ is already known to be of finite type and the schemes X and Y are Noetherian.

Theorem 3.2.15 (The Valuation Criterion for Properness)**.** *Let $\pi : X \to Y$ be a morphism of finite type with X and Y Noetherian. Then π is proper if and only if for every DVR R with morphisms $f : U \to X$ and $g : C \to Y$ where $U = \operatorname{Spec} K$ and $C = \operatorname{Spec} R$ which forms a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & X \\ {\scriptstyle\iota}\downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle\pi} \\ C & \xrightarrow[\ g\ ]{} & Y \end{array}$$

there exist a unique morphism $C \to X$ making the entire diagram commutative where $\iota : U \hookrightarrow C$ is the natural inclusion.

This criterion comes in handy for us since all of our schemes later on are going to be Noetherian and the morphisms between them are going to be of finite type.

The following definition demonstrates why properness is a nice property to have.

Proposition 3.2.16 ([15], Proposition 11.5.4)**.**

1. *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Their composition $g \circ f : X \to Z$ is also proper.*

2. *Properness is preserved under pullback.*

3. *Properness is local on the target.*

4. *Suppose we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & Y \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & Z & \end{array}$$

   where α is proper and β is separated. Then π is proper.

3.3 Flat Morphisms

Similar to properness, flatness is another property crucial for defining models of curves. In this section, we will review the definition for flat morphisms following the treatment in Hartshorne [8]. We will also give a criterion to check when is affine schemes define over finitely generated R-algebras flat where R is a DVR.

To define flatness of morphisms between schemes, we will first recall the definition of flat modules.

Definition 3.3.1 (Flat Modules)**.** Let R be a ring. An R-module M is said to be *flat over* R or simply *flat* if the functor $N \mapsto M \otimes_R N$ is exact for $N \in \mathrm{Mod}(R)$ i.e. whenever we have an exact sequence of R-modules

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

the induced sequence

$$0 \longrightarrow M \otimes_R N' \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_R N \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_R N'' \longrightarrow 0$$

is also exact.

Remark 4. For an arbitary R-module M, the above functor $N \mapsto M \otimes_R N$ is always right exact. Flatness condition requires it to be exact.

Proposition 3.3.2 ([8], Proposition III.9.1A)**.**

1. *An R-module M is flat if and only if for every finitely generated ideal $I \subseteq R$, the map $I \otimes_R M \to M$ is injective.*

2. *Base extension: If M is a flat R-module, and $R \to S$ is ring map, then $M \otimes_R S$ is a flat S-module.*

3. *Transitivity: If S is a flat R-algebra, and N is a flat S-module, then N is also flat as an R-module.*

4. *Localization: M is flat over R if and only if M_p is flat over R_p for all $p \in \operatorname{Spec} R$.*

We now give the definition for flat morphisms of schemes.

Definition 3.3.3 (Flat Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *flat* over Y at a point $x \in X$, if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,y}$-module, where $y = \pi(x)$ and the $\mathcal{O}_{Y,y}$-module structure of $\mathcal{F}_x$ is given by the natural local ring map $\pi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. We say $\mathcal{F}$ is *flat over* Y if it is flat at every point of X. We say X is *flat over* Y if $\mathcal{O}_X$ is flat over Y.

Remark 5. In particular, given a morphism between affine schemes $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$. Proposition 3.3.2 part 4 says: π is flat if and only if the induced ring map $B \to A$ turns A into a flat B-module.

We shall show that gluing schemes which are flat over a fixed scheme S results in a flat scheme over S.

Proposition 3.3.4. *Suppose we have a family of schemes $\{X_\alpha\}_I$ and a family of morphisms $\{\psi_{\alpha,\beta}\}_I$ satisfy Theorem 3.1.1 and so they glue to a scheme X (i.e. there are open subsets $X_{\alpha,\beta} \subseteq X_\alpha$ for all $\beta \neq \alpha$ in I). Suppose for each $\alpha \in I$, we have a flat morphism $f_\alpha : X_\alpha \to Y$ such that $f_\beta \circ \phi_{\alpha,\beta} = f_\alpha|_{X_{\alpha,\beta}}$ for a fixed scheme Y. Then the induced map $f : X \to Y$ is also flat.*

Proof. For each $x \in X$, there exist $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. By assumption

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{U_{\alpha_x},x} \cong \mathcal{O}_{X_{\alpha_x},\overline{x}}$$

is a flat $\mathcal{O}_{Y,y}$ module where $\overline{x} \in X_{\alpha_x}$ is the corresponding point of $x \in U_{\alpha_x}$ and $y = f_{\alpha_x}(\overline{x})$. By Theorem 3.1.1, $f|_{U_{\alpha_x}}(x) = f_{\alpha_x}(\overline{x}) = y$ and so $\mathcal{O}_X$ is flat over Y at x. Since x was arbitrarily chosen, we have X is flat over Y. $\square$

Let us recall the definition for open immersions.

Definition 3.3.5 (Open immersions)**.** An *open immersion* is a morphism of schemes $f : X \to Y$ which induces an ismorphism of X with an open subscheme of Y.

We can now state some additional properties for flat morphisms.

Proposition 3.3.6 ([8], Proposition III.9.2)**.**

1. *Open immersions are flat.*

2. *Let $f : X \to Y$ be a morphism of schemes, let $\mathcal{F}$ be an flat $\mathcal{O}_X$-module over Y, and let $g : Y' \to Y$ be any morphism. Let $X' := X \times_Y Y'$, let $\pi_1 : X' \to X$ and $\pi_2 : X' \to Y'$ be the projection morphisms, and let $\mathcal{F}' := \pi_1^*(\mathcal{F})$. Then $\mathcal{F}'$ is flat over Y'.*

Proof. The proof follows from 2. in Proposition 3.3.2. $\square$

We immediately obtain the following corollary from 2. in Proposition 3.3.6.

Corollary 3.3.6.1. *The base change of a flat morphism is flat.*

Proof. Let $f : X \to Y$ be a flat morphism, $g : Y' \to Y$ any morphism and $\pi_1 : X' := X' \times_Y Y' \to X$ the projection map. Since $f : X \to Y$ is flat i.e. $\mathcal{O}_X$ is flat over Y, by 2. in Proposition 3.3.6, $\pi_1^*\mathcal{O}_X$ is flat over Y'. We note that

$$\pi_1^*\mathcal{O}_X = \pi_1^{-1}\mathcal{O}_X \otimes_{\pi_1^{-1}\mathcal{O}_X} \mathcal{O}_{X'} \cong \mathcal{O}_{X'}$$

is the structure sheaf of X' and thus the projection morphism $\pi_2 : X' \to S'$ is flat. $\square$

Since we will be requiring our model of curves to be flat over the prime spectrum of their defining rings later on, we should give ways to verify flatness for affine schemes. From this point and onward, all of our schemes are assumed to be Noetherian.

Definition 3.3.7 (Associated Primes)**.** Let R be a ring and M be an R-module. A prime $\mathfrak{p} \in \operatorname{Spec} R$ is *associated to* M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted as $\operatorname{Ass}_R(M)$.

Definition 3.3.8 (Associated Points)**.** A point x of a (Noetherian) scheme is an *associated point* of X if the maximal ideal $\mathfrak{m}_x$ is an associated prime of 0 in the local ring $\mathcal{O}_{X,x}$, or in other words, if every element of $\mathfrak{m}_x$ is a zero divisor.

Remark 6. In the statement of Definition 3.3.8, the ring R is $\mathcal{O}_{X,x}$ and the R-module M is also $\mathcal{O}_{X,x}$ i.e. $\mathcal{O}_{X,x}$ is considered as a module over itself according to Definition 3.3.7.

We now recall what is means for schemes to be integral/reduced.

Definition 3.3.9 (Integral Schemes)**.** A scheme X is *integral* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 3.3.10 (Reducedness)**.** A ring R is called *reduced* if the nilradical $\mathcal{N}(R) = 0$ i.e. R has no non-zero nilpotent elements. A scheme X is said to be *reduced* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is reduced.

Proposition 3.3.11 ([8], Proposition II.3.1)**.** *A scheme is integral if and only if it is both reduced and irreducible.*

We now give a criterion for checking flatness over an integral, regular of dimension 1 base ring.

Proposition 3.3.12 ([8], Proposition III.9.7)**.** *Let $\pi : X \to Y$ be a morphism of schemes, with Y integral and regular of dimension 1. Then π is flat if and only if every associated point $x \in X$ maps to the generic point of Y. In particular, if X is reduced, this says that every irreducible component of X dominates Y.*

We will now prove a series of lemmas and propositions to demonstrate how flatness can be checked for an affine variety of the form $V(f)$ over R where R is a DVR and $f \in R[x_1, ..., x_n]$.

Lemma 3.3.13. *Let X be a integral scheme. Then its only associated point of X is its generic point.*

Proof. X is integral implies X is irreducible by Proposition 3.3.11. We know that X has a unique generic point η. Pick any point $x \in X$ and an affine open subscheme $U \subseteq X$ containg x where $U = \operatorname{Spec} A$. By abusing of notation, we will denote the prime ideal in A corresponding to x as x. We then have $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x} = A_x$ which is an integral domain since X is integral. This means $\mathfrak{m}_x \subseteq A_x$ contains non-zero elements (which are not zero divisors). Therefore the only associated point of X is η as $\mathfrak{m}_\eta = (0) \subset \mathcal{O}_{X,\eta} = \operatorname{Frac} A$. □

Proposition 3.3.14. *Let R be a DVR with field of fractions K and X an integral R-scheme of finite type over R whose generic fiber X_η/K is nonempty. Then X is flat over R.*

Proof. By Lemma 3.3.13 the only associated point of X is its unique generic point η. Let η_R be the generic point of $\operatorname{Spec} R$. Suppose that under the natural morphism $\pi : X \to \operatorname{Spec} R$ we have $\pi(\eta) = \mathfrak{p} \neq \eta_R$ where $\mathfrak{p}$ is the special point of $\operatorname{Spec} R = \{\eta_R, \mathfrak{p}\}$. We see that

$$\pi(X) = \pi(\mathrm{cl}_X\{\eta\}) \subset \mathrm{cl}_{\operatorname{Spec} R}(\pi(\eta)) = \mathrm{cl}_{\operatorname{Spec} R}(\{\mathfrak{p}\}) \subsetneq \operatorname{Spec} R$$

since the generic point η_R is the only non-closed point in $\operatorname{Spec} R$. This implies $\pi^{-1}(\eta_R)$ is empty. However as topological spaces, $X_\eta = X \times_{\operatorname{Spec} R} K$ is homeomorphic to $\pi^{-1}(\eta_R)$ by Proposition 3.2.6 thus contradicting $X_\eta \neq \emptyset$. Therefore we must have $\pi(\eta) = \eta_R$ and by Proposition 3.3.12, X is flat over R. □

For the sake of simplified notations, we will first consider the simple case when there are only two variables, x and y.

We recall that in a UFD, an element is prime if and only if it is irreducible.

Proposition 3.3.15. *Let R be a DVR and $\mathcal{C} \subset \mathbb{A}^2_R$ an affine scheme defined by a single polynomial equation*

$$f(x,y) = 0 \quad \text{for some } f(x,y) = \sum a_{i,j} x^i y^j \in R[x,y].$$

Assume that f is not constant. Since R is a PID, $R[x,y]$ is a UFD. If we factorize $f = \prod_{i=1}^r p_i^{e_i}$ in $R[x,y]$, where the p_i's are distinct irreducibles (p_i is not a unit multiple of p_j for any $i \neq j$), then the associated points of $V(f)$ are precisely the (p_i)'s.

Proof. By assumption, we have

$$V(f) = \bigcup_{i=1}^r V(p_i^{e_i}).$$

Consider any one of the reduced components $V(p_i^{e_i})$. The generic point corresponds to the unique minimal prime $\sqrt{(0)} = (p_i) \subset (R[x,y]/(f))/(p_i^{e_i})$ where (0) is the zero ideal of $(R[x,y]/(f))/(p_i^{e_i})$. Such point is indeed an associated point since p_i is a zero divisor in the local ring $(R[x,y]/(f))_{(p_i)}$ as

$$p_i \cdot \left(p_i^{e_i - 1} \prod_{j \neq i} p_j^{e_j} \right) = 0.$$

Since R is an integral domain, then so is $R[x, y]$. Hence if $ab = 0$ in $R[x, y]/(f)$ for $a, b \in R[x, y]/(f)$, then we must have $ab = f$ in $R[x, y]$ which means the zero divisors of $R[x, y]/(f)$ only arise from the nontrivial factors of f. Our goal is to find the associated prime ideals in $R[x, y]/(f)$ i.e. prime ideals of the form $\mathfrak{p} = \operatorname{Ann}(d)$ for some element $d \in R[x, y]/(f)$. Note that if $\mathfrak{p} \neq 0$, then by our above observation on the zero divisors of $R[x, y]/(f)$, such a d must be a nontrivial factor of f which implies $f/d = g$ for some $g \in R[x, y]$. We claim that $\operatorname{Ann}(d) = (g)$. It is clear that $(g) \subseteq \operatorname{Ann}(d)$ since $g \in \operatorname{Ann}(d)$. Conversely if $h \in \operatorname{Ann}(d)$ then $dh = 0$ in $R[x, y]/(f)$ i.e. $dh = f\ell$ for some $\ell \in R[x, y]$ which implies $dh = dg\ell$ and so $h = g\ell \in (g)$. Finally, for $\mathfrak{p}$ to be prime, it is necessary that g has to be irreducible in $R[x, y]$ which means $g = p_i$ for some i. □

Proposition 3.3.16. *Keep the same notations and assumptions as in Proposition 3.3.15, then the inclusion* $R \overset{\iota}{\hookrightarrow} R[x, y]/(f)$ *is flat if and only if the ideal* $I = (a_{i,j}) = (1)$.

Proof. We first recall from the definition of affine schemes that the zero ideal (0_R) of R corresponds to the generic point $\eta_R \in \operatorname{Spec} R$. We note that ι is injective since it is the natural inclusion map

$$\begin{aligned} \iota : R &\longrightarrow R[x, y]/(f) \\ a &\longmapsto [a] \end{aligned}$$

where $[a]$ denotes the equivalence class of a in the quotient ring $R[x, y]/(f)$.

For the forward direction, let us denote $\mathfrak{m}$ as the maximal ideal of R with uniformizer π. Suppose $I \neq (1)$, that is, $a_{i,j} \in \mathfrak{m}$ for all i and j. Then $f = \pi^r g$ for $r > 0$ and $g \in R[x, y]$ has at least one coefficient which is a unit. We have $V(f) = V(\pi^r) \cup V(g)$. By Proposition 3.3.15, (π) is an associated point of $V(f)$ which does not map to the generic point η_R under the map $V(f) \to \operatorname{Spec} R$. Otherwise $(\pi) \mapsto \eta_R$ would imply that $\iota^{-1}((\pi)) = (0_R) \subset R$ which is impossible since ι is injective and $\pi \neq 0$. Therefore by Proposition 3.3.12, the map $\mathcal{C} = V(f) \to \operatorname{Spec} R$ is not flat.

Conversely, assume that $I = (a_{i,j}) = (1)$ and that $V(f) \to \operatorname{Spec} R$ is not flat. By Proposition 3.3.12, some associated point α of $V(f)$ gets sent to the special point $(\pi) \in \operatorname{Spec} R$. By Proposition 3.3.15, $\alpha = (p_i)$ for some irreducible $p_i \in R[x, y]/(f)$ and $p_i \mid f$. This implies $\iota^{-1}((p_i)) = (\pi)$. But p_i being irreducible forces $p_i = u \cdot \pi$ for some unit $u \in R$ which implies $\pi \mid f$ contradicting $I = (1)$. Hence $V(f) \to \operatorname{Spec} R$ must be flat. □

We realize that none of our proofs depended on the number of variables. Hence we can naturally generalize the result in Proposition 3.3.16 to any finite number of variables.

Corollary 3.3.16.1. *The statement for Proposition 3.3.16 can be generalized in a similar way for* $V(f)$ *where* $f \in R[x_1, ..., x_n]$.

As we can already imagine from all that, checking flatness is closely related to checking an ideal being prime. Although it is much stronger to say an ideal is prime, we can in fact check primeness for ideals generated by a single polynomial over any integral domain using the following Theorem.

Theorem 3.3.17 ([14], Theorem A)**.** *Let R be any integral domain. Let $f = \sum_{i=0}^{d} a_i x^i \in R[x]$ be a non-constant polynomial and $I = (a_0, ..., a_d) \subseteq R$. The ideal $(f) \subseteq R[x]$ is prime if and only if f is irreducible over $K = \operatorname{Frac} R$ and $I^{-1} = R$ where*

$$I^{-1} = (R :_K I) = \{r \in K : rI \in R\}.$$

Remark 7. In particular, if f in Theorem 3.3.17 is monic in x, then $I^{-1} = R$. This is because $I = (1) = R$. For any $r \in I^{-1}$, we have $1 \cdot r \in R$ and so $r \in R$.

3.3.1 Primary Decomposition

In the previous section, we have determined what the associated points of

$$\operatorname{Spec} R[x_1, x_2, ..., x_n]/(f)$$

look like when R is a DVR and f is a single irreducible polynomial. However this is not yet sufficient for us as later on, since the affine schemes that we will be studying are often cut out by more than a single polynomial generator. This leads us to the notion of primary decomposition, for which, we will follow the treatment in [7] in this section.

We first begin with the definition for primary ideals.

Definition 3.3.18 (Primary Ideals)**.** An ideal $Q \subseteq R$ is said to be *primary* if whenever $fg \in Q$, either $f \in Q$ or $g^n \in Q$ for some $n \geq 1$.

We now define P-primary and what a primary decomposition is for an ideal.

Definition 3.3.19 (Primary Decomposition)**.** Let R be a Noetherian ring and $P \subset R$ a prime ideal. An ideal $Q \subset P$ is called *primary* to P or P*-primary* if P is the radical of Q and for any elements $f, g \in R$ with $fg \in Q$ but $g \notin P$ we have $f \in Q$; equivalently, $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\mathfrak{p}$ is its radical and the localization map $R/\mathfrak{q} \to R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}$ is a monomorphism.

If we write an ideal $I \subseteq R$ as

$$I = \bigcap_{i=1}^{n} Q_i \tag{3.3.1}$$

where

1. each Q_i is primary to $\sqrt{Q_i}$,

2. removing any Q_i from the intersection will change the equality,

3. the prime ideals $\sqrt{Q_i}$ are all distinct from each other,

then we say that (3.3.1) is a *primary decomposition* of I. The Q_i's are called *primary components* of I.

Remark 8. Some authors might require a primary decomposition to only satisfy point 1 in Definition 3.3.19 and call a primary decomposition which satisfies points 2 and 3 an *irredundant* primary decomposition.

The following lemma will show us that the above definitions are equivalent, and we can use whichever one we want.

Lemma 3.3.20. *If an ideal Q is P-primary for some prime ideal P in R, then Q itself is a primary ideal in R. Conversely, the radical of a primary ideal Q is prime and Q is $\sqrt{Q}$-primary.*

Proof. Take $f, g \in R$ such that $fg \in Q$, and suppose $f \notin Q$. Since Q is P-primary, we have $\sqrt{Q} = P$. If $f \notin P$ as well then we are done. Suppose $f \in P$. Since $\sqrt{Q} = P$, by definition $f^m \in Q$ for some $m > 1$.

Conversely, if Q is a primary ideal, let us take $fg \in \sqrt{Q}$. We have $(fg)^n = f^n g^n \in Q$ for some $n \geq 1$. Since Q is primary, we either have $f^n \in Q$ which implies $f \in \sqrt{Q}$ or $(g^n)^m = g^{nm} \in Q$ for some $m \geq 1$ which implies $g \in \sqrt{Q}$. Now if $fg \in Q$ but $f \notin \sqrt{Q}$, we have $g \in \sqrt{Q}$ as $fg \in \sqrt{Q}$ and $\sqrt{Q}$ is prime. □

We will now show a few properties regarding P-primarility (P-primary ideals) as well as the notion for an ideal being irreducible.

Proposition 3.3.21. *If $Q_1, ..., Q_n$ are P-primary ideals of R for some prime ideal P then $\bigcap_{i=1}^n Q_i$ is also P-primary.*

Proof. We set $Q := \bigcap_{i=1}^n Q_i$. We want to show that $\sqrt{Q} = P$. Since each $\sqrt{Q_i} \subseteq P$, we have $\sqrt{Q} \subseteq P$. Conversly if P' is any prime ideal that contains Q, since $Q_1 \cdots Q_n \subseteq Q \subseteq P'$, it follows that $Q_i \subseteq P'$ for some i. Thus $P = \sqrt{Q_i} \subseteq P'$. This shows that $P \subseteq \sqrt{Q}$. Hence $\sqrt{Q} = P$.

To show that Q is P-primary, by our lemma, it is suffices to show that Q is a primary ideal. Let us take any $f, g \in R$ with $fg \in Q$ and $f \notin Q$. Hence for some j we have $f \notin Q_j$. Since Q_j is primary, we have $g^n \in Q_j$ for some $n \geq 1$. This implies $g \in \sqrt{Q_j} = P = \sqrt{Q}$ and so $g^m \in Q$ for some $m \geq 1$ which shows that Q is primary and thus P-primary. □

Definition 3.3.22 (Irreducible Ideals)**.** A proper ideal $I \subseteq R$ is said to be *irreducible* if for any ideals J and K in R such that $I = J \cap K$ either $J = I$ or $K = I$.

Lemma 3.3.23. *In a Noetherian ring, every irreducible ideal is primary.*

Proof. Let R be a Noetherian ring, and let Q be an irreducible ideal of R. Take $a, b \in R$ such that $ab \in Q$ but $b \notin Q$. For each $n \geq 1$, consider the set

$$A_n := (Q : (a^n)) = \{r \in R : ra^n \in Q\}.$$

We see that A_n is an ideal of R for each $n \geq 1$ and $A_i \subseteq A_j$ whenever $i \leq j$ since if $r \in A_i$, then $a^{j-i}ra^i = ra^j \in Q$ i.e. $r \in A_j$. We get an ascending chain of ideals

$$A_1 \subseteq A_2 \subseteq \cdots .$$

Since R is Noetherian, there exist some N such that $A_N = A_m$ for all $m \geq N$. Let us consider the ideals $I := (a^n) + Q$ and $J := (b) + Q$ for $n \geq N$. It is clear that $Q \subseteq I \cap J$. Conversely, if we take $y \in I \cap J$ and write $y = ra^n + q$ for some $r \in R$ and $q \in Q$, as $aJ = (ab) + aQ \subseteq Q$ it follows that $ay \in Q$. Thus $ra^{n+1} = ay - aq \in Q$. This implies that $r \in A_{n+1} = A_n$, and so $y = ra^n + q \in Q$ which shows that $Q = I \cap J$. Since Q is irreducible, we must have $Q = I$ or $Q = J$. Using the fact that $b \notin Q$, we have $Q \neq J$ which shows that $Q = I = (a^n) + Q$, and hence $a^n \in Q$. By definition, Q is primary. □

Proposition 3.3.24. *Every ideal in a Noetherian ring R admits a primary decomposition.*

Proof. Let I be an ideal in R. Since R is Noetherian, by Lemma 3.3.23, in order to prove that I has a primary decomposition, it is suffices to show that every proper ideal of R is the finite intersection of irreducible ideals. Towards a contradiction, let $\mathcal{I}$ be the set of all proper ideals of R which cannot be written as a finite intersection of irreducible ideals. Since R is a Noetherian ring and the set $\mathcal{I}$ is nonempty, $\mathcal{I}$ must contain a maximal element by Zorn's lemma, say $J \in \mathcal{I}$. Since J belongs to $\mathcal{I}$ it cannot be irreducible itself, hence $J = I_1 \cap I_2$ for some ideals I_1 and I_2 properly containing J. The maximality of J in $\mathcal{I}$ implies both I_1 and I_2 can be written as a finite intersection of irreducible ideals in R. However, this immediately implies that J can be written as a finite intersection of irreducible ideals in R contradicting J belonging to $\mathcal{I}$. Thus the assertion of $\mathcal{I} \neq \emptyset$ is absurd which implies every proper ideal of R admits a finite irreducible decomposition.

Furthermore, any primary decomposition can be turned into an irredundant primary decomposition by dropping unnecessary primary ideals from the intersection and successively replacing all primary ideals with the same radical by their intersection. □

We will soon see that the primary components of an ideal $(f_1, ..., f_m) \subseteq R[x_1, ..., x_n]$ play the same role as the prime factors p_i of a single generator $f \in R[x_1, ..., x_n]$ as in Proposition 3.3.15 with the following proposition.

Proposition 3.3.25. *The associated primes of an ideal $I \subseteq R$ are exactly the radicals of the primary components in a primary decomposition of I.*

Proof. We want to show that the set of radical ideals of the primary components in a primary decomposition of I is

$$\{(I : Rc) \in \mathrm{Spec}(R) : c \in R \setminus I\}$$

where

$$(I : Rc) = \{r \in R : rc \in I\}$$

denotes the ideal quotient. We first write

$$I = \bigcap_{i=1}^{n} Q_i$$

as an irredundant primary decomposition of I, and set $P_j := \sqrt{Q_j}$ for each j. We fix a j and define

$$J := \bigcap_{i \neq j} Q_i$$

and observe that $I = J \cap Q_j$ is strictly contained in J. Since R is Noetherian, there exists $m \geq 1$ such that $P_j^m \subseteq Q_j$, and so $JP_j^m \subseteq J \cap P_j^m \subseteq J \cap Q_j = I$. Assume that m is the minimal positive integer such that $JP_j^m \subseteq I$. Take $c \in JP_j^{m-1} \subseteq J \cap P_j^{m-1}$ such that $c \notin I$. The fact that $c \in J$, along with $c \notin I$ ensures that $c \notin Q_j$. So if $r \in R$ satisfies $rc \in I \subseteq Q_j$, then the fact that Q_j is primary guarantees that $r \in \sqrt{Q_j} = P_j$. Hence $(I : Rc) \subseteq P_j$. Conversely, note that $cP_j \subseteq JP_j^m \subseteq I$ (since $c \in JP_j^{m-1}$), which implies that $P_j \subseteq (I : Rc)$. Hence $P_j = (I : Rc)$ as desired.

Now fix $c \in R \setminus I$ with $P := (I : Rc)$ prime. Note that there exist a j such that $c \notin Q_j$. Consider the ideal

$$K := \prod_{c \notin Q_i} Q_i.$$

We see that

$$cK = Kc \subseteq \bigcap_{i=1}^{n} Q_i = I.$$

Therefore by the definition of $(I : Rc)$ we have $K \subseteq (I : Rc) = P$, and the fact that P is prime ensures that $Q_i \subseteq P$ for some i with $c \notin Q_i$. Thus $\sqrt{Q_i} \subseteq P$. On the other hand, take $x \in P$, and observe that $xc \in I \subseteq Q_i$ by the definition of $(I : Rc)$. Because Q_i is primary and $c \notin Q_i$, it follows that $x \in \sqrt{Q_i}$. Hence $P = \sqrt{Q_i}$. □

Finally, Proposition 3.3.25 allows us to generalize Proposition 3.3.16 to ideals with more than one generators.

Theorem 3.3.26. *Let R be a DVR, then $R \hookrightarrow R[x_1, ..., x_n]/(f_1, ..., f_m)$ is flat if and only if the uniformizer $(\pi) \not\supseteq \sqrt{Q_i}$ for any primary component Q_i of $(f_1, ..., f_m)$ in a primary decomposition.*

Proof. Let $I = (f_1, ..., f_m)$ and let $I = \bigcap_{i=1}^r Q_i$ be a primary decomposition of I. By Proposition 3.3.25, the associated points of $V(I)$ are precisely $\sqrt{Q_i} \in V(I) = \operatorname{Spec} R[x_1, ..., x_n]/I$ for $i = 1, ..., r$. By Proposition 3.3.12, the map $\iota : R \hookrightarrow R[x_1, ..., x_n]/I$ is flat if and only if $\iota^{-1}(\sqrt{Q_i}) \neq (\pi)$ for all i if and only if $\sqrt{Q_i} \not\subseteq (\pi)$ for all i. □

Remark 9. Computing primary decomposition of ideals in $R[x_1, ..., x_n]$ is in general very difficult. In the special case where I is already a prime ideal however, we just have to check that $I \not\subseteq (\pi)$.

For modules over a valuation ring, the following lemma provides a convenient criterion to test flatness.

Lemma 3.3.27 ([13] Lemma 15.22.11)**.** *A module M over a valuation ring is flat if and only if M is torsion free i.e. $0 \in M$ is the only torsion element.*

3.4 Curves

In this section, we will be defining hyperelliptic curves, models of curves, and semistable reduction type.

3.4.1 Smoothness

We will begin by recalling the definition for tangent spaces, regular points and smoothness for schemes. We will need these concepts to talk about 'smooth curves' and 'singular points'. We will also recall a important way of checking smoothness for schemes over a base known as the Jacobian criterion.

Definition 3.4.1. Let X be a scheme and $x \in X$. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ be the residue field. Then $\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is in a natural way, a $k(x)$-vector space. Its dual space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the *(Zariski) tangent space to X at x*. We denote it by $T_{X,x}$.

We have the following proposition about tangent spaces.

Proposition 3.4.2 ([9], Proposition IV.2.2)**.** *Let X be a locally Noetherian scheme. For any $x \in X$, we have* $\dim_{k(x)} T_{X,x} \geq \dim \mathcal{O}_{X,x}$.

Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $k = A/\mathfrak{m}$. Using [9, Corollary II.5.14(b)] we know that $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$. We can now give the definition for a Noetherian local ring to be regular.

Definition 3.4.3 (Regular Noetherian Local Rings)**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring. We say that A is *regular* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$. By Nakayama's lemma, A is regular if and only if $\mathfrak{m}$ is generated by $\dim A$ elements.

The definition for regular points for locally Noetherian schemes.

Definition 3.4.4 (Regular Points)**.** Let X be a locally Noetherian scheme, and let $x \in X$ be a point. We say that X is *regular at* x, or that x is a *regular point of* X, if $\mathcal{O}_{X,x}$ is regular, that is, $\dim \mathcal{O}_{X,x} = \dim_{k(x)} T_{X,x}$. We say that X is *regular* if it is regular at all of its points. A point $x \in X$ which is not regular is called a *singular point of* X. A scheme that is not regular is said to be *singular.*

Remark 10. Any DVR is regular. Conversely if $(A, \mathfrak{m})$ is a Noetherian regular local ring of dimension 1, then by definition $\mathfrak{m}$ is generated by a single element, and A is a PID.

In fact we only have to worry about regularity at the closed points of a Noetherian scheme.

Proposition 3.4.5 ([9], Corollary IV.2.17)**.** *Let X be a Noetherian scheme. Then X is regular if and only if it is regular at its closed points.*

At a first glance, the definition for regularity looks abstract. However for varieties, regularity can be checked explicitly using their defining polynomials.

Theorem 3.4.6 (Jacobian Criterion)**.** *Let k be a field. Let $X = V(I)$ be a closed subvariety of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, ..., x_n]$ where $I = (f_1, ..., f_m) \subseteq k[x_1, ..., x_n]$. Let $x \in X$ be a closed point. Let us consider the matrix*

$$J_x := \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

in $M_{r \times n}(k)$. Then X is regular at x if and only if

$$\operatorname{rank}(J_x) = n - \dim X.$$

We can now define smoothness, a term we have mentioned in our introduction.

Definition 3.4.7 (Smooth Points and Singular/Smooth Locus)**.** Let X/k be an algebraic variety over a field k. Let $\overline{k}$ be the algebraic closure of k and $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. We say that X is *smooth at* $x \in X$ if the points of $X_{\overline{k}}$ lying above x are all regular points of $X_{\overline{k}}$. We say that X is *smooth over* k or *non-singular over* k if it is smooth at all of its points (i.e., $X_{\overline{k}}$ is regular). We denote $\operatorname{Sing}(X)$ as the set of non-smooth (singular) points of X and we call it the *singular locus of* X. We call $X \setminus \operatorname{Sing}(X)$ the *smooth locus of* X.

Remark 11. Usually in algebraic geometry, smoothness is used for morphism of schemes and regularity is a property for points on a scheme. Over an algebraically closed field, the two notion coincides (cf. [15, Exercise 13.2.J]). Since we are only talking about smoothness/regularity for schemes defined over algebraically closed fields in this chapter, we will be using these two terms interchangeably.

We can in fact compute the Zariski tangent space of a variety by using the Jacobian matrix.

Lemma 3.4.8 ([11], Lemma 3.3)**.** *Let* $X = \operatorname{Spec}(k[x_1, ..., x_n]/(f_1, ..., f_m))$ *be an affine variety and* $p \in X$ *a smooth point of* X*. The Zariski tangent space* $T_{X,p}$ *is equal to the kernel of the Jacobian matrix* $\ker\left(\frac{\partial f_i}{\partial x_j}(p)\right)$ *(*$T_{X,p}$ *is identified with a subspace of the tangent space of* $\mathbb{A}^n$ *at* p*).*

Remark 12. Smoothness of an algebraic variety X/k can be verified by applying the Jacobian criterion and Proposition 3.4.5 to $X_{\overline{k}}$ (see [9, Exercise IV.3.20]).

The following proposition will guide us towards how to define nodal singularities in Section 3.4.3.

Proposition 3.4.9 ([9], Proposition IV.2.27)**.** *Let* X *be an algebraic variety over an algebrically closed field* k*, and let* $x \in X$ *be a smooth point of* X*. Let* $\mathfrak{m}_x$ *denote the maximal ideal of* $\mathcal{O}_{X,x}$*. Then we have*

$$\widehat{\mathcal{O}_{X,x}} \cong k[\![t_1, ..., t_d]\!],$$

with $d = \dim X$ *(see Chapter 2 for the* $\widehat{\cdot}$ *notation).*

Example 1. Consider the variety in $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ defined by the equation

$$V : y^2 = x^3 + x$$

where k is an algebraically closed field with $\operatorname{char}(k) \neq 2, 3$. Let us denote $f(x, y) = y^2 - x^3 - x$. By Theorem 3.4.6, in order for a point $p = (x_0, y_0) \in V$ to be singular, we require that

$$\frac{\partial f}{\partial x}(p) = -3x_0^2 - 1 = 0 = 2y_0 = \frac{\partial f}{\partial y}(p).$$

This can only happen when $x_0 = \pm i/\sqrt{3}$ and $y_0 = 0$ where $i^2 = -1$ in k. We see that the points $(\pm i/\sqrt{3}, 0)$ are not on V since $0 \neq \pm 2i/3\sqrt{3}$.

3.4.2 Hyperelliptic Curves

In the introduction, we briefly gave the informal definition for the central object of this thesis, which are hyperelliptic curves. We called that definition as the "affine model" of the curve which is not the way we will be thinking about hyperelliptic curves. As promised, in this section, we will give their non-singular projective definition (non-singular completion). Later on in the thesis, whenever we write down a hyperelliptic curve (affine model or not), what we are really considering is its non-singular completion by gluing together two charts.

We recall a *curve* is a one-dimensional algebraic variety over a field. We also recall *genus* is a number which counts how many 'holes' a topological surface has. Genus is also a number defined for non-singular curves and they serves as an important invariant which classifies curves. Since we do not need to use this invariant explicitly, we will not be formally defining it in this thesis. For a more rigorous treatment on genus of curves, see [9, Section 7.3.2].

Remark 13. By base changing a non-singular projective curve defined over a subfield $L \subseteq \mathbb{C}$ to $\mathbb{C}$, it can then be viewed as a two dimensional real Riemann surface. The genus of the curve is precisely the genus of its Riemann surface structure over L.

Definition 3.4.10 (Hyperelliptic Curves)**.** Let C be a non-singular geometrically connected curve over a field k, of genus $g \geq 1$. We say that C is a *hyperelliptic curve* if there exists a finite separable morphism $C \to \mathbb{P}^1_k$ of degree 2.

Remark 14. We have in fact, defined a restricted class of hyperelliptic curves. The more general definition only requires that there exists a finite separable morphism $C \to X$ of degree 2 with X being a smooth projective conic (cf. [9, Definition VII.4.7]). For this thesis, Definition 3.4.10 is sufficient for us.

The following proposition allows us to see why our definition in the introduction is correct.

Proposition 3.4.11 ([9], Proposition VII.4.24)**.** *Let C be a hyperelliptic curve of genus g over a field k, with a separable morphism $f : X \to \mathbb{P}^1_k$ of degree 2. We have the following statements:*

1. *The function field* $\operatorname{Frac} \mathcal{O}_C(C)$ *admits a presentation*

$$k(x)[y]/(y^2 + Q(x)y - P(x)), \quad P, Q \in k[x],$$

with

$$2g + 1 \leq \max\{2 \deg Q(x), \deg P(x)\} \leq 2g + 2.$$

We can take $Q(x) = 0$ if $char(k) \neq 2$.

2. *The curve C is the union (gluing) of two affine open schemes*

$$U = \operatorname{Spec} k[x, Y]/(Y^2 + Q(x)Y - P(x)),$$

and

$$V = \operatorname{Spec} k[u, V]/(V^2 + Q'(u)V - P'(u))$$

where $Q'(u) = Q(1/u)u^{g+1}$, $P'(u) = P(1/u)u^{2g+2}$, and the two open subschemes glue along $D(x) \cong D(u)$ with the relation $x = 1/u$ and $Y = x^{g+1}V$.

3.4.3 Models of Curves and Semistable Reduction Type

We have mentioned in the introduction that we want nodal singularities for the models of our hyperelliptic curves. We now define what we mean by 'model' and 'nodal singularities'.

Definition 3.4.12 (Models of Curves)**.** Let R be a DVR, with field of fractions K. Let η denotes the generic point of $\operatorname{Spec} R$. Let C be a non-singular and connected projective curve over K. A *model* of $C \to \operatorname{Spec} K$ is a proper and flat scheme $\mathcal{C} \to \operatorname{Spec} R$ together with an isomorphism of K-schemes $\mathcal{C}_\eta \cong C$.

Remark 15. We could have given the more general definition where we replace $\operatorname{Spec} R$ with a Dedekind scheme S of dimension 1 (i.e. a Noetherian and integral scheme of dimension 1 whose stalks are all regular local rings) and $\mathcal{C} \to S$ a fibered surface (cf. [9], Definition X.1.1). However, the definition we have is enough for us to work with.

We now define nodal singularities and semistable reduction type.

Definition 3.4.13 (Double Points Over Algebraically Closed Fields)**.** Let k be an algebraically closed field and X a one dimensional algebraic scheme over k i.e. the structure morphism $X \to \operatorname{Spec} k$ is of finite type. A point $x \in X$ is an *ordinary double point* if and only if

$$\widehat{\mathcal{O}_{X,x}} \cong k[\![x,y]\!]/(xy).$$

Remark 16. By Proposition 3.4.9, any smooth point $x \in X$ will satisfy $\widehat{\mathcal{O}_{X,x}} \cong k[\![t]\!]$.

Definition 3.4.14 (Double Points)**.** Let k be any field and X a one dimensional algebraic scheme over k.

1. We recall that $X_{\overline{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. We say a closed point $x \in X$ is a *node*, or an *ordinary double point*, or *defines a nodal singularity* if there exist an ordinary double point $\overline{x} \in X_{\overline{k}}$ (in the sense of Definition 3.4.13) mapping to x.
2. We say the singularities of X are *at-worst-nodal* if all closed points of X are either in the smooth locus of X/k or are ordinary double points.

We will later show that for a very specific case of curves, a singular point being a node can be checked using the Jacobian matrix. Before going there, let us first consider a simple example.

Example 2. Let Y be the plane cubic curve given by the equation

$$Y : y^2 = x^2(x+1)$$

over a field k (where $\operatorname{char}(k) \neq 2$). The origin $O \in Y$ is a nodal singularity. To see this let us consider the completion

$$\widehat{\mathcal{O}_{Y,O}} \cong k[\![x,y]\!]/(y^2 - x^2 - x^3).$$

We note that $y^2 - x^2$ factors as $(y+x)(y-x)$. We will first construct $g, h \in k[\![x,y]\!]$ such that

$$g = y + x + g_2 + g_3 + \cdots$$
$$h = y - x + h_2 + h_3 + \cdots$$

where h_i, g_i are homogeneous of degree i such that $y^2 - x^2 - x^3 = gh$. For g_2 and h_2 we want

$$(y-x)g_2 + (y+x)h_2 = -x^3.$$

This can be done since $y-x$ and $y+x$ generates the maximal ideal of $k[\![x,y]\!]$ i.e. $(y-x, y+x) = (x,y)$. Moreover, we can pick g_2 and h_2 to be homogeneous of degree 2 since $-x^3$ is homogeneous of degree 3. Similarly we can also find homogeneous g_3 and h_3 of degree 3 in $k[\![x,y]\!]$ such that

$$(y-x)g_3 + (y+x)h_3 = -g_2h_2.$$

Inductively, each g_i and h_i are obtained by solving

$$(y-x)g_i + (y+x)h_i = -g_{i-1}h_{i-1}.$$

This can be done since g_{i-1} and h_{i-1} are both homogeneous of degree $i-1$.

Therefore $\widehat{\mathcal{O}_{Y,O}} \cong k[\![x,y]\!]/(gh)$. Since g and h begin with linearly independent linear terms, we can define an automorphism sending g and h to x and y respectively. This shows that $\widehat{\mathcal{O}_{Y,O}} \cong k[\![x,y]\!]/(xy)$ and by definition O is a nodal singularity of Y.

Let Z be the plane cubic curve given by the equation

$$Z : y^2 = x^3$$

over k. The origin $O \in Y$ is not an ordinary double point i.e. not a node (it is called a cusp).

Remark 17. In proving that $Y : y^2 = x^2(x+1)$ has a nodal singularity at the origin, we were essentially finding the Taylor series for $\sqrt{1+x}$, since over the completion $\widehat{\mathcal{O}_{Y,O}}$ we have the factorization

$$y^2 - x^2(x+1) = (y - x\sqrt{x+1})(y + x\sqrt{x+1}).$$

These two factors are the g and h we found. The reason this works is because $\mathrm{char}(k) \neq 2$ since the Taylor series of $\sqrt{1+x}$ involves division by 2.

The following proposition will demonstrate how to check for nodal singularities using the Jacobian matrix for curves that we are interested in later on.

Proposition 3.4.15. *Let $C = C_1 \cup C_2$ be a union of two smooth irreducible curves where $C_1 = V(f_1, ..., f_r)$ and $C_2 = V(g_1, ..., g_s)$ where $f_i, g_j \in k[x_1, ..., x_n]$. Suppose $C_1 \cap C_2 = \{p\}$*

is a single point. Let

$$J_1 := \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1\le i\le r, 1\le j\le n}, \quad \textit{and} \quad J_2 := \left(\frac{\partial g_i}{\partial x_j}(p)\right)_{1\le i\le s, 1\le j\le n}$$

be the corresponding Jacobian matrices of C_1 and C_2 at p respectively. Suppose $\ker J_1 = \mathrm{Span}_k\{v_1\}$ *and* $\ker J_2 = \mathrm{Span}_k\{v_2\}$ *where v_1 and v_2 are linearly independent over k. Then p is a nodal singularity of C.*

Proof. Without loss of generality, we may assume that $p = O$ the origin, since we can always make a suitable affine change of coordinates. Let $\mathfrak{m}_1 \subset R/I_1$ and $\mathfrak{m}_2 \subset R/I_2$ be the images of the maximal ideal $(x_1, ..., x_n) \subseteq R$ corresponding to p for R/I_1 and R/I_2 respectively where $I_1 = (f_1, ..., f_r)$, $I_2 = (g_1, ..., g_s)$ and $R = k[x_1, ..., x_n]$. Since C_1 and C_2 are of dimension 1, we have $r, s \ge n-1$. Note that $\dim_k(\ker(J_1)) = 1 = \dim_k(\ker(J_2))$ implies $T_{C_1,p} = \mathrm{Span}_k\{v_1\}$ and $T_{C_2,p} = \mathrm{Span}_k\{v_2\}$ by Lemma 3.4.8 (these vector spaces are viewed as subspaces of the tangent space of $\mathbb{A}_n$ at p). Due to v_1 and v_2 being linearly independent, we have the dual spaces of $T_{C_1,p}$ and $T_{C_2,p}$, namely $\mathfrak{m}_1/\mathfrak{m}_1^2$ and $\mathfrak{m}_2/\mathfrak{m}_2^2$ are also being spanned by linearly independent variables. Therefore we may make another affine change of coordinates so that

$$\begin{aligned} f_1 &= x_2 + \text{h.o.t.} \\ f_2 &= x_3 + \text{h.o.t.} \\ &\vdots \\ f_{n-1} &= x_n + \text{h.o.t.} \end{aligned}$$

and $f_n, f_{n+1}, ..., f_r$ are all of the form $\mathrm{Span}_k\{x_2, ..., x_n\} + \text{h.o.t.}$ as well as

$$\begin{aligned} g_1 &= x_1 + \text{h.o.t.} \\ g_2 &= x_3 + \text{h.o.t.} \\ &\vdots \\ g_{n-1} &= x_n + \text{h.o.t.} \end{aligned}$$

and $g_n, g_{n+1}, ..., g_s$ are all of the form $\mathrm{Span}_k\{x_1, x_3, ..., x_n\} + \text{h.o.t.}$ where h.o.t. is the abbreviation for “higher order terms”. This gives $\mathfrak{m}_1/\mathfrak{m}_1^2 = (x_1)$ and $\mathfrak{m}_2/\mathfrak{m}_2^2 = (x_2)$ by computing the kernel of the Jacobian matrices after applying our change of coordinates. Next, since $C_1 \cap C_2 = \{p\}$, we have $I_1 + I_2 = (x_1, ..., x_n) \subset R$ and so $R/(I_1 + I_2) = k$. Finally, in order to compute $\mathcal{O}_{C,p} = (R/(I_1 \cap I_2))_{\mathfrak{m}}$, we will first consider the short exact sequence

$$0 \to R/(I_1 \cap I_2) \to (R/I_1) \oplus (R/I_2) \to R/(I_1 + I_2) \to 0.$$

where $\mathfrak{m} = (x_1, ..., x_n) \subseteq R/(I_1 \cap I_2)$ and the third map $(R/I_1) \oplus (R/I_2) \to R/(I_1 + I_2)$ is given by the difference map $(r + I_1, s + I_2) \mapsto (r - s) + (I_1 + I_2)$.

By localizing at the maximal ideals of p for every term, we get

$$0 \to (R/(I_1 \cap I_2))_{\mathfrak{m}} \to (R/I_1)_{\mathfrak{m}_1} \oplus (R/I_2)_{\mathfrak{m}_2} \to k \to 0.$$

Since $\mathfrak{m}_1/\mathfrak{m}_1^2 = (x_1)$ and $\mathfrak{m}_2/\mathfrak{m}_2^2 = (x_2)$ we have $(R/I_1)_{\mathfrak{m}_1} \cong k[x_1]$ and $(R/I_2)_{\mathfrak{m}_2} \cong k[x_2]$. Thus the sequence becomes

$$0 \to \mathcal{O}_{C,p} \to k[x_1] \oplus k[x_2] \to k \to 0$$

where the third map is given by

$$\begin{aligned} k[x_1] \oplus k[x_2] &\longrightarrow k \\ (u(x_1), v(x_2)) &\longmapsto u(0) - v(0) \end{aligned}$$

Consider the natural injection

$$\begin{aligned} k[x_1, x_2]/(x_1x_2) &\longrightarrow k[x_1] \oplus k[x_2] \\ q(x_1, x_2) &\longmapsto (q(x_1, 0), q(0, x_2)) \end{aligned}$$

The image of this map is $\{(u(x_1), v(x_2)) : u(0) = v(0)\} = \ker(k[x_1] \oplus k[x_2] \to k)$. By the first isomorphism theorem, we must have $\mathcal{O}_{C,p} \cong k[x_1, x_2]/(x_1x_2)$ and so

$$\widehat{\mathcal{O}_{C,p}} \cong k[\![x_1, x_2]\!]/(x_1x_2).$$

Therefore by definition, p is a nodal singularity of C. $\square$

We can finally define what it means for a model of a curve to be semistable.

Definition 3.4.16 (Geometric Reducedness)**.** Let X be a scheme defined over a field K. We say X is *geometrically reduced* if $X_{\overline{K}} = X \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{K}$ is reduced.

Definition 3.4.17 (Semistable Models)**.** Keeping the notations in Definition 3.4.12, we say a model $\mathcal{C}/R$ of C/K is *semistable* if its special fiber $\mathcal{C}_s$ is geometrically reduced and is at-worst-nodal. When such a model exists, we say C/K is *semistable* or has *semistable reduction.*

Definition 3.4.18 (Potential Semistable Reduction)**.** Keeping the above notations, we say C/K has *potential semistable reduction* if there exist a finite extension $L \supseteq K$ such that $C_L = C \times_{\operatorname{Spec} K} \operatorname{Spec} L \to \operatorname{Spec} L$ has semistable reduction.

3.5 Blow-ups

Resolving singularities is a big theme in algebraic geometry. The blow-up construction is a way of turning a scheme/variety with worse-than-nodal singularities into nodal singularities. One might already sense that this construction will be useful for us since our desired semistable models are required to have nodal singularities after taking the special fiber. In this section, we will introduce the blow-up construction and demonstrate how will it benefit us for finding semistable models in the next chapter.

3.5.1 Blowing Up Affine Varieties

We will begin by defining blow-ups of affine spaces at their origins over a field K.

Let $O = (0, ..., 0)$ be the origin of $\mathbb{A}^n$. Consider the product $\mathbb{A}^n \times \mathbb{P}^{n-1} = \mathbb{A}^n \times_{\operatorname{Spec} K} \mathbb{P}^{n-1}$, which is a quasi-projective variety i.e., a open subset of the projective variety $\mathbb{P}^n \times \mathbb{P}^{n-1}$ (cf. [8] Exercise I.2.14). Let $x_1, ..., x_n$ be the affine coordinates of $\mathbb{A}^n$, and $y_1, ..., y_n$ be the homogeneous coordinates of $\mathbb{P}^{n-1}$. The closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ are defined by polynomials in x_i and y_j, which are homogeneous with respect to the y_j's.

We now define the *blow-up of* $\mathbb{A}^n$ *at the point* O to be the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i : i, j = 1, ..., n\}$. We obtain the commutative diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{A}^n \times \mathbb{P}^{n-1} \\ & \overset{\phi}{\searrow} & \downarrow \pi \\ & & \mathbb{A}^n \end{array}$$

The map $\pi : \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ is the natural projection and ϕ is the restriction of π at $X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$. We will make the following observations for X:

(i) Let $O \neq P \in \mathbb{A}^n$ be a point, and write $P = (a_1, ..., a_n)$ with $a_i \neq 0$ for some i. We take any $P \times [y_1 : \cdots : y_n] \in \phi^{-1}(P) \subset X$. Then for each j, we have $y_j = (a_j/a_i)y_i$. Hence $[y_1 : \cdots : y_n]$ is uniquely determined as a point in $\mathbb{P}^{n-1}$. We see that for each non-zero $P \in \mathbb{A}^n$, $\phi^{-1}(P)$ consists of a single point. In fact, ϕ gives an isomorphism of $X - \phi^{-1}(O)$ onto $\mathbb{A}^n - O$. The morphism $\psi : (\mathbb{A}^n - O) \to (X - \phi^{-1}(O))$ given by

$$\psi(P) := (a_1, ..., a_n) \times [a_1 : \cdots : a_n]$$

defines an inverse to ϕ.

(ii) $\phi^{-1}(O) \cong \mathbb{P}^{n-1}$ since $\phi^{-1}(O)$ consists of all points $O \times Q$ with $Q = [y_1 : \cdots : y_n] \in \mathbb{P}^{n-1}$, subject to no restriction.

(iii) A line L through O in $\mathbb{A}^n$ can be given by the parametric equations $x_i = a_i t$ for $i = 1, ..., n$, where $a_i \in K$ are not all zero, and $t \in \mathbb{A}^1$. Now consider the line $L' =$

$\phi^{-1}(L-O)$ in $X-\phi^{-1}(O)$. It is given parametrically by

$$\begin{cases} x_i = a_i t \\ y_i = a_i t \end{cases}$$

with $t \in \mathbb{A}^1 - 0$. But the y_i are homogeneous coordinates in $\mathbb{P}^{n-1}$, so we can equally well describe L' by the equations

$$\begin{cases} x_i = a_i t \\ y_i = a_i \end{cases}$$

for $t \in \mathbb{A}^1 - 0$. Note that these new equations for L' make sense also for $t = 0$, and give the closure $\bar{L}'$ of L' in X. Now $\bar{L}'$ meets $\phi^{-1}(O)$ in the point $Q = [a_1, ..., a_n] \in \mathbb{P}^{n-1}$, so we see that sending L to Q gives a 1-1 correspondence between lines through O in $\mathbb{A}^n$ and points in $\phi^{-1}(O)$.

(iv) X is irreducible. Indeed, X is the union of $X - \phi^{-1}(O)$ and $\phi^{-1}(O)$. The first piece is isomorphic to $\mathbb{A}^n - O$, hence irreducible. On the other hand, we have just seen that every point of $\phi^{-1}(O)$ is in the closure of some subset (the line L') of $X - \phi^{-1}(O)$. Hence $X - \phi^{-1}(O)$ is dense in X, and X is irreducible.

We can now give the definition of the blow-up of an affine variety at one of its point.

Definition 3.5.1 (Blowing Up At A Point)**.** If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, we define the *blow-up of* Y *at the point* O to be $\tilde{Y} = \mathrm{cl}_X(\phi^{-1}(Y - O))$, where $\phi : X \to \mathbb{A}^n$ is the blow-up of $\mathbb{A}^n$ at the point O described above. Abusing of notations, by writing $\phi : \tilde{Y} \to Y$, we mean the restriction of $\phi : X \to \mathbb{A}^n$ to $\tilde{Y}$. To blow up any other point P of Y, we just make a linear change of coordinates sending P to O.

Note that ϕ induces an isomorphism of $\tilde{Y} - \phi^{-1}(O)$ to $Y - O$, so that ϕ is birational morphism of $\tilde{Y}$ to Y. It seems like this definition apparently depends on the embedding of Y in $\mathbb{A}^n$, but in fact, the construction of blowing up is intrinsic.

We will now give an example to demonstrate how blow-up constructions resolve singularities.

Example 3. Let Y be the plane cubic curve given in Example 2. We will blow up Y at the nodal singularity $O \in Y$. Let t, u be homogeneous coordinates for $\mathbb{P}^1$. Following the above construction, the blow-up of $\mathbb{A}^2$ at O denoted as X, is defined by the equation $xu = ty$ inside $\mathbb{A}^2 \times \mathbb{P}^1$. It is isomorphic to $\mathbb{A}^2$ everywhere except that the origin O, where O has been replaced by a copy of $\mathbb{P}^1$ corresponding to the slopes of lines through O in $\mathbb{A}^2$. We will call this $\phi^{-1}(O) = \mathbb{P}^1$ the *exceptional divisor*, and denote it as E.

We obtain the inverse image $\phi^{-1}(Y)$ in X by considering the equations $y^2 = x^2(x+1)$ and $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1$. Now $\mathbb{A}^2 \times \mathbb{P}^1$ is covered by the open charts

$$U_t = \{(p, [1 : u]) : p \in \mathbb{A}^2, u \in k\}$$

and

$$U_u = \{(p, [t : 1]) : p \in \mathbb{A}^2, t \in k\}$$

which we will consider separately.

Remark 18. The more natural way of writing out the chart is

$$U_t = \{(p, [t : u]) : p \in \mathbb{A}^2, t \in k^\times, u \in k\}.$$

However we recall that over a projective space, we are free to scale our projective coordinates by units over our base ring (nonzero elements in $k^\times$ in this case). Hence the point $[t : u]$ is equal to $[1 : u/t]$ in $\mathbb{P}^1$ for $t \in k^\times$. Since for any fixed $t_0 \in k^\times$, the map $u \mapsto u/t_0$ gives a bijection between k and itself, we are allowed to just treat $t = 1$ on the chart U_t. The same can be said for U_u as well.

On the first chart $t \neq 0$, we set $t = 1$, and use u as an affine parameter (recalling that $\{[1 : u] : u \in k\} \cong \mathbb{A}^1$). We then have the equations

$$\begin{cases} y^2 = x^2(x+1) \\ y = xu \end{cases}$$

in $\mathbb{A}^3$ with coordinates x, y, u. Substituting the second equation into the first, we get

$$x^2u^2 - x^2(x+1) = 0$$

which factors as

$$x^2(u^2 - x - 1) = 0.$$

Thus we obtain two irreducible components, one defined by $x = 0$, $y = 0$, u arbitrary, which is E, and the other defined by $u^2 = x + 1$, $y = xu$. This is $\tilde{Y}$ since $\phi^{-1}(Y - O)$ lies in it and it is also an irreducible component. Note that $\tilde{Y}$ meets E at the points $u = \pm 1$. These points correspond to the slope of the two branches of Y at O.

To see more clearly on why the equations $u^2 = x+1$ and $y = xu$ defines $\tilde{Y}$, we may move to the other chart $u \neq 0$. We can then assume that $u = 1$ and the equations $y^2 = x^2(x+1)$ and $xu = ty$ becomes

$$\begin{cases} y^2 = x^2(x+1) \\ x = ty. \end{cases}$$

Substituting again, we get

$$y^2 - t^2y^2(x+1) = 0$$

which factors as

$$y^2(1 - t^2(x+1)) = 0.$$

This again gives us two irreducible components $x = y = 0$, t arbitrary and

$$1 = \left(\frac{x}{y}\right)^2 (x+1) = t^2(x+1), \qquad x = ty.$$

The first component is the exceptional divisor E on the patch $\{u \neq 0\}$. The second component has to be part of $\tilde{Y} = \text{cl}_X(\phi^{-1}(Y - O))$ since again, this component and $\tilde{Y}$ are both irreducible. Note that in the second component, there is no point with $t = 0$, hence all of its points are already in the variety defined by $u^2 = x + 1$ and $y = xu$ in $\mathbb{A}^3$ presented on the other chart $t \neq 0$. Finally, we see that $\tilde{Y}$ intersects E at the points $x = y = 0$ and $[t : u] = [1 : 1]$ or $[1 : -1]$. Using the fact that points in $\phi^{-1}(O)$ are in 1-1 correspondence with lines through O in $\mathbb{A}^2$, one may check that the points $\{(O, [1 : \pm 1])\}$ corresponds to the lines $y = \pm x$ whose slopes are ± 1. We then note that the lines $y = \pm x$ are the tangents of the two branches of Y at O.

The effect of blowing up is thus to separate out branches of Y passing through O according to their slopes. If the slopes are different, the corresponding branches in $\tilde{Y}$ no longer meet in X. Instead, they meet E at points corresponding to the different slopes, thus resolving the original singularity at O.

Our next step is to define blow-ups along closed subvarieties which are more than just a single point.

Let $X \subset \mathbb{A}^n$ be any affine variety and $Y \subset X$ any closed subvariety. We may choose a set of generators $f_1, ..., f_r \in K(X)$ for the ideal of Y in X and set $U = X \setminus Y = X \setminus V(f_1, ..., f_r)$. Consider the well-defined morphism

$$\begin{aligned} \phi : U &\longrightarrow \mathbb{P}^{r-1} \\ x &\longmapsto \phi(x) = [f_1(x) : \cdots : f_r(x)] \end{aligned}$$

obtained by composition with the quotient morphism $\mathbb{A}^r - O \twoheadrightarrow \mathbb{P}^{r-1}$.

Definition 3.5.2 (Blowing Up Subvarieties)**.** With the above setup, let

$$\Gamma_\phi = \{(x, \phi(x)) : x \in U\} \subset X \times \mathbb{P}^{r-1}$$

be the graph of ϕ. We define the *blow-up of X along Y*, denoted $\text{Bl}_Y(X)$, to be the closure of Γ_ϕ inside $X \times \mathbb{P}^{r-1}$, together with the natural projection $\varphi : \text{Bl}_Y(X) \to X$ onto the first coordinate called the *blow-up morphism*. We call Y the *center* of the blow-up and $E =$

$\varphi^{-1}(Y)$ the *exceptional divisor*. If $Z \subset X$ is another subvariety, then $\mathrm{Bl}_{Z\cap Y}(Z) \subset \mathrm{Bl}_Y(X)$ is called the *strict transform* of Z in the blow-up of X along Y.

Remark 19. If $X \subset \mathbb{P}^n$ is a projective variety and $Y \subset X$ is any closed subvariety, we can similarly define the blow-up of X along Y by taking a collection $F_1, ..., F_r$ of homogeneous polynomials of the same degree generating an ideal with saturation $I(Y)$ and letting $\mathrm{Bl}_Y(X)$ be the closure of the graph of the rational map $\phi : X - Y \to \mathbb{P}^{r-1}$ given by

$$x \mapsto [F_0(x) : \cdots : F_r(x)]$$

in $X \times \mathbb{P}^{r-1}$.

Similar to the one point case, we have an induced isomorphism $U \cong \Gamma_\phi$ (for blowing up a point we had $X - O \cong \phi^{-1}(X - O)$.) Since X is irreducible and $Y \neq X$, we have that U is a non-empty open subset of an irreducible space, and so U is also irreducible. Thus Γ_ϕ is irreducible, and so is $\mathrm{Bl}_Y(X)$. Therefore φ is a birational morphism, because it induces an isomorphism on a non-empty (dense) open subset.

We may check that the construction is well-defined, meaning that it is independent of the choice of generators for the ideal of Y in X.

Proposition 3.5.3 ([10] Prop. 1)**.** *If we choose a different set of generators for the ideal $I_X(Y) \subset K(X)$, we obtain a variety $\mathrm{Bl}_Y(X)'$ which is isomorphic to $\mathrm{Bl}_Y(X)$ through an isomorphism ψ making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Bl}_Y(X) & \xrightarrow{\ \psi\ } & \mathrm{Bl}_Y(X)' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & X & \end{array}$$

Proof. Let $f'_1, ..., f'_s \in I_X(Y)$ be another set of generators. Since both sets generate the ideal, we can find $g_{11}, ..., g_{rs} \in K(X)$ and $h_{11}, ..., h_{sr} \in K(X)$ such that

$$f_i = \sum_{j=1}^{s} g_{ij} f'_j \quad \text{in } K(X) \text{ for each } i = 1, ..., r, \text{ and} \tag{3.5.1}$$

$$f'_j = \sum_{k=1}^{r} h_{jk} f_k \quad \text{in } K(X) \text{ for each } j = 1, ..., s. \tag{3.5.2}$$

Now define

$$\begin{aligned} \psi : \mathrm{Bl}_Y(X) &\longrightarrow \mathrm{Bl}_Y(X)' \\ (x, y) = (x, y_1 : \cdots : y_r) &\longmapsto (x, y') = \left(x, \sum_{k=1}^{r} h_{1k}(x) y_k : \cdots : \sum_{k=1}^{r} h_{sk}(x) y_k \right). \end{aligned}$$

We check that ψ is well-defined. Let $(x, y) \in \Gamma_\phi$. Since

$$y = [y_1 : \cdots : y_r] = [f_1(x) : \cdots : f_r(x)] \in \mathbb{P}^{r-1},$$

we can find $0 \neq \lambda \in K$ such that $y_i = \lambda f_i(x)$ for all $i = 1, ..., r$, one of them at least being non-zero. Plugging equations (4.1.69) into (3.5.1), we obtain the new equations

$$f_i(x) = \sum_{j=1}^{s} g_{ij}(x) \left(\sum_{k=1}^{r} h_{jk}(x) f_k(x) \right)$$

for each $i = 1, ..., r$. We can multiply the previous equation by λ and obtain the relations

$$y_i = \sum_{j=1}^{s} g_{ij}(x) \left(\sum_{k=1}^{r} h_{jk}(x) y_k \right) = \sum_{j=1}^{s} g_{ij}(x) y'_j.$$

So if $y' = 0$, then $y = 0$, which is a contradiction with $y \in \mathbb{P}^{r-1}$. The above relation remains valid in the closure, and so the same holds for any $(x, y) \in \mathrm{Bl}_Y(X)$. Moreover, by construction we have $\psi(x, y) \in \mathrm{Bl}_Y(X)'$ for all $(x, y) \in \Gamma_\phi$, and so for all $(x, y) \in \mathrm{cl}_{X \times \mathbb{P}^{r-1}}(\Gamma_\phi) = \mathrm{Bl}_Y(X)$. Hence ψ is well-defined. To check that it is an isomorphism, we construct ψ^{-1} in the exact same way (changing the roles of the set of generators). The commutativity of the diagram follows in the exact same fashion. □

Remark 20. Using the notations from Definition 3.5.2 and the definition of the graph, it follows immediately that the points

$$(x, y) = (x, [y_1 : \cdots : y_r]) = (x, \phi(x)) = (x, [f_1(x) : \cdots : f_r(x)]) \in \Gamma_\phi$$

satisfy $y = \phi(x)$ in $\mathbb{P}^{r-1}$ i.e. $y_i = \lambda f_i(x)$ for some $\lambda \in k^\times$ for all i. Therefore we get the equations

$$y_i f_j(x) = y_j f_i(x) \tag{3.5.3}$$

for all i, j and $(x, y) \in \Gamma_\phi$. Since the equations (3.5.3) also hold for points in the closure $\mathrm{Bl}_Y(X)$ of Γ_ϕ, we get the inclusion

$$\mathrm{Bl}_Y(X) \subset \{(x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } i, j = 1, ..., r\}. \tag{3.5.4}$$

We will discuss this more formally later as (3.5.4) allows us to write down explicit equations for computing the blow-up of arbitrary affine varieties along their affine subvarieties.

Example 4. In particular, if we take X to be the blow-up of the affine space at the origin using Definition 3.5.1, it follows that the blow-up $\mathrm{Bl}_O(\mathbb{A}^n)$ we get from Definition 3.5.2 satisfies $\mathrm{Bl}_O(\mathbb{A}^n) \subset X$ where $O = Y = V(x_1, ..., x_n)$ by Remark 20. It is in fact a closed subset since they are both closed in the product $\mathbb{A}^n \times \mathbb{P}^{n-1}$. We have that X and $\mathrm{Bl}_O(\mathbb{A}^n)$ are both irreducible and they share the same non-empty open subset U, so they are birationally equivalent. By irreducibility of X, this implies that $X = \mathrm{Bl}_O(\mathbb{A}^n)$. Otherwise, we could write $X = \mathrm{Bl}_O(\mathbb{A}^n) \cup (X - U)$ which is a union of two proper closed subsets. Therefore, Definition 3.5.2 generalizes the blowing up at a single point case in a very natural way.

The following proposition states that blowing up varieties is a local construction.

Proposition 3.5.4 ([10] Theorem 1)**.** *Let X be an arbitrary variety, $Y \subseteq X$ is a subvariety, $U \subset X$ is a non-empty open subset and $\varphi : \mathrm{Bl}_Y(X) \to X$ the blow-up morphism. Then $\varphi^{-1}(U) \subseteq \mathrm{Bl}_Y(X)$ is the blow-up of U along $Y \cap U$.*

Proof. By irreducibility of Y, $U \cap Y$ is dense in Y. We know that if a polynomial vanishes in a set of points, then it also vanishes in the closure of the set of points. This gives us the inclusion $I_X(U \cap Y) \subseteq I_X(Y)$. Conversely, since $U \cap Y \subseteq Y$, we also get the direction $I_X(Y) \subseteq I_X(U \cap Y)$ which shows that they are actually equal.

Let $f_1, ..., f_r \in K(X)$ be a set of generators of $I_X(Y) = I_X(U \cap Y)$. Let us consider the morphism $\phi : X \setminus Y \to \mathbb{P}^{r-1}$ used to construct the blowing up of X along Y and $\phi_U : U \setminus (U \cap Y) \to \mathbb{P}^{r-1}$ used to construct the blowing up of U along $Y \cap U$. The equality $I_X(Y) = I_X(U \cap Y)$ implies ϕ_U is just the restriction of ϕ to the open subset $U \setminus (U \cap Y)$.

Therefore, $\Gamma_{\phi_U} = \Gamma_\phi \cap (U \times \mathbb{P}^{r-1})$. But $\varphi^{-1}(U)$ is precisely the set of points $(a, b) \in \mathrm{cl}_{X \times \mathbb{P}^{r-1}}(\Gamma_\phi)$ such that $a \in U$. Hence we get

$$\begin{aligned}\varphi^{-1}(U) &= \mathrm{cl}_{X \times \mathbb{P}^{r-1}}(\Gamma_\phi) \cap (U \times \mathbb{P}^{r-1}) \\ &= \mathrm{cl}_{U \times \mathbb{P}^{r-1}}(\Gamma_\phi \cap (U \times \mathbb{P}^{r-1})) \\ &= \mathrm{cl}_{U \times \mathbb{P}^{r-1}}(\Gamma_{\phi_U}) \\ &= \mathrm{Bl}_{Y \cap U}(U).\end{aligned}$$

□

3.5.2 General Definition of Blow-up

There is more than one method of computing blow-ups of schemes along their closed subschemes and each method has its own advantages/disadvantages. In the affine case, Definition 3.5.2 is good for explicit computations and it will be our main tool for computing blow-ups when constructing our semistable models. For general schemes, blow-ups are usually computed via global Proj constructions which we will define later in Section 3.5.3. The global Proj construction method is good for proving theoretical results such as "blow-up maps are proper" but it is more abstract and harder to work with concretely.

In this section, we will give a universal blow-up definition following [6] which unifies every method of computing blow-ups. We start with a definition.

Definition 3.5.5 (Cartier Subschemes)**.** Let X be any scheme, $Y \subseteq X$ a subscheme. We say that Y is a *Cartier subscheme* in X if for all $p \in X$ there exist an affine open neighborhood $U = \operatorname{Spec} A$ of p in X such that $Y \cap U = V(f) \subset U$ for some non-zero function $f \in A$. More generally, we say that Y is a *regular subscheme* if it is locally the zero locus of a regular sequence of functions on X.

We can now give the universal blow-up definition for schemes.

Definition 3.5.6 (General Definition of Blow-ups)**.** Let X be a scheme, $Y \subseteq X$ a subscheme. The *Blow-up of X along Y*, denoted $\phi : \mathrm{Bl}_Y(X) \to X$, is the morphism to X characterized by the properties:

1. The inverse image $\phi^{-1}(Y)$ is a Cartier subscheme in $\mathrm{Bl}_Y(X)$.
2. ϕ is universal with respect to this property; that is; if $\psi : Z \to X$ is any morphism such that $\psi^{-1}(Y)$ is a Cartier subscheme in Z, there exist a unique morphism $f : Z \to \mathrm{Bl}_Y(X)$ such that $\psi = \phi \circ f$.

As before, the inverse image $E = \phi^{-1}(Y)$ of Y in $\mathrm{Bl}_Y(X)$ is called the *exceptional divisor* of the blow-up, and Y the *center* of the blow-up.

With this general definition, it is unclear that if the blow-up defined this way actually exists. The following proposition says that blow-up of varieties following Definition 3.5.2 actually agrees with Definition 3.5.6.

Proposition 3.5.7 ([6] Prop IV-18.)**.** *Let A be any commutative unital ring. Take $X = \operatorname{Spec} A$ and let*

$$Y = V(f_1, ..., f_m) \subseteq X$$

be a closed subscheme. The blow-up of X along Y is the closure in $X \times_A \mathbb{P}_A^{m-1} = \mathbb{P}_A^{m-1}$ of the graph of the morphism

$$\alpha_{(f_1,...,f_m)} : X \setminus Y \longrightarrow \mathbb{P}_A^{m-1}.$$

Remark 21. The morphism $\alpha_{(f_1,...,f_m)}$ is obtained by restricting the inclusion

$$X \hookrightarrow \operatorname{Spec} A[x_1, ..., x_m]$$

given by the ring homomorphism

$$\begin{aligned} A[x_1, ..., x_m] &\longrightarrow A \\ x_i &\longmapsto f_i \end{aligned}$$

onto the open subset $U = X \setminus V(f_1, ..., f_m)$. The proposition reduces to Definition 3.5.2 when A is a finitely generated, reduced and integral K-algebra. We notice that $\mathbb{P}_A^{m-1} = X \times_K \mathbb{P}_K^{m-1}$ in that case.

The following lemma states that blow-ups are preserved under 'flat' base changes. This will come handy for us when we are trying to obtain new models for our curves over a DVR via blow-up.

Lemma 3.5.8 ([13] Lemma 70.13.3)**.** *Let S be a scheme. Let $X_1 \to X_2$ be a flat morphism of schemes over S. Let $Y_2 \subseteq X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Then there exist a commutative diagram*

$$\begin{array}{ccc} \mathrm{Bl}_{Y_1}(X_1) \cong \mathrm{Bl}_{Y_2}(X_2) \times_{X_2} X_1 & \longrightarrow & \mathrm{Bl}_{Y_2}(X_2) \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of schemes over S.

As we have seen previously in Section 3.5.1, the blowing up along a closed subscheme leaves the complement of the subscheme unchanged. The following lemma states this more precisely.

Lemma 3.5.9 ([13] Lemma 31.32.4)**.** *Let X be a scheme and $Y \subseteq X$ a closed subscheme of X. Let $\phi : \mathrm{Bl}_Y(X) \to X$ be the blow-up morphism. Then the restriction map*

$$\phi\big|_{\phi^{-1}(X \setminus Y)} : \phi^{-1}(X \setminus Y) \to X \setminus Y$$

is an isomorphism.

We also get a generalization of blowing up being a local construction similar to Proposition 3.5.4.

Proposition 3.5.10 ([6] Prop. IV-21)**.** *Let X be any scheme, $Y \subset X$ a subscheme and $\phi : \mathrm{Bl}_Y(X) \to X$ the blow-up of X along Y. Let $\alpha : X' \to X$ be any morphism and set $Y' := \alpha^{-1}(Y) \subseteq X'$. Let W be the closure of $\pi_1^{-1}(X' \setminus Y')$ in $X' \times_X \mathrm{Bl}_Y(X)$ where π_1 is the natural projection onto the first coordinates, then $\pi_1\big|_W : W \to X'$ is the blow-up of X' along Y'.*

Remark 22. As we will see later, this proposition is often used for the case where X' is a open subscheme of X. In this case,

$$\phi^{-1}(X') \cong \mathrm{Bl}_{X' \cap Y}(X')$$

by Lemma 3.5.8 since the open immersion $X' \hookrightarrow X$ is flat.

Similar to Definition 3.5.2, we also have that blow-ups preserves irreducibility for affine schemes by a similar reasoning.

Lemma 3.5.11 ([10] Lemma 3)**.** *If $X = \operatorname{Spec} A$ is an affine scheme and $Y = \operatorname{Spec}(A/I)$ is a closed subscheme of X, then the open subscheme $\mathrm{Bl}_Y(X) - \phi^{-1}(Y)$ of the blow-up $\mathrm{Bl}_Y(X)$ is dense.*

This lemma tells us that blow-ups of affine schemes preserve irreducibility: every non-empty open subset of an irreducible space is irreducible. By Lemma 3.5.9, the blow-up induces an isomorphism outside of the center, so the complement of the exceptional divisor $\mathrm{Bl}_Y(X) - \phi^{-1}(Y)$ is also irreducible. By Lemma 3.5.11, it is dense in the blow-up. Since irreducibility is preserved by taking closures, we conclude that

$$\mathrm{cl}_{\mathrm{Bl}_Y(X)}(\mathrm{Bl}_Y(X) - \phi^{-1}(Y)) = \mathrm{Bl}_Y(X)$$

is irreducible.

3.5.3 Blow-ups via Proj Constructions

We shall now give the global Proj definition of blow-ups which in turn allows us to show that the blow-up maps we consider, are proper. In the first part of this section, we will recall a few definitions which are all borrowed from [8].

Definition 3.5.12 (Sheaf Associated To A Module)**.** Let A be a ring and let M be an A-module. We define the *sheaf associated to* M on $\operatorname{Spec} A$, denoted by $\tilde{M}$, as follows. For each prime ideal $\mathfrak{p} \subset A$, let $M_{\mathfrak{p}}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_{\mathfrak{p}}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_{\mathfrak{q}}$. We make $\tilde{M}$ into a sheaf by using the obvious restriction maps.

Definition 3.5.13 (Quasi-coherent Sheaves)**.** Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$, such that for each i there is an A_i-module M_i with $\mathcal{F}\big|_{U_i} \cong \tilde{M}_i$. We say that $\mathcal{F}$ is *coherent* if furthermore each M_i can be taken to be a finitely generated A_i-module.

We will later see that the global Proj construction on graded coherent sheaves naturally gives us proper morphisms. The following proposition makes sure that the sheaves we encounter for defining blow-ups are all coherent.

Proposition 3.5.14 ([8] Prop. II.5.9)**.** *Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathcal{I}_Y$ is a quasi-coherent sheaf of ideals on X. If X is Noetherian, it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

We will now define the global Proj construction of a sheaf of graded algebras $\mathcal{F}$ over a scheme X. For simplicity, we assume X is a Noetherian scheme, $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, which has a structure of a sheaf of graded $\mathcal{O}_X$-algebras. Thus $\mathcal{F} \cong \bigoplus_{d \geq 0} \mathcal{F}_d$, where $\mathcal{F}_d$ is the homogeneous part of degree d. We assume furthermore that $\mathcal{F}_0 = \mathcal{O}_X$, that $\mathcal{F}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathcal{F}$ is locally generated by $\mathcal{F}_1$ as an $\mathcal{O}_X$-algebra (it follows that $\mathcal{F}_d$ is coherent for all $d \geq 0$.)

Definition 3.5.15 (Global Proj)**.** Let X be a scheme and $\mathcal{F}$ a sheaf of graded $\mathcal{O}_X$-algebras satisfying the above conditions. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\mathcal{F}(U)$ be the graded A-algebra $\Gamma(U, \mathcal{F}|_U)$. Then we consider $\operatorname{Proj} \mathcal{F}(U)$ and its natural morphism $\pi_U : \operatorname{Proj} \mathcal{F}(U) \to U$. If $f \in A$, and $U_f = \operatorname{Spec} A_f$, then using the fact that $\mathcal{F}$ is quasi-coherent, we see that $\operatorname{Proj} \mathcal{F}(U_f) \cong \pi_U^{-1}(U_f)$. It follows that if U, V are two open affine subsets of X, then $\pi_U^{-1}(U \cap V)$ is naturally isomorphic to $\pi_V^{-1}(U \cap V)$. These isomorphisms allow us to glue the schemes $\operatorname{Proj} \mathcal{F}(U)$ together. Thus we obtain a scheme $\mathbf{Proj}\, \mathcal{F}$ together with a morphism $\pi : \mathbf{Proj}\, \mathcal{F} \to X$ such that for each open affine $U \subseteq X$, $\pi^{-1}(U) \cong \operatorname{Proj} \mathcal{F}(U)$. Furthermore the invertible sheaves $\mathcal{O}(1)$ on each $\operatorname{Proj} \mathcal{F}(U)$ are compatible under this construction, so they glue together to give an invertible sheaf $\mathcal{O}(1)$ on $\mathbf{Proj}\, \mathcal{F}$, canonically determined by this construction.

Definition 3.5.16 (Description of Blow-ups Using Proj)**.** Let X be a Noetherian scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Consider the sheaf of graded algebras $\mathcal{F} = \bigoplus_{d \geq 0} \mathcal{I}^d$, where $\mathcal{I}^d$ is the d-th power of the ideal $\mathcal{I}$, and we set $\mathcal{I}^0 = \mathcal{O}_X$. Then $X, \mathcal{F}$ satisfies our condition to define a **Proj** construction. We define $\tilde{X} = \mathbf{Proj}\, \mathcal{F}$ to be the *blowing-up* of X *with respect to the coherent sheaf of ideals* $\mathcal{I}$. If Y is a closed subscheme of X corresponding to $\mathcal{I}$, then we also call $\tilde{X}$ the *blowing-up* of X along Y, *or with center* Y.

The following theorem guarantees that the above definition is indeed a blow-up satisfying Definition 3.5.6.

Theorem 3.5.17 ([6] Theorem IV-23)**.** *Let X be a scheme and $Y \subset X$ a closed subscheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Y. If $\mathcal{F}$ is the sheaf of graded $\mathcal{O}_X$-algebras*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{I}^n = \mathcal{O}_X \oplus \mathcal{I}^1 \oplus \mathcal{I}^2 \oplus \cdots ,$$

then the scheme $\mathbf{Proj}\, \mathcal{F} \to X$ is the blow-up of X along Y.

Finally, the following Proposition in Hartshorne says that the blow-up maps we are interested in are proper since our schemes are always going to be Noetherian.

Proposition 3.5.18 ([8] Prop. II.7.10)**.** *Let $X, \mathcal{F}$ satisfy our conditions in Definition 3.5.16. Let $\pi : \mathbf{Proj}\, \mathcal{F} \to X$ be the natural projection morphism. Then π is a proper morphism. In particular, it is separated and of finite type.*

With Proposition 3.5.18 we have that the blow-up morphism $\tilde{X} \to X$ is proper where $\tilde{X}$ is the blowing-up of X along a closed subscheme Y.

Remark 23. In the case where $X = \operatorname{Spec} R$ is affine, we have $\mathbf{Proj}\,\mathcal{F} \cong \operatorname{Proj} A$ where A is the *Rees algebra*

$$A = R \oplus I \oplus I^2 \oplus \cdots$$

and I is the ideal of $Y \subseteq X$.

We have promised in Remark 20 that we will describe the explicit equations for blowing up varieties. To do this, we will need to recall the definition of regular sequences in a ring.

Definition 3.5.19 (Regular Sequence)**.** Let R be a ring and let $f_1, ..., f_r \in R$ be elements of R. We say $f_1, ..., f_r$ forms a regular sequence of R if $(f_1, ..., f_r) \subseteq R$ is a proper ideal of R and for $i = 1, ..., r$, f_i is not a zero divisor of $R/(f_1, ..., f_{i-1})$.

Lemma 3.5.20. *Let $X = \operatorname{Spec} R$ be a affine scheme and $Y \subseteq X$ a closed subscheme with defining ideal $I \subseteq R$. If $I = (f_1, ..., f_r) \subset R$ is generated by a regular sequence $f_1, ..., f_r$ then*

$$\operatorname{Bl}_Y(X) = V(y_i f_j - y_j f_i) \subseteq X \times \mathbb{P}^{r-1}$$

where $y_1, ..., y_r$ are the homogeneous coordinates of $\mathbb{P}^{r-1}$.

Proof. See [6, Exercise IV-26.]. □

3.5.4 Blowing Up Arithmetic Schemes

In this section, we will specialize to the case where we are blowing up arithmetic schemes. We define *arithmetic schemes* to be schemes that are separated and of finite type over Dedekind domains. We are in particular interested in arithmetic schemes defined over DVRs.

Our idea of blowing up an arithmetic scheme defined over a DVR is to have its generic fiber unchanged after the blow-up. By Proposition 3.2.8, this can be done by blowing up at subschemes that are completely contained in the special fiber since we know that blow-ups only affects the subscheme that we are blowing up at. In the situation for curves, we can produce new models for them via this method. We will demonstrate our idea with the example given in [12, Remark IV.7.7] where Silverman blows up an arithmetic surface over a DVR at the origin.

Let R be a DVR with uniformizer π and residue field k. Let $C \subset \mathbb{A}^2_R$ be an arithmetic surface defined by a single equation

$$f(x, y) = 0 \qquad \text{for some polynomial } f \in R[x, y].$$

In the scheme language, we have $C = \operatorname{Spec} R[x, y]/(f)$. We assume f is not constant and C is flat over R.

We will treat the uniformizer π as another "coordinate function" i.e. the affine space $\mathbb{A}_R^2 = \operatorname{Spec} R[x,y]$ has three "coordinate functions" π, x, and y. In order to calculate the special fiber C_π, we will set $\pi = 0$.

Let us now assume that the special fiber C_π has a singularity at the origin i.e.,

$$f(0,0) \equiv \frac{\partial f}{\partial x}(0,0) \equiv \frac{\partial f}{\partial y}(0,0) \equiv 0 \pmod{\pi}.$$

Let $\mathfrak{m} = (\pi, x, y) \in C$ be the singular point on the special fiber of C. It is clear that π, x, y forms a regular sequence in $R[x,y]$ and thus by Lemma 3.5.20, the *blow-up of C at $\mathfrak{m}$* is formed by taking the following three schemes/charts and gluing them together.

We let $f_0 = \pi, f_1 = x, f_2 = y$ and u_0, u_1, u_2 be the variables in $\mathbb{P}^2$. By our definition of the blow up, $\mathrm{Bl}_{\mathfrak{m}}(C)$ lies inside the subscheme defined by the equations

$$f_i u_j = f_j u_i \quad \text{for } 0 \le i, j \le 2.$$

in $(\operatorname{Spec} R[x,y]) \times \mathbb{P}^2$. Writing them out explicitly, we have

$$\begin{aligned} f_0 u_1 &= f_1 u_0 \\ f_0 u_2 &= f_2 u_0 \\ f_1 u_2 &= f_2 u_1 \end{aligned}$$

or

$$\begin{aligned} \pi u_1 &= x u_0 \\ \pi u_2 &= y u_0 \\ x u_2 &= y u_1. \end{aligned}$$

Similar to Example 3, we will describe the three charts.
Chart 1. We set $u_0 = 1$ and we obtain:

$$\begin{aligned} \pi u_1 &= x \\ \pi u_2 &= y \\ x u_2 &= y u_1 \text{ (redundant).} \end{aligned}$$

Let ν_0 be the largest integer so that

$$f(\pi u_1, \pi u_2) = \pi^{\nu_0} f_0(u_1, u_2) \qquad \text{with } f_0(u_1, u_2) \in R[u_1, u_2].$$

In other words, factor out a power of π so that the coefficient of f_0 are in R and at least one coefficient is a unit. Then the first coordinate chart for the blow-up C at $\mathfrak{m}$ is the scheme

$C_0 \subset \mathbb{A}^2_R = \operatorname{Spec} R[u_1, u_2]$ defined by

$$C_0 : \operatorname{Spec} R[u_1, u_2]/(f_0(u_1, u_2)).$$

Chart 2. We set $u_1 = 1$:

$$\begin{aligned} \pi &= xu_0 \\ \pi u_2 &= yu_0 \text{ (redundant)} \\ xu_2 &= y. \end{aligned}$$

We substitute these into the polynomial $f(x, y)$. This means we do two things. First, we replace y by xu_2. Second, we take each coefficient a of $f(x, y)$ and replace the largest power of π dividing a by that power of xu_0. For example, if $\pi^2 \mid a$ and $\pi^3 \nmid a$, then we would replace a by $(xu_0)^2\pi^{-2}a$. We factor out the largest possible power of x to get

$$f(x, xu_2) = x^{\nu_1} f_1(x, u_2) \qquad \text{with } f_1(x, u_2) \in R[x, u_2].$$

The second coordinate chart of the blow-up is the scheme

$$C_1 : \operatorname{Spec} R[x, u_0, u_2]/(\pi - xu_0, f_1(x, u_2)).$$

Note that u_0 is a new variable, treated exactly like the other variables x and u_2. The scheme C_1 is the closed subscheme of $\mathbb{A}^3_R = \operatorname{Spec} R[x, u_0, u_2]$ defined by the two equations $\pi = xu_0$ and $f_1(x, u_2) = 0$.

Chart 3. We set $u_2 = 1$:

$$\begin{aligned} \pi u_1 &= xu_0 \text{ (redundant)} \\ \pi &= yu_0 \\ x &= yu_1. \end{aligned}$$

Substituting yu_1 as x into $f(x, y)$ as what we did for chart 1 and 2 while pulling out the largest power of y gives

$$f(yu_1, y) = y^{\nu_2} f_2(y, u_1) \qquad \text{with } f_2(y, u_1) \in R[y, u_1].$$

Then the third coordinate chart of the blow-up is the scheme

$$C_2 : \operatorname{Spec} R[y, u_0, u_1]/(\pi - yu_0, f_2(y, u_1)).$$

To see how the three charts glue together, we will first relabel our homogeneous variables on each chart:

For chart 1, we have

$$u_{0,0} = u_0, \qquad u_{0,1} = \frac{u_1}{u_0}, \qquad u_{0,2} = \frac{u_2}{u_0}.$$

For chart 2, we have

$$u_{1,0} = \frac{u_0}{u_1}, \qquad u_{1,1} = u_1, \qquad u_{1,2} = \frac{u_2}{u_1}.$$

For chart 3, we have

$$u_{2,0} = \frac{u_0}{u_2}, \qquad u_{2,1} = \frac{u_1}{u_2}, \qquad u_{2,2} = u_2.$$

For the transition maps, in order to map from C_0 to C_1, we need to solve for $u_{1,0}, u_{1,1}, u_{1,2}$ in terms of $u_{0,0}, u_{0,1}, u_{0,2}$. Using our defining equations for each chart, we observe that

$$u_{1,0} = \frac{u_0}{u_1} = \frac{\pi}{x} = \frac{1}{u_{0,1}}, \quad u_{1,1} = u_1 = u_{0,1}, \quad u_{1,2} = \frac{u_2}{u_1} = \frac{y}{x} = \frac{u_{0,2}}{u_{0,1}}. \tag{3.5.5}$$

These equations defines a birational map $C_0 \to C_1$ which is defined everywhere except at the points of C_0 with $u_{0,1} = 0$. Similarly, we get a birational map $C_0 \to C_2$ by using the equations

$$u_{2,0} = \frac{u_0}{u_2} = \frac{\pi}{y} = \frac{1}{u_{0,2}}, \quad u_{2,1} = \frac{u_1}{u_2} = \frac{x}{y} = \frac{u_{0,1}}{u_{0,2}}, \quad u_{2,2} = u_2 = u_{0,2}, \tag{3.5.6}$$

and a birational map $C_1 \to C_2$ using

$$u_{2,0} = \frac{u_0}{u_2} = \frac{\pi}{y} = \frac{u_{1,0}}{u_{1,2}}, \quad u_{2,1} = \frac{u_1}{u_2} = \frac{x}{y} = \frac{1}{u_{1,2}}, \quad u_{2,2} = u_2 = u_{1,2}. \tag{3.5.7}$$

These birational maps allow us to glue the three coordinate charts together, and the resulting scheme is the blow-up of C at $\mathfrak{m}$.

In order to find the special fiber of the blow-up, we take the special fiber of each of the coordinate charts and then glue them together. The special fiber of a coordinate chart is calculated by setting $\pi = 0$ and looking at the resulting curve defined over the residue field k. The first coordinate chart is the easiest, and we find that its special fiber is

$$\tilde{C}_0 = \operatorname{Spec} k[u_1, u_2]/(\tilde{f}_0(u_1, u_2)).$$

In other words, $\tilde{C}_0$ is the curve defined in $\mathbb{A}_k^2$ given by the single equation $\tilde{f}_0 = 0$ where $\tilde{f}_0$ is the reduction of f_0 modulo $\pi = 0$ (for its coefficients).

Similarly, the special fiber of C_1 is obtained by setting $\pi = 0$, which means that

$$\tilde{C}_1 = \operatorname{Spec} k[x, u_0, u_2]/(xu_0, \tilde{f}_1(x, u_2)).$$

Again, u_0 is to be treated as another variable, so $\tilde{C}_1$ consists of two pieces, one obtained by setting $u_0 = 0$ and the other obtained by setting $x = 0$. Of course, each piece may consist of several components, or a piece could be empty.

Finally $\tilde{C}_2$ is given by

$$\tilde{C}_2 = \operatorname{Spec} k[y, u_0, u_1]/(yu_0, \tilde{f}_2(y, u_1)),$$

so $\tilde{C}_2$ also consists of two pieces, one with $u_0 = 0$ and the other with $y = 0$.

Remark 24. Similar to Example 3, Silverman's way of describing the blow-up is to first compute the total space $\mathrm{Bl}_{\mathfrak{m}}(\mathbb{A}^2_R)$. Afterwards, we take the inverse image of C under the blow-up map $\phi : \mathrm{Bl}_{\mathfrak{m}}(\mathbb{A}^2_R) \to \mathbb{A}^2_R$ and remove components that intersect the exceptional divisor $E = \phi^{-1}(\mathfrak{m})$. Finally, the irreducible component that remains has to be the strict transform

$$\mathrm{Bl}_{\mathfrak{m}}(C) = \mathrm{cl}_{\mathrm{Bl}_{\mathfrak{m}}(\mathbb{A}^2_R)}(\phi^{-1}(C) - E)$$

since blow-up of affine schemes preserve irreduciblity by Lemma 3.5.11.

We recall that one of the main purpose for this blow-up construction is to produce a model for C over R. The following proposition will demonstrate this.

Proposition 3.5.21. $\mathrm{Bl}_{\mathfrak{m}}(C)$ *has the same generic fiber as* C*.*

Proof. We have $K = \operatorname{Frac} R$. The extension of $\mathfrak{m} = (\pi, x, y)$ from $R[x, y]$ to $K[x, y]$ yields the unit ideal (since π is a unit in K), reflecting that the corresponding subscheme is disjoint from the generic fiber. Hence we have

$$\{\mathfrak{m}\} \times_{\operatorname{Spec} R} \operatorname{Spec} K = \operatorname{Spec}(K/(\pi, x, y)) = \operatorname{Spec}(K/(1)) = \emptyset. \tag{3.5.8}$$

The generic fiber of C is

$$C_{(0)} = C \times_{\operatorname{Spec} R} \operatorname{Spec} K.$$

The open immersion $\iota : \operatorname{Spec} K \to \operatorname{Spec} R$ is flat by 1. in Proposition 3.3.6. By Corollary 3.3.6.1, $\pi_2^* \mathcal{O}_{\operatorname{Spec} K}$ is flat over C where $\pi_1 : C_{(0)} \to C$ and $\pi_2 : C_{(0)} \to \operatorname{Spec} K$ are the projection morphisms. But by the same arguments used in the proof of Corollary 3.3.6.1, we have $\pi_2^* \mathcal{O}_{\operatorname{Spec} K} = \mathcal{O}_{C_{(0)}}$. Thus π_1 is a flat morphism of schemes over R. We note that

$$\pi_1^{-1}(\{\mathfrak{m}\}) = \{\mathfrak{m}\} \times_{\operatorname{Spec} R} \operatorname{Spec} K = \emptyset$$

using (3.5.8) and so by Lemma 3.5.8, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\emptyset}(C_{(0)}) \cong C_{(0)} & \longrightarrow & \mathrm{Bl}_{\mathfrak{m}}(C) \\ \downarrow & & \downarrow \\ C_{(0)} & \longrightarrow & C \end{array}$$

where

$$\begin{aligned} C_{(0)} &\cong \mathrm{Bl}_{\emptyset}(C_{(0)}) \\ &\cong \mathrm{Bl}_{\mathfrak{m}}(C) \times_C C_{(0)} \\ &\cong \mathrm{Bl}_{\mathfrak{m}}(C) \times_C C \times_{\operatorname{Spec} R} \operatorname{Spec} K \\ &\cong \mathrm{Bl}_{\mathfrak{m}}(C) \times_{\operatorname{Spec} R} \operatorname{Spec} K \end{aligned}$$

i.e. $C_{(0)}$ is the generic fiber of $\mathrm{Bl}_{\mathfrak{m}}(C)$. □

Remark 25. In the proof of Proposition 3.5.21, we demonstrated the usage of the fact that blow-ups commute with 'flat base changes' (namely, Lemma 3.5.8). We could have given a much shorter proof by noting that the blow-up is an isomorphism away from the center, which is $V(\pi, x, v)$. This center lies completely within the special fiber and by Proposition 3.2.8, must be disjoint from the generic fiber.

Chapter 4

Hyperelliptic Curves Over Local Fields of Residue Characteristic 2

4.1 The Main Problem

Before stating our main strategy on obtaining semistable models, we will borrow some terminology from [1].

Let K be a finite extension of $\mathbb{Q}_2$. Denote v as the normalised valuation of K and let $\mathcal{O}_K$ be its ring of integers with uniformizer π. We also let $k = \mathcal{O}_K/(\pi)$ denote the residue field. Let C/K be a hyperelliptic curve given by the affine model

$$y^2 = f(x) = c \prod_{r \in \mathcal{R}} (x - r),$$

where $f \in K[x]$ is separable, $\mathcal{R}$ is the set of roots of $f(x)$ in K^{sep}.

Definition 4.1.1 (Clusters)**.** A *cluster of C* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = \{x \in \bar{K} : v(x - z) \geq d\}$ for some $z \in \bar{K}$ and $d \in \mathbb{Q}$.

For a cluster $\mathfrak{s}$ with $\#\mathfrak{s} > 1$, its *depth* $d_{\mathfrak{s}}$ is the maximal d for which $\mathfrak{s}$ is cut out by such a disc, that is, $d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r - r')$. If moreover $\mathfrak{s} \neq \mathcal{R}$, then its *relative depth* is $\delta_{\mathfrak{s}} := d_{\mathfrak{s}} - d_{P(\mathfrak{s})}$, where $P(\mathfrak{s})$ is the smallest cluster with $\mathfrak{s} \subsetneq P(\mathfrak{s})$ (the *parent cluster*).

We refer to this data as the *cluster picture* of C.

With this definition, we can now state our main strategy for explicitly constructing semistable models of hyperelliptic curves satisfying the following assumptions.

Condition 1 (Main Assumptions)**.** Let C be a hyperelliptic curve over a finite extension K of $\mathbb{Q}_2$ given by

$$C : y^2 = f(x),$$

where $f(x) \in \mathcal{O}_K[x]$ is monic and has degree 5. Suppose

1. $f(x)$ splits completely over $\mathcal{O}_K$ with five distinct roots.

2. K has ramification index e over $\mathbb{Q}_2$,

3. the reduction of $f(x)$ has 2 distinct double roots,

4. the depth of the clusters of $f(x)$ corresponding to each double root are $\geq 4e$.

In this setting, the two double roots after reduction give two worse-than-nodal singularities on the special fiber. Our goal is then to demonstrate that C/K has potential semistable reduction by constructing an explicit semistable model over a finite extension of K.

The method is to explicitly construct a model for C over $\mathcal{O}_L$ which has semistable reduction by gluing together 'local elliptic models', where $\mathcal{O}_L$ is the ring of integers of some finite extension L/K. The semistable condition is obtained by first moving each worse-than-nodal singularity to the singular locus of some scheme over $\mathcal{O}_L$ on the special fiber. Next, we do arithmetic blow-ups at suitable closed subschemes corresponding to these singular loci. We then show that we get at-worst-nodal singularities on the special fibers of these blow-up schemes over $\mathcal{O}_L$.

Recall the open subset U_S of an affine scheme $\operatorname{Spec} A$ given by

$$U_S = \{\mathfrak{p} \in \operatorname{Spec} A : f(\mathfrak{p}) \neq 0 \text{ for all } f \in S\}$$

is isomorphic to $\operatorname{Spec} A[S^{-1}]$ where $f(\mathfrak{p})$ is the image of f in the field of fractions of the residue class ring $A/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$.

The initial model over $\mathcal{O}_K$ for C is given by the equation

$$y^2 = (x-\gamma)\prod_{j=1}^{2}(x-\beta_{j,1})(x-\beta_{j,2}), \tag{4.1.1}$$

whose reduction is

$$y^2 = (x-\bar{\gamma})(x-\bar{\beta}_1)^2(x-\bar{\beta}_2)^2,$$

where

- $\bar{\beta}_j \in k$ is the common reduction of $\beta_{j,1}$ and $\beta_{j,2}$,
- β_j is a lift of $\bar{\beta}_j$ to the maximal unramified subextension of $K/\mathbb{Q}_2$ such that

$$v(\beta_j - \beta_{j,k}) \geq 4e. \tag{4.1.2}$$

Such a β_j exists by hypothesis (4).

More specifically, the model over $\mathcal{O}_K$ is given by two affine patches

$$C^0 = \operatorname{Spec} \mathcal{O}_K[x,y]/(y^2 - f(x)),$$
$$C^\infty = \operatorname{Spec} \mathcal{O}_K[u,v]/(v^2 - g(u)),$$

where

$$g(u) = u^6 f(1/u),$$
$$x = 1/u \tag{4.1.3}$$
$$y = v/u^3. \tag{4.1.4}$$

Applying a Suitable Change of Coordinates

By the hypothesis in Condition 1, the $\bar{\gamma}$ and $\bar{\beta}_j$ are distinct in k. Thus, there are 2 worse-than-nodal singular points on the special fiber of C, namely, $(\bar{\beta}_j, 0)$ for $j = 1, 2$.

Let $i \in \{1, 2\}$. Making the substitution $x \to X_i + \beta_i$ moves the singularity $(\bar{\beta}_i, 0)$ to the origin, and we obtain the model over $\mathcal{O}_K$ given by two affine patches

$$C_i^0 = \operatorname{Spec} \mathcal{O}_K[X_i, y]/(y^2 - f_i(X_i)), \tag{4.1.5}$$

where

$$f_i(X_i) = f(X_i + \beta_i) = (X_i - \gamma_i) \prod_{j=1}^{2} (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)}),$$

and

$$\gamma_i = \gamma - \beta_i, \tag{4.1.6}$$
$$\alpha_{j,k}^{(i)} = \beta_{j,k} - \beta_i, \tag{4.1.7}$$
$$\alpha_j^{(i)} = \beta_j - \beta_i, \tag{4.1.8}$$
$$\alpha_{j,k}^{(i)} - \alpha_j^{(i)} = \alpha_{j,k}^{(j)}. \tag{4.1.9}$$

We note that (4.1.9) is true since $\alpha_{j,k}^{(i)} - \alpha_j^{(i)} = \beta_{j,k} - \beta_j = \alpha_{j,k}^{(j)}$.

Let us recall from Condition 1 that e was the ramification index of $K/\mathbb{Q}_2$. By the choice of β_i,

$$v(\alpha_{i,k}^{(i)}) = v(\beta_{i,k} - \beta_i) \geq 4e$$

holds for all i, k.

The localization of C_i^0 away from S_i, where

$$S_i = \left\{ X_i - \alpha_j^{(i)} : j = 1, 2, j \neq i \right\},$$

is given by

$$\tilde{C}_i^0 = \operatorname{Spec} \mathcal{O}_K[X_i, Y_i][S_i^{-1}]/(Y_i^2 - \tilde{f}_i(X_i)), \tag{4.1.10}$$

where

$$\tilde{f}_i(X_i) = (X_i - \gamma_i)(X_i - \alpha_{i,1}^{(i)})(X_i - \alpha_{i,2}^{(i)}) \prod_{j=1, j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2},$$

using the substitution $y \to Y_i \prod_{j=1,j\neq i}^{2}(X_i - \alpha_j^{(i)})$.

The localization of C^∞ away from T_i, where

$$T_i = \{\beta_j u - 1 : j = 1, 2, j \neq i\},$$

is given by

$$\tilde{C}_i^\infty = \operatorname{Spec} \mathcal{O}_K[u, v][T_i^{-1}]/(v^2 - g(u)). \tag{4.1.11}$$

Remark 26. Gluing the $\tilde{C}_i^\infty$ along the open sets U_{T_i}, gives us C^∞. We let U_T be the glued open subset in C^∞ corresponding to U_{T_i}.

Let L be the finite extension of K given by adjoining all the $\sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}$ for $1 \leq i \leq 2$ and denote by $\mathcal{O}_L$ the ring of integers of L and k_L its residue field.

Replacing $Y_i \to Y_i' + \sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}X_i$ yields the equation

$${Y_i'}^2 + a_1 Y_i' X_i = (X_i^3 + a_4 X_i + a_6) \prod_{j=1,j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} + a_2 X_i^2 \tag{4.1.12}$$

where the constants a_i's are given by

$$a_1 = 2\sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}, \tag{4.1.13}$$

$$a_2 = (\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i) \cdot \left(1 - \prod_{j=1,j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2}\right), \tag{4.1.14}$$

$$a_4 = \alpha_{i,1}^{(i)}\alpha_{i,2}^{(i)} + \gamma_i\alpha_{i,1}^{(i)} + \gamma_i\alpha_{i,2}^{(i)}, \tag{4.1.15}$$

$$a_6 = -\alpha_{i,1}^{(i)}\alpha_{i,2}^{(i)}\gamma_i. \tag{4.1.16}$$

Since $v(\alpha_{i,k}^{(i)}) \geq 4e$ we have

$$v(a_1) = v\left(2\sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}\right) = e, \tag{4.1.17}$$

$$v(a_4) = v(\alpha_{i,1}^{(i)}\alpha_{i,2}^{(i)} + \gamma_i\alpha_{i,1}^{(i)} + \gamma_i\alpha_{i,2}^{(i)}) \geq 4e, \tag{4.1.18}$$

$$v(a_6) = v(-\alpha_{i,1}^{(i)}\alpha_{i,2}^{(i)}\gamma_i) \geq 8e. \tag{4.1.19}$$

As we have

$$\alpha_j^{(i)} \equiv \alpha_{j,1}^{(i)} \equiv \alpha_{j,2}^{(i)} \pmod{\pi^{4e}} \tag{4.1.20}$$

by (4.1.2), it follows that

$$a_2/(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i) = 1 - \prod_{j=1, j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} \in \pi^{4e}\mathcal{O}_K[X_i][S_i^{-1}]. \qquad (4.1.21)$$

Remark 27. We note that the coefficients a_1, a_2, a_4, a_6 depend on the indexing $i = 1, 2$ as well. We suppress this dependence in the notation for simplicity.

The Local Elliptic Curves

We apply the transformations $Y_i' \to \pi^{3e}\bar{Y}_i, X_i \to \pi^{2e}\bar{X}_i$ to the equation (4.1.12) to obtain the model over $\mathcal{O}_L$ given by

$$F_i(\bar{X}_i, \bar{Y}_i) = \bar{Y}_i^2 + a_1\pi^{-e}\bar{Y}_i\bar{X}_i - \left((\bar{X}_i^3 + a_4\pi^{-4e}\bar{X}_i + a_6\pi^{-6e}) \prod_{j=1, j\neq i}^{2} \frac{(\pi^{2e}\bar{X}_i - \alpha_{j,1}^{(i)})(\pi^{2e}\bar{X}_i - \alpha_{j,2}^{(i)})}{(\pi^{2e}\bar{X}_i - \alpha_j^{(i)})^2} + a_2\pi^{-2e}\bar{X}_i^2\right). \qquad (4.1.22)$$

Consider also

$$G_i(\bar{U}_i, \bar{V}_i, X_i) = \bar{V}_i^2 + a_1\pi^{-e}\bar{V}_i\bar{U}_i - \left((\bar{U}_i + a_4\pi^{-4e}\bar{U}_i^3 + a_6\pi^{-6e}\bar{U}_i^4) \prod_{j=1, j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} + a_2\pi^{-2e}\bar{U}_i^2\right). \qquad (4.1.23)$$

Define the affine schemes

$$\operatorname{Spec} A_i = \operatorname{Spec} \mathcal{O}_L[\bar{X}_i, \bar{Y}_i][\bar{S}_i^{-1}]/(F_i(\bar{X}_i, \bar{Y}_i)), \qquad (4.1.24)$$
$$\operatorname{Spec} B_i = \operatorname{Spec} \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]/(G_i(\bar{U}_i, \bar{V}_i, X_i), \pi^{2e} - \bar{U}_iX_i), \qquad (4.1.25)$$

where

$$\bar{S}_i = \left\{\pi^{2e}\bar{X}_i - \alpha_j^{(i)} : j = 1, 2, j \neq i\right\}, \qquad (4.1.26)$$
$$\bar{T}_i = \left\{X_i - \alpha_j^{(i)}, X_i + \beta_i : j = 1, 2, j \neq i\right\}. \qquad (4.1.27)$$

Remark 28. By the relation (4.1.20), reducing modulo π we obtain

$$\frac{(\pi^{2e}\bar{X}_i - \alpha_{j,1}^{(i)})(\pi^{2e}\bar{X}_i - \alpha_{j,2}^{(i)})}{(\pi^{2e}\bar{X}_i - \alpha_j^{(i)})^2} = \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} = 1.$$

Thus by taking the special fibers of Spec A_i and Spec B_i, we see that their defining equations F_i and G_i become $\bar{Y}_i^2 + c_i\bar{Y}_i\bar{X}_i = p_i(\bar{X}_i)$ and $\bar{V}_i^2 + d_i\bar{V}_i\bar{U}_i = q_i(\bar{U}_i)$ respectively where c_i

and d_i are constants and p_i and q_i are polynomials of degree 3 in $\bar{X}_i$ and $\bar{U}_i$ respectively. Therefore F_i and G_i become the equations of elliptic curves when we take the special fibers of $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_i$.

From now on, we will mainly focus on studying the two schemes $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_i$.

A Not-Yet Semistable Scheme Over $\mathcal{O}_L$

We shall show that the new schemes we have obtained glue to a $\mathcal{O}_L$-scheme which has its generic fiber isomorphic to C over L.

Proposition 4.1.2. *The schemes* $\operatorname{Spec} A_i$ *and* $\operatorname{Spec} B_i$ *for* $i = 1, 2$ *together with* C^0 *and* C^∞ *glue to a new scheme* $\mathcal{C}$ *over* $\mathcal{O}_L$*.*

Proof. The localizations $A_i[\bar{X}_i^{-1}]$ and $B_i[\bar{U}_i^{-1}]$ are $\mathcal{O}_L$-isomorphic using the relations

$$\bar{X}_i = 1/\bar{U}_i, \tag{4.1.28}$$

$$\bar{Y}_i = \bar{V}_i/\bar{U}_i^2, \tag{4.1.29}$$

$$\pi^{2e}\bar{X}_i = X_i. \tag{4.1.30}$$

For $i = 1, 2$, let $\mathcal{C}_i$ denote the scheme over $\mathcal{O}_L$ obtained by gluing together $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_i$ along the open subsets corresponding to the localizations

$$\operatorname{Spec} A_i[\bar{X}_i^{-1}] \cong \operatorname{Spec} B_i[\bar{U}_i^{-1}] \tag{4.1.31}$$

via the relations (4.1.28)–(4.1.30). More concretely, we have the $\mathcal{O}_L$-algebra isomorphisms

$$A_i[\bar{X}_i^{-1}] \longrightarrow B_i[\bar{U}_i^{-1}]$$

$$\bar{X}_i \longmapsto 1/\bar{U}_i \tag{4.1.32}$$

$$\bar{Y}_i \longmapsto \bar{V}_i/\bar{U}_i^2 \tag{4.1.33}$$

and

$$B_i[\bar{U}_i^{-1}] \longrightarrow A_i[\bar{X}_i^{-1}]$$

$$\bar{U}_i \longmapsto 1/\bar{X}_i \tag{4.1.34}$$

$$\bar{V}_i \longmapsto \bar{Y}_i/\bar{X}_i^2 \tag{4.1.35}$$

$$X_i \longmapsto \pi^{2e}\bar{X}_i \tag{4.1.36}$$

These in turn, give us the gluing maps $\operatorname{Spec} A_i[\bar{X}_i^{-1}] \cong \operatorname{Spec} B_i[\bar{U}_i^{-1}]$.

By the gluing construction (Theorem 3.1.1), for each $i = 1, 2$, the scheme $\mathcal{C}_i$ is covered by two open charts with one being isomorphic to $\operatorname{Spec} A_i$ and the other being isomorphic to $\operatorname{Spec} B_i$. Abusing of notation, we will just write $\operatorname{Spec} B_i \subseteq \mathcal{C}_i$ as the open chart isomorphic to Spec B_i. We now want to glue $\mathcal{C}_1$ and $\mathcal{C}_2$ along open subsets of Spec B_1 and Spec B_2. We have the relations

$$X_1 + \beta_1 = X_2 + \beta_2, \tag{4.1.37}$$

$$(X_1 - \alpha_2^{(1)})Y_1 = (X_2 - \alpha_1^{(2)})Y_2, \tag{4.1.38}$$

$$X_i = \pi^{2e}\bar{X}_i, \tag{4.1.39}$$

$$Y_i = \pi^{3e}\bar{Y}_i + \sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)} \cdot X_i, \tag{4.1.40}$$

$$\bar{X}_i = 1/\bar{U}_i, \tag{4.1.41}$$

$$\bar{Y}_i = \bar{V}_i/\bar{U}_i^2 \tag{4.1.42}$$

for $i = 1, 2$. Writing out the corresponding coordinate ring maps for the gluing maps, we have

$$B_1 \longrightarrow B_2$$

$$\bar{U}_1 \longmapsto \frac{\pi^{2e}\bar{U}_2}{\pi^{2e} - \alpha_1^{(2)}\bar{U}_2} \tag{4.1.43}$$

$$\bar{V}_1 \longmapsto \frac{(\pi^{2e}\bar{V}_2 + \pi^e\ell_2 - \pi^e\ell_1)\bar{U}_2}{\pi^{2e} - \alpha_1^{(2)}\bar{U}_2} \tag{4.1.44}$$

and

$$B_2 \longrightarrow B_1$$

$$\bar{U}_2 \longmapsto \frac{\pi^{2e}\bar{U}_1}{\pi^{2e} - \alpha_2^{(1)}\bar{U}_1} \tag{4.1.45}$$

$$\bar{V}_2 \longmapsto \frac{(\pi^{2e}\bar{V}_1 + \pi^e\ell_1 - \pi^e\ell_2)\bar{U}_1}{\pi^{2e} - \alpha_2^{(1)}\bar{U}_1} \tag{4.1.46}$$

where $\ell_i = \sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}$. We first note that by definition of B_i (equations (4.1.25) and (4.1.27)), the ring element

$$X_i - \alpha_j^{(i)} = \frac{\bar{U}_i}{\pi^{2e} - \alpha_j^{(i)}\bar{U}_i} \in B_i$$

is invertible for $i, j = 1, 2$ and $j \neq i$. Hence the above maps are well-defined. Using the relation $\alpha_i^{(j)} - \alpha_j^{(i)} = 0$ the maps are indeed $\mathcal{O}_L$-algebra isomorphisms. Thus we glue $\mathcal{C}_1$ and $\mathcal{C}_2$ along Spec B_1 and Spec B_2 to get a scheme $\mathcal{C}_0$ over $\mathcal{O}_L$.

Now, we glue $\mathcal{C}_0$ along the open subset $\operatorname{Spec} B_i[\bar{U}_i^{-1}] \subset \operatorname{Spec} B_i$ together with C^∞ along the open subset $\mathcal{U} := (\tilde{\mathcal{C}}_1^\infty \cap \tilde{\mathcal{C}}_2^\infty) \setminus \{u = 0\} \subseteq C^\infty$ (see definition (4.1.11)) using the relations

$$X_1 + \beta_1 = x = X_2 + \beta_2, \tag{4.1.47}$$

$$(X_1 - \alpha_2^{(1)})Y_1 = y = (X_2 - \alpha_1^{(2)})Y_2 \tag{4.1.48}$$

together with the relations (4.1.3)–(4.1.4), (4.1.28)–(4.1.30) and (4.1.40) to obtain a scheme $\mathcal{C}_\infty$ over $\mathcal{O}_L$. More concretely, the corresponding coordinate ring maps for the gluing maps are

$$\begin{aligned} \mathcal{O}_L[\mathcal{U}] &\longrightarrow B_i[\bar{U}_i^{-1}] \\ u &\longmapsto \frac{\bar{U}_i}{\pi^{2e} + \beta_i \bar{U}_i} && (4.1.49) \\ v &\longmapsto \frac{(\pi^{2e} - \alpha_j^{(i)}\bar{U}_i)}{\bar{U}_i} \cdot \frac{(\pi^{3e}\bar{V}_i + \ell_i \pi^{2e}\bar{U}_i)}{\bar{U}_i^2} \cdot \frac{\bar{U}_i^3}{(\pi^{2e} + \beta_i \bar{U}_i)^3} && (4.1.50) \end{aligned}$$

and

$$\begin{aligned} B_i[\bar{U}_i^{-1}] &\longrightarrow \mathcal{O}_L[\mathcal{U}] \\ \bar{U}_i &\longmapsto \frac{\pi^{2e} u}{1 - \beta_i u} && (4.1.51) \\ \bar{V}_i &\longmapsto \frac{\pi^e v}{(1 - \beta_j u)(1 - \beta_i u)^2} - \frac{\ell_i \pi^e u}{1 - \beta_i u} && (4.1.52) \end{aligned}$$

for any $i = 1, 2$ and $j \neq i$ where $\mathcal{O}_L[\,\cdot\,]$ denotes the coordinate ring. We note that the maps are well defined since the ring element

$$X_i + \beta_i = \frac{\bar{U}_i}{\pi^{2e} + \beta_i \bar{U}_i} \in B_i$$

is invertible. Using the relation $\alpha_j^{(i)} = \beta_j - \beta_i$ again, we may check that the maps are inverse of each other and thus are $\mathcal{O}_L$-algebra isomorphisms.

Finally, we glue C^0 to the open subset $C^\infty \subset \mathcal{C}_\infty$ via Proposition 3.4.11 (relations (4.1.3)–(4.1.4)) to include the point $x = 0$. This gives us our desired scheme $\mathcal{C}$ over $\mathcal{O}_L$. □

Remark 29. The gluing maps that are given above all have integer coefficients. Indeed, β_1 and β_2 are chosen to be in $\mathcal{O}_K$ since (4.1.2) must hold true. The constants γ_i's, $\alpha_{j,k}^{(i)}$'s, and $\alpha_j^{(i)}$'s are all differences of integral elements using (4.1.6)–(4.1.8). Finally, the square root $\sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}$ is the algebraic number we have used to make our finite extension from K to L and thus it is in $\mathcal{O}_L$.

Proposition 4.1.3. *The generic fiber $\mathcal{C}_L$ of the scheme $\mathcal{C}$ is L-isomorphic to C.*

Proof. The relations

$$X_i = \pi^{2e}\bar{X}_i, \tag{4.1.53}$$

$$Y_i = \pi^{3e}\bar{Y}_i + \pi^{2e}\sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)} \cdot \bar{X}_i, \tag{4.1.54}$$

$$\bar{X}_i = \frac{X_i}{\pi^{2e}}, \tag{4.1.55}$$

$$\bar{Y}_i = \frac{Y_i - \sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)} \cdot X_i}{\pi^{3e}}, \tag{4.1.56}$$

give an L-isomorphism

$$\begin{aligned}(\operatorname{Spec} A_i)_L &= \operatorname{Spec} L[\bar{X}_i, \bar{Y}_i][\bar{S}_i^{-1}]/(F_i(\bar{X}_i, \bar{Y}_i)) \\ &\cong \operatorname{Spec} L[X_i, Y_i][S_i^{-1}]/(Y_i^2 - \tilde{f}_i(X_i, Y_i)) = (\tilde{C}_i^0)_L.\end{aligned}$$

Now, the defining equation $\pi^{2e} - \bar{U}_i X_i$ in $\operatorname{Spec} B_i$ implies $\bar{U}_i \neq 0$ in $(\operatorname{Spec} B_i)_L$ since π^{2e} is a unit in L. Hence for $i, j = 1, 2$ and $i \neq j$, the relations

$$u = \frac{\bar{U}_i}{\pi^{2e} + \beta_i \bar{U}_i}, \tag{4.1.57}$$

$$v = \left(\frac{\pi^{2e} - \alpha_j^{(i)}\bar{U}_i}{\bar{U}_i}\right)\left(\frac{\pi^{3e}\bar{V}_i + \ell_i \pi^{2e}\bar{U}_i}{\bar{U}_i^2}\right)\left(\frac{\bar{U}_i^3}{(\pi^{2e} + \beta_i \bar{U}_i)^3}\right), \tag{4.1.58}$$

$$\bar{U}_i = \frac{\pi^{2e} u}{1 - \beta_i u}, \tag{4.1.59}$$

$$\bar{V}_i = \left(\frac{v}{(1 - \beta_j u)u^2} - \ell_i \cdot \left(\frac{1 - \beta_i u}{u}\right)\right) \cdot \frac{\pi^e u^2}{(1 - \beta_i u)^2}, \tag{4.1.60}$$

give an L-isomorphism

$$(\operatorname{Spec} B_i)_L \cong (\tilde{C}_i^\infty)_L$$

where $\ell_i = \sqrt{-(\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i)}$. From the above and Remark 26 we get that $\mathcal{C}_L \cong C/L$ since $\mathcal{C}_L$ is the gluing of the $(\mathcal{C}_i)_L$'s and each $(\mathcal{C}_i)_L$ is then the gluing of $(\operatorname{Spec} A_i)_L$ and $(\operatorname{Spec} B_i)_L$. □

We will now show that $\mathcal{C}$ is proper over $\mathcal{O}_L$. Note that it is not important whether $\mathcal{C}$ is a model for C over L at this moment as we will soon see that $\mathcal{C}$ is not yet semistable.

Proposition 4.1.4. *The scheme $\mathcal{C}$ is proper over $\mathcal{O}_L$.*

Proof. We first note that $\mathcal{C}$ is of finite type over $\mathcal{O}_L$ since it is covered by $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_i$ for $i = 1, 2$ which are of finite type over $\mathcal{O}_L$. Similarly, $\mathcal{C}$ is also Noetherian since each $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_j$ are Noetherian. Hence we can use the valuative criterion for properness (Theorem 3.2.15) to show $\mathcal{C}$ is proper over $\mathcal{O}$. Let R be a DVR with valuation v and $F = \operatorname{Frac} R$.

Suppose P is an F-point of $\mathcal{C}$. Then there is an inclusion $K_P \hookrightarrow F$, where K_P is the residue field of P, considered as a point of $\mathcal{C}$. Without loss of generality, we are in one of the following two cases:

1. P lies in C^0 and corresponds to a section sending $x \mapsto x_0 \in F$ where $v(x_0) \geq 0$
2. P lies in C^∞ and corresponds to a section sending $u \mapsto u_0 \in F$ where $v(u_0) > 0$

Case (1): If $v(x_0) > 0$ then the F-point P corresponds to a section sending $x \mapsto x_0 \in R$, and hence by the relation $y^2 - f(x) = 0$, we have that $y \in R$. Now if $v(x_0) = 0$, let $1 \leq i \leq 2$ be such that $v(x_0 - \beta_i) \geq 0$ and $v(x_0 - \beta_j) = 0$ for all $j \neq i$. Consider the affine patch $\mathcal{C}_i$. The F-point P corresponds to a section sending $\bar{X}_i \mapsto \bar{x}_i \in R$ or $\bar{U}_i \to \bar{u}_i \in R$.

In the first case, the relation $F_i(\bar{X}_i, \bar{Y}_i) = 0$ shows that $\bar{Y}_i \mapsto \bar{y}_i \in R$.

In the second case, $G_i(\bar{U}_i, \bar{V}_i) = 0$ shows that $\bar{V}_i \mapsto \bar{v}_i \in R$. We have that $\bar{x}_i = \pi^{2e}/\bar{u}_i$. If $v(\bar{u}_i) \leq v(\pi^{2e})$, then $\bar{X}_i \mapsto \bar{x}_i = \pi^{2e}/\bar{u}_i \in R$.

If $v(\bar{u}_i) > v(\pi^{2e})$, then $U_i \mapsto \bar{u}_i/\pi^{2e}$, where $X_i U_i = 1$ and $U_i = \bar{U}_i/\pi^{2e}$. However, $v(\bar{u}_i/\pi^{2e}) > 0$ and so we have $v(u_0) > 0$ as

$$U_i = 1/(x - \beta_i) = u/(1 - \beta_i u),$$

so we are in case (2).

Thus, we have shown the existence of an R-point which gives the F-point P.

Case (2): Consider the affine patch C^∞. The F-point P corresponds to a section sending $u \mapsto u_0 \in R$, and hence by the relation $v^2 - g(u) = 0$, we have that $v \in R$. This shows the existence of a R-point which gives the F-point P.

In all cases, the R-point exhibited which gives the F-point P is unique because such R-point is unique on each chart. If P lies in both charts, then the two exhibited R-points on different charts are identified together when we glue C^0 and C^∞. By the valuative criterion for properness, the scheme $\mathcal{C}$ is proper over $\mathcal{O}$. □

By Proposition 4.1.3 and 4.1.4 we have that the generic fiber of the proper $\mathcal{O}_L$-scheme $\mathcal{C}$ is isomorphic to C over L. Our next step is to see why $\mathcal{C}$ is not yet semistable.

In the following proposition, we will first show that $\operatorname{Spec} A_i$ is semistable. Afterwards, we will examine the special fiber of $\operatorname{Spec} B_i$ to see where the failure of semistability occurs.

Proposition 4.1.5. *The scheme* $\operatorname{Spec} A_i$ *over* $\mathcal{O}_L$ *has semistable reduction.*

Proof. Let $\bar{a}_i = a_i \pi^{-ie}$ and $[\bar{a}_i]$ be the image of $\bar{a}_i$ in k_L. We recall that $v(a_1) = e$, hence $[\bar{a}_1] \neq 0$. Moreover, it holds that $[\bar{a}_2] = [\bar{a}_6] = 0$. Note also that each term in the product over $j = 1, 2, j \neq i$ of (4.1.22) reduces to 1.

If $v(a_4) > 4e$, then $[\bar{a}_4] = 0$. The special fiber of $\operatorname{Spec} A_i$ is a singular curve over k given by the Weierstrass equation

$$\bar{Y}_i^2 + [\bar{a}_1]\bar{X}_i\bar{Y}_i = \bar{X}_i^3,$$

which has a nodal singularity at $\bar{X}_i = \bar{Y}_i = 0$.

If $v(a_4) = 4e$, then $[\bar{a}_4] \neq 0$. The special fiber of $\operatorname{Spec} A_i$ is a non-singular genus 1 curve over k_L given by the Weierstrass equation

$$\bar{Y}_i^2 + [\bar{a}_1]\bar{X}_i\bar{Y}_i = \bar{X}_i^3 + [\bar{a}_4]\bar{X}_i.$$

□

In a similar fashion as the above proof, the special fiber of $\operatorname{Spec} B_i$ is given by the equations

$$\begin{aligned}\bar{V}_i^2 + [\bar{a}_1]\bar{U}_i\bar{V}_i = \bar{U}_i + [\bar{a}_4]\bar{U}_i^3,\\ X_i\bar{U}_i = 0,\\ X_i \neq \alpha_j^{(i)}, j = 1, 2, j \neq i\end{aligned}$$

over k. We note that the closed subscheme $\bar{U}_i = \bar{V}_i = 0$ is a non-reduced component on the special fiber. This can be seen by the fact that both relations $X_i\bar{U}_i = 0$ and $\bar{V}_i^2 + [\bar{a}_1]\bar{U}_i\bar{V}_i = \bar{U}_i + [\bar{a}_4]\bar{U}_i^3$ contains this subscheme. This shows that $\mathcal{C}$ is not semistable over $\mathcal{O}_L$ as the special fiber of $\operatorname{Spec} B_i$ over k has a non-reduced component which is certainly worse-than nodal i.e. $\operatorname{Spec} B_i$ is not semistable over $\mathcal{O}_L$.

We claim that a single arithmetic blow-up of $\operatorname{Spec} B_i$ at the subscheme $\operatorname{Spec} B_i \cap V(\bar{U}_i, \bar{V}_i, \pi^e)$ will give us a semistable model of C over $\mathcal{O}_L$ (by taking the strict transform of the blow-up map and then glue with $\operatorname{Spec} A_i$). The extra generator π^e will ensure that the blow-up leaves the generic fiber unchanged similar to Proposition 3.5.21.

4.1.1 A Semistable Model via One Arithmetic Blow-up

As we have already mentioned, we will blow-up $\operatorname{Spec} B_i$ in Section 4.1 along the closed susbscheme given generated by $\bar{U}_i$, $\bar{V}_i$, and π^e over $\mathcal{O}_L$. We will first rewrite B_i as

$$B_i = \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]/(G_i(\bar{U}_i, \bar{V}_i, X_i, W_i), \pi^{2e} - \bar{U}_iX_i, (X_i - \alpha_j^{(i)})W_i - 1)$$

where

$$G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = \bar{V}_i^2 + \bar{a}_1\bar{V}_i\bar{U}_i - \Big((\bar{U}_i + \bar{a}_4\bar{U}_i^3 + \bar{a}_6\bar{U}_i^4)(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 + \bar{a}_2\bar{U}_i^2\Big) \tag{4.1.61}$$

for $i \neq j$ and $\bar{a}_i = a_i\pi^{-ie}$.

Let us define

$$D_i^{\text{tot}} := \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]/(\bar{U}_i, \bar{V}_i, \pi^e)$$

and

$$D_i := \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]/(G_i(\bar{U}_i, \bar{V}_i, X_i, W_i), \pi^{2e} - \bar{U}_i X_i, (X_i - \alpha_j^{(i)})W_i - 1, \bar{U}_i, \bar{V}_i, \pi^e).$$

We note that $\operatorname{Spec} D_i = \operatorname{Spec} D_i^{\mathrm{tot}} \cap \operatorname{Spec} B_i$ is a closed subscheme of $\operatorname{Spec} B_i$. Consider the ambient affine space $\mathbb{A}^4_{\mathcal{O}_L} = \operatorname{Spec} \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]$ and let

$$\varphi_i : \mathrm{Bl}_{\operatorname{Spec} D_i^{\mathrm{tot}}}(\mathbb{A}^4_{\mathcal{O}_L}) \to \mathbb{A}^4_{\mathcal{O}_L}$$

be the blow-up map of the ambient space at $\operatorname{Spec} D_i^{\mathrm{tot}}$. Since the generators $\bar{U}_i, \bar{V}_i, \pi^e$ form a regular sequence for $\mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]$, we can again, describe the blow-up of the total space using Lemma 3.5.20. We will follow Section 3.5.4 to describe the strict transform $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ as a scheme in $\mathbb{A}^4_{\mathcal{O}_L} \times \mathbb{P}^2$ with homogeneous coordinates r_i, s_i, t_i for $\mathbb{P}^2$ corresponding to the three generators $\bar{U}_i, \bar{V}_i, \pi^e$ of D_i at on their respective charts (the homogeneous coordinates u_0, u_1, u_2 in Section 3.5.4 correspond to the coordinates r_i, s_i, t_i here).

Let us denote

$$E_i := \varphi_i^{-1}(\operatorname{Spec} D_i)$$

as the exceptional divisor of the blow-up $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$. Writing it out, we have

$$E_i = (\operatorname{Spec} D_i) \times \mathbb{P}^2 = \{(p, [r_i : s_i : t_i]) : p \in \operatorname{Spec} D_i, [r_i : s_i : t_i] \in \mathbb{P}^2\}.$$

More concretely, the closed points of the exceptional divisor is the set

$$\begin{aligned} \{(\bar{U}_i, &\bar{V}_i, X_i, W_i, [r_i : s_i : t_i]) : \\ &G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = 0, \bar{U}_i X_i = \pi^{2e}, (X_i - \alpha_j^{(i)})W_i = 1, \bar{U}_i = \bar{V}_i = \pi^e = 0\} \end{aligned} \tag{4.1.62}$$

where $[r_i : s_i : t_i] \in \mathbb{P}^2$.

The 1st Chart

On the first chart $r_i = 1$, the equations defining the graph Γ_i of $\varphi_i^{-1}(\operatorname{Spec} B_i)$ are given by

$$\begin{cases} G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = 0, & (4.1.63) \\ X_i \bar{U}_i = \pi^{2e}, & (4.1.64) \\ (X_i - \alpha_j^{(i)})W_i = 1, & (4.1.65) \\ \bar{U}_i t_i = s_i \bar{V}_i, & (4.1.66) \\ \bar{U}_i = \pi^e s_i, & (4.1.67) \\ \bar{V}_i = \pi^e t_i. & (4.1.68) \end{cases}$$

We note that the (4.1.66) is redundant since cross multiplying the equations (4.1.67) and (4.1.68) together gives

$$\bar{U}_i\pi^e t_i = \bar{V}_i\pi^e s_i$$

which has two components $\pi^e = 0$ and $\bar{U}_i t_i = s_i\bar{V}_i$. We note that the component corresponds to $\pi^e = 0$ lies completely within the exceptional divisor E_i and thus the blow-up must lie on the other component corresponding to $\bar{U}_i t_i = s_i\bar{V}_i$ which coincides with (4.1.66).

Substituting (4.1.67) and (4.1.67) into and give the equations

$$\begin{aligned} G_i(\pi^e s_i, \pi^e t_i, X_i, W_i) =& \pi^{2e}(t_i^2 + \bar{a}_1 s_i t_i - \bar{a}_2 s_i^2) \\ & - \pi^e(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(s_i + \bar{a}_4\pi^{2e}s_i^3 + \bar{a}_6\pi^{3e}s_i^4), \end{aligned}$$

and

$$\pi^e X_i s_i = \pi^{2e}.$$

Now our equations defining $\mathrm{Bl}_{\mathrm{Spec}\, D_i}(\mathrm{Spec}\, B_i)$ on the chart $r_i = 1$ are

$$\begin{cases} G_i(\pi^e s_i, \pi^e t_i, X_i, W_i) = 0, \\ \pi^e X_i s_i = \pi^{2e}, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = \pi^e s_i, \\ \bar{V}_i = \pi^e t_i. \end{cases}$$

We note that the equation $G_i(\pi^e s_i, \pi^e t_i, X_i, W_i) = 0$ factors as

$$\begin{aligned} &\pi^e[\pi^e(t_i^2 + \bar{a}_1 s_i t_i - \bar{a}_2 s_i^2) \\ &\quad - (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})(s_i W_i^2 + \bar{a}_4\pi^{2e}s_i^3 W_i^2 + \bar{a}_6\pi^{3e}s_i^4 W_i^2)] = 0. \end{aligned} \tag{4.1.69}$$

The equation (4.1.69) gives two irreducible components, one is defined by

$$\begin{cases} \pi^e = 0, \\ X_i\bar{U}_i = \pi^{2e}, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = 0, \\ \bar{V}_i = 0 \end{cases}$$

which is precisely E_i by (4.1.62). The other component is

$$\begin{cases} \pi^e(t_i^2+\bar{a}_1s_it_i-\bar{a}_2s_i^2)-(X_i-\alpha_{j,1}^{(i)})(X-\alpha_{j,2}^{(i)})(s_iW_i^2+\bar{a}_4\pi^{2e}s_i^3W_i^2+\bar{a}_6\pi^{3e}s_i^4W_i^2), \\ \pi^eX_is_i=\pi^{2e}, \\ (X_i-\alpha_j^{(i)})W_i=1, \\ \bar{U}_i=\pi^es_i, \\ \bar{V}_i=\pi^et_i \end{cases}$$

which has to then contain $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ on this chart.

By the exact same argument for $\pi^eX_is_i=\pi^{2e}$ or $\pi^e(X_is_i-\pi^e)=0$, we must have that $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ is contained in

$$\begin{cases} \pi^e(t_i^2+\bar{a}_1s_it_i-\bar{a}_2s_i^2)=(X_i-\alpha_{j,1}^{(i)})(X_i-\alpha_{j,2}^{(i)})(s_iW_i^2+\bar{a}_4\pi^{2e}s_i^3W_i^2+\bar{a}_6\pi^{3e}s_i^4W_i^2), & (4.1.70) \\ X_is_i=\pi^e, & (4.1.71) \\ (X_i-\alpha_j^{(i)})W_i=1, & (4.1.72) \\ \bar{U}_i=\pi^es_i, & (4.1.73) \\ \bar{V}_i=\pi^et_i & (4.1.74) \end{cases}$$

on this chart.

Now substituting (4.1.71) into (4.1.70) gives

$$X_is_i(t_i^2+\bar{a}_1s_it_i-\bar{a}_2s_i^2)=s_i(X_i-\alpha_{j,1}^{(i)})(X_i-\alpha_{j,2}^{(i)})W_i^2(1+\bar{a}_4\pi^{2e}s_i^2+\bar{a}_6\pi^{3e}s_i^3). \quad (4.1.75)$$

The equation (4.1.75) again gives two components, where the component corresponding to $s_i=0$ yields $\pi^e=X_is_i=0$ and subsequently $\bar{U}_i=\bar{V}_i=0$ which completely lies within E_i. Hence the blow-up must lie on

$$\begin{cases} X_i(t_i^2+\bar{a}_1s_it_i-\bar{a}_2s_i^2)=(X_i-\alpha_{j,1}^{(i)})(X_i-\alpha_{j,2}^{(i)})W_i^2(1+\bar{a}_4\pi^{2e}s_i^2+\bar{a}_6\pi^{3e}s_i^3), \\ X_is_i=\pi^e, \\ (X_i-\alpha_j^{(i)})W_i=1, \\ \bar{U}_i=\pi^es_i, \\ \bar{V}_i=\pi^et_i. \end{cases}$$

If we show that the closed subscheme parameterized by

$$\begin{aligned} \Gamma_{r_i}=\{&(\pi^es_i,\pi^et_i,X_i,W_i,1,s_i,t_i): \\ &X_i(t_i^2+\bar{a}_1s_it_i-\bar{a}_2s_i^2)=(X_i-\alpha_{j,1}^{(i)})(X_i-\alpha_{j,2}^{(i)})W_i^2(1+\bar{a}_4\pi^{2e}s_i^2+\bar{a}_6\pi^{3e}s_i^3), \\ &X_is_i=\pi^e,(X_i-\alpha_j^{(i)})W_i=1\}\subseteq\Gamma_i \end{aligned}$$

is irreducible, then similar to Example 3, we will know that this has to be our blow-up on this chart. We will show this in a more concrete example later.

Taking the special fiber by reducing modulo π, we get

$$\tilde{\Gamma}_{r_i} = \{(0,0,X_i,1,s_i,t_i) : X_i(t_i^2 + \bar{a}_1 s_i t_i - \bar{a}_2 s_i^2) = 1, X_i s_i = 0\}.$$

We see that this is reduced and non-singular as it is isomorphic to $\operatorname{Spec} k_L[X_i,t_i]/(X_i t_i^2 - 1)$ for all $i = 1,2$ by observing that X_i is nonzero and hence $s_i = 0$.

Remark 30. Similar to Remark 28, the relation $(X_i - \alpha_j^{(i)})W_i = 1$ is no longer required after reducing modulo π since

$$\frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} = 1$$

over k due to the relation

$$\alpha_j^{(i)} \equiv \alpha_{j,1}^{(i)} \equiv \alpha_{j,2}^{(i)} \pmod{\pi}$$

which comes from (4.1.20). More concretely, let us consider any $\mathcal{O}_L$-algebra of the form

$$\mathcal{O}_L[z_1,...,z_n,x,y]/(f_1,...,f_m,(x-\alpha_j^{(i)})y-1) =: R.$$

We have

$$R \otimes_{\mathcal{O}_L} k \cong \frac{k[\,[z_1],...,[z_n],[x],[y]\,]}{([f_1],...,[f_m],[(x-\alpha_j^{(i)})y-1])}.$$

where the bracket $[\cdot]$ denotes the reduction modulo $\pi = 0$ in k. We note that

$$[(x-\alpha_j^{(i)})y-1] = ([x]-[\alpha_j^{(i)}])[y]-[1] = [0]$$

and

$$[(x-\alpha_{j,1}^{(i)})(x-\alpha_{j,2}^{(i)})] = ([x]-[\alpha_{j,1}^{(i)}])([x]-[\alpha_{j,2}^{(i)}]) = ([x]-[\alpha_j^{(i)}])^2.$$

Therefore

$$[(x-\alpha_{j,1}^{(i)})(x-\alpha_{j,2}^{(i)})y^2] = ([x]-[\alpha_j^{(i)}])^2[y]^2 = (([x]-[\alpha_j^{(i)}])[y])^2 = ([-1])^2 = [1].$$

Remark 31. When we write a set of points

$$V = \{(x_1,...,x_n) : f_i(x_1,...,x_n) = g_i(x_1,...,x_n), i = 1,...,m\}$$

what we have is really the affine scheme

$$V = \operatorname{Spec} \mathcal{O}_L[x_1, ..., x_n]/(f_i - g_i)_{1 \le i \le m}$$

which includes the generic points of its irreducible components. Since we do not work with the generic points, the first way of writing V is just to emphasize the closed points.

The 2nd Chart

On the second chart $s_i = 1$, the equations defining Γ_i are given by

$$\begin{cases} G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = 0, & (4.1.76) \\ X_i \bar{U}_i = \pi^{2e}, & (4.1.77) \\ (X_i - \alpha_j^{(i)}) W_i = 1, & (4.1.78) \\ \bar{U}_i t_i = \bar{V}_i, & (4.1.79) \\ \bar{U}_i r_i = \pi^e, & (4.1.80) \\ \bar{V}_i r_i = \pi^e t_i. & (4.1.81) \end{cases}$$

Similar to the previous chart, the last equation $\bar{V}_i r_i = \pi^e t_i$ is redundant. Substituting (4.1.79) into (4.1.76) we get

$$\bar{U}_i[\bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) - (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)}) W_i^2 (1 + \bar{a}_4 \bar{U}_i^2 + \bar{a}_6 \bar{U}_i^3)] = 0.$$

This again gives two irreducible components. The first component being

$$\begin{cases} \bar{U}_i = 0, \\ X_i \bar{U}_i = \pi^{2e}, \\ (X_i - \alpha_j^{(i)}) W_i = 1, \\ \bar{V}_i = 0, \\ \pi^e = 0 \end{cases}$$

which is E_i by (4.1.62). The other component is

$$\begin{cases} \bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)}) W_i^2 (1 + \bar{a}_4 \bar{U}_i^2 + \bar{a}_6 \bar{U}_i^3), & (4.1.82) \\ X_i \bar{U}_i = \pi^{2e}, & (4.1.83) \\ (X_i - \alpha_j^{(i)}) W_i = 1, & (4.1.84) \\ \bar{V}_i = \bar{U}_i t_i, & (4.1.85) \\ \bar{U}_i r_i = \pi^e & (4.1.86) \end{cases}$$

which has to contain $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ on this chart. Substituting (4.1.86) into (4.1.83) we get

$$\bar{U}_i(X_i - \bar{U}_i r_i^2) = 0$$

which also has two components with the first being

$$\begin{cases} (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 = 0, \\ \bar{U}_i = 0, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{V}_i = 0, \\ \pi^e = 0. \end{cases}$$

We note that this component lies completely within E_i. Thus we only have to consider the other component

$$\begin{cases} \bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\bar{U}_i^2 + \bar{a}_6\bar{U}_i^3), \\ X_i = \bar{U}_i r_i^2, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{V}_i = \bar{U}_i t_i, \\ \bar{U}_i r_i = \pi^e \end{cases}$$

which has to contain our blow-up on this chart. We will show that the closed subscheme parameterized by

$$\begin{aligned} \Gamma_{s_i} = \{(\bar{U}_i, &\bar{U}_i t_i, X_i, W_i, r_i, 1, t_i) : \\ &\bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\bar{U}_i^2 + \bar{a}_6\bar{U}_i^3), \\ &X_i = \bar{U}_i r_i^2, (X_i - \alpha_j^{(i)})W_i = 1, \bar{U}_i r_i = \pi^e\} \subseteq \Gamma_i \end{aligned}$$

is irreducible and thus defines the blow-up on this chart in a concrete example later on.

Taking the special fiber by reducing modulo π while using Remark 30 we get

$$\begin{aligned} \tilde{\Gamma}_{s_i} = \{(\bar{U}_i, &\bar{U}_i t_i, X_i, r_i, 1, t_i) : \\ &\bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (1 + \bar{a}_4\bar{U}_i^2 + \bar{a}_6\bar{U}_i^3), X_i = \bar{U}_i r_i^2, \bar{U}_i r_i = 0\}. \end{aligned}$$

We first note that $\bar{U}_i \neq 0$, otherwise we would get $0 = 1$ from the first equation. This gives $r_i = 0$ and thus $X_i = 0$. Hence we get

$$\tilde{\Gamma}_{s_i} \cong \operatorname{Spec} k_L[\bar{U}_i, t_i]/(\bar{U}_i t_i^2 + \bar{a}_1\bar{U}_i t_i - \bar{a}_2\bar{U}_i - \bar{a}_4\bar{U}_i^2 - \bar{a}_6\bar{U}_i^3 - 1).$$

By Remark 30, we note that

$$\bar{a}_2 \equiv (\alpha_{i,1}^{(i)} + \alpha_{i,2}^{(i)} + \gamma_i) \cdot (1-1) \equiv 0 \pmod{\pi}.$$

Similarly, we also have $\bar{a}_6 \equiv 0 \pmod{\pi^{4e}}$ by (4.1.19). The defining equation for $\tilde{\Gamma}_{s_i}$ becomes

$$\bar{U}_i t_i^2 + \bar{a}_1 \bar{U}_i t_i - \bar{a}_4 \bar{U}_i^2 - 1 = 0.$$

Let

$$S_i(\bar{U}_i, t_i) := \bar{U}_i t_i^2 + \bar{a}_1 \bar{U}_i t_i - \bar{a}_4 \bar{U}_i^2 - 1.$$

We see that there are no points $p = (\bar{U}_i, t_i)$ satisfying

$$S_i(p) = \frac{\partial S_i}{\partial \bar{U}_i}(p) = \frac{\partial S_i}{\partial t_i}(p) = 0.$$

Indeed, since we are in characteristic 2, the equation

$$\frac{\partial S_i}{\partial t_i}(p) = 2\bar{U}_i t_i + \bar{a}_1 \bar{U}_i = \bar{a}_1 \bar{U}_i = 0$$

would lead to $\bar{U}_i = 0$ since $\bar{a}_1 \not\equiv 0 \pmod{\pi}$ which is a contradiction. Thus $\tilde{\Gamma}_{s_i}$ is non-singular. Now we want to show that $\tilde{\Gamma}_{s_i}$ is reduced. It is sufficient to show that S_i is irreducible in $k_L[\bar{U}_i][t_i]$ as $k_L[\bar{U}_i][t_i]/(S_i)$ will then be a reduced ring (an integral domain). Suppose S_i is reducible and write

$$S_i = (m_i(\bar{U}_i)t_i + n_i(\bar{U}_i)) \cdot (p_i(\bar{U}_i)t_i + q_i(\bar{U}_i))$$

for $m_i, n_i, p_i, q_i \in k_L[\bar{U}_i]$. We require that

$$\begin{aligned} m_i p_i &= \bar{U}_i, \\ m_i q_i + n_i p_i &= \bar{a}_1 \bar{U}_i, \\ n_i q_i &= -\bar{a}_4 \bar{U}_i^2 - 1. \end{aligned}$$

These equations force n_i and q_i to both be of degree 1 in $\bar{U}_i$. Otherwise, if one of n_i or q_i is of degree 2, then one of m_i or p_i will also be of degree 2. Thus it would be impossible for $m_i p_i = \bar{U}_i$ to hold. Now both n_i and q_i being degree 1 force m_i and p_i to both be of degree 0 which is absurd. Therefore S_i must be irreducible over k_L. Hence we conclude that $\tilde{\Gamma}_{s_i}$ is both reduced and non-singular.

The 3rd Chart

On the third chart $t_i = 1$, we have the equations defining Γ_i are given by

$$\begin{cases} G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = 0, & (4.1.87) \\ X_i \bar{U}_i = \pi^{2e}, & (4.1.88) \\ (X_i - \alpha_j^{(i)}) W_i = 1, & (4.1.89) \\ \bar{U}_i = s_i \bar{V}_i, & (4.1.90) \\ \bar{U}_i r_i = \pi^e s_i, & (4.1.91) \\ \bar{V}_i r_i = \pi^e. & (4.1.92) \end{cases}$$

As before, (4.1.91) is redundant. Substituting (4.1.90) into (4.1.87) gives

$$\bar{V}_i(\bar{V}_i(1 + \bar{a}_1 s_i - \bar{a}_2 s_i^2) - (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)}) W_i^2 (s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3)) = 0.$$

This again has two irreducible components, with the first one being

$$\begin{cases} \bar{V}_i = 0, \\ X_i \bar{U}_i = \pi^{2e}, \\ (X_i - \alpha_j^{(i)}) W_i = 1, \\ \bar{U}_i = 0, \\ \pi^e = 0. \end{cases}$$

We note that this is E_i by (4.1.62). The other component is

$$\begin{cases} \bar{V}_i(1 + \bar{a}_1 s_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)}) W_i^2 (s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3), & (4.1.93) \\ X_i \bar{U}_i = \pi^{2e}, & (4.1.94) \\ (X_i - \alpha_j^{(i)}) W_i = 1, & (4.1.95) \\ \bar{U}_i = s_i \bar{V}_i, & (4.1.96) \\ \bar{V}_i r_i = \pi^e & (4.1.97) \end{cases}$$

which then has to contain $\mathrm{Bl}_{\mathrm{Spec}\, D_i}(\mathrm{Spec}\, B_i)$ on this chart. Substituting both (4.1.96) and (4.1.97) into (4.1.94) we get

$$\bar{V}_i(X_i s_i - \bar{V}_i r_i^2) = 0$$

which has two components. The first component is

$$\begin{cases} (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 s_i = 0, \\ \bar{V}_i = 0, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = 0, \\ \pi^e = 0 \end{cases}$$

which is contained in E_i. The other component is

$$\begin{cases} \bar{V}_i(1 + \bar{a}_1 s_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3), & (4.1.98) \\ X_i s_i = \bar{V}_i r_i^2, & (4.1.99) \\ (X_i - \alpha_j^{(i)})W_i = 1, & (4.1.100) \\ \bar{U}_i = s_i \bar{V}_i, & (4.1.101) \\ \bar{V}_i r_i = \pi^e & (4.1.102) \end{cases}$$

which has to contain our blow-up on this chart. Now consider the relation coming from (4.1.98):

$$\bar{V}_i = s_i(\bar{a}_2 s_i \bar{V}_i - \bar{a}_1 \bar{V}_i + (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4 s_i^2 \bar{V}_i^2 + \bar{a}_6 s_i^3 \bar{V}_i^3)).$$

Substituting this into (4.1.99) we get

$$X_i s_i = s_i(\bar{a}_2 s_i \bar{V}_i - \bar{a}_1 \bar{V}_i + (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4 s_i^2 \bar{V}_i^2 + \bar{a}_6 s_i^3 \bar{V}_i^3))r_i^2.$$

This again gives two components with the first one being

$$\begin{cases} \bar{V}_i = 0, \\ s_i = 0, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = 0, \\ \pi^e = 0 \end{cases}$$

which lies completely within E_i. The other component, which has to contain our blow-up, is given by the equations

$$\begin{cases} \bar{V}_i(1+\bar{a}_1 s_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3), \\ X_i = (\bar{a}_2 s_i \bar{V}_i - \bar{a}_1 \bar{V}_i + (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4 s_i^2 \bar{V}_i^2 + \bar{a}_6 s_i^3 \bar{V}_i^3))r_i^2, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = s_i \bar{V}_i, \\ \bar{V}_i r_i = \pi^e \end{cases}$$

We will show that the subscheme Γ_{t_i} defined by the above equations in Γ_i is irreducible and hence it is our blow-up on this chart in a concrete example later.

Taking the special fiber and using Remark 30, the subvariety over k is

$$\begin{aligned} \tilde{\Gamma}_{t_i} = \{(s_i\bar{V}_i, \bar{V}_i, X_i, r_i, s_i, 1) : & \bar{V}_i(1+\bar{a}_1 s_i - \bar{a}_2 s_i^2) = (s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3), \\ & X_i = r_i^2(1 - \bar{a}_1 \bar{V}_i + \bar{a}_2 s_i \bar{V}_i + \bar{a}_4 s_i^2 \bar{V}_i^2 + \bar{a}_6 s_i^3 \bar{V}_i^3), \bar{V}_i r_i = 0\}. \end{aligned}$$

We will use `Magma` to see that this is reduced, but also singular for our concrete example later. We will also show that the singular point (only at the origin) turns out to be nodal.

In the next section, we will demonstrate this construction for a concrete example of C that we have promised to give. We will then glue $\mathrm{Bl}_{\mathrm{Spec}\, D_i}(\mathrm{Spec}\, B_i)$ with $\mathrm{Spec}\, A_i$ for the example to get a proper and flat scheme $\mathcal{C}_i$ over $\mathcal{O}_L$ for $i = 1, 2$. Finally, we will show that $\mathcal{C}_1$ and $\mathcal{C}_2$ will glue to a semistable model for our C over $\mathcal{O}_L$.

4.1.2 A Concrete Example

We now apply our construction of a semistable model on

$$C : y^2 = x^5 - 2x^4 - 129x^3 + 130x^2 + 3905x \tag{4.1.103}$$

as a numerical example over $\mathcal{O}_K$ the ring of integers of $K = \mathbb{Q}(\sqrt{5})$. We will consider C as a curve over the completion of K at the prime $\pi = 2$ i.e. the local field $\mathrm{Frac}\,\widehat{(\mathcal{O}_K)_{(2)}}$ where $(\mathcal{O}_K)_{(2)}$ is localized at the maximal ideal (2). Since the rational prime 2 is inert over K, we get $e = 1$ in this example. Since all of our coefficients are over $\mathcal{O}_K$, we may for now work over K instead of the completion.

Let $f(x) := x^5 - 2x^4 - 129x^3 + 130x^2 + 3905x$. We see that f factors as

$$f(x) = x(x - (z - 2^3))(x - (z + 2^3))(x - (\bar{z} - 2^3))(x - (\bar{z} + 2^3))$$

over $\mathcal{O}_K$ where $z = \frac{1+\sqrt{5}}{2}$ and $\bar{z} = \frac{1-\sqrt{5}}{2}$. Taking the special fiber, we get

$$C_2 : y^2 = x(x-z)^2(x-\bar{z})^2$$

over $k = \mathcal{O}_K/(2) = \{0, 1, z, \bar{z}\}$. We note that C satisfies the assumptions of Condition 1 since the clusters all have depth $\geq 4e = 4$.

Let

$$\beta_{1,1} = z - 2^3, \quad \beta_{1,2} = z + 2^3, \quad \beta_{2,1} = \bar{z} - 2^3, \quad \beta_{2,2} = \bar{z} + 2^3.$$

By picking

$$\beta_1 = z - 2^3, \quad \text{and} \quad \beta_2 = \bar{z} - 2^3$$

we have that $v_2(\beta_j - \beta_{j,k}) \geq v_2(2^4) = 4 = 4e$. Computing the required constants, we obtain

$$\begin{aligned} \gamma = 0, \quad \gamma_1 = 2^3 - z, \quad \gamma_2 = 2^3 - \bar{z}, \\ \alpha_{1,1}^{(1)} = 0, \quad \alpha_{1,1}^{(2)} = \sqrt{5}, \quad \alpha_{1,2}^{(1)} = 16, \\ \alpha_{1,2}^{(2)} = 16 + \sqrt{5}, \quad \alpha_{2,1}^{(1)} = -\sqrt{5}, \quad \alpha_{2,1}^{(2)} = 0, \\ \alpha_{2,2}^{(1)} = 16 - \sqrt{5}, \quad \alpha_{2,2}^{(2)} = 16. \end{aligned}$$

These then give

$$\alpha_1^{(1)} = 0 = \alpha_2^{(2)}, \quad \text{and} \quad \alpha_1^{(2)} = \sqrt{5} = -\alpha_2^{(1)}.$$

We note that

$$\ell_1 := \sqrt{-(\alpha_{1,1}^{(1)} + \alpha_{1,2}^{(1)} + \gamma_1)} = \sqrt{\frac{-47 + \sqrt{5}}{2}}, \quad \text{and} \quad \ell_2 := \sqrt{-(\alpha_{2,1}^{(2)} + \alpha_{2,2}^{(2)} + \gamma_2)} = \sqrt{\frac{-47 - \sqrt{5}}{2}}.$$

Thus we form the extension

$$L = K(\ell_1, \ell_2),$$

which is of degree 4 over K. Using `Magma` we can check that $2\mathcal{O}_L$ is no longer a prime and $2\mathcal{O}_L = \mathfrak{p}^4$ for some prime ideal $\mathfrak{p} \subset \mathcal{O}_L$.

```
> OL := Integers(L);
> Factorization(2*OL);
[
<Prime Ideal of OL
Two element generators:
[[[2, 0], [0, 0]], [[0, 0], [0, 0]]]
[[[4, 2], [0, -1]], [[-1, 1], [-1, 0]]], 4>
]
```

Hence k_L will be the residue field of $\mathcal{O}_L$ at the prime $\mathfrak{p}$ which we can see that it remains to be $\mathrm{GF}(2^2)$.

```
> ResidueClassField(Factorization(2*OL)[1,1]);
Finite field of size 2^2
Mapping from: RngOrd: OL to GF(2^2)
given by the ideal Prime Ideal of OL
Two element generators:
[[[2, 0], [0, 0]], [[0, 0], [0, 0]]]
[[[1, 0], [1, -1]], [[2, -3], [1, 1]]]
```

For $i = 1$, the constants a_k's are

$$a_1 = 2\ell_1, \quad a_2 = -\ell_1\left(1 - \frac{(X_1 - \alpha_{2,1}^{(1)})(X_1 - \alpha_{2,2}^{(1)})}{(X_1 - \alpha_2^{(1)})^2}\right), \quad a_4 = 120 - 8\sqrt{5}, \quad a_6 = 0.$$

For $i = 2$, the constats a_k's are

$$a_1 = 2\ell_2, \quad a_2 = -\ell_2\left(1 - \frac{(X_2 - \alpha_{1,1}^{(2)})(X_2 - \alpha_{1,2}^{(2)})}{(X_2 - \alpha_1^{(2)})^2}\right), \quad a_4 = 120 + 8\sqrt{5}, \quad a_6 = 0.$$

For $i = 1, 2$, $\pi = 2$ and $e = 1$, plugging in our blow-up construction from Section 4.1.1, we get three schemes which we will glue together using the birational maps defined via the homogeneous coordinates r_i, s_i, t_i on the three different charts (see Section 3.5.4):

$$\begin{aligned}\Gamma_{r_i} = \{(\bar{U}_i, \bar{V}_i, X_i, W_i, s_i, t_i) :\\ X_i(t_i^2 + \bar{a}_1 s_i t_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\pi^{2e}s_i^2 + \bar{a}_6\pi^{3e}s_i^3),\\ X_i s_i = \pi^e, (X_i - \alpha_j^{(i)})W_i = 1, \bar{U}_i = \pi^e s_i, \bar{V}_i = \pi^e t_i\},\end{aligned}$$

$$\begin{aligned}\Gamma_{s_i} = \{(\bar{U}_i, \bar{V}_i, X_i, W_i, r_i, t_i) :\\ \bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\bar{U}_i^2 + \bar{a}_6\bar{U}_i^3),\\ X_i = \bar{U}_i r_i^2, (X_i - \alpha_j^{(i)})W_i = 1, \bar{U}_i r_i = \pi^e, \bar{V}_i = \bar{U}_i t_i\},\end{aligned}$$

and

$$\Gamma_{t_i} = \{(\bar{U}_i, \bar{V}_i, X_i, W_i, r_i, s_i) :$$
$$\bar{V}_i(1 + \bar{a}_1 s_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3),$$
$$X_i = (\bar{a}_2 s_i \bar{V}_i - \bar{a}_1 \bar{V}_i + (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4 s_i^2 \bar{V}_i^2 + \bar{a}_6 s_i^3 \bar{V}_i^3))r_i^2,$$
$$(X_i - \alpha_j^{(i)})W_i = 1, \bar{U}_i = s_i \bar{V}_i, \bar{V}_i r_i = \pi^e\}$$

for $j = 1, 2$ and $j \neq i$ where $\bar{a}_i = 2^{-i} a_i$. Their corresponding special fibers are

$$\tilde{\Gamma}_{r_i} = \{(0, 0, X_i, 1, s_i, t_i) : X_i(t_i^2 + [\bar{a}_1]s_i t_i - [\bar{a}_2]s_i^2) = 1, X_i s_i = 0\},$$

$$\tilde{\Gamma}_{s_i} = \{(\bar{U}_i, \bar{V}_i, X_i, r_i, t_i) :$$
$$\bar{U}_i(t_i^2 + [\bar{a}_1]t_i - [\bar{a}_2]) = (1 + [\bar{a}_4]\bar{U}_i^2 + [\bar{a}_6]\bar{U}_i^3),$$
$$X_i = \bar{U}_i r_i^2, \bar{U}_i r_i = 0, \bar{V}_i = \bar{U}_i t_i\},$$

and

$$\tilde{\Gamma}_{t_i} = \{(\bar{U}_i, \bar{V}_i, X_i, r_i, s_i) :$$
$$\bar{V}_i(1 + [\bar{a}_1]s_i - [\bar{a}_2]s_i^2) = (s_i + [\bar{a}_4]s_i^3 \bar{V}_i^2 + [\bar{a}_6]s_i^4 \bar{V}_i^3),$$
$$X_i = r_i^2(1 - \bar{a}_1 \bar{V}_i + \bar{a}_2 s_i \bar{V}_i + [\bar{a}_4]s_i^2 \bar{V}_i^2 + [\bar{a}_6]s_i^3 \bar{V}_i^3), \bar{U}_i = s_i \bar{V}_i, \bar{V}_i r_i = 0\}$$

respectively where the $[\bar{a}_i]$'s are their corresponding residue classes in the residue field $k = \mathrm{GF}(2^2)$. We recall from Proposition 4.1.5 that $\bar{a}_1 \neq 0$ and $\bar{a}_2 = \bar{a}_6 = 0$ in k.

Verifying the Blow-up

Before further examine the special fibers, we can do a sanity check to show that Γ_{r_i}, Γ_{s_i} and Γ_{t_i} are indeed the blow-ups on the three standard charts respectively. We first argue that Γ_{s_i} does not intersect the exceptional divisor E_i given as (4.1.62). We note that any point on E_i has the U_i coordinate being 0. By assuming $U_i = 0$ on Γ_{s_i}, we obtain the equations $X_i = 0$ and $\frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} = 0$ or equivalently

$$\frac{\alpha_{j,1}^{(i)} \alpha_{j,2}^{(i)}}{(\alpha_j^{(i)})^2} = 0. \tag{4.1.104}$$

For $i = 1$, we have

$$\frac{\alpha_{2,1}^{(1)} \alpha_{2,2}^{(1)}}{(\alpha_2^{(1)})^2} = \frac{5 - 16\sqrt{5}}{5}$$

which is a unit in $\mathcal{O}_L$ and thus can not be 0 hence yielding no solution for (4.1.104). Similarly, for $i = 2$ we have

$$\frac{\alpha_{1,1}^{(2)}\alpha_{1,2}^{(2)}}{(\alpha_1^{(2)})^2} = \frac{5 + 16\sqrt{5}}{5}$$

which is again, a unit in $\mathcal{O}_L$ and hence yielding no solution for (4.1.104). This shows that no point on Γ_{s_i} has the U_i coordinate being 0 i.e. Γ_{s_i} does not intersect E_i. This tells us the closed subscheme Γ_{s_i} has to be our blow-up on the chart $s_i = 1$ since

$$\mathrm{Bl}_{\mathrm{Spec}\, D_i}(\mathrm{Spec}\, B_i) = \mathrm{cl}_{\mathrm{Bl}_{\mathrm{Spec}\, D_i^{\mathrm{tot}}}(\mathbb{A}^4_{\mathcal{O}_L})}\left(\varphi_i^{-1}(\mathrm{Spec}\, B_i) - E_i\right).$$

Now using the maps

$$r_i = \frac{1}{s_i} \qquad \text{and} \qquad t_i = \frac{t_i}{s_i}$$

from chart $s_i = 1$ to chart $r_i = 1$, the defining equations for Γ_{s_i}:

$$\begin{cases} \bar{U}_i(t_i^2 + \bar{a}_1 t_i - \bar{a}_2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\bar{U}_i^2 + \bar{a}_6\bar{U}_i^3), \\ X_i = \bar{U}_i r_i^2, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{V}_i = \bar{U}_i t_i, \\ \bar{U}_i r_i = \pi^e \end{cases}$$

become

$$\begin{cases} \pi^e s_i\left(\frac{t_i^2}{s_i^2} + \bar{a}_1\frac{t_i}{s_i} - \bar{a}_2\right) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\pi^{2e}s_i^2 + \bar{a}_6\pi^{3e}s_i^3), \\ X_i s_i = \pi^e, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ V_i = \pi^e t_i, \\ U_i = \pi^e s_i \end{cases}$$

where the first equation can be rewritten as

$$X_i(t_i^2 + \bar{a}_1 s_i t_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(1 + \bar{a}_4\pi^{2e}s_i^2 + \bar{a}_6\pi^{3e}s_i^3)$$

since we are looking at the open set where both s_i and r_i are invertible. Thus we precisely get Γ_{r_i} after the substitutions. Note that we can also obtain Γ_{s_i} from the defining equations for Γ_{r_i} with the substitutions $s_i = \frac{1}{r_i}$ and $t_i = \frac{t_i}{r_i}$. Therefore Γ_{r_i} and Γ_{s_i} are birationally equivalent on the dense open subset $r_i \neq 0$ and $s_i \neq 0$. This shows that Γ_{r_i} must be the blow-up on the first chart $r_i \neq 0$ or $r_i = 1$.

Now to go from the first chart $r_i = 1$ to the third chart $t_i = 1$, we consider the maps

$$t_i = \frac{1}{r_i} \quad \text{and} \quad s_i = \frac{s_i}{r_i}.$$

Applying this substitution, the defining equations for Γ_{r_i} become

$$\begin{cases} \bar{V}_i(1 + \bar{a}_1 s_i - \bar{a}_2 s_i^2) = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2(s_i + \bar{a}_4 s_i^3 \bar{V}_i^2 + \bar{a}_6 s_i^4 \bar{V}_i^3), \\ X_i s_i = \bar{V}_i r_i^2, \\ (X_i - \alpha_j^{(i)})W_i = 1, \\ \bar{U}_i = s_i \bar{V}_i, \\ \bar{V}_i r_i = \pi^e \end{cases}$$

which is not quite our Γ_{t_i} yet, as we are required to remove an extra component corresponding to the exceptional divisor E_i (this was at the last step of the previous section).

By setting $U_i = V_i = \pi^e = 0$, the defining equations for Γ_{t_i} become

$$\begin{cases} (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 s_i = 0, \\ (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 r_i^2 = X_i \end{cases}$$

or equivalently

$$\begin{cases} s_i = 0, \\ (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 r_i^2 = X_i. \end{cases}$$

Therefore Γ_{s_i} intersects the exceptional divisor E_i along the irreducible scheme given by

$$V(s_i, (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 r_i^2 - X_i). \tag{4.1.105}$$

Now we examine the blow-up on the first chart Γ_{r_i}, and suppose $X_i = 0$. We note that the defining equations of Γ_{r_i} becomes

$$\begin{cases} \frac{\alpha_{j,1}^{(i)}\alpha_{j,2}^{(i)}}{(\alpha_j^{(i)})^2}(1 + \bar{a}_4 \pi^{2e} s_i^2 + \bar{a}_6 \pi^{3e} s_i^3) = 0, \\ \pi^e = 0 \end{cases}$$

or equivalently

$$\frac{\alpha_{j,1}^{(i)}\alpha_{j,2}^{(i)}}{(\alpha_j^{(i)})^2} = 0.$$

We have already shown that there are no solution for this equation and so $X_i \neq 0$ on Γ_{r_i}. Keeping this in mind, and intersect Γ_{r_i} with $U_i = V_i = \pi^e = 0$ to obtain the equations

$$\begin{cases} X_i t_i^2 = (X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2, \\ s_i = 0, \\ X_i \neq 0. \end{cases}$$

With the transition map $t_i = \frac{1}{r_i}$, this is precisely the scheme (4.1.105) we get when intersecting Γ_{t_i} with E_i where the points corresponding to $X_i = 0$ is removed. We note that $X_i = 0$ only gives one point $(U_i, V_i, X_i, s_i, t_i) = (0,0,0,0,0)$ on $\Gamma_{t_i} \cap E_i$ which we will soon show that it is a nodal singularity on the special fiber of Γ_{t_i}. This shows that Γ_{r_i} and Γ_{t_i} are birationally equivalent on the dense open set $r_i \neq 0$ and $t_i \neq 0$ since the hyperplane $X_i \neq 0$ does not intersect Γ_{r_i}. Denote Ω_{r_i} and Ω_{t_i} as the first and the third standard charts of $\operatorname{Spec} B_i \times \mathbb{P}^2$. Taking the closure of $\Gamma_{t_i} \cap \Omega_{r_i}$ in Ω_{t_i} only adds the origin to the set and thus the closure is precisely Γ_{t_i}. This shows that Γ_{t_i} has to be the blow-up on the third chart Ω_{t_i}.

Examining the Special Fibers

Now we will examine the special fibers of our blow-up. Using `Magma` we can do a sanity check to see that $\tilde{\Gamma}_{r_i}$ and $\tilde{\Gamma}_{s_i}$ are indeed both reduced and non-singular.

```
// flag == true if the curve Γ̃_{r_i} is reduced and non-singular.
> flag := true;
> F := FiniteField(4); // Defining GF(2^2).
> A<X,s,t> := AffineSpace(F,3);
// Looping over every possible value of [ā1] and [ā2].
> for a1,a2 in F do
>     C:=Scheme(A, [X*(t^2+a1*s*t-a2*s^2)-1,X*s]); // Defining Γ̃_{r_i}.
>     Dim := Dimension(C); // Checking if Γ̃_{r_i} is a curve or not.
>     Red := IsReduced(C);
>     Sing := IsSingular(C);
      // Checking if dim(Γ̃_{r_i}) = 1 and Γ̃_{r_i} is reduced and non-singular.
>     if not(Dim eq 1) or not(Red) or Sing then
>         flag := false;
>         a1,a2;
>     end if;
> end for;
> flag;
```

```
true // flag == true.
```

```
// flag == true if the curve Γ̃_{s_i} is reduced and non-singular.
> flag := true;
> F := FiniteField(4);
> A<U,V,X,r,t> := AffineSpace(F,5);
// Looping over every possible value of [ā_1], [ā_2], [ā_4], and [ā_6].
> for a1,a2,a4,a6 in F do
      // Using the fact that [ā_1] ≠ 0 and [ā_2] = [ā_6] = 0.
>     if a1 ne 0 and a2 eq 0 and a6 eq 0 then
          // Defining Γ̃_{s_i}.
>         C:=Scheme(A, [
>         U*(t^2+a1*t-a2)-(1 + a4*U^2+a6*U^3),
>         X-U*r^2,
>         U*r,
>         V-U*t]);
>         Dim := Dimension(C); // Checking if Γ̃_{s_i} is a curve or not.
>         Red := IsReduced(C);
>         Sing := IsSingular(C);
          // Checking if dim(Γ̃_{s_i}) = 1 and Γ̃_{s_i} is reduced and non-singular.
>         if not(Dim eq 1) or not(Red) or Sing then
>             a1,a2,a4,a6;
>         end if;
>     end if;
> end for;
> flag;
true // flag == true.
```

Finally, for $\tilde{\Gamma}_{t_i}$, we will first compute $[\bar{a}_1]$ and $[\bar{a}_4]$. For $i = 1$, we note that $\bar{a}_1 = \ell_1$ and $\bar{a}_4 = 30 - 2\sqrt{5}$. It is immediate that $[\bar{a}_4] = 0$. For $[\bar{a}_1]$, Magma tells us that it is $\bar{z} \in k = \{0, 1, z, \bar{z}\}$:

```
> OL := Integers(L);
> k,f := ResidueClassField(Factorization(2*OL)[1,1]);
> f(OL!(L0.1)); // L0.1 is ℓ_1 and f compute its residue class in k.
k.1^2 // k.1 represents z = (1+√5)/2 and k.1^2 is its conjugate.
```

Now we see that the curve $\tilde{\Gamma}_{t_1}$ is reduced but also singular with a singularity at the origin.

```
> A<U,V,X,r,s> := AffineSpace(k,5);
// Defining the coefficients [ā_k]'s.
> a1 := k.1^2;
> a2 := 0;
> a4 := 0;
> a6 := 0;
// Defining Γ̃_{t_1}.
> C:=Scheme(A, [
> V*(1+a1*s-a2*s^2)-(s+a4*s^3*V^2+a6*s^4*V^3),
> X-r^2*(1-a1*V+a2*s*V+a4*s^2*V^2+a6*s^3*V^3),
> U-s*V,
> V*r]);
> Dimension(C);
> IsReduced(C);
> IsSingular(C);
1 // Γ̃_{t_1} is indeed a curve.
true // It is reduced.
true // But also singular.
```

We now compute the singular points.

```
> S := SingularSubscheme(C);
// Computing the singular points over k = GF(2^2).
> PointsOverSplittingField(S);
{@ (0, 0, 0, 0, 0) @}
Finite field of size 2^2
// We see that the only singular point is the origin.
```

The following computation shows us that $\tilde{\Gamma}_{t_1}$ is the union of two varieties.

```
// Computing the irreducible components of Γ̃_{t_1}.
> Irred := IrreducibleComponents(C);
> Irred[1];
> Irred[2];
// The first curve.
Scheme over GF(2^2) defined by
U + k.1*V + k.1*s,
V*s + k.1*V + k.1*s,
X,
```

```
r
// The second curve.
Scheme over GF(2^2) defined by
U,
V,
X + r^2,
s
```

The following shows that $\tilde{\Gamma}_{t_1}$ is actually the union of two smooth curves which intersects at the origin, creating a singularity.

```
> Dimension(Irred[1]);
> IsReduced(Irred[1]);
> IsSingular(Irred[1]);
> Dimension(Irred[2]);
> IsReduced(Irred[2]);
> IsSingular(Irred[2]);
1
true
false
1
true
false
```

We will use Proposition 3.4.15 to check that the origin is actually a node for $\tilde{\Gamma}_{t_1}$. Writing out the equation given by `Magma`, we have

$$\tilde{\Gamma}_{t_1} = V(\bar{U}_1 + z\bar{V}_1 + zs_1, \bar{V}_1 s_1 + z\bar{V}_1 + zs_1, X_1, r_1) \cup V(\bar{U}_1, \bar{V}_1, \bar{X}_1 + r_1^2, s_1).$$

Let $x_1 := \bar{U}_1, x_2 := \bar{V}_1, x_3 := X_1, x_4 := r_1, x_5 := s_1$, and write $f_1 := \bar{U}_1 + z\bar{V}_1 + zs_1, ..., f_4 := r_1, g_1 := \bar{U}_1, ..., g_4 := \bar{X}_1 + r_1^2, s_1$. Computing the corresponding Jacobian matrices evaluated at $p = (0, 0, 0, 0, 0)$ we get

$$J_1 = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \le i \le 4, 1 \le j \le 5} = \begin{pmatrix} 1 & z & 0 & 0 & z \\ 0 & z & 0 & 0 & z \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$J_2 = \left(\frac{\partial g_i}{\partial x_j}(p)\right)_{1\leq i\leq 4, 1\leq j\leq 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $\ker(J_1) = \mathrm{Span}_k \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \end{pmatrix} \right\}$ and $\ker(J_2) = \mathrm{Span}_k \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\}$. It is clear that the two spanning vectors are orthogonal to each other, and thus by Proposition 3.4.15, the only singular point $p = (0, 0, 0, 0, 0)$ is a node of $\tilde{\Gamma}_{t_1}$. We can also verify this using `Magma`:

```
> P := PointsOverSplittingField(S);
> P[1];
// Magma function to check if a singularity on a curve is nodal or not.
> IsNode(C,P[1]);
(0, 0, 0, 0, 0)
true
```

For $i = 2$, we have $[\bar{a}_1] = z$ and $[\bar{a}_4] = 0$. `Magma` tells us that the computation is almost identical besides a minor difference in the coefficients of one of the two smooth components:

```
// Computing the irreducible components of Γ̃_{t_2}.
> Irred := IrreducibleComponents(C);
> Irred[1];
> Irred[2];
// The four coefficients here are z̄ instead of z.
Scheme over GF(2^2) defined by
U + k.1^2*V + k.1^2*s,
V*s + k.1^2*V + k.1^2*s,
X,
r
// The second curve is identical.
Scheme over GF(2^2) defined by
U,
V,
X + r^2,
s
```

This difference in the coefficients for the first component does not affect anything when computing the Jacobian matrices at the origin and thus the origin is still the only singularity of $\tilde{\Gamma}_{t_2}$ which is a node.

The Semistable Model

Similar to the example in Section 3.5.4, the gluing maps between r_i, s_i, t_i is exactly the same as the gluing maps described for u_0, u_1, u_2 in that section. Therefore we obtain a new scheme $\mathcal{B}_i = \mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ by gluing together $\Gamma_{r_i}, \Gamma_{s_i}, \Gamma_{t_i}$. Our next goal is to use Proposition 3.5.18 to conclude that our new scheme obtained by gluing up $\mathcal{B}_i$ with $\operatorname{Spec} A_i$ is proper over $\mathcal{O}_L$.

Proposition 4.1.6. *The schemes* $\operatorname{Spec} A_i$ *and* $\mathcal{B}_i$ *for* $i = 1, 2$ *glue to a proper scheme over* $\mathcal{O}_L$.

Proof. Using the exact same maps for gluing $\operatorname{Spec} A_i$ and $\operatorname{Spec} B_i$ from Proposition 4.1.2, the blow-up $\mathcal{B}_i$ and $\operatorname{Spec} A_i$ glue to a scheme $\mathcal{C}_i'$ over $\mathcal{O}_L$.

Using the exact same maps (given by the relations (4.1.37)–(4.1.42)) for gluing $\mathcal{C}_1$ and $\mathcal{C}_2$, the two schemes $\mathcal{C}_1'$ and $\mathcal{C}_2'$ glue to a new scheme $\mathcal{C}_0'$ over $\mathcal{O}_L$. Similarly, we can glue $\mathcal{C}_0'$ with C^∞ and C^0 (to include the point $x = 0$) by using the relations (4.1.47)–(4.1.48) from Proposition 4.1.2 as well as (4.1.3)–(4.1.4) to obtain a new scheme $\mathcal{C}'$ over $\mathcal{O}_L$.

Let us now recall the subscheme $\operatorname{Spec} D_i \subseteq \operatorname{Spec} B_i$ given by

$$D_i = \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]/(G_i(\bar{U}_i, \bar{V}_i, X_i, W_i), \pi^{2e} - \bar{U}_i X_i, (X_i - \alpha_j^{(i)})W_i - 1, \bar{U}_i, \bar{V}_i, \pi^e)$$

where

$$G_i(\bar{U}_i, \bar{V}_i, X_i, W_i) = \bar{V}_i^2 + \bar{a}_1 \bar{V}_i \bar{U}_i - \Big((\bar{U}_i + \bar{a}_4 \bar{U}_i^3 + \bar{a}_6 \bar{U}_i^4)(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})W_i^2 + \bar{a}_2 \bar{U}_i^2\Big)$$

for $i \neq j$ and $\bar{a}_i = a_i \pi^{-ie}$. We first note that

$$G_i(\bar{U}_i, \bar{V}_i, X_i, W_i), \pi^{2e} - \bar{U}_i X_i \in (\bar{U}_i, \bar{V}_i, \pi^e) \subset \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i, W_i]$$

and so

$$\begin{aligned} \operatorname{Spec} D_i &= V(\bar{U}_i, \bar{V}_i, \pi^e) \\ &\subseteq \operatorname{Spec} B_i \\ &= \operatorname{Spec} \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]/(G_i(\bar{U}_i, \bar{V}_i, X_i, W_i), \pi^{2e} - \bar{U}_i X_i) \end{aligned}$$

is just the vanishing of the coordinates $\bar{U}_i, \bar{V}_i$ and π^e on $\operatorname{Spec} B_i$. This implies $\operatorname{Spec} D_i$ is a closed subscheme of $\mathcal{C}$ as well. We note that by Lemma 3.5.9, the blow-up $\mathrm{Bl}_{\operatorname{Spec} D_i}(\mathcal{C})$ leaves anything outside of $\operatorname{Spec} B_i$ unchanged since $\operatorname{Spec} D_i$ lies completely within $\operatorname{Spec} B_i$. Hence by Proposition 3.5.10, the blow-up $\mathrm{Bl}_{\operatorname{Spec} D_i}(\mathcal{C})$ is just the gluing of $\mathrm{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ with $\operatorname{Spec} A_i$, C^∞ and C^0 for $i = 1, 2$ which is precisely $\mathcal{C}'$. By Proposition 4.1.4 and Proposition 3.5.18, it follows that $\mathcal{C}'$ is proper. □

Keeping the same notations as in the above proof, we let $\mathcal{C}'$ be our newly glued scheme. To conclude that $\mathcal{C}'$ is another model of C over L, we are left to show that $\mathcal{C}'$ is flat over $\mathcal{O}_L$ and $(\mathcal{C}')_L \cong C/L$.

Unlike the general definition for $\operatorname{Spec} B_i$ given in Section 4.1, we can in fact show that $\operatorname{Spec} B_i$ for our concrete example is irreducible for $i = 1, 2$, or equivalently, to show that the ideal

$$(G_i(\bar{U}_i, \bar{V}_i, X_i), X_i\bar{U}_i - 4) \subseteq \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]$$

is prime for $i = 1, 2$.

Proposition 4.1.7. *The schemes* $\operatorname{Spec} B_i$ *are irreducible.*

Proof. Recall that we have

$$\operatorname{Spec} B_i = \operatorname{Spec} \mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]/(G_i(\bar{U}_i, \bar{V}_i, X_i), \pi^{2e} - \bar{U}_i X_i)$$

where

$$G_i(\bar{U}_i, \bar{V}_i, X_i) = \bar{V}_i^2 + a_1\pi^{-e}\bar{V}_i\bar{U}_i - \left((\bar{U}_i + a_4\pi^{-4e}\bar{U}_i^3 + a_6\pi^{-6e}\bar{U}_i^4)\prod_{j=1, j\neq i}^{2} \frac{(X_i - \alpha_{j,1}^{(i)})(X_i - \alpha_{j,2}^{(i)})}{(X_i - \alpha_j^{(i)})^2} + a_2\pi^{-2e}\bar{U}_i^2\right).$$

Using commutative algebra, we have the isomorphism

$$\frac{\mathcal{O}_L[\bar{U}_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]}{(G_i(\bar{U}_i, \bar{V}_i, X_i), \pi^{2e} - \bar{U}_i X_i)} \cong \frac{\mathcal{O}_L[4/X_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]}{(G_i(4/X_i, \bar{V}_i, X_i))}$$

i.e. we may replace U_i with $\frac{4}{X_i}$ since $\pi^{2e} = 4$. Hence to prove our claim, it is sufficient to show that $(G_i(4/X_i, \bar{V}_i, X_i))$ is prime in $\mathcal{O}_L[4/X_i, \bar{V}_i, X_i][\bar{T}_i^{-1}]$ for $i = 1, 2$.

For our concrete example, we have

$$G_1(4/X_1, \bar{V}_1, X_1) = \bar{V}_1^2 + \frac{\sqrt{24 - 2z}}{4X_1}\bar{V}_1 + \frac{4X_1^3 + (4\sqrt{5} - 64)X_1^2 - 1024X_1 + 992\sqrt{5} - 7840}{X_1^3(X_1 + \sqrt{5})}$$

and

$$G_2(4/X_2, \bar{V}_2, X_2) = \bar{V}_2^2 + \frac{\sqrt{24 + 2z}}{4X_2}\bar{V}_2 + \frac{4X_2^3 - (4\sqrt{5} + 64)X_2^2 - 1024X_2 - 992\sqrt{5} - 7840}{X_2^3(X_2 - \sqrt{5})}$$

where $z = \frac{1+\sqrt{5}}{2}$. Using `Magma`, we may check that both G_1 and G_2 are irreducible over $\operatorname{Frac} \mathcal{O}_L[4/X_i, X_i][\bar{T}_i^{-1}] \cong \operatorname{Frac} L[X_i]$ with respect to the variable $\bar{V}_i$:

```
> LL<X> := PolynomialRing(L,1);
> FF := FieldOfFractions(LL);
> x := FF!X;
> RR<V> := PolynomialRing(FF,1);
// z = √5.
// L0.1 corresponds to ℓ1.
> Factorization(V^2 + (4*L0.1/x)*V + (4*x^3 + (4*z - 64)*x^2 - 1024*x
> + 992*z - 7840)/(x^3*(x+z)));
// L.1 corresponds to ℓ2.
> Factorization(V^2 + (4*L.1/x)*V + (4*x^3 - (4*z + 64)*x^2 - 1024*x
> - 992*z - 7840)/(x^3*(x+z)));
[
<V^2 + 4*L0.1/$.1*V + (4*$.1^3 + (4*z - 64)*$.1^2 - 1024*$.1 + (992*z -
7840))/($.1^4 + z*$.1^3), 1>
]
[
<V^2 + 4*L.1/$.1*V + (4*$.1^3 + (-4*z - 64)*$.1^2 - 1024*$.1 + (-992*z -
7840))/($.1^4 + z*$.1^3), 1>
]
// This shows that the two polynomials given in V
// are irreducible over Frac L[X].
```

Since both G_1 and G_2 are monic in V_1 and V_2 respectively, by Theorem 3.3.17, both (G_1) and (G_2) are prime in $\mathcal{O}_L[X_1, 4/X_1, \bar{V}_1][\bar{T}_1^{-1}]$ and $\mathcal{O}_L[X_2, 4/X_2, \bar{V}_2][\bar{T}_2^{-1}]$ respectively. $\square$

Remark 32. We could have showed that $G_1(4/X_1, \bar{V}_1, X_1)$ and $G_2(4/X_1, \bar{V}_1, X_1)$ are irreducible over $\operatorname{Frac} L[X_1]$ and $\operatorname{Frac} L[X_2]$ by hand. For example when $i = 1$, using the quadratic formula for $G_1(4/X_1, \bar{V}_1, X_1)$ for solving $\bar{V}_1$ gives

$$\bar{V}_1 = \frac{-\frac{\sqrt{24-2z}}{4X_1} \pm \sqrt{X_1 \cdot f_1(X_1)}}{2}$$

where

$$\begin{aligned} f_1(X_1) = & X_1^4 + \left(\frac{15+3\sqrt{5}}{2}\right) X_1^3 \\ & + (496 + 15\sqrt{5})X_1^2 + \left(\frac{3845 - 1483\sqrt{5}}{2}\right) X_1 + (1240 - 1960\sqrt{5}). \end{aligned}$$

Since $\sqrt{X_1 \cdot f_1(X_1)} \notin \operatorname{Frac} L[X_1]$ we conclude that $G_1(4/X_1, \bar{V}_1, X_1)$ can not be reducible as a polynomial in $(\operatorname{Frac} L[X_1])\,[\bar{V}_1]$.

Proposition 4.1.8. *The scheme $\mathcal{C}'$ is flat over $\mathcal{O}_L$.*

Proof. By definition, we can see that flatness is a stalk-local property. Hence we only have to check for flatness on the affine open charts. We first note that for $i = 1, 2$ single defining polynomial $F_i(\bar{X}_i, \bar{Y}_i)$ for $\operatorname{Spec} A_i$ given in (4.1.22) has a unit coefficient. By Corollary 3.3.16.1, the affine schemes $\operatorname{Spec} A_i$ are flat over $\mathcal{O}_L$ for $i = 1, 2$.

It remains to show that $\mathcal{B}_i$ is flat over $\mathcal{O}_L$ for $i = 1, 2$. We can in fact go even further and show this on the three affine charts Γ_{r_i}, Γ_{s_i}, and Γ_{t_i} of $\mathcal{B}_i$. By Lemma 3.5.11 and Proposition 4.1.7, the blow-ups $\operatorname{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$ is irreducible for $i = 1, 2$. For each $i = 1, 2$, the three charts are irreducible since they are open subsets of the irreducible space $\operatorname{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i)$. The primary decomposition of their ideals are just themselves. Hence by Theorem 3.3.26, we just need to verify that they are not contained in the ideal (π). Indeed, each defining ideal of Γ_{r_i}, Γ_{s_i}, and Γ_{t_i} contains a generator with unit coefficients and thus none of the ideals can be contained in (π). □

Proposition 4.1.9. *The scheme $\mathcal{C}'$ has the same generic fiber as C over L.*

Proof. By Proposition 4.1.3, it is suffices to show that $\mathcal{C}'$ and $\mathcal{C}$ has the same generic fiber over L. Similar to Proposition 3.5.21 the ideal $(\bar{U}_i, \bar{V}_i, \pi^e)$ contains a unit π^e when considered as an ideal over L, and thus

$$(\mathcal{B}_i)_L = (\operatorname{Bl}_{\operatorname{Spec} D_i}(\operatorname{Spec} B_i))_L \cong (\operatorname{Spec} B_i)_L.$$

Since $\operatorname{Spec} A_i$ is unchanged as well, we have that $(\mathcal{C}')_L \cong \mathcal{C}_L$ since taking coproducts (gluing) commute with base changing. □

Thus we have established that $\mathcal{C}'$ is indeed a new model of C over L. The following proposition will show that $\mathcal{C}'$ is a semistable model of C over L.

Proposition 4.1.10. *The model $\mathcal{C}'$ of C over L is semistable.*

Proof. By construction, the two worse-than-nodal singularities on the original curve C_π were identified with the singular loci of $(\operatorname{Spec} B_i)_\pi$ for $i = 1, 2$. Hence the singularity condition of $(\mathcal{C}')_\pi$ is solely determined by the singularity condition on its open subsets corresponding to $\operatorname{Spec} A_i$ and $\mathcal{B}_j$ for $i, j = 1, 2$. By Definition 3.4.4, a point being singular is a stalk-local property. Since taking coproducts (gluing) commute with base changing, we just have to check the special fibers of each one of $\operatorname{Spec} A_i$ and $\mathcal{B}_j$, individually. By Proposition 4.1.5, $\operatorname{Spec} A_i$ is already semistable over $\mathcal{O}_L$. Breaking down the special fiber of $\mathcal{B}_i$ further apart by its three affine charts $\tilde{\Gamma}_{r_i}$, $\tilde{\Gamma}_{s_i}$, and $\tilde{\Gamma}_{t_i}$ over k_L, we have check that only $\tilde{\Gamma}_{t_i}$ has a nodal singularity at one point. Thus the special fiber of $\mathcal{C}'$ only possess at-worst-nodal singularities which proves that $\mathcal{C}'$ is semistable by Definition 3.4.17. □

Remark 33. At last, let us examine how do the charts describing the special fiber of the blow-up $\mathrm{Bl}_{\mathrm{Spec}\, D_i}(\mathrm{Spec}\, B_i)$ glue together. For both $i = 1, 2$, the two components of the special fiber $\tilde{\Gamma}_{t_i}$ were

$$\mathcal{I}_{i,1} := V(\bar{U}_i, \bar{V}_i, X_i + r_i^2, s_i)$$

and

$$\mathcal{I}_{i,2} := V([\bar{a}_4]\bar{V}_i^2 s_i^3 + [\bar{a}_1]\bar{V}_i s_i + \bar{V}_i + s_i, \bar{U}_i + \bar{V}_i s_i, X_i, r_i)$$

These are two curves intersecting at a single point where $\mathcal{I}_{i,1}$ has genus 0 and $\mathcal{I}_{i,2}$ has either genus 0 or 1 depending whether $[\bar{a}_4] = 0$ or $[\bar{a}_4] \neq 0$ respectively (see `Magama` code in Appendix A for these computations for generic $[\bar{a}_1]$ and $[\bar{a}_4]$.)

We recall the special fiber of the first chart $\tilde{\Gamma}_{r_i}$ was

$$V(\bar{U}_i, \bar{V}_i, X_i t_i^2 - 1, s_i).$$

Using the transition map $t_i = 1/r_i$ from chart $r_i = 1$ to chart $t_i = 1$, the equation becomes $X_i + r_i^2 = 0$ (since we are in characteristic 2). This tells us that $\tilde{\Gamma}_{r_i}$ is identified with the irreducible component $\mathcal{I}_{i,1}$ on the third chart.

Similarly, the special fiber of the second chart $\tilde{\Gamma}_{s_i}$ was

$$V(\bar{U}_i(t_i^2 + [\bar{a}_1]t_i) - 1 - [\bar{a}_4]\bar{U}_i^2), \bar{V}_i - \bar{U}_i t_i, X_i, r_i).$$

Using the transition map $t_i = 1/s_i$, the equations become

$$\begin{cases} \bar{U}_i \left(\frac{1}{s_i^2} + \frac{[\bar{a}_1]}{s_i} \right) = 1 + [\bar{a}_4]\bar{U}_i^2, \\ \bar{U}_i = \bar{V}_i s_i, \\ X_i, \\ r_i. \end{cases}$$

Make the substitution $\bar{U}_i = \bar{V}_i s_i$ gives

$$\begin{cases} \bar{V}_i + [\bar{a}_1]\bar{V}_i s_i = s_i + [\bar{a}_4]\bar{V}_i^2 s_i^3, \\ \bar{U}_i = \bar{V}_i s_i, \\ X_i, \\ r_i. \end{cases}$$

We note that this is precisely the second component $\mathcal{I}_{i,2}$ of $\tilde{\Gamma}_{t_i}$. Therefore, we conclude that $\tilde{\Gamma}_{r_i}$ is identified with the component $\mathcal{I}_{i,1}$ of $\tilde{\Gamma}_{t_i}$ and $\tilde{\Gamma}_{s_i}$ is identified with the other component $\mathcal{I}_{i,2}$ of $\tilde{\Gamma}_{t_i}$.

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Appendix A

Code

GitHub link for the `Magma` codes.