Toric analysis of symmetric differentials on A_n-singularities

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Abstract

Du Val singularities appear in the classification of algebraic surfaces and other areas of algebraic geometry. Wahl's concept of local Euler characteristics of sheaves helps in describing the properties of these singularities. We consider the sheaf of symmetric differentials and compute one ingredient of the local Euler characteristic: the codimension of those symmetric differentials that extend to the resolution of the singularity in the space of those that are regular around it. Singularities of type A_n can be described with toric varieties. We use Klyachko's theory of toric vector bundles to express this codimension as a lattice point count in a rational polytope. For symmetric differentials of symmetric degree m at A_n-singularities we explicitly determine these polytopes and find expressions for the counts in terms of Ehrhart's quasi-polynomials. We also analyse the behaviour of this quantity as a function of n.

Keywords: Du Val singularities; Toric bundles; Ehrhart quasi-polynomials.

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Chapter 1

Introduction

An important part of the study of algebraic varieties involves the study of their singularities. A quotient singularity on a surface over $\mathbb{C}$ is interpreted analytically as a quotient of an open subset V of $\mathbb{C}^2$ by a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$. The *du Val singularities* are a class of quotient singularities. They are characterised locally over $\mathbb{C}$ by the equations

$$\begin{aligned}
A_n &: \ x^2+y^2+z^{n+1}=0; \\
D_n &: \ x^2+y^2z+z^{n-1}=0(\text{ for } n\geq 4); \\
E_6 &: \ x^2+y^3+z^4=0; \\
E_7 &: \ x^2+y^3+yz^3=0; \\
E_8 &: \ x^2+y^3+z^5=0.
\end{aligned}$$

The du Val singularities are important objects in the study of algebraically quasi-hyperbolic surfaces, which contain only finitely many curves of genus 0 or 1. Bogomolov and de Oliveira (see [2]) studied the algebraic quasi-hyperbolicity of surfaces via the sheaf of symmetric differentials on the surfaces. One sufficient condition for a projective surface X to be algebraically quasi-hyperbolic is if the growth of $h^0(X, S^m\Omega_X)$ with m is to the order of m^3. For smooth projective surfaces $X \subset \mathbb{P}^3$, however, one has that $\mathrm{H}^0(X, S^m\Omega_X) = 0$ for all m (see [12]). The fundamental observation by Bogomolov and de Oliveira [2] is that for a singular surface $X \subset \mathbb{P}^3$ with many du Val singularities, the minimal resolution $\phi : Y \to X$ with exceptional divisor ϵ may have that $h^0(Y, S^m\Omega_Y)$ grows with m nonetheless. Informally, the presence of singularities may help in proving algebraic quasi-hyperbolicity. One ingredient in establishing this result is a relation between reflexive sheaves on Y and X. Let $\mathcal{F}$ be a reflexive sheaf on Y. Outside of the singular locus of X, we have that $X \setminus \{s\}$ and $Y \setminus \epsilon$ are isomorphic, so the push-forward $\phi_*(\mathcal{F})$ of $\mathcal{F}$ to X is reflexive outside the singular locus as well. If the singular locus of X is 0-dimensional, we can extend $\phi_*(\mathcal{F})$ uniquely to a reflexive sheaf on X by taking the double dual $(\phi_*\mathcal{F})^{\vee\vee}$, see [9]. Blache [1] showed for a locally free (and hence reflexive) sheaf $\mathcal{F}$ on Y that the difference in Euler characteristic between $\mathcal{F}$ and $(\phi_*(\mathcal{F}))^{\vee\vee}$ can be written as a sum of local terms at the singularities of X, if those singularities are quotient singularities.

$$\chi(X, (\phi_*\mathcal{F})^{\vee\vee}) = \chi(Y, \mathcal{F}) + \sum_{s \in \mathrm{Sing(X)}} \chi_{\mathrm{loc}}(s, \mathcal{F}). \tag{1.1}$$

The local term $\chi_{\mathrm{loc}}(s, \mathcal{F})$ is called the *local Euler characteristic* of $\mathcal{F}$ at s as introduced by Wahl in [14]. It is defined as follows. For a sufficiently small open neighborhood X° of s on X and a corresponding open set $Y^\circ = \phi^{-1}(X^\circ) \subset Y$, with the exceptional fiber $\epsilon_s = \phi^{-1}(s)$ above s, we write

$$\begin{aligned}
\chi^0_{\mathrm{loc}}(s, \mathcal{F}) &:= \dim \mathrm{H}^0(Y^\circ - \epsilon_s, \mathcal{F}) / \mathrm{H}^0(Y^\circ, \mathcal{F}), \\
\chi^1_{\mathrm{loc}}(s, \mathcal{F}) &:= \dim \mathrm{H}^1(Y^\circ, \mathcal{F}),
\end{aligned}$$

and define

$$\chi_{\mathrm{loc}}(s, \mathcal{F}) = \chi^0_{\mathrm{loc}}(s, \mathcal{F}) + \chi^1_{\mathrm{loc}}(s, \mathcal{F}).$$

A surface $X \subset \mathbb{P}^3$ of degree d with sufficiently many A_1 singularities can be shown to be algebraically quasi-hyperbolic. The exact computation of χ^0_{loc} allows precise statements about exactly how many singularities are required as a function of d and for which m one is guaranteed that $\mathrm{H}^0(Y, S^m\Omega_Y)$ is non-trivial, see [2] and [4].

We recognize that up to a change of coordinates, the local equation of an A_n-singularity can be written as $xz - y^{n+1} = 0$. As we will see in Chapter 3, this is a toric variety and its minimal desingularization is also a toric variety. Furthermore, the symmetric powers of the cotangent bundle and its reflexive hull are toric vector bundles.

In our case, we consider a singular toric variety X with an A_n singularity s and open dense torus $T \subset X$. Its minimal desingularization Y is also a toric variety and the birational morphism $Y \to X$ allows us to identify the torus in Y with T.

Since $\dim T = 2$, the action of T on itself induces a bigrading on $\mathrm{H}^0(T, S^m\Omega_T)$. We can identify $\mathrm{H}^0(X, (S^m\Omega_X)^{\vee\vee})$ wth $\mathrm{H}^0(Y - \epsilon, S^m\Omega_Y)$, where ϵ is the exceptional divisor of $\phi : Y \to X$, and by considering $T \subset Y$, we can consider both as subspaces of $\mathrm{H}^0(T, S^m\Omega_T)$ and hence decompose them using the bigrading into finite dimensional summands. These can then be described in terms of Klyachko's filtered vector spaces. We observe that $\mathrm{H}^0(Y - \epsilon, S^m\Omega_Y)$ and $\mathrm{H}^0(Y, S^m\Omega_Y)$ only differ in finitely many bigraded summands. The codimension of one in the other can be expressed as a lattice point count of a dilation by $(m + 1)$ of a rational polytope $\mathcal{P}(n)$ depending only on n. Furthermore, we analyse the asymptotic behaviour, which yields the result:

$$\lim_{n\to\infty} \mathrm{Vol}(\mathcal{P}(n)) = \frac{2\pi^2}{9} - 2.$$

For an A_1-singularity s, Bruin, Thomas and Várilly-Alvarado computed $\chi^0_{\mathrm{loc}}(s, S^m\Omega^1_{Y^\circ})$, see [4]. We generalize the result to an A_n-singularity for arbitrary n. The main result of this thesis is the following Theorem.

Theorem 1.1. *Let X be a surface with a singularity s of type A_n, and let Y be a minimal resolution of X with exceptional divisor ϵ. The quantity $\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) = \dim \mathrm{H}^0(Y - \epsilon, S^m\Omega_Y)/\,\mathrm{H}^0(Y, S^m\Omega_Y)$ is a quasi-polynomial in m with period dividing $\mathrm{lcm}(1, 2, \ldots, n+2)$ that we can explicitly determine.*

The quasi-polynomial mentioned in the main theorem is the Ehrhart quasi-polynomial of the rational polytope $\mathcal{P}(n)$, which we explicitly construct using Klyachko filtration. Moreover, we show that $\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y)$ is independent of n whenever $m < n$. Furthermore, with the asymptotic behaviour of the polytope $\mathcal{P}(n)$, we obtain the expression:

$$\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) \sim (\frac{2\pi^2}{9} - 2)(m+1)^3 + O(m).$$

Chapter 2 briefly reviews the basics of algebraic geometry with an emphasis on locally free sheaves and sheaves of symmetric differentials. In Chapter 3, we review basic toric geometry, providing an introduction to toric vector bundles on toric varieties and presenting Klyachko's classification of toric vector bundles. Chapter 4 introduces Ehrhart's theory of lattice point counting in polytopes and applies the Klyachko filtration to construct the polytope, which captures the information of the symmetric differentials. With Ehrhart's work, we prove the main theorem. In Chapter 5, we present explicit computations for A_2 and A_3-singularities. In Section 5.3, we demonstrate that when $m < n$, the quantity χ^0_{loc} is independent of n.

Chapter 2

Preliminaries

In this chapter, we review some basic notions concerning sheaves on algebraic varieties that we use in later chapters. The reader is assumed to have basic knowledge of the notion of schemes.

Unless otherwise specified, all schemes in this chapter are locally Noetherian, and all rings are commutative with unity. The reader may take the field k to be the field of complex numbers $\mathbb{C}$, although the presented content remains valid for any algebraically closed field k with characteristic 0.

2.1 Locally free sheaves

Let X be a scheme over k. Recall that the structure sheaf $\mathcal{O}_X$ on X is a sheaf of rings that assigns to each open subset $U \subset X$ the ring $\mathcal{O}_X(U)$ of regular functions on U. If $X = \operatorname{Spec} R$ is affine, we have $\mathcal{O}_X(X) = R$. For any $f \in R$, let $D(f)$ be the distinguished open set $X \setminus V(f)$, where $V(f)$ is the closed subscheme of X here f vanishes. As we now describe $\mathcal{O}_X(D(f)) = R_f$, where $R_f = R[\frac{1}{f}]$ is the localization of R at f.

On an affine scheme $X = \operatorname{Spec} R$, an R-module M defines a sheaf $\widetilde{M}$. Recall that the distinguished open sets $\{D(f) = \operatorname{Spec} R_f : f \in \operatorname{Spec} R\}$ form a basis for the Zariski topology on X, see [13, Section 3.5]. Writing $\widetilde{M}(D(f)) = M \otimes R_f$ as the localization of M at f, the sections of $\widetilde{M}$ are described constructively on distinguished open subsets $D(f) \subset X$ by $\widetilde{M}(D(f))$. This gives a sheaf on the basis of the Zariski topology, which leads to a sheaf of modules on the affine scheme X, see [13, Section 2.5].

Definition 2.1. [8, Section 2.5] A sheaf of $\mathcal{O}_X$-modules on X is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for any open subset $U \subset X$. Moreover, given the inclusion of open subsets $V \subset U$, the restriction morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction morphism of the structure sheaf.

Definition 2.2. Let X be a scheme over k. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is a *quasicoherent* sheaf if, for any affine open subset $U = \operatorname{Spec} R \subset X$, there is an R-module M such that $\mathcal{F}|_U = \widetilde{M}$.

Definition 2.3. Let $\mathcal{F}$ be a quasicoherent sheaf on X. We say $\mathcal{F}$ is a *coherent* sheaf if for any affine open subset $U = \operatorname{Spec} R$, the sections $\mathcal{F}(U)$ form a finitely generated R-module.

Definition 2.4. Let X be a scheme over k. A quasicoherent sheaf $\mathcal{F}$ is *locally free* if there exists an open covering $\{U_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i}$ is a sheaf of free $\mathcal{O}_X|_{U_i}$-modules for each open cover U_i. We say $\mathcal{F}$ has rank r if for each U_i the restriction $\mathcal{F}|_{U_i}$ is isomorphic to $(\mathcal{O}_X|_{U_i})^{\oplus r}$.

Definition 2.5. Let X be a scheme over k, and let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on X. Then the *tensor product sheaf* $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ of $\mathcal{E}$ and $\mathcal{F}$ is the sheafification of the presheaf $(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})_{\text{pre}}$ which assigns to each open subset $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$. We write $\mathcal{E}^{\otimes d}$ for the d-th tensor power of $\mathcal{E}$.

Definition 2.6. [13, Section 2.3] Let X be a scheme over k, and let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on X. The *sheaf-hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is the sheaf of abelian groups on X which assigns to each open subset $U \subset X$ the abelian group $\operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U, \mathcal{F}|_U)$.

Definition 2.7. Let X be a scheme over k, and let $\mathcal{E}$ be a locally free sheaf on X. The *sheaf of non-commutative algebras* $T(\mathcal{E})$ is the sheafification of the presheaf defined as follows.

$$U \subset X \longmapsto \bigoplus_{i\geq 0} \mathcal{E}^{\otimes i}(U) \text{ for any open subset } U \subset X.$$

For two sections $s_1 \in \mathcal{E}^{\otimes i}(U)$ and $s_2 \in \mathcal{E}^{\otimes j}(U)$ we get $s_1 \otimes s_2 \in \mathcal{E}^{\otimes(i+j)}(U)$, so by extending the tensor product bilinearly, we obtain the non-commutative algebra $T(\mathcal{E})(U)$.

Definition 2.8. Let X be a scheme over k, and $\mathcal{E}$ is a locally free sheaf on X. The sheaf $S^m(\mathcal{E})$ is the sheafification of the presheaf $S^m(\mathcal{E})_{\text{pre}}$ which assigns to each open subset $U \subset X$ the $\mathcal{O}_X(U)$-module $S^m(\mathcal{E}(U))$ where S^m is the d-th symmetric power of $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$.

Definition 2.9. Let X be a scheme over k, and $\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves on X.The *sheaf of commutative algebras* $S(\mathcal{E})$ is the sheafification of the presheaf defined as follows.

$$U \subset X \longmapsto \bigoplus_{i\geq 0} S^i(\mathcal{E}(U)) \text{ for any open subset } U \subset X.$$

For two sections $s_1 \in S^i(\mathcal{E}(U))$ and $s_2 \in S^j(\mathcal{E}(U))$ we get $s_1 s_2 \in S^{i+j}(\mathcal{E})(U)$, we obtain the commutative algebra $S(\mathcal{E})(U)$.

Proposition 2.10. *Let $\mathcal{E}$ and $\mathcal{F}$ be two locally free sheaves on a scheme X, suppose* $\operatorname{rank} \mathcal{E} = e$ *and* $\operatorname{rank} \mathcal{F} = f$*. Then $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ are locally free of rank ef. The symmetric power $S^m\mathcal{E}$ is also locally free of rank $\binom{m+e-1}{m}$.*

Proof: Being locally free is a local property, so it is sufficient to choose a sufficiently small affine open cover such that for each U in this cover, we have $\mathcal{E}|_U \cong \mathcal{O}_X|_U^{\oplus e}$ and $\mathcal{F} \cong \mathcal{O}_X|_U^{\oplus f}$. This reduces the question to the affine case. Suppose $X = \operatorname{Spec} R$, and $\mathcal{E} \cong \widetilde{R^{\oplus e}}$ and $\mathcal{F} \cong \widetilde{R^{\oplus f}}$. The result follows from basic algebra. We present only the proof that $S^m\mathcal{E}$ is locally free:

Pick an open subset U such that $\mathcal{E} \cong \mathcal{E}|_U \cong \mathcal{O}_X|_U^{\oplus e}$. Consider the sections of the presheaf $\operatorname{Sym}^d(\mathcal{E})_{\text{pre}}$ on U such that $\operatorname{Sym}^d(\mathcal{E})_{\text{pre}}(U)$ is a free $\mathcal{O}_X(U)$-module, and assume further that U is affine. We have the isomorphism:

$$\varphi : S^m(\mathcal{E})_{\text{pre}}(U) \xrightarrow{\cong} S^m\mathcal{O}_X(U)^{\oplus e}.$$

Note that $S^m(\mathcal{E})_{\text{pre}}|_U$ is a presheaf on U. By the universal property of sheafification, its sheafification $S^m(\mathcal{E})|_U$ is the same as the sheafification of $S^m(\mathcal{E})_{\text{pre}}$ restricted to U. Thus, there is an induced map:

$$\varphi' : S^m(\mathcal{E})|_U(U) \to S^m\mathcal{O}_X(U)^{\oplus e}.$$

This is an isomorphism as we can easily verify on the stalks. Therefore, the sheaf $S^m(\mathcal{E})$ is locally free. Its rank is the rank of $S^m(\mathcal{O}_X|_U)^{\oplus e}$, which is $\binom{m+e-1}{m}$. $\square$

The study of sheaves of modules is related to vector bundles. Classically, a vector bundle on a scheme X over k of rank r is a scheme $\mathcal{E}$ with a surjective morphism of schemes $\pi : \mathcal{E} \to X$ with a *zero section* $\xi : X \to \mathcal{E}$ and two additional morphisms

$$\begin{aligned} \mathcal{E} \times_X \mathcal{E} &\to \mathcal{E}, \text{ called addition and} \\ k \times \mathcal{E} &\to \mathcal{E}, \text{ called multiplication,} \end{aligned}$$

satisfying the following properties. As suggested by the name, the addition morphism is both commutative and associative; and the scalar multiplication is distributive over addition. For each $x \in X$, the *fiber* $\mathcal{E}(x) = \pi^{-1}(x)$ is an affine space $\mathbb{A}_k^r$. See 2.2.G(a) and 14.2.2. of [13]. The addition and scalar multiplication structure restricts to the fibers

$$\begin{aligned} \mathcal{E}(x) \times \mathcal{E}(x) &\to \mathcal{E}(x), \\ k \times \mathcal{E}(x) &\to \mathcal{E}(x), \end{aligned}$$

This gives a vector space structure for the fibre over x, and $\xi(x)$ is the zero vector in $\mathcal{E}(x)$. Furthermore, there exists an open covering $\{U_i\}_{i \in I}$ of X such that $\psi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times k^r$ is an isomorphism for each $i \in I$. The pair (U_i, ψ_i) is referred to as a trivialization of $\mathcal{E}$ over U_i. Additionally, we have transition morphisms defined on the intersections of the trivializing open subsets as follows:

Definition 2.11. Given two trivializations (U_i, ψ_i) and (U_j, ψ_j) of a vector bundle $\mathcal{E}$ on X. Let $V = \operatorname{Spec} A \subset U_i \cap U_j$, we define the *transition* function on V as an

$$\psi_{ij} := \left(\psi_i \circ \psi_j^{-1}\right)|_{U_i \cap U_j} : V \times \mathbb{A}^r \to V \times \mathbb{A}^r.$$

Note $\psi_{ij} \in \mathrm{GL}_r(A)$. The transition morphisms satisfy the *cocycle condition*:

$$\psi_{ik}|_{U_i \cap U_j \cap U_k} = \psi_{jk} \circ \psi_{ij}|_{U_i \cap U_j \cap U_k}.$$

Definition 2.12. Let X be a scheme over k, and let $\mathcal{E}$ be a vector bundle on X of rank r. We say $\mathcal{E}$ is a trivial vector bundle if $E \cong X \times k^r$.

Definition 2.13. A morphism of vector bundles $(\mathcal{E}, \pi_1) \to (\mathcal{F}, \pi_2)$ over X is a map $\varphi : \mathcal{E} \to \mathcal{F}$ such that the following diagram is commutative,

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & \mathcal{F} \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

and $\varphi_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is a linear map for each $x \in X$.

Definition 2.14. The vector bundle $(\mathcal{E}, \pi)$ defines a sheaf on X such that on each open subset $U \subset X$, the sections are $\mathrm{H}^0(U, \mathcal{E}) = \{s : U \to \mathcal{E} : \pi \circ s = \mathrm{id}_U\}$.

We present the following proposition to indicate the relationship between locally free sheaves and vector bundles.

Proposition 2.15. *[8, Chapter 2, Exercise 5.18] Let X be a scheme. There is a one-to-one correspondence between isomorphism classes of locally free sheaves of rank r on X and isomorphism classes of vector bundles of rank r on X.*

Proof: The correspondence is constructive. Let $\mathcal{E}$ be a vector bundle of rank r on X. Given a collection of trivializations $(\psi_i, U_i)_{i \in I}$ of $\mathcal{E}$, the sections on U_i can be expressed as $s : U_i \to U_i \times \mathbb{A}^r$. This leads us to the corresponding ring map:

$$s^\sharp : \mathcal{O}_X(U_i) \otimes_{k[x_1,..,x_r]} k[x_1, \ldots, x_r] \to \mathcal{O}_X(U_i),$$

where $\mathcal{O}_X(U_i) \otimes_{k[x_1,..,x_r]} k[x_1, \ldots, x_r] = \mathcal{O}_X(U_i)[x_1, \ldots, x_n]$, this gives the isomorphism $\mathcal{E}|_{U_i} = \mathcal{O}_X^{\oplus r}$. The vector bundle $\mathcal{E}$ is a locally free sheaf on X of rank r.

Conversely, given a locally free sheaf $\mathcal{F}$ of rank r. Let $\underline{\mathrm{Spec}} S(\mathcal{F})$ be the *relative spectrum* of the sheaf of $\mathcal{O}_X$-algebra $S(\mathcal{F})$. There is a natural projection $\pi : \underline{\mathrm{Spec}} S(\mathcal{F}) \to X$ such that for any affine open $V \subset X$, one has $\pi^{-1}(V) \cong \operatorname{Spec} S(\mathcal{F})(V)$. See [8, Section 2.5, Exercise 5.16,1.17.] and [13, Chapter 18].

Choose a affine open subset $U \subset X$ such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X(U)$-module with basis $\{x_1, \ldots, x_r\}$. Note that on the open subset U, the symmetric algebra

$$S(\mathcal{F}(U)) = S(\oplus_{i=1}^r x_i \mathcal{O}_X(U)) \cong \mathcal{O}_X(U)[x_1, \ldots, x_r].$$

Thus we have an isomorphism

$$\psi : \pi^{-1}(U) \cong \operatorname{Spec} \mathcal{O}_X[x_1, , x_r] \xrightarrow{\cong} U \times \mathbb{A}^r.$$

Therefore, the data $(\underline{\operatorname{Spec}} S(\mathcal{F}), \pi)$ defines a vector bundle of rank r on X. $\square$

Definition 2.16. Let X be an integral Noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. The *dual* of $\mathcal{F}$ is the sheaf $\mathcal{F}^\vee$ defined as $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The sheaf $\mathcal{F}$ is *reflexive* if $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$.

Definition 2.17. Let X be an integral Noetherian scheme, and let $\mathcal{F}$ be a coherent sheaf on X. The double dual $\mathcal{F}^{\vee\vee}$ of a coherent sheaf $\mathcal{F}$ on X is called the *reflexive hull* of $\mathcal{F}$. If $\mathcal{F}$ is locally free, then $\mathcal{F}^{\vee\vee} \cong \mathcal{F}$. We also write $\hat{\mathcal{F}}$ for $\mathcal{F}^{\vee\vee}$.

Definition 2.18. Let $\varphi : X \to Y$ be a morphism of schemes, and consider $\mathcal{F}$ as a sheaf on X. The *pushforward* or *direct image* of $\mathcal{F}$ on Y, denoted as $\varphi_*\mathcal{F}$, is a sheaf defined by assigning to any open subset $V \subset Y$ the value $\mathcal{F}(\varphi^{-1}(V))$.

We cite a weaker version of an important theorem by Hartshorne about reflexive sheaves:

Proposition 2.19. *[9, Proposition 1.6] Let $\mathcal{F}$ be reflexive sheaf on a normal integral scheme X. For each open subset $U \subset X$, and for each closed subscheme $V \subset U$ of codimension at least 2; let $i : U \setminus V \hookrightarrow U$ be the inclusion map. We have $i_*\mathcal{F}|_{U\setminus V} \cong \mathcal{F}|_U$.*

2.2 Differentials and cotangent sheaves

In this section, we will introduce the sheaf of Kähler differentials. The main references are [8, Section 2.8] and [13, Chapter 22].

Definition 2.20. Let R be a ring and A an R-algebra, and let M be an A-module. An R-derivation of A into M is a map $\mathrm{d} : A \to M$ such that d satisfies the following properties:

(i) d is additive,

(ii) d satisfies the *Leibniz rule*: $\mathrm{d}(xy) = x\,\mathrm{d}y + y\,\mathrm{d}x$ for all $x, y \in A$, and

(iii) d is zero on R: $\mathrm{d}r = 0$ for all $r \in R$.

Note d is A-linear. We write the collection of R-derivations of A into M as $\mathrm{Der}_R(A, M)$.

Definition 2.21. [8, Proposition 8.1 A, Section 2.8] Let R be a ring, and A an R-algebra. Define the map $e : A \otimes_R A \to A$ such that $e(x \otimes y) = xy$. Denote the kernel of e as I_e. Then the A-module I_e/I_e^2 is the module of Kähler differentials of A over R together with the map $\mathrm{d} : A \otimes_R A \to I_e/I_e^2$ by $\mathrm{d}(x) = \overline{1 \otimes x - x \otimes 1} + I_e^2$. We write module of *Kähler differentials* of the A-algebra over R as $\Omega_{A/R}$.

Proposition 2.22. *The module of Kähler differentials has the universal property: for any R-derivation $\mathrm{d}' : A \to M$ for some A-module, there exists a unique A-module homomorphism $f : \Omega_{A/R} \to M$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "\mathrm{d}'"] \arrow[dr, "\mathrm{d}"'] & & M \\ & \Omega_{A/R} \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 2.23. *[8, Proposition 8.4A, Section 2.8] Let A be an R-algebra, and let $I \subset A$ be an ideal. Then we have a natural exact sequence*

$$I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A (A/I) \to \Omega_{(A/I)/R} \to 0,$$

where for any $\delta(\bar{a}) = \mathrm{d}a \otimes 1$ for all $\bar{a} \in I/I^2$.

Definition 2.24. [8, Page 175] Let $\varphi : X \to Y$ be a morphism of schemes. Consider the diagonal map $\Delta : X \hookrightarrow X \times_Y X$. The map Δ is a locally closed embedding. Then the *sheaf of Kähler differentials* of X over Y is $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I} = \ker\left(\mathcal{O}_{\Delta(X)} \to \varphi^*\mathcal{O}_X\right)$.

Definition 2.25. A k-variety is a reduced, separated scheme over k of finite type.

Definition 2.26. Let X be a k-variety, and let $p \in X$ be a closed point. Then X is smooth at p if $\mathcal{O}_{X,p}$ is a regular local ring. The variety X is *smooth* if and only if $\mathcal{O}_{X,p}$ is a regular local ring for all closed $p \in X$.

Example 2.27. [13, 22.2.E] Let X be an irreducible k-variety. Locally, it is the spectrum of a k-algebra $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let the *Jacobian* J be the matrix $(\frac{\partial f_i}{\partial x_j})_{1 \le i \le m, 1 \le j \le n}$. Then $\operatorname{coker} J = \Omega_{A/k}$. In particular, we can write

$$\Omega_{A/k} = (\oplus_{i=1}^n A \,\mathrm{d}x_i)/(\oplus_{j=1}^m \mathrm{d}f_j).$$

The variety X is smooth if and only if $\operatorname{rank} J(p) = n - \dim X$ for all k-valued point $p \in X$.

Remark 2.28. Suppose X is a projective hypersurface, i.e. $X = V(f) \subset \mathbb{P}^n$ where f is a homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$. Then X is smooth if and only if all partial derivatives $\frac{\partial f}{\partial x_i}$ do not vanish simultaneously on X. Conversely, we say X is singular at $p \in X$ if all partial derivatives of f vanish at p.

Remark 2.29. [8, Remark 8.9.2] In the local setting, consider $U = \operatorname{Spec} A$ as a subset of Y, and $V = \operatorname{Spec} B$ as a subset of $\varphi^{-1}(U) \subset X$. The restriction of the diagonal morphism Δ onto V induces a ring homomorphism: $B \otimes_A B \to B$. Consequently, we observe that the module of Kähler differentials I/I^2 defines the sheaf $(\mathcal{I}/\mathcal{I}^2)|_{V \times_U V}$. Therefore, we can conclude that $\Omega_{U/V} = \widetilde{\Omega}_{B/A}$, where $\widetilde{\Omega}_{B/A}$ denotes the sheaf associated with the module of Kähler differentials over the ring homomorphism $B \to A$.

Proposition 2.30. *[13, 22.3.9] Let X be a smooth k-variety, and let p be a closed point on X. Let $\mathfrak{m} \subset \mathcal{O}_{X,p}$ be the maximal ideal in the local ring. The Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ of X at p is isomorphic to the fiber $\Omega_{X/k} \otimes_A k(p)$ of $\Omega_{X/k}$ at p.*

Proposition 2.30 explains why the sheaf of Kähler differentials is also called the cotangent sheaf.

Remark 2.31. Given a scheme X, the global sections of $S^m\Omega_X$ are called the regular symmetric differentials of *symmetric degree* m on X. If X is smooth, then Ω_X is locally free, and so is $S^m\Omega_X$.

Let X be an algebraic surface with a du Val singularity $s \in X$ as described in the introduction. Note that the sheaf $S^m\Omega_X$ is not locally free at s. We write for reflexive hull $\hat{S}^m\Omega_X$. We have

$$\mathrm{H}^0(X - \{s\}, S^m\Omega_X) \cong \mathrm{H}^0(X - \{s\}, \hat{S}^m\Omega_X).$$

Since s is a codimension 2 closed subset of X, by Theorem 2.19, we have

$$\begin{tikzcd}
\mathrm{H}^0(X - s, S^m\Omega_X) \arrow[rr, "\cong"] \arrow[dr, dashed] & & \mathrm{H}^0(X - s, \hat{S}^m\Omega_X) \arrow[dl, "\cong"] \\
& \mathrm{H}^0(X, \hat{S}^m\Omega_X) &
\end{tikzcd}$$

Let Y be the minimal resolution of the singularity s on X by blow-ups, and let ϵ be the exceptional divisor. The isomorphism between $X - \{s\}$ and $Y - \epsilon$ induces an isomorphism as follows

$$\mathrm{H}^0(Y - \epsilon, S^m\Omega_Y) \cong \mathrm{H}^0(X, \hat{S}^m\Omega_X).$$

Chapter 3

Toric vector bundles

In this chapter, we review toric varieties over $\mathbb{C}$. A toric variety is a variety that contains a torus as a dense open subvariety, with an action of that torus that restricts to the normal multiplication action of the torus on itself.

We briefly review how toric varieties are described by combinatorial objects, called cones and fans. See [5] and [6] for a more complete treatment.

Next, we describe vector bundles on toric varieties that are equivariant under the torus action, following [10]. As with toric varieties themselves, we see that such vector bundles can be described in terms of the same combinatorial data that determines the toric variety itself.

3.1 Toric varieties

An algebraic torus of dimension n is an algebraic variety T isomorphic to $(\mathbb{C}^*)^n$. A character χ of T is a morphism of varieties $\chi : T \to \mathbb{C}^*$ that is also a group homomorphism. The collection of all characters with pointwise multiplication forms the character group $M \cong \mathbb{Z}^n$ of the torus T. An element $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ can be identified with the character

$$\begin{aligned} \chi^u : T \cong (\mathbb{C}^*)^n &\to \mathbb{C}^*, \\ (t_1, \ldots, t_n) &\mapsto t_1^{u_1} \ldots t_n^{u_n}. \end{aligned}$$

An *one-parameter subgroup* of T is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. The collection of one-parameter subgroups forms a group N isomorphic to $\mathbb{Z}^n$. We can identify M as $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. See [5, Section 1.1 pp 11-12]

A *toric variety* of dimension n is a normal variety X containing an algebraic torus $T = (\mathbb{C}^*)^n$ as a Zariski-dense open subset, and with an action of T on X that on T restricts to the natural multiplication action of T on itself.

Definition 3.1. Let $N \cong \mathbb{Z}^n$ be a lattice, then $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a real vector space. Let S be a finite subset of $N_{\mathbb{R}}$. A *convex polyhedral cone* spanned by S is the following set

$$\sigma := \text{Cone}(S) = \{\sum_{v \in S} \lambda_v v | \lambda_v \geq 0\},$$

A convex polyhedral cone with one generator is called a *ray*.

Definition 3.2. A convex polyhedral cone σ in the vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is referred to as *rational* if it can be expressed as $\sigma = \text{Cone}(S)$ for some subset $S \subset N$. If σ is a rational convex polyhedral cone and ρ is a ray of σ, we will slightly abuse notation and use ρ to denote both the ray itself and its minimal generating vector in N.

Definition 3.3. A convex cone σ is called *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. Unless specified otherwise, we use the term cone to refer to a rational strongly convex cone.

Let N be a lattice, and let $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We denote $M_{\mathbb{R}}$ as the vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$. Following the notation in [5], we use $\langle \cdot, \cdot \rangle$ to represent the natural pairing between the real vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. In other words, for $u \in M$ and $v \in N$, we write $\langle u, v \rangle = u(v)$.

Definition 3.4. The *dual cone* $\sigma^{\vee}$ of $\sigma \subset N_{\mathbb{R}}$ is the set

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

The dual of a strongly convex rational polyhedral cone is not necessarily a strongly convex rational polyhedral.

Definition 3.5. Let $\sigma \subset N_{\mathbb{R}}$ be a cone, and let $u \neq 0 \in M_{\mathbb{R}}$. The hyperplane $H_u := \{v \in N_{\mathbb{R}} | \langle u, v \rangle = 0\}$ is called a *supporting plane* of σ if σ is contained in *supporting half-space* defined as $H_u^+ := \{v \in N_{\mathbb{R}} | \langle u, v \rangle \geq 0\}$.

Definition 3.6. Let σ be a cone. A *face* τ of σ is an intersection of σ with a supporting plane H_u of σ.

Note that any face of a convex polyhedral cone is itself a convex polyhedral cone. Furthermore, the intersection of two faces is also a face, and the face of a face is also a face. A maximal face of a cone, which is not the whole cone itself, is called a facet.

Theorem 3.7. *[5, Proposition 1.2.17] Let $\sigma \subset N_{\mathbb{R}}$ be cone. The set $S_\sigma = \sigma^{\vee} \cap M$ is a finitely generated additive semigroup.*

The semigroup S_σ defines a $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$ with generators $\{\chi^u \mid u \in S_\sigma\}$ and identity $\chi^0 = 1$, where χ^u represents the character associated with $u \in S_\sigma$. The multiplication is given as follows,

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}.$$

We define the *affine toric variety* X_σ as follows:

$$X_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma].$$

Example 3.8. Let $\{e_1, e_2\}$ be the standard basis of $N_\mathbb{R}$, and let $\{e_1^*, e_2^*\}$ be the dual basis of $\{e_1, e_2\}$. Consider the cone $\sigma := \operatorname{Cone}(e_2, 2e_1+e_2) \subset N_\mathbb{R}$. See Figure 3.1. The dual cone $\sigma^\vee$ is generated by $\{e_1^*, e_2^*, -e_1^*+2e_2^*\}$, and the semigroup S_σ is generated by $\{\chi^{(1,0)}, \chi^{(0,1)}, \chi^{(-1,2)}\}$.

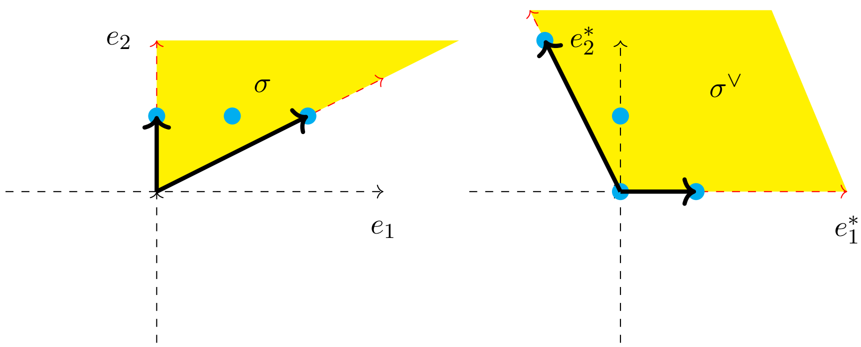

Figure 3.1: The cone of a toric variety with an A_1-singularity.

Taking $x = \chi^{(1,0)}, y = \chi^{(0,1)}$ and $z = \chi^{(-1,2)}$, we see that the $\mathbb{C}$-algebra is $\mathbb{C}[x, y, z]/(xz-y^2)$. The toric variety is

$$X_\sigma = \operatorname{Spec} \mathbb{C}[x, y, z]/(xz - y^2) \subset \mathbb{A}^3_\mathbb{C}.$$

This affine toric variety has an A_1-singularity at the origin $(x, y, z) = (0, 0, 0)$.

An affine toric variety is determined by a cone. We can describe a toric variety in terms of its affine toric subvarieties and the way they intersect. This leads us to consider the collection of their cones, together with information reflecting how they intersect. Such a collection is called a fan and a toric variety is determined by its fan. We proceed with the formal definition.

Definition 3.9. A *fan* Σ is a collection of strongly convex rational polyhedral cones satisfying the following properties:

(a) Each face of a cone in Σ is in Σ.

(b) The intersection of two cones in Σ is a face of both cones.

The conditions in Definition 3.9 represent the compatibility conditions required for gluing the affine toric varieties associated with each cone in Σ. For detailed information on the gluing process of toric varieties, we refer the reader to [6, Section 1.4]. Each cone σ in Σ defines an affine toric variety $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ as an open subvariety of X_Σ. To illustrate these concepts we provide an example of a toric variety related to Example 3.8. We denote $\Sigma(1)$ as the collection of rays in Σ.

Example 3.10. Let $\{e_1, e_2\}$ be the standard basis of $N_{\mathbb{R}}$, and let $\{e_1^*, e_2^*\}$ be the dual basis of $\{e_1, e_2\}$. Consider the fan Σ with maximal cones $\text{Cone}(e_2, e_1+e_2)$ and $\text{Cone}(e_1+e_2, 2e_1+e_2)$. See Figure 3.6. The ray $\tau = \text{Cone}(e_1 + e_2)$ is the intersection of σ_1 and σ_2. Taking the dual cone, we have $\sigma_1^\vee \cup \sigma_2^\vee \subset \tau^\vee$.

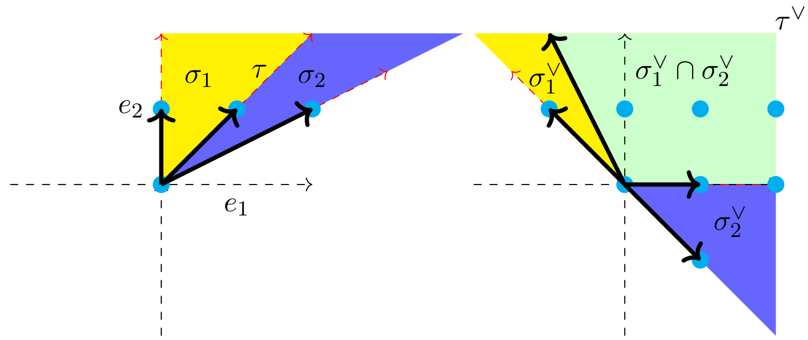

Figure 3.2: The fan of the toric variety X_Σ.

The toric variety X_Σ is obtained by gluing U_{σ_1} and U_{σ_2} along U_τ. We have

$$\begin{aligned} U_{\sigma_1} &= \operatorname{Spec} \mathbb{C}[x^{-1}y, x] \cong \mathbb{C}^2, \\ U_{\sigma_2} &= \operatorname{Spec} \mathbb{C}[x^{-1}y^2, xy^{-1}] \cong \mathbb{C}^2, \\ U_\tau &= \operatorname{Spec} \mathbb{C}[x^{-1}y, xy^{-1}, x, y] \cong \mathbb{C} \times \mathbb{C}^*. \end{aligned}$$

We verify that $U_\tau \subset U_{\sigma_1} \cap U_{\sigma_2}$ by examining the coordinate rings. The coordinate ring of U_τ can be expressed as $\mathbb{C}[x^{-1}y, xy^{-1}, x]$, which corresponds to the localization of $\mathbb{C}[x^{-1}y, x]$ at the element $x^{-1}y$. Similarly, it can also be written as $\mathbb{C}[xy^{-1}, x^{-1}y, y]$, corresponding to the localization of $\mathbb{C}[x^{-1}y^2, xy^{-1}]$ at the element xy^{-1}.

The transition map is induced by identifying the coordinate rings with subrings of one larger ring $\mathbb{C}[x, x^{-1}, y, y^{-1}]$. This yields

$$\begin{aligned} \varphi_{12} : \mathbb{C}[x, x^{-1}y, xy^{-1}] &\xrightarrow{\cong} \mathbb{C}[y, xy^{-1}, x^{-1}y] \\ x^{-1}y &\mapsto x^{-1}y, \\ x &\mapsto y(xy^{-1}). \end{aligned}$$

In particular, the variety X_Σ is smooth and is the resolution of an A_1-singularity on X_σ in Example 3.8.

Theorem 3.11. *[5, Theorem 3.1.19 (a)]. The toric variety X_Σ defined by the fan Σ is smooth if and only if, for every cone $\sigma \in \Sigma$, the minimal set of ray generators of σ forms a part of a $\mathbb{Z}$-basis of the lattice N.*

Given a toric variety defined by the fan Σ. We say a fan Σ' is a *refinement* of another fan Σ if every cone of Σ' is contained in a cone of Σ.

Theorem 3.12. *[5, Theorem 11.1.9] Let X_Σ be a singular toric variety defined by a fan Σ. There is a refinement Σ' of Σ by subdividing the non-smooth cones such that the toric variety $X_{\Sigma'}$ is a resolution of singularities.*

We verify that the toric variety X_Σ in Example 3.10 is isomorphic to the blow-up of the affine variety $X = V(xz - y^2) \subset \mathbb{A}^3_{\mathbb{C}}$ at the origin. Let Y be the strict transform of the blow-up $\widetilde{X} \subset \mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ with coordinates $(x, y, z; a : b : c)$. The blow-up $\widetilde{X} = V(bx - ay, cx - az, cy - bz, xz - y^2)$ is covered by its image in each of the affine charts of $\mathbb{A}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, and the strict transform Y is glued together in a way consistent with the gluing of the affine toric varieties of X_Σ. We see

$$\begin{aligned}
Y_1 &:= Y \cap \{a \neq 0\} \cong \operatorname{Spec} \mathbb{C}\left[x, \frac{b}{a}, \frac{c}{a}\right] / \left(\frac{c}{a} - \frac{b^2}{a^2}\right) \cong \mathbb{C}^2, \\
Y_2 &:= Y \cap \{b \neq 0\} \cong \operatorname{Spec} \mathbb{C}\left[y, \frac{a}{b}, \frac{c}{b}\right] / \left(\frac{ac}{b^2} - 1\right) \cong \mathbb{C} \times \mathbb{C}^*, \\
Y_3 &:= Y \cap \{c \neq 0\} \cong \operatorname{Spec} \mathbb{C}\left[z, \frac{a}{c}, \frac{b}{c}\right] / \left(\frac{a}{c} - \frac{b^2}{c^2}\right) \cong \mathbb{C}^2.
\end{aligned}$$

We denote

$$\begin{aligned}
A &:= \operatorname{Spec} \mathbb{C}\left[x, \frac{b}{a}, \frac{c}{a}\right] / \left(\frac{c}{a} - \frac{b^2}{a^2}\right), \\
B &:= \operatorname{Spec} \mathbb{C}\left[y, \frac{a}{b}, \frac{c}{b}\right] / \left(\frac{ac}{b^2} - 1\right), \\
C &:= \operatorname{Spec} \mathbb{C}\left[z, \frac{a}{c}, \frac{b}{c}\right] / \left(\frac{a}{c} - \frac{b^2}{c^2}\right).
\end{aligned}$$

We can identify Y_1 with U_{σ_1}, and Y_3 with U_{σ_2} and Y_2 with U_τ, with the relations $bx = ay, cx = az$ and $cy = bz$, and the fact that

$$A_{\frac{b}{a}} \cong B \cong C_{\frac{b}{c}}.$$

The transition maps are given as follows:

$$\mathbb{C}[x, \tfrac{b}{a}, \tfrac{a}{b}] \xrightarrow{x \mapsto y\frac{a}{b}, \frac{b}{a} \mapsto \frac{a}{b}} \mathbb{C}[y, \tfrac{a}{b}, \tfrac{b}{a}] \xrightarrow{=} \mathbb{C}[y, \tfrac{c}{b}, \tfrac{b}{c}] \xleftarrow{y\frac{c}{b} \leftarrow\!\shortmid z, \frac{c}{b} \leftarrow\!\shortmid \frac{b}{c}} \mathbb{C}[z, \tfrac{b}{c}, \tfrac{c}{b}].$$

This gives us a variety that is obtained by gluing two copies of $\mathbb{C}^2$ along their common subvariety $\mathbb{C} \times \mathbb{C}^*$, which is the same as the toric variety X_Σ described in Example 3.6.

Theorem 3.13. *[5, Theorem 3.1.5] Let Σ be a fan in $N_{\mathbb{R}}$. The toric variety X_Σ is normal and separable.*

3.2 Equivariant vector bundles on toric varieties

Consider a fan $\Sigma \subset N_{\mathbb{R}}$ such that the associated toric variety X_Σ is smooth. Let M be the dual lattice of N, and let $T = \operatorname{Spec} \mathbb{C}[M]$ denote the torus. Suppose T acts linearly on a vector space V. Then for each character χ^u the T-eigenspace $V_u = \{v \in V : t \cdot u = \chi^u(t)v \text{ for all } t \in T\}$ and we can decompose V as follows:

$$V = \bigoplus_{u \in M} V_u,$$

This decomposition is known as the *isotypical* decomposition. An element $v \in V$ is referred to as torus-invariant if $t \cdot v = v$ for all $t \in T$. The torus-invariant elements of V form sub-vector space V_0.

Let X be an affine toric variety. We observe that the action of the torus T on X induces an action on the coordinate ring $\mathbb{C}[X]$ of X, which is a finitely generated $\mathbb{C}$-algebra. Using the theory of algebraic groups, one can show that $\mathbb{C}[X]$ decomposes as a direct sum of graded subspaces with respect to the T-action. In other words, $\mathbb{C}[X]$ can be written as

$$\mathbb{C}[X] = \bigoplus_{u \in M} \mathbb{C}[X]_u,$$

where M is the lattice to the character group of T, and $\mathbb{C}[X]_u$ is the graded subspace corresponding $u \in M$.

Definition 3.14. A vector bundle $\pi : \mathcal{E} \to X_\Sigma$ with a T-action is called a *toric vector bundle* if the action of T on $\mathcal{E}$ is compatible with the action of T on X_Σ.

Let Σ be a fan, and suppose $\mathcal{E}$ is a toric vector bundle on a toric variety X_Σ. Let E be the fiber of $\mathcal{E}$ over e_T where e_T is the identity of the torus T. The action of the torus T on X_Σ induces an action of T on $\mathrm{H}^0(X_\Sigma, \mathcal{E})$ as follows: for each $t \in T$ and $s \in \mathrm{H}^0(X_\Sigma, \mathcal{E})$ we have

$$(t \cdot s)(x) := t(s(t^{-1}x)).$$

We can decompose $\mathrm{H}^0(X, \mathcal{E})$ into a direct sum of isotypical subspaces:

$$\mathrm{H}^0(X, \mathcal{E}) \cong \bigoplus_{u \in M} \mathrm{H}^0(X, \mathcal{E})_u,$$

where $\mathrm{H}^0(X, \mathcal{E})_u = \{s \in \mathrm{H}^0(X, \mathcal{E}) : t \cdot s = \chi^u(t)s \text{ for all } t \in T\}$.

Proposition 3.15. *[10, Proposition 2.1.i] Toric vector bundles on affine toric varieties are trivial as vector bundles.*

Referring to [11, Section 2.1], it is stated that for any toric variety X, sections in $\mathrm{H}^0(X, \mathcal{E})_u$ that agree on e_T must also agree on T and, consequently, on the entire toric variety X. Consequently, for any cone $\sigma \in \Sigma$, we consider the open affine variety U_σ. We have that the evaluation map of sections in $\mathrm{H}^0(U_\sigma, \mathcal{E})_u$ at e_T establishes an injective map.

$$\begin{aligned} \mathrm{eval}_{e_T} : \mathrm{H}^0(U_\sigma, \mathcal{E})_u &\hookrightarrow E \\ s &\mapsto s(e_T). \end{aligned}$$

Denote the image $\mathrm{eval}_{e_T}(\mathrm{H}^0(U_\sigma, \mathcal{E}))$ as E_u^σ. In addition, let $u' \in \sigma^\vee \cap M$. One has the inclusion of $\mathrm{H}^0(U_\sigma, \mathcal{E})_u$ in $\mathrm{H}^0(U_\sigma, \mathcal{E})_{u-u'}$ by multiplying $\chi^{u'}$:

$$\mathrm{mult}_{\chi^{u'}} : \mathrm{H}^0(U_\sigma, \mathcal{E})_u \hookrightarrow \mathrm{H}^0(U_\sigma, \mathcal{E})_{u-u'}.$$

Composing the inclusion with evaluation at e_T gives an inclusion $E_u^\sigma \subset E_{u-u'}^\sigma$. Let ρ be a ray in Σ, the space E_u^ρ depends only on $i := \langle \rho, u \rangle$. Therefore, we define

$$E^\rho(i) := E_u^\rho \text{ for any } u \text{ such that } \langle \rho, u \rangle = i.$$

Definition 3.16. For a fan $\Sigma \subset N_\mathbb{R}$ and a vector space E, a Σ-filtration of E is a collection of subvector spaces $E^\rho(i)$ for every ray $\rho \in \Sigma(1)$ and $i \in \mathbb{Z}$, such that $E^\rho(i) \subseteq E^\rho(j)$ whenever $i \geq j$, with $E^\rho(i) = E$ for i sufficiently small and $E^\rho(i) = 0$ for i sufficiently large. It also satisfies the following compatibility condition:

Suppose M is the dual lattice of N. Let $\sigma \in \Sigma$ and $M_\sigma = M/(\sigma^\perp \cap M)$. Then, there exists a decomposition $E = \bigoplus_{[u] \in M_\sigma} E_{[u]}$ such that each component $E_{[u]}$ satisfies the following properties:

$$E^\rho(i) = \sum_{[u], \langle u, \rho \rangle \geq i} E_{[u]}, \text{ for all } \rho \in \sigma(1).$$

Definition 3.17. Let X be a smooth toric variety associated with the fan Σ. The *category of toric vector bundles* on X consists of the collection of toric vector bundles on X, and the morphisms are defined as equivariant morphisms of toric vector bundles.

Definition 3.18. Let Σ be a fan in $N_\mathbb{R}$, and let M be the dual lattice of N. We define the category of finite-dimensional vector spaces with a family of decreasing Σ-filtered vector spaces. This category consists of vector spaces E equipped with a decreasing filtration $E^\rho(i)$ indexed by the rays of Σ, satisfying the following compatibility condition:
In this category, the morphisms are defined as linear maps $\varphi : E \to F$ that preserve the filtrations. This means that for each $\rho \in \Sigma(1)$ and $i \in \mathbb{Z}$, the restriction $\varphi|_{E^\rho(i)} : E^\rho(i) \to F^\rho(i)$ is a well-defined linear map.

Theorem 3.19 (Klyachko [10],1990)**.** *Let Σ be a fan. The category of toric vector bundles on the toric variety X_Σ is equivalent to the category of Σ-filtered vector spaces as defined in Definition 3.18.*

We also refer to a decreasing filtration as described in Theorem 3.19 as the Klyachko filtration.

Theorem 3.20. *[10, Theorem 4.1.1(i)] Let X_Σ be a toric variety, and let $\mathcal{E}$ be a toric vector bundle on X_Σ. We have*

$$\mathrm{H}^0(X_\Sigma, \mathcal{E})_u = \cap_{\rho\in\Sigma(1)} E^\rho(\langle u, \rho\rangle).$$

Example/Proposition 3.21. [10, Example 2.3, (5)] Let X_Σ be a smooth toric variety with the cotangent bundle Ω_{X_Σ}. The filtrations of Ω_{X_Σ} are defined by the vector space $E = M \otimes \mathbb{C}$ and a family of decreasing filtrations $E^\rho(i)$ indexed by the rays $\rho \in \Sigma(1)$.

$$E^\rho(i) = \begin{cases} E, & \text{for } i < 0, \\ (\mathrm{Span}\{\rho\})^\perp, & \text{for } i = 0, \\ 0, & \text{for } i > 0. \end{cases}$$

Example 3.22. In this example, we examine the cotangent bundle on the variety $X := \mathbb{C}^2$. The variety X is an affine toric variety with the defining cone $\sigma = \mathrm{Cone}(e_1, e_2) \subset N_\mathbb{R} \cong \mathbb{R}^2$, where e_1 and e_2 represent the standard basis vectors. We denote the standard dual basis as e_1^* and e_2^*. The torus $T := (\mathbb{C}^*)^2$ is defined as the torus. The dual cone $\sigma^\vee$ is given by $\mathrm{Cone}(e_1^*, e_2^*)$. Since X is affine, we have

$$\mathrm{H}^0(X, \Omega_X) = \mathbb{C}[x, y]\,\mathrm{d}x + \mathbb{C}[x, y]\,\mathrm{d}y.$$

Let ρ_1 be the ray generated by the vector $e_1 = (1, 0)$, and let ρ_2 be the ray generated by $e_2 = (0, 1)$. We compute

$$\mathrm{H}^0(U_{\rho_1}, \Omega_X) = \mathbb{C}[x, y, y^{-1}]\,\mathrm{d}x + \mathbb{C}[x, y, y^{-1}]\,\mathrm{d}y,$$
$$\mathrm{H}^0(U_{\rho_2}, \Omega_X) = \mathbb{C}[x, x^{-1}, y]\,\mathrm{d}x + \mathbb{C}[x, x^{-1}, y]\,\mathrm{d}y.$$

For any $t \in T$, the action of t on $\mathrm{d}\chi^u$ is given by

$$t \cdot \mathrm{d}\chi^u = \mathrm{d}t \cdot \chi^u = \mathrm{d}\chi^{-u}(t)\chi^u = \chi^{-u}(t)\,\mathrm{d}\chi^u.$$

We identify $\Omega_X(e_T) = M \otimes \mathbb{C}$ with $\mathrm{H}^0(T, \Omega_X)_{(0,0)}$ by the following map

$$\begin{aligned} M \otimes \mathbb{C} &\to \mathrm{H}^0(T, \Omega_X)_{(0,0)} = \mathrm{Span}_{\mathbb{C}}\{\frac{\mathrm{d}x}{x}, \frac{\mathrm{d}y}{y}\}. \\ u &\mapsto \frac{\mathrm{d}\chi^u}{\chi^u}. \end{aligned}$$

We only consider ρ_1 here since the filtration for ρ_2 can be computed analogously. We pick $u = (1, 0)$, $(0, 0)$, and $(-1, 0)$, which correspond to the vector spaces $E^{\rho_1}(i)$ for $i = 1$, 0, and -1, respectively. By inspection, we see that $\mathrm{H}^0(U_{\rho_1}, \Omega_X)_{(1,0)}$ is zero. The only torus-invariant sections in $\mathrm{H}^0(U\rho_1, \Omega_X)_{(0,0)}$ are scalar multiples of $\frac{\mathrm{d}y}{y}$. For $u = (-1, 0)$, we observe that $\mathrm{H}^0(U\rho_1, \Omega_X)_{(-1,0)}$ is generated by $\mathrm{d}x$. Since $E^{\rho_1}(-1) \supset E^{\rho_1}(0)$ and $\mathrm{d}x$ and $\frac{dy}{y}$ are linearly independent when evaluated at e_T, by a dimension count, we conclude that $E^{\rho_1}(-1) = E$. Since the filtration is decreasing, this gives the filtration of Ω_X on $X = \mathbb{C}^2$ as described in Proposition 3.21.

In the following examples, we give an explicit computation of $\mathrm{H}^0(X_\sigma, \Omega_{X_\sigma})$ and $\mathrm{H}^0(X_\Sigma, \Omega_\Sigma)$ for the toric varieties in Example 3.8 and 3.10.

Example 3.23. Let $\sigma = \mathrm{Cone}(e_2, 2e_1 + e_2)$, and let $\rho_1 = \mathrm{Cone}(e_2)$ and $\rho_3 = \mathrm{Cone}(2e_1 + e_2)$. Note that we denote the ray generated by $2e_1 + e_2$ by ρ_3 to be consistent with later discussions. Note that the toric variety $X := X_\sigma$ is singular at the origin O, thus we will take the reflexive hull $\hat{\Omega}_X$. By Proposition 2.19, one has

$$\mathrm{H}^0(X, \hat{\Omega}_X) \cong \mathrm{H}^0(X - \{O\}, \Omega_X).$$

By Corollary 3.20 the global sections of the of the cotangent bundle Ω_X on $X - \{O\}$ are given by the following:

$$\mathrm{H}^0(X - \{O\}, \Omega_X) = \bigoplus_{(a,b)\in\mathbb{Z}^2} E^{\rho_1}(\rho_1(a, b)) \cap E^{\rho_3}(\rho_3(a, b))$$

Consider the toric variety $Y := X_\Sigma$ where Σ is the fan as in Example 3.10. The toric variety X_Σ is smooth. Let $\rho_2 = \mathrm{Cone}(e_1 + e_2) \in \Sigma(1)$. We have

$$\mathrm{H}^0(Y, \Omega_Y) = \bigoplus_{(a,b)\in\mathbb{Z}^2} E^{\rho_1}(\rho_1(a, b)) \cap E^{\rho_2}(\rho_2(a, b)) \cap E^{\rho_3}(\rho_3(a, b))$$

This gives us a method to determine which sections in $\mathrm{H}^0(Y - \epsilon, \Omega_Y)$ extend to all of Y. Since both spaces are graded, their quotient is also graded, and we have:

$$\frac{\mathrm{H}^0(Y - \epsilon, \Omega_Y)}{\mathrm{H}^0(Y, \Omega_Y)} = \bigoplus_{(a,b)\in\mathbb{Z}^2} \frac{E^{\rho_1}(\rho_1(a, b)) \cap E^{\rho_3}(\rho_3(a, b))}{E^{\rho_1}(\rho_1(a, b)) \cap E^{\rho_2}(\rho_2(a, b)) \cap E^{\rho_3}(\rho_3(a, b))}.$$

The left-hand side is a quotient space of an infinite-dimensional $\mathbb{C}$-vector space, while the right-hand side is an (infinite) direct sum of finite-dimensional spaces. We will show that only finitely many of them are non-trivial, resulting in a finite direct sum.

We denote $\rho_i(a,b) = \langle (a,b), \rho_i \rangle = (i-1)a + b$ for $i = 1,2,3$. For each bigraded space with grading (a,b) on the right-hand side, it is trivial if either $E^{\rho_1}(\rho_1(a,b))$ or $E^{\rho_3}(\rho_3(a,b))$ is trivial, or if $E^{\rho_2}(\rho_2(a,b))$ is the full space E.

The vector space $E^{\rho_1}(\rho_1(a,b))$ is zero when $b > 0$, and the vector space $E^{\rho_3}(\rho_3(a,b))$ is zero when $2a + b > 0$. The two inequalities define the shaded region in Figure 3.3.

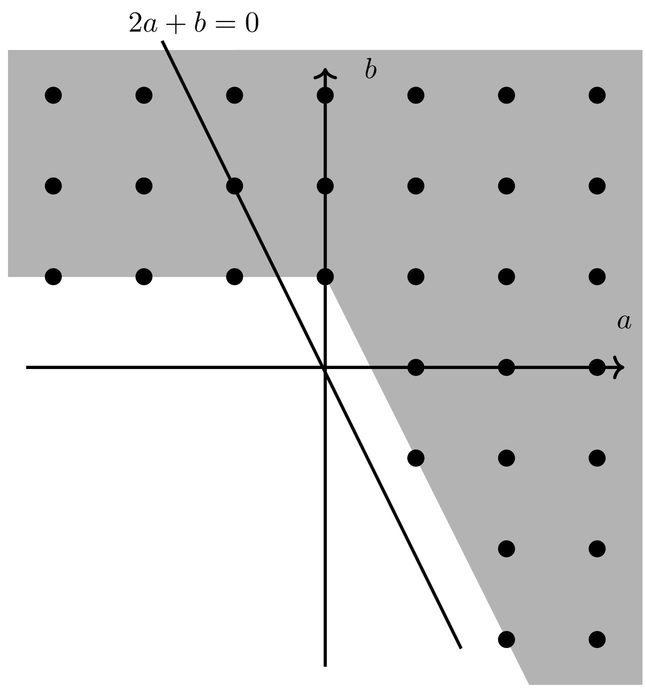

Figure 3.3: $b > 0$ or $2a + b > 0$ in the (a,b)-plane.

Furthermore, the vector space $E^{\rho_2}(\rho_2(a,b))$ is the full space whenever $\rho_1(a,b) = a+b < 0$. This condition eliminates the lattice points in the shaded region shown in Figure 3.4.

To sum up, all the lattice points in the shaded region are shown in Figure 3.5. The shaded region consists of all the pairs of integers (a,b) such that the space

$$\frac{E^{\rho_1}(\rho_1(a,b)) \cap E^{\rho_3}(\rho_3(a,b))}{E^{\rho_1}(\rho_1(a,b)) \cap E^{\rho_2}(\rho_2(a,b)) \cap E^{\rho_3}(\rho_3(a,b))}$$

is trivial.

The only lattice point left is the origin, where the vector spaces are

$$\begin{aligned} E^{\rho_1}(\rho_1(0,0)) &= \mathrm{Span}_{\mathbb{C}}\{(1,0)\}, \\ E^{\rho_2}(\rho_2(0,0)) &= \mathrm{Span}_{\mathbb{C}}\{(-1,1)\}, \\ E^{\rho_3}(\rho_3(0,0)) &= \mathrm{Span}_{\mathbb{C}}\{(-1,2)\}. \end{aligned}$$

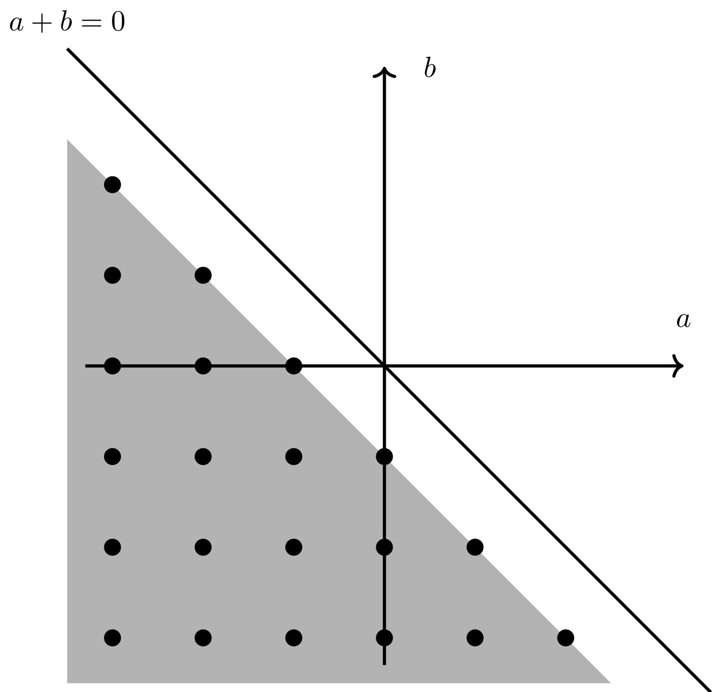

Figure 3.4: $a+b<0$ in the (a,b)-plane.

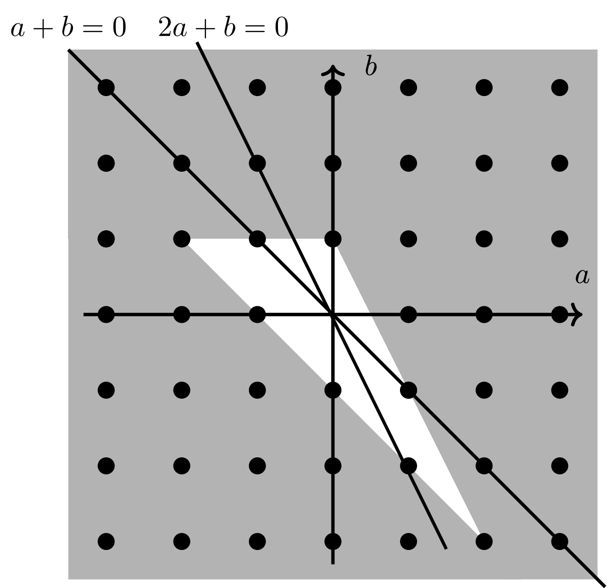

Figure 3.5: (a,b)-plane.

The intersection $E^{\rho_1}(0) \cap E^{\rho_3}(0) = 0$, implying that all Kähler differentials on $Y - \epsilon$ are regular as sections of $\mathrm{H}^0(Y, \Omega_Y)$.

This type of computation can be generalized to A_n-singularities and higher symmetric powers of the cotangent sheaves. We introduce the following lemma which describes the varieties with an A_n-singularity as a toric variety.

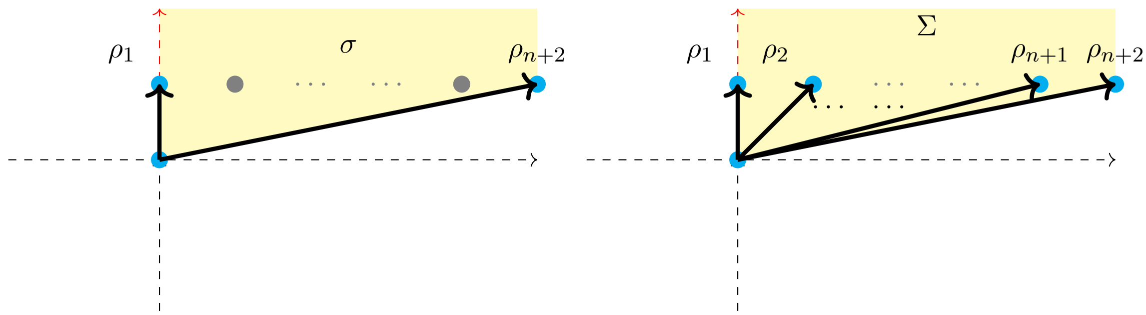

Figure 3.6: The cone σ and the fan Σ

Lemma 3.24. *Let $\sigma = \mathrm{Cone}(e_2, (n+1)e_1 + e_2)$. The affine toric variety X_σ has an A_n-singularity at the origin.*

Let Σ be the fan with maximal cones $\{\mathrm{Cone}((i-1)e_1 + e_2, ie_1 + e_2) : i = 1, \dots, n+2\}$. See Figure 3.6. The toric variety X_Σ is the resolution of the A_n-singularity on the affine toric variety X_σ.

Proof: The dual cone of σ is $\sigma^\vee = \mathrm{Cone}(e_1^*, -e_1^* + (n+1)e_2^*) \subset M_\mathbb{R}$. The associated semigroup S_σ is generated by $\{\chi^{(1,0)}, \chi^{(0,1)}, \chi^{(-1,n+1)}\}$, so the affine toric variety $X_\sigma = \mathrm{Spec}\,\mathbb{C}[x, y, x^{-1}y^{n+1}]$. By writing $z = x^{-1}y^{n+1}$, we obtain

$$\mathbb{C}[x, y, x^{-1}y^{n+1}] \cong \mathbb{C}[x, y, z]/(xz - y^{n+1}),$$

and we see that X_σ has an A_n-singularity at the origin.

The fan Σ can be regarded as a refinement of the fan that defines X_σ. Consequently, X_Σ serves as a resolution of X_σ. For further detail into this refinement process, see [5, Example 10.1.5, Example 10.1.9, and Exercise 10.1.5]. □

Example 3.25. Let n be any positive integer, and let σ and Σ be the cone and fan described in Lemma 3.24. We write $X := X_\sigma$ and $Y := X_\Sigma$. The dense torus T contained in X corresponds to the open subvariety defined by $xz \neq 0$. The rays e_2 and $(n+1)e_1 + e_2$ correspond to the divisors $x = 0$ and $z = 0$ on X.

As we saw in Example 3.22, for each u in the character group of T, Klyachko shows that the filtrations $E^\rho(\rho(u))$ measure the regularity along the divisor corresponding to the ray ρ for the sections in $\mathrm{H}^0(T, S^m\Omega_T)_u$. The desingularization Y of X is given by the fan spanned by the rays $e_2, e_1 + e_2, \dots, (n+1)e_1 + e_2$. The additional rays $ie_1 + e_2$ for $i = 1, ..., n$ correspond to the components of the exceptional divisor ϵ.

Chapter 4

Symmetric differentials and polytopes

4.1 Klyachko Filtration of Symmetric Differentials

In this section, we focus on the explicit computation of the sections of symmetric differentials on the toric desingularization of a variety with an A_n-singularity. Proposition 2.10 indicates that symmetric powers of a vector bundle are again vector bundles. Our primary approach will involve the computation of sections of symmetric differentials through the utilization of Corollary 3.20.

To commence our exploration, we reference the following results from [7]:

Proposition 4.1. *[7, Corollary 3.2] Let $\mathcal{E}$ be a toric vector bundle on a smooth toric variety X. Let $\{E^{\rho(i)}\}$ be the Klyachko filtration of $\mathcal{E}$. For any $m \in \mathbb{N}_{\geq 0}$, the Klyachko filtration for $\mathcal{E}^{\otimes m}$ is given by*

$$F^{\rho}(i) = \sum_{i_1+i_2+\cdots+i_m=i} E^{\rho}(i_1) \otimes E^{\rho}(i_2) \otimes \cdots \otimes E^{\rho}(i_m).$$

Proposition 4.2. *[7, Corollary 3.5] Let $\mathcal{E}$ be a toric vector bundle on a smooth toric variety X. Let $\{E^{\rho}(i)\}$ be the Klyachko filtration of $\mathcal{E}$. For any $m \in \mathbb{N}_{\geq 0}$, the Klyachko filtration for $S^m\mathcal{E}$ is given by*

$$F^{\rho}(i) = \sum_{i_1+i_2+\cdots+i_m=i} \operatorname{Im}\left(E^{\rho}(i_1) \otimes E^{\rho}(i_2) \otimes \cdots \otimes E^{\rho}(i_m) \to S^m E\right).$$

With no ambiguity, we write

$$\sum_{i_1+i_2+\ldots+i_m=i} \operatorname{Im}\left(E^{\rho}(i_1) \otimes E^{\rho}(i_2) \otimes \ldots \otimes E^{\rho}(i_m) \to S^m E\right) = \sum_{i_1+i_2+\ldots+i_m=i} E^{\rho}(i_1)E^{\rho}(i_2)\cdots E^{\rho}(i_m).$$

Proposition 4.2 provides us with a direct approach to explicitly compute the Klyachko filtration of the symmetric differentials. This becomes particularly advantageous as we have already obtained the Klyachko filtration of the cotangent bundle in Example/Proposition 3.21. Using Proposition 4.2, we can expand our computations to include the symmetric differentials and determine their corresponding Klyachko filtration.

We fix the notation σ and Σ for the cone and fan as described in Lemma 3.24, and we write $X := X_\sigma$ and $Y := X_\Sigma$ from now on.

Proposition 4.3. *Let Σ and Y as described above. The locally free sheaf $S^m\Omega_Y$ as a toric vector bundle on Y corresponds to the following data:*

(a) A vector space S^mE where $E = M \otimes \mathbb{C} \cong \mathbb{C}^2$.

(b) For each ray $\rho \in \Sigma(1)$ there is a decreasing filtration of vector spaces $F^\rho(i)$ such that

$$F^\rho_m(i) = \begin{cases} S^mE & \text{for } i \leq -m-1; \\ (\rho^\perp)^{i+m}S^{-i}E & \text{for } -m \leq i \leq 0; \\ 0 & \text{for } i \geq 1. \end{cases} \tag{4.1}$$

The term $(\rho^\perp)^{i+m}S^{-i}E$ corresponds to the image of the map defined by multiplying each symmetric differential in $S^{-i}E$ by $(\rho^\perp)^{i+m}$ to obtain S^mE.

Proof: This follows directly from Proposition 4.2, where we replace each $E^\rho(i_j)$ appearing in the summation with the corresponding terms from the Klyachko filtration of the cotangent sheaf, as described in Example 3.21. By substituting the explicit expressions of the Klyachko filtration in place of $E^\rho(i_j)$, we can obtain the desired result. $\square$

Example 4.4. Recall from Example 3.23 that we considered the cone $\sigma := \mathrm{Cone}\{e_2, 2e_1 + e_2\}$, which defines a surface X_σ with an A_1-singularity. Using Corollary 3.20, we computed the sections of Ω_Y, where Y is the resolution of this singularity. Let ρ_i denote the ray generated by $(i-1, 1)$ for $i = 1, 3$. The affine toric variety X_σ associated with this cone also has an A_1-singularity. Therefore, we can apply Corollary 3.20 again to compute the sections of the symmetric differentials on Y.

$$\mathrm{H}^0(Y - \epsilon, S^m\Omega_Y) = \bigoplus_{(a,b)\in\mathbb{Z}^2} F^{\rho_1}_m(\rho_1(a,b)) \cap F^{\rho_3}_m(\rho_3(a,b));$$
$$\mathrm{H}^0(Y, S^m\Omega_Y) = \bigoplus_{(a,b)\in\mathbb{Z}^2} F^{\rho_1}_m(\rho_1(a,b)) \cap F^{\rho_2}_m(\rho_2(a,b)) \cap F^{\rho_3}_m(\rho_3(a,b)).$$

We now adapt the computation for A_n-singularities. We take the cone σ and the fan Σ as in Lemma 3.24. Fixing $(a,b) \in \mathbb{Z}^2$, Proposition 4.3(b) indicates that the dimension of each vector space $F_m^{\rho_i}(\rho_i(a,b))$ is given by

$$\dim F_m^{\rho_i}(\rho_i(a,b)) = \begin{cases} 0 & \text{if } \rho_i(a,b) \geq 1, \\ -\rho_i(a,b)+1 & \text{if } -m \leq \rho_i(a,b) \leq 1, \\ m+1 & \text{if } \rho_i(a,b) \leq -m-1. \end{cases} \tag{4.2}$$

We will compute the vector space $F_m^{\rho_i}(\rho_i(a,b)) \cap F_m^{\rho_j}(\rho_j(a,b))$ for $i,j = 1,2,\ldots,n+2$ and $i \neq j$, when both $\rho_i(a,b)$ and $\rho_j(a,b)$ are in the range of $-m$ to 0. Otherwise, the intersection is easy to determine. Recall from 4.1 that the space $F_m^{\rho_i}(\rho_i(a,b))$ consists of symmetric differentials in $S^m E$ having a factor of $(\rho_i^{\perp})^{-\rho_i(a,b)+m}$.

Note that for any $\omega \in F_m^{\rho_i}(\rho_i(a,b)) \cap F_m^{\rho_j}(\rho_j(a,b))$, it is either 0 or must have a factor of $(\rho_i^{\perp})^{-\rho_i(a,b)+m} \cdot (\rho_j^{\perp})^{-\rho_j(a,b)+m}$. By counting the total symmetric degree, we find that the dimension of the intersection is given by $\max\{m+1-(\rho_i(a,b)+m)-(\rho_j(a,b)+m)\,,0\}$. Extending this computation, we can compute the dimensions of any intersection of the vector spaces in the filtration. We record for future use the following proposition for the filtrations introduced here for the resolution of an A_n singularity.

Proposition 4.5. *Let $(a,b) \in \mathbb{Z}^2$, suppose $\rho_i(a,b) \leq 0$ for all i. Let $d_i(a,b) = \max\{0, -\rho_i(a,b)+m\}$, then*

$$\dim F_m^{\rho_1}(\rho_1(a,b)) \cap F_m^{\rho_{n+2}}(\rho_{n+2}(a,b)) = \max\{0, m+1-d_1(a,b)-d_{n+2}(a,b)\},$$
$$\dim \bigcap_{i=0}^{n+1} F_m^{\rho_i}(\rho_i(a,b)) = \max\{0, m+1-\sum_{i=1}^{n+2} d_i(a,b)\}.$$

Proof: For each $i = 1,\ldots,n+2$, we can express the pairing $\langle (a,b), \rho_i\rangle$ as $\rho_i(a,b) = (i-1)a+b$. If $\rho_i(a,b) \leq -m-1$, then $F_m^{\rho_i}(\rho_i(a,b)) = S^m E$, so in the intersection this space does not restrict the result.

For $-m \leq \rho_i(a,b) \leq 0$, the vector space $F_m^{\rho_i}(\rho_i(a,b))$ consists of the symmetric vectors divisible by $(\rho_i^{\perp})^{-i+m}$. Since for each i, taking the intersection of the space $S^m E$ with each of the spaces $F_m^{\rho_i}(\rho_i(a,b))$ will reduce the dimension by $-\rho_i(a,b)+m$. Thus, we can conclude the result by an inductive argument. $\square$

4.2 Rational polytopes and Ehrhart Theory

In this section, we establish a connection between the computation of dimensions of regular symmetric differentials on Y and $Y-\epsilon$ and the enumeration of lattice points in a 3-dimensional rational polytope. Counting lattice points in polytopes is a well-studied problem in combinatorics. We refer to [3, Chapter 1 and 2] to introduce the concepts of convex sets and polytopes.

Definition 4.6. An open half-space in $\mathbb{R}^d$ determined by the vector $\mathbf{a} \in \mathbb{R}^d$ and a constant b is defined by the set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^d, \mathbf{a} \cdot \mathbf{x} > b\}$. A closed half-space in $\mathbb{R}^d$ determined by the vector $\mathbf{a} \in \mathbb{R}^d$ and a constant b is defined by the set $\{\mathbf{x} : \mathbf{x} \in \mathbb{R}^d, \mathbf{a} \cdot \mathbf{x} \geq b\}$.

If $H := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} = b\}$ is a hyperplane in $\mathbb{R}^d$, then H gives two closed half-spaces

$$\begin{aligned} H_1 &= \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{x} \geq b\}, \\ H_2 &= \{\mathbf{x} \in \mathbb{R}^d : -\mathbf{a} \cdot \mathbf{x} \geq -b\}. \end{aligned}$$

Definition 4.7. A convex polytope in $\mathbb{R}^d$ is an intersection of finitely many closed half-spaces.

Similar to the supporting planes of a cone that we defined in Chapter 2, we can also define the supporting plane of a convex polytope $\mathcal{P}$:

Definition 4.8. Let H be a hyperplane in $\mathbb{R}^d$. We say H is a *supporting hyperplane* of a convex polytope $\mathcal{P}$ if the following holds:

(a) H intersects $\mathcal{P}$;

(b) $\mathcal{P}$ is contained in one of the closed half-space of H. We call the intersection $\mathcal{P} \cap H$ a *face* of $\mathcal{P}$.

Theorem 4.9. *[3, Corollary 8.4] Any face of a polytope is a polytope.*

Definition 4.10. Let $\mathcal{P}$ be a polytope in $\mathbb{R}^d$. Pick a point x_0 in $\mathcal{P}$ and define the dimension to be the dimension of the vector space spanned by $V_{X,x_0} = \{x - x_0 : x \in X\}$. The dimension of an affine space is the dimension of a vector space parallel to the affine space.

Definition 4.11. Let $\mathcal{P}$ be a polytope in $\mathbb{R}^d$. Faces of dimension 0 are called *vertices* of $\mathcal{P}$; faces of dimension 1 are called *edges* of $\mathcal{P}$.

Definition 4.12. The *convex hull* $\mathrm{Conv}(A)$ of a finite set A of points in $\mathbb{R}^n$ is the smallest convex set that contains A.

One easily checks that any bounded convex polytope is the convex hull of its vertices. If $V_1, \ldots, V_k$ are the vertices of a closed convex polytope, we write $\mathcal{P}(V_1, \ldots, V_k)$.

Definition 4.13. A *dilation* of a polytope $\mathcal{P}$ with a *dilation factor* λ, denote by $\lambda\mathcal{P}$ is defined by $\{\lambda x : x \in \mathcal{P}\}$.

Definition 4.14. A *rational polytope* in $\mathbb{R}^n$ is a polytope with vertices in $\mathbb{Q}^n$. The *denominator* of $\mathcal{P}$, denoted by $d(\mathcal{P})$, is defined as the smallest integer k such that $k\mathcal{P} \subset \mathbb{Z}^n$. In other words, $d(\mathcal{P})$ is the lowest common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.

Definition 4.15. A quasi-polynomial of degree n is a function $p\colon \mathbb{N} \to \mathbb{R}$ of form $p(m) = \sum_{i=0}^{n} c_i(m)m^i$, where there exists a positive integer d such that $c_i(m+d) = c_i(m)$ for all i and $m \in \mathbb{N}$. The *period* of a quasi-polynomial is the smallest positive integer d such that $c_i(m+d) = c_i(m)$.

Theorem 4.16 (Ehrhart,1962 and McMullen, 1978)**.** *Let $\mathcal{P} \subset \mathbb{R}^n$ be a rational polytope. There is a quasi-polynomial $Q(\mathcal{P})$ of degree $\dim \mathcal{P}$ and period dividing $d(\mathcal{P})$ such that $Q(\mathcal{P})(\lambda) = \#(\lambda\mathcal{P} \cap \mathbb{Z}^n)$ for $\lambda \in \mathbb{N}_{\geq 0}$. The leading coefficient of $Q(\mathcal{P})$ is the volume of $\mathcal{P}$.*

4.3 Explicit construction of polytopes

We recall the toric variety $Y = X_\Sigma$ from Lemma 3.24, and we also recall the Klyachko filtration of the symmetric differentials as given in Proposition 4.3.

Proposition 4.17. *There exist only finitely many lattice points $(a,b) \in \mathbb{Z}^2$ such that the finite-dimensional quotient vector space*

$$Z_m(a,b) := \left(F_m^{\rho_1}(\rho_1(a,b)) \bigcap F_m^{\rho_{n+2}}(\rho_{n+2}(a,b))\right) / \bigcap_{i=1}^{n+2} F_m^{\rho_i}(\rho_i(a,b)).$$

is non-trivial.

The proof of Proposition 4.17 follows from the following construction. If either of the values $\rho_1(a,b)$ or $\rho_{n+2}(a,b)$ is strictly greater than 0, then the intersection of the vector spaces is trivial:

$$F_m^{\rho_1}(\rho_1(a,b)) \cap F_m^{\rho_{n+2}}(\rho_{n+2}(a,b)) = 0.$$

Thus, the quotient space $Z_m(a,b)$ is also trivial. On the other hand, if we have the following condition:

$$\bigcap_{i=1}^{n+2} F_m^{\rho_i}(\rho_i(a,b)) = F_m^{\rho_1}(\rho_1(a,b)) \cap F_m^{\rho_{n+2}}(\rho_{n+2}(a,b)),$$

then the space $Z_m(a,b)$ is trivial again. The proof of Proposition 4.17 is explicitly constructive. We will show that the lattice points (a,b) for which $Z_m(a,b)$ is non-trivial lie inside a bounded polytope.

Definition 4.18. We define the vector space $V_m^{(a,b)}$ as $\left(F_m^{\rho_1}(\rho_1(a,b)) \cap F_m^{\rho_{n+2}}(\rho_{n+2}(a,b))\right)$ and $W_m^{(a,b)} = \bigcap_{i=1}^{n+2} F_m^{\rho_i}(\rho_i(a,b))$. Furthermore, we denote $z_m(a,b)$ as the dimension of the quotient vector space $Z_m(a,b)$.

We subdivide the (a,b)-plane into finitely many convex polygonal regions where we can compute $\dim V_m(a,b)$ and $\dim W_m(a,b)$ relatively easily. See Figure 4.1.

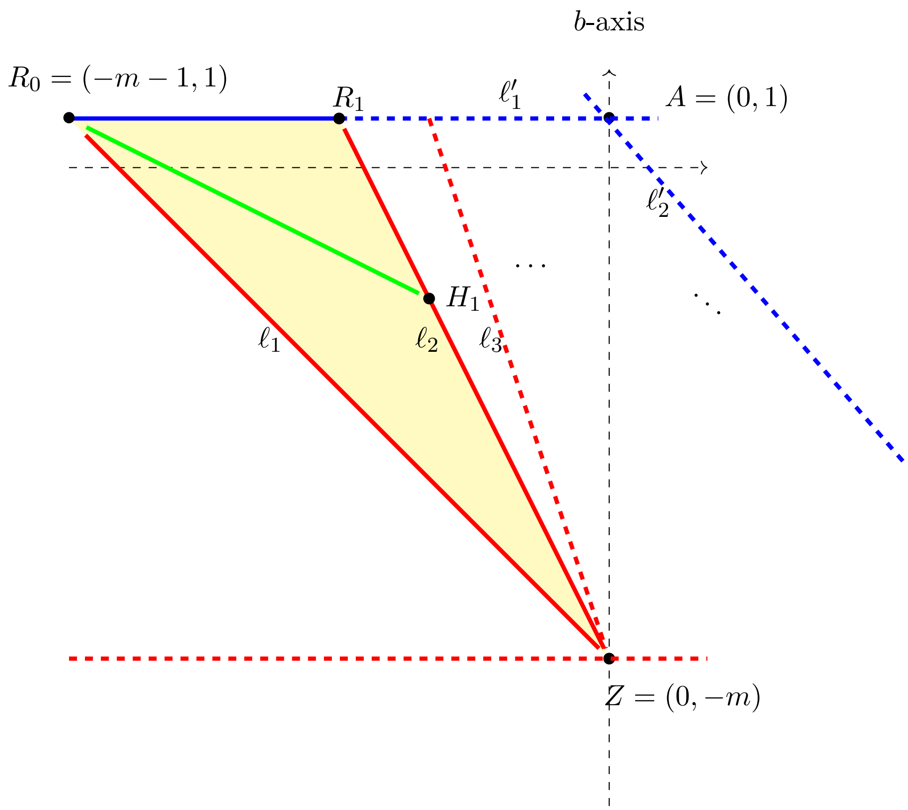

Figure 4.1: Polygonal region part 1 (a).

Let $\ell_i := \rho_{i+1}(a,b) + m = 0$ and $\ell'_i = \rho_{i+1}(a,b) - 1 = 0$ for $i = 1, \ldots, n+1$, and let $Z = (0, -m)$. We denote the intersection point of the line $b = 1$ and ℓ_i as R_{i-1}, for each $i = 1, \ldots, n+1$.

The polygonal region $\mathrm{Conv}(R_0, R_1, Z)$ in the (a,b)-plane is defined by the inequalities

$$\begin{aligned}
\rho_1(a,b) - 1 &\leq 0, \\
\rho_1(a,b) + m &\geq 0, \\
\rho_2(a,b) + m &\geq 0, \\
\rho_j(a,b) + m &\leq 0 \text{ for } j = 3, \ldots, n+1.
\end{aligned}$$

Since $a \leq 0$ for all (a,b) in this region, we determine the expressions for the vector spaces as follows using Proposition 4.3:

$$\begin{aligned}
F_m^{\rho_1}(\rho_1(a,b)) &= (\rho_1^{\perp})^{\rho_1(a,b)+m} S^{-\rho_1(a,b)} E, \\
F_m^{\rho_2}(\rho_2(a,b)) &= (\rho_2^{\perp})^{\rho_2(a,b)+m} S^{-\rho_2(a,b)} E, \\
F_m^{\rho_j}(\rho_j(a,b)) &= S^m E \text{ for all } j = 3, \ldots, n+1.
\end{aligned}$$

Therefore, using Corollary 4.5 for each (a,b) in this polygonal region, we express dimensions as follows:

$$\begin{aligned}\dim V_m^{(a,b)} &= 1-b,\\ \dim W_m^{(a,b)} &= \max\{0,(m+1)-(b+m)-(a+b+m)\}.\end{aligned}$$

Let H_1 be the intersection of the line $(m+1)-(b+m)-(a+b+m)=0$ with ℓ_2, we have

$$H_1 = \left(-\frac{m+1}{3}, -\frac{m-2}{3}\right).$$

The line $(m+1)-(b+m)-(a+b+m)=0$ intersects ℓ_1 at R_0. We also define H_0 to be the same point to maintain consistency with H_1 and later constructions. Then, the line segment $\overline{R_0H_1}$ distinguishes the regions where $\dim W_m(a,b)=0$ or $\dim W_m(a,b)=(m+1)-(b+m)-(a+b+m)$. Thus, the codimension is given by:

$$z_m(a,b) = \begin{cases} 1-b \text{ for } (a,b)\in \operatorname{Conv}(R_0,H_1,R_1),\\ a+b+m \text{ for } (a,b)\in \operatorname{Conv}(R_0,H_1,Z).\end{cases}$$

We denote the intersection of ℓ_i with ℓ_i' by P_i. Refer to Figure 4.2 for an illustration of the points P_0, P_1, and P_2. Similarly, we can analyze the polygonal region extending beyond the b-axis. The dashed line through Z and P_0 is defined by the equation $b+m=0$.

The line segment $\overline{P_0P_2}$ is defined by the equation $\rho_{n+2}(a,b)+1=0$, and for any (a,b) above this line, all vector spaces become trivial. The two lines $\overline{ZP_1}$ and $\overline{ZP_2}$ correspond to the line segments $\overline{R_0Z}$ and $\overline{R_1Z}$ in Figure 4.1, respectively. Inside the polygonal region $\operatorname{Conv}(Z,P_1,P_2)$ in the (a,b)-plane, we observe the reversed inequalities compared to part (a):

$$\begin{aligned}&\rho_1(a,b)+m\le 0,\\ &\rho_2(a,b)+m\le 0,\\ &\rho_j(a,b)+m\ge 0 \text{ for } j=3,\dots,n+1.\end{aligned}$$

We determine the vector spaces accordingly using Proposition 4.3:

$$\begin{aligned}&F_m^{\rho_1}(\rho_1(a,b)) = F_m^{\rho_2}(\rho_2(a,b)) = S^mE,\\ &F_m^{\rho_j}(\rho_j(a,b)) = (\rho_j^{\perp})^{\rho_j(a,b)+m}S^{-\rho_j(a,b)}E \text{ for } j=3,\dots,n+1.\end{aligned}$$

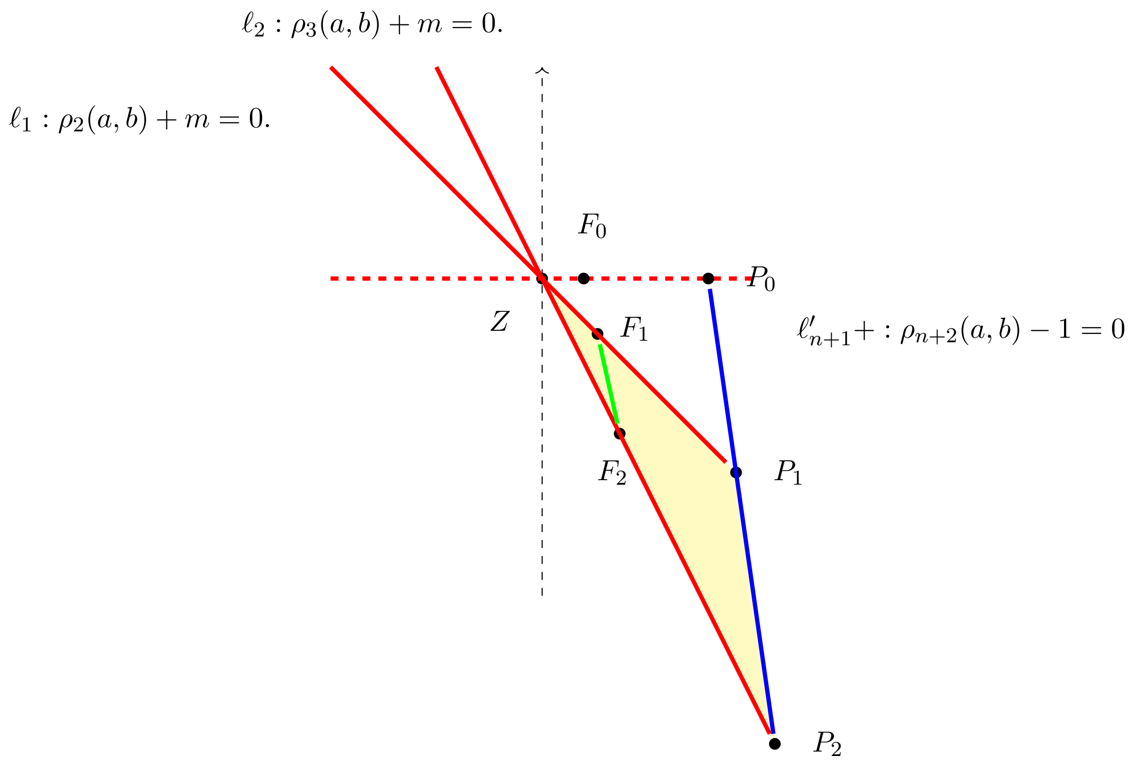

Figure 4.2: Polyhedron region part 1 (b).

For each (a,b) within the polygonal region $\mathrm{Conv}(Z, P_1, P_2)$, the dimensions of the vector spaces are as follows:

$$\dim V_m^{(a,b)} = m + 1 - ((n+1)a + b + m) = 1 - (n+1)a - b,$$
$$\dim W_m^{(a,b)} = \max\{0, -\frac{(n+3)n}{2}a - nb - m(n-1) + 1\}.$$

The line $-\frac{(n+3)n}{2}a - nb - m(n-1) + 1 = 0$ intersects ℓ_1 and ℓ_2 at F_1 and F_2, respectively. The line segment $\overline{F_1F_2}$ divides the polygonal region $\mathrm{Conv}(Z, P_1, P_2)$ in the (a,b)-plane into two separate polygonal regions: $\mathrm{Conv}(Z, F_1, F_2)$ and $\mathrm{Conv}(F_1, F_2, P_2, P_1)$. The dimension of the quotient space for each (a,b) is given by:

$$z_m(a,b) = \begin{cases} 1 - (n+1)a - b \text{ for } (a,b) \in \mathrm{Conv}(F_1, F_2, P_2, P_1), \\ \frac{(n+2)(n-1)}{2} + (n-1)b + m(n-1) \text{ for } (a,b) \in \mathrm{Conv}(Z, F_1, F_2). \end{cases}$$

Using this expression, we can interpolate the function $z_m(a,b)$ to obtain values for real coordinates a and b. This allows us to define a rational polytope in $\mathbb{R}^3$ with coordinates (a, b, z_m).

For any integer k in the range $0, \ldots, n-2$, we can identify a corresponding region in the (a,b)-plane, which is defined by the following linear relations. See Figure 4.3 for an illustration.

$$\begin{aligned}b+1&=0,\\ \rho_{k+2}(a,b)+m&=0,\\ \rho_{k+3}(a,b)+m&=0,\\ \rho_{n+2}(a,b)+1&=0.\end{aligned}$$

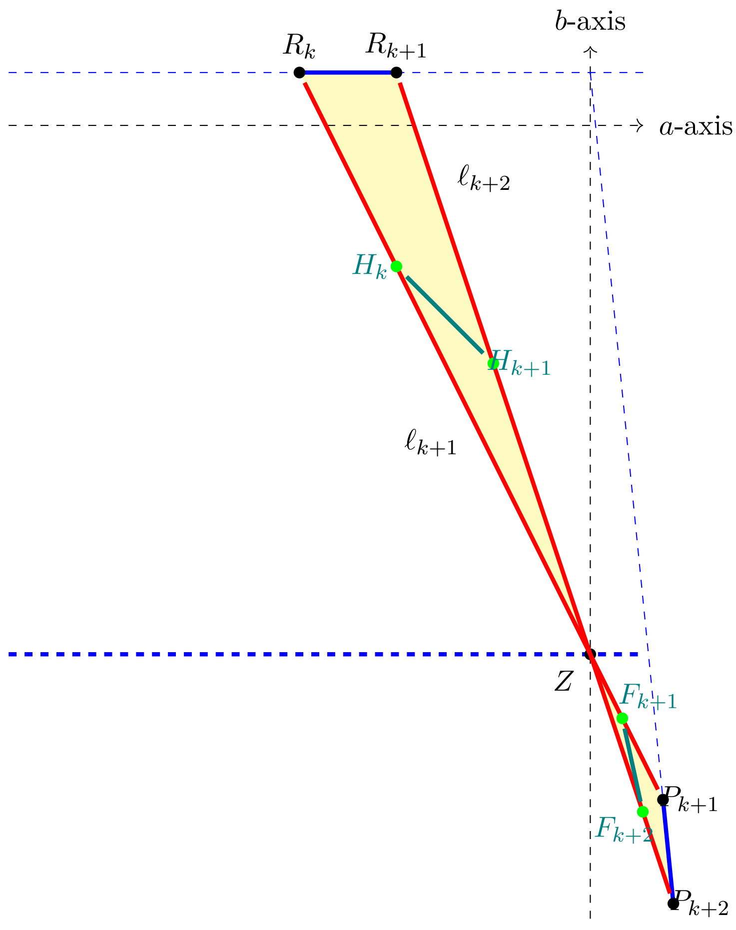

Figure 4.3: Polyhedron for arbitrary k.

The line segments $\overline{R_kP_{k+1}}$ and $\overline{R_{k+1}P_{k+2}}$ are defined by the equations $\rho_{k+1}(a,b)+m=0$ and $\rho_{k+2}(a,b)+m=0$, respectively. We determine the vector spaces, for $(a,b)\in \mathrm{Conv}(Z,R_k,R_{k+1})$, and obtain

$$F_m^{\rho_i}(\rho_i(a,b))=\begin{cases}(\rho_i^{\perp})^{\rho_i(a,b)+m}S^{-\rho_i(a,b)}E & \text{for } 1\le i\le k+2,\\ S^mE & \text{for } k+3\le i\le n+2.\end{cases}$$

For $(a,b) \in \mathrm{Conv}(Z, P_{k+1}, P_{k+2})$, we have

$$F_m^{\rho_i}(\rho_i(a,b)) = \begin{cases} (\rho_i^{\perp})^{\rho_i(a,b)+m} S^{-\rho_i(a,b)} E & \text{for } k+3 \le i \le n+2, \\ = S^m E & \text{for } 1 \le i \le k+2. \end{cases}$$

Using this, we compute the dimensions of the vector spaces $V_m(a,b)$ and $W_m(a,b)$, and encode them in Equation 4.3 and Equation 4.4.

$$\dim V_m^{(a,b)} = \begin{cases} 1-b & \text{for } (a,b) \in \mathrm{Conv}(R_k, R_{k+1}, Z); \\ 1-(n+1)a-2b & \text{for } (a,b) \in \mathrm{Conv}(P_{k+1}, P_{k+2}, Z). \end{cases} \tag{4.3}$$

$$\dim W_m(a,b) = \begin{cases} \max\{0, (m+1) - \sum_{i=1}^{k+2}(i-1)a + b + m\} & \text{for } (a,b) \in \mathrm{Conv}(R_k, R_{k+1}, Z); \\ \max\{0, (m+1) - \sum_{i=k+3}^{n+2}(i-1)a + b + m\} & \text{for } (a,b) \in \mathrm{Conv}(P_{k+1}, P_{k+2}, Z). \end{cases} \tag{4.4}$$

The line $(m+1) - \sum_{i=1}^{k+2}(i-1)a + b + m = 0$ intersects ℓ_{k+1} and ℓ_{k+2} at H_k and H_{k+1}, respectively. Similarly, the line $(m+1) - \sum_{i=k+3}^{n+2}(i-1)a + b + m = 0$ intersects ℓ_{k+1} and ℓ_{k+2} at F_{k+1} and F_{k+2}, respectively. The line segments $\overline{H_k H_{k+1}}$ and $\overline{F_{k+1} F_{k+2}}$ serve as thresholds for determining whether the vector space $W_m^{(a,b)}$ is zero. Then we can express the dimension of $W_m(a,b)$ based on the polygonal regions in the (a,b)-plane, as described in Figure 4.3. See Equation 4.5.

$$\dim W_m^{(a,b)} = \begin{cases} 0 & \text{for } (a,b) \in C_1; \\ -\frac{(k+2)(k+1)}{2} a - (k+2)b - (k+1)m + 1 & \text{for } (a,b) \in C_2; \\ \left(\frac{(n+k+3)(-n+k)}{2}\right) a - (n-k)b - (n-1-k)m + 1 & \text{for } (a,b) \in C_3; \\ 0 & \text{for } (a,b) \in C_4. \end{cases} \tag{4.5}$$

where

$$\begin{aligned} C_1 &= \mathrm{Conv}(R_k, R_{k+1}, H_k, H_{k+1}); \\ C_2 &= \mathrm{Conv}(H_k, H_{k+1}, Z); \\ C_3 &= \mathrm{Conv}(F_{k+1}, F_{k+2}, Z); \\ C_4 &= \mathrm{Conv}(P_{k+1}, P_{k+2}, F_{k+1}, F_{k+2}). \end{aligned}$$

For $0 \le k \le n-2$, the coordinates of the vertices are obtained by taking the intersections of the half-spaces. The function $z_m(a,b)$ extends to a piecewise-linear function on the (a,b)-

plane. We also include $z_m(a,b)$ as the third coordinate of the vertices. With a slight abuse of notation, we regard the vertices as points in $\mathbb{R}^3$ with coordinates $(a,b,z_m(a,b))$. The coordinates of the vertices are listed in the following table:

Table 4.1: Coordinates of the vertices.

Vertex	Coordinate
R_k	$\Big(-\frac{1}{k+1}(m+1), 0+1, 0\Big)$
H_k	$\Big(-\frac{2(m+1)}{(k+2)(k+1)}, -\frac{k(m+1)}{k+2}+1, \frac{k(m+1)}{k+2}\Big)$
P_k	$\Big(\frac{(m+1)}{(n+1-k)}, -\frac{(n+1)(m+1)}{n+1-k}+1, 0\Big)$
F_k	$\Big(\frac{2(m+1)}{(n+2-k)(n+1-k)}, \frac{(-(k-n)^2+k-3n-2)(m+1)}{(n+2-k)(n+1-k)}+1, \frac{(n-k)(m+1)}{n-k+2}\Big)$
Z	$\Big(0, -(m+1)+1, 0\Big)$
A	$\Big(0, 0+1, 0\Big)$

Remark 4.19. We can perform a shift of the polytopes by 1 unit along the b-axis, such that each component of the coordinates has a factor of $(m+1)$. This shift will be useful when we apply Ehrhart theory to count the lattice points in the polytopes in the next chapter.

Observe that when $k=0$, we obtain the same information as in Figure 4.1 and Figure 4.2, where the coordinates of H_0 and R_0 coincide. Similarly, when we set $k=n-2$, then F_n and P_n coincide. There are additional symmetries in this polytope construction, which we will address later in this section.

Although we assumed that $k=0,\dots,n-2$, the coordinates for R_n, H_n, P_0, and F_0 can also be included in the table. We complete the polyhedron construction with the last polytope before concluding this section. See Figure 4.4.

First, we consider the polygonal region $\mathrm{Conv}(R_{n-1}, R_n, Z)$. Following the same procedure as in Figure 4.3, we obtain:

$$\begin{aligned}
\dim V_m^{(a,b)} &= 1-b,\\
\dim W_m^{(a,b)} &= \max\{0, (m+1) - \sum_{i=1}^{n+1}(i-1)a+b+m\}\\
&= \max\{0, -\frac{(n+1)n}{2}a-(n+1)b-mn+1\}.
\end{aligned}$$

Let H_{n-1} and H_n be the intersection points of line $(m+1)-\sum_{i=1}^{n+1}(i-1)a+b+m=0$ with ℓ_n and ℓ_{n+1} in the (a,b)-plane, respectively. We compute the coordinates as in $\mathbb{R}^3$ by adding $z_m(a,b)$ as the third coordinate, we obtain

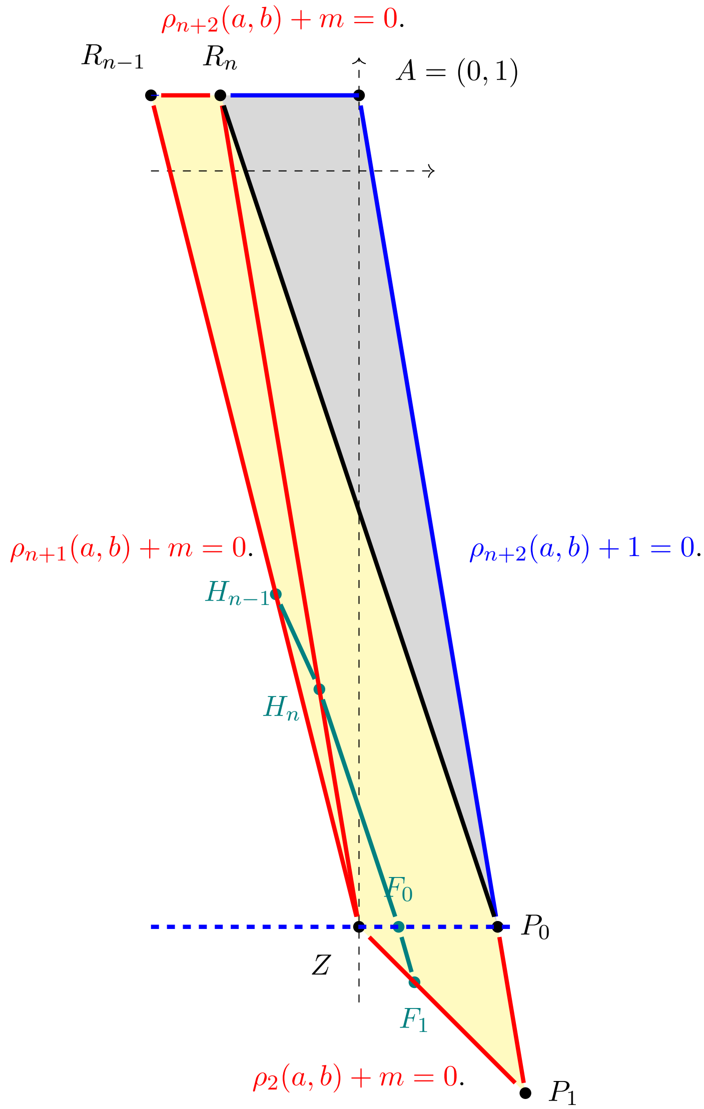

Figure 4.4: Central polyhedron

$$H_{n-1}=\left(-\frac{2(m+1)}{(n+1)n},-\frac{(n-1)(m+1)}{n+1}+1,\frac{(n-1)(m+1)}{n+1}\right), \tag{4.6}$$

$$H_n=\left(-\frac{2(m+1)}{(n+2)(n+1)},-\frac{(n)(m+1)}{n+2}+1,\frac{(n)(m+1)}{n+2}\right). \tag{4.7}$$

Similarly, for the polygonal region $\mathrm{Conv}(P_0,P_1,Z)$, we have the inequalities:

$$\rho_1(a,b)+m\le 0,$$
$$\rho_j(a,b)+m\ge 0 \text{ for all } j\ge 2.$$

This implies that

$$F_m^{\rho_i}(\rho_i(a,b)) = \begin{cases} S^m E & \text{if } i = 1; \\ (\rho_i^{\perp})^{\rho_i(a,b)+m} S^{-\rho_i(a,b)} E & \text{for all } 2 \leq i \leq n+2. \end{cases}$$

As a result, we obtain the dimensions:

$$\begin{aligned} \dim V_m^{(a,b)} &= 1 - (n+1)a - b - m, \\ \dim W_m^{(a,b)} &= \max\{0, -\frac{(n+2)(n+1)}{2}a - (n+1)b - mn + 1\}. \end{aligned}$$

Let F_0 and F_1 be the intersection points of the line $-\frac{(n+2)(n+1)}{2}a-(n+1)b-mn+1=0$ with ℓ_1 and ℓ_2, respectively. Using this, we can determine that $\dim V_m^{(a,b)}/W_m^{(a,b)}$ is equal to $\dim V_m^{(a,b)}$ above the line segment $\overline{F_0F_1}$ and $\dim V_m^{(a,b)} - \dim W_m^{(a,b)}$ below $\overline{F_0F_1}$. By adding $z_m(a,b)$ as the third coordinate, we have the coordinates of the points as follows:

$$F_0 = \left(\frac{2(m+1)}{(n+2)(n+1)}, \frac{(-n^3-3n-2)(m+1)}{(n+2)(n+1)} + 1, \frac{n(m+1)}{(n+2)} \right), \tag{4.8}$$

$$F_1 = \left(\frac{2(m+1)}{(n+1)(n)}, \frac{(-(n-1)^2-3n-1)(m+1)}{(n+1)(n)} + 1, \frac{(n-1)(m+1)}{(n+1)} \right). \tag{4.9}$$

Now it remains to consider the polygonal region $\mathrm{Conv}(R_n, A, P_0, Z)$. In this range, we have the following inequalities:

$$\begin{aligned} -m &\leq \rho_1(a,b) \leq 1, \\ -m &\leq \rho_i(a,b) \leq 0 \text{ for all } i \geq 2. \end{aligned}$$

These inequalities determine the dimensions of the vector spaces as follows:

$$\dim V_m^{(a,b)} = \max\{0, (m+1) - (b+m) - ((n+1)a + b + m)\}, \tag{4.10}$$

$$\dim W_m^{(a,b)} = \max\{0, (m+1) - \sum_{i=1}^{n+2} (i-1)a + b + m\}. \tag{4.11}$$

The shaded region is where $(m+1)-(b)-((n+1)a+b+m) < 0$ and hence all vector spaces will be trivial above the line segment $\overline{R_nP_0}$. Therefore, it is natural to consider only the polygonal region $\mathrm{Conv}(R_n, P_0, Z)$. Similarly, the line segment $\overline{H_nF_0}$ is given by the equation $(m+1) - \sum_{i=1}^{n+2}(i-1)a + b + m = 0$.

The shaded region represents the region where $(m+1)-(b)-((n+1)a+b+m)<0$, all vector spaces will be trivial above the line segment $\overline{R_nP_0}$. Therefore, it is natural to consider only the polygonal region $\mathrm{Conv}(R_n, P_0, Z)$. Similarly, the line segment $\overline{H_nF_0}$ is given by the equation $(m+1)-\sum_{i=1}^{n+2}(i-1)a+b+m=0$.

The coordinates of H_{n-1}, H_n, F_0, F_1 in $\mathbb{R}^3$ match the ones in Table 4.1. The points $\{R_k, H_k, F_k, P_k, Z : k=0,\ldots,n\}$ define a 3-dimensional rational polytope.

Remark 4.20. We observe that the polytope $\mathcal{P} \subset \mathbb{R}^3$ can be partitioned into smaller 3-dimensional polytopes using the construction described above. We can express $\mathcal{P}$ as the union of the following polytopes:

$$\mathcal{P} = \left(\bigcup_{i=0}^{n-1} \mathcal{P}(R_i, R_{i+1}, H_i, H_{i+1}, Z)\right) \cup \mathcal{P}(R_n, P_0, H_n, F_n, Z) \cup \left(\bigcup_{j=0}^{n-1} \mathcal{P}(P_j, P_{j+1}, F_j, F_{j+1}, Z)\right). \tag{4.12}$$

This partitioning satisfies the construction and covers the entire polytope $\mathcal{P}$. We call $\mathcal{P}(R_n, P_0, H_n, F_n, Z)$ the central polytope. Therefore, this confirms Proposition 4.17 that $z_m(a,b)$ is non-zero only in a bounded polytope in $\{(a,b) : a, b \in \mathbb{Z}^2\}$.

Proposition 4.21. *There is an affine transformation ϕ that preserves lattice points such that for all $k=0,\ldots,n$ we have*

$$\begin{aligned}
\phi \cdot R_k &= P_{n-k},\\
\phi \cdot H_k &= F_{n-k},\\
\phi \cdot Z &= Z.
\end{aligned}$$

Proof: Following Remark 4.19, we first apply a shift ψ in the negative b-axis direction by one unit, which can be represented as the affine transformation:

$$\psi : \begin{pmatrix} a \\ b \\ z \end{pmatrix} \mapsto \begin{pmatrix} a \\ b-1 \\ z \end{pmatrix}.$$

By inspection of the coordinates in Table 4.1, we identify a linear transformation $f \in \mathrm{GL}_2(\mathbb{Z})$ whose matrix representation is

$$f := \begin{pmatrix} -1 & 0 & 0 \\ (n+1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The determinant $\det f = -1$, so the linear transformation is volume-preserving. In particular, f maps integer points to integer points. Moreover, f is an involution, so $f^{-1} = f$ preserves volume and lattice points.

Therefore, we define the affine transformation $\phi := \psi^{-1} f \psi$. For each $k = 0, \dots, n$ we check:

$$\phi \cdot \psi(R_k) = \begin{pmatrix} -1 & 0 & 0 \\ (n+1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{k}(m+1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{m+1}{k+1} \\ -\frac{(n+1)(m+1)}{k+1} \\ 0 \end{pmatrix} = \psi(P_{n-k}),$$

$$\phi \cdot \psi(H_k) = \begin{pmatrix} -1 & 0 & 0 \\ (n+1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2(m+1)}{(k+2)(k+1)} \\ -\frac{k(m+1)}{k+2} \\ \frac{k(m+1)}{k+2} \end{pmatrix} = \begin{pmatrix} \frac{2(m+1)}{(k+2)(k+1)} \\ -\frac{(k^2+k+2n+2)(m+1)}{(k+2)(k+1)} \\ \frac{k(m+1)}{k+2} \end{pmatrix} = \psi(F_{n-k}),$$

$$\phi \cdot \psi(Z) = \begin{pmatrix} -\frac{2(m+1)}{(k+2)(k+1)} \\ -\frac{k(m+1)}{k+2} \\ \frac{k(m+1)}{k+2} \end{pmatrix} \begin{pmatrix} 0 \\ -m-1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -m-1 \\ 0 \end{pmatrix} = \psi(Z).$$

This establishes the claim in the proposition. □

Remark 4.22. Proposition 4.21 shows that the polytope $\mathcal{P}$ admits a symmetry. We can therefore get full information on lattice point counts in $\mathcal{P}$ by considering the convex parts $\mathrm{Conv}(R_k, H_k, R_{k+1}, H_{k+1}, Z)$ and the polytope on $\mathrm{Conv}(R_n, P_0, Z)$ in $\mathbb{R}^3$.

By Proposition 4.21 and Remark 4.22, we can simplify our analysis by focusing on the coordinates obtained after applying the affine transformation ψ described in the proposition. Consequently, we will adopt the adjusted table of coordinates for the vertices of any polytope constructed according to the procedure above from this point onward.

Table 4.2: Coordinates of the adjusted vertices.

Vertex	Coordinate
R_k	$\left(-\frac{1}{k+1}(m+1), 0, 0\right)$
H_k	$\left(-\frac{2(m+1)}{(k+2)(k+1)}, -\frac{k(m+1)}{k+2}, \frac{k(m+1)}{k+2}\right)$
P_k	$\left(\frac{(m+1)}{(n+1-k)}, -\frac{(n+1)(m+1)}{n+1-k}, 0\right)$
F_k	$\left(\frac{2(m+1)}{(n+2-k)(n+1-k)}, \frac{(-(k-n)^2+k-3n-2)(m+1)}{(n+2-k)(n+1-k)}, \frac{(n-k)(m+1)}{n-k+2}\right)$
Z	$\left(0, -(m+1), 0\right)$

We refer to Figure 5.1 and Figure 5.2 as examples of the polytope constructed from an A_2 and an A_3-singularity, respectively. For the SageMath code that generates these polytopes, please refer to Appendix A.

4.4 Main Theorem

We briefly introduced Ehrhart theory and lattice point counting in the previous sections. We showed an application of computing the dimension of a quotient space as a lattice point count inside a polytope. With no ambiguity, for fixed n and m, we write $\mathcal{P}$ for the polytope we construct using the procedure in the previous sections. We will prove Theorem 1.1.

We can extend the function $z_m(a,b)$, as defined in Definition 4.18, to a piecewise-linear function on $\mathbb{R}^2$. By doing so, we find that the dimension of $V_m(a,b)/W_m(a,b)$ can be expressed as $\max(0, \lfloor z_m(a,b) \rfloor)$. This observation, as stated in Proposition 4.17, allows us to represent the quantity $\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y)$ in terms of counting lattice points within a dilation of a polytope with a dilation factor of $m+1$.

We recall the main theorem and present its proof here.

Theorem (Main)**.** *Let X be a surface with a singularity s of type A_n, and let Y be a minimal resolution of X. The quantity $\chi^0_{loc}(s, S^m\Omega_Y) = \dim \mathrm{H}^0(Y-\epsilon, S^m\Omega_Y)/\,\mathrm{H}^0(Y, S^m\Omega_Y)$ is a quasi-polynomial in m with period dividing* $\mathrm{lcm}(1, 2, \ldots, n+2)$ *that we can explicitly determine.*

Proof: Define the ground polytope by taking the intersection of $\mathcal{P}$ with the (a,b)-plane:

$$\mathcal{P}_g := \mathcal{P} \cap \{(a,b,z) \in \mathbb{R}^3 : z = 0\}.$$

Note that $\mathcal{P}_g$ is the intersection of $\mathcal{P}$ with $z = 0$. We recognize that $\mathcal{P}_g$ is a 2-dimensional polytope with the same vertices but without the third coordinate. Therefore, Theorem 4.16 applies to the rational polytope $\mathcal{P}_g$ as well.

Let $Q(\mathcal{P})$ and $Q(\mathcal{P}_g)$ be the quasi-polynomials of the rational polytopes $\mathcal{P}$ and $\mathcal{P}_g$, respectively. By Corollaries 4.5 and 3.20, the number of lattice points in $\mathcal{P}$ is given by

$$\mathcal{P} \cap \mathbb{Z}^3 = \sum_{(a,b) \in \mathcal{P}_g \cap \mathbb{Z}^2} (z_m(a,b) + 1) = \chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) + Q(\mathcal{P}_g).$$

Therefore, for any A_n-singularity and symmetric differential of degree m, the codimension count is given by

$$\chi^0_{\mathrm{loc}}(S^m\Omega_Y) = \dim \frac{\mathrm{H}^0(Y-\epsilon, S^m\Omega_Y)}{\mathrm{H}^0(Y, S^m\Omega_Y)} = Q(\mathcal{P}) - Q(\mathcal{P}_g).$$

Since $\mathcal{P}_g$ is a 2-dimensional rational polytope with a dilation factor of $(m+1)$, its quasi-polynomial $Q(\mathcal{P}_g)$ is a quadratic quasi-polynomial.

Note that the coordinates of the vertices R_i have denominators $\{i+1 : i = 0, \ldots, n\}$, and the coordinates of the vertices H_i have denominators $\{(i+2)(i+1) : i = 0, \ldots, n\}$. Therefore, the denominator $d(\mathcal{P})$ of the polytope $\mathcal{P}$ divides $(n+2)!$. More precisely, we expect the period to be even smaller than $\mathrm{lcm}(\{1, 2, \ldots, n+2\})$. Thus, we see that both

$Q(\mathcal{P})$ and $Q(\mathcal{P}g)$ have a period dividing $\operatorname{lcm}(\{1, 2, \ldots, n+2\})$. Therefore, we conclude that $\chi^0_{\mathrm{loc}}(S^m\Omega_Y)$ is a cubic quasi-polynomial with a dilation factor of $(m+1)$ and a period that divides $\operatorname{lcm}(\{1, 2, \ldots, n+2\})$. $\square$

Chapter 5

Explicit computation of quasi-polynomials and limit behaviour

In this chapter, we present explicit volume computations to study the limit behavior of χ^0_{loc} as $n \to \infty$.

5.1 Explicit computation for $n = 2$ and 3

The computation of $\chi^0_{\mathrm{loc}}(S^m\Omega_X)$ for X with an A_1-singularity is indeed already in [4]. As an extension, we will now provide the explicit formula for $\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y)$ where s is an A_2 or A_3-singularity.

When $n = 2$, the coordinates of all vertices of $\psi(\mathcal{P})$ are

Table 5.1: Coordinates of the vertices of $\psi(\mathcal{P})$ for $n = 2$.

Vertex	Coordinate	Vertex	Coordinate
R_0	$\Big(-(m+1),0,0\Big)$	P_0	$\Big(\frac{m+1}{3},-m-1,0\Big)$
R_1	$\Big(-\frac{m+1}{2},0,0\Big)$	P_1	$\Big(\frac{(m+1)}{2},-\frac{3(m+1)}{2},0\Big)$
R_2	$\Big(-\frac{m+1}{3},0,0\Big)$	P_2	$\Big((m+1),-3(m+1),0\Big)$
H_0	$\Big(-(m+1),0,0\Big)$	F_0	$\Big(\frac{(m+1)}{6},-(m+1),\frac{m+1}{2}\Big)$
H_1	$\Big(-\frac{(m+1)}{3},-\frac{(m+1)}{3},\frac{(m+1)}{3}\Big)$	F_1	$\Big(\frac{(m+1)}{3},-\frac{4(m+1)}{3},\frac{(m+1)}{3}\Big)$
H_2	$\Big(-\frac{(m+1)}{6},-\frac{(m+1)}{2},\frac{(m+1)}{2}\Big)$	F_2	$\Big(m+1,-3(m+1),0\Big)$
Z	$\Big(0,-(m+1),0\Big)$		

By setting $m = 0$, we plot the polytope with SAGE, see Figure 5.1

From the table, it can be observed that the period of the quasi-polynomial divides 6. By evaluating it at different values of m and performing polynomial interpolation, we eventually

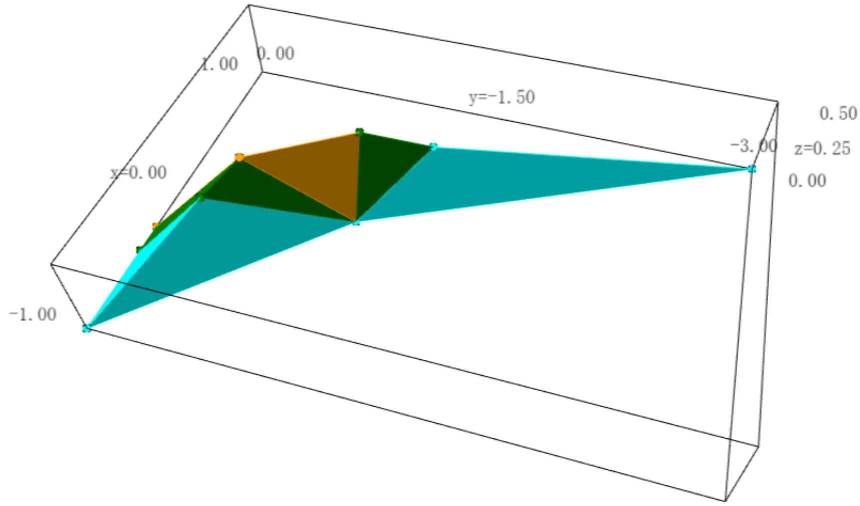

Figure 5.1: Polytope when $n = 2$

obtain the Ehrhart quasi-polynomial with a leading coefficient of $\frac{29}{216}$ and a period of 6, as follows:

$$\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) = \begin{cases} \frac{29}{216}(m+1)^3 - \frac{23}{72}(m+1) + \frac{5}{27} & \text{if } m \equiv 0 \bmod 6, \\ \frac{29}{216}(m+1)^3 - \frac{5}{18}(m+1) - \frac{14}{27} & \text{if } m \equiv 1 \bmod 6, \\ \frac{29}{216}(m+1)^3 - \frac{5}{24}(m+1) & \text{if } m \equiv 2 \bmod 6, \\ \frac{29}{216}(m+1)^3 - \frac{5}{18}(m+1) + \frac{14}{27} & \text{if } m \equiv 3 \bmod 6, \\ \frac{29}{216}(m+1)^3 - \frac{23}{72}(m+1) - \frac{5}{27} & \text{if } m \equiv 4 \bmod 6, \\ \frac{29}{216}(m+1)^3 - \frac{1}{6}(m+1) & \text{if } m \equiv 5 \bmod 6. \end{cases}$$

We may expand the quasi-polynomial as follows,

$$\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) = \begin{cases} \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m & \text{if } m \equiv 0 \bmod 6, \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m - \frac{143}{216} & \text{if } m \equiv 1 \bmod 6, \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{7}{36}m - \frac{2}{27} & \text{if } m \equiv 2 \bmod 6, \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m + \frac{3}{8} & \text{if } m \equiv 3 \bmod 6, \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m - \frac{10}{27} & \text{if } m \equiv 4 \bmod 6, \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{17}{72}m - \frac{7}{216} & \text{if } m \equiv 5 \bmod 6. \end{cases}.$$

For $n \geq 3$, the period increases drastically as n increases, as expected in Theorem 1.1. When $n = 3$, the Ehrhart quasi-polynomial has a leading coefficient of $\frac{809}{5400}$ and a period of $\operatorname{lcm}(2,3,4,5) = 60$. For large values of n, the computation can be performed using SAGE or Magma.

Let's focus on the case when $n = 3$. Below, we list the vertices of the polytope $\mathcal{P}$, and we include a plot of the polytope $\mathcal{P}$ when $n = 3$ using SAGE. However, it is important to note that the quasi-polynomial, in this case, has a period of 60, which we will present in Appendix A.

Table 5.2: Coordinates of the vertices of $\psi(\mathcal{P})$ for $n = 3$.

Vertex	Coordinate	Vertex	Coordinate
R_0	$\big(-(m+1), 0, 0\big)$	P_0	$\big(\frac{m+1}{4}, -m-1, 0\big)$
R_1	$\big(-\frac{m+1}{2}, 0, 0\big)$	P_1	$\big(\frac{(m+1)}{3}, -\frac{4(m+1)}{3}, 0\big)$
R_2	$\big(-\frac{m+1}{3}, 0, 0\big)$	P_2	$\big(\frac{(m+1)}{2}, -2(m+1), 0\big)$
R_3	$\big(-\frac{m+1}{4}, 0, 0\big)$	P_3	$\big((m+1), -4(m+1), 0\big)$
H_0	$\big(-(m+1), 0, 0\big)$	F_0	$\big(\frac{(m+1)}{10}, -(m+1), \frac{3(m+1)}{5}\big)$
H_1	$\big(-\frac{(m+1)}{3}, -\frac{(m+1)}{3}, \frac{(m+1)}{3}\big)$	F_1	$\big(\frac{(m+1)}{6}, -\frac{7(m+1)}{6}, \frac{(m+1)}{2}\big)$
H_2	$\big(-\frac{(m+1)}{6}, -\frac{(m+1)}{2}, \frac{(m+1)}{2}\big)$	F_2	$\big(\frac{(m+1)}{3}, -\frac{5(m+1)}{3}, \frac{(m+1)}{3}\big)$
H_3	$\big(-\frac{(m+1)}{10}, -\frac{3(m+1)}{5}, \frac{3(m+1)}{5}\big)$	F_3	$\big(m+1, -4(m+1), 0\big)$
Z	$\big(0, -(m+1), 0\big)$		

The plot of the polytope $\mathcal{P}$ generated by SAGE:

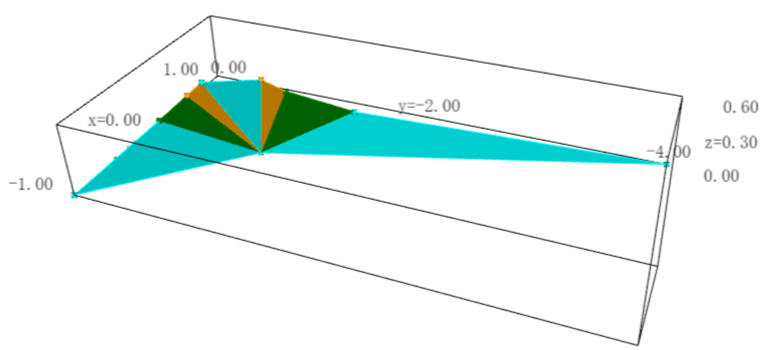

Figure 5.2: Polytope when $n = 3$

In Figure 5.1, one can observe the sheared symmetry described in Proposition 4.21. The image may be misleading due to the shearing, which increases with n. However, as we have computed, the affine transformation ϕ preserves lattice points. Therefore, the lattice point counts in the corresponding polytopes still match.

5.2 Independence of lattice point counts from n when $n > m$

In Section 4.3, we have explicitly computed the vertices of the polytopes for arbitrary $n > 0$. We denote the polytope corresponding to the A_n-singularity as $\mathcal{P}_n$. Let $\{Z, R_i, H_i, P_i, F_i : i = 0, \dots, n\}$ be the vertices of $\mathcal{P}_n$ as discussed in the previous section. Additionally, let $\{Z, R'_i, H'_i, P'_i, F'_i : i = 0, \dots, n+1\}$ be the vertices of the polytope $\mathcal{P}_{n+1}$.

From this point forward, for any polytope $\mathcal{A}$, we denote its intersection with the ground plane $\{(a, b, z) : a, b \in \mathbb{R}, z = 0\}$ as $\mathcal{A}_g$.

We observe that the polytopes $\mathcal{P}_{n+1}$ and $\mathcal{P}_n$ have the same subpolytopes

$$\mathcal{P}(Z, R'_i, R'_{i+1}, H'_i, H'_{i+1}) = \mathcal{P}(Z, R_i, R_{i+1}, H_i, H_{i+1}) \text{ for } i \leq n. \tag{5.1}$$

Therefore, according to Proposition 4.21, for a fixed value of m, the difference between the number of lattice points in $\mathcal{P}_{n+1}$ and the number of lattice points in $\mathcal{P}_n$ is equivalent to the difference between the lattice points in the following polytopes:

$$\begin{aligned}
\mathcal{A} &:= \mathcal{P}(R_{n-1}, R_n, H_{n-1}, H_n, Z, F_0, F_1, P_0, P_1), \\
\mathcal{B} &:= \mathcal{P}(R_n, P_0, F_0, H_n, Z).
\end{aligned}$$

We denote the polytope $\mathcal{P}(R'_n, R'_{n+1}, H'_n, H'_{n+1}, Z)$ as $\mathcal{A}_1$, the polytope $\mathcal{P}(R'_{n+1}, P'_0, F'_0, H'_{n+1}, Z)$ as $\mathcal{A}_2$, and the polytope $\mathcal{P}(P'_0, P'_1, F_1, F'_0, Z)$ as $\mathcal{A}_3$. We recognize that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$.

Proposition 5.1. *Let n_2, n_1 and m be positive integers. Suppose $n_2 > n_1$, if $m < n_1$ then*

$$\sum_{(a,b) \in (\mathcal{P}_{n_2})_g} \dim \frac{V_m^{(a,b)}}{W_m^{(a,b)}} = \sum_{(a,b) \in (\mathcal{P}_{n_1})_g} \dim \frac{V_m^{(a,b)}}{W_m^{(a,b)}}.$$

i.e., the number of lattice points in the half-open polytope $\mathcal{P} \setminus \mathcal{P}_g$ does not depend on n if $m < n$.

Proof: It is sufficient to demonstrate that the number of lattice points in $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is the same when $m < n$. By equation (5.1), we need to show that

$$\#\mathcal{A} \cap \{(a, b, z) : a, b, z \in \mathbb{Z}, z > 0\} = \#\mathcal{B} \cap \{(a, b, z) : a, b, z \in \mathbb{Z}, z > 0\}.$$

By Proposition 4.21, the affine transformation ϕ preserves lattice points and

$$\phi(\mathcal{A}_1) = \mathcal{A}_3.$$

Then we have

$$\#\mathcal{A}_1 \cap \mathbb{Z}^3 = \#\mathcal{A}_3 \cap \mathbb{Z}^3.$$

For $m < n$, the polytope $\mathcal{A}_1$ has vertices

$$\begin{aligned}
&R_n : \left(-\frac{1}{n+1}(m+1), 1, 0\right), R_{n+1} : \left(-\frac{1}{n+2}(m+1), 1, 0\right);\\
&H_n : \left(-\frac{2(m+1)}{(n+2)(n+1)}, -\frac{n(m+1)}{(n+2)} + 1, -\frac{n(m+1)}{(n+2)}\right);\\
&H_{n+1} : \left(-\frac{n(m+1)}{n+2}, -\frac{(n+1)(m+1)}{(n+3)} + 1, \frac{(n+1)(m+1)}{(n+3)}\right);\\
&Z : (0, -m, 0).
\end{aligned}$$

Since $m < n$, we have

$$\begin{aligned}
-1 < -\frac{(m+1)}{n+1} < -\frac{(m+1)}{(n+2)} \leq 0,\\
-1 < -\frac{2(m+1)}{(n+2)(n+1)} < -\frac{n(m+1)}{n+2} < 0.
\end{aligned}$$

We observe that the only vertex in the polytope $\mathcal{A}_1$ with an integral first coordinate is Z. However, the vertex Z belongs to the ground polytope $(\mathcal{A}_1)_g$. Therefore, there are no lattice points in $(\mathcal{A}_1)\setminus(\mathcal{A}_1)_g$. Consequently, there are no lattice points in $(\mathcal{A}_3)\setminus(\mathcal{A}_3)_g$. The number of lattice points in $\mathcal{A} \setminus \mathcal{A}_g$ is the same as the number of lattice points in $\mathcal{A}_2 \setminus (\mathcal{A}_2)_g$.

Now it remains to study $\mathcal{A}_2$ and $\mathcal{B}$. Note that $\mathcal{A}_2$ and $\mathcal{B}$ share a similarity in that they both represent the middle polytope discussed in Remark 4.20 for $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$, respectively. Without loss of generality, let us examine $\mathcal{A}_2$ first. Recall the first component of the coordinates of the vertices of $\mathcal{A}_2$ as shown in Table 4.1. When $m < n$, we observe that

$$-1 < -\frac{m+1}{n+1} < -\frac{2(m+1)}{(n+1)(n+2)} < \frac{2(m+1)}{(n+1)(n+2)} < \frac{m+1}{n+1} < 1.$$

If there exist lattice points, they must lie in the plane $\{(a, b, z) \in \mathbb{R}^3 \mid a = 0\}$. By taking intersections with the half-spaces, we obtain two line segments: $\overline{G_1 G_2}$ and $\overline{G_2 Z}$, where

$$\begin{aligned}
G_1 :&(0, -\frac{m}{2} + \frac{1}{2}, 0),\\
G_2 :&(0, -\frac{(n+1)m - 1}{(n+2)}, \frac{n(m+1)}{(n+2)}).
\end{aligned}$$

See Figure 5.3.

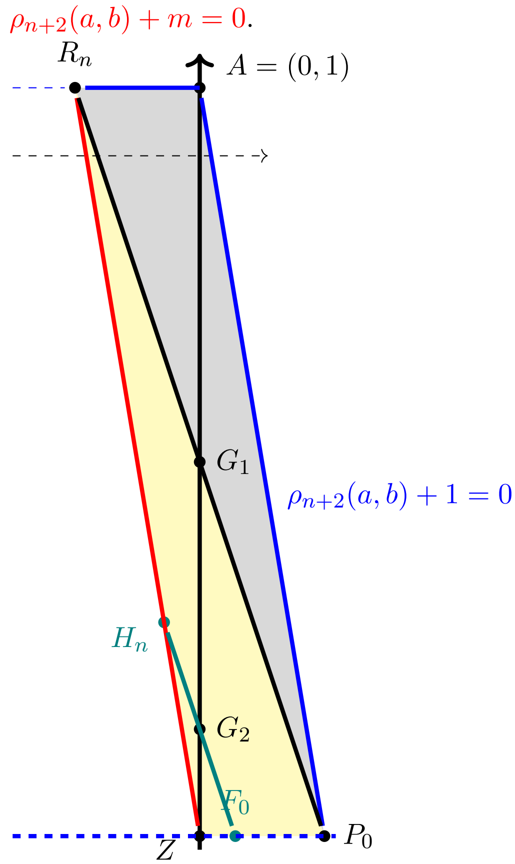

Figure 5.3: Base polyhedron of $\mathcal{A}_2$

Note $Z \in \mathbb{Z}^3$ has b-coordinate equal to $-m$ and

$$-\frac{(n+1)m-1}{n+2} - (-m) = \frac{m+1}{n+2} < 1.$$

Then there are no lattice points lying below the line segment $\overline{G_2 Z}$. And all the lattice points must lie below the line segment $\overline{G_1 G_2}$, which is

$$\overline{G_1G_2} : z = -2b - m + 1, b \in \left[-\frac{(n+1)m-1}{n+2}, -\frac{m-1}{2}\right].$$

There are at most $\lfloor \frac{n(m+1)}{2n+4} \rfloor$ possible integer values for b. We see that

$$-\frac{(n+1)m-1}{(n+2)} = -\frac{(n+2)m-m-1}{n+2} = -m + \frac{m+1}{n+2} \notin \mathbb{Z}.$$

So, the possible integral values for b are

$$\{-\frac{m-1}{2}, -\frac{m-1}{2}-1, \dots, -m+1\} \text{ if } m \text{ is odd };$$
$$\{-\frac{m}{2}, -\frac{m}{2}-1, \dots, -m+1\} \text{ if } m \text{ is even}.$$

Therefore, the number of lattice points in $\#\mathcal{A}_2 \cap \{(a,b,z) \in \mathbb{Z}^3, z > 0\}$ is

$$\sum_{b=-m+1}^{\lfloor -\frac{m-1}{2} \rfloor} -2b - m + 1 = \begin{cases} \frac{m^2-1}{4} & \text{if } m \text{ is odd;} \\ \frac{m^2}{4} & \text{if } m \text{ is even.} \end{cases} \tag{5.2}$$

We see that the number of lattice points in $\#\mathcal{A}_2 \cap \{(a,b,z) \in \mathbb{Z}^3, z > 0\}$ is independent of n, so the same results holds for all poistive integer n greater than m. □

5.3 Volume computation and limit behaviour

From (4.16), we know that the leading term of the quasi-polynomial of a polytope is determined by its volume. In the case of the polytope $\mathcal{P}$, we can obtain the leading coefficient of the quasi-polynomial $Q(\mathcal{P})$ by computing the volume of $\mathcal{P}$ with $(m+1)$ set to 1, that is, $m = 0$. As discussed in Remark 4.20, we can compute the volumes of the polytopes in the union:

$$\left(\bigcup_{i=0}^{n-1} \mathcal{P}(R_i, R_{i+1}, H_i, H_{i+1}, Z)\right) \cup \mathcal{P}(R_n, P_0, H_n, F_n, Z) \cup \left(\bigcup_{j=0}^{n-1} \mathcal{P}(P_j, P_{j+1}, F_j, F_{j+1}, Z)\right).$$

Note that by Corollary 4.21, we only need to compute the volumes of the polytopes $\mathcal{P}(R_n, P_0, H_n, F_0, Z)$ and $\mathcal{P}(R_i, R_{i+1}, H_i, H_{i+1}, Z)$ for $i = 0, \dots, n$. This is because the affine transformation ϕ preserves volumes.

Since we have extended the coordinates of the vertices of $\mathcal{P}$ in $\mathbb{R}^3$ by adding $z_m(a,b)$ as the third coordinate to the (a,b)-plane, we denote $\underline{\text{Conv}}(V_1, V_2, \dots, V_r)$ as the projection of the convex set $\text{Conv}(V_1, \dots, V_r)$ onto the (a,b)-plane, for $V_1, \dots, V_r$ in $\mathbb{R}^3$. Therefore, the volume of $\mathcal{P}$ is given by:

$$\text{Vol}(\mathcal{P}) = 2\sum_{i=0}^{n-1} \iint_{\underline{\text{Conv}}(R_i, R_{i+1}, Z)} z_0(a,b)\,\mathrm{d}a\,\mathrm{d}b + \iint_{\underline{\text{Conv}}(R_n, P_0, Z)} z_0(a,b)\,\mathrm{d}a\,\mathrm{d}b. \tag{5.3}$$

We begin with $\mathcal{P}(R_k, R_{k+1}, H_k, H_{k+1}, Z)$ for $k = 0, \dots, n-1$.
The volume of the polytope $\mathcal{P}(R_k, R_{k+1}, H_k, H_{k+1}, Z)$ is given by the integral

$$\text{Vol}(\mathcal{P}(R_k, R_{k+1}, H_k, H_{k+1}, Z)) = \iint_{\underline{\text{Conv}}(R_k, R_{k+1}, Z)} z_0(a,b)\,\mathrm{d}a\,\mathrm{d}b.$$

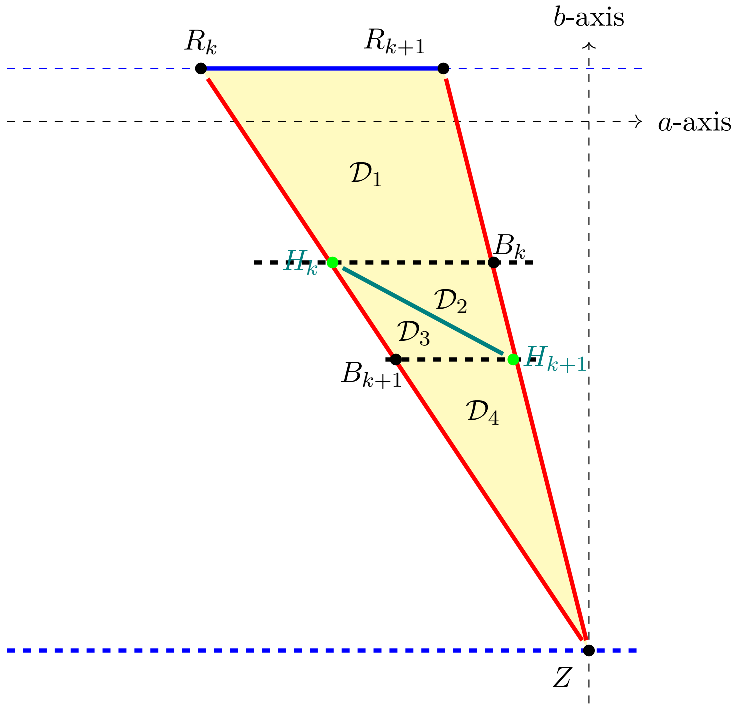

Figure 5.4: Base plane of $\mathcal{P}(R_k, R_{k+1}, H_k, H_{k+1}, Z)$

As shown in the figure, we can partition the polygonal region $\underline{\mathrm{Conv}}(R_k, R_{k+1}, Z)$ into four sub-polygonal regions $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$. The integral can be easily computed for each $\mathcal{D}_i$.

We recall (4.3) and (4.4), the dimension of the quotient space is given by:

$$z_0(a,b) = \begin{cases} 1-b & \text{for } (a,b) \in \mathcal{D}_1 \cup \mathcal{D}_2, \\ \frac{(k+2)(k+1)}{2}a + (k+1)b & \text{for } (a,b) \in \mathcal{D}_3 \cup \mathcal{D}_4. \end{cases}$$

Interpreting $\mathcal{D}_i$ as the ranges of a and b, we have

$$\mathcal{D}_1 = \begin{cases} -\frac{b}{k+1} \leq a \leq \frac{b}{k+2}, \\ -\frac{k}{k+2} + 1 \leq b \leq 1, \end{cases}$$

$$\mathcal{D}_2 = \begin{cases} \frac{-2((k+2)b-1)}{(k+2)(k+1)} \leq a \leq -\frac{b}{k+2}, \\ -\frac{(k+1)}{(k+3)} + 1 \leq b \leq -\frac{k}{k+1} + 1. \end{cases}$$

$$\mathcal{D}_3 = \begin{cases} -\frac{b}{k+1} \leq a \leq \frac{-2((k+2)b-1)}{(k+2)(k+1)}, \\ -\frac{k+1}{k+3} + 1 \leq b \leq -\frac{k}{k+1} + 1. \end{cases}$$

$$\mathcal{D}_4 = \begin{cases} -\frac{b}{k+1} \leq a \leq \frac{b}{k+2}, \\ 0 \leq b \leq -\frac{k+1}{k+3} + 1. \end{cases}$$

Let I_i be the integral of $z_0(a,b)$ over $\mathcal{D}_i$, we get

$$\begin{aligned}
I_1 &= \int_{\frac{-k}{k+2}+1}^{1} \int_{\frac{-b}{k+1}}^{\frac{-b}{k+2}} (1-b)\,\mathrm{d}a\,\mathrm{d}b = \frac{k^2(k+6)}{6(k+3)(k+2)^2(k+1)},\\
I_2 &= \int_{\frac{-2((k+2)b+1)}{(k+2)(k+1)}}^{-\frac{b}{k+2}} \int_{-\frac{(k+1)}{(k+3)}+1}^{-\frac{k}{k+1}+1} (1-b)\,\mathrm{d}a\,\mathrm{d}b = \frac{2(3k^2+9k+2)}{3(k+1)(k+3)^2(k+2)^4},\\
I_3 &= \int_{\frac{b}{k+1}}^{\frac{-2((k+2)b+1)}{(k+2)(k+1)}} \int_{-\frac{k+1}{k+3}+1}^{-\frac{k}{k+1}+1} \frac{(k+2)(k+1)}{2}a + (k+1)b\,\mathrm{d}a\,\mathrm{d}b = \frac{2(3k^2+8k+2)}{3(k+3)^3(k+2)^2(k+1)},\\
I_4 &= \int_{-\frac{b}{k+1}}^{\frac{b}{k+2}} \int_{0}^{-\frac{k+1}{k+3}+1} \frac{(k+2)(k+1)}{2}a + (k+1)b\,\mathrm{d}a\,\mathrm{d}b = \frac{4k+2}{3(k+3)^3(k+2)(k+1)}.
\end{aligned}$$

Summing up the integrals, we get

$$\mathrm{Vol}(\mathcal{P}(R_k, R_{k+1}, H_k, H_{k+1}, Z) = I_1 + I_2 + I_3 + I_4 = \frac{k^2+5k+2}{6(k+2)^2(k+1)(k+3)}.$$

We proceed similarly to compute the volume of the central polytope. Again, we subdivide the ground plane of the central polytope $\mathcal{P}(R_n, H_n, P_0, F_0, Z)$ into three parts $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$ as shown in Figure 5.5:

The volume of $\mathcal{P}(R_n, H_n, P_0, F_0, Z)$ is

$$\mathrm{Vol}(\mathcal{P}(R_n, H_n, P_0, F_0, Z)) = \sum_{i=1}^{3} \iint_{\mathcal{B}_i} z_0(a,b)\,\mathrm{d}a\,\mathrm{d}b.$$

By (4.10), we write

$$\begin{aligned}
\dim V_0^{(a,b)} &= 1-(b)-((n+1)a+b),\\
\dim W_0^{(a,b)} &= \begin{cases} 0 & \text{for } (a,b) \in \mathcal{B}_1 \cup \mathcal{B}_2,\\ -\frac{(n+2)(n+1)}{2}a-(n+2)b+1 & \text{for } (a,b) \in \mathcal{B}_3. \end{cases}
\end{aligned}$$

We have

$$z_0(a,b) := \begin{cases} 1-(n+1)a-2b \text{ for } (a,b) \in \mathcal{B}_1 \cup \mathcal{B}_2,\\ \frac{(n+1)n}{2}a - nb \text{ for } (a,b) \in \mathcal{B}_3. \end{cases}$$

Interpreting each $\mathcal{B}_i$ as the ranges of (a,b) in the (a,b)-plane, we have

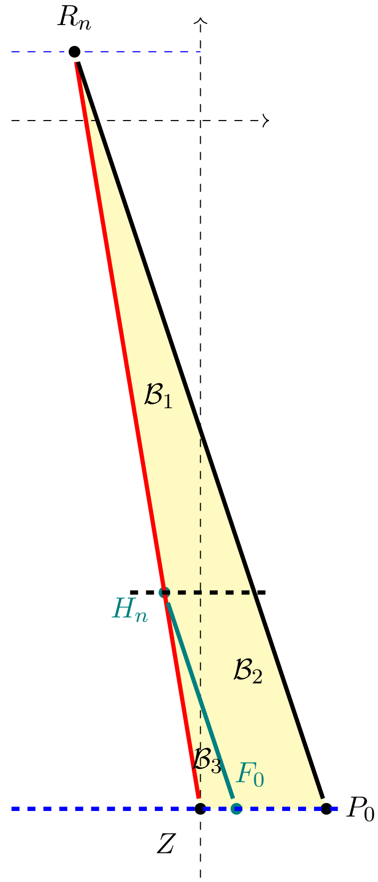

Figure 5.5: Base plane of $\mathcal{P}(R_n, H_n, P_0, F_0, Z)$

$$\mathcal{B}_1 = \begin{cases} -\frac{b}{n+1} \leq a \leq \frac{-2b+1}{n+1}, \\ \frac{-n}{n+2} + 1 \leq b \leq 1. \end{cases} \qquad \mathcal{B}_2 = \begin{cases} \frac{-2((n+2)b-1)}{(n+2)(n+1)} \leq a \leq \frac{-2b+1}{n+1}, \\ 0 \leq b \leq 1. \end{cases}$$

$$\mathcal{B}_3 = \begin{cases} -\frac{b}{(n+1)} \leq a \leq \frac{-2((n+2)b-1)}{(n+2)(n+1)}, \\ 0 \leq b \leq -\frac{n}{(n+2)} + 1. \end{cases}$$

We compute the integrals

$$I'_1 = \iint_{\mathcal{B}_1} h(a,b) \, \mathrm{d}a \, \mathrm{d}b = \frac{n^3}{6(n+1)(n+2)^3},$$
$$I'_2 = \iint_{\mathcal{B}_2} h(a,b) \, \mathrm{d}a \, \mathrm{d}b = \frac{n^2}{(n+1)(n+2)^3},$$
$$I'_3 = \iint_{\mathcal{B}_3} h(a,b) \, \mathrm{d}a \, \mathrm{d}b = \frac{4n}{3(n+1)(n+2)^3}.$$

The volume of the central polytope $\mathcal{P}(R_n, P_0, F_0, H_n, Z)$ is

$$\mathrm{Vol}(\mathcal{P}(R_n, P_0, F_0, H_n, Z) = I'_1 + I'_2 + I'_3 = \frac{n(n+4)}{6(n+1)(n+2)^2}.$$

By Equation 5.3, we can compute the total volume of the polytope $\mathcal{P}$:

$$\mathrm{Vol}(\mathcal{P})(n) = 2\sum_{k=0}^{n-1} \frac{k^2+5k+2}{6(k+2)^2(k+1)(k+3)} + \frac{n(n+4)}{6(n+1)(n+2)^2}. \tag{5.4}$$

We provide a table of the volumes with respect to A_n-singularities as n varies:

Table 5.3: Table of volumes with respect to small n.

Singularity type	Volume of $\mathcal{P}$	Period
A_1	$\frac{11}{108}$	6
A_2	$\frac{29}{216}$	6
A_3	$\frac{809}{5400}$	60
A_4	$\frac{143}{900}$	30
A_5	$\frac{10903}{66150}$	210
A_6	$\frac{178873}{1058400}$	420
A_7	$\frac{204929}{1190700}$	2520

Remark 5.2. From Equation 5.4, we compute the limit as n goes to infinity:

$$\lim_{n\to\infty} \mathrm{Vol}(\mathcal{P}_n) = \frac{2\pi^2}{9} - 2.$$

Therefore, we have

$$\lim_{n\to\infty} \chi^0_{\mathrm{loc}}(s, S^m\Omega_Y) = \left(\frac{2\pi^2}{9} - 2\right)(m+1)^3 + O(m).$$

Recalling Table 5.3, we observe that the volumes converge to the limit rather quickly. Furthermore, Proposition 5.1 shows that for a fixed m, the quantity χ^0_{loc} tends to stabilize as n becomes sufficiently large.

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Appendix A

Sage Code for polytope computation

The computation is done using SageMath version 9.8.

First, we need the following class and function to assist with the arithmetic of quasi-polynomials:

```
class quasipol:
    def __init__(self, L):
        self.n=len(L)
        self.L=L
    def __getitem__(self,i):
        return self.L[i % self.n]
    def __add__(self,other):
        m=lcm(self.n,other.n)
        return quasipol([self[i]+other[i] for i in range(m)])
    def __sub__(self,other):
        m=lcm(self.n,other.n)
        return quasipol([self[i]-other[i] for i in range(m)])
    def __rmul__(self,scalar):
        return quasipol([scalar*s for s in self.L])
    def __call__(self,a):
        return self[a](a)
    def __repr__(self):
        return "Quasipolynomial of period {} with components {}".format(self.n,self.L)


def reduce_period(P):
    def test(d):
        for i in range(d):
          for j in range(i+d,P.n,d):
            if P[i] != P[j]:
                return False
```

```
        return True
    for d in ZZ(P.n).divisors():
        if test(d):
          return quasipol(P.L[:d])
return P
```

The following function takes an input integer n and returns the polytopes $\mathcal{P}_n$ and $(\mathcal{P}_n)_g$

```
def polylist(n):
    m=0
    D = [((-m-1)/(i+1),0,0) for i in range (0,n+1)]
    H = [(-(m+1)/((i+2)*(i+1)/2),-i*(m+1)/(i+2),i*(m+1)/(i+2)) for i in range (0,n+1)]
    H_g = [(-(m+1)/((i+2)*(i+1)/2),-i*(m+1)/(i+2),0) for i in range (0,n+1)]
    Z = (0,-m-1,0)
    P = [( (m+1)/(n+1-i),-(n+1)*(m+1)/(n+1-i),0) for i in range (0,n+1)]
    F = [((m+1)/( (n+2-i)*(n+1-i)/2 ), (-(i-n)^2+i-3*n-2)*(m+1)/((n+2-i)*(n+1-i)),
        (n-i)*(m+1)/(n-i+2) ) for i in range (0,n+1)]
    F_g = [((m+1)/( (n+2-i)*(n+1-i)/2 ), (-(i-n)^2+i-3*n-2)*(m+1)/((n+2-i)*(n+1-i)), 0)
          for i in range (0,n+1)]
    P_out = D+P
    P_in = H + F
    P_in_g = H_g+F_g

    PList = [Polyhedron([ P_out[i],P_out[i+1],P_in[i],P_in[i+1],P_in_g[i],
            P_in_g[i+1],Z ],backend='normaliz') for i in range (0,2*n+1)]
    ground = Polyhedron([(-5*n,-5*n,0),(5*n,-5*n,0),(-5*n,5*n,0),(5*n,5*n,0)])
    GList = [P.intersection(ground) for P in PList]

    return PList, GList
```

For $n = 3$, we compute:

```
    n = 3
    PP,PG = polylist(n)
```

To plot the polytope $\mathcal{P}_3$, we use:

```
sum(P.plot() for P in PP)
```

This produces Figure 5.2 with the default coloring scheme.

The polytopes we obtain from the function `polylist` are unions of a list of subpolytopes with intersecting faces. To compute $\chi^0_{\mathrm{loc}}(s, S^m\Omega_Y)$ for an A_3-singularity s, we need to apply the principle of inclusion-exclusion to account for lattice points lying on common faces of different subpolytopes. We compute the intersecting faces:

```
I=[PP[i-1].intersection(PP[i]) for i in range(1,len(PP))]
IG=[PP[i-1].intersection(PG[i]) for i in range(1,len(PG))]
```

The quasi-polynomial computation is given by the following:

```
Quasipol_PP=reduce_period(sum([quasipol(P.ehrhart_quasipolynomial())
            for P in PP],quasipol([0])) -sum([quasipol(P.ehrhart_quasipolynomial())
            for P in I],quasipol([0])))
Quasipol_PG=reduce_period(sum([quasipol(P.ehrhart_quasipolynomial())
            for P in PG],quasipol([0]))-sum([quasipol(P.ehrhart_quasipolynomial())
            for P in IG],quasipol([0])))
answer=Quasipol_PP - Quasipol_PG
answer
```

We obtain the quasi-polynomial:

```
Quasipolynomial of period 60 with components
[809/5400*t^3 - 1/5*t, 809/5400*t^3 - 509/1800*t + 359/2700,
809/5400*t^3 - 97/225*t - 227/675, 809/5400*t^3 - 127/600*t - 41/100,
809/5400*t^3 - 88/225*t - 16/675, 809/5400*t^3 - 73/360*t + 31/108,
809/5400*t^3 - 7/25*t + 8/25, 809/5400*t^3 - 581/1800*t - 343/2700,
809/5400*t^3 - 97/225*t - 173/675, 809/5400*t^3 - 103/600*t + 33/100,
809/5400*t^3 - 14/45*t + 8/27, 809/5400*t^3 - 509/1800*t - 791/2700,
809/5400*t^3 - 8/25*t - 1/25, 809/5400*t^3 - 581/1800*t + 143/2700,
809/5400*t^3 - 88/225*t - 416/675, 809/5400*t^3 - 11/120*t - 1/4,
809/5400*t^3 - 88/225*t + 416/675, 809/5400*t^3 - 581/1800*t + 1207/2700,
809/5400*t^3 - 8/25*t + 1/25, 809/5400*t^3 - 509/1800*t - 559/2700,
809/5400*t^3 - 14/45*t - 8/27, 809/5400*t^3 - 103/600*t + 17/100,
809/5400*t^3 - 97/225*t + 173/675, 809/5400*t^3 - 581/1800*t - 1007/2700,
809/5400*t^3 - 7/25*t - 8/25, 809/5400*t^3 - 73/360*t + 23/108,
809/5400*t^3 - 88/225*t + 16/675, 809/5400*t^3 - 127/600*t - 9/100,
809/5400*t^3 - 97/225*t + 227/675, 809/5400*t^3 - 509/1800*t + 991/2700,
809/5400*t^3 - 1/5*t, 809/5400*t^3 - 509/1800*t - 991/2700,
809/5400*t^3 - 97/225*t - 227/675, 809/5400*t^3 - 127/600*t + 9/100,
809/5400*t^3 - 88/225*t - 16/675, 809/5400*t^3 - 73/360*t - 23/108,
809/5400*t^3 - 7/25*t + 8/25, 809/5400*t^3 - 581/1800*t + 1007/2700,
809/5400*t^3 - 97/225*t - 173/675, 809/5400*t^3 - 103/600*t - 17/100,
809/5400*t^3 - 14/45*t + 8/27, 809/5400*t^3 - 509/1800*t + 559/2700,
809/5400*t^3 - 8/25*t - 1/25, 809/5400*t^3 - 581/1800*t - 1207/2700,
809/5400*t^3 - 88/225*t - 416/675, 809/5400*t^3 - 11/120*t + 1/4,
809/5400*t^3 - 88/225*t + 416/675, 809/5400*t^3 - 581/1800*t - 143/2700,
809/5400*t^3 - 8/25*t + 1/25, 809/5400*t^3 - 509/1800*t + 791/2700,
809/5400*t^3 - 14/45*t - 8/27, 809/5400*t^3 - 103/600*t - 33/100,
809/5400*t^3 - 97/225*t + 173/675, 809/5400*t^3 - 581/1800*t + 343/2700,
809/5400*t^3 - 7/25*t - 8/25, 809/5400*t^3 - 73/360*t - 31/108,
809/5400*t^3 - 88/225*t + 16/675, 809/5400*t^3 - 127/600*t + 41/100,
809/5400*t^3 - 97/225*t + 227/675, 809/5400*t^3 - 509/1800*t - 359/2700]
```