Applications of Full State Feedback Control in Macroeconomic Policy

by

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Abstract

This thesis demonstrates the utility of feedback controls systems in guiding monetary policy by designing a state feedback control system from scratch to demonstrate its effectiveness. We first demonstrate the strong parallel between feedback control systems and the formulation of monetary policy. The advantages of full state feedback over other techniques such as PID and model predictive control are discussed. In addition, data-driven machine learning control and its potential for augmented monetary policy is also discussed. For the purposes of this undergraduate thesis, an enhanced version of the three equation keynesian economic model will be used to design a corresponding state feedback controller. The parameters of the state feedback control system can be varied to reflect various prominent economies in the 21st century. With our control system – a one percent inflation shock will be utilized with parameters that reflect the United States economy to obtain the optimal interest rate policy for the Federal Reserve. The process for designing the state feedback control system is fully outlined.
This thesis will be defended by August of 2023.

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1 Introduction

The concept of control theory has been criticized in the context of designing macroeconomic policy since the early 1970s – citing that control theory is impractical given the stochastic nature of the real economy [1]. Many made the case that economic models do not exhibit time invariant behavior, making them impractical for optimal control [2]. However, these criticisms were later revised, realizing that economic systems are not explicitly stochastic [3]. In addition, as we will see with certain control techniques, the economic model need not be time invariant.

When the Federal Reserve (FED) – the central bank of the US - conducts monetary policy with respect to the discrepancy between actual and targeted macroeconomic performance, the central bank is imposing control over the economy based on feedback such as inflation, and unemployment data. This approach is explicitly that of a feedback control system – a process which regulates the behavior of a dynamic system by continuously monitoring the output and adjusting the input. The system compares the targeted and actual output, and adjusts the input to minimize the discrepancy. This process continues in a loop and hopefully the actual output tracks the targeted output, resulting in a stable and controlled system [4].

Hawkins et al. demonstrated the strong parallel between PID controllers and monetary policy in his 2014 paper – "These characteristics of monetary policy rule development are strikingly similar to controller development in engineering and suggest that these fields share a common framework" [5]. This thesis is an augmentation and extension of this line of work. Full state feedback – another method employed in feedback control systems offers crucial advantages over other strategies such as PID control as it allows the user to select the eigenvalues directly according to desired characteristics such as overshoot percentage, and settling time [4], [6].

Full state feedback is ubiquitous in modern control systems, and it offers several advantages such as optimal control, robustness, and stability. Full state feedback uses all available state information which can lead to better performance. It is particularly useful in applications where the system dynamics are complex or difficult to model. In addition, full state feedback can be used to design controllers for a wide range of systems, including linear, non-linear, and time-varying systems, with multiple inputs and outputs. This makes it a very versatile technique as we continue to modify and augment the economic model and our control system [4].

Professionals who are well versed in this realm of research, would make the case that data driven machine learning control would be a far better fit for the FED, given its access to vast amounts of data. However, in pursuit of such an approach, it was found that it is too complex and impractical for an undergraduate thesis. Alternatively, an enhanced economic model was used to design a corresponding state feedback controller. The goal for this thesis is to demonstrate the efficacy of feedback control systems in guiding monetary policy through a proof of concept. Any excess time will be allocated in augmenting the economic model and possibly incorporating the use of machine learning.

We will start by deriving the transfer function for the system from an enhanced three-equation keynesian model. Then we establish the controllability of the system. Once the system is deemed controllable, the state feedback control law will be applied to design our controller. Optimization techniques such as eigenvalue selection, linear quadratic regulation and genetic algorithms will be used to select the gains for the state feedback control law. To test the feedback control system, an inflation shock will be applied to the system, and the evolution of the input (interest rates) and the output (inflation) will be observed. The input for the system is the optimal interest rate policy for the Federal Reserve.

The state feedback control system will be considered effective if:

1. Stability: The closed-loop system should be stable, meaning that the system’s interest rates (output) should not oscillate and diverge uncontrollably.
2. Controllability: The system should be controllable, meaning that there exists a control law which can convert the system's initial state of inflation to the target rate in a finite time.
3. Performance: The control system should be able to correct a 1% inflation shock within 5-10 years without the need to apply too high or low of an interest rate. The applied interest rates should be within -0.5% to 3%.

2 Theory

In the pursuit of framing monetary policy as a feedback control system, we must understand the objectives of such policies. The Federal Reserve possesses a dual mandate: To foster maximum employment and target an inflation rate of 2% [7]. Thus, we will need an equation that will capture the relationship between unemployment and inflation – precisely captured by the Phillips Curve [8].

2.1 The Three-Equation New Keynesian Model

2.1.1 The Phillips Curve

The Phillips curve depicts an inverse relationship – illustrating the policy trade-off between the unemployment rate and the inflation rate. As unemployment decreases, wages rise to compete for the job vacancies. As wages rise, firms pass the increased labor costs to consumers – leading to inflation. Conversely, when unemployment increases, wages fall due to a competitive job market. The figure below portrays this relationship [8].

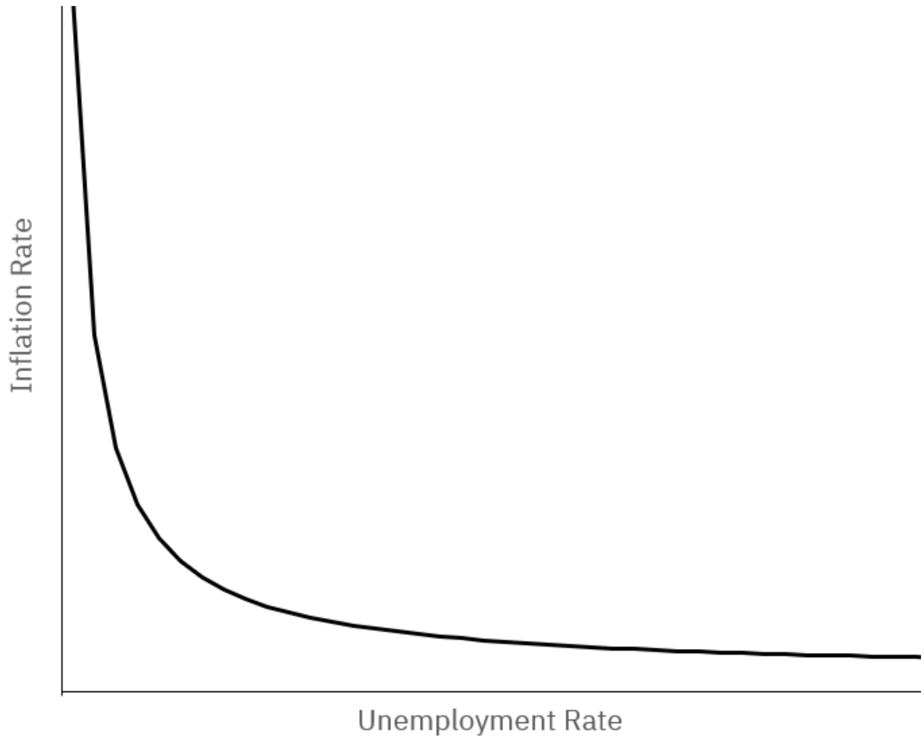


Figure 2.1: A plotted image of the Phillips curve. The horizontal axis represents the unemployment rate and the vertical axis depicts the inflation rate.

Given the trade-off between the inflation and unemployment rate, the central bank must optimize by minimizing a loss function subject to the Phillip's Curve:

$$\pi_1 = \pi_0 + \alpha(y_1 - y_0) \tag{1}$$

Where π_0 represents the current inflation rate and π_1 is the following period inflation rate. The difference between the next period output y_1 and the target output y_e is multiplied by α – a factor of proportionality [8].

While the Phillips curve was developed and used in the 1950s and 1960s, it has been criticized by economists for its practicality in the 21st century. A stable relationship between unemployment and inflation is idealistic and purely empirical. In addition, the Phillips curve does not account for uncertainty and structural changes in the economy [9]. However, it is an exceedingly simple model, making it very practical for the purpose of this thesis.

In addition to a relationship between inflation and unemployment, in order to conduct monetary policy, we need a model which describes how people spend and invest their money with respect to interest rates. This is precisely captured by the investment savings curve [8].

2.1.2 The Investment Saving Curve

The investing savings (IS) curve is a relationship between aggregate investment and aggregate saving in an economy. The IS curve assumes that investments and savings are equivalent in an economy. As the interest rate increases, the cost of borrowing increases which reduces the aggregate investment in the economy. The IS equation is as follows:

$$y_1 = A_0 - ar_0 \tag{2}$$

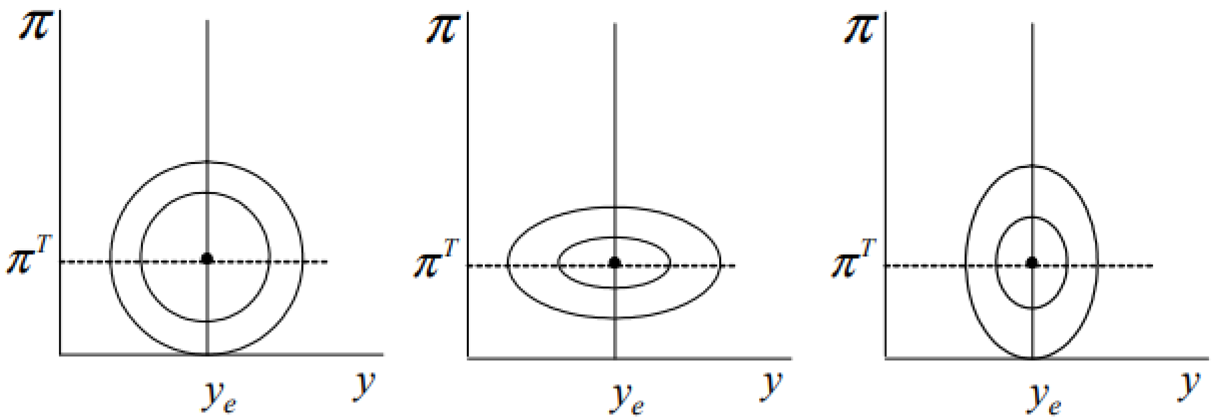
Where A_0 is the current level of autonomous expenditure, a is a proportionality factor and r_0 is the current interest rate [8]. The IS curve is also an idealistic model as it assumes that investment spending is solely dependent on the interest rate [9]. Once again, its simple form makes it a practical choice for this thesis.

2.1.3 The Elliptical Central Bank Loss Function

The central bank's loss function is a representation of its policy objectives and preferences. The loss function allows the central bank to determine optimal policy given the trade-off between inflation and unemployment:

$$L = (y_1 - y_e)^2 + \beta(\pi_1 - \pi^T)^2 \tag{3}$$

β is the crucial parameter in the central bank loss function: $\beta > 1$, is a characteristic of inflation-aversion, $\beta < 1$ is associated with unemployment-aversion and $\beta = 1$ is a characteristic of a balanced policy. The mathematical significance of β is graphically demonstrated in the figure below [8].



(a) Balanced: $\beta = 1$ (b) Inflation-averse: $\beta > 1$ (c) Unemployment-averse: $\beta < 1$

Figure 2.2: A graphical demonstration of the effect of β on the elliptical loss function. As L decreases, the radius of the ellipse contracts. $\beta > 1$ contracts the inflation range and expands the unemployment domain. Conversely, $\beta < 1$ contracts the unemployment domain and expands the inflation range [8].

Together, 2.1.1, 2.1.2, and 2.1.3 form the new Keynesian model for macroeconomics. Given its simplicity, this model will serve as a practical start to designing our controller using state-space methods.

2.2 Feedback Control Systems

In a feedback control system, a process or operation is monitored and adjusted based on feedback information. An output of a system is measured and as information to adjust the input to achieve a desired output. These systems are used in engineering to regulate and optimize the performance of systems [4]. In feedback control systems relevant to this thesis, there are three key components:

1. Controller: Processes the measurement information and adjusts the input to achieve the desired output. In our case, this would be the central bank [4].
2. Plant: The process or system that is being controlled – the economy [4].
3. Feedback Elements: These elements allow the controller to continuously monitor and adjust the process. Our feedback elements are the inflation and unemployment rate [4].

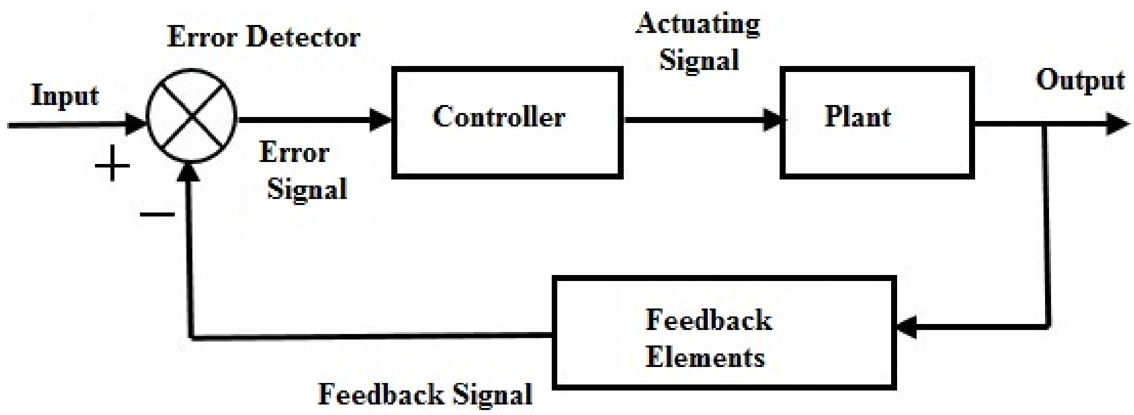


Figure 2.3: A block diagram depicting the principle of a feedback control system along with its key components [10].

For example, to enumerate, our error detector in figure 2.3 above, could be subject to a 1% inflation shock. The controller would absorb this information and send an actuating signal (a change in the interest rate) to the plant which represents our economic system. The updated inflation rate is the output and is fed back into the controller until stability is achieved and hopefully the actual output tracks the desired output.

2.3 Full State Feedback Control System

2.3.1 State Feedback Control

State feedback control is a control technique which uses the complete state of the system, including all the system internal variables and their derivatives. The state is used to compute the optimal control action that drives the system to the desired state. There are several methods of control, such as proportional-integral-derivative (PID) control, which is a feedback control system that uses an error between a predetermined setpoint and the measured process variable to compute the control action. Unfortunately, PID control does not take a system's internal dynamics, rendering it ineffective for economic systems [4].

Model predictive control (MPC) is another control technique which would seem promising at first glance. MPC utilizes a model of the system to predict future behaviour and compute a corresponding optimal control action. However, MPC requires a detailed and very predictable model of the system [11].

Considering the evolution of central banks like the FED and their vastly growing access to economic data – one can definitely make the case that data-driven machine learning control would be a superior and futureproof choice at scale. In the context of monetary policy, a data-driven machine learning control system would be trained on historical data to identify causal patterns and relationships [12]. This would allow the control system to predict at least the probability of changes in economic indicators such as inflation, GDP, and unemployment rates – identifying pending economic crises – and mitigate them by determining its corresponding optimal monetary policy. This would be especially effective with the advent and wide adoption of CBDCs, giving the control system access to data regarding virtually every single transaction in the economy. Alternatively, the FED could get access to data on existing transactions by mandating all commercial banks to report all transactional data to the FED through congressional action. The FED could significantly augment monetary policy by incorporating this technique. In fact, the use of data-driven machine learning control in the pursuit of augmented monetary policy was the initial proposal for this thesis. However, given a deep dive into this topic – it has proved to be impractical for a 4 month undergraduate thesis and would require significantly more manpower with expertise in mathematics, economics and engineering. Alternatively, the three-equation new keynesian model will be used and a corresponding state feedback controller will be designed as a proof of concept to demonstrate the utility of feedback control systems in guiding monetary policy. If time permits, more complex economic models and machine learning could be introduced.

State feedback control is an appropriate commencement for guiding monetary policy compared to other methods for the following reasons:

- Better control performance: All the system's internal states are considered when computing the control action, enabling precise and accurate control [4].
- Flexibility and Robustness: Can be applied to a wide range of systems including non-linear and time-varying systems with multiple inputs. In addition, state feedback control is robust to disturbances and uncertainties in the system - making it exceedingly appropriate for complex economic models [4].
- Ease of implementation: Standard control theory and techniques can be applied, making design and implementation extremely practical [4].

2.3.1 State Space Equations

State space representation is built around the state vector *x(t)*. This is a vector of all the state variables – a set of variables that describe the current state of the system. The derivative of the state vector is a linear combination of the state vector, *x(t)* ,and the control signal, *u(t)* :

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

This is precisely known as the matrix form of the state equation. *A* is the dynamics matrix and it describes the relationship between the internal states – the underlying dynamics of the system. *B* is the control matrix, describing how the control signal enters into the system and which states they affect [13].

The output vector *y(t)* is a vector of the outputs of the system and is a linear combination of the state vector, *x(t)*, and the control signal, *u(t)* :

$$y(t) = Cx(t) + Du(t) \tag{5}$$

C is the sensor matrix, describing how the states are combined to obtain the outputs. *D* is the direct term, important for systems where any inputs are directly related to the output. In our case, the output *y(t)* would be the inflation rate and the input/control signal *u(t)* is the interest rate [13].

2.3.2 Controllability

State feedback control can only be used if the system is controllable – meaning that there exist control signals which allows the system to reach any state in a finite amount of time. The state equation (4) is controllable if and only if and only if:

$$rank[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = n \tag{6}$$

I.e. if the matrix enclosed above is full rank. It is known as the controllability matrix and is often denoted by *P* for conciseness. *n* is the dimension of the state vector [13].

2.4 Transfer Function and Frequency Domain Analysis

A transfer function is expressed as the ratio of a system's output signal to its input signal in the frequency domain. In certain applications, transfer functions and frequency domain analysis are preferred over time domain analysis. Frequency domain analysis is imperative for stability analysis and controller design. Furthermore, the criteria for stability can be simply applied in the frequency domain [4].

Although it is possible to design our controller in the time domain by putting the economic system directly into state space representation, the transfer function simplifies our computation significantly. Convolution in the time domain is equivalent to multiplication in the frequency domain, giving more insight into the system while making analysis simpler and more intuitive. In addition, the transfer function represents differential equations as simple algebraic functions in the frequency domain [4].

2.5 Controller Design and Optimization

Once we derive our transfer function we can find its corresponding state space representation and continue.

2.5.1 State Feedback Control Law

We can verify the controllability of the system if the determinant of P is non-zero. If the system is controllable, we can apply the following state feedback control law:

$$u(t) = -Kx(t) + r(t) \tag{7}$$

In order to achieve desired performance characteristics, this control law is applied to the state equations (4) and (5), giving:

$$\dot{x}(t) = (A - BK)x(t) + Br(t) \tag{8}$$

$$y(t) = Cx(t) + Du(t) \tag{9}$$

Where *r(t)* is the input, and *K* is the gain matrix. For our case, we have the interest rate as the input and the inflation rate as the output. This is a single input and output case, meaning that *K* is a 1x*n* row vector [13].

The state space system model can be constructed and we can perform simulations on it through MATLAB by using the **ss** command [14].

We have three options when selecting the gains for the *K* vector:

1. Shaping the dynamic response through eigenvalue selection.
2. Linear Quadratic Regulator (LQR).
3. Genetic algorithm.

2.5.2 Eigenvalue Selection

The eigenvalues of a system determine its stability and dynamic response. This method is done manually and involves placing the poles directly. This gives us the ability to shape and choose the characteristics of the dynamic response. Characteristics such as rise time, peak time, percentage overshoot and settling time can be chosen with eigenvalue selection [13].

2.5.3 The Linear Quadratic Regulator

LQR is a common technique for eigenvalue selection. The goal of LQR is to provide the optimal feedback gain matrix that minimizes a cost function. LQR offers customizability as it allows the cost function to be tailored to meet certain performance specifications such as settling time. This technique can be performed on MATLAB by using the **lqr** command [14].

The economic model proposed in section 2.1 is a linear system – which qualifies for the use of LQR. However, if we were to propose a non-linear economic model, we need an alternative technique for optimal gain selection [13].

2.5.4 Genetic Algorithm

The Genetic Algorithm is based on the principles of natural selection and genetic recombination and can be used to optimize non-linear state feedback control systems. The genetic algorithm would generate a population of potential K matrices. Each K matrix is evaluated for its performance relative to a cost function. The best performing solution is selected to reproduce and programmed to generate new offspring with combinations and mutations of the best performing K matrices. This process is repeated over a multitude of generations, leading the K matrices to converge to an optimal K matrix[13].

Genetic algorithms can handle non-linear, high dimensional cost functions, making it suitable for complex economic systems. Furthermore, genetic algorithms can run through a wide range of potential solutions, which can allow it to converge to a truly globally optimal solution, rather than a local one. The only disadvantage to using genetic algorithms is their computational cost. They may require significant computational power and large amounts of time to converge to an optimal solution[15].

2.5.5 Weighing Matrices

In the optimization of control systems, weighing matrices are crucial for achieving desired system performance and defining the control objectives. Weighing matrices are positive semi-definite diagonal matrices that capture the relative importance of each state and control input in the overall cost function. For our control system, we have two weighing matrices. One for the states, Q_x, and one for the control signal u, Q_u. If we look at this through the lens of the Federal Reserve dual mandate, Q_x captures the relative importance of bringing inflation and its derivatives back to target and Q_u captures the importance of maximizing the employment rate [13].

3 Methods, Analysis, Results and Discussion

3.1 Methods

3.1.1 Gap Based Notation and Model Enhancement

Our current form of the three equation keynesian model is not practical. We will modify the notation such that the discrepancy between the targeted and actual interest rate, inflation and output is considered.

Starting with PC, we subtract the inflation target π^T from both sides of equation (1):

$$\pi_1 - \pi^T = \pi_0 - \pi^T + \alpha(y_1 - ye) \quad \Rightarrow \quad g_\pi[n] = g_\pi[n-1] + \alpha g_y[n] \tag{7}$$

Where $g_\pi[n]$ is the inflation gap at time *n* and $g_\pi[n-1]$ is the inflation gap at time *n*-1. Similarly, for the IS curve and loss function, we have:

$$g_y[n] = -a g_r[n] \tag{8}$$

$$L[n] = g_y[n]^2 + \beta g_\pi[n]^2 \tag{9}$$

Where $g_r[n]$ is the rate gap at time *n* and $g_y[n]$ is the output gap at time *n*.

The real economy is difficult to model. Although state feedback control doesn't require an extremely detailed and accurate model to function, a decent understanding of system dynamics and accurate measurements of state variables are still required. The economy has a large degree of unpredictability. Factors such as changes in consumer behavior, natural disasters or political events significantly impact the economy and may not be captured by the state variables or be represented by the system dynamics. Furthermore, the effects of monetary policy can take time to affect the economy. However, state feedback control is most effective when the system response is immediate. This also creates risk of over tightening or loosening of monetary policy due to lagging indicators. In addition, state feedback control relies on relatively accurate modeling of system dynamics and may not be well tailored for black swan events. Massive changes to the economic model, such as those temporarily caused by the COVID-19 pandemic could potentially destabilize the system. These are risks that come with using the simple version of the New Keynesian model. We will remedy these risks by enhancing our model in this section.

One of the biggest caveats of the New Keynesian model is that it does not accurately depict the dynamic interplay between output and inflation for the PC and the output and rate interplay for the IS curve [16]. For example, the output and rate gaps are shown for the United States in figure 2 below – contrary to the instantaneous proportionality proposed by the IS curve. Clearly, there is a lack of empirical support for the instantaneous response suggested by equation (8) above [17]. In addition, the New Keynesian model is a discrete-time model. Continuous-time models are preferred over discrete-time models for state feedback control as they provide more accurate representations of real world systems by capturing continuous and smooth dynamics. Furthermore, continuous-time models allow for a wider range of control design and analysis techniques.

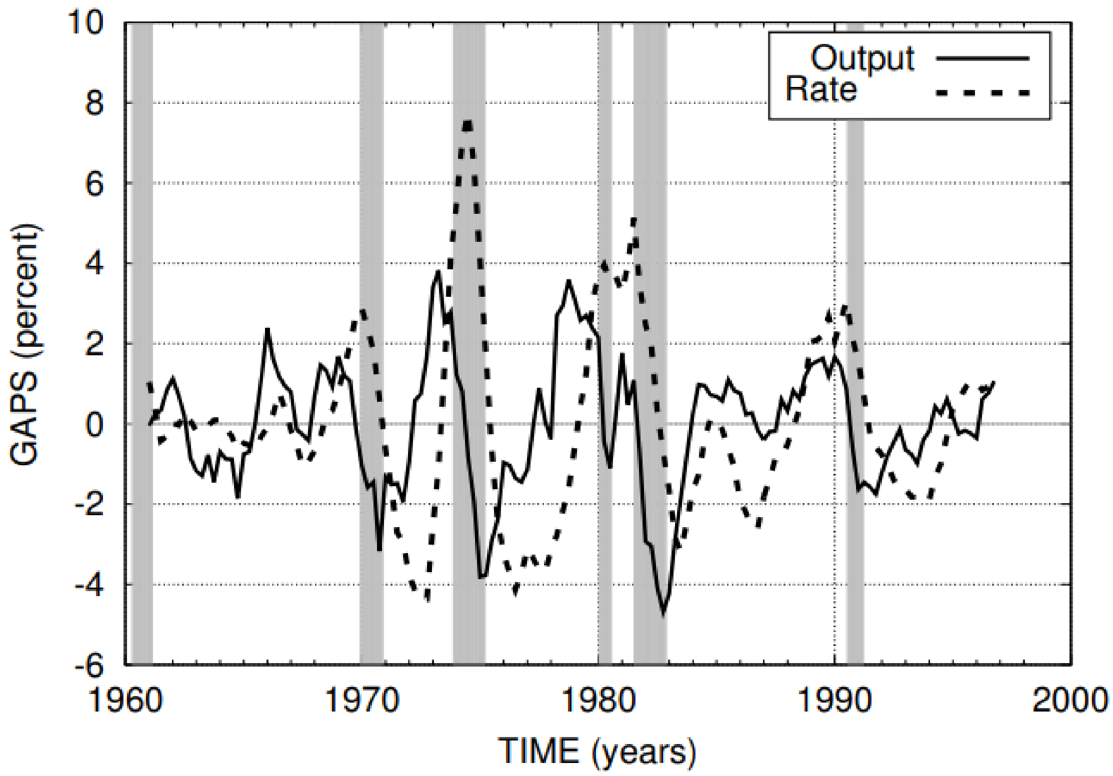


Figure 3.1: The output and rate gaps for the United States. The data for this figure was obtained from the Federal Reserve Bank of St. Louis Economic Data (FRED) website [17].

To account for this we will modify the IS curve to relax the assumption of instantaneous proportionality by allowing time dependence. We start with the zeroth-order differential equation for the IS curve at equilibrium [17]:

$$g_y = J_R r \tag{10}$$

Where J_R is a negative valued constant. Now we introduce time dependency by introducing the first derivatives of g_y and r on both sides of the equation without violating the equality allowing for an anelastic economy [17].

$$\tau_r \dot{g_y} + g_y = \tau_r J_U \dot{r} + J_R r \tag{11}$$

J_R = J_r is the equilibrium proportionality and J_U represents any instantaneous response. The difference $J_R - J_U$ is the time dependent component of the response and will recover equation (10) above if it is zero valued [17].

Similarly, our current model of the PC is not representative of an anelastic economy. We can see some dynamic interplay between output and inflation in figure 3.2 below [16].

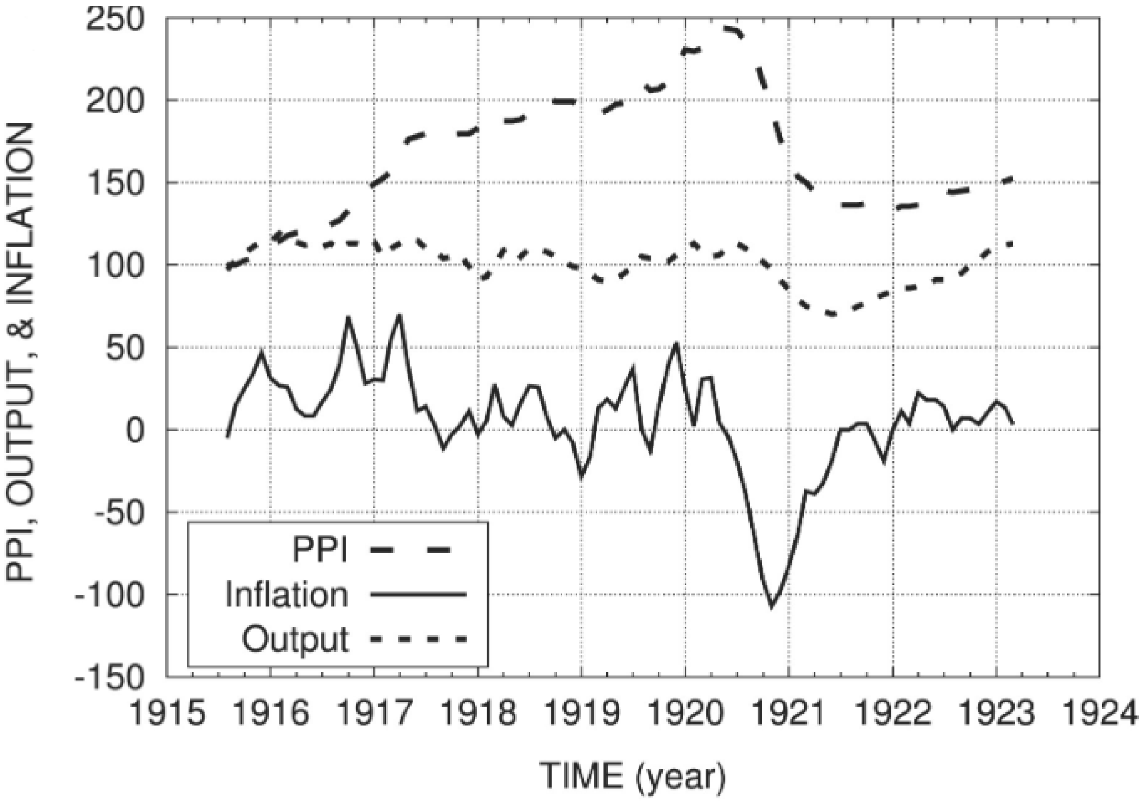


Figure 3.2: The output and inflation gaps for the United States. The data for this figure was obtained from the Federal Reserve Bank of St. Louis Economic Data (FRED) website [16].

Raymond J. Hawkins has derived a second order differential equation which he introduces in his paper 'Macroeconomic susceptibility, inflation, and aggregate supply' in 2018, which much more accurately depicts the relationship between inflation and output seen in figure 3.3 below – capturing many of the high and low frequency features [16].

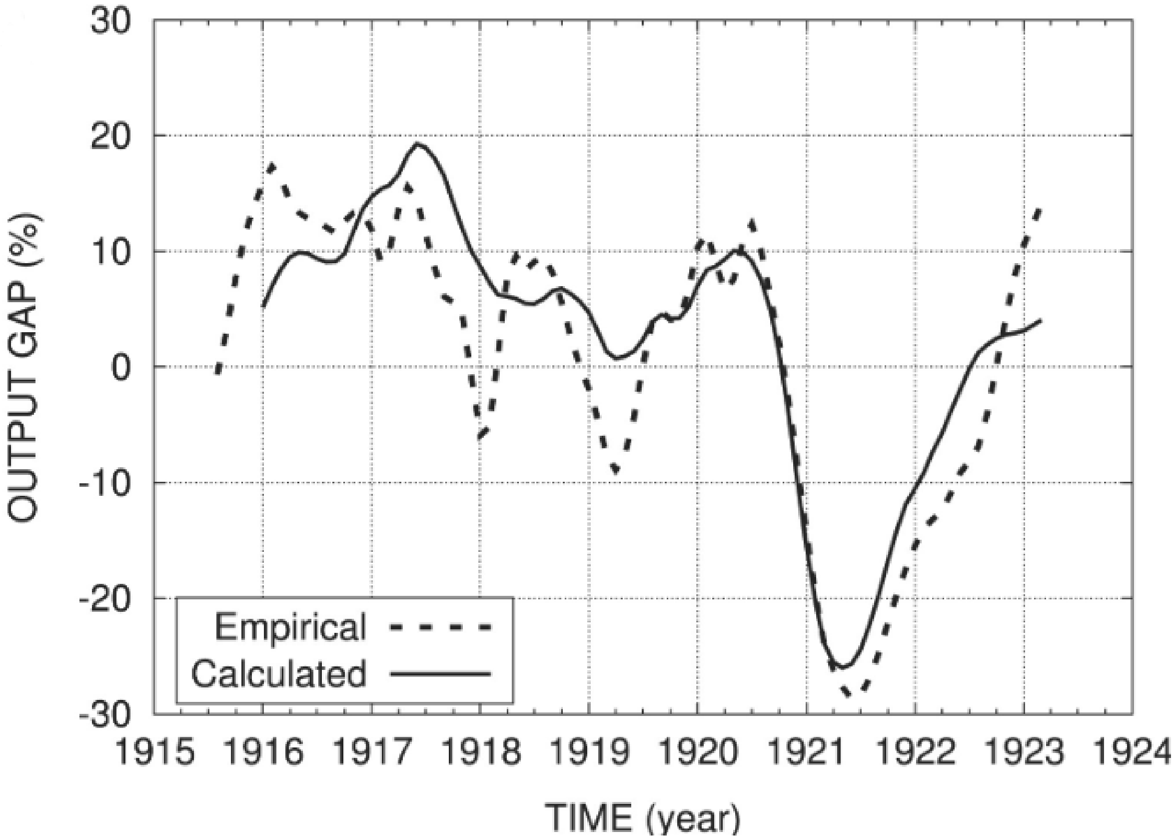


Figure 3.3: The empirical output gap vs the calculated output gap using equation (12) for the United States. The data for this figure was obtained from the Federal Reserve Bank of St. Louis Economic Data (FRED) website [16].

The differential equation is essentially an enhanced PC equation as it has the same form but contains information about the first and second derivatives of the output with respect to inflation. The second order differential equation is as follows [16]:

$$\ddot{g_y}(t) + \dot{g_y}(t) + \omega_0 g_y(t) = g_\pi(t)/m + \Gamma(t)/m \tag{12}$$

γ is the damping rate for a macroeconomic shock, ω_0 is the resonant frequency of the macroeconomy, *m* is the reluctance of an economic variable to alter its rate of change and $\Gamma(t)$ is the fluctuation of the output. Equation (12) is a langevin equation – a stochastic differential equation which describes how a system evolves subject to a combination of random and deterministic forces [16]. We will not be going over the derivation of equation (12) as it is complex and not the focus of this thesis.

3.1.2 Frequency Domain Analysis and Solving for the Transfer Function

For both our IS curve and PC equations, we can describe the relationship between the input and output of each equation through a response function. In the frequency domain, our IS curve and PC – equation (11) and (12) can be represented respectively as follows [16], [17]:

$$G_y(s) = \chi_r G_r(s) \tag{13}$$

$$G_\pi(s) = \chi_y G_y(s) \tag{14}$$

Where χ_r describes the output of the economy given a certain interest rate as input, and χ_y describes the inflation rate given a certain output for the economy. Since we are designing a control system which takes the interest rates at time *t* as the input and outputs the corresponding inflation rate at time *t*, we need to combine equations (13) and (14) by substituting equation (13) into equation (14), yielding [16], [17]:

$$G_\pi(s) = \chi_y(s)\chi_r(s)G_r(s) \tag{15}$$

$$\Rightarrow \qquad \tfrac{G_\pi(s)}{G_r(s)} = \chi_y(s)\chi_r(s)$$

$$\Rightarrow \qquad T(s) = \chi_y(s)\chi_r(s) \tag{16}$$

Where $T(s)$ is the system transfer function. We now need to find χ_y and χ_r to solve for the transfer function $T(s)$. The response function for equation (13) – χ_r – is the solution to equation (11). We will perform a laplace transform to put equation (11) into the frequency domain, and solve for χ_r. The laplace transform of equation (11) is [16], [17]:

$$\tau_r s G_y(s) + G_y(s) = \tau_r J_u s G_r(s) + J_r G_r(s) \tag{17}$$

$$\Rightarrow \qquad G_y(s)(\tau_r s + 1) = G_r(s)(\tau_r J_u s + J_r)$$

$$\Rightarrow \qquad \tfrac{G_y(s)}{G_r(s)} = \tfrac{\tau_r J_u s + J_r}{\tau_r s + 1}$$

$$\Rightarrow \qquad \chi_r(s) = \tfrac{\tau_r J_u s + J_r}{\tau_r s + 1} \tag{18}$$

The output response function for equation (13) – $\chi_y(t)$ – is also provided by Hawkins in his 2018 paper. In fact, this was the response function used to calculate the curve in figure 3.3 in section 3.1 [16].

$$\chi_y(t) = \frac{1}{\omega_1 m} e^{-\gamma t/2} \sinh(\omega_1 t) \tag{19}$$

Which χ_y is the overdamped Green's function of equation (12) [16]. Taking the laplace transform of equation (19) gives us:

$$\chi_y(t) = \frac{1}{m((s+\frac{\gamma}{2})^2 - \omega_1^2)} \tag{20}$$

Therefore, the transfer function is [16]:

$$T(s) = (\frac{\tau_r J_u s + J_r}{\tau_r s + 1})(\frac{1}{m((s+\frac{\gamma}{2})^2 - \omega_1^2)})$$

$$\Rightarrow \qquad T(s) = \frac{\tau_r J_u s + J_r}{(\tau_r s + 1) m((s+\frac{\gamma}{2})^2 - \omega_1^2)} \tag{21}$$

However, J_U is often not observable and thus [16]:

$$T(s) = \frac{J_r}{m(\tau_r s^3 + (\tau_r \gamma + 1)s^2 + (\frac{\tau_r \gamma^2}{4} - \omega_1^2 \tau_r + \gamma)s + \frac{\gamma^2}{4} - \omega_1^2)} \tag{22}$$

3.1.3 Transfer Function to State Space Representation and Controllability

In equation (4) and (5), since our system has inflation as the output *y(t)*, and the input/control signal *u(t)* as the interest rate, we will set the elements of the state vector, x_1, x_2, and x_3 to be inflation, with its first and second derivatives respectively. Therefore, with x_1 being identical to the output *y(t)*, we can immediately simplify our control system by setting the sensor matrix C to [1 0 0] and the direct term *D* to 0.

Now we will convert from transfer function form to state space representation. We will start by taking the inverse laplace transform of equation (22) to convert to the time domain. Note that $g_\pi = \pi_g$:

$$J_r r(t) = m\tau_r \frac{d^3}{dt^3}\pi_g(t) + m(\tau_r\gamma + 1)\frac{d^2}{dt^2}\pi_g(t) + m(\frac{\tau_r\gamma^2}{4} - \omega_1^2\tau_r + \gamma)\frac{d}{dt}\pi_g(t) + (\frac{\gamma^2}{4} - \omega_1^2)\pi_g(t) \quad (23)$$

Now we let:

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = \pi_g & \Rightarrow & \dot{x}_1 = \frac{d\pi_g}{dt} \\ x_2 = \frac{d\pi_g}{dt} & \Rightarrow & \dot{x}_2 = \frac{d^2\pi_g}{dt^2} \\ x_3 = \frac{d^2\pi_g}{dt^2} & \Rightarrow & \dot{x}_3 = \frac{d^3\pi_g}{dt^3} \end{matrix}$$

By observation:

$$\dot{x_3} = -\frac{\tau_r\gamma+1}{\tau_r}x_3 - \frac{\frac{\tau_r\gamma^2}{4}-\omega_1^2\tau_r+\gamma}{\tau_r}x_2 - \frac{\frac{\gamma^2}{4}-\omega_1^2}{\tau_r}x_1 + \frac{J_r}{m\tau_r}r(t) \quad (24)$$

Now we will fully define our state space equations. If we compare this to equation (4), we can see that:

$$\dot{x}(t) = Ax(t)+Bu(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\frac{\gamma^2}{4}-\omega_1^2}{\tau_r} & -\frac{\frac{\tau_r\gamma^2}{4}-\omega_1^2\tau_r+\gamma}{\tau_r} & -\frac{\tau_r\gamma+1}{\tau_r} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{J_r}{m\tau_r} \end{bmatrix} u(t) \quad (25)$$

If we compare the system we have defined so far to equation (5). We can see that:

$$y(t) = Cx(t) + Dx(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) = x_1 \quad (26)$$

Now that we have completed state space representation, we shall establish the controllability of the system. Using Symbolab, the controllability matrix (P) for our system is:

$$P = [\ B\ AB\ A^2B\ ]$$

$$\Rightarrow \quad P = \begin{bmatrix} 0 & 0 & \frac{J_r}{m\tau_r} \\ 0 & \frac{J_r}{m\tau_r} & -\frac{J_r(\tau_r\gamma+1)}{m\tau_r^2} \\ \frac{J_r}{m\tau_r} & -\frac{J_r(\tau_r\gamma+1)}{m\tau_r^2} & \frac{J_r(3\tau_r^2\gamma^2+4\omega_1^2\tau_r^2+4\tau_r\gamma+4)}{4m\tau_r^3} \end{bmatrix} \quad (27)$$

To check controllability, we calculate the determinant of P:

$$\det(P) = -\frac{J_r^3}{\tau_r^3 m^3} \tag{28}$$

The controllability matrix P is independent of the transfer function coefficients and is a product of its diagonal elements. If P has a nonzero determinant, it is controllable. Therefore, the system is controllable for any real and non-zero values of J_r, m, and τ_r. This is an immaculate result, as it implies that any state space realization in this form is controllable [13]. Thus, we can significantly modify our economic model without forgoing controllability.

3.1.4 Controller Design and Optimization

Since the system is controllable, we can apply the state feedback control law – equation (7). Our system is third order and thus:

$$u(t) = r(t) - [\ k_1\ \ k_2\ \ k_3\ ]x(t) \tag{29}$$

In order to achieve the target inflation rate, we will utilize the control signal *u(t)* and perform gain selection. We define the state space control system by using the **ss** object in MATLAB discussed in section 2.5.1.

We will start by assigning values to our parameters. The following values are based on the US economy and were sourced from the same papers which we sourced our economic models from [16], [17].

Parameter	τ_r (year)	J_r (year)	γ (1/year)	ω_1 (1/year)	$m\,(10^{-4})$
Value	18.18	-2.527	4.370	0.009300	640.36

Table 3.1: The data for this figure was chosen with respect to data from the Federal Reserve Bank of St. Louis Economic Data (FRED) website [16], [17].

For gain selection, and optimizing the control system, we have the three options discussed in section 2.5. A copy of the matlab scripts with all of their respective details are included in the appendix. A combination of eigenvalue selection, the LQR and genetic algorithm were employed. For each optimization technique, equal weighing was used for the weighing matrices Q_x and Q_u discussed in section 2.5.5.

Subsequently, relative weighting was used with the superior optimization technique. To simulate our control system, a 1% inflation shock was used for the initial condition using the **initial** function in MATLAB. The interest rate was obtained by multiplying each state variable by its corresponding optimal gain value and summing the results.

3.2 Results, Analysis and Discussion

3.2.1 Controlled Response Using Eigenvalue Selection

First, eigenvalue selection was used to choose the desired pole locations for the closed loop system. The poles were chosen such that the inflation rate gap converges as quickly as possible with the use of a reasonable rate hike whilst remaining stable.

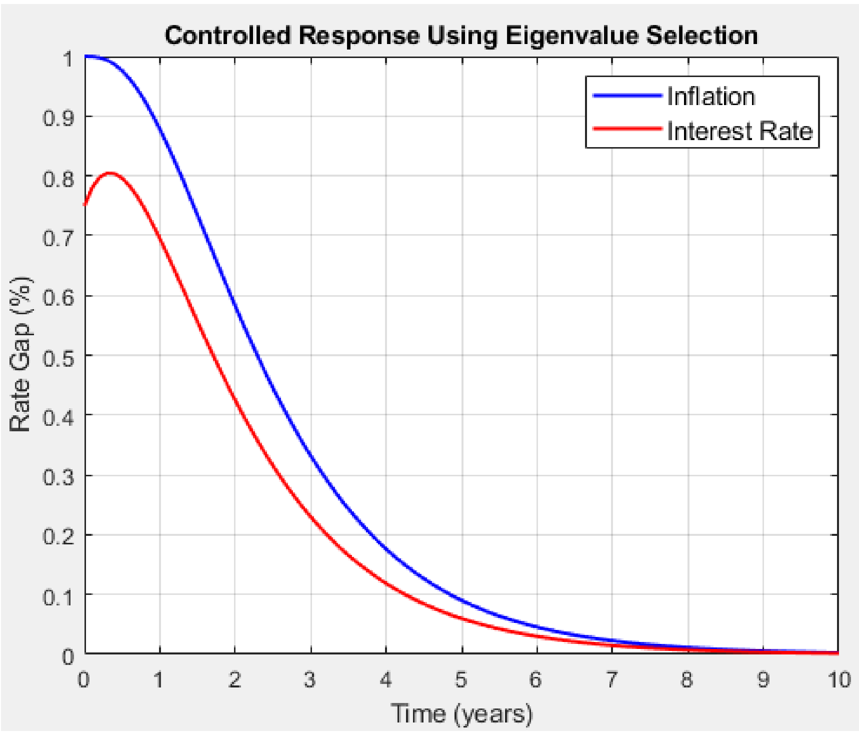


Figure 3.4: The controlled inflation response and optimal overnight federal funds rate using eigenvalue selection after a 1% inflation shock.

Right away, with eigenvalue selection alone, we were able to achieve our goal set in the thesis proposal and the introduction of this thesis. The control system was able to correct for a 1% inflation shock within 5-10 years while applying a maximum interest rate of 0.8% with an initial rate hike of 75 basis points. This is on par with historical data in the economics literature. Historical data suggests that a rate hike of 25 to 100 basis points is necessary to correct for a 1% inflation shock within a few quarters to a few years [18].

3.2.2 Controlled Response Using the LQR

Next, we utilized a well established control technique – the LQR. For this technique we used the MATLAB **lqr** function to find the optimal control gains that minimize a quadratic cost function.

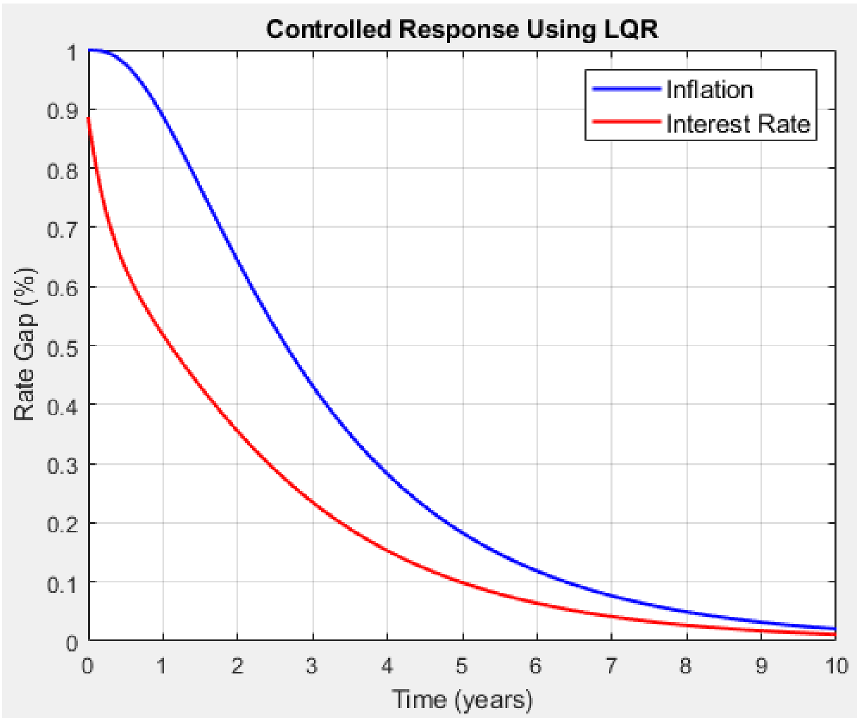


Figure 3.5: The controlled inflation response and optimal overnight federal funds rate using the LQR after a 1% inflation shock.

Interestingly, using LQR arguably gave us a worse result compared to eigenvalue selection. With eigenvalue selection, we were able to converge the inflation gap to within 0.1% within 5 years with a more modest initial rate hike compared to achieving a 0.18% inflation gap within 5 years with LQR. Although the initial rate hike is higher for the LQR response, one could argue that the economy incurs a higher cost by spending more time in a higher interest rate environment in the first year of the eigenvalue selection response, and thus maximum employment isn't being utilized. Therefore, eigenvalue selection may have outperformed LQR in controlling inflation, but not necessarily at the mercy of the unemployment rate.

3.2.3 Controlled Response Using the Genetic Algorithm

We also used the genetic algorithm due to its ability to handle complex systems and run through a wide range of potential solutions, allowing it to converge to a globally optimal solution rather than a local one. By using the Optimtool in the optimization app for MATLAB, we employed a genetic algorithm by using the **ga** function. The genetic algorithm employed a population-based search strategy and incorporated mutation and crossover operations to explore the solution space. For the cost function, we initially used the LQR cost function and modified it to see if we can get more optimal solutions.

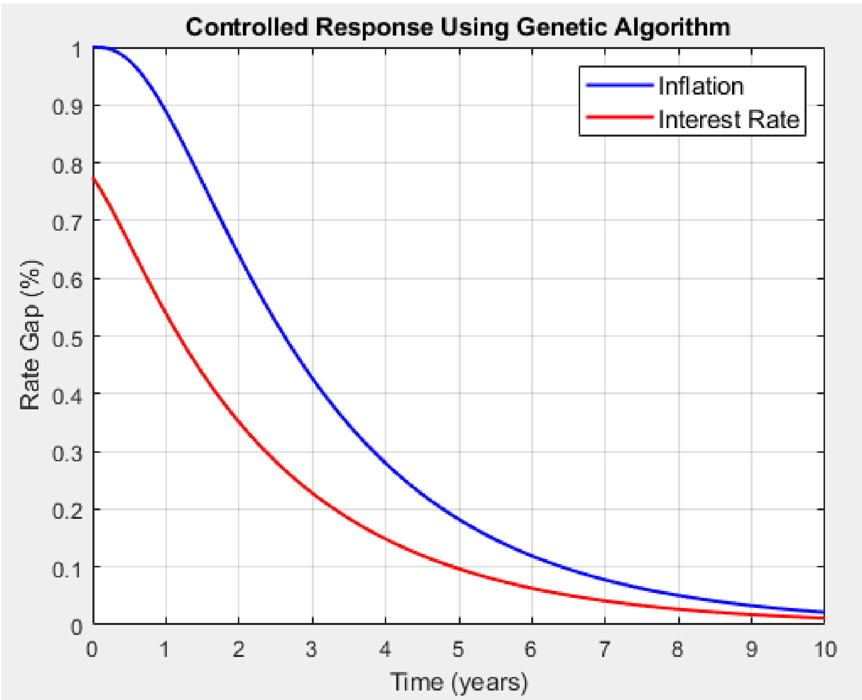


Figure 3.6: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock. MATLAB cost function: cost = sum(sum(x.^2 * Qx)) + sum(u.^2 * Qu).

Astoundingly, the genetic algorithm was able to reproduce the exact same controlled inflation response while using a much more optimal overnight federal funds rate despite using the same cost function. The genetic algorithm response only uses an initial rate hike of ~77 basis points along with a much smoother descent versus the ~88 basis point rate hike for the LQR response.

Although these are reasonable controlled responses, it takes about 10 years for the inflation gap to converge to 0%. We would ideally want to converge the inflation gap to 0% within a quicker amount of time. The culprit lies in the cost function. After experimenting with the genetic algorithm it was found that using a quadratic cost function for the control signal element incurs too high of a cost on a prolonged rate hike, leading to an inefficient inflation control policy. To remedy this, we used a quartic cost function for the control signal element. As we can see in figure 3.7, this significantly improved the performance of the control system, allowing the inflation response to converge to 0% within 7-8 years. However, after also experimenting with a sextic and octic cost function, we found that the sextic cost function for the control signal element generates the most optimal results (figure 3.8), without incurring too high of a cost on the economy, and thus maximizing the employment rate during the inflation control process. With the sextic cost function we were able to converge the inflation gap response to 0% within 6 years and to 0.2% within just 4 years.

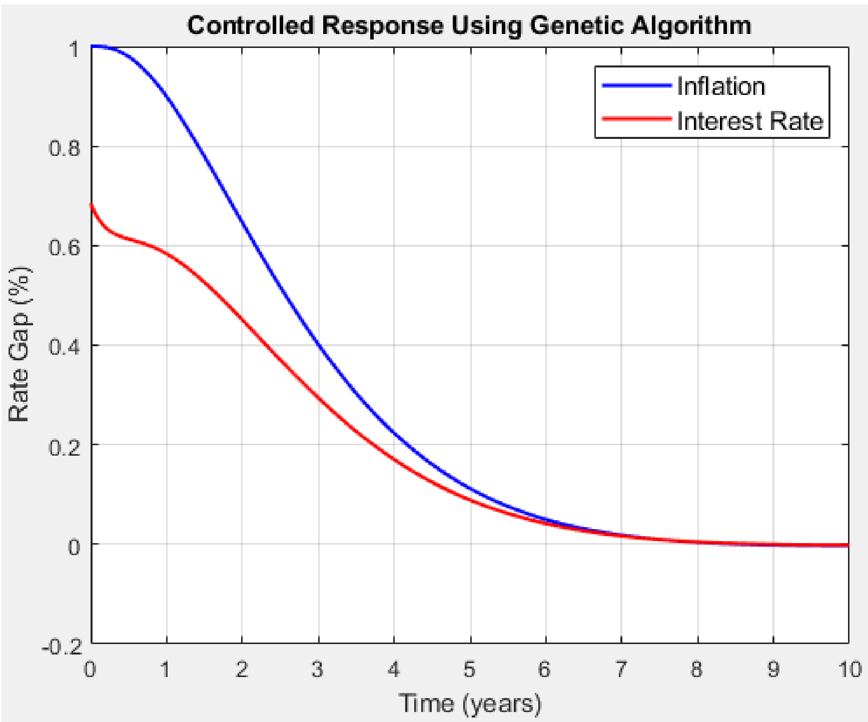


Figure 3.7: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock. MATLAB cost function: cost = sum(sum(x.^2 * Qx)) + sum(u.^4 * Qu).

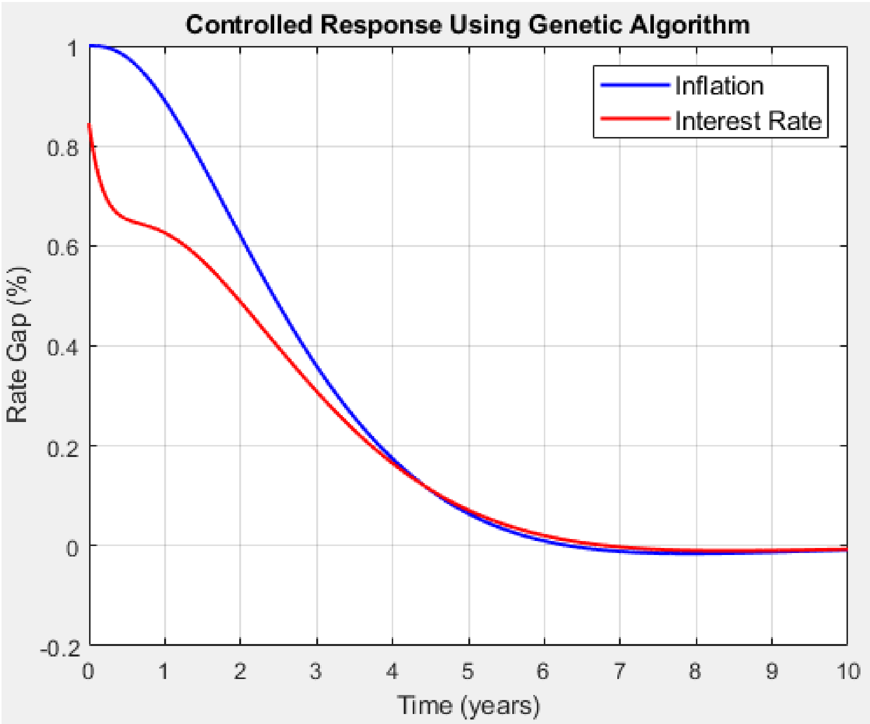


Figure 3.8: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock. MATLAB cost function: cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu).

For the rest of our analysis, we will proceed with the genetic algorithm along with the sextic cost function due to its superior performance.

3.2.4 Controlled Response with Relative Weighing

Now that the superior optimization technique is established, we can explore the effects of relative weighing. To demonstrate relative weighing, each constant of the weighing matrices was varied by a factor of 5 ceteris paribus. Relative weighing is utilitarian for the FED, as it allows for flexibility in optimal control with respect to the general state of the economy and especially the strength of the employment market. For instance, in the case of a dovish FED, a more circumspective approach can be taken during a fragile employment market. In this case, we will apply a cost emphasis on the interest rate.

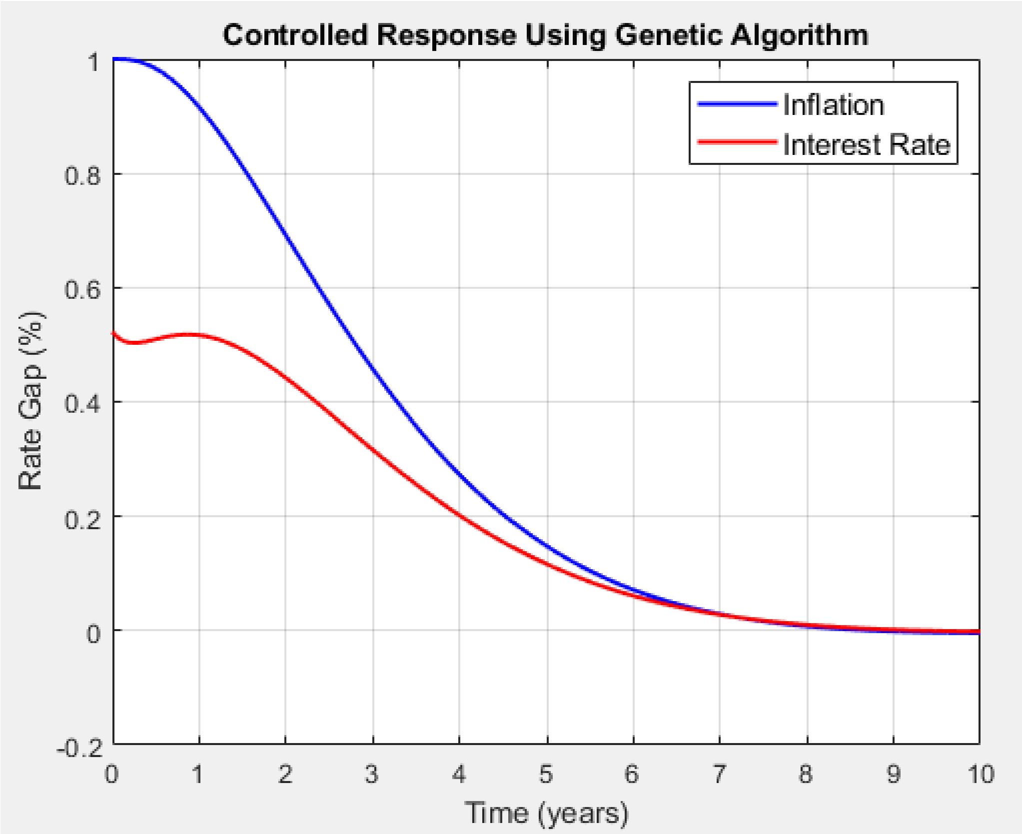


Figure 3.9: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock with cost emphasis on the control signal. MATLAB cost function:
cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu).

The result is elegant. Only a ~52 basis point rate hike is initially applied, but it is held for about 1.5 years before slowly descending to 0%. However, the trade off with convergence time is also clear, as it takes 8 years for inflation to converge to 0%.

In the case of an exceptionally strong employment market, the FED can be more hawkish in its approach for taming inflation. In this instance, we can apply a cost emphasis on inflation.

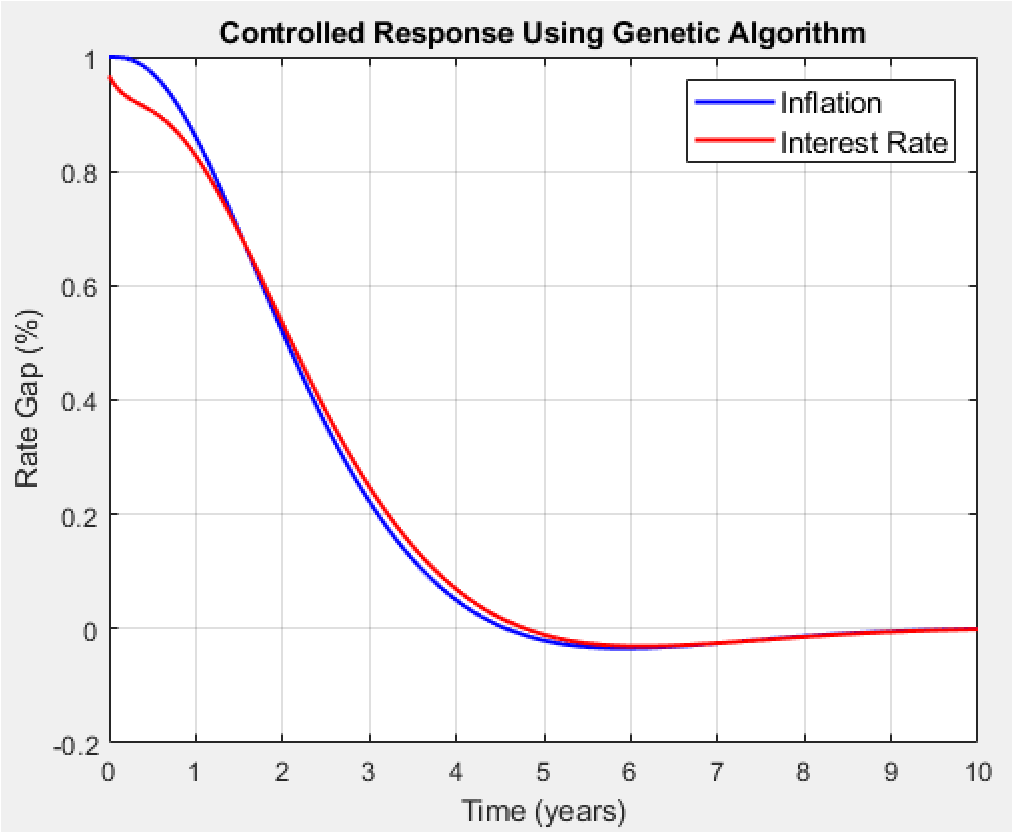


Figure 3.10: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock with cost emphasis on inflation. MATLAB cost function: cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu).

If we compare the results to the equal cost emphasis approach in figure 3.8, we can see a much more aggressive initial rate hike of ~96 basis points along with more time being spent in a high interest rate environment. In addition, we have a convergence time of 4.5 years compared to the convergence time of 6 years in figure 3.8.

With this control system, we also have the ability to apply a cost emphasis on the derivative of inflation to slow the descent of the inflation. There are no incentives to inflation stability, but it is one of the capabilities of our control system and is worth showcasing.

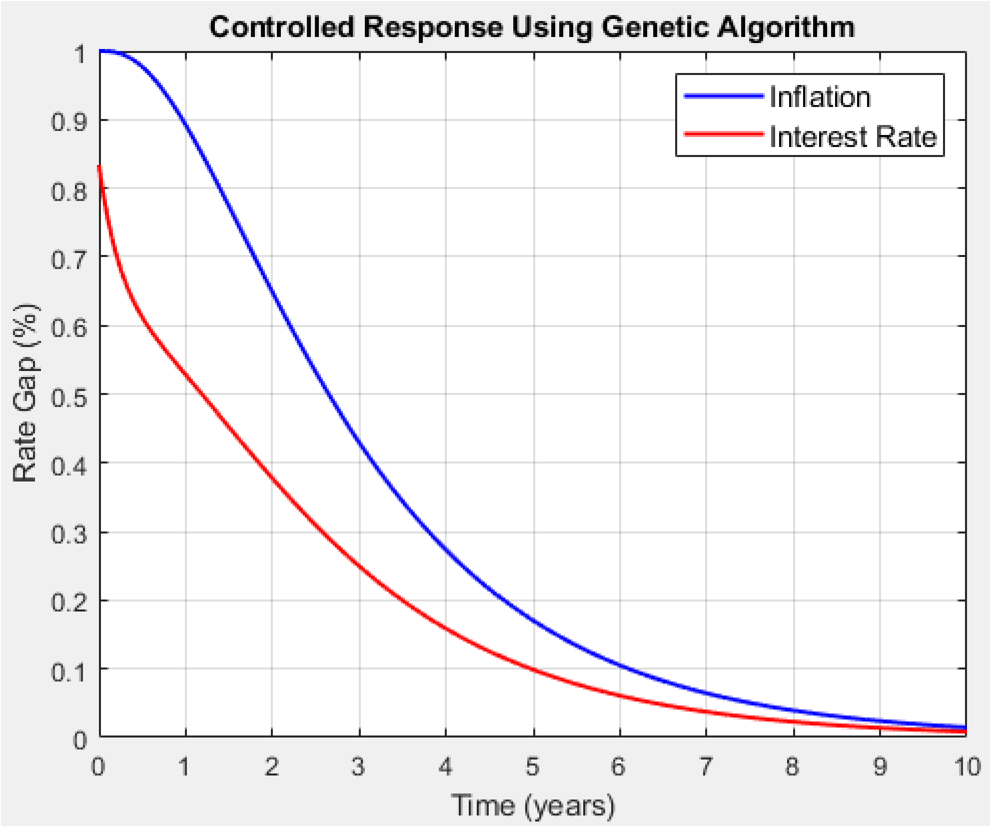


Figure 3.11: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock with cost emphasis on the derivative of inflation. MATLAB cost function:
cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu).

The downside for this cost emphasis is quite severe due to its large trade off with convergence time despite a relatively large initial rate hike of ~83 basis points. This result makes sense as the controller is working to quickly decrease the interest rate to limit change in the inflation rate.

3.2.3 Discretized Controlled Response

The federal open market committee (FOMC) currently holds 8 regularly scheduled meetings per year [19]. We are using a continuous time model. Although this wouldn't be a problem if the FOMC had daily scheduled meetings for changing interest rates, it is still worth considering a discretized version of our model for practical implementation with the FOMCs current schedule.

Creating a discretized system is not as simple as sampling the original curves 8 times a year. The system dynamics change when the discretization takes place. In the discretized version of the genetic algorithm, the continuous-time dynamics of the system were transformed into a discrete time representation by using the zero-order hold (ZOH) method. Details of the MATLAB script are in the appendix. The discrete time model took into account the discrete nature of control inputs and system states, which influenced the behavior of the system over time.

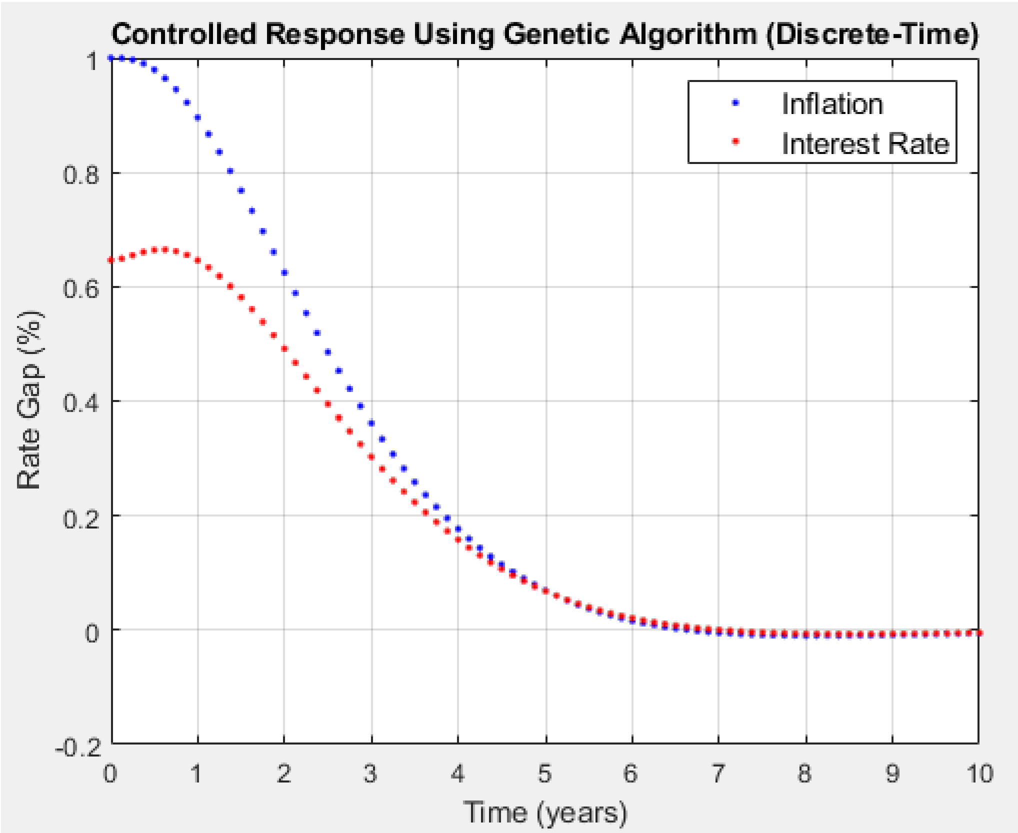


Figure 3.12: The controlled inflation response and optimal overnight federal funds rate using the genetic algorithm after a 1% inflation shock in discrete time. MATLAB cost function: cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu).

If we compare this to the continuous time control system in figure 3.8, we can see that they have identical performance and characteristics but the discrete time response has a much lower initial rate hike – incurring lower initial costs on the economy. Thus, one could argue that remarkably – the discrete-time model creates a more optimal control solution.

4 Conclusion

With the fabrication of our effective state feedback controller, this thesis demonstrated the utility of feedback control systems in guiding monetary policy. We elucidated how fundamentally – the formulation of monetary policy is a feedback control system – emphasizing the advantages of full state feedback over other techniques such as PID and model predictive control. Although data-driven machine learning control was acknowledged as a potentially superior approach, it was deemed impractical for the scope of this undergraduate thesis.

Most importantly, our results indicated that we were successful in fulfilling our goals in the thesis proposal and introduction section of this paper. The closed loop system was stable, controllable and was able to correct a 1% inflation shock within 4.5 years without the need to apply too high or low of a rate hike. However, our controller isn't without its limitations. With the FEDs current sampling rate for inflation, it is difficult to feed the controller with the initial conditions of inflation. In addition, access to the first and second derivatives of inflation is limited. In order to fully utilize control systems used in engineering, central banks such as the FED need to significantly improve their ability to gather data on inflation.

The introduction of relative weighing demonstrates the flexibility of the control system in adapting to different economic conditions and policy objectives. By adjusting the relative weights of the cost function, the control system can prioritize inflation, derivatives of inflation and interest rate relatively based on the prevailing economic environment. This allows for a more tailored and nuanced approach to monetary policy. Furthermore, the consideration of a discretized version of the control system highlights the practical implementation of the model with the FOMC's schedule.

Overall, this thesis provides valuable insights into the application of feedback control systems in guiding monetary policy. It lays the foundation for further exploration and potential enhancements. I believe that this work is utilitarian, important, fulfilling and appropriate for an Engineering Physics undergraduate thesis given its multidisciplinary nature. Central banks are due for a technological revolution given their technological insufficiency relative to the advent of modern technologies. State feedback control systems are a useful addition to the central banking monetary policy toolbox - enabling central banks to adjust policy settings more accurately and precisely based on real time feedback data. Albeit, we must proceed with caution, carefully considering the public's confidence in central bank operations and the need for regulatory compliance and financial stability. As central banking continues to evolve, especially in the realm of data acquisition, it is very likely that feedback control systems such as state feedback control and data-driven machine learning control will play an important role in the fabrication of monetary policy.

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Appendix A

Appendix A1 : Matlab Script for State Feedback Control System With Eigenvalue Selection

```
% Define the State Space Representation
A = [0 1 0; 0 0 1; -0.2626 -5.015 -4.425];
B = [0; 0; -2.171];
C = [1 0 0];
D = 0;

% Define the desired pole locations / eigenvalues
desiredPoles = [-1.5; -0.7; -1.8];

% Calculate the state feedback gain matrix K
K = place(A, B, desiredPoles)

% Create the controlled state-space model
sysControlled = ss(A - B * K, B, C, D);

% Define the initial condition
x0 = [1; 0; 0];  % Initial state condition

% Define the simulation time
t = linspace(0, 10, 100);  % Time vector

% Simulate the controlled response
[y, ~, x] = initial(sysControlled, x0, t);

u= -(K*x');

% Plot the controlled response
plot(t, y, 'b', 'LineWidth', 1.5);hold on;
plot(t, u, 'r', 'LineWidth', 1.5);hold off;
xlabel('Time (years)');
ylabel('Rate Gap (%)');
legend({'Inflation','Interest Rate'},'FontSize',11)
title('Controlled Response Using Eigenvalue Selection');
xlim([0 10]);
ylim([-0.1 1]);
grid on;

clear;
```

Figure A.1: This MATLAB script demonstrates the implementation of a state feedback control system using eigenvalue selection to achieve the desired closed-loop pole locations. The state space representation of the system is defined with the system matrices A,B, C and D. The desired pole locations are specified, and the state feedback gain matrix K is calculated using the place() function. The controlled state space model is created by subtracting the product of B and K from A. The initial condition and simulation time are defined and the controlled response is simulated using the initial() function. The output response and control input are obtained from the simulation results. Finally, the controlled response is plotted, showing the time evolution of the inflation and interest rate gaps – allowing for the analysis and evaluation of the system's performance.

Appendix A2 : Matlab Script for State Feedback Control System With LQR

```
% Define the State Space Representation
A = [0 1 0; 0 0 1; -0.2626 -5.015 -4.425];
B = [0; 0; -2.171];
C = [1 0 0];
D = 0;

% Define the quadratic cost matrices
Qx = diag([1 1 1]);  % State cost matrix
Qu = 1;  % Control cost / unemployment cost

% Calculate the optimal gains using LQR
[K, ~, ~] = lqr(A, B, Qx, Qu)

% Create the state-space model
sys = ss(A - B * K, B, C, D);
[V,D]=eig(A-B*K);
p=diag(D)

% Define the time vector
t = 0:0.01:10;

% Set the initial condition
x0 = [1; 0; 0];

% Simulate the controlled response
r = zeros(size(t));
[y, ~, x] = lsim(sys, r, t, x0);

% Apply control signal constraint
u = -K * x';

% Plot the results
plot(t, y, 'b', 'LineWidth', 1.5);hold on;
plot(t, u, 'r', 'LineWidth', 1.5);hold off;
xlabel('Time (years)');
ylabel('Rate Gap (%)');
legend({'Inflation','Interest Rate'},'FontSize',11)
title('Controlled Response Using LQR');
grid on;

clear;
```

Figure A.2: This MATLAB script implements a state feedback control system using the Linear Quadratic Regulator (LQR) method. The state space representation of the system is defined with the A, B, C and D matrices. The quadratic cost matrices Qx and Qu are specified to define the cost function for the LQR controller. The optimal gains K are computed using the lqr() function. The state-space model is created by subtracting the product of B and K from A. The time vector and initial condition are defined, and the controlled response is simulated using the lsim() function. The control input is obtained by multiplying the state vector x with the gain matrix K. The results are plotted, showing the time evolution of the inflation and interest rate gaps – allowing for the analysis and evaluation of the system's performance.

Appendix A3 : Matlab Script for State Feedback Control System With GA

```matlab
% Define the State Space Representation
A = [0 1 0; 0 0 1; -0.2626 -5.015 -4.425];
B = [0; 0; -2.171];
C = [1 0 0];
D = 0;

% Define the quadratic cost matrices
Qx = diag([1 1 1]);  % State cost matrix
Qu = 1;  % Control cost

% Define the time vector
t = 0:0.01:10;

% Set the initial condition
x0 = [1; 10; 0];

% Define the options for the genetic algorithm
options = optimoptions('ga', 'Display', 'iter', 'PopulationSize', 1000, 'MaxGenerations', 500);

% Define the cost function for optimization
costFunction = @(K) calculateCost(K, A, B, C, D, Qx, Qu, t, x0);

% Run the genetic algorithm to optimize the control gains
[K_opt, ~, ~] = ga(costFunction, numel(B), [], [], [], [], [], [], [], options);

% Create the state-space model with optimized gains
sys = ss(A - B * K_opt, B, C, D);

% Simulate the controlled response
r = zeros(size(t));
[y, ~, x] = lsim(sys, r, t, x0);

% Apply control signal
u = -K_opt * x';

% Plot the results
plot(t, y, 'b', 'LineWidth', 1.5);hold on;
plot(t, u, 'r', 'LineWidth', 1.5);hold off;
xlabel('Time (years)');
ylabel('Rate Gap (%)');
legend({'Inflation','Interest Rate'},'FontSize',11)
title('Controlled Response Using Genetic Algorithm');
grid on;

% Define the cost function for optimization
function cost = calculateCost(K, A, B, C, D, Qx, Qu, t, x0)
    % Create the state-space model with current gains
    sys = ss(A - B * K, B, C, D);

    % Simulate the controlled response
    r = zeros(size(t));
    [y, ~, x] = lsim(sys, r, t, x0);

    % Calculate the control signal
    u = -K * x';

    % Calculate the cost based on the quadratic cost matrices
    cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu);
end
```

Figure A.3: This MATLAB script demonstrates the implementation of a control system using a genetic algorithm to optimize the control gains for a given state space representation. The state space representation of the system is defined by matrices A, B, C, and D. Matrix A represents the system dynamics, matrix B relates the control input to the state variables, matrix C defines the output equation, and matrix D represents the direct feedthrough term. The script starts by defining the quadratic cost matrices Qx and Qu. These cost matrices are used to construct a cost function that quantifies the performance of the control system. Next, the time vector and initial condition are specified to define the duration of the simulation and the initial state of the system. The options for the genetic algorithm are set using the **optimoptions** function. These options include parameters like display settings, population size, and the maximum number of generations. The script defines a cost function named **calculateCost**, which takes the control gains, system matrices, cost matrices, time vector, and initial condition as inputs. This cost function evaluates the performance of the control system by simulating the controlled response using the state-space model with the given control gains. It calculates the cost based on the quadratic cost matrices by summing the squares of the state variables and applying a power of 6 to the control input. The genetic algorithm is then executed using the **ga** function with the specified options. It searches for the optimal control gains that minimize the cost function. The cost function is repeatedly evaluated for different sets of control gains, and the algorithm iteratively refines the search to converge towards the optimal solution. After the genetic algorithm optimization process is completed, the state-space model is reconstructed using the optimized control gains. The controlled response is simulated by applying the optimized control gains to the system and calculating the resulting outputs and control signals. The inflation and interest rate gaps over time are plotted to visualize the performance of the controlled system.

Appendix A4 : Matlab Script for State Feedback Control System With GA in Discrete Time

```matlab
% Define the State Space Representation
A = [0 1 0; 0 0 1; -0.2626 -5.015 -4.425];
B = [0; 0; -2.171];
C = [1 0 0];
D = 0;

% Define the quadratic cost matrices
Qx = diag([1 1 1]);  % State cost matrix
Qu = 1;  % Control cost

% Define the time vector
t = 0:0.125:10;  % 8 data points per unit time

% Set the initial condition
x0 = [1; 0; 0];

% Define the options for the genetic algorithm
options = optimoptions('ga', 'Display', 'iter', 'PopulationSize', 1000, 'MaxGenerations', 500);

% Define the cost function for optimization
costFunction = @(K) calculateCost(K, A, B, C, D, Qx, Qu, t, x0);

% Run the genetic algorithm to optimize the control gains
[K_opt, ~, ~] = ga(costFunction, numel(B), [], [], [], [], [], [], [], options);

% Create the discrete-time state-space model with optimized gains
Ts = 0.125;  % Sampling time for 8 data points per unit time
sys_d = c2d(ss(A - B * K_opt, B, C, D), Ts, 'zoh');

% Simulate the controlled response in discrete-time
r_d = zeros(size(t));
[y_d, ~, x_d] = lsim(sys_d, r_d, t, x0);

% Apply control signal in discrete-time
u_d = -K_opt * x_d';

% Plot the results with dotted lines
plot(t, y_d, 'b.', 'LineWidth', 1.5, 'MarkerSize', 6); hold on;
plot(t, u_d, 'r.', 'LineWidth', 1.5, 'MarkerSize', 6); hold off;
xlabel('Time (years)');
ylabel('Rate Gap (%)');
legend({'Inflation','Interest Rate'}, 'FontSize', 11);
title('Controlled Response Using Genetic Algorithm (Discrete-Time)');
grid on;

% Define the cost function for optimization
function cost = calculateCost(K, A, B, C, D, Qx, Qu, t, x0)
    % Create the state-space model with current gains
    sys = ss(A - B * K, B, C, D);

    % Simulate the controlled response
    r = zeros(size(t));
    [y, ~, x] = lsim(sys, r, t, x0);

    % Calculate the control signal
    u = -K * x';

    % Calculate the cost based on the quadratic cost matrices
    cost = sum(sum(x.^2 * Qx)) + sum(u.^6 * Qu);
end
```

Figure A.4: This MATLAB script is identical to the MATLAB script in figure A.3. However, after the genetic algorithm optimization process, a discrete-time state-space model with the optimized gains is constructed using the **c2d** function. The sampling time, Ts, is specified to determine the discretization of the continuous-time system. The controlled response is simulated in discrete-time by applying the optimized control gains to the system and calculating the resulting outputs and control signals. The inflation and interest rate gaps over time are plotted, utilizing dotted lines to distinguish them.