Vector Partition Functions

by

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Abstract

The problem of enumerating vector partitions is the d-dimensional analogue of the well-studied coin exchange problem. Given a set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^d$, the vector partition function yields the number of solutions to $\mathbf{a}_1 x_1 + \ldots, \mathbf{a}_n x_n = \mathbf{b}$ as a function of $\mathbf{b}$. One can view this as the enumeration of integer points in the polytope $\{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ where A is the matrix whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. The vector partition function p_A associated to the matrix A takes $\mathbf{b}$ as input and returns the corresponding number of vector partitions. Sturmfels (1994) showed that vector partition function can be represented explicitly as a piecewise quasi-polynomial (roughly a polynomial with periodic coefficients) whose domains of quasi-polynomiality are the maximal cones (chambers) of a fan (called the chamber complex) associated to the matrix A. In addition, Sturmfels and De Loera (2003) showed that if A is unimodular (every square submatrix has determinant $0, \pm 1$), then the quasi-polynomials are actually each polynomials.

We show that for certain chambers of A (which we call external chambers) the associated quasi-polynomial arises from a coin exchange problem, and is univariate after an appropriate change of variables. Additionally, we show that if A is unimodular, then the polynomial associated to an external chamber is given by a negative binomial coefficient which depends on a single facet of the chamber. We also show that one can easily calculate linear factors of polynomials associated to other chambers of A (which we call semi-external chambers) in the case that A is unimodular.

The Littlewood-Richardson and Kronecker coefficients are two different sets of structure constants associated to the Schur polynomials. Rassart (2004) and Mishna, Rosas, Sundaram (2021) have considered vector partition function approaches to computing Littlewood-Richardson and Kronecker coefficients respectively. We exploit Rassart's approach in order to derive a new determinantal formula for the Littlewood-Richardson coefficients associated to GL_3. We also use it to give a novel geometrical interpretation of a well-known stability result. Additionally, we address some answers related to symmetries of the Littlewood-Richardson coefficients, partially by computing the chamber complex for the Littlewood-Richardson coefficients associated to GL_4. In our work on Kronecker coefficients, we use the vector partition function approach to create a computational tool for Kronecker coefficients with partition lengths bounded by $2, 4$, and 8. Additionally, we obtain vanishing conditions and generate a stable face of the Kronecker polyhedron. Finally, we obtain new upper bounds for the Kronecker coefficients, which in some cases seem to be the best known.

Keywords: Vector partition functions, Littlewood-Richardson coefficients, Kronecker coefficients

Dedication

Pentru bunicii mei cu mult drag.

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Symbol	Meaning
$c^{\nu}_{\lambda,\mu}$	Littlewood-Richardson coefficient associated to λ, μ, ν
col	Column space
conv	Convex hull
det	Determinant
$\mathbf{e}_i$	The i^{th} standard basis vector
$g_{\lambda,\mu,\nu}$	Kronecker coefficient associated to λ, μ, ν
GL_k	General linear group of degree k
ker	Kernel
Id	Identity element in symmetric group
$\ell(\lambda)$	Length of partition λ
Λ_n	Ring of symmetric polynomials
$L_{\mathcal{P}}(t)$	Ehrhart quasi-polynomial of polytope $\mathcal{P}$
M_γ	Ray matrix of cone γ
$\mathbb{N}$	Set of non-negative integers
$[n]$	Set of positive integers $\{1, \ldots, n\}$
$\mathrm{pos}_{\mathbb{R}}$	Positive hull
p_A	Vector partition function of A
p_A^{γ}	Quasi-polynomial arising from p_A and associated to chamber γ
$\mathbb{R}$	Set of real numbers
s_λ	Schur polynomial associated to λ
$\mathfrak{S}_n$	Symmetric group of order $n!$
S°	interior of subset S
$\sigma^\vee$	Dual of cone σ
$SL(k)$	Special linear group of degree k
$\mathbf{u}^{\mathbf{v}}$	For vectors $\mathbf{u}, \mathbf{v}$ of same dimension n: $\prod_{i=1}^{n} u_i^{v_i}$
$\mathbb{Z}$	Set of integers
$\mathbf{0}$	The zero vector
$\mathbf{1}$	The all-ones vector

Letting the days go by,
let the water hold me down.
Into the blue again,
after the money's gone.
Once in a lifetime,
water flowing underground.
Same as it ever was.
Same as it ever was.

DAVID BYRNE

Sometimes I go about in pity for myself
and all the while
a great wind carries me across the sky.

OJIBWE PROVERB

Here's a little song I wrote,
you might want to sing it note for note -
don't worry, be happy.

BOBBY MCFERRIN

Chapter 1

Background

1.1 Introduction

The vector partition enumeration problem is the d-dimensional analogue of the *coin exchange* problem that students of mathematics often study in an introductory discrete mathematics course. We state the problem here to help motivate vector partitions, and because it plays a role in some of the proofs of Chapter 2. We note that throughout this text $\mathbb{N}$ denotes the set of non-negative integers.

Definition 1.1.1 (Coin exchange problem)**.** Let $a_1, \ldots, a_n$ be positive integers. The problem of computing the number of solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ of

$$a_1 x_1 + \cdots + a_n x_n = b$$

for a non-negative integer b is called the *coin exchange problem*.

Replacing each a_i with a vector $\mathbf{a}_i \in \mathbb{Z}^d$, and also replacing b with a vector $\mathbf{b} \in \mathbb{Z}^d$, we obtain the equation

$$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = \mathbf{b}. \tag{1.1}$$

Let us consider the problem of enumerating the number of solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ of Eq. (1.1). Equivalently, this is the problem of enumerating the number of solutions $\mathbf{x} \in \mathbb{N}^n$ of the equation

$$A\mathbf{x} = \mathbf{b} \tag{1.2}$$

where A is the $d \times n$ integral matrix whose columns are $\mathbf{a}_1, \ldots \mathbf{a}_n$. The number of solutions $\mathbf{x} \in \mathbb{N}^n$ of Eq. (1.1) may be infinite – for example if $n = 2$, $\mathbf{a}_2 = -\mathbf{a}_1$, and $\mathbf{b} = \mathbf{0}$. In order to ensure that this does not occur, one imposes the condition that $\ker(A) \cap \mathbb{R}^n_{\geq 0} = \{\mathbf{0}\}$. Under these conditions, we say that the problem of counting the number of solutions $\mathbf{x} \in \mathbb{N}^n$ of Eq. (1.1) is the *vector partition enumeration problem.* Additionally, the *vector partition function* $p_A(\mathbf{b})$ associated to the matrix A is the function that yields the number of solutions $\mathbf{x} \in \mathbb{N}^n$ to Eq. (1.2) as a function of $\mathbf{b}$.

We note that the vector partition enumeration problem can indeed be viewed as the d-dimensional analogue of the coin exchange problem, since if $d = 1$, the condition $\ker(A) \cap \mathbb{R}^n_{\geq 0} = \{\mathbf{0}\}$ ensures that either each of the the entries of A are positive integers, or that each of the entries of A are negative integers. The second of these cases can be reduced to the first by negating both sides, and so we see without loss of generality that the $d = 1$ case of the vector partition enumeration problem is indeed the coin exchange problem.

The coin exchange problem remains an active area of study (see for example [10]) with many intriguing open problems ([8, Chapter 1] provides a nice summary). However, it does not fully capture the geometry associated to vector partition functions, which can be very complex. Vector partition functions can be described explicitly as piecewise quasi-polynomials (essentially polynomials whose coefficients are periodic functions) whose domains of quasi-polynomiality are maximal cones (called chambers) of a fan associated to the matrix A (called the chamber complex of A). As an example to illustrate the complexity of the geometry we refer the reader to the problem of enumerating the number of chambers associated to Kostant's partition function described in [53]. Although the matrices associated to Kostant's partition function are easy to describe, computing the sequence counting the number of chambers of these matrices remains an open problem – in fact, only the first seven instances have been computed.

Vector partition functions appear in many problems associated to algebraic combinatorics and representation theory. In particular, in the study of Littlewood-Richardson coefficients and Kronecker coefficients, vector partition functions can play a prominent role (see for example [15, 62, 75, 74]). Studying these different coefficients through the vector partition function lens has proven fruitful and led to many non-trivial results. For example, in [16] Briand, Rosas, and Orellana use computational results from [15] to disprove a saturation conjecture of Mulmuley. As another example, Rassart [75] proves that the Littlewood-Richardson coefficients associated to the general linear group GL_k can be described by a piecewise polynomial. Using the geometric description of the resulting piecewise polynomial in the $k = 3$ case, Briand and Rosas then discovered a novel linear symmetry of the Littlewood-Richardson coefficients associated to GL_3 and explicitly computed the full list of symmetries in this case.

1.2 Summary of contribution

The main contributions of this work can roughly be split into two categories. The first are directly related to the theory of vector partition functions, and the second are related to algebraic combinatorial problems with associated vector partition functions. Contributions of the first category are described in Chapter 2 and contributions of the second category are described in Chapters 3 and 4.

We begin by describing the contributions in the first category. We define a particular type of column of the matrix A, which we call an external column. Our main result (Theorem 2.3.4) shows that (up to some lattice requirements) the quasi-polynomial associated to a chamber γ containing external columns can be obtained by considering a simpler vector partition function problem. This vector partition function is simpler in the sense that its dimension is smaller than the original vector partition function by the number of external columns in γ. In the case that the number of external columns in γ is maximal (without being a trivial case), we are able to obtain further results. We define such a chamber to be an external chamber and show that (subject to the same lattice condition) the quasi-polynomial formulae for such chambers arise from a coin exchange problem, and are thus Ehrhart (univariate) quasi-polynomial functions after an appropriate change of variables (Theorem 2.3.5). Moreover, we are able to characterize exactly when the quasi-polynomial is a polynomial, in which case the formula takes on a particularly nice form – as a negative binomial coefficient (Theorem 2.4.2). We then apply this result to unimodular matrices in Corollaries 2.4.5 and 2.4.8. We also conjecture that a result of Baldoni and Vergne [3] on computing linear factors of polynomials may be generalized. This would potentially allow one to compute linear factors of polynomials associated to Littlewood-Richardson coefficients. Finally, we give a pair of examples illustrating how these formulae can be used to give combinatorial results in Section 2.6. Namely, we derive a novel result related to multigraph counting (Theorem 2.6.4), and also rederive a well-known result related to Kostant's partition function (Theorem 2.6.9). These results appear in Chapter 2. This work is gathered in an article (in preperation) for which I am sole author [86].

We now describe the contributions of the second category, which appear in Chapters 3 and 4.

In Chapter 3, we use a vector partition function approach developed by Rassart [75] in order to obtain some novel results for the Littlewood-Richardson coefficients associated to GL_3. Namely, we obtain a determinantal formula for these coefficients (Theorem 3.6.4), give a novel geometrical interpretation for some well-known stability results (Theorem 4.6.4) In addition, we have explicitly computed the fan associated to the Littlewood-Richardson coefficients associated to GL_4 , and used this result in order to compute the linear symmetries of the corresponding Littlewood-Richardson coefficients (Theorem 3.7.2). This chapter is based on work with Briand and Rosas [21]. We reported the computation of the chamber complex in the GL_4 case in [20]. We also include a section on the computation of linear factors of polynomials arising from the Littlewood-Richardson coefficients. This section is based on ongoing work and is thus somewhat speculative – it is made up of empirical results and conjectures.

In Chapter 4, we give some results related to Kronecker coefficients by using a vector partition function approach due to Mishna, Rosas, and Sundaram [62]. We develop a computational tool to compute Kronecker coefficients $g_{\lambda,\mu,\nu}$ with $\ell(\lambda) \leq 8, \ell(\mu) \leq 2, \ell(\nu) \leq 4$.

Additionally, we obtain vanishing conditions on the Kronecker coefficients (Theorem 4.5.5). We also compute a stable face of the Kronecker polyhedron associated to the vector partition approach (Theorem 4.6.4). Finally, we give upper bounds on the Kronecker coefficients, which in some cases seem to be the best known (Corollaries 4.7.6 and 4.7.8). This work is jointly done with Mishna [63] and submitted for publication.

In the current chapter, we have no novel contributions – the aim is solely to build the necessary theory for the following chapters. We begin by introducing the polyhedra, cones, and fans in Section 1.3. Next, in Section 1.4, we give a short discussion of Ehrhart theory. Finally, we introduce symmetric polynomials in Section 1.6 in order to establish the background for Chapters 3 and 4.

1.3 Polyhedra and cones

The main references we follow in this section are Beck and Robbins [8], Cox, Little and Schenck [29], and Fulton [38]. Our goal is to define the objects required to understand the geometry associated to vector partition functions. Although many of the definitions we give can be made more abstract, we choose to take a more concrete approach and define everything within the vector space $\mathbb{R}^d$.

An *affine hyperplane* $H \subseteq \mathbb{R}^d$ is the subspace of $\mathbb{R}^d$ defined by a single linear equation of the form $m_1x_1 + \cdots + m_dx_d = b$ for some $m_1, \ldots, m_d, b \in \mathbb{R}$ with at least one of $m_1, \ldots, m_d$ non-zero. If $b = 0$, then H is a hyperplane. A *closed half-space* is a subset of $\mathbb{R}^d$ defined by a single linear inequality of the form $m_1x_1 + \cdots + m_dx_d \geq b$ or $m_1x_1 + \cdots + m_dx_d \leq b$. It is often useful to rewrite the hyperplane equation using dot products, in which case we see that each hyperplane can be defined by a single vector in $\mathbb{R}^d$. More precisely, for $\mathbf{m} \in \mathbb{R}^d$, $\mathbf{m} \neq 0$, we define the hyperplane $H_{\mathbf{m}}$

$$H_{\mathbf{m}} := \{\mathbf{u} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u} = 0\},$$

which divides $\mathbb{R}^d$ into the closed half-spaces

$$H_{\mathbf{m}}^+ := \{\mathbf{u} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u} \geq 0\} \text{ and } H_{\mathbf{m}}^- := \{\mathbf{u} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u} \leq 0\}.$$

If $S, S' \subseteq \mathbb{R}^d$ are sets with the property that $S \subseteq H_m^+$ and $S' \subseteq H_m^-$, we say that H *separates* S and S'.

A *polyhedron* $\mathcal{P}$ is a set obtained by taking the intersection of finitely many half-spaces – namely

$$\mathcal{P} = \bigcap_{\mathbf{m} \in S} H_{\mathbf{m}}^+$$

for some finite subset $S \subset \mathbb{R}^d$. The *ambient dimension* of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ is d and its *dimension* is the dimension of the affine space $\{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in \mathcal{P}, \lambda \in \mathbb{R}\}$. If the dimension of $\mathcal{P}$ is k, we may sometimes say that $\mathcal{P}$ is a *k-polyhedron.* If the intersection of two polyhedra $\mathcal{P}_1, \mathcal{P}_2$ is a k-polyhedron, then we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ *intersect k-dimensionally.*

A hyperplane $H_{\mathbf{m}} \subset \mathbb{R}^d$ with $\mathbf{m} \neq \mathbf{0}$ is called a *supporting hyperplane* of $\mathcal{P}$ if $\mathcal{P}$ lies in one of the half-spaces $H^+_{\mathbf{m}}$ or $H^-_{\mathbf{m}}$ defined by $H_{\mathbf{m}}$, and the intersection $\mathcal{P} \cap H$ is non-empty. A *face* of a polyhedron $\mathcal{P}$ is a set of the form $\mathcal{P} \cap H_{\mathbf{m}}$ for some supporting hyperplane $H_{\mathbf{m}} \subset \mathbb{R}^d$. We note that faces of polyhedra are themselves polyhedra since $H_{\mathbf{m}} = H^+_{\mathbf{m}} \cap H^+_{-\mathbf{m}}$ for each $\mathbf{m} \in \mathbb{R}^d$, $\mathbf{m} \neq \mathbf{0}$. Faces of dimension 0 are called *vertices* and faces of co-dimension 1 (relative to the polyhedron) are called *facets.*

A *polytope* is a bounded polyhedron. While we can define polytopes by intersections of half-spaces, they can also be defined by their set of vertices. More precisely, if the vertices of a polytope $\mathcal{P}$ are $\mathbf{v_1}, \ldots, \mathbf{v_k}$, then

$$\mathcal{P} = \mathrm{conv}(\mathbf{v_1}, \ldots, \mathbf{v_k})$$

where

$$\mathrm{conv}(\mathbf{v_1}, \ldots, \mathbf{v_k}) := \{\lambda_1 \mathbf{v_1} + \cdots + \lambda_k \mathbf{v_k} : \lambda_1, \ldots, \lambda_k \geq 0,\ \lambda_1 + \cdots + \lambda_k = 1\}$$

is the *convex hull* of points $\mathbf{v_1}, \ldots, \mathbf{v_k} \in \mathbb{R}^d$. A polytope is called *rational* if its vertices are in $\mathbb{Q}^d$, and *integral* if each of its vertices are in $\mathbb{Z}^d$. Inheriting the language from polyhedra, if a polytope has dimension k, we say that it is a *k-polytope.* It is not trivial to prove that the half-space and vertex definitions of polytopes are equivalent, and this duality plays an important role in their study. For a detailed proof, we refer the reader to [55, Appendix A].

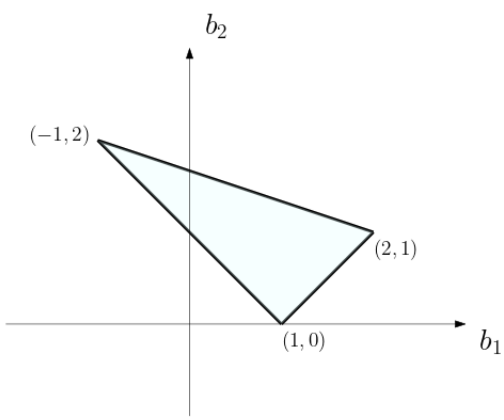


Figure 1.1: The polytope $\mathcal{P}$ defined by the inequalities $b_1 + 3b_2 \leq 5$, $b_1 + b_2 \geq 1$, and $b_1 - b_2 \leq 1$. Equivalently $\mathcal{P} = \mathrm{conv}\left((1,0), (-1,2), (2,1)\right)$.

The central geometric object of our work is a type of unbounded polyhedron, known as a *convex polyhedral cone* $\sigma \in \mathbb{R}^d$. It is the positive convex hull of a set of points

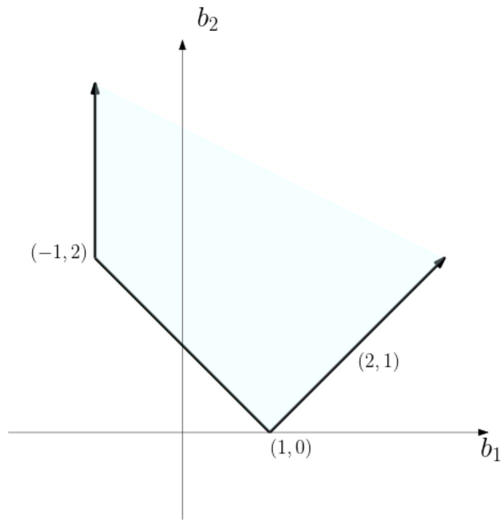


Figure 1.2: The (unbounded) polyhedron defined by the inequalities $b_1 \geq -1$, $b_1 + b_2 \geq 1$, and $b_1 - b_2 \leq 1$.

$\mathbf{u_1}, \ldots, \mathbf{u_k} \in \mathbb{R}^d$, that is:

$$\sigma = \mathrm{pos}_{\mathbb{R}}(\mathbf{u}_1, \ldots, \mathbf{u}_k) := \{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_k \mathbf{u}_k : \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0\}$$

and we say that σ is *generated* by $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^d$. Additionally, we define $\mathrm{pos}_{\mathbb{R}}(\emptyset) = \{\mathbf{0}\}$.

Definition 1.3.1 (Dual cone)**.** Let σ be a convex polyhedral cone in $\mathbb{R}^d$. We say that

$$\sigma^\vee := \{\mathbf{m} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u} \geq 0 \text{ for all } \mathbf{u} \in \sigma\}$$

is the *dual cone of* σ.

Since each $\mathbf{u} \in \sigma$ can be represented as a non-negative linear combination of the generators $\mathbf{u}_1, \ldots, \mathbf{u}_k$ of σ,

$$\sigma^\vee = \{\mathbf{m} \in \mathbb{R}^d : \mathbf{m} \cdot \mathbf{u}_j \geq 0 \text{ for all } j = 1, \ldots, k\}.$$

Therefore, we see that the dual cone $\sigma^\vee$ is an intersection of finitely many half-spaces, and is thus a polyhedron. In fact, $\sigma^\vee$ is a convex polyhedral cone, and $(\sigma^\vee)^\vee = \sigma$. We remark that the convex polyhedral cone σ is also technically a polyhedron since it is the dual of a convex polyhedral cone. Therefore, the definitions associated to polyhedra also apply to convex polyhedral cones. Faces of convex polyhedral cones are themselves also convex polyhedral cones, and each proper face of a convex polyhedral cone is the intersection of all facets containing it.

For a convex polyhedral cone σ with supporting hyperplane H, we say that $\mathbf{u}$ is an *inner facet normal* of σ if $\mathbf{u} \in H^+$ is normal to H (i.e normal to each element of H), and that $\mathbf{u}$

is an *outer facet normal* of σ if $\mathbf{u} \in H^-$ is normal to H. An important duality we exploit is that the rays of the dual cone $\sigma^\vee$ correspond to facets of σ and vice-versa. We describe this duality formally in the following proposition (which is described partially in Proposition 1.2.8 of [29] and partially in the discussion immediately following it).

Proposition 1.3.2. *[29, Proposition 1.28] Let $\sigma \subseteq \mathbb{R}^d$ be the polyhedral cone $\sigma = H^+_{\mathbf{m}_1} \cap H^+_{\mathbf{m}_2} \cap \ldots \cap H^+_{\mathbf{m}_s}$. Then $\sigma^\vee = \operatorname{pos}_{\mathbb{R}}(\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s)$. In particular, $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s$ are inner facet normals of σ if and only if $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s$ generate rays of $\sigma^\vee$.*

A convex polyhedral cone $\sigma \subseteq \mathbb{R}^d$ is *rational* if it can be generated by a finite number of integer points, and it is *pointed* if it contains no 1-dimensional subspace. In general in this thesis, we deal with convex rational pointed polyhedral cones, so we simply call these *cones* for short.

The faces of a cone are also cones, and faces of dimension 1 are called *rays*. The dual of a cone $\sigma \subseteq \mathbb{R}^d$ is a rational convex polyhedral cone, but need not be pointed unless σ is d-dimensional, in which case $\sigma^\vee$ is indeed also a cone.

Given a ray r, we say that $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d$ is a *ray generator* of r if $r = \operatorname{pos}_{\mathbb{R}}(\mathbf{w})$. If additionally, $\mathbf{w} \in \mathbb{Z}^d$ and $\gcd(\{w_i : 1 \leq i \leq d\}) = 1$, then we say that $\mathbf{w}$ is the minimal ray generator of r. Any set of generators of a cone σ contains some minimal subset of generators, which still generate σ. This subset consists of exactly the vectors that generate its rays. We call such a subset a *minimal generating set* of σ, and call its elements *ray generators of σ*. If a minimal generating set S of σ additionally has the property that each element is a minimal ray generator, then we say that S is the set of *minimal ray generators* of σ. Each cone σ has a unique set of minimal ray generators. We say that σ is *simplicial* if its set of ray generators is linearly independent. A *triangulation* of a cone σ is a collection of simplicial cones $\sigma_1, \ldots, \sigma_m$ such that $\cup_{i=1}^m \sigma_i = \sigma$, and for each $1 \leq i < j \leq m$ the intersection $\sigma_i \cap \sigma_j$ of any pair of cones σ_i, σ_j is a face of both σ_i and σ_j. It is a non-trivial fact that any cone admits a triangulation with no new ray generators – that is the union of the minimal ray generators of the σ_i is the set of minimal ray generators of σ. We refer the reader to [55, Appendix B] for the full details of the proof.

Remark 1.3.3. We have deviated slightly from the notation of [29] and [38]. In [38] what we call "ray generators" are called "minimal generators". In [29] what we refer to as the "minimal ray generator" is simply called the "ray generator", and what we call "ray generators" of σ are called "minimal generators" of σ. Our choice is dictated by our introduction of the terms "external ray generators" and "minimal external ray generators" for which either notation scheme (i.e that of [29] or [38]) would cause confusion.

Example 1.3.4. Consider the cone

$$\sigma = \operatorname{pos}_{\mathbb{R}}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right) \subseteq \mathbb{R}^2$$

and its dual cone

$$\sigma^\vee = \mathrm{pos}_{\mathbb{R}}\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right)$$

which are illustrated in Figure 1.3. The faces of σ are:

2d face the whole cone σ,

1d face the ray/facet $\{t\begin{bmatrix}1\\1\end{bmatrix} : t \geq 0\} = \{(b_1, b_1) : b_1 \geq 0\}$,

1d face the ray/facet $\{t\begin{bmatrix}2\\1\end{bmatrix} : t \geq 0\} = \{(2b_2, b_2) : b_2 \geq 0\}$,

0d face the origin.

We note that the dual cone $\sigma^\vee$ is indeed generated by the inner facet normals of σ.

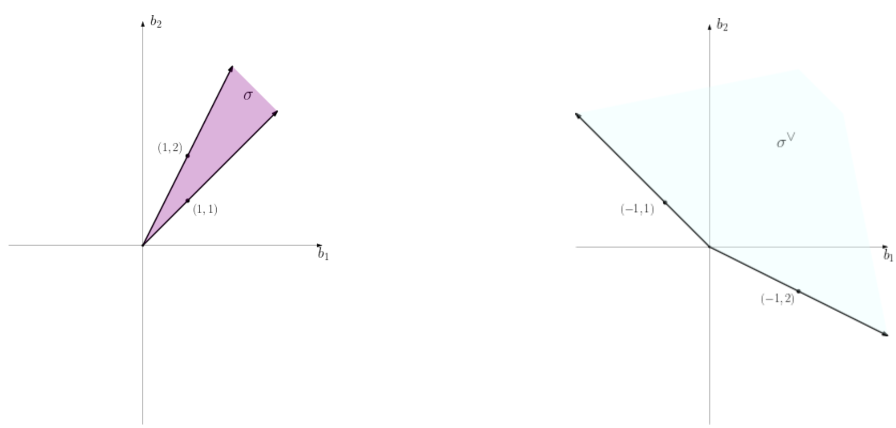


Figure 1.3: A cone σ shaded in dark magenta and its dual $\sigma^\vee$ shaded in light cyan. The purpose of the labelled points is to explicitly describe the direction of the rays containing them.

A *fan* Σ is a set of cones such that if $\sigma \in \Sigma$, then every face of σ is in Σ, and for all $\sigma_1, \sigma_2 \in \Sigma$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. Maximal cones of Σ are called *chambers*. We note that a fan is defined by its chambers, since the fan is the set of chambers along with all of their faces. Therefore, fans can be expressed more succinctly simply by giving the list of chambers. Figure 1.4 illustrates a fan comprised of seven cones: the chambers $\sigma_1 := \mathrm{pos}_{\mathbb{R}}((0,1),(1,2)), \sigma_2 := \mathrm{pos}_{\mathbb{R}}((1,2),(1,1)), \sigma_3 := \mathrm{pos}_{\mathbb{R}}((1,1),(1,0))$, the rays $r_1 := \mathrm{pos}_{\mathbb{R}}((0,1)), r_2 := \mathrm{pos}_{\mathbb{R}}((1,2)), r_3 := \mathrm{pos}_{\mathbb{R}}((1,1)), r_4 := \mathrm{pos}_{\mathbb{R}}((1,0))$, and the origin $v := \mathrm{pos}_{\mathbb{R}}(\emptyset)$.

1.4 Ehrhart theory

Ehrhart theory is the study of enumerating integer points in dilations of rational polytopes. It has many applications in algebraic combinatorics, optimization, graph theory, and alge-

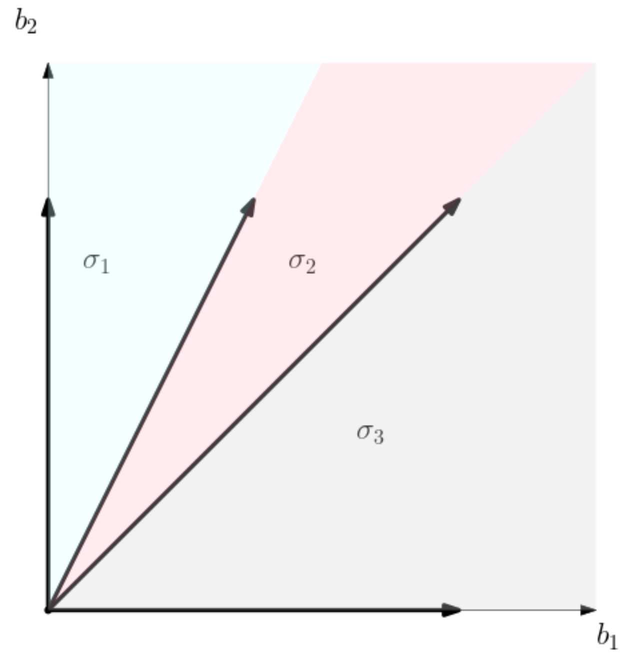


Figure 1.4: The fan $\Sigma \subseteq \mathbb{R}^2$ defined by the chambers σ_1 (shaded in cyan), σ_2 (shaded in pink), σ_3 (shaded in grey). The cones of σ are the three 2-dimensional cones $\sigma_1, \sigma_2, \sigma_3$, the four 1-dimensional rays/facets r_1, r_2, r_3, r_4, and the 0-dimensional vertex v (the origin).

braic geometry (among others[1]). For an excellent introduction to the topic, we refer the reader to the survey *The many aspects of counting lattice points in polytopes* by De Loera [31], which gives several examples of applications as well as an overview of the theory. For a more in-depth treatment, the books *Computing the continuous discretely* by Beck and Robbins [8] and *Integer points in polyhedra* by Barvinok [5] provide excellent sources of information. The study of this problem provides the geometrical intuition for understanding vector partition functions, which are our central object of study.

In order to link the enumeration of integer points in polytopes to vector partition functions, we begin by describing a well-known embedding that makes the correspondence clear. Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ described by ℓ linear inequalities, we introduce slack variables $s_1, \ldots, s_\ell$ in order to rewrite each of the ℓ inequalities as equalities. Then we can write

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : [A|I_{\ell\times\ell}] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}, \mathbf{s} \geq \mathbf{0}\}$$

[1] In [31], De Loera states: “Counting lattice points in (four dimensional) convex bodies is something that credit card cyber-thieves would care about too!” before explaining that the factorization of numbers arising in RSA encryption can be viewed as counting the number of integer points in a 4-dimensional polytope.

where $\mathbf{s} = (s_1, \ldots, s_\ell)$. Since $\mathcal{P}$ is bounded, we can translate it into the cone $\mathbf{x} \geq 0$ to obtain a translation of the polytope, $\mathcal{P}'$ of the form

$$\mathcal{P}' = \{\mathbf{x} \in \mathbb{R}^d : [A|I_{\ell\times\ell}] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}, \mathbf{x}, \mathbf{s} \geq \mathbf{0}\}.$$

Next we may embed $\mathcal{P}'$ into $\mathbb{R}^{d+\ell}$ via the mapping $\mathbf{x} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ to obtain a polytope $\mathcal{P}'' \subseteq \mathbb{R}^{d+\ell}$ defined by

$$\{\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{d+\ell} : [A|I_{\ell\times\ell}] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}, \mathbf{x}, \mathbf{s} \geq \mathbf{0}\}.$$

The polytope $\mathcal{P}''$ has essentially the same properties as $\mathcal{P}$. We note in particular that $\mathcal{P}''$ has the same number of integer points as $\mathcal{P}$ (i.e $\#(\mathcal{P} \cap \mathbb{Z}^d) = \#(\mathcal{P}'' \cap \mathbb{Z}^{d+\ell})$). Therefore, without loss of generality we may view any polytope $\mathcal{P}$ as a set of the form

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

and so we see that the enumeration of points in the polytope $\mathcal{P}$ is equivalent to the vector partition enumeration problem with the matrix A and vector $\mathbf{b}$.

Let $\mathcal{P}$ be the polytope defined as the convex hull of points $\mathbf{v_1}, \ldots, \mathbf{v_k}$, and let t be a positive integer. The polytope $t\mathcal{P}$, called the t^{th} *dilate* of $\mathcal{P}$ is the set

$$t\mathcal{P} = \operatorname{conv}(t\mathbf{v}_1, \ldots, t\mathbf{v}_k).$$

If the polytope is represented as $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, then

$$t\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = t\mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}.$$

Let $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$ be the number of integer points in the t^{th} dilation of $\mathcal{P}$. The sequence of integer points in successive dilations of $\mathcal{P}$

$$(L_{\mathcal{P}}(t))_{t\geq 1}$$

is encoded in the coefficients of the power series

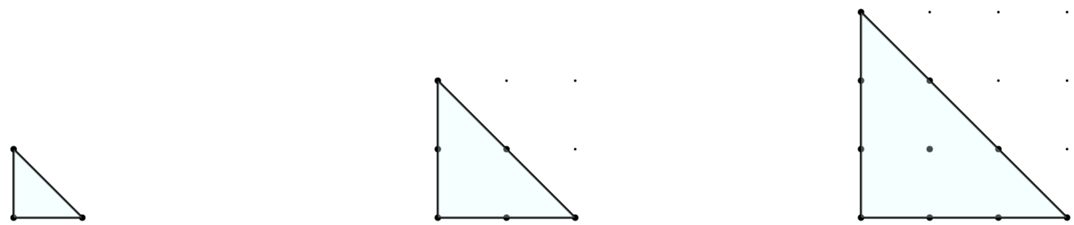
$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t\geq 1} \#(t\mathcal{P} \cap \mathbb{Z}^d) z^t$$

called the *Ehrhart series* of the polytope $\mathcal{P}$.

Example 1.4.1. Let $\Delta := \operatorname{conv}((0,0),(1,0),(0,1))$. Then, as Figure 1.5 illustrates,

$$\operatorname{Ehr}_{\Delta}(z) = 1 + 3z + 6z^2 + 10z^3 + \cdots = \sum_{t \geq 0} \binom{t+1}{2} z^t.$$

We note that for this polytope Δ, the coefficients of the series are the *triangular numbers.*



(a) Polytope Δ has 3 integer points

(b) Polytope 2Δ has 6 integer points

(c) Polytope 3Δ has 10 integer points

Figure 1.5: The first three dilations of the polytope Δ. The integer points in each polytope are emphasized by the thick discs. The initial terms of $Ehr_{\Delta}(z)$ are $1+3z+6z^2+10z^3+\ldots$.

In the above example, we see that the function $L_{\Delta}(t) = \binom{t+1}{2}$ is a polynomial in t. We now describe the general form of the function $L_{\Delta}(t)$, which can be viewed as a polynomial with periodic coefficients. More precisely, a *univariate quasi-polynomial* $f(t)$ is a function of the form

$$f(t) := \sum_{k=1}^{d} c_k(t) t^k$$

where $c_1(t), \ldots, c_d(t) : \mathbb{Z} \to \mathbb{Q}$ are periodic functions in t. The *degree* of the quasi-polynomial f is d (here we assume that c_d is not the zero function) and the *period* of f is the lowest common multiple of the periods of $c_1, \ldots, c_n$.

Theorem 1.4.2 (Ehrhart, 1962 [34])**.** *Let $\mathcal{P}$ be a rational d-polytope. Then $L_{\mathcal{P}}(t)$ is a quasi-polynomial of degree d in t. Moreover, if $\mathcal{P}$ is an integral d-polytope, then $L_{\mathcal{P}}(t)$ is a polynomial.*

We call $L_{\mathcal{P}}(t)$ the *Ehrhart quasi-polynomial* of $\mathcal{P}$, and if $L_{\mathcal{P}}(t)$ is actually polynomial we call it the *Ehrhart polynomial* of $\mathcal{P}$.

Ehrhart quasi-polynomials have many nice, and often surprising properties. We highlight one of these properties that we will exploit in the proof of Theorem 2.5.3. As a notational point, for a set $S \subseteq \mathbb{R}^k$, we use S° to denote the interior of S.

Theorem 1.4.3 (Ehrhart-Macdonald reciprocity)**.** *Let $\mathcal{P}$ be a rational polytope. Then*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim(\mathcal{P})} L_{\mathcal{P}^\circ}(t).$$

In words, evaluating the Ehrhart quasi-polynomial at a negative integral value $-t$ (for some positive integer t) has a geometrical meaning – namely, it is (up to sign) the number of integer points in the interior of the dilated polytope $t\mathcal{P}$.

1.5 Vector partition functions

Here we give a brief primer on vector partition functions, generally following the notation of [84]. Throughout the rest of this section A will denote a $d \times n$ matrix of rank d with integer entries and

$$\ker(A) \cap \mathbb{R}^n_{\geq 0} = \{\mathbf{0}\}.$$

We begin by recalling the definition of the vector partition function.

Definition 1.5.1. The *vector partition function of A*

$$p_A : \mathbb{Z}^d \to \mathbb{N}$$

is defined by

$$p_A(\mathbf{b}) := \#\{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}.$$

One can view this problem as enumerating the number of "partitions" of the vector $\mathbf{b}$ whose parts are the columns of A – hence the name vector partition function.

Remark 1.5.2. Recall that the condition $\ker(A) \cap \mathbb{R}^n_{\geq 0} = \{\mathbf{0}\}$ is imposed so that p_A is indeed a function. Otherwise we may have that $p_A(\mathbf{b})$ is not finite for some $\mathbf{b} \in \mathbb{Z}^d$.

Given a matrix M with columns $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k$, we define the *cone associated to M*, denoted $\mathrm{pos}_{\mathbb{R}}(M)$ to be the set $\{\lambda_1\mathbf{m}_1 + \lambda_2\mathbf{m}_2 + \cdots + \lambda_k\mathbf{m}_k : \lambda_1, \ldots, \lambda_k \geq 0\}$. In words, $\mathrm{pos}_{\mathbb{R}}(M)$ is the cone generated by the columns of M.[2] For any $s \subseteq \{1, 2, \ldots, n\}$, we define A_s to be the submatrix of A composed of the columns of s. For any s satisfying $|s| = \mathrm{rank}(A_s) = \mathrm{rank}(A) = d$, we say that $\mathrm{pos}_{\mathbb{R}}(A_s)$ is a *simplicial cone of A*. The *chamber complex* of A is the fan obtained as the common refinement of the simplicial cones of A (viewed as fans). Explicitly, defining cone($\mathbf{b}$) to be the intersection of all simplicial cones of A containing $\mathbf{b}$ - that is,

$$\mathrm{cone}(\mathbf{b}) := \{ \bigcap_{s \subseteq [n]} \mathrm{pos}_{\mathbb{R}}(A_s) : |s| = \mathrm{rank}(A_s) = d, \mathbf{b} \subseteq \mathrm{pos}_{\mathbb{R}}(A_s)\}$$

the chamber complex of A is the set of cones $\{\mathrm{cone}(\mathbf{b}) : \mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A)\}$ along with all of their faces.

[2]Some authors use $\mathrm{pos}_{\mathbb{R}}(M)$ to indicate the cone generated by the rows of M.

The cones of maximal dimension of the chamber complex are called *geometrical chambers* - we call them *chambers* for short[3]. Equivalently, these are the cones of dimension d of the chamber complex.

We list some facts about the chamber complex that we exploit throughout this thesis. These facts can be derived directly from the definition of the chamber complex.

1. Any chamber is exactly the intersection of all simplicial cones containing it.

2. For any $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A)$, the intersection of all simplicial cones containing $\mathbf{b}$ is a cone of the chamber complex – in particular if the intersection is d-dimensional, it is a chamber.

3. For a given chamber γ, if $\mathbf{b} \in \gamma^\circ$ (where γ° denotes the interior of γ), then γ is the intersection of all simplicial cones containing $\mathbf{b}$.

4. If a chamber γ of A intersects a simplicial cone σ of A d-dimensionally, then $\gamma \subseteq \sigma$.

5. Let $j \in \{1, \ldots, n\}$. If column $\mathbf{a}_j$ is in a chamber γ, then $\mathbf{a}_j$ is a ray generator of γ.

The vector partition function p_A can be described explicitly as a piecewise function whose domains are the chambers of A. The functions that are valid on the chambers are *quasi-polynomials*, which are finite sums of the form

$$q(z_1, \ldots, z_k) = \sum_{(i_1,\ldots,i_k)\in S} c_{i_1,\ldots,i_k}(z_1, \ldots, z_k) z_1^{i_1}, \ldots, z_k^{i_k}$$

where $S \subset \mathbb{Z}^k$ is finite and the $c_{i_1,\ldots,i_k}$ are non-zero periodic functions in $(z_1, \ldots, z_k)$. That is, there exist positive integers $n_1, \ldots, n_k$ such that $c_{i_1,\ldots,i_k}(z_1, \ldots, z_k) = c_{j_1,\ldots,j_k}(z'_1, \ldots, z'_k)$ whenever $z_\ell \equiv z'_\ell \mod n_\ell$ for $\ell = 1, \ldots, k$. The *degree* of the quasi-polynomial q is the maximum over the sums $i_1 + \cdots + i_k$, and the *period* of q is the minimal positive integer N such that $c_{i_1,\ldots,i_k}(z_1, \ldots, z_k) = c_{j_1,\ldots,j_k}(z_1, \ldots, z_k)$ whenever $i_\ell \equiv j_\ell \mod N$ for $\ell = 1, \ldots, k$ for all $(i_1, \ldots, i_k) \in S$. We remark that a quasi-polynomial in one variable is indeed a univariate quasi-polynomial as defined in Section 1.4, and also that a quasi-polynomial with period equal to one is just a polynomial. To indicate the quasi-polynomial associated to a chamber γ, we write p_A^γ.

The explicit characterization of the form of the vector partition function is due to Sturmfels, and we record it in the following theorem.

Theorem 1.5.3 (Sturmfels, 1994 [84])**.** *Let A be a $d \times n$ matrix of rank d. The vector partition function of A, p_A, is a piecewise quasi-polynomial of degree $n - d$ whose domains of quasi-polynomiality are the maximal cones (chambers) in the chamber complex of A.*

[3] In the literature chambers are usually defined to be the interiors of what we call chambers (i.e the maximal cells of the chamber complex). We use this terminology since we are, much more often than not, interested in the closed sets and not their interiors.

Remark 1.5.4. For each $\mathbf{b} \in \mathbb{Z}^d$, the Ehrhart quasi-polynomial $L_{\mathcal{P}}(t)$ associated to the polytope $\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is equal to $p_A(t\mathbf{b})$ (viewed as a function of t). Thus, we can draw intuition for the previous result from Ehrhart theory in order to understand the quasi-polynomial nature of p_A. Geometrically, for all $\mathbf{b}$ in the interior of a given chamber, the "general shape" of the polytope $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ is the same, and only shifts when we cross into another chamber.

Barvinok's algorithm [7] allows one to explicitly compute the piecewise quasi-polynomial p_A in polynomial time for fixed dimension n. Multiple implementations of Barvinok's algorithm exist: *Latte* [54] developed by De Loera, Hemmecke, Tauzer, and Yoshida can be used to (among many other things) compute the quasi-polynomial $p_A(t\mathbf{b})$ for any $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A)$ and is integrated in *Sagemath*; *Barvinok* [88], developed by Koeppe, Verdoolaege, and Woods can be used to compute the full piecewise quasi-polynomial p_A[4]. While Barvinok's algorithm is polynomial time for fixed dimension n, the problem of computing the vector partition function quickly becomes intractable as the dimension grows. For example, we were unable to compute the vector partition function associated to the 4×30 matrix $A^{3,3}$ described in Section 4.4.3.

The generating function formulation (due to Euler [36]) is in terms of the coefficient of the term $\mathbf{x}^{\mathbf{b}}$ in the Taylor series expansion of a product of geometric series:

$$p_A(\mathbf{b}) = [\mathbf{x}^{\mathbf{b}}] \prod_{j=1}^{n} \frac{1}{1 - \mathbf{x}^{\mathbf{a}_j}}, \tag{1.3}$$

with the convention that for vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, $\mathbf{u}^{\mathbf{v}}$ denotes the product $\prod_{i=1}^{d} u_i^{v_i}$. The rational function $\prod_{j=1}^{n} \frac{1}{1-\mathbf{x}^{\mathbf{a}_j}}$ is called the *vector partition generating function of A*.

We give the following example for two reasons: firstly it can be worked out by hand, and secondly it serves to motivate Lemma 2.3.3 in the following chapter.

Example 1.5.5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{N}^2$. We wish to compute the vector partition function

$$\begin{aligned} p_A(\mathbf{b}) &= \#\{\mathbf{x} \in \mathbb{N}^3 : A\mathbf{x} = \mathbf{b}\} \\ &= \#\{(x_1, x_2, x_3) \in \mathbb{N}^3 : x_1 + x_3 = b_1,\ x_2 + x_3 = b_2\}. \end{aligned}$$

For any choice of x_3 with $0 \leq x_3 \leq \min(b_1, b_2)$ there is a unique solution for x_1 and x_2 (namely $x_1 = b_1 - x_3,\ x_2 = b_2 - x_3$). Since there are $\min(b_1, b_2) + 1$ choices of x_3, we find

[4] The algorithm implemented by Koeppe, Verdoolaege, and Woods is called the *Barvinok-Woods* algorithm. It is based on the original formulation of Barvinok.

that

$$p_A(\mathbf{b}) = \min(b_1, b_2) + 1.$$

We can represent this as a piecewise quasi-polynomial (more precisely piecewise polynomial in this case):

$$p_A(\mathbf{b}) = \begin{cases} b_1 + 1 \text{ if } b_2 \geq b_1 \geq 0 \\ b_2 + 1 \text{ if } b_1 \geq b_2 \geq 0. \end{cases}$$

We now present a slightly different approach to this problem, the purpose of which is to motivate our approach in the following chapter. We note that throughout this thesis we use $\mathbf{1}$ to denote the all-ones vector, and $\mathbf{e}_i$ to denote the ith standard basis vector (in the following case since the dimension is 2, $\mathbf{1} = (1, 1), \mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)$). Consider a scenario in which we are given A as well as the two chambers of A

$$\sigma_1 := \{(b_1, b_2) \in \mathbb{R}^2 : b_1 \geq b_2 \geq 0\} = \{\lambda_1 \mathbf{e_2} + \lambda_2 \mathbf{1} : \lambda_1, \lambda_2 \geq 0\},$$

and

$$\sigma_2 := \{(b_1, b_2) \in \mathbb{R}^2 : b_2 \geq b_1 \geq 0\} = \{\lambda_1 \mathbf{e_1} + \lambda_2 \mathbf{1} : \lambda_1, \lambda_2 \geq 0\}.$$

Let us compute the quasi-polynomial $p_A^{\sigma_1}$ associated to the chamber σ_1. First note that x_1 may be viewed as a slack variable, so that

$$p_A^{\sigma_1}(\mathbf{b}) = \#\{(x_2, x_3) \in \mathbb{N}^2 : x_3 \leq b_1,\ x_2 + x_3 = b_2\} \tag{1.4}$$

for all $\mathbf{b} \in \mathbb{Z}^2 \cap \sigma_1$. Exploiting the fact that $\mathbf{b} \in \sigma_1$, we have $b_2 \leq b_1$, and so

$$\begin{aligned} &x_2 + x_3 = b_2 \\ \Longrightarrow\ &x_2 + x_3 \leq b_1 \\ \Longrightarrow\ &x_3 \leq b_1 \end{aligned}$$

and so the inequality $x_3 \leq b_1$ is not necessary in Eq. (1.4). In particular,

$$p_A^{\sigma_1}(\mathbf{b}) = \#\{(x_2, x_3) \in \mathbb{N}^2 : x_2 + x_3 = b_2\},$$

and so

$$p_A(\mathbf{b}) = p_B(b_2),$$

where $B = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Therefore, we have reduced the problem of computing $p_A^{\sigma_1}(\mathbf{b})$ for $\mathbf{b} \in \mathbb{N}^2$ to the coin exchange problem of computing $p_B(b_2)$ for $b_2 \in \mathbb{N}$. In fact, viewing $b_2 \in \mathbb{N}$ as a variable, the quasi-polynomial $p_B(b_2)$ is simply the number of partitions of b_2 into two non-negative integer parts x_2, x_3. That is,

$$p_B(b_2) = \binom{b_2 + 2 - 1}{2 - 1} = b_2 + 1.$$

One can apply the same approach to compute that $p_A^{\sigma_2}(\mathbf{b}) = p_B(b_1) = b_1 + 1$.

In the previous example, we draw attention to the fact that the chamber σ_1 (respectively σ_2) contains $\mathbf{e}_1$ ($\mathbf{e}_2$) as a ray generator, and that the quasi-polynomial p_A restricted to σ_1 (σ_2) does not depend on b_1 (b_2). In Chapter 2 we shall see how one can exploit this in order to compute quasi-polynomial formulae for particular chambers.

1.6 Symmetric polynomials

We mainly follow the classic text *Symmetric functions and Hall polynomials* [56] by I.G. Macdonald in this section.

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a weakly decreasing sequence of non-negative integers. The elements $\lambda_1, \ldots, \lambda_n$ of the partition are its *parts*, and the *length* of the partition, denoted $\ell(\lambda)$ is the number of non-zero parts of λ. Additionally, we say that λ is a partition of $N \in \mathbb{N}$, denoted $\lambda \vdash N$ if $\sum_{i=1}^{n} \lambda_i = N$. In this case, we also say that the *size* of λ, denoted $|\lambda|$, is N. For any positive integer n, $\delta^{(n)} := (n-1, n-2, \ldots, 1, 0)$ is the n^{th} *staircase partition*.

For independent variables $x_1, \ldots, x_n$, we consider a subring of $\mathbb{Z}[x_1, \ldots, x_n]$. Namely, we define Λ_n to be the subset of polynomials of $\mathbb{Z}[x_1, \ldots, x_n]$ that are invariant under permutation of the variables $x_1, \ldots, x_n$. The elements of Λ_n are called *symmetric polynomials*. We note that Λ_n is indeed a subring of $\mathbb{Z}[x_1, \ldots, x_n]$ since $1 \in \Lambda_n$ and the difference and product of a pair of symmetric polynomials are both symmetric polynomials themselves.

Example 1.6.1. The function $f(x_1, x_2) = x_1^2 x_2$ is not a symmetric polynomial in Λ_2 since $f(x_2, x_1) = x_1 x_2^2 \neq f(x_1, x_2)$. On the other hand $g(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$ is a symmetric polynomial in Λ_2.

The ring Λ_n has additional structure – it has a natural grading by polynomial degree

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$$

where Λ_n^k denotes the group of homogeneous symmetric polynomials of degree k along with the 0 polynomial. One may also view Λ_n as a module over $\mathbb{Z}$, for which there are six widely used bases in the literature. We first describe the monomial basis in order to establish the connection between partitions and symmetric polynomials.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length at most n, define

$$m_\lambda(x_1, \ldots, x_n) := \sum_{\alpha} \mathbf{x}^\alpha$$

where α runs over all distinct permutations of the parts of λ. For example, the function f in Example 1.6.1 is $m_{(2,1)}(x_1, x_2)$. We remark without proof that the set

$$\{m_\lambda(x_1, \ldots, x_n) : \ell(\lambda) \leq n\}$$

forms a basis for Λ_n that is called the *monomial basis*.

We now define another $\mathbb{Z}$-basis of Λ_n, the *Schur basis*, which plays an important role in Chapters 3 and 4. For a partition λ of length at most n, the *alternant* $a_\lambda(x_1, \ldots, x_n)$ is defined as

$$a_\lambda(x_1, x_2, \ldots, x_n) := \det (x_i^{\lambda_j})_{1 \leq i,j \leq n}.$$

The alternant is *skew-symmetric* - that is for all permutations $\sigma \in \mathfrak{S}_n$, we have

$$a_\lambda(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^{\mathrm{sgn}(\sigma)} a_\lambda(x_1, \ldots, x_n)$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of permutation σ. The set A_n of skew-symmetric polynomials is a $\mathbb{Z}$-module, of which the set of alternants $\{a_{\lambda+\delta} : \ell(\lambda) \leq n\}$ forms a basis. The *Schur polynomial* $s_\lambda(x_1, \ldots, x_n)$ is defined as the ratio of alternants

$$s_\lambda(x_1, \ldots, x_n) := \frac{a_{\lambda+\delta^{(n)}}(x_1, x_2, \ldots, x_n)}{a_{\delta^{(n)}}(x_1, x_2, \ldots, x_n)}. \tag{1.5}$$

The alternant $a_{\delta^{(n)}}$ is the Vandermonde determinant

$$\prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Additionally, for each $1 \leq i < j \leq n$, the polynomial $(x_i - x_j)$ divides $a_{\lambda+\delta^{(n)}}(x_1, x_2, \ldots, x_n)$ in $\mathbb{Z}[x_1, \ldots, x_n]$ since setting $x_j := x_i$ causes the matrix $(x_i^{\lambda_j})_{1 \leq i,j \leq n}$ to have two equal columns, and thus determinant zero. Therefore, the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is indeed a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$. In fact, $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial in Λ_n since it is the ratio of skew-symmetric polynomials.

Example 1.6.2. We compute the Schur polynomial $s_{(2,1,0)}(x_1, x_2, x_3)$:

$$\begin{aligned} s_{(2,1,0)}(x_1,x_2,x_3) &= \frac{\begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}} \\ &= \frac{x_1^4x_2^2 - x_1^2x_2^4 - x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_3^4 - x_2^2x_3^4}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \\ &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2. \end{aligned}$$

The following result is well-known, see for example [57].

Theorem 1.6.3. *The set of Schur polynomials*

$$\{s_\lambda(x_1,\ldots,x_n) : \ell(\lambda) \le n\} \tag{1.6}$$

forms a basis for the $\mathbb{Z}$*-module* Λ_n.

While the Schur polynomials form a basis for Λ_n, the problem of computing structure constants (the coefficients to express a given symmetric polynomial in the Schur basis) is not straightforward. In fact, the study of such coefficients provides some of the most rich and interesting problems in algebraic combinatorics. We consider two sets of coefficients that naturally arise in this manner in Chapter 3 (the Littlewood-Richardson coefficients) and Chapter 4 (the Kronecker coefficients).

Remark 1.6.4. In this section we have not discussed *symmetric functions* which can roughly be viewed similarly to symmetric polynomials with an infinite number of variables (so that polynomials are replaced with infinite sums). For this work, it will always be sufficient to work within Λ_n, for some n large enough.

Chapter 2

External chambers of vector partition functions

2.1 Summary of contribution

In this chapter we define *external columns*, which are certain distinguished columns of A. Our main result shows that (up to a lattice condition) for a chamber γ of A with k external columns, the quasi-polynomial p_A^γ can be obtained from a vector partition function whose dimension is k less than that of A. We also define a chamber of a vector partition function which we call an *external chamber*. Our main result applied in this case yields that the quasi-polynomial associated to such a chamber arises from a coin exchange problem. By considering this case in more detail, we are able to obtain negative binomial coefficient formulae in specific cases, as well as to classify when p_A^γ is polynomial. Unless explicitly stated, all of the results in this chapter are novel.

In Section 2.2, we introduce objects which we call external columns, rays, facets, and chambers. We then deduce some basic properties of these objects.

In Section 2.3, we prove our main result, Theorem 2.3.4, which states that the quasi-polynomial for an external chamber (that obeys some lattice conditions) arises from a coin exchange problem. We then use this result in order to show that the quasi-polynomial is also the Ehrhart quasi-polynomial associated to a single ray of the chamber (Theorem 2.3.5).

In Section 2.4, we show that if such a quasi-polynomial is actually polynomial, then it must actually be given by a negative binomial coefficient (Theorem 2.4.2). We then use this result to study a class of matrices called *unimodular* matrices for which we derive some results (Corollary 2.4.5 and Corollary 2.4.8).

In Section 2.5, we re-derive a known result (Theorem 2.5.3) involving linear factors of polynomials associated to certain chambers for unimodular matrices. Our aim is to suggest a generalization of this result (Conjecture 2.5.4) that can be used to compute linear factors of polynomials arising from Littlewood-Richardson coefficients. We address this connection in Section 3.8.

In Section 2.6, we consider two applications of these results. In 2.6.1, we give an application to the exact enumeration of multigraphs with a given degree sequence. In 2.6.2 we show that one can handily apply Theorem 2.5.3 to re-derive known results related to Kostant's partition function.

2.2 External columns and chambers

In this section, we introduce the main objects of study of this chapter: external columns, rays, facets, and chambers. Additionally, we describe some properties of these objects that will be necessary for our main result, Theorem 2.3.4. Throughout this chapter A will denote a $d \times n$ matrix with integer entries, of rank d, and satisfying

$$\ker(A) \cap \mathbb{R}^n_{\geq 0} = \{\mathbf{0}\}.$$

We denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

2.2.1 External columns

Here, we often refer to a *chamber of the matrix* A instead of saying a *chamber in the chamber complex* of A for short. Given a $d \times n$ matrix M, by $M_{\hat{i},\hat{j}}$ we denote the $(d-1) \times (n-1)$ submatrix obtained by removing row i and column j. We also denote by $M_{\hat{i},\cdot}$ the $(d-1) \times n$ matrix obtained by removing row i, and by $M_{\cdot,\hat{j}}$ the $d \times (n-1)$ matrix obtained by removing column j. Similarly for a vector $\mathbf{v}$, by $\mathbf{v}_{\hat{i}}$, we denote $\mathbf{v}$ with the ith coordinate removed.

Definition 2.2.1. Let $\mathbf{a}_j$ be the jth column of A for some $j \in \{1, \ldots, n\}$. We define $\mathbf{a}_j$ to be an *external column* of A if $\mathbf{a}_j \notin \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$.

If $\mathbf{a}_j$ is an external column of A, then the cone $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ is a proper subset of $\mathrm{pos}_{\mathbb{R}}(A)$.

Proposition 2.2.2. *Let* $\mathbf{a}_j$ *be a column of* A *for some* $j \in \{1, \ldots, n\}$*. Then* $\mathbf{a}_j$ *is an external column of* A *if and only if* $\mathbf{a}_j$ *is a ray generator of* $\mathrm{pos}_{\mathbb{R}}(A)$ *and no other column of* A *is in the span of* $\mathbf{a}_j$*.*

Proof. Let $\mathbf{a}_j$ be an external column of A. Since $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}}) \neq \mathrm{pos}_{\mathbb{R}}(A)$, $\mathbf{a}_j$ is part of a minimal generating set of $\mathrm{pos}_{\mathbb{R}}(A)$, and thus is a ray generator of $\mathrm{pos}_{\mathbb{R}}(A)$. Also, no other column $\mathbf{c}$ of A is in its span. Otherwise, either $\mathbf{c}$ is a positive multiple of $\mathbf{a}_j$ in which case $\mathbf{a}_j \in \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ or $\mathbf{c}$ is a negative multiple of $\mathbf{a}_j$ in which case $\ker(A) \cap \mathbb{R}^d_{\geq 0} \neq \{\mathbf{0}\}$.

Conversely, let $\mathbf{a}_j$ be a ray generator of $\mathrm{pos}_{\mathbb{R}}(A)$ with no other column of A in its span. Then $\mathbf{a}_j$ is in a minimal generating set of $\mathrm{pos}_{\mathbb{R}}(A)$, and since no other column of A is in its span, $\mathbf{a}_j \notin \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$. Therefore, $\mathbf{a}_j$ is an external column of A. □

A straightforward consequence of this proposition is that external columns of A lie on facets of $\mathrm{pos}_{\mathbb{R}}(A)$. The following proposition will be useful in the following section, and is also a straightforward consequence of Proposition 2.2.2.

Proposition 2.2.3. *Let $\mathbf{a}_j$ be an external column of A for some $j \in \{1, \ldots, n\}$, and let $\mathrm{pos}_{\mathbb{R}}(A_s)$ for some $s \subseteq \{1, \ldots, n\}$. If $\mathbf{a}_j \in \mathrm{pos}_{\mathbb{R}}(A_s)$, then $\mathbf{a}_j$ is a ray generator of $\mathrm{pos}_{\mathbb{R}}(A_s)$. In particular, $j \in s$.*

One can also define the external columns in terms of the vector partition function p_A. They are exactly the columns of A for which $p_A(t\mathbf{a}_j) \leq 1$ for all non-negative integers t (i.e for which the Ehrhart quasi-polynomial $p_A(t\mathbf{a}_j)$ has degree 0).

2.2.2 External chambers

We now introduce external chambers, the main objects of study in this chapter.

Definition 2.2.4. Let γ be a chamber of A. We define γ to be an *external chamber* of A if all but one ray of γ is generated by an external column of A. Further, we define the rays that are generated by external columns of A to be *external rays* of γ and the other ray to be the *internal ray* of γ. Moreover, we define any generator of an external ray to be an *external ray generator* of γ and any ray generator of an internal ray to be an *internal ray generator* of γ.

Remark 2.2.5. The reader may wonder what happens in the case that γ is generated solely by external columns. This case is degenerate: A has a single chamber and $p_A(\mathbf{b}) \leq 1$ for all $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A) \cap \mathbb{Z}^d$.

Example 2.2.6. Consider the following matrix

$$A^{2,2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

given in [62] that is a member of a family of matrices $A^{m,n}$ that we study in Chapter 4 in our work on Kronecker coefficients. Call its columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. The chamber complex of $A^{2,2}$ is defined by the three chambers

$$\gamma_1 = \mathrm{pos}_{\mathbb{R}}\left(\mathbf{a}_1, \mathbf{a}_3\right), \quad \gamma_2 = \mathrm{pos}_{\mathbb{R}}\left(\mathbf{a}_3, \mathbf{a}_4\right), \quad \gamma_3 = \mathrm{pos}_{\mathbb{R}}\left(\mathbf{a}_2, \mathbf{a}_3\right).$$

The external columns of T are $\mathbf{a}_1$ and $\mathbf{a}_2$, and so we see that γ_1 and γ_3 are external chambers while γ_2 is not. The columns and chambers of $A^{2,2}$ are depicted in Figure 2.1.

The next proposition follows directly from the definition of the chamber complex of A.

Proposition 2.2.7. *If a chamber γ of A contains d linearly independent columns of A, say $\mathbf{a}_1, \ldots, \mathbf{a}_d$ then $\gamma = \mathrm{pos}_{\mathbb{R}}\left(\mathbf{a}_1, \ldots, \mathbf{a}_d\right)$.*

Proposition 2.2.8. *External chambers of A are simplicial.*

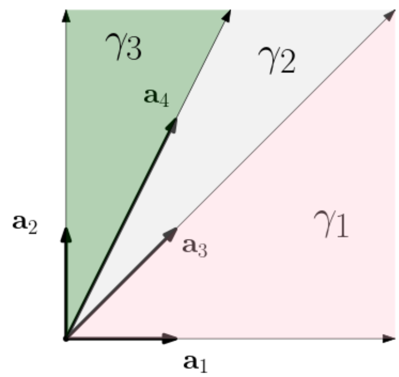


Figure 2.1: The columns and chambers of $A^{2,2}$.

Proof. Let γ be an external chamber of A. If γ is not simplicial, then it must contain at least d external columns, say $\mathbf{a}_1, \ldots, \mathbf{a}_k$ for some $k \geq d$. By definition, γ is contained in some simplicial cone $\mathrm{pos}_{\mathbb{R}}(A_s)$ for some $s \subseteq [n]$ with $|s|= \mathrm{rank}(A_s) = d$. By Proposition 2.2.3, $\{1, \ldots, k\} \subseteq s$, so $k \leq d$. Thus $k = d$, and we find that

$$\mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_d) \subseteq \gamma \subseteq \mathrm{pos}_{\mathbb{R}}(A_s) = \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_d). \tag{2.1}$$

Therefore $\gamma = \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_d)$ which is a contradiction since γ must have an internal ray generator. $\square$

Remark 2.2.9. In Theorem 2.3.4, we make a statement about simplicial chambers of A which contain external columns (and satisfy a lattice condition). At one point of this work we suspected that it is sufficient for a chamber of A to contain at least one external column in order to be simplicial. This is not the case as the following counterexample illustrates. Consider the matrix C below

$$C = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 & 2 \\ 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using *Barvinok*, we compute that the 4-dimensional cone

$$\gamma := \mathrm{pos}_{\mathbb{R}} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 19 \\ 4 \\ 7 \\ 0 \end{bmatrix} \right) \tag{2.2}$$

is a chamber of C, and contains the external column $(2, 1, 1, 1)$ of C. However, γ is not simplicial as the 6 generators given in Eq. (2.2) form a minimal generating set of γ which is 4-dimensional.

The following lemma will prove useful in terms of computing external chambers. Additionally, it plays a key role in the results of Section 2.3. Figure 2.2 provides an illustration of some of the elements of the proof, and may be a useful visual guide.

Lemma 2.2.10. *Let $\mathbf{a}_j$ be an external column of A for some $j \in \{1, \ldots, n\}$. Then any chamber of A containing $\mathbf{a}_j$ has a single facet f not containing $\mathbf{a}_j$. Moreover, if H is the supporting hyperplane of f, then H separates $\mathbf{a}_j$ and $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$.*

Proof. Since $\mathbf{a}_j$ is an external column of A, by definition $\mathbf{a}_j \notin \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$. Let $f_1, \ldots, f_k$ denote the faces of $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ of dimension $d-1$. We note that these faces are not necessarily facets of $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$: if $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ is $(d-1)$-dimensional, then $k = 1$ and $f_1 = \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$. However, the proof proceeds in the same way regardless of whether $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ is of dimension d or $d-1$.

For $i = 1, \ldots, k$, the faces f_i can be described as $f_i = \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}}) \cap H_i$ for some supporting hyperplanes $H_1, \ldots, H_k \subset \mathbb{R}^d$. Without loss of generality, let $H_1, H_2, \ldots, H_\ell$ be the set of hyperplanes that separate $\mathbf{a}_j$ and the cone $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ and also do not contain $\mathbf{a}_j$. For $i = 1, \ldots, \ell$, let κ_i be the cone generated by $\mathbf{a}_j$ and the columns of $A_{\cdot,\hat{j}}$ that generate f_i. Since f_i is generated by columns of $A_{\cdot,\hat{j}}$ spanning H_i, it follows that κ_i is a union of simplicial cones of A – that is, $\kappa_i = \cup_{s \in S} \mathrm{pos}_{\mathbb{R}}(A_s)$ for some $S \subset \mathcal{P}(\{1, \ldots, n\})$ with each $s \in S$ satisfying $|s| = \mathrm{rank}(A_s) = d$.

Let γ be a chamber of A containing $\mathbf{a}_j$. Then γ has a d-dimensional intersection with a simplicial cone of A that is contained in κ_{i^*} for some $1 \leq i^* \leq \ell$, and so $\gamma \subseteq \kappa_{i^*}$. Let τ be a facet of γ not contained in f_{i^*}, so τ is not contained in $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$. By the definition of the chamber complex, τ must be contained in a facet τ' of a simplicial cone σ of A. Thus, we can write $\tau' = \sigma \cap H'$ for some hyperplane $H' \subseteq \mathbb{R}^d$. Then $\gamma \cap H'$ defines a proper face of γ containing the facet τ, and so $\tau = \gamma \cap H'$. Since τ' is generated by columns of A and not contained in $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$, $\mathbf{a}_j$ is a ray generator of τ'. Thus, $\mathbf{a}_j \in H'$, and since $\mathbf{a}_j \in \gamma$ by assumption, $\mathbf{a}_j \in \tau$. Finally, we see that the unique facet of γ not containing $\mathbf{a}_j$ is $\gamma \cap H_{i^*} = \gamma \cap f_{i^*}$ and H_{i^*} is a separating hyperplane of $\mathbf{a}_j$ and $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$. $\square$

We define the cone generated by the external ray generators of an external chamber γ to be an *external facet* of A. The following result illustrates how to identify external facets and compute the corresponding external chambers. We impose the somewhat artificial condition that A should have at least two chambers to avoid the degenerate case referred to in Remark 2.2.5.

Proposition 2.2.11 (Constructing external chambers)**.** *Assume that the chamber complex of A contains at least two chambers, and let f be a $(d-1)$-dimensional cone in the chamber*

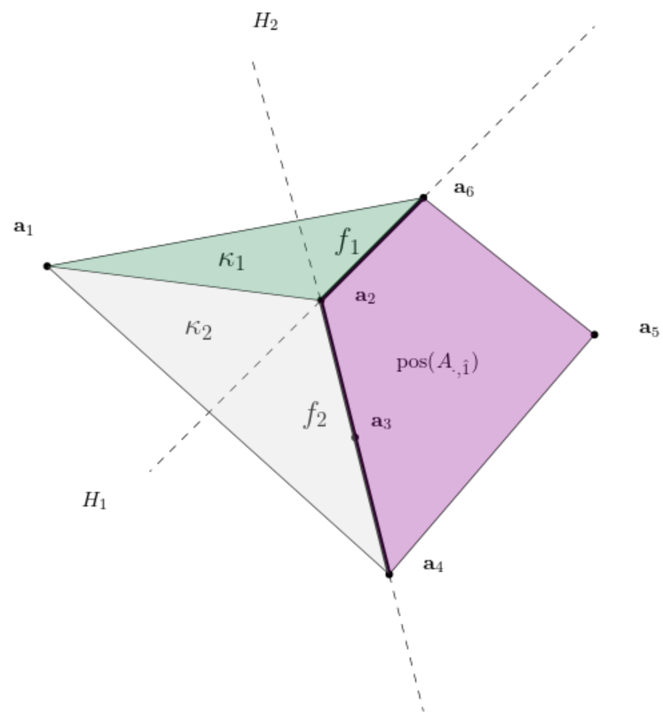


Figure 2.2: A sketch of some of the elements in the proof of Lemma 2.2.10. Here A is a 3×6 matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_6$. The polytope with vertices labelled $\mathbf{a}_1, \mathbf{a}_4, \mathbf{a_5}, \mathbf{a_6}$ represents a 2-dimensional cross-section of the 3-dimensional cone $\text{pos}_{\mathbb{R}}(A)$. Each of the points labelled by a column $\mathbf{a}_j$ of A represent the ray generated by $\mathbf{a}_j$. From the picture we see that $\mathbf{a}_1, \mathbf{a}_4, \mathbf{a_5}, \mathbf{a_6}$ are the external columns of A. The cone $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$ is shaded in dark magenta. The hyperplanes H_1 and H_2 separate $\mathbf{a}_1$ and $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$. Any chamber γ containing $\mathbf{a}_1$ must be contained in one of the cones κ_1 (shaded dark green) or κ_2 (shaded light grey) and the unique facet of γ not containing $\mathbf{a}_1$ is equal to $\gamma \cap f_1$ or $\gamma \cap f_2$.

complex. Then f is an external facet of A if and only if f is a facet of $\text{pos}_{\mathbb{R}}(A)$ containing exactly $d-1$ columns of A. Moreover, if the columns of A generating f are $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$, then the unique external chamber containing f is

$$\gamma := \bigcap_{k=0}^{n-d} \text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{a}_{d+k}).$$

Proof. We begin with the forward direction. Suppose that f is an external facet of A, and assume towards a contradiction that f is not a facet of $\text{pos}_{\mathbb{R}}(A)$ containing exactly $d-1$ columns of A. Since f is an external facet of A, it is a facet of an external chamber γ' of A. Since γ' is simplicial, it has d facets, say $f_1, \ldots, f_{d-1}, f_d = f$. Assume moreover that f_i is the unique facet of γ' not containing $\mathbf{a}_i$ for each $i = 1, \ldots, d-1$. Let $\boldsymbol{\iota}_1, \ldots, \boldsymbol{\iota}_d$ be the inner facet normals (with respect to γ') corresponding to the facets $f_1, \ldots, f_d$, and $H_1, \ldots, H_d$ be the corresponding supporting hyperplanes. Since f is not a facet of A containing exactly $d-1$ columns of A, there are two options to consider

1. f is not a facet of $\text{pos}_{\mathbb{R}}(A)$,

2. f is a facet of $\text{pos}_{\mathbb{R}}(A)$, but contains more than $d-1$ columns of A.

In the first case there is some column $\mathbf{a}$ of A with $\boldsymbol{\iota}_d \cdot \mathbf{a} \leq 0$ and $\mathbf{a} \notin \gamma$. By Lemma 2.2.10, the hyperplane H_j separates the column $\mathbf{a}_j$ from the cone $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{j}})$ for each $j = 1, \ldots, d-1$. Therefore, $\boldsymbol{\iota}_j \cdot \mathbf{a} \leq 0$ for each $j = 1, \ldots, d-1$. Then $-\mathbf{a} \in \gamma'$, since $\boldsymbol{\iota}_j \cdot (-\mathbf{a}) \geq 0$ for each $j = 1, \ldots, d$. Now, let $\text{pos}_{\mathbb{R}}(A_s)$ be a simplicial cone of A for some $s \subseteq \{1, \ldots, n\}$ with $|s| = \text{rank}(A_s) = d$ so that $\gamma' \subseteq \text{pos}_{\mathbb{R}}(A_s)$. Since $-\mathbf{a} \in \gamma'$, $-\mathbf{a} \in \text{pos}_{\mathbb{R}}(A_s)$, and so $-\mathbf{a} = \sum_{i \in s} \lambda_i \mathbf{a}_i$ for some $\lambda_i \geq 0$. But then, $\sum_{i \in s} \lambda_i \mathbf{a}_i + \mathbf{a} = \mathbf{0}$, and so $\ker(A) \cap \mathbb{R}^d_{\geq 0} \neq \{\mathbf{0}\}$. This is a contradiction. Therefore, f is indeed a facet of $\text{pos}_{\mathbb{R}}(A)$, and f contains exactly the $d-1$ columns of A, $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$.

In the second case, there is some column $\mathbf{a} \in f$ with $\mathbf{a} \notin \{\mathbf{a}_1, \ldots \mathbf{a}_{d-1}\}$. However, since $\mathbf{a}$ is a column of A, it generates a 1-dimensional cone of the chamber complex of A, and so $f = \text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots \mathbf{a}_{d-1})$ cannot be a cone of the chamber complex. This contradicts the fact that f is an external facet.

We now prove the reverse direction. If f is a facet of $\text{pos}_{\mathbb{R}}(A)$ containing exactly $d-1$ columns of A, then each of these columns is a ray generator of $\text{pos}_{\mathbb{R}}(A)$ and no pair is linearly dependent. Therefore, each of $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ are external columns of A.

Finally, we show that γ is indeed a chamber of A. First note that none of the columns $\mathbf{a}_d, \ldots, \mathbf{a}_n$ lie on f, and so $\text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{a}_{d+k})$ is a simplical cone of A for each $k = 0, \ldots, n-d$. Therefore γ is the intersection of simplicial cones, and since each of these simplicial cones lie on the same side of the facet f, the cone γ must be d-dimensional. Consider the point

$$\mathbf{b} := \mathbf{a}_1 + \cdots + \mathbf{a}_{d-1}. \tag{2.3}$$

Since $\{\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}\}$ is a linearly independent set, and f is a facet of $\text{pos}_{\mathbb{R}}(A)$, the formulation of (2.3) is the unique way to represent $\mathbf{b}$ as a $\mathbb{N}$-linear combination of the columns of A. Therefore, any simplicial cone $\text{pos}_{\mathbb{R}}(A_s)$ of A containing $\mathbf{b}$ must contain each of the external columns $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$. Moreover, $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ are each ray generators of $\text{pos}_{\mathbb{R}}(A_s)$ by Proposition 2.2.3. Therefore, γ is a chamber of A since it is a d-dimensional cone obtained as the intersection of all simplicial cones containing $\mathbf{b}$. Furthermore, γ is the unique external chamber containing f, since any other d-dimensional cone containing $\mathbf{b}$ and obtained by an intersection of simplicial cones of A (necessarily containing all of $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ as ray generators) must contain γ as a subset.

By Lemma 2.2.10, there is a unique facet of γ not containing $\mathbf{a}_i$ for each $1 \leq i \leq d-1$, and so γ is simplicial. Therefore, $\gamma = \text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{v})$ for some ray generator $\mathbf{v}$. Since A has at least two chambers, it must have some column $\mathbf{c} \notin \gamma$. As well, $\mathbf{c} \notin f$, so $\gamma' := \text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{c})$ is a simplicial cone of A, and $\gamma \subsetneq \gamma'$. Therefore, it follows that $\mathbf{v} \in \text{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{c})$ and so $\mathbf{v}$ is an internal ray generator for γ. Thus, γ is an external chamber of A and f is an external facet of A. $\square$

The previous result allows us to compute external chambers without having to compute the entire chamber complex of A, which can be computationally intensive. We note however that in some cases there are no external chambers. For example, for the matrix

$$K_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

the chamber complex of K_4 has 48 chambers, none of which are external. The matrix K_4 is part of a family of matrices associated to Kostant's partition. We study the associated vector partition functions in Section 2.6.2.

Finally, we remark that external facets of A are exactly the facets f of $\mathrm{pos}_{\mathbb{R}}(A)$ for which $p_A(\mathbf{b}) \leq 1$ for all $\mathbf{b} \in f$.

2.2.3 A vector partition function preserving transformation

We now prove some results that allow us to transform the matrix A while preserving the vector partition function (up to an appropriate change of variables) and the structure of the chamber complex. We use these results in Section 2.3 in order to transform A into a form well-suited for analysis (described in Lemma 2.3.3).

Proposition 2.2.12. *Let $M \in \mathbb{Q}^{d\times d}$ be an invertible matrix with integer entries. Then $p_A(\mathbf{b}) = p_{MA}(M\mathbf{b})$ for all $\mathbf{b} \in \mathbb{Z}^d$.*

Proof. The matrices A and M both have integer entries, and so MA also has integer entries. Since M is invertible, MA also has rank d, and $\ker(A) = \ker(MA)$. Therefore p_{MA} is well defined, and since $A\mathbf{x} = \mathbf{b} \iff MA\mathbf{x} = M\mathbf{b}$, it follows that $p_A(\mathbf{b}) = p_{MA}(M\mathbf{b})$ as required. □

For a cone $\sigma \subseteq \mathbb{R}^d$ and invertible matrix $M \in \mathbb{Q}^{d\times d}$, define the cone $M\sigma := \{M\mathbf{b} : \mathbf{b} \in \sigma\}$. We note that $\{\mathbf{u}_1, \dots \mathbf{u}_k\}$ is a generating set of σ if and only if $\{M\mathbf{u}_1, \dots M\mathbf{u}_k\}$ is a generating set of $M\sigma$.

Proposition 2.2.13. *Let $M \in \mathbb{Q}^{d\times d}$ be an invertible matrix. The cone γ is a chamber of A if and only if $M\gamma$ is a chamber of MA. Moreover, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a minimal generating set of γ if and only if $\{M\mathbf{u}_1, \dots, M\mathbf{u}_k\}$ is a minimal generating set of $M\gamma$.*

Proof. Assume that γ is a chamber of A. We show that $M\gamma$ is indeed a chamber of MA. Since γ is a chamber of A, it is the intersection of all simplicial cones containing it, so for some

$$S := \{s \subseteq [n] : |s| = \mathrm{rank}(A_s) = d, \gamma \subseteq \mathrm{pos}_{\mathbb{R}}(A_s)\} \tag{2.4}$$

we can write γ as

$$\gamma = \bigcap_{s \in S} \mathrm{pos}_{\mathbb{R}}(A_s).$$

For any $s \subseteq [n]$, the cone $\mathrm{pos}_{\mathbb{R}}(A_s)$ is a simplicial cone of A if and only if $\mathrm{pos}_{\mathbb{R}}(MA_s)$ is a simplicial cone of MA since $\mathrm{rank}(A_s) = \mathrm{rank}\left((MA)_s\right)$. Further, $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A_s)$ if and only if $M\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}\left((MA)_s\right)$ for all $\mathbf{b} \in \mathbb{R}^d$. Thus,

$$M\gamma = \bigcap_{s \in S} \mathrm{pos}_{\mathbb{R}}((MA)_s).$$

The cones $\mathrm{pos}_{\mathbb{R}}(A_s)$ with $s \in S$ are also the exact set of simplicial cones of A containing $\mathbf{b}$ for any point $\mathbf{b} \in \gamma^\circ$. Therefore, the cones $\mathrm{pos}_{\mathbb{R}}((MA)_s)$ are exactly the set of simplicial cones of MA containing $M\mathbf{b} \in M\gamma^\circ$. As γ is d-dimensional, so is $M\gamma$, and so $M\gamma$ is a chamber of MA (since it is the d-dimensional intersection of simplicial cones containing a point in $\mathrm{pos}_{\mathbb{R}}(MA)$).

Since M is invertible, the reverse implication also holds – that is, if $M\gamma$ is a chamber of MA, then γ is a chamber of A.

Finally, since generating sets of γ map to generating sets of $M\gamma$ (and vice-versa), minimal generating sets must also map to minimal generating sets (and vice-versa). Therefore, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a minimal generating set of γ if and only if $\{M\mathbf{u}_1, \ldots, M\mathbf{u}_k\}$ is a minimal generating set of $M\gamma$ as required.

□

Remark 2.2.14. The previous proposition does not always hold if we replace "minimal generating set" with "minimal ray generators". In the case that γ is a simplicial cone with minimal ray generators $\mathbf{v}_1, \ldots, \mathbf{v}_d$, then $M\mathbf{v}_1, \ldots, M\mathbf{v}_d$ are minimal ray generators of $M\gamma$ if and only if M is invertible over $\mathbb{Z}$ (equivalently $\det(M) = \pm 1$).

Proposition 2.2.15. *Let $M \in \mathbb{Q}^{d\times d}$ be an invertible matrix. Let $\mathbf{a}_j$ be a column of A for some $j \in \{1, \ldots, n\}$. Then $\mathbf{a}_j$ is an external column of A if and only if $M\mathbf{a}_j$ is an external column of MA.*

Proof. Assume without loss of generality that $j = 1$. Points $\mathbf{b}$ in $\mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$ map to points in $\mathrm{pos}_{\mathbb{R}}(MA_{\cdot,\hat{1}})$ under the invertible mapping $\mathbf{b} \mapsto M\mathbf{b}$:

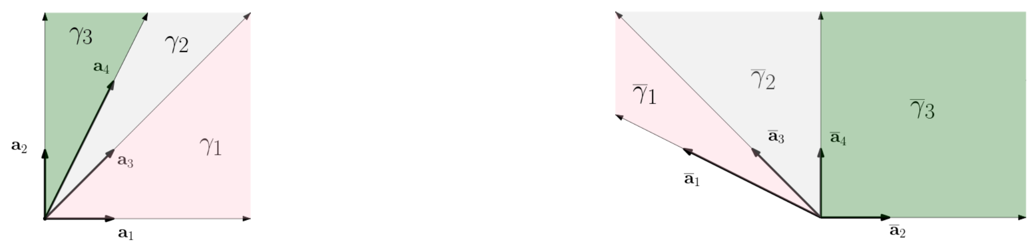
$$\begin{aligned}
\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}}) &\iff \mathbf{b} = \sum_{i=2}^{n} \lambda_i \mathbf{a}_i && \text{(for } \lambda_2, \ldots, \lambda_n \geq 0\text{)} \\
&\iff M\mathbf{b} = \sum_{i=2}^{n} \lambda_i M\mathbf{a}_i \\
&\iff M\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(MA_{\cdot,\hat{1}})
\end{aligned}$$

Therefore $\mathbf{a}_1 \notin \mathrm{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$ if and only if $M\mathbf{a}_1 \notin \mathrm{pos}_{\mathbb{R}}(MA_{\cdot,\hat{1}})$, and so $\mathbf{a}_1$ is an external column if and only if $M\mathbf{a}_1$ is an external column. $\square$

Proposition 2.2.16. *Let $M \in \mathbb{Q}^{d\times d}$ be an invertible matrix. Then γ is an external chamber of A if and only if $M\gamma$ is an external chamber of MA.*

Proof. This follows immediately from Propositions 2.2.13 and 2.2.15. $\square$

The following example illustrates how to compute the quasi-polynomial associated to the chamber γ_3 of the matrix $A^{2,2}$ given in Example 2.2.6 by hand: we first apply an appropriate transformation M so that the external column of $M\gamma_3$ is a standard basis vector, allowing us to reduce dimension by removing a redundant equation (as in Example 1.5.5). As we shall see in Section 2.3, this process can be applied to any simplicial chamber γ containing external columns, and the appropriate transformation M is related to the dual cone $\gamma^\vee$.



(a) The columns and chambers of $A^{2,2}$. (b) The columns and chambers of $\overline{A^{2,2}}$.

Figure 2.3: An illustration of the transformation $A^{2,2} \mapsto MA^{2,2}$ on the columns and chambers.

Example 2.2.17. Let $M \in \mathbb{Q}^{2\times 2}$ be the invertible matrix

$$M = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},$$

and recall the matrix $A^{2,2}$ from Example 2.2.6 given explicitly below

$$A^{2,2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

We reproduce an illustration of the columns and chambers of $A^{2,2}$ in Figure 2.3a. Consider the matrix $\overline{A^{2,2}} := MA^{2,2}$:

$$\overline{A^{2,2}} = \begin{bmatrix} -2 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

and the vector

$$\overline{\mathbf{b}} = M\mathbf{b} = \begin{bmatrix} b_2 - 2b_1 \\ b_1 \end{bmatrix}.$$

Denote the columns of $\overline{A^{2,2}}$ by $\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3, \bar{\mathbf{a}}_4$, noting that $\bar{\mathbf{a}}_j = M\mathbf{a}_j$ for each $j = 1, 2, 3, 4$. The chambers of $\overline{A^{2,2}}$ are

$$\overline{\gamma}_1 = \mathrm{pos}_{\mathbb{R}}\left(\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_3\right), \quad \overline{\gamma}_2 = \mathrm{pos}_{\mathbb{R}}\left(\bar{\mathbf{a}}_3, \bar{\mathbf{a}}_4\right), \quad \overline{\gamma}_3 = \mathrm{pos}_{\mathbb{R}}\left(\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_4\right).$$

The columns and chambers of $\overline{A^{2,2}}$ are illustrated in Figure 2.3b. One should check that $\overline{\gamma}_j = M\gamma_j$ for each $j = 1, 2, 3$. Recall also that the external columns of $A^{2,2}$ are $\mathbf{a}_1$ and $\mathbf{a}_2$, and that its external chambers are γ_1 and γ_3. As Figure 2.3b illustrates, the external columns of $\overline{A^{2,2}}$ are $\bar{\mathbf{a}}_1 = M\mathbf{a}_1$ and $\bar{\mathbf{a}}_2 = M\mathbf{a}_2$, and its external chambers are $\overline{\gamma}_1 = M\gamma_1$ and $\overline{\gamma}_3 = M\gamma_3$. As a whole, Figure 2.3 illustrates the correspondence between columns and chambers of $A^{2,2}$ and $\overline{A^{2,2}}$.

The chamber γ_3 of $A^{2,2}$ has now been mapped to the chamber $\overline{\gamma}_3$ of $\overline{A^{2,2}}$ which is simply the positive quadrant (defined by $\bar{b}_1, \bar{b}_2 \geq 0$). Additionally, the external column of γ_3 has been mapped to the standard basis vector $\mathbf{e}_1$.

We now utilize the same approach as in Example 1.5.5 in order to compute the quasi-polynomial $p_A^{\gamma_3}$. That is: we observe that one of the equations arising from $A\mathbf{x} = \mathbf{b}$ can be written as an inequality by noting that one of the x_i is a slack variable, and then we show that this inequality is implied when $\mathbf{b}$ is in the appopriate chamber.

Consider some point $\mathbf{b} \in \gamma_3 \cap \mathbb{Z}^2$. Any solution $\mathbf{x} \in \mathbb{N}^4$ to the equation $A^{2,2}\mathbf{x} = \mathbf{b}$, also satisfies $\overline{A^{2,2}}\mathbf{x} = \overline{\mathbf{b}}$, and so

$$\begin{aligned} \bar{b}_1 &= -2x_1 - x_2 + x_4 \\ \bar{b}_2 &= x_1 + x_2 + x_3 \end{aligned}$$

and since $x_4 \geq 0$, it can be viewed as a slack variable, so the number $\mathbf{x} \in \mathbb{N}^4$ satisfying the previous pair of equations is equal to the number of $(x_1, x_2, x_3) \in \mathbb{N}^3$ satisfying:

$$\bar{b}_1 \geq -2x_1 - x_2 \tag{2.5}$$

$$\bar{b}_2 = x_1 + x_2 + x_3. \tag{2.6}$$

Since $\mathbf{b} \in \gamma_3$, it follows that $\overline{\mathbf{b}} \in \overline{\gamma}_3$, which is the positive quadrant

$$\bar{b}_1 \geq 0 \tag{2.7}$$

$$\bar{b}_2 \geq 0 \tag{2.8}$$

and so, we see that Inequality (2.7) implies Inequality (2.5). In other words, for $\mathbf{b} \in \gamma_3 \cap \mathbb{Z}^2$, we find that

$$\begin{aligned} p_A(\mathbf{b}) &= \#\{\mathbf{x} \in \mathbb{N}^4 : A^{2,2}\mathbf{x} = \mathbf{b}\} \\ &= \#\{\mathbf{x} \in \mathbb{N}^4 : \overline{A^{2,2}}\mathbf{x} = \overline{\mathbf{b}}\} \\ &= \#\{(x_1, x_2, x_3) \in \mathbb{N}^3 : x_1 + x_2 + x_3 = \overline{b}_2\} \end{aligned}$$

which (as in Example 1.5.5) is a coin exchange problem. In this case, we see that $p_A^{\gamma_3}(\mathbf{b})$ is just the number of ways of partitioning $\overline{b}_2$ into three non-negative integer parts x_1, x_2, x_3 - and so:

$$\begin{aligned} p_A^{\gamma_3}(\mathbf{b}) &= \binom{\overline{b}_2 + 2}{2} \\ &= \binom{b_1 + 2}{2}. \end{aligned}$$

As we shall see in the following section, the procedure illustrated in Example 2.2.17 forms the essence of our approach. That is: given an external chamber γ of A (satisfying some lattice requirements), we construct an invertible matrix $M \in \mathbb{Q}^{d \times d}$ with integer entries (we call this matrix, described in the preamble to Theorem 2.3.5, the dual ray matrix) that maps each of the external columns of A in γ to positive multiples of the standard basis vectors. From this form, each of the corresponding variables can be viewed as slack variables in order to obtain some $1 \times (n - d + 1)$ matrix B with the property that

$$\begin{aligned} \#\{\mathbf{x} \in \mathbb{N}^d : A\mathbf{x} = \mathbf{b}\} &= \#\{\mathbf{x} \in \mathbb{N}^d : MA\mathbf{x} = M\mathbf{b}\} \\ &= \#\{\mathbf{x} \in \mathbb{N}^d : B\mathbf{x} = (M\mathbf{b})_d\}. \end{aligned}$$

This reduces the problem of computing the quasi-polynomial p_A^{γ} associated to the external chamber to that of solving a coin exchange problem.

2.3 Dimension reduction and determinantal formula

In this section, we consider chambers of A that contain external columns. Up to a lattice condition, we show that the quasi-polynomial p_A^{γ} for such a chamber γ can be obtained via a vector partition function of lower dimension. In particular, if γ has k external columns, then (up to a change of variables) $p_A^{\gamma} = p_B^{\gamma'}$ for a matrix B of k fewer rows and columns than A, and chamber γ' of B. When this result is applied to external chambers, we find that B has a single row, so that p_A^{γ} is obtained from a coin exchange problem. As a consequence, such a p_A^{γ} is a univariate quasi-polynomial. Indeed, p_A^{γ} is precisely the Ehrhart quasi-polynomial associated to the internal ray of γ.

Let $\mathcal{L}(A)$ denote the lattice generated by the columns of A, and $\text{pos}_{\mathbb{N}}(A)$ denote the affine semigroup generated by the columns of A - that is,

$$\mathcal{L}(A) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a}_i : \lambda_i \in \mathbb{Z} \right\},$$
$$\text{pos}_{\mathbb{N}}(A) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{a}_i : \lambda_i \in \mathbb{N} \right\}.$$

Often we informally identify a matrix with its set of columns, so that we may write $\text{pos}_{\mathbb{N}}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ in lieu of $\text{pos}_{\mathbb{N}}(A)$.

Definition 2.3.1. Let $s \subseteq \{1, \ldots, n\}$. If

$$\text{pos}_{\mathbb{N}}(A_s) = \mathcal{L}(A) \cap \text{pos}_{\mathbb{R}}(A_s)$$

then we say that the set of columns $\{\mathbf{a}_j : j \in s\}$ is *A-lattice minimal.*[1]

Clearly, the condition $\text{pos}_{\mathbb{N}}(A_s) \subseteq \mathcal{L}(A) \cap \text{pos}_{\mathbb{R}}(A_s)$ is always satisfied, so to verify A-lattice minimality, one only has to prove the reverse inclusion: $\mathcal{L}(A) \cap \text{pos}_{\mathbb{R}}(A_s) \subseteq \text{pos}_{\mathbb{N}}(A_s)$.

Example 2.3.2. Consider the matrix C below:

$$C = \begin{array}{c} \begin{matrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{matrix} \\ \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

We prove that the singleton set $\{\mathbf{c}_1\}$ is C-lattice minimal. Assume towards a contradiction that there is some element $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{L}(C) \cap \text{pos}_{\mathbb{R}}(\mathbf{c}_1)$ that is not in $\text{pos}_{\mathbb{N}}(\mathbf{c}_1)$. Then $u_1 \equiv 1 \mod 2$, and $u_2 = u_3 = 0$. Since $\mathbf{u} \in \mathcal{L}(C)$,

$$\mathbf{u} = m_1\mathbf{c}_1 + m_2\mathbf{c}_2 + m_3\mathbf{c}_3 + m_4\mathbf{c}_4$$

for some integers m_1, m_2, m_3, m_4. Therefore,

$$2m_1 + m_4 \equiv 1 \mod 2$$
$$2m_2 - m_4 \equiv 0 \mod 2$$

which is a contradiction since m_4 cannot be both even and odd. A similar argument shows that the singleton set $\mathbf{c}_2$ is also C-lattice minimal.

[1]Equivalently, the semigroup $\text{pos}_{\mathbb{R}}(A_s)$ is *saturated* in the lattice $\mathcal{L}(A)$ (i.e if $c\mathbf{v} \in \text{pos}_{\mathbb{R}}(A)$ for a positive integer c and $\mathbf{v} \in \mathcal{L}(A)$, then $\mathbf{v} \in \text{pos}_{\mathbb{R}}(A)$). See [67, Proposition 1.1] for details.

On the other hand, the set $\{\mathbf{c}_1, \mathbf{c}_2\}$ is not C-lattice minimal since the element

$$\mathbf{v} := (1,1,0) = \mathbf{c}_2 - \mathbf{c}_3 + \mathbf{c}_4$$

is in $\mathcal{L}(C) \cap \mathrm{pos}_{\mathbb{R}}(\mathbf{c}_1, \mathbf{c}_2)$ but is not in $\mathrm{pos}_{\mathbb{N}}(\mathbf{c}_1)$.

The following technical lemma shows that under certain conditions a quasi-polynomial p_A^γ can be obtained from the vector partition function of B for a submatrix B of A. The proof involves several intermediate results and is worked out in detail in Appendix A.

Lemma 2.3.3. *Let γ be a chamber of A. Assume without loss of generality that the external columns of A in γ are $\mathbf{a}_1, \ldots, \mathbf{a}_\ell$ for some $\ell \in \{0, \ldots, d-1\}$. Also assume that $\mathbf{a}_i = k_i \mathbf{e}_i$ for each $i \in \{1, \ldots, \ell\}$ and some positive integers $k_1, \ldots, k_\ell$. Finally assume that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ is A-lattice minimal. Let B be the matrix obtained by removing the first ℓ rows and columns of A. Then there exists a chamber γ' of B such that*

$$p_A^\gamma(\mathbf{b}) = p_B^{\gamma'}(b_{\ell+1}, \ldots, b_d)$$

for all $\mathbf{b} = (b_1, \ldots, b_n) \in \mathrm{pos}_{\mathbb{N}}(A) \cap \gamma$.

In the previous lemma, the condition that the external columns γ are given by positive integer multiples of standard basis vectors may appear contrived. However, we now show that if A has a simplicial chamber γ with A-lattice minimal external columns, one can always apply an appropriate change of variables so that the pair A and γ are in the form of Lemma 2.3.3.

For a chamber γ of A fix an ordering $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of the minimal ray generators of γ. We define the *ray matrix* M_γ of γ to be the matrix whose rows are the minimal ray generators of γ. If γ is a simplicial cone with minimal ray generators $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$, let $\mathbf{w}_1, \ldots, \mathbf{w}_d$ be minimal ray generators of $\gamma^\vee$ so that $\mathbf{w}_i$ is the internal facet normal of the sole facet of γ not containing $\mathbf{v}_i$. We abuse notation by setting $M_{\gamma^\vee}$ to be the matrix whose rows are $\mathbf{w}_1, \ldots, \mathbf{w}_d$ (so that the order of rows of $M_{\gamma^\vee}$ is set by the ordering of minimal ray generators of γ). We call $M_{\gamma^\vee}$ the *dual ray matrix.*

The matrix $M_\gamma^\vee$ maps the external columns of γ to positive integer multiples of standard basis vectors (as is necessary for Lemma 2.3.3). Moreover, $M_\gamma^\vee$ maps the chamber γ to the positive orthant (defined by $b_1, \ldots, b_d \geq 0$). This is not a necessary condition for Lemma 2.3.3, but is nice to have. In particular, if $M_\gamma^\vee \gamma$ is the positive orthant in $\mathbb{R}^d$ (i.e the cone defined by each variable being non-negative), then the chamber γ' of B obtained by Lemma 2.3.3 is the positive orthant in $\mathbb{R}^{d-\ell}$ (see Appendix A for details).

Theorem 2.3.4. *Let A be a $d \times n$ matrix of rank d with integer entries, and let γ be a chamber of A that is simplicial. Without loss of generality assume that $\mathbf{a}_1, \ldots, \mathbf{a}_\ell$ are the external columns of γ. Assume additionally that $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ is A-lattice minimal. Let B be* *the matrix obtained by removing the first ℓ rows and columns from $M_{\gamma^\vee} A$. Then*

$$p_A^\gamma(\mathbf{b}) = p_B^{\gamma'}\left((M_{\gamma^\vee}\mathbf{b})_{\ell+1}, \ldots, (M_{\gamma^\vee}\mathbf{b})_d\right)$$

for all $\mathbf{b} \in \gamma \cap \operatorname{pos}_{\mathbb{N}}(A)$. Moreover, γ' is the positive orthant in $\mathbb{R}^{d-\ell}$.

Proof. Since γ is simplicial, the dual ray matrix $M := M_{\gamma^\vee}$ is well defined. Consider the matrix MA with columns $\mathbf{m}_1 := M\mathbf{a}_1, \ldots, \mathbf{m}_n := M\mathbf{a}_n$. Our goal is to apply Lemma 2.3.3 with the matrix MA, chamber $M\gamma$, and columns $\mathbf{m}_1, \ldots, \mathbf{m}_\ell$ so we show that each of its conditions are met.

The matrix $M_{\gamma^\vee}$ is invertible over $\mathbb{Q}$ and has integer entries, and thus satisfies the conditions of Propositions 2.2.12–2.2.16. Therefore, the vector partition functions of A and $M_{\gamma^\vee} A$ are the same up to a change of variables. More precisely,

$$p_A^\gamma(\mathbf{b}) = p_{MA}^{M\gamma}(M\mathbf{b}) \tag{2.9}$$

for all $\mathbf{b} \in \gamma \cap \operatorname{pos}_{\mathbb{N}}(A)$.

By Proposition 2.2.13, $M\gamma$ is a chamber of MA, and $\mathbf{m}_1, \ldots, \mathbf{m}_\ell$ are ray generators of $M\gamma$ (which is the positive orthant defined by the inequalities $b_1, \ldots, b_d \geq 0$). By Proposition 2.2.15 they are each external columns of MA. Additionally, the set of external columns is MA-lattice minimal. Up to re-ordering of the columns, $\mathbf{m}_1 = k_1'\mathbf{e}_1, \ldots, \mathbf{m}_\ell = k_\ell'\mathbf{e}_\ell$ for some positive integers $k_1', \ldots, k_\ell'$. Therefore MA, $M\gamma$, and $\mathbf{m}_1, \ldots, \mathbf{m}_\ell$ do indeed meet the conditions of Lemma 2.3.3, and so

$$p_{MA}^{M\gamma}(M\mathbf{b}) = p_B^{\gamma'}((M\mathbf{b})_{\ell+1}, \ldots, (M\mathbf{b})_d) \tag{2.10}$$

for each $\mathbf{b} \in \gamma \cap \operatorname{pos}_{\mathbb{N}}(A)$ where γ' is the positive orthant in $\mathbb{R}^{d-\ell}$. Putting together (2.9) and (2.10) yields the result. $\square$

We note that, in the previous result, the condition that γ must be simplicial is used so that the dual ray matrix is defined, which in turn yields that the chamber γ' is the positive orthant in $\mathbb{R}^{d-\ell}$. However, one can replace the role of the dual ray matrix with any linear mapping $M \in \mathbb{Q}^{d\times d}$ with integer entries that sends the external columns of γ to positive scalar multiples of the standard basis. In particular, one can replace $M_{\gamma^\vee}$ in the above proof with the mapping $M_{\sigma^\vee}$ for some simplicial cone σ of A containing the columns $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$. Then, we no longer need the condition that γ is simplicial. However, γ' will not be the positive orthant in $\mathbb{R}^{d-\ell}$.

In essence, (up to A-lattice minimality) one can reduce the dimension of the vector partition function for a particular chamber by the number of external columns present in that chamber.

In particular, if γ is an external chamber whose external columns are A-lattice minimal, then B is a $1 \times (n-d)$ matrix, and so the quasi-polynomial p_A^γ arises from a coin exchange problem. By exploiting this fact, we prove that p_A^γ can also be obtained from the Ehrhart quasipolynomial associated to the single internal ray of γ after an appropriate change of variables.

Theorem 2.3.5. *Let A be a $d \times n$ matrix of rank d with integer entries. Let γ be an external chamber of A, and without loss of generality assume that the external columns of γ are $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}$. Assume additionally that the set $\{\mathbf{a}_1, \dots, \mathbf{a}_{d-1}\}$ is A-lattice minimal. Denote by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d-1} \in \mathbb{Z}^d$ the external ray generators corresponding to $\mathbf{a}_1, \dots, \mathbf{a}_{d-1}$ respectively, and let $\mathbf{v}_d \in \mathbb{Z}^d$ be an internal ray generator. If $f(t) := p_A(t\mathbf{v}_d)$ is the Ehrhart quasi-polynomial associated to the polytope $A\mathbf{x} = \mathbf{v}_d$, $\mathbf{x} \geq \mathbf{0}$, then the quasi-polynomial $p_A^\gamma(\mathbf{b})$ associated to γ is equal to*

$$p_A^\gamma(\mathbf{b}) = f\left(\frac{\det(\mathbf{v}_1, \dots, \mathbf{v}_{d-1}, \mathbf{b})}{\det(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)}\right)$$

for all $\mathbf{b} \in \gamma \cap \operatorname{pos}_{\mathbb{N}}(A)$.

Proof. Since γ is an external chamber, by Proposition 2.2.8, it is simplicial, and so the dual ray matrix of γ exists. Let $M := M_{\gamma^\vee}$ following the same ordering as the ray generators $\mathbf{v}_1, \dots, \mathbf{v}_d$. Let $\mathbf{b} \in \gamma \cap \operatorname{pos}_{\mathbb{N}}(A)$. Then

$$\mathbf{b} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d \tag{2.11}$$

for some $\lambda_1, \dots, \lambda_d \geq 0$, and so

$$M\mathbf{b} = \lambda_1 k_1 \mathbf{e}_1 + \cdots + \lambda_d k_d \mathbf{e}_d.$$

By Theorem 2.3.4, we have

$$p_A^\gamma(\mathbf{b}) = p_B(\lambda_d k_d)$$

where B is the $1 \times (n-d+1)$ matrix obtained by removing the first $d-1$ rows and columns of the matrix MA. On the other hand, by setting $\lambda_1 = \ldots = \lambda_{d-1} = 0$ in Eq. (2.11), we find that $p_A^\gamma(\lambda_d \mathbf{v}_d) = p_B(\lambda_d k_d)$ as well. Therefore, $p_A^\gamma(\mathbf{b}) = p_A^\gamma(\lambda_d \mathbf{v}_d) = f(\lambda_d)$. Finally, by Cramer's rule

$$\lambda_d = \frac{\det(\mathbf{v}_1, \dots, \mathbf{v}_{d-1}, \mathbf{b})}{\det(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d)}$$

and so as quasi-polynomials,

$$p_A^\gamma(\mathbf{b}) = f\left(\frac{\det(\mathbf{v}_1, \dots, \mathbf{v}_{d-1}, \mathbf{b})}{\det(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d))}\right)$$

as required.

$\square$

Example 2.3.6. Recall the matrix from our running example

$$A^{2,2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

whose columns we denote by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. The chamber

$$\gamma_1 = \operatorname{pos}_{\mathbb{R}}(\mathbf{a}_1, \mathbf{a}_3)$$

is an external chamber with external column $\mathbf{a}_1$. From Figure 2.1, we can see that the column $\mathbf{a}_3$ is an internal ray generator for γ_1. In particular we can take $\mathbf{v}_1 = \mathbf{a}_1$ to be the external ray generator and $\mathbf{v}_2 = \mathbf{a}_3$ to be the internal ray generator. Let $f(t) = p_{A^{2,2}}(t\mathbf{a}_3)$ be the Ehrhart quasi-polynomial associated to the internal ray of γ_1. We can compute using *Latte*:

$$f(t) = \begin{cases} \frac{(t+2)^2}{4} & \text{if } t \equiv 0 \mod 2 \\ \frac{(t+1)(t+3)}{4} & \text{if } t \equiv 1 \mod 2. \end{cases}$$

By Theorem 2.3.5 we deduce that

$$\begin{aligned} p_{A^{2,2}}^{\gamma_1}(\mathbf{b}) &= f\left(\frac{\det(\mathbf{a}_1, \mathbf{b})}{\det(\mathbf{a}_1, \mathbf{a}_2)}\right) \\ &= f\left(\frac{\det\left(\begin{bmatrix} 1 & b_1 \\ 0 & b_2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)}\right) \\ &= \begin{cases} \frac{(b_2+2)^2}{4} & \text{if } b_2 \equiv 0 \mod 2 \\ \frac{(b_2+1)(b_2+3)}{4} & \text{if } b_2 \equiv 1 \mod 2. \end{cases} \end{aligned}$$

This agrees with previous computations [62], and the output of *Barvinok*.

In the previous example, we could have also applied Theorem 2.3.4 to prove that for all $\mathbf{b} \in \gamma_3$,

$$p_{A^{2,2}}^{\gamma_1}(\mathbf{b}) = p_B(b_2)$$

where $B = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$, and then solved the corresponding coin exchange problem. Example 2.6.3 in Section 2.6 provides a slightly more involved application of the determinant formula.

2.4 Unimodular case

In the previous section we showed that if γ is an external chamber of A whose external columns form an A-lattice minimal set, then the quasi-polynomial p_A^γ is equal to p_B for a $1 \times k$ matrix B with integer entries. Next, we exploit this this fact in order to characterize exactly when p_A^γ is a polynomial. Moreover, we show that this polynomial is given by a negative binomial coefficient and is easy to compute, without explicitly computing the chamber γ. For a class of matrices (called *unimodular* matrices), this result immediately allows us to prove that the polynomial p_A^γ for an external chamber γ is given by a negative binomial coefficient that is readily computable.

Lemma 2.4.1. *Let $B = [b_{1,1}, \ldots, b_{1,k}]$ be a $1 \times k$ integer matrix for some positive integer k, and assume that $\ker(B) \cap \mathbb{R}^k_{\geq 0} = \{\mathbf{0}\}$. Then p_B is a polynomial of degree $k-1$ on $\mathrm{pos}_{\mathbb{N}}(B)$ if and only if each of the k entries of B are equal to some non-zero integer β. In this case,*

$$p_B(b) = \binom{\frac{b}{\beta} + k - 1}{k-1}$$

for all $b \in \mathrm{pos}_{\mathbb{N}}(B)$.

Proof. We begin by proving the reverse implication. Assume B is a $1 \times k$ integer matrix with each of the k entries equal to some non-zero integer β. Then for any $b \in \mathrm{pos}_{\mathbb{N}}(B)$, $p_B(b)$ is the number of ways of partitioning b/β into k equal non-negative integral parts. Therefore,

$$p_B(b) = \binom{\frac{b}{\beta} + k - 1}{k-1} \tag{2.12}$$

is a polynomial in b for $b \in \mathrm{pos}_{\mathbb{N}}(B)$.

We now prove the forward implication. Suppose p_B is a polynomial of degree $k-1$. We may assume that $k \geq 2$, since if $k = 1$, B has a single entry. We note further that the entries of B must be either all positive or all negative or else $\ker(B) \cap \mathbb{R}^k_{\geq 0} \neq \{\mathbf{0}\}$. We assume that all entries are positive, noting that the negative case follows a similar argument. For any $1 \leq j \leq k$, the vector partition function $p_{B_{\cdot,\hat{j}}}$ is a polynomial of degree $k-2$ since it is the difference of two polynomials:

$$p_{B_{\cdot,\hat{j}}}(b) = p_B(b) - p_B(b - b_{1,j})$$

for all $b \in \mathcal{L}(B) \cap \mathbb{N}$. In particular, by repeated application of this fact, it follows that for each 1×2 submatrix of B, the vector partition function is a polynomial of degree 1.

Assume towards a contradiction that B has two distinct entries, say, without loss of generality, $b_{1,1}$ and $b_{1,2}$. Let $B' = [b_{1,1}, b_{1,2}]$ be the 1×2 submatrix consisting of the two distinct entries, so that $p_{B'}$ is a polynomial of degree 1. Also, $p_{B'}(0) = p_B(\min(b_{1,1}, b_{1,2})) = 1$, so $p_{B'}(b) = 1$ for all $b \in \mathbb{N}$ since $p_{B'}$ is linear. However, $p_{B'}(b_{1,1}b_{1,2}) \geq 2$ since both

$\mathbf{x} = (b_{1,2}, 0)$ and $\mathbf{x} = (0, b_{1,1})$ are solutions to $B'\mathbf{x} = b_{1,1}b_{1,2}$ with $\mathbf{x} \in \mathbb{N}^2$. This contradicts that $p_{B'}$ is a polynomial of degree 1, and thus that p_B is a polynomial of degree $k-1$. Therefore, the entries of B must be the same as required.

$\square$

For a facet f of a cone $\sigma \subset \mathbb{R}^m$, we call an inner/outer facet normal $\boldsymbol{\iota} \in \mathbb{Z}^m$ of f a *minimal inner/outer facet normal* if $\boldsymbol{\iota}$ is a minimal generator of the ray $\{t\boldsymbol{\iota} : t \geq 0\}$. By Proposition 1.3.2, $\boldsymbol{\iota}$ is a ray generator of $\sigma^\vee$. Therefore, if f is a facet of a simplicial chamber γ of A, then $\boldsymbol{\iota}$ is a row of the dual ray matrix $M_{\gamma^\vee}$. This observation allows us to characterize exactly when p_A^γ is a polynomial on $\mathrm{pos}_{\mathbb{N}}(A)$ if γ is an external chamber whose external columns form an A-lattice minimal set.

Theorem 2.4.2. *Let γ be an external chamber of A, with external facet f, and let $\boldsymbol{\iota}$ be the minimal inner facet normal of f. Assume without loss of generality that $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ are the external columns of γ. Let $\mathbf{a}_{d+\ell}$ be a column of A for some $\ell \in \{0, \ldots, n-d\}$. Finally assume that $\{\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}\}$ is A-lattice minimal. Then p_A^γ is a polynomial on $\gamma \cap \mathrm{pos}_{\mathbb{N}}(A)$ if and only if*

$$\boldsymbol{\iota} \cdot \mathbf{a}_j = \begin{cases} 0 \text{ if } \mathbf{a}_j \in f \\ \beta \text{ if } \mathbf{a}_j \notin f \end{cases}$$

for each $j = 1, \ldots, n$, for some positive integer β. Moreover, if p_A^γ is a polynomial on $\mathrm{pos}_{\mathbb{N}}(A)$, then

$$p_A^\gamma(\mathbf{b}) = \binom{\frac{\boldsymbol{\iota} \cdot \mathbf{b}}{\beta} + n - d}{n-d} \tag{2.13}$$

$$= \binom{\frac{\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{b})}{\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{a}_{d+\ell})} + n - d}{n-d} \tag{2.14}$$

for each $\mathbf{b} \in \gamma \cap \mathrm{pos}_{\mathbb{N}}(A)$.

Proof. We have $\boldsymbol{\iota} \cdot \mathbf{a}_j = 0$ for each $j = 1, \ldots, d-1$. Let $M := M_{\gamma^\vee}$ be the dual ray matrix of γ so that the first $d-1$ rows appear in the same order as the corresponding $d-1$ external columns of γ. By Theorem 2.3.4, for each $\mathbf{b} \in \gamma$,

$$p_A^\gamma(\mathbf{b}) = p_B((M\mathbf{b})_d)$$

where B is the $1 \times (n-d+1)$ matrix obtained by removing the first $d-1$ rows and columns from MA. Additionally, the last row of M is simply $\boldsymbol{\iota}$ since $\boldsymbol{\iota}$ is the only minimal ray generator of $\gamma^\vee$ not corresponding to a column in f. Therefore, the last row of MA is

$\boldsymbol{\iota}^T A$, and so we have

$$\begin{aligned} B_{1,j} &= (MA)_{d,d-1+j} \\ &= \boldsymbol{\iota} \cdot \mathbf{a}_{d-1+j} \end{aligned}$$

for each $j = 1, \ldots, n-d+1$. By Lemma 2.4.1, p_B is polynomial on $\mathrm{pos}_{\mathbb{N}}(B)$ if and only if each of these entries is equal to some positive integer β. Since $p_A^{\gamma}(\mathbf{b}) = p_B((M\mathbf{b})_d)$ for all $\mathbf{b} \in \gamma \cap \mathrm{pos}_{\mathbb{N}}(A)$, p_A^{γ} is polynomial if and only if p_B is polynomial.

We now prove that Eq. (2.13) and Eq. (2.14) hold if p_A^{γ} is polynomial. In this case, for all $\mathbf{b} \in \gamma \cap \mathrm{pos}_{\mathbb{N}}(A)$,

$$\begin{aligned} p_A^{\gamma}(\mathbf{b}) &= p_B((M\mathbf{b})_d) \\ &= p_B(\boldsymbol{\iota} \cdot \mathbf{b}) \\ &= \binom{\frac{\boldsymbol{\iota}\cdot\mathbf{b}}{\beta} + n - d}{n-d} \end{aligned}$$

and so Eq. (2.13) holds. Since $\mathbf{b} \in \gamma$, by Proposition 2.2.11, $\mathbf{b}$ is in the simplicial cone of A, $\mathrm{pos}_{\mathbb{R}}(A_s)$, where $s = \{1, \ldots, d-1, d+\ell\}$. Therefore, $\mathbf{b} = \lambda_1 \mathbf{a}_1 + \cdots + \lambda_{d-1}\mathbf{a}_{d-1} + \lambda_d \mathbf{a}_{d+\ell}$ for some $\lambda_1, \ldots, \lambda_d \geq 0$. Then

$$\begin{aligned} \boldsymbol{\iota} \cdot \mathbf{b} &= \lambda_d(\boldsymbol{\iota} \cdot \mathbf{a}_{d+\ell}) && (2.15) \\ &= \lambda_d && (2.16) \\ &= \frac{\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{b})}{\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{a}_{d+\ell})} && (2.17) \end{aligned}$$

where the last equality follows from Cramer's rule. Eq. (2.14) now follows by plugging in Eq. (2.17) into Eq. (2.13). □

We note also that in the previous result the columns $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ can be replaced by any ray generators $\mathbf{v}_1, \ldots, \mathbf{v}_{d-1}$ with $\mathrm{pos}_{\mathbb{R}}(\mathbf{v}_i) = \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_i)$ for $i = 1, \ldots, d-1$.

Remark 2.4.3. We note that if a column of A is an internal ray generator $\mathbf{v}$ of γ (equivalently some column of A is in γ but is not an external column of γ), then

$$p_A(\mathbf{v}) = n - d + 1.$$

This is exactly the number of simplicial cones of A that contain γ as a subset since there are $n - (d-1)$ choices of d^{th} column to add to the $d-1$ columns on the external facet. For each such simplicial cone $\mathrm{pos}_{\mathbb{R}}(A_s)$ (i.e with $\gamma \subseteq \mathrm{pos}_{\mathbb{R}}(A_s)$), there is exactly one solution $\mathbf{x} \in \mathbb{N}^n$ to $A\mathbf{x} = \mathbf{v}$ with $x_i = 0$ for all $i \notin s$ (since the columns of A in s form a basis of $\mathbb{R}^d$ and $\mathbf{v} \in \mathrm{pos}_{\mathbb{R}}(A_s)$). Therefore, we find that each solution to $A\mathbf{x} = \mathbf{v}$ is of this form, and that no other solutions $\mathbf{x} \in \mathbb{N}^n$ exist.

We suspect that the A-lattice minimality condition in the previous theorem can be removed (intuitively we view A-lattice minimality as "nice" from the periodic point of view, so we expect that removing this property on the external columns introduces periodicity).

We now introduce unimodular matrices – these matrices have the special property that p_A is a piecewise polynomial, so p_A^γ is polynomial for each chamber.

A full rank $d \times n$ matrix A with integer entries is *unimodular* if every $d \times d$ submatrix of A has determinant 1, -1, or 0 (see for example [80, Section 19.1]). If A is unimodular, then $\mathrm{pos}_{\mathbb{N}}(A) = \mathrm{pos}_{\mathbb{R}}(A) \cap \mathbb{Z}^d$.

In [53], De Loera and Sturmfels introduce a generalization of matrix unimodularity given by a geometrical criterion. Both of these definitions appear in this section, so we distinguish them by refering to the older definition simply as unimodular and the one introduced by De Loera and Sturmfels as DeLS-unimodular. A $d \times n$ matrix A with integer entries is defined to be *DeLS-unimodular* if the polyhedron $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ associated to the vector partition function $p_A(\mathbf{b})$ has only integral vertices whenever $\mathbf{b}$ is in the lattice spanned by the columns of A. Under these conditions, p_A is piecewise polynomial by the following result of De Loera and Sturmfels. We remark that unimodular matrices are DeLS-unimodular.

Theorem 2.4.4 (De Loera, Sturmfels 2003 [53])**.** *Let A be a $d \times n$ DeLS-unimodular matrix of rank d. Then p_A is a piecewise polynomial of degree $n-d$ on $\mathcal{L}(A) \cap \mathrm{pos}_{\mathbb{R}}(A)$ and is zero everywhere else on $\mathbb{Z}^d \cap \mathrm{pos}_{\mathbb{R}}(A)$.*

If A is DeLS-unimodular, then each subset of columns of A is A-lattice minimal. If A is unimodular then it is also DeLS-unimodular, and therefore if A is a unimodular matrix, then the subset of columns of A are also A-lattice minimal. The following corollary now follows immediately.

Corollary 2.4.5. *Let A be a $d \times n$ DeLS-unimodular matrix of rank d, and γ be an external chamber of A with external columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{d-1}$. Let $\mathbf{a}_{d+\ell}$ be a column of A for some $\ell \in \{0, \ldots, n-d\}$. Then*

$$p_A^\gamma(\mathbf{b}) = \binom{\frac{\det(\mathbf{a}_1,\ldots,\mathbf{a}_{d-1},\mathbf{b})}{\det(\mathbf{a}_1,\ldots,\mathbf{a}_{d-1},\mathbf{a}_{d+\ell})} + n - d}{n-d} \tag{2.18}$$

for all $\mathbf{b} \in \mathrm{pos}_{\mathbb{N}}(A) \cap \gamma$.

Example 2.4.6. Consider the following DeLS-unimodular matrix

$$D = \begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix}.$$

that we have obtained by multiplying the matrix in the running example of [53] by two. This multiplication has no effect on the chamber complex (i.e the chamber complex of D is the same as that in their running example). However,

$$\mathcal{L}(D) := \{(b_1, b_2, b_3) \in \mathbb{Z}^3 : b_1, b_2, b_3 \equiv 0 \mod 2\}$$

in our example, whereas in their running example, the lattice spanned by the matrix is $\mathbb{Z}^3$.

The first three columns are external and the other two are not. Additionally, the chamber

$$\gamma := \operatorname{pos}_{\mathbb{R}} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

is external with minimal internal ray generator

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We now compute the polynomial associated to γ using Corollarly 2.4.5 with the first column of D playing the role of $\mathbf{a}_{d+\ell}$. Let d denote the ratio of determinants – that is,

$$d := \det \left(\begin{bmatrix} 0 & 0 & b_1 \\ 0 & 2 & b_2 \\ 2 & 0 & b_3 \end{bmatrix} \right) \Big/ \det \left(\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \right) = \frac{b_1}{2}.$$

Then

$$\begin{aligned} p_D^\gamma(\mathbf{b}) &= \binom{d+2}{2} \\ &= \binom{\frac{b_1}{2}+2}{2} \end{aligned}$$

for all $\mathbf{b} \in \operatorname{pos}_{\mathbb{N}}(D) \cap \gamma$. Since $b_1 \equiv 0 \mod 2$, the resulting polynomial does indeed yield integers. Finally, remark that we could have also used the fourth or fifth columns of D in the place of the first column (only the external columns of γ cannot be used).

In the case that A is unimodular (not just DeLS-unimodular), we can further simplify the expression given in Theorem 2.4.4. We begin with the following useful lemma.

Lemma 2.4.7. *Let A be a $d \times n$ unimodular matrix of rank d. Let f be a facet of* $\operatorname{pos}_{\mathbb{R}}(A)$ *with minimal inner facet normal $\boldsymbol{\iota}$. Let $\mathbf{c}$ be a column of A. Then*

$$\boldsymbol{\iota} \cdot \mathbf{c} = \begin{cases} 0 \text{ if } \mathbf{c} \in f \\ 1 \text{ if } \mathbf{c} \notin f. \end{cases}$$

Proof. If $\mathbf{c} \in f$ then $\boldsymbol{\iota} \cdot \mathbf{c} = 0$, so assume that $\mathbf{c} \notin f$. In this case, there are linearly independent columns $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ of A lying on f so that $\mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{c})$ is a simplicial cone of A. Since A is unimodular, the matrix M whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{c}$ must have determinant ± 1. Thus M is invertible over $\mathbb{Z}$, and in particular, there exists a vector $\mathbf{v}$ in $\mathbb{Z}^d$ such that $\mathbf{c} \cdot \mathbf{v} = 1$. Since $\mathbf{c}, \mathbf{v} \in \mathbb{Z}^d$ with $\mathbf{c} \cdot \mathbf{v} = 1$, we see that $\gcd(c_1, \ldots, c_d) = 1$. Finally since $\boldsymbol{\iota} \in \mathbb{Z}^d$ as well, $\boldsymbol{\iota} \cdot \mathbf{c}$ is integral, and since $\boldsymbol{\iota}$ is minimal, $\gcd(\iota_1, \ldots, \iota_d) = 1$, and so $\boldsymbol{\iota} \cdot \mathbf{c} = 1$ as required. □

Corollary 2.4.8. *Let A be a $d \times n$ unimodular matrix of rank d, f be a facet of A containing exactly $d-1$ columns of A, and $\boldsymbol{\iota}$ be the minimal inner normal of f. Moreover, let $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}$ be the external columns of A on f, and let γ be the external chamber containing f. Then the polynomial $p_A^\gamma(\mathbf{b})$ associated to γ is*

$$p_A^\gamma(\mathbf{b}) = \binom{\boldsymbol{\iota} \cdot \mathbf{b} + n - d}{n - d}. \tag{2.19}$$

$$= \binom{|\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{b})| + n - d}{n - d} \tag{2.20}$$

Remark 2.4.9. We note that $|\det(\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{b})|$ is the continuous volume of the paralleliped

$$\Pi := \{\lambda_1 \mathbf{a}_1 + \cdots + \lambda_{d-1} \mathbf{a}_{d-1} + \lambda_d \mathbf{b} : 0 \leq \lambda_1, \ldots, \lambda_d \leq 1\}$$

generated by $\mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, \mathbf{b}$.

2.5 Semi-external chambers

In this section we rederive a known result on linear factors appearing in polynomials associated to certain chambers of vector partition functions whose associated matrix is unimodular. The theorem in this section (Theorem 2.5.3) was originally proven in 2008 by Baldoni and Vergne [3, Corollary 14]. We worked out this result prior to knowing about their paper, but keep it in the document (along with the proof) since we believe that it may be possible to generalize – see Conjecture 2.5.4 for the exact statement.[2]

In Section 3.8 we show that the generalization suggested by Conjecture 2.5.4 can aid in the computation of linear factors of polynomials associated to the Littlewood-Richardson coefficients.

Definition 2.5.1. For a $d \times n$ matrix A of rank d, we define a chamber of A to be *semi-external* if it intersects a facet of $\mathrm{pos}_{\mathbb{R}}(A)$ $(d-1)$-dimensionally.

[2]In [3] the result of Theorem 2.5.3 is given as a corollary of a result of Dahmen and Micchelli. It is also stated that this result follows from "reciprocity relations for the vector partition function", which is the approach we take. In [3] the explicit proof using this approach is not given.

Example 2.5.2. Shown in Figure 2.4 is a 2-dimensional projection of the 3-dimensional chamber complex of a matrix K_3 with 6 columns. The intersection of each column with the slice is a vertex labeled with the appropriate column number. Additionally, the intersection of each chamber is given by a labeled (with Roman numerals) 2-dimensional region bounded by edges. The facets of $\mathrm{pos}_{\mathbb{R}}(K_3)$ correspond to the three line segments bounded by vertices $1, 6$, vertices $1, 4$, and vertices $4, 6$. Of the seven chambers, we see that Chambers I, III, IV, V, VI are semi-external, and the rest are not. We study this in more detail in Example 2.6.7 of Section 2.6.2.

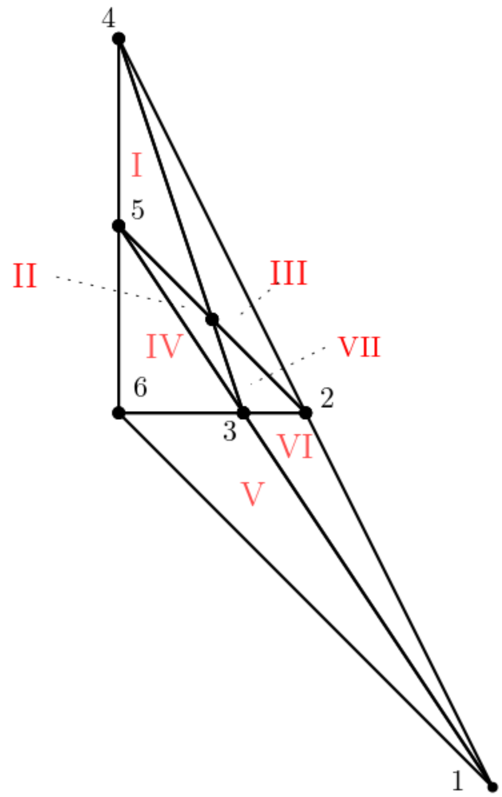


Figure 2.4: A 2-dimensional projection of a 3-dimensional chamber complex of a matrix with 6 columns. Chambers correspond to the 2-dimensional regions labeled with roman numerals. Chambers I, III, IV, V, VI are semi-external.

We note that external chambers are also semi-external chambers.

Theorem 2.5.3 (Baldoni, Vergne, 2008 [3]). *Let A be a $d \times n$ unimodular matrix of rank d, let f be a facet of $\mathrm{pos}_{\mathbb{R}}(A)$ with inner facet normal $\boldsymbol{\iota}$, and let γ be a semi-external chamber of A intersecting f $(d-1)$-dimensionally. Let k be the number of columns of A not in f. Then $p_A^{\gamma}(\mathbf{y})$ has linear factors*

$$(\boldsymbol{\iota} \cdot \mathbf{y}) + i$$

for $i = 1, \ldots, k-1$.

Proof. Let $\mathbf{v} \in \gamma^\circ \cap \mathbb{Z}^d$ such that $\boldsymbol{\iota} \cdot \mathbf{v} = i$ for some $i \in \{1, \dots, k-1\}$ (notice that such a $\mathbf{v}$ does exist since γ is d-dimensional). Let $\mathcal{P}$ be the polytope defined by

$$\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{v},\ \mathbf{x} \geq \mathbf{0}\}.$$

Our aim is to show that the interior $\mathcal{P}^\circ$ of $\mathcal{P}$, has no integer points, and then exploit the Ehrhart reciprocity result of Theorem 1.4.3 in order to compute evaluations of the polynomial p_A^γ on the hyperplane $(\boldsymbol{\iota} \cdot \mathbf{y}) + i = 0$.

Without loss of generality let $\mathbf{a}_1, \dots, \mathbf{a}_k$ be the columns of A not on f so that $\mathbf{a}_j \cdot \mathbf{v} = 1$ for each $j = 1, \dots, k$ by assumption. For each integral point $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{P}$, it follows that

$$\begin{aligned} \mathbf{v} &= x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n} \\ \implies \boldsymbol{\iota} \cdot \mathbf{v} &= \boldsymbol{\iota} \cdot (x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n}) \\ \implies i &= x_1 + \cdots + x_k. \end{aligned}$$

In particular, since $x_1, \dots, x_k$ are all non-negative integers and since $i \leq k-1$, it follows that $x_j = 0$ for some $j \in \{1, \dots, k\}$.

On the other hand, we claim that for each $j = 1, \dots, k$, there is some integral point in $\mathcal{P}$ with $x_j > 0$. If this is not the case, then $\mathbf{v} - \mathbf{a}_j \notin \operatorname{pos}_{\mathbb{R}}(A)$. Then for some minimal inner facet normal $\boldsymbol{\iota}'$ of $\operatorname{pos}_{\mathbb{R}}(A)$, we find that

$$\boldsymbol{\iota}' \cdot (\mathbf{v} - \mathbf{a}_j) < 0 \tag{2.21}$$
$$\implies \boldsymbol{\iota}' \cdot \mathbf{v} < 1 \tag{2.22}$$
$$\implies \boldsymbol{\iota}' \cdot \mathbf{v} = 0 \tag{2.23}$$

where the last inequality arises since $\boldsymbol{\iota}' \cdot \mathbf{v}$ is an integer. This is a contradiction since $\mathbf{v} \in \gamma^\circ$ so γ does not lie on a facet of $\operatorname{pos}_{\mathbb{R}}(A)$.

For each $j = 1, \dots, k$, define $H_j \subseteq \mathbb{R}^n$ to be the hyperplane $x_j = 0$. The polytope $\mathcal{P} \cap H_j$ is a proper face of $\mathcal{P}$ for each j since $\mathcal{P} \nsubseteq H_j$ for any $j \in \{1, \dots, k\}$. Therefore, each integral point of $\mathcal{P}$ must lie on a proper face of $\mathcal{P}$, since it must lie on at least one of the H_j. Therefore $\mathcal{P}^\circ$ contains no integer points, and since $L_{\mathcal{P}}(t) = p_A^\gamma(t\mathbf{v})$, it follows by Ehrhart reciprocity that

$$\begin{aligned} L_{\mathcal{P}}(-1) &= p_A^\gamma(-\mathbf{v}) \\ &= 0. \end{aligned}$$

For all $\alpha \in \mathbb{R}$, let $\tilde{H}_\alpha$ be the affine space defined by the equation $\boldsymbol{\iota} \cdot \mathbf{y} = \alpha$. Consider the set $S := \tilde{H}_i \cap \gamma^\circ$. The set S contains the $(d-1)$-dimensional translated cone $\mathbf{v} + f := \{\mathbf{v} + \mathbf{w} : \mathbf{w} \in f\}$. As we previously showed, any $\mathbf{u} \in S \cap \mathbb{Z}^d$ satisfies $p_A^\gamma(-\mathbf{u}) = 0$.

Therefore, p_A^γ vanishes on the integer points of the translated $(d-1)$-dimensional cone $-\mathbf{v} - f := \{-\mathbf{v} - \mathbf{w} : \mathbf{w} \in f\} \subseteq \tilde{H}_{-i}$. Moreover, by the Combinatorial Nullstellensatz, the polynomial p_A^γ vanishes on the entire affine space $\tilde{H}_{-i}$, and so $\boldsymbol{\iota} \cdot \mathbf{y} + i$ appears as a linear factor of p_A^γ. □

We suspect that Theorem 2.5.3 can be generalized since we have found evidence of linear factors appearing in non-unimodular cases, and also since the unimodularity of A is not fully exploited in the previous theorem. In particular, we suspect that the key is the dot product condition of Lemma 2.4.7. Motivated by this, we make the following conjecture.

Conjecture 2.5.4. *Let A be a $d \times n$ matrix of rank d with integer entries, let f be a facet of $\operatorname{pos}_{\mathbb{R}}(A)$ with inner facet normal $\boldsymbol{\iota}$, and let γ be a semi-external chamber of A intersecting γ $(d-1)$-dimensionally. Assume that $\boldsymbol{\iota} \cdot \mathbf{c} = 1$ for each column $\mathbf{c}$ of A not on f and that $p_A^\gamma(\mathbf{y})$ is a polynomial. Let k be the number of columns of A not on f. Then $p_A^\gamma(\mathbf{y})$ has linear factors*

$$(\boldsymbol{\iota} \cdot \mathbf{y}) + i$$

for $i = 1, \ldots, k-1$ for all chambers of A intersecting f $(d-1)$-dimensionally.

The issue in the proof of Theorem 2.5.3 if only the assumptions of Conjecture 2.5.4 are taken lies in proving that there is some integral point in $\mathcal{P}$ with $x_j > 0$. We suspect that this can be remedied by a more careful choice of $\mathbf{v}$.

Theorem 2.5.3 and Corollary 2.4.8 suggest the following procedure to compute linear factors associated to semi-external chambers of a full rank unimodular matrix A.

1. Compute minimal ray generators of the dual cone $\operatorname{pos}_{\mathbb{R}}(A)^\vee$, and call these $\boldsymbol{\iota}_1, \ldots, \boldsymbol{\iota}_m$ (these are the minimal inner facet normals of $\operatorname{pos}_{\mathbb{R}}(A)$),
2. For each $\ell = 1, \ldots, m$, compute the number k of columns $\mathbf{c}$ of A such that $\boldsymbol{\iota}_\ell \cdot \mathbf{c} = 1$,
3. For any chamber γ of A intersecting the corresponding facet f_ℓ $(d-1)$-dimensionally

$$\boldsymbol{\iota} \cdot \mathbf{y} + 1, \ldots, \boldsymbol{\iota} \cdot \mathbf{y} + n - \ell + 1$$

occur as linear factors of $p_A^\gamma(\mathbf{y})$. In the case that $k = d-1$ (so the facet is external), we know that the polynomial is

$$p_A^\gamma(\mathbf{y}) = \binom{\boldsymbol{\iota} \cdot \mathbf{y} + n - d}{n - d}$$

and we may also compute the associated external chamber via Proposition 2.2.11.

Remark 2.5.5. Corollary 2.4.8 can also be derived from the result of Baldoni and Vergne (our Theorem 2.5.3), since in this case Theorem 2.5.3 predicts the linear factors

$$\boldsymbol{\iota} \cdot \mathbf{b} + 1, \ldots, \boldsymbol{\iota} \cdot \mathbf{b} + n - d - 1.$$

The constant can then be computed by observing that $p_A^\gamma(\mathbf{0}) = 1$.

2.6 Examples

2.6.1 Multigraph counting

In this section we consider the problem of enumerating the number of labelled multigraphs with vertices $v_1, \ldots, v_m$ and a given sequence $d_1, \ldots, d_m$ so that $\deg(v_i) = d_i$ for $1 \le i \le m$. In order that this may be encoded as a vector partition function, we allow multiple edges between any pair of vertices but do not allow loops. Most known results for enumerating the number of graphs or multigraphs with a given degree sequence are asymptotic (see for example [6, 40, 60]). We give an exact result for a relatively simple case, that we found by identifying external chambers of the corresponding vector partition functions. We have not seen this result in the literature, nor any attempts to approach this problem via the vector partition function formulation. This is somewhat surprising since it is well known that the enumeration of simple graphs with a given degree sequence can be viewed as counting integer points in polytopes.

For any positive integer m, define $M_m(d_1, \ldots, d_m)$ to be the number of multigraphs on the vertex set $v_1, \ldots, v_m$ with degree sequence $(d_1, \ldots, d_m)$. We note that we do not assume that the degree sequence is monotonically decreasing unless explicitly stated.

For each pair of distinct vertices v_i and v_j $(1 \le i \ne j \le m)$, let $x_{i,j}$ denote the number of edges joining v_i and v_j. A multigraph on the vertex set $v_1, \ldots, v_m$ has degree sequence $(d_1, \ldots, d_m)$ if the following m linear equations are satisfied:

$$\sum_{\substack{i=1 \\ i \ne j}}^{m} x_{i,j} = d_j \quad \text{for all } j = 1, \ldots, m. \tag{2.24}$$

The number of edges between any pair of vertices is a non-negative integer, and so one can describe $M_m(d_1, \ldots, d_m)$ as the number of solutions $\mathbf{x} = (x_{1,2}, \ldots, x_{m-1,m}) \in \mathbb{N}^m$ satisfying the linear equations of (2.24). This description leads to the following vector partition function formulation.

Proposition 2.6.1. *Let m be a positive integer, and let G_m denote the incidence matrix of the complete graph K_m. Then*

$$M_m(d_1, \ldots, d_m) = p_{G_m}(d_1, \ldots, d_m)$$

for any degree sequence $(d_1, \ldots, d_m) \in \mathbb{N}^m$ on the lattice $d_1 + d_2 + \cdots + d_m \equiv 0 \mod 2$.

Example 2.6.2. Let us compute $M_4(5, 4, 3, 2)$ which is the number of multigraphs on the vertex set $\{v_1, v_2, v_3, v_4\}$ and degree sequence $\mathbf{d} = (5, 4, 3, 2)$. By Proposition 2.6.1, this is equivalent to computing $p_{G_6}(\mathbf{d})$, and thus of enumerating the number of integer solutions

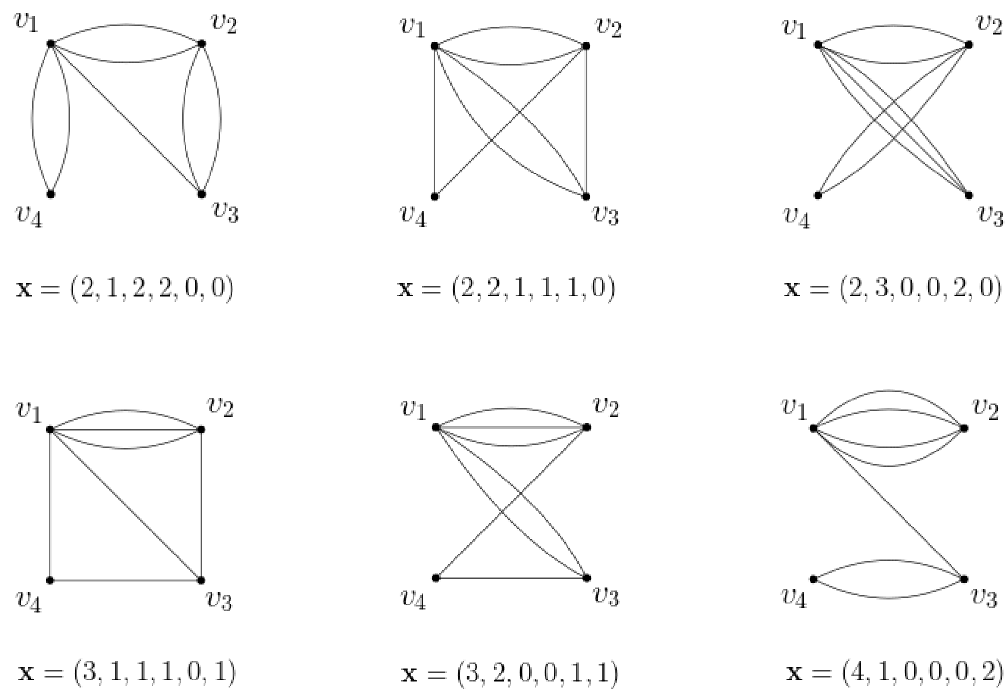


Table 2.1: The multigraphs with vertices v_1, v_2, v_3, v_4 and degree sequence $\mathbf{d} = (5,4,3,2)$. Each multigraph is labeled by the corresponding solution $\mathbf{x} \in \mathbb{N}^6$ of $G_4\mathbf{x} = \mathbf{d}$.

$\mathbf{x} = (x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}) \in \mathbb{Z}^6$ in the polytope $G_4\mathbf{x} = \mathbf{d}, \mathbf{x} \geq \mathbf{0}$. Using *Latte* we compute the solutions explicitly; there are six of them. In Table 2.1 we give each of the solutions $\mathbf{x} \in \mathbb{Z}^6$ along with the corresponding multigraph.

Our goal is to study a particular external chamber of the vector partition function p_{G_m}.

Example 2.6.3. Let $m = 6$. Then

$$G_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the chamber γ of G_6 defined by minimal ray generators

$$\mathbf{v}_1 := (3,1,1,1,1,1),\ \mathbf{v}_2 := (1,1,0,0,0,0),\ \mathbf{v}_3 := (1,0,1,0,0,0),$$
$$\mathbf{v}_4 := (1,0,0,1,0,0),\ \mathbf{v}_5 := (1,0,0,0,1,0),\ \mathbf{v}_6 := (1,0,0,0,0,1)$$

is external with $\mathbf{v}_1$ the sole internal ray generator as $\mathbf{v}_2, \ldots, \mathbf{v}_6$ are external columns of G_6.

Our aim is to compute the quasi-polynomial $p^{\gamma}_{G_6}$. The matrix G_6 is not DeLS-unimodular (and therefore also not unimodular), since $\mathbf{1} \in \mathcal{L}(G_6)$, but the polytope defined by $G_6\mathbf{x} = \mathbf{1}$, $\mathbf{x} \geq 0$ is not integral – for example one of its vertices is:

$$(1/2, 1/2, 0, 0, 0, 1/2, 0, 0, 0, 0, 0, 0, 1/2, 1/2, 1/2).$$

Therefore we cannot use Corollary 2.4.5 or Corollary 2.4.8. We illustrate two methods, one using Theorem 2.3.5 and another using Theorem 2.4.2.

We begin by showing that the set of external columns $\{\mathbf{v}_2, \dots, \mathbf{v}_6\}$ is G_6-lattice minimal: if $\mathbf{u} \in \mathrm{pos}_{\mathbb{R}}(\mathbf{v}_2, \dots, \mathbf{v}_6) \cap \mathcal{L}(G_6)$, then

$$\begin{aligned} \mathbf{u} &= \lambda_2 \mathbf{v}_2 + \cdots + \lambda_6 \mathbf{v}_6 \\ &= (\lambda_2 + \cdots + \lambda_6, \lambda_2, \dots, \lambda_6) \end{aligned}$$

for some $\lambda_2, \dots, \lambda_6 \geq 0$. Since $\mathbf{u} \in \mathcal{L}(G_6)$ each of $\lambda_2, \dots, \lambda_6$ must also be integral, and so $\mathbf{u} \in \mathrm{pos}_{\mathbb{N}}(G_6)$. Therefore, $\mathrm{pos}_{\mathbb{R}}(\mathbf{v}_2, \dots, \mathbf{v}_6) \cap \mathcal{L}(G_6) \subseteq \mathrm{pos}_{\mathbb{N}}(G_6)$. The reverse inclusion is immediate, so the columns $\{\mathbf{v}_2, \dots, \mathbf{v}_6\}$ do indeed form a G_6-lattice minimal set.

Using Latte, we compute that

$$f(t) := p_{G_6}(tv_1) = \binom{t+9}{9},$$

and so by Theorem 2.3.5,

$$p^{\gamma}_{G_6} = f\left(\frac{\det d, v_2, v_3, v_4, v_5, v_6}{\det v_1, v_2, v_3, v_4, v_5, v_6}\right) = \binom{\frac{-d_1+d_2+d_3+d_4+d_5+d_6}{2} + 9}{9}$$

on the lattice $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \equiv 0 \mod 2$.

Since $p^{\gamma}_{G_6}$ is a polynomial, we see that applying Theorem 2.4.2 is a simpler approach, since no computer aid is required. For the external facet of γ generated by $\mathbf{v}_2, \dots, \mathbf{v}_6$, the minimal inner facet normal is $\boldsymbol{\iota} = (-1, 1, 1, 1, 1, 1)$. We note that $\boldsymbol{\iota} \cdot \mathbf{g} = 2$ for all columns $\mathbf{g}$ of G_6 not on the external facet. Therefore,by Theorem 2.4.2,

$$\begin{aligned} p^{\gamma}_{G_6}(\mathbf{d}) &= \binom{\frac{\boldsymbol{\iota}\cdot\mathbf{d}}{2} + 9}{9} \\ &= \binom{\frac{-d_1+d_2+d_3+d_4+d_5+d_6}{2} + 9}{9}. \end{aligned}$$

We prove combinatorially that this result is true in general.

Theorem 2.6.4. *Let m be a positive integer, and let $(d_1, \ldots, d_m) \in \mathbb{N}^m$ be monotonically decreasing. If $d_1 + d_m \geq \sum_{i=2}^{m-1} d_i$, then*

$$M_m(d_1, \ldots, d_m) = \binom{e - d_1 + \binom{m-1}{2} - 1}{\binom{m-1}{2} - 1} \tag{2.25}$$

where e is the number of edges $\sum_{i=1}^{m} \frac{d_i}{2}$ of any multigraph with degree sequence $(d_1, \ldots, d_m)$.

Proof. Since $d_1 + d_m \geq \sum_{i=2}^{m-1} d_i$, we see that $2(d_1 + d_m) \geq \sum_{i=1}^{m} d_i$, so $d_1 + d_m \geq \frac{\sum_{i=1}^{m} d_i}{2}$ and so $d_m \geq e - d_1$. Now consider distributing edges between the vertices $v_2, v_3, \ldots, v_m$. There are $e - d_1$ edges to distribute, and $\binom{m-1}{2}$ vertex pairs. Since $d_i \geq d_m \geq e - d_1$ for any $2 \leq i \leq m$, we may distribute these edges in any way possible, and no vertex will be incident to too many edges. One can see that the number of such choices is given by Eq. (2.25). This leaves a single way to distribute the remaining edges from v_1 to $\{v_2, v_3, \ldots, v_m\}$. □

Remark 2.6.5. Recall that in Example 2.6.2 we computed $M_4(5, 4, 3, 2)$ using *Latte*. Since $d_1 + d_4 = 5 + 2 \geq 4 + 3 = d_2 + d_3$, we can also apply Theorem 2.6.4 to compute this value. Here $e = 7, d_1 = 5$ and $m = 4$, and so by Theorem 2.6.4:

$$M_4(5, 4, 3, 2) = \binom{7 - 5 + \binom{4-1}{2} - 1}{\binom{4-1}{2} - 1} = 6$$

agreeing with the value computed with *Latte*.

Although Theorem 2.6.4 in the end has a simple combinatorial proof, the vector partition function approach yields the correct inequalities to consider for which the formula becomes simple. We also give a geometric proof of Theorem 2.6.4. Recall from Example 2.6.3, that the matrices G_m are not DeLS-unimodular in general, so we cannot use Corollary 2.4.5 or Corollary 2.4.8 to compute the polynomial. Instead, we use Theorem 2.4.2.

Lemma 2.6.6. *Let $m \geq 3$ be an integer. The cone defined by the minimal ray generators*

$$\gamma := \{(m-3)\mathbf{e}_1 + \sum_{i=2}^{m} \mathbf{e}_i\} \cup \{\mathbf{e}_1 + \mathbf{e}_i : 2 \leq i \leq m\}$$

is an external chamber of G_m. Moreover the ray generator in the first set is the sole internal ray generator, and the ray generators in the second set are external columns. Finally the second set is G_m-lattice minimal.

Proof. The G_m-lattice minimality of the second set follows similarly to the proof in Example 2.6.3.

We now prove that γ is an external chamber of G_m. The vector $\mathbf{e}_1 + \mathbf{e}_i$ is a column of G_m corresponding to the edge $(1, i)$ of the complete graph K_m. Additionally, every other column $\mathbf{c}$ must have a non-zero entry c_j for some $j \notin \{1, i\}$. Therefore, $\mathbf{e}_1 + \mathbf{e}_i$ is not in the cone generated by the other $\binom{m}{2} - 1$ columns of G_m, and is thus an external ray generator. The cone f generated by $\{\mathbf{e}_1 + \mathbf{e}_i : 2 \le i \le m\}$ is the $(n-1)$-dimensional intersection of the cone $\mathrm{pos}_{\mathbb{R}}(G_m)$ with the hyperplane $d_1 - \sum_{i=2}^{m} d_i$, and is thus a facet of $\mathrm{pos}_{\mathbb{R}}(G_m)$. No other columns of G_m appear in the facet f, so f is an external facet. By Proposition 2.2.11, the external chamber $\tilde{\gamma}$ containing f is the intersection of all simplicial cones of G_m containing each of the columns $\mathbf{e}_1 + \mathbf{e}_i$ for $2 \le i \le m$. For j, k with $1 < j < k < m$, set $\sigma_{j,k} := \mathrm{pos}_{\mathbb{R}}(f, \mathbf{e}_j + \mathbf{e}_k)$ where we overload notation by denoting the ray generators of f by f. Then

$$\tilde{\gamma} = \bigcap_{1<j<k\le m} \sigma_{j,k}.$$

We now prove that $\gamma = \tilde{\gamma}$.

The cone $\sigma_{j,k}$ is defined by the inequalities

$$\begin{aligned} d_i &\ge 0 \text{ for all } i \ne 1, j, k \\ \sum_{i=2}^{m} d_i &\ge d_1 \\ d_1 + d_j &\ge \sum_{i=2, i\ne j}^{m} d_i \\ d_1 + d_k &\ge \sum_{i=2, i\ne k}^{m} d_i \end{aligned}$$

and so $\tilde{\gamma}$ is defined by the inequalities

$$\sum_{i=2}^{m} d_i \ge d_1 \tag{2.26}$$

$$d_1 + d_l \ge \sum_{i=2, i\ne l}^{m} d_i \text{ for all } 2 \le l \le m \tag{2.27}$$

where the inequalities $d_i \ge 0$ for $2 \le i \le m$ follow implicilty from Inequalities (2.26) and (2.27). This is the same set of inequalities defining γ. □

The inequality defining the external facet f is $\sum_{i=2}^{m} d_i \ge d_1$. Thus we see that for column k corresponding to the vector $\mathbf{e}_i + \mathbf{e}_j$ $(1 \le i < j \le m)$ the entry $(1, k)$ of $M_\gamma^\vee G_m$ is

$$(-1, 1, \ldots, 1) \cdot (\mathbf{e}_i + \mathbf{e}_j) = \begin{cases} 0 \text{ if } i = 1 \\ 2 \text{ if } i \ne 1 \end{cases}$$

and so by Theorem 2.4.2,

$$p_{G_m}(\mathbf{d}) = \binom{\frac{-d_1+d_2+\cdots+d_m}{2} + \binom{m-1}{2} - 1}{\binom{m-1}{2} - 1}$$

on the chamber defined by Inequalities (2.26) and (2.27).

This concludes our geometric proof of Theorem 2.6.4.

2.6.2 Kostant's Partition Function

Here we give an example where the matrices are unimodular, but where external chambers do not exist in each dimension (for example there are no external chambers in dimension 4). However, we are able to exploit Theorem 2.5.3 in order to handily reproduce a known result of Rassart [74].

For any positive integer m, let K'_m be the adjacency matrix of the directed tournament on $m+1$ vertices, and define K_m to be the matrix obtained by removing the last row of K'_m. The matrices K_m are unimodular and of rank m. The corresponding vector partition functions p_{K_m} are the *Kostant's partition functions for the root system* A_{m-1}.

Although the matrices are easy to describe, it is still open to compute the number of chambers of K_m (a problem posed by Kirilov [48]). This number has been computed for $m \leq 7$, and the piecewise polynomials p_{K_m} have been explicitly computed for $m \leq 6$ [53]. In this section, we use Theorem 2.5.3 in order to compute linear factors of polynomials of p_{K_m} whose chambers intersect facets $(m-1)$-dimensionally. We begin with an illustrative example.

Example 2.6.7. Consider the case of directed tournaments on 4 vertices. The corresponding vector partition function is p_{K_3} where

$$K_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The minimal inner facet normals of $\mathrm{pos}_{\mathbb{R}}(K_3)$ are

$$\boldsymbol{\iota}_1 := (1,0,0,),\ \boldsymbol{\iota}_2 := (1,1,0),\ \boldsymbol{\iota}_3 := (1,1,1).$$

A projection of the chamber complex of K_3 onto the plane $\{(x,y) : 3x+2y+z=1\}$ is illustrated in Figure 2.5. This plane was chosen to have normal vector $\boldsymbol{\iota}_1+\boldsymbol{\iota}_2+\boldsymbol{\iota}_3$. From the figure we see that there are three facets of the cone $\mathrm{pos}_{\mathbb{R}}(K_3)$, of which a single facet – the one generated by columns 1 and 6 of K_3 and with inner facet normal $\boldsymbol{\iota}_2$ - is external. We conclude from Corollary 2.4.5 that for the corresponding external chamber γ generated by columns 1,6 the associated polynomial is given by $p^{\gamma}_{A_3}(\mathbf{b}) = \binom{b_1+b_2+b_3+3}{3}$. Table 2.6.7 illustrates how one can use Theorem 2.5.3 in order to find linear factors for $p^{\gamma'}_{K_3}$ for any chamber γ' intersecting one of the three facets 2-dimensionally. We abuse notation by writing the linear factors $l(\mathbf{b})+1,\ldots,l(\mathbf{b})+k$ in the form $\binom{l(\mathbf{b})+k}{k}$.

Table 2.2: Linear factors associated to facets of $\mathrm{pos}_{\mathbb{R}}(K_3)$.

facet normal	inequality	columns in facet	# missing columns	linear factors
$\boldsymbol{\iota}_1$	$b_1 \geq 0$	4,5,6	3	$\binom{b_1+2}{2}$
$\boldsymbol{\iota}_2$	$b_1 + b_2 \geq 0$	1,6	4	$\binom{b_1+b_2+3}{3}$
$\boldsymbol{\iota}_3$	$b_1 + b_2 + b_3 \geq 0$	1,2,4	3	$\binom{b_1+b_2+b_3+2}{2}$

Proposition 2.6.8. *The minimal inner facet normals of* $\mathrm{pos}_{\mathbb{R}}(K_m)$ *are*

$$(1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1). \tag{2.28}$$

Proof. The minimal inner facet normals of $\mathrm{pos}_{\mathbb{R}}(K_m)$ generate the dual cone $\mathrm{pos}_{\mathbb{R}}(K_m)^\vee$. Also, $\mathbf{v} \in \mathrm{pos}_{\mathbb{R}}(K_m)^\vee$ if and only if $\mathbf{v} \cdot \mathbf{c} \geq 0$ for each column $\mathbf{c}$ of K_m. We now show that the second condition is equivalent to the condition $v_i \geq v_{i+1} \geq 0$ for each $i = 1, \dots, m-1$. Indeed, each column of K_m containing a -1 entry contains a $+1$ entry before it, and for each $j = 2, \dots, m$ there is a column of K_m containing a -1 at entry j. Therefore the dual cone is defined precisely by the inequalities $v_1 \geq v_2 \geq \dots \geq v_m \geq 0$. Equivalently the dual cone is generated by the vectors of (2.28), and so these are the inner facet normals of $\mathrm{pos}_{\mathbb{R}}(K_m)$. Finally, we note that they are indeed minimal. $\square$

We now give linear factors for polynomials associated to chambers intersecting facets of $\mathrm{pos}_{\mathbb{R}}(K_m)$ $(m-1)$-dimensionally. We remark once again that this result is originally due to Rassart [74]. Our purpose in describing it is to illustrate the utility of Theorem 2.5.3 - that is, simply knowing the facet normals of the cone $\mathrm{pos}_{\mathbb{R}}(K_m)$ yields enough information to compute linear factors of some of the polynomials of p_{K_m}.

Theorem 2.6.9 (Rassart, 2004 [74])**.** *Let γ be a chamber of K_m intersecting a facet f of* $\mathrm{pos}_{\mathbb{R}}(K_m)$ $(m-1)$*-dimensionally. Let $\boldsymbol{\iota}$ be the minimal inner facet normal of f so that $\boldsymbol{\iota}$ is a vector obtained by taking k ones followed by $m-k$ zeroes for some $1 \leq k \leq m$. Then the polynomial $p^{\gamma}_{K_m}(\mathbf{b})$ has linear factors*

$$\boldsymbol{\iota} \cdot \mathbf{b} + 1, \dots, \boldsymbol{\iota} \cdot \mathbf{b} + (m+1-k)k - 1.$$

Proof. We utilize the correspondence of columns of K_m and edges of the complete graph on $m+1$ vertices in order to count the number of columns of K_m not in f. A column c of K_m does not lie in f if it is not perpendicular to $\boldsymbol{\iota}$, which is equivalent to c having exactly one non-zero entry in the first k entries. The set of such columns corresponds to the set of edges with exactly one end in the first k vertices of the complete graph, and one end in the remaining $m+1-k$ vertices. Therefore there are $(m+1-k)(k)$ such columns. The result now follows by application of Theorem 2.5.3. $\square$

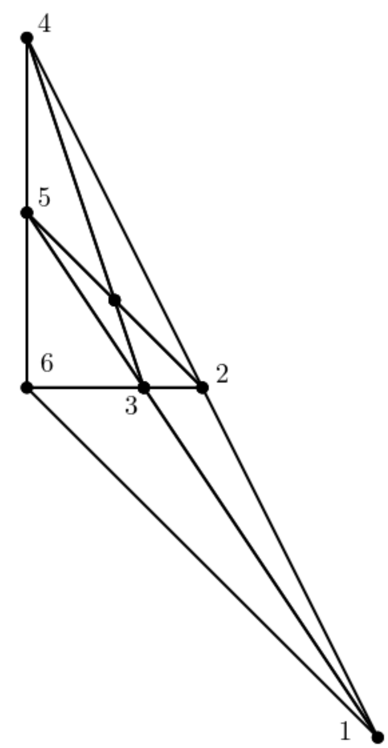


Figure 2.5: A projection of the chamber complex of K_3 via $(x, y, z) \rightarrow (\frac{x}{3x+2y+z}, \frac{y}{3x+2y+z})$. Here vertex i is obtained by projecting the ith column of K_3 (given in Example 2.6.7). The sole unnumbered vertex is the projection of the ray generated by $(1, 1, -1)$ obtained in the refinement process.

Chapter 3

Littlewood-Richardson coefficients

3.1 Background

The *Littlewood-Richardson coefficients* are the structure constants appearing from the ordinary multiplication of Schur functions (equivalently Schur polynomials with sufficiently many variables). That is, for partitions λ, μ, ν, the Littlewood-Richardson (LR) coefficients $c^{\nu}_{\lambda,\mu}$ are the coefficients appearing in the equation

$$s_\lambda s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu.$$

Recall that the Schur polynomials form a basis for the $\mathbb{Z}$-module of symmetric polynomials in n variables Λ_n, so that the Littlewood-Richardson coefficients are indeed well defined.

From this definition, one can show that a particular LR coefficient $c^{\nu}_{\lambda,\mu}$ can only be non-zero when $|\lambda|+|\mu|= |\nu|$. One can also understand the Littlewood-Richardson coefficients from the representation theory perspective. The irreducible polynomial representations of the complex general linear group GL_k are labelled by partitions of length $\leq k$. Denote by W_α the irreducible representation labelled by partition α. Given any triple of partitions λ, μ, ν the Littlewood–Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is defined as the multiplicity of W_ν in the tensor product $W_\lambda \otimes W_\mu$. We often make reference to the Littlewood-Richardson coefficients of GL_k in the text, by which we mean that the partitions λ, μ, ν each have length at most k.

The LR coefficients are non-negative integers, and have various combinatorial interpretations, called *Littlewood-Richardson rules*. Some examples are Littlewood-Richardson tableaux [52], Berenstein-Zelevinsky triangles [9], hives [50], and Littleman paths [51]. As a historical note, the first of these Littlewood-Richardson rules was initially conjectured, and partially proven in 1934, but its proof would wait 40 years, until 1974. Amusingly, as Gordon James relates in [44]:

> Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there.

Finding a proof of the Littlewood-Richardson rule was not the only long-standing problem in this area – the *Saturation Conjecture* (now Theorem) states that if $c^{N\nu}_{N\lambda,N\mu} \neq 0$ (where $N\lambda$ denotes the multiplication of each part of λ by N) for some positive integer N, then $c^{\nu}_{\lambda,\mu} \neq 0$. This result was finally proven true by Knutson and Tao in 1999 [50].

Another interesting phenomenon that appears for Littlewood-Richardson coefficients is that of *stability*. For partitions $\alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k)$ of length at most k, denote the component-wise sum $(\alpha_1 + \beta_1, \dots, \alpha_k + \beta_k)$ by $\alpha + \beta$. A triple of partitions λ^*, μ^*, ν^* of length at most k is a *stable triple* if $c^{\nu^*}_{\lambda^*,\mu^*} > 0$ and the sequence

$$(c^{\nu+N\nu^*}_{\lambda+N\lambda^*,\mu+N\mu^*})_{N\geq 0}$$

converges for any partitions λ, μ, ν of length at most k for which $c^{\nu}_{\lambda,\mu} > 0$. Moreover, the limit of this sequence is called the *stable value*. For certain triples of partitions λ^*, μ^*, ν^*, this phenomenon was studied by Murnaghan (see [65]). However, the systematic study only began in 2014 due to Stembridge (see [83]). In [78, Theorem 4.6], Sam and Snowden characterize the set of stable triples λ^*, μ^*, ν^* - namely they show that they are exactly the partition triples for which $c^{\nu^*}_{\lambda^*,\mu^*} = 1$.

Littlewood-Richardson coefficients have many applications in physics. The classification of orbital states of particles in the nuclear shell model is one such application where the LR coefficients represent the total angular momentum [35]. The LR coefficients also have applications in boundary conformal theory and integrable 2-d field theory [28, 37]. Other notable references include [39, 57, 66, 76, 79, 89].

For an excellent (although now slightly dated) overview of results relating to the LR coefficients the reader is directed to [87]. Additionally, the *Symmetric Functions Catalog* [1] maintained by Per Alexandersson provides a great resource for the current state of research.

Our work on Littlewood-Richardson coefficients is based on the approach of Rassart [75], who used the combinatorial interpretation of LR coefficients as counting hives in order to relate them to vector partition functions. Using this approach (and some additional facts), he proved that the LR coefficients associated to GL_k are given by a piecewise polynomial Φ_k whose domains of polynomiality are the maximal cones of a fan denoted by $\mathcal{LR}_k$. Additionally, Rassart explicitly computes the piecewise polynomial Φ_3 associated to the GL_3 case.

The next three sections do not contain novel results. In Section 3.2 we introduce hives and explain their connection with LR coefficients. We then describe how to construct the vector partition function formulation from the hive interpretation in Section 3.3. We also explicitly describe the piecewise polynomial Φ_3 in Section 3.4. The remainder of the chapter is new results, summarized in Section 3.5 and developed in Sections 3.6, 3.7, and 3.8.

3.2 Hives

Here we follow the notation of [75]. For $k \in \mathbb{N}$, a *k-hive* is a triangular array of numbers $a_{i,j}$ such that $0 \leq i, j \leq k$ and $i + j \leq k$. If $a_{i,j} \in \mathbb{Z}$ for all i, j we say that $(a_{i,j})$ is *integral.* We are interested in integral k-hives that also satisfy two sets of conditions: *hive conditions* HC and *boundary conditions* BC.

The hive conditions are given by the following set of inequalities that hold for all $i, j \geq 0$ such that $i + j \leq k - 2$:

$$\begin{aligned} a_{i+1,j} + a_{i,j+1} &\geq a_{i,j} + a_{i+1,j+1} \\ a_{i+1,j} + a_{i+1,j+1} &\geq a_{i+2,j} + a_{i,j+1} \\ a_{i,j+1} + a_{i+1,j+1} &\geq a_{i+1,j} + a_{i,j+2} \end{aligned}$$

The boundary conditions are given by a triple of partitions (λ, μ, ν) each of length $\leq k$. They are:

$$\begin{aligned} a_{0,0} &= 0 \\ a_{0,j} &= \lambda_1 + \cdots + \lambda_j \text{ for all } 0 \leq j \leq k \\ a_{i,0} &= \nu_1 + \cdots + \nu_i \text{ for all } 0 \leq i \leq k \\ a_{m,k-m} &= |\lambda| + \mu_1 + \cdots + \mu_m \text{ for all } 0 \leq m \leq k \end{aligned}$$

For $m = 0$, $a_{0,k} = |\lambda|$ which agrees with our definition for $a_{0,j}$ with $j = k$. Similarly, for $m = k$, $a_{k,0} = |\lambda| + |\mu| = |\nu|$ and thus this agrees with $a_{i,0}$ with $i = k$.

Theorem 3.2.1 (Knutson, Tao, 1999 [50])**.** *Given partitions λ, μ, ν of length at most k, $c^{\nu}_{\lambda,\mu}$ is the number of integral k-hives satisfying the hive conditions, and boundary conditions given by (λ, μ, ν).*

Example 3.2.2. A generic integral 3-hive satisfying BC is shown in Figure 3.1.

$$\begin{array}{cccc} 0 & \lambda_1 & \lambda_1 + \lambda_2 & \lambda_1 + \lambda_2 + \lambda_3 \\ \nu_1 & a_{1,1} & |\lambda| + \mu_1 & \\ \nu_1 + \nu_2 & |\lambda| + \mu_1 + \mu_2 & & \\ |\nu| & & & \end{array}$$

Figure 3.1: A 3-hive satisfying the boundary conditions for λ, μ, ν

For $\lambda = (5,3,0), \mu = (4,2,0), \nu = (7,5,2)$, the following are the only integral 3-hives satisfying HC and BC. Therefore, $c^{\nu}_{\lambda,\mu} = 3$.

$$\begin{array}{cccc} 0 & 5 & 8 & 8 \\ 7 & 10 & 12 & \\ 12 & 14 & & \\ 14 & & & \end{array} \qquad \begin{array}{cccc} 0 & 5 & 8 & 8 \\ 7 & 11 & 12 & \\ 12 & 14 & & \\ 14 & & & \end{array} \qquad \begin{array}{cccc} 0 & 5 & 8 & 8 \\ 7 & 12 & 12 & \\ 12 & 14 & & \\ 14 & & & \end{array}$$

3.3 A vector partition function for Littlewood-Richardson coefficients

Using the hive interpretation of Littlewood-Richardson coefficients, if the length of the partitions λ, μ, ν are fixed by some k, we can express the Littlewood-Richardson coefficients as something resembling a vector partition function. This is achieved by writing the boundary conditions and hive conditions in matrix form (introducing slack variables to express the hive conditions as equalities). This section provides a partial summary of the work done by Rassart in [75].

We begin by describing the process of building the vector partition function formulation from the hive conditions in the $k = 3$ case. We do not give the $k = 1$ or $k = 2$ cases since in this case, the Littlewood-Richardson coefficient is always either 0 or 1 (in these cases each of the hive entries are determined by the boundary conditions). Given λ, μ, ν, Figure 3.1 illustrates a 3-hive satisfying the boundary conditions BC. We note that in this case, the only unknown is $a_{1,1}$. We first rewrite each of the hive conditions, isolating $a_{1,1}$ in each inequality:

$$\begin{array}{rrr} a_{1,1} \le a_{0,1} + a_{1,0} - a_{0,0} & -a_{1,1} \le a_{0,2} - a_{0,1} - a_{1,2} & -a_{1,1} \le a_{2,0} - a_{1,0} - a_{2,1} \\ -a_{1,1} \le a_{0,1} - a_{1,0} - a_{0,2} & a_{1,1} \le a_{0,2} + a_{1,2} - a_{0,3} & -a_{1,1} \le a_{2,1} - a_{2,0} - a_{1,2} \\ -a_{1,1} \le a_{1,0} - a_{2,0} - a_{0,1} & -a_{1,1} \le a_{1,2} - a_{2,1} - a_{0,2} & a_{1,1} \le a_{2,0} + a_{2,1} - a_{3,0} \end{array}$$

There are 9 defining inequalities for $a_{1,1}$. Substituting in the boundary conditions and introducing slack variables $s_1, s_2, \ldots, s_9 \in \mathbb{N}$, we obtain the system:

$$\begin{array}{rrr} a_{1,1} + s_1 = \lambda_1 + \nu_1 & -a_{1,1} + s_2 = -\lambda_1 - \lambda_3 - \mu_1 & -a_{1,1} + s_3 = -|\lambda| - \mu_1 - \mu_2 + \nu_2 \\ -a_{1,1} + s_4 = -\lambda_2 - \nu_1 & a_{1,1} + s_5 = \lambda_1 + \lambda_2 + \mu_1 & -a_{1,1} + s_6 = \mu_2 - \nu_1 - \nu_2 \\ -a_{1,1} + s_7 = -\lambda_1 - \nu_2 & -a_{1,1} + s_8 = -\lambda_1 - \lambda_2 - \mu_2 & a_{1,1} + s_9 = |\lambda| + \mu_1 + \mu_2 - \nu_3. \end{array}$$

Finally, we rewrite this as the matrix equation:

$$E_3(a_{1,1}, s_1, \ldots, s_9)^T = B_3(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3)^T$$

where

$$E_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is the number of solutions

$$\mathbf{x} = (a_{1,1}, s_1, \ldots, s_9)^T \ \in \ \mathbb{N}^{10}$$

to the equation

$$E_3\mathbf{x} = B_3(\lambda|\mu|\nu)$$

where $(\lambda|\mu|\nu)$ denotes the column vector $(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3)^T$.

We have encoded the hive conditions into a linear system for which the number of solutions is equal to the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$. This formulation can be generalized for any $k \geq 3$. Up to a permutation of rows, one obtains matrices E_k and B_k, which are the coefficient matrices for the hive inequalities and their corresponding slack variables; and (λ, μ, ν) respectively. For λ, μ, ν with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq k$, one can express the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ as the number of solutions $\mathbf{x} \in \mathbb{N}^d$ to the equation

$$E_k\mathbf{x} = B_k(\lambda|\mu|\nu) \tag{3.1}$$

where d is the number of undetermined hive values plus the number of slack variables introduced (one for each inequality).

Define $\mathcal{T}_k := \{(\lambda, \mu, \nu) : \ell(\lambda), \ell(\mu), \ell(\nu) \leq k, |\lambda| = |\mu| = |\nu|\}$, and the function

$$\begin{aligned} \Phi_k : \mathcal{T}_k &\to \mathbb{N} \\ (\lambda, \mu, \nu) &\mapsto c^{\nu}_{\lambda,\mu}. \end{aligned}$$

Rewriting Eq. (3.1) using vector partition functions, we obtain

$$c^{\nu}_{\lambda,\mu} = \Phi_k(\lambda, \mu, \nu) = p_{E_k}(B_k(\lambda, \mu, \nu)). \tag{3.2}$$

Therefore, one can compute LR coefficients by computing the vector partition function associated to E_k, finding the chamber containing the point $\mathbf{p} := B_k(\lambda, \mu, \nu)$, and finally using the appropriate quasi-polynomial to evaluate $\mathbf{p}$. However, this approach is inefficient since the dimension of the chamber complex of E_k, $3\binom{k}{2}$, is larger than $3k-1$ (the dimension of the space containing $\mathcal{T}_k$). Additionally, one would ideally like an explicit description of the function Φ_k. In fact, Rassart proves that such a description not only exists, but that it is also particularly nice.

Theorem 3.3.1 (Rassart, 2004 [75])**.** *The function Φ_k is a piecewise polynomial of degree at most $\binom{k-1}{2}$ whose domains of polynomiality are the maximal cones of a fan $\mathcal{LR}_k$.*

We note that the matrices E_k are not necessarily unimodular (nor DeLS-unimodular), so that the polynomiality of Φ_k is not immediately evident from the vector partition formulation of Eq. (3.2). For example, as we illustrate in Section 3.8, the matrix E_4 is neither unimodular nor DeLS-unimodular.

We now summarize Rassart's process for obtaining the fan $\mathcal{LR}_k$ from the chamber complex of E_k. For ease of notation, we let Γ_k denote the chamber complex of E_k, and let τ_k be cone defined by the inequalities $\lambda_k, \mu_k \geq 0$. Figure 3.2 provides an illustration of the process with each arrow labelled by the corresponding step outlined below.

1. Intersect Γ_k with the column space of B_k, $\mathrm{col}(B_k)$, to obtain the fan $\mathcal{C}_k$.

2. Replace each cone σ in $\mathcal{C}_k$ defined by inequalities $\{\boldsymbol{\iota}_i \cdot B_k(\lambda|\mu|\nu)^T \geq 0 : i = 1, \ldots, m\}$ with the cone $\rho_k(\sigma)$ defined by the inequalities $\{B_k^T \boldsymbol{\iota}_i \cdot (\lambda|\mu|\nu)^T \geq 0 : i = 1, \ldots, m\}$ in order to obtain a fan $B_k^* \mathcal{C}_k \subseteq \mathbb{R}^{3k}$. Rassart calls this the *rectified complex* - following this nomenclature, we call the process of replacing σ with $\rho_k(\sigma)$ *rectifying* the cone σ.

3. Take the intersection of $B_k^* \mathcal{C}_k$ with the cone τ_k to obtain the fan $\mathcal{LR}_k \subseteq \mathbb{R}^{3k}$.[1]

[1] In Rassart's original formulation, the intersection is taken with two cones – one defining partition inequalities, and another defining conditions for the LR coefficient to be non-zero. However, Pak and Vallejo [71] have shown that each of the partition inequalities except $\lambda_k, \mu_k, \nu_k \geq 0$ are implied by the hive inequalities, and Briand [14] has shown that $\nu_k \geq 0$ is also implied. The non-negativity conditions are also implied since each cone of Γ_k is in $\mathrm{pos}_{\mathbb{R}}(E_k)$.

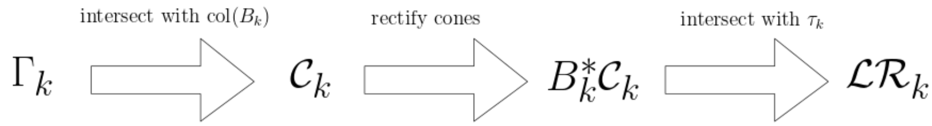


Figure 3.2: Obtaining the fan $\mathcal{LR}_k$ from the chamber complex Γ_k of E_k.

The chambers of $\mathcal{LR}_k$ each arise as a set of the form

$$\rho_k\left(\gamma \cap \operatorname{col}(B_k)\right) \cap \tau_k,$$

for some chamber γ of E_k. We remark however that not all of the chambers of E_k map to chambers of $\mathcal{LR}_k$. Indeed, as we shall see in Section 3.8, the chamber complex of E_3 has 21 chambers, of which only 18 map to chambers of $\mathcal{LR}_3$. Now, consider some chamber $\gamma' \in \mathcal{LR}_k$ with

$$\gamma' := \rho_k(\gamma \cap \operatorname{col}(B_k)) \cap \tau_k$$

for some chamber γ of E_k. Then

$$\Phi_k^{\gamma'}(\lambda, \mu, \nu) = p_{E_k}^{\gamma}(B_k(\lambda|\mu|\nu))$$

and so $p_{E_k}^{\gamma}$ is a polynomial. In words, the quasipolynomials associated to chambers of E_k that map to chambers of $\mathcal{LR}_k$ via Rassart's procedure are actually polynomials.

The union of the cones of $\mathcal{LR}_k$ is called the *positive Horn cone*, and we denote it by H_k^+. The positive Horn cone H_k^+ admits an explicit description – namely, it is the set of points in $\mathbb{R}^{3k}$ satisfying $|\lambda|+|\mu|=|\nu|$, each of the partition inequalities $\lambda_i \geq \lambda_{i+1} \geq 0$, $\mu_i \geq \mu_{i+1} \geq 0$, $\nu_i \geq \nu_{i+1} \geq 0$ for $i = 1, \ldots, k-1$, as well as a set of inequalities called *Horn inequalities*. These inequalities have been described explicitly by Klyachko, as well as (independently) Knutson and Tao (we direct the reader to [47, Theorem 2.3] for the formulation). We remark that not all Horn inequalities define facets of H_k^+. Those that do are called *essential Horn inequalities*.

3.4 The piecewise polynomial Φ_3

As previously stated, the piecewise polynomial Φ_3 is given explicitly by Rassart in [75]. We reproduce it here following the the same notation. The rays of the fan $\mathcal{LR}_3$ are generated by the following minimal ray generators:

$$\begin{aligned}
\mathbf{a}_1 &= (1,1,1 \mid 0,0,0 \mid 1,1,1) & \mathbf{a}_2 &= (0,0,0 \mid 1,1,1 \mid 1,1,1)\\
\mathbf{b} &= (2,1,0 \mid 2,1,0 \mid 3,2,1) & \mathbf{c} &= (1,1,0 \mid 1,1,0 \mid 2,1,1)\\
\mathbf{d}_1 &= (1,1,0 \mid 1,0,0 \mid 1,1,1) & \mathbf{d}_2 &= (1,0,0 \mid 1,1,0 \mid 1,1,1)\\
\mathbf{e}_1 &= (1,1,0 \mid 0,0,0 \mid 1,1,0) & \mathbf{e}_2 &= (0,0,0 \mid 1,1,0 \mid 1,1,0)\\
\mathbf{f} &= (1,0,0 \mid 1,0,0 \mid 1,1,0) & &\\
\mathbf{g}_1 &= (1,0,0 \mid 0,0,0 \mid 1,0,0) & \mathbf{g}_2 &= (0,0,0 \mid 1,0,0 \mid 1,0,0)
\end{aligned}$$

and the chambers (along with their polynomials) are given in Table 3.1. We have followed the notation of [75]. The purpose of pairing off minimal ray generators (i.e having $\mathbf{a}_1, \mathbf{a}_2$) is to emphasize the effect of the linear symmetry of the LR coefficients obtained by interchanging λ and μ (this will be discussed in more detail in Section 3.7). The ray generators that have no pair ($\mathbf{b}, \mathbf{c}, \mathbf{f}$) are each stabilized by the symmetry. Throughout this section $\mathbf{e}_1, \mathbf{e}_2$ denote the vectors given above, unless it is explicitly specified that we are dealing with standard basis vectors.

We note that $\mathcal{LR}_3$ is the fan defined by the chambers $\kappa_1, \ldots, \kappa_{18}$ - that is the cones of $\mathcal{LR}_3$ are the cones $\kappa_1, \ldots, \kappa_{18}$ along with all of their faces. Similarly, H_3^+ is the cone generated by each of the minimal ray generators, that is: $\mathsf{H}_3^+ = \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{g}_2)$.

The polynomials of Φ_3 were computed by interpolation by Rassart. We note that the description of Φ_3 given in [75] contains a couple of typos: ν_3 appears in Chambers 13 and 14 (in [75]) instead of ν_1 (which is correct).

Example 3.4.1. Let $\lambda = (5,3,0),\ \mu = (4,2,0),\ \nu = (7,5,2)$, and let $\mathbf{p} = (\lambda|\mu|\nu)$. Then $\mathbf{p}$ is in κ_1 since

$$\mathbf{p} = 2\mathbf{b} + \mathbf{e}_1.$$

Evaluating using the polynomial $\Phi_3^{\kappa_1}$ associated to κ_1, we find that

$$\begin{aligned}
c^{\nu}_{\lambda,\mu} &= 1 - \lambda_2 - \mu_2 + \nu_1\\
&= 1 - 3 - 2 + 7\\
&= 3.
\end{aligned}$$

This matches the answer found by counting hives in Example 3.2.2. The point $\mathbf{p}$ is also in Chambers $3, 4, 6, 7, 9, 10, 11, 13, 15, 17$, and 18. One can check that the result is invariant to the choice of chamber containing $\mathbf{p}$.

Table 3.1: The piecewise-polynomial Φ_3

Chamber	Ray generators	Polynomial
κ_1	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2$	$1-\lambda_2-\mu_2+\nu_1$
κ_2	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{g}_1, \mathbf{g}_2$	$1+\nu_2-\nu_3$
κ_3	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_1, \mathbf{g}_2$	$1+\lambda_1+\mu_1-\nu_1$
κ_4	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}$	$1+\nu_1-\nu_2$
κ_5	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1+\lambda_2+\mu_2-\nu_3$
κ_6	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1-\lambda_3-\mu_3+\nu_3$
κ_7	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{g}_1$	$1+\lambda_3+\mu_1-\nu_3$
κ_8	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_2, \mathbf{g}_2$	$1+\lambda_1+\mu_3-\nu_3$
κ_9	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_2$	$1+\lambda_1-\lambda_2$
κ_{10}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{g}_1$	$1+\mu_1-\mu_2$
κ_{11}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{g}_1, \mathbf{g}_2$	$1-\lambda_2-\mu_3+\nu_2$
κ_{12}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}_2, \mathbf{e}_2, \mathbf{g}_1, \mathbf{g}_2$	$1-\lambda_3-\mu_2+\nu_2$
κ_{13}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{f}, \mathbf{g}_1$	$1-\lambda_1-\mu_3+\nu_1$
κ_{14}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_2$	$1-\lambda_3-\mu_1+\nu_1$
κ_{15}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1+\mu_2-\mu_3$
κ_{16}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_2, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$	$1+\lambda_2-\lambda_3$
κ_{17}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_2$	$1+\lambda_1+\mu_2-\nu_2$
κ_{18}	$\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1$	$1+\lambda_2+\mu_1-\nu_2$

Briand, Rosas, and I have computed the fan $\mathcal{LR}_4$ [20] by using the GIT-fan library developed by Boehm, Keicher and Ren [12] in *Singular* [32][2]. In this case, there are 67769 chambers and 515 rays (i.e 1-dimensional cones of the fan $\mathcal{LR}_4$). In Section 3.7, we illustrate how the explicit description of $\mathcal{LR}_4$ aids in computing symmetries of the LR coefficients. Additionally, in Section 3.8 we use this explicit description in order to make some conjectures about the linear factors of the polynomials of Φ_k.

3.5 Summary of original contributions

In the remainder of this chapter, we present novel results related to Littlewood-Richardson coefficients.

The main result of Section 3.6 is a unifying determinantal formula for the Littlewood-Richardson coefficients of GL_3 (Theorem 3.6.4) starting from Rassart's description of the fan $\mathcal{LR}_3$. We also describe a different result characterizing the polynomials of GL_3 via integral points on an interval. This interpretation then enables us to find a geometrical interpretation for a well-known stability result (Proposition 3.6.9) for the LR coefficients

[2]The chamber complex of A can be obtained as the GIT-fan of A along with the 0 ideal of the ring $\mathbb{C}[x_1, \ldots, x_n]$.

associated to GL_3. This is our second main result. The two are connected by the observation that a single ray of $\mathcal{LR}_3$ has a non-constant Ehrhart polynomial.

In [19], Briand and Rosas computed the full group of linear symmetries of the Littlewood-Richardson coefficients of GL_3. Surprisingly, they found that the number of such symmetries is 288 - significantly more than the 24 well-known symmetries in the literature. As previously mentioned, Briand, Rosas, and I have computed the fan $\mathcal{LR}_4$. In Section 3.7 we use the explicit description of the fan $\mathcal{LR}_4$ in order show that in the GL_4 case the only linear symmetries are the 24 well-known ones (Theorem 3.7.2). Additionally, we present *cell diagrams* through which one can view the effect of the linear symmetries on the cones of the chamber complex of $SL(3)$ (obtained by removing the rays $\mathbf{a}_1, \mathbf{a}_2$ from the fan $\mathcal{LR}_3$).

In Section 3.8, we study the appearance of linear factors arising in the polynomials of Φ_k. The chambers and associated polynomials of Φ_k arise from the vector partition function p_{E_k}. In Conjecture 3.8.1 we hypothesize a simple connection between the facets of H_k^+ and $\mathrm{pos}_{\mathbb{R}}(E_k)$ that, in conjunction with Conjecture 2.5.4, would allow one to compute these linear factors. Notably, in the case that a facet of $\mathrm{pos}_{\mathbb{R}}(E_k)$ is external, the formula for the corresponding external chamber of E_k can be computed via Theorem 2.4.2. In such a case we would obtain a negative binomial coefficient formula for the associated chamber of H_k^+ (see Example 3.8.3).

3.6 Novel results in the $k = 3$ case

In this section, we consider the GL_3 case in more detail. In particular, we exploit the linearity of the polynomials of Φ_3 and properties of the ray generators in order to obtain a determinantal formula, as well as some stability results.

3.6.1 Determinantal formula

The polynomials for each chamber of $\mathcal{LR}_3$ were computed by interpolation by Rassart. By studying the geometry closely, we are able to find a determinantal formula that yields a new interpretation for the LR coefficients associated to GL_3. This interpretation shows that the LR coefficients of GL_3 can be viewed as continuous volumes of a parallelepiped.

Our main observation is that the only minimal ray generator of the fan $\mathcal{LR}_3$ that has a non-constant Ehrhart polynomial is $\mathbf{b}$, for which the the Ehrhart polynomial is

$$\Phi_3(t\mathbf{b}) = t + 1.$$

Indeed, for any of the other minimal ray generators $\mathbf{v} \in \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2\}$, the Ehrhart polynomial is constant, i.e $\Phi_3(t\mathbf{v}) = 1$. Recall from the previous chapter that external columns $\mathbf{a}_j$ of A (for some $j \in \{1, \ldots, n\}$) can also be characterized as exactly the columns of A for which the associated Ehrhart quasi-polynomial $p_A(t\mathbf{a}_j)$ has degree

0. In fact, the external columns generate exactly the set of 1-dimensional cones (rays) of the chamber complex of A whose associated Ehrhart quasi-polynomial has degree 0. In this sense, we may view the minimal ray generators $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2\}$ as "external" and the minimal ray generator $\mathbf{b}$ as internal. Moreover, each of the chambers of $\mathcal{LR}_3$ is simplicial, and contains $\mathbf{b}$ as a ray generator. Therefore, we may also (in some sense) view each of the chambers as "external." Theorem 3.6.4 in this section, as well as Example 3.8.3 in Section 3.8 suggest that the results of the previous section may be generalized beyond vector partition functions. Unfortunately, we have not yet been able to prove this generalization.

We note that the Ehrhart polynomials associated to the ray generators can easily be verified by evaluating Rassart's polynomials at the appropriate vectors. However, one can also compute the Ehrhart polynomials directly from the hive interpretation.[3] We note that what we have called Ehrhart polynomials here are generally called *stretched Littlewood-Richardson coefficients* in the literature.

It will also be useful to introduce the auxilliary function $\Psi_3 : \mathsf{H}_3^+ \cap \mathbb{Z}^9 \to \mathbb{N}$ defined by

$$\Psi_3(\lambda, \mu, \nu) := \Phi_3(\lambda, \mu, \nu) - 1.$$

We note that Ψ_3 is a piecewise polynomial as well, and moreover, the polynomials of Ψ_3 are linear forms. Also, $\Psi_3(\mathbf{b}) = 1$, and $\Psi_3(\mathbf{v}) = 0$ for all $\mathbf{v} \in \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2\}$.

Recall that each of the chambers of $\mathcal{LR}_3$ is simplicial. Thus, for any chamber $\gamma \in \mathcal{LR}_3$, each point $\mathbf{p} \in \gamma \cap \mathbb{Z}^9$ can be expressed uniquely as a linear combination of the minimal ray generators of γ. For such a minimal ray generator $\mathbf{v}$ of γ, denote by $t^{\gamma}_{\mathbf{p},\mathbf{v}}$ the coefficient of $\mathbf{v}$ in this expression. Recall also that $\mathbf{b}$ is a ray generator for each of the 18 chambers of $\mathcal{LR}_3$. Thus, given a point $\mathbf{p} = (\lambda|\mu|\nu) \in \gamma$, the coefficient $t^{\gamma}_{\mathbf{p},\mathbf{b}}$ associated to $\mathbf{b}$ is well-defined regardless of the choice of γ. In fact, $t^{\gamma}_{\mathbf{p},\mathbf{b}}$ is invariant to the choice of chamber γ containing $\mathbf{p}$.

Lemma 3.6.1. *Let $\mathbf{p} \in H_3^+ \cap \mathbb{Z}^9$, such that $\mathbf{p} \in \gamma$ for some chamber γ in $\mathcal{LR}_3$. Then $t^{\gamma}_{\mathbf{p},\mathbf{b}}$ is invariant to the choice of γ containing* $\mathbf{p}$.

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the chambers of $\mathcal{LR}_3$ containing the point $\mathbf{p}$. Then

$$\gamma' := \bigcap_{i=1}^{m} \gamma_i \in \mathcal{LR}_3$$

is a face of each of $\gamma_1, \gamma_2, \ldots, \gamma_m$ since $\mathcal{LR}_3$ is a fan. In particular, γ' is simplicial and generated by the common minimal ray generators of $\gamma_1, \ldots, \gamma_m$. Each of $\gamma', \gamma_1, \ldots, \gamma_m$ are simplicial, so there is a unique way to express $\mathbf{p}$ as a linear combination of the minimal ray

[3] The fact that $\Phi_3(t\mathbf{b}) = t+1$ can also be viewed as a special case of a nice result proved by Ikenmeyer [42] that $\Phi_k(t(\lambda|\mu|\nu)) = t + 1$ whenever $c^{\nu}_{\lambda,\mu} = 2$.

generators of each of these cones. This unique choice is given by the linear combination of minimal ray generators of γ' in each case, and so $t^{\gamma}_{\mathbf{p},\mathbf{b}}$ is indeed invariant to the choice of chamber γ containing $\mathbf{p}$. $\square$

Motivated by this, we define the $\mathbf{b}$-*coefficient* of a point $\mathbf{p}$, denoted $t_{\mathbf{p},\mathbf{b}}$ to be the coefficient of $\mathbf{b}$ when $\mathbf{p}$ is expressed as a linear combination of the generating vectors of γ for any chamber γ containing $\mathbf{p}$.

Proposition 3.6.2. *Let λ, μ, ν be partitions of length ≤ 3, and let $\mathbf{p} := (\lambda|\mu|\nu)$. Then $c^{\nu}_{\lambda,\mu} = t_{\mathbf{p},\mathbf{b}} + 1$.*

Proof. Let $\gamma \in \mathcal{LR}_3$ be a chamber containing $\mathbf{p}$, with generators $\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_7$, so that $p = t_{\mathbf{p},\mathbf{b}}\mathbf{b} + t^{\gamma}_{\mathbf{p},\mathbf{v}_1}\mathbf{v}_1 + \cdots + t^{\gamma}_{\mathbf{p},\mathbf{v}_7}\mathbf{v}_7$. Then

$$
\begin{aligned}
c^{\nu}_{\lambda,\mu} &= \Phi_3(\mathbf{p}) && (3.3)\\
&= \Psi_3(t_{\mathbf{p},\mathbf{b}}\mathbf{b} + t^{\gamma}_{\mathbf{p},\mathbf{v}_1}\mathbf{v}_1 + \cdots + t^{\gamma}_{\mathbf{p},\mathbf{v}_7}\mathbf{v}_7) + 1 && (3.4)\\
&= t_{\mathbf{p},\mathbf{b}}\Psi_3(\mathbf{b}) + t^{\gamma}_{\mathbf{p},\mathbf{v}_1}\Psi_3(\mathbf{v}_1) + \cdots + t^{\gamma}_{\mathbf{p},\mathbf{v}_7}\Psi_3(\mathbf{v}_7) + 1 && (3.5)\\
&= t_{\mathbf{p},\mathbf{b}} + 1 && (3.6)
\end{aligned}
$$

as required. We note that (3.5) follows by the linearity of Ψ_3. $\square$

Example 3.6.3. Recall from Example 3.4.1 that for $\lambda = (5,3,0), \mu = (4,2,0), \nu = (7,5,2)$ the point $\mathbf{p} = (5,3,0,4,2,0,7,5,2)$ is in κ_1, and that $\mathbf{p} = 2\mathbf{b} + \mathbf{e}_1$. Thus, $t_{\mathbf{p},\mathbf{b}} = 2$, and so $c^{\nu}_{\lambda,\mu} = 2 + 1 = 3$.

We now rewrite the $\mathbf{b}$-coefficient by exploiting Cramer's rule. In order to do this, it is convenient to project the fan $\mathcal{LR}_3$ into $\mathbb{R}^8$ by dropping the last coordinate to obtain a fan $\tilde{\mathcal{LR}}_3 \subseteq \mathbb{R}^8$. Since $\mathcal{LR}_3 \subseteq \mathbb{R}^9$ lies in the 8-dimensional subspace defined by the equation $|\lambda|+|\mu|= |\nu|$, the last coordinate can be recovered. For each of the relevant objects in $\mathbb{R}^9$, we define the analogous objects in $\mathbb{R}^8$ obtained by deletion of the final coordinate by putting a ˜ overhead. That is, we denote the union of the chambers of $\tilde{\mathcal{LR}}_3$, by $\tilde{\mathsf{H}}^+_3$, the chambers of $\tilde{\mathcal{LR}}_3$ by $\tilde{\kappa_1}, \ldots, \tilde{\kappa_{18}}$, and so on.

Then, $\mathbf{p} \in \gamma$ if and only if $\tilde{\mathbf{p}} \in \tilde{\gamma}$. Additionally, $t_{\tilde{\mathbf{p}},\tilde{\mathbf{b}}}$ is well-defined (i.e doesn't depend on choice of chamber of $\tilde{\mathcal{LR}}_3$ containing $\tilde{\mathbf{p}}$), and is equal to $t_{\mathbf{p},\mathbf{b}}$. Fix an ordering of the ray generators of $\tilde{\mathcal{LR}}_3$, say

$$\tilde{\mathbf{a}_1} < \tilde{\mathbf{a}_2} < \tilde{\mathbf{b}} < \tilde{\mathbf{c}} < \tilde{\mathbf{d}}_1 < \tilde{\mathbf{d}}_2 < \tilde{\mathbf{e}_1} < \tilde{\mathbf{e}_2} < \tilde{\mathbf{f}} < \tilde{\mathbf{g}_1} < \tilde{\mathbf{g}_2}.$$

Let λ, μ, ν be partitions of length ≤ 3, and let $\tilde{\mathbf{p}} = (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2)$. For a given chamber $\tilde{\gamma} \in \tilde{\mathcal{LR}}_3$ containing $\tilde{\mathbf{p}}$, let $D_{\tilde{\gamma}}$ be the matrix with the generators of $\tilde{\gamma}$ as its columns (respecting the given ordering). Let i be the column index corresponding to $\tilde{\mathbf{b}}$, and let $D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}}$ be the matrix $D_{\tilde{\gamma}}$ with column i replaced by $\tilde{\mathbf{p}}$.

Then, by Cramer's rule, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is

$$c^{\nu}_{\lambda,\mu} = \frac{\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})}{\det(D_{\tilde{\gamma}})} + 1.$$

Furthermore, we may note that for any chamber $\tilde{\gamma} \in \tilde{\mathcal{LR}}_3$, $\det(D_{\tilde{\gamma}}) = \pm 1$, and that the signs of $\det(D_{\tilde{\gamma}})$ and $\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})$ are the same, so this expression simplifies to $|\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})|+1$. We record this result in the following theorem. Here $\tilde{\nu} = (\nu_1, \nu_2)$.

Theorem 3.6.4. *Let λ, μ, ν be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 3$, $\tilde{\mathbf{p}} := (\lambda|\mu|\tilde{\nu})$, and $\tilde{\gamma} \in \tilde{\mathcal{LR}}_3$ be a chamber containing $\tilde{\mathbf{p}}$. Then*

$$c^{\nu}_{\lambda,\mu} = |\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})|+1.$$

Let $\tilde{\mathbf{v}_1}, \ldots, \tilde{\mathbf{v}_7}$ denote the minimal ray generators of $\tilde{\gamma}$ apart from $\tilde{\mathbf{b}}$. Then $|\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})|$ represents the volume of the parallelepiped

$$\Pi^{\tilde{\gamma}}_{\tilde{\mathbf{p}}} := \{t_0\tilde{\mathbf{p}} + t_1\tilde{\mathbf{v}_1} + \cdots + t_7\tilde{\mathbf{v}_7} : 0 \leq t_0, \ldots, t_7 \leq 1\}$$

and so the Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu} = \Phi_3(\mathbf{p})$ can be viewed as a continuous volume.

Example 3.6.5. Consider our running example with $\lambda = (5,3,0)$, $\mu = (4,2,0)$, $\nu = (7,5,2)$, $\mathbf{p} = (5,3,0,4,2,0,7,5,2)$ (so that $\tilde{\mathbf{p}} = (5,3,0,4,2,0,7,5)$) and $\gamma = \kappa_1$. We deduce that

$$D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}} = \begin{array}{c} \begin{array}{cccccccc} \tilde{\mathbf{a}_1} & \tilde{\mathbf{a}_2} & \tilde{\mathbf{p}} & \tilde{\mathbf{c}} & \tilde{\mathbf{d}_1} & \tilde{\mathbf{d}_2} & \tilde{\mathbf{e}_1} & \tilde{\mathbf{e}_2} \end{array} \\ \begin{pmatrix} 1 & 0 & 5 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 7 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

One can check that $|\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})| = 2$ as expected, and so $c^{\nu}_{\lambda,\mu} = 2 + 1 = 3$.

We now consider some consequences of our determinantal formula. Firstly, we are able to obtain a straightforward proof of the following theorem of King, Tollu, and Toumazet.

Corollary 3.6.6 (King, Tollu, Toumazet, 2004 [46])**.** *Let λ, μ, ν be partitions of length at most 3. Then*

$$c^{N\nu}_{N\lambda,N\mu} = 1 + N(c^{\nu}_{\lambda,\mu} - 1) \tag{3.7}$$

for any positive integer N.

Proof. This follows from the properties of the determinant. □

Rassart reproves this result with a case by case analysis on each of the 18 polynomials corresponding to the chambers. However, this follows directly from Theorem 3.6.4 by determinant properties. Letting $\tilde{\mathbf{p}} = (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2)$, and $\tilde{\gamma} \in \widetilde{\mathcal{LR}}_3$ a chamber containing $\tilde{\mathbf{p}}$, we have:

$$\begin{aligned} c^{N\nu}_{N\lambda,N\mu} &= |\det(D^{N\tilde{\mathbf{p}}}_{\tilde{\gamma}})|+1 \\ &= N|\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})|+1 \\ &= N(c^{\nu}_{\lambda,\mu} - 1) + 1. \end{aligned}$$

Secondly, we can deduce the the Saturation Theorem for the GL_3 case since if

$$c^{N\nu}_{N\lambda,N\mu} \;=\; |\det(D^{N\tilde{\mathbf{p}}}_{\tilde{\gamma}})| \;+\; 1 \;>\; 0$$

then clearly $c^{\nu}_{\lambda,\mu} = |\det(D^{\tilde{\mathbf{p}}}_{\tilde{\gamma}})|+1 > 0$ as well.

Finally, consider (λ, μ, ν) with $c^{\nu}_{\lambda,\mu} = 1$. It is clear that $\tilde{\mathbf{p}} = (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2)$ must be in the facet of $\tilde{\gamma}$ generated by all vectors of $\tilde{\gamma}$ with the exception of $\tilde{\mathbf{b}}$. Thus, $N\tilde{\mathbf{p}}$ must also lie in this facet for all positive integers N, and so $c^{\nu}_{\lambda,\mu} = 1 \implies c^{N\nu}_{N\lambda,N\mu} = 1$ as well. From a geometrical point of view, we see that in this case the paralleliped $\Pi^{\tilde{\gamma}}_{\tilde{\mathbf{p}}}$ is not 8-dimensional so its volume is 0.

3.6.2 An alternative interpretation of Φ_3

In this section we show how to obtain the polynomials of Φ_3 from the hive inequalities. Our main purpose is to prove a result on stability (Proposition 3.6.9). However the intermediate result given in this section (Eq. (3.8)) is of independent interest.

The piecewise polynomial Φ_3 can also be well understood from the hive inequalities, which we reproduce here (with some rearrangement, and by using the equality $|\lambda|+|\mu|= |\nu|$):

$$a_{1,1} \leq \underbrace{\lambda_1 + \nu_1}_{f_1} \tag{H1}$$

$$a_{1,1} \leq \underbrace{\nu_1 + \nu_2}_{f_2} \tag{H2}$$

$$a_{1,1} \leq \underbrace{\lambda_1 + \lambda_2 + \mu_1}_{f_3} \tag{H3}$$

$$a_{1,1} \geq \underbrace{\lambda_2 + \nu_1}_{g_1} \tag{h1}$$

$$a_{1,1} \geq \underbrace{\nu_1 + \nu_3 - \mu_3}_{g_2} \tag{h2}$$

$$a_{1,1} \geq \underbrace{-\mu_2 + \nu_1 + \nu_2}_{g_3} \tag{h3}$$

$$a_{1,1} \geq \underbrace{\lambda_1 + \nu_2}_{g_4} \tag{h4}$$

$$a_{1,1} \geq \underbrace{\lambda_1 + \lambda_2 + \mu_2}_{g_5} \tag{h5}$$

$$a_{1,1} \geq \underbrace{\lambda_1 + \lambda_3 + \mu_1}_{g_6} \tag{h6}$$

where the inequalities (H1)–(H3) are upper bounds on $a_{1,1}$ and the inequalities (h1)–(h6) are lower bounds on $a_{1,1}$. For each $i = 1, 2, 3$, let $f_i(\lambda, \mu, \nu)$ denote the the ith upper bound of $a_{1,1}$, and for each $j = 1, \dots, 6$ let $g_j(\lambda, \mu, \nu)$ denote the jth lower bound of $a_{1,1}$. In [21], we show that each of the 18 chambers corresponds to a particular choice of (i, j). For such a choice, say (i^*, j^*), the associated chamber κ_{i^*,j^*} (this will be one of the chambers $\kappa_1, \dots, \kappa_{18}$ described in Table 3.1 - we index it here by the pair i^*, j^* to emphasize the correspondence) is given by the following inequalities

$$\begin{aligned}
g_{j^*}(\lambda, \mu, \nu) &\leq f_{i^*}(\lambda, \mu, \nu) \\
f_{i^*}(\lambda, \mu, \nu) &\leq f_i(\lambda, \mu, \nu) && \text{for each } i = 1, 2, 3 \\
g_{j^*}(\lambda, \mu, \nu) &\geq g_j(\lambda, \mu, \nu) && \text{for each } j = 1, \dots, 6.
\end{aligned}$$

Together with the hive inequalities, we find that for $(\lambda|\mu|\nu) \in \kappa_{i^*,j^*}$, the possible values of $a_{1,1}$ are exactly the integer points on the interval $[g_{j^*}(\lambda, \mu, \nu), f_{i^*}(\lambda, \mu, \nu)]$. Each such choice of $a_{1,1}$ corresponds to an integral k-hive satisfying BC and HC, and so by Theorem 3.2.1,

$$\begin{aligned}
c^{\nu}_{\lambda\mu} &= [f_{i^*}(\lambda, \mu, \nu), g_{j^*}(\lambda, \mu, \nu)] \cap \mathbb{Z} \\
&= f_{i^*}(\lambda, \mu, \nu) - g_{j^*}(\lambda, \mu, \nu) + 1.
\end{aligned}$$

This formula is valid for each $(\lambda|\mu|\nu) \in \gamma$ and is a linear polynomial. Therefore it is exactly the polynomial of Φ_3 associated to γ - that is: $\Phi_3^\gamma = f_{i^*}(\lambda,\mu,\nu) - g_{j^*}(\lambda,\mu,\nu) + 1$.

As an example, if we choose inequalities (H1) and (h1), we find that

$$\begin{aligned} f_1(\lambda,\mu,\nu) - g_1(\lambda,\mu,\nu) &= (\lambda_1+\nu_1) - (\lambda_2+\nu_1) \\ &= \lambda_1 - \lambda_2, \end{aligned}$$

which corresponds to the chamber κ_9 since the associated polynomial is exactly $1+\lambda_1-\lambda_2$. From this view, we see that

$$\Phi_3(\lambda,\mu,\nu) = \min_{\ell\in\{1,\ldots,18\}} \Phi_3^{\kappa_\ell}(\lambda,\mu,\nu), \tag{3.8}$$

since

$$\Phi_3(\lambda,\mu,\nu) = 1 + \min_{i\in\{1,2,3\}} f_i(\lambda,\mu,\nu) - \max_{j\in\{1,\ldots,6\}} g_j(\lambda,\mu,\nu).$$

Conversely, for any choice of $\mathbf{p} := (\lambda|\mu|\nu) \in \mathsf{H}_3^+$ one can compute the set of chambers containing $\mathbf{p}$ as follows. Let m be the minimum value over the evaluations $f_i(\lambda,\mu,\nu)$ (for $i = 1,2,3$), and M be the maximum value over the evaluations $g_j(\lambda,\mu,\nu)$ (for $j = 1,\ldots,6$). Then compute the sets

$$\begin{aligned} S_1 &:= \{i : f_i(\lambda,\mu,\nu) = m\} \subseteq \{1,2,3\} \\ S_2 &:= \{j : g_j(\lambda,\mu,\nu) = M\} \subseteq \{1,\ldots,6\}. \end{aligned}$$

The set of chambers $\kappa_{i,j}$ containing $(\lambda|\mu|\nu)$ is exactly the set for which $(i,j) \in S_1 \times S_2$.

Example 3.6.7. For $\lambda = (5,3,0), \mu = (4,2,0), \nu = (7,5,2)$,

$$f_1(\lambda,\mu,\nu) = f_2(\lambda,\mu,\nu) = f_3(\lambda,\mu,\nu) = 12,$$

$$g_1(\lambda,\mu,\nu) = g_3(\lambda,\mu,\nu) = g_4(\lambda,\mu,\nu) = g_5(\lambda,\mu,\nu) = 10,$$

and

$$g_2(\lambda,\mu,\nu) = g_6(\lambda,\mu,\nu) = 9.$$

The minimum over the f_i is 12 and is achieved by f_1, f_2, f_3 and the maximum over the g_j is 10 which is achieved by g_1, g_3, g_4, g_5. Therefore $(\lambda|\mu|\nu)$ is in exactly the set of chambers of $\mathcal{LR}_3$ associated to some $(i,j) \in \{1,2,3\} \times \{1,3,4,5\}$. Equivalently, this is the set of chambers of $\mathcal{LR}_3$ with polynomial in the set $\{1+f_i(\lambda,\mu,\nu)-g_j(\lambda,\mu,\nu) : i \in \{1,2,3\},\ j \in \{1,3,4,5\}\}$, which is the same set of 12 chambers described in Example 3.2.2.

We remark that Eq. (3.8) implicitly addresses an interesting question – what happens if the point $(\lambda|\mu|\nu)$ is evaluated with the "wrong" polynomial of Φ_3 - i.e if we compute $\Phi_3^\gamma(\mathbf{p})$ for a polynomial Φ_3^γ associated to a chamber γ not containing $\mathbf{p}$. Eq. (3.8) tells us that this

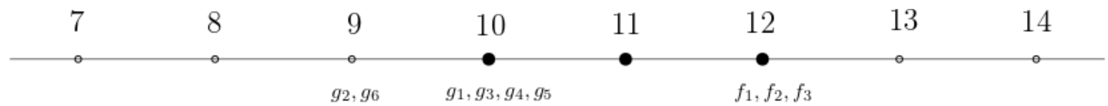


Figure 3.3: Evaluations of f_i and g_j for $\lambda = (5,3,0)$, $\mu = (4,2,0)$, $\nu = (7,5,2)$. Each f_i and g_j appears directly below the corresponding evaluation $f_i(\lambda,\mu,\nu)$ or $g_j(\lambda,\mu,\nu)$. The integer points on the interval $[\max_j g_j(\lambda,\mu,\nu), \min_i f_i(\lambda,\mu,\nu)] = [10,12]$ are indicated by large filled in discs, while the integer points are indicated by smaller open circles. The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is given by the number of integer points on this interval, and is therefore equal to 3.

evaluation is at least as large as the evaluation $\Phi_3^{\gamma^*}$ for some chamber γ^* containing $\mathbf{p}$. We exploit this fact in our discussion of stability in Section 3.6.3.

We note that by our previous discussion, the statement of Eq. (3.8) can be strengthened. Let $(\lambda|\mu|\nu) \in \mathsf{H}_3^+$, and let γ, γ' be chambers of $\mathcal{LR}_3$ such that $(\lambda|\mu|\nu) \in \gamma$ and $(\lambda|\mu|\nu) \notin \gamma'$. Then

$$\Phi_3^{\gamma}(\lambda,\mu,\nu) < \Phi_3^{\gamma'}(\lambda,\mu,\nu).$$

Notably, one can compute the exact set of chambers containing $(\lambda|\mu|\nu)$ without explicitly using geometry, by computing $\Phi_3^{\kappa_i}(\lambda,\mu,\nu)$ for each $i = 1,\ldots,18$. The chambers containing $(\lambda|\mu|\nu)$ are exactly those for which the minimimum value of $\Phi_3^{\kappa_i}$ is attained.

3.6.3 Stability

Our work in Section 3.6.2 allows us to give a geometrical perspective for a well-known stability result. We begin with the following straight-forward proposition.

Proposition 3.6.8. *Let* $\mathbf{p}, \mathbf{r} \in H_3^+ \cap \mathbb{Z}^9$ *be in a common chamber* $\gamma \in \mathcal{LR}_3$ *with* $t_{\mathbf{b},\mathbf{r}} = 0$*, then* $\Phi_3(\mathbf{p}+\mathbf{r}) = \Phi_3(\mathbf{p})$.

Proof. We proceed by exploiting the linearity of Ψ_3^{γ}:

$$\begin{aligned}
\Phi_3(\mathbf{p}+\mathbf{r}) &= \Psi_3^{\gamma}(\mathbf{p}+\mathbf{r}) + 1 \\
&= \Psi_3^{\gamma}(\mathbf{p}) + \Psi_3^{\gamma}(\mathbf{r}) + 1 \\
&= t_{\mathbf{b},\mathbf{p}} + t_{\mathbf{b},\mathbf{r}} + 1 \\
&= t_{\mathbf{b},\mathbf{p}} + 1 \\
&= \Phi_3(\mathbf{p}).
\end{aligned}$$

□

The first part of the following proposition follows immediately from the result of Sam and Snowden – that is, that the set of stable triples (λ,μ,ν) are exactly those partition triples for which $c^{\nu}_{\lambda,\mu} = 1$ [78, Theorem 4.6]). In particular if $\mathbf{b} = (\lambda|\mu|\nu)$, the condition $t_{\mathbf{b},\mathbf{r}} = 0$ appearing in the statement of the proposition is equivalent to $c^{\nu}_{\lambda,\mu} = 1$. Our contribution is the geometrical interpretation (which we give in detail in the discussion immediately following the result).

Proposition 3.6.9. *Let $\mathbf{p}, \mathbf{r} \in H_3^+ \cap \mathbb{Z}^9$ with $t_{\mathbf{b},\mathbf{r}} = 0$. Then there exists a positive integer k_0 such that $\Phi_3(\mathbf{p} + (k+1)\mathbf{r}) = \Phi_3(\mathbf{p} + k\mathbf{r})$ for all integers $k \geq k_0$. Moreover, the stable value is*

$$\min_{\gamma \in \mathcal{LR}_3} \{\Phi_3^{\gamma}(\mathbf{p}) : \mathbf{r} \in \gamma\}.$$

Proof. The key is to observe that there exists a positive integer k' such that $\mathbf{p} + k'\mathbf{r}$ and $\mathbf{r}$ are in the same chamber γ^*. Since γ^* is a cone, $\mathbf{p} + k\mathbf{r} \in \gamma^*$ for all $k \geq k'$. Also,

$$\begin{aligned} \Phi_3(\mathbf{p} + k\mathbf{r}) &= 1 + \Psi_3^{\gamma^*}(\mathbf{p}) + k\Psi_3^{\gamma^*}(\mathbf{r}) && (3.9) \\ &= 1 + \Psi_3^{\gamma^*}(\mathbf{p}). && (3.10) \end{aligned}$$

where Eq. (3.10) follows from Eq. (3.9) since $\Psi_3^{\gamma^*}(\mathbf{r}) = t_{\mathbf{b},\mathbf{r}} = 0$. By the same argument $\Phi_3(\mathbf{p} + k'\mathbf{r})$ is also equal to the expression in (3.10). Thus, taking k_0 as k' confirms the first part of the result.

Since $\mathbf{p} + k\mathbf{r} \in \gamma^*$, by Eq. (3.8), it follows that

$$\Phi_3^{\gamma^*}(\mathbf{p} + k\mathbf{r}) \leq \Phi_3^{\gamma}(\mathbf{p} + k\mathbf{r})$$

for all chambers $\gamma \in \mathcal{L}R_3$, and so

$$\Psi_3^{\gamma^*}(\mathbf{p} + k_0\mathbf{r}) = \min_{\gamma \in \mathcal{LR}_3} \{\Psi_3^{\gamma'}(\mathbf{p} + k_0\mathbf{r}) : \mathbf{r} \in \gamma\}. \tag{3.11}$$

Since $\mathbf{r}$ is in each of the chambers appearing in the set in the right-hand side of (3.11), and since $\Psi_3(\mathbf{r}) = 0$,

$$\Psi_3^{\gamma^*}(\mathbf{p}) = \min_{\gamma \in \mathcal{LR}_3} \{\Psi_3^{\gamma}(\mathbf{p}) : \mathbf{r} \in \gamma\}$$

and thus

$$\Phi_3^{\gamma^*}(\mathbf{p}) = \min_{\gamma \in \mathcal{LR}_3} \{\Phi_3^{\gamma}(\mathbf{p}) : \mathbf{r} \in \gamma\}.$$

□

The geometrical interpretation of Proposition 3.6.9 is that the sequence $(\Phi_3(\mathbf{p} + k\mathbf{r}))_{k\geq 0}$ stabilizes exactly at the first integral value of k for which $\mathbf{p} + k\mathbf{r}$ is in a chamber containing $\mathbf{r}$. Additionally, the previous result reveals an interpretation of evaluating a point with the "wrong" polynomial. More precisely, the stable value $\Phi_3(\mathbf{p} + k\mathbf{r})$ is given by the evaluation $\Phi_3^{\gamma^*}(\mathbf{p})$ of the point $\mathbf{p}$ using the polynomial $\Phi_3^{\gamma^*}$ associated to the chamber $\gamma^* \in \mathcal{LR}_3$ which does not necessarily contain $\mathbf{p}$. We have exploited the linearity of the piecewise-polynomial Ψ_3 in order to obtain the result, but it is worth asking to what extent such a result may generalize or whether a weaker formulation exists for the general GL_k case or for vector partition functions.

Example 3.6.10. Let $\lambda = (5, 3, 1), \mu = (6, 3, 2), \nu = (10, 6, 4)$, so that

$$\mathbf{p} = (5, 3, 1, 6, 3, 2, 10, 6, 4),$$

and let

$$\mathbf{r} = (1, 0, 0, 1, 0, 0, 2, 0, 0).$$

Then, $t_{\mathbf{b},\mathbf{r}} = 0$. The sequence $\Phi_3(\mathbf{p} + k\mathbf{r})$ for $k = 0, \ldots, 6$ is illustrated in Table 3.2. We see from the table that the stable value is 7. We now compute this value by using Proposition 3.6.9. We first compute that $\mathbf{r}$ is only in Chambers $\kappa_2, \kappa_3, \kappa_5, \kappa_6, \kappa_{11}, \kappa_{12}, \kappa_{15}$, and κ_{16}. Next, we compute the value $\Phi_3^{\gamma}(\mathbf{p})$ in each of these chambers, to find that the minimimum value of 7 occurs in Chambers 11 and 15. Table 3.3 gives the values of $\Phi_3(\mathbf{p})$ for each chamber containing $\mathbf{r}$, with the minimum values in bold font.

Table 3.2: Stability of $\Phi_3(\mathbf{p} + k\mathbf{r})$

k	$\mathbf{p} + k\mathbf{r}$	λ	μ	ν	$c^{\nu}_{\lambda,\mu} = \Phi_3(\mathbf{p} + k\mathbf{r})$
0	(5, 3, 1, 6, 3, 2, 10, 6, 4)	(5, 3, 1)	(6, 3, 2)	(10, 6, 4)	2
1	(6, 3, 1, 7, 3, 2, 12, 6, 4)	(6, 3, 1)	(7, 3, 2)	(12, 6, 4)	3
2	(7, 3, 1, 8, 3, 2, 14, 6, 4)	(7, 3, 1)	(8, 3, 2)	(14, 6, 4)	4
3	(8, 3, 1, 9, 3, 2, 16, 6, 4)	(8, 3, 1)	(9, 3, 2)	(16, 6, 4)	5
4	(9, 3, 1, 10, 3, 2, 18, 6, 4)	(9, 3, 1)	(10, 3, 2)	(18, 6, 4)	6
5	(10, 3, 1, 11, 3, 2, 20, 6, 4)	(10, 3, 1)	(11, 3, 2)	(20, 6, 4)	**7**
6	(11, 3, 1, 12, 3, 2, 22, 6, 4)	(11, 3, 1)	(12, 3, 2)	(22, 6, 4)	**7**

Table 3.3: Evaluation of $\Phi_3(\mathbf{p})$ in chambers containing $\mathbf{r}$

Chamber κ_i containing $\mathbf{r}$	κ_2	κ_3	κ_5	κ_6	κ_{11}	κ_{12}	κ_{15}	κ_{16}
Evaluation $\Phi_3^{\kappa_i}(\mathbf{p})$	9	72	9	72	**7**	74	**7**	74

3.7 Symmetries of the Littlewood-Richardson coefficients

The study of symmetries associated to the Littlewood-Richardson coefficients is an area of active research (see [19, 30, 41, 73]). In this section, we compute the set of linear symmetries associated to the general linear group GL_4.

Let $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_k)$ be partitions of length at most k. We define a *linear symmetry* of the LR coefficient $c^{\nu}_{\lambda,\mu}$ to be a linear mapping M on $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k)$ such that if $M(\lambda, \mu, \nu) = (\lambda^*, \mu^*, \nu^*)$ and λ^*, μ^*, ν^* are partitions, then $c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\lambda^*,\mu^*}$. In order to describe the linear symmetries, we introduce some notation. For a partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, we write $\alpha^\square$ for a partition obtained by taking the conjuage of α within a rectangle – that is $\alpha^\square$ denotes any partition of the form $(j - \alpha_k, \ldots, j - \alpha_1)$ for some integer $j \geq \alpha_1$. In the description of the maps, we do not explicitly give the j.

There are 24 well known-symmetries of the Littlewood-Richardson coefficients associated to GL_k. They are generated by the linear maps:

$$(\lambda, \mu, \nu) \mapsto (\mu, \lambda, \nu) \tag{3.12}$$

$$(\lambda, \mu, \nu) \mapsto (\lambda^\square, \mu^\square, \nu^\square) \tag{3.13}$$

$$(\lambda, \mu, \nu) \mapsto (\nu^\square, \mu, \lambda^\square) \tag{3.14}$$

$$(\lambda, \mu, \nu) \mapsto (\lambda - (\lambda_k - \mu_k)\mathbf{1}, \mu - (\mu_k - \lambda_k)\mathbf{1}, \nu) \tag{3.15}$$

and form a group isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_3$. We follow [21] in calling these the *general symmetries* of the Littlewood-Richardson coefficients.

The first three of these symmetries (and thus the 6 symmetries that they generate) can be derived from the interpretation of LR coefficients as intersections of triples of Schubert varieties [85] (there are also combinatorial proofs given in [71]). The last of these has long been known – for example via Schur functions [18].

In a recent preprint, Briand and Rosas [19] studied the linear symmetries of the LR coefficients associated to the special linear group $SL(3)$, and surprisingly found 144 symmetries (this translates to 288 linear symmetries of the LR coefficients associated to GL_3). In addition to the general symmetries, they found the following symmetry:

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) \mapsto (\lambda_1 + \mu_1 - \nu_2, \lambda_2 + \mu_1 - \nu_2, \lambda_3, \nu_2, \mu_2, \mu_3, \nu_1, \mu_1, \nu_3)$$

that together with the general symmetries generates the full 288 element symmetry group, which is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$ where $\wr$ denotes the wreath product (for the definition of wreath product, see for example [[33], page 187]). In [21], we show that these linear symmetries act on the rays (1-dimensional cones) of $\mathcal{L}R_3$ by permutations that fix each of the sets

$$\{\mathbf{a}_1, \mathbf{a}_2\}, \{\mathbf{b}\}, \{\mathbf{c}, \mathbf{f}\}, \{\mathbf{d}_1, \mathbf{e}_2, \mathbf{g}_1\}, \{\mathbf{d}_2, \mathbf{e}_1, \mathbf{g}_2\}$$

and allow the interchange of the sets $\{\mathbf{d}_1, \mathbf{e}_2, \mathbf{g}_1\}, \{\mathbf{d}_2, \mathbf{e}_1, \mathbf{g}_2\}$. These are exactly the permutations of rays that preserve the chamber complex. We also introduce objects called *cell diagrams* that capture the 144 symmetries of the $SL(3)$ case. These diagrams encode each of the cones of the fan associated to the LR coefficients of $SL(3)$ (which is obtained by deleting the generating rays $\mathbf{a}_1, \mathbf{a}_2$ from each of the cones of $\mathcal{LR}_3$). They encode each of the 9 generating rays of this fan (i.e $\mathbf{b}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, \mathbf{g}_1, \mathbf{g}_2$) via *items* which are one of: a vertex of a hexagon, a left (“West”) pointing triangle, a right (“East”) pointing triangle, or a central dot. See Table 3.4 for the exact correspondence. Each of the cones of the fan is then given by the set of all items associated to the rays of the cone.

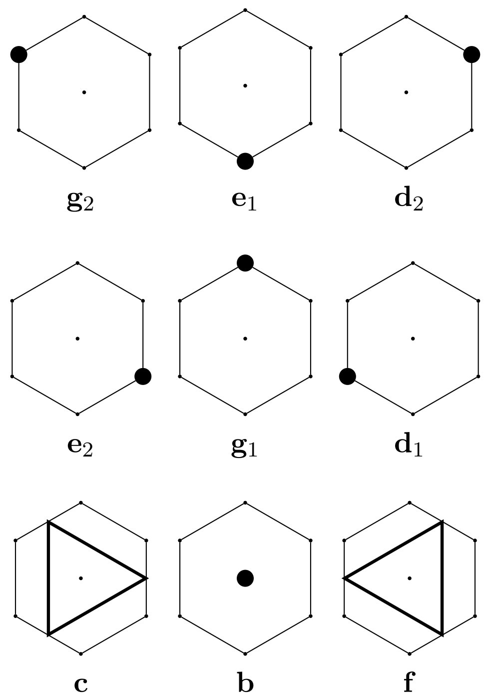


Table 3.4: The cell diagrams of the nine minimal rays of the $SL(3)$ chamber complex

In order to describe how the linear symmetries act on the cell diagrams, we define two triangles T_N and T_S. Let T_N be the triangle joining the vertices of the hexagon corresponding to rays $\mathbf{d}_1, \mathbf{e}_2, \mathbf{g}_1$, and let T_S be the triangle joining the vertices of the hexagon corresponding to rays $\mathbf{g}_2, \mathbf{d}_2, \mathbf{e}_1$ (see Table 3.4 for more details – the triangle T_N points "North" and the triangle T_S points "South"). Then the group of linear symmetries associated to SL_3 is generated by:

1. the symmetries of the hexagon,
2. the symmetries of T_N (keeping all other items unchanged),
3. the symmetries of T_S (keeping all other items unchanged),
4. the involution which maps the West pointing triangle to the East pointing triangle and vice-versa.

We remark that a different interpretation of the symmetries of the LR coefficients associated to $SL(3)$ appears in [30]. There the symmetries are encoded by actions on two 3×3 matrices (transposition of both matrices, simultaneous permutation of rows, simultaneous permutation of columns, and exchange of the two matrices). An advantage of the cell diagrams is that they encode information about the cone itself – for example the number of items is the dimension of the cone (which is not easy to discern in the array-pair interpretation).

The computation of the 288 symmetries in the $k = 3$ case was made through the following observation.

Theorem 3.7.1 (Briand, Rosas, 2020 [19])**.** *A linear symmetry of the LR coefficients associated to GL_k induces a permutation of the chambers of $\mathcal{LR}_k$.*

Recall that we have computed the fan $\mathcal{LR}_4$. By using Theorem 3.7.1 in this case, we are able to compute the linear symmetries in the $k = 4$ case.

Theorem 3.7.2. *There are 24 linear symmetries of the Littlewood-Richardson coefficients associated to GL_4. They are the general symmetries.*

Question 3.7.3. *Is it true that for all positive integers $k > 3$ that the only linear symmetries of the LR coefficients associated to GL_k are exactly the 24 general symmetries? This seems intuitively true for two reasons:*

1. *As k grows, the fan $\mathcal{LR}_k$ has many more chambers and rays, therefore it seems less likely that there exist permutations of the rays that preserve each of the chambers.*

2. *The polynomials of the Φ_3 case are all of degree 1 so it is unsurprising that they interact well with linear symmetries. However, for $k > 3$, the polynomials of Φ_k are not linear.*

Remark 3.7.4. Other symmetries of the Littlewood-Richardson coefficients (that are not linear) have also been studied. For example, a result of Coquereaux and Zuber [28] relating to the LR coefficients of $SL(3)$ states that for any λ, μ with $\ell(\lambda), \ell(\mu) \leq 2$,

$$\sum_{\nu} c_{\lambda,\mu}^{\nu} = \sum_{\nu'} c_{\lambda,\mu}^{\nu'}. \tag{3.16}$$

The symmetry underlying Eq. (3.16) is not linear, but piecewise linear (see [28] for details).

3.8 Linear factors of Littlewood-Richardson polynomials

In this section, we discuss our attempts to apply the work of Chapter 2 to the study of Littlewood-Richardson coefficients. Our aim is to study the appearance of linear factors in the polynomials of Φ_k by studying the matrices E_k and their associated vector partition functions p_{E_k}. Ideally, we would like to prove the following conjecture.

Conjecture 3.8.1. *Let f be a facet of H_k^+ which is not defined by one of the two hyperplanes $\lambda_k = 0$ or $\mu_k = 0$. Let $\boldsymbol{\iota}$ be an inner normal of f. Let W denote the subspace of $\mathbb{R}^{3k}$ defined* *by* $|\lambda|+|\mu|=|\nu|$*. Then* f *corresponds to a facet* f' *of* $\mathrm{pos}_{\mathbb{R}}(E_k)$ *with inner normal* $\boldsymbol{\iota}$ *by the following relation*

$$(\boldsymbol{\iota}')^T B_k - \boldsymbol{\iota} \in W.$$

Moreover, for any chamber γ *of* $\mathcal{LR}_k$ *intersecting* f $(3k-1)$*-dimensionally, the polynomial* Φ_k^γ *has linear factors*

$$\boldsymbol{\iota} \cdot (\lambda|\mu|\nu) + i$$

for $i = 1, \ldots, k-1$ *where* k *is the number of columns of* E_k *not on* f*.*

We first discuss the correspondence of facets of $\mathrm{pos}_{\mathbb{R}}(E_k)$ and facets of H_k^+ described by Conjecture 3.8.1. By applying Rassart's procedure, we can relate the positive Horn cone to $\mathrm{pos}_{\mathbb{R}}(E_k)$ by the following equation

$$\mathsf{H}_k^+ = \rho_k(\mathrm{pos}_{\mathbb{R}}(E_k) \cap \mathrm{col}(B_k)) \cap \tau_k.$$

Consider a facet f' of $\mathrm{pos}_{\mathbb{R}}(E_k)$ with minimal inner normal $\boldsymbol{\iota}'$, so $f' = H_{\boldsymbol{\iota}'} \cap \mathrm{pos}_{\mathbb{R}}(E_k)$. Such a facet describes one of the inequalities $\boldsymbol{\iota}' \cdot \mathbf{b} \geq 0$ which defines the cone $\mathrm{pos}_{\mathbb{R}}(E_k)$. It may (or may not) also describe an inequality of the form $\boldsymbol{\iota}' \cdot (B_k(\lambda|\mu|\nu)) \geq 0$ for the cone $pos(E_k) \cap \mathrm{col}(B_k)$. In such a case we see that the rectification process produces the inequality

$$\left((\boldsymbol{\iota}')^T B_k\right) \cdot (\lambda|\mu|\nu) \geq 0$$

If this is not already implied by the inequalities of τ_k, then this inequality defines a facet f of the positive Horn cone H_k^+ with inner normal $\boldsymbol{\iota} := (\boldsymbol{\iota}')^T B_k$. Our claim is that every facet H_k^+ with the exception of the one defined by $\lambda_k = 0$ and the one defined by $\mu_k = 0$ is obtained in this manner. We have checked this result numerically for the cases $k = 3, 4, 5, 6$.

Ideally, we would like to apply the result of Baldoni and Vergne (Theorem 2.5.3) in order to compute linear factors of polynomials associated to semi-external chambers of E_k. However, as we remarked in Section 3.3, the matrices E_k are not unimodular in general, and so Theorem 2.5.3 does not apply. For example in the $k = 4$ case, the matrix

$$E_4 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

is not unimodular as the submatrix obtained by taking the set of columns with indices $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 20, 21\}$ has determinant 2. In fact, E_4 is not even DeLS-unimodular.

On the other hand, we do know that for each of the chambers γ of E_k for which $\rho_k(\gamma \ \cap \ \mathrm{col}(B_k)) \cap \tau_k$ is a chamber of $\mathcal{LR}_k$, that $p^{\gamma}_{E_k}$ is a polynomial. Therefore, subject to proving Conjecture 2.5.4, one only needs to verify that the dot product condition of Conjecture 2.5.4 holds for a given facet of $\mathrm{pos}_{\mathbb{R}}(E_k)$ in order to compute linear factors. We have verified that this condition does indeed hold for the cases $k = 3, 4, 5, 6$.

We now illustrate some evidence for Conjecture 2.5.4. As stated earlier, the fan $\mathcal{LR}_4$ has 67769 chambers – we abuse notation slightly by calling them $\kappa_1, \dots, \kappa_{67769}$. It should be understood from context if we are talking about chambers of the GL_3 or GL_4 case.

Example 3.8.2. Consider the facet f of H_4^+ corresponding to the essential Horn inequality

$$\lambda_2 + \mu_2 \geq \nu_3.$$

Exactly 12 of the ray generators of $\mathcal{LR}_4$ lie on f. The chamber κ_{67579} with minimal ray generators

$$\begin{array}{lll} (0,0,0,0,1,0,0,0,1,0,0,0) & (0,0,0,0,1,1,1,1,1,1,1,1) & (1,0,0,0,0,0,0,0,1,0,0,0) \\ (1,0,0,0,1,0,0,0,1,1,0,0) & (1,0,0,0,1,1,0,0,1,1,1,0) & (1,0,0,0,1,1,1,0,1,1,1,1) \\ (1,1,0,0,1,0,0,0,1,1,1,0) & (1,1,1,0,1,0,0,0,1,1,1,1) & (1,1,1,1,0,0,0,0,1,1,1,1) \\ (2,1,1,0,2,1,1,0,3,2,2,1) & (4,2,1,0,4,2,1,0,5,4,3,2) & \end{array}$$

meets f 10-dimensionally (the only minimal ray generator of κ_{67579} not lying on f is $(4,2,1,0,4,2,1,0,5,4,3,2)$). In this case, we find by interpolation that the associated polynomial $\Phi_4^{\kappa_{67579}}(\lambda|\mu|\nu)$ is

$$\frac{1}{6}(\lambda_2+\mu_2-\nu_3+1)(\lambda_2+\mu_2-\nu_3+2)(-3\lambda_1-2\lambda_2-3\lambda_4-2\mu_1-3\mu_3+3\mu_4+3\nu_1+3\nu_2+2\nu_3+3).$$

The facet f corresponds to the facet f' of $\mathrm{pos}_{\mathbb{R}}(E_4)$ with minimal inner normal

$$\boldsymbol{\iota}' := (0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0)$$

since $(\boldsymbol{\iota}')^T B_4(\lambda|\mu|\nu)$ is the Horn inequality corresponding to the facet f. We find that exactly 3 columns of E_4 do not lie on f. We note that the facet f' satisfies the conditions of Conjecture 2.5.4 that predicts the factors $\lambda_2+\mu_2-\nu_3+1$ and $\lambda_2+\mu_2-\nu_3+2$.

Example 3.8.3. Consider the facet f of H_4^+ corresponding to the essential Horn inequality

$$\lambda_1+\lambda_3+\mu_1+\mu_2 \geq \nu_1+\nu_3.$$

The facet f is contained in the chamber κ_{67709} of $\mathcal{LR}_4$ with minimal ray generators

$$\begin{array}{lll} (0,0,0,0,1,0,0,0,1,0,0) & (0,0,0,0,1,1,1,0,1,1,1) & (0,0,0,0,1,1,1,1,1,1,1) \\ (1,0,0,0,0,0,0,0,1,0,0) & (1,1,0,0,0,0,0,0,1,1,0) & (1,1,0,0,1,0,0,0,1,1,1) \\ (1,1,0,0,1,1,0,0,2,1,1) & (1,1,0,0,1,1,1,0,2,1,1) & (1,1,1,0,0,0,0,0,1,1,1) \\ (1,1,1,1,0,0,0,0,1,1,1) & (4,3,1,0,3,2,1,0,6,4,3) & \end{array}$$

We have computed by interpolation that the polynomial Φ_4^γ is

$$\Phi_4^\gamma(\lambda,\mu,\nu) = \binom{\lambda_1+\lambda_3+\mu_1+\mu_2-\nu_1-\nu_3+3}{3}.$$

We note that the facet f corresponds to the facet f' of E_4 with minimal inner normal

$$\boldsymbol{\iota}' := (0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,1,0,0).$$

since $\boldsymbol{\iota}' B_4(\lambda|\mu|\nu)$ is the essential Horn inequality corresponding to the facet f. One should not be surprised to find that f' is an external facet of E_4. In this case we do not need to rely on Conjecture 2.5.4, as the formula is given by applying Theorem 2.4.2 (having checked the E_4-minimality of the external columns on f').

In the $k = 4$ case, there are 21 facets of H_4^+ which correspond to external facets of E_4. These are defined by essential Horn inequalities (coincidentally this is exactly the number of external facets of E_3 - however, not of H_3^+). We list the linear forms associated to the essential Horn inequalities:

$$\begin{array}{rr}
-\lambda_1 - \lambda_2 - \mu_3 - \mu_4 + \nu_1 + \nu_2 & -\lambda_1 - \lambda_3 - \mu_2 - \mu_4 + \nu_1 + \nu_2 \\
-\lambda_1 - \lambda_3 - \mu_3 - \mu_4 + \nu_1 + \nu_3 & -\lambda_1 - \lambda_4 - \mu_1 - \mu_4 + \nu_1 + \nu_2 \\
-\lambda_1 - \lambda_4 - \mu_2 - \mu_4 + \nu_1 + \nu_3 & -\lambda_2 - \lambda_3 - \mu_2 - \mu_3 + \nu_1 + \nu_2 \\
-\lambda_2 - \lambda_3 - \mu_2 - \mu_4 + \nu_1 + \nu_3 & -\lambda_2 - \lambda_3 - \mu_3 - \mu_4 + \nu_2 + \nu_3 \\
-\lambda_2 - \lambda_4 - \mu_1 - \mu_3 + \nu_1 + \nu_2 & -\lambda_2 - \lambda_4 - \mu_1 - \mu_4 + \nu_1 + \nu_3 \\
-\lambda_2 - \lambda_4 - \mu_2 - \mu_3 + \nu_1 + \nu_3 & -\lambda_2 - \lambda_4 - \mu_2 - \mu_4 + \nu_2 + \nu_3 \\
-\lambda_3 - \lambda_4 - \mu_1 - \mu_2 + \nu_1 + \nu_2 & -\lambda_3 - \lambda_4 - \mu_1 - \mu_3 + \nu_1 + \nu_3 \\
-\lambda_3 - \lambda_4 - \mu_2 - \mu_3 + \nu_2 + \nu_3 & \lambda_2 + \lambda_3 + \mu_1 + \mu_2 - \nu_2 - \nu_3 \\
\lambda_1 + \lambda_3 + \mu_1 + \mu_3 - \nu_2 - \nu_3 & \lambda_1 + \lambda_3 + \mu_1 + \mu_2 - \nu_1 - \nu_3 \\
\lambda_1 + \lambda_2 + \mu_2 + \mu_3 - \nu_2 - \nu_3 & \lambda_1 + \lambda_2 + \mu_1 + \mu_3 - \nu_1 - \nu_3 \\
\lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - \nu_2 &
\end{array}$$

The polynomial associated to each corresponding chamber is indeed given by the appropriate negative binomial coefficient.

We now reconstruct the $k = 3$ case to further illustrate the facet correspondence described by Conjecture 3.8.1. Recall that E_3 is the matrix

$$E_3 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The cone $\mathrm{pos}_{\mathbb{R}}(E_3)$ has 21 facets. In fact, E_3 has 21 chambers, each of which is external – thus the facets of $\mathrm{pos}_{\mathbb{R}}(E_3)$ are all external facets, and there is a 1 to 1 correspondence between facets of $\mathrm{pos}_{\mathbb{R}}(E_3)$ and chambers of E_3. For each of these facets, we compute the minimal inner facet normal $\boldsymbol{\iota}'$ and then compute $\boldsymbol{\iota}^T B_3$. We then identify whether or not $\boldsymbol{\iota} := \boldsymbol{\iota}^T B_3$ is an inner facet normal for some facet of H_3^+.

Table 3.5 illustrates the correspondence of facets of $\mathrm{pos}_{\mathbb{R}}(E_3)$ and facets of H_3^+. The facets of $\mathrm{pos}_{\mathbb{R}}(E_3)$ are represented by their corresponding minimal inner facet normals in the second column. The linear forms potentially defining facets of H_3^+ are given in the third column. We note that Facets 7, 17, and 21 of $\mathrm{pos}_{\mathbb{R}}(E_3)$ do not map to facets of H_3^+, but we give the corresponding linear forms regardless. In the fourth column we illustrate the $\mathcal{LR}_3$ chamber containing the facet of H_3^+ or indicate if there is no such chamber (since the facet of $\mathrm{pos}_{\mathbb{R}}(E_3)$ does not map to a facet of H_3^+). In total there are 22 facets of H_3^+, and, as expected, there are two which are not mapped to, which are defined by the equations $\lambda_3 = 0$ and $\mu_3 = 0$.

Table 3.5: Correspondence of facets of $\mathrm{pos}_{\mathbb{R}}(E_3)$ and facets of H_3^+.

#	$\mathrm{pos}_{\mathbb{R}}(E_3)$ inner facet normal	Potential H_3^+ facet linear form	$\mathcal{LR}_3$ chamber
1	(1, 1, 0, 0, 0, 0, 0, 0, 0)	$-\lambda_3 - \mu_1 + \nu_1$	κ_{14}
2	(1, 0, 1, 0, 0, 0, 0, 0, 0)	$\lambda_1 + \mu_3 - \nu_3$	κ_8
3	(1, 0, 0, 1, 0, 0, 0, 0, 0)	$\lambda_1 - \lambda_2$	κ_9
4	(1, 0, 0, 0, 0, 1, 0, 0, 0)	$\lambda_1 + \mu_2 - \nu_2$	κ_{17}
5	(1, 0, 0, 0, 0, 0, 1, 0, 0)	$\nu_1 - \nu_2$	κ_4
6	(1, 0, 0, 0, 0, 0, 0, 1, 0)	$-\lambda_2 - \mu_2 + \nu_1$	κ_1
7	(1, 0, 0, 0, 0, 0, 0, 0, 0)	$\lambda_1 + \nu_1$	None
8	(0, 1, 0, 0, 1, 0, 0, 0, 0)	$\lambda_2 - \lambda_3$	κ_{16}
9	(0, 1, 0, 0, 0, 0, 0, 0, 1)	$\lambda_2 + \mu_2 - \nu_3$	κ_5
10	(0, 0, 1, 0, 1, 0, 0, 0, 0)	$-\lambda_3 - \mu_2 + \nu_2$	κ_{12}
11	(0, 0, 1, 0, 0, 0, 0, 0, 1)	$\nu_2 - \nu_3$	κ_2
12	(0, 0, 0, 1, 1, 0, 0, 0, 0)	$\lambda_1 + \mu_1 - \nu_1$	κ_3
13	(0, 0, 0, 1, 0, 0, 0, 0, 1)	$-\lambda_2 - \mu_3 + \nu_2$	κ_{11}
14	(0, 0, 0, 0, 1, 1, 0, 0, 0)	$-\lambda_3 - \mu_3 + \nu_3$	κ_6
15	(0, 0, 0, 0, 1, 0, 1, 0, 0)	$\lambda_2 + \mu_1 - \nu_2$	κ_{18}
16	(0, 0, 0, 0, 1, 0, 0, 1, 0)	$\mu_1 - \mu_2$	κ_{10}
17	(0, 0, 0, 0, 1, 0, 0, 0, 0)	$\lambda_1 + \lambda_2 + \mu_1$	None
18	(0, 0, 0, 0, 0, 1, 0, 0, 1)	$-\mu_2 - \mu_3$	κ_{15}
19	(0, 0, 0, 0, 0, 0, 1, 0, 1)	$-\lambda_1 - \mu_3 + \nu_1$	κ_{13}
20	(0, 0, 0, 0, 0, 0, 0, 1, 1)	$\lambda_3 + \mu_1 - \nu_3$	κ_7
21	(0, 0, 0, 0, 0, 0, 0, 0, 1)	$-\mu_3 + \nu_1 + \nu_2$	None

The problem of computing linear factors of the stretched Littlewood-Richardson polynomials $P^{\nu}_{\lambda,\mu}(t) := c^{t\nu}_{t\lambda,t\mu}$ has been studied by King, Tollu, and Toumazet [47, Section 6]. They offer conjectures based on what they call the *negatively stretched Littlewood-Richardson coefficients*.

Last year, in [23], Chaput and Ressayre proved an interesting result relating particular Littlewood-Richardson coefficents and binomial coefficients. For positive integers p, q and a partition α of length at most k define $\alpha(p,q)$ to be the partition

$$\alpha(p,q) := (p\alpha_1, \ldots, p\alpha_1, p\alpha_2, \ldots, p\alpha_2, \ldots, p\alpha_k, \ldots, p\alpha_k)$$

where each $p\alpha_i$ appears exactly q times.

Theorem 3.8.4 (Chaput, Ressayre, 2022 [23]). *Let λ, μ, ν be partitions of length at most k with $c^{\nu}_{\lambda,\mu} = 2$. Then*

$$c^{\nu(p,q)}_{\lambda(p,q),\mu(p,q)} = \binom{p+q}{q}$$

for any positive integers p, q.

Finally, the following conjecture about the matrices E_k may be interesting to prove in order to better understand the facets of $\mathrm{pos}_{\mathbb{R}}(E_k)$.

Conjecture 3.8.5. *Each of the columns of E_k is an external column.*

Chapter 4

Kronecker Coefficients

4.1 Background

The *Kronecker coefficients* $g_{\lambda,\mu,\nu}$ are the structure constants in the decomposition of a tensor product of irreducible representations of the symmetric group into irreducible representations:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda g_{\lambda,\mu,\nu} V_\lambda.$$

Consequently, they can be expressed using Schur polynomials

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\lambda,\mu,\nu} s_\mu[X] s_\nu[Y], \tag{4.1}$$

where $X := (x_1, \dots, x_m), Y := (y_1, \dots, y_n), XY := (x_1y_1, x_1y_2, \dots, x_my_n)$. Here, the Schur polynomials are indexed by partitions λ, μ, ν with at most mn, m, n parts, respectively.

Since their introduction in 1938 by Murnaghan, the Kronecker coefficients have proved to be among the most intriguing objects in algebraic combinatorics. After several decades of research, many open questions about the Kronecker coefficients remain. They are all non-negative integers, but have no known combinatorial interpretation, unlike the Littlewood-Richardson coefficients. One might view Kronecker coefficients as a generalization of the Littlewood-Richardson coefficients, hence the resistance to a clear interpretation is surprising, particularly in view of the publicity they have received[1].

The basic problem of computing the Kronecker coefficient $g_{\lambda,\mu,\nu}$ for general partitions λ, μ, ν is #P-hard (in the bitlength of the size of the partitions) and contained in GapP [22][2]. Baldoni, Vergne and Walter distribute code [2] for use with Maple mathematical software to compute Kronecker coefficients for partitions λ, μ, ν of bounded lengths ($\ell(\lambda), \ell(\mu), \ell(\nu) \leq 3$;

[1] "One of the main problems in the combinatorial representation theory of the symmetric group is to obtain a combinatorial interpretation for the Kronecker coefficients." [81]

[2] Problems in GapP can be expressed as the difference of two functions which are in #P.

and $\ell(\lambda) \le 6, \ell(\mu) \le 2, \ell(\nu) \le 3$). There are at least two packages that handle partition sets without a bound on length, such as [25] and the SF Maple package of Stembridge [82]. These two packages are prohibitably computationally expensive except for some small or particular cases.

Even the problem of approximating the Kronecker coefficients is non-trivial and very few useful bounds are known. Pak and Panova [68, Corollary 3.4] determine a bound for partitions λ, μ, ν with $\ell(\lambda) \le l, \ell(\mu) \le m, \ell(\nu) \le n$:

$$g_{\lambda,\mu,\nu} \le \prod_{i=1}^{l} \binom{\lambda_i - i + mn}{mn - i}. \tag{4.2}$$

More recently, in [70], they obtained the following bound in $N = |\lambda| = |\mu| = |\nu|$ via *contingency tables*

$$g_{\lambda,\mu,\nu} \le \left(1 + \frac{lmn}{N}\right)^N \left(1 + \frac{N}{lmn}\right)^{lmn}. \tag{4.3}$$

We remark that both of these bounds are polynomial in the length of the partitions – however, the degree is generally not optimal. For example, when $l = 4, m = 2, n = 2$, we find that if $\lambda = (\frac{N}{4}, \frac{N}{4}, \frac{N}{4}, \frac{N}{4})$, both bounds are of the order $O(N^{16})$ whereas the actual growth is (more precisely) $O(N^2)$ [15].

By $h(u)$, we denote the hook-length of the box u in the Ferrers diagram of λ. The hook length formula [70, Corollary 3.2] also gives a bound:

$$g_{\lambda,\mu,\nu} \le \min(f^\lambda, f^\mu, f^\nu),$$

where $f^\alpha := \frac{k!}{\prod_{u \in [\lambda]} h(u)}$ for a partition α of length k.

Some progress has been made to understand conditions on λ, μ, ν for which $g_{\lambda,\mu,\nu} = 0$. Denote by $k\alpha$ the partition obtained by multiplying each part of α by k. Recall that the Littlewood-Richardson coefficients satisfy a *saturation property*:

$$c^{\nu}_{\lambda,\mu} = 0 \iff c^{k\nu}_{k\lambda,k\mu} = 0.$$

The Kronecker coefficients do not satisfy such a property universally:

$$g_{(1,1),(1,1),(1,1)} = 0, \quad \text{but} \quad g_{(2,2),(2,2),(2,2)} = 1.$$

Even deciding "$g_{\lambda,\mu,\nu} = 0$?" is NP-hard [43]. There are numerous vanishing conditions known – typically expressed as inequalities in the parts of λ, μ, ν which guarantee that the coefficient $g_{\lambda,\mu,\nu}$ is zero. A classical result of Murnaghan and Littlewood (appearing for example in [45]) is that for any non-zero Kronecker coefficient $g_{\lambda,\mu,\nu}$, it follows that $\overline{\lambda} \le \overline{\mu} + \overline{\nu}$, where $\overline{\gamma}$ is the partition obtained by deleting the first part of partition γ. Consider the set of points constructed by concatenating partitions of fixed length. Those points that come from partitions giving a non-zero Kronecker coefficient have a nice geometry. Specifically,

$$Kron_{l,m,n} := \{(\lambda, \mu, \nu) \in \mathbb{Z}^{l+m+n} : g_{\lambda,\mu,\nu} \neq 0,\ \ell(\lambda) \leq l,\ \ell(\mu) \leq m, \ell(\nu) \leq n\}$$

is a finitely generated semigroup in $\mathbb{Z}^{l+m+n}$ that generates a rational polyhedral cone. Following Manivel [58], we call this cone the *Kronecker polyhedron* and denote it $PKron_{l,m,n}$. In [49] the cone $PKron_{l,m,n}$ is computed explicitly for small values of l, m, n, and it seems the number of inequalities increases rapidly. While this set is theoretically computable for any positive integers l, m, n, it is quickly computationally infeasible to do so. Another set of vanishing conditions valid for triples of partitions of any lengths were given recently by Ressayre in [77, Theorems 1 & 2].

Just as the Littlewood-Richardson coefficients, the Kronecker coefficients also exhibit the phenomenon of stability. A classic result of Murnaghan states that for partitions (λ, μ, ν) the sequence $\left(g_{\lambda+(k),\mu+(k),\nu+(k)}\right)_{k\geq 0}$ eventually stabilizes. Since then, many other partition triples α, β, γ with this property have been identified– that is, the values of the sequence $(g_{\lambda+k\gamma,\mu+k\alpha,\nu+k\beta})_{k\geq 0}$ stabilize for any choice of λ, μ, ν. Such triples (α, β, γ) are called *stable triples*. Stabilization phenomenon have been studied in [17, 58, 59, 72, 83].

Applications of Kronecker coefficients extend beyond the realm of algebraic combinatorics. The *Geometric Complexity Theory* (GCT) program, developed by Mulmuley and Sohoni, with the goal of solving P versus NP, relies heavily on the computation of Kronecker coefficients as one of its main ingredients (see [11, 43, 64]). More specifically, problems of positivity (as discussed in the Appendix of [16] entitled *Erratum to the saturation hypothesis (SH) in "Geometric Complexity Theory VI"* and contributed by Mulmuley) related to the previously described saturation problems play an important role.

Kronecker coefficients appear in quantum computing where they encode the relationship between composite systems and their subsystems [25, 26, 27]. As in the case of GCT, being able to determine the positivity of Kronecker coefficients is useful. In the context of quantum computing, non-zero Kronecker coefficients correspond to *admissible spectral triples* which play an important role in the study of bipartite quantum states in quantum information theory [24].

4.1.1 Kronecker coefficients and vector partition functions

Here, we address many of these fundamental questions on Kronecker coefficients using a detailed analysis of Eq. (4.1). The first step is to deduce an expression for $g_{\lambda,\mu,\nu}$ using coefficient extraction of multivariate Taylor series of rational functions. This formulation allows us to represent Kronecker coefficients as a signed sum of vector partition function evaluations.

We reformulate the expression for $g_{\lambda,\mu,\nu}$ given in [62, Theorem 26] as Theorem 4.1.1 below. Once positive integers m, n are chosen, the main ingredients in this approach are:

1. a matrix $A^{m,n}$ and its vector partition function $p_{A^{m,n}}$;
2. vectors α, β of length $m-1$ and $n-2$ respectively;
3. linear functions r_s, r_t;
4. linear functions $l_s(\cdot;\sigma), l_t(\cdot;\sigma)$ defined for each $\sigma \in \mathfrak{S}_{mn}$.

The quantities α, β, and the linear functions r_s, r_t, l_s, l_t (which all depend on m, n) are explicitly given in the discussion after Theorem 4.1.1 and defined (implicitly) in [62]. The matrices $A^{m,n}$ are given implicitly in [62]; we give explicitly only the cases $m=2$, $n = 3, 4$ used in our work. For given m, n and σ, we call the function in the parts of λ, μ, ν given by

$$\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma) := (r_s(\mu,\nu) + \alpha - l_s(\lambda;\sigma),\ r_t(\mu,\nu) + \beta - l_t(\lambda;\sigma))$$

the *vector partition function input of* σ. Additionally, we refer to the quantity

$$\operatorname{sgn}(\sigma)\ p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma)\Big)$$

as the *contribution of the alternant term associated to* σ. In general it will be clear to which m, n we refer, but we explicitly state this when needed.

Theorem 4.1.1 (Mishna, Rosas, Sundaram, 2021, [62])**.** *Let m, n be positive integers. Then for any partitions λ, μ, ν with $\ell(\lambda) \le mn,\ \ell(\mu) \le m,\ \ell(\nu) \le n$, we have*

$$g_{\lambda,\mu,\nu} = \sum_{\sigma \in \mathfrak{S}_{mn}} \operatorname{sgn}(\sigma)\ p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma)\Big). \tag{4.4}$$

The following expressions are implicit in [62], however we have computed here the explicit formulations. With the exception of the vector partition function $p_{A^{m,n}}$, they constitute the necessary ingredients of Theorem 4.1.1. The expressions are valid for all integers u, v with $1 \le u \le m-1$ and $1 \le v \le n-2$. The components of the vectors α, β are:

$$\alpha_0 = \frac{1}{2}\,(nm + n - m - 2)(n-1)(m-1)$$
$$\alpha_u = \frac{1}{2}\,\Big(u^2 n - 2\,unm + 2\,nm^2 - u^2 + u - n - 2\,m + 2\Big)(n-1)$$
$$\beta_v = \frac{1}{12}\,\Big(8\,n^2 m^2 - 6\,vnm + 5\,n^2 m - 10\,nm^2 + 6\,v^2 - 12\,vn + 6\,vm - 19\,nm + 2\,m^2 + 18\,v + 14\,m\Big)(m-1).$$

The components of the vectors $r_s(\mu,\nu)$ and $r_t(\mu,\nu)$ are:

$$r_s(\mu,\nu)_0 = |\nu| - \nu_1 + \binom{n-1}{2}$$

$$r_s(\mu,\nu)_u = \sum_{i=u+1}^{m} \mu_i + |\nu| - \nu_1 + \binom{m-u}{2} + \binom{n-1}{2}$$

$$r_t(\mu,\nu)_v = \sum_{i=2}^{m}(i-1)\mu_i + (m-1)\sum_{j=2}^{v+1}\nu_j + m\sum_{j=v+2}^{n}\nu_j + \binom{m}{3} + (m-1)\binom{n-1}{2} + \binom{n-v-1}{2}.$$

The components of the vectors $l_s(\lambda;\sigma)$ and $l_t(\lambda;\sigma)$ are:

$$l_s(\lambda;\sigma)_0 = \sum_{i=m+1}^{mn}\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right)$$

$$l_s(\lambda;\sigma)_u = \sum_{i=u+1}^{m+u(n-1)}\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right) + 2\sum_{i=m+u(n-1)+1}^{mn}\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right)$$

$$\begin{aligned} l_t(\lambda;\sigma)_v = {} & \sum_{i=2}^{m}(i-1)\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right) + (m-1)\sum_{i=m+1}^{m+v}\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right) + m\sum_{i=m+v+1}^{m+n-1}\left(\lambda_{\sigma(i)} + \delta^{(mn)}_{\sigma(i)}\right) \\ & + \sum_{i=1}^{m-1}\sum_{j=1}^{v}(i+m-1)\left(\lambda_{\sigma(m+i(n-1)+j)} + \delta^{(mn)}_{\sigma(m+i(n-1)+j)}\right) + \sum_{i=1}^{m-1}\sum_{j=v+1}^{n-1}(i+m)\lambda_{m+i(n-1)+j}. \end{aligned}$$

The identity permutation in $\mathfrak{S}_{mn}$ is denoted by Id. It is convenient to have an explicit derivation in the case when $\sigma = Id$:

$$l_s(\lambda;Id)_0 = \sum_{i=m+1}^{mn}\lambda_i + \frac{1}{2}(mn-m-1)(n-1)m$$

$$\begin{aligned} l_s(\lambda;Id)_u = {} & \sum_{i=u+1}^{m+u(n-1)}\lambda_i + 2\sum_{i=m+u(n-1)+1}^{mn}\lambda_i + \frac{1}{2}(m-u)\Big(2mn-u-m-1\Big) \\ & + (n-1)^2 m - \binom{n}{2} + \frac{1}{2}(n-1)(m-1)\Big(n(m-1)-m\Big) \\ & + \frac{1}{2}(n-1)(m-u)\Big(mn-u(n-1)-m-1\Big) \end{aligned}$$

$$\begin{aligned} l_t(\lambda;Id)_v = {} & \sum_{i=2}^{m}(i-1)\lambda_i + (m-1)\sum_{i=m+1}^{m+v}\lambda_i + m\sum_{i=m+v+1}^{m+n-1}\lambda_i \\ & + \sum_{i=1}^{m-1}\sum_{j=1}^{v}(i+m-1)\lambda_{m+i(n-1)+j} + \sum_{i=1}^{m-1}\sum_{j=v+1}^{n-1}(i+m)\lambda_{m+i(n-1)+j} \\ & + \frac{m}{12}\left(8\,m^2 - 3\,m + 1\right)n^2 - m\,(m+1)\,(10\,m + 6\,v - 1)\,n \\ & + 2\,m\left(3\,v^2 + 3\,vm + 2\,m^2 + 6\,v + 3\,m + 1\right). \end{aligned}$$

For $u = 0, \ldots, m-1$, the constant term (with respect to $\lambda_1, \ldots, \lambda_{mn},\ \mu_1, \ldots, \mu_m,\ \nu_1, \ldots, \nu_n$) of the i^{th} coordinate of $r_s(\mu,\nu) + \alpha - l_s(\lambda;Id)$ is 0, and for all $1 \le v \le n-2$, the constant term of the j^{th} coordinate of $r_t(\mu,\nu)+\beta-l_t(\lambda;Id)$ is also 0. In other words both $r_s(\mu,\nu)+\alpha-l_s(\lambda;Id)$ and $r_t(\mu,\nu)+\beta-l_t(\lambda;Id)$ are linear forms whose variables are the parts of λ,μ,ν (and thus so is the vector partition function input $\mathbf{b}^{m,n}(\lambda,\mu,\nu;Id)$).

The expression in Eq. (4.4) writes the Kronecker coefficient $g_{\lambda,\mu,\nu}$ as a signed sum over permutations. As we shall see, the single term associated with the identity permutation is the largest, and can be used to derive properties about the Kronecker coefficient. Specifically, for partitions λ,μ,ν with $\ell(\mu)\leq m,\ \ell(\nu)\leq n\ \ell(\lambda)\leq mn$, the *atomic Kronecker coefficient* $\tilde{g}^{m,n}_{\lambda,\mu,\nu}$ is the coefficient obtained by taking only the contribution of the alternant term corresponding to the identity permutation in Eq. (4.4) - that is,

$$\tilde{g}^{m,n}_{\lambda,\mu,\nu} := p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;Id)\Big). \tag{4.5}$$

Atomic Kronecker coefficients were introduced in [62], where it was proven that in the $m=n=2$ case they provide an upper bound for the Kronecker coefficients. These authors also conjecture that they provide an upper bound in general [61]. This seems to be justified in each computation we have made (in the $m=2,n=3,4$ cases). Interestingly, the atomic Kronecker coefficients depend on the values m,n and not just the indexing partitions. As an example (given also in [62]), consider $\lambda=(12,7,4,1),\mu=(12,12),\nu=(12,12)$. If we set, $m=n=2$, the atomic Kronecker coefficient $\tilde{g}^{2,2}_{\lambda,\mu,\nu}$ is 32 - however, by padding λ and ν with zeroes (i.e. representing λ,ν as $\lambda=(12,7,4,1,0,0),\nu=(12,12,0)$), we find that the atomic Kronecker coefficient $\tilde{g}^{2,3}_{\lambda,\mu,\nu}$ in this case is 8793. The atomic Kronecker coefficients are expressed using a single partition function, which is polynomial time computable for a fixed dimension. However, the dimension grows very quickly as a function of m,n.

4.2 Vector partition functions and Kronecker coefficients

The central formula, Eq. (4.4), was developed by Mishna, Rosas and Sundaram [62]. It is deduced from the formula using Schur polynomials, determinant formulas for Schur polynomials and, a variable substitution. We reproduce some of the details here to establish notation.

4.2.1 From Schur polynomials to vector partition generating functions

Recall from Chapter 1 the *staircase partition* $\delta^{(k)}=(k-1,k-2,\ldots,1,0)$. Also recall that the *alternant* $a_\lambda(x_1,\ldots,x_k)$ is the anti-symmetric polynomial

$$a_\lambda(x_1,\ldots,x_k) := \det(x_i^{\lambda_j})_{1\leq i,j\leq k}.$$

An expression for the Kronecker coefficients involving alternants is

$$\frac{a_{\delta^{(m)}}[X]a_{\delta^{(n)}}[Y]}{a_{\delta^{(mn)}}[XY]}a_{\lambda+\delta^{(mn)}}[XY] = \sum_{\mu,\nu} g_{\lambda,\mu,\nu} S(a_{\mu+\delta^{(m)}}[X])S(a_{\nu+\delta^{(n)}}[Y]), \tag{4.6}$$

where

$$X = (1, x_1, \dots, x_{m-1}), Y = (1, y_1, \dots, y_{n-1}),$$
$$XY = (1, x_1, \dots, x_{m-1}, y_1, \dots, y_{n-1}, x_1y_1, x_1y_2, \dots, x_{m-1}y_{n-1}),$$

and

$$S(a_\alpha(z_1, \dots, z_k)) = \prod_{i=1}^{k} z_i^{\alpha_i},$$

for a partition α of length at most k. We note that in each of X, Y, XY of our starting point, Eq. (4.1), one of the indeterminates has been replaced with 1. This can be done since the Schur polynomials are homogeneous (each term is of the same degree).

The ratio of alternants $\frac{a_{\delta^{(m)}}[X]a_{\delta^{(n)}}[Y]}{a_{\delta^{(mn)}}[XY]}$ simplifies to the rational function

$$\frac{a_{\delta^{(m)}}[X]a_{\delta^{(n)}}[Y]}{a_{\delta^{(mn)}}[XY]} = \frac{1}{\mathcal{ABCDEF}}$$

with the following polynomials:

$$\mathcal{A} = \prod_{j=1}^{n}\prod_{i=1}^{m}(x_i - y_j) \tag{4.7}$$

$$\mathcal{B} = \prod_{j=1}^{n}\prod_{i=1}^{m}(1 - x_iy_j) \tag{4.8}$$

$$\mathcal{C} = \prod_{i=1}^{m-1} x_i^{n-1} \prod_{j=1}^{n-1} y_j^{m-1} \prod_{i=1}^{m-1} (1 - x_i) \prod_{j=1}^{n-1} (1 - y_j)^{m-1} \tag{4.9}$$

$$\mathcal{D} = \prod_{k=1,k\neq i}^{m-1} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (x_k - x_iy_j) \prod_{k=1,k\neq i}^{m-1} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (y_k - x_iy_j) \tag{4.10}$$

$$\mathcal{E} = \prod_{j\neq l=1}^{n-1} \prod_{1\leq i<k\leq m-1} (x_iy_j - x_ky_l) \tag{4.11}$$

$$\mathcal{F} = \prod_{i=1}^{m-1} x_i^{\binom{n-1}{2}} \prod_{j=1}^{n-1} y_j^{\binom{m-1}{2}} \prod_{1\leq i<k\leq m-1} (x_i - x_k)^{n-1} \prod_{1\leq j<l\leq n-1} (y_j - y_l)^{m-1}. \tag{4.12}$$

After the variable substitution

$$x_i = s_1s_2 \dots s_i(t_1t_2 \dots t_{n-2})^i \quad \text{for } 1 \leq i \leq m-1, \tag{4.13}$$

$$\text{and } y_j = s_0s_1 \dots s_{m-1}(t_1t_2 \dots t_{n-2})^{m-1}t_1t_2 \dots t_{j-1} \quad \text{for } 1 \leq j \leq n-1 \tag{4.14}$$

the rational function $\frac{1}{\mathcal{ABCDEF}}$ can be written as the product

$$\mathbf{s}^{\alpha}\mathbf{t}^{\beta}F_{m,n}(s_0, s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-2})$$

where $F_{m,n}(s_0, s_1, \ldots, s_{m-1},\ t_1, \ldots, t_{n-2})$ is a vector partition generating function. After the variable substitution, the terms $S(a_{\mu+\delta^{(m)}}[X])$ and $S(a_{\nu+\delta^{(n)}}[Y])$ become $\mathbf{s}^{r_s(\mu,\nu)}$ and $\mathbf{t}^{r_t(\mu,\nu)}$ respectively. Finally, the term of the determinant

$$a_{\lambda+\delta^{(mn)}}(1, s_0, \ldots, s_{m-1}, t_1, \ldots, t_{n-2})$$

corresponding to permutation σ becomes $\mathbf{s}^{l_s(\lambda;\sigma)}\mathbf{t}^{l_t(\lambda;\sigma)}$.

For a monomial M and variable x, by $\deg_x(M)$ we denote the exponent of x in the monomial M.

Proposition 4.2.1. *Let $u \in \{1, s_0, \ldots, s_{m-1}, t_1, \ldots, t_{n-2}\}$. Then*

$$\begin{aligned}\deg_u(1) \le \deg_u(x_1) \le \ldots \le \deg_u(x_{m-1})& \\ \le \deg_u(y_1) \le \ldots \le \deg_u(y_{n-1})& \\ \le \deg_u(x_1y_1) \le \deg_u(x_1y_2) \le \ldots \le \deg_u(x_{m-1}y_{n-1}).& \quad (4.15)\end{aligned}$$

4.2.2 The vector partition functions $p_{A^{m,n}}$

By $\mathcal{P}_A(\mathbf{b})$ we denote the set $\mathcal{P}_A(\mathbf{b}) := \{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}$ of vector partitions of $\mathbf{b}$, so that $p_A(\mathbf{b})$ is the cardinality of $\mathcal{P}_A(\mathbf{b})$. By exploiting some of the properties of the matrices $A^{m,n}$ given in Corollary 30 of [62] (Properties 1–5 in the list below), we can deduce properties of the corresponding vector partition functions $p_{A^{m,n}}$ without explicitly computing the associated piecewise quasi-polynomials:

(i) each entry of $A^{m,n}$ is a non-negative integer;

(ii) the largest entry of $A^{m,n}$ is $2m-1$;

(iii) the number of columns of $A^{m,n}$ is $\binom{mn}{2} - \binom{n}{2} - \binom{m}{2}$;

(iv) the number of rows of $A^{m,n}$ is $m+n-2$;

(v) each of the standard basis vectors appears as a column of $A^{m,n}$, and so its rank is $m+n-2$.

(vi) the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ each appear exactly once as a column of $A^{m,n}$, and the standard basis vectors $\mathbf{e}_{n+1}, \ldots, \mathbf{e}_{m+n-2}$ each appear exactly $m-1$ times as a column of $A^{m,n}$.

4.3 Plan for the rest of the chapter

The rest of this chapter consists of our work on Kronecker coefficients. These results are novel unless explicitly stated otherwise.

Our main aim is to apply Theorem 4.1.1 to study some of the main questions of Kronecker coefficients: exact computation, vanishing conditions, stability, and upper bounds. In [62] the authors focused on the $m = n = 2$ case; we adapt the main ideas of that article to general m, n.

Once an expression of the vector partition function $p_{A^{m,n}}$ as a piecewise quasi-polynomial has been computed, the complexity of using this form to determine the Kronecker coefficient comes from the large number $(mn)!$ of terms in the sum. Significantly fewer than the $(mn)!$ terms are needed (either due to vanishing or cancellation): when $m = n = 2$ only 7 of the 24 terms are needed, and when $m = 2, n = 3$ at most 482 are needed. However, we do not know how many terms are needed in general for a given m, n.

Using this to compute $g_{\lambda,\mu,\nu}$ is efficient for small m and n, and we have developed a *Sagemath* tool for computing Kronecker coefficients $g_{\lambda,\mu,\nu}$ for $l \leq 8, m \leq 2, n \leq 4$. The exact formulas are given in Section 4.4. This section is split into two subsections – in 4.4.1 we describe the more restricted case $\ell(\mu) \leq 2, \ell(\nu) \leq 3, \ell(\lambda) \leq 6$, and in 4.4.2 we describe the general case.

In Section 4.5, we show vanishing conditions (conditions on λ, μ, ν ensuring that the coefficient in question is 0) on the atomic Kronecker coefficient give vanishing conditions for the Kronecker coefficients. We subsequently deduce explicit conditions. These are given in Theorem 4.5.5. For each m, n we obtain a set of $m + n - 2$ conditions for partitions λ, μ, ν with $\ell(\mu) \leq m,\ \ell(\nu) \leq n,\ \ell(\lambda) \leq mn$. Our conditions have the advantage of being easy to compute and implement practically.

By considering the set of partition triples (λ, μ, ν) satisfying the equation

$$\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0} \tag{4.16}$$

we obtain a stable face of the Kronecker cone $PKron_{m,n,mn}$. Additionally, each triple (λ, μ, ν) satisfying the above equation is stable (Theorem 4.6.4). Eq. (4.16) is natural to consider from the point of view of the expression for the Kronecker coefficient given in Eq. (4.4). In this case, the contribution of the alternant term associated to the identity permutation is 1, and the contribution of all other alternant terms is 0 (and so the atomic Kronecker coefficient and Kronecker coefficient are both equal to 1). These results are described in Section 4.6.

The atomic Kronecker coefficient can be bounded from above using binomial coefficients (Theorem 4.7.5). By bounding each of the terms of Eq. (4.4) we are able to obtain upper bounds for the Kronecker coefficients which seem to be best known in certain cases. This is described in Section 4.7, and the main results are Corollaries 4.7.8 and 4.7.6.

4.4 Explicit computation of Kronecker coefficients

When the partition lengths are sufficiently small, it is computationally feasible to determine the vector partition functions needed to compute individual Kronecker coefficients $g_{\lambda,\mu,\nu}$. We provide explicit formulas for two cases here, starting from Eq. (4.4), rewritten below:

$$g_{\lambda,\mu,\nu} = \sum_{\sigma\in\mathfrak{S}_{mn}} \operatorname{sgn}(\sigma)\ p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma)\Big).$$

We compute $p_{A^{m,n}}$ first for $m = 2, n = 3$, then $m = 2, n = 4$ (the $m = n = 2$ case appears in [62]). We note that, to compute a coefficient, it is best to minimize the choice of m and n that bound the lengths of μ and ν. The first optimization comes from trying to identify which terms in the sum are zero. Recall, in the $m = n = 2$ case, only 7 of the terms are needed since of the original $4! = 24$ terms in the right hand side: 13 of them always evaluate to zero for partitions λ, μ, ν, and another 4 of them cancel pairwise. To eliminate terms in other cases, we consider restrictions imposed by positivity in the linear algebra system, and the partition inequalities on the parts of the partitions.

4.4.1 Exact expressions for $g_{\lambda,\mu,\nu}$ when $\ell(\lambda) \leq 6,\ \ell(\mu) \leq 2,\ \ell(\nu) \leq 3$

The matrix $A^{2,3}$ is determined in [62, Example 5]:

$$A^{2,3} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{bmatrix}$$

Using *Barvinok* it is straightforward to determine that the corresponding vector partition function $p_{A^{2,3}}$ is of degree 8 and has 34 chambers. At most 482 of the 720 terms of the alternant $a_{\lambda+\delta^{(6)}}$ yield a non-zero contribution to the Kronecker coefficient computation. The most non-zero terms we have found for any given coefficient is 288. This occurs for $\mu = (99, 99),\ \nu = (66, 66, 66),\ \lambda = (87, 87, 24, 0, 0, 0)$. It is less clear how to find cancelling pairs as in the $m = n = 2$ case, so this remains a place for potential optimization – each term represents a vector partition function evaluation, which in the worst case means searching through all chambers. The formula is as follows.

Proposition 4.4.1. *Let λ, μ, ν be partitions with $\ell(\lambda) \leq 6,\ \ell(\mu) \leq 2,\ \ell(\nu) \leq 3$. Then the Kronecker coefficient is given by*

$$g_{\lambda,\mu,\nu} = \sum_{\sigma\in\mathfrak{S}_6} \operatorname{sgn}(\sigma) p_{A^{2,3}}\left(\nu_2+\nu_3+6-l_s(\lambda;\sigma)_1, \mu_2+\nu_2+\nu_3+11-l_t(\lambda;\sigma)_1, \mu_2+\nu_2+2\nu_3+13-l_t(\lambda;\sigma)_2\right).$$

and the atomic Kronecker coefficient is given by

$$\tilde{g}^{2,3}_{\lambda,\mu,\nu} = p_{A^{2,3}}\,(\nu_2 + \nu_3 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6, \quad \mu_2 + \nu_2 + \nu_3 - \lambda_2 - \lambda_3 - \lambda_4 - 2\lambda_5 - 2\lambda_6, \\ \mu_2 + \nu_2 + 2\nu_3 - \lambda_2 - \lambda_3 - 2\lambda_4 - 2\lambda_5 - 3\lambda_6)\,. \quad (4.17)$$

Our implementation seems to be significantly faster at computing single Kronecker coefficients in the $\ell(\lambda) \leq 6, \ell(\mu) \leq 2, \ell(\nu) \leq 3$ case than that of Baldoni, Vergne, and Walter [2]. However, they are able to compute dilated Kronecker coefficients and, more generally, expressions that hold over the entire chamber, while our code does not do either.

4.4.2 Exact expressions for $g_{\lambda,\mu,\nu}$ when $\ell(\lambda) \leq 8,\ \ell(\mu) \leq 2,\ \ell(\nu) \leq 4$

It is straightforward to determine $A^{2,4}$ following the same method

$$A^{2,4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}.$$

The corresponding vector partition function $p_{A^{2,4}}$ is of degree 17 with 4328 chambers. It took roughly 20 days to compute it on the Compute Canada *Cedar* research cluster. The vector partition function is available in .sobj format and in .txt format. The .txt format is the raw output from *Barvinok*.

Out of the $8! = 40320$ terms of the alternant $a_{\lambda+\delta^{(8)}}$, at most 28322 yield a non-zero contribution to the Kronecker coefficient. It is not apparent if they can be grouped for cancellation as in the $m = n = 2$ case.

Proposition 4.4.2. *Let λ, μ, ν be partitions with $\ell(\lambda) \leq 8,\ \ell(\mu) \leq 2,\ \ell(\nu) \leq 4$. Then the Kronecker coefficient is given by*

$$g_{\lambda,\mu,\nu} = \sum_{\sigma \in \mathfrak{S}_8} \operatorname{sgn}(\sigma) p_{A^{2,4}}(\nu_2 + \nu_3 + \nu_4 + 15 - l_s(\lambda;\sigma)_1, \mu_2 + \nu_2 + \nu_3 + \nu_4 + 24 - l_s(\lambda;\sigma)_2, \\ \mu_2 + \nu_2 + 2\nu_3 + 2\nu_4 + 32 - l_t(\lambda;\sigma)_1, \mu_2 + \nu_2 + \nu_3 + 2\nu_4 + 27 - l_t(\lambda;\sigma)_2), \quad (4.18)$$

and the atomic Kronecker coefficient is given by

$$\tilde{g}^{2,4}_{\lambda,\mu,\nu} = p_{A^{2,4}}(\nu_2 + \nu_3 + \nu_4 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8, \\ \mu_2 + \nu_2 + \nu_3 + \nu_4 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - 2\lambda_6 - 2\lambda_7 - 2\lambda_8, \\ \mu_2 + \nu_2 + 2\nu_3 + 2\nu_4 - \lambda - \lambda_3 - 2\lambda_4 - 2\lambda_5 - 2\lambda_6 - 3\lambda_7 - 3\lambda_8, \\ \mu_2 + \nu_2 + \nu_3 + 2\nu_4 - \lambda_2 - \lambda_3 - \lambda_4 - 2\lambda_5 - 2\lambda_6 - 2\lambda_7 - 3\lambda_8) \quad (4.19)$$

Example 4.4.3. This formula gives the same result for the following example, taken from [4]. For $\lambda = (6,4,4,1)$, $\mu = (12,3)$, $\nu = (5,4,3,3)$, we compute that $g_{\lambda,\mu,\nu} = 4$. The authors of [4] computed this via a combinatorial rule – in this case the Kronecker coefficient is counting combinatorial objects called *Kronecker tableaux*. We note that in this case the atomic Kronecker coefficient $g^{(4,4)}_{\lambda,\mu,\nu}$ is 45310.

Example 4.4.4. Let $\lambda = (57,57,57,33,33,33,10,0)$, $\mu = (140,140)$, $\nu = (70,70,70,70)$, we compute that $g_{\lambda,\mu,\nu} = 391$. We were unable to compute this example with the package *SF* (it ran into a memory error after using 203146718216 bytes), nor the *Sagemath* symmetric functions package (which also ran into a memory error). It cannot be computed by the Maple package of Baldoni, Vergne, and Walter [2] which specifically handles the cases $\ell(\lambda), \ell(\mu), \ell(\nu) \leq 3$ and $\ell(\lambda) \leq 6, \ell(\mu) \leq 2, \ell(\nu) \leq 3$.

4.4.3 Some notes on the $\ell(\lambda) \leq 9, \ell(\mu) \leq 3, \ell(\nu) \leq 3$ case

The $A^{3,3}$ matrix is straightforward to compute, it has 4 rows and 30 columns. However obtaining the piecewise quasi-polynomial representation of the vector partition function was not computationally feasible: we had no results after roughly 30 days on the Compute Canada research cluster Cedar at which time the computation was terminated by the server.

4.4.4 External chambers

Given our work in Chapter 2, it is natural to ask whether the matrices $A^{m,n}$ have external chambers in general. Unfortunately, while $A^{2,2}$, $A^{2,3}$ and $A^{3,3}$ each have the external chamber $\mathrm{pos}_{\mathbb{R}}(\mathbf{e}_1, \ldots, \mathbf{e}_{m+n-3}, \mathbf{1})$, this pattern does not extend beyond these cases. Indeed, the matrices $A^{m,n}$ have no external chambers for $m \geq 4$ or $n \geq 4$.

4.5 Vanishing conditions

A key to our analysis is a dominance property of vector partition functions (not to be confused with dominance order of partitions). We use this property to prove Theorem 4.5.5, a generalization of some non-vanishing conditions for the Kronecker coefficients given in [13]. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$. We say that $\mathbf{u}$ *dominates* $\mathbf{v}$ if $u_i \geq v_i$ for each $1 \leq i \leq k$, and we denote this by $\mathbf{u} \succeq \mathbf{v}$.

Lemma 4.5.1. *Let m, n be positive integers, if $\mathbf{a} \succeq \mathbf{b}$, then $p_{A^{m,n}}(\mathbf{a}) \geq p_{A^{m,n}}(\mathbf{b})$*

Proof. Each of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{m+n-2}$ is a column of $A^{m,n}$. Without loss of generality assume that columns $1, \ldots, m+n-2$ are the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{m+n-2}$ in the same order. It follows that any vector partition $\mathbf{x} \in S_{A^{m,n}}(\mathbf{b})$ can be mapped to a unique vector partition $\mathbf{x}' \in S_{A^{m,n}}(\mathbf{a})$ by taking $\mathbf{x}'_i := \mathbf{x}_i + (a_i - b_i)\mathbf{e}_i$ for each $1 \leq i \leq m+n-2$. This forms an injective map from $\mathcal{P}_A(\mathbf{b})$ to $\mathcal{P}_A(\mathbf{a})$. $\square$

Lemma 4.5.2. *Let m, n be positive integers. Let $\sigma_1, \sigma_2 \in \mathfrak{S}_{mn}$ such that*

$$(l_s(\lambda;\sigma_1), l_t(\lambda;\sigma_1)) \succeq (l_s(\lambda;\sigma_2), l_t(\lambda;\sigma_2))$$

for all partitions λ with $\ell(\lambda) \leq mn$. Then

$$p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_1)\Big) \leq p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_2)\Big)$$

for all partitions λ, μ, ν with $\ell(\lambda) \leq mn, \ell(\mu) \leq m, \ell(\nu) \leq n$.

Proof. Multiplication by -1 reverses domination. The domination of one vector over another is preserved if we subtract the same vector from both sides, and if we add a positive vector to the larger one. Thus, for any partitions λ, μ, ν with $\ell(\lambda) \leq mn, \ell(\mu) \leq m, \ell(\nu) \leq n$, we find that

$$\begin{aligned}
\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_2) &= \Big(r_s(\mu,\nu), r_t(\mu,\nu)) + (\alpha,\beta) - (l_s(\lambda;\sigma_2), l_t(\lambda;\sigma_2)\Big) \\
&\succeq \Big(r_s(\mu,\nu), r_t(\mu,\nu)) + (\alpha,\beta) - (l_s(\lambda;\sigma_1), l_t(\lambda;\sigma_1)\Big) \\
&= \mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_1).
\end{aligned}$$

Then, by Lemma 4.5.1 we have that $p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_2)\Big) \geq p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma_1)\Big)$ as required. $\square$

The previous lemma induces a poset structure on $\mathfrak{S}_{mn}$ via the relation $\sigma_2 \geq \sigma_1$ if and only if

$$(l_s(\lambda;\sigma_1), l_t(\lambda;\sigma_1)) \succeq (l_s(\lambda;\sigma_2), l_t(\lambda;\sigma_2))$$

for all partitions λ with $\ell(\lambda) \leq mn$. Figure 4.1 illustrates the poset for the $m = n = 2$ case, showing only the permutations associated to the 7 alternant terms which contribute to the Kronecker coefficient. The poset in the figure is the *dependency digraph for the monomials in P_λ* given in [62, Figure 4]. However, there the poset is computed by comparing the contributions of the alternant terms as opposed to the vector partition function inputs. Our approach allows us to compute the posets for larger m, n than the previous method. In such cases comparing the contributions is infeasible (either due to the large number of chambers or the difficulty of computing the vector partition function as a piecewise quasi-polynomial).

In the following lemma we show that the identity permutation is a maximal element of the poset for any positive integers m, n (in fact it is unique, and thus the maximal element).

Lemma 4.5.3. *For all $\sigma \in \mathfrak{S}_{mn}$ and partitions λ of length at most mn,*

$$(l_s(\lambda;\sigma), l_t(\lambda;\sigma)) \succeq (l_s(\lambda;Id), l_t(\lambda;Id)).$$

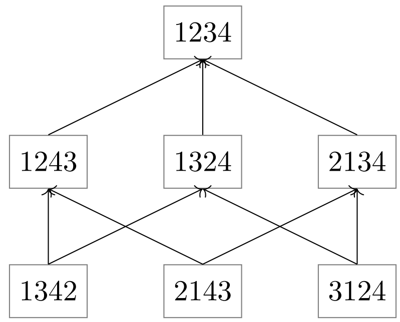


Figure 4.1: The poset of contributing alternant terms in the $m = n = 2$ case. Each alternant term is given by its permutation in one line notation.

Proof. The alternant $a_{\lambda+\delta^{(mn)}}$ is the determinant of the matrix $(z_i^{\lambda_j})_{1\le i,j\le mn}$ where z_i is the i^{th} variable in XY. The k^{th} coordinate of $(l_s(\lambda;\sigma), l_t(\lambda;\sigma))$ is

$$(l_s(\lambda;\sigma), l_t(\lambda;\sigma))_k = \sum_{i=1}^{mn} (\lambda_i + mn - i) \deg_u(z_{\sigma^{-1}(i)}) \tag{4.20}$$

where u is the k^{th} element of $(1, s_0, \ldots, s_{m-1}, t_1, \ldots, t_{n-2})$. Since $(\lambda_1 + mn - 1, \lambda_2 + mn - 2, \ldots, \lambda_{mn})$ is a monotonically decreasing sequence, and $\deg_u$ monotonically increasing over $(1, x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1},$
$x_1y_1, \ldots, x_my_n)$, the above expression is minimized for the term obtained by the change of variables from

$$1^{\lambda_1+mn-1} x_1^{\lambda_2+mn-2} \ldots \, (x_my_n)^{\lambda_{mn}}$$

corresponding to the identity permutation. □

Combining the previous two lemmas yields the following result relating the atomic Kronecker coefficient $\tilde{g}^{m,n}_{\lambda,\mu,\nu}$ with the Kronecker coefficient $g_{\lambda,\mu,\nu}$, from which vanishing conditions (given in Theorem 4.5.5) can be derived.

Lemma 4.5.4. *Let λ, μ, ν be partitions with $\ell(\lambda) \le mn, \ell(\mu) \le m, \ell(\nu) \le n$ for some positive integers m, n. If $\tilde{g}^{m,n}_{\lambda,\mu,\nu} = 0$, then $g_{\lambda,\mu,\nu} = 0$.*

Proof. If $\tilde{g}^{m,n}_{\lambda,\mu,\nu} = 0$ then for any $\sigma \in \mathfrak{S}_{mn}$,

$$\begin{aligned} 0 = \tilde{g}^{m,n}_{\lambda,\mu,\nu} &= p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu; Id)\Big) \quad \text{by Eq. (4.5),} \\ &\ge p_{A^{m,n}}\Big(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma)\Big) \quad \text{by Lemmas 4.5.2 and 4.5.3,} \\ &\ge 0, \end{aligned}$$

and thus $p_{A^{m,n}}\left(\mathbf{b}^{m,n}(\lambda,\mu,\nu;\sigma)\right) = 0$. Since it is true for all σ, all the terms in the sum in Eq. (4.4) vanish, and so $g_{\lambda,\mu,\nu} = 0$. $\square$

The converse of this lemma is not true. For instance in [62, Example 8], Mishna, Rosas, and Sundaram show that for $\lambda = (12, 7, 4, 1), \mu = (12, 12), \nu = (12, 12)$ the Kronecker coefficient is 0, but the atomic Kronecker coefficient is 32. Notably, each of the 7 necessary contributing terms of the alternant $a_{\lambda+\delta^{(4)}}$ are non-zero, and no pair cancels out.

Since $\tilde{g}^{m,n}_{\lambda,\mu,\nu}$ is given by a single vector partition function evaluation $p_{A^{m,n}}(\mathbf{b})$, we know that it is 0 exactly when $\mathbf{b}$ is not in the cone generated by the columns of $A^{m,n}$. This occurs if and only if $b_i < 0$ for some $1 \leq i \leq m+n-2$. Since $\mathbf{b} = (r_s(\mu,\nu)+\alpha-l_s(\lambda, Id), r_t(\mu,\nu)+\beta-l_t(\lambda, Id))$ for the atomic Kronecker coefficient $\tilde{g}^{m,n}_{\lambda,\mu,\nu}$, we get a set of vanishing conditions for the Kronecker coefficient $g_{\lambda,\mu,\nu}$. We express the conditions using the contrapositive (i.e. we give conditions imposed on λ, μ, ν if the Kronecker coefficient is non-zero) since the set of λ, μ, ν satisfying them forms a cone.

Theorem 4.5.5. *Let m, n be positive integers and λ, μ, ν be partitions so that $\ell(\lambda) \leq mn$, $\ell(\mu) \leq m$, $\ell(\nu) \leq n$. If $g_{\lambda,\mu,\nu} \neq 0$ then each of the following inequalities hold:*

$$\sum_{k=1}^{m} \lambda_k \geq \nu_1;$$

For all a satisfying $1 \leq a \leq m-1$:

$$\sum_{k=1}^{a} \lambda_k - \sum_{k=m+n}^{m+(a+1)(n-1)} \lambda_k \geq \nu_1 - \sum_{k=a+1}^{m} \mu_k$$

For all b satisfying $1 \leq b \leq n-2$:

$$m\lambda_1 + \sum_{k=2}^{m}(m-k+1)\lambda_k + \sum_{k=m+1}^{m+b}\lambda_k - \sum_{i=1}^{m-1}\sum_{j=1}^{b}(i-1)\lambda_{m+i(n-1)+j} - \sum_{i=1}^{m-1}\sum_{j=b+1}^{n-1} i\lambda_{m+i(n-1)+j} \geq m\nu_1 + \sum_{k=2}^{b+1}\nu_k - \sum_{k=2}^{m}(k-1)\mu_k.$$

Remark 4.5.6. When $m = n = 2$, Theorem 4.5.5 reduces to vanishing conditions given by Bravyi in [13]. This case was worked out explicitly in [62, Proposition 5].

An inequality $n \cdot x \leq 0$ is *essential* for a cone τ if $\{x : n \cdot x = 0\} \cap \tau$ is a facet of τ, and each $p \in \tau$ satisfies the inequality ($n \cdot p \leq 0$ for all $p \in \tau$).

Remark 4.5.7. Klyachko [49] gives the full list of 41 essential inequalities in the $m = 2, n = 3$ case. In this case, none of our inequalities appear on Klyachko's list. Thus, while our inequalities are easy to compute and use practically, regrettably none are essential inequalities for the cone $PKron_{2,3,6}$. Thus, one should not expect, for general m, n, that the inequalities given by Theorem 4.5.5 are essential.

Ressayre determined two sets of vanishing conditions for the Kronecker coefficients for any lengths l, m, n which are essential, [77, Theorems 1 & 2].

Theorem 4.5.8 (Ressayre, 2019, [77])**.** *Let e, f be two positive integers, and let λ, μ, ν be partitions of N with*

$$\ell(\mu) \leq e+1,\ \ell(\nu) \leq f+1,\ \ell(\lambda) \leq e+f+1 \tag{4.21}$$

If $g_{\lambda,\mu,\nu} \neq 0$, then

$$N + \lambda_1 + \lambda_{e+j} \leq \mu_1 + \nu_1 + \nu_j$$

for all $2 \leq j \leq f+1$.

These are quite strong, although there is likely a smaller error in these conditions, given the following example we found.

Example 4.5.9. Upon setting $e = 1, f = 3, n = 4$ and $j = 4$ in Eq. (4.21), we have that $\ell(\mu) \leq 2, \ell(\nu) \leq 4, \ell(\lambda) \leq 5$ and the Kronecker coefficient $g_{\lambda,\mu,\nu}$ should be 0 if, furthermore,

$$|\lambda| + \lambda_1 + \lambda_5 > \mu_1 + \nu_1 + \nu_4. \tag{4.22}$$

Consider $\lambda = (1,1,1,1,0), \mu = (2,2), \nu = (2,2)$. Inequality (4.22) is satisfied, but $g_{\lambda,\mu,\nu} = 1$, not 0. A second example is given by $\lambda = (4), \mu = (2,2), \nu = (2,2)$.

4.6 Stability

By considering a set of conditions implying that the atomic Kronecker coefficient and Kronecker coefficient are both equal to 1, we are able to obtain a stable face of the Kronecker polyhedron $PKron_{m,n,mn}$ for each m, n. Moreover, each partition triple (λ, μ, ν) satisfying these conditions is a stable triple. We note that elements of this approach appear in [61] for the case $m = n = 2$.

Proposition 4.6.1. *If $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$, then $g_{\lambda,\mu,\nu} = \tilde{g}^{m,n}_{\lambda,\mu,\nu} = 1$.*

Proof. When $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$, $\mathbf{b}^{m,n}(\lambda, \mu, \nu; \sigma)$ has at least one negative coordinate for each $\sigma \in \mathfrak{S}_{mn},\ \sigma \neq Id$, and so

$$\begin{aligned} g_{\lambda,\mu,\nu} &= \tilde{g}^{m,n}_{\lambda,\mu,\nu} \\ &= p_{A^{m,n}}(\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id)) \\ &= 1. \end{aligned}$$

□

The condition $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$ yields $m + n - 2$ equations involving the parts of λ, μ, ν. By also including the equations $|\lambda| = |\mu| = |\nu|$, we obtain relatively simple expressions for each part of μ and ν in the parts of λ.

Proposition 4.6.2. *Let λ, μ, ν be partitions of the same positive integer N with $\ell(\mu) \leq m, \ell(\nu) \leq n, \ell(\lambda) \leq mn$. Then $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$ if and only if (λ, μ, ν) satisfy the following equations:*

$$\mu_u = \lambda_u + \sum_{i=m+(u-1)(n-1)+1}^{m+u(n-1)} \lambda_i \qquad \text{for } u = 1, \ldots, m \tag{4.23}$$

$$\nu_1 = \sum_{i=1}^{m} \lambda_i \tag{4.24}$$

$$\nu_v = \sum_{i=0}^{m-1} \lambda_{m+(n-1)i+v-1} \qquad \text{for } v = 2, \ldots, n. \tag{4.25}$$

The proof of this appears in Appendix B.1. The following result follows directly from Propositions 4.6.1 and 4.6.2.

Corollary 4.6.3. *Let λ, μ, ν be partitions of N with $\ell(\mu) \leq m, \ell(\nu) \leq n, \ell(\lambda) \leq mn$, such that λ, μ, ν satisfy Eqs.* (4.23)–(4.25). *Then $g_{\lambda,\mu,\nu} = \tilde{g}^{m,n}_{\lambda,\mu,\nu} = 1$.*

We can say more about the partition triples (λ, μ, ν) satisfying Eqs. (4.23)–(4.25). We follow [59] for notation. A triple of partitions (λ, μ, ν) is called *weakly stable* if $g_{k\lambda,k\mu,k\nu} = 1$ for each positive integer k. Recall that a triple of partitions (λ, μ, ν) is *stable* if for any partitions α, β, γ the sequence $(g_{\alpha+k\lambda,\beta+k\mu,\gamma+k\nu})_{k\geq 0}$ stabilizes.

For given positive integers l, m, n, the *weight lattice* $W_{l,m,n}$ is the sublattice of $\mathbb{Z}^{l+m+n}$ defined by the equations $|\lambda|= |\mu|= |\nu|$. In [59], Manivel defines a *stable face* of the cone $PKron_{l,m,n}$ to be a face of $PKron_{l,m,n}$ whose intersection with $W_{l,m,n}$ is a subset of $SKron_{l,m,n}$ - the set of all weakly stable triples (λ, μ, ν) with $\ell(\lambda) \leq l,\ \ell(\mu) \leq m, \ell(\nu) \leq n$. A stable face is *maximal* if it is maximal in $SKron_{l,m,n}$.

We note that the set of triples (λ, μ, ν) satisfying Eqs. (4.23)–(4.25) along with the partition inequalities (for any partition α of length k, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0$) generate a cone $\tau_{m,n}$. By Corollary 4.6.3, each λ, μ, ν in the intersection $\tau_{m,n} \cap W_{mn,m,n}$ is weakly stable. In fact, as the next theorem shows, they are actually stable.

Theorem 4.6.4. *Each triple λ, μ, ν satisfying Eqs.* (4.23) – (4.25) *is a stable triple. Moreover, the cone $\tau_{m,n}$ is a stable face of $PKron_{mn,m,n}$.*

The proof of the previous theorem is given in Appendix B.2. It relies on the connection between *additive tableaux* and stable faces given in [58, Propositions 7 and 9].

Example 4.6.5. Let $\lambda = (10, 8, 5, 3, 2, 2), \mu = (17, 12), \nu = (18, 7, 5)$. One can check that λ, μ, ν satisfy Eqs (4.23)–(4.25). Further we have checked that $g_{k\lambda,k\mu,k\nu} = 1$ for all positive integers k computing the quasi-polynomial $g_{k\lambda,k\mu,k\nu}$ via the code of Baldoni, Vergne and Walter. We now give an example to illustrate the stability of λ, μ, ν. For $\alpha = (34, 27, 20, 12, 4, 3), \beta = (70, 30), \nu = (43, 39, 18)$, the sequence $(g_{\alpha+k\lambda,\beta+k\mu,\gamma+k\nu})_{k\geq 1}$

stabilizes at 44729 at $k = 6$. The sequence from $k = 0$ to 6 is 2566, 18028, 36174, 43896, 44638, 44713, 44729.

We note that the stable face $\tau_{m,n}$ is not maximal in general. For example, $\tau_{3,3}$ is contained in the stable faces F_2^-, F_5^-, F_7^- and F_8 from [59, Example 2]. In particular, F_5^- is the (maximal) stable facet defined by the intersection of $PKron_{3,3,9}$ and the equation

$$\mu_2 + 2\mu_3 + 2\nu_2 + 3\nu_3 = \lambda_2 + 2\lambda_3 + 2\lambda_4 + 3\lambda_5 + 3\lambda_6 + 4\lambda_7 + 4\lambda_8 + 5\lambda_9$$

which is $\mathbf{b}^{3,3}(\lambda, \mu, \nu; Id)_4 = 0$.

We remark also that a couple well-known results are implied by Theorem 4.6.4. When λ, μ, ν are each rectangular partitions of lengths mn, m, n respectively (that is $\lambda_1 = \cdots = \lambda_{mn}$, $\mu_1 = \cdots = \mu_m$, $\nu_1 = \cdots = \nu_n$), the Kronecker coefficient is 1 (and the triple (λ, μ, ν) is stable). It is straightforward to check that λ, μ, ν satisfy Eqs. (4.23)–(4.25). The case $\mu = \lambda$ and $\ell(\nu) = 1$ (so $\nu = (|\lambda|)$) also satisfies the same equations (and again the Kronecker coefficient in this case is 1, and the partition triple (λ, μ, ν) is stable).

4.7 Bounds

The atomic Kronecker coefficients are given by a single vector partition function evaluation $p_{A^{m,n}}(\mathbf{b})$. By constructing a companion matrix to $A^{m,n}$, we are able to obtain a simpler vector partition function for which the evaluations can be computed by hand and whose evaluations bound $p_{A^{m,n}}$ from above. By bounding each of the terms of Eq.(4.4), we are then able to obtain upper bounds for the Kronecker coefficients.

4.7.1 A bound in terms of atomic Kronecker coefficients

In [62], Mishna, Rosas and Sundaram show that in the $m = n = 2$ case, the atomic Kronecker coefficient $\tilde{g}^{2,2}_{\lambda,\mu,\nu}$ bounds the corresponding Kronecker coefficient $g_{\lambda,\mu,\nu}$ from above, and in [61] they conjecture that this is the case in general. Since we do know that the atomic Kronecker coefficient is the largest term in the sum, we can use this to give a general weaker bound.

Proposition 4.7.1. *Let λ, μ, ν be partitions with $\ell(\mu) \leq m, \ell(\nu) \leq n, \ell(\lambda) \leq ln$. Then*

$$g_{\lambda,\mu,\nu} \leq \frac{(mn)!}{2} \tilde{g}^{m,n}_{\lambda,\mu,\nu}.$$

Proof. Splitting the sum in Eq. (4.4) in two halves, one for the permutations with positive sign, and one for the permutations with negative sign, we bound each of the negative sign terms above by 0 and each of the positive terms by the atomic term (by Lemmas 4.5.1 and 4.5.3). $\square$

Finally, we remark that there is still much work to do in obtaining effective upper bounds for the Kronecker coefficients. Recall from Example 4.4.3 that for $\lambda = (6, 4, 4, 1), \mu = (12, 3), \nu = (5, 4, 3, 3)$ the Kronecker coefficient is $g_{\lambda,\mu,\nu} = 4$. In this case our bounds give $g_{\lambda,\mu,\nu} \leq 4.72 \cdot 10^{11}$. Similarly, recall from Example 4.4.4 that for the triple of partitions $\lambda = (57, 57, 57, 33, 33, 33, 10, 0), \mu = (140, 140), \nu = (70, 70, 70, 70)$ the Kronecker coefficient is $g_{\lambda,\mu,\nu} = 391$. In this case our bound yields $g_{\lambda,\mu,\nu} \leq 1.08 \cdot 10^{67}$.

4.7.2 Estimating atomic Kronecker Coefficients

We can approximate the partition function of a matrix A by replacing its columns with standard basis vectors so that the rank is preserved. Partition functions of such matrices are easy to write using binomial coefficients. Lemma 4.7.3 describes the replacement process and Proposition 4.7.4 is the resulting bound. The following proposition sets up Lemma 4.7.3.

Proposition 4.7.2. *Let A be a $d \times n$ matrix with integer entries and* $\ker(A) \cap \mathbb{R}^m_{\geq 0} = \{\mathbf{0}\}$. *Let $\mathbf{c}$ be a column of A, and let $\mathbf{c}'$ be a $1 \times n$ vector. Let A' be the matrix obtained by replacing column $\mathbf{c}$ with $\mathbf{c}'$. If $p_{A'}(\mathbf{c}) \geq p_A(\mathbf{c})$, then*

$$p_{A'}(\mathbf{b}) \geq p_A(\mathbf{b})$$

for all $\mathbf{b} \in \mathbb{N}^d$.

Proof. Let j be the index at which column $\mathbf{c}$ appears in A (and thus column $\mathbf{c}'$ appears in A'). Partition the set of vector partitions $\mathcal{P}_A(\mathbf{b})$ of $\mathbf{b}$ into $U_1 := \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, x_j = 0\}$ and $U_2 := \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, x_j > 0\}$. The set U_1 is equal to the set $\{x : A'\mathbf{x} = \mathbf{b}, x_j = 0\}$. Also $\mathbf{c} \in \{A'\mathbf{x} : x_j > 0\}$ since $p_{A'}(\mathbf{c}) \geq p_A(\mathbf{c})$, and so $|U_2| \leq |\{x : A'\mathbf{x} = \mathbf{b}, x_j > 0\}|$. Thus $p_A(\mathbf{b}) = |U_1| + |U_2| \leq p_{A'}(\mathbf{b})$ as required. □

The following Lemma describes how to replace columns of $A^{m,n}$ with standard basis vectors via the previous proposition.

Lemma 4.7.3. *Let A be a $d \times n$ matrix with non-negative integer entries and each standard basis vector $\mathbf{e}_1, \ldots, \mathbf{e}_d$ appearing as a column of A. Let $\mathbf{c}$ be a column of A, and let $I = \{k : c_k > 0, 1 \leq k \leq n\}$ be the set of non-zero coordinates of $\mathbf{c}$. Let $E^{(i)}$ denote the matrix obtained by replacing column $\mathbf{c}$ with $\mathbf{e}_i$ for some $i \in I$. Then*

$$p_{E^{(i)}}(\mathbf{b}) \geq p_A(\mathbf{b})$$

for all $\mathbf{b} \in \mathbb{N}^d$.

Proposition 4.7.4. *Let E be a $d \times n$ matrix such that the columns of E are formed by taking i_j copies of each standard basis vector $\mathbf{e}_i$ where $i_1, \ldots, i_d \geq 0$. Then*

$$p_A(\mathbf{b}) = \prod_{i=1}^{k} \binom{b_i + i_j - 1}{i_j - 1}.$$

Proof. For each component i we must take a total of b_i copies of the standard basis vector $\mathbf{e}_i$. We can think of this problem as distributing b_i balls to the i_j different columns of A which are the copies of $\mathbf{e}_i$. This is counted by the i^{th} term in the given product of binomial coefficients. □

By application of Lemma 4.7.3 and Proposition 4.7.4 we obtain binomial coefficient bounds for the atomic Kronecker coefficients, and thus the Kronecker coefficients as well. The technical details of the proof appear in Appendix B.3 where we work out explicitly which columns have which non-zero coordinates. We note that there are many choices of column replacements that can be made, and different choices provide better bounds for certain choices of λ, μ, ν. The formulation of Theorem 4.7.5 represents a single choice whose advantage is that it is relatively simple to explain.

Theorem 4.7.5. *Let m, n be positive integers, and λ, μ, ν be partitions with $\ell(\lambda) \leq mn$, $\ell(\mu) \leq m$, $\ell(\nu) \leq n$. Then:*

$$\tilde{g}^{m,n}_{\lambda,\mu,\nu} \leq \binom{b_1 + c_1}{b_1}\binom{b_2 + c_2}{b_2}\binom{b_{m+n-2} + c_3}{b_{m+n-2}} \prod_{i=3}^{m} \binom{b_i + f_1(i)}{b_i} \prod_{j=1}^{n-3} \binom{b_{m+j} + f_2(j)}{b_{m+j}}. \tag{4.26}$$

where $\mathbf{b} = (b_1, \ldots, b_{m+n-2}) = \mathbf{b}^{m,n}(\lambda, \mu, \nu; Id)$ *and*

$$\begin{aligned} c_1 &= (m^2 - 1)(n-1) - 1 \\ c_2 &= (m-1)(n-1)^2 - 1 \\ c_3 &= \binom{m-1}{2}(n-1) + (m-1) - 1 \\ f_1(i) &= 2\binom{n-1}{2}(i-2) - 1 \\ f_2(j) &= (n-j-1)(m-1) - 1 \end{aligned}$$

Corollary 4.7.6. *Theorem 4.7.5 in combination with Proposition 4.7.1 gives:*

$$g_{\lambda,\mu,\nu} \leq \frac{(mn)!}{2} R_1$$

where R_1 is the expression on the right-handside of Inequality (4.26).

We also give a weaker general bound which depends only on m, n and the size N of the partitions λ, μ, ν. We do this by bounding the coordinates of $\mathbf{b}^{m,n}$ by multiples of N.

Corollary 4.7.7. *Let m, n be positive integers, and λ, μ, ν, be partitions of N with lengths at most mn, m, n respectively.*

$$\tilde{g}^{m,n}_{\lambda,\mu,\nu} \le \binom{N+c_1}{N}\binom{2N+c_2}{2N}\binom{(2m-1)N+c_3}{(2m-1)N}\prod_{i=3}^{m}\binom{2N+f_1(i)}{2N}\prod_{j=1}^{n-3}\binom{(2m-1)N+f_2(j)}{(2m-1)N}. \tag{4.27}$$

where c_1, c_2, c_3, f_1, f_2 are as in Theorem 4.7.5.

Proof. Recall that for each component of

$$\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = (r_s(\mu, \nu) + \alpha - l_s(\lambda; Id), r_t(\mu, \nu) + \beta - l_t(\lambda; Id)$$

the constant terms cancel. Therefore each b_i of Theorem 4.7.5 is bounded above by linear combination in the parts of μ, ν appearing. Explicitly, we find that $b_1 \le N$, $b_i \le 2N$ for $2 \le i \le m$ and $b_{m+j} \le (2m-1)N$ for $1 \le j \le n-2$. □

As before, combining the previous result with Proposition 4.7.1, we obtain the following bound for the Kronecker coefficients.

Corollary 4.7.8. *Let m, n be positive integers, and λ, μ, ν be partitions of N of lengths at most mn, m, n respectively. Then*

$$g_{\lambda,\mu,\nu} \le \frac{(mn)!}{2} R_2$$

where R_2 is the expression on the right-hand side of Inequality (4.27).

The bound given in line (4.7.8) is $O(N^d)$, where d is the difference between the number of columns and rows of $A^{m,n}$ - that is:

$$d = \binom{mn}{2} - \binom{n}{2} - \binom{m}{2} - n - m + 2$$

whereas the bound given in [70] is $O(N^{(mn)^2})$. We note that this analysis holds for the case when $\ell(\mu) = m, \ell(\nu) = n, \ell(\lambda) = mn$. For example, the bound given by Pak and Panova is stronger if $\ell(\lambda) = \ell(\mu) = \ell(\nu) = m$ since in this case their bound is $O(N^{m^3})$, while our bound is $O(N^{\binom{m^2}{2} - 2\binom{m}{2} - 2m + 2})$. If $\ell(\mu) = m,\ \ell(\nu) = n$ are fixed, we find that the exponent x given by our $O(N^x)$ expression is smaller when

$$\ell(\lambda) > \frac{mn}{2} - \left(\frac{m^2 + n^2 + m + n - 4}{2mn}\right) - \frac{1}{2} \tag{4.28}$$

Source	Bound
Corollary 4.7.6	$1.42 \cdot 10^{16}$
Corollary 4.7.8	$5.38 \cdot 10^{45}$
Pak and Panova, Inequality (4.2) [69]	$2.84 \cdot 10^{27}$
Pak and Panova, Inequality (4.3) [70]	$1.13 \cdot 10^{54}$

Table 4.1: Upper bound comparison for $g_{15^3\,10^5\,5, 35^2\,30, 40\,30^2}$

and larger when the inequality is flipped. We note that for $\ell(\mu), \ell(\nu) \geq 2$ (i.e. $m, n \geq 2$), the expression on the right-hand side of (4.28) is smaller than $\frac{mn}{2}$.

We give the explicit bound in the $m = n = 3$ case for which there is no efficient computational tool.

Corollary 4.7.9. *For all partitions λ, μ, ν of N with $\ell(\mu), \ell(\nu) \leq 3, \ell(\lambda) \leq 9$.*

$$g_{\lambda,\mu,\nu} \leq \frac{9!}{2}\binom{b_1+15}{15}\binom{b_2+7}{7}\binom{b_3+1}{1}\binom{b_4+3}{3}$$

where

$$\begin{aligned}
b_1 &= \nu_2 + \nu_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 - \lambda_9 \\
b_2 &= \mu_2 + \mu_3 + \nu_2 + \nu_3 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - 2\lambda_6 - 2\lambda_7 - 2\lambda_8 - 2\lambda_9 \\
b_3 &= \mu_3 + \nu_2 + \nu_3 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - 2\,\lambda_8 - 2\,\lambda_9 \\
b_4 &= \mu_2 + 2\,\mu_3 + 2\,\nu_2 + 3\,\nu_3 - \lambda_2 - 2\,\lambda_3 - 2\,\lambda_4 - 3\,\lambda_5 - 3\,\lambda_6 - 4\,\lambda_7 - 4\,\lambda_8 - 5\,\lambda_9.
\end{aligned}$$

Example 4.7.10. Table 4.1 presents bounds on $g_{\lambda,\mu,\nu}$ in the $\ell(\lambda) \leq 9, \ell(\mu), \ell(\nu) \leq 3$ case for the partitions $\lambda = (15, 15, 15, 10, 10, 10, 10, 10, 5)$, $\mu = (35, 35, 30)$, $\nu = (40, 30, 30)$.

The bound given by Inequality (4.2) by Pak and Panova is better on some examples. From our experience our bound is the better choice when λ is close to rectangular due to the large coefficients on small parts of λ.

Chapter 5

Outlook

In this thesis, we have derived results about the quasi-polynomials arising from vector partition functions. We have then used these results in order to study of combinatorial objects which count integral points in polytopes (multigraphs with a given degree sequence, and Kostant's partition function for the root system A_{m-1}). Also, we have studied two sets of algebraic combinatorial coefficients – the Littlewood-Richardson and Kronecker coefficients – which can be understood (less directly) through a vector partition function lens.

One potentially interesting avenue of research is to study other combinatorial objects that are related to vector partition functions in order to obtain results like the one we obtained for multigraphs: Theorem 2.6.4. Recall that in this case we identified a sequence of external chambers of the corresponding vector partition functions p_{G_m}. Moreover, the inequalities defining the external chamber for each m depend solely on m. By exploiting this, we were able to obtain a geometric result enumerating multigraphs (Theorem 2.6.4).

The goal is to apply the same process for other combinatorial objects – that is, to identify "persisting" external chambers (i.e a sequence of external chambers whose inequalities only depend on the size of the objects being enumerated). Then one can use the reduction to coin exchange problems via Theorem 2.3.4 (or Theorem 2.4.2 in the case that its conditions are satisfied) to potentially obtain exact enumeration results. In the case of Theorem 2.6.4 the combinatorial proof is simpler than the geometrical one. However, the vector partition function approach is still valuable in such a case since the computation of the vector partition function (or just of external chambers) yields the inequalities one should consider, as well as instances of the general formula which should be proven. It would be particularly interesting to identify a case where the geometrical proof is simpler than the combinatorial proof.

1. Compute external chambers for small instances (if they exist).

2. Identify the first few terms of a sequence of "persisiting" external chambers.

3. Prove an exact enumeration result for the number of combinatorial objects satisfying the inequalities given by the "persisting" external chambers using either the geometrical approach or a combinatorial approach.

Of course, one can also consider applying this process for other (non-external) chambers, in which case we may not obtain an exact formula, but could potentially compute linear factors.

For the Littlewood-Richardson coefficients, our main aim is to complete the study of the appearance of linear factors in the polynomials of Φ_k. Namely, we would like to prove or disprove Conjecture 3.8.1 - the conjecture relating the facets of E_k and $\mathcal{LR}_k$ and thus the associated linear factors. Additionally, we hope to use the hive interpretation to understand the facets of E_k in order to compute the number of linear factors combinatorially. Finally, we remark that the Littlewood-Richardson coefficients can also be expressed as an alternating sum of vector partition functions (via Kostant's partition function). It may be interesting to research "atomic LR coefficients" in the same mannner as we have done for the Kronecker coefficients via the approach of Mishna, Rosas, and Sundaram.

We also leave some potentially interesting problems related to our work on Kronecker coefficients via the approach of Mishna, Rosas, and Sundaram.

1. Compute the piecewise quasi-polynomial $p_{A^{3,3}}$.

2. Prove that the atomic Kronecker coefficient bounds the Kronecker coefficient (to remove the $\frac{(mn)!}{2}$ term in the upper bound of Corollaries 4.7.6 and 4.7.8).

3. Study the properties of the posets of alternant terms.

We also quickly remark that, while computing the full piecewise quasi-polynomial $p_{A^{3,3}}$ was not feasible for us, we could compute the piecewise quasi-polynomial for the external chamber. It may be interesting to try computations for Kronecker coefficients for which each alternant term is in the external chamber (or yields 0).

Finally, we leave some questions about Kronecker coefficients which are not directly related to the approach of Mishna, Rosas, and Sundaram. Like the Littlewood-Richardson coefficients, for partitions λ, μ, ν of bounded lengths $\ell(\lambda) \leq mn, \ell(\mu) \leq m, \ell(\nu) \leq n$, the Kronecker coefficients can be described by a piecewise quasi-polynomial $\kappa_{m,n}$ (this is not a piecewise polynomial as in the LR coefficient case however) whose domains of quasi-polynomiality are governed by a fan. The function $\kappa_{2,2}$ has been explicitly computed by Briand, Orellana, and Rosas [15], and the software of Baldoni, Vergne and Walter [2] allows one to explicitly compute chambers and quasi-polynomials of $\kappa_{2,3}$.

Briand, Rosas and I have done some work to better understand the piecewise quasi-polynomial Kronecker function $\kappa_{2,2}$. Namely, we found a poset structure of the chambers associated chamber complex which is related to objects analagous to external rays (i.e rays of the chamber complex which have constant Ehrhart quasi-polynomial). Surprisingly, we have also found that an analogue of the determinantal formula of Theorem 2.3.5 can be used to compute one of the quasi-polynomials of the Kronecker function $\kappa_{2,2}$. It would be curious to justify why this should happen in order to understand the extent to which Theorem 2.3.5 can be generalized.

In [2], Baldoni, Vergne, and Walter identify that on some faces of the fan of the $\kappa_{2,3}$ case the Kronecker coefficients are given by a polynomial. Additionally, they remark on the high number of linear factors of this polynomial.[1] One possible line of research would be to study the linear factors appearing on such faces – specifically to check whether or not Theorem 2.5.3 can be applied in such a case.

[1]From Baldoni, Vergne, Walter [2]: "We summarize our results in Table 3. We find that, remarkably, the symbolic function on $\mathfrak{c}_{v_{F_I}}$ is polynomial, instead of merely quasipolynomial (first row). It is a striking fact that this polynomial function is divisible by 7 linear factors with constant values 1, 2, 3, 4, 5, 6, 7 on the face F_I."

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Appendix A

Lemma 2.3.3 Proof

A.1 Statement and notation

We begin by recalling Lemma 2.3.3, written here as Lemma A.1.1 below.

Lemma A.1.1. *Let γ be a chamber of A. Assume without loss of generality that the external columns of A in γ are $\mathbf{a}_1, \ldots, \mathbf{a}_\ell$ for some $\ell \in \{0, \ldots, d-1\}$. Also assume that $\mathbf{a}_i = k_i \mathbf{e}_i$ for each $i \in \{1, \ldots, \ell\}$ and some positive integers $k_1, \ldots, k_\ell$. Finally assume that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ is A-lattice minimal. Let B be the matrix obtained by removing the first ℓ rows and columns of A. Then there exists a chamber γ' of B such that*

$$p_A^\gamma(\mathbf{b}) = p_B^{\gamma'}(b_{\ell+1}, \ldots, b_d) \tag{A.1}$$

for all $\mathbf{b} = (b_1, \ldots, b_n) \in \operatorname{pos}_{\mathbb{N}}(A) \cap \gamma$.

To prove this lemma, we proceed by induction on ℓ. The base case $\ell = 0$ is clear. We assume henceforth that $\ell \geq 1$. For the inductive step there are two things that we need to show (labelled [C] and [V] below):

[C] The matrix $A_{\hat{1},\hat{1}}$ also satisfies the hypotheses of Lemma 2.3.3, that is:

- (a) there is a chamber γ' of $A_{\hat{1},\hat{1}}$ which is simplicial. Furthermore, for all $\mathbf{b} \in \gamma$, it follows that $\mathbf{b}_{\hat{1}} \in \gamma'$,
- (b) the columns $\{(\mathbf{a}_2)_{\hat{1}}, \ldots, (\mathbf{a}_\ell)_{\hat{1}}\}$ are external columns of $A_{\hat{1},\hat{1}}$,
- (c) the set $\{(\mathbf{a}_2)_{\hat{1}}, \ldots, (\mathbf{a}_\ell)_{\hat{1}}\}$ is $A_{\hat{1},\hat{1}}$-lattice minimal.

[V] The vector partition function of $A_{\hat{1},\hat{1}}$ respects Eq. (A.1), that is: $p_A^\gamma(\mathbf{b}) = p_{A_{\hat{1},\hat{1}}}^{\gamma'}(\mathbf{b}_{\hat{1}})$ for all $\mathbf{b} \in \operatorname{pos}_{\mathbb{N}}(A) \cap \gamma$.

By proving [C] and [V], we show that we can iteratively remove the ℓ rows and columns of A corresponding to the external columns of γ.

Having outlined our plan, we begin by introducing some notation.

Throughout this appendix, we assume that γ is a simplicial chamber of A. We additionally assume that γ has external columns $\mathbf{a}_1, \dots, \mathbf{a}_\ell$ satisfying $\mathbf{a}_i = k_i \mathbf{e}_i$ for some positive integers $k_1, \dots, k_\ell$, and that the set $\{\mathbf{a}_1, \dots, \mathbf{a}_\ell\}$ is A-lattice minimal.

We also use $\tilde{A}$ to denote the matrix $A_{\hat{1},\hat{1}}$, and we overload notation by denoting the columns of $\tilde{A}$ by $\tilde{\mathbf{a}}_2, \dots, \tilde{\mathbf{a}}_n$ (in order to keep the indexing consistent between A and $\tilde{A}$). Similarly for a subset $\tilde{s} \subseteq \{2, \dots, n\}$, we write $\tilde{A}_{\tilde{s}}$ to indicate the submatrix of $\tilde{A}$ whose columns are $\{\tilde{\mathbf{a}}_i : i \in \tilde{s}\}$. Finally, by B, we denote the matrix obtained by removing the first ℓ rows and columns of A. We write the general form of the matrices A and $\tilde{A}$ below:

$$A = \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_\ell & \mathbf{a}_{\ell+1} & \dots & \mathbf{a}_n \end{array} \\ \left(\begin{array}{cccc|c} k_1 & & & & \\ & k_2 & & & * \\ & & \ddots & & \\ & & & k_\ell & \\ \hline & & 0 & & B \end{array}\right) \end{array} \qquad \tilde{A} = \begin{array}{c} \begin{array}{cccccc} \tilde{\mathbf{a}}_2 & \dots & \tilde{\mathbf{a}}_\ell & \tilde{\mathbf{a}}_{\ell+1} & \dots & \tilde{\mathbf{a}}_n \end{array} \\ \left(\begin{array}{ccc|c} k_2 & & & \\ & \ddots & & * \\ & & k_\ell & \\ \hline & 0 & & B \end{array}\right) \end{array}.$$

A.2 Proof of [C]

A.2.1 The simplicial cones of $\tilde{A}$

We now prove some results which relate the simplicial cones of A with those of $\tilde{A}$ with a view towards proving the conditions of [C].

Recall that $\mathrm{pos}_{\mathbb{R}}(A_s)$ is a simplicial cone of A if and only if $|s| = \mathrm{rank}(A_s) = d$.

Proposition A.2.1. *Let $\tilde{s} \subseteq \{2, \dots, n\}$, and let $s := \{1\} \cup \tilde{s}$. Then $\mathrm{pos}_{\mathbb{R}}(A_s)$ is a simplicial cone of A if and only if $\mathrm{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$ is a simplicial cone of $\tilde{A}$.*

Proof. This result follows immediately from the observation that $\mathrm{rank}(A_s) = \mathrm{rank}(\tilde{A}_{\tilde{s}}) + 1$. □

In the following proposition we use $\tilde{\mathbf{b}}$ to denote $\mathbf{b}_{\hat{1}}$.

Proposition A.2.2. *Let $\mathbf{b} \in \gamma$. Let $\mathrm{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$ be a simplicial cone of $\tilde{A}$ for some subset $\tilde{s} \subseteq \{2, \dots, n\}$, and let $s := \{1\} \cup \tilde{s}$. Then $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A_s)$ if and only if $\tilde{\mathbf{b}} \in \mathrm{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$.*

Proof. The forward direction (if $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(A_s)$ then $\tilde{\mathbf{b}} \in \mathrm{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$) is clear, so we prove only the reverse direction.

Since $\tilde{\mathbf{b}} \in \mathrm{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$,

$$\tilde{\mathbf{b}} = \sum_{i \in \tilde{s}} \lambda_i \tilde{\mathbf{a}}_i$$

for some $\lambda_i \geq 0$. Let

$$\mathbf{c} := \sum_{i \in \tilde{s}} \lambda_i \mathbf{a}_i.$$

There are two cases to consider.

Case 1: If $c_1 \leq b_1$, then

$$\begin{aligned} \mathbf{b} &= (b_1 - c_1)\mathbf{e}_1 + \mathbf{c} \\ &= \frac{b_1 - c_1}{k_1}\mathbf{a}_1 + \sum_{i \in \tilde{s}} \lambda_i \mathbf{a}_i \\ &= \sum_{i \in s} \lambda_i \mathbf{a}_i && \text{(setting } \lambda_1 := \frac{b_1 - c_1}{k_1}\text{)} \end{aligned}$$

and so $\mathbf{b} \in \operatorname{pos}_{\mathbb{R}} A_s$.

Case 2: If $c_1 > b_1$, then

$$\begin{aligned} \mathbf{c} &= \mathbf{b} + (c_1 - b_1)\mathbf{e}_1 \\ &= \mathbf{b} + \frac{c_1 - b_1}{k_1}\mathbf{a}_1 \end{aligned}$$

and since $\mathbf{a}_1, \mathbf{b} \in \gamma$, it follows that $\mathbf{c} \in \gamma$. Moreover, by definition $\mathbf{c} \in \operatorname{pos}_{\mathbb{R}}(A_{\hat{1}})$. Therefore, $\mathbf{c}$ lies on the unique facet f of γ not containing $\mathbf{a}_1$. Since f is a face of γ, if the sum of any two vectors in γ is in f, then both of those vectors must be in f (see [38, Section 1.2] for example). Therefore, $\mathbf{b}$, $\frac{c_1 - b_1}{k_1}\mathbf{a}_1 \in f$ and in particular, $\mathbf{a}_1 \in f$. This is a contradiction since f is the unique facet of γ not containing $\mathbf{a}_1$, and so Case 2 cannot occur.

Therefore, $\mathbf{b} \in \operatorname{pos}_{\mathbb{R}}(A_s)$ as required. □

A.2.2 (a) Simplicial chamber

Let S denote the subset of $\mathcal{P}([n])$ such that $s \in S$ if and only if $\operatorname{pos}_{\mathbb{R}}(A_s)$ is a simplicial cone satisfying $\gamma \subseteq \operatorname{pos}_{\mathbb{R}}(A_s)$, and let $\tilde{S} := \{s \setminus \{1\} : s \in S\}$.

Proposition A.2.3. *The cone*

$$\tilde{\gamma} := \bigcap_{\tilde{s} \in \tilde{S}} \operatorname{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}}) \tag{A.2}$$

is a chamber of $\tilde{A}$. Additionally, $\mathbf{b} \in \gamma$ if and only if $\tilde{\mathbf{b}} \in \tilde{\gamma}$.

Proof. Recall that we can represent γ as the intersection of all simplicial cones of A containing $\mathbf{b}$ for some $\mathbf{b} \in \gamma^\circ$. Consider $\tilde{\mathbf{b}} := \mathbf{b}_{\hat{1}}$. By Proposition A.2.2, it follows that the simplicial cones of $\tilde{A}$ containing $\tilde{\mathbf{b}}$ are exactly the simplicial cones appearing on the right-hand side of (A.2). Therefore, $\tilde{\gamma}$ is a cone in the chamber complex of $\tilde{A}$. By construction, $b \in \gamma$ if and only if $\tilde{b} \in \tilde{\gamma}$.

In order to prove that $\tilde{\gamma}$ is a chamber of $\tilde{A}$, we must show that it is $(d-1)$-dimensional. Let G be the matrix whose columns are the minimal ray generators of γ (up to column permutation) so that the first column of G is $\mathbf{a}_1 = k_1\mathbf{e}_1$. Then, the columns of $G_{\hat{1},\hat{1}}$ are exactly the minimal ray generators of $\tilde{\gamma}$. Since $\operatorname{rank}(G) = \operatorname{rank}(G_{\hat{1},\hat{1}}) + 1$, it follows that $\tilde{\gamma}$ is indeed $(d-1)$-dimensional, and thus a chamber of $\tilde{A}$. $\square$

We note that $\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell \in \tilde{\gamma}$. In the next section we show that each of these columns is an external column of $\tilde{A}$.

A.2.3 (b) External columns

Proposition A.2.4. *Each of the columns $\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell$ are external columns of $\tilde{A}$.*

Proof. We give the proof for $\tilde{\mathbf{a}}_2$ noting that the other cases follow similarly.

Assume towards a contradiction that $\tilde{\mathbf{a}}_2$ is not an external column of $\tilde{A}$. Since a cone can be triangulated into simplicial cones with no new ray generators, there is some subset $s' \subseteq \{3, \ldots, n\}$ such that

$$\tilde{\mathbf{a}}_2 \in \operatorname{pos}_{\mathbb{R}}(\tilde{A}_{s'})$$

and $\{\tilde{\mathbf{a}}_i : i \in s'\}$ is linearly independent. Additionally, since $\tilde{\mathbf{a}}_2$ is part of a linearly dependent set, we see that s' can be extended to some set $\tilde{s} \subseteq \{3, \ldots, n\}$ with $\operatorname{rank}(\tilde{A}_{\tilde{s}}) = |\tilde{s}| = d-1$. Therefore, $\tilde{\mathbf{a}}_2$ is in the simplicial cone $\operatorname{pos}_{\mathbb{R}}(\tilde{A}_{\tilde{s}})$ of $\tilde{A}$, and so by Proposition A.2.2, $\mathbf{a}_2 \in \operatorname{pos}_{\mathbb{R}}(A_s)$ where $s = \tilde{s} \cup \{1\}$. This is a contradiction since $\mathbf{a}_2$ is an external column. $\square$

A.2.4 (c) A-lattice minimality

Recall that a set of columns of A whose indices lie in s for some $s \subseteq [n]$ is A-lattice minimal if

$$\operatorname{pos}_{\mathbb{N}}(A_s) = \mathcal{L}(A) \cap \operatorname{pos}_{\mathbb{R}}(A_s).$$

We note that $\operatorname{pos}_{\mathbb{N}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell) \subseteq \mathcal{L}(A) \cap \operatorname{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell)$ for any choice of columns, so one only needs to show the reverse inclusion.

Proposition A.2.5. *The set of columns $\{\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell\}$ is $\tilde{A}$-lattice minimal.*

Proof. We need to prove that $\operatorname{pos}_{\mathbb{R}}(\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell) \cap \mathcal{L}(\tilde{A}) \subseteq \operatorname{pos}_{\mathbb{N}}(\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell)$.

Let $\tilde{\mathbf{b}} \in \operatorname{pos}_{\mathbb{R}}(\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell) \cap \mathcal{L}(\tilde{A})$. Since $\tilde{\mathbf{b}} \in \mathcal{L}(\tilde{A})$, there exist integers $d_2, \ldots, d_n$ such that

$$\tilde{\mathbf{b}} = \sum_{i=2}^{n} d_i \tilde{\mathbf{a}}_i.$$

and since $\tilde{\mathbf{b}} \in \operatorname{pos}_{\mathbb{R}}(\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell)$, there exist $\lambda_2, \ldots, \lambda_n \geq 0$ such that

$$\tilde{\mathbf{b}} = \sum_{i=2}^{n} \lambda_i \tilde{\mathbf{a}}_i.$$

Let

$$\mathbf{c} := \sum_{i=2}^{n} d_i \mathbf{a}_i \in \mathcal{L}(A),$$

and let

$$\mathbf{c}' := \sum_{i=2}^{n} \lambda_i \mathbf{a}_i \in \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell).$$

By construction, $c_2 = c_2', \ldots, c_d = c_d'$. Let N be a positive integer satisfying $k_1 N > c_1' - c_1$, and let $\mathbf{b} := \mathbf{c} + N\mathbf{a}_1$, so that $b_1 = c_1 + k_1 N$. Then $\mathbf{b} \in \mathcal{L}(A)$ since $\mathbf{a}_1, \mathbf{c} \in \mathcal{L}(A)$ and N is an integer. Additionally,

$$\begin{aligned}\mathbf{b} &= \mathbf{c}' + (k_1 N + c_1 - c_1')\mathbf{e}_1 \\ &= \mathbf{c}' + \frac{k_1 N + c_1 - c_1'}{k_1}\mathbf{a}_1\end{aligned}$$

and since $\mathbf{a}_1, \mathbf{c}' \in \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell)$ and $\frac{k_1 N + c_1 - c_1'}{k_1} > 0$, it follows that $\mathbf{b} \in \mathrm{pos}_{\mathbb{R}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell)$.

By the A-lattice minimality of $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ it follows that $\mathbf{b} \in \mathrm{pos}_{\mathbb{N}}(\mathbf{a}_1, \ldots, \mathbf{a}_\ell)$, and so

$$\mathbf{b} = m_1 \mathbf{a}_1 + \cdots + m_\ell \mathbf{a}_\ell$$

for some non-negative integers $m_1, \ldots, m_\ell$. Therefore, $\tilde{\mathbf{b}} = m_2 \tilde{\mathbf{a}}_2 + \cdots + m_\ell \tilde{\mathbf{a}}_\ell$, and thus $\tilde{\mathbf{b}} \in \mathrm{pos}_{\mathbb{R}}(\tilde{\mathbf{a}}_2, \ldots, \tilde{\mathbf{a}}_\ell)$ as required. $\square$

A.3 Proof of [V]

Let $\tilde{\gamma}$ be defined as in (A.2) - equivalently

$$\tilde{\gamma} = \{\mathbf{b}_{\hat{1}} : \mathbf{b} \in \gamma\}.$$

Before proceeding to the proof we make a quick remark about A-lattice minimality which we exploit. If $\{\mathbf{a}_1, \ldots, \mathbf{a}_\ell\}$ is A-lattice minimal and each $\mathbf{a}_j$ for $j = 1, \ldots, \ell$ is an external column, then the singleton sets $\{\mathbf{a}_j\}$ are also A-lattice minimal.

Lemma A.3.1. *Let* $\mathbf{b} \in \mathrm{pos}_{\mathbb{N}}(A) \cap \gamma$*, and let* $\tilde{\mathbf{b}} := \mathbf{b}_{\hat{1}}$*. Then*

$$p_A^{\gamma}(\mathbf{b}) = p_{\tilde{A}}^{\tilde{\gamma}}(\tilde{\mathbf{b}}).$$

Proof. The following proof proceeds in two stages. In the first stage, we use the assumption that $\mathbf{b} \in \mathrm{pos}_{\mathbb{N}}(A)$. In the second stage we use geometrical methods exploiting the fact that $\mathbf{b} \in \gamma$.

Denote the entry at the rth row and cth column of A by $a_{r,c}$. We note that $a_{2,1} \ldots, a_{n,1} = 0$, because the first column is $k_1 \mathbf{e}_1$.

Let $\mathbf{b} = (b_1, \ldots, b_d) \in \text{pos}_{\mathbb{N}}(A) \cap \gamma$. Consider some $\tilde{\mathbf{x}} = (x_2, \ldots, x_n) \in \mathbb{N}^{n-1}$ satisfying

$$
\begin{aligned}
a_{1,2}x_2 + \cdots + a_{1,n}x_n &\leq b_1 && \text{(A.3)}\\
a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 && \text{(A.4)}\\
\vdots \qquad\qquad \vdots \quad & \quad \vdots \\
a_{d,2}x_2 + \cdots + a_{d,n}x_n &= b_d. && \text{(A.5)}
\end{aligned}
$$

For such $\tilde{\mathbf{x}}$,

$$
\begin{aligned}
\mathbf{u} &:= \mathbf{b} - x_2\mathbf{a}_2 - \ldots - x_n\mathbf{a}_n \\
&= ((\underbrace{b_1 - (a_{1,2}x_2 + \cdots + a_{1,n}x_n}_{m}), 0, \ldots, 0) \\
&= m\mathbf{e}_1
\end{aligned}
$$

where m is a non-negative integer. So, $\mathbf{u} \in \text{pos}_{\mathbb{R}}(\mathbf{e}_1)$ and since $\mathbf{b} \in \text{pos}_{\mathbb{N}}(A) \subseteq \mathcal{L}(A)$, and $\mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathcal{L}(A)$, it follows that $\mathbf{u} \in \mathcal{L}(A)$ as well. By hypothesis

$$
\begin{aligned}
\mathcal{L}(A) \cap \text{pos}_{\mathbb{R}}(\mathbf{e}_1) &= \text{pos}_{\mathbb{N}}(\mathbf{a}_1) \\
&= \text{pos}_{\mathbb{N}}(k_1\mathbf{e}_1)
\end{aligned}
$$

hence $k_1|m$. Any such $\tilde{\mathbf{x}}$ extends uniquely to the solution

$$
\mathbf{x} = \left(\frac{m}{k_1}, x_2, \ldots, x_n\right) \in \mathbb{N}
$$

of $A\mathbf{x} = \mathbf{b}$. We note that this is the unique choice for x_1, because

$$
A\mathbf{x} = \begin{bmatrix} k_1x_1 + b_1 - m \\ b_2 \\ \vdots \\ b_d \end{bmatrix}
$$

and so $k_1x_1 + b_1 - m = b_1$ implying that $x_1 = m/k_1$. The previous argument describes an injective map from the set of solutions $\mathbf{x}' \in \mathbb{N}^{d-1}$ satisfying Lines (A.3)–(A.5) to the set of solutions $\mathbf{x} \in \mathbb{N}^d$ of $A\mathbf{x} = \mathbf{b}$. This map is clearly a bijection since the inverse map is the projection $\mathbf{x} = (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$.

We now show that the conditions $\mathbf{b} \in \gamma$ and $A_{\hat{1},\cdot}\mathbf{x} = \mathbf{b}_{\hat{1}}$ are sufficient to imply Inequality (A.3). The second of these assumptions will be used to obtain Eq. (A.7) from Eq. (A.6). Equations (A.4) – (A.5) are equivalent to $A_{\hat{1},\cdot}\mathbf{x} = \mathbf{b}_{\hat{1}}$ since $a_{2,1}, \ldots, a_{n,1} = 0$.

By Lemma 2.2.10, γ has a unique facet $f = \gamma \cap H$ not containing $k_1\mathbf{e}_1$, where H is a supporting hyperplane of γ separating $\mathbf{a}_1$ from $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$. Let $\boldsymbol{\iota}$ denote an inner normal of H with respect to the cone γ. Since $k_1\mathbf{e}_1 \in H^+$, we have that $\boldsymbol{\iota} \cdot (k_1\mathbf{e}_1) > 0$ and so $\iota_1 > 0$. By definition, we also have that $\boldsymbol{\iota}^T\mathbf{b} \geq 0$ for all $\mathbf{b} \in \gamma$, and since H separates $\mathbf{a}_1$ from $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}})$, we find that $\text{pos}_{\mathbb{R}}(A_{\cdot,\hat{1}}) \subseteq H^-$ and so $\boldsymbol{\iota}^T A_{\cdot,\hat{1}}\mathbf{x}_{\hat{1},\cdot} \leq 0$.

We have

$$\begin{aligned}
\boldsymbol{\iota}^T \mathbf{b} &= \iota_1 b_1 + \boldsymbol{\iota}_{\hat{1}} \cdot \mathbf{b}_{\hat{1}} && \text{(A.6)}\\
&= \iota_1 b_1 + \boldsymbol{\iota}_{\hat{1}}^T A_{\hat{1},\cdot} \mathbf{x} && \text{(A.7)}\\
&= \iota_1 b_1 + \boldsymbol{\iota}^T A_{\cdot,\hat{1}} x_{\hat{1}} - (\iota_1 a_{1,2} x_2 + \cdots + \iota_1 a_{1,n} x_n) && \text{(A.8)}\\
&= \iota_1 \Big(b_1 - (a_{1,2} x_2 + \cdots + a_{1,n} x_n) \Big) + \boldsymbol{\iota}^T A_{\cdot,\hat{1}} \mathbf{x}_{\hat{1}} && \text{(A.9)}
\end{aligned}$$

where Eq.(A.8) follows since $a_{2,1} = \cdots = a_{d,1} = 0$ as the first column of A is $k_1 \mathbf{e}_1$. Rearranging Eq. (A.9) to solve for b_1, we have

$$\begin{aligned}
b_1 &= \frac{\boldsymbol{\iota}^T \mathbf{b} - \boldsymbol{\iota}^T A_{\cdot,\hat{1}} \mathbf{x}_{\hat{1}}}{\iota_1} + (a_{1,2} x_2 + \cdots + a_{1,n} x_n)\\
&\geq a_{1,2} x_2 + \cdots + a_{1,n} x_n.
\end{aligned}$$

Therefore, we see that Equations (A.4) – (A.5) do indeed imply Inequality (A.3). Since $\mathbf{x}'$ satisfying Inequality (A.3) and Equations (A.4) – (A.5) extends to a solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$, we find that for $\mathbf{b} \in \gamma \cap \mathrm{pos}_{\mathbb{N}}(A)$, the sets $\{\mathbf{x} \in \mathbb{N}^n : A\mathbf{x} = \mathbf{b}\}$ and $\{\mathbf{x} \in \mathbb{N}^n : A_{\hat{1},\cdot} \mathbf{x} = \mathbf{b}_{\hat{1}}\}$ are equal. Therefore, $p_A(\mathbf{b}) = p_{A_{\hat{1},\cdot}}(\mathbf{b}_{\hat{1}})$. Finally since the first column of $A_{\hat{1},\cdot}$ is all zeroes, we have that $p_{A_{\hat{1},\cdot}}(\mathbf{b}_{\hat{1}}) = p_{A_{\hat{1},\hat{1}}}(\mathbf{b}_{\hat{1}})$, and so

$$p_A(\mathbf{b}) = p_{A_{\hat{1},\hat{1}}}(\mathbf{b}_{\hat{1}}) = p^{\tilde{\gamma}}_{\tilde{A}}(\tilde{\mathbf{b}})$$

completing the proof. □

Now that we have proven [C] and [V], we see that we can indeed iteratively remove the first ℓ rows and columns of A. This concludes our proof of Lemma 2.3.3.

Appendix B

Kronecker coefficient proofs

B.1 Proof of Proposition 4.6.2

We begin by restating Proposition 4.6.2.

Proposition B.1.1. *Let λ, μ, ν be partitions of the same positive integer N with $\ell(\mu) \leq m, \ell(\nu) \leq n, \ell(\lambda) \leq mn$. Then $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$ if and only if (λ, μ, ν) satisfy the following equations:*

$$\mu_u = \lambda_u + \sum_{i=m+(u-1)(n-1)+1}^{m+u(n-1)} \lambda_i \qquad \textit{for } u = 1, \ldots, m \tag{B.1}$$

$$\nu_1 = \sum_{i=1}^{m} \lambda_i \tag{B.2}$$

$$\nu_v = \sum_{i=0}^{m-1} \lambda_{m+(n-1)i+v-1} \qquad \textit{for } v = 2, \ldots, n. \tag{B.3}$$

Proof. One can check that for λ, μ, ν respecting Eqs. (4.23) – (4.25) we do indeed get $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$.

The set of solutions (λ, μ, ν) to $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id) = \mathbf{0}$ over $\mathbb{R}^{m+n+mn}$ with $|\lambda| = |\mu| = |\nu|$ is $\ker(Q)$ for a matrix Q whose rows are given by the equations $|\mu| = |\lambda|$, $|\nu| = |\lambda|$ and the coordinate-wise equalities $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id)_i = 0$ for $1 \leq i \leq m + n - 2$. Below we give the matrix Q' obtained from Q by removing all columns indexed by λ. The row corresponding to coordinate i of $\mathbf{b}^{m,n}(\lambda, \mu, \nu; Id)$ is indexed by the s or t variable from which the equation arises. The matrix Q' is

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccc} \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{m-1} & \mu_m & \nu_1 & \nu_2 & \nu_3 & \dots & \nu_{n-1} & \nu_n \end{array} \\
\begin{array}{c} |\mu|=|\lambda| \\ |\nu|=|\lambda| \\ s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{m-1} \\ t_1 \\ t_2 \\ \vdots \\ t_{n-2} \end{array} &
\begin{pmatrix}
1 & 1 & \dots & \dots & \dots & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\
0 & 0 & \dots & \dots & \dots & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\
0 & 0 & \dots & \dots & \dots & 0 & 0 & 1 & 1 & \dots & \dots & 1 \\
0 & 1 & 1 & \dots & \dots & 1 & 0 & 1 & 1 & \dots & \dots & 1 \\
0 & 0 & 1 & \dots & \dots & 1 & 0 & 1 & 1 & \dots & \dots & 1 \\
\vdots & & & & & & \vdots & & & & & \vdots \\
0 & 0 & 0 & \dots & 0 & 1 & 0 & 1 & 1 & \dots & \dots & 1 \\
0 & 1 & 2 & \dots & m-2 & m-1 & 0 & m-1 & m & m & \dots & m \\
0 & 1 & 2 & \dots & m-2 & m-1 & 0 & m-1 & m-1 & m & \dots & m \\
\vdots & & & & & & \vdots & & & & & \vdots \\
0 & 1 & 2 & \dots & m-2 & m-1 & 0 & m-1 & m-1 & m-1 & \dots & m-1
\end{pmatrix}
\end{array}
$$

and its rank is $m+n$. Therefore Q also has rank $m+n$, and so $\ker(Q)$ has dimension mn and co-dimension $m+n$. Since the set of λ, μ, ν respecting Eqs. (4.23) – (4.25) also has co-dimension $m+n$, we see that the two systems of linear equations are equivalent. $\square$

B.2 Proof of Theorem 4.6.4

In [58], Manivel gives a description of the stable faces of the Kronecker polyhedron in terms of a particular type of standard tableau. A standard tableau T of shape $m \times n$ is *additive* if there exist increasing sequences $x_1 < x_2 < \cdots < x_m, y_1 < y_2 < \cdots < y_n$ with the property that

$$T(i,j) < T(l,k) \iff x_i + x_j < x_l + x_k.$$

For an $m \times n$ additive tableau T and partition λ of length at most mn, Manivel defines the partitions $a_T(\lambda)$ and $b_T(\lambda)$ as follows:

$$a_T(\lambda)_i = \sum_{j=1}^{m} \lambda_{T(i,j)} \text{ for } i = 1, \dots, m$$

$$b_T(\lambda)_j = \sum_{i=1}^{n} \lambda_{T(i,j)} \text{ for } j = 1, \dots, n.$$

Then $(\lambda, a_T(\lambda), b_T(\lambda))$ is a stable triple [58, Proposition 7] and the set $\{(\lambda, a_T(\lambda), b_T(\lambda)) : \ell(\lambda) \leq mn\}$ is a face of the Kronecker polyhedron of minimal dimension [58, Proposition 9]. We now restate Theorem 4.6.4, and then show that $\tau_{m,n}$ can be described by an additive tableau, thus proving that each λ, μ, ν satisfying Eqs. (4.23) – (4.25) is a stable triple.

Theorem B.2.1. *Each triple λ, μ, ν satisfying Eqs.* (4.23) – (4.25) *is a stable triple. Moreover, the cone $\tau_{m,n}$ is a stable face of $PKron_{mn,m,n}$.*

Proof. Consider the tableau

$$T = \begin{bmatrix} 1 & m+1 & m+2 & \dots & m+n-1 \\ 2 & m+(n-1)+1 & m+(n-1)+2 & \dots & m+2(n-1) \\ 3 & m+2(n-1)+1 & m+2(n-1)+2 & \dots & m+3(n-1) \\ \vdots & \vdots & \vdots & & \vdots \\ m & m+(m-1)(n-1)+1 & m+(m-1)(n-1)+2 & \dots & mn \end{bmatrix}$$

defined by

$$\begin{aligned} T_{i,1} &= i && \text{for } i = 1, \dots, m \\ T_{i,j} &= m + i(n-1) + j && \text{for } i = 2, \dots, m, j = 1, \dots, n. \end{aligned}$$

It is straightforward to check that for any λ with $\ell(\lambda) \leq mn$, $a_T(\lambda)$ and $b_T(\lambda)$ are the partitions μ and ν defined by Eqs. (4.23) – (4.25). We now show that T is an additive tableau.

Consider the sequences

$$x_i = (i-1)(n-1) \text{ for } i = 1, \dots, m$$

and

$$y_1 = 0, y_j = (m-1)(n-1) + j - 1 \text{ for } j = 2, \dots, m.$$

If $T_{i,j} < T_{k,l}$, we have three main cases to consider.

1. If $l = 1$, then $j = 1$, and so $i < k$. In this case $x_i + y_1 < x_k + y_1$ since $x_i < x_k$.
2. If $l \geq 2$ and $j = 1$, then

$$\begin{aligned} x_i + y_1 &\leq (m-1)(n-1) \\ &< (m-1)(n-1) + 1 \\ &\leq x_k + y_2 \\ &\leq x_k + y_l \end{aligned}$$

3. If $l, j \geq 2$, then $T_{i,j} < T_{k,l}$ if and only if $i < k$ or ($i = k$ and $j < l$).
   If $i < k$, then

$$\begin{aligned} x_i + y_j &= (i-1)(n-1) + (m-1)(n-1) + j - 1 \\ &< i(n-1) + (m-1)(n-1) + 1 \\ &\leq x_{i+1} + y_2 \\ &\leq x_k + y_l \end{aligned}$$

   so $x_i + y_j < x_k + y_l$.
   If $i = k$ and $j < l$, then $x_i + y_j < x_k + y_l$ since $y_j < y_l$.

Therefore T is an additive tableau and $\tau_{m,n}$ is the face associated to T. Thus, we conclude that each triple of partitions λ, μ, ν satisfying Eqs. (4.23) – (4.25) is stable, and that $\tau_{m,n}$ is a stable face of $PKron_{mn,m,n}$. $\square$

Remark B.2.2. Manivel [58] introduced the *(T, λ)-reduced Kronecker coefficient* $g_{T,\lambda}(\alpha, \beta, \gamma)$ to be the stable value of the sequence $(g_{\alpha+k\lambda,\beta+k\mu,\gamma+k\nu})_{k\geq 0}$. He also shows that the (T, λ)-reduced Kronecker coefficient counts integral points in a polytope $P_{T,\lambda}$ (and thus may be written as a vector partition function). It may be interesting to compare (T, λ)-reduced Kronecker coefficients (for the T given above) and atomic Kronecker coefficients for a given m, n (although the choice of λ is not a priori obvious).

B.3 Proof of Theorem 4.7.5

We begin by restating Theorem 4.7.5.

Theorem B.3.1. *Let m, n be positive integers, and λ, μ, ν be partitions with $\ell(\lambda) \leq mn$, $\ell(\mu) \leq m$, $\ell(\nu) \leq n$. Then:*

$$\tilde{g}^{m,n}_{\lambda,\mu,\nu} \leq \binom{b_1 + c_1}{b_1}\binom{b_2 + c_2}{b_2}\binom{b_{m+n-2} + c_3}{b_{m+n-2}}\prod_{i=3}^{m}\binom{b_i + f_1(i)}{b_i}\prod_{j=1}^{n-3}\binom{b_{m+j} + f_2(j)}{b_{m+j}}, \quad \text{(B.4)}$$

where $\mathbf{b} = (b_1, \ldots, b_{m+n-2}) = \mathbf{b}^{m,n}(\lambda, \mu, \nu; Id)$ and

$$\begin{aligned} c_1 &= (m^2 - 1)(n - 1) - 1 \\ c_2 &= (m - 1)(n - 1)^2 - 1 \\ c_3 &= \binom{m-1}{2}(n - 1) + (m - 1) - 1 \\ f_1(i) &= 2\binom{n-1}{2}(i - 2) - 1 \\ f_2(j) &= (n - j - 1)(m - 1) - 1 \end{aligned}$$

Proof. For each of column $\mathbf{c}$ of $A^{m,n}$ we analyze which of the indices $1 \leq h \leq m+n-2$ are non-zero, in order to understand which of the standard basis vectors $e_1, \ldots, e_{m+n-2}$ we may use to replace $\mathbf{c}$ with. The columns of $A^{m,n}$ arise from the binomials of $\mathcal{A}, \ldots, \mathcal{F}$ from lines (4.8) – (4.12) after the substitution to s, t variables. Explicitly, if column $\mathbf{c}$ of $A^{m,n}$ corresponds to the binomial $1 - \prod_{u=0}^{m-1}\prod_{v=1}^{n-2} s_u^{p_u} t_v^{r_v}$ for non-negative integers $p_0, \ldots, p_{m-1}, r_1, \ldots, r_{n-2}$, then

$$c_k = \begin{cases} p_{k-1} \text{ if } 1 \leq k \leq m \\ r_{k-m} \text{ if } m + 1 \leq k \leq m + n - 2. \end{cases}.$$

If c_k is non-zero, then by Proposition 4.7.3, we can replace column $\mathbf{c}$ by e_k. In the following discussion, we analyze each column of $A^{m,n}$ to find which standard basis vectors may be used to replace it in order to obtain bounds. The information is collected in Table B.1. We note that we only three cases – when replacement can be done via the standard basis vector e_1, when it can be done via e_2, or when any of the standard basis vectors e_{m+b} for $b = 1, \ldots, n-2$ can be used. We use this approach in order to keep the number of cases relatively low.

column origin	binomial	# columns	e_1	e_2	e_{m+b}
$\mathcal{A}$	$1 - \frac{y_j}{x_i}$	$(m-1)(n-1)$	✓		
$\mathcal{B}$	$1 - x_i y_j$	$(m-1)(n-1)$	✓	✓	✓
$\mathcal{C}$	$1 - x_i$	$(m-1)(n-1)$		✓	✓
$\mathcal{C}$	$1 - y_j$	$(m-1)(n-1)$	✓	✓	✓
$\mathcal{D}$	$1 - \frac{x_i y_j}{x_k}$	$2\binom{m-1}{2}(n-1)$	✓	✓	✓
$\mathcal{D}$	$1 - \frac{x_i y_j}{y_k}$	$2\binom{n-1}{2}(m-1)$		✓	
$\mathcal{E}$	$1 - \frac{x_k y_l}{x_i y_j}$	$2\binom{n-1}{2}\binom{m-1}{2}$			
$\mathcal{F}$	$1 - \frac{x_k}{x_i}$	$(n-1)\binom{m-1}{2}$			✓
$\mathcal{F}$	$1 - \frac{y_l}{y_j}$	$(m-1)\binom{n-1}{2}$			

Table B.1: Columns of $A^{m,n}$ and the standard basis vectors which can be used to replace them.

For each column type (defined by the form of the binomial it arose from), we provide the number of such columns and illustrate which of the standard basis vectors in the set $\{e_1, e_2\} \cup \{e_{m+b}\}_{1 \leq b \leq n-2}$ can be used to replace such a column.

Columns arising from $\mathcal{A}$

The binomials of $\mathcal{A}$ are of the form $1 - \frac{y_j}{x_i}$ for $1 \leq i \leq m-1,\ 1 \leq j \leq n-1$. There are $(m-1)(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given i, j is

$$1 - s_0(s_{i+1} \ldots s_{m-1})(t_1 \ldots t_{n-2})^{m-1-i}(t_1 \ldots t_{j-1}).$$

Here s_0 appears for each choice of i, j. Therefore we can replace any of the columns arising from $\mathcal{A}$ by e_1. We thus choose e_1 to replace all such columns.

Columns arising from $\mathcal{B}$

The binomials of $\mathcal{B}$ are of the form $1 - x_i y_j$ for $1 \leq i \leq m-1,\ 1 \leq j \leq n-1$. There are $(m-1)(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given i, j is

$$1 - (s_1 \ldots s_i)(t_1 \ldots t_{n-2})^i(s_0 \ldots s_{m-1})(t_1 \ldots t_{n-2})^{m-1}(t_1 \ldots t_{j-1}).$$

Each s_a and t_b, $(0 \leq a \leq m-1,\ 1 \leq b \leq n-2)$ appear for all choices of i, j. Therefore we can replace any of the columns arising from $\mathcal{B}$ by any of the standard basis vectors $e_1, \ldots, e_{m+n-2}$. We choose e_1 to replace all such columns.

Columns arising from $\mathcal{C}$

There are two types of binomials arising from $\mathcal{C}$.

The first type of binomial of $\mathcal{C}$ is of the form $1 - x_i$ for $1 \le i \le m-1$. In this case each binomial is raised to the power $n-1$, so there are $(m-1)(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given i is

$$1 - (s_1 \dots s_i)(t_1 \dots t_{n-2})^i.$$

Here s_1 and t_b $(1 \le b \le n-2)$ appear for all choices of i. Therefore we can replace any of the columns of the first type arising from $\mathcal{C}$ by e_2 or e_{m+b} for $1 \le b \le n-2$. We choose e_2 to replace all such columns.

The second type of binomial of $\mathcal{C}$ is of the form $1 - y_j$ for $1 \le j \le n-1$. In this case each binomial is raised to the power $m-1$, so there are $(m-1)(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given j is

$$1 - (s_0 \dots s_{m-1})(t_1 \dots t_{n-2})^{m-1}(t_1 \dots t_{j-1})$$

Here each s_a, t_b $(0 \le a \le m-1,\ 1 \le b \le n-2)$ appear for all choices of j. Therefore we can replace any of the columns of the second type arising from $\mathcal{C}$ by any of the standard basis vectors $e_1, \dots, e_{m+n-2}$. We choose e_1 to replace all such columns.

Columns arising from $\mathcal{D}$

There are two types of binomials arising from $\mathcal{D}$. The first type of binomial of $\mathcal{D}$ is of the form $1 - \frac{x_i y_j}{x_k}$ for $1 \le i \le m-1, 1 \le k \le m-1, 1 \le j \le n-1$ with $k \ne i$. There are $2\binom{m-1}{2}(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given choice of i, j, k is

$$1 - (s_0 \dots s_i)(s_{k+1} \dots s_{m-1})(t_1 \dots t_{n-2})^{m+i-k-1}(t_1 \dots t_{j-1})$$

Here s_0, s_1 and t_b $(1 \le b \le n-2)$ each appear for all choices of i, j, k. Therefore, we can replace any of the columns of the first type arising from $\mathcal{D}$ by e_1, e_2 or e_{m+b} for $1 \le b \le n-2$. We choose e_1 to replace all such columns.

The second type of binomial of $\mathcal{D}$ is of the form $1 - \frac{x_i y_j}{y_k}$ for $1 \le i \le m-1, 1 \le j \le n-1, 1 \le k \le n-1$ with $k \ne j$. There are $2\binom{n-1}{2}(m-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given choice of i, j, k is

$$1 - (s_1 \dots s_i)(t_1 \dots t_{k-1})^{i-1}(t_k \dots t_{n-2})^i(t_1 \dots t_{j-1})$$

Here s_1 appears for each choice of i, j, k. Therefore we can replace any of the columns of the first type arising from $\mathcal{D}$ by e_2. We thus choose e_2 to replace all such columns.

Columns arising from $\mathcal{E}$

The binomials of $\mathcal{E}$ are of the form $1-\frac{x_k y_l}{x_i y_j}$ for $1 \leq i < k \leq m-1, 1 \leq j \leq n-1, 1 \leq l \leq n-1$ with $j \neq l$. There are $2\binom{n-1}{2}\binom{m-1}{2}$ columns of this type. After the variable substitution, the binomial in s, t arising from a given choice of i, j, k, l is

$$1 - (s_{i+1} \dots s_k)(t_1 \dots t_{j-1})^{k-i-1}(t_j \dots t_{n-2})^{k-i}(t_1 \dots t_{l-1})$$

Here none of the variables appear for each choice of i, j, k, l. However, for any $q = 2, \dots, m-1$, we may consider the set of rows for which s_q appears and no s_r with $r > q$ appears. In each of these cases using e_{q+1} to replace the row is a sensible choice for any choice of q. There are

$$2\binom{n-1}{2}(q-1)$$

such columns for each $2 \leq q \leq m-1$ (note that $q \neq 1$ since $i+1 \geq 2$).

Columns arising from $\mathcal{F}$

There are two types of binomials arising from $\mathcal{F}$. The first type of binomial of $\mathcal{F}$ is of the form $1 - \frac{x_k}{x_i}$ for $1 \leq i < k \leq m-1$. Each of these binomials is raised to the power $n-1$, so there are $\binom{m-1}{2}(n-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given choice of i, k is

$$1 - (s_{i+1} \dots s_k)(t_1 \dots t_{n-2})^{k-i}$$

Here t_b ($1 \leq b \leq n-2$) appears for all choices of i, k. Therefore we can replace any of the columns arising from $\mathcal{A}$ by e_{m+b} for $1 \leq b \leq n-2$. We choose e_{m+n-2} to replace all such columns.

The second type of binomial of $\mathcal{F}$ is of the form $1 - \frac{y_l}{y_j}$ for $1 \leq j < l \leq n-1$. Each of these binomials is raised to the power $m-1$, so there are $\binom{n-1}{2}(m-1)$ columns of this type. After the variable substitution, the binomial in s, t arising from a given choice of j, l is

$$1 - (t_j \dots t_{l-1})$$

Here none of the variables appear for all choices of j, l. However, for any $q = 1, \dots, n-2$ we may consider the set of rows for which t_q appears but no t_r appears with $r < q$. In each of these cases using e_{m+q} to replace the row is a sensible choice for any choice of q. There are

$$(n-1-q)(m-1)$$

such columns for each $1 \leq q \leq n-2$ (note that $q < l$, so $q \neq n-1$).

Replacing each column via the process described above produces the bound given in Theorem 4.7.5, where c_1+1 is the number of columns replaced by e_1, c_2+1 is the number of comlumns replaced by e_2, c_3+1 is the number of columns replaced by e_{m+n-2}, $f_1(i)+1$ is the number of columns replaced by e_i (for $3 \leq i \leq m$) and $f_2(j)+1$ is the number of columns replaced by e_{m+j} (for $1 \leq j \leq n-3$). We note that the added ones appear since c_1, c_2, c_3, f_1, f_2 have incorporated the subtraction by one necessary for the negative binomial coefficient. $\square$

Appendix C

Code

C.1 The AlternantTerm class

```
class AlternantTerm:
  def __init__(self, m, n, perm):
    self.m = m
    self.n = n
    self.perm = perm


  @staticmethod
  def dominates(v1, v2):
    """
    Checks if a tuple v1 'dominates' another tuple v2. That is, checks if
    each partial sum of v1 is at least the partial sum of v2. The partial
    sums of a tuple are just the sums of the first k elements for all k from
    1 to the length of v1.

    :param v1: a tuple
    :param v2: a tuple
    :return: a bool
    """
    l = len(v1)
    return all([sum(v1[:i]) >= sum(v2[:i]) for i in range(1, l+1)])

  def __le__(self, alternant_term):
    """
    Returns True if the vector partition function evaluation of self is
    always at most the vector partition function evaluation of alternant_term
    regardless of the choice of lmbda, mu, nu. We do this by checking
    domination of the s & t vectors of both.

    :param alternant_term: an AlternantTerm
    :return: a bool
    """

    bool1 = all([self.dominates(self.s()[k], alternant_term.s()[k]) for k in
    range(self.m)])
    bool2 = all([self.dominates(self.t()[k], alternant_term.t()[k]) for k in
    range(1, self.n-1)])
```

```
    return bool1 and bool2

  def __lt__(self, alternant_term):
    """
    Returns True if the vector partition function evaluation of self is always less than the vector partition function evaluation of alternant_term regardless of the choice of
    lmbda, mu, nu. We do this by checking domination of the s & t vectors of both.

    :param alternant_term: an AlternantTerm
    :return: a bool
    """

    if self <= alternant_term and self != alternant_term:
      return True

    else:
      return False

  def __ge__(self, alternant_term):
    """
    Returns True if the vector partition function evaluation of self is always at least the vector partition function evaluation of alternant_term regardless of the choice of lmbda, mu, nu. We do this by checking domination of the s & t vectors of both.

    :param alternant_term: an AlternantTerm
    :return: a bool
    """
    return not self < alternant_term

  def __gt__(self, altnerant_term):
    """
    Returns True if the vector partition function evaluation of self is always greater than the vector partition function evaluation of alternant_term regardless of the choice of lmbda, mu, nu. We do this by checking domination of the s & t vectors of both.

    :param alternant_term: an AlternantTerm
    :return: a bool
    """
    return not self <= alternant_term

  def b_value_dominates(self, alternant_term, lmbda, mu, nu):
    """
    Checks if the b value associated to self and the partitions lmbda, mu, nu is at least as large as the b value as associated to alternant_term and the partitions lmbda, mu, nu in each coordinate.

    :param alternant_term: an AlternantTerm
    :param lmbda: a partition of length m*n
    :param mu: a partition of length m
    :param nu: a partition of length n
    :return: a bool
    """
    b1 = self.vpf_input(lmbda, mu, nu)
```

```
    b2 = alternant_term.vpf_input(lmbda, mu, nu)

    if all([x > y for (x,y) in zip(b1, b2)]):
      return True

    else:
      return False

  def delta(self):
    """
    Returns the partition delta_mn = (mn-1, mn-2, ..., 1, 0).

    :return: a list of length m*n
    """
    m,n = self.m, self.n
    return tuple([m*n - i - 1 for i in range(m*n)])

  def part_sum(self, part_vector, lmbda):
    """
    Returns a linear combination of the parts of (lambda + delta_mn).

    :param part_vector: an iterable of length m*n
    :param lmbda: a partition of length m*n
    :return: a non-negative integer
    """
    delta = self.delta()

    ps = 0
    for i, coeff in enumerate(part_vector):
          ps += coeff * (lmbda[i] + delta[i])
    return ps

  def indices_to_parts(self, indices):
    """
    Converts a list of indices into a binary vector whose k-th index is 1 if k+1 is in the indices list (the discrepency is due to Pythonic notation).

    :param indices: a list of non-negative integers
    :return: a binary vector of length m*n
    """
    m,n = self.m, self.n
    return vector([i+1 in indices for i in range(m*n)])

  def x_indices(self):
    """
    Computes the parts of lambda coming from the exponents of x for the alternant term 'self' from the LHS of the Jacobi-Trudi identity.

    :return: a dictionary, key=positive integer in 1,..,m-1, val=list of non-negative integers
    """
    m, n, perm = self.m, self.n, self.perm
    x_inds = {}
    for i in range(1, m):
          x_inds[i] = [perm[i]] + perm[m + (i) * (n - 1):m + (i + 1) * (n - 1)]
    return x_inds
```

```python
    def y_indices(self):
        """
        Computes the parts of lambda coming from the exponents of y for the alternant term 'self' from the LHS of the Jacobi-Trudi identity.

        :return: a dictionary, key=positive integer in 1,..,n-1, val=list of non-negative integers
        """
        m, n, perm = self.m, self.n, self.perm
        y_indices = {}
        for j in range(1, n):
                y_indices[j] = [perm[m-1+j]]

        for k in range(1, m):
                y_indices[j].append(perm[m-1+j+k*(n-1)])

        return y_indices

    def x(self):
        """
        Represents the output of x_indices using a binary vector.

        :return: a dictionary where key is an integer from 0 to m-1 & val is binary vector of length m-1 (?)
        """
        x_parts = {}
        for i in range(1, self.m):
                x_parts[i] = self.indices_to_parts(self.x_indices()[i])
        return x_parts

    def y(self):
        """
        Represents the output of y_indices using a binary vector.

        :return: a dictionary where key is an integer from 0 to n-1 & val is binary vector of length m*n.
        """
        y_parts = {}
        for i in range(1, self.n):
                y_parts[i] = self.indices_to_parts(self.y_indices()[i])
        return y_parts

    def s(self):
        """
        Computes the parts of lambda from the vector l_s. We note that no constants appear here since they can be computed if we know the parts of lambda - each lambda_i comes with m*n - i. For example the output {0: (0, 0, 1, 1), 1: (0, 1, 1, 2)} corresponds to the l_s(lambda) = (lambda_3 + lambda_4 + 1, lambda_2 + lambda_3 + 2lambda_4 + 3)

        :return: a dictionary where key is an integer from 0 to m-1 & val is a vector of length m*n with non-negative integer coordinates.
        """
        m,n = self.m, self.n
        s_parts = {}
        s_parts[0] = sum(self.y()[j] for j in range(1, n))

        for k in range(1, m):
```

```
            s_parts[k] = sum(self.x()[i] for i in range(k,m)) + sum(self.y()[j] for j in range(1, n))

        return s_parts

    def t(self):
        """
        Computes the parts of lambda from the vector l_t. note that no constants appear here since they can be computed if we know the parts of lambda - each lambda_i comes with m*n - i.

        :return: a dictionary where key is an integer from 1 to n-2 & val is a vector of length m*n with non-negative integer coordinates.
        """
        m, n = self.m, self.n
        t_parts = {}
        for l in range(1, n - 1):
            t_parts[l] = (sum(i * self.x()[i] for i in range(1, m)) +
            sum(self.y()[j] for j in range(l + 1, n)) +
            (m - 1) * sum(self.y()[j] for j in range(1, n)))

        return t_parts

    def sign(self):
        """
        Returns the sign associated to 'self'. If this sign is positive, the alternant term will always make a non-negative contribution to the Kronecker coefficient. If it is negative the contribution will always be non-positive.

        :return: +/- 1
        """
        return self.perm.sign()

    def vpf_input(self, lmbda, mu, nu):
        """
        Returns the vpf input associated to alternant term 'self' and the partitions lmbda, mu, nu. This is the value which will be evaluated by the vector partition function in order to determine the contribution of alternant term 'self' to the Kronecker coefficient.

        :return: a list of length m+n-2.
        """
        return [r-l for r,l in zip(self.rhs_powers(mu, nu), self.lhs_powers(lmbda))]

    def symbolic_vpf_input(self):
        """
        Computes the vpf input for the alternant term 'self' as a function of the parts of lmbda, mu, nu. For example, in the K22 case, for at = K22.alternant_terms[0] (the atomic KC) the output is [-l3 - l4 + n2, -l2 - l3 - 2*l4 + m2 + n2]


        :return: a list of length m + n - 2 whose entries are degree 1 functions in the parts of lmbda, mu, nu.
        """
        m,n = self.m, self.n
```

```
        lmbda = var('␣'.join([f'l{i}' for i in range(1, m*n+1)]))
        mu = var('␣'.join([f'm{i}' for i in range(1, m+1)]))
        nu = var('␣'.join([f'n{i}' for i in range(1, n+1)]))
        return self.vpf_input(lmbda, mu, nu)

    def alpha_beta(self):
        """
        Computes (alpha, beta).

        :return: a list of length m+n-2
        """
        n,m = self.n, self.m

        the_constants = []

        # 1st coordinate comes from s0
          alpha_0 = 1/2*(m*n + n - m - 2)*(n-1)*(m-1)
        the_constants.append(alpha_0)

        # coordinates 1,...,m-1 come from sa
        for u in range(1,m):
            alpha_u = 1/2*(u^2*n - 2*u*n*m + 2*n*m^2 - u^2 + u - n - 2*m +
            2)*(n-1)
              the_constants.append(alpha_u)

        # coordinates m+1,..,n+m-2 come from tb
        for v in range(1,n-1):
              beta_v = 1/12*(8*n^2*m^2 - 6*v*n*m + 5*n^2*m - 10*n*m^2 + 6*v^2 -
              12*v*n + 6*v*m - 19*n*m + 2*m^2 + 18*v + 14*m)*(m-1)
              the_constants.append(beta_v)

        return the_constants

    def lhs_powers(self, lmbda):
        """
        Returns a list which is the evaluation ((l_s(lmbda;sigma),
        l_t(lmbda;sigma)) where 'sigma' is the permutation of self (i.e
        self.perm).

        :return: a list of non-negative integers
        """
        m,n = self.m, self.n
        powers = []
        for i in range(m):
              powers.append(self.part_sum(self.s()[i], lmbda))

        for j in range(1, n-1):
          powers.append(self.part_sum(self.t()[j], lmbda))

        return powers

    l_st = lhs_powers  # this is the name we use in the paper.

    def rhs_powers(self, mu, nu, no_constants = False):
        """
        Returns a list which is the evaluation (r_s(mu, nu), r_t(mu, nu)) +
        (alpha, beta) where 'sigma' is the permutation of self (i.e self.perm).
```

```
        :return: a list of non-negative integers
        """
        return [x+y for x,y in zip(self.r_st(mu, nu), self.alpha_beta())]

    def r_st(self, mu, nu):
        """
        Computes (r_s(mu, nu), r_t(mu, nu)).

        :param mu: a partition of length at most m
        :param nu: a partition of length at most n

        :return: a list of length m+n-2
        """
        m, n = self.m, self.n

        # 1st coordinate comes from s0
        mu_part = 0
        nu_part = sum(nu) - nu[0]


        constant_part = binomial(n - 1, 2)
        r_st_coords = [mu_part + nu_part + constant_part]

        # coordinates 1 <= u <= m  comes from s1 to sn
        for u in range(1, m):
            mu_part = sum([mu[i] for i in range(u, m)])
            nu_part = sum(nu) - nu[0]

            constant_part = binomial(m - u, 2) + binomial(n - 1, 2)
            r_st_coords.append(mu_part + nu_part + constant_part)

        # coordinates m+1 to m+n-2  (i.e. m+v for 1 <= v <= n-2) comes from t1 to tm-2
        for v in range(1, n-1):
            mu_part = sum([i*mu[i] for i in range(1, m)])
            nu_part = (m-1)*sum([nu[j] for j in range(1, n)]) + sum([nu[j] for j in range(v+1, n)])

            constant_part = binomial(m, 3) + (m-1)*binomial(n-1, 2) + binomial(n-v-1, 2)
            r_st_coords.append(mu_part + nu_part + constant_part)

            return r_st_coords

    def lmn_coeff_vectors(self):
        """
        Returns the [FI\lambda IN]

        :return: a tuple
        """
        m, n = self.m, self.n

        coeff_vectors = []

        lmbda = var(' '.join([f'l{i}' for i in range(1, m*n+1)]))
        mu = var(' '.join([f'm{i}' for i in range(1, m+1)]))
        nu = var(' '.join([f'n{i}' for i in range(1, n+1)]))
```

```
    for expr in self.symbolic_vpf_input():
      lmbda_coeffs = [expr.coefficient(lmbda[i]) for i in range(m*n)]
      mu_coeffs = [expr.coefficient(mu[i]) for i in range(m)]
      nu_coeffs = [expr.coefficient(nu[i]) for i in range(n)]

      coeff_vectors.append(tuple(lmbda_coeffs + mu_coeffs + nu_coeffs))

    return coeff_vectors
```

C.2 The KroneckerComputer class

```
load('AlternantTerm.sage')
load('PiecewiseQuasipolynomial.sage')

from itertools import combinations
from collections import Counter

class KroneckerComputer:
  def __init__(self, m, n, alternant_terms, vector_partition_function):
    self.m = m
    self.n = n
    self.alternant_terms = alternant_terms
    self.vector_partition_function = vector_partition_function

  def alternant_terms_by_evaluation(self, lmbda, mu, nu):
    """
    Returns a record of the vector partition evaluations associated to each AlternantTerm when computing the Kronecker coefficient of lmbda, mu, nu.

    :param lmbda: partition of length at most m*n
    :param mu: partition of length at most m
    :param nu: partition of length at most n
    :return: dictionary, key=integer, val=list of alternant terms
    """
    d = {}
    vp = self.vector_partition_function
    for alternant_term in self.alternant_terms:
      b = alternant_term.vpf_input(lmbda, mu, nu)
      val = vp.evaluate(b)*alternant_term.sign()
      d.setdefault(val, []).append(alternant_term)
    return d

  def atomic_kronecker_coefficient(self, lmbda, mu, nu):
    """
    Returns the atomic Kronecker coefficient for partitions lmbda, mu, nu.

    :param lmbda: partition of length at most m*n
    :param mu: partition of length at most m
    :param nu: partition of length at most n
    :return: a non-negative integer
    """
    atomic_alternant_term = self.alternant_terms[0]
    b = atomic_alternant_term.vpf_input(lmbda, mu, nu)
    vp = self.vector_partition_function
```

```
    return vp.evaluate(b)

  def bounds(self, lmbda, mu, nu):
    """
    Returns the bounds from Corollary 6.6 of our paper `Estimating and computing Kronecker coefficients.'

    :param lmbda: partition of length at most m*n
    :param mu: partition of length at most m
    :param nu: partition of length at most n
    """
    m,n = self.m, self.n

    c1 = (n-1)*(m^2-1) - 1
    c2 = (m-1)*(n-1)^2 - 1
    c3 = (n-1)*binomial(m-1, 2) + (m-1) - 1

    def f1(i):
      return 2*binomial(n-1, 2)*(i-2) - 1

    def f2(j):
      return (n-j-1)*(m-1) - 1

    at = self.alternant_terms[0]
    b = at.vpf_input(lmbda, mu, nu)

    P1 = (binomial(b[0] + c1, b[0])*binomial(2*b[1] + c2, b[1])*
    binomial(b[-1] + c3, b[-1]))
    P2 = product([binomial(b[i-1] + f1(i), b[i-1]) for i in range(3,m)])
    P3 = (product([binomial(b[m+j-1] + f2(j), b[m+j-1])
    for j in range(1, n-2)]))

    return factorial(m*n)*P1*P2*P3/2

  def bounds_simple(self, N):
    """
    Returns the bounds from Corollary 6.8 of our paper `Estimating and computing Kronecker coefficients'.

    :param lmbda: partition of length at most m*n
    :param mu: partition of length at most m
    :param nu: partition of length at most n
    :return: a non-negative rational number
    """
    m,n = self.m, self.n

    c1 = (m^2-1)*(n-1) - 1
    c2 = (m-1)*(n-1)^2 - 1
    c3 = binomial(m-1, 2)*(n-1) + (m-1) - 1

    def f1(i):
      return 2*binomial(n-1, 2)*(i-2) - 1

    def f2(j):
      return (n-j-1)*(m-1) - 1

    P1 = binomial(N + c1, N)*binomial(2*N + c2, 2*N)*binomial((2*m - 1)*N + c3, (2*m - 1)*N)
```

```
    P2 = product([binomial(2*N + f1(i), 2*N) for i in range(3,m+1)])
    P3 = product([binomial((2*m-1)*N + f2(j), (2*m-1)) for j in range(1,
    n-2)])

    return factorial(m*n)*P1*P2*P3/2

  def kronecker_coefficient(self, lmbda, mu, nu):
    """
    Computes the Kronecker coefficient for lambda, mu, nu.

    :param lmbda: a partition of length m*n
    :param mu: a partition of length m
    :param nu: a partition of length n
    :return: non-negative integer
    """
    kc = 0
    vp = self.vector_partition_function

    for alternant_term in self.alternant_terms:
      b = alternant_term.vpf_input(lmbda, mu, nu)
      vp_val = vp.evaluate(b)
      kc += vp_val*alternant_term.sign()

    return kc

  def kronecker_coefficient_table(self, lmbda, mu, nu, non_zero_only = True,
  include_st = False):
    """
    Returns the Kronecker coefficient and the number of alternant terms with
    a non-zero contribution, as well as a table storing
    - the permutation for each alternant term
    - the s & t vectors for the alternant term (optional, by default this is
    not shown)
    - the b value for that alternant term at lmbda, mu, nu
    - the evaluation of the b value by the vector partition function
    One can optionally show only the alternant terms with a non-zero
    contribution (this is the default)
    or all of the alternant terms.

    :param lmbda: a partition of length m*n
    :param mu: a partition of length m
    :param nu: a partition of length n
    :param non_zero_only: a bool
    :param include_st: a bool
    :return: a non-negative integer, a table, and a positive integer
    """
    kc = 0
    vp = self.vector_partition_function
    num_terms = 0

    table_rows = []
    for alternant_term in self.alternant_terms:
      b = alternant_term.vpf_input(lmbda, mu, nu)
      vp_val = vp.evaluate(b)
      kc += vp_val*alternant_term.sign()

      if vp_val != 0:
        num_terms += 1
```

```
            if vp_val != 0 or not non_zero_only:

                if include_st:
                    table_rows.append([alternant_term.perm,
                    alternant_term.s(), alternant_term.t(),
                    tuple(b), vp_val*alternant_term.sign()])

                else:
                    table_rows.append([alternant_term.perm,
                    tuple(b), vp_val*alternant_term.sign()])

        table_rows.sort(key=lambda row: -row[-1]^2)

        if include_st:
            header = [['Permutation', 's', 't', 'b value', 'Signed evaluation']]

        else:
            header = [['Permutation', 'b value', 'Signed evaluation']]

        table_rows = header + table_rows

        return kc, num_terms, table(table_rows, header_row=True)

    def kronecker_coefficient_chamber_table(self, lmbda, mu, nu):
        """
        Returns a table storing
        - the permutation for each alternant term
        - the b value for that alternant term at lmbda, mu, nu
        - a list of the chambers in which this b value takes place
        - the evaluation of the b value by the vector partition function

        :param lmbda: a partition of length m*n
        :param mu: a partition of length m
        :param nu: a partition of length n
        :return: a non-negative integer, a table, and a positive integer
        """
        kc = 0
        vp = self.vector_partition_function
        num_terms = 0

        table_rows = []
        for alternant_term in self.alternant_terms:
            b = alternant_term.vpf_input(lmbda, mu, nu)
            vp_val, chambers = vp.evaluate_with_chambers(b)
            kc += vp_val*alternant_term.sign()

            if vp_val != 0:
                num_terms += 1

                table_rows.append([alternant_term.perm, tuple(b),
                chambers, vp_val*alternant_term.sign()])

    table_rows.sort(key=lambda row: -row[-1]^2)
    header = [['Permutation', 'b value', 'chambers', 'Signed evaluation']]
    table_rows = header + table_rows

    return kc, num_terms, table(table_rows, header_row=True)
```

```
    def murnaghan_inequality(self):
        """
        Computes Murnaghan's vanishing condition.

        :return: a list containing a single tuple
        """
        m,n = self.m, self.n

        ineq = [0 for i in range(m*n + m + n)]

        for i in range(1, m*n):
            ineq[i] = -1

        for j in range(m*n + 1, m*n + m):
            ineq[j] = 1

        for k in range(m*n + m + 1, m*n + m + n):
            ineq[k] = 1

        return [tuple([0] + ineq)]

    def num_terms(self):
        """
        Computes number of alternant terms associated to self. Note for K22, K23,
        K24 we have filtered out the alternant terms that never have a non-zero
        contribution to the Kronecker coefficient.

        :return: a positive integer
        """
        return len(self.alternant_terms)

    @staticmethod
    def pad_partition(partition, k):
        """
        Takes a partition of length at most k, and returns the version of the
        partition padded with enough zeroes to make the length k.

        :param partition: a partition of length at most k
        :param k: a positive integer
        :return: a 'padded' partition of length k
        """
        q = k - len(partition)
        return partition + [0 for i in range(q)]

    def partition_equalities(self):
        """
        Returns the tuples associated to the equalities
        |lambda| = |mu| = |nu|

        :return: a list of equalities
        """
        m,n = self.m, self.n

        partition_equalities = []
        starter = [0 for i in range(m*n + m + n)]

        # |lambda| = |mu|
```

```
    eq = [x for x in starter]
    for i in range(m*n+m):
      eq[i] = (-1)^int(i < m*n)
      partition_equalities.append(eq)

    eq = [x for x in starter]
    for i in range(m*n, m*n+m+n):
      eq[i] = (-1)^int(i < m*n + m)
      partition_equalities.append(eq)

      return [tuple([0] + pq) for pq in partition_equalities]

  def partition_inequalities(self):
    """
    Returns the inequalities alpha_i >= alpha_{i+1} >= 0 for each of the partitions lambda, mu, nu.

    :return: a list of inequalities
    """
    m,n = self.m, self.n

    partition_inequalities = []
    starter = [0 for i in range(m*n + m + n)]  # a list from which we will generate all the tuples

    # lambda inequalities
    for i in range(m*n-1):
      ineq = [x for x in starter]
      ineq[i] = 1
      ineq[i+1] = -1
      partition_inequalities.append(ineq)

    ineq = [x for x in starter]
    ineq[m*n-1] = 1
    partition_inequalities.append(ineq)

    # mu inequalities
    for i in range(m*n, m*n + m - 1):
      ineq = [x for x in starter]
      ineq[i] = 1
      ineq[i+1] = -1
      partition_inequalities.append(ineq)

    ineq = [x for x in starter]
    ineq[m*n+m-1] = 1
    partition_inequalities.append(ineq)

    # nu inequalities
    for i in range(m*n+m, m*n + m + n - 1):
      ineq = [x for x in starter]
      ineq[i] = 1
      ineq[i+1] = -1
      partition_inequalities.append(ineq)

    ineq = [x for x in starter]
    ineq[-1] = 1
    partition_inequalities.append(ineq)

```

```
        # need to put a zero in front of each for the RHS of the inequality.
        partition_inequalities1 = [tuple([0] + pi) for pi in partition_inequalities]

        return partition_inequalities1

    def poset(self):
        """
        Returns a poset of the alternant terms where at1 > at2 iff each of the s & t vectors of at1 are > the s & t vectors of at2.

        :return: a Poset
        """
        l = len(self.alternant_terms)
        return Poset((list([0..l-1]), lambda i,j : self.alternant_terms[i] <= self.alternant_terms[j]))

    def stable_face(self):
        """
        Returns the stable face from Theorem 5.4 (defined by the inequalities 40--42 of Proposition 5.2).

        :return: a Cone
        """
        atomic_at = self.alternant_terms[0]
        eq_vectors = []

        for coeff_vector in atomic_at.lmn_coeff_vectors():
            eq_vector = tuple([0] + list(coeff_vector))
            eq_vectors.append(eq_vector)

        P = Polyhedron(eqns=eq_vectors + self.partition_equalities(), ieqs=self.partition_inequalities(), base_ring=QQ)

        return Cone(P)

    def vanishing_conditions(self):
        """
        Returns the set of (non)-vanishing conditions from Theorem 4.4.

        :return: list of inequalities
        """
        atomic_at = self.alternant_terms[0]
        return [c >= 0 for c in atomic_at.symbolic_vpf_input()]

    def vanishing_conditions_cone(self):
        """
        Computes a cone such that any triple (lambda, mu, nu) outside the cone must have Kronecker coefficient equal to zero.

        :return: a Cone
        """
        m,n = self.m, self.n

        atomic_at = self.alternant_terms[0]
        ineq_vectors = []

        for coeff_vector in atomic_at.lmn_coeff_vectors():
```

```
            ineq_vector = tuple([0] + list(coeff_vector))
            ineq_vectors.append(ineq_vector)

        P = Polyhedron(ieqs=ineq_vectors+self.partition_inequalities(), eqns =
            self.partition_equalities(), base_ring=QQ)

    return Cone(P)
```