Factorizations of the canonical full cycle and symmetric q, t-polynomials

by

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Abstract

The aim of our study is to examine the relationship between minimal factorizations of the canonical full cycle that arise in lattices of non-crossing partitions, as well as the statistics attached to these objects. We shall first study properties of these factorizations that will help us establish statistic-preserving bijections among these objects. We also study some special subsets of these factorizations that come from their relationship with parking functions. Furthermore, some of the statistics on these objects give rise to interesting symmetric q,t-polynomials that we also examine.

Keywords: non-crossing partition; parking function; Dyck path; minimal factorizations; q, t-polynomials

Dedicated to the loving memory of my mother-in-law, J Nancy Devtha.

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Chapter 1

Introduction

Let n be a non-negative integer and let $[n] \coloneqq \{0, 1, 2, \ldots, n\}$. Recall the famous *Catalan numbers* as $C_n = \frac{1}{n+1}\binom{2n}{n}$. The first instance of Catalan numbers dates back to 1730, where they were studied by a Mongolian mathematician Mingantu. Since then, several people have studied them and discovered plethora of objects counted by them. Stanley [36] compiled a comprehensive collection of over two hundred objects counted by these numbers and discussed their properties and application in various areas of mathematics. He also provides some bijections among these objects. One of the most fascinating objects counted by Catalan numbers are the *non-crossing set partitions* of the set $[n]$.

Recall a set partition of $[n]$ is a collection of non-empty subsets (*blocks*) of $[n]$ that are pairwise disjoint and whose union is $[n]$. A non-crossing partition of $[n]$ is a set partition such that if B and B' are blocks and if $0 \leq a < b < c < d \leq n$ with $a, c \in B$ and $b, d \in B'$, then $B = B'$. Non-crossing partitions were introduced and studied by Kreweras [21] in 1972. Simion [31] published a survey article detailing their connection to enumerative combinatorics, topology and geometric combinatorics. Non-crossing partitions also play a vital role in laying the foundations of studying noncommutative random variables (free probability) [26]. In recent years, they appear in diverse areas such as low-dimensional topology and geometric group theory. A paper by McCammond [25] details some of these connections to other areas. The non-crossing partitions are naturally identified with elements of the symmetric group $\mathfrak{S}_{[n]}$ by sending blocks to cycles such that each block is an increasing cycle. There is a well-known ordering on the set of non-crossing partitions and we denote the corresponding lattice of non-crossing partitions by $\mathcal{NC}_{n+1}$. Biane [5] and Brady [7] observed that the maximal chains of this lattice can be considered as factorizations of the full cycle $c \coloneqq (0, 1, 2, \ldots, n)$ into a product of a minimal number of transpositions. Such factorizations are counted by the tree numbers $(n+1)^{n-1}$. Factorizations of permutations in general have been studied by numerous authors. See the survey of Goulden and Jackson [15]. We consider specific subsets of these factorizations. One subset, *maximal factorizations*, occur as the maximal elements of a poset on the *Hurwitz graph*. They were first studied by Adin and Roichman [1], who were interested in calculating the radius of the Hurwitz graph. They prove that the cardinality of the set of maximal factorizations of c is the n^{th} Catalan number. But they do not mention any connection to other classes of factorizations counted by the Catalan number. The *increasing factorizations*, as defined in [19], are also factorizations of the full cycle c into minimal number of transpositions such that the *lower sequence* is increasing. This subset also contains Catalan number of elements. In Chapter 4 we connect these distinct factorizations.

Another class of objects that are counted by the tree numbers with subclasses of objects counted by Catalan numbers are *parking functions*. Parking functions are a class of combinatorial objects that have a wealth of connection with other combinatorial and algebraic objects. An extensive survey can be found in Haglund's book [17] and in the work of Yan [37]. There is a simple bijection between parking functions and minimal factorizations of c. We explore this connection in Chapter 2. We are interested in subclasses of them counted by Catalan numbers. One subclass of objects is a collection of parking functions with a certain *reading word*. Haglund [17] in his book calls these specific parking functions *Maxdinv parking functions*. Using the bijection between parking functions and minimal factorizations of c, we find the factorizations corresponding to Maxdinv parking functions and call them the *reading word factorizations*. Although Maxdinv parking functions are studied in Haglunds book, we do not find anything related to factorizations in his work. In this thesis, we study reading word factorizations, and we believe we are doing this for the first time.

It is of natural interest to study bijections between classes of objects which have the same cardinality, in particular, objects counted by Catalan numbers. It is also of interest to study how different properties behave between objects under these bijection. This drives us to study some connection, if there are any, among the different subsets of factorizations of the canonical full cycle mentioned above. This is a primary goal of this thesis.

We are also interested in two variable polynomials that record certain statistics over these factorizations. A q, t-Catalan polynomial is a polynomial in two variables q and t with rational coefficients such that when we set $q = t = 1$, it evaluates to a Catalan number. Garsia and Haiman introduced some q, t-Catalan polynomials in connection to Macdonald polynomials and their theory of diagonal harmonics [12]. Later, Haglund introduced a specific sequence of q, t-Catalan polynomials, denoted by $\tilde{C}_n(q, t)$ for $n \geq 0$, that have attracted a lot of attention. He gave two different combinatorial formulas as sum over all *Dyck paths*, one graded by the *area* and *bounce* statistics on Dyck paths, and the other graded by the area and *dinv* statistics. From the works of Garsia, Haiman and Haglund, we know that $\tilde{C}_n(q, t)$ is symmetric in q and t for $n \geq 0$, though their methods are neither combinatorial nor elementary. One of the most important open problems is to prove the symmetry of $\tilde{C}_n(q, t)$ combinatorially. The problem of symmetry of the polynomials $\tilde{C}_n(q, t)$ inspired the study of other symmetric q, t-Catalan polynomials. Adin and Roichman proved that the q, t-polynomial graded by *inversions* over all maximal factorizations is a symmetric q, t-Catalan polynomial that has an interesting form. This is motivation for us to study q, t-polynomials over other subsets of factorizations and find if these polynomials have nice properties. We also look at q, t-polynomials graded by other statistics over factorizations and provide two new ways of generating symmetric q, t-polynomials.

1.1 Thesis outline

We organize the thesis as follows. In Chapter 2, we give some necessary background on factorizations of full cycles, non-crossing partitions and give some characterizations of them. Parking functions are equinumerous with the factorizations of the canonical full cycle. We recall some facts regarding parking functions that will help us in our study and present a known algorithm to find a factorization corresponding to a given parking function. We also introduce maximal factorizations, discuss inversion statistics on factorizations, and also define a q, t-Catalan polynomial $C_n(q, t)$ in Section 2.2.

In Chapter 3, we introduce increasing factorizations and reading word factorizations, and find elementary facts about these objects. We also define various statistics on these factorizations. In Section 3.3, we enumerate maximal factorizations and increasing factorizations with first factor $(0\ 1)$, $(0\ n)$ and with last factors $(n-1\ n)$. It turns out that these are also counted by the Catalan numbers C_{n-1}. The reading word factorizations with first factor $(0\ 1)$, $(0\ n)$ are also counted by C_{n-1}. As for reading word factorizations with last factor $(0\ n)$, we get a nice pattern and produce a conjecture that we discuss.

In Chapter 4, we give a new bijection between maximal factorizations and increasing factorizations that preserves some interesting statistics. Adin and Roichman show that their q, t-Catalan polynomial, where the two variables count left and right *inversions*, satisfy a natural recursion. We give a new proof of this by finding a bijection between maximal factorizations and increasing factorizations that map both inversions to natural statistics on increasing factorizations. Section 4.2 mainly focuses on the q, t-Catalan polynomial and two functions *mateflipre*v and *mateflipsort*. We prove that these two functions are involutions on parking functions and increasing parking functions respectively and mention consequences of that.

Chapter 2

Factorizations, non-crossing partitions and parking functions

We provide essential background in this chapter. Most of the contents of this chapter are not original material. We refer to the work of Adin and Roichman [1], and others will be referenced as needed. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. Let $\mathfrak{S}_{[n]}$ represent the symmetric group on $[n] \coloneqq \{0, 1, \ldots, n\}$ and $c = (0\ 1\ \ldots\ n) \in \mathfrak{S}_{[n]}$ be the canonical full cycle. Let T denote the set of all transpositions in $\mathfrak{S}_{[n]}$. We assume the reader is familiar with some basics of the symmetric group $\mathfrak{S}_{[n]}$. We mostly use the cycle notation to represent the elements of the symmetric group $\mathfrak{S}_{[n]}$, but in some places use one-line notation as well. We multiply cycles from left to right. For example, $(0\ 1\ 4)(1\ 2) = (0\ 2\ 1\ 4)$. Note that fixed points are often suppressed.

2.1 Minimal factorizations of the canonical full cycle

Our main objects of study are factorizations of the canonical full cycle c into transpositions. We first give a *join-cut* analysis of multiplication by a transposition. When we write transpositions $(a\ b)$, we assume $a < b$.

Let $\sigma \in \mathfrak{S}_{[n]}$ and let $\kappa(\sigma)$ be the number of cycles in the disjoint cycle representation of σ. Let $\tau = (a\ b)$ be a transposition. When we consider the product $\sigma\tau$, we have two possible outcomes. Firstly, if a and b belong to different cycles in σ, then multiplying by τ will join the cycles containing a and b. In this case, we call τ a *join* for σ, and we note that $\kappa(\sigma\tau) = \kappa(\sigma) - 1$. On the other hand, if a and b are in the same cycle in σ, then multiplying by τ will cut the cycle containing a and b into two and we call τ a *cut* for σ. When τ is a cut for σ, we note that $\kappa(\sigma\tau) = \kappa(\sigma) + 1$. The following example illustrates what we have discussed. Let $\sigma = (0\ 4\ 3)(1\ 5\ 2)$, $\tau_1 = (0\ 2)$ and $\tau_2 = (1\ 2)$ be permutations in $\mathfrak{S}_{[5]}$.

$$\sigma\tau_1 = (0\ 4\ 3\ 2\ 1\ 5) \text{ and}$$
$$\sigma\tau_2 = (0\ 4\ 3)(1\ 5)(2).$$

Here $\kappa(\sigma) = 2, \kappa(\sigma\tau_1) = 1$ and $\kappa(\sigma\tau_2) = 3$. Hence τ_1 is a join for σ, while τ_2 is a cut.

For a given $\sigma \in \mathfrak{S}_{[n]}$, we define $\ell_T(\sigma)$ to be the minimal number of factors needed when σ is written as product of transpositions. Since $\kappa(\sigma\tau) = \kappa(\sigma) - 1$ if and only if τ is a join for σ, it follows that $\ell_T(c) \geq n$. On the other hand $c = (0\ 1)(0\ 2)\dots(0\ n)$, so $\ell_T(c) = n$. From this, it follows that in a factorization of c every transposition is a join when the number of factors is n, the minimum number. Note that in a minimal factorization, there cannot be two repeated factors, since every factor is a join. We now formally define minimal factorizations of the canonical full cycle.

Definition 2.1 (Minimal factorizations of the canonical full cycle)**.** For $c = (0\ 1\ \dots\ n) \in \mathfrak{S}_{[n]}$, let $\tau_1, \tau_2, \dots, \tau_n$ be a sequence of transpositions such that

$$c = \tau_1\tau_2\dots\tau_n.$$

Then we call $(\tau_1, \tau_2, \dots, \tau_n)$ a minimal factorization of c. Let $\mathcal{F}_n$ denote the set of all minimal factorizations of the canonical full cycle c.

A lot of people in the past have studied factorizations of permutations. Stanley [33] studied the factorizations of length k of any permutation in $\mathfrak{S}_{[n]}$ into full cycles for any k. Goulden and Jackson [15] (a comprehensive survey) worked on *transitive factorizations* of arbitrary permutations of arbitrary length. In [15], they use a join-cut technique to enumerate transitive factorizations of an arbitrary element (not discussed here), and they show that such problems have rich connections to geometry. We will be only looking at the specific case when the target permutation is the canonical full cycle c, the factors are transpositions, and the length of the factorization is minimal. We have the following theorem that counts the number of elements in $\mathcal{F}_n$.

Theorem 2.2. *The number of elements in $\mathcal{F}_n$ is $(n+1)^{n-1}$.*

A proof of the theorem is given by Dénes [9] who proved it using an indirect counting argument using trees. Below we give the proof by establishing a bijection between trees and factorizations of cyclic permutations. A classical theorem by Cayley tells that the number of vertex-labelled trees on $n+1$ vertices $\mathcal{T}_n$ is enumerated by $(n+1)^{n-1}$. The proof presented here is Moszkowski's description of Dénes argument [27].

For a tuple of transpositions $f = (\tau_1, \tau_2, \dots, \tau_k)$, we let $G(f)$ be the graph with vertices $[n]$ and edges labelled $1, 2, \dots, n$ such that there is an edge between a, b labelled i if $\tau_i = (a\ b)$.

Proof of Theorem 2.2. We will show that if $\tau_1, \tau_2, \dots, \tau_n$ are transpositions of $\mathfrak{S}_{[n]}$, then the product of transpositions $\pi \coloneqq \tau_1\tau_2\dots\tau_n$ is equal to a cyclic permutation of length $n+1$ if and only if the graph $G(\tau_1, \tau_2, \dots, \tau_n)$ is a tree. For the forward implication, observe that for $0 \leq i \leq n$, there exists a path in $G(\tau_1, \tau_2, \dots, \tau_n)$ that joins i and $\pi(i)$. Since π is a cyclic permutation, powers of π can move any symbol i to any symbol k. This implies that the graph $G(\tau_1, \tau_2, \ldots, \tau_n)$ is connected. Note that $G(\tau_1, \tau_2, \ldots, \tau_n)$ is a graph on $n+1$ vertices, has n edges, and is connected. This shows that $G(\tau_1, \tau_2, \ldots, \tau_n)$ is a tree. This argument gives another justification that the product of less than n transpositions in $\mathfrak{S}_{[n]}$ cannot be a cyclic permutation, as observed before Definition 2.1.

Conversely suppose that $G(\tau_1, \tau_2, \ldots, \tau_n)$ is a tree. Consider the graph $G(\tau_1, \tau_2, \ldots, \tau_n)$ with edge τ_n removed. The graph is the union of two disjoint trees, say T_1 and T_2. Every edge of one of the trees commutes with every edge of the other as transpositions. By induction, the trees T_1 and T_2 represent cycles. That is, $\tau_1\tau_2 \ldots \tau_{n-1} = C_1 \cdot C_2$, where the vertices of T_1 are the elements in C_1, and vertices of T_2 are the elements in C_2. Here the cycles C_1 and C_2 are disjoint. Suppose that $\tau_n = (a\ b)$. We must have a in one of the cycles and b in the other, say $a \in C_1$ and $b \in C_2$. Then we have $\tau_1\tau_2 \ldots \tau_n = C_1 \cdot C_2 \cdot (a\ b)$. We know that $a \in C_1, b \in C_2$ and $C_1 \cap C_2 = \varnothing$. Hence τ_n is a join and so $\tau_1\tau_2 \ldots \tau_n$ is a cyclic permutation. Thus $f \mapsto G(f)$ is a bijection between n transpositions in $\mathfrak{S}_{[n]}$ such that the product is a cyclic permutation and the set of vertex-labelled, edge-labelled trees on $n+1$ vertices.

Let $\mathcal{F}_n(\mathcal{C})$ denote the set of all n-tuples of transpositions in $\mathfrak{S}_{[n]}$ such that the product of the transpositions is a given cyclic permutation $\mathcal{C}$. By symmetry, the cardinality $|\mathcal{F}_n(\mathcal{C})|$ does not depend on the choice of the cyclic permutation $\mathcal{C}$. So $|\mathcal{F}_n(\mathcal{C})| = |\mathcal{F}_n|$ for any $\mathcal{C}$. The number of vertex-labelled and edge-labelled trees is clearly $|\mathcal{T}_n|n!$, and the number of elements in $\bigcup \mathcal{F}_n(\mathcal{C})$ is $|\mathcal{F}_n|n!$, where the union is over all cyclic permutations $\mathcal{C} \in \mathfrak{S}_{[n]}$. Hence, from the bijection, we have $|\mathcal{T}_n|n! = |\mathcal{F}_n|n!$. Thus we have $|\mathcal{F}_n| = |\mathcal{T}_n| = (n+1)^{n-1}$. □

The number $(n+1)^{n-1}$ is well-known and arises in the study of other combinatorial objects. One of them is *parking functions*, where the number of parking functions of length n is also $(n+1)^{n-1}$, which we discuss in Section 2.5. Minimal factorizations of the canonical full cycle have connections with *non-crossing partitions*, which we will see next.

2.2 Non-crossing partitions

This section briefly covers some basics of non-crossing partitions of the set $[n]$. We discuss the connection of non-crossing partitions to the factorizations of the canonical full cycle; namely, factorizations are equinumerous with the set of all maximal chains in the lattice of non-crossing partitions. Let us begin by defining what non-crossing partitions are. Recall the definition of a partition. A collection of non-empty sets $\{B_1, B_2, \ldots, B_k\}$ is a partition of $[n]$ if $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $\cup_j B_j = [n]$.

Definition 2.3 (Non-crossing partitions)**.** For a given positive integer n, let $\mathcal{B} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\}$ be a partition of the set $[n]$ where each $\mathcal{B}_i$ is a *block* of $\mathcal{B}$ for $1 \leq i \leq k$. Two blocks $\mathcal{B}_i \neq \mathcal{B}_j$ are said to *cross* if there exist $0 \leq a < b < c < d \leq n$ such that $\{a, c\} \subseteq \mathcal{B}_i$ and $\{b, d\} \subseteq \mathcal{B}_j$. If $\mathcal{B}_i$ and $\mathcal{B}_j$ do not cross for all $1 \leq i < j \leq k$, we say that

$\mathcal{B}$ is a non-crossing partition of $[n]$. Denote the set of all non-crossing partitions of $[n]$ by NC_{n+1}.

The non-crossing partitions were first introduced by Kreweras who investigated them under the refinement ordering [21].

Recall the Catalan numbers C_n for $n \geq 0$ from the introduction chapter. The following lemma regarding the number of non-crossing partitions of $[n]$ is our first result, and we give a small proof. The Catalan numbers satisfy the following well-known recurrence (See Theorem 2.28 in [22]):

$$C_0 = 1 \text{ and } C_{n+1} = C_0 C_n + C_1 C_{n-1} + \ldots + C_n C_0 \text{ for } n \geq 0. \tag{2.1}$$

Lemma 2.4. *The number of non-crossing partitions NC_{n+1} is counted by the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}$.*

Proof. Let $f(n+1) = |NC_{n+1}|$. Since $f(n+1)$ denotes the number of all non-crossing partitions of $[n]$, it is sufficient to show that $f(n+1)$ satisfies the same recurrence as C_{n+1}. The base case is NC_0. There is just one trivial non-crossing partition, and therefore $f(0) = 1$.

Now let us consider the set NC_{n+1}, and let $NC_{n+1,k}$ be the set of all non-crossing partitions of $[n]$ such that k is the least element in the block containing n. Clearly $NC_{n+1} = \bigcup_{k=0}^{n} NC_{n+1,k}$. Define $\phi_k : NC_{n+1,k} \longrightarrow NC_k \times NC_{n-k}$ to be

$$\phi_k(\pi) = \pi_1 \times \pi_2,$$

where π_1 is the set of blocks of π containing $\{0, 1, \ldots, k-1\}$ and π_2 is the set of blocks of π containing $\{k, k+1, \ldots, n-1\}$. Note that for an element in $NC_{n+1,k}$, the elements k and n are in the same block. If a block B contains elements from both $\{0, 1, \ldots, k-1\}$ and $\{k, k+1, \ldots, n-1\}$, then B will cross the block containing k and n, which is not possible. Thus any element in $NC_{n+1,k}$ cannot contain a block with elements from both $\{0, 1, \ldots, k-1\}$ and $\{k, k+1, \ldots, n-1\}$. Also, it is easy to see that ϕ_k is a bijection for each k. Since $0 \leq k \leq n$ and $NC_{n+1} = \bigcup_{k=0}^{n} NC_{n+1,k}$, we have

$$f(n+1) = \sum_{k=0}^{n} f(k) f(n-k).$$

Thus $f(n)$ satisfies the same recurrence as C_n and hence $|NC_{n+1}| = C_{n+1}$. □

We now give a very important connection between the set of non-crossing partitions of $[n]$ and the symmetric group $\mathfrak{S}_{[n]}$. A cycle $(a_1\ a_2\ \ldots\ a_k) \in \mathfrak{S}_{[n]}$ is *increasing* if $a_1 < a_2 < \ldots < a_n$. We can identify each non-crossing partition of $[n]$ with an element of $\mathfrak{S}_{[n]}$ by sending blocks to cycles such that each cycle is increasing. That is, if $\mathcal{B} = \{i_1, i_2, \ldots, i_k\}$, where $i_j < i_{j+1}$ for all j, then the cycle associated to $\mathcal{B}$ is $(i_1\ i_2\ \ldots\ i_k)$.

Example 2.5. For the set $[10]$, a non-crossing partition is $p = \{\{0, 1, 3\}, \{4, 5, 6, 10\}, \{7, 9\}, \{2\}, \{8\}\}$. Figure 2.1 gives a diagram representation of it. The corresponding element in $\mathfrak{S}_{[10]}$ is $\pi = (0\ 1\ 3)(4\ 5\ 6\ 10)(7\ 9)$. Here (2) and (8) are fixed points. A pictorial representation of this partition is given in Figure 2.1. Notice that the convex hull of the blocks do not intersect except at vertices.

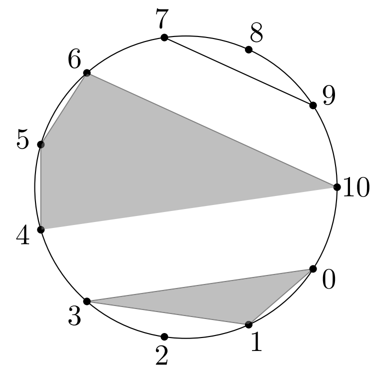


Figure 2.1: A non-crossing partition of [10]

Recall that we call a poset a *join-semilattice* if there exists a least upper bound for any two elements in the poset. Similarly, we call a poset as *meet-semilattice* if there exists a greatest lower bound for any two elements in the poset. A poset that is both a join-semilattice and meet-semilattice is called a *lattice*. A *sublattice* of a lattice L is a subset of the lattice L with the same meet and join operations in L. We refer the reader to Chapter 3 in [35] for basic poset terminology.

Define Π_{n+1} to be the set of all partitions of $[n]$. Under the *refinement* ordering, Π_{n+1} is a poset, and we write the poset as $(\Pi_{n+1}, \leq_r)$. That is, if $\pi_1, \pi_2 \in \Pi_{n+1}$, we say $\pi_1 \leq \pi_2$ if each block of π_1 is contained in a block of π_2. The minimal element is the trivial set partition $\{\{0\}, \{1\}, \ldots, \{n\}\}$ and the maximum element is $\{\{0, 1, \ldots, n\}\}$. This poset is fundamental in poset theory and in algebraic combinatorics [6]. Since $NC_{n+1} \subseteq \Pi_{n+1}$, we can define the induced refinement ordering on NC_{n+1} that makes it into a lattice, as follows.

Definition 2.6 (Non-crossing partition lattice)**.** Given two non-crossing partitions $\mathcal{B}$ and $\mathcal{B}'$ of $[n]$, we say $\mathcal{B} \leq \mathcal{B}'$ if each block of $\mathcal{B}$ is contained in a block of $\mathcal{B}'$. This defines a partial ordering on the set of all non-crossing partitions, and we call the corresponding lattice the *non-crossing partition poset* and denote it by $\mathcal{NC}_{n+1}$.

Figure 2.2 gives the Hasse diagram of the poset $\mathcal{NC}_4$. This poset $\mathcal{NC}_{n+1}$ is in fact a lattice. But note that this is *not* a sublattice of Π_{n+1} because the join operations do not agree. This can be observed from the example in Figure 2.2. Note that the join of $\{\{0, 2\}, \{1\}, \{3\}\}$ and $\{\{1, 3\}, \{0\}, \{2\}\}$ in the lattice of Π_4 is $\{\{0, 2\}, \{1, 3\}\}$, whereas in $\mathcal{NC}_4$ the join of $\{\{0, 2\}, \{1\}, \{3\}\}$ and $\{\{1, 3\}, \{0\}, \{2\}\}$ is $\{\{0, 1, 2, 3\}\}$. On the other hand,

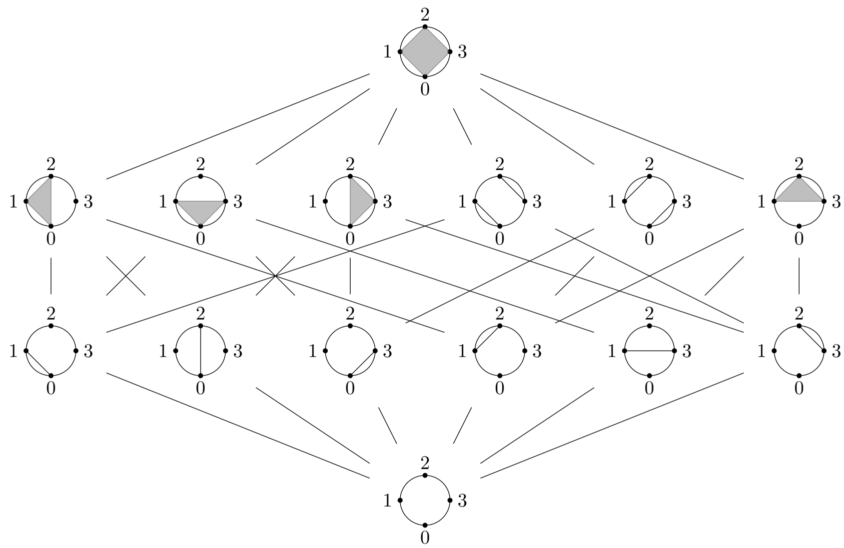


Figure 2.2: Hasse diagram of $\mathcal{NC}_4$

$\mathcal{NC}_{n+1}$ is a meet-sublattice of Π_{n+1}. To find the meet of two non-crossing partitions, we define two elements a and b to be in a block of the meet of two non-crossing partitions σ and π if they lie in the same block of σ and π. This is indeed a non-crossing partition, and is a refinement of σ and π. To find the join of σ and π, we only need to worry about the blocks that overlap or cross when taking the join. Thus, to find the join of σ and π, we take the convex hulls of all blocks of σ and π, as in Figure 2.1, superimpose them on top of each other, and take the convex hull of the connected components.

2.3 A partial ordering on $\mathfrak{S}_{[n]}$ and connections to the lattice $\mathcal{NC}_{n+1}$

Recall the definition of Cayley graph. Let G be a group and S be a subset of G. The Cayley graph associated to G is a directed graph such that:

- The elements of the group G are precisely the vertices of the graph.
- For every $g, h \in G$, insert an oriented edge from g to h if and only if there is an element $s \in S$ such that $gs = h$. We label this edge from g to h with the element s.

Recall the function ℓ_T from Section 2.1. Let id represent the identity permutation in $\mathfrak{S}_{[n]}$. Let $d : \mathfrak{S}_{[n]} \times \mathfrak{S}_{[n]} \longrightarrow \mathbb{Z}$ be the distance function in the Cayley graph of $\mathfrak{S}_{[n]}$ with the set of all transpositions of $\mathfrak{S}_{[n]}$ as the generating set.

Notice that for $\sigma \in \mathfrak{S}_{[n]}$, we have $\ell_T(\sigma) = d(id, \sigma)$. It can be readily seen that d is a metric on $\mathfrak{S}_{[n]}$, and by the triangle inequality, it follows that for $\sigma, \pi \in \mathfrak{S}_{[n]}$, we have $\ell_T(\sigma) \leq \ell_T(\pi) + \ell_T(\pi^{-1}\sigma)$. Also, observe that $\ell_T(\sigma) = \ell_T(\pi) + \ell_T(\pi^{-1}\sigma)$ if and only if there exists a minimal factorization of σ as a product of transpositions which has the minimal factorization of π as a prefix. Another way to see this equality is that π is on a minimal length path from id to σ.

We can define a partial ordering on $\mathfrak{S}_{[n]}$ by declaring that $\pi \leq \sigma$ if $\ell_T(\pi) + \ell_T(\pi^{-1}\sigma) = \ell_T(\sigma)$. This indeed satisfies the conditions for being a partial order (reflexive, antisymmetric, and transitive), and thus puts a poset structure on $\mathfrak{S}_{[n]}$. We denote this poset by $(\mathfrak{S}_{[n]}, \leq)$. We will now give some lemmas, which are due to Brady [7], which will help us to show there is a poset isomorphism between the non-crossing partition lattice and the poset of permutations less than or equal to the cycle $(0\ 1\ \ldots\ n)$.

For $\pi \in \mathfrak{S}_{[n]}$, the *support* of π is $\{i \in [n] : \pi(i) \neq i\}$. We say σ and π are *disjoint* if their supports are disjoint. We make three simple observations.

Observation 2.7. Let $\sigma \in \mathfrak{S}_{[n]}$, and $(i\ j)$ be a transposition where $0 \leq i < j \leq n$. Then $(i\ j) \leq \sigma$ if and only if i and j belong to the same cycle of σ.

The above observation follows from a simple join-cut argument.

Observation 2.8. Let $\pi, \sigma \in \mathfrak{S}_{[n]}$. If π and σ are disjoint, then $\ell_T(\pi\sigma) = \ell_T(\pi) + \ell_T(\sigma)$.

Observation 2.9. If σ is a cycle of $\tau \in \mathfrak{S}_{[n]}$, then $\sigma \leq \tau$.

Lemma 2.10. *For $\pi, \sigma \in \mathfrak{S}_{[n]}$, if $\pi \leq \sigma$, then every cycle of π is contained in a cycle of σ.*

Proof. Suppose there exists elements a, b in different cycles of σ but in the same cycle of π. We have $\ell_T((a\ b)) = 1$, and from Observation 2.7, we have $\ell_T((a\ b)\pi) = \ell_T(\pi) - 1$. Thus we have $\ell_T(\pi) = \ell_T((a\ b)) + \ell_T((a\ b)\pi)$. So $(a\ b) \leq \pi$. By a similar computation $(a\ b) \not\leq \sigma$ from Observation 2.7. By transitivity we have $\pi \not\leq \sigma$. □

Lemma 2.11. *Let $\pi_1, \pi_2, \sigma_1, \sigma_2 \in \mathfrak{S}_{[n]}$. If σ_1 and σ_2 are disjoint, and $\pi_1 \leq \sigma_1$ and $\pi_2 \leq \sigma_2$, then $\pi_1\pi_2 \leq \sigma_1\sigma_2$.*

Proof. By Lemma 2.10, we have that π_1 and π_2 are disjoint. Also since $\pi_1 \leq \sigma_1$ and $\pi_2 \leq \sigma_2$,

$$\ell_T(\pi_1) + \ell_T(\pi_1^{-1}\sigma_1) = \ell_T(\sigma_1). \tag{2.2}$$

$$\ell_T(\pi_2) + \ell_T(\pi_2^{-1}\sigma_2) = \ell_T(\sigma_2). \tag{2.3}$$

But

$$\ell_T((\pi_1\pi_2)^{-1}\sigma_1\sigma_2) = \ell_T(\pi_1^{-1}\sigma_1\pi_2^{-1}\sigma_2),$$

since only π_1, σ_1 and π_2, σ_2 do not commute. So

$$\ell_T((\pi_1\pi_2)^{-1}\sigma_1\sigma_2) = \ell_T(\pi_1^{-1}\sigma_1) + \ell_T(\pi_2^{-1}\sigma_2). \tag{2.4}$$

But also

$$\begin{aligned} \ell_T(\pi_1\pi_2) &= \ell_T(\pi_1) + \ell_T(\pi_2), \text{ and} \\ \ell_T(\sigma_1\sigma_2) &= \ell_T(\sigma_1) + \ell_T(\sigma_2). \end{aligned} \tag{2.5}$$

Adding (2.2) and (2.3), and using (2.4) and (2.5) gives

$$\ell_T(\pi_1\pi_2) + \ell_T((\pi_1\pi_2)^{-1}\sigma_1\sigma_2) = \ell_T(\sigma_1\sigma_2).$$

Thus, we have $\pi_1\pi_2 \leq \sigma_1\sigma_2$. □

For a permutation $\sigma \in \mathfrak{S}_{[n]}$, the cycle structure of σ defines a partition of the set $[n]$, where each block of the partition is the elements of the corresponding cycle of σ. We denote this partition by $\{\sigma\}$. As a result, we can think of Lemma 2.10 as a poset map $\sigma \mapsto \{\sigma\}$ from $(\mathfrak{S}_{[n]}, \leq) \to (\Pi_{n+1}, \leq_r)$. We write $\{\sigma\} \subseteq \{\tau\}$ if every block of σ is contained in a block of τ.

Lemma 2.12. *Let $k \geq 3$ and $a_1, a_2, \ldots, a_k \in [n]$, with $a_1 \leq a_i$ for all $1 \leq i \leq k$. Then $0 \leq a_1 < a_2 < \ldots < a_k \leq n$ if and only if $(a_1\ a_2\ \ldots\ a_k) \leq (0\ 1\ \ldots\ n) = c$.*

Proof. Suppose that $0 \leq a_1 < a_2 < \ldots < a_k \leq n$. We have $\tau := (a_1\ a_2\ \ldots\ a_k)^{-1}(0\ 1\ \ldots\ n) = (a_1\ a_k\ \ldots\ a_2)(0\ 1\ \ldots\ n)$ is given by

$$\tau = (0\ 1\ \ldots\ a_1\ a_k+1\ \ldots\ n)(a_k\ a_{k-1}+1\ \ldots\ a_k-1)(a_{k-1}\ a_{k-2}+1\ \ldots\ a_{k-1}-1)\ldots(a_2\ a_1+1\ \ldots\ a_2-1).$$

This means that $\ell_T(\tau) = n-k+1$. Since $\ell_T(c) = n$ and $\ell_T((a_1\ a_2\ \ldots\ a_k)) = k-1$, we see that $(a_1\ a_2\ \ldots\ a_k)$ and c satisfy the ordering condition.

Conversely, suppose that $(a_1\ a_2\ \ldots\ a_k) \leq c$ and $a_1, \ldots, a_k$ are out of order. Then there exists $t \geq 2$ such that $a_t > a_{t+1}$. Set $i = a_1, k = a_t$, and $j = a_{t+1}$, so $i < j < k$. Let $\pi = (i\ k\ j)$. Then we have $\pi^{-1} = (i\ j\ k)$ which can be written as product of transpositions $(i\ j)(i\ k)$. In the product $\pi^{-1}c$, first consider the product $(i\ k)c$. Then $\pi_1 := (i\ k)(0\ 1\ \ldots\ n) = (0\ 1\ \ldots\ i\ k+1\ k+2 \ldots\ n)(i+1\ i+2\ \ldots\ j\ \ldots\ k)$. Since i and j belong to two different cycles in π_1, the product $(i\ j)\pi_1$ will join the two cycles to give one cycle. This forces $\ell_T(\pi^{-1}c) = n$, so $\ell_T(\pi) + \ell_T(\pi^{-1}c) = 2+n > \ell_T(c)$. Therefore $\pi \nleq c$. But

$$\begin{aligned} \pi^{-1}(a_1\ a_2\ \ldots\ a_k) &= (a_1\ a_{t+1}\ a_t)(a_1\ a_2\ \ldots\ a_k) \\ &= (a_1\ a_{t+2}\ \ldots\ a_k)(a_{t+1})(a_2\ a_3\ \ldots\ a_t). \end{aligned}$$

We have $\ell_T(\pi) = 2, \ell_T(\pi^{-1}(a_1\ a_2\ \ldots\ a_k)) = k-3$, and $\ell_T((a_1\ a_2\ \ldots\ a_k)) = k-1$. Thus $\ell_T(\pi) + \ell_T(\pi^{-1}(a_1\ a_2\ \ldots\ a_k)) = \ell_T((a_1\ a_2\ \ldots\ a_k))$. This implies that $\pi \leq (a_1\ a_2\ \ldots\ a_k)$. Since $\pi \nleq c$ and $\pi \leq (a_1\ a_2\ \ldots\ a_k)$, we have $(a_1\ a_2\ \ldots\ a_k) \nleq c$, which is a contradiction. This proves the converse. □

We say an element $\sigma \in \mathfrak{S}_{[n]}$ is *increasing* if each cycle of σ is increasing. By relabelling elements we get the next corollary, which follows from Lemma 2.12.

Corollary 2.13. *Let $\sigma, \tau \in \mathfrak{S}_{[n]}$ be cycles such that $\{\sigma\} \subseteq \{\tau\}$ with τ increasing. Then $\sigma \leq \tau$ if and only if σ is increasing.*

Definition 2.14. A permutation $\pi \in \mathfrak{S}_{[n]}$ has a crossing if there exists $a < b < c < d$ such that $(a\ c)(b\ d) \leq \pi$. If π has no crossing, then π is *non-crossing.*

Lemma 2.15. *Let δ be an increasing cycle in $\mathfrak{S}_{[n]}$. For $\pi \in \mathfrak{S}_{[n]}$, if $\pi \leq \delta$, then π is non-crossing.*

Proof. We may assume by relabelling that $\delta = (0\ 1\ \dots\ m)$. Assume that the statement is not true; that is, π has a crossing. Then there exists $0 \leq a < b < c < d \leq m$ such that $(a\ c)(b\ d) \leq \pi \leq \delta$. Therefore, $(a\ c)(b\ d) \leq \delta$. Now consider

$$\pi' \coloneqq (b\ d)\delta = (0\ 1\ 2\ \dots\ a\ \dots\ b-1\ b\ d+1\ \dots\ m)(b+1\ b+2\ \dots\ c\ \dots\ d-1\ d).$$

Since a and c are in different cycles of π', we see that $(a\ c)$ is a join for π', and $(a\ c)\pi'$ is a cycle. This means that $\ell_T([(a\ c)(b\ d)]^{-1}(0\ 1\ \dots\ m)) = m$. Thus we have $\ell_T((a\ c)(b\ d)) + \ell_T([(a\ c)(b\ d)]^{-1}(0\ 1\ \dots\ m)) = 2+m > m = \ell_T((0\ 1\ \dots\ m))$. This implies that $(a\ c)(b\ d) \nleq \delta$, which is a contradiction. □

Theorem 2.16 (Brady [7])**.** *Let $\sigma \in \mathfrak{S}_{[n]}$, and let δ be an increasing cycle. Then $\sigma \leq \delta$ if and only if*

(i) $\{\sigma\} \subseteq \{\delta\}$,

(ii) σ *is increasing, and*

(iii) σ *is non-crossing.*

Proof. Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be disjoint cycles of σ.

Suppose that $\sigma \leq \delta$. Then for each σ_i, by Observation 2.9 we have $\sigma_i \leq \sigma \leq \delta$. Thus *(i)* follows from Lemma 2.10. By Corollary 2.13, σ_i is an increasing cycle, proving *(ii)*. Also, *(iii)* follows from Lemma 2.15.

Conversely assume that σ satisfies *(i), (ii)* and *(iii)*. We can assume without loss of generality that $\delta = (0\ 1\ \dots\ m)$. We will prove that $\sigma \leq \delta$ by induction on m. The base case is easy to check.

Since $\sigma = \sigma_1\sigma_2 \dots \sigma_k$, we have $\sigma_1^{-1}\sigma = \sigma_2\sigma_3 \dots \sigma_k$. Suppose $\sigma_1 = (a_1\ a_2\ \dots\ a_l)$ where we may assume $a_1 = 0$. By *(ii)* it follows that $0 = a_1 < a_2 < \dots < a_l \leq m$. Note $\sigma_1^{-1} = (a_1\ a_l\ \dots\ a_2)$. Then

$$\sigma_1^{-1}\delta = (0\ 1\ \dots\ a_1\ a_l+1\ \dots\ m)(a_1+1\ a_1+2\ \dots a_2)\dots(a_{l-1}+1\ a_{l-1}+2\ \dots\ a_l). \quad (2.6)$$

We claim that for $p = 2, \dots, k$, each σ_p is contained in a cycle of $\sigma_1^{-1}\delta$. To see why, suppose $i, j \in \sigma_p$. Then since σ_1 and σ_p are disjoint, we know $i, j \neq a_1, a_2, \dots, a_l$. This implies that there exists r and s such that

$$a_1 < a_2 < \dots < a_r < i < \dots < a_s < j \dots \leq m.$$

But then by Observation 2.7, we have $(i\ j) \leq \sigma_p$ and $(a_r\ a_s) \leq \sigma_1$. So by Lemma 2.11, we have $(i\ j)(a_r\ a_s) \leq \sigma_p\sigma_1 \leq \sigma$. So σ has a crossing, which is a contradiction. Thus, each σ_p is contained in a cycle of $\sigma_1^{-1}\delta$.

We let τ_j be the j^{th} cycle of $\sigma_1^{-1}\delta$ in (2.6). Note that for each j we have τ_j is increasing by (2.6). Now we apply induction to each cycle τ_j, and all σ_i for $i = 2, 3, \dots, m$ where $\{\sigma_i\} \subseteq \{\tau_j\}$. For $i \geq 2$, let A_j be the set of indices such that if $a \in A_j$, then $\{\sigma_a\} \subseteq \{\tau_j\}$. That is, $A_j = \{a : \{\sigma_a\} \subseteq \{\tau_j\}\}$. By induction $\prod_{a \in A_j} \sigma_a \leq \tau_j$. Then by Lemma 2.11, we have

$$\sigma_2\sigma_3 \dots \sigma_k = \sigma_1^{-1}\sigma \leq \sigma_1^{-1}\delta. \tag{2.7}$$

Note from (2.6) that

$$\ell_T(\sigma_1^{-1}\delta) = m + 1 - l = m - (l - 1) = \ell_T(\delta) - \ell_T(\sigma_1) \tag{2.8}$$

But then from (2.7)

$$\begin{aligned} \ell_T(\sigma_1^{-1}\sigma) &= \ell_T((\sigma_1^{-1}\sigma)^{-1}\sigma_1^{-1}\delta) + \ell_T(\sigma_1^{-1}\delta) \\ &= \ell_T(\sigma^{-1}\delta) + \ell_T(\sigma_1^{-1}\delta). \end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned} \ell_T(\delta) &= \ell_T(\sigma_1) + \ell_T(\sigma_1^{-1}\delta) \qquad \text{(by (2.8))} \\ &= \ell_T(\sigma_1) + \ell_T(\sigma_1^{-1}\sigma) - \ell_T(\sigma^{-1}\delta) \qquad \text{(by (2.9))} \\ &= \ell_T(\sigma) - \ell_T(\sigma^{-1}\delta) \quad \text{(because } \sigma_1 \leq \sigma\text{)}. \end{aligned}$$

Thus $\sigma \leq \delta$. □

Theorem 2.17 (Biane [5], Brady [7]). *Suppose $\sigma, \tau \in \mathfrak{S}_{[n]}$. Then $\sigma \leq \tau \leq c$ if and only if*

(i) $\{\sigma\} \subseteq \{\tau\}$,

(ii) all cycles of σ and τ are increasing,

(iii) σ and τ are non-crossing.

Proof. Suppose $\sigma \leq \tau \leq c$. Then *(i)* follows from Lemma 2.10, and *(ii)* and *(iii)* follow from Theorem 2.16.

To prove the converse, we have the following. Criterias *(ii)* and *(iii)* imply that $\sigma \le c$ and $\tau \le c$ by Theorem 2.16. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_t$ be the disjoint cycle representation of σ, and $\tau = \tau_1\tau_2\ldots\tau_k$ be the disjoint cycle representation of τ. Then by *(i)*, we have $\{\sigma\} \subseteq \{\tau\}$. So let $A_1 = \{i : \{\sigma_i\} \subseteq \{\tau_1\}\}$. Since τ_1 is increasing, we apply Theorem 2.16 to $\prod_{i\in A_1}\sigma_i$ and τ_1 to get $\prod_{i\in A_1}\sigma_i \le \tau_1$. We do the same for $\tau_2,\ldots,\tau_k$. Namely for each $j = 2,\ldots,k$ we define A_j analogously to A_1 and conclude that $\prod_{i\in A_j}\sigma_i \le \tau_j$. Then Lemma 2.11 implies that $\sigma \le \tau$. □

Let $(\mathfrak{S}_{[n]}, \le_c)$ denote the poset of all permutations π such that $\pi \le c$, where $c = (0\ 1\ \ldots\ n)$. We then have the following corollary.

Corollary 2.18. *The map* $\sigma \mapsto \{\sigma\}$ *from* $(\mathfrak{S}_{[n]}, \le_c) \to \mathcal{NC}_{n+1}$ *is a poset isomorphism.*

For a given $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$, define the partial products as

$$\sigma_j := \tau_1\tau_2\cdots\tau_j \text{ for } 1 \le j \le n$$

and the empty product

$$\sigma_0 := id.$$

Corollary 2.19. *If* $\tau_1,\ldots,\tau_n \in \mathcal{F}_n$*, and* σ_j *a partial product, then cycles of* σ_j *are non-crossing and increasing. Furthermore* $id = \sigma_0 \le \sigma_1 \le \ldots \le \sigma_n = c$ *is a maximal chain in* $(\mathfrak{S}_{[n]}, \le_c)$.

Proof. Clearly $\ell_T(\sigma_{i+1}) = i + 1 = \ell_T(\sigma_i) + 1 = \ell_T(\sigma_i) + \ell_T(\tau_{i+1}) = \ell_T(\sigma_i) + \ell_T(\sigma_i^{-1}\sigma_{i+1})$. So $\sigma_i \le \sigma_{i+1}$ for all i. The result follows from Corollary 2.18. □

Example 2.20. From Figure 2.2, when viewed as partitions, note that each element is a non-crossing partition of [3]. The minimal element of the poset is $\{\{0\},\{1\},\{2\},\{3\}\}$. This is a trivial non-crossing partition of [3]. $\{\{0,3\},\{1\},\{2\}\}$ is another non-crossing partition of [3] and we can see that $\{\{0\},\{1\},\{2\},\{3\}\} < \{\{0,3\},\{1\},\{2\}\}$. One maximal chain is $\{(\{0\},\{1\},\{2\},\{3\}), (\{0,3\},\{1\},\{2\}), (\{0,3\},\{1,2\}), \{0,1,2,3\}\}$. The corresponding element of this maximal chain in $\mathcal{F}_3$ is the product of transpositions $(0\ 3), (1\ 2)$ and $(1\ 3)$, which we write as $((0\ 3),(1\ 2),(1\ 3))$. When we view the elements of the non-crossing partition of [3] as permutations, the same maximal chain is the chain of partial products $id < (0\ 3) < (0\ 3)(1\ 2) < (0\ 1\ 2\ 3)$. Note that the i-th element in this chain is the partial product $\sigma_i = \tau_1\tau_2\ldots\tau_i$.

In $\mathcal{NC}_{n+1}$, we have π' covers π if and only if π' is obtained from π by merging two blocks (viewed as sets). From Corollary 2.18, this is equivalent to multiplying the permutation associated to π by a join transposition. For example, $((0\ 1\ 2\ 5)(7\ 10\ 11))(5\ 7) = (0\ 1\ 2\ 5\ 7\ 10\ 11)$ and to be clear $(0\ 1\ 2\ 5\ 7\ 10\ 11)$ covers $((0\ 1\ 2\ 5)(7\ 10\ 11))$. Another example is $((0\ 1\ 2\ 6)(3\ 5))(3\ 6) = (0\ 1\ 2\ 3\ 5\ 6)$ and here, $(0\ 1\ 2\ 3\ 5\ 6)$ covers $((0\ 1\ 2\ 6)(3\ 5))$.

Example 2.21. For the element $\pi = (0\ 1\ 3)(4\ 5\ 6\ 10)(7\ 9) \in \mathfrak{S}_{[10]}$ from Example 2.5, we see $\ell_T(\pi) = 6$. Note that $\pi^{-1} = (0\ 3\ 1)(4\ 10\ 5\ 6)(7\ 9)$ and hence $\pi^{-1}c = (0\ 4)(1)(2\ 3)(5)(6)(7\ 10)(8\ 9)$. Thus $\ell_T(\pi) + \ell_T(\pi^{-1}c) = 10 = \ell_T(c)$. Suppose that we have $\pi = (0\ 3\ 1)(5\ 6\ 4) \in \mathfrak{S}_{[6]}$ which is not increasing, then $\ell_T(\pi) = 4$. We can see that $\pi^{-1} = (0\ 1\ 3)(5\ 4\ 6)$ and $\pi^{-1}c = (0\ 2\ 3\ 1\ 4)(5)(6)$. So we have $\ell_T(\pi^{-1}c) = 4$ which makes $\ell_T(\pi) + \ell_T(\pi^{-1}c) = 8 \neq 10 = \ell_T(c)$.

2.4 The Hurwitz graph

In the previous two sections, we saw two different notions of $\mathcal{F}_n$, one as a sequence of transpositions whose product is c, and one as maximal chains in the non-crossing partition lattice which is equivalent to viewing $\mathcal{F}_n$ as a sequence of partial products. In this section, we will see the *Hurwitz graph* ties these two notions together more tightly.

The *Hurwitz graph* is an undirected graph $G_T(n+1)$ whose vertex set is the set of factorization $\mathcal{F}_n$, and two vertices share an edge if they differ in exactly one element when viewed as a maximal chains. Equivalently, when the maximal chain is viewed as a sequence of partial products, adjacent factorizations have exactly one different partial product. Figure 2.3 is an example of Hurwitz graph $G_T(4)$.

For simplicity, we write the transposition as ab instead of $(a\ b)$ in Figure 2.3 and in Figure 2.4, and we let id denote the identity element.

Example 2.22. In Figure 2.3 of the Hurwitz graph $G_T(4)$, let us take two the vertices $03, 23, 12$ and $03, 13, 23$. The element $03, 23, 12$ is the chain of partial products

$$id < (0\ 3) < \underbrace{(0\ 2\ 3)}_{=(0\ 3)(2\ 3)} < \underbrace{(0\ 1\ 2\ 3)}_{=(0\ 2\ 3)(1\ 2)},$$

and the element $03, 13, 23$ is the chain of partial products

$$id < (0\ 3) < \underbrace{(0\ 1\ 3)}_{=(0\ 3)(1\ 3)} < \underbrace{(0\ 1\ 2\ 3)}_{=(0\ 1\ 3)(2\ 3)}.$$

Notice that the elements differ only in the second partial product ((0 2 3) compared to (0 1 3)), so they share an edge in $G_T(4)$.

Now we will define the *Hurwitz actions* on a factorization seen as a list of transpositions. Given $\sigma = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$ and for all $1 \leq i \leq n$, we define the *left* Hurwitz action $\mathsf{HL}_i(\sigma)$ and the *right* Hurwitz action $\mathsf{HR}_i(\sigma)$ as follows:

$$\mathsf{HL}_i(\sigma) := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_i^{\tau_{i+1}}, \tau_{i+2}, \ldots, \tau_n)$$
$$\mathsf{HR}_i(\sigma) := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}^{\tau_i}, \tau_i, \tau_{i+2}, \ldots, \tau_n)$$

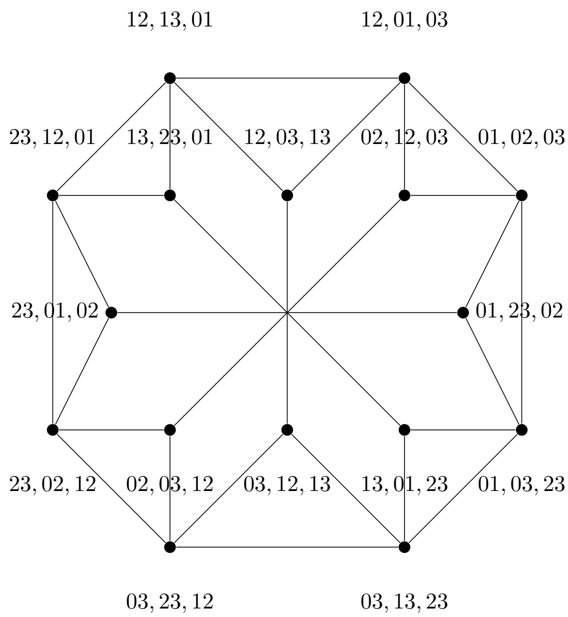


Figure 2.3: Hurwitz graph $G_T(4)$

where $g^h = h^{-1}gh$.

We note that the Hurwitz actions satisfy $\mathsf{HR}_i(.) = \mathsf{HL}_i^{-1}(.)$. Also note, the actions preserve the product of the transpositions. That is, we still get an element of $\mathcal{F}_n$.

The following example will further illustrate these actions. Let $\sigma = ((2\ 3), (2\ 6), (1\ 7), (1\ 8), (4\ 5), (0\ 1), (4\ 6), (0\ 9), (2\ 7)) \in \mathcal{F}_9$. Then we have

$$\mathsf{HL}_3(\sigma) = ((2\ 3), (2\ 6), (1\ 8), (7\ 8), (4\ 5), (0\ 1), (4\ 6), (0\ 9), (2\ 7)) \text{ and}$$
$$\mathsf{HR}_3(\sigma) = ((2\ 3), (2\ 6), (7\ 8), (1\ 7), (4\ 5), (0\ 1), (4\ 6), (0\ 9), (2\ 7)).$$

Also,

$$\mathsf{HL}_3^{-1}(\sigma) = ((2\ 3), (2\ 6), (7\ 8), (1\ 7), (4\ 5), (0\ 1), (4\ 6), (0\ 9), (2\ 7)).$$

Observation 2.23. Two elements $p, q \in \mathcal{F}_n$ share an edge in $G_T(n)$ if and only if there exists $1 \leq i \leq n-1$ such that $q = \mathsf{HR}_i(p)$ or $q = \mathsf{HL}_i(p)$. Note that this means q and p differ exactly at the i^{th} partial product, which was defined before Corollary 2.19.

This observation connects the definition of edges obtained from viewing $\mathcal{F}_n$ as maximal chains and viewing them as factorizations, and is due to Adin and Roichman [1].

Example 2.24. In the Hurwitz graph $G_T(4)$ (Figure 2.3), $12, 13, 01$ and $12, 01, 03$ share an edge because $12, 01, 03 = \mathsf{HL}_2(12, 13, 01)$. On the contrary $12, 13, 01$ does not share an edge with $02, 12, 03$ because $02, 12, 03 \neq \mathsf{HL}_i(12, 13, 01)$ for any i and also $02, 12, 03 \neq \mathsf{HR}_i(12, 13, 01)$ for any i .

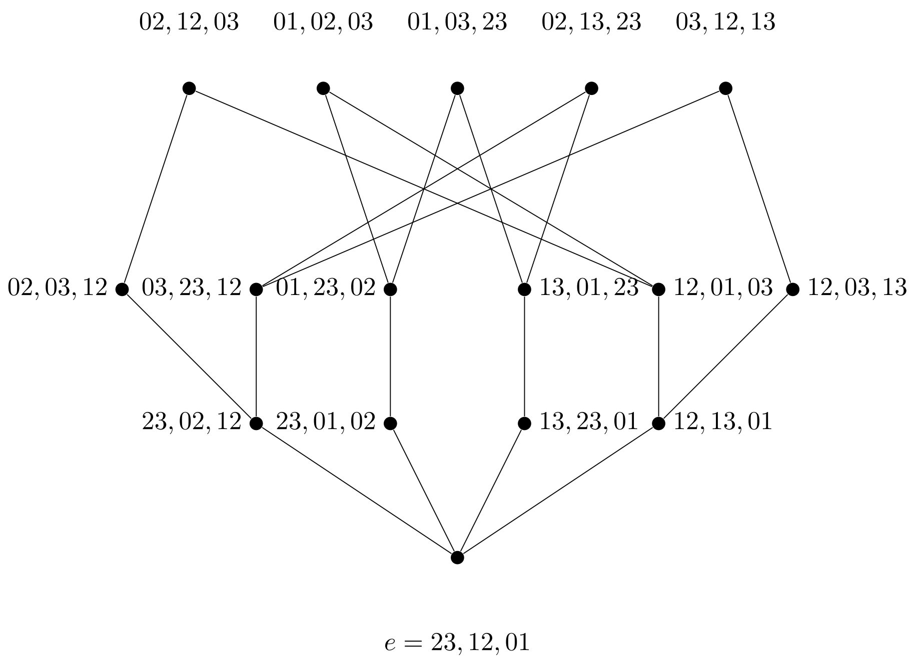


Figure 2.4: Hasse diagram of Weak($\mathcal{F}_3$)

We can now define an ordering on $\mathcal{F}_n$ which we call the *weak ordering*. This puts a poset structure on $\mathcal{F}_n$.

Definition 2.25 (Weak order)**.** For $p, q \in \mathcal{F}_n$, we let $p \leq_W q$ if p is on a geodesic (a shortest path between two vertices) in $G_T(n+1)$ from $e := ((n-1\ n), (n-2\ n-1), \ldots, (1\ 2), (0\ 1))$ to q. We call this the *weak order*. The poset $(\mathcal{F}_n, \leq_W)$ will be denoted by Weak($\mathcal{F}_n$). The element $e := ((n-1\ n), (n-2\ n-1), \ldots, (1\ 2), (0\ 1))$ is the unique minimal element in Weak($\mathcal{F}_n$).

Figure 2.4 gives the Hasse diagram of Weak($\mathcal{F}_3$). The pairs of elements $\{(02, 12, 03), (03, 12, 13)\}$ and $\{(23, 02, 12), (12, 13, 01)\}$ shows that this poset is neither a meet semilattice nor a join semilattice.

We recall some basic definitions concerning posets which are important for our discussion. In a poset, a *chain* $\mathcal{C}$ is a subset where any two elements are comparable. The length of the chain $\mathcal{C}$ is $|\mathcal{C}| - 1$. A *maximal chain* in a poset is a chain that is not contained properly in a larger chain. We say that a poset P is *graded* if there is a well-defined function $rank : P \longrightarrow \mathbb{N}$ satisfying $rank(x) > rank(y)$ if $x > y$ and $rank(x) = rank(y) + 1$ if x covers y. We call a poset P as *ranked* if every maximal chain in the poset has the same length. Note that every ranked poset is graded, but the converse is not true.

The weak poset Weak($\mathcal{F}_n$) has several properties. As found in [1], one of the properties of Weak($\mathcal{F}_n$) is that this poset is graded. We will find the rank function in Section 2.4.1. The rank function of Weak($\mathcal{F}_n$) is well-defined since it is the distance function in the Hurwitz graph. Moreover, the poset Weak($\mathcal{F}_n$) is ranked. This is not explained in [1], and we will prove this also in Section 2.4.1.

One main property of Weak($\mathcal{F}_n$) is that it contains a Catalan number of maximal elements (which we will prove in Chapter 3). We call these maximal elements *maximal factorizations* and denote them by $\mathcal{F}_n^{max}$. Since it contains Catalan number of elements, they are of prime importance to us. In Chapter 3 we will discuss how these maximal elements can be identified with elements of *increasing factorizations* from Section 3.1. In the following section, we will introduce maximal factorizations and characterize them.

2.4.1 Maximal factorizations

The maximal factorizations were described as the maximal elements of the weak poset Weak($\mathcal{F}_n$). These are all the elements that are maximal distance away from e in Weak($\mathcal{F}_n$). But it is not actually clear how we can find them. In this section, we give the rank function of Weak($\mathcal{F}_n$), and this will help us find them easily. We also look at the *inversion statistics* over factorizations.

Let $f = (\tau_1, \tau_2, \ldots, \tau_n)$ be a factorization of the canonical full cycle c and recall the partial products $\sigma_j = \tau_1\tau_2 \ldots \tau_j$ for $0 \leq j \leq n$. Notice that $\sigma_j(i) \neq \sigma_{j+1}(i)$ for exactly two values of $0 \leq i \leq n$, and $\sigma_{j+1}(i) > \sigma_j(i)$ for exactly one value of i, where $1 \leq j \leq n$. Now let us define a set A_j as follows.

$$A_j = \{i \in [n-1] : \sigma_j(i) > \sigma_{j-1}(i)\}.$$

If the range of i in A_j included n, then it follows that $|A_j| = 1$. But the range does not and, hence we will prove the following lemma.

Lemma 2.26. *For $j \geq 1$, we have $|A_j| = 1$. Furthermore $A_i \cap A_j = \emptyset$ for all $i \neq j$.*

Proof. From the argument preceeding this lemma, we have

$$|A_j| \leq 1 \quad \text{for all} \ \ 1 \leq j \leq n. \tag{2.10}$$

Now, we have $\sigma_n(i) = c(i) = i + 1$ and $\sigma_0(i) = i$. This implies that

$$i \in \bigcup_{j=1}^{n} A_j \quad \text{for all} \ \ 0 \leq i < n, \ \text{and thus} \ \left|\bigcup_{j=1}^{n} A_j\right| \geq n. \tag{2.11}$$

We conclude that

$$\left|\bigcup_{j=1}^{n} A_j\right| = n.$$

Since the union of all A_j's should contain precisely n elements and $|A_j| \in \{0,1\}$, it is only possible that

$$|A_j| = 1 \quad \text{for all } \ 1 \leq j \leq n,$$

and $A_i \cap A_j = \emptyset$ for all $i \neq j$. □

Thus if $m \in \mathcal{F}_n$, we have $x_0, x_1, \ldots, x_{n-1}$ is a permutation of $[n-1]$, where $x_i \in A_{i+1}$. Set $\pi_m = (x_0, x_1, \ldots, x_{n-1})$. Here π_m is written in one-line notation. Note that $\pi_m \in \mathfrak{S}_{[n-1]}$ and we index the permutation beginning at 0.

Definition 2.27. Define the map $\phi : \mathcal{F}_n \to \mathfrak{S}_{[n-1]}$ to be

$$\phi(m) \coloneqq \pi_m = (x_0, x_1, \ldots, x_{n-1}) \quad \forall m \in \mathcal{F}_n.$$

We emphasize that we index $\phi(m)$ from 0.

Example 2.28. Let $m = ((0\ 3), (2\ 3), (0\ 4), (0\ 5), (1\ 2)) \in \mathcal{F}_5$. Let us find $\phi(m)$ using the A_i's. It is easier to write down all the partial products first and then use them as needed.

$$\sigma_0 = id, \sigma_1 = (0\ 3), \sigma_2 = (0\ 2\ 3), \sigma_3 = (0\ 2\ 3\ 4), \sigma_4 = (0\ 2\ 3\ 4\ 5), \sigma_5 = (0\ 1\ 2\ 3\ 4\ 5) = c.$$

Therefore by definition of A_i,

$$A_1 = \{i : \sigma_1(i) > \sigma_0(i)\} = \{0\},$$
$$A_2 = \{i : \sigma_2(i) > \sigma_1(i)\} = \{2\}.$$

Similarly, we have

$$A_3 = \{3\},$$
$$A_4 = \{4\},$$
$$A_5 = \{1\}.$$

Thus, $\phi(m) = (0, 2, 3, 4, 1)$.

Observation 2.29. For every $m \in \mathcal{F}_n$ and for every $0 \leq j \leq n-1$, if $\tau_{j+1} = (a\ b)$ with $a < b$,

$$\phi(m)(j) = \sigma_j^{-1}(a),$$

where σ_j is the j^{th} partial product of m.

Example 2.30. Let us take the same factorization m from the previous example. Then, we have

$$\phi(m)(0) = 0 = \sigma_0^{-1}(0),$$
$$\phi(m)(1) = 2 = \sigma_1^{-1}(2),$$
$$\phi(m)(2) = 3 = \sigma_2^{-1}(0),$$
$$\phi(m)(3) = 4 = \sigma_3^{-1}(0),$$
$$\phi(m)(4) = 1 = \sigma_4^{-1}(1).$$

The following three lemmas are by Adin and Roichman [1] and are useful to find the rank function of Weak($\mathcal{F}_n$).

Lemma 2.31. *Let* $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$ *and let* $1 \leq i < j \leq n$*. If* τ_i *and* τ_j *do not commute, then the pair* (τ_i, τ_j) *has one of the three forms:* $((a\ b), (a\ c)), ((a\ c), (b\ c))$ *or* $((b\ c), (a\ b))$ *for some* $a < b < c$*.*

Proof. Note that if τ_i and τ_j do not commute, then necessarily they should have exactly one element common since they cannot be equal. So either their least element is in common, greater element is in common, or the least element of one is the greatest element in the other. Let us assume that the pair (τ_i, τ_j) has their least element in common and is of the form $((a\ c), (a\ b))$ where $a < b < c$. Because $\tau_i = (a\ c)$, the elements a and c are in the same cycle of the partial product σ_{j-1}. Let x be the preimage of a in this cycle. If $x < a$, then in the partial product σ_{j-1}, the cycle containing a and c is of the form $(\ldots\ x\ a\ \ldots\ c)$ where $x < a < c$. Let $(\ldots\ y\ b\ z\ \ldots)$ be the cycle containing b in σ_{j-1}. Then a cycle in σ_j is of the form $(\ldots\ x\ b\ z\ \ldots\ y\ a\ \ldots\ c)$. This contradicts Corollary 2.19.

Similarly, if $x > a$, the cycle containing a and c in σ_{j-1} is of the form $(a\ \ldots\ c\ \ldots\ x)$, that is, a is the smallest element of the cycle, then a cycle of σ_j is $(a\ \ldots\ c\ \ldots\ x\ b\ z\ \ldots\ y)$, which is also a contradiction by Corollary 2.19. Thus if $(a\ b)$ and $(a\ c)$ are members of $(\tau_1, \tau_2, \ldots, \tau_n)$, the factor $(a\ b)$ appears before $(a\ c)$. This completes the first case, and the others can be proved similarly. □

Lemma 2.32. *Let* $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$ *and* $1 \leq j \leq n$*. If the transposition* τ_j *is* $(a\ b)$*, then* $i := \phi(m)(j-1)$ *satisfies* $a \leq i < b$*.*

Proof. From the definition of A_j, there exists a unique i such that $i = \sigma_0(i) \geq \sigma_1(i) \geq \sigma_2(i) \geq \ldots \geq \sigma_n(i) = i+1$, for all $1 \leq i \leq n$, except for one step j where $i = \sigma_{j-1}(i) < \sigma_j(i)$. But from Observation 2.29 we have $i = \phi(m)(j-1) = \sigma_{j-1}^{-1}(a)$ where a is defined as the lower element of τ_j. This means $\sigma_{j-1}(i) = a$. Since $\sigma_j = \sigma_{j-1}(a\ b)$, we have $\sigma_j(i) = \sigma_{j-1}(a\ b)(i) = b$. Now the result follows. □

We now state and prove a lemma regarding the composition of ϕ function with HR_i and HL_i. Recall that a *simple reflection* is a transposition of two consecutive numbers; that is, $s_i = (i\ i+1)$ for $i \in \mathbb{N}$. The n simple reflections in $\mathfrak{S}_{[n]}$ are $s_0 = (0\ 1), s_1 = (1\ 2), \ldots,$ and $s_{n-1} = (n-1\ n)$.

Lemma 2.33. *For every* $m \in \mathcal{F}_n$ *and* $1 \leq i \leq n-1$*,* $\phi(\mathsf{HR}_i(m)) = \phi(m)$ *if there exists* $a < b < c$ *such that* $\tau_i = (a\ c)$ *and* $\tau_{i+1} = (b\ c)$ *or* $\phi(\mathsf{HR}_i(m)) = s_{i-1}\phi(m)$ *otherwise. Similarly we have* $\phi(\mathsf{HL}_i(m)) = \phi(m)$ *if there exists* $a < b < c$ *such that* $\tau_i = (a\ b)$ *and* $\tau_{i+1} = (a\ c)$ *or* $\phi(\mathsf{HL}_i(m)) = s_{i-1}\phi(m)$ *otherwise.*

Proof. We will prove the result only for HR_i as the proof is similar for HL_i.

Recall that for $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$, the right Hurwitz action HR_i on m gives us $(\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}^{\tau_i}, \tau_i, \tau_{i+2}, \ldots, \tau_n)$. This means that the k-th transposition of $\mathsf{HR}_i(m)$ is the same as the k-th transposition of m for all $k \neq i$ or $i+1$. If we let σ_j be the partial product of first j transpositions of m and σ'_j be the partial product of first j transpositions of $\mathsf{HR}_i(m)$, we have that $\sigma'_j = \sigma_j$ for all $j \neq i$ since there is an edge from $\mathsf{HR}_i(m)$ to m. Therefore, it follows that $\phi(\mathsf{HR}_i(m))(k) = \phi(m)(k)$ for all $k \neq i-1$ or i. Because of this, we have $\phi(\mathsf{HR}_i(m)) = \phi(m)$ or $\phi(\mathsf{HR}_i(m)) = s_{i-1}\phi(m)$. Hence we must show that $\phi(\mathsf{HR}_i(m)) = \phi(m)$ if and only if $\tau_i = (a\ c)$ and $\tau_{i+1} = (b\ c)$ for some a, b, c where $a < c$ and $b < c$. By Lemma 2.31 there are four possible cases for the pair τ_i and τ_{i+1}: (1) τ_i and τ_{i+1} are disjoint, or (2) there exists $a < b < c$ such that (i) $\tau_i = (a\ b)$ and $\tau_{i+1} = (a\ c)$, or (ii) $\tau_i = (a\ c)$ and $\tau_{i+1} = (b\ c)$, or (iii) $\tau_i = (b\ c)$ and $\tau_{i+1} = (a\ b)$. Let us look at the cases one by one. We noted earlier that $\sigma_j = \sigma'_j$ for all $j \neq i$. Crucial to our analysis is that $\sigma_{i-1} = \sigma'_{i-1}$.

Case (1): We have τ_i and τ_{i+1} are disjoint, so $\tau_i = (a\ b)$ and $\tau_{i+1} = (c\ d)$ where a, b, c, d are distinct. Then $\phi(m)(i-1) = \sigma_{i-1}^{-1}(a)$ and $\phi(\mathsf{HR}_i(m))(i-1) = \sigma'^{-1}_{i-1}(c)$. But we have $\sigma_{i-1} = \sigma'_{i-1}$, so $\sigma_{i-1}^{-1}(a) \neq \sigma'^{-1}_{i-1}(c)$. Thus $\phi(m)(i-1) \neq \phi(\mathsf{HR}_i(m))(i-1)$.

Case (2): We have $\phi(m)(i-1) = \sigma_{i-1}^{-1}(d)$ and $\phi(\mathsf{HR}_i(m))(i-1) = \sigma'^{-1}_{i-1}(d') = \sigma_{i-1}^{-1}(d')$ because $\sigma_{i-1} = \sigma'_{i-1}$, where $d = min(\tau_i)$ and $d' = min(\tau_{i+1}^{\tau_i})$. Therefore $\phi(\mathsf{HR}_i(m))(i-1) \neq \phi(m)(i-1)$ if and only if $d \neq d'$.

In Case (2)(i), we have $\tau_i = (a\ b)$ and $\tau_{i+1} = (a\ c)$. So $d = min(\tau_i) = a$ and $\tau_{i+1}^{\tau_i} = (b\ c)$. This implies that $d' = b$. Therefore $\phi(m)(i-1) \neq \phi(\mathsf{HR}_i(m))(i-1)$.

In Case (2)(ii), we have $\tau_i = (a\ c)$ and $\tau_{i+1} = (b\ c)$. So $d = min(\tau_i) = a$ and $\tau_{i+1}^{\tau_i} = (a\ b)$. This implies that $d' = a$. Therefore $\phi(m)(i-1) = \phi(\mathsf{HR}_i(m))(i-1)$.

In Case (2)(iii), we have $\tau_i = (b\ c)$ and $\tau_{i+1} = (a\ b)$. So $d = min(\tau_i) = b$ and $\tau_{i+1}^{\tau_i} = (a\ c)$. This implies that $d' = a$. Therefore $\phi(m)(i-1) \neq \phi(\mathsf{HR}_i(m))(i-1)$.

Of the four Cases (1), (2)(i), (2)(ii) and (2)(iii), only in case (2)(ii) we have $\phi(m)(i-1) = \phi(\mathsf{HR}_i(m))(i-1)$. Therefore only in this case we have $\phi(m) = \phi(\mathsf{HR}_i(m))$. This proves our claim. □

Lemma 2.34. *The element e is the unique element that satisfies $\phi(e) = (n-1, n-2, \ldots, 1, 0)$.*

Proof. We have $e = ((n-1, n), (n-2, n-1), \ldots, (1, 2), (0, 1))$ where $\tau_j = (n-j, n-j+1)$ for $1 \leq j \leq n-1$. The partial product $\sigma_j = \tau_1\tau_2 \ldots \tau_j = (n-j, n-j+1, \ldots, n)$. This implies that $\phi(e)(n-j) = \sigma_{j-1}^{-1}(n-j) = n-j$ (from Observation 2.29).

Now assume $m = (t_1, t_2, \ldots, t_n) \in \mathcal{F}_n$ is such that $\phi(m) = (n-1, n-2, \ldots, 1, 0)$. We will prove that $m = e$. We have $\phi(m) = (n-1, n-2, \ldots, 1, 0)$, which implies that $\phi(m)(0) = n-1$. This means that $\sigma_1(n-1) > \sigma_0(n-1)$, and we know $\sigma_0 = id$. Thus we have $\sigma_0(n-1) = n-1$, from which we conclude that $\sigma_1(n-1) = n$. This implies that $\tau_1 = \sigma_1 = (n-1\ n)$. From this, it follows that $\tau_2 \dots \tau_n = (n-1\ n)(0\ 1\ \dots\ n) = (0\ 1\ \dots\ n-1)$. This means that we have $\tau_2, \dots, \tau_n \in \mathfrak{S}_{[n-1]}$ and $m' = (\tau_2, \tau_3, \dots, \tau_n) \in \mathcal{F}_{n-1}$. Note that in this case we have $\phi(m') = (n-2, n-3, \dots, 1, 0) \in \mathfrak{S}_{[n-1]}$. If we repeat the same and proceed, we will finally end up with $m = ((n-1\ n), (n-2\ n-1), \dots, (0\ 1)) = e$. □

Recall that $\mathrm{Weak}(\mathcal{F}_n)$ is graded, so let $\mathrm{rank}(\cdot)$ be rank function. Set $\mathrm{rank}(e) = 0$. Let $\mathrm{rank}(w)$ denote the rank of the element w in the poset $\mathrm{Weak}(\mathcal{F}_n)$. For $\pi \in \mathfrak{S}_{[n-1]}$, let $\mathrm{inv}(\pi)$ denote its *inversion number*. Recall that in a permutation, an inversion is a pair of numbers such that the smallest number appears to the left of a larger number in the permutation, and the inversion number is the number of inversion pairs. That is, $\mathrm{Inv}(\pi) \coloneqq \{(i,j) : i < j \text{ and } \pi(i) < \pi(j)\}$ and $\mathrm{inv}(\pi) = |\mathrm{Inv}(\pi)|$ [1]. We will now prove the following theorem that gives the rank of an element in $\mathcal{F}_n$.

Theorem 2.35. *For any $m \in \mathcal{F}_n$,*

$$\mathrm{rank}(m) = \mathrm{inv}(\phi(m)).$$

Proof. We will prove this by induction on $\mathrm{inv}(\phi(m))$.

Base case: Suppose that $\mathrm{inv}(\phi(m)) = 0$. This means that $\phi(m) = (n-1, n-2, \dots, 1, 0)$ and therefore by Lemma 2.34, we have $m = e$. Thus $\mathrm{rank}(m) = \mathrm{inv}(\phi(m)) = 0$.

Induction: Assume that the statement is true for all $m' \in \mathcal{F}_n$ whenever $\mathrm{inv}(\phi(m')) \leq k-1$. We will show that if $m \in \mathcal{F}_n$, then $\mathrm{rank}(m) = k$ if $\mathrm{inv}(\phi(m)) = k$. Since $\mathrm{inv}(\phi(m)) > 0$, there exists $0 \leq i \leq n-2$ such that $\mathrm{inv}(\phi(m)) > \mathrm{inv}(s_i\phi(m))$. Now, by Lemma 2.33, the only way $\phi(\mathsf{HR}_{i+1}(m)) = \phi(\mathsf{HL}_{i+1}(m)) = \phi(m)$ is if both of the following hold simultaneously: *(i)* there exists $a < b < c$ such that $\tau_{i+1} = (a\ c)$ and $\tau_{i+2} = (b\ c)$, and *(ii)* there exists $a < b < c$ such that $\tau_{i+1} = (a\ b)$ and $\tau_{i+2} = (a\ c)$. The only way for them both to hold is if $\tau_{i+1} = \tau_{i+2}$, which is impossible. Thus at least one of $\phi(\mathsf{HR}_{i+1}(m))$ or $\phi(\mathsf{HL}_{i+1}(m))$ is $s_i\phi(m)$. Set $m' = \mathsf{HR}_{i+1}(m)$ or $\mathsf{HL}_{i+1}(m)$, whichever gives $\phi(m') = s_i\phi(m)$. Then m' reduces the number of inversions by 1 from m and hence $\mathrm{inv}(\phi(m')) = \mathrm{inv}(s_i\phi(m)) = k-1$. By induction hypothesis $\mathrm{rank}(m') = \mathrm{inv}(\phi(m'))$. Since $m' \sim m$, $\mathrm{rank}(m) \leq k-1+1 = k$.

But then again by Lemma 2.33, the inv value changes only by 0 or 1 on each arc in a geodesic from e to m in the Hurwitz graph $G_T(n)$. This means $\mathrm{rank}(m) \geq \mathrm{inv}(\phi(m)) = k$ from which we conclude that $\mathrm{rank}(m) = \mathrm{inv}(\phi(m)) = k$. □

Corollary 2.36. *Let $m \in \mathrm{Weak}(\mathcal{F}_n)$ be a maximal element. Then*

$$\phi(m) = (0, 1, \dots, n-1).$$

[1] Note that this is not the standard definition of inversion. The standard definition has the largest element to the left. But obviously our definition has the same count, i.e., both definitions give the same number of permutations with i inversions for any i.

Since $\text{Weak}(\mathcal{F}_n)$ is graded, this is equivalent to showing that every maximal chain in $\text{Weak}(\mathcal{F}_n)$ has the same length.

Proof. We assume that for $m \in \mathcal{F}_n$, $\text{rank}(m) < \binom{n}{2}$, and we find a factorization m' that covers it. This is sufficient to prove our claim.

Let $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$ be a factorization such that $\text{rank}(m) < \binom{n}{2}$. Then $\phi(m) \neq (0, 1, 2, \ldots, n-1)$. Hence there exists $i \geq 1$ such that $\phi(m)(i-1) > \phi(m)(i)$. By Lemma 2.33, if $\tau_i = (a\ c)$ and $\tau_{i+1} = (b\ c)$, then $\phi(\mathsf{HL}_i(m)) = s_{i-1}\phi(m)$. Otherwise, $\phi(\mathsf{HR}_i(m)) = s_{i-1}\phi(m)$. Therefore, set m' to be either $\phi(\mathsf{HL}_i(m))$ or $\phi(\mathsf{HR}_i(m))$, whichever gives $s_{i-1}\phi(m)$. Then by the observation preceding Example 2.24, the factorizations m and m' share an edge in the Hurwitz graph. But then, $\text{inv}(\phi(m')) = \text{inv}(\phi(m)) + 1$ and thus $\text{rank}(m') = \text{rank}(m) + 1$, which completes the proof. □

This also proves that every maximal chain in $\text{Weak}(\mathcal{F}_n)$ has same length. We denote the set of all maximal elements in $\text{Weak}(\mathcal{F}_n)$ by $\mathcal{F}_n^{max}$. Note that $(0, 1, \ldots, n-1) \in \mathfrak{S}_{[n-1]}$ is the element with maximum number of inversions, and for any $m \in \mathcal{F}_n^{max}$, we have $\phi(m) = (0, 1, \ldots, n-1)$.

Example 2.37. For $n = 3$, the elements of the set $\mathcal{F}_n^{max}$ are $((0\ 1), (0\ 2), (0\ 3))$, $((0\ 2), (1\ 2), (0\ 3))$, $((0\ 1), (0\ 3), (2\ 3))$, $((0\ 3), (1\ 2), (1\ 3))$ and $((0\ 3), (1\ 3), (2\ 3))$.

Example 2.38. To explain the characterization of $\mathcal{F}_n^{max}$, let $f = ((0\ 2), (1\ 2), (0\ 3), (0\ 7), (4\ 7), (5\ 6), (5\ 7)) \in \mathcal{F}_7^{max}$. Let us begin with the empty list $\phi(f) = (\star, \star, \ldots, \star)$. The first partial product is $(0\ 2)$ and the least element in the product is 0. Thus we add 0 to the list to get $\phi(f) = (0, \star, \ldots, \star)$. Next, the second partial product is $(0\ 2)(1\ 2)$, the least element in the product which is not already in $\phi(f)$ is 1 and thus $\phi(f) = (0, 1, \star, \ldots, \star)$. The next partial product is $(0\ 2)(1\ 2)(0\ 3)$. The unique least element that is not already a part of $\phi(f)$ is 2. Thus $\phi(f) = (0, 1, 2, \star, \ldots, \star)$. Continuing in the same way till we get the full product, we get $\phi(f) = (0, 1, 2, 3, 4, 5, 6)$. On the contrary, let us take $f' = ((2\ 3), (0\ 1), (3\ 4), (4\ 5), (1\ 5), (5\ 6), (6\ 7))$. Since $\phi(f') = (2, 0, 3, 4, 1, 5, 6)$, we conclude that $f' \notin \mathcal{F}_7^{max}$.

We can now define some inversion statistics for elements in $\mathcal{F}_n$.

Definition 2.39 (Right inversions, left inversions and neutral inversions)**.** For a factorization $m = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{F}_n$, the set of *inversions* of m is $\text{Inv}(m) := \text{Inv}(\phi(m))$. Recall that $\phi(m)$ is indexed by $[n-1]$. If $\tau_i = (a\ b)$, define $I_i = [a, b] \subseteq \mathbb{R}$, the closed interval in $\mathbb{R}$. For each inversion $(i, j) \in \text{Inv}(m)$, define (i, j) to be:

- a *right* inversion if $I_{i+1} \subseteq I_{j+1}$;
- a *left* inversion if $I_{j+1} \subseteq I_{i+1}$; and
- a *neutral* inversion otherwise.

Recall that $\phi(m)$ is indexed from 0 to $n-1$. For $m \in \mathcal{F}_n$, we denote the set of all right inversions by $\mathrm{Inv}_R(m)$, left inversions by $\mathrm{Inv}_L(m)$, and neutral inversions by $\mathrm{Inv}_N(m)$. Denote the cardinalities of each of these three sets by $\mathrm{inv}_R(m)$, $\mathrm{inv}_L(m)$ and $\mathrm{inv}_N(m)$ respectively.

Example 2.40. Let $m = ((5\ 6), (0\ 4), (1\ 3), (1\ 4), (2\ 3), (5\ 7), (0\ 5)) \in \mathcal{F}_7$. Then we have

$$\phi(m) = (5, 0, 1, 3, 2, 6, 4) \text{ and so,}$$
$$\mathrm{Inv}(m) = \mathrm{Inv}\ (\phi(m)) = \{(0,5), (1,2), (1,3),$$
$$(1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,5), (3,6), (4,5), (4,6)\}.$$

For example, the pair $(0,5) \in \mathrm{Inv}(\phi(m))$ because $\phi(m)(0) = 5 \leq 6 = \phi(m)(5)$. We have the following right, left and neutral inversions.

$$\mathrm{Inv}_R(m) = \{(0,5), (1,6), (2,3), (2,6), (3,6), (4,6)\},$$
$$\mathrm{Inv}_L(m) = \{(1,2), (1,3), (1,4), (2,4)\},$$
$$\mathrm{Inv}_N(m) = \{(1,5), (2,5), (3,5), (4,5)\}.$$

Notice that from our example, $\mathrm{Inv}(m)$ is the disjoint union of the three sets $\mathrm{Inv}_R(m)$, $\mathrm{Inv}_L(m)$ and $\mathrm{Inv}_N(m)$. This is clearly always true.

It is useful to encode the joint statistics of $\mathrm{inv}_R(m)$ and $\mathrm{inv}_L(m)$ in a two variable polynomials. Let

$$\mathcal{F}_n^{max}(q,t) := \sum_{w \in \mathcal{F}_n^{max}} q^{\mathrm{inv}_R(w)} t^{\mathrm{inv}_L(w)}.$$

Recall that a *q,t*-Catalan polynomial is a two variable polynomial in q and t such that when $q = t = 1$, the polynomial reduces to a Catalan number C_n. One example of q,t-Catalan polynomial were the ones briefly discussed in Chapter 1. The following definition gives an example of another one.

Definition 2.41. For $n \geq 1$, define $C_n(q,t)$ as:

$$C_n(q,t) := \sum_{k=0}^{n-1} q^k t^{n-k-1} C_k(q,t) C_{n-k-1}(q,t), \text{with } C_0(q,t) = 1. \tag{2.12}$$

The first few polynomials $C_n(q,t)$ are

$C_0(q,t) = 1.$
$C_1(q,t) = 1.$
$C_2(q,t) = q + t.$
$C_3(q,t) = q^3 + q^2t + qt^2 + qt + t^3.$
$C_4(q,t) = q^6 + q^5t + q^4t^2 + q^4t + 2q^3t^3 + q^3t + q^2t^4 + 2q^2t^2 + qt^5 + qt^4 + qt^3 + t^6.$

We can notice that $C_0(1,1) = 1, C_1(1,1) = 1, C_2(1,1) = 2, C_3(1,1) = 5, C_4(1,1) = 14$, which are the first five Catalan numbers. Note that $C_n(q,t)$ is a q,t-Catalan polynomial because

$$C_n(1,1) = \sum_{k=0}^{n-1} C_k(1,1)C_{n-k-1}(1,1),$$

with $C_0(1,1) = 1$. This is precisely the recurrence for the Catalan numbers given in (2.1). Clearly, the polynomial $C_n(q,t)$ is symmetric in q and t for all $n \geq 0$.

We have the following theorem of Adin-Roichman.

Theorem 2.42 (Adin-Roichman [1])**.** *For every positive integer n,*

$$\mathcal{F}_n^{max}(q,t) = C_n(q,t).$$

This theorem tells us that the joint distribution of inv_R and inv_L statistics over maximal factorizations is given as the coefficients of $C_n(q,t)$. Thus $\mathcal{F}_n^{max}(q,t)$ is also a symmetric q,t-Catalan polynomial. We give an alternate original proof of this theorem in Chapter 4. Our proof will use the connection with *increasing factorizations* and the *area* statistic which are easier to understand.

2.5 Parking functions

This section briefly introduces parking functions, their properties and some well-known results.

Definition 2.43 (Parking function)**.** Let n be a positive integer. A parking function of length n is a sequence $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ such that the weakly increasing rearrangement $(a_1', a_2', \ldots, a_n')$ satisfies the condition $a_i' \leq i-1$ for all $1 \leq i \leq n$. We denote the set of all parking functions of length n by $\mathcal{P}_n$.

Example 2.44. We have the set $\mathcal{P}_3 = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,0,2), (0,2,0), (2,0,0), (0,1,1), (1,0,1), (1,1,0), (0,1,2), (0,2,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0)\}$. Note that $|\mathcal{P}_3| = 16$.

By definition, any permutation of the elements of a parking function of length n is again a parking function of length n.

We explain the name "parking function" for the set $\mathcal{P}_n$. Suppose there are n numbered parking spots in a linear lot, say $0, 1, \ldots, n-1$, in this order. Suppose $a_1, a_2, \ldots, a_n$ are preferred parking spots for the cars $c_1, c_2, \ldots, c_n$ where $a_i \in [n-1]$. Each car, in order, drives to its preferred spot and parks if the spot is empty. If the preferred spot is occupied, it moves to the next spot available. If no spot is available when car c_i tries to park, the car leaves the lot. Then the preference sequence $a_1, a_2, \ldots, a_n$ is a parking function of length n if and only if all cars can park.

It is best to look at the equivalence of Definition 2.43 and the description given in the previous paragraph. Let $p = (a_1, a_2, \ldots, a_n)$ be a sequence of non-negative integers such that for at least one i, we have $a'_i > i-1$. We then want to show that not all cars can park. This follows because the $n-i$ cars corresponding to $a'_{i+1}, a'_{i+2}, \dots, a'_n$ must park in fewer than $n - a'_i$ spots. By our assumption, and the fact that $n-i \geq n-a'_i$, our claim follows.

Conversely, assume that the sequence of non-negative integers $(a_1, a_2, \dots, a_n)$ does not allow all cars to park. Once all the cars have attempted to park, and the ones that cannot park leave, assume that the empty spot with largest label is $i-1$. This means that more than $n-i$ cars wanted to park in spots $i, i+1, \dots, n-1$, and all cars must have had a preferred spot greater than or equal to i (if not one of the cars would have parked in spot $i-1$). Thus at least $n-i+1$ cars had their preferred spots from i to $n-1$. Now we can see that $a'_i, a'_{i+1}, \dots, a'_n$ are all greater than or equal to i. So $a'_i \geq i$. This completes the equivalence.

The permutation obtained by reading the cars in spots $0, 1, 2, \dots, n-1$ in order is called as the *parking order*. Figure 2.6 is an example of the parking function $p = (2, 3, 0, 2, 6, 0, 0, 6)$ with its parking order. The parking order is written along the left side.

The following proposition was proved analytically by A. G. Konheim and B. Weiss in 1966 in their paper [20] but we give a simple proof due to Pollack using the parking scenario.

Theorem 2.45 (Pyke [29], Konheim-Weiss [20])**.** *For a positive integer n, the number of parking functions of length n is $(n+1)^{n-1}$.*

The proof of the theorem can be found in [11], but they credit Pollack for the proof.

Proof. We play the parking game again, but this time we play the game on a circular lot with $n+1$ spots numbered clockwise from 0. Let $a_1, a_2, \dots, a_n \in \{0, 1, \dots, n\}^n$ be a preference sequence for cars $c_1, c_2, \dots, c_n$. In this modified game, each car enters the lot at 0, goes to its preferred parking spot, and parks if the spot is empty. If not, it moves to the next available spot clockwise. Since we have n cars and $n+1$ spots, all cars can park. It remains the case that $a_1, a_2, \dots, a_n$ is a parking function if and only if spot n is empty. Notice if $a_1, a_2 \dots, a_n$ leaves the spot i empty, then $a_1+1, a_2+1 \dots, a_n+1 \pmod{n+1}$ leaves spot $i+1 \pmod{n+1}$ empty. Thus precisely one of $(a_1, a_2 \dots, a_n), (a_1+1, a_2+1 \dots, a_n+1), \dots, (a_1+n, a_2+n \dots, a_n+n)$ is a parking function and the set $[n]^n$ is partitioned by these orbits. Hence, the number of parking functions is $(n+1)^n/(n+1) = (n+1)^{n-1}$. □

For a given parking function $p = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, the sequence $(n-a_1, n-a_2, \dots, n-a_n)$ is called the *major sequence*. We denote the set of all major sequences of length n by $\mathcal{M}_n$. Alternatively, we say $(b_1, b_2, \dots, b_n) \in \mathbb{N}^n$ is a major sequence if the weakly increasing rearrangement $(b'_1, b'_2, \dots, b'_n)$ satisfies the condition $i \leq b'_i \leq n$ for all i. Clearly $|\mathcal{M}_n| = |\mathcal{P}_n|$.

A *Dyck path* of length n is a lattice path with a sequence of East $(1,0)$ and North $(0,1)$ steps from $(0,0)$ to (n,n) such that the path lies weakly below the diagonal $y = x$. The set

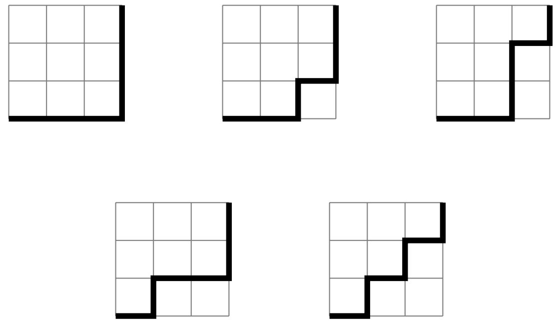


Figure 2.5: The five elements of $\mathcal{D}_3$

of all Dyck paths of length n is denoted by $\mathcal{D}_n$. Figure 2.5 has $\mathcal{D}_3$ and note that $|\mathcal{D}_3| = 5$. We have the following well-known theorem.

Theorem 2.46. *For a positive integer n, we have $|\mathcal{D}_n| = C_n$.*

One of the earliest proof is by André [2], which uses the reflection principle. There are lots of proofs of this theorem. See for example [3], [10]. One is similar to our proof that $|NC_n| = C_n$.

The classic "first return to the diagonal" decomposition for Dyck paths can be used to show that they are counted by Catalan numbers. Consider a Dyck path that terminates at $(n+1, n+1)$. Let $k \geq 0$ be the smallest number such that the point $(k+1, k+1)$ of the line $y = x$ is on the path. Then to the right of $(k+1, k+1)$ is a Dyck path of size $n-k$. To the left of the point $(k+1, k+1)$ is a path that lies strictly under the line $y = x$. Thus removing its first horizontal and last vertical step gives a Dyck path of length k. This decomposition can be used to show that the number of Dyck paths satisfy the recurrence of (2.1). The details are similar to the proof of Lemma 2.4, so we omit them.

A *labelled* Dyck path is a Dyck path in which each east $(1, 0)$ steps is labelled from $1, 2, \ldots, n$ such that the labels in each row are increasing left to right.

Given a parking function of length n, we can associate it to a labelled Dyck path as follows. Let D be a Dyck path with e_i east steps after the $(i-1)$-st north step, where e_i counts the number of occurrences of i in the parking function $p = (a_1, a_2, \ldots, a_n)$. Now label the e_i east steps after the $(i-1)$-st north step by the positions of the letter i in increasing order in p. See Figure 2.6.

Observation 2.47. A sequence of non-negative integers $p = (a_1, a_2, \ldots, a_n)$ is a parking function if and only if $|\{i : a_i < k\}| \geq k$ for all $1 \leq k \leq n$. This follows from the connection with the labelled Dyck path, but also follows from the definition of parking functions.

The connection between parking functions and factorizations is as follows. Recall that $\mathcal{F}_n$ denotes the set of all factorizations of the canonical full cycle c, and Proposition 2.2 gives that $|\mathcal{F}_n| = (n+1)^{n-1}$. There is indeed a relation between $\mathcal{F}_n$ and $\mathcal{P}_n$. Consider the

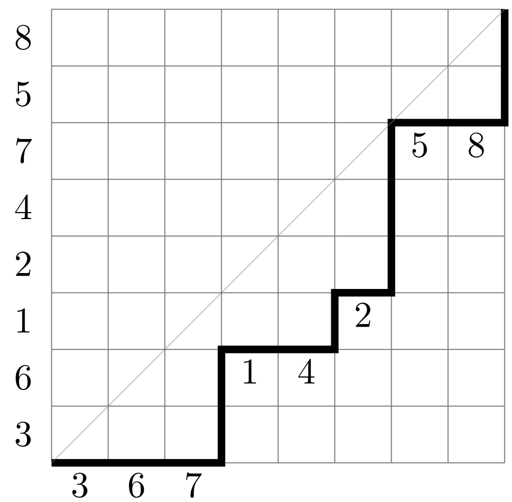


Figure 2.6: Parking function $p = (2, 3, 0, 2, 6, 0, 0, 6)$ with parking order 36124758

two mappings $L : \mathcal{F}_n \to \mathbb{N}^n$ and $U : \mathcal{F}_n \to \mathbb{N}^n$ that send factorizations $f \in \mathcal{F}_n$ to the *lower* and the *upper* sequences respectively. That is,

$$(a_1\ b_1)(a_2\ b_2)\dots(a_n\ b_n) \overset{L}{\longmapsto} (a_1, a_2, \dots, a_n), \tag{2.13}$$

$$(a_1\ b_1)(a_2\ b_2)\dots(a_n\ b_n) \overset{U}{\longmapsto} (b_1, b_2, \dots, b_n). \tag{2.14}$$

Theorem 2.48 (Stanley [34], Biane [5])**.** *The function L bijectively maps $\mathcal{F}_n$ into $\mathcal{P}_n$ and the function U bijectively maps $\mathcal{F}_n$ into $\mathcal{M}_n$.*

Sketch of proof. The proof for the forward direction of the bijectivity is given in [34], where Stanley views factorizations as maximal chains of $\mathcal{NC}_{n+1}$. We provide a different proof here which is due to Irving-Rattan [19].

Given any factorization $f = ((a_1\ b_1), (a_2\ b_2), \dots, (a_n\ b_n)) \in \mathcal{F}_n$, finding the corresponding parking function is done by using L that sends f to its lower sequence $(a_1, a_2, \dots, a_n)$. We claim that $(a_1, a_2, \dots, a_n)$ is a parking function. Recall the proof of Theorem 2.2. For a factorization $f = (\tau_1, \tau_2, \dots, \tau_n)$, the graph $G(f)$ (which has an edge between a and b labelled i if $\tau_i = (a\ b)$) is a tree. Now let K be the set of vertices which are greater than or equal to k. So $|K| = n - (k-1) = n - k + 1$. The induced subgraph of $G(f)$ on K is a forest. So the number of edges is at most $n - k$. But then the number of edges in $G(f)$ with one edge in $0, 1, \dots, k-1$ is at least $n - (n-k) = k$. Thus $\left|\{i : a_i \le k\}\right| \ge k$. Therefore by Observation 2.47, we see that $(a_1, a_2, \dots, a_n)$ is a parking function.

The reverse direction of the bijectivity of $L : \mathcal{F}_n \to \mathcal{P}_n$ is explicitly proved in [5]. We give an algorithm to find the factorization in $\mathcal{F}_n$ corresponding to a parking function in $\mathcal{P}_n$ and discuss why it works in Section 2.6.

The bijectivity of $U : \mathcal{F}_n \to \mathcal{M}_n$ observed by Irving-Rattan [19], follows from bijectivity of L and the involution on $\mathcal{F}_n$ that first swaps i and $n-i$ terms and then reverses the order of the transpositions. $\square$

Thus, from a factorization, it is trivial to find the associated parking function under L. Determining the factorization from the parking function under L^{-1} is more difficult. As previously mentioned in the proof, and we address this in Section 2.6.

For the factorization $f = (a_1\ b_1)(a_2\ b_2)\ldots(a_n\ b_n) \in \mathcal{F}_n$, we have $p = (a_1, a_2, \ldots, a_n) \in \mathcal{P}_n$, and we call the major sequence $(b_1, b_2, \ldots, b_n)$ the *mate* of p and denote it as $mate(p) \in \mathcal{M}_n$. Similar to the correspondence of parking functions with Dyck paths, we can do the same for mate of a parking function analogously. The corresponding Dyck path of the mate will lie weakly above the line $y = x$ and have labelled east steps.

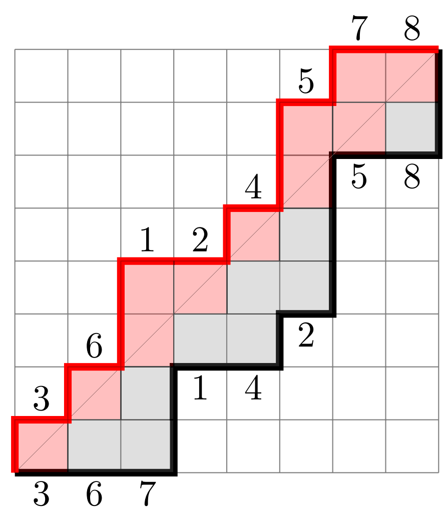


Figure 2.7: The parking function $p = (2, 3, 0, 2, 6, 0, 0, 6)$ with its area (in black) and $mate(p) = q = (4, 4, 1, 5, 7, 2, 8, 8)$ with its area (in red).

2.5.1 The area statistic

The goal of the upcoming two sections is to introduce two well-known statistics on parking functions that are of prime importance in our discussion of q,t-polynomials in Chapter 4.

Recall that the number of vertex labelled trees on n vertices (denoted by $\mathcal{T}_n$) is given by $(n+1)^{n-1}$. This also counts the number of parking functions of length n. Because of this, one naturally expects correspondence between the two. Several people have found such bijections (see [4] and [30]). In fact, substructures of these are equidistributed. We look at the tree inversion statistic, which we denote by $tinv(T)$ (we do not give a definition here) for a tree T. The *inversion enumerator* on trees is given by $I_n(q) \coloneqq \sum_{T \in \mathcal{T}_n} q^{tinv(T)}$. They were introduced by Mallows and Riordan [24] who proved that these satisfy a nice recurrence (which we do not give here). Kreweras later proved that

$$I_n(q) := \sum_{p \in \mathcal{P}_n} q^{\binom{n}{2} - (a_1 + a_2 + \ldots + a_n)}, \tag{2.15}$$

where $p = (a_1, a_2, \ldots, a_n)$. Accordingly, let $p = (a_1, a_2, \ldots, a_n) \in \mathcal{P}_n$ be a parking function of length n. Define the *area* of the parking function p to be

$$area(p) := \binom{n}{2} - \sum_{i=1}^{n} a_i \tag{2.16}$$

The statistic is so named because $area(p)$ is the total number of full squares between the diagonal line $y = x$ and the (labelled) Dyck path d corresponding to the parking function p. See Figure 2.6.

We can also define the area of the mate of p as follows. If $q = (b_1, b_2, \ldots, b_n)$ is the mate of the parking function p, then,

$$area(q) := \sum_{i=1}^{n} b_i - \binom{n}{2} \tag{2.17}$$

Equivalently, $area(q)$ is the total number of squares that lies between the labelled Dyck path d corresponding to the parking function p and the diagonal $y = x$, including the ones that lie on the diagonal.

Example 2.49. For the parking function $p = (2, 3, 0, 2, 6, 0, 0, 6)$ from Figure 2.6,

$$area(p) = \binom{8}{2} - (2 + 3 + 0 + 2 + 6 + 0 + 0 + 6) = 9 \tag{2.18}$$

Also, we have the area vector is $(0, 1, 2, 1, 2, 2, 0, 1)$ and hence

$$area(p) = 0 + 1 + 2 + 1 + 2 + 2 + 0 + 1 = 9 \tag{2.19}$$

For the parking function p, its mate is $q = (4, 4, 1, 5, 7, 2, 8, 8)$.

$$area(q) = (4 + 4 + 1 + 5 + 7 + 2 + 8 + 8) - \binom{8}{2} = 11 \tag{2.20}$$

which is also evident from Figure 2.7.

Definition 2.50. For a factorization $f \in \mathcal{F}_n$, define $\text{area}_L(f) := \text{area}(L(f))$ and $\text{area}_U(f) := \text{area}(U(f))$.

Consider the q,t-polynomial $\mathcal{F}_n^{area}(q,t) \coloneqq \sum_{f\in\mathcal{F}_n} q^{area_L(f)}t^{area_U(f)}$. From (2.15) and Theorem 2.48, we have $\mathcal{F}_n^{area}(q,t)$ with $t=1$ is the inversion enumerator. It turns out that there is another statistic on trees which is connected to the area statistics called the *coinversions* (denoted by *coinv*) for a vertex labelled tree on n vertices. They were introduced and studied by Irving and Rattan [19]. In their work, they show that the joint distribution of the tree inversions (*tinv* and *coinv*) is the same as the joint distribution of lower and upper area statistics over all factorizations. That is,

$$\sum_{T\in\mathcal{T}_n} q^{tinv(T)}t^{coinv(T)} = \sum_{f\in\mathcal{F}_n} q^{area_L(f)}t^{area_U(f)}. \tag{2.21}$$

While we do not go into detail about tree inversions here, the result in (2.21) shows the joint area statistics have an interesting combinatorial connections with trees. In this thesis we will be interested in the joint area statistics in (2.21) over subsets of minimal factorizations. One pivotal subset of the minimal factorizations $\mathcal{F}_n$ is the set of all increasing factorizations. We define them next.

Definition 2.51. Let $f=(a_1\ b_1)(a_2\ b_2)\dots(a_n\ b_n)\in\mathcal{F}_n$ be a minimal factorization of $c=(0\ 1\ \dots\ n)$, with $a_i<b_i$ for $1\le i\le n$. We say f is an *increasing factorization* if $a_1\le a_2\le\cdots\le a_n$. We denote the set of all increasing factorizations of c by $\mathcal{F}_n^{inc}$.

We can then define the following q,t-polynomial.

Definition 2.52. For $n\ge 1$, define

$$\mathcal{F}_n^{inc}(q,t) \coloneqq \sum_{f\in\mathcal{F}_n^{inc}} q^{\mathrm{area}_L(f)}t^{\mathrm{area}_U(f)}.$$

It transpires that $\mathcal{F}_n^{inc}(q,t)$ satisfies the definition of q,t-Catalan polynomial in Definition 2.41, so it is also connected to $\mathcal{F}_n^{max}(q,t)$. We give a direct proof of this in Section 3.1.

2.5.2 The dinv statistic

This section deals with the introduction of the *dinv* statistic on parking functions.

Definition 2.53. Given a (labelled or unlabelled) Dyck path d, let $a_i(d)$ denote the number of full squares in column i between the diagonal $y=x$ and d. We call $a_i(d)$ as the *length of the column* i and call $(a_1(d),a_2(d),\dots,a_n(d))$ the *area vector* of d. Hence $\sum_i a_i(d)$ gives the area of the Dyck path d.

There are two definitions of the dinv statistic. One is for Dyck paths and another is for parking functions. We will define them both.

Definition 2.54. Let $d \in \mathcal{D}_n$ be a Dyck path of length n. Define,

$$\begin{aligned} dinv(d) = &\big|\{(i,j) : 1 \le i < j \le n \quad a_i(d) = a_j(d)\}\big| \\ &+ \big|\{(i,j) : 1 \le i < j \le n, \quad a_i(d) = a_j(d) + 1\}\big|, \end{aligned} \tag{2.22}$$

where $(a_1(d), a_2(d), \ldots, a_n(d))$ is the area vector corresponding to a Dyck path d as defined in Definition 2.53.

The pairs (i, j) that contribute to $dinv(d)$ are called *diagonal inversion pairs*. This statistic counts the number of pairs of columns of d that have the same length or that differ by length one, with the longer column to the left of the shorter ones.

Example 2.55. For the underlying Dyck path of the parking function p in Example 2.49 (also refer Figure 2.7), the area vector is $(0, 1, 2, 1, 2, 2, 0, 1)$. The inversion pairs are $(1, 7), (2, 4), (2, 8), (4, 8), (3, 5), (3, 6), (5, 6)$ (corresponding to columns having same length) and $(2, 7), (4, 7), (3, 4), (3, 8), (5, 8), (6, 8)$ (corresponding to columns having lengths that differ by one) and $dinv(d) = 13$.

We extend the definition of dinv for Dyck paths to parking functions of length n as follows. Recall that we can associate a labelled Dyck path to a parking function as in Figure 2.6, and we say $label(j) = i$ if i is the horizontal step in column j. For example, for the parking function in Figure 2.6, we have $label(3) = 7$, and $label(4) = 1$.

Definition 2.56. Let $p \in \mathcal{P}_n$ be a parking function of length n.

$$\begin{aligned} dinv(p) = &\big|\{(i,j) : 1 \le i < j \le n \quad a_i(p) = a_j(p) \text{ and } label(i) < label(j)\}\big| \\ &+ \big|\{(i,j) : 1 \le i < j \le n, \quad a_i(p) = a_j(p) + 1 \text{ and } label(i) > label(j)\}\big|. \end{aligned} \tag{2.23}$$

Example 2.57. For the parking function $p = (2, 3, 0, 2, 6, 0, 0, 6)$ from Example 2.49 and Figure 2.7), the inversion pairs are $(1, 7), (2, 8), (4, 8), (3, 4), (2, 7)$ and so $dinv(p) = 5$.

We have now seen the area statistic and dinv statistic. The following theorem from [17] gives a nice relation between the two.

Theorem 2.58. *The distributions of area and dinv are the same over all parking functions. That is,*

$$\sum_{p \in \mathcal{P}_n} q^{area(p)} = \sum_{p \in \mathcal{P}_n} q^{dinv(p)}$$

One natural question that arises in the theory of multivariate polynomials is about the symmetry of the polynomials. The q, t-polynomial $\mathcal{A}_n(q, t) := \sum_{p \in \mathcal{P}_n} q^{area(p)} t^{dinv(p)}$ is symmetric in q and t. An elementary combinatorial proof for the symmetry of $\mathcal{A}_n(q, t)$ is still an open problem (see [17]). Loehr, in his thesis, details about the *Hilbert series* of *diagonal harmonics* (denoted by $H_n(q, t)$). He mentions that the q, t-polynomial $\mathcal{A}_n(q, t)$

is conjectured to be same as $H_n(q,t)$. Then the symmetry of $H_n(q,t)$ implies proof of the theorem. That is, $\mathcal{A}_n(q,t) = \mathcal{A}_n(t,q)$ and thus $\mathcal{A}_n(q,1) = \mathcal{A}_n(1,q)$. These topics are beyond the scope of this thesis and we refer the reader to [23]. We also direct the reader to [8] by Carlsson and Mellit for the proof of symmetry of $\mathcal{A}_n(q,t)$, and [18] by Haglund and Xin for an exposition of the proof by Carlsson and Mellit.

Our interest in the *dinv* statistic is because of its connection to the *reading word factorization*. We study more combinatorics about them in Chapter 3.

2.6 Parking functions and factorizations

We have seen that there is a bijection between parking functions and factorizations of full cycles (Theorem 2.48). When given a factorization, finding the corresponding parking function is trivial, the converse - finding the factorization corresponding to a parking function - is less obvious. In this section, we provide a method of constructing this factorization of the full cycle $c = (0\ 1\ \dots\ n)$. Let $f = (\tau_1, \tau_2, \dots, \tau_n)$ be a factorization of c and $G(f)$ be the vertex-labelled and edge-labelled graph on $0, 1, \dots, n$ with $\{a, b\}$ as an edge labelled i if $\tau_i = (a\ b)$. Recall from Theorem 2.2 that $G(f)$ is a tree.

Theorem 2.59. *There is a unique embedding of $G(f)$ in the plane with (1) vertex i at $(i, 0)$, (2) edges drawn above the x-axis such that the edges do not cross, and (3) increasing rotators at each vertex, where the* rotator *at a vertex is the counter clockwise list of edges in order beginning at x-axis.*

More generally, define an *arch diagram* of size n to be a tree with edges labelled $1, 2, \dots, n$ embedded in the plane such that

1. its vertices are at $(0,0), (1,0), \dots, (n,0)$;
2. all edges are drawn above the x-axis such that they do not cross;
3. the *rotators* at a vertex v - that is, the list of edges encountered beginning at the x-axis and moving counter-clockwise around v - are all increasing.

Let $\mathbb{A}_n$ be the set of arch diagrams of size n. We have the following theorem.

Theorem 2.60 (Goulden-Yong [14])**.** *Given an arch diagram $\mathcal{A} \in \mathbb{A}_n$, define a sequence of transpositions $\tau_1, \tau_2, \dots, \tau_n$ by $\tau_i = (a\ b)$ if edge i is between $(a, 0)$ and $(b, 0)$. Then this construction defines a bijection $F : \mathbb{A}_n \to \mathcal{F}_n$.*

Let A be the inverse of F. The fact that this function A (or F) is well-defined and bijective is not obvious. The phrasing above is due to Irving-Rattan [19], however they state that their arch diagrams are essentially Goulden and Yong's "circle chord" diagrams.

Our interest in arch diagrams is that the inverse of $L : \mathcal{F}_n \to \mathcal{P}_n$ is most easily described by building the arch diagram from a parking function.

Remark 2.61. We can describe a bijection $\mathcal{F}_n \longrightarrow \mathcal{T}_n$, where $\mathcal{T}_n$ is the set of labelled trees on $[n]$. Let $f \in \mathcal{F}_n$. By Theorem 2.60, we can find $A(f)$ which is the arch diagram corresponding to the factorization f. Remove all vertex labels except the vertex 0. Now "push" all edge labels on to the vertices in the direction away from the vertex 0. This gives a map $PUSH\colon \{A(f) : f \in \mathcal{F}_n\} \longrightarrow \mathcal{T}_n$. $PUSH$ is bijective, although this is not obvious. This correspondence is described in detail in Irving-Rattan [19], Goulden-Yong [14], but this is essentially Moszkowski's bijection [27]. Thus, this gives another proof that $|\mathcal{F}_n| = (n+1)^{n-1}$.

Example 2.62. For $f = ((9\ 10), (0\ 3), (5\ 8), (1\ 2), (6\ 8), (1\ 3), (4\ 5), (7\ 8), (0\ 9), (4\ 9)) \in \mathcal{F}_{10}$, the unique arch diagram of f is given in Figure 2.8.

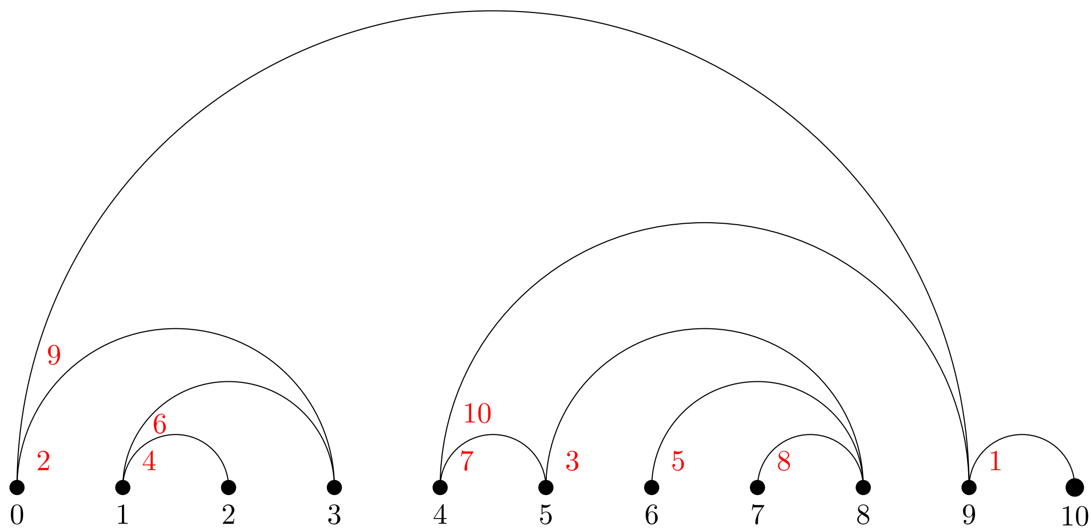


Figure 2.8: Arch diagram $A(f)$ for $f = (\underset{1}{(9\ 10)}, \underset{2}{(0\ 3)}, \underset{3}{(5\ 8)}, \underset{4}{(1\ 2)}, \underset{5}{(6\ 8)}, \underset{6}{(1\ 3)}, \underset{7}{(4\ 5)}, \underset{8}{(7\ 8)}, \underset{9}{(0\ 9)}, \underset{10}{(4\ 9)}) \in \mathcal{F}_{10}$. Note that this is the tree as mentioned in Theorem 2.59.

Notice that the arch diagram is easily obtained from a factorization, and vice-versa. In order to find L^{-1} - that is, produce a factorization from the parking functions - we give an algorithm to find the arch diagram from a parking function, thus finding L^{-1}.

Algorithm 2.63. Let $p = (a_1, a_2, \ldots, a_n)$ be a parking function of length n.

- Initialize:
  - Arrange the vertices at the points $(0, 0), (1, 0), \ldots, (n, 0)$.
  - For each $a_i \in p$ for $1 \le i \le n$, attach a half-edge with label i at $(a_i, 0)$.
  - $(n, 0)$ is the last available vertex. Since $0 \le a_i \le n - 1$, no half-edge begins at $(n, 0)$.

– If $a_i = a_j$, then the half-edges are attached such that the rotator at the vertex is increasing (i.e. the half-edge j comes after i when reading the edges counter-clockwise around $(a_i, 0)$ beginning at x-axis).

- Iterate:

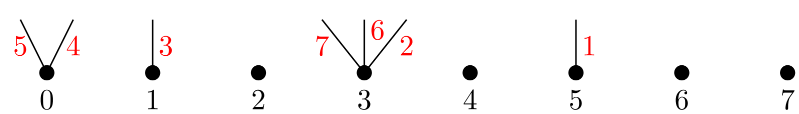
  – Let v be the largest vertex with at least one incident half-edge. If there is no such v, we terminate the algorithm.

  – Extend the half-edge with smallest label at v to connect it with the first possible vertex available on the right with the conditions that (1) the rotators remain increasing when the edge is complete and (2) the edges do not cross and the underlying graph has no cycles.

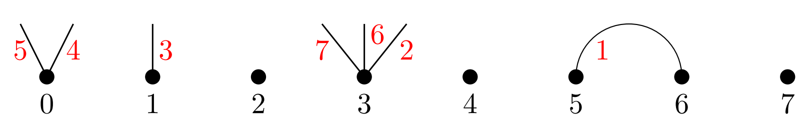
Theorem 2.65 gives the working of the algorithm. A more general version of the algorithm can be found in [19], where the authors give the algorithm to find the arch diagram given a parking function p and a *unimodal cycle* σ. The resulting factorization has product σ. We say a full cycle $(0\ s_1\ \dots\ s_n)$ is *unimodal* if there exists some N such that $s_N = n$ and $0 < s_1 < s_2 < \cdots < s_N > s_{N+1} > s_{N+2} > \cdots > s_n$. It is best to give an example to show step by step how the algorithm works. We only consider the canonical full cycle. Notice that this also produces the mate of p. We give an example before the sketch of the proof.

Example 2.64. Let us find the mate of $p = (5, 3, 1, 0, 0, 3, 3)$. We will follow the steps in the algorithm in order.

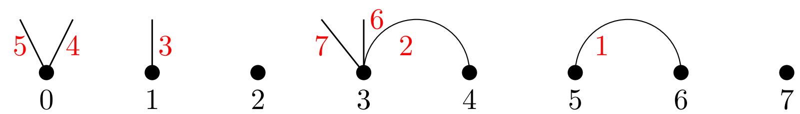
Step 1: Initializing

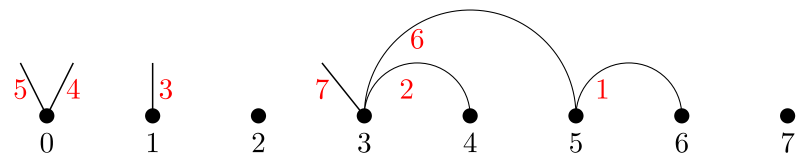


Step 2: Iteration

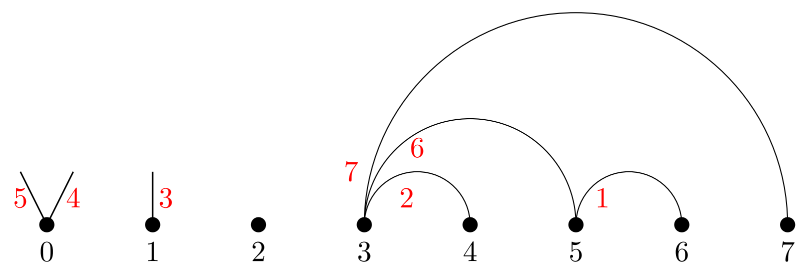


The first available half-edge is at vertex 5. The next available vertex which is edge-free is 6, so we complete the edge.

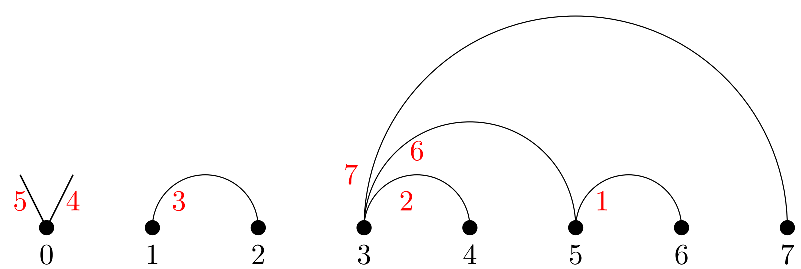




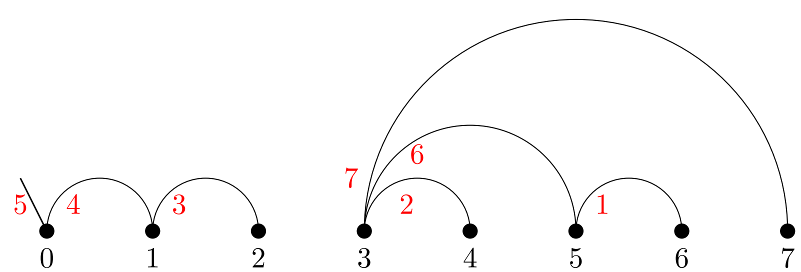
The edge 6 cannot end at 4 as it creates a parallel edge. So it looks at the next available vertex, which is 5. The edge 6 can be completed at vertex 5 because once the edge is completed, the rotator at vertex 5 is increasing.



Edge 7 has the possibility of ending at vertex 5, but then the rotator at vertex 5 will not be increasing and also this would not be a tree. Similarly it cannot end at vertex 6 for the same reason. So it ends at vertex 7.



Edge 3 ends at vertex 2 because vertex 2 is free of edges, and it is the first possible vertex available.



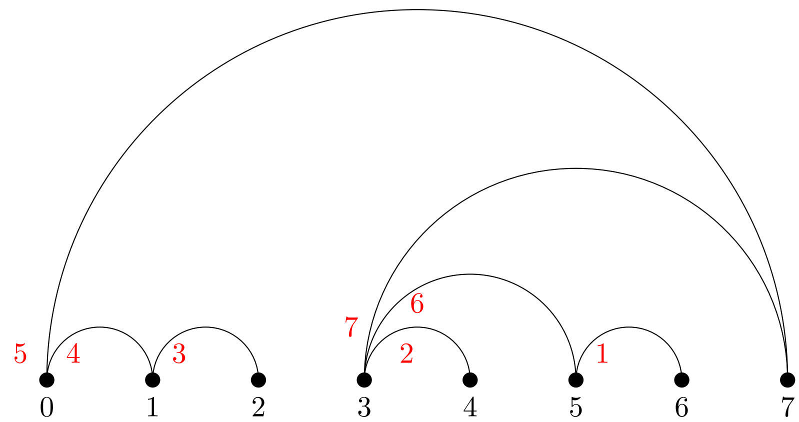


Figure 2.9: Graphical illustration of finding the mate of $p = (5, 3, 1, 0, 0, 3, 3)$. This produces $mate(p) = (6, 4, 2, 1, 7, 5, 7)$ and factorization $f = ((5\ 6), (3\ 4), (1\ 2), (0\ 1), (0\ 7), (3\ 5), (3\ 7))$.

Thus the corresponding factorization of p is

$$f = ((5\ 6), (3\ 4), (1\ 2), (0\ 1), (0\ 7), (3\ 5), (3\ 7)).$$

Theorem 2.65. *Algorithm 2.63 gives the arch diagram of f.*

Sketch of proof. Let $p = (a_1, a_2, \ldots, a_n)$ be a parking function. The iteration step always completes a half-edge (if any exists), or the algorithm terminates. We show that the algorithm can always complete an edge at the i^{th} step if $i < n$.

Base case: When $i = 0$, no edges have been completed, so we are considering the least half-edge on the right most vertex. This first edge can be completed because $max(p) = n-1$. So we can always complete the edge.

Now suppose that we are at the $i + 1$ stage of the iteration and $i < n$, so i edges have been completed. By induction, the completed edges form an acyclic graph with non-crossing edges satisfying the increasing rotator condition.

Suppose that the half-edge labelled j is being considered and is at vertex v. By Observation 2.47, we have $v \leq n - i - 1$. So $\left|\{v, \ldots, n\}\right| \geq n - (n - i - 2) = i + 2$. This implies that there are i completed edges and at least $i + 2$ vertices $v, \ldots, n$. Thus the completed edges on $v, \ldots, n$ comprise a forest with at least two components, and with non-crossing edges satisfying the increasing rotator condition.

Consider the left-most component to the right of vertex v whose left-most vertex it r and the right-most vertex is t. It is pictured in Figure 2.10. Note that the path between r and t may not have length two, but the argument in the contrary case is the same as when the path has length two. Here i_1 and i_2 are the outermost edge labels, and we have $i_2 < i_1$ (from the increasing rotator condition). Now the label j on our half-edge is such that either $j > i_1$, or $i_1 > j > i_2$, or $i_2 > j$. If $j > i_1$, then connect v to r; or if $i_1 > j > i_2$, then connect

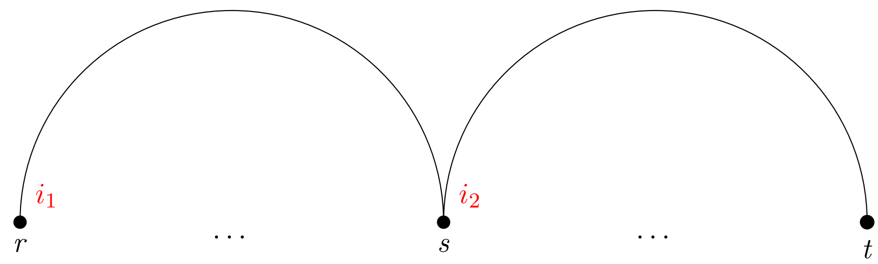


Figure 2.10: The left-most component to the right of vertex v.

v to s; or if $i_2 > j$, connect v to t. The correct choice preserves increasing rotators and the non-crossing criteria. Note that the new edge connects two different connected components by construction, and thus it does not create a cycle. Thus the $i+1$ step can be completed. This completes the proof. □

Corollary 2.66. *For any $f \in \mathcal{F}_n$, if $(0\ 1)$ is one of the factors of f, then it is the leftmost (first) factor that contains 0. Similarly, if $(0\ n)$ is one of the factors of f, then it is the rightmost (last) factor that contains 0.*

This follows from the increasing rotator condition at vertex 0 from Theorem 2.59. Since the rotators are increasing, if $(0\ 1)$ is one of the factors of f, then it will be the first edge encountered counter-clockwise at 0, and similarly if $(0\ n)$ is one of the factors of f, then it will be the last edge encountered counter-clockwise at 0.

In this chapter, we discussed maximal factorizations. One can immediately ask if there are other subsets of factorizations that are interesting. In the next chapter we answer this question.

Chapter 3

Some factorizations and similarities among them

We introduce two other collections of factorizations of the canonical full cycle, *increasing factorizations* and *reading word factorizations*. We refer to the work of Gewurz and Merola [13], Irving and Rattan [19], and Haglund [17]. In the final section, we will see some connections among the different collection of factorizations.

3.1 Increasing factorizations

Recall $c = (0\ 1\ \ldots\ n) \in \mathfrak{S}_{[n]}$ is the canonical full cycle, and $\mathcal{F}_n^{inc}$ denotes the set of all increasing factorizations as defined in Definition 2.51. Obviously $\mathcal{F}_n^{inc} \subseteq \mathcal{F}_n$. As mentioned in Chapter 1, these are also counted by the Catalan number as the next proposition gives.

Proposition 3.1. *For a given positive integer n, the number of increasing factorizations of the full cycle $c = (0\,1\ \ldots\ n)$ is counted by the Catalan number C_n.*

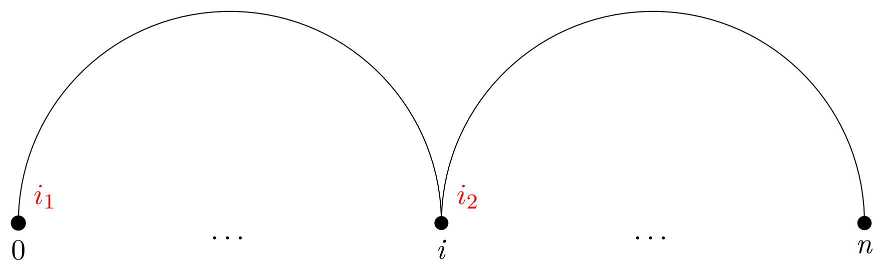
Proof. This proposition was proved explicitly in [13]. Recall that $L : \mathcal{F}_n \longrightarrow \mathcal{P}_n$ in (2.13) is a bijection. Hence it can be also proved via the fact that the number of (weakly) increasing parking functions are easily interpreted as unlabelled ordinary Dyck paths and Theorem 2.46. $\square$

Example 3.2. For $n = 3$, we have $c = (0\,1\,2\,3)$ and $\mathcal{F}_3^{inc} = \{(0\,1)(0\,2)(0\,3), (0\,2)(0\,3)(1\,2), (0\,1)(0\,3)(2\,3), (0\,3)(1\,2)(1\,3), (0\,3)(1\,3)(2\,3)\}$. Note that $|\mathcal{F}_3^{inc}| = 5 = C_3$.

We now make the following two observations. Recall Theorem 2.59 and Corollary 2.66.

Observation 3.3. If $(0\ n)$ is the first factor of any factorization $f \in \mathcal{F}_n$, then it is the only factor containing 0 since the rotator at vertex 0 has to be increasing (Theorem 2.59), and the edge corresponding to $(0\ n)$ in the arch diagram $A(f)$ will be the last edge containing 0. Because $(0\ n)$ is the first factor, it follows that it is the only factor that contains 0.

Observation 3.4. For any factorization $f \in \mathcal{F}_n^{inc}$, the transposition $(0\ n)$ is a factor of f. This is in fact easy to see by looking at the arch diagram of f. Suppose that there is an increasing factorization which does not have $(0\ n)$ as a factor. Then the only possibility for the arch diagram is to have the following form. Here i_1 and i_2 are labels of the edges. Note that what is pictured is the unique path from 0 to n. The picture assumes that the path has length 2, but the argument works for any path length from 0 to n. The key is $i_2 < i_1$. We give the argument for two archs, but the general case is the same.



If you look at vertex i, by the increasing rotator condition for arch diagrams $i_2 < i_1$, but this implies that $(i\ n)$ must occur before $(0\ i)$ in f. But this means that f will not be an increasing factorization. Thus $(0\ n)$ must be a factor of any $f \in \mathcal{F}_n^{inc}$.

Recall the q, t-polynomial defined in Equation (2.52). We have the following theorem.

Theorem 3.5 (Irving-Rattan [19])**.** *For all* $n \geq 1$*,*

$$\mathcal{F}_n^{inc}(q,t) = \sum_{k=0}^{n-1} q^k t^{n-k} \mathcal{F}_k^{inc}(q,t)\mathcal{F}_{n-k-1}^{inc}(q,t), \text{with } \mathcal{F}_0^{inc}(q,t) = 1. \tag{3.1}$$

from which we have $\mathcal{F}_n^{inc}(q,t) = t^n C_n(q,t)$*.*

Proof. Let $f \in \mathcal{F}_n^{inc}$. By Observation 3.4, the transposition $(0\ n)$ is a factor of f. Let us see the effect of removing $(0\ n)$ from the factorization f. While the arch diagram in Figure 2.9 is not that of an increasing factorization, it is still illustrative to look at nonetheless. Removing the arch corresponding to $(0\ n)$ gives us two arch diagrams, one on the vertices $(0, 1, \ldots, k)$ for some k, and the other on the vertices $(k+1, k+2, \ldots, n)$ since the underlying graph of an arch diagram is a tree. These correspond to two factorizations: f_1, which is a factorization of $(0\ 1\ \ldots\ k)$ and f_2, which is a factorization of $(k+1\ k+2\ \ldots\ n)$. So $f_1 \in \mathcal{F}_k^{inc}$. Note that $f_2 \notin \mathcal{F}_{n-k-1}^{inc}$, but if we subtract $k+1$ from each element of f_2, we get $\tilde{f}_2 = (0\ 1\ \ldots\ n-k-1)$, which does belong to $\mathcal{F}_{n-k-1}^{inc}$.

This sets up a function α such that

$$f \mapsto (f_1, \tilde{f}_2) \in \mathcal{F}_k^{inc} \times \mathcal{F}_{n-k-1}^{inc}. \tag{3.2}$$

It can be observed that the map α is well-defined and injective. To see α is surjective, we construct the inverse map. Given any element $(f_1, \tilde{f}_2) \in \mathcal{F}_k^{inc} \times \mathcal{F}_{n-k-1}^{inc}$, one can always add

$k+1$ to each element of $\tilde{f}_2$ to get say f_2. Replacing the factor $(0\ n)$ needs some care. Suppose that we start with $(f_1, (0\ n), f_2)$. This is not an increasing factorization because $(0\ n)$ is not in the correct place. Note that the product of $f_1, (0\ n), f_2$ is $(0\ 1\ \ldots\ n)$. Since f_1 contains no factor of n, the transposition $(0\ n)$ commutes with all factors of f_1 except the factors containing 0, which occur as the first few factors of f_1. Thus if $\tilde{f}_1$ is f_1 with $(0\ n)$ occurring after all factors with 0, then the product $\tilde{f}_1, f_2 = (0\ 1\ \ldots\ n)$. Thus $(\tilde{f}_1, f_2) \in \mathcal{F}_n^{inc}$, and clearly $\alpha((\tilde{f}_1, f_2)) = (f_1, \tilde{f}_2)$.

Comparing Definition 2.41 to (3.1), it now suffices to show that

$$\begin{aligned}\text{area}_L(f) &= \text{area}_L(f_1) + \text{area}_L(\tilde{f}_2) + k \text{ and}\\ \text{area}_U(f) &= \text{area}_U(f_1) + \text{area}_U(\tilde{f}_2) + n - k.\end{aligned}$$

We show the first. Let $f = ((a_1\ b_1), (a_2\ b_2), \ldots, (a_n\ b_n))$ and let $(0\ n)$ be the t-th factor. That is, $(a_t\ b_t) = (0\ n)$.

$$\begin{aligned}\text{area}_L(f) &= \binom{n}{2} - \sum_{i=1}^{n} a_i \\ &= \binom{k}{2} - \sum_{i=1, i\neq t}^{k+1} a_i + \left(\left(\binom{n}{2} - \binom{k}{2}\right) - \sum_{i=k+2}^{n} a_i\right) - \underbrace{a_t}_{=0} \\ &= \text{area}_L(f_1) + \binom{n}{2} - \binom{k}{2} - \sum_{i=k+2}^{n} (a_i - (k+1)) - (n-k-1)(k+1) - \underbrace{a_t}_{=0} \\ &= \text{area}_L(f_1) + k(n-k) + (n-k-1) + \left(\binom{n-k-1}{2} - \sum_{i=k+2}^{n} (a_i - (k+1))\right) \\ &\qquad - (n-k-1)(k+1) \\ &= \text{area}_L(f_1) + \text{area}_L(\tilde{f}_2) + k.\end{aligned} \tag{3.3}$$

Similarly, we can show that

$$\text{area}_U(f) = \text{area}_U(f_1) + \text{area}_U(\tilde{f}_2) + n - k. \tag{3.4}$$

Note that $\mathcal{F}_n^{inc}(q,t)$ is not the same as $C_n(q,t)$. This is because of the difference in the power of t in the recursive sum in Definition 2.41 and (3.1). Note that the power of t in Definition 2.41 is $n-k-1$, whereas in Equation (3.1), the power of t is $n-k$. Thus $\mathcal{F}_n^{inc}(q,t) = t^n C_n(q,t)$. □

3.2 Reading word factorizations

In this section, we define the reading word of a parking function and study reading word factorizations. In the next section, we shall see a number of facts about these factorizations, and we give a new conjecture that connects a subset of these factorizations to some specific Dyck paths that are counted by a sequence in OEIS with very few combinatorial interpretations.

Definition 3.6 (Reading word of a parking function)**.** Let p be a parking function and d be the associated labelled Dyck path. We define the reading word of p to be the permutation obtained by reading the labels in d along diagonals in SW direction starting from the diagonal farthest from the line $y = x$ and then moving closer to the line $y = x$. We denote it by $read(p)$.

Example 3.7. For the parking function p in Figure 2.7, we have $read(p) = [2, 4, 7, 8, 1, 6, 5, 3]$.

We can easily show that $read(p) = [n, n-1, \ldots, 2, 1]$ if and only if $dinv(p) = dinv(d)$, where d is the unlabelled Dyck path corresponding to p.

Comparing the definitions (2.22) and (2.23) of $dinv(d)$ and $dinv(p)$, we have the following. Let d_1 and d_2 denote the two sets in (2.22) and let d_1' and d_2' denote the two sets in (2.23). Notice that $dinv(p) = dinv(d)$ if and only if we have $|d_1| + |d_2| = |d_1'| + |d_2'|$. This means that $d_1 = d_1'$ and $d_2 = d_2'$ (because $d_1' \subseteq d_1$ and $d_2' \subseteq d_2$). If $(i, j) \in d_1$, then $a_i = a_j$ and $i < j$. Then $(i, j) \in d_1'$ if and only if $label(i) < label(j)$. On the other hand, if $(i, j) \in d_2$, we have $a_i = a_j + 1$ and $i < j$. Then $(i, j) \in d_2'$ if and only if $label(i) > label(j)$. These are true if and only if the smallest label in the diagonal farther from the diagonal $y = x$ is greater than the largest element in the diagonal closer to $y = x$. This is true if and only if the reading word of p is $[n, n-1, \ldots, 2, 1]$. Observe that no two parking functions with the same unlabelled Dyck path have the same reading word.

Because of this we have that the number of parking functions with reading word $=$ $[n, n-1, \ldots, 2, 1]$ is also counted by Catalan number. We can find the factorizations corresponding to these kind of parking functions using Algorithm 2.63 and call them *reading word factorizations*. Denote such factorizations of length n by $\mathcal{F}_n^{read}$.

We have now looked at maximal factorizations, increasing factorizations and reading word factorizations in the previous sections. Let us denote the set of parking functions corresponding to these factorizations by $\mathcal{P}_n^{max}, \mathcal{P}_n^{inc}$ and $\mathcal{P}_n^{read}$ respectively. In the following section, we are interested in some special factorizations that are subsets of the above mentioned sets and we will enumerate them. This gives some nice similarities among the three different collection of factorizations.

3.3 Some similarities among the different collection of factorizations.

This section contains original propositions that we prove one by one. We also give a conjecture about a specific collection of reading word factorizations at the end of the section.

Proposition 3.8. *Let n be a positive integer. Then we have the following.*

1. *The number of factorizations $f \in \mathcal{F}_n^{inc}$ such that $(0\ n)$ is the first factor is C_{n-1}.*
2. *The number of factorizations $f \in \mathcal{F}_n^{max}$ such that $(0\ n)$ is the first factor is C_{n-1}.*
3. *The number of factorizations $f \in \mathcal{F}_n^{read}$ such that $(0\ n)$ is the first factor is C_{n-1}.*

Before we begin the proof of these propositions, we recall Observation 3.3.

Proof of Proposition 3.8. *1.* Let Ω_n^{inc} be the subset of $\mathcal{F}_n^{inc}$ such that the first factor is $(0\ n)$. We give a bijection between the two sets Ω_n^{inc} and $\mathcal{F}_{n-1}^{inc}$. Given a factorization $f \in \Omega_n^{inc}$, by Observation 3.3, it has to be the only factor that contains 0. If we remove the factor $(0\ n)$, we will get a factorization $f' \in \mathcal{F}_{n-1}^{inc}$ such that f' is a factorization of $(1\ 2\ \dots\ n)$. Note that f' is increasing. Conversely, given any factorization $f' \in \mathcal{F}_{n-1}^{inc}$, shift up each element by 1, so that we get a factorization of $(1\ 2\ \dots\ n)$ instead of $(0\ 1\ \dots\ n-1)$. Now if we prepend this new factorization of $(1\ 2\ \dots\ n)$ with the factor $(0\ n)$, we indeed get a factorization $f \in \Omega_n^{inc}$ since $(0\ n)$ is the only factor of f with 0, and it is the first factor. This establishes a bijection between Ω_n^{inc} and $\mathcal{F}_{n-1}^{inc}$ as claimed.

2. The idea of this proof is similar to the first one except for one complication that we will discuss below. Let Ω_n^{max} be the subset of $\mathcal{F}_n^{max}$ such that the first factor is $(0\ n)$. We once again give a bijection between Ω_n^{max} and $\mathcal{F}_{n-1}^{max}$. Let $f \in \Omega_n^{max}$. By Observation 3.3, the transposition $(0\ n)$ has to be the only factor that contains 0. If we remove the factor $(0\ n)$, we obtain a factorization f' of the cycle $(1\ 2\ \dots\ n)$. In contrast to *1*, it is not obvious that this is still a maximal factorization, so we should verify that it is a maximal factorization. But note that for f, we have $\phi(f) = (0, 1, \dots, n-1)$. Since we have removed the first factor, which is the only factor that contains a 0, and f' is a factorization of the cycle $(1\ 2\ \dots\ n)$, we can conclude that $\phi(f') = (1, 2, \dots, n-1)$ and thus $f' \in \mathcal{F}_{n-1}^{max}$. Conversely suppose that $f' \in \mathcal{F}_{n-1}^{max}$ be a maximal factorization of the cycle $(0\ 1\ \dots\ n-1)$. Shift up each element by 1 so that we obtain a factorization of $(1\ 2\ \dots\ n)$. Note that $\phi(f')$ is also shifted by 1 to get $(1, 2, \dots, n-1)$. Let us add the factor $(0\ n)$. Once again this gives us a factorization $f \in \Omega_n^{max}$. This is because $(0\ n)$ is the only factor with a 0 and it has to be the first factor. f is indeed maximal because we get $\phi(f) = \phi((0\ n), f') = (0, 1, \dots, n-1)$.

3. Let Ω_n^{read} be the subset of $\mathcal{F}_n^{read}$ such that the first factor is $(0\ n)$. As before, we will give a bijection between Ω_n^{read} and $\mathcal{F}_{n-1}^{read}$. Any factorization $f \in \Omega_n^{read}$ contains $(0\ n)$ as the first factor and by Observation 3.3, this has to be the only factor that contains 0. Removing $(0\ n)$ gives us a factorization f' of $(1\ 2\ \dots\ n)$. Also note $read(L(f)) = [n, n-1, \dots, 2, 1]$ and $read(L(f')) = [n, n-1, \dots, 3, 2]$. This confirms that $f' \in \mathcal{F}_{n-1}^{read}$. On the other hand, suppose that $f' \in \mathcal{F}_{n-1}^{read}$. This means that $read(f') = [n, n-1, \dots, 2, 1]$. Once again if we shift the elements up by 1, we get f' as a reading word factorization of $(1\ 2\ \dots\ n)$ such that the reading word becomes $[n+1, n, \dots, 3, 2]$. If we prepend the factor $(0\ n)$, we get a factorization of $(0\ 1\ \dots\ n)$ and since it is the first factor, it is only possible that the reading word becomes $[n, n-1, \dots, 2, 1]$. □

In fact, something more general holds.

Proposition 3.9. *Let $\Omega_{k,n}^{inc}$ be the subset of $\mathcal{F}_n^{inc}$ such that $(0\ k)$ is the first factor of the factorizations in $\mathcal{F}_n^{inc}$. Define $\Omega_{k,n}^{max}$ and $\Omega_{k,n}^{read}$ similarly. Then $|\Omega_{k,n}^{inc}| = |\Omega_{k,n}^{max}| = |\Omega_{k,n}^{read}| = C_{k-1}C_{n-k}$.*

Sketch of proof. If $f = (\tau_1, \tau_2, \dots, \tau_n) \in \mathcal{F}_n$ and $\tau_1 = (0\ k)$, then we have

$$\tau_2\tau_3 \dots \tau_n = (1\ 2\ \dots\ k)(0\ k+1\ \dots\ n),$$

and this factorization is minimal so every factor τ_i has support in $\{1, 2, \dots, k\}$ or $\{0, k+1, \dots, n\}$. Suppose that $\Omega_{k,n}^A \subseteq \mathcal{F}_n^A$ where A refers to *inc* or *max*, then we find a function $\psi_{k,n}^A : \Omega_{k,n}^A \to \mathcal{F}_{k-1}^A \times \mathcal{F}_{n-k}^A$ by considering transpositions in the list $\tau_2, \dots, \tau_n$ whose support is in $\{1, 2, \dots, k\}$. Notice that by the increasing rotator condition for the arch diagram of f, we have $(0\ a)$ is not a factor for $a < k$. Also notice for $i = 2, 3, \dots, n$ we have $\tau_i \neq (a\ b)$ where $0 < a < k < b$ because of the non-crossing condition.

Suppose that $A = inc$ and $(\tau_1, \tau_2, \dots, \tau_n) \in \Omega_{k,n}^{inc}$. Since the factorizations are increasing, there exists i such that $\tau_2, \dots, \tau_i$ each contain a 0, while no other factors contain 0. Then $\tau_{i+1}, \dots, \tau_{i+k-1}$ will all have support in $\{1, 2, \dots, k\}$, while the $n-k$ transpositions $\tau_2, \dots, \tau_i, \tau_{i+k}, \dots, \tau_n$ will have support in $\{0, k+1, \dots, n\}$. By a suitable relabelling, we have $(\tau_{i+1}, \dots, \tau_{i+k-1}) \in \mathcal{F}_{k-1}^{inc}$ and $(\tau_2, \dots, \tau_i, \tau_{i+k}, \dots, \tau_n) \in \mathcal{F}_{n-k}^{inc}$. This defines our function $\psi_{k,n}^{inc}$. If is not difficult to see that this construction is reversible.

Now suppose $A = max$. Recall by Corollary 2.36 that for $m \in \mathcal{F}_n^{max}$, we have $\phi(m) = (0, 1, \dots, n-1)$. We claim the factors $\tau_2, \dots, \tau_k$ all have support in $\{1, 2, \dots, k\}$. This is because of the following. For $i \in \{2, 3, \dots, k\}$, suppose $\tau_i = (a\ b)$. We will look at the possible cases for a and b. We have that $\tau_1 = (0\ k)$ by assumption.

Case 1: If $0 < a < k$ and $b > k$, as noted above, the edges in the arch diagram of the factorization corresponding to τ_1 and τ_i will cross, which is a contradiction. Thus this case cannot happen.

Case 2: If $a = k$ and $b > k$, then we see that the arch diagram of the factorization fails the increasing rotator condition at vertex $a = k$ because of τ_1 and τ_i. So this case cannot happen as well.

Case 3: Suppose $a > k$ and $b > k$. Pick $i \in \{2, 3, \ldots, k\}$ as the least index such that $a > k$ and $b > k$. Then a must occur in position $i - 1$ in $\phi((\tau_1, \ldots, \tau_k))$ because this is the first time a appears in a transposition. For if a appeared in an earlier transposition, then by cases 1 and 2 it follows that $\tau_j = (0\ a)$ for some $j < i$. But if this happens, then the factorization will fail the increasing rotator condition at vertex a because the factor $\tau_j = (0\ a)$ appears before $\tau_i = (a\ b)$. So a occurs in position $i - 1$ in $\phi((\tau_1, \ldots, \tau_k))$. Now since $i \leq k < a$, we see that $\phi((\tau_1, \tau_2, \ldots, \tau_n)) \neq (0, 1, \ldots, n - 1)$, a contradiction. So this case is not possible.

Case 4: If $a = 0$, then b has to be greater that k, because if $b < k$, the arch diagram of the factorization will fail the increasing rotator condition at 0, as observed in the first paragraph. So we have $b > k$. But then if $a = 0$ and $b > k$, then this will force k to appear in one of the position $0, 1, \ldots, k-1$ in $\phi((\tau_1, \tau_2, \ldots, \tau_n))$ since then $0, k, b$ will be in the same cycle in $\tau_1 \ldots \tau_i$, a contradiction.

Thus, the only possibility is that $1 \leq a < b \leq k$. That is, the factors $\tau_2, \ldots, \tau_k$ all have support in $\{1, 2, \ldots, k\}$. And $\tau_{k+1}, \ldots, \tau_n$ has support in $\{0, k + 1, \ldots, n\}$. It can also be seen that $\tau_2, \ldots, \tau_k$ and $\tau_{k+1}, \ldots, \tau_n$ are maximal factorizations of their respective products. Thus $(\tau_2, \ldots, \tau_k) \in \mathcal{F}^{max}_{k-1}$ and $(\tau_{k+1}, \ldots, \tau_n) \in \mathcal{F}^{max}_{n-k}$ after relabelling, giving us an element in $\mathcal{F}^{max}_{k-1} \times \mathcal{F}^{max}_{n-k}$, defining our function $\psi^{max}_{k,n}$.

Consider the case $\Omega^{read}_{k,n} \subseteq \mathcal{F}^{read}_n$. Note that Proposition 3.8 shows $\left|\Omega^{read}_{n,n}\right| = C_{n-1}$. For $k = 1$, we can show that $\left|\Omega^{read}_{1,n}\right| = C_0 C_{n-1} = C_{n-1}$ by a proof similar to the one in Proposition 3.8. To show $\left|\Omega^{read}_{k,n}\right| = C_{k-1} C_{n-k}$ for $k \geq 2$, we construct a function $\psi^{read}_{k,n}$: $\Omega^{read}_{k,n} \to \mathcal{F}^{read}_{k-1} \times \Omega^{read}_{1,n-k+1}$ as follows. Let B be the set of indices $i \in \{2, 3, \ldots, n\}$ such that τ_i has support in $\{1, 2, \ldots, k\}$, and let B' be the set of indices $j \in \{2, 3, \ldots, n\}$ such that τ_j has support in $\{0, k + 1, \ldots, n\}$. Note that $B \cup B' = \{2, 3, \ldots, n\}$. Now we have

$$\begin{aligned}\prod_{j \in \{1\} \cup B'} \tau_j &= (0\ k)(0\ k+1\ \ldots\ n) \\ &= (0\ k\ k+1\ \ldots\ n).\end{aligned}$$

For all $j \in \{1\} \cup B'$ let τ'_j be the transposition obtained from τ_j by subtracting $k - 1$ from each element. We omit the details, but it follows that $\prod_{j \in \{1\} \cup B'} \tau'_j \in \Omega^{read}_{1,n-k+1}$. We also have $\prod_{i \in B} \tau_i \in \mathcal{F}^{read}_{k-1}$. Thus we obtain an element in $\mathcal{F}^{read}_{k-1} \times \Omega^{read}_{1,n-k+1}$, and this defines our function $\psi^{read}_{k,n}$.

The fact that the above constructions are reversible are details left to the reader. □

We make the following observation which will be helpful.

Observation 3.10. If (0 1) is the first factor of any factorization $f \in \mathcal{F}_n$, then it is the only factor containing 1 since the rotator at vertex 1 has to be increasing (Theorem 2.59), and the edge corresponding to (0 1) will be the last edge in the rotator. Because (0 1) is the first factor, it follows that it is the only factor that contains 1.

Proposition 3.11. *Let n be a positive integer. Then we have the following.*

7. *The number of factorizations $f \in \mathcal{F}_n^{max}$ such that $(0\ n)$ is the last factor is C_{n-1}.*

8. *The number of factorizations $f \in \mathcal{F}_n^{inc}$ such that $(n-1\ n)$ is the last factor is C_{n-1}.*

9. *The number of factorizations $f \in \mathcal{F}_n^{max}$ such that $(n-1\ n)$ is the last factor is C_{n-1}.*

Due to the increasing rotator condition on the vertex n, any factorization $f \in \mathcal{F}_n$ that contains $(0\ n)$ as the last factor has $(0\ n)$ as the unique factor containing n. The only increasing factorization that contains $(0\ n)$ as the last factor is the factorization $((0\ 1), (0\ 2), \ldots, (0\ n))$. For a maximal factorization $f \in \mathcal{F}_n^{max}$ though, if $(0\ n)$ is the last factor, the removal of $(0\ n)$ gives a factorization f' of $(0\ 1\ \ldots\ n{-}1)$ and $\phi(f') = (0, 1, \ldots, n{-}2)$. So similar proofs to the previous two propositions will work. For (*8*) and (*9*), it is easy to apply the increasing rotator condition at vertex $n-1$ and complete the bijections.

In light of the previous propositions, one might think that the number of factorizations in $\mathcal{F}_n^{inc}, \mathcal{F}_n^{max}$ or $\mathcal{F}_n^{read}$ with $(0\ x)$ as the first factor is Catalan for $x \in \{2, 3\ldots, n-1\}$. But this is not true. For example, the number of factorizations in $\mathcal{F}_7^{inc}$ with (0 4) as the first factor is 28, which is not a Catalan number. We now give a theorem that gives a deeper connection between the increasing factorizations and the maximal factorizations. This result is novel to this thesis.

Theorem 3.12. *For $n > 0$, the number of factorizations f such that $f \in \mathcal{F}_n^{inc}$ and $f \in \mathcal{F}_n^{max}$ is 2^{n-1}.*

Before we give the proof, let us look at an example.

Example 3.13. For $n = 5$, the factorizations that are in both $\mathcal{F}_n^{inc}$ and $\mathcal{F}_n^{max}$ are as follows:

$$
\begin{aligned}
&((0\,1),(0\,2),(0\,3),(0\,4),(0\,5)),((0\,1),(0\,2),(0\,3),(0\,5),(4\,5)),\\
&((0\,1),(0\,2),(0\,5),(3\,4),(3\,5)),((0\,1),(0\,2),(0\,5),(3\,5),(4\,5)),\\
&((0\,1),(0\,5),(2\,3),(2\,4),(2\,5)),((0\,1),(0\,5),(2\,3),(2\,5),(4\,5)),\\
&((0\,1),(0\,5),(2\,5),(3\,4),(3\,5)),((0\,1),(0\,5),(2\,5),(3\,5),(4\,5)),\\
&((0\,5),(1\,2),(1\,3),(1\,4),(1\,5)),((0\,5),(1\,2),(1\,3),(1\,5),(4\,5)),\\
&((0\,5),(1\,2),(1\,5),(3\,4),(3\,5)),((0\,5),(1\,2),(1\,5),(3\,5),(4\,5)),\\
&((0\,5),(1\,5),(2\,3),(2\,4),(2\,5)),((0\,5),(1\,5),(2\,3),(2\,5),(4\,5)),\\
&((0\,5),(1\,5),(2\,5),(3\,4),(3\,5)),((0\,5),(1\,5),(2\,5),(3\,5),(4\,5)).
\end{aligned}
$$

There are $2^{5-1} = 16$ of them.

Looking at Example 3.13, we can see $(0\,1)$ is the first factor in half of the factorizations while $(0\,5)$ (which is $(0\,n)$ in this case) is the first factor in the other half. In each case it is easy to see the effect on the target, which is c, of multiplying each factorization by $(0\,1)$ or $(0\,n)$, respectively. In the first half of the factorizations, if $(\tau_1, \tau_2, \ldots, \tau_5)$ is a factorization, so $\tau_1 = (0\ 1)$, then $(\tau_2, \tau_3, \tau_4, \tau_5)$ is a factorization of $(0\ 2\ 3\ 4\ 5)$. Similarly, if $(\tau_1, \tau_2, \ldots, \tau_5)$ is a factorization, so $\tau_1 = (0\ 5)$, then $(\tau_2, \tau_3, \tau_4, \tau_5)$ is a factorization of $(0\ 1\ 2\ 3\ 4)$. We structure our proof on these observations.

Proof of Proposition 3.12. Let $\mathcal{I}_n$ be the collection of all factorizations f such that $f \in \mathcal{F}_n^{inc}$ and $f \in \mathcal{F}_n^{max}$.

First observe that $\mathcal{I}_1 = \{(0\ 1)\}$ and so the claim is true for $n = 1$.

We define $\psi : \mathcal{I}_n \longrightarrow \mathcal{I}_{n-1}$ and show that ψ is a two-to-one map. This along with the base case $n = 1$ proves the result.

Let $f = ((a_1\ b_1), (a_2\ b_2), \ldots, (a_n\ b_n)) \in \mathcal{I}_n$. Observe that at least one $a_i = 0$. But since $f \in \mathcal{F}_n^{inc}$, we have $a_1 = 0$. Now let $a_1, a_2, \ldots, a_k$ be the a_i's that are 0. There are two possibilities:

1. Suppose $k > 1$. Then $f = ((0\ b_1), (0\ b_2), \ldots, (0\ b_k), (a_{k+1}\ b_{k+1}), \ldots, (a_n\ b_n))$. Since f is also an element of $\mathcal{F}_n^{max}$, we have $\phi(f) = (0, 1, \ldots, n)$. But $\phi(((0\ b_1), (0\ b_2), \ldots, (0\ b_k),$ $(a_{k+1}\ b_{k+1}), \ldots, (a_n\ b_n))) = (0, b_1, \ldots, n-1)$. This implies that $b_1 = 1$.

2. Suppose $k = 1$. Since $f \in \mathcal{F}_n^{inc}$, we conclude by Observation 3.4 that $(a_1\ b_1) = (0\ n)$. So $b_1 = n$.

Accordingly, define two sets $\mathcal{I}_n^1$ and $\mathcal{I}_n^2$ such that $\mathcal{I}_n^1 = \{f \in \mathcal{I}_n : \text{the first factor of } f \text{ is } (0\ 1)\}$ and $\mathcal{I}_n^2 = \{f \in \mathcal{I}_n : \text{the first factor } f \text{ is } (0\ n)\}$. From 1. and 2. above, $\mathcal{I}_n^1$ and $\mathcal{I}_n^2$ partition $\mathcal{I}_n$. We define ψ based on whether $f \in \mathcal{I}_n^1$ or $\mathcal{I}_n^2$ and we show that ψ is bijective in both cases.

Let $f = ((a_1\ b_1), (a_2\ b_2), \ldots, (a_n\ b_n)) \in \mathcal{I}_n$. We define $\psi : \mathcal{I}_n \longrightarrow \mathcal{I}_{n-1}$ as follows.

Case 1: $f \in \mathcal{I}_n^1$. Define

$$\psi(f) = ((a_2'\ b_2'), (a_3'\ b_3'), \ldots, (a_n'\ b_n')),$$

where $a_i' = a_i - 1$ if $a_i \neq 0$ and $b_i' = b_i - 1$. If $a_k = 0$ for any k, we let $a_k' = a_k = 0$. Notice that $(0\ 1)(a_2\ b_2) \ldots (a_n\ b_n) = (0\ 1\ \ldots\ n)$. This means $(a_2\ b_2)(a_3\ b_3) \ldots (a_n\ b_n) = (0\ 2\ 3\ \ldots\ n)$ and thus $(a_2'\ b_2')(a_3'\ b_3') \ldots (a_n'\ b_n') = (0\ 1\ 2\ \ldots\ n-1)$. Therefore $\psi(f) \in \mathcal{F}_{n-1}$.

But notice that $\psi(f) \in \mathcal{F}_{n-1}^{inc}$. Also notice $\phi(\psi(f)) = (0, 1, 2, \ldots, n-2)$. This follows from $\phi(((a_2\ b_2), (a_3\ b_3), \ldots, (a_n\ b_n))) = (0, 2, 3, \ldots, n-1)$ and hence $\phi(((a_2'\ b_2'), (a_3'\ b_3'), \ldots,$ $(a_n'\ b_n'))) = (0, 1, \ldots, n-2)$.

Hence $\psi(f)$ is well-defined. It is clear that ψ is invertible which proves that ψ is bijective in this case.

Case 2: $f \in \mathcal{I}_n^2$. Define

$$\psi(f) = ((a_2 - 1\ b_2 - 1), (a_3 - 1\ b_3 - 1), \ldots, (a_n - 1\ b_n - 1)).$$

Notice that ψ is well-defined in this case because $a_i > 0$ for all $i \geq 2$. We have

$$(0\ n)(a_2\ b_2) \ldots (a_n\ b_n) = (0\ 1\ \ldots\ n).$$

Multiplying by the factor $(0\ n)$ on the left, we get

$$(a_2\ b_2)(a_3\ b_3) \ldots (a_n\ b_n) = (1\ 2\ \ldots\ n).$$

Scaling down each element by 1, we get

$$(a_2 - 1\ b_2 - 1)(a_3 - 1\ b_3 - 1) \ldots (a_n - 1\ b_n - 1) = (0\ 1\ \ldots\ n - 1).$$

Note that $\psi(f) \in \mathcal{F}_{n-1}^{inc}$, and it is easy to check that $\phi(\psi(f)) = (0, 1, 2, \ldots, n-2)$. Hence $\psi(f) \in \mathcal{I}_{n-1}$, and it is obvious that ψ is invertible in this case. Thus ψ is bjiective in this case as well.

From both the cases, it follows that $\psi : \mathcal{I}_n \longrightarrow \mathcal{I}_{n-1}$ is a two-to-one map. □

Define the subset $\mathcal{F}_n^{\text{read},(0\,n)} \subset \mathcal{F}_n^{read}$ with $(0\ n)$ as the last factor. A Sage program to find $\mathcal{F}_n^{\text{read},(0\,n)}$ finds that the number of such factorizations of length 1 to 9 (finding $\mathcal{F}_9^{\text{read},(0\,9)}$ takes about 9 hours) is $1, 1, 1, 2, 4, 9, 22, 57$ and 154. Searching OEIS [32], this appears as sequence A287709. The entry there is described in terms of Dyck paths, but drawn differently from how we have done, which we now describe.

A Dyck path of length $2n$ (or semilength n) can also be seen as a walk in $\mathbb{Z}^2$ that begins at the origin, has steps $(1, 1)$ and $(1, -1)$, never goes below the y-axis, and terminates at $(2n, 0)$. A *peak* in a Dyck path is a step $(1, 1)$ immediately followed by $(1, -1)$. The *height* of a peak is the y-coordinate of the point on the path after the step $(1, 1)$ has been taken.

With this, sequence A287709 counts the number of Dyck paths of semilength $n-1$ such that each peak at height y is preceded by at least one peak at height $y - 1$. We denote this set by $\mathcal{D}_{n-1}^{\text{peak}}$.

Conjecture 3.14. *For $n \geq 2$, we have $|\mathcal{F}_n^{\text{read},(0\,n)}| = |\mathcal{D}_{n-1}^{\text{peak}}|$.*

Let us look at the specific example of the Conjecture 3.14 for the case $n = 6$. There are 9 members of $\mathcal{F}_6^{\text{read},(0\,6)}$, and they are

$$\begin{aligned}
&(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),\\
&(0,1),(0,3),(0,4),(0,5),(2,3),(0,6),\\
&(0,1),(0,2),(0,4),(3,4),(0,5),(0,6),\\
&(0,1),(0,2),(4,5),(0,3),(0,4),(0,6),\\
&(0,1),(0,4),(0,5),(2,4),(3,4),(0,6),\\
&(0,1),(0,3),(4,5),(0,4),(2,3),(0,6),\\
&(0,1),(0,2),(3,4),(0,5),(3,5),(0,6),\\
&(0,1),(0,2),(3,5),(4,5),(0,3),(0,6),\\
&(0,1),(0,5),(2,5),(3,5),(4,5),(0,6)
\end{aligned} \tag{3.5}$$

We draw the fourth member of the list in Figure 3.1 as a pair of labelled Dyck path as described in Section 2.5. There it is clear that the reading word for the lower sequence of fourth factorization is $[6,5,4,3,2,1]$. The 9 members of $\mathcal{D}_5^{\text{peak}}$ are given in Figure 3.2. The

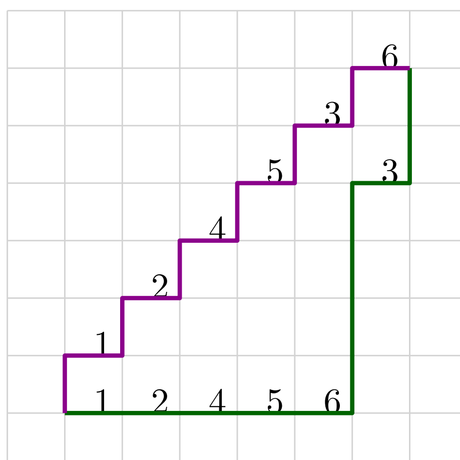


Figure 3.1: The fourth member of $\mathcal{F}_6^{\text{read},(0\,6)}$ given in (3.5). The drawing makes it clear that its lower sequence has reading word $[6,5,4,3,2,1]$.

sequence A287709, as stated in the OEIS, has a generating series. Define $D_{\text{peak}}(x)$ to be the generating series

$$D_{\text{peak}}(x) = \sum_{n\geq 0} |\mathcal{D}_n^{\text{peak}}| x^n,$$

where $|\mathcal{D}_0^{\text{peak}}| = 1$. Evidently

$$D_{\text{peak}}(x) = 1 + x + x^2 + 2x^3 + 4x^3 + 9x^4 + 22x^5 + 57x^6 + \cdots .$$

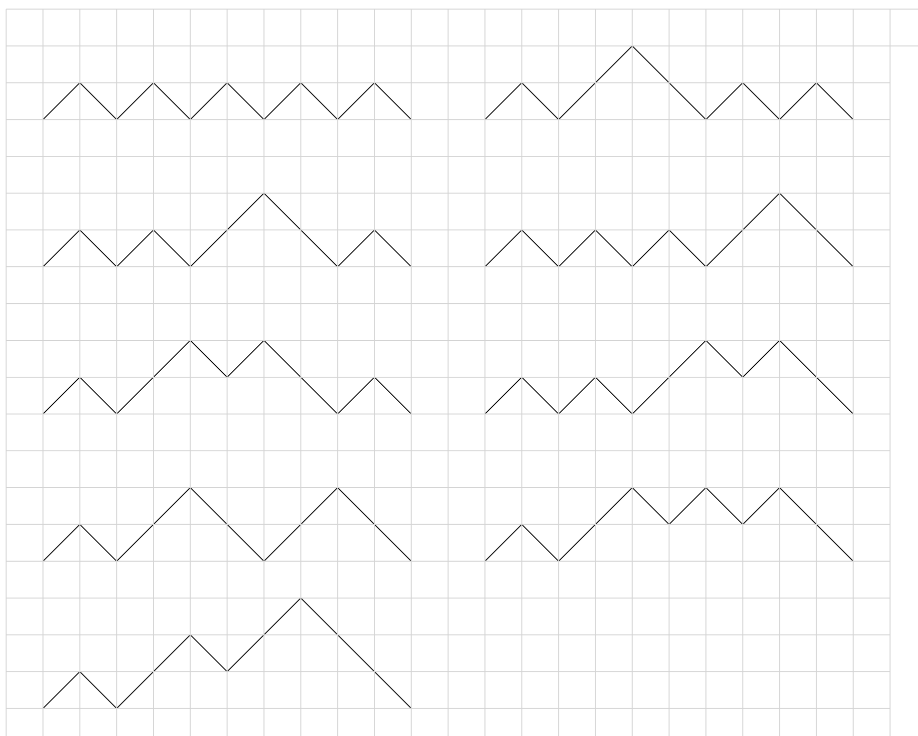


Figure 3.2: The members of $\mathcal{D}_5^{\text{peak}}$. In the last lattice path, notice that there are three peaks. The one at height 3 is preceded by a peak at height 2. Also notice in the penultimate path that there are three peaks at height 2. The second peak at height 2 has a peak at height 1 preceding it, though not immediately preceding it.

Let $U_k(x)$, for $k \geq 0$ be the *Chebyshev polynomials of the second kind.* They can be defined by the recursion

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x) \text{ for } k \geq 1,$$

where $U_0(x) = 1$ and $U_1(x) = 2x$. The OEIS gives

$$D_{\text{peak}}(x) = 1 + \sum_{k \geq 0} \frac{x^{k+1}}{2xU_{k+1}(x)}.$$

Currently, we have no proof of Conjecture 3.14.

Chapter 4

Proof of Theorem 2.42 and symmetric q,t-polynomials

In Section 4.1, we give the promised proof of Theorem 2.42. We use Theorem 3.5 to prove this as increasing factorizations are easier to understand. Adin and Roichman [1] give a different proof. Also, we will look at other q, t-Catalan polynomials in Section 4.2.

4.1 Proof of Theorem 2.42

We give a key connection between maximal factorizations and increasing factorizations in this section first.

A vertex labelled tree is said to be an *alternating* tree if the vertices along any path in the tree are of the form $\ldots > i < j > k < \ldots$. We have the following two lemmas which are needed for our proof of Theorem 2.42. Recall that for any $m \in \mathcal{F}_n$, Theorem 2.60 gives the arch diagram $A(m)$. We say $A(m)$ is alternating if vertices along every path in $A(m)$ is alternating.

Lemma 4.1. *If $m \in \mathcal{F}_n^{max}$, then the arch diagram $A(m)$ is alternating.*

Proof. Suppose not. Then there exists $a < b < c$ such that both $\{a, b\}$ and $\{b, c\}$ are edges in the arch diagram $A(m)$. By the increasing rotator condition at vertex b, we have the edge $\{b, c\}$ occurring before the edge $\{a, b\}$. Let $\tau_j = (b\ c)$ and $\tau_k = (a\ b)$ for $j < k$. By Corollary 2.36, we know that $\phi(m) = (0, 1, \ldots, n-1)$. We claim that the pair $(\tau_j, \tau_k) = ((b, c), (a, b))$ does not correspond to an inversion in $\phi(m)$. To see why, suppose that it does correspond to an inversion pair. Then by Lemma 2.32, we have $b \leq \phi(m)(j-1) < c$ and $a \leq \phi(m)(k-1) < b$, thus implying $\phi(m)(k-1) < \phi(m)(j-1)$. This is a contradiction to the definition of inversion. This shows that the corresponding arch diagram $A(m)$ is alternating. □

Lemma 4.2 (Adin and Roichman [1])**.** *Every element in $\mathcal{F}_n^{max}$ contains $(0\ n)$ as a factor.*

Proof. Suppose that $m = (\tau_1, \tau_2, \ldots, \tau_n)$ is an element in $\mathcal{F}_n^{max}$. It is sufficient to show that $\{0, n\}$ is an edge in $A(m)$.

Suppose that $\{0, n\}$ is not an edge in the arch diagram $A(m)$. Then consider the path from vertex 0 to vertex n. Since $\{0, n\}$ is not an edge, there exists some $k < n$ such that $\{0, k\}$ is an edge. Lemma 4.1 shows that $A(m)$ is alternating, and so the next vertex in the path should be some $j < k$ with edge $\{j, k\}$. Now since the path ends at n, at some instance, an edge must cross the edge $\{0, k\}$ to reach n, which contradicts that $A(m)$ is non-crossing. □

Define the functions $\zeta_n : \mathcal{F}_n^{inc} \longrightarrow \mathcal{F}_n^{max}$ inductively as follows. Since (0 1) is in both $\mathcal{F}_1^{inc}$ and $\mathcal{F}_1^{max}$, we define $\zeta_1((0\ 1)) = (0\ 1)$. Recall from (3.2) the decomposition of factorization $f \in \mathcal{F}_n^{inc}$ into two factorizations $f_1 \in \mathcal{F}_k^{inc}$ and $\tilde{f}_2 \in \mathcal{F}_{n-k-1}^{inc}$ for some k.

For $n > 1$, we define $\zeta_n(f)$ to be

$$\zeta_n : f \longmapsto (\zeta_k(f_1), (0\ n), \zeta_{n-k-1}(\tilde{f}_2)') = (f_1', (0\ n), f_2') = f', \tag{4.1}$$

where f_2' is $\zeta_{n-k-1}(\tilde{f}_2)$ shifted by $k + 1$. Then we have the following proposition.

Proposition 4.3. *ζ_n is a bijection for all n. Furthermore if $f' = \zeta_n(f)$, then f and f' have the same factors (though not necessarily in the same order).*

Proof. We prove this by induction on n.

Base case: For $n = 1$, we have $\mathcal{F}_1^{inc} = \{(0, 1)\} = \mathcal{F}_1^{max}$. Hence in the base case, we see that ζ_1 is a bijection, and $\zeta_1(f)$ has the same factors as f for all $f \in \mathcal{F}_n^{inc}$.

Induction: Assume that the statement is true for all $k < n$. We will now prove the result for the case n.

From (4.1) and the induction hypothesis, we have ζ_k is a bijection from $\mathcal{F}_k^{inc}$ to $\mathcal{F}_k^{max}$ and ζ_{n-k-1} is a bijection from $\mathcal{F}_{n-k-1}^{inc}$ to $\mathcal{F}_{n-k-1}^{max}$ such that all factors are the same. What remains is to check that $f_1', (0\ n), f_2' = f'$ is indeed a maximal factorization because from induction the factors of f' are the same as the factors of f. This can be done with the help of the ϕ function. Recall that $f \in \mathcal{F}_n^{max}$ if and only if $\phi(f) = (0, 1, \ldots, n-1)$. Since $f_1' \in \mathcal{F}_k^{max}$, we have $\phi(f_1') = (0, 1, \ldots, k-1)$ and since $\tilde{f}_2{}' \in \mathcal{F}_{n-k-1}^{max}$, we have $\phi(\tilde{f}_2) = (0, 1, \ldots, n-k-2)$ and hence $\phi(f_2') = (k + 1, k + 2, \ldots, n - 1)$. Considering the product $f_1'(0\ n)$, we get $\phi(f_1', (0\ n)) = (0, 1, \ldots, k)$. This implies that $\phi(f') = (0, 1, \ldots, n - 1)$ and thus $f' \in \mathcal{F}_n^{max}$.

The inverse of ζ_n, that is ζ_n^{-1}, is easily defined because ζ_k is a bijection on $\mathcal{F}_k^{inc}$ and ζ_{n-k-1} a bijection on $\mathcal{F}_{n-k-1}^{inc}$. By Lemma 4.2, each element of $\mathcal{F}_n^{max}$ contains $(0\ n)$ as a factor. If $(0\ n)$ is the $(k + 1)^{st}$ factor of f', we let f_1' be the first k factors, and f_2' be the remaining $n - k - 1$ factors after $(0\ n)$. Note that $\phi(f_1') = (0, 1, \ldots, k - 1)$ and $\phi(f_2') = (k, k + 1, \ldots, n - 2)$. We can now inductively use ζ_k on f_1' and ζ_{n-k-1} on $\tilde{f}_2'$ to find the corresponding elements in $\mathcal{F}_k^{inc}$ and $\mathcal{F}_{n-k-1}^{inc}$, where $\tilde{f}_2'$ is f_2' shifted down by k. □

By Theorem 3.5, coefficients of $q^i t^j$ of $t^n C_n(q,t)$ is the number of factorizations $f \in \mathcal{F}_n^{inc}$ such that $(\text{area}_L(f), \text{area}_U(f)) = (i,j)$, where $\text{area}_L(f)$ and $\text{area}_U(f)$ are the lower and upper area defined in Definition 2.50. Theorem 2.42 asserts that the coefficients of $q^i t^j$ in $C_n(q,t)$ is the number of factorizations $f \in \mathcal{F}_n^{max}$ with $(\text{inv}_R(f), \text{inv}_L(f)) = (i,j)$, where inv_L, inv_R are defined in Definition 2.39. Therefore, to prove Theorem 2.42, it suffices to show that $(\text{area}_L(f), \text{area}_U(f) - n) = (\text{inv}_R(\zeta_n(f)), \text{inv}_L(\zeta_n(f)))$ for all $f \in \mathcal{F}_n^{inc}$.

Theorem 4.4. *The bijection $\zeta_n : \mathcal{F}_n^{inc} \longrightarrow \mathcal{F}_n^{max}$ defined in (4.1) maps the area statistic to the inversion statistics. More precisely,*

$$(\text{area}_L(f), \text{area}_U(f) - n) \longmapsto (\text{inv}_R(\zeta_n(f)), \text{inv}_L(\zeta_n(f))).$$

Proof. We prove this by induction on n.

Base case: For $n = 1$, we have $\mathcal{F}_1^{inc} = \{(0,1)\} = \mathcal{F}_1^{max}$. Also,

$$(\text{area}_L((0\ 1)), \text{area}_U((0\ 1)) - 1) = (0,0) = (\text{inv}_R((0\ 1)), \text{inv}_L((0\ 1))).$$

Hence the result is true for the base case.

Induction: Assume that the result is true for all $k < n$.

Recall the bijection ζ_n in equation (4.1). If $f \in \mathcal{F}_n^{inc}$, find $f_1 \in \mathcal{F}_k^{inc}$ and $\tilde{f}_2 \in \mathcal{F}_{n-k}^{inc}$, then

$$\begin{aligned} \zeta_n(f) &= (\zeta_k(f_1), (0\ n), \zeta_{n-k-1}(\tilde{f}_2)') \\ &= (f_1', (0\ n), f_2') \\ &= f'. \end{aligned}$$

By induction,

$$\begin{aligned} &(\text{area}_L(f_1), \text{area}_U(f_1) - k) \longmapsto (\text{inv}_R(f_1'), \text{inv}_L(f_1')) \text{ and} \\ &(\text{area}_L(\tilde{f}_2), \text{area}_U(\tilde{f}_2) - (n-k-1)) \longmapsto (\text{inv}_R(\tilde{f}_2'), \text{inv}_L(\tilde{f}_2')) \end{aligned} \tag{4.2}$$

We have already shown in the proof of Theorem 3.5 that $\text{area}_L(f) = \text{area}_L(f_1) + \text{area}_L(\tilde{f}_2) + k$, and $\text{area}_U(f) = \text{area}_U(f_1) + \text{area}_U(\tilde{f}_2) + n - k$.

For the other side, that is for inversions, we have the following. Recall that $f' = \zeta_n(f) = (f_1', (0\ n), f_2')$, where $f_1' = \zeta_k(f_1)$ and $f_2' = \zeta_{n-k-1}(\tilde{f}_2)$ shifted by $k+1$. Here $f_1 \in \mathcal{F}_k^{inc}$ and $\tilde{f}_2 \in \mathcal{F}_{n-k-1}^{inc}$ for some k. For $\zeta_n(f) = (f_1', (0\ n), f_2')$, we have $f_1' = \tau_1, \tau_2, \ldots, \tau_k$ and $\tilde{f}_2' = t_{k+2}, t_{k+3}, \ldots, t_n$ for some $t_{k+2}, t_{k+3}, \ldots, t_n$ such that f_2' is the shifted $\tilde{f}_2'$. Now we have the following computation for the inversions:

$\text{inv}_R(f') = \text{inv}_R(f_1') + \text{inv}_R(f_2') +$ (number of right inversions between $(0\ n)$ in $f_1', (0\ n), f_2'$ and other transpositions) and

$$\text{inv}_L(f') = \text{inv}_L(f_1') + \text{inv}_L(f_2') + \text{ (number of left inversions between } (0\ n) \text{ in } f_1', (0\ n), f_2' \text{ and other transpositions).}$$

If $f_1' = \tau_1, \tau_2, \ldots, \tau_k$, then for each $1 \leq i \leq k$, we have $\tau_i = (a\ b)$ where $0 \leq a < b \leq k$. Also $f_2' = \tau_{k+2}, \tau_{k+3}, \ldots, \tau_n$ and for each $k+1 < j \leq n$ we have $\tau_j = (c\ d)$ where $k+1 \leq c < d \leq n$. By Definition 2.39, each (τ_i, τ_{k+1}) is a right inversion for $1 \leq i \leq k$ (there are k of them), and each (τ_{k+1}, τ_j) for $k+2 \leq j \leq n$ is a left inversion (there are $n-k-1$ of them).

Therefore,

$$\begin{gathered} \text{inv}_R(f') = \text{inv}_R(f_1') + \text{inv}_R(f_2') + k \text{ and} \\ \text{inv}_L(f') = \text{inv}_L(f_1') + \text{inv}_L(f_2') + n - k - 1. \end{gathered}$$

From Equation (4.2) and (3.4),

$$\begin{aligned} \text{inv}_L(f') &= \text{inv}_L(f_1') + \text{inv}_L(f_2') + n - k - 1 \\ &= (\text{area}_U(f_1) - k) + (\text{area}_U(\tilde{f}_2) - (n-k-1)) + n - k - 1 \\ &= \text{area}_U(f_1) + \text{area}_U(\tilde{f}_2) - k \\ &= \text{area}_U(f) - n. \end{aligned} \tag{4.3}$$

Similarly $\text{inv}_R(f') = \text{area}_L(f)$, proving the result. □

4.2 Symmetric q, t-polynomials

The q, t-Catalan polynomial $C_n(q,t)$ that we introduced in Chapter 2 is just one of many q, t-Catalan polynomials that are being studied. We recall that q, t-Catalan polynomial are polynomials in variables q and t such that when $q = t = 1$, the polynomial reduces to the Catalan numbers C_n. Recall from Chapter 1 that some of them were first introduced by Garsia and Haiman [12] as rational functions in q and t with non-negative integer coefficients that sum to Catalan number C_n in the context of algebraic geometry and Macdonald polynomials. Haglund in his paper [16] provided a combinatorial formula for his q, t-Catalan polynomial as $\tilde{C}_n(q,t) = \sum q^{area(d)} t^{bounce(d)}$, where the sum is over all Dyck paths, and where *bounce* is another statistic on Dyck paths that we will not define here (See Chapter 3 in [17]). Haglund soon came up with a different formula for q, t-Catalan polynomial where $\tilde{C}_n(q,t) = \sum q^{dinv(d)} t^{area(d)}$. The polynomial $\tilde{C}_n(q,t)$ is symmetric in q and t but the proof is not easy. A simple combinatorial proof for the symmetry of $\tilde{C}_n(q,t)$ is yet to be found. Haglund extended his definition of q, t-polynomial to q, t-polynomials over all parking functions graded by area and dinv.

Pappe, Paul and Schilling [28] study two different symmetric q, t-Catalan polynomials one of which is graded by area and depth statistics over all Dyck paths and another graded by dinv and dinv of *depth* statistic. They in fact show that both these polynomials are symmetric in q and t. Pappe et al. show that their q,t-polynomial graded by area and depth statistics satisfy the recursion of the q,t-Catalan polynomial in Definition 2.41 originally introduced by Adin and Roichman [1]. So their polynomial is equal to the one in Definition 2.41.

These works by various authors motivate us to understand the underlying reason why some of these polynomials are symmetric.

Lemma 4.5. *If* $(a_1\ \ b_1)(a_2\ \ b_2)\ldots(a_n\ \ b_n) = (0\ \ 1\ \ \ldots\ \ n) \in \mathfrak{S}_{[n]}$*, then we have* $(n-b_1\ n-a_1)(n-b_2\ n-a_2)\ldots(n-b_n\ n-a_n) = (n\ n-1\ \ldots\ 0)$.

Proof. We have

$$(a_1\ b_1)(a_2\ b_2)\ldots(a_n\ b_n) = (0\ 1\ \ldots\ n)$$

Conjugation by $\alpha = (0\ n)(1\ n-1)\ldots$ on the right side gives us,

$$\begin{aligned}\alpha(0\ 1\ \ldots\ n)\alpha^{-1} &= ((0\ n)(1\ n-1)\ldots)(0\ 1\ \ldots\ n)((0\ n)(1\ n-1)\ldots)\\ &= (0\ n\ \ldots\ 1)\\ &= (n\ n-1\ \ldots\ 0)\end{aligned}$$

And the left side gives us,

$$\begin{aligned}\alpha(a_1\ b_1)(a_2\ b_2)\ldots(a_n\ b_n)\alpha^{-1} &= \alpha(a_1\ b_1)\alpha^{-1}\alpha(a_2\ b_2)\alpha^{-1}\alpha\ldots\alpha^{-1}\alpha(a_n\ b_n)\alpha^{-1}\\ &= (\alpha(a_1\ b_1)\alpha^{-1})(\alpha(a_2\ b_2)\alpha^{-1})\ldots(\alpha(a_n\ b_n)\alpha^{-1})\\ &= (n-a_1\ n-b_1)(n-a_2\ n-b_2)\ldots(n-a_n\ n-b_n)\end{aligned}$$

which proves the lemma. □

4.2.1 *mateflipre v*

In Section 2.5, we defined the mate of a parking function. In this section, we define two new functions called *flip* and *rev*, and our main goal is to prove that the composition of these three functions *mateflipre v* is an involution on $\mathcal{P}_n$.

Definition 4.6. Let $s = (a_1, a_2, \ldots, a_n)$ be any sequence of length n. Then the map *flip* takes the sequence $s = (a_1, a_2, \ldots, a_n)$ to the image $flip(s) = (n-a_1, n-a_2, \ldots, n-a_n)$.

Definition 4.7. Let $r = (a_1, a_2, \ldots, a_n)$ be a sequence of length n. The map *rev* maps r to the *reverse* of r. That is $rev(r) = (a_n, a_{n-1}, \ldots, a_1)$.

Now that we have defined all the required terms, we will prove the following proposition. Here *mateflipre v* is the composition of the functions *rev* ∘ *flip* ∘ *mate*, where *mate* is applied first.

Proposition 4.8. *The map mateflipre v is an involution on* $\mathcal{P}_n$*.*

Proof. Let $p = (a_1, a_2, \ldots, a_n)$ be a parking function of length n. We know that $mate(p) = (b_1, b_2, \ldots, b_n)$, where

$$(a_1\ b_1)(a_2\ b_2)\ldots(a_n\ b_n) = (0\ 1\ \ldots\ n). \tag{4.4}$$

And by Theorem 2.48 this is unique. Thus we have

$$\begin{aligned} mateflip(p) &= (n - b_1, n - b_2, \ldots, n - b_n) \\ \Rightarrow mateflipre v(p) &= (n - b_n, n - b_{n-1}, \ldots, n - b_1) = q \end{aligned}$$

We now need to show that $mateflipre v(q) = p$.

Let $mate(q) = (c_n, c_{n-1}, \ldots, c_1)$. We have

$$(n - b_n\ c_n)(n - b_{n-1}\ c_{n-1})\ldots(n - b_1\ c_1) = (0\ 1\ \ldots\ n). \tag{4.5}$$

Hence we can see that

$$\begin{aligned} mateflip(q) &= (n - c_n, n - c_{n-1}, \ldots, n - c_1) \\ \Rightarrow mateflipre v(q) &= (n - c_1, n - c_2, \ldots, n - c_n) \end{aligned}$$

It is sufficient to show that $n - c_i = a_i$ for all $1 \leq i \leq n$.

Applying Lemma 4.5 to (4.5), we have

$$(n - c_n\ b_n)(n - c_{n-1}\ b_{n-1})\ldots(n - c_1\ b_1) = (n\ n-1\ \ldots\ 0).$$

Taking the inverse of both the sides gives us,

$$(n - c_1\ b_1)(n - c_2\ b_2)\ldots(n - c_n\ b_n) = (0\ 1\ \ldots\ n).$$

From (4.4) and by Theorem 2.48, we conclude that

$$n - c_i = a_i \text{ for all } 1 \leq i \leq n. \tag{4.6}$$

□

Example 4.9. Let $p = (3, 2, 4, 6, 0, 0, 7, 1, 7)$ be a parking function of length 9. For this parking function, $mate(p) = (5, 3, 5, 9, 2, 6, 8, 2, 9)$, $mateflip(p) = (4, 6, 4, 0, 7, 3, 1, 7, 0)$ and $mateflipre v(p) = (0, 7, 1, 3, 7, 0, 4, 6, 4)$. Let $q = (0, 7, 1, 3, 7, 0, 4, 6, 4)$. If we apply *mateflipre v* to q, we should get back p. Let us see if that is the case. $mate(q) = (2, 8, 2, 9, 9, 3, 5, 7, 6)$, $mateflip(p) = (7, 1, 7, 0, 0, 6, 4, 2, 3)$ and thus $mateflipre v(q) = (3, 2, 4, 6, 0, 0, 7, 1, 7)$ which is indeed the parking function p we started with.

Proposition 4.10.

$$P_n^{stat}(q, t) := \sum_{p \in \mathcal{P}_n} q^{stat(p)} t^{stat(mateflipre v(p))} \tag{4.7}$$

is symmetric in q and t. Here stat is any statistic defined on elements of $\mathcal{P}_n$.

The proof follows immediately from Proposition 4.8. We emphasize that the polynomial is graded by the same statistic. We note that the following q,t-Catalan polynomials are symmetric as well.

$$P^{stat}_{n,A}(q,t) \coloneqq \sum_{p\in A} q^{stat(p)} t^{stat(mateflipre v(p))} \tag{4.8}$$

where $A \in \{\mathcal{P}^{inc}_n, \mathcal{P}^{read}_n\}$. But note that *mateflipre v* is not necessarily an involution on A for $A \in \{\mathcal{P}^{inc}_n, \mathcal{P}^{read}_n\}$. Proposition 4.10 explains the symmetry of the q,t-polynomial graded by lower and upper area defined in Equation (2.52). This also potentially produces new symmetric polynomials such as polynomials graded by *dinv* of parking function p and *dinv* of *mateflipre v* of p. The *dinv* statistic is interesting because it not only depends on the shape of the Dyck path (unlike *area*), but it also depends on the labelling corresponding to the parking function. One natural q,t-polynomial that we have from factorizations is the polynomial

$$\sum_{p\in\mathcal{P}_n} q^{dinv(p)} t^{dinv(mateflip(p))}.$$

This is natural because *mateflip* is effectively just the $mate(p)$ turned into a parking function so *dinv* can be applied. This polynomial is not symmetric, whereas our polynomial in (4.7) graded by *dinv* statistic is symmetric. Let us look at the following example which explains this.

Example 4.11. For $n = 3$, we have

$$\sum_{p\in\mathcal{P}_3} q^{dinv(p)} t^{dinv(mateflip(p))} = q^3 + q^2t^2 + q^2t + q^2 + 2qt^2 + 3qt + q + t^3 + 2t + 3.$$

This polynomial in q and t is not symmetric. However, the polynomial

$$\sum_{p\in\mathcal{P}_3} q^{dinv(p)} t^{dinv(mateflipre v(p))} = q^3t^3 + 3q^2t + 3qt^2 + 3qt + 6.$$

is symmetric. So while the first polynomial is more natural from factorizations point of view, (since *mateflip* really gives the "parking function" of the *mate*), the latter is symmetric.

4.2.2 *mateflipsort*

In this subsection, we will define *sort* and prove that *mateflipsort*, which is the composition of *mate*, *flip* and *sort* (below) is an involution on $\mathcal{P}^{inc}_n$.

Definition 4.12. For a sequence $s = (a_1, a_2, \ldots, a_n)$, the image $sort(s)$ is the rearrangement of the sequence s to $s' = (a'_1, a'_2, \ldots, a'_n)$ such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.

Proposition 4.13. *The map mateflipsort is an involution on* $\mathcal{P}^{inc}_n$.

Let us look at the following example.

Example 4.14. Let $p = (0, 0, 1, 3, 3, 3, 4, 7) \in \mathcal{P}_8^{inc}$. The mate of this parking function is $mate(p) = (2, 8, 2, 5, 6, 8, 5, 8)$. Then $mateflip(p) = (6, 0, 6, 3, 2, 0, 3, 0)$ and composing with *sort* gives us $matefipsort(p) = (0, 0, 0, 2, 3, 3, 6, 6)$, call this q. Notice that q is a parking function and it makes sense to find its mate, i.e., $mate(q) = (1, 5, 8, 5, 4, 5, 7, 8)$. Applying *flip* and *sort* in order gives us $mateflip(q) = (7, 3, 0, 3, 4, 3, 1, 0)$ and thus $mateflipsort(q) = (0, 0, 1, 3, 3, 3, 4, 7)$, which is the parking function p that we started with.

To explain why *mateflipsort* is an involution, let us define another function $\Lambda : \mathcal{P}_n^{inc} \longrightarrow \mathcal{P}_n^{inc}$. Let $p \in \mathcal{P}_n^{inc}$ and let $\overline{p} = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_{n-1}})$ be the parking function p with the last 0 removed. Let k be the largest number such that $p_1' = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_k})$ is still a parking function. Note that k could be 0 (so that p_1' is empty). Set $p_2' = (\overline{a_{k+2}} - (k+1), \ldots, \overline{a_{n-1}} - (k+1))$. Set $P = mateflipsort(p_1')$ and $Q = mateflipsort(p_2')$ with a 0 inserted. Now define $\Lambda : \mathcal{P}_n^{inc} \longrightarrow \mathcal{P}_n^{inc}$ by $\Lambda(p) = Q, \overline{P}$, where $\overline{P}$ is P shifted up by $n - k$.

We look at the following example first that explains how Λ gives the same output as *mateflipsort*.

Example 4.15. Let us take the parking function $p = (0, 0, 1, 3, 3, 3, 4, 7) \in \mathcal{P}_8^{inc}$ from Example 4.14. Here the parking function $\overline{p} = (0, 1, 3, 3, 3, 4, 7)$. The largest number k such that p_1' is still a parking function is 2. That is, $p_1' = (0, 1)$ and $p_2' = (0, 0, 0, 1, 4)$. Then we have, $P =$*mateflipsort*$(p_1') = (0, 0)$ and $Q =$*mateflipsort*$(p_2') = (0, 0, 0, 2, 3, 3)$. So $\Lambda = (0, 0, 0, 2, 3, 3, 6, 6)$.

Proof of Proposition 4.13. We claim that $\Lambda =$ *mateflipsort*$(\star)$. From there, the recursive definition of Λ can be used to provide an easy proof of Proposition 4.13, so we omit the details. From Λ, it is easy to see that *mateflipsort* is an involution on $\mathcal{P}_n^{inc}$. □

As a consequence of this, we have the following proposition.

Proposition 4.16.

$$\tilde{P}_n^{stat}(q, t) := \sum_{p \in \mathcal{P}_n^{inc}} q^{stat(p)} t^{stat(mateflipsort(p))} \tag{4.9}$$

is symmetric in q and t. Here stat is any statistic defined on elements of $\mathcal{P}_n^{inc}$.

From Sections 4.2.1 and 4.2.2, we see that *mateflipre v* and *mateflipsort* are two new ways of generating *q,t*-symmetric polynomials. We can then investigate if these symmetric polynomials have a natural interpretation beyond *mateflipre v*.

In this thesis, we looked at minimal factorizations of the canonical full cycle c, denoted by $\mathcal{F}_n$, and three different subsets of $\mathcal{F}_n$, namely maximal factorizations, increasing factorizations and reading word factorizations. These collections were of interest primarily because each of them are counted by the famous Catalan numbers. This motivated us to see if there were some nice connections, and in Section 4.1, we established a statistic preserving bijective map ζ_n between the increasing factorizations $\mathcal{F}_n^{inc}$, and the maximal factorizations $\mathcal{F}_n^{max}$, which preserved the inversion and area statistics. Although the reading word parking functions were studied before, we looked at them from a new perspective, that is, as factorizations. We used Algorithm 2.63 to find the factorizations that were corresponding to the reading word parking functions and gave a conjecture about them (Conjecture 3.14). These motivate us to question about the existence of other subsets of $\mathcal{F}_n$ counted by Catalan numbers. It will also be nice to find a bijection between the reading word factorizations $\mathcal{F}_n^{read}$ and $\mathcal{F}_n^{inc}$ or $\mathcal{F}_n^{max}$ that preserve certain statistics. Further to this, we can study about the joint distribution of known statistics, like inversions, area, dinv, etc. over the different class of factorizations by looking at the q, t-polynomials graded by these statistics. Of course, it is possible to look at these q, t-polynomials. Determining if the polynomials we introduced have other nice properties, like for example, which of them have nice generating series that satisfy any recursions, etc., would be interesting.

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