On Cuts and Cycles in Planar Graphs

by

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Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Spring 2023

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Abstract

A partition is said to be *balanced* if the sizes of the sets in the partition differ by at most one. The *size* of a graph partition is the number of edges that have their end-vertices in different parts of the partition. A folklore conjecture claims that, for any planar graph G on n vertices, a minimum balanced bipartition of G has size at most n. We confirm this conjecture.

A *Hamiltonian* graph is one that contains a cycle passing through all its vertices. A *pancyclic* graph is one that contains cycles of all possible lengths. In 1971-72, Bondy made the meta-conjecture that, barring a simple family of exceptions, any nontrivial condition on a graph that implies that the graph is Hamiltonian also implies that the graph is pancyclic. This meta-conjecture implies, as a special case, that every 4-connected planar graph is pancyclic. Identifying 4-connected planar graphs that do not contain a 4-cycle as a family of exceptions, Malkevitch conjectured that a 4-connected planar graph is pancyclic if it contains a 4-cycle. We show that every 4-connected planar graph contains at least $\lceil \frac{n}{2} \rceil + 1$ cycles of pairwise distinct lengths. We also show that every 4-connected planar graph not containing a 4-cycle contains at least $\lceil \frac{5n}{6} \rceil + 2$ cycles of pairwise distinct lengths.

Keywords: planar graphs; connectivity; planar triangulations; graph partitions; balanced partitions; outerplanar graphs; Hamiltonicity; pancyclicity

To my family.

Acknowledgements

I would like to thank my supervisor, Bojan Mohar, without whose guidance, support and trust this thesis would never have materialized. I am specially grateful to him for believing in me and allowing me the time and liberty to grow and mature as an independent researcher. I would also like to thank Ladislav Stacho and Matt DeVos for their supply of problems, discussions, advice and encouragement. I would like to acknowledge the financial support and counselling received from Simon Fraser University over the course of my program.

I would never have gotten to undertake this journey if it weren't for Jim Geelen, Levent Tunçel, Joseph Cheriyan, Robin Thomas and Dana Randall. I am grateful to them for inspiring, advising, training and vouching for me.

A shout-out to all the people who helped make it a rewarding experience in more ways than I can count - Dale Yamaura, Sofia Leposavic, Christie Carlson and Stacey Openshaw for taking care of everything else school; Petra Menz for welcoming me and my family into her home and looking in on us from time to time; Sebastián González Hermosillo de la Maza, Seyyed Aliasghar Hosseini, Alexandra Wesolek, Aniket Mane, Peter Bradshaw and Bianca Ng for making me not want to go home after a long day; Nishad Kothari, Akash Kumar and Anand Louis for continuing to indulge me; Annapoorna Shanbhag, Siddharth Shenoy, Chaitra Nayini, Mukesh Maguluri, Ananya Naidu, Nikhil Naidu, Ipsa Bhardwaj and Amit Kumar Singh for being our family away from home.

Finally, I would like to thank my family − Ma, Papa and Priyam for their unwavering love and support, Harshil and Jayitri for filling our lives with unbounded happiness at a time when we needed it most, and Ambica for staying by my side through everything.

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Chapter 1

Introduction

Planar graphs have attracted mathematicians since decades before they were characterized by Kazimierz Kuratowksi around 1930. Despite all that is known about them and the apparent ease of taking a problem to pen and paper, several problems have been found to be notoriously hard to solve for this class of graphs and those that were have added to the lot of problems and conjectures that remain unsettled. In this thesis, we attack two of these conjectures.

We report singificant progress in both cases - one of the conjectures is settled in full, and for the other we present significant partial results with proofs that draw on techniques and lemmas of independent interest. In both cases, the proofs are constructive and yield simple algorithms to find instances of the graph (sub)structure(s) in question that satisfy the promised optimality bounds. In this chapter, we introduce the two conjectures and provide details on background and state-of-the-art in each case. The next two chapters present our findings on the two conjectures in full. We state future research directions and open problems for both in the conclusion of this thesis.

1.1 Balanced partitions

Graph partitioning problems seek to find a partition of the vertex set that optimizes a given parameter under some given conditions. Given a partition $V(G) = V_1 \cup ... \cup V_k$ of the vertex-set, we indicate the number of edges with both ends in V_i by $e(V_i)$ and the *size of the partition* by $e(V_1, ..., V_k) = |E(G))| - \sum_{i=i}^{k} e(V_i)$.

A classic example of graph partitioning problem is the *max-cut* problem or the *maximum bipartite subgraph* problem which requires one to find a partition $V(G) = V_1 \cup V_2$ that maximises $e(V_1, V_2)$. In contrast to this, Bollobás and Scott ([2]) considered the problem of finding a partition minimizing $\max\{e(V_1), e(V_2)\}$. Several other partitioning problems have been considered under different constraints (see [15], [27], [44]), and not just for graphs but digraphs (see [18], [20], [26]) and hypergraphs as well (see [19], [21]).

Graph partitioning comes up naturally in the study and design of complex networks like social networks, transportation networks, etc., and several real-world problems may be modeled as graph partitioning problems each with its own specification. More recently, graph partitioning has been applied extensively in VLSI design.

A partition is said to be *balanced* if the sizes of the sets in the partition differ by at most one. Following up on [2], Bollobás and Scott [4] proved that almost every regular graph with m edges admits a balanced bipartition $V(G) = V_1 \cup V_2$ such that $\max\{e(V_1), e(V_2)\} \leq m/4$. In [43], Xu, Yan and Yu proved that every graph minimum degree at least 5 admits a balanced bipartition such that $\max\{e(V_1), e(V_2)\} \leq m/3$. Bollobás and Scott conjectured in [3] that every graph with minimum degree at least 2 admits such a bipartition. In that paper, they also posed the following balanced bipartition problem: for a graph G with n vertices and m edges, what is the maximum and the minimum size of a balanced bipartition?

In [16], Fan, Xu, Yu and Zhou proved that minimum balanced bipartitions of any graph G with n vertices and m edges have size at most $\frac{1}{2}(m + \lceil \frac{n}{2} \rceil - |M|)$, where M is a maximum matching in the complement graph G^c. They also proved that if G is planar, then there exists a corresponding upper bound of $n-2$ if G is triangle-free, and $n+1$ if G is Hamiltonian. Of these, the latter was improved to n for minimum balanced bipartitions of any planar graph G without separating triangles by Olsen and Revsbæk in [33]. As observed in K_4 and an infinite family of planar graphs given by Fan, Xu, Yu and Zhou in [16], minimum balanced bipartitions can have size equal to n. In the same paper, Fan, Xu, Yu and Zhou mention a folklore conjecture which claims that, for the class of planar graphs, we cannot do any worse. We settle this conjecture in the affirmative in this thesis.

1.2 Pancyclicity

In 1971-72, Bondy (see [5], [6]) proposed his now famed meta-conjecture that any nontrivial condition on a graph which guarantees the existence of a *Hamiltonian cycle* (a cycle passing through all the vertices) also guarantees that the given graph is *pancyclic* (contains cycles of all possible lengths), with possibly a simple family of exceptions. This meta-conjecture has since attracted a considerable amount of research.

On the affirmative side, Bondy himself proved that Ore's sufficient condition for Hamiltonicity also implies pancyclicity, unless the graph is the complete balanced bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Bauer and Schmeichel ([1]), relying on previous results of Schmeichel and Hakimi ([36]), have shown that the sufficient conditions for Hamiltonicity of Bondy ([7]), Chvátal ([9]) and Fan ([14]) all imply pancyclicity, barring a small family of exceptions. Motivated by the classical theorem of Chvátal and Erdös ([10]) that a graph with connectivity $\kappa(G)$ at least as large as its independence number $\alpha(G)$ must be Hamiltonian, Keevash and Sudakov ([25]) proved the weaker result that $\kappa(G) \geq 600\alpha(G)$ is sufficient for pancyclicity. On the negative side, Zamfirescu ([45]) recently showed that Thomassen's sufficient condition for Hamiltonicity in planar graphs ([39]) does not guarantee pancyclicity; in fact, one can construct (an infinite family of) graphs satisfying the said sufficient condition to make the number of missing cycle lengths arbitrarily large.

Hamiltonicity is a classic problem that has real applications in diverse fields such as computer graphics, electronic circuit design, genome mapping, and operations research. For instance, when mapping genomes scientists must combine many tiny fragments of genetic code (called "reads") into one single genomic sequence (a "superstring"). This can be done by finding a Hamiltonian path or cycle, where each of the reads are considered nodes in a graph and each overlap (place where the end of one read matches the beginning of another) is considered to be an edge. Pancyclicity is a natural extension of this classic problem.

In his paper, along with proving Ore's condition being sufficient for pancyclicity, Bondy conjectured that Tutte's result about Hamiltonicity of 4-connected planar graphs could be similarly strengthened to imply pancyclicity. Malkevitch (see [30]) pointed out a simple family of exceptions to this conjecture (line graphs of cyclically 4-edge-connected, cubic, planar graphs of girth 5) no member of which contains a cycle of length 4. However, it is suspected that that might be the only cycle length absent in a 4-connected planar graph. Malkevitch then revised Bondy's conjecture to add the extra condition that the graph contain a 4-cycle in order to be pancyclic.

It is known that every planar graph with minimum degree $\delta \geq 4$ must contain cycles of length $3, 5$ ([42]) and 6 ([17]). Using Tutte paths, Nelson ([40]), Thomas and Yu ([38]) and Sanders ([34]) showed that every 4-connected planar graph on n vertices contains cycles of length $n-1, n-2$ and $n-3$. This was further strengthened by Chen, Fan and Yu. ([8]) to include the lengths lengths $n-4, n-5$ and $n-6$, and by Cui, Hu and Wang. ([11]) to include the length $n-7$. Recently, Lo ([28]) showed that every planar Hamiltonian graph with $\delta \geq 4$ has cycles of lengths $\lfloor \frac{n}{2} \rfloor, ..., \lceil \frac{n}{2} \rceil + 3$. We add to this body of results by showing in this thesis that every 4-connected planar graph contains at least $\lceil \frac{n}{2} \rceil + 1$ cycles of pairwise distinct lengths. We also show that every 4-connected planar graph contains at least $\lceil \frac{5n}{6} \rceil + 2$ cycles of pairwise distinct lengths when it does not contain any 4-cycles. We acknowledge that a result similar to the former has been obtained independently by Lo ([29]).

Chapter 2

Minimum balanced bipartitions of planar graphs

2.1 A folklore conjecture

A partition is said to be *balanced* if the sizes of the sets in the partition differ by at most one. Balanced bipartition problems in graphs ask for a balanced partition of the vertex-set of a graph into two sets with various requirements. In [3], Bollobás and Scott posed the following balanced bipartition problem: for a graph G with n vertices and m edges, what is the maximum and the minimum size of (the edge-cut in) a balanced bipartition of $V(G)$? While a substantial amount of work has been done on both the maximum and the minimum balanced bipartition problems, in this chapter, we will focus on the latter.

In [16], Fan, Xu, Yu and Zhou proved that minimum balanced bipartitions of any graph G with n vertices and m edges have size at most $\frac{1}{2}(m + \lceil \frac{n}{2} \rceil - |M|)$, where M is a maximum matching in the complement graph G^c. They also proved that if G is planar, then there exists a corresponding upper bound of $n-2$ if G is triangle-free, and $n+1$ if G is Hamiltonian. Of these, the latter was improved to n for minimum balanced bipartitions of any planar graph G without separating triangles by Olsen and Revsbæk in [33]. As observed in K_4 and an infinite family of planar graphs given by Fan, Xu, Yu and Zhou in [16], minimum balanced bipartitions can have size equal to n. In the same paper, Fan, Xu, Yu and Zhou mention a folklore conjecture which claims that, for the class of planar graphs, we cannot do any worse.

Conjecture 1. *A minimum balanced bipartition of any planar graph G has size at most $|V(G)|$.*

In this chapter, we prove this conjecture. Our main result is the following theorem ([37]).

Theorem 2. *If G is a plane triangulation, then there exists a balanced bipartition (V_1, V_2) of $V(G)$ such that both $G[V_1]$ and $G[V_2]$ are connected near-triangulations, and the total number of blocks in $G[V_1]$ and $G[V_2]$ exceeds the total number of internal vertices by at most 2.*

The theorem above makes a stronger statement about balanced bipartitions of planar triangulations from which the conjecture follows as a direct corollary. Moreover, the proof that we give is constructive and may be used to derive an algorithm for obtaining a balanced bipartition which respects the conjectured bound for any planar graph G.

The layout of the rest of the chapter is as follows. In Section 2.2, we define the graph-terminology used in the chapter and prove some preliminary edge-counting statements. In the following two sections, we introduce the two key tools used in the proof − the partitioning lemmas (Section 2.3) and the separating-triangle-decomposition of planar triangulations (Section 2.4), and set up the proof infrastructure. In Section 2.5, we describe a basic construction which uses this infrastructure to obtain a balanced bipartition respecting the conjectured bound for any planar triangulation G barring a special case. Later, in Section 2.6, we present variants of the same basic construction which we use to handle the said special case. Finally, we conclude by putting everything together in Section 2.7.

2.2 Preliminaries

The graphs in this chapter are planar and do not contain loops or parallel edges.

For any graph G, a *bipartition* (V_1, V_2) of $V(G)$ is defined as a pair of disjoint subsets V_1 and V_2 of $V(G)$ such that $V_1 \cup V_2 = V(G)$. A bipartition (V_1, V_2) of $V(G)$ is said to be a *balanced bipartition* if $|V_1|$ and $|V_2|$ differ by at most 1.

A graph G is said to be *connected* if, for every pair of vertices $u, v \in V(G)$, there exists a $u - v$ path in G. A single vertex forms a connected graph in the trivial sense. A graph that is not connected is said to be *disconnected*.

A graph G is said to be *k-connected* if it has at least $k + 1$ vertices and, for every $(k-1)$-subset V' of $V(G)$, $G - V'$ is connected. A $(k-1)$-subset of $V(G)$ which when removed from the graph leaves the graph disconnected, is called a *$(k-1)$-vertex-cut* (or a *$(k-1)$-cut*) in G. A 1-cut is also known as a *cut-vertex*. A *k-separation* (G_1, G_2) in G is defined as a pair of subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, $E(G_1) \cap E(G_2) = \emptyset$, and $|V(G_1) \cap V(G_2)| = k$. Note that if, for a k-separation (G_1, G_2), $V(G_1) \cap V(G_2)$ forms a k-cut in G, then both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ must be nonempty.

To simplify the notation in this chapter, we define a graph G to be *biconnected* if G is 2-connected or $G = K_2$. A biconnected graph is 2-connected if and only if it has 3 or more vertices.

A *block* in a graph G is a maximally biconnected subgraph of G.

Two edges in a graph G are said to be *disjoint* if they are not incident with a common vertex.

A graph is said to be *planar* if it can be drawn in the plane in such a way that its edges intersect only at their endpoints. A graph so drawn in the plane is called a *plane graph*. For any plane graph G, the regions of $\mathbb{R}^2 \backslash G$ are called the *faces* of G. If G is finite, then exactly one of its faces is unbounded and is called the *infinite face* of G.

A *closed walk* in a graph is defined as a sequence of vertices, starting and ending at the same vertex, such that every pair of consecutive vertices in the sequence is adjacent to each other in the graph. Each face of a plane graph G is bounded by a closed walk in G called the *boundary* of the face. Any vertex of G not lying on the boundary of the infinite face of G is said to be an *internal vertex* of G. Since a planar graph can also be drawn on a sphere in such a way that its edges intersect only at their endpoints, under a suitably chosen stereographic projection from the sphere to the plane, any face boundary in a plane graph G may be "designated" as the infinite face boundary.

A *planar triangulation* is a graph that is maximally planar. Every face in any drawing of a planar triangulation in the plane is bounded by three edges. Every planar triangulation corresponds to a unique *plane triangulation* upto isomorphism, so the terms are often used interchangeably.

A *near-triangulation* is a plane graph G such that every face of G except at most one is bounded by three edges. Thus, a plane triangulation is also a near-triangulation. In this chapter, we designate the face not bounded by three edges in a near-triangulation as the infinite face.

Readers are referred to [13] for any terminology that we may have missed and the notation used in this chapter.

We now prove a couple of edge-counting statements.

Proposition 3. *If G is a connected near-triangulation with $|V(G)| \geq 2$, b blocks, and i internal vertices, then $|E(G)| = 2|V(G)| - 2 + i - b$.*

Proof. The proof is by induction on b. Let G be a connected near-triangulation with $|V(G)| \geq 2$.

If G is a single edge, then $2|V(G)| - 2 + i - b = 4 - 2 + 0 - 1 = 1 = |E(G)|$. If G is 2-connected, then the number of edges missing from G is the number of non-intersecting edges required to triangulate its unbounded face, which is $|V(G)| - i - 3$, and so $|E(G)| = 3|V(G)| - 6 - (|V(G)| - i - 3) = 2|V(G)| - 2 + i - 1$. Thus, the proposition holds for the base case of the induction ($b = 1$).

Now suppose that G has $b \geq 2$ blocks. Consider a leaf block G_1 in G which is separated from the rest of the graph by a cut-vertex u. Then, $G_2 := G - (V(G_1) - u)$ is a connected near-triangulation with at least 2 vertices and $b - 1$ blocks. By the induction hypothesis, $|E(G_2)| = 2|V(G_2)| - 2 + i_2 - (b - 1)$, where i_2 is the number of internal vertices in G_2. Similarly, $|E(G_1)| = 2|V(G_1)| - 2 + i_1 - 1$, where i_1 is the number of internal vertices in

G_1. This gives us that

$$\begin{aligned}|E(G)| &= |E(G_1)| + |E(G_2)| \\ &= 2(|V(G_1)| + |V(G_2)|) - 4 + (i_1 + i_2) - b \\ &= 2(|V(G)| + 1) - 4 + i - b \\ &= 2|V(G)| - 2 + i - b,\end{aligned}$$

which concludes the induction. □

Corollary 4. *If G is a plane triangulation with $|V(G)| \geq 4$, and (V_1, V_2) is a bipartition of $V(G)$ with $|V_1|, |V_2| \geq 2$ such that both $G[V_1], G[V_2]$ are near-triangulations together containing a total of b blocks and i internal vertices, then $e(V_1, V_2) = |V(G)| + b - i - 2$.*

Proof. Let G be a plane triangulation with $|V(G)| \geq 4$, and let (V_1, V_2) be a bipartition of $V(G)$ with $|V_1|, |V_2| \geq 2$. For $k = 1, 2$, let $G[V_k]$ be a near-triangulation with b_k blocks and i_k internal vertices, and let $i = i_1 + i_2, b = b_1 + b_2$. Then, using Proposition 3,

$$\begin{aligned}e(V_1, V_2) &= |E(G)| - (|E(G[V_1])| + |E(G[V_2])|) \\ &= 3|V(G)| - 6 - (2(|V_1| + |V_2|) - 4 + (i_1 + i_2) - (b_1 + b_2)) \\ &= 3|V(G)| - 6 - (2|V(G)| - 4 + i - b) \\ &= |V(G)| - 2 - i + b.\end{aligned}$$

□

2.3 Partitioning lemmas

In this section, we prove a set of partitioning lemmas that will be used as "navigation" and "dissection" tools in the main proof. The first of these applies to 4-connected plane triangulations and yields as a corollary a proof of the conjecture for 4-connected planar graphs. This lemma helps in both navigating a plane triangulation in the attempt to find an "ideal partitioning site" and in affecting the actual partitioning.

Lemma 5. *If G is a 4-connected plane triangulation or K_4, and e_1, e_2, are disjoint edges in G, then for every $2 \leq k \leq |V(G)| - 2$ there exists a bipartition (V_1, V_2) of $V(G)$ such that*

(i) $|V_1| = k, |V_2| = |V(G)| - k$,

(ii) $e_1 \in E(G[V_1]), e_2 \in E(G[V_2])$, *and*

(iii) *both $G[V_1], G[V_2]$ are biconnected near-triangulations.*

Proof. The proof is by induction on k. Let G be a 4-connected plane triangulation with $|V(G)| \geq 4$ and disjoint edges $e_1, e_2 \in E(G)$. Let $e_1 = u_1v_1, e_2 = u_2v_2$, where $u_1, u_2, v_1, v_2 \in V(G)$.

If $k = 2$ or $G = K_4$, define $V_1 := \{u_1, v_1\}, V_2 := V(G) - V_1$. Then $G[V_1]$ is a biconnected near-triangulation and, since G is 4-connected or K_4, $G[V_2]$ is a biconnected near-triangulation too (the only possible non-triangular face being the one bounded by $N_G(\{u_1, v_1\}) - \{u_1, v_1\}$). Thus, the lemma holds for K_4 and the base case of the induction.

Now suppose that $2 < k \leq n-2$. By the induction hypothesis, there exists a bipartition (V_1', V_2') of $V(G)$ such that $|V_1'| = k-1, e_i \in E(G[V_i'])$ for $i = 1, 2$, and both $G[V_1'], G[V_2']$ are biconnected near-triangulations. For every edge xy on the boundary of the infinite face of $G[V_1']$ there exists a vertex z on the boundary of the infinite face of $G[V_2']$ such that the vertex-set $\{x, y, z\}$ bounds a triangular face of G (we say that the vertex z forms a triangular face of G with the edge xy). Let Z be the set of all vertices on the boundary of the infinite face of $G[V_2']$ each of which forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$. Since $|V_2'| \geq 3$, $|Z| \geq 2$ for otherwise one of the edges on the boundary of the infinite face of $G[V_1']$ is on the boundary of only one triangular face in G; $Z \neq \{u_2, v_2\}$, for otherwise u_2 and v_2 have a common neighbor (say x') on the boundary of the infinite face of $G[V_1']$ such that $\{u_2, v_2, x'\}$ forms a 3-cut in G. Consider a vertex $z \in Z - \{u_2, v_2\}$. If $G[V_2' - z]$ is biconnected for any such vertex z, then the bipartition $(V_1, V_2) := (V_1' \cup z, V_2' - z)$ satisfies $(i) - (iii)$ and the induction is complete.

So we may assume that each such vertex z is contained in a 2-cut $\{z, z'\}$ in $G[V_2']$. Since $G[V_2']$ is a biconnected near-triangulation, for every such 2-cut $\{z, z'\}$, the vertex z' also lies on the boundary of the infinite face of $G[V_2']$ and is, hence, adjacent to z, but the edge zz' does not lie on the boundary of the infinite face. Let $(H_{zz'}, J_{zz'})$ denote the 2-separation in $G[V_2']$ corresponding to the 2-cut $\{z, z'\}$. Then either $V(H_{zz'}) - V(J_{zz'})$ or $V(J_{zz'}) - V(H_{zz'})$ contains neither u_2 nor v_2; without loss of generality, let $(V(H_{zz'}) - V(J_{zz'})) \cap \{u_2, v_2\} = \emptyset$. Note that $V(H_{zz'}) - V(J_{zz'})$ contains one or more vertices that lie on the boundary of the infinite face of $G[V_2']$. As before, if any of these vertices w forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$, then it is contained in a 2-cut $\{w, w'\} \subset V(H_{zz'})$ and the corresponding 2-separation $(H_{ww'}, J_{ww'})$ is non-crossing with $(H_{zz'}, J_{zz'})$, i.e., $H_{ww'} \subset H_{zz'}$ and $J_{ww'} \supset J_{zz'}$. In particular, for any such 2-separation $(H_{ww'}, J_{ww'})$, $|V(H_{ww'})| < |V(H_{zz'})|$.

Consider such a 2-separation $(H_{\bar{w}\bar{w}'}, J_{\bar{w}\bar{w}'})$ with $|V(H_{\bar{w}\bar{w}'})|$ minimal (if none of the vertices in $V(H_{zz'}) - V(J_{zz'})$ forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$, then $(H_{\bar{w}\bar{w}'}, J_{\bar{w}\bar{w}'}) = (H_{zz'}, J_{zz'})$). Since no vertex in $V(H_{\bar{w}\bar{w}'}) - V(J_{\bar{w}\bar{w}'})$ forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$, there exists a vertex p on that boundary which is adjacent to every vertex in $V(H_{\bar{w}\bar{w}'})$ that lies on the boundary of the infinite face of $G[V_2']$ (including $\bar{w}$ and $\bar{w}'$). But then the vertex-set

$\{p, \bar{w}, \bar{w}'\}$ forms a 3-cut in G, a contradiction to G being 4-connected. This concludes the induction. □

Corollary 6. *If G is a 4-connected planar graph or $G \subseteq K_4$, then for every $1 \leq k \leq |V(G)| - 1$ there exists a bipartition (V_1, V_2) of $V(G)$ with $|V_1| = k$ such that $e(V_1, V_2) \leq |V(G)|$.*

Proof. Observe that it suffices to prove the corollary for the cases when G is a 4-connected plane triangulation or $G \subseteq K_4$. It is easily seen that the corollary holds if $G \subseteq K_4$, so we may assume that G is a 4-connected plane triangulation.

The corollary is trivially true for $k = 1, n - 1$ since the maximum degree of a vertex in G is at most $|V(G)| - 1$.

Suppose $2 \leq k \leq |V(G)| - 2$. By Lemma 5, there exists a bipartition (V_1, V_2) of $V(G)$ such that $|V_1| = k$, and both $G[V_1]$ and $G[V_2]$ are biconnected near-triangu-lations. Then, using Corollary 4, $e(V_1, V_2) = |V(G)| - 2 - i + 2 = |V(G)| - i \leq |V(G)|$, where i is the number of internal vertices in $G[V_1]$, $G[V_2]$ taken together. □

The next pair of lemmas applies to plane triangulations which may not be 4-connected and are primarily used for affecting the actual partitioning once an ideal partitioning site has been located.

Lemma 7. *If G is a plane triangulation and the vertex-set $\{a, b, c\} \subseteq V(G)$ bounds the infinite face of G, then for every $2 \leq k \leq |V(G)| - 1$ there exists a bipartition (V_1, V_2) of $V(G)$ such that*

(i) $|V_1| = k, |V_2| = |V(G)| - k$,

(ii) $\{a, b\} \subseteq V_1, c \in V_2$,

(iii) $G[V_1]$ *is a biconnected near-triangulation with the edge ab on the boundary of the infinite face, and*

(iv) $G[V_2]$ *is a connected near-triangulation with the vertex c on the boundary of the infinite face.*

Proof. The proof is by induction on k and very similar to that of Lemma 5. Let G be a plane triangulation and let $\{a, b, c\} \subseteq V(G)$ be the vertex-set that bounds the infinite face of G.

If $k = 2$, define $V_1 := \{a, b\}, V_2 := V(G) - V_1$. Then $G[V_1]$ is a biconnected near-triangulation with the edge ab on the boundary of the infinite face. Since G is a plane triangulation and hence 3-connected, $G[V_2]$ is a connected near-triangulation too with the only possible non-triangular face being the one bounded by the vertex-set $N_G(\{a, b\}) - \{a, b\}$ containing c. Thus, the lemma holds for the base case of the induction.

Now suppose that $2 < k \leq n-1$. By the induction hypothesis, there exists a bipartition (V_1', V_2') of $V(G)$ such that $|V_1'| = k-1$, $G[V_1']$ is a biconnected near-triangulation with the edge ab on the boundary of the infinite face, and $G[V_2']$ is a connected near-triangulation with the vertex c on the boundary of the infinite face. For every edge xy on the boundary of the infinite face of $G[V_1']$ there exists a vertex z on the boundary of the infinite face of $G[V_2']$ such that the vertex z forms a triangular face of G with the edge xy. Let Z be the set of all vertices on the boundary of the infinite face of $G[V_2']$ each of which forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$. Since $|V_2'| \geq 2$, $|Z| \geq 2$ for otherwise one of the edges on the boundary of the infinite face of $G[V_1']$ is on the boundary of only one triangular face in G. Consider a vertex $z \in Z - \{c\}$. If $G[V_2' - z]$ is connected for any such vertex z, then the bipartition $(V_1, V_2) := (V_1' \cup z, V_2' - z)$ satisfies $(i) - (iii)$ and the induction is complete.

So we may assume that each such vertex z is a cut-vertex in $G[V_2']$. Then there exists a leaf-block B which does not contain the vertex c, not even as the cut-vertex (say u) connecting the block to the rest of $G[V_2']$. Observe that there exists at least one vertex $v \in V(B), v \neq u$, such that v lies on the boundary of the infinite face of $G[V_2']$. But since neither $v = c$, nor is v a cut-vertex in $G[V_2']$, v does not form a triangular face of G with an edge on the boundary of the infinite face of $G[V_1']$. This implies that v lies on the boundary of a face in G that is not a triangle, a contradiction since G is a plane triangulation. This concludes the induction. □

Remark 1. Notice that the proofs of Lemma 5 and Lemma 7 are constructive in nature and underline the algorithm for achieving the desired biparitition in each case by describing the next vertex to be included in one of the two parts. Notice that the algorithm underlined by the proof of Lemma 7, in particular, ensures for any common neighbor $v \neq c$ of a and b that all the vertices in the component of $G - \{a, b, v\}$ not containing c are included in V_1 if v is; it also ensures that v is included in V_1 if any vertex of the component of $G - \{a, b, v\}$ containing c is.

Lemma 8. *Let G be a plane triangulation such that $|V(G)| \geq 4$ and the vertex-set $\{a, b, c\} \subset V(G)$ bounds the infinite face of G. If (V_1, V_2) is a bipartition of $V(G)$ such that*

(i) $\{a, b\} \subseteq V_1, c \in V_2$,

(ii) $|V_2| \geq 2$, *and*

(iii) *both $G[V_1]$ and $G[V_2]$ are connected near-triangulations,*

then there exists a vertex $v \in V_2 - c$ such that both $G[V_1 \cup v]$ and $G[V_2 - v]$ are connected near-triangulations, and the total number of blocks in $G[V_1 \cup v]$ and $G[V_2 - v]$ exceeds that in $G[V_1]$ and $G[V_2]$ by at most 1.

Proof. Let G be a plane triangulation such that $|V(G)| \geq 4$ and and the vertex-set $\{a,b,c\} \subset V(G)$ bounds the infinite face of G. Let (V_1, V_2) be a bipartition of $V(G)$ such that $\{a,b\} \subseteq V_1, c \in V_2, |V_2| \geq 2$, and both $G[V_1]$ and $G[V_2]$ are connected near-triangulations.

Consider a leaf-block B of $G[V_2]$. Let u denote the cut-vertex connecting B to the rest of $G[V_2]$ if the latter is not biconnected; if $G[V_2]$ is biconnected, let $u = c$. Then B contributes a vertex $z \neq u$ to the boundary of the infinite face of $G[V_2]$ such that z forms a triangular face of G with an edge xy on the boundary of the infinite face of $G[V_1]$; if not, either one of the edges on the boundary of the infinite face of $G[V_1]$ is on the boundary of only one triangular face in G, or u lies on the boundary of a face in G that is not a triangle.

If there exists such a vertex $z \in V(B)$ that is contained in at most a single 2-cut in B, then we may choose $v = z$ and the additional block (if any) in $G[V_1 \cup v]$ and $G[V_2 - v]$ appears in $G[V_2 - v]$. So we may assume that for each such vertex $z \in V(B)$, there exist two or more distinct 2-separations (H, J) in $G[V_2]$ such that $z \in (V(H) \cap V(J)) \subset V(B)$ and z is adjacent to $z' := (V(H) \cap V(J)) - z$ but the edge zz' does not lie on the boundary of the infinite face of $G[V_2]$. For all but one of these 2-separations, $u \notin V(J) \cap V(H)$; without loss of generality, let $u \in V(J) - V(H)$ whenever $u \notin V(J) \cap V(H)$. Let (H_z, J_z) be one of these 2-separations with $|V(H_z)|$ minimal, one for which z is not contained in a 2-cut in B with any vertex from $V(H_z) - V(J_z)$. Then, a similar 2-separation (H_w, J_w) for a vertex $w \in V(H_z) - V(J_z)$ is non-crossing with (H_z, J_z), i.e., $H_w \subset H_z$ and $J_w \supset J_z$. In particular, for any such 2-separation (H_w, J_w), $|V(H_w)| < |V(J_w)|$.

Consider such a 2-separation $(H_{\bar{w}}, J_{\bar{w}})$ with $|V(H_{\bar{w}})|$ minimal (if none of the vertices in $V(H_z) - V(J_z)$ forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1]$, then $(H_{\bar{w}}, J_{\bar{w}}) = (H_z, J_z)$); let $V(H_{\bar{w}}) \cap V(J_{\bar{w}}) = \{\bar{w}, \bar{w}'\}$. Let Q denote the set of vertices $q \in V(H_{\bar{w}}) - V(J_{\bar{w}})$ that lie on the boundary of the infinite face of $G[V_2]$; note that $Q \neq \emptyset$. Since no vertex in Q forms a triangular face of G with an edge on the boundary of the infinite face of $G[V_1]$, there exists a vertex p on that boundary which is adjacent to every vertex in $Q \cup \{\bar{w}, \bar{w}'\}$. Let C denote the cycle formed by the edge $\bar{w}\bar{w}'$ and the segment of the boundary of the infinite face of $G[V_2]$ formed by the vertex-set $Q \cup \{\bar{w}, \bar{w}'\}$. Let xx' denote a chord of C of minimal length contained in $G[V_2]$ (i.e., $x, x' \in V(C), xx' \in E(G[V_2]) - E(C)$, and the smaller of the two $x-x'$ paths along C has minimal length); if $G[V_2]$ does not contain any chords of C, let $xx' = \bar{w}\bar{w}'$. Then there exists a 2-separation $(H_{xx'}, J_{xx'})$ in $G[V_2]$ such that $V(H_{xx'}) \cap V(J_{xx'}) = \{x, x'\}, u \in V(J_{xx'}) - V(H_{xx'})$ and $(V(H_{xx'}) - V(J_{xx'})) \cap V(C) \neq \emptyset$. Since every vertex in $V(C) \cap V(H_{xx'})$ is adjacent to p and $G[V_2]$ does not contain any chords of the cycle $(C - (V(J_{xx'}) - V(H_{xx'}))) \cup xx'$, we may choose v to be any vertex in $(V(H_{xx'}) - V(J_{xx'})) \cap V(C)$ and the additional block (if any) $G[V_1 \cup v]$ and $G[V_2 - v]$ appears in $G[V_1 \cup v]$. □

Corollary 9. *Let G be a plane triangulation such that the vertex-set $\{a,b,c\} \subseteq V(G)$ bounds the infinite face of G. If (V_1', V_2') is a bipartition of $V(G)$ such that $\{a,b\} \subseteq V_1', c \in V_2'$, and* *both $G[V_1']$ and $G[V_2']$ are connected near-triangulations, then for every $0 \leq k \leq |V_2'| - 1$ there exists a bipartition (V_1, V_2) of $V(G)$ such that*

(i) $V_1' \subseteq V_1, c \in V_2 \subseteq V_2'$,

(ii) $|V_1| = |V_1'| + k$, *and*

(iii) both $G[V_1]$ and $G[V_2]$ are connected near-triangulations with the total number of blocks in $G[V_1]$ and $G[V_2]$ exceeding that in $G[V_1']$ and $G[V_2']$ by at most k.

Proof. The proof is by induction on k. Let G be a plane triangulation such that the vertex-set $\{a, b, c\} \subseteq V(G)$ bounds the infinite face of G. Let (V_1', V_2') be a bipartition of $V(G)$ such that $\{a, b\} \subseteq V_1', c \in V_2'$, and both $G[V_1']$ and $G[V_2']$ are connected near-triangulations with a total of b blocks. If $k = 0$ or $|V_2'| = 1$, then $(V_1, V_2) = (V_1', V_2')$. So we may assume that $|V_2'| \geq 2$.

Now suppose that $0 < k \leq |V_2'| - 1$. By the induction hypothesis, there exists a bipartition (V_1'', V_2'') of $V(G)$ such that $V_1' \subseteq V_1'', c \in V_2'' \subseteq V_2', |V_1''| = |V_1'| + k - 1$, and both $G[V_1'']$ and $G[V_2'']$ are connected near-triangulations with a total of at most $b + k - 1$ blocks. Then, by Lemma 8, there exists a vertex $v \in V_2'' - c$ such that both $G[V_1'' \cup v]$ and $G[V_2'' - v]$ are connected near-triangulations with a total of at most $b + k$ blocks, and we may choose $(V_1, V_2) = (V_1'' \cup v, V_2'' - v)$.

This concludes the induction. □

2.4 Setup

In this section, we establish the key elements of the "map" used by the navigation lemma from the previous section to identify an ideal partitioning site. For such a map to be feasible, we need to get out of our way a pathological case which is handled by the following proposition.

Proposition 10. *If G is a plane triangulation with $|V(G)| = n$ and $T \subseteq G$ is a separating triangle such that each of the two components of $G - V(T)$ contains at least $\lfloor \frac{n}{2} \rfloor - 1$ vertices, then there exists a balanced bipartition (V_1, V_2) of $V(G)$ such that $G[V_1]$ is a biconnected near-triangulation and $G[V_2]$ is a connected near-triangulation with the number of blocks exceeding the total number of internal vertices in $G[V_1]$ and $G[V_2]$ by at most 1.*

Proof. Let G be a plane triangulation with $|V(G)| = n$ and let T be a separating triangle in G such that each of the two components of $G - V(T)$ contains at least $\lfloor \frac{n}{2} \rfloor - 1$ vertices. Since $2(\lfloor \frac{n}{2} \rfloor - 1) + 3 = 2\lfloor \frac{n}{2} \rfloor + 1$, n is odd and each component of $G - V(T)$ contains exactly $\lfloor \frac{n}{2} \rfloor - 1$ vertices.

Let $V(T) = \{a, b, c\}$ and let U_1, U_2 denote the sets of vertices in the two components of $G - V(T)$. Consider the balanced bipartition $(U_2 \cup \{b, c\}, U_1 \cup a)$ of $V(G)$. Observe that $G[U_2 \cup \{b, c\}]$ is a biconnected near-triangulation with exactly $|U_2| - |U_2 \cap N_G(a)|$ internal vertices, while $G[U_1 \cup a]$ is a connected near-triangulation with exactly $|U_1 \cap N_G(b) \cap N_G(c)|$ blocks. By Corollary 4, if this bipartition does not qualify as the bipartition (V_1, V_2), then it must be the case that

$$|U_2| - |U_2 \cap N_G(a)| \leq |U_1 \cap N_G(b) \cap N_G(c)| - 2.$$

Similarly, if none of the analogously defined balanced bipartitions $(U_2 \cup \{a, c\}, U_1 \cup b)$ and $(U_2 \cup \{a, b\}, U_1 \cup c)$ qualifies as the bipartition (V_1, V_2), we get that

$$\begin{aligned}
&|U_2| - |U_2 \cap N_G(b)| \leq |U_1 \cap N_G(a) \cap N_G(c)| - 2, \text{ and} \\
&|U_2| - |U_2 \cap N_G(c)| \leq |U_1 \cap N_G(a) \cap N_G(b)| - 2.
\end{aligned}$$

Observe that every vertex in U_2 except at most one is adjacent to at most two of the three vertices a, b and c. Thus,

$$\begin{aligned}
&2|U_2| + 1 \geq |U_2 \cap N_G(a)| + |U_2 \cap N_G(b)| + |U_2 \cap N_G(c)| \\
\Rightarrow &|U_2| - 1 \leq 3|U_2| - (|U_2 \cap N_G(a)| + |U_2 \cap N_G(b)| + |U_2 \cap N_G(c)|).
\end{aligned} \tag{2.1}$$

Similarly, none of the common neighbors of a and b in U_1 except at most one is adjacent to c, and likewise for the pairs b, c and a, c. Thus,

$$|U_1 \cap N_G(a) \cap N_G(c)| + |U_1 \cap N_G(a) \cap N_G(c)| + |U_1 \cap N_G(a) \cap N_G(c)| \leq |U_1| + 2$$

Then, adding the first three inequalities, we get that

$$\begin{aligned}
&3|U_2| - (|U_2 \cap N_G(a)| + |U_2 \cap N_G(b)| + |U_2 \cap N_G(c)|) \\
\leq &|U_1 \cap N_G(b) \cap N_G(c)| + |U_1 \cap N_G(a) \cap N_G(c)| + |U_1 \cap N_G(a) \cap N_G(b)| - 6 \\
\leq &|U_1| - 4,
\end{aligned}$$

a contradiction to (2.1) above since $|U_1| = |U_2|$. Thus, one of the three balanced bipartitions described above qualifies as the balanced bipartition (V_1, V_2) described in the proposition statement. □

In the remainder of this section and in the following two sections, we assume that if G is a plane triangulation with $|V(G)| = n$, then G does not contain a separating triangle T such that each of the two components of $G - V(T)$ contains at least $\lfloor \frac{n}{2} \rfloor - 1$ vertices.

Now, to build the said "map", we use the *separating-triangle-decomposition* of a plane triangulation into 4-connected pieces defined by Jackson and Yu in [22]. It is described as follows. Let G be a plane triangulation with $|V(G)| = n$ and let T be a separating triangle in G. Then we can identify two graphs H_T and I_T in G such that both H_T and I_T have at least 4 vertices, $H_T \cup I_T = G$, $H_T \cap I_T = T$, and each of H_T and I_T is a plane triangulation with $V(T)$ bounding a face in it; for notational convenience, we choose the labels so that $|V(I_T)| \leq |V(H_T)|$. T is referred to as a *marker triangle* in H_T and I_T. This procedure is now recursively iterated on H_T and I_T until we obtain a collection $\mathcal{T}$ of plane triangulations each without any separating triangles. These triangulations are referred to as the *pieces* of G. Note that each separating triangle in G will occur as a marker triangle in exactly two pieces of G. Let D be a graph defined on the set $\mathcal{T}$ such that $T_1, T_2 \in \mathcal{T}$ are joined by an edge if they have a marker triangle in common. It follows from the decomposition theory given by Cunningham and Edmonds in [12] that D is a tree, and that the set $\mathcal{T}$ and the tree D are uniquely defined by G. D is referred to as the *separating-triangle-decomposition-tree* of G. We will mostly refer to D as the *std-tree* of G.

For each edge e of D, let T_e be the common marker triangle between the pieces joined by e and let (H_{T_e}, I_{T_e}) be the subgraph-pair as described above (we will henceforth refer to T_e as e's marker triangle and to (H_{T_e}, I_{T_e}) as T_e's (or e's) *std-subgraph-pair*). Due to Proposition 10, we may assume that $|V(H_{T_e})| \geq \lfloor \frac{n}{2} \rfloor + 2$ and $|V(I_{T_e})| \geq \lfloor \frac{n}{2} \rfloor + 2$ are not both true. Then, since $2(\lfloor \frac{n}{2} \rfloor + 1) - 3 < n$, exactly one of H_{T_e} and I_{T_e} has at most $\lfloor \frac{n}{2} \rfloor + 1$ vertices, and is thus "smaller" than the other which has at least $\lceil \frac{n}{2} \rceil + 2$ vertices. Recalling that we choose the labels so that I_{T_e} is the smaller subgraph, we orient each edge e in D away from piece contained in I_{T_e} to get the directed std-tree $\overrightarrow{D}$. It is easy to see that exactly one piece in $\overrightarrow{D}$ is incident with all edges oriented inward. We call this piece the *sink*. By construction, the sink represents a triangulation $S \subseteq G$ with no separating triangles; we call this triangulation the *sink triangulation* of G. Due to Proposition 11 below, we may assume that the sink in $\overrightarrow{D}$ has degree at least 2 (S contains at least 2 distinct triangles that are separating in G). Since $|V(S)| \geq 4$ and there does not exist a plane triangulation without separating triangles on 5 vertices, due to Proposition 13 below, we may assume that indeed $|V(S)| \geq 6$ and, hence, that S is 4-connected.

The proofs that follow use some additional terminology which is defined as follows. For any T in G with $V(T) = \{a, b, c\}$, we denote by J_T the smaller std-subgraph I_T of G if T is separating in G, and the triangle T otherwise (this latter case represents the unique "empty" side of T when it is non-separating); we denote the set $V(J_T) - \{a, b, c\}$ by U_T.

For each $x \in \{a, b, c\}$, the number of internal vertices in $J_T - x$ is denoted by $i_{T,x}$. As observed earlier in Proposition 10, if T is separating in G then $i_{T,x} = |U_T| - |N_G(x) \cap U_T|$. Since every vertex in U_T except at most one is adjacent to at most two of the three vertices a, b, c, we get that $|N_G(a) \cap U_T| + |N_G(b) \cap U_T| + |N_G(c) \cap U_T| \leq 2|U_T| + 1$; then $i_{T,a} + i_{T,b} + i_{T,c} = 3|U_T| - |N_G(a) \cap U_T| - |N_G(b) \cap U_T| - |N_G(c) \cap U_T| \geq |U_T| - 1$. Note that this holds trivially if T is non-separating with $U_T = \emptyset$ and $i_{T,a} = i_{T,b} = i_{T,c} = 0$.

For each $x \in \{a, b, c\}$, the number of blocks in $J_T - (\{a, b, c\} - \{x\})$ containing at least two vertices is denoted by $b_{T,x}$, and the leaf block in $J_T - (\{a, b, c\} - \{x\})$ containing x is denoted by $B_{T,x}$. As also observed in Proposition 10, if T is separating in G then $b_{T,x} = |U_T \cap N_G(y) \cap N_G(z)|$, where $\{y, z\} = \{a, b, c\} - \{x\}$. Since each cut-vertex in $J_T - \{y, z\}$ is contained in exactly two blocks, and the vertex set $N_G(x) \cap U_T$ is contained in $B_{T,x}$, we get that $b_{T,x} - 1 \leq i_{T,x}$ and $|U_T| - |V(B_{T,x}) - x| \leq i_{T,x}$. Again, this holds trivially if T is non-separating with $b_{T,a} = b_{T,b} = b_{T,c} = 0$, and $B_{T,a}, B_{T,b}$ and $B_{T,c}$ being single-vertex graphs containing the vertices a, b and c , respectively. Since, if T is separating every block in $J_T - (\{a, b, c\} - \{x\})$ contains at least two vertices, and if T is non-separating $J_T - (\{a, b, c\} - \{x\})$ is a single-vertex graph (which, in the proofs that follow, is invariably contained in another block), we will refer to $b_{T,x}$ as just the number of blocks in $J_T - (\{a, b, c\} - \{x\})$.

Proposition 11. *If G is a plane triangulation with $|V(G)| = n$ such that the sink in its directed std-tree has degree 1, then there exists a balanced bipartition (V_1, V_2) of $V(G)$ such that both V_1 and V_2 induce biconnected near-triangulations.*

Proof. Let G be a plane triangulation with $|V(G)| = n$ such that its directed std-tree $\overrightarrow{D}$ has a leaf-piece that is incident with an inward-oriented edge e. Let T_e be e's marker triangle with $V(T_e) = \{a, b, c\}$, let (H_{T_e}, I_{T_e}) be e's std-subgraph-pair where H_{T_e} is the 4-connected leaf-piece towards which e is oriented, and let $k = \lceil \frac{n}{2} \rceil - (|V(I_{T_e})| - 1)$. Since $|V(I_{T_e})| \leq \lfloor \frac{n}{2} \rfloor + 1$, we get that $k \geq 0$; since $|V(H_{T_e})| \geq \lceil \frac{n}{2} \rceil + 2$, we also get that $|V(H_{T_e})| - k \geq \lceil \frac{n}{2} \rceil + 2 - k = |V(I_{T_e})| + 1 \geq 5$ (since $|V(I_{T_e})| \geq 4$).

Choose a vertex $d \in N_{H_{T_e}}(c) - \{a, b\}$. By Lemma 5, there exists a bipartition (U_1, U_2) of $V(H_{T_e})$ such that $|U_1| = 2 + k, \{a, b\} \subseteq U_1, \{c, d\} \subseteq U_2$, and both $G[U_1], G[U_2]$ are biconnected near-triangulations. Then $G[U_1] \cup (I_{T_e} - c)$ is a biconnected near-triangulation containing exactly $\lceil \frac{n}{2} \rceil$ vertices, while $G[U_2]$ is a biconnected near-triangulation containing exactly $\lfloor \frac{n}{2} \rfloor$ vertices. So we may choose $(V_1, V_2) = (V(G) - U_2, U_2)$. □

Proposition 12. *If G is a plane triangulation with $|V(G)| = n$ such that the sink triangulation S of G contains at least 2 distinct triangles that are separating in G and, for some triangle $T \subseteq S$, $|V(J_T)| = \lfloor \frac{n}{2} \rfloor + 1$, then there exists a balanced bipartition (V_1, V_2) of $V(G)$ such that $G[V_1]$ is a biconnected near-triangulation and $G[V_2]$ is a connected near-triangulation with the number of blocks exceeding the total number of internal vertices in $G[V_1]$ and $G[V_2]$ by at most 1.*

Proof. Let G be a plane triangulation with $|V(G)| = n$ such that the sink triangulation S of G contains at least 2 distinct triangles that are separating in G and, for some triangle $T \subseteq S$, $|V(J_T)| = \lfloor \frac{n}{2} \rfloor + 1$. Note that, since $|V(S)| \geq 4$, $|V(G)| \geq 6$ and T shares its edges with exactly three distinct triangles in S.

Let $V(T) = \{u_1, u_2, u_3\}$. For $i = 1, 2, 3$, let T_i denote the triangle sharing the vertices $V(T) - u_i$ with T in S. Consider the balanced bipartitions $(W_i, V(G) - W_i)$, where $W_i :=$

$V(J_T) - u_i$, for $i = 1, 2, 3$. Observe that $G[W_i]$ is a biconnected near-triangulation with exactly i_{T,u_i} internal vertices, while $G[V(G) - W_i]$ is a connected near-triangulation with at most $|U_{T_i}| + 1$ blocks. By Corollary 4, if none of these three bipartitions qualifies as the bipartition (V_1, V_2), then it must be the case that

$$i_{T,u_i} \leq |U_{T_i}| + 1 - 2,$$

for $i = 1, 2, 3$. Adding the three inequalities corresponding to $i = 1, 2, 3$, we get that

$$\begin{aligned} i_{T,u_1} + i_{T,u_2} + i_{T,u_3} &\leq |U_{T_1}| + |U_{T_2}| + |U_{T_3}| - 3 \\ &\leq |V(G)| - |U_T| - |V(S)| - 3, \\ &\leq |V(G)| - |U_T| - 7. \end{aligned}$$

Since $i_{T,u_1} + i_{T,u_2} + i_{T,u_3} \geq |U_T| - 1$, we get that

$$\begin{aligned} |U_T| - 1 &\leq |V(G)| - |U_T| - 7 \\ &= n - |U_T| - 7, \end{aligned}$$

a contradiction since $2|U_T| - 1 = 2\lfloor \frac{n}{2} \rfloor - 5 \geq n - 6$. Thus, one of the three balanced bipartitions described above qualifies as the balanced bipartition (V_1, V_2) described in the proposition statement. $\square$

Proposition 13. *If G is a plane triangulation with $|V(G)| = n$ such that the sink triangulation S of G contains exactly 4 vertices, then there exists a balanced bipartition (V_1, V_2) of $V(G)$ such that both V_1 and V_2 induce connected near-triangulations, and the total number of blocks in $G[V_1]$ and $G[V_2]$ exceeds the total number of internal vertices by at most 2.*

Proof. Let G be a plane triangulation with $|V(G)| = n$ and let S be the sink triangulation of G, where $V(S) = \{s_1, s_2, s_3, s_4\}$. For $i = 1, ..., 4$, let T_i denote the triangle with the vertex set $V(T_i) = V(S) - s_i$.

If, for some pair $i, j \in \{1, ..., 4\}, i \neq j, |U_{T_i}| + |U_{T_j}| + 2 = \lceil \frac{n}{2} \rceil$, then $V_1 := U_{T_i} \cup U_{T_j} \cup (V(S) - \{s_i, s_j\})$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where both V_1 and V_2 induce biconnected near-triangulations. So we may assume that $|U_{T_i}| + |U_{T_j}| + 2 \neq \lceil \frac{n}{2} \rceil$ for all pairs $i, j \in \{1, ..., 4\}, i \neq j$. This implies that there exists a bipartition $(U_{T_w}, \{U_{T_x}, U_{T_y}, U_{T_z}\})$ of $\{U_{T_1}, U_{T_2}, U_{T_3}, U_{T_4}\}$ such that, either

(i) $|U_{T_w}| + |U_{T_i}| + 2 > \lceil \frac{n}{2} \rceil$ for every $i \in \{x, y, z\}$, or

(ii) $|U_{T_w}| + |U_{T_i}| + 2 < \lceil \frac{n}{2} \rceil$ for every $i \in \{x, y, z\}$.

Case (i): Note that in this case, $|U_{T_i}| + |U_{T_j}| + 2 < \lceil \frac{n}{2} \rceil$ for all pairs $i, j \in \{x, y, z\}, i \neq j$. Without loss of generality, let $|U_{T_z}| \geq |U_{T_x}|, |U_{T_y}|$.

Due to Proposition 12, we may assume that $|U_{T_w}| + 2 < \lfloor \frac{n}{2} \rfloor$. By Lemma 7, there exists a bipartition $(V_1', V(J_{T_z}) - V_1')$ of $V(J_{T_z})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_w}|, V_1' \cap V(T_z) = \{s_x, s_y\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_z} - V_1'$ is a connected near-triangulation with at most $|U_{T_z} - V_1'| \leq |U_{T_z}| - 1$ blocks. If $i_{T_w,s_z} \geq |U_{T_z}| - 1$, then $V_1 := V_1' \cup U_{T_w}$ and $V_2 := V(G) - V_1$ form the required balanced bipartition. So we may assume that $|U_{T_z}| - 1 \geq i_{T_w,s_z} + 1$.

Without loss of generality, let $b_{T_y,s_z} \geq b_{T_x,s_z}$. By Lemma 7, there exists a bipartition $(V_1', V(J_{T_w}) - V_1')$ of $V(J_{T_w})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_y}| - |U_{T_z}| - 1, V_1' \cap V(T_w) = \{s_x, s_y\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_w} - V_1'$ is a connected near-triangulation. Suppose $V_1' \cap V(B_{T_w,s_z}) = \emptyset$. Then, by Remark 1, the number of blocks in $J_{T_w} - V_1'$ is at most $i_{T_w,s_z} + 1$. Then $V_1 := U_{T_y} \cup U_{T_z} \cup s_w \cup V_1' = U_{T_y} \cup U_{T_z} \cup V(T_z) \cup V_1'$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_z}| + b_{T_y,s_z} - 1$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $i_{T_w,s_z} + 1 + b_{T_x,s_z} \leq |U_{T_z}| - 1 + b_{T_y,s_z}$ blocks. So we may assume that $V_1' \cap V(B_{T_w,s_z}) \neq \emptyset$.

Let $k = \lceil \frac{n}{2} \rceil - |U_{T_y}| - |U_{T_z}| - 1 - |U_{T_w} - V(B_{T_w,s_z})|$. By Lemma 7, there exists a bipartition $(V_1'', V(J_{T_x}) - V_1'')$ of $V(J_{T_x})$ such that $|V_1''| = k, V_1'' \cap V(T_x) = \{s_w, s_y\}, G[V_1'']$ is a biconnected near-triangulation, and $J_{T_x} - V_1''$ is a connected near-triangulation; note that since $k \geq 3$, V_1'' includes at least one vertex from U_{T_x} so that the number of blocks in $J_{T_x} - V_1''$ is at most $|U_{T_x}| - 1$. Then $V_1 := U_{T_y} \cup U_{T_z} \cup s_x \cup (U_{T_w} - V(B_{T_w,s_z})) \cup V_1'' = U_{T_y} \cup U_{T_z} \cup V(T_z) \cup (U_{T_w} - V(B_{T_w,s_z})) \cup V_1''$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_z}|$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $1 + |U_{T_x}| - 1 \leq |U_{T_z}|$ blocks. This concludes case (i).

Case (ii): Note that in this case, $|U_{T_i}| + |U_{T_j}| + 2 > \lceil \frac{n}{2} \rceil$ for all pairs $i, j \in \{x, y, z\}, i \neq j$. Without loss of generality, let $|U_{T_z}| \geq |U_{T_x}|, |U_{T_y}|$.

Suppose $i_{T_x,s_z} + i_{T_y,s_z} \leq |U_{T_z}| - 1$. Let $k = \lceil \frac{n}{2} \rceil - |U_{T_w}| - |U_{T_z}| - |V(T_z)|$. If $k \leq |U_{T_x} - V(B_{T_x,s_z})|$, then by Lemma 7 and Remark 1, there exists a bipartition $(V_1', V(J_{T_x}) - V_1')$ of $V(J_{T_x})$ such that $|V_1'| = k + 2, V_1' \cap V(T_x) = \{s_w, s_y\}, V_1' \cap V(B_{T_x,s_z}) = \emptyset, G[V_1']$ is a biconnected near-triangulation, and $J_{T_x} - V_1'$ is a connected near-triangulation containing the block B_{T_x,s_z}. Then $V_1 := U_{T_w} \cup U_{T_z} \cup V(T_z) \cup V_1'$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_z}|$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $i_{T_x,s_z} + i_{T_y,s_z} + 2 \leq |U_{T_z}| + 1$ blocks. If $k > |U_{T_x} - V(B_{T_x,s_z})|$, then by Lemma 7 there exists a bipartition $(V_1', V(J_{T_y}) - V_1')$ of $V(J_{T_y})$ such that $|V_1'| = k - |U_{T_x} - V(B_{T_x,s_z})| + 2, V_1' \cap V(T_y) = \{s_w, s_x\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_y} - V_1'$ is a connected near-triangulation. Then, $V_1 := U_{T_w} \cup U_{T_z} \cup V(T_z) \cup (U_{T_x} - V(B_{T_x,s_z})) \cup V_1'$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_z}|$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $1 + |U_{T_y}| - 1 \leq |U_{T_z}|$

blocks. So we may assume that $i_{T_x,s_z} + i_{T_y,s_z} \geq |U_{T_z}|$, and hence that one of i_{T_x,s_z} and i_{T_y,s_z} is at least $\lceil \frac{|U_{T_z}|}{2} \rceil$. Without loss of generality, let $i_{T_x,s_z} \geq \lceil \frac{|U_{T_z}|}{2} \rceil$.

Now suppose that $i_{T_x,s_z} \geq i_{T_z,s_x}$.

If $\lceil \frac{n}{2} \rceil - |U_{T_x}| \leq |U_{T_z} - V(B_{T_z,s_x})| + 2$, then by Lemma 7 and Remark 1 there exists a bipartition $(V_1', V(J_{T_z}) - V_1')$ of $V(J_{T_z})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_x}|, V_1' \cap V(T_z) = \{s_w, s_y\}, V_1' \cap V(B_{T_z,s_x}) = \emptyset, G[V_1']$ is a biconnected near-triangulation, and $J_{T_z} - V_1'$ is a connected near-triangulation containing the block B_{T_z,s_x}. Then $V_1 := U_{T_x} \cup V_1'$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least i_{T_x,s_z} internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $2 + i_{T_z,s_x} - 1 \leq i_{T_x,s_z} + 1$ blocks. So we may assume that $\lceil \frac{n}{2} \rceil - |U_{T_x}| > |U_{T_z} - V(B_{T_z,s_x})| + 2$.

If $\lceil \frac{n}{2} \rceil - |U_{T_x}| \geq |U_{T_z}| - \lceil \frac{|U_{T_z}|}{2} \rceil + 2$, then by Lemma 7 there exists a bipartition $(V_1', V(J_{T_z}) - V_1')$ of $V(J_{T_z})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_x}|, V_1' \cap V(T_z) = \{s_w, s_y\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_z} - V_1'$ is a connected near-triangulation. Then $V_1 := U_{T_x} \cup V_1'$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $i_{T_x,s_z} \geq \lceil \frac{|U_{T_z}|}{2} \rceil$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $1 + \lceil \frac{|U_{T_z}|}{2} \rceil$ blocks. So we may assume that $\lceil \frac{n}{2} \rceil - |U_{T_x}| < |U_{T_z}| - \lceil \frac{|U_{T_z}|}{2} \rceil + 2 = \lfloor \frac{|U_{T_z}|}{2} \rfloor + 2$.

Let $k' = \lceil \frac{n}{2} \rceil - |U_{T_x}| - |V(J_{T_z}) - V(B_{T_z,s_x})|$; note that $k' \leq \lfloor \frac{|U_{T_z}|}{2} \rfloor - 1$. Let $V_1' = V(J_{T_z}) - V(B_{T_z,s_x})$ so that $G[V_1'], J_{T_z} - V_1' = B_{T_z,s_x}$ and $G[U_{T_x} \cup V_1']$ are all biconnected near-triangulations, while $G - (U_{T_x} \cup V_1')$ is a connected near-triangulation with exactly 2 blocks one of which is B_{T_z,s_x}. Then, by Corollary 9, there exists a partition $(V_1'', V(J_{T_z}) - V_1'')$ of $V(J_{T_z})$ such that $V_1' \subset V_1'', s_x \in V(J_{T_z}) - V_1'' \subset V(B_{T_z,s_x}), |V_1''| = |V_1'| + k'$, and both $G[V_1'']$ and $J_{T_z} - V_1''$ are connected near-triangulations with a total of at most $k' + 2$ blocks. Then $V_1 := U_{T_x} \cup V_1''$ and $V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a connected near-triangulation with at least $i_{T_x,s_z} \geq \lceil \frac{|U_{T_z}|}{2} \rceil$ internal vertices and $G[V_2]$ is a connected near-triangulation, and the total number of blocks in $G[V_1]$ and $G[V_2]$ is at most $k' + 2 + 1 \leq \lfloor \frac{|U_{T_z}|}{2} \rfloor + 2 \leq \lceil \frac{|U_{T_z}|}{2} \rceil + 2$ blocks.

Observe that since $|U_{T_z}| \geq |U_{T_x}|$, we get that $i_{T_x,s_z} \geq \lceil \frac{|U_{T_z}|}{2} \rceil \geq \lceil \frac{|U_{T_x}|}{2} \rceil$. Then, a similar argument can be made if $i_{T_z,s_x} \geq i_{T_x,s_z}$ instead. This concludes case (ii). $\square$

The map is now ready for deploying the partitioning lemmas from Section 2.3.

2.5 Constructing the bipartition

In this section, we develop a basic construction that yields a balanced bipartition of size at most $|V(G)|$ for any plane triangulation G barring a special case. The construction essentially outlines the steps of an algorithm which first navigates the 4-connected sink triangulation S of G to find an ideal partitioning site, partially building the bipartition in the process, and then partitions the smaller std-subgraph of a single triangle in S (if needed) to balance the two parts.

We first describe the said algorithm *Construct Bipartition*. The input to this algorithm is a plane triangulation G such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G. Roughly speaking, the algorithm attempts to construct the bipartition from scratch by starting with the smaller side of one of the two separating triangles with the largest smaller sides in each part, and then adding more smaller sides to the two parts as it navigates the sink triangulation S of the graph. The algorithm is guided in its navigation of the sink triangulation by Lemma 5, which keeps the two parts biconnected until either the algorithm finishes or the *split configuration* is reached. The split configuration is reached when none of the two parts can accept any more smaller sides without going over the prescribed size of $\lceil \frac{|V(G)|}{2} \rceil$, but there remain vertices which have not yet been added to any part. The algorithm ensures that (in the split configuration) it partitions the smaller side of at most one triangle $T_{par} \subset S$ in order to balance the two parts. In doing so, the parts may not necessarily remain biconnected but they are still connected near-triangulations. However, at the end of the entire process, the algorithm has collected enough internal vertices in the two parts so that the total number of blocks in the two near-triangulations exceeds the number of internal vertices by at most 2. The algorithm ends by returning the required bipartition (V_1, V_2) of $V(G)$, except for a special case which we handle separately (the algorithm returns a "null" partition in this case).

The algorithm uses the following terminology associated with the split configuration. The split configuration consists of a vertex $v_{last} \in V(S)$ which is the only vertex in $V(S)$ that has not yet been added to any part of the bipartition, and triangles $T_{last_1}, T_{last_2} \subseteq S$ such that $V(T_{last_1}) \cap V(T_{last_2}) = v_{last}$. Claim 21 provides a more detailed description of the split configuration.

Algorithm: Construct Bipartition

Input: A plane triangulation G with $|V(G)| = n$, such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G.

(CB1) Pick two distinct triangles $T_1, T_2 \subset S$ such that, for every other triangle $T \subset S$, $|V(J_T)| \leq |V(J_{T_1})|, |V(J_{T_2})|$. Choose T_1 and T_2 with as few vertices in common as possible. If $|V(T_1) \cap V(T_2)| = 2$, then return $(\emptyset, \emptyset)$.

Let $|V(J_{T_1})| \geq |V(J_{T_2})|$. For $i = 1, 2$, let $V(T_i) = \{a_i, b_i, c_i\}$, and let $\{b_i, c_i\} \subseteq V(T_i) - V(T_{3-i})$. For $i = 1, 2$, intialize $V_i = V(J_{T_i}) - a_i$ and $V_{S,i} = \{b_i, c_i\}$.

(CB2) For $i = 1, 2$, let $T'_i \subset S$ be the triangle that shares the edge $b_i c_i$ with T_i in S.

If $|U_{T_i}| + |U_{T'_i}| + 2 > \lceil \frac{n}{2} \rceil$ for $i = 1$ or 2, then by Lemma 7 there exists a partition $(V'_1, V(J_{T'_i}) - V'_1)$ of $V(J_{T'_i})$ where $|V'_1| = \lceil \frac{n}{2} \rceil - |U_{T_i}|, V'_1 \cap V(T'_i) = \{b_i, c_i\}, G[V'_1]$ is a biconnected near-triangulation, and $J_{T'_i} - V'_1$ is a connected near-triangulation; update $V_1 = U_{T_i} \cup V'_1, V_2 = V(G) - V_1, V_{S,1} = V(S) \cap V_1, V_{S,2} = V(S) \cap V_2$, initialize $T_{par} = T'_i$, and return (V_1, V_2).

Otherwise, for $i = 1, 2$, update $V_i = V_i \cup U_{T'_i}$.

(CB3) For $i = 1, 2$, let $T_{b_i}, T_{c_i} \subset S$ be the triangles sharing the edges $a_i b_i$ and $a_i c_i$ with T_i, respsectively.

If $|V_i| + 1 + |U_{T_{b_i}}| + |U_{T_{c_i}}| > \lceil \frac{n}{2} \rceil$ for some $i \in \{1, 2\}$, then update $V_{3-i} = V(G) - V_i - a_i - U_{T_{b_i}} - U_{T_{c_i}}, V_{S,3-i} = V(S) \cap V_{3-i}$, initialize $v_{last} = a_i, \{T_{last_1}, T_{last_2}\} = \{T_{b_i}, T_{c_i}\}$, and go to (CB5).

Otherwise update $V_1 = V_1 \cup a_1 \cup U_{T_{b_1}} \cup U_{T_{c_1}}, V_{S,1} = V_{S,1} \cup a_1$; if $|V_1| < \lceil \frac{n}{2} \rceil$, then go to (CB4), otherwise update $V_2 = V(G) - V_1, V_{S,2} = V(S) \cap V_2$, and return (V_1, V_2).

(CB4) By Lemma 5 there exists a vertex $v \in V(S) - (V_{S,1} \cup V_{S,2})$ such that both $S[V_{S,1} \cup v]$ and $S - (V_{S,1} \cup v)$ are biconnected near-triangulations. Let N_v denote the set of vertices on the boundary of the infinite face of $S[V_{S,1}]$ that are adjacent to v, and let $u, w \in N_v$ be the two vertices such that $|N_v \cap N_S(u)| = |N_v \cap N_S(w)| = 1$. Let $T_u, T_w \subset S$ be the triangles such that $\{u, v\} \subset V(T_u), \{v, w\} \subset V(T_w)$ and $V_{S,1} \cap V(T_u) = u, V_{S,1} \cap V(T_w) = w$.

If $|V_1| + 1 + |U_{T_u}| + |U_{T_w}| \leq \lceil \frac{n}{2} \rceil$, then update $V_1 = V_1 \cup v \cup U_{T_u} \cup U_{T_w}, V_{S,1} = V(S) \cap V_1$; if $|V_1| < \lceil \frac{n}{2} \rceil$, then repeat (CB4), otherwise update $V_2 = V(G) - V_1, V_{S,2} = V(S) \cap V_2$, and return (V_1, V_2).

Otherwise update $V_2 = V(G) - V_1 - v - U_{T_u} - U_{T_w}, V_{S,2} = V(S) \cap V_2$, and initialize $v_{last} = v, \{T_{last_1}, T_{last_2}\} = \{T_u, T_w\}$.

(CB5) G is now in the *split configuration.*

Without loss of generality, let $|U_{T_{last_1}}| \geq |U_{T_{last_2}}|$. For some $i \in \{1, 2\}, |V_i| + 1 + |U_{T_{last_2}}| \leq \lceil \frac{n}{2} \rceil$. By Lemma 7, there exists a bipartition $(V'_1, V(J_{T_{last_1}}) - V'_1)$ of $V(J_{T_{last_1}})$ such that $|V'_1| = \lceil \frac{n}{2} \rceil - (|V_i| + 1 + |U_{T_{last_2}}|) + 2, V'_1 \cap V(T_{last_1}) = \{v_{last}, V(T_{last_1}) \cap V_{S,i}\}, G[V'_1]$ is a biconnected near-triangulation, and $J_{T_{last_1}} - V'_1$ is a connected near-triangulation. Update $V_1 = V_i \cup v_{last} \cup U_{T_{last_2}} \cup V'_1, V_2 = V(G) - V_1, V_{S,1} = V(S) \cap V_1, V_{S,2} = V(S) \cap V_2$, initialize $T_{par} = T_{last_1}$, and return (V_1, V_2).

We will now prove a series of claims which will establish that the algorithm is well-defined and always returns the required bipartition (V_1, V_2) of $V(G)$, except when $|V(T_1) \cap V(T_2)| = 2$. As mentioned earlier, these proofs assume that the input graph G to the algorithm is a plane triangulation with $|V(G)| = n$, such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G.

Claim 14. *Triangles T_1 and T_2 exist.*

Proof. This follows directly from the assumption that S contains at least 2 distinct triangles that are separating in G. $\square$

Claim 15. *Triangles T_1, T_2, T_1' and T_2' are distinct.*

Proof. This follows directly from the observation that the edges b_1c_1 and b_2c_2 have no vertices in common. $\square$

Claim 16. *If $|U_{T_i}| + |U_{T_i'}| + 2 > \lceil \frac{n}{2} \rceil$ for some $i \in \{1, 2\}$, then Algorithm Construct Bipartition returns a bipartition (V_1, V_2) of $V(G)$ such that $|V_1| = \lceil \frac{n}{2} \rceil, G[V_1]$ is a biconnected near-triangulation, and $G[V_2]$ is a connected near-triangulation with the number of blocks exceeding the total number of internal vertices in $G[V_1]$ and $G[V_2]$ by at most* 1.

Proof. This follows from the observations that $V(T_{3-i}) \subseteq V_2$ which implies that all the vertices in $U_{T_{3-i}}$ are internal vertices in $G[V_2]$, and that the number of blocks in $G[V_2]$ is at most $1 + |U_{T_i'}| \leq 1 + |U_{T_{3-i}}|$. $\square$

Claim 17. *For any triangle $T \subset S, S - V(T)$ is 2-connected.*

Proof. Let $T \subset S$ be a triangle with $V(T) = \{a, b, c\}$. Suppose for the sake of contradiction that $S - V(T)$ is not 2-connected. Since $|V(S)| \geq 6$ and $S - \{b, c\}$ is 2-connected, we get that a is in a 2-cut in $S - \{b, c\}$ and hence has at least three neighbors in the cycle C that forms the boundary of the infinite face of $S - \{b, c\}$. Since $V(C) = (N_S(b) \cup N_S(c)) - \{b, c\}$ and $a \in V(C)$, if a has at least three neighbors in C then either a shares at least two neighbors with b none of which is c, or a shares at least two neighbors with c none of which is b. In either case, we get a separating triangle in S, a contradiction to S being 4-connected. $\square$

Claim 18. *Each of $S[V_{S,1}], S - V_{S,1}, S[V_{S,2}]$ and $S - V_{S,2}$ is biconnected.*

Proof. This follows directly from Claim 17 and Lemma 5, since after the first two vertices every new vertex added to $V_{S,1}$ or $V_{S,2}$ respects one of these. $\square$

Claim 19. *If $v \in V(S) - (V_{S,1} \cup V_{S,2})$ is a vertex such that both $S[V_{S,1} \cup v]$ and $S - (V_{S,1} \cup v)$ are biconnected near-triangulations, and if N_v denotes the set of vertices on the boundary of the infinite face of $S[V_{S,1}]$ that are adjacent to v, then there exist exactly two vertices $u, w \in N_v$ such that $|N_v \cap N_S(u)| = |N_v \cap N_S(w)| = 1$.*

Moreover, if $T_u, T_w \subset S$ are the triangles such that $\{u, v\} \subset V(T_u), \{v, w\} \subset V(T_w)$ and $V_{S,1} \cap V(T_u) = u, V_{S,1} \cap V(T_w) = w$, then T_u and T_w are distinct, and $(U_{T_u} \cup U_{T_w}) \cap (V_1 \cup V_2) = \emptyset$.

Proof. By Lemma 5, v forms a triangular face of S with an edge on the boundary of the infinite face of $S[V_{S,1}]$. If $|V_{S,1}| = 2$, then the first part of the claim holds trivially. So we may assume that $|V_{S,1}| > 2$ which, since S is 4-connected, implies that $N_S(v) \cap V_{S,1}$ induces a subpath P of the boundary of the infinite face of $S[V_{S,1}]$. Then u and w are the end-vertices of this subpath P.

The triangles T_u and T_w are distinct for otherwise the vertex-set $\{v, u, w\}$ induces a separating triangle in S, a contradiction.

The last part of the claim follows from the observation that if $V_{S,1} \cup V_{S,2} \neq V(S)$, then for any triangle $T \subset S$ and any $j \in \{1, 2\}$, if $U_T \cap V_j \neq \emptyset$ then $U_T \subset V_j$ and $|V(T) \cap V_j| \geq 2$. $\square$

Claim 20. $V_{S,1} \cap V_{S,2} = \emptyset = V_1 \cap V_2$.

Proof. The first equality follows directly from the construction of $V_{S,1}$ and $V_{S,2}$. The second equality follows from the first and the observations that, for any triangle $T \subset S$, (i) if $U_T \subset V_i$ for some $i \in \{1, 2\}$, then $|V(T) \cap V_i| \geq 2$, and (ii) if $U_T \not\subset V_i$ for $i = 1, 2$, then $T = T_{par}$ and U_T is partitioned between V_1 and V_2. $\square$

Claim 21. *Let $v_{last} \in V(S)$ be the only vertex in $V(S)$ that is not added to any part of the bipartition when the algorithm is in the split configuration. For $i = 1, 2$, let $N_S(v_{last}) \cap V_{S,i} = \{w_{i,1}, ..., w_{i,d_i}\}$, where $w_{i,1}, ..., w_{i,d_i}$ are taken in clockwise order around v_{last}. Then,*

(i) $|N_S(v_{last})| = d_1 + d_2$ *and* $w_{1,1}, ..., w_{1,d_1}, w_{2,1}, ..., w_{2,d_2}$ *occur clockwise in that order around* v_{last}*,*

(ii) $d_1, d_2 \geq 2$*, and*

(iii) $\{V(T_{last_1}), V(T_{last_2})\} = \{\{v_{last}, w_{1,1}, w_{2,d_2}\}, \{v_{last}, w_{2,1}, w_{1,d_1}\}\}$.

Proof. Since $v_{last} \in V(S)$ is the only vertex in $V(S)$ that is not added to any part of the bipartition $N_S(v_{last}) = \bigcup_{i=1,2} \{w_{i,1}, ..., w_{i,d_i}\}$ and hence $|N_S(v_{last})| = d_1 + d_2$. Recall that, for some $j \in \{1, 2\}$, $v_{last} \in V(S) - (V_{S,1} \cup V_{S,2})$ is picked so that both $S[V_{S,j} \cup v_{last}]$ and $S - (V_{S,j} \cup v_{last})$ are biconnected near-triangulations, and that $N_S(v_{last}) \cap V_{S,j}$ induces a subpath of the boundary of the infinite face of $S[V_{S,j}]$. Let $C \subset S$ be the cycle that forms the boundary of the face of $S - v_{last}$ that contains v_{last}. Then $C[\{w_{j,1}, ..., w_{j,d_j}\}]$ is connected for otherwise v_{last} is a cut-vertex in $S - V_{S,j}$, a contradiction to $S - V_{S,j}$ being biconnected. Thus, $C[\{w_{i,1}, ..., w_{i,d_i}\}]$ is connected for $i = 1, 2$, and $w_{1,1}, ..., w_{1,d_1}, w_{2,1}, ..., w_{2,d_2}$ occur clockwise in that order around v_{last}.

Again, since for some $j \in \{1, 2\}$, both $S[V_{S,j} \cup v_{last}]$ and $S - V_{S,j}$ are 2-connected, $d_1, d_2 \geq 2$.

This follows directly from the observation that the algorithm reaches the split configuration when, for some $j \in \{1,2\}$, $|V_j|+1+|U_{T_{last_1}}|+|U_{T_{last_2}}| > \lceil \frac{n}{2} \rceil$, where $T_{last_1}, T_{last_2} \subset S$ are distinct triangles such that, for some $\{x,y\} = \{w_{j,1}, w_{j,d_j}\}$, $\{x, v_{last}\} \subset V(T_{last_1})$, $\{y, v_{last}\} \subset V(T_{last_2})$ and $V_{S,j} \cap V(T_{last_1}) = x$, $V_{S,j} \cap V(T_{last_2}) = y$. □

Claim 22. *In the split configuration, for some $i \in \{1,2\}$, $|V_i| + 1 + |U_{T_{last_2}}| \leq \lceil \frac{n}{2} \rceil$.*

Proof. Note that the algorithm reaches the end configuration when, for some $j \in \{1,2\}$, $|V_j| < \lceil \frac{n}{2} \rceil$, $|V_j|+1+|U_{T_{last_1}}|+|U_{T_{last_2}}| > \lceil \frac{n}{2} \rceil$, and $V_{3-j} = V(G) - V_j - v_{last} - U_{T_{last_1}} - U_{T_{last_2}}$. The last two inequalities give us that $|V_{3-j}| < \lceil \frac{n}{2} \rceil$, and the first and the third inequalities give us that $|V_{3-j}| + 1 + |U_{T_{last_1}}| + |U_{T_{last_2}}| > \lceil \frac{n}{2} \rceil$.

Suppose $|V_i| + 1 + |U_{T_{last_2}}| > \lceil \frac{n}{2} \rceil$ for $i = 1,2$. Then $|V_i| + |U_{T_{last_2}}| \leq |V_i| + |U_{T_{last_1}}| < \lfloor \frac{n}{2} \rfloor \leq \lceil \frac{n}{2} \rceil$, a contradiction. □

Claim 23. *Let G be a plane triangulation with $|V(G)| = n$, such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G. If $T_1, T_2 \subset S$ are distinct triangles such that $|V(T_1) \cap V(T_2)| \leq 1$ and, for every other triangle $T \subset S$, $|V(J_T)| \leq |V(J_{T_1})|, |V(J_{T_2})|$, then Algorithm Construct Bipartition returns a bipartition (V_1, V_2) of $V(G)$ with $|V_1| = \lceil \frac{n}{2} \rceil$ such that $G[V_1]$ is a biconnected near-triangulation, and $G[V_2]$ is a connected near-triangulation with the number of blocks exceeding the total number of internal vertices in $G[V_1]$ and $G[V_2]$ by at most 1.*

Proof. Note that, by Claim 20 $V_1 \cap V_2 = \emptyset$, and by construction $V_1 \cup V_2 = V(G)$ and $|V_1| = \lceil \frac{n}{2} \rceil$. Moreover, $S[V_{S,1}]$ and $S[V_{S,2}]$ are biconnected near-triangulations at all times, and before $J_{T_{last_1}}$ is partitioned (if at all), for every triangle $T \subset S, T \neq T_{last_1}$, and for some $j \in \{1,2\}$, $U_T \subset V_j$ and $|V(T) \cap V_j| \geq 2$. This gives us that $G[V_1]$ and $G[V_2]$ remain biconnected near-triangulations before $J_{T_{last_1}}$ is partitioned. After the bipartition of $V(J_{T_{last_1}})$ following Lemma 7, by construction, $G[V_1]$ is a biconnected near-triangulation, while $G[V_2]$ is a connected near-triangulation with at most $|U_{T_{last_1}}| + 1$ blocks. Then to complete the proof it suffices to show that, for some $j, k \in \{1,2\}$, $V(T_j) \subseteq V_k$, which implies that $G[V_k]$ contains all the vertices in U_{T_j} as internal vertices, and hence that the total number of internal vertices in $G[V_1]$ and $G[V_2]$ is at least $|U_{T_j}| \geq |U_{T_{last_1}}|$.

Upto the choice of labels, let $\{b_i, c_i\} \subset V_i$ for $i = 1,2$. If $|V(T_1) \cap V(T_2)| = 1$, then for some $i \in \{1,2\}$, $a_1 = a_2 \in V_i$ and hence $V(T_i) \subseteq V_i$. So we may assume that $|V(T_1) \cap V(T_2)| = 0$ and $a_1 \neq a_2$. But then, by step (CB3) of the algorithm, if $|V_i|+1+|U_{T_{b_i}}|+|U_{T_{c_i}}| > \lceil \frac{n}{2} \rceil$ for some $i \in \{1,2\}$, we get that $V(T_{3-i}) \subseteq V_{3-i}$, otherwise we get that $V(T_i) \subseteq V_i$. □

This concludes this section.

2.6 Special case

In this section, we deal with the special case that is "skipped" by the algorithm in the preceding section. The proof in this section still uses Algorithm Construct Bipartition but with minor variations based on certain "local" observations. The following remarks describe these variants of Algorithm Construct Bipartition.

Remark 2. Note that Algorithm Construct Bipartition does not depend on the assumption that triangles T_1 and T_2 have the largest subgraphs J_{T_1} and J_{T_2}, respectively, in its operation. Thus, it can be run for any pair of distinct triangles $\bar{T}_1, \bar{T}_2 \subset S$ such that $|V(\bar{T}_1) \cap V(\bar{T}_2)| \leq 1$. Similarly, since the proofs of Claim 15 and Claims 17-22 do not make that assumption either, they hold if the algorithm is run for the triangles $\bar{T}_1$ and $\bar{T}_2$ described above. Then, the bipartition (V_1, V_2) of $V(G)$ returned by the algorithm satisfies the conclusions of Claims 16 and 23 if either both $G[V_1]$ and $G[V_2]$ are biconnected, or the two together have at least $|U_{T_{par}}|$ internal vertices, where $T_{par} \subset S$ is the triangle for which the subgraph $J_{T_{par}}$ is partitioned by the algorithm.

Remark 3. Note that if $\bar{T}_1, \bar{T}_2 \subset S$ are distinct triangles such that $V(\bar{T}_i) = \{p_1, p_2, q_i\}$ for $i = 1, 2$, then Algorithm Construct Bipartition can be run for $\bar{T}_1$ and $\bar{T}_2$ by replacing steps (CB1)-(CB3) with the following steps:

(CB1') For $i = 1, 2$, intialize $V_i = V(J_{\bar{T}_i}) - p_{3-i}, V_{S,i} = \{p_i, q_i\}$.

(CB2') For $i = 1, 2$, let $\bar{T}'_i \subset S$ be the triangle that shares the edge $p_i q_i$ with $\bar{T}_i$ in S.

If $|U_{\bar{T}_i}| + |U_{\bar{T}'_i}| + 2 > \lceil \frac{n}{2} \rceil$ for $i = 1$ or 2, then by Lemma 7 there exists a partition $(V'_1, V(J_{\bar{T}'_i}) - V'_1)$ of $V(J_{\bar{T}'_i})$ where $|V'_1| = \lceil \frac{n}{2} \rceil - |U_{\bar{T}_i}|, V'_1 \cap V(\bar{T}'_i) = \{p_i, q_i\}, G[V'_1]$ is a biconnected near-triangulation, and $J_{\bar{T}'_i} - V'_1$ is a connected near-triangulation; update $V_1 = U_{\bar{T}_i} \cup V'_1, V_2 = V(G) - V_1, V_{S,1} = V(S) \cap V_1, V_{S,2} = V(S) \cap V_2$, initialize $T_{par} = \bar{T}'_i$, and return (V_1, V_2).

Otherwise, for $i = 1, 2$, update $V_i = V_i \cup U_{\bar{T}'_i}$.

Moreover, Claim 15 and Claims 17-22 hold in this case as well, and the bipartition returned by the algorithm satisfies the conclusions of Claims 16 and 23 if either both $G[V_1]$ and $G[V_2]$ are biconnected, or the two together have at least $|U_{T_{par}}|$ internal vertices, where $T_{par} \subset S$ is the triangle whose std-subgraph $J_{T_{par}}$ is partitioned by the algorithm.

The next lemma settles the special case.

Lemma 24. *Let G be a plane triangulation with $|V(G)| = n$, such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G. If $|V(T_1) \cap V(T_2)| = 2$ for every pair of distinct triangles $T_1, T_2 \subset S$ with the property that $|V(J_{T_1})|, |V(J_{T_2})| \geq |V(J_T)|$ for every triangle $T \subset S, T \neq T_1, T_2$,* *then there exists a balanced bipartition* (V_1, V_2) *of* $V(G)$ *such that both* $G[V_1]$ *and* $G[V_2]$ *are connected near-triangulations with the total number of blocks in* $G[V_1]$ *and* $G[V_2]$ *exceeding the total number of internal vertices by at most* 2.

Proof. Let G be a plane triangulation with $|V(G)| = n$, such that the sink triangulation S of G is 4-connected with at least 6 vertices and contains at least 2 distinct triangles that are separating in G; also, let $|V(T_1) \cap V(T_2)| = 2$ for every pair of distinct triangles $T_1, T_2 \subset S$ with the property that $|V(J_{T_1})|, |V(J_{T_2})| \geq |V(J_T)|$ for every triangle $T \subset S, T \neq T_1, T_2$. Let $T_1, T_2 \subset S$ be one such pair of distinct triangles; note that both T_1 and T_2 are separating in G. Let $V(T_1) \cap V(T_2) = \{p, q\}$, and for $i = 1, 2$, let $V(T_i) - V(T_{3-i}) = r_i$. If $|U_{T_1}| + |U_{T_2}| + 2 = \lceil \frac{n}{2} \rceil$, then $V_1 := U_{T_1} \cup U_{T_2} \cup \{p, q\}$ and $V_2 := V(G) - V_2$ form a balanced biparition of $V(G)$ where both $G[V_1]$ and $G[V_2]$ are biconnected near-triangulations. So we may assume that either

(i) $|U_{T_1}| + |U_{T_2}| + 2 < \lceil \frac{n}{2} \rceil$, or

(ii) $|U_{T_1}| + |U_{T_2}| + 2 > \lceil \frac{n}{2} \rceil$.

Let $T_{p_1}, T_{q_1} \subset S$ be the triangles that share the edges pr_1, qr_1 with T_1 in S, respectively. Similarly, let $T_{p_2}, T_{q_2} \subset S$ be the triangles that share the edges pr_2, qr_2 with T_2 in S, respectively. Note that since S is 4-connected, the triangles $T_{p_1}, T_{q_1}, T_{p_2}$ and T_{q_2} are all distinct.

Case (i): Since S is a 4-connected triangulation with at least 6 vertices, S does not contain the edge $r_1 r_2$ and the boundary of the infinite face of $S - \{p, q\}$ contains at least 4 verties. Then there exists a triangle $T_3 \subset S$ distinct from $T_1, T_2, T_{p_1}, T_{q_1}, T_{p_2}$ and T_{q_2} such that $V(T_3) \cap \{p, q\} = \emptyset$ and $|V(T_3) \cap \{r_1, r_2\}| \leq 1$. Let xy be an edge of T_3 such that $\{x, y\} \cap \{p, q, r_1, r_2\} = \emptyset$.

Now, we may assume that $|U_{T_1}| + |U_{T_2}| + 3 + \min\{|U_{T_{p_1}}|, |U_{T_{q_1}}|\} > \lceil \frac{n}{2} \rceil$ for otherwise, by Remark 2, running Algorithm Construct Bipartition for triangles $\bar{T}_1 := T_1$ and $\bar{T}_2 := T_3$ with $b_1 c_1 := pq$ and $b_2 c_2 := xy$ leads to a solution (V_1, V_2) with $V(J_{T_1})$ contained in one of the two parts, and hence at least $|U_{T_1}|$ internal vertices (say $|U_{T_{p_1}}| \leq |U_{T_{q_1}}|$; if $|U_{T_1}| + |U_{T_2}| + 3 + |U_{T_{p_1}}| + |U_{T_{q_1}}| \leq \lceil \frac{n}{2} \rceil$, then the algorithm includes U_{T_2} followed by $r_1 \cup U_{T_{p_1}} \cup U_{T_{q_1}}$ in the part initialized with $V(J_{T_1}) - r_1$, otherwise it includes U_{T_2} followed by $r_1 \cup U_{T_{p_1}}$ in that part, and then partitions $U_{T_{q_1}}$ between the two parts). Similarly, we may assume that $|U_{T_1}| + |U_{T_2}| + 3 + \min\{|U_{T_{p_2}}|, |U_{T_{q_2}}|\} > \lceil \frac{n}{2} \rceil$.

Without loss of generality, let $|U_{T_{p_1}}| \leq |U_{T_{q_1}}|$. By Lemma 7, there exists a partition $(V_1', V(J_{T_{p_1}}) - V_1')$ of $V(J_{T_{p_1}})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_1}| - |U_{T_2}| - 1, V_1' \cap V(T_{p_1}) = \{p, r_1\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_{p_1}} - V_1'$ is a connected near-triangulation. If $|U_{T_1}| \geq |U_{T_{p_1}}| + |U_{T_{q_1}}|$, then $V_1 := U_{T_1} \cup U_{T_2} \cup q \cup V_1', V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_1}|$ internal vertices and $G[V_2]$ is a connected near-triangulation with at most $|U_{T_{p_1}}| + |U_{T_{q_1}}| + 1$

blocks. So we may assume that $|U_{T_1}| < |U_{T_{p_1}}| + |U_{T_{q_1}}|$. Similarly, we may assume that $|U_{T_2}| < |U_{T_{p_2}}| + |U_{T_{q_2}}|$. Adding these two inequalities, we get that

$$|U_{T_1}| + |U_{T_2}| \le |U_{T_{p_1}}| + |U_{T_{q_1}}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| - 2. \tag{2.2}$$

Let $V(T_{p_1}) = \{p, r_1, t_1\}$ and $V(T_{q_2}) = \{q, r_2, t_2\}$; note that $t_1 \neq t_2$. By Remark 3, if the algorithm is run for triangles $\bar{T}_1 := T_1$ and $\bar{T}_2 := T_2$ with $p_1q_1 := pr_1$ and $p_2q_2 := qr_2$, then when the algorithm first executes step (CB4), it includes either t_1 in the part containing $U_{T_1} \cup U_{T_{p_1}} \cup \{p, r_1\}$ or t_2 in the part containing $U_{T_2} \cup U_{T_{q_2}} \cup \{q, r_2\}$. Say it does the former and the part containing t_1 is V_1. Then V_1 cannot include t_2 for otherwise q or r_2 would be a cut-vertex in $V_{S,2}$, unless $V_{S,1}$ includes $V(S) - \{q, r_2\}$ entirely in which case $\lceil \frac{n}{2} \rceil \le |U_{T_2}| + |\{q, r_2\}| + |U_{T_{q_2}}| \le |U_{T_2}| + 2 + |U_{T_1}|$, a contradiction. Thus, $V(J_{T_{p_1}})$ and $V(J_{T_{q_2}})$ are entirely contained in different parts of the bipartition (V_1, V_2) returned by the algorithm, and together $G[V_1]$ and $G[V_2]$ have at least $|U_{T_{p_1}}| + |U_{T_{q_2}}|$ internal vertices. If $|U_{T_{p_1}}| + |U_{T_{q_2}}| \ge |U_{T_2}|$, then by Remark 3, this bipartition satisfies the conclusion of the lemma. So we may assume that $|U_{T_{p_1}}| + |U_{T_{q_2}}| < |U_{T_2}|$. Similarly, we may assume that $|U_{T_{p_2}}| + |U_{T_{q_1}}| < |U_{T_2}|$. Adding the two inequalities, we get that

$$|U_{T_{p_1}}| + |U_{T_{q_2}}| + |U_{T_{p_2}}| + |U_{T_{q_1}}| \le |U_{T_1}| + |U_{T_2}| - 2,$$

a contradiction to (2.2).

Case (ii): We first prove the following claim.

Claim 25. *Either* $|U_{T_1}| + |U_{T_{p_1}}| + 2 > \lceil \frac{n}{2} \rceil$ *or* $|U_{T_1}| + |U_{T_{q_1}}| + 2 > \lceil \frac{n}{2} \rceil$.

Proof. Suppose for the sake of contradiction that neither is true. If $|U_{T_1}| + |U_{T_{p_1}}| + 2 = \lceil \frac{n}{2} \rceil$, then $V_1 := U_{T_1} \cup U_{T_{p_1}} \cup \{p, r_1\}, V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where both $G[V_1]$ and $G[V_2]$ are biconnected near-triangulations. So we may assume that $|U_{T_1}| + |U_{T_{p_1}}| + 2 < \lceil \frac{n}{2} \rceil$. Similarly, we may assume that $|U_{T_1}| + |U_{T_{q_1}}| + 2 < \lceil \frac{n}{2} \rceil$.

Let $V(T_{p_1}) = \{p, r_1, t_1\}$ and $V(T_{q_1}) = \{q, r_1, s_1\}$; note that the vertices r_2, s_1 and t_1 are all distinct. Recall that $b_{T_{p_1}, t_1}$ denotes the number of blocks in $J_{T_{p_1}} - \{p, r_1\}$. Without loss of generality, let $b_{T_{p_1}, t_1} \ge b_{T_{q_1}, s_1}$. By Lemma 7, there exists a bipartition $(V_1', V(J_{T_2}) - V_1')$ of $V(J_{T_2})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_1}| - |U_{T_{p_1}}| - 1, V_1' \cap V(T_2) = \{p, q\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_2} - V_1'$ is a connected near-triangulation with at most $|U_{T_2}|$ blocks. If $b_{T_{p_1}, t_1} + |U_{T_1}| \ge b_{T_{q_1}, s_1} + |U_{T_2}| + 1$, then $V_1 := U_{T_1} \cup U_{T_{p_1}} \cup r_1 \cup V_1', V_2 := V(G) - V_1'$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_1}| + b_{T_{p_1}, t_1} - 1 \ge |U_{T_2}| + b_{T_{q_1}, s_1}$ internal vertices and $G[V_2]$ is a connected near-triangulation with at most $1 + |U_{T_2}| + b_{T_{q_1}, s_1}$ blocks. So we may assume that $b_{T_{p_1}, t_1} + |U_{T_1}| \le b_{T_{q_1}, s_1} + |U_{T_2}|$, and hence that $b_{T_{p_1}, t_1} = b_{T_{q_1}, s_1}$ and $|U_{T_1}| = |U_{T_2}|$.

Now, if $|U_{T_1}| + |U_{T_{p_1}}| + |U_{T_{q_1}}| + 3 \le \lceil \frac{n}{2} \rceil$, then by Lemma 7 there exists a bipartition $(V_1', V(J_{T_2}) - V_1')$ of $V(J_{T_2})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_1}| - |U_{T_{p_1}}| - |U_{T_{q_1}}| - 1, V_1' \cap$

$V(T_2) = \{p, q\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_2} - V_1'$ is a connected near-triangulation with at most $|U_{T_2}|$ blocks. Then $V_1 := U_{T_1} \cup U_{T_{p_1}} \cup U_{T_{q_1}} \cup r_1 \cup V_1', V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_1}|$ internal vertices, and $G[V_2]$ is a connected near-triangulation with at most $1 + |U_{T_2}| = 1 + |U_{T_1}|$ blocks. So it must be the case that $|U_{T_1}| + |U_{T_{p_1}}| + |U_{T_{q_1}}| + 3 > \lceil \frac{n}{2} \rceil$. Similarly, it must be the case that $|U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + 3 > \lceil \frac{n}{2} \rceil$. But then $|V(G)| \geq |U_{T_1}| + |U_{T_{p_1}}| + |U_{T_{q_1}}| + |U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + |V(S)| \geq 2(\lceil \frac{n}{2} \rceil + 1) \geq |V(G)| + 2$, a contradiction. $\square$

Without loss of generality, let $|U_{T_1}| + |U_{T_{p_1}}| + 2 > \lceil \frac{n}{2} \rceil$. Then $|U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + 3 < \lceil \frac{n}{2} \rceil$. Recall that i_{T_1,r_1} denotes the number of internal vertices in $J_{T_1} - r_1$, and B_{T_1,r_1} denotes the leaf-block in $J_{T_1} - \{p, q\}$ that contains r_1. By Lemma 7, there exists a bipartition $(V_1', V(J_{T_2}) - V_1')$ of $V(J_{T_2})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_1}|, |V_1'| \geq 3, V_1' \cap V(T_2) = \{p, q\}, G[V_1']$ is a biconnected near-triangulation, and $J_{T_2} - V_1'$ is a connected near-triangulation with at most $|U_{T_2}| - 1$ blocks; here $|V_1'| \geq 3$ follows from Proposition 12. If $i_{T_1,r_1} \geq |U_{T_2}| - 1$, then $V_1 := U_{T_1} \cup V_1', V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least i_{T_1,r_1} internal vertices and $G[V_2]$ is a connected near-triangulation with at most $1 + |U_{T_2}| - 1 \leq i_{T_1,r_1} + 1$ blocks. So we may assume that $i_{T_1,r_1} \leq |U_{T_2}| - 2$.

If $|U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + 3 + |U_{T_1} - V(B_{T_1,r_1})| \geq \lceil \frac{n}{2} \rceil$, then by Lemma 7 and Remark 1 there exists a bipartition $(V_1', V(J_{T_1}) - V_1')$ of $V(J_{T_1})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |U_{T_2}| - |U_{T_{p_2}}| - |U_{T_{q_2}}| - 1, V_1' \cap V(T_1) = \{p, q\}, V_1' \cap V(B_{T_1,r_1}) = \emptyset, G[V_1']$ is a biconnected near-triangulation, and $J_{T_1} - V_1'$ is a connected near-triangulation with at most $1 + i_{T_1,r_1} - 1 = i_{T_1,r_1}$ blocks (since $|U_{T_1}| + |U_{T_{p_1}}| + 2 > \lceil \frac{n}{2} \rceil$, we get that $|U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + 3 < \lfloor \frac{n}{2} \rfloor \leq \lceil \frac{n}{2} \rceil$, and hence $|V_1'| \geq 3$). Then $V_1 := U_{T_2} \cup U_{T_{p_2}} \cup U_{T_{q_2}} \cup r_2 \cup V_1', V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_1]$ is a biconnected near-triangulation with at least $|U_{T_2}|$ internal vertices and $G[V_2]$ is a connected near-triangulation with at most $1 + i_{T_1,r_1} \leq |U_{T_2}| - 1$ blocks. So we may assume that $|U_{T_2}| + |U_{T_{p_2}}| + |U_{T_{q_2}}| + 3 + |U_{T_1} - V(B_{T_1,r_1})| < \lceil \frac{n}{2} \rceil$.

If $|U_{T_{p_1}}| + |V(B_{T_1,r_1})| \geq \lceil \frac{n}{2} \rceil$, then by Lemma 7 there exists a bipartition $(V_1', V(J_{T_{p_1}}) - V_1')$ of $V(J_{T_{p_1}})$ such that $|V_1'| = \lceil \frac{n}{2} \rceil - |V(B_{T_1,r_1})| + 1, (V(J_{T_{p_1}}) - V_1') \cap V(T_{p_1}) = \{p, t_1\}, J_{T_{p_1}} - V_1'$ is a biconnected near-triangulation, and $G[V_1']$ is a connected near-triangulation with at most $|U_{T_{p_1}}|$ blocks. Then $V_1 := V(B_{T_1,r_1}) \cup V_1', V_2 := V(G) - V_1$ form a balanced bipartition of $V(G)$ where $G[V_2]$ is a biconnected near-triangulation with at least $|U_{T_2}|$ internal vertices and $G[V_1]$ is a connected near-triangulation with at most $1 + |U_{T_{p_1}}| \leq 1 + |U_{T_2}|$ blocks. So we may assume that $|U_{T_{p_1}}| + |V(B_{T_1,r_1})| < \lceil \frac{n}{2} \rceil$.

Now run Algorithm Construct Bipartition for triangles $\bar{T}_1 := T_1$ and $\bar{T}_2 := T_{p_1}$ with $p_1 q_1 := pq$ and $p_2 q_2 := r_1 t_1$ as explained in Remark 3 with the following modification: in step (CB1') initialize $V_1 = V(J_{T_1}) - V(B_{T_1,r_1}), V_2 = (V(J_{T_{p_1}}) - p) \cup V(B_{T_1,r_1}), V_{S,1} = \{p, q\}, V_{S,2} = \{r_1, t_1\}$. Observe that V_2 cannot include r_2 for otherwise p or q would be a cut-vertex in $V_{S,1}$, unless $V_{S,2}$ includes $V(S) - \{p, q\}$ entirely in which case $|U_{T_2}| + 2 + |U_{T_1} - V(B_{T_1,r_1})| \geq \lceil \frac{n}{2} \rceil$, a contradiction; indeed V_1 includes every vertex in $U_{T_{p_2}} \cup U_{T_{q_2}} \cup U_{T_2} \cup V(T_2) \cup (U_{T_1} - V(B_{T_1,r_1}))$. Then the bipartition (V_1, V_2) returned by the algorithm leads to a total of at most $b + 3$ blocks and at least $|U_{T_2}|$ internal vertices in $G[V_1]$ and $G[V_2]$, where $b \leq |U_{T_{par}}| \leq |U_{T_2}|$ (recall that $T_{par} \subset S$ is the triangle whose std-subgraph $J_{T_{par}}$ is partitioned by the algorithm). We may assume that equality holds throughout, i.e. $b = |U_{T_{par}}| = |U_{T_2}|$, for otherwise (V_1, V_2) already satisfies the conclusion of the lemma. Then $T_{par} = T_{q_1}$, since all pairs of triangles with the largest smaller std-subgraphs share an edge and $T_{par} \neq T_2, T_{p_1}$; also, since T_{q_1} does not contain the edge pq or r_1t_1, it is partitioned in the split configuration so that $v_{last} = s_1$ and $T_{last_1} = T_{q_1}$. Following the description in Claim 21, let $T_{last_2} \subset S$ be the other triangle in the split configuration with $|U_{T_{last_2}}| \leq |U_{T_{last_1}}|$. Then $|U_{T_{last_2}}| \leq |U_{T_{last_1}}| - 1$ as T_{last_2} does not share an edge with T_1. Since the partition of $V(J_{T_{last_1}})$ leads to exactly $b = |U_{T_{last_1}}|$ blocks, it must be the case that the vertex-set $U_{T_{last_2}} \cup v_{last}$ is included in one part of the bipartition and the vertex-set $U_{T_{last_1}}$ in the other; without loss of generality, let $U_{T_{last_1}} \subset V_1$. We also get that $N_G(v_{last}) \cap N_G(r_1) \cap U_{T_{last_1}} = U_{T_{last_1}}$. Since $|U_{T_{last_1}}| \geq |U_{T_{last_2}}| + 1$, an alternative solution is to include $U_{T_{last_2}} \cup v_{last}$ in V_1 and then partition $V(J_{T_{last_1}})$ (in the split configuration, if $|V_i| + 1 + |U_{T_{last_2}}| \leq \lceil \frac{n}{2} \rceil$ for $i = 1, 2$, then the algorithm chooses the part to include $v_{last} \cup U_{T_{last_2}}$ in arbitrarily). If the new partition of $V(J_{T_{last_1}})$ leads to $|U_{T_{last_1}}| - 1$ or fewer blocks, then the resulting bipartition (V_1, V_2) satisfies the conclusion of the lemma, otherwise we get that $|U_{T_{last_1}}| = |U_{T_{last_2}}| + 1$. Since $J_{T_{last_1}} - \{q, v_{last}\}$ forms a biconnected near-triangulation, we get that $1 = |U_{T_{last_1}}| = |U_{T_2}|$ (since, even in the alternative solution, the partition of $V(J_{T_{last_1}})$ leads to $|U_{T_{last_1}}|$ blocks). This implies that $\lceil \frac{n}{2} \rceil < |U_{T_1}| + |U_{T_2}| + 2 = |U_{T_1}| + 3$, and hence that $|V(J_{T_1})| > \lceil \frac{n}{2} \rceil$. Since $V(J_{T_1})| \leq \lfloor \frac{n}{2} \rfloor + 1$, we get that n is even and $|V(J_{T_1})| = \lfloor \frac{n}{2} \rfloor + 1$. But then the required bipartition exists by Proposition 12. This concludes case (ii). □

2.7 Proving the conjecture

In this final section, we give a proof of Theorem 2 and show that the conjecture follows as a corollary.

Proof of Theorem 2. Let G be a plane triangulation with $|V(G)| = n$. Then we may make the following assumptions, in that order, for otherwise the required bipartition (V_1, V_2) of $V(G)$ exists by the parenthesised statement.

(i) There exists a separating triangle in G (Corollary 6).

(ii) There does not exist a separating triangle $T \subset G$ such that each of the two components of $G - V(T)$ contains at least $\lfloor \frac{n}{2} \rfloor - 1$ vertices (Proposition 10).

(iii) The sink in the directed std-tree of G has degree > 1, i.e., the sink triangulation S of G contains at least 2 distinct triangles that are separating in G (Proposition 11).

(iv) S has at least 6 vertices and, hence, is 4-connected (Proposition 13).

(v) For every pair of distinct triangles $T_1, T_2 \subset S$, if $|V(J_{T_1})|, |V(J_{T_2})| \geq |V(J_T)|$ for every triangle $T \subset S, T \neq T_1, T_2$, then $|V(T_1) \cap V(T_2)| = 2$ (Claim 23).

By Lemma 24, there exists under assumptions (i)-(v) a balanced bipartition (V_1, V_2) of $V(G)$ satisfying the conclusion of the theorem. $\square$

Corollary 26. *If G is a planar graph, then a minimum balanced bipartition (V_1, V_2) of $V(G)$ has $e(V_1, V_2) \leq |V(G)|$.*

Proof. It suffices to prove the corollary for all planar triangulations G, and that follows directly from Theorem 2 and Corollary 4. $\square$

Chapter 3

Pancyclicity in 4-connected planar graphs

3.1 A special case of Bondy's meta-conjecture

The *cycle spectrum* of a simple graph G on n vertices, denoted $\mathcal{C}(G)$, is the set of distinct lengths of cycles in G. G is said to be *Hamiltonian* if $n \in \mathcal{C}(G)$; it is said to be *pancyclic* if $|\mathcal{C}(G)| = n - 2$. In 1971-72, Bondy (see [5], [6]) proposed his now famed meta-conjecture "Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic". Bondy further allowed that "There may be a simple family of exceptional graphs". He provided several results in support of his meta-conjecture starting with the extension of Ore's condition which states that any graph G on n vertices satisfying $d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices u and v is Hamiltonian. In this case, the complete bipartite graphs $K_{\frac{n}{2},\frac{n}{2}}$ form the family of exceptions (see [5]).

In the same paper, Bondy conjectured that Tutte's result about Hamiltonicity of 4-connected planar graphs may be similarly extended. Malkevitch (see [30]) pointed out a simple family of exceptions to this conjecture (line graphs of cyclically 4-edge-connected, cubic, planar graphs of girth 5) no member of which contains a cycle of length 4. It is suspected that that might be the only cycle length absent in a 4-connected planar graph. Malkevitch then revised Bondy's conjecture to the following form.

Conjecture 27 (Malkevitch, [31])**.** *A* 4*-connected planar graph is pancyclic if it contains a cycle of length* 4*.*

In joint work with Bojan Mohar, we obtain the following partial results in this chapter in relation to the conjecture by Malkevitch.

Theorem 28. *If G is a* 4*-connected planar graph on n vertices and e is an edge in G, then G contains at least $\lceil \frac{n}{2} \rceil + 1$ cycles of pairwise distinct lengths each containing e.*

Theorem 29. *For any integer $k \geq 1$, there exists a 4-connected planar graph G_k on $3k+3$ vertices containing an edge e such that G_k contains at most $2k+2$ cycles of pairwise distinct lengths each containing e.*

Theorem 30. *If G is a 4-connected planar graph not containing any 4-cycles, then G contains at least $\lceil \frac{5n}{6} \rceil + 2$ cycles of pairwise distinct lengths.*

Our proofs for the above theorems are constructive and yield simple algorithms for finding the promised set of cycles.

The layout of the rest of the chapter is as follows. In Section 3.2, we define the graph-terminology used in the chapter and introduce two analogous weight functions to be used in the proof of Theorem 30. In Section 3.3, we prove Theorems 28 and 29. In the following two sections, we prove an upper bound pertaining to large faces in 4-connected planar graphs not containing any 4-cycles (Section 3.4) and prove Theorem 30 by relating the size of the cycle spectrum of such graphs to this bound (Section 3.5). Finally, in Section 3.6, we prove some results that provide additional evidence/motivation towards a proof of the existence of an almost complete cycle spectrum in 4-connected planar graphs not containing any 4-cycles.

3.2 Preliminaries

All graphs in this chapter are planar and do not contain loops or parallel edges. Readers are referred to [13] for any terminology that we may have missed and the notation used in this chapter.

A graph G is said to be *k-connected* if it has at least $k+1$ vertices and, for every $(k-1)$-subset V' of $V(G)$, $G - V'$ is connected.

A graph is *planar* if it can be drawn in the plane in such a way that its edges intersect only at their endpoints. A graph so drawn in the plane is called a *plane graph.* For any plane graph G, the regions of $\mathbb{R}^2 \backslash G$ are called the *faces* of G, the set of them denoted $F(G)$. If G is finite, then exactly one of its faces is unbounded and is called the *infinite face* of G.

Given a plane graph G, its *dual graph* G^* is defined as follows. Corresponding to each face F of G there is a vertex F^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*. Two vertices F_1^* and F_2^* are joined by the edge e^* in G^* if and only if their corresponding faces F_1 and F_2 are separated by the edge e in G. It is easy to see that $(G^*)^* = G$. For the sake of conveninence, we will use $df(\cdot)$ and $dv(\cdot)$ to indicate dual elements, e.g. $F^* = dv(F)$ and $F = df(F^*)$.

A *closed walk* in a graph is defined as a sequence of vertices, starting and ending at the same vertex, such that every pair of consecutive vertices in the sequence is adjacent to each other in the graph. Each face of a plane graph G is bounded by a closed walk in G called the *boundary* of the face. A face F of G whose boundary consists of k vertices (including repetitions) is said to be a face of *size* $|F| = k$, or simply a *k-face*. We also refer to 3-faces as *triangular faces*.

Since a planar graph can also be drawn on a sphere in such a way that its edges intersect only at their endpoints, under a suitably chosen stereographic projection from the sphere to the plane, any face boundary in a plane graph G may be "designated" as the infinite face boundary. Any vertex of a plane graph G not lying on the boundary of its infinite face is said to be an *internal vertex* of G. An *outerplanar* graph is a planar graph that can be drawn in the plane with no internal vertices. A *chord* of a plane outerplanar graph is an edge not contained in the boundary of its infinite face.

A *planar triangulation* is a graph that is maximally planar. Every face in any drawing of a planar triangulation in the plane is bounded by three edges. Every planar triangulation corresponds to a unique *plane triangulation* up to isomorphism, so the terms are often used interchangeably.

A plane Hamiltonian graph G with a Hamiltonian cycle C can be thought of as a union of two 2-connected plane outerplanar graphs G_0 and G_1, each with C bounding one of its faces, such that $G_0 \cap G_1 = C$. Let F_0 and F_1, respectively, be the faces of G_0 and G_1 bounded by C that are not in $F(G)$; we designate these as the infinite faces of G_0 and G_1. For any plane Hamiltonian graph G with a Hamiltonian cycle C, we define the quadruple (G, C, G_0, G_1) distinguishing G_0 from G_1 using the assumption that $|E(G_1)| \geq |E(G_0)|$. For each $i \in \{0, 1\}$, let c_i be the number of chords in G_i.

The *internal dual* of a plane graph G with dual graph G^* and outer face dual to the vertex $v^* \in V(G^*)$ is the graph $G^* - v^*$. It is easy to see that the internal dual of a 2-connected outerplanar graph is a tree. Let T_0 and T_1 be the internal duals of the graphs G_0 and G_1, respectively, in (G, C, G_0, G_1). Since $(F(G_0) - F_0) \cup (F(G_1) - F_1) = F(G)$, we get that $V(T_0) \cup V(T_1) = V(G^*)$.

We now introduce two analogous weight functions and establish an easy identity for each. For each $i \in \{0, 1\}$, $v \in V(T_i)$, let $df(v)$ be the corresponding dual face in G. Analogously, for each $Q \in F(G)$, let $dv(Q)$ be the corresponding dual vertex in T_0 or T_1. For each $v \in V(T_0) \cup V(T_1)$, let $w : V(T_0) \cup V(T_1) \to \mathbb{Z}$ be defined as $w(v) = |df(v)| - 2$.

Proposition 31. *Let G be a plane Hamiltonian graph on n vertices and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1), for each $i \in \{0, 1\}$,* $\sum_{v \in V(T_i)} w(v) = n - 2$.

Proof. Let G be a plane Hamiltonian graph on n vertices and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1), for each $i \in \{0, 1\}$,

$$\begin{aligned}
\sum_{v \in V(T_i)} w(v) &= \sum_{v \in V(T_i)} |df(v)| - 2|V(T_i)| \\
&= 2|E(G_i)| - n - 2|V(T_i)| \\
&= 2(n + |E(T_i)|) - n - 2|V(T_i)| \\
&= n - 2.
\end{aligned}$$

□

For each $e \in E(G)$ and the faces $Q_e, Q'_e \in F(G)$ incident with e, let $w' : E(G) \to \mathbb{R}$ be defined as $w'(e) = \frac{w(dv(Q_e))}{|Q_e|} + \frac{w(dv(Q'_e))}{|Q'_e|}$. Then the following is an easy corollary of Proposition 31.

Corollary 32. *If G is a plane Hamiltonian graph on n vertices, then $\sum_{e \in E(G)} w'(e) = 2(n-2)$.*

Proof. Let G be a plane Hamiltonian graph on n vertices and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1),

$$\begin{aligned}
\sum_{e \in E(G)} w'(e) &= \sum_{e \in E(G)} \sum_{\substack{Q \in F(G); \\ e \in E(Q)}} \frac{w(dv(Q))}{|Q|} \\
&= \sum_{Q \in F(G)} \sum_{\substack{e \in E(G); \\ e \in E(Q)}} \frac{w(dv(Q))}{|Q|} \\
&= \sum_{Q \in F(G)} w(dv(Q)) \\
&= \sum_{v \in V(T_1) \cup V(T_2)} w(v) \\
&= \sum_{v \in V(T_1)} w(v) + \sum_{v \in V(T_2)} w(v) \\
&= 2(n-2),
\end{aligned}$$

where the last equality follows from Proposition 31. □

3.3 The general case

In this section, we prove Theorem 28 as a corollary of the following lemma and then provide a construction for the family of graphs described in Theorem 29.

Lemma 33. *If G is a 2-connected plane outerplanar graph on n vertices containing c chords and e is an edge contained in the boundary of its infinite face, then G contains at least $c+1$ cycles of pairwise distinct lengths each containing the edge e.*

Proof. Let G be a 2-connected plane outerplanar graph on n vertices containing c chords, and let e be an edge contained in the boundary of its infinite face F. Let F' be the face of G sharing the edge e with F, and let T be the internal dual of G.

Consider a sequence of subtrees $T^0 \subsetneq ... \subsetneq T^c$ of T_i such that $T^0 = dv(F')$ and, for each $i \in \{1, ..., c\}$, $|T^i| = |T^{i-1}| + 1$. For each $i \in \{0, ..., c\}$, let G^i be the plane subgraph of G with T^i as its internal dual and let C^i be the boundary of its infinite face; it is easy to see that G^i is a 2-connected plane outerplanar graph on $2 + \sum_{v \in T^i} (|df(v)| - 2)$ vertices and $e \in E(C^i)$. Thus, the cycles $C^0, ..., C^c$ each contain the edge e and have pairwise distinct lengths. □

We need a weak form of the following result by Sanders ([35]) to prove Theorem 28. The result will be used in full in Section 3.5.

Theorem 34 (Sanders, [35])**.** *If G is a 4-connected planar graph and $e_1, e_2 \in E(G)$, then there exists a Hamiltonian cycle in G containing both e_1 and e_2.*

Proof of Theorem 28. It suffices to prove the theorem for any 4-connected plane graph. Let G be a 4-connected plane graph on n vertices and let e be an edge in G.

By Theorem 34, there exists a Hamiltonian cycle C in G containing the edge e. Since G contains at least $2n$ edges, in the quadruple (G, C, G_0, G_1), G_1 is a 2-connected plane outerplanar graph with at least $\lceil \frac{n}{2} \rceil$ chords. Then, by Lemma 33, G_1 contains at least $\lceil \frac{n}{2} \rceil + 1$ cycles of pairwise distinct lengths in G_1 each containing the edge e. □

Remark 4. Note that even though Theorem 34 is used in a weak form in the proof above, G must be 4-connected for a bound linear in n. As evidence of this fact, one might consider the planar triangulations constructed by Moon and Moser ([32]) where the length of the longest cycle is sublinear in n ($\mathcal{O}(n^{\log_3 2})$).

Proof of Theorem 29. For any integer $k \geq 1$, consider the graph G_k on $3k + 3$ vertices constructed as follows. Let $P_u = u_1u_2...u_k, P_w = w_1w_2...w_k$ and $P_v = v_1v_2...v_{k+1}$ be three disjoint paths of lengths $k-1, k-1$ and k respectively. For each $i \in \{1, ..., k\}$, connect both u_i and w_i to each of v_i and v_{i+1} with edges. Now take two additional vertices v_0 and v_{k+2} and connect v_0 to u_1, v_1 and w_1, and v_{k+2} to u_k, v_{k+1} and w_k with edges. Finally, connect v_0 to v_{k+2} with an edge; label this last edge e. It is easy to see that G_k is 4-connected and can be embedded in the plane (see Fig. 3.1).

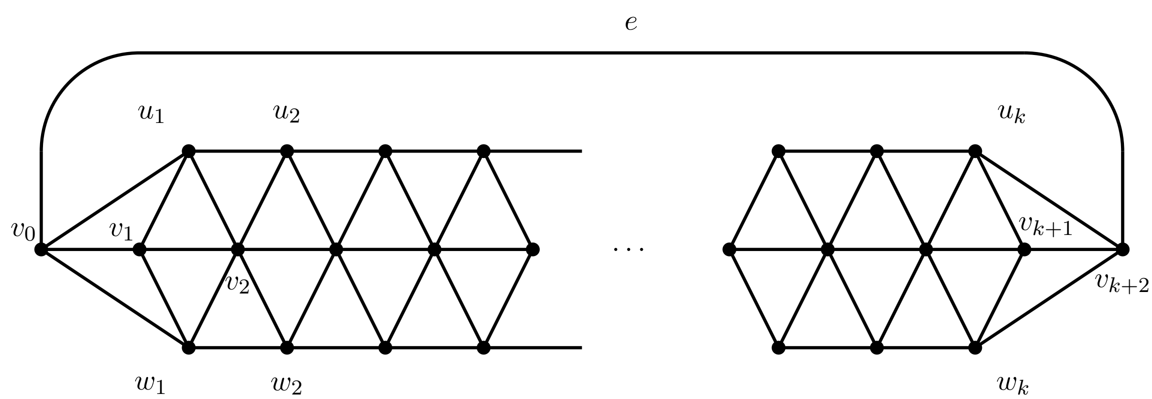

Figure 3.1: 4-connected planar graph G_k on $3k+3$ vertices. Each of the two faces incident with the edge e has size $k+2$.

The shortest path between the vertices v_0 and v_{k+2} in $G_k - e$ is of length $k+1$, so any cycle in G_k containing e has length $\geq k+2$ (in fact, for every $l \geq k+2$, there exists a cycle of length l in G_k containing e; to see this, take a Hamiltonian cycle C in G_k containing e and enumerate the cycles in the plane outerplanar graph formed by any one side of C starting with the $(k+2)$-face as described in the proof of Lemma 33). Thus, G_k contains at most $(3k+3)-(k+1) = 2k+2$ cycles of pairwise distinct lengths each containing e. $\square$

3.4 Excluding C_4

Let G be a plane Hamiltonian graph and C be a Hamiltonian cycle in G. For every $i \in \{0,1\},\ j \in \{5,6\}$, we define the following in the quadruple (G, C, G_0, G_1).

- $f_i^{\geq j}$: the number of faces of size $\geq j$ in $F(G_i) - F_i$;
- $s_i^{>j}$: the sum $\sum_{\substack{F \in F(G_i) - F_i; \\ |F| > j}} |F| - j = \sum_{\substack{F \in F(G_i) - F_i; \\ |F| \geq j}} |F| - j$;
- $f^{\geq j}$: the number of faces of size $\geq j$ in $F(G)$;
- $s^{>j}$: the sum $\sum_{\substack{F \in F(G); \\ |F| > j}} |F| - j = \sum_{\substack{F \in F(G); \\ |F| \geq j}} |F| - j$.

Note that $f^{\geq j} = f_0^{\geq j} + f_1^{\geq j}$ and $s^{>j} = s_0^{>j} + s_1^{>j}$.

In a plane graph G, for any vertex $v \in V(G)$, we define a 5-*fan incident with* v as a maximal set $\emptyset \neq J \subset F(G)$ of 5-faces each incident with v such that, in the cyclic list enumerating all faces incident with v in clockwise order, all the faces in the set appear together.

We define a 5-*block* as a maximal set $\emptyset \neq J \subset F(G)$ of 5-faces such that $G^*_J := G^*[\{dv(Q) : Q \in J\}]$ is connected; we refer to G^*_J as the *dual* 5-*block* of J. A 5-block is said to be *trivial* if it is a singleton set, and *non-trivial* otherwise; a 5-block J is said to be *acyclic* if its dual 5-block is acyclic. A *shared edge of a* 5-*block* is an edge that is shared between two members of the said 5-block; an *unshared edge of a* 5-*block* is an edge that is incident with only one member of the said 5-block. We also define some special 5-blocks (note that a 5-fan may or may not be a 5-block).

We define a 5-*flower* as a 5-block consisting of six members $\{P_0, P_1, ..., P_5\}$, where P_0 shares an edge with every other member of the set and all members of the set except P_0 are pairwise edge-disjoint. Alternatively, a 5-flower is a 5-block for which the dual 5-block is a $K_{1,5}$.

Finally, we define a 5-*tree* as an acyclic 5-block J such that, for each vertex u in the dual 5-block G^*_J, $d_{G^*_J}(u) \in \{0, 1, 2, 5\}$, each of the three degree classes forms an independent set in G^*_J, and no vertex of degree 1 is adjacent to a vertex of degree 2 in G^*_J. Note that every trivial 5-block and every 5-flower is a 5-tree. A simple calculation shows that a 5-tree for which the dual 5-block contains p vertices of degree 5 (and hence $p-1$ vertices of degree 2) contains $p+(p-1)+(5p-2(p-1)) = 5p+1$ members and $3(p-1)+4(5p-2(p-1)) = 5(3p+1)$ unshared edges.

In this section, we provide an upper bound for $s^{>5}$ when the said graph G has minimum degree 4 and does not contain any 4-cycles. We use the discharging method to do that.

Lemma 35. *If G is a plane Hamiltonian graph on $n \geq 5$ vertices with $\delta(G) \geq 4$ and not containing any 4-cycles, then $s^{>5} \leq \frac{n}{3} - 10$.*

Proof. Let G be a plane Hamiltonian graph on $n \geq 5$ vertices not containing any 4-cycles. For each vertex $v \in V(G)$, set the initial charge $c(v)$ to be $d_G(v)-4$. For each face $Q \in F(G)$, set the initial charge $c(Q)$ to be $|Q|-4$. A rearrangement of Euler's formula $|V(G)|-|E(G)|+|F(G)| = 2$ gives us that

$$\sum_{v \in V(G)} (d_G(v) - 4) + \sum_{Q \in F(G)} (|Q| - 4) = -8.$$

In other words, the total charge on all the vertices and faces is -8.

We now use a discharging procedure defined by the rules R1-R5 below, leading to a final charge distribution where we will compare the total charge with -8 to conclude some information about the structure of G.

(R1) Each triangular face receives 1/3 charge from every face it is adjacent to.

(R2) Each face of size ≥ 6 donates 1/3 charge to every 5-face it is adjacent to.

(R3) Each vertex of degree $d \geq 6$ donates 2/3 charge to every 5-fan incident with it.

(R4) Each vertex of degree 5 donates $2/3$ charge to the 5-fan of size ≥ 2 incident with it (if any), and $1/3$ charge to every 5-fan of size 1 incident with it.

(R5) This rule is executed after each of (R1)-(R4) has been executed. Each 5-block accumulates the charge from all its members at a single arbitrarily chosen member, resetting the charge at every other member to zero.

For every $z \in V(G) \cup F(G)$, let $c'(z)$ denote the new charge after the execution of the discharging procedure. For any $Z \subseteq V(G) \cup F(G)$, let $c(Z) = \sum_{z \in Z} c(z)$ and $c'(Z) = \sum_{z \in Z} c'(z)$.

Claim 36. *For each $v \in V(G)$, $c'(v) \geq 0$.*

Proof. For any vertex $v \in V(G)$, there can be at most $\lfloor \frac{d_G(v)}{2} \rfloor$ 5-fans incident with v. If $d_G(v) \geq 6$, then $c'(v) = d_G(v) - 4 - \frac{2}{3}\lfloor \frac{d_G(v)}{2} \rfloor \geq \frac{2}{3}d_G(v) - 4 \geq 0$ by (R3). If $d_G(v) = 5$, then either there are no 5-fans of size ≥ 2 and at most two 5-fans of size 1 incident with v, or there is one 5-fan of size 2 and at most one 5-fan of size 1 incident with v; so, by (R4), $c'(v) \geq d_g(v) - 4 - \max\{2 \cdot \frac{1}{3}, \frac{2}{3} + \frac{1}{3}\} = 0$. If $d_G(v) = 4$, then $c'(v) = c(v) = d_G(v) - 4 = 0$. $\square$

Claim 37. *Let Q be a face of G. Then*

(i) $c'(Q) = 0$ *if* $|Q| = 3$, *and*

(ii) $c'(Q) \geq \frac{m_Q}{3} + \frac{2}{3}(|Q| - 6)$ *if* $|Q| \geq 6$ *and* m_Q *of* Q*'s edges are shared with other faces of size* ≥ 6.

Proof. (i) follows directly from (R1). By (R1) and (R2), a face Q of size ≥ 6 with m_Q of its edges shared with other faces of size ≥ 6 donates at most $\frac{|Q| - m_Q}{3}$ charge so that $c'(Q) \geq |Q| - 4 - \frac{|Q| - m_Q}{3} = \frac{m_Q}{3} + \frac{2}{3}|Q| - 4 = \frac{m_Q}{3} + \frac{2}{3}(|Q| - 6)$, which gives us (ii). $\square$

Claim 38. *Let $J \subset F(G)$ be a 5-block in G and let $U \subset V(G)$ be the set of vertices that are each incident with a member of J. If $c'(J) < 0$ then all of the following are true:*

(i) J *is a* 5*-tree,*

(ii) each unshared edge of J *is shared with a triangular face of* G, *and*

(iii) $\sum_{u \in U} d_G(u) \leq 4|U| + 1$.

Moreover, either $c'(J) = -2/3$ *and* $\sum_{u \in U} d_G(u) = 4|U|$, *or* $c'(J) = -1/3$ *and* $\sum_{u \in U} d_G(u) = 4|U| + 1$ *with the vertex of degree* 5 *in* U *incident with a single member of* J.

Proof. Let $J \subset F(G)$ be a 5-block in G with $c'(J) < 0$ and $Q \in J$ be a 5-face in G. Let G^*_J be the dual 5-block of J with $a = |J| = V(G^*_J)$ and $b = E(G^*_J)$; note that b denotes the number of shared edges of J. Let l be the number of unshared edges of J that are each shared with a face $P \in F(G) - J$ such that $|P| \geq 6$. Then, by (R1)-(R4), the members of

J donate $\frac{5a-2b-l}{3}$ charge, and receive $\frac{l}{3}+c_v$ charge, where c_v is the total charge received by the members of J from the all the vertices they are incident with. This gives us that $c'(J) = c(J) - \frac{5a-2b-l}{3} + \frac{l}{3} + c_v = a(5-4) - \frac{5a-2b-l}{3} + \frac{l}{3} + c_v = \frac{2}{3}(b-a+l) + c_v$. Since $c'(J) < 0$ and G_J^* is connected, we get that $b = a-1$ and $l = 0$ (i.e., J is acyclic, and each unshared edge of J is shared with a triangular face of G) so that $c'(J) = c_v - \frac{2}{3}$.

Suppose that J is non-trivial. Let xy be an edge that is shared between two members P_1 and P_2 of J, where $x, y \in V(G)$. Since $c'(J) = c_v - \frac{2}{3} \geq -2/3$, if either x or y has degree ≥ 5 then, by (R4), $c'(J) \geq 0$, a contradiction; additionally, since J is acyclic, not all faces incident with x (or y) are 5-faces. So each of x and y has degree 4 and is incident with one triangular face and three 5-faces. Let P_3 and P_4 be the 5-faces incident with, respectively, x and y in addition to P_1 and P_2. Then P_3 shares an edge with one of P_1 and P_2 that is adjacent to the edge xy at x; similarly, P_4 shares an edge with one of P_1 and P_2 that is adjacent to the edge xy at y. Thus, every shared edge of J is adjacent at each of its ends to exactly one other shared edge of J and, consequently, all shared edges in J form a set of disjoint cycles in G. Further, since every pair of adjacent shared edges belongs to a member of J, the corresponding pair of dual edges is also adjacent; as a result, for any cycle formed by shared edges of J, the set of edges dual to those in the cycle forms either a star (i.e., the cycle forms the boundary of a member of J) or a graph containing a cycle. Since J is acyclic it must be the case that every cycle formed by shared edges of J forms the boundary of a member of J.

Let $\mathcal{C}$ be the set of disjoint cycles formed by the shared edges of J. If $|\mathcal{C}| = 1$, then J is a 5-flower. So we may assume that $|\mathcal{C}| > 1$. Since the cycles in $\mathcal{C}$ are disjoint, the edges of the dual 5-block G_J^* can be partitioned so that each part forms a $K_{1,5}$. To ensure that G_J^* is connected, each part contains an edge that is adjacent to an edge in another part. If u is a vertex in G_J^* at which edges from distinct parts are adjacent to each other, then $D_{G_J^*}(u) = 2$ because the corresponding dual edges are contained in distinct cycles in $\mathcal{C}$ and must, therefore, be pairwise disjoint. Thus, in this case, J is a 5-tree.

Let $U' \subset V(G)$ be the set of vertices that are each incident with a member of J but not with a shared edge of J. We have shown above that $\sum_{u \in U-U'} d_G(u) = 4|U-U'|$. By (R3) and (R4), $\sum_{u \in U'} d_G(u) \leq 4|U'|+1$, for otherwise $c_v \geq 2/3$ and $c'(J) \geq 0$, a contradiction; if $\sum_{u \in U'} d_G(u) = 4|U'|$, then $c_v = 0$ and $c'(J) = -2/3$, and if $\sum_{u \in U'} d_G(u) = 4|U'|$, then $c_v = 1/3$ and $c'(J) = -1/3$. Thus, we get that $\sum_{u \in U} d_G(u) \leq 4|U-U'| + 4|U'| + 1 = 4|U|+1$; we also get that $c'(J) = -2/3$ if $\sum_{u \in U} d_G(u) = 4|U|$, and $c'(J) = -1/3$ if $\sum_{u \in U} d_G(u) = 4|U|+1$ in which case the vertex of degree 5 in U is incident with a single member of J. This is true in each case whether J is trivial, a 5-flower or a 5-tree. $\square$

We focus on faces in G of size ≥ 6. By (ii) in Claim 37, such faces hold a total charge of $c^+ = \frac{2}{3}(s^{>6} + m)$, where m is the number of edges in G that are each shared between two such faces. By Claims 36-38 above, negative charge may occur only within 5-trees that share each of their unshared edges with a triangular face of G; such a 5-block may have a charge of $-2/3$ or $-1/3$, if at all negative, depending on the degree sum of the vertices its members are incident with. Since the total charge held by G is -8, we may assume that there are at least $r = \frac{c^{+}+}{2/3} = s^{>6} + m + 12$ such 5-trees for otherwise $c'(G) > -8$. Let $\bar{p}$ be the average number of vertices of degree 5 contained in the dual 5-block of each of these 5-trees.

Let $W' = \sum_{e \in E(G)} w'(e)$. We will now estimate the contribution towards W' by edges of G based on their incidence with faces of different sizes and then invoke Corollary 32. We start with the edges in $E(G)$ incident with faces of size ≥ 6. The contribution of all of such edges towards W' due to the faces of size ≥ 6 is given by $W'_1 = 4f^{\geq 6} + s^{>6}$. Since only m of these edges are each shared between two faces of size ≥ 6, the other face incident with each of the remaining $m' = 6f^{\geq 6} + s^{>6} - 2m$ edges is either a triangular face or a 5-face; as a result, the contribution of these m' edges towards W' due to the faces of size ≤ 5 is at least $W'_2 = \min\{\frac{1}{3}, \frac{3}{5}\} \cdot m' = \frac{m'}{3}$. The edges in $E(G)$ that are not incident with faces of size ≥ 6 are incident with either 5-faces and triangular faces or just 5-faces. Since there are at least r 5-trees with an average of $\bar{p}$ vertices of degree 5 per dual 5-block, the contribution of these edges due to the 5-faces is at least $W'_3 = 3 \cdot r(5\bar{p} + 1)$. The same r 5-trees lead to a contribution of at least $W'_4 = \frac{1}{3} \cdot r(5(3\bar{p} + 1))$ by their unshared edges due to the triangular faces they are incident with. Then, by Corollary 32, we get that

$$\begin{aligned}
2(n-2) &= W' \\
&\geq W'_1 + W'_2 + W'_3 + W'_4 \\
&= 4f^{\geq 6} + s^{>6} + \frac{m'}{3} + 3 \cdot r(5\bar{p} + 1) + \frac{1}{3} \cdot r(5(3\bar{p} + 1)) \\
&= 4f^{\geq 6} + s^{>6} + \frac{m'}{3} + r(20\bar{p} + \frac{14}{3}) \\
&\geq 4f^{\geq 6} + s^{>6} + \frac{m'}{3} + \frac{14r}{3} \qquad (3.1) \\
&= 4f^{\geq 6} + s^{>6} + \frac{6f^{\geq 6} + s^{>6} - 2m}{3} + \frac{14}{3}(s^{>6} + m + 12) \\
&= 6f^{\geq 6} + 6s^{>6} + 4m + 56 \\
&\geq 6f^{\geq 6} + 6s^{>6} + 56 \qquad (3.2) \\
&= 6s^{>5} + 56, \qquad (3.3)
\end{aligned}$$

where (3.1) follows from the observation that each of the r 5-trees may be trivial and hence contain no vertices of degree 5 in its dual 5-block ($\bar{p} = 0$), (3.2) from the observation that $m \geq 0$, and (3.3) from the observation that $f^{\geq 6} + s^{>6} = s^{>5}$. The final inequality then gives us that

$$s^{>5} \leq \frac{n}{3} - 10.$$

□

3.5 Generating the cycle spectrum

In any plane Hamiltonian graph G with a Hamiltonian cycle C, we define a *leaf-triangle of* (G, C) as a triangular face of G whose boundary contains exactly two edges of C, and its *tip* as the vertex shared between the said pair of edges. Such a face is then contained in either G_0 or G_1 in (G, C, G_0, G_1), and is dual to a leaf vertex of either T_0 or T_1.

In a plane graph, we define a *well-triangulated* vertex as one of even degree d and incident with at least $d/2$ triangular faces.

In this section, we provide a lower bound on the size of the cycle spectrum of a 4-connected planar graph G in terms of the number of vertices. We do that by identifying a well-triangulated vertex v in G and enumerating cycles of pairwise distinct lengths starting with one of the faces incident with the said vertex. In doing so, we keep a count of the lengths skipped and relate it to bound obtained in the preceding section which, in turn, gives us the lower bound we seek.

We start with the enumeration lemma and follow it up with a corollary that uses Theorem 34 to identify a special Hamiltonian cycle C with respect to v. In (G, C, G_0, G_1), the enumeration lemma uses faces predominantly from a single side of C and relates the lower bound to the corresponding $s_i^{>5}$, which the corollary replaces with $s^{>5}$. Theorem 41 then establishes the final result in a much stronger form, drawing on the proof of Lemma 35 in the preceding section to confirm the existence of the said vertex v as well as translate the bound to in terms of the number of vertices. For the sake on consistency, we still state a formal proof of Theorem 30 towards the end of the section.

Lemma 39. *Let G be a plane Hamiltonian graph on $n \geq 5$ vertices with $\delta \geq 4$ and not containing any 4-cycles and C be a Hamiltonian cycle in G. If, for (G, C, G_0, G_1), for some $i \in \{0, 1\}$, there exists a face $Q_i \in F(G) \cap F(G_i)$ of size ≥ 5 that is adjacent to two leaf-triangles of (G, C), then there exists a set of cycles $\mathcal{C}_i$ in G of pairwise distinct lengths, each of size at least $|Q_i|$ and containing all the vertices in $V(Q_i)$ that are not the tips of the said leaf-triangles, such that $|\mathcal{C}_i| \geq n - 5 - s_i^{>5}$.*

Proof. Let G be a plane Hamiltonian graph on $n \geq 5$ vertices with $\delta \geq 4$ not containing any 4-cycles and C be a Hamiltonian cycle in G. Suppose, for (G, C, G_0, G_1), for some $i \in \{0, 1\}$, there exists a face $Q_i \in F(G) \cap F(G_i)$ of size ≥ 5 that is adjacent to two leaf-triangles L and L' of (G, C). Let $e \in E(L)$ and $e' \in E(L')$ be the edges of these leaf-triangles that are not contained in C, $y \in V(L)$ and $y' \in V(L')$ be the vertices that are not contained in Q_i, and $x \in V(L)$ and $x' \in V(L')$ be the tips. Note that if $x = y$ then $e \in E(G_i)$ and if $x \neq y$ then $e \in E(G_{1-i})$ (likewise for x', y' and e').

Suppose $e, e' \in E(G_i)$. Consider a sequence of subtrees $T_i^0 \subsetneq T_i^3 \subsetneq T_i^6 \subsetneq ... \subsetneq T_i^{3c_i}$ of T_i such that $T_i^0 = dv(Q_i)$, $\{dv(L), dv(L')\} \subset V(T_i^6)$, $T_i^{3c_i} = T_i$ and, for each $j \in \{1, ..., c_i\}$, $|T_i^{3j}| = |T_i^{3j-3}| + 1$. For each $j \in \{0, ..., c_i\}$, let G_i^{3j} be the plane subgraph of G_i with T_i^{3j} as its internal dual and let C_i^{3j} be the boundary of its outer face; it is easy to see that G_i^{3j} is a 2-connected outerplanar graph on $\sum_{v \in T_i^{3j}} w(v) + 2$ vertices and, whenever $j \geq 2$, $E(C_i^{3j}) \cap E(L) = E(C) \cap E(L)$ and $E(C_i^{3j}) \cap E(L') = E(C) \cap E(L')$. For each $j \in \{3, ..., c_i\}$, let $G_i^{3j-1} = G_i^{3j} - x$ and $G_i^{3j-2} = G_i^{3j} - x - x'$; note that G_i^{3j-1} and G_i^{3j-2} too are both 2-connected outerplanar graphs on, respectively, $|G_i^{3j}| - 1$ and $|G_i^{3j}| - 2$ vertices. For the sake of completeness, let $G_i^1 = G_i^2 = G_i^3$ and $G_i^4 = G_i^5 = G_i^6$. Thus, for each $j \in \{3, ..., c_i\}$ and $z = V(T_i^{3j}) - V(T_i^{3j-3})$, if $w(z) > 1$, then $|G_i^{3j-1}|, |G_i^{3j-2}| > |G_i^k|$ for all $k \in \{0, 1, ..., 3j-3\}$; in other words, between $|G_i^{3j-3}|$ and $|G_i^{3j}|$, we skip at most $w(z) - 3$ sizes. Let $\mathcal{G} := \{G_i^k : 0 \leq k \leq 3c_i\}$. Since each graph $H \in \mathcal{G}$ gives us a cycle of length $|H|$, and G invariably contains cycles of length 3 and 5 (which makes the number of cycle lengths missing up to $|G_i^0| = |G_i^3| - 1 = |G_i^6| - 2$ at most $w(dv(Q_i)) - 2$), the number of cycle lengths that are not present in any graph in $\mathcal{G}$ is at most

$$\begin{aligned}
\sum_{\substack{v \in T_i; \\ w(v) > 1}} (w(v) - 3) + 1 &= \sum_{\substack{v \in T_i; \\ w(v) > 1}} (|df(v)| - 5) + 1 \\
&= \sum_{\substack{F \in F(G_i) - F_i; \\ |F| > 3}} (|F| - 5) + 1 \\
&= \sum_{\substack{F \in F(G_i) - F_i; \\ |F| \geq 5}} (|F| - 5) + 1 \\
&= s_i^{>5} + 1.
\end{aligned}$$

Now suppose $e \in E(G_i)$ and $e' \in E(G_{1-i})$. Let $R_i \in F(G) \cap F(G_i)$ be the other face that is adjacent to L'. Consider a sequence of subtrees $T_i^3 \subsetneq T_i^6 \subsetneq ... \subsetneq T_i^{3c_i+3}$ of T_i such that $T_i^3 = dv(Q_i)$, $dv(L) \in V(T_i^6)$, $T_i^{3c_i+3} = T_i$ and, for each $j \in \{1, ..., c_i\}$, $|T_i^{3j+3}| = |T_i^{3j}| + 1$. Let $r \in \{4, ..., c_i+1\}$ be such that $V(T_i^{3r}) - V(T_i^{3r-3}) = dv(R_i)$. For each $j \in \{1, ..., c_i+1\}$, let G_i^{3j} be the plane subgraph of G_i with T_i^{3j} as its internal dual and, for each $j \in \{0, ..., c_i+1\}$, let H_i^{3j} be the plane subgraph of G given by

$$H_i^{3j} = \begin{cases} G_i^3 & \text{if } j = 0, \\ (V(G_i^{3j}) \cup y', E(G_i^{3j}) \cup \{e', x'y'\}) & \text{if } 0 < j < r, \\ G_i^{3j} & \text{if } j \geq r; \end{cases}$$

let C_i^{3j} be the boundary of the outer face of H_i^{3j}. Then, for each $j \in \{0, ..., c_i+1\}$, H_i^{3j} is a 2-connected outerplanar graph where $E(C_i^{3j}) \cap E(L) = E(C) \cap E(L)$ and $E(C_i^{3j}) \cap E(L') =$

$\{e', x'y'\}$ with $|C_i^{3j}| = \sum_{v \in T_i^{3j}} w(v) + 3$ whenever $2 \le j < r$, and $E(C_i^{3j}) \cap E(L) = E(C) \cap E(L)$ and $E(C_i^{3j}) \cap E(L') = E(C) \cap E(L')$ with $|C_i^{3j}| = \sum_{v \in T_i^{3j}} w(v) + 2$ whenever $j \ge r$. For each $j \in \{3, ..., c_i + 1\}$, let $H_i^{3j-1} = H_i^{3j} - x$ and $H_i^{3j-2} = H_i^{3j} - x - y'$ whenever $j < r$, and let $H_i^{3j-1} = H_i^{3j} - x$ and $H_i^{3j-2} = (H_i^{3j} - x) \cup e'$ whenever $j \ge r$; let the outer face boundaries of H_i^{3j-1} and H_i^{3j-2} be, respectively, C_i^{3j-1} and C_i^{3j-2}, where $|C_i^{3j-1}| = |C_i^{3j}| - 1$ and $|C_i^{3j-2}| = |C_i^{3j}| - 2$. For the sake of completeness, let $H_i^1 = H_i^2 = H_i^3$ and $H_i^4 = H_i^5 = H_i^6$. Thus, for each $j \in \{3, ..., c_i + 1\} - \{r\}$ and $z = V(T_i^{3j}) - V(T_i^{3j-3})$, if $w(z) > 1$, then $|C_i^{3j-1}|, |C_i^{3j-2}| > |C_i^k|$ for all $k \in \{0, 1, ..., 3j-3\}$; in other words, between $|C_i^{3j-3}|$ and $|C_i^{3j}|$, we skip at most $w(z) - 3$ sizes. Since $|C_i^{3r-3}| = \sum_{v \in T_i^{3r-3}} w(v) + 3$ and $|C_i^{3r}| = \sum_{v \in T_i^{3r}} w(v) + 2$, between the two of them we skip sizes (at most $w(dv(R_i)) - 4$ many of them) only if $w(dv(R_i)) > 4$. Then, similar to the previous case, we get that the number of cycle lengths missing up to $|H_i^0| = |H_i^3| - 1 = |H_i^6| - 2$ is at most $w(dv(Q_i)) - 2)$, and the number of cycle lengths that are not present in any graph in $\mathcal{G} := \{H_i^k : 0 \le k \le 3c_i + 3\}$ is at most $s_i^{>5} + 1$.

Finally, suppose $e, e' \in E(G_{1-i})$. Let $R_i, S_i \in F(G) \cap F(G_i)$ be the other faces that are adjacent to, respectively, L' and L. Consider a sequence of subtrees $T_i^6 \subsetneq T_i^9 \subsetneq ... \subsetneq T_i^{3c_i+6}$ of T_i such that $T_i^6 = dv(Q_i)$, $T_i^{3c_i+6} = T_i$ and, for each $j \in \{2, ..., c_i+1\}$, $|T_i^{3j+3}| = |T_i^{3j}| + 1$. Let $r, s \in \{4, ..., c_i+2\}$ be such that $V(T_i^{3r}) - V(T_i^{3r-3}) = dv(R_i)$ and $V(T_i^{3s}) - V(T_i^{3s-3}) = dv(S_i)$. Since $\delta \ge 4, r \ne s$; without loss of generality, we may assume that $r < s$. For each $j \in \{2, ..., c_i + 2\}$, let G_i^{3j} be the plane subgraph of G_i with T_i^{3j} as its internal dual and, for each $j \in \{0, ..., c_i + 2\}$, let H_i^{3j} be the plane subgraph of G given by

$$H_i^{3j} = \begin{cases} G_i^6 & \text{if } j = 0, \\ (V(G_i^6) \cup y', E(G_i^6) \cup \{e', x'y'\}) & \text{if } j = 1, \\ (V(G_i^{3j}) \cup \{y', y\}, E(G_i^{3j}) \cup \{e', x'y', e, xy\}) & \text{if } 2 \le j < r, \\ (V(G_i^{3j}) \cup \{y\}, E(G_i^{3j}) \cup \{e, xy\}) & \text{if } r \le j < s, \\ G_i^{3j} & \text{if } j \ge s; \end{cases}$$

let C_i^{3j} be the boundary of the outer face of H_i^{3j}. Then, for each $j \in \{0, ..., c_i + 2\}$, H_i^{3j} is a 2-connected outerplanar graph where $E(C_i^{3j}) \cap E(L) = \{e, xy\}$ and $E(C_i^{3j}) \cap E(L') = \{e', x'y'\}$ with $|C_i^{3j}| = \sum_{v \in T_i^{3j}} w(v) + 4$ whenever $2 \le j < r$, $E(C_i^{3j}) \cap E(L) = \{e, xy\}$ and $E(C_i^{3j}) \cap E(L') = E(C) \cap E(L')$ with $|C_i^{3j}| = \sum_{v \in T_i^{3j}} w(v) + 3$ whenever $r \le j < s$, and $E(C_i^{3j}) \cap E(L) = E(C) \cap E(L)$ and $E(C_i^{3j}) \cap E(L') = E(C) \cap E(L')$ with $|C_i^{3j}| = \sum_{v \in T_i^{3j}} w(v) + 2$ whenever $j \ge s$. For each $j \in \{3, ..., c_i + 2\}$, let $H_i^{3j-1} = H_i^{3j} - y$ and $H_i^{3j-2} = H_i^{3j} - y - y'$ whenever $j < r$, let $H_i^{3j-1} = H_i^{3j} \cup e$ and $H_i^{3j-2} = (H_i^{3j} - y') \cup e$ whenever $r \le j < s$, and let $H_i^{3j-1} = H_i^{3j} \cup e$ and $H_i^{3j-2} = H_i^{3j} \cup \{e, e'\}$ whenever $j \ge s$;

let the outer face boundaries of H_i^{3j-1} and H_i^{3j-2} be, respectively, C_i^{3j-1} and C_i^{3j-2}, where $|C_i^{3j-1}| = |C_i^{3j}|-1$ and $|C_i^{3j-2}| = |C_i^{3j}|-2$. For the sake of completeness, let $H_i^1 = H_i^2 = H_i^3$ and $H_i^4 = H_i^5 = H_i^6$. Thus, for each $j \in \{3, ..., c_i+2\} - \{r, s\}$ and $z = V(T_i^{3j}) - V(T_i^{3j-3})$, if $w(z) > 1$, then $|C_i^{3j-1}|, |C_i^{3j-2}| > |C_i^k|$ for all $k \in \{0, 1, ..., 3j-3\}$; in other words, between $|C_i^{3j-3}|$ and $|C_i^{3j}|$, we skip at most $w(z) - 3$ sizes. Since $|C_i^{3r-3}| = \sum_{v \in T_i^{3r-3}} w(v) + 4$ and $|C_i^{3r}| = \sum_{v \in T_i^{3r}} w(v) + 3$, between the two of them we skip sizes (at most $w(dv(R_i)) - 4$ many of them) only if $w(dv(R_i)) > 4$; similarly, between $|C_i^{3s-3}|$ and $|C_i^{3s}|$ we skip sizes (at most $w(dv(S_i)) - 4$ many of them) only if $w(dv(S_i)) > 4$. Then, similar to the previous cases, we get that the number of cycle lengths missing up to $|H_i^0| = |H_i^3| - 1 = |H_i^6| - 2$ is at most $w(dv(Q_i)) - 2)$, and the number of cycle lengths that are not present in any graph in $\mathcal{G} := \{H_i^k : 0 \leq k \leq 3c_i + 6\}$ is at most $s_i^{>5} + 1$.

Thus, in each case, the number of missing cycle lengths is at most $s_i^{>5}+1$. Also, for each length $\geq |Q_i|$ noted as present in some member of $\mathcal{G}$, we have a cycle that passes through all the vertices in $V(Q_i)$ which are not the tips of L or L' (we start with the boundary of Q_i as a cycle of size $|Q_i|$, and all the subsequent outer face boundaries retain the vertex-set $V(Q_i)$ with the exception of possibly one or both of the vertices x and x'). This excludes the lengths 3 and possibly 5 (when $|Q_i| \geq 6$). Thus, we have a set of at least $n-2-(s_i^{>5}+1+2)$ cycles as claimed in the theorem statement. □

Corollary 40. *Let G be a 4-connected plane graph on $n \geq 5$ vertices not containing any 4-cycles. If there exists a well-triangulated vertex v of degree 4 in G incident in cyclic order with faces Q_0, R, Q_1 and R' where R and R' are triangular faces, then, for some $i \in \{0, 1\}$, there exists a set of cycles $\mathcal{C}$ in G of pairwise distinct lengths, each of size at least $|Q_i|$ and containing all the vertices in $V(Q_i) - N_{Q_i}(v)$ such that $|\mathcal{C}| \geq n - 5 - s^{>5}/2$.*

Proof. Let G be a 4-connected plane graph on $n \geq 5$ vertices not containing any 4-cycles, and let v be a well-triangulated vertex v of degree 4 in G incident in cyclic order with faces Q_0, R, Q_1 and R' where R and R' are triangular faces. Let e and e' be the edges of R and R', respectively, that are not incident with v.

Since G is 4-connected, by Theorem 34, there exists a Hamiltonian cycle C in G containing both e and e'. This cycle contains v and, hence, exactly two of the edges incident with it, one from each of R and R', which makes both R and R' leaf-triangles of (G, C). Additionally, the faces Q_0 and Q_1 lie on different sides of C and each of them is adjacent to both R and R'. Without loss of generality, we may assume that $Q_0 \in F(G_0)$ and $Q_1 \in F(G_1)$ in (G, C, G_0, G_1). Since, for (G, C, G_0, G_1), $s^{>5} = s_0^{>5} + s_1^{>5}$, either $s_0^{>5}$ or $s_1^{>5}$ is at most $s^{>5}/2$. The corollary then follows from Lemma 39. □

Theorem 41. *Let G be a 4-connected plane graph on $n \geq 5$ vertices not containing any 4-cycles. Then there exists a set $U \subset V(G)$ of at least three vertices and a set of cycles $\mathcal{C}$*

in G of pairwise distinct lengths, each of size at least $|U|+2$ and containing all the vertices in U, such that $|\mathcal{C}| \geq \frac{5}{6}(n-2)-2$. Moreover, the number of pairwise distinct cycle lengths present in G is at least $\lceil \frac{5}{6}(n-2) \rceil$.

Proof. Let G be a 4-connected plane graph on $n \geq 5$ vertices not containing any 4-cycles. By the proof of Lemma 35, there exist at least twelve 5-trees in G each of which shares its unshared edges with triangular faces in G and has its members incident with vertices all but one of which are of degree 4. Of these degree 4 vertices, every vertex not incident with a shared edge of the corresponding 5-tree is a well-triangulated vertex with a 5-face for at least one of the non-triangular faces incident with it. Let v be one such vertex with faces Q_0, R, Q_1 and R', where R and R' are triangular faces and Q_0 is a 5-face. Then, by Corollary 40, for some $i \in \{0,1\}$, there exists a set of cycles $\mathcal{C}$ in G of pairwise distinct lengths, each containing all the vertices in $U := V(Q_i) - N_{Q_i}(v)$ and of size at least $|Q_i| = |U|+2$ and such that

$$\begin{aligned} |\mathcal{C}| &\geq n-5-\frac{s^{>5}}{2} \\ &\geq n-5-\frac{n}{6}+5 \\ &= \frac{5n}{6}, \end{aligned}$$

where the second inequality follows from Lemma 35. Note that since $|Q_i| \geq 5$, we get that $|U| \geq 3$.

Moreover, by the proof of Lemma 39, this bound assumed exclusion of cycle lengths 3 and 5 which are clearly present in G. Thus, the number of pairwise distinct cycle lengths present in G is at least $\lceil \frac{5n}{6} \rceil + 2$. □

Proof of Theorem 30. Since every 4-connected planar graph has a unique embedding in the plane, it suffices to prove the theorem for any 4-connected plane graph not containing any 4-cycles. The proof then follows directly from Theorem 41. □

3.6 Some additional results

In this section, we prove some additional results that were discovered in trying to establish an almost complete cycle spectrum for 4-connected planar graphs not containing any 4-cycles. In view of the proof of Theorem 30, we believe that these results may lead to a proof strategy that is not too far from the one already used.

Proposition 42. *If G is a plane graph on $n \geq 5$ vertices not containing any 4-cycles then $|E(G)| \leq \frac{15}{7}(n-2)$, with equality achieved if and only if every face of G has size 3 or 5 and every edge of G is incident with one of each.*

Proof. Let G be a plane graph on $n \geq 5$ vertices not containing any 4-cycles, and let G' be a plane triangulation obtained by adding edges to the faces of G arbitrarily. Let $\bar{e} = |E(G')| - |E(G)| = 3n - 6 - |E(G)|$ be the total number of edges added. A face of G of size $f > 3$ contributes $f - 3$ to $\bar{e}$ and is replaced by $f - 2$ triangular faces; in other words, each edge added to this face increases the number of triangular faces by $\frac{f-2}{f-3}$ on average. Since $f \geq 5$, every edge added to G increases the number of triangular faces by at most $\frac{3}{2}$ on average. Thus, in constructing G' from G as described above the number of triangular faces added is at most $\frac{3}{2}\bar{e}$ which makes the number of triangular faces in G at least $2n - 4 - \frac{3}{2}\bar{e}$.

Since every edge in G can be in at most one triangular face, the number of triangular faces in G can be at most $\frac{|E(G)|}{3}$. Thus, $\frac{|E(G)|}{3} \geq 2n - 4 - \frac{3}{2}\bar{e} = 2n - 4 - \frac{3}{2}(3n - 6 - |E(G)|)$. Rearranging, we get that

$$\begin{aligned}
\frac{3}{2}|E(G)| - \frac{|E(G)|}{3} \leq \frac{9}{2}(n-2) - 2n + 4 \iff \\
\frac{7}{6}|E(G)| \leq \frac{5}{2}n - 5 \iff \\
|E(G)| \leq \frac{15}{7}(n-2).
\end{aligned}$$

To achieve equality, G must have exactly $2n - 4 - \frac{3}{2}\bar{e} = \frac{|E(G)|}{3}$ triangular faces, which makes every non-triangular face of size 5 and every edge incident with exactly one triangular face. In the other direction, if G is a plane graph with every face of size 3 or 5 and every edge incident with one of each, then Euler's formula gives us that

$$\begin{aligned}
n - |E(G)| + \left(\frac{|E(G)|}{3} + \frac{|E(G)|}{5}\right) = 2 \iff \\
|E(G)| = \frac{15}{7}(n-2).
\end{aligned}$$

□

Leaf-path decomposition of a tree: We define this as a decomposition of a tree T into a set of paths $\mathcal{P}(T) = \{P_1, ..., P_k\}$ such that:

- $\bigcup_{i=1}^{k} V(P_i) = V(T)$, $\bigcup_{i=1}^{k} E(P_i) \subset E(T)$,
- at least one leaf of every path in $\mathcal{P}(T)$ is also a leaf of T, and
- exactly one path in $\mathcal{P}(T)$ ends in leaves which are (possibly identical and) both leaves of T as well.

Proposition 43. *There exists a leaf-path decomposition for every tree T.*

Proof. Let T be a tree. We prove this by induction on the number of leaves k in T. If $k \leq 2$, then $\mathcal{P}(T) = \{T\}$, so we may assume that $k > 2$. Let v be a leaf of T, and let P_v be the maximal path in T with v as one end and with every vertex other than v of degree 2 in T. Let $T' = T - V(P)$. T' has one less leaf than T and, by the inductive hypothesis, there exists a leaf-path decomposition $\mathcal{P}(T')$ for T'. Then $\mathcal{P}(T) = \mathcal{P}(T') \cup \{P_v\}$ is a leaf-path decomposition for T. $\square$

Proposition 44. *Let G be a plane Hamiltonian graph on $n \geq 5$ vertices not containing any 4-cycles and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1), for any $i \in \{0, 1\}, c_i \leq \frac{5}{7}(n-3)$, with equality achieved if and only if every face of G_i except its outer face has size 3 or 5 and every edge of G_i is incident with a triangular face.*

Proof. Let G be a plane Hamiltonian graph on $n \geq 5$ vertices not containing any 4-cycles and C be a Hamiltonian cycle in G. For any $i \in \{0, 1\}$, consider the plane outerplanar graph G_i in (G, C, G_0, G_1). Since each edge in G_i is incident with at most one triangular face of G_i (also a triangular face of G), the number of non-triangular faces of G_i not including its outer face is at least $c_i + 1 - \frac{n+c_i}{3}$. Since each of these faces is also a face of G and has size at least 5, we get that

$$\begin{aligned} c_i &\leq n - 3 - 2(c_i + 1 - \frac{n + c_i}{3}) \iff \\ \frac{7}{3}c_i &\leq \frac{5}{3}n - 1 \iff \\ c_i &\leq \frac{5}{7}(n-3). \end{aligned}$$

To achieve equality, G_i must have exactly $c_i+1-\frac{n+c_i}{3}$ non-triangular faces (not including its outer face) each of size 5, which makes every edge incident with exactly one triangular face. In the other direction, if G_i is a plane outerplanar graph not containing any 4-cycles such that every face except its outer face has size 3 or 5 and every edge is incident with a triangular face, then Euler's formula gives us that

$$\begin{aligned} n - (n + c_i) + (\frac{n + c_i}{3} + \frac{c_i}{5} + 1) = 2 &\iff \\ c_i &= \frac{5}{7}(n-3). \end{aligned}$$

Note that c_0 and c_1 cannot both equal $\frac{5}{7}(n-3)$ as then $|E(G)| = \frac{10}{7}(n-3) + n = \frac{17}{7}(n-2) + \frac{4}{7}$, a contradiction to Proposition 42 $\square$

Recall from Section 3.5 that for any plane Hamiltonian graph G with a Hamiltonian cycle C, a leaf-triangle of (G, C) is defined as a triangular face of G whose boundary contains exactly two edges of C. For each $i \in \{0, 1\}$, let t_i be the number of leaf-triangles of (G, C) contained in G_i.

Lemma 45. *Let G be a plane Hamiltonian graph on $n \geq 5$ vertices not containing any 4-cycles and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1), for each $i \in \{0, 1\}$, $t_i \geq s_i^{>5} + 2c_i - n + 4$.*

Proof. Let G be a plane Hamiltonian graph on $n \geq 5$ vertices not containing any 4-cycles and C be a Hamiltonian cycle in G. For any $i \in \{0, 1\}$, consider the internal dual T_i of G_i in (G, C, G_0, G_1). In any leaf-path decomposition of T_i into $\mathcal{P}(T_i)$, if we remove the vertices dual to the leaf-triangles of (G, C) each path in $\mathcal{P}(T_i)$ is reduced to one with at least one leaf that is dual to a face of G of size ≥ 5. Since no two triangular faces in G share an edge, we get that $f_i^{\geq 5} \geq \frac{1}{2}(|V(T_i)| - t_i) = \frac{1}{2}(c_i + 1 - t_i)$. Then, by Proposition 31,

$$\begin{aligned} n - 2 &= (f_i^{\geq 5}(5 - 2) + s_i^{>5}) + (|V(T_i)| - f_i^{\geq 5})(3 - 2) \\ &= 3f_i^{\geq 5} + s_i^{>5} + c_i + 1 - f_i^{\geq 5} \\ &= 2f_i^{\geq 5} + s_i^{>5} + c_i + 1 \\ &\geq s_i^{>5} + 2c_i + 2 - t_i \iff \\ t_i &\geq s_i^{>5} + 2c_i - n + 4. \end{aligned}$$

□

Corollary 46. *Let G be a plane Hamiltonian graph on $n \geq 5$ vertices containing at least $2n$ edges but not containing any 4-cycles and C be a Hamiltonian cycle in G. Then for (G, C, G_0, G_1),*

(i) $t_1 \geq s_1^{>5} + 4$, *and*

(ii) $t_0 + t_1 \geq s^{>5} + 8$.

Proof. Let G be a plane Hamiltonian graph on $n \geq 5$ vertices containing at least $2n$ edges but not containing any 4-cycles and C be a Hamiltonian cycle in G so that, for (G, C, G_0, G_1), $c_0 + c_1 \geq n$, and $c_1 \geq n/2$. Then, by Lemma 45,

$$\begin{aligned} t_1 &\geq s_1^{>5} + 2c_1 - n + 4 \geq s_1^{>5} + 4, \text{and} \\ t_0 + t_1 &\geq s_0^{>5} + s_1^{>5} + 2(c_0 + c_1) - 2n + 8 \geq s_0^{>5} + s_1^{>5} + 8 = s^{>5} + 8. \end{aligned}$$

□

Remark 5. Observe that in Propositions 42 and 44, equality is achieved under conditions which imply that either $s^{>5} = 0$ or $s_i^{>5} = 0$ for some $i \in \{0, 1\}$. Similarly, Corollary 46 establishes a substantial lower bound on the number of leaf-triangles in (G, C). In contrast, the proof of Theorem 30 we provide makes use of only two such triangles to enumerate all but roughly $\min\{s_0^{>5}, s_1^{>5}\} \leq s^{>5}/2$ lengths.

Chapter 4

Conclusion

In the course of research done for this thesis, we were met with several open problems (some of them new) as compelling avenues for follow-up research. We mention some of them here.

Observe that the bound in the Conjecture 1 about minimum balanced bipartitions improves on the more general tight bound obtained by Fan, Xu, Yu and Zhou in [16] by a factor of roughly $\frac{3}{2}$. This prompted us to think whether it is possible to refine the latter based on the (orientable) genus of the graph. We pose this possibility as the following question.

Question 47. *Given a graph G on n vertices and of (orientable) genus g, does there exist a function $f(n, g)$ linear in n such that a minimum balanced bipartition of G has size at most $f(n, g)$?*

A possible first step could be to obtain/prove a tight bound for toroidal triangulations, possibly by trying to extend the proof for the planar case to the torus. That together with the planar bound might be indicative of what the said function f might look like.

Concerning pancyclicity, there are the conjectures pertaining to 4-connected planar graphs that are still unsettled − a complete cycle spectrum in 4-connected planar graphs containing a 4-cycle, and an almost complete cycle spectrum in the ones not containing any 4-cycles (with 4 being the only length missing). However, in light of the proof technique used, the following questions might make worthy intermediate steps.

Question 48. *Given a 4-connected planar graph G on n vertices, there exist in G at least $\lambda n + c$ cycles of pairwise distinct lengths containing any given edge e of G, where c is a constant. We know from Theorems 28 and 29 that $\lambda \in [\frac{1}{2}, \frac{2}{3}]$. We conjecture that $\lambda = \frac{2}{3}$.*

Question 49. *Given a 2-connected plane outerplanar graph on n vertices with exactly $\frac{n}{2}$ chords, there exist in G at least $\frac{n}{2} + c$ cycles of pairwise distinct lengths (where c is a constant) such that these lengths form a continuous interval in $3, ..., n$*

Additionally, there are some other sufficient conditions for Hamiltonicity that are close to the problem tackled in this thesis and thus make reasonable candidates to be inspected for pancyclicity. We mention one which concerns a popular subclass of planar graphs below.

Question 50. *Every* 3*-connected planar cubic graph with faces of size at most* 6 *is Hamiltonian ([23]). Is such a graph also pancyclic?*

The class of graphs in Question 50 refers to comprises of Barnette-Goodey graphs and contains as a subclass the more popular Fullerene graphs. Fullerene graphs have long been conjectured to be Hamiltonian and [23] verifies that as a special case. In light of Bondy's meta-conjecture, one could either examine either Fullerene graphs or Barnette-Goodey graphs for pancyclicity, or try to replace the computer-assisted proof in [23] with a combinatorial (and possibly simpler/shorter) one.

Then there are some other not-so-close sufficient conditions for Hamiltonicity that could be verified and/or checked for implying pancyclicity. We conclude this thesis by stating two of these below.

Theorem 51 (Kawarabayashi, Ozeki, [24])**.** *Every* 5*-connected toroidal graph is Hamiltonian-connected.*

Conjecture 52 (Thomassen, [41])**.** *Every* 4*-connected line graph is Hamiltonian.*

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