Graph colorings with local restrictions

by

Peter Anthony Bradshaw

M.Sc., Simon Fraser University, 2020
B.Sc., University of Kansas, 2016

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Fall 2022

Declaration of Committee

Name:	**Peter Anthony Bradshaw**
Degree:	**Doctor of Philosophy**
Thesis title:	**Graph colorings with local restrictions**
Committee:	**Chair:** Petr Lisonek Professor, Mathematics

Bojan Mohar
Co-supervisor
Professor, Mathematics

Ladislav Stacho
Co-supervisor
Professor, Mathematics

Luis Goddyn
Committee Member
Professor, Mathematics

Matt DeVos
Examiner
Professor, Mathematics

Maya Stein
External Examiner
Professor, Mathematics
Universidad de Chile

Abstract

A graph coloring is an assignment of a label, usually called a color, to each vertex of a graph. In nearly all applications of graph coloring, the colors on a graph's vertices must avoid certain forbidden local configurations. In this thesis, we will consider several problems in which we aim to color the vertices of a graph while obeying more complex local restrictions presented to us by an adversary.

The first problem that we will consider is the *list coloring* problem, in which we seek a proper coloring of a graph in which every vertex receives a color from a prescribed list given to that vertex by an adversary. We will consider this problem specifically for bipartite graphs, and we will take a modest step toward a conjecture of Alon and Krivelevich on the number of colors needed in the list at each vertex of a bipartite graph in order to guarantee the existence of a proper list coloring.

The second problem that we will consider is *single-conflict coloring*, in which we seek a graph coloring that avoids a forbidden color pair prescribed by an adversary at each edge. We will prove an upper bound on the number of colors needed for a single-conflict coloring in a graph of bounded degeneracy. We will also consider a special case of this problem called the *cooperative coloring* problem, and we will find new results on cooperative colorings of forests.

The third problem that we will consider is the *hat guessing game*, which is a graph coloring problem in which each coloring of the neighborhood of a vertex v determines a single forbidden color at v, and we aim to color our graph so that no vertex receives the color forbidden by the coloring of its neighborhood. We will prove that the number of colors needed for such a coloring in an outerplanar graph is bounded, and we will extend our method to a large subclass of planar graphs.

Finally, we will consider the *graph coloring game*, a game in which two players take turns properly coloring the vertices of a graph, with one player attempting to complete a proper coloring, and the other player attempting to prevent a proper coloring. We will show that if a graph G has a proper coloring in which the *game coloring number* of each bicolored subgraph is bounded, then the *game chromatic number* of G is bounded. As a corollary, we will obtain upper bounds for the game chromatic numbers of certain graph products and answer a question of X. Zhu.

Keywords: list coloring; single-conflict coloring; hat guessing game; graph coloring game; Lovász Local Lemma

Acknowledgements

I have been extremely fortunate to meet countlessly many good people during my journey to this point, and I have only been able to complete this program because of people who graciously helped me when they didn't need to, who are too many to list here completely. I am grateful to Stan Reimer for encouraging me and supporting me as I strove to grow as a young person and as a mathematician. I am grateful to Seung Kyung Park and JiSun Huh for showing me how deep, challenging, and rewarding math can be. I am grateful to Margaret Bayer and Jeremy Martin for being such inspirational educators and for working so hard to help me develop an interest in discrete math. I am grateful to my advisors Laco and Bojan for constantly supporting and encouraging me as I struggled to grow as a researcher. I am grateful to everyone with whom I had the pleasure of collaborating, especially Tomás Masařík, Jana Novotná, Kevin Halasz, Seyyed Aliasghar Hosseini, Jérémie Turcotte, Lyn Ge, Pavol Hell, Hehui Wu, Yaobin Chen, Hao Ma, and Alex Clow. I am also grateful to the many friends with whom I have shared some of my happiest moments.

I would also like to give special thanks to my wife, Jiamin, who stayed by my side while I spent years following my esoteric pursuits, and to my parents and sister who have patiently tolerated, supported, and encouraged me throughout my entire life.

Finally, thank you Bobo for being such a good dog. We miss you.

Table of Contents

List of Figures

Chapter 1

Introduction

A *graph* is an abstract object consisting of a set of elements called *vertices*, as well as a set of *edges*, which consist of vertex pairs. A graph is a natural theoretical model for any structure whose objects have pairwise relationships, such as a computer network, a social network, or a logistical network. In these applications, vertices typically represent the members of a network, and edges represent connections between these network members, such as direct communication links, acquaintanceships, or shipping routes between facilities.

A *graph coloring* is an assignment of a color to each vertex of a graph. Graph coloring is often used as a theoretical tool used to represent assignments of properties, roles, locations, or times to the members of a network. For example, in a social network consisting of employees, a graph coloring may represent a job assignment to each worker. Applications of graph coloring often require that a graph coloring satisfy a certain property, such as always assigning distinct colors to any two vertices joined by an edge. Given a graph representing a group of employees, for example, if each edge in the graph represents a conflict between two workers, then this restriction ensures that no two conflicting employees are assigned to the same job.

Graph coloring is one of the oldest concepts in graph theory, first considered by Francis Guthrie in 1852, who conjectured that every map can be colored using four colors so that no two contiguous regions use the same color [59]. Graph coloring is especially popular nowadays in computer science due to its numerous applications therein [61]. One important application of graph coloring in computer science is the *register allocation* problem [26], which asks whether a compiler can store a given set of variables in a limited number of registers, which the computer can access faster than other memory. A computer's architecture typically forbids two variables stored in the same register from being accessed at the same time. Therefore, the register allocation problem is often modelled as a graph coloring problem in which vertices represent variables, and two vertices are adjacent if the two corresponding variables require simultaneous access. Since computation speed depends on efficient use of registers [30], a better understanding of graph coloring has the potential to allow for the design of faster computers, which would have a positive impact in all fields using computational approaches, including protein structure prediction [52], renewable energy methods [5], and green transportation infrastructure [62].

In this thesis, we will consider several problems in which we are required to color the vertices of a graph while obeying certain local restrictions given to us by an adversary. Before we introduce the problems that we will consider in this thesis, we establish some definitions and notation.

1.1 Definitions and notation

Our definitions and notation follow Diestel [28]. A *graph* G is a pair (V, E), where V is a finite set, and E is a collection of unordered pairs of elements from V. We say that V is the *vertex set* of G, and we call the elements of V *vertices*. Similarly, we say that E is the *edge set* of G, and we call the elements of E *edges*. In this thesis, we do not allow a graph $G = (V, E)$ to have edges of the form $\{v, v\}$ for some $v \in V$ (often called *loops*). Unless otherwise noted, we do not allow repeated elements in E (often called *parallel edges*). A graph with no parallel edges is sometimes called a *simple* graph for the sake of clarity.

Given a graph G, we often write $V(G)$ and $E(G)$ for the vertex set and edge set of G, respectively. For two vertices $u, v \in V(G)$, if $\{u, v\} \in E(G)$, then we say that u and v are *adjacent*. For an edge $\{u, v\} \in E(G)$, we will often simply write uv for short. If H is a graph satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that H is a *subgraph* of G. If H is a subgraph of G such that for each vertex pair $u, v \in V(H)$ it holds that $uv \in E(G)$ if and only if $uv \in E(H)$, then H is an *induced subgraph* of G. Two graphs G and G' are *isomorphic* if there exists a bijection $\phi : V(G) \to V(G')$ such that for each pair $u, v \in V(G)$, $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G')$. If G and G' are isomorphic, we write $G \cong G'$. If G contains no subgraph isomorphic to H, then G is *H-free*.

An *directed graph* (or *digraph*) is a graph G in which each edge $uv \in E(G)$ has an order (u, v) or (v, u). An ordered edge in a directed graph is called an *arc*. Given a digraph G and a vertex $v \in V(G)$, the *out-degree* of v is the number of arcs of the form (v, u) in G, where $u \in V(G)$. If G has an arc (v, u), then we say that u is an *out-neighbor* of v. We write $N^+(v)$ for the set of out-neighbors of v. Similarly, the *in-degree* of v is the number of arcs of the form (u, v) in G, where $u \in V(G)$. If G has an arc (u, v), then we say that u is an *in-neighbor* of v. We write $N^-(v)$ for the set of in-neighbors of v. We often call a graph with unordered edges an *undirected graph*. An *orientation* of an undirected graph G is an assignment of an order or direction to each edge of G.

Given a graph G and a vertex $v \in V(G)$, we write $N(v)$ for the set of vertices $u \in V(G)$ for which u and v form an edge—that is, $N(v) = \{u \in V(G) : uv \in E(G)\}$. We say that $N(v)$ is the *neighborhood* of v and that the elements of $N(v)$ are *neighbors* of v. The vertex v is *universal* if $N(v) = V(G) \setminus \{v\}$. If $e \in E(G)$ and $v \in e$, then we say that v is *incident* to e. We define the *degree* of v as the number of edges $e \in E(G)$ to which v is incident, and we note that $\deg(v) = |N(v)|$ (unless we allow G to have parallel edges). A vertex of degree 1 is called a *leaf*. We often write Δ for the maximum degree over all vertices in a graph, and we often write δ for the minimum degree.

A graph G is *d-regular* if every vertex of G has degree d, and G is *regular* if it is d-regular for some d.

A *path* is a graph G with a vertex set $V(G) = \{v_1, \dots, v_t\}$ and an edge set $E(G) = \{v_i v_{i+1} : 1 \leq i \leq t-1\}$, and the vertices v_1 and v_t are called the *endpoints* of the path. A graph G is *connected* if for every vertex pair $u, v \in V(G)$, G contains a path with endpoints u and v as a subgraph. A *cycle* is a connected graph in which every vertex has degree exactly 2. A *component* of a graph G is a maximal connected subgraph of G.

A graph G is k-degenerate if every nonempty subgraph H of G has a vertex of degree at most k. If d is the minimum integer such that G is k-degenerate for each value $k \geq d$, then we say that d is the *degeneracy* of G. A 1-degenerate graph is called a *forest* or an *acyclic* graph. Alternatively, a graph G is a forest if and only if G contains no cycle as a subgraph. A connected forest is called a *tree*.

A graph G for which $uv \in E(G)$ for each pair $u, v \in V(G)$ is called a *clique* or a *complete graph*. A clique of t vertices is called a *t-clique* and is denoted as K_t. The *clique number* of a graph G is the maximum integer ω for which G contains a subgraph isomorphic to K_ω. A K_3 is called a *triangle*. A graph G for which $E(G) = \emptyset$ is called an *independent set*.

A graph G is *planar* if, roughly speaking, G can be drawn on the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with no crossing edges. More formally, let each edge $e \in E(G)$ have an associated open interval $I_e \subseteq \mathbb{R}$, with distinct edges having disjoint intervals, and let each vertex $v \in V(G)$ have a distinct value $x_v \in \mathbb{R} \setminus \bigcup_{e \in E(G)} \overline{I_e}$. (Here, $\overline{I_e}$ denotes the closure of I_e, obtained from I_e by adding the two endpoints of the open interval.) Then, a continuous injection $\pi : \bigcup_{v \in V(G)} \{x_v\} \cup \bigcup_{e \in E(G)} I_e \to S^2$ is a *planar embedding* if for each edge $uv \in E(G)$ and each $\varepsilon > 0$, $\pi(I_{uv})$ contains a point at a distance of at most ε from $\pi(x_u)$ and a point at a distance of at most ε from $\pi(x_v)$. The graph G is planar if it has a planar embedding.

In addition to the sphere, we will also consider graph drawings on more complex surfaces. However, the formal definition of a surface is highly technical, so we give an informal definition instead. For a formal definition of surfaces, see Hatcher [45]. Informally, we say that a *handle* is added to the sphere by flattening a small region of the sphere and then removing the interior of an $r \times r$ square in this flattened region with a boundary parametrized by a function

$$f(t) = \begin{cases} (x_0, y_0) + (rt, 0) & t \in [0,1], \\ (x_0, y_0) + (0, r(t-1)) & t \in (1,2), \\ (x_0, y_0 + r) + (r(t-2), 0) & t \in [2,3], \\ (x_0 + r, y_0) + (0, r(t-3)) & t \in (3,4), \end{cases}$$

and then identifying the point $f(t)$ with $f(2+t)$ for all $t \in [0, 2)$. Similarly, we say that a *cross cap* is added to the sphere again by flattening a small region of the sphere and then removing the interior of a circle of some radius r in this flattened region with a boundary parametrized by a function $f(t) = (x_0, y_0) + r(\cos t, \sin t)$ for $t \in [0, 2\pi]$, and then identifying the point $f(t)$ with $f(t+\pi)$ for all $t \in [0, \pi]$. If we add multiple handles or cross caps to the sphere, then we require that their associated squares and circles be disjoint, and we say that a sphere possibly with handles and cross caps added is called a *surface*. If a surface S is constructed from the sphere by adding h handles and c cross caps, then we say that the *Euler genus* of S is $2h + c$. Note that the sphere has Euler genus 0. We define a graph embedding on a surface similarly to a planar embedding; that is, roughly speaking, a graph embedding is a drawing of a graph on a surface with no crossing edges. For a formal definition of a graph embedding on a surface, see Hatcher [45]. If a graph G can be embedded on a surface of Euler genus g with no crossing edges, then we say that the Euler genus of G is at most g. A planar graph has Euler genus 0. For an example of a graph with Euler genus 1, consider the complete graph K_5. A simple exercise shows that K_5 has no planar embedding. However, we may draw K_5 on the sphere with only one pair of edges crossing, and then we may replace a small neighborhood of this crossing in the sphere with a cross cap, yielding an embedding of K_5 on a surface of Euler genus 1.

Given a graph G, a *graph coloring* of G is a map $\phi : V(G) \to \mathbb{N}$. Given a graph coloring ϕ, we often refer to the elements in the codomain of ϕ as *colors*, and for each element c in the image of ϕ, we say that the vertex set $\phi^{-1}(c)$ is a *color class* of ϕ. A graph coloring ϕ of G is *proper* if for each edge $uv \in E(G)$, $\phi(u) \neq \phi(v)$. In other words, ϕ is proper if and only if each color class of ϕ is an independent set. If the image of a proper coloring ϕ contains exactly k colors, then we say that ϕ is a *k-coloring*. If G admits a k-coloring, then we say that G is *k-colorable*. A 2-colorable graph is called *bipartite*. Given a 2-coloring ϕ of a bipartite graph G, we refer to the color classes of ϕ as *partite sets*. A *complete bipartite graph* is a graph with two partite sets A and B and an edge set $\{ab : a \in A, b \in B\}$. We write $K_{m,n}$ for a complete bipartite graph with partite sets of sizes m and n. A graph of the form $K_{1,n}$ is called a *star*. Similiar to a graph coloring, an *edge coloring* of a graph G is a map $\phi : E(G) \to \mathbb{N}$. We say that ϕ is *proper* if whenever $e, e' \in E(G)$ are incident edges, $\phi(e) \neq \phi(e')$.

We write $\chi(G)$ for the *chromatic number* of G, which is defined as the least integer k for which G is k-colorable. The chromatic number of every graph G is well defined, as a greedy argument shows that $\chi(G) \leq \Delta + 1$, where Δ is the maximum degree of G.

All logarithms that we use in this thesis are natural (with base e) unless otherwise specified. Suppose $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ are functions. If there exists a positive value $C \in \mathbb{R}$ such that $|f(n)| \leq C|g(n)|$ for all $n \in \mathbb{N}$, then we write $f = O(g)$ and $g = \Omega(f)$. If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then we write $f = o(g)$ and $g = \omega(f)$. In particular, if $f = o(1)$, then $\lim_{n\to\infty} f(n) = 0$, and if $f = \omega(1)$, then $\lim_{n\to\infty} f(n) = \infty$. If an event A depends on some unbounded parameter so that the probability of A satisfies $\Pr(A) = 1 - o(1)$, then we say that A occurs *asymptotically almost surely*, or *a.a.s.* for short.

Throughout the thesis, we will often omit floors and ceilings when they have no effect on our arguments.

1.2 Chromatic number

The notion of a graph's chromatic number goes back to Guthrie's map coloring problem from 1852, and the study of the relationship between a graph's chromatic number and its other properties is one of the foundational problems in graph theory. In particular, a great deal of graph theoretic research has focused on the relationship between a graph's maximum degree and its chromatic number. In 1941, Brooks [24] proved that if G is a graph of maximum degree $\Delta \geq 3$ and is not a clique, then $\chi(G) \leq \Delta$. In 1977, Borodin and Kostochka [17] made a conjecture of a similar flavor, postulating that if G is a graph of maximum degree $\Delta \geq 9$ with clique number at most $\Delta - 1$, then $\chi(G) \leq \Delta - 1$. Reed proved that Borodin and Kostochka's conjecture holds whenever $\Delta \geq 10^{14}$, and he claims that a more careful analysis can prove the conjecture for $\Delta \geq 1000$ [69]. Reed also conjectured a more general relationship between a graph's maximum degree, clique number, and chromatic number [68]. Reed's conjecture states that if G is a graph of maximum degree Δ and clique number ω, then $\chi(G) \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$. While Reed's conjecture is still open, Reed has proven the weaker result that there exists a constant $\varepsilon > 0$ such that $\chi(G) \leq \lceil (1-\varepsilon)(\Delta+1) + \varepsilon\omega \rceil$ holds for all graphs G of maximum degree Δ [68, Corollary 2].

One particular branch of research on the relationship between a graph's maximum degree and chromatic number considers the special case that the clique number of a graph G is at most 2—that is, that G is triangle-free. In 1996, Johansson [50] proved that if a graph G of maximum degree Δ is triangle-free, then $\chi(G) \leq \frac{9\Delta}{\log \Delta}$. Molloy [65] later reduced the constant asymptotically in this upper bound with the following theorem:

Theorem 1.2.1. *If G is a triangle-free graph of maximum degree Δ, then $\chi(G) \leq (1 + o(1))\frac{\Delta}{\log \Delta}$.*

Theorem 1.2.1 is best possible up to a factor of $2 + o(1)$, since random constructions show that triangle-free graphs G exist satisfying $\chi(G) \geq (\frac{1}{2} + o(1))\frac{\Delta}{\log \Delta}$ [41]. However, improving the coefficient in Theorem 1.2.1 seems to be a difficult problem for several reasons. First, the best known lower bound for the independence number of a Δ-regular triangle-free graph on n vertices is $(1 + o(1))\frac{n \log \Delta}{\Delta}$, which was established in 1983 by Shearer [71] and has not been improved since then apart from the $o(1)$ function [72]. Since a graph on n vertices with chromatic number χ has independence number at least n/χ, a reduction of the coefficient in Theorem 1.2.1 to $1 - \varepsilon$ for some $\varepsilon > 0$ would also give an improvement to Shearer's result, namely an increase of the $1 + o(1)$ coefficient to $\frac{1}{1-\varepsilon}$. Second, Molloy has pointed out that due to similarities between random regular graphs and triangle-free graphs, finding an efficient algorithm that colors a triangle-free graph with $(1 - \varepsilon)\frac{\Delta}{\log \Delta}$ colors for any $\varepsilon > 0$ would solve a major open problem in the theory of colorings of random graphs (see [1, 65, 79] for details). Therefore, it seems that Theorem 1.2.1 is best possible without a major advance in current knowledge.

While this thesis does not directly study the chromatic numbers of graphs, the existing research on chromatic number discussed above provides important foundation and context for our results.

1.3 List coloring

One graph coloring problem that we will consider in the thesis is the *list coloring problem*, defined as follows. Consider a graph G, and suppose that each vertex $v \in V(G)$ has an associated list $L(v) \subseteq \mathbb{N}$. Then, an *$L$-coloring* of G is a proper coloring $\phi : V(G) \to \mathbb{N}$ such that $\phi(v) \in L(v)$ for each $v \in V(G)$. Given a graph G and a list assignment L, the list coloring problem asks whether G has an L-coloring.

If G has an L-coloring for each list assignment L satisfying $|L(v)| \geq k$ for each vertex $v \in V(G)$, then we say that G is *k-choosable*. The *list chromatic number*, *choice number*, or *choosability* of a graph, written $\mathrm{ch}(G)$, is the minimum integer k for which G is k-choosable. The choosability of every graph G is well defined, as a greedy argument shows that $\mathrm{ch}(G) \leq \Delta + 1$, where Δ is the maximum degree of G. If G is k-choosable, then G has an L-coloring when $L(v) = \{1, \dots, k\}$ for each vertex $v \in V(G)$; therefore, $\chi(G) \leq \mathrm{ch}(G)$. When determining the choosability of a graph G, we may imagine that the lists $L(v)$ for vertices $v \in V(G)$ are chosen by an adversary who wishes to make finding an L-coloring as difficult as possible.

The list coloring problem was first considered by Erdős, Rubin, and Taylor [36], who proved that every connected graph G of maximum degree $\Delta \geq 3$ satisfies $\mathrm{ch}(G) \leq \Delta$, except when $G \cong K_{\Delta+1}$. Similarly to the chromatic number, a great deal of research on choice number has investigated the relationship between this parameter and a graph's maximum degree and clique number. For instance, Choi, Kierstead, Rabern, and Reed [27] showed that Borodin and Kostochka's conjecture discussed above holds for choosability in graphs of large maximum degree, proving that if G has maximum degree $\Delta \geq 10^{20}$, then $\mathrm{ch}(G) \leq \Delta - 1$ whenever G contains no subgraph isomorphic to K_Δ. Furthermore, the results of Johansson and Molloy for graphs of bounded clique number discussed above also hold in the setting of list coloring.

One specific subclass of triangle-free graphs that has been the target of list coloring research is the class of bipartite graphs. While the chromatic number of a bipartite graph G is at most 2, Erdős, Rubin, and Taylor [36] showed that the choosability of a bipartite graph G may be arbitrarily large by proving that $\mathrm{ch}(K_{n,n}) = (1 + o(1)) \log_2 n$. Later, Alon showed that in addition to complete bipartite graphs, *every* bipartite graph G with large minimum degree δ has large choosability, proving the lower bound $\mathrm{ch}(G) \geq \Omega\left(\frac{\log \delta}{\log \log \delta}\right)$ [6] and later $\mathrm{ch}(G) \geq (\frac{1}{2} - o(1)) \log_2 \delta$ [7]. Saxton and Thomason [70] later proved a better lower bound $\mathrm{ch}(G) \geq (1 - o(1)) \log_2 \delta$, and the fact that $\mathrm{ch}(K_{n,n}) = (1 + o(1)) \log_2 n$ shows that this bound is best possible up to the $o(1)$ function.

In [10], Alon and Krivelevich show that there exists an absolute constant d_0 such that if G is a random bipartite graph of expected average degree $d > d_0$ obtained from two partite sets A and B of size n by adding each edge in $\{ab : a \in A, b \in B\}$ independently with probability $p = d/n$, then a.a.s. $\text{ch}(G) \le (1 + o(1)) \log_2 d$. They made the following conjecture, which asserts that a similar upper bound holds for the choosability of every bipartite graph.

Conjecture 1.3.1 ([10])**.** *If G is a bipartite graph of maximum degree Δ, then* $\text{ch}(G) = O(\log \Delta)$.

While Conjecture 1.3.1 asks a very natural question about a fundamental property of a highly-studied graph class, surprisingly, researchers have made very little progress toward an answer. In particular, the currently best-known upper bound for the choosability of a bipartite graph of maximum degree Δ is $(1 + o(1))\frac{\Delta}{\log \Delta}$, which Molloy [65] proved for all triangle-free graphs, as discussed above.

In Chapter 2, we will prove the following upper bound for the choosability of a bipartite graph, which improves the currently best-known coefficient of $1 + o(1)$ when Δ is large.

Theorem 1.3.2. *If G is a bipartite graph of sufficiently large maximum degree Δ, then* $\text{ch}(G) < 0.797\frac{\Delta}{\log \Delta}$.

On one hand, Theorem 1.3.2 makes only a modest improvement to the coefficient of the previously-known upper bound and is still far away from the conjectured bound of $O(\log \Delta)$. On the other hand, since the bound of $(1+o(1))\frac{\Delta}{\log \Delta}$ in Theorem 1.2.1 for triangle-free graphs seems to be a very difficult to improve with current knowledge, as discussed above, Theorem 1.3.2 gives evidence that the list coloring problem is significantly easier in bipartite graphs than in triangle-free graphs. Therefore, Theorem 1.3.2 gives a good step toward Conjecture 1.3.1 by reducing the upper bound on the choosability of bipartite graphs well below what is believed possible with current methods for triangle-free graphs and therefore showing evidence of a fundamental difference between the list coloring problem in bipartite graphs and triangle-free graphs.

1.4 Single-conflict coloring

Another graph coloring problem that we will consider is the *single-conflict coloring* problem, which is defined as follows. Let G be a graph, and let $C = \{1, \dots, k\}$ be a set of colors. Suppose that f is a function that maps each edge $uv \in E(G)$ to an ordered forbidden color pair $f(u, v) = (c_1, c_2)$, with $c_1, c_2 \in C$. Then, we say that a (not necessarily proper) coloring $\phi : V(G) \to C$ is a single-conflict coloring with respect to f and C if $f(u, v) \neq (\phi(u), \phi(v))$ for each edge (u, v) of G. We call the image of an edge (u, v) under f a *conflict*, and we call f a *conflict function*. If a graph G always has a single-conflict coloring for any conflict function f when $C = \{1, \dots, k\}$, then we say that the *single-conflict chromatic number* of G is at most k, and we write $\chi_{\leftrightarrow}(G) \le k$. Hence, the single-conflict chromatic number of G is the minimum integer k for which the property above holds.

A simple argument shows that if G has a single-conflict coloring for any conflict function C when $C = \{1, \dots, k\}$, then the same holds when $C = \{1, \dots, k+1\}$. When determining the single-conflict chromatic number of a graph, we may imagine that the graph's conflict function is chosen by an adversary who wishes to make finding a single-conflict coloring as difficult as possible. When we consider the single-conflict coloring problem, we will typically allow graphs to have parallel edges.

Note that a conflict is an ordered pair; that is, if $f(u, v) = (\texttt{red}, \texttt{blue})$, then it is forbidden to use `red` at u and `blue` at v, but it is acceptable to use `blue` at u and `red` at v. Hence, we consider the edge set $E(G)$ to be a set containing ordered vertex pairs, and we write that f is a mapping $f : E(G) \to C^2$. If $(u, v) \in E(G)$ is an edge with conflict $f(u, v) = (c_1, c_2)$, then we will also say that $(v, u) \in E(G)$ is an edge with conflict $f(v, u) = (c_2, c_1)$. Furthermore, if we allow parallel edges in G, then if an edge $(u, v) \in E(G)$ appears with multiplicity m, we let f map (u, v) to a set of m conflicts in C^2, one for each parallel edge.

The single-conflict coloring problem is a specific kind of *independent transversal* problem, which is defined as follows. Given a graph H with a vertex partition $V_1 \cup \dots \cup V_r$, we say that an independent transversal on H is an independent set I in H such that I contains exactly one vertex from each part V_i. Given a graph G with a conflict function $f : E(G) \to \{1, \dots, k\}$, we can transform the single-conflict coloring problem on G into an independent transversal problem as follows. We define a graph H with a vertex set $V(H) = V \times [k]$, an edge $(u, f(u))(v, f(v))$ for each edge $uv \in E(G)$, and with a vertex partition consisting of a part $\{v\} \times [k]$ for each vertex $v \in V$. Then, given such a graph H constructed from G and f, an independent transversal on H with respect to the parts described above gives a single-conflict coloring of G, and any single-conflict coloring of G can be transformed into an independent transversal on H. Certain other graph coloring problems can also be naturally described as independent transversal problems. For example, DP-coloring (also called correspondence coloring) is a recent generalization of list coloring invented by Dvořák and Postle [32]. One way of defining the DP-chromatic number $\chi_{DP}(G)$ of a graph G is with the following statement: $\chi_{DP}(G) \leq k$ if and only if every graph H forming a k-sheeted covering space of G with a projection $p : H \to G$ has an independent transversal with respect to the partition $\bigcup_{v \in V(G)} p^{-1}(v)$ of $V(H)$ (see Hatcher [45] for relevant definitions).

The concept of a single-conflict coloring of a graph was considered in several equivalent settings by Erdős, Gyárfás, and Łuczak [35], Dvořák and Postle [32], and Fraigniaud, Heinrich, and Kosowski [40]. The notion of the single-conflict chromatic number was later introduced by Dvořák, Esperet, Kang, and Ozeki [31], who proved the following upper bounds.

Theorem 1.4.1. *If G is a graph of maximum degree Δ, then* $\chi_{\nleftrightarrow}(G) \leq \left\lceil 2\sqrt{\Delta} \right\rceil$.

Theorem 1.4.2. *If G is a simple graph of Euler genus g, then* $\chi_{\nleftrightarrow}(G) = O((g+1)^{1/4} \log(g+2))$.

Dvořák et al. prove Theorem 1.4.1 using Bernshteyn's Local Cut Lemma [13], and a simple argument using Rosenfeld counting can also prove the result (see [76] for an introduction to this method). In the case that G has edge-multiplicity $o(\Delta)$, Glock and Sudakov [44], as well as Kang and Kelly [53], were able to reduce the upper bound of Theorem 1.4.1 to $(1+o(1))\sqrt{\Delta}$.

As for lower bounds on the single-conflict chromatic number, Dvořák, Esperet, Kang, and Ozeki [31] showed that that if a graph G has degeneracy d, then $\chi_{\nleftrightarrow}(G) = \Omega\left(\sqrt{\frac{d}{\log d}}\right)$. Furthermore, Molloy [64] showed that for every d-degenerate graph of chromatic number $d+1$, $\chi_{\nleftrightarrow}(G) \geq (1+o(1))\sqrt{d+1}$. Dvořák, Esperet, Kang, and Ozeki [31] posed the following question, which asks whether these lower bounds are close to best possible.

Question 1.4.3. *Suppose G is a d-degenerate graph on n vertices. Is it true that $\chi_{\nleftrightarrow}(G) = O(\sqrt{d}\log n)$?*

The reason that Dvořák, Esperet, Kang, and Ozeki ask Question 1.4.3 with an "error" factor as large as $\log n$ is that the $\log n$ factor is still small enough that a positive answer would give an alternative proof of Theorem 1.4.2, as shown in [31]. We will give an affirmative answer to Question 1.4.3 for simple graphs in Chapter 3 with the following theorem, which also shows that the error factor can be reduced to roughly $\sqrt{\log \Delta}$.

Theorem 1.4.4. *If G is a d-degenerate simple graph of maximum degree Δ, then*

$$\chi_{\nleftrightarrow}(G) \leq \left\lceil 2\sqrt{d\left[1+\log((d+1)\Delta)\right]}\right\rceil.$$

Theorem 1.4.4 gives a large class of d-degenerate graphs G satisfying $\chi_{\nleftrightarrow}(G) = O(d^{\frac{1}{2}+o(1)})$, containing in particular d-degenerate simple graphs G with maximum degree $\Delta = \exp(d^{o(1)})$. While we are unable to prove that the logarithmic error factor in Theorem 1.4.2 is best possible, we note that an upper bound of less than $d+1$ is unachievable, as Kostochka and Zhu [57] give examples of d-degenerate graphs G that satisfy $\chi_{\nleftrightarrow}(G) = d+1$.

1.4.1 Adapted colorings and cooperative colorings

One special case of the single-conflict coloring problem is the *adapted coloring* problem, which is obtained from the single-conflict coloring problem by requiring that each conflict be monochromatic. Specifically, the adapted-coloring problem can be defined as follows. Given a graph G with a (not necessarily proper) edge-coloring $\psi : E(G) \to \{1,\dots,k\}$, an *adapted coloring* of G is a (not necessarily proper) vertex coloring $\phi : V(G) \to \{1,\dots,k\}$ such that $\phi(u) = \phi(v) = \psi(uv)$ does not hold for any edge $uv \in E(G)$. In other words, the forbidden condition in an adapted coloring of G is an edge $e \in E(G)$ for which both endpoints are colored $\psi(e)$. The adapted coloring problem was first introduced by Hell and Zhu [49].

Adapted colorings are equivalent to *cooperative colorings*, which are defined as follows. Given a family $\mathcal{G} = \{G_1,\dots,G_k\}$ of graphs on a common vertex set V, a cooperative coloring on $\mathcal{G}$ is defined as a family of sets $R_1,\dots,R_k \subseteq V$ such that for each $1 \leq i \leq k$, R_i is an independent set of

G_i, and $V = \bigcup_{i=1}^k R_i$. The term "cooperative coloring" and this formulation of the adapted coloring problem first appear in [2]. A cooperative coloring problem may be translated into an adapted coloring problem by coloring the edges of each graph $G_i \in \mathcal{G}$ with the color i and then considering the union of all graphs in $\mathcal{G}$. Overall, this gives us the following observation.

Observation 1.4.5. *Given a family $\mathcal{G} = \{G_1, \dots, G_k\}$ of graphs on a common vertex set, the cooperative coloring problem on $\mathcal{G}$ is equivalent to the adapted coloring problem on the edge-colored multigraph $G = \bigcup_{i=1}^k G_i$ in which each edge originally from G_i is colored with the color i.*

In [2], Aharoni, Berger, Chudnovsky, Havet, and Jiang ask how many graphs of maximum degree d must belong to a family $\mathcal{G}$ on a common vertex set in order to guarantee the existence of a cooperative coloring. It is straightforward to show that Theorem 1.4.1 implies that a graph family $\mathcal{G}$ containing k graphs of maximum degree d on a common vertex set V has a cooperative coloring whenever $k \geq 4d$. However, a theorem of Haxell [46] for independent transversals shows that it is sufficient to let $k \geq 2d$, as in the following theorem.

Theorem 1.4.6. *If $\mathcal{G}$ is a family of graphs maximum degree d on a common vertex set, and if $|\mathcal{G}| \geq 2d$, then $\mathcal{G}$ has a cooperative coloring.*

Furthermore, the following result of Aharoni, Berger, Chudnovsky, Havet, and Jiang [2] shows that if $\mathcal{T}$ is a family of 1-degenerate graphs (i.e. forests) of maximum degree d on a common vertex set, then $\mathcal{G}$ has a cooperative coloring even when $|\mathcal{T}|$ is small compared to d.

Theorem 1.4.7 ([2])**.** *If $\mathcal{T}$ is a family of forests of maximum degree d on a common vertex set V, then there exists a value $k = (1 + o(1)) \log_{4/3} d$ such that if $|\mathcal{T}| \geq k$, then $\mathcal{T}$ has a cooperative coloring.*

Aharoni, Berger, Chudnovsky, Havet, and Jiang [2] gave a lower bound of the form $\Omega(\log \log d)$ for the minimum number of forests required in $\mathcal{T}$ to guarantee the existence of a cooperative coloring by constructing a family of $\Omega(\log \log d)$ forests of maximum degree d that do not admit a cooperative coloring. In Chapter 3, we will find a better lower bound on the number of forests needed for a cooperative coloring that nearly matches the upper bound in Theorem 1.4.7. Namely, we will prove the following.

Theorem 1.4.8. *For each sufficiently large value d, there exists a family $\mathcal{T}$ of $(1 + o(1)) \frac{\log d}{\log \log d}$ forests of maximum degree d on a common vertex set that does not admit a cooperative coloring.*

In addition, using a stronger version of Theorem 1.4.4, we will extend Theorem 1.4.7 to families of graphs of bounded degeneracy at the expense of a constant factor, as follows.

Corollary 1.4.9. *Let $\mathcal{G}$ be a family of $m \geq 13(1 + k \log(kd))$ graphs on a common vertex set V. If each graph $G \in \mathcal{G}$ is at most k-degenerate and of maximum degree d, then $\mathcal{G}$ has a cooperative coloring.*

1.5 The hat guessing game

The *hat guessing game* is defined as follows. We have a graph G, and a *player* resides at each vertex of G. For each vertex $v \in V(G)$, the player at v can see exactly those players at the neighbors of v. In particular, a player cannot see himself. An *adversary* possesses a large collection of hats of different colors. When the game starts, the adversary places a hat on the head of each player, and then each player privately guesses the color of his hat. The players win the game if at least one player correctly guesses the color of his hat; otherwise, the adversary wins. Before the game begins, the players may come together to devise a guessing strategy, but this strategy is known to the adversary, and the adversary may choose a hat assignment with the strategy of the players in mind. The hat guessing game was first considered for complete graphs by Winkler [77] and later for general graphs by Butler, Hajiaghayi, Kleinberg, and Leighton [25].

The hat guessing game is typically studied with the following two assumptions. First, it is assumed that the adversary possesses enough hats of each color so that no color will ever run out while the adversary is assigning hats to players. Second, it is assumed that the players follow a deterministic strategy to guess their hat colors; that is, the guess of a player at a vertex v is uniquely determined by the hat colors at neighbors of v. Given a graph G, we say that the *hat guessing number* of G is the maximum number k of hat colors such that the players on G have a strategy that guarantees that at least one player will correctly guess his hat color when each player is given a hat with a color from the set $\{1, \dots, k\}$. We write $\mathrm{HG}(G)$ for the hat guessing number of G. In other words, if $\mathrm{HG}(G) \geq k$, then there exists a strategy for players on the graph G such that for any hat color assignment $V(G) \to \{1, \dots, k\}$, at least one player will correctly guess the color of his hat.

We may describe the hat guessing game formally as a graph coloring problem as follows. Let G be a graph, and let $S = \{1, \dots, k\}$ be a set of colors. We define a *hat guessing strategy* on G to be a family $\Gamma = \{\Gamma_v\}_{v \in V(G)}$ of functions, where each function is a mapping $\Gamma_v : S^{N(v)} \to S$; that is, each function Γ_v takes a coloring of $N(v)$ as input and returns a color from S as output. We say that the strategy Γ is a *winning strategy* if, for every (not necessarily proper) graph coloring $\phi : V(G) \to S$, there exists a vertex $v \in V(G)$ with neighbors $(u_1, \dots, u_t)$ such that Γ_v maps $(\phi(u_1), \dots, \phi(u_t))$ to $\phi(v)$. It is clear that the players win the hat guessing game on G with hat color set S if and only if there exists a winning strategy Γ on G. Kohkas and Latyshev show in [55] that with optimal play, the winner of the hat guessing game does not change even when each vertex $v \in V(G)$ receives a hat from an arbitrary list L_v of size $|S|$, rather than from the set S. Furthermore, they show that if each vertex $v \in V(G)$ has a list L_v of possible hat colors, then only the size of each list L_v affects whether or not the players have a winning strategy on G. Note that if we consider the hat guessing game as a graph coloring problem in which we seek a hat assignment that defeats the players on a graph G, the players in fact play the adversarial role by choosing hat guessing strategies that require as many colors to defeat as possible.

We find it useful to give a simple example of a winning strategy in the hat guessing game. We consider the hat guessing game played on K_2 in which the adversary has only red hats and blue hats. We refer to the players as Alice and Bob. It is straightforward to see that Alice and Bob have the following winning strategy: Alice will guess the color of Bob's hat, and Bob will guess the opposite color of Alice's hat. This way, if Alice and Bob receive the same hat color, then Alice will guess correctly. On the other hand, if Alice and Bob receive different hat colors, then Bob will guess correctly. This shows that $\mathrm{HG}(K_2) \geq 2$, and in fact, $\mathrm{HG}(K_2) = 2$ [39].

While the rules of the hat guessing game are simple, establishing upper bounds for the hat guessing numbers achieved by graphs in large classes is surprisingly difficult. In contrast to other game-related graph parameters such as game coloring number [81] and cop number [51, 19], no upper bound is known for the hat guessing number of graphs of bounded treewidth or graphs of bounded Euler genus. Farnik [38] asked whether the hat guessing number of a graph G is bounded by some function of the degeneracy of G, but this question remains unanswered, and graphs of degeneracy d and hat guessing number at least $2^{2^{d-1}}$ have been constructed [48]. In particular, the following more specific question, appearing in [18] and [56], remains unanswered:

Question 1.5.1. *Is the hat guessing number of planar graphs bounded above by some universal constant?*

While Question 1.5.1 has not yet been answered, Kokhas and Latyshev [56] have shown examples of planar graphs G for which $\mathrm{HG}(G) \geq 14$. For some restricted graph classes, such as cliques [39], cycles [73], complete bipartite graphs [8, 43], graphs of bounded degree [38], graphs of bounded treedepth [48], embedded graphs of large girth [18], cliques joined at a single cut-vertex, and split graphs [47], bounds for the hat guessing number have been determined.

In Chapter 4, we will take a major step toward answering Question 1.5.1 by proving an affirmative answer for a large class of planar graphs. For outerplanar graphs, we will prove the following result.

Theorem 1.5.2. *If G is an outerplanar graph, then* $\mathrm{HG}(G) < 2^{125000}$.

Recently, Knierim, Martinsson, and Steiner [54] improved this upper bound to 40.

We will also prove an upper bound for the hat guessing number of *layered planar graphs*, which we roughly define as planar graphs that can be obtained by beginning with a 2-connected outerplanar graph G_1, and then for some value τ and $1 \leq i \leq \tau$, adding a 2-connected outerplanar graph G_{i+1} to some interior face of G_i and adding non-crossing edges between G_i and G_{i+1}. We also use the term *layered planar graph* to describe a subgraph of such a planar graph. For layered planar graphs, we have the following result.

Theorem 1.5.3. *If G is a layered planar graph, then* $\log_2 \log_2 \log_2 \log_2 \log_2 \mathrm{HG}(G) < 149$.

Theorems 1.5.2 and 1.5.3 are the first results that show upper bounds for the hat guessing number of large topologically defined graph classes. The main ingredients for the proofs of Theorems 1.5.2 and 1.5.3 will be a vertex partition lemma from Bosek, Dudek, Farnik, Grytczuk, and Mazur [18], as well as a new theorem that bounds the hat guessing number of graphs that admit a vertex partition with a certain tree-like structure. The proof of this new theorem uses an argument based on a Turán-type edge density problem, and the combination of our edge-density argument and the lemma of Bosek et al. is what causes our upper bounds to be so large.

1.6 The graph coloring game

The *graph coloring game* is a game played on a finite graph G with perfect information by two players, Alice and Bob. In the graph coloring game, Alice and Bob take turns coloring vertices of G, with Alice moving first. On each player's turn, the player chooses an uncolored vertex $v \in V(G)$ and colors v using a color from a predetermined set $\{1, \dots, k\}$. Each player must color G properly on each turn; that is, a player may not color a vertex v with a color that appears in the neighborhood of v. Alice wins the game if each vertex of G is properly colored, and Bob wins the game if every color of $\{1, \dots, k\}$ appears in the neighborhood of some uncolored vertex v, as this means that v can never be properly colored. The *game chromatic number* of G, written $\chi_g(G)$, is the minimum integer k for which Alice has a winning strategy in the graph coloring game on G when playing with a color set $\{1, \dots, k\}$. When determining the game chromatic number of a graph G, we assume that Bob is Alice's adversary and wishes to make it as difficult as possible to complete a proper coloring on G.

The game chromatic number was introduced by Bodlaender [15] in 1990 and has received considerable attention since its invention. It is straightforward to show that for a graph G of chromatic number $\chi(G)$ and maximum degree $\Delta(G)$, the following inequality holds:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1.$$

The upper bound of $\Delta(G)+1$ is far from optimal in many cases, however. For instance, when G is a forest, $\chi_g(G) \leq 4$ [37], and when G is planar, $\chi_g(G) \leq 17$ [83]. Furthermore, $\chi_g(G)$ can be bounded above by other parameters of G. For instance, when G has treewidth at most w, $\chi_g(G) \leq 3w + 2$ [81], and when G has genus at most g, $\chi_g(G) \leq \lfloor \frac{1}{2}(3\sqrt{1 + 48g} + 23) \rfloor$ [81]. Furthermore, Dinski and Zhu [29] show that $\chi_g(G)$ is bounded above by a function of the *acyclic chromatic number* of G, written $\chi_a(G)$, which is the minimum number of colors needed to give G a proper coloring in which every bicolored subgraph of G is a forest. Dinski and Zhu give the following upper bound:

$$\chi_g(G) \leq \chi_a(G)(\chi_a(G) + 1).$$

Similar to the graph coloring game, the *graph marking game* is also a game played on a finite graph G with perfect information by two players, Alice and Bob. In the graph marking game, first considered by Faigle et al. [37], the players take turns, with Alice moving first, and on a player's turn, the player chooses an unmarked vertex $v \in V(G)$ and marks v with a black pen. The game ends when all vertices in G have been marked. After a play of the graph marking game, each vertex $v \in V(G)$ receives a *score* equal to the number of neighbors of v that were already marked at the time that v was marked. A play of the graph marking game on G is then given a score equal to the maximum score over all vertices of $V(G)$, plus one. Alice's goal in the graph marking game is to minimize the score of the play, and Bob's goal is to maximize the score of the play. The *game coloring number* of G, written $\text{col}_g(G)$, is the minimum integer t for which Alice has a strategy to limit the score of a play on G to t. It is straightforward to show that $\chi_g(G) \leq \text{col}_g(G)$ [80].

When attempting to find an upper bound for the game chromatic number of a graph G, it is often convenient to consider the graph marking game on G and find an upper bound for $\text{col}_g(G)$. The reason for this is that the game coloring number satisfies certain convenient properties that are not satisfied by the game chromatic number. For instance, when H is a subgraph of G, Wu and Zhu [78] show that $\text{col}_g(H) \leq \text{col}_g(G)$. On the other hand, Tuza and Zhu [75] show that the "cocktail party graph," obtained from the complete bipartite graph $K_{n,n}$ by deleting a perfect matching, has a game chromatic number of n, but the game chromatic number drops to 2 if a single isolated vertex is added to the graph. Therefore, many upper bounds for the game chromatic number of certain graph classes, such as the bounds for planar graphs and graphs of bounded treewidth given above, are obtained by studying the graph marking game.

In [82], Zhu asked whether the game chromatic number of the Cartesian product of two graphs is bounded whenever each graph's game coloring number is bounded:

Question 1.6.1. *Suppose both* $\text{col}_g(G_1)$ *and* $\text{col}_g(G_2)$ *are bounded by a constant. Is it true that* $\chi_g(G_1 \square G_2)$ *is bounded by a constant?*

In Chapter 5, we will show a relationship between the game chromatic number of a graph G and the properties of the bicolored subgraphs of G with respect to some fixed proper coloring. Our method will generalize the method of Dinski and Zhu [29] used to prove the inequality $\chi_g(G) \leq \chi_a(G)(\chi_a(G) + 1)$. As a corollary, we will obtain the following theorem, which answers Question 1.6.1 in the affirmative.

Theorem 1.6.2. *Let* G_1 *and* G_2 *be graphs, and let* $t = \max\{\text{col}_g(G_1), \text{col}_g(G_2)\}$. *Then,*

$$\chi_g(G_1 \square G_2) \leq t^2((t^2 - 1)t + 1) = t^5 - t^3 + t^2.$$

1.7 The Lovász Local Lemma

Since we consider graph coloring problems with local restrictions, one tool that we will frequently use is the Lovász Local Lemma. This lemma was first introduced by Erdős and Lovász [34] and can be used in the following general setting. Suppose that we perform a random experiment and wish to avoid certain bad events. (For example, we might randomly color the vertices of a graph and hope to avoid monochromatic edges.) The Lovász Local Lemma roughly states that if the probability of each bad event is not too large, and if each bad event is not dependent with too many other bad events, then there is a positive probability that our experiment will avoid all bad events. If our experiment is a random graph coloring, for example, then the Lovász Local Lemma gives us conditions under which there exists a coloring avoiding certain undesired properties. We will use the following form of the lemma, which appears in [66, Chapter 19].

Lemma 1.7.1. *Let $A_1, \dots, A_n$ be a set of bad events, and for each $i = 1, \dots, n$, let D_i denote the set of events A_j, $j \neq i$, with which A_i is dependent. If there exist real numbers $x_1, \dots, x_n \in [0, 1)$ such that for each $i = 1, \dots, n$,*

$$\Pr(A_i) \leq x_i \prod_{A_j \in D_i} (1 - x_j),$$

then with positive probability, no bad event A_i occurs.

By setting each $x_i = \frac{1}{D+1}$ in Lemma 1.7.1, the Lovász Local Lemma has the following simpler form.

Lemma 1.7.2. *Let $A_1, \dots, A_n$ be a set of bad events of probability at most p, and for each $i = 1, \dots, n$, let A_i be dependent with at most D other events A_j. If*

$$pe(D + 1) \leq 1,$$

then with positive probability, no bad event A_i occurs.

In almost all applications of the Lovász Local Lemma, a set $\mathcal{P}$ of mutually independent variables in a probability space Ω is given. (For example, in graph coloring, these mutually independent variables are typically the randomly assigned colors of vertices.) In this setting, each bad event A_i is determined by some unique minimal subset $S \subseteq \mathcal{P}$ of random variables, and two bad events A_i and A_j are dependent if and only if the subsets of $\mathcal{P}$ determining those bad events intersect [67].

The first proofs of the Lovász Local Lemma are nonconstructive; that is, given a set of bad events determined by $\mathcal{P}$, these proofs do not explicitly determine an assignment to the variables of $\mathcal{P}$ that avoids all bad events. However, Moser and Tardos [67] developed a randomized algorithm, often called the *entropy compression* algorithm, that finds an assignment to the variables of $\mathcal{P}$ that avoids all bad events a.a.s. in linear time whenever the conditions of the Lovász Local Lemma are satisfied. In the setting of graph coloring, this means that whenever the existence of a specific coloring of a graph G is proven using the Lovász Local Lemma, a randomized algorithm exists that finds such a coloring of G a.a.s. in $O(|V(G)|)$ time. In particular, all coloring results in this thesis proven with the Lovász Local Lemma have an associated efficient randomized algorithm.

1.8 Attributions

The content of Section 2 is part of ongoing work with Bojan Mohar and Ladislav Stacho.

The work in Chapter 3 appears in [23] and [20]. The manuscript [23] is joint work with Tomáš Masařík and has been submitted to Journal of Graph Theory. The manuscript [20] has been submitted to the Electronic Journal of Combinatorics. I am grateful to Jana Novotná and Ladislav Stacho for helpful discussions on the content of [23], and to Ross Kang for giving feedback on this manuscript. I am also grateful to an anonymous referee for providing useful feedback on [20].

The work in Chapter 4 has been published in Journal of Combinatorial Theory, Series B [22]. I am grateful to Bojan Mohar for reading an earlier draft of [22] and for suggesting Lemma 4.5.1 as a way to improve a previous version of Theorem 4.5.2. I am also grateful to an anonymous referee for giving valuable feedback that improved the presentation of the manuscript.

The work in Chapter 5 has been published in Journal of Graph Theory [21]. I am grateful to Bojan Mohar for his helpful advice regarding the organization and presentation of the results in [21], and for pointing out an error in an earlier version of that manuscript. I am also grateful to the referees for their helpful comments.

I am grateful to the examining committee for reading this thesis and for offering helpful feedback, which improved the overall quality.

Chapter 2

List colorings of bipartite graphs

2.1 Introduction

In this chapter, we prove that a bipartite graph G of sufficiently large maximum degree Δ satisfies $\mathrm{ch}(G) < 0.797\frac{\Delta}{\log \Delta}$ (Theorem 1.3.2). This result establishes the best known upper bound on the choosability of a bipartite graph in terms of its maximum degree and takes a modest step toward Alon and Krivelevich's conjectured upper bound of $O(\log \Delta)$ (Conjecture 1.3.1). Our method is based on an approach of Alon, Cambie, and Kang [9] related to the coupon collector problem.

The approach of Alon, Cambie, and Kang [9] can be summarized as follows. If G is a bipartite graph of maximum degree Δ with partite sets A and B, and if each vertex $v \in A$ receives a color uniformly at random from its list $L(v)$, then the probability that a given vertex $w \in B$ has no available color from its list $L(w)$ is small enough to apply the Lovász Local Lemma and find a proper L-coloring of G whenever each color list has at least $(1 + o(1))\frac{\Delta}{\log \Delta}$ colors. In fact, their method implies that if the vertices in A have color lists of size $\omega(1)$, then there exists a function $o(1)$ so that by giving the vertices in B lists of size $(1 + o(1))\frac{\Delta}{\log \Delta}$, G has a proper L-coloring.

The method that we will use to prove Theorem 1.3.2 is very similar to the approach of Alon, Cambie, and Kang, but we will improve their upper bound on list sizes by choosing a particular non-uniform distribution on each list $L(v)$ for $v \in A$. Our proof also uses the Lovász Local Lemma and hence yields an efficient randomized algorithm via Moser and Tardos's entropy compression method [67, 74]. Given the apparent difficulty of designing an efficient algorithm to list-color a Δ-regular triangle-free graph with $(1 - \varepsilon)\frac{\Delta}{\log \Delta}$ available colors at each vertex for any $\varepsilon > 0$, as discussed in the introduction, Theorem 1.3.2 provides evidence that the list-coloring problem is fundamentally easier in bipartite graphs than in triangle-free graphs. Hence, our result takes a step toward solving Conjecture 1.3.1.

2.2 The choice number of bipartite graphs

In our proofs, we will omit floors and ceilings, as they will have no effect on our arguments. Before proving our main result, we will need a lemma about the coupon collector problem, which takes place in the following setting. Let $L, L_1, \dots, L_\Delta$, be lists, each of exactly k colors. We may imagine that the colors in L represent differently colored coupons that a coupon collector wishes to gather and that the collector has a set of Δ unopened boxes, each with one coupon inside. We further assume that the coupon contained in the ith box must be of a color from L_i, for each i ($1 \le i \le \Delta$). We are interested in the probability that the coupon collector will be able to gather coupons of all k colors in L.

In order to estimate the probability that the coupon collector successfully obtains coupons of all colors in L, we need to define some parameters. We let $0 < p < 1$ be a positive number, possibly dependent on Δ. For each value i ($1 \le i \le \Delta$), we define a probability distribution $P_i : L_i \to [0, p)$. Since P_i is a probability distribution, we require that $\sum_{c \in L_i} P_i(c) = 1$, and for a color $c \notin L_i$, we write $P_i(c) = 0$. We also define independent random variables $\phi_1, \dots, \phi_\Delta$ so that for each i and $c \in L_i$, $\phi_i = c$ with probability $P_i(c)$. For each $c \in L$, we write $\rho(c) = \sum_{i=1}^{\Delta} P_i(c)$. Then, we have the following lemma, which gives us an upper bound on the probability that every color in L appears at some random variable ϕ_i, i.e. the probability that for each of the k colors $c \in L$, the coupon collector obtains a coupon of color c. The ideas in this lemma are similar to those in the coupon collection argument of Alon, Cambie, and Kang [9].

Lemma 2.2.1. *Let $0 < \varepsilon \le 1$ and $0 \le a \le 1$ be fixed, and let $k = \left\lceil \frac{a\Delta}{(1-p)(\log \Delta - 4 \log\log \Delta)} \right\rceil$. Suppose that there exists a set $L^* \subseteq L$ of size at least εk such that the average value $\rho(c)$ for $c \in L^*$ satisfies*

$$\frac{1}{|L^*|} \sum_{c \in L^*} \rho(c) \le a\Delta/k.$$

Then, when Δ is sufficiently large, $\Pr(L \subseteq \{\phi_1, \dots, \phi_\Delta\}) < \exp(-\log^2 \Delta)$.

Proof. First, we will show that

$$\Pr(L \subseteq \{\phi_1, \dots, \phi_\Delta\}) \le \exp\left(- \sum_{c \in L} \exp\left(-\frac{1}{1-p} \rho(c) \right) \right). \tag{2.1}$$

Consider a color $c \in L$, and let B_c be the event that $\phi_i = c$ for some value i ($1 \le i \le \Delta$), i.e. the event that the coupon collector obtains a coupon of color c. Since the variables ϕ_i are independent, $\Pr(B_c) = 1 - \prod_{i=1}^{\Delta}(1 - P_i(c))$. Applying the inequality $1 - x \ge e^{\frac{-x}{1-x}} \ge e^{\frac{-x}{1-p}}$ for $x < p$, we see that

$$\Pr(B_c) \le 1 - \exp\left(-\frac{1}{1-p} \sum_{i=1}^{\Delta} P_i(c) \right) = 1 - \exp\left(-\frac{1}{1-p} \rho(c) \right).$$

Furthermore, it is well known that the individual coupon collection events $\{B_c : c \in L\}$ are negatively correlated (see e.g. [9, Section 3]), so the probability of the event $\bigcap_{c\in L} B_c$, or equivalently the event $L \subseteq \{\phi_1, \dots, \phi_\Delta\}$, is at most

$$\prod_{c\in L}\left(1-\exp\left(-\frac{1}{1-p}\rho(c)\right)\right) \le \exp\left(-\sum_{c\in L}\exp\left(-\frac{1}{1-p}\rho(c)\right)\right),$$

proving (2.1).

By possibly taking a subset of L^*, we assume without loss of generality that $\varepsilon < 1$ and that $|L^*| = \varepsilon k$. We write $L_* = L \setminus L^*$. By (2.1),

$$\Pr\left(L \subseteq \{\phi_1, \dots, \phi_\Delta\}\right) \le \exp\left(-\sum_{c\in L^*}\exp\left(-\frac{1}{1-p}\rho(c)\right) - \sum_{c\in L_*}\exp\left(-\frac{1}{1-p}\rho(c)\right)\right). \tag{2.2}$$

Since the function $f(x) = e^{-x}$ is convex, and since $\frac{1}{\varepsilon k}\sum_{c\in L^*}\rho(c) \le a\Delta/k$, it follows that

$$\sum_{c\in L^*}\exp\left(-\frac{1}{1-p}\rho(c)\right) \ge \varepsilon k \exp\left(-\frac{a\Delta}{(1-p)k}\right).$$

Furthermore, as $\sum_{c\in L_*}\rho(c) \le \sum_{c\in L}\rho(c) < \Delta$,

$$\sum_{c\in L_*}\exp\left(-\frac{1}{1-p}\rho(c))\right) > (1-\varepsilon)k\exp\left(-\frac{\Delta}{(1-p)(1-\varepsilon)k}\right).$$

Therefore, the argument of the outer exponential in (2.2) is less than

$$-\varepsilon k\exp\left(-\frac{a\Delta/k}{(1-p)}\right) - (1-\varepsilon)k\exp\left(-\frac{\Delta/k}{(1-p)(1-\varepsilon)}\right) = -(\varepsilon+o(1))k\exp\left(-\frac{a\Delta/k}{(1-p)}\right).$$

Now, if we substitute our value of k, then the argument of the outer exponential function in (2.2) becomes $-(\varepsilon+o(1))\cdot\frac{\Delta}{(1-p)\log\Delta}\cdot\exp(4\log\log\Delta - \log\Delta) < -\log^2\Delta$, so the lemma holds. $\square$

2.2.1 A warmup: $\mathrm{ch}(G) \le \left(\frac{4}{5}+o(1)\right)\frac{\Delta}{\log\Delta}$

We will first prove the following weaker result. The main ideas in the proof of the following theorem are very similar to those in our proof of Theorem 1.3.2, which achieves the smaller coefficient of 0.797. However, the ideas of the following proof avoid some technical details from the proof of Theorem 1.3.2, so we present this result first as a warmup.

Theorem 2.2.2. *If G is a bipartite graph of maximum degree Δ, then* $\mathrm{ch}(G) \le (\frac{4}{5}+o(1))\frac{\Delta}{\log\Delta}$.

Proof. We fix an arbitrarily small value $\gamma > 0$ and assume that the maximum degree Δ of G is sufficiently large with respect to γ. Without loss of generality, we may assume that G is Δ-regular.

We let each vertex $v \in V(G)$ have a list $L(v)$ of $k = \left\lceil \frac{(4/5+\gamma)\Delta}{(1-1/\sqrt{\Delta})(\log \Delta - 4 \log\log \Delta)} \right\rceil$ colors, represented as integers in increasing order. We will show that G has a proper list coloring.

We partition $V(G)$ into two partite sets A and B. Our strategy will be to create a probability distribution on each list $L(v)$ for $v \in A$, and we will use these distributions to color all vertices $v \in A$. Then, we will use Lemma 2.2.1 and the Lovász Local Lemma to show that with positive probability, each vertex $w \in B$ still has an available color even after all vertices in A have been colored.

For each vertex $u \in V(G)$, we write $L(u) = (c_1, \dots, c_k)$ as an increasing integer sequence, and for each color $c \in L(u)$, we write $I(u, c) = i$ if $c = c_i$—that is, if c is in the ith position in $L(v)$. We say that $I(u, c)$ is the *index* of c in u. For each vertex $w \in B$, we define the *weight* of w as

$$Z(w) = \sum_{v \in N(w)} |L(v) \cap L(w)|.$$

Clearly, for each vertex $w \in B$, $Z(w) \leq \Delta k$. For each vertex $v \in A$ and $c \in L(v)$, we write

$$P_v(c) = \frac{8/5}{k(1 - \frac{3}{5k})} \left(1 - \frac{3}{4} \cdot \frac{I(v,c)}{k}\right).$$

For convenience, we will define $I(v, c) = \frac{4}{3}k$ for $c \notin L(v)$ so that $P_v(c) = 0$ for these colors c. Observe that $\sum_{c \in L(v)} P_v(c) = 1$. For each $w \in B$ and $c \in L(w)$, we write $\rho_w(c) = \sum_{v \in N(w)} P_v(c)$. For each $v \in A$ and $c \in L(v)$, we will use c to color v with probability $P_v(c)$. Then, we will use the Lovász Local Lemma to show that with positive probability, our random coloring of A can be extended to a proper list coloring of G. Observe that each color in $L(v)$ is used with a probability of (much) less than $1/\sqrt{\Delta}$.

Now, consider a vertex $w \in B$, and write $z = \frac{Z(w)}{\Delta k}$. For a color $c \in L(w)$ and a constant $\varepsilon > 0$, if $I(w, c) \geq (1 - \varepsilon)k$, then since our color lists are sorted in increasing order,

$$\sum_{v \in N(w)} I(v, c) \geq (z - \varepsilon)\Delta k.$$

Therefore,

$$\rho_w(c) = \sum_{v \in N(w)} P_v(c) = \frac{8/5 + o(1)}{k} \sum_{v \in N(w)} \left(1 - \frac{3}{4} \cdot \frac{I(v,c)}{k}\right) \leq \left(\frac{8}{5} + o(1)\right)\left(1 - \frac{3}{4}(z - \varepsilon)\right)\frac{\Delta}{k}.$$

Hence, for the last εk colors $c \in L(w)$ (i.e. those of largest index), the average value of $\rho_w(c)$ is at most $\left(\frac{8}{5} + o(1)\right)\left(1 - \frac{3}{4}y + \frac{3}{4}\varepsilon\right)\frac{\Delta}{k}$. On the other hand,

$$\sum_{c \in L(w)} \sum_{v \in N(w)} I(v, c) > \Delta \cdot \frac{1}{2}(zk)^2 + \frac{4}{3}(1 - z)k\Delta,$$

as this sum is minimized when for each vertex $v \in N(w)$, $L(v)$ contains exactly zk colors of $L(w)$. Therefore, the average value $\rho_w(c)$ over all colors $c \in L(w)$ satisfies

$$\begin{aligned}\frac{1}{k}\sum_{c\in L(w)}\rho_w(c) = \sum_{c\in L(w)}\sum_{v\in N(w)}P_v(c) &= \frac{8/5+o(1)}{k^2}\sum_{c\in L(w)}\sum_{v\in N(w)}\left(1-\frac{3}{4}\cdot\frac{I(v,c)}{k}\right)\\ &< \frac{8/5+o(1)}{k^2}\left(Z(w)-\frac{3}{8}z^2k\Delta\right)\\ &= \left(\frac{8}{5}+o(1)\right)z\left(1-\frac{3}{8}z\right)\frac{\Delta}{k}.\end{aligned}$$

Hence, we can always find a dense subset $L^*(w) \subseteq L(w)$ of size at least εk for which the average value $\rho(c)$ for $c \in L^*(w)$ is at most $\min\left\{\left(1-\frac{3}{4}z+\frac{3}{4}\varepsilon\right), z\left(1-\frac{3}{8}z\right)\right\}\cdot\left(\frac{8}{5}+o(1)\right)\frac{\Delta}{k} < \left(\frac{4}{5}+\gamma\right)\frac{\Delta}{k}$, where the inequality holds whenever ε is sufficiently small and Δ is sufficiently large with respect to γ.

Now, for each vertex $w \in B$, we define a bad event B_w as the event that after A is randomly colored, no color in $L(w)$ is available—that is, that every color in $L(w)$ appears in $N(w)$. By applying Lemma 2.2.1 with our value ε, as well as with $a = \frac{4}{5}+\gamma$, $L = L(w)$, $L^* = L^*(w)$, and $\{L_1, \dots, L_k\} = \{L(v) : v \in N(w)\}$, we find that $\Pr(B_w) < \exp(-\log^2\Delta)$. It is easy to see that if no bad event occurs, then every vertex of G can be successfully colored. Since each bad event occurs with probability less than $\exp(-\log^2\Delta)$ and is dependent with fewer than Δ^2 other bad events, it follows from the Lovász Local Lemma (Lemma 1.7.2) that with positive probability, no bad event occurs provided that Δ is large enough so that $e\Delta^2\exp(-\log^2\Delta) \le 1$. Thus, the proof is complete. □

2.2.2 Breaking the $\frac{4}{5}$ coefficient

In this subsection, we show that the $\frac{4}{5}+o(1)$ coefficient from Theorem 2.2.2 can be reduced to 0.797 using a similar coupon collection argument to that in Theorem 2.2.2. While this improvement is minimal, the fact that the $\frac{4}{5}+o(1)$ coefficient can be broken with a similar argument suggests that perhaps a more involved application of similar ideas can reduce the coefficient even more.

Before we prove that this lower coefficient can be achieved, we summarize the method used in Theorem 2.2.2 and observe which parts of the method give room for improvement. In our proof of Theorem 2.2.2, we consider a vertex $w \in B$, and we hope to show that after randomly coloring all vertices in A, the probability that w has no available color is small. In order to show this, we aim to show that for some dense set of colors $c \in L(w)$, the values $\rho_w(c)$ are small. We write $z\Delta k$ for the weight of w, and we roughly describe two cases.

In the first case, if z is large, then the colors $c \in L(w)$ must appear at the lists $L(v)$ for neighbors $v \in N(w)$ with high frequency. Consequently, the colors c of large index must also have fairly large indices $I(v, c)$ for many neighbors $v \in N(w)$. Since the probability of c being used to color v becomes small when $I(v, c)$ is large, this means that colors $c \in L(w)$ of large index must have small values $\rho_w(c)$. Specifically, we see in the proof of Theorem 2.2.2 that these colors c of large index in w approximately satisfy $\rho_w(c) \leq \frac{8}{5}(1 - \frac{3}{4}z)\frac{\Delta}{k}$.

In the second case, if z is small, then for each neighbor $v \in N(w)$, $L(v)$ on average does not contain many colors from $L(w)$. Therefore, the average value $\rho_w(c)$ for all colors $c \in L(w)$ is small. Specifically, we see in the proof that the average value $\rho_w(c)$ is at most roughly $\frac{8}{5}z(1 - \frac{3}{8}z)\frac{\Delta}{k}$.

In both cases, we can find a dense set of colors $c \in L(w)$ for which the average value $\rho(c)$ is at most $(\frac{4}{5} + o(1))\frac{\Delta}{k}$, with the upper bound being achieved when z is close to $\frac{2}{3}$. Now, let us consider the extremal case when this value $(\frac{4}{5} + o(1))\frac{\Delta}{k}$ is achieved in more detail. When we compute the upper bound $\frac{8}{5}z(1 - \frac{3}{8}z)\frac{\Delta}{k}$ for the average value $\rho(c)$ over all colors $c \in L(w)$, we assume that the values $|L(v) \cap L(w)|$ are the same for each neighbor $v \in N(w)$ and that the indices $I(v, c)$ for $c \in L(w)$ are as low as possible. Therefore, in our extremal case, we assume that for each neighbor $v \in N(w)$, $L(v)$ contains roughly $\frac{2}{3}k$ colors from $L(w)$ occupying the first $\frac{2}{3}k$ indices at v. However, if this is the case, then we should be able to slightly increase the probabilities $P_v(c')$ for the colors $c' \in L(v)$ with indices close to k without increasing the probabilities $P_v(c)$ of colors $c \in L(v) \cap L(w)$, as the colors $c' \in L(v)$ with large index should not belong to $L(w)$. This will allow us to decrease the probabilities $P_v(c)$ of the colors $c \in L(v)$ with smaller index, which will reduce $P_v(c)$ for colors $c \in L(v) \cap L(w)$ and allow us to reduce our coefficient below $\frac{4}{5}$. On the other hand, if increasing the probabilities $P_v(c')$ for colors $c' \in L(v)$ of large index causes the probabilities $P_v(c)$ of many colors in $c \in L(v) \cap L(w)$ to increase, then this implies that the colors in $L(v) \cap L(w)$ for neighbors $v \in N(w)$ are not arranged like in the extremal case, and the method of Theorem 2.2.2 should still give a coefficient lower than $\frac{4}{5}$.

Now, we are ready to prove our improved coefficient.

Theorem 2.2.3. *If G is a bipartite graph of sufficiently large maximum degree Δ, then* $\mathrm{ch}(G) < 0.797\frac{\Delta}{\log \Delta}$.

Proof. We assume that the maximum degree Δ of G is sufficiently large. Without loss of generality, we may assume that G is Δ-regular. We let each vertex $v \in V(G)$ have a list $L(v)$ of $k = \left\lceil \frac{0.7969\Delta}{(1-1/\sqrt{\Delta})(\log \Delta - 4 \log\log \Delta)} \right\rceil$ colors, represented as integers in increasing order. We will show that G has a proper list coloring.

We partition $V(G)$ into two partite sets A and B. Again, we will randomly color the vertices of A and then use the Lovász Local Lemma to show that B can be colored with positive probability. We define the weight $Z(w)$ of each vertex $w \in B$ as before, and we also define $I(u, c)$ for each $u \in V(G)$ and each $c \in L(u)$ as before. For vertices $v \in A$ and $w \in B$, we write $|L(v) \cap L(w)| = \ell_{v,w}$.

We define a function $f : [1, \infty) \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 - \frac{3}{4k}x & \text{if } x \leq \frac{9}{10}k \text{ or } x > k \\ \frac{13}{40} & \text{if } \frac{9}{10}k < x \leq k. \end{cases}$$

We write C for the average value of $f(i)$ for $i \in \{1, \dots, k\}$ and observe that $C = \frac{503}{800} + o(1)$. For each $c \in L(v)$, we write

$$P_v(c) = \frac{1}{Ck} f\left(I(v,c)\right).$$

Again, for convenience, we will define $I(v,c) = \frac{4}{3}k$ for $c \notin L(v)$ so that $P_v(c) = 0$ for $c \notin L(v)$. Observe that $\sum_{c \in L(v)} P_v(c) = 1$. For each $w \in B$ and $c \in L(w)$, we again write $\rho_w(c) = \sum_{v \in N(w)} P_v(c)$. For each $v \in A$ and $c \in L(v)$, we will use c to color v with probability $P_v(c)$.

Now, consider a vertex $w \in B$, and write $z = \frac{Z(w)}{\Delta k}$. Define $0 \leq y \leq 1$ so that exactly $y\Delta$ neighbors $v \in N(w)$ satisfy $\ell_{v,w} > \frac{9}{10}k$. Define α so that $\sum_{v \in N'(w)} \left(\ell_{v,w} - \frac{9}{10}k\right) = \alpha y k \Delta$. Observe $0 \leq \alpha \leq \frac{1}{10}$. We write $N'(w)$ for the set of $y\Delta$ neighbors $v \in N(w)$ for which $\ell_{v,w} > \frac{9}{10}k$, and we write $N''(w) = N(w) \setminus N'(w)$ for the remaining set of $(1-y)\Delta$ neighbors of w.

Now, consider a color $c \in L(w)$ of index at least $(1-\varepsilon)k$ with respect to $L(w)$. We upper bound $\rho_w(c)$ as follows, using the fact that f is decreasing and $\frac{3}{4}$-Lipschitz.

$$\begin{aligned}
\rho_w(c) &= \frac{1}{Ck} \sum_{v \in N(w)} f(I(v,c)) \leq \frac{1}{Ck} \sum_{v \in N(w)} f(\ell_{v,w} - \varepsilon k) \\
&\leq \frac{3\varepsilon}{4Ck}\Delta + \frac{1}{Ck} \sum_{v \in N(w)} f(\ell_{v,w}) \\
&= \frac{3\varepsilon}{4Ck}\Delta + \frac{1}{Ck} \left(\sum_{v \in N''(w)} f(\ell_{v,w}) + \sum_{v \in N'(w)} f(\ell_{v,w}) \right) \\
&= \frac{3\varepsilon}{4Ck}\Delta + \frac{1}{Ck} \left(\sum_{v \in N''(w)} \left(1 - \frac{3}{4k}\ell_{v,w}\right) + \sum_{v \in N'(w)} \left(1 - \frac{3}{4k}\ell_{v,w} + \frac{3}{4k}\ell_{v,w} - \frac{27}{40}\right) \right) \\
&= \frac{3\varepsilon}{4Ck}\Delta + \frac{1}{Ck} \left(\sum_{v \in N(w)} \left(1 - \frac{3}{4k}\ell_{v,w}\right) + \sum_{v \in N'(w)} \left(\frac{3}{4k}\ell_{v,w} - \frac{27}{40}\right) \right) \\
&= \frac{3\varepsilon}{4Ck}\Delta + \frac{1}{Ck} \left(\Delta\left(1 - \frac{3}{4}z\right) + \frac{3}{4k} \sum_{v \in N'(w)} \left(\ell_{v,w} - \frac{9}{10}k\right) \right) \\
&= \frac{\Delta}{Ck} \left(1 - \frac{3}{4}z + \frac{3}{4}\alpha y + \frac{3}{4}\varepsilon\right).
\end{aligned}$$

Hence, for any constant $\varepsilon > 0$, the average value of $\rho_w(c)$ for the last εk colors $c \in L(w)$ is at most $\frac{\Delta}{Ck}\left(1 + \frac{3}{4}(-z + \alpha y + \varepsilon)\right)$.

On the other hand, the average value $\rho_w(c)$ over all colors $c \in L(w)$ satisfies

$$\begin{aligned}
\frac{1}{k}\sum_{c\in L(w)}\rho_w(c) &= \frac{1}{Ck^2}\sum_{v\in N(w)}\sum_{c\in L(w)} f(I(v,c)) \le \frac{1}{Ck^2}\sum_{v\in N(w)}\sum_{i=1}^{\ell_{v,w}} f(i) \\
&= \frac{1}{Ck^2}\left(\sum_{v\in N''(w)}\sum_{i=1}^{\ell_{v,w}}(1-\frac{3}{4k}i) + \sum_{v\in N'(w)}\left(\sum_{i=1}^{\frac{9}{10}k}(1-\frac{3}{4k}i) + \sum_{i=\frac{9}{10}k+1}^{\ell_{v,w}}\frac{13}{40}\right)\right) \\
&< \frac{1}{Ck^2}\sum_{v\in N''(w)}\sum_{i=1}^{\ell_{v,w}}(1-\frac{3}{4k}i) + \frac{1}{Ck^2}\sum_{v\in N'(w)}\left(\frac{9}{10}k - \frac{3}{4k}\cdot\frac{1}{2}(\frac{9}{10}k)^2 + \frac{13}{40}(\ell_{v,w}-\frac{9}{10})\right) \\
&= \frac{1}{Ck^2}\sum_{v\in N''(w)}\sum_{i=1}^{\ell_{v,w}}(1-\frac{3}{4k}i) + \frac{y\Delta}{Ck}\left(\frac{477}{800}+\frac{13}{40}\alpha\right)
\end{aligned}$$

Since $\sum_{v\in N''(w)}\ell_{v,w} = k\Delta(z-(\frac{9}{10}+\alpha)y)$, the average value $\ell_{v,w}$ for $v\in N'(w)$ is $\overline{\ell} := \frac{k(z-(\frac{9}{10}+\alpha)y)}{1-y}$. Furthermore, the sum above will be maximized if all values $\ell_{v,w}$ for $v \in N'(w)$ are equal to $\overline{\ell}$. Hence,

$$\begin{aligned}
\frac{1}{k}\sum_{c\in L(w)}\rho_w(c) &< \frac{(1-y)\Delta}{Ck^2}\left(\overline{\ell} - \frac{3}{4k}\cdot\frac{1}{2}\overline{\ell}^2\right) + \frac{y\Delta}{Ck}\left(\frac{477}{800}+\frac{13}{40}\alpha\right) \\
&= \frac{\Delta}{Ck}\left[\left(z-\left(\frac{9}{10}+\alpha\right)y\right)\left(1-\frac{3}{8}\cdot\frac{z-\left(\frac{9}{10}+\alpha\right)y}{1-y}\right) + y\left(\frac{477}{800}+\frac{13}{40}\alpha\right)\right].
\end{aligned}$$

Hence, writing $g(\alpha,y,z) = z - \left(\frac{9}{10}+\alpha\right)y$, we can always find a dense subset $L^*(w)\subseteq L(w)$ of size at least εk for which the average value $\rho(c)$ for $c\in L^*(w)$ is at most

$$\frac{\Delta}{Ck}\min\left\{1+\frac{3}{4}(-z+\alpha y+\varepsilon), g(\alpha,y,z)\left(1-\frac{3}{8}\cdot\frac{g(\alpha,y,z)}{1-y}\right)+y\left(\frac{477}{800}+\frac{13}{40}\alpha\right)\right\}.$$

We would like to show that this quantity is less than $\frac{0.7969\Delta}{k}$ when ε is sufficiently small and Δ is sufficiently large. To establish this upper bound, we first observe that if $z-\alpha y > 0.66535$, then $\frac{\Delta}{Ck}(1+\frac{3}{4}(-z+\alpha y+\varepsilon)) < (0.7968+\frac{3}{4}\varepsilon+o(1))\frac{\Delta}{k}$, which is smaller than $\frac{0.7969\Delta}{k}$ when ε is sufficiently small and Δ is sufficiently large. Hence, we may assume that $z-\alpha y \le 0.66535$. Since $y\le 1$ and $\alpha \le 0.1$, this implies in particular that $z < 0.8$. Furthermore, since $z \ge 0.9y$, we thus may assume that $y < 0.9$. We would like to show that under these constraints,

$$\frac{\Delta}{Ck}\left(g(\alpha,y,z)\left(1-\frac{3}{8}\cdot\frac{g(\alpha,y,z)}{1-y}\right)+y\left(\frac{477}{800}+\frac{13}{40}\alpha\right)\right) < \frac{0.7969\Delta}{k},$$

which will prove our upper bound. To this end, we execute the following commands in Maple:

```
f := (a, y, z) -> 800/503*(z - (0.9 + a)*y)*
(1 + (-1)*0.375*(z - (0.9 + a)*y)/(1 - y)) + 800/503*y*(477/800 + 13/40*a)
```

```
with(Optimization)

Maximize(f(a, y, z), {0 <= a, a <= 0.1, 0 <= y, y <= 0.9, 0 <= z, z <= 1,
-a*y + z <= 0.66535})
```

This gives us the following output:

```
[0.796309237086130106, [a = 0.100000000000000, y = 0.202933582180192,
z = 0.685643358218019]]
```

As a result, we find that under our constraints on α, y, and z, our expression is less than $(0.7964 + o(1))\frac{\Delta}{k}$, which is certainly less than our upper bound when Δ is sufficiently large. Hence, we can always find a dense subset $L^*(w) \subseteq L(w)$ of colors for which the average value $\rho(c)$ for $c \in L^*(w)$ is less than $\frac{0.7969\Delta}{k}$.

As before, for each vertex $w \in B$, we define a bad event B_w to be the event that no color of $L(w)$ is available after A is colored. By applying Lemma 2.2.1 with our value ε, as well as with $a = 0.7969$, $L = L(w)$, $L^* = L^*(w)$, and $\{L_1, \dots, L_k\} = \{L(v) : v \in N(w)\}$, we find that $\Pr(B_w) < \exp(-\log^2 \Delta)$. As before, we apply the Lovász Local Lemma (Lemma 1.7.2) when Δ is sufficiently large to find that with positive probability, no bad event occurs. Hence, we find our proper coloring of G, and the proof is complete. $\square$

Chapter 3

Single-conflict colorings

3.1 Introduction

In this section, we allow a graph G to have *parallel edges*, which are edges that appear in the set $E(G)$ more than once. In other words, we allow the edge set $E(G)$ of a graph to be a multiset. Recall that the degree of a vertex v is defined as the number of edges incident to v. Thus, in a graph with parallel edges, a vertex v may have fewer than $\deg v$ distinct neighbors.

Recall that a *single-conflict coloring* is defined as follows. Let G be a graph, and let $\{1, \dots, k\}$ be a set of colors. Suppose that f is a function that maps each edge (u, v) of G to a forbidden color pair $f(u, v) = (c_1, c_2)$, with $c_1, c_2 \in \{1, \dots, k\}$. Then, we say that a (not necessarily proper) coloring $\phi : V(G) \to \{1, \dots, k\}$ is a single-conflict coloring with respect to f and $\{1, \dots, k\}$ if $f(u, v) \neq (\phi(u), \phi(v))$ for each edge (u, v) of G. We call the image of an edge (u, v) under f a *conflict*, and we call f a *conflict function.* If k is the minimum integer for which a graph G always has a single-conflict coloring for the color set $\{1, \dots, k\}$ and any conflict function f, then we say that k is the *single-conflict chromatic number* of G, and we write $\chi_{\nleftrightarrow}(G) = k$.

In the first section of this chapter, we consider the single-conflict coloring problem in its most general setting, in which the conflict at a given may be any ordered pair from the color set $\{1, \dots, k\}$. In the second section of this chapter, we will consider the problems of *adapted colorings* and *co-operative colorings*, in which we seek a single-conflict coloring for a graph in which each conflict is monochromatic.

3.2 General conflicts

In this section, we will prove the following theorem, which implies Theorem 1.4.4 and also implies Corollary 1.4.9 as a corollary.

Theorem 3.2.1. *If G is a d-degenerate graph with maximum degree Δ and edge-multiplicity at most μ, then*

$$\chi_{\nleftrightarrow}(G) \leq \left\lceil \sqrt{d} \cdot 2^{\mu/2+2} \sqrt{\mu} \sqrt{1 + \log((d+1)\Delta)} \right\rceil.$$

One key tool that we will use to prove this theorem is an application of the Lovász Local Lemma in which each vertex $v \in V$ receives a random *inventory* S_v of colors, and then a color c is deleted from S_v if c also belongs to the inventory S_w of some in-neighbor w of v with respect to a given edge orientation. This technique of applying the Lovász Local Lemma to random color inventories is also used by Bernshteyn, Kostochka, and Zhu [14] for fractional DP-colorings and by Aharoni, Berger, Chudnovsky, Havet, and Jiang [2] for cooperative colorings of forests.

We will fix some preliminaries. Suppose G is a d-degenerate graph on n vertices with a linear vertex-ordering $v_1, \dots, v_n$ in which each vertex v_j has at most d neighbors v_i satisfying $i < j$. Then, there exists an orientation of $E(G)$ of maximum out-degree d obtained by giving each edge $v_i v_j$ with $i < j$ an orientation (v_j, v_i). Therefore, the class of d-degenerate graphs may be considered as a subclass of directed graphs of maximum out-degree d, and we will often consider this more general class rather than the class of d-degenerate graphs. For a vertex $v \in V(G)$, we write $E^+(v)$ for the set of arcs outgoing from v, and we write $E^-(v)$ for the set of arcs incoming to v.

3.2.1 Uniquely restrictive conflicts

In this subsection, we will consider the single-conflict coloring problem with *uniquely restrictive* conflict functions, defined as follows. Consider a color set $C = \{1, \dots, k\}$ and a directed graph G with a conflict function $f : E(G) \to C^2$. First, given a vertex $v \in V(G)$ and an arc $e \in E(G)$ containing v, we say that the (v, e) *conflict color* is the color appearing in the entry of $f(e)$ corresponding to the position of v in e. We write $\mathrm{cc}(v, e)$ for the (v, e) conflict color. Then, we have the following definition.

Definition 3.2.2. Let $w \in V(G)$. Suppose that for each pair of parallel arcs $e_1, e_2 \in E^-(w)$ such that $\mathrm{cc}(w, e_1) = \mathrm{cc}(w, e_2)$, it holds that $\mathrm{cc}(v, e_1) = \mathrm{cc}(v, e_2)$, where v is the second endpoint of e_1 and e_2. Then, we say that f is *uniquely restrictive* at w. Furthermore, if f is uniquely restrictive at each vertex $w \in V(G)$, then we simply say that f is *uniquely restrictive*.

An informal way of describing unique restrictiveness would be to say that if we color a vertex $w \in V(G)$ with some color, say `red`, then we only want this choice of `red` at w to contribute to the exclusion of at most one color possibility at each in-neighbor of w. We note that unique restrictiveness is a rather natural idea, as the conflict functions that represent adapted coloring and proper coloring problems are uniquely restrictive; indeed, in both of these settings, choosing the color `red` at a vertex v can only contribute to the exclusion of the color `red` at neighbors of v. Furthermore, DP-coloring problems always give uniquely restrictive conflict functions when represented as single-conflict coloring problems, since the conflicts between any two vertices form a matching in $C \times C$.

With this definition in place, we have the following theorem, which gives an upper bound on the number of colors needed for a single-conflict coloring of a d-degenerate graph whose conflict function is uniquely restrictive. Since any conflict function on a simple graph is uniquely restrictive, this following theorem implies Theorem 1.4.4. Our main tool for this theorem will be the application of the Lovász Local Lemma used by Aharoni, Berger, Chudnovsky, Havet, and Jiang [2], in which each vertex receives a random inventory of colors.

Theorem 3.2.3. *Let G be a directed graph of maximum degree Δ with a maximum out-degree of at most d. Let C be a set of k colors, and let each arc $e \in E(G)$ have an associated conflict $f(e) \in C^2$. If f is uniquely restrictive, and if*

$$k \geq 2\sqrt{d\,[1 + \log((d+1)\Delta)]},$$

then G has a single-conflict coloring with respect to f and C.

Proof. First, we note that since every subgraph of G has an average degree of at most $2d$, G is $(2d)$-degenerate and hence has a single-conflict coloring whenever $k \geq 2d + 1$. Therefore, we may assume in our proof that $k \leq 2d$.

First, for each vertex $v \in V(G)$, we define a color inventory S_v, and for each color $c \in C$, we add c to S_v independently with probability $p = \frac{k}{2d} \leq 1$. Next, we let S'_v be a copy of S_v. (We will need these copies for technical reasons related to the Lovász Local Lemma.) Then, for each vertex $v \in V(G)$, we consider each outgoing arc e of v, and we write $e = (v, w)$. If, for some color $c \in S_v$, we have

$$f(e) \in \{(c, c') : c' \in S_w\},$$

then we delete c from S'_v. In other words, if the color c at v contributes to the forbidden pair $f(v, w) = (c, c')$ of an outgoing arc $(v, w) \in E^+(v)$, and if $c' \in S_w$, then we delete c from S'_v. Then, for each vertex $v \in V(G)$, we let B_v denote the bad event that after this process, S'_v is empty. We observe that if no bad event occurs, then we may arbitrarily color each vertex v with a color from S'_v to obtain a single-conflict coloring of G. Indeed, if some arc (v, w) is colored with a forbidden pair (c, c') where $c \in S'_v$ and $c' \in S'_v$, then it must follow that c was actually deleted from S'_v, a contradiction.

Now, given a vertex $v \in V(G)$, we calculate the probability that the bad event B_v occurs. For a given color $c \in C$, we write b_c for the number of arcs $e \in E^+(v)$ for which $c = \mathrm{cc}(v, e)$. If c does not belong to S'_v, then either c was never added to S_v, or c was added to S_v and then deleted from S'_v. The probability that c was never added to S_v is equal to $1 - p$, and the probability that c was added to S_v and then deleted from S'_v is at most $b_c p^2$. Therefore, the total probability that $c \notin S'_v$ is at most $1 - p + b_c p^2$. Furthermore, since f is uniquely restrictive, the probabilities of any two given colors being absent from S'_v are independent. Therefore, the probability of the bad event B_v is at most

$$\prod_{c \in C} \left(1 - \left(p - b_c p^2\right)\right) < \exp\left(-\sum_{c \in C}\left(p - b_c p^2\right)\right) = \exp\left(-pk + p^2 \sum_{c \in C} b_c\right) = \exp\left(-pk + p^2 d\right).$$

Substituting $p = \frac{k}{2d}$, we see that

$$\Pr(B_v) < \exp\left(-\frac{k^2}{4d}\right).$$

Furthermore, as the bad event B_v involves $d+1$ vertices (namely v and at most d out-neighbors of v), each of maximum degree Δ, B_v is dependent with fewer than $(d+1)\Delta$ other bad events. Note that since we use unmodified inventories S_w to determine whether the copy S'_v is empty, we prevent the dependencies of B_v from spreading past the out-neighbors of v. Therefore, using the Lovász Local Lemma (Lemma 1.7.2), we see that G receives a single-conflict coloring with positive probability as long as the following inequality holds:

$$e(d+1)\Delta \exp\left(-\frac{k^2}{4d}\right) \leq 1.$$

This inequality holds whenever

$$k \geq 2\sqrt{d[1 + \log((d+1)\Delta)]},$$

which completes the proof. $\square$

If G does not have parallel edges, then any conflict function $f : E(G) \to C^2$ must be uniquely restrictive. Then, Theorem 3.2.3 tells us that

$$\chi_{\nleftrightarrow}(G) \leq 2\left\lceil \sqrt{d\,(1 + \log((d+1)\Delta))} \right\rceil,$$

which gives an affirmative answer to Question 1.4.3 for simple graphs.

3.2.2 Non-uniquely restrictive conflicts

In this section, we will consider single-conflict colorings with general conflict functions, rather than only those with uniquely restrictive conflict functions. To this end, we establish the following definition. Given a directed graph G with a conflict function $f : E(G) \to C^2$, we define the *restrictiveness* of f at v as the maximum value r_v for which there exists an r_v-tuple of parallel arcs in $E^+(v)$ whose conflicts form a set

$$\{(c_1, c^*), (c_2, c^*), \ldots, (c_{r_v}, c^*)\},$$

where the first entry in each conflict corresponds to v, where $c^* \in C$ is any single color, and where $c_1, \ldots, c_{r_v}$ are all distinct colors. Then, we say that the *restrictiveness* of f is the maximum restrictiveness r_v of f at v, taken over all vertices $v \in V(G)$. The restrictiveness r of a uniquely restrictive conflict function satisfies $r = 1$. If f is a conflict function on a graph G of edge-multiplicity at most μ, then the restrictiveness r of f satisfies $r \leq \mu$.

Theorem 3.2.3 gives an upper bound on number of colors needed for a single-conflict coloring given a conflict function with restrictiveness $r = 1$. In this section, we will show in the following theorem that we can also find an upper bound on the number of colors needed for a single-conflict coloring given a conflict function whose restrictiveness r is known but may be greater than 1. Since $r \leq \mu$ for any graph G with edge multiplicity at most μ, the following theorem also proves Theorem 3.2.1, giving an upper bound for $\chi_{\leftrightarrow}(G)$ of d-degenerate graphs G with small edge-multiplicity.

Theorem 3.2.4. *Let G be a directed graph of maximum degree Δ with a maximum out-degree of at most d. Let C be a set of k colors, and let each arc $e \in E(G)$ have an associated conflict $f(e)$. If the restrictiveness of f is at most r, and if*

$$k \geq \sqrt{d} \cdot 2^{r/2+2}\sqrt{r}\sqrt{1 + \log((d+1)\Delta)},$$

then G has a single-conflict coloring with respect to f and C.

Proof. We will use a similar probabilistic method as in Theorem 3.2.3, but we will need to work much harder to show that the Lovász Local Lemma still applies. We write $C = \{1, \dots, k\}$ for our color set. We define our color inventories S_v and S'_v as well as our bad events B_v in the same way as Theorem 3.2.3, but this time we will use the probability value

$$p = \frac{k}{2^{r+3}rd}.$$

Again, we can assume that $k \leq 2d$, so we assume that $p \leq \frac{1}{4}$.

Now, given a vertex $v \in V(G)$, we calculate the probability that the bad event B_v occurs. For each $1 \leq t \leq k$ and each subset

$$\{c_1, \dots, c_t\} \subseteq C,$$

we write $b(c_1, \dots, c_t)$ to denote the number of t-tuples of parallel outgoing arcs from v whose conflicts form a set

$$\{(c_1, c^*), (c_2, c^*), \dots, (c_t, c^*)\},$$

where the first entry in each conflict corresponds to v, and where $c^* \in C$ is any single color. In other words, $b(c_1, \dots, c_t)$ denotes the number of ways that $c_1, \dots, c_t$ might all be simultaneously deleted from S'_v due to a single color on the list of an out-neighbor of v.

Now, suppose that the bad event B_v occurs, so that S'_v is empty. It must have happened that for some value $0 \leq t \leq k$, exactly t colors were added to S_v, and then those t colors were all deleted from S'_v. If P_t is the probability that exactly t colors were added to S_v and then all deleted from S'_v, then

$$\Pr(B_v) \leq P_0 + P_1 + \cdots + P_k.$$

We aim to estimate each P_t, but in order to make our proof as easy to understand as possible, we will start with small values of t.

First, when $t = 0$, it is straightforward to calculate that $P_0 = (1-p)^k$.

Next, when $t = 1$, the probability that a given color $c \in C$ is the only color added to S_v is equal to $(1-p)^{k-1}p$. Then, the probability that c is subsequently deleted from S'_v is at most $p \cdot b(c)$. Therefore,

$$P_1 \leq (1-p)^{k-1}p^2 \cdot (b(1) + b(2) + \dots b(k)) .$$

Next, when $t = 2$, the probability that two given colors $c, c' \in C$ are the only two colors added to S_v is equal to $(1-p)^{k-2}p^2$. Then, if c and c' are deleted from S'_v, there are two possible reasons. First, it is possible that there is an out-neighbor w of v that can be reached from v by two distinct outgoing arcs e and e' satisfying $f(e) = (c, c^*)$ and $f(e') = (c', c^*)$ for some color $c^* \in C$, and that since incidentally $c^* \in S_w$, c and c' were both deleted from S'_v. The probability of such a subsequent event is at most $p \cdot b(c, c')$. Second, it is possible that there exist two (possibly identical) out-neighbors w and w' of v that can be reached by two distinct arcs e and e' satisfying $f(e) = (c, c^*)$ and $f(e) = (c', \tilde{c})$, and that since incidentally $c^* \in S_w$ and $\tilde{c} \in S_{w'}$, c and c' were deleted from S'_v. The probability of such a subsequent event is at most $p^2 b(c) b(c')$. Therefore,

$$P_2 \leq (1-p)^{k-2}p^3 \left(\sum_{1 \leq i < j \leq k} b(c_i, c_j) \right) + (1-p)^{k-2}p^4 \left(\sum_{1 \leq i < j \leq k} b(c_i) b(c_j) \right) .$$

Note that the first term in the upper bound of P_2 bounds the probability that S'_v became empty because of the presence of a single color $c^* \in S_w$ for a single out-neighbor w of v, and the second term bounds the probability that S'_v became empty because of two colors $c^* \in S_w$ and $\tilde{c} \in S_{w'}$ for two (possibly identical) out-neighbors w, w' of v.

Now, we consider a general value $0 \leq t \leq k$. The probability that some set $\{c_1, \dots, c_t\}$ is exactly the set added to S_v is equal to $(1-p)^{k-t}p^t$. For some value $z \leq t$, suppose that S'_v becomes empty because of the presence of z colors at $S_{w_1}, \dots, S_{w_z}$ for z out-neighbors $w_1, \dots, w_z$ of v. Then it must follow that there exists a partition $q_1 + \dots + q_z = t$ such that for each $1 \leq i \leq z$, some color in S_{w_i} caused a set C_i of q_i colors to be deleted from S'_v. Since we consider the outgoing arcs of v one at a time, the sets C_i are disjoint, as otherwise a color would be deleted from S'_v twice. We will also assume without loss of generality that the sets $C_1, \dots, C_z$ are in lexicographic order when considered as increasing sequences. Since the restrictiveness of f is at most r, we may assume that $q_i \leq r$ for each i. We may also assume that each value q_i in this partition is at least 1, as when some $q_i = 0$, we may cover this case with a smaller value of z. Therefore, when $t \geq 1$, we may bound P_t

as follows:

$$P_t \le (1-p)^{k-t} p^t \left(\sum_{z=1}^{t} p^z \sum_{\substack{q_1+\cdots+q_z=t \\ 1\le q_i \le r \text{ for } 1\le i\le z}} \sum_{\substack{|C_i|=q_i \text{ for } 1\le i\le z \\ C_i \text{ pairwise disjoint}}} b(C_1)b(C_2)\dots b(C_z) \right).$$

We will make the following notational simplification. If we have sets $A_1,\dots,A_m$ that are pairwise disjoint and such that $|A_i| = a_i$ for $1 \le i \le m$, then we write $[A_1,\dots,A_m] = (a_1,\dots,a_m)$. Also, if A satisfies $A \subseteq \{1,\dots,k\}$, $j \in A$, and $A \cap \{1,\dots,j-1\} = \emptyset$, then we write $\ell(A) = j$. For each fixed integer partition $q_1+\cdots+q_z = t$, we make the following claim.

Claim 3.2.5.

$$\sum_{[C_1,\dots,C_z]=(q_1,\dots,q_z)} b(C_1)b(C_2)\dots b(C_z) \le 2^{zr} \sum_{1\le i_1<i_2<\cdots<i_z\le k} b(i_1)b(i_2)\dots b(i_z).$$

Proof of Claim 3.2.5. We write

$$T = \sum_{[C_1,\dots,C_z]=(q_1,\dots,q_z)} b(C_1)b(C_2)\dots b(C_z)$$

for the sum that we are bounding.

We induct on z. For the base case, when $z=1$, we must show that

$$T = \sum_{|C_1|=t} b(C_1) \le 2^r \sum_{1\le i\le k} b(i).$$

We have

$$T = \sum_{|C_1|=t} b(C_1) \le \sum_{j=1}^{k} \sum_{C_1 \ni j} b(C_1).$$

Let j be fixed. Consider an outgoing arc $e=(v,w)$ from v for which $\mathrm{cc}(v,e) = j$, and write $f(e) = (j, c^*)$. As the restrictiveness of f is at most r, there exist at most 2^r sets of parallel edges joining v and w whose conflicts have c^* in the entry corresponding to w. It follows that

$$\sum_{C_1 \ni j} b(C_1) \le 2^r b(j).$$

Then the base case follows immediately.

Now, consider a value $z \geq 2$. We have

$$T = \sum_{j=1}^{k} \sum_{[C_2,\ldots,C_z]=(q_2,\ldots,q_z)} b(C_2)\ldots b(C_z) \sum_{\substack{|C_1|=q_1 \\ C_1 \cap C_i = \emptyset \text{ for } 2 \leq i \leq z \\ \ell(C_1)=j}} b(C_1).$$

Since $C_2, S_3, \ldots$ are disjoint with C_1, and since the sets C_i are in lexicographic order, it is equivalent to write

$$T = \sum_{j=1}^{k} \sum_{\substack{[C_2,\ldots,C_z]=(q_2,\ldots,q_z), \\ \ell(C_i)>j \text{ for } 2\leq i\leq z}} b(C_2)\ldots b(C_z) \sum_{\substack{|C_1|=q_1 \\ C_1 \cap C_i = \emptyset \text{ for } 2 \leq i \leq z \\ \ell(C_1)=j}} b(C_1).$$

By the same argument used in the base case, for fixed j, we have

$$\sum_{\substack{|C_1|=q_1, \\ \ell(C_1)=j}} b(C_1) \leq \sum_{C_1 \ni j} b(C_1) \leq 2^r b(j).$$

Therefore, it follows that

$$T \leq 2^r \sum_{j=1}^{k} b(j) \sum_{\substack{[C_2,\ldots,C_z]=(q_2,\ldots,q_z), \\ C_i \not\ni j \text{ for } 2\leq i\leq z}} b(C_2)\ldots b(C_z).$$

Now, we may apply the induction hypothesis to the sum

$$\sum_{\substack{[C_2,\ldots,C_z]=(q_2,\ldots,q_z), \\ \ell(C_i)>j \text{ for } 2\leq i\leq z}} b(C_2)\ldots b(C_z),$$

after which we have that

$$\begin{aligned} T &\leq 2^r \sum_{j=1}^{k} b(j) 2^{(z-1)r} \sum_{j+1\leq i_2<\cdots<i_z\leq k} b(i_2)\ldots b(i_z) \\ &= 2^{zr} \sum_{1\leq i_1<\cdots<i_z\leq k} b(i_1)\ldots b(i_z). \end{aligned}$$

This completes induction. $\diamondsuit$

From now on, we will simply write

$$\sigma_z = \sum_{1\leq i_1<\cdots<i_z\leq k} b(i_1)\ldots b(i_z).$$

By Claim 3.2.5,

$$P_t \leq (1-p)^{k-t} p^t \left(\sum_{z=1}^{t} (2^r p)^z \sum_{\substack{q_1+\cdots+q_z=t \\ 1\leq q_i \leq r \text{ for } 1\leq i\leq z}} \sigma_z \right).$$

As $1 \leq q_i \leq r$ for each i, the number of integer partitions in the second sum is at most r^z. Thus we have

$$P_t \leq (1-p)^{k-t} p^t \left(\sum_{z=1}^{t} (2^r rp)^z \sigma_z \right).$$

It follows that

$$\begin{aligned}
\Pr(B_v) &\leq P_0 + \sum_{t=1}^{k} (1-p)^{k-t} p^t \left(\sum_{z=1}^{t} (2^r rp)^z \sigma_z \right) \\
&= P_0 + \sum_{t=1}^{k} \sum_{z=1}^{t} (1-p)^{k-t} p^t \left((2^r rp)^z \sigma_z \right) \\
&= P_0 + \sum_{z=1}^{k} \sum_{t=z}^{k} (1-p)^{k-t} p^t \left((2^r rp)^z \sigma_z \right) \\
&< P_0 + (1-p)^k \sum_{z=1}^{k} \left((2^r rp)^z \sigma_z \right) \sum_{t=z}^{\infty} \left(\frac{p}{1-p} \right)^t
\end{aligned}$$

Since $p \leq \frac{1}{4}$, we crudely estimate

$$\sum_{t=z}^{\infty} \left(\frac{p}{1-p} \right)^t < (2p)^z \sum_{t=0}^{\infty} (2p)^t \leq 2^{z+1} p^z.$$

Therefore,

$$\begin{aligned}
\Pr(B_v) &< P_0 + (1-p)^k \sum_{z=1}^{k} 2(2^{r+1} rp^2)^z \sigma_z \\
&= P_0 + (1-p)^k \sum_{z=1}^{k} (2^{r/2+1} \sqrt{r} p)^{2z} \sigma_z. \qquad (3.1)
\end{aligned}$$

Now, we claim that

$$\Pr(B_v) < \prod_{i=1}^{k} \left((1 - p + (2^{r/2+1} \sqrt{r} p)^2 b(i) \right).$$

For convenience, we will write each factor as $\alpha + \beta_i$, where $\alpha = 1 - p$ and $\beta_i = (2^{r/2+1}\sqrt{r}p)^2 b(i)$. It suffices to show that each term of (3.1) has a dominating term in the expansion of the product $\prod_{i=1}^{k} (\alpha + \beta_i)$.

First, it is clear that $P_0 = \alpha^k$. Now, for each $1 \le z \le k$, we claim that

$$(1-p)^k (2^{r/2+1}\sqrt{r}p)^{2z}\sigma_z$$

is bounded above by $[\alpha^{k-z}]\prod_{i=1}^k(\alpha+\beta_i)$—that is, the sum of the terms in $\prod_{i=1}^k(\alpha+\beta_i)$ in which α has an exponent of $k-z$. Indeed,

$$\begin{aligned}
[\alpha^{k-z}]\prod_{i=1}^{k}(\alpha+\beta_i) &= \alpha^{k-z}\sum_{1\le i_1<\cdots<i_z\le k}\beta_{i_1}\ldots\beta_{i_z} \\
&= \alpha^{k-z}(2^{r/2+1}\sqrt{r}p)^{2z}\sigma_z \\
&> \alpha^{k}(2^{r/2+1}\sqrt{r}p)^{2z}\sigma_z \\
&= (1-p)^{k}(2^{r/2+1}\sqrt{r}p)^{2z}\sigma_z.
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
\Pr(B_v) &< \prod_{i=1}^{k}(\alpha+\beta_i) \\
&= \prod_{i=1}^{k}\left((1-p+(2^{r/2+1}\sqrt{r}p)^2 b(i)\right) \\
&\le \prod_{i=1}^{k}\exp\left(-p+(2^{r/2+1}\sqrt{r}p)^2 b(i)\right) \\
&= \exp\left(-kp+(2^{r/2+1}\sqrt{r}p)^2\sum_{i=1}^{k} b(i)\right) \\
&= \exp\left(-kp+(2^{r/2+1}\sqrt{r}p)^2 d\right).
\end{aligned}$$

Now, substituting $p = \frac{k}{2^{r+3}rd}$, we have that

$$\Pr(B_v) < \exp\left(-\frac{k^2}{2^{r+4}rd}\right).$$

As B_v is a bad event that involves $d+1$ vertices, and as G has a maximum degree of Δ, it follows that B_v is dependent with fewer than $(d+1)\Delta$ other bad events. Therefore, by the Lovász Local Lemma (Lemma 1.7.2), G has a single-conflict coloring as long as

$$e\cdot(d+1)\Delta\cdot\exp\left(-\frac{k^2}{2^{r+4}rd}\right)\le 1,$$

which holds whenever

$$k \ge \sqrt{d}\cdot 2^{r/2+2}\sqrt{r}\sqrt{1+\log((d+1)\Delta)}.$$

This completes the proof. $\square$

3.3 Cooperative colorings

3.3.1 Introduction

Recall that given a family $\mathcal{G} = \{G_1, \ldots, G_k\}$ of graphs on a common vertex set V, a *cooperative coloring* on $\mathcal{G}$ is defined as a family of sets $R_1, \ldots, R_k \subseteq V$ such that for each $1 \leq i \leq k$, R_i is an independent set of G_i, and $V = \bigcup_{i=1}^k R_i$. The notion of a cooperative coloring can be naturally generalized to the notion of a *cooperative list coloring*, defined as follows. Consider a graph family $\mathcal{G} = \{G_1, \ldots, G_k\}$ in which each graph G_i has a vertex set V_i that may or may not share vertices with the vertex sets V_j of the other graphs $G_j \in \mathcal{G}$. We write $V = V_1 \cup \cdots \cup V_k$. Then, we say that a *cooperative list coloring* of $\mathcal{G}$ is a family of vertex subsets $R_1, \ldots, R_k$ such that for each value $1 \leq i \leq k$, it holds that $R_i \subseteq V_i$ and R_i is an independent set of G_i, and such that $V = \bigcup_{i=1}^k R_i$. Every list coloring problem on a graph G with a list function L can be transformed into a cooperative list coloring problem as follows. For each color $c \in \bigcup_{v \in V(G)} L(v)$, we define the graph G_c to be the subgraph of G induced by those vertices $v \in V(G)$ for which $c \in L(v)$. Then, finding a list coloring on G is equivalent to finding a cooperative list coloring on the family $\mathcal{G} = \{G_c : c \in \bigcup_{v \in V(G)} L(v)\}$. The cooperative list coloring problem can be transformed into an independent transversal problem in a similar way to the cooperative coloring problem.

In the setting of cooperative colorings, we may naturally ask how many graphs of maximum degree d are necessary in a graph family $\mathcal{G}$ on a common vertex set in order to guarantee the existence of a cooperative coloring. (Note that in this section, we will use d rather than Δ for the maximum degree of a vertex in a graph $G_i \in \mathcal{G}$. We do this both to follow the conventions of previous research and also because a vertex $v \in V$ may have more than d neighbors when all graphs in $\mathcal{G}$ are considered.) Theorem 1.4.6 tells us that $\mathcal{G}$ is guaranteed a cooperative coloring whenever $|\mathcal{G}| \geq 2d$, and when d is large, Loh and Sudakov [60] have shown that a lower bound of the form $|\mathcal{G}| \geq (1 + o(1))d$ also guarantees the existence of a cooperative coloring on $\mathcal{G}$. On the other hand, Aharoni, Holzman, Howard, and Sprüssel [4] have constructed families containing $d + 1$ graphs of maximum degree d spanning a common vertex set that do not admit a cooperative coloring.

For a graph class $\mathcal{H}$, Aharoni, Berger, Chudnovsky, Havet, and Jiang [3] defined the parameter $m_{\mathcal{H}}(d)$ to be the minimum value m for which the following holds: If $\mathcal{G}$ is a family of at least m graphs of $\mathcal{H}$ of maximum degree at most d that span a common vertex set, then $\mathcal{G}$ must have a cooperative coloring. When $\mathcal{H}$ is the family of all graphs, they write $m(d) = m_{\mathcal{H}}(d)$. The discussion above implies that $m(d) \leq 2d$ for all values $d \geq 1$, and $m(d) \leq d + o(d)$ asymptotically when d is large. Note that all asymptotics in this paper will be with respect to the parameter d, which will always be an upper bound for the maximum degree of each graph in a given graph class.

In a similar fashion to Aharoni, Berger, Chudnovsky, Havet, and Jiang, we will define the parameter $\ell_{\mathcal{H}}(d)$ for a graph class $\mathcal{H}$ as follows. We say $\ell_{\mathcal{H}}(d)$ is the minimum value ℓ such that if $\mathcal{G}$ is a family of graphs from $\mathcal{H}$ of maximum degree at most d whose vertex sets are subsets of a universal vertex set V, and if each vertex $v \in V$ belongs to at least ℓ graphs in $\mathcal{G}$, then $\mathcal{G}$ has a cooperative list coloring. It is straightforward to show that for any graph class $\mathcal{H}$ and for any value d, $m_{\mathcal{H}}(d) \leq \ell_{\mathcal{H}}(d)$. When $\mathcal{H}$ is the class of all graphs, we write $\ell(d) = \ell_{\mathcal{H}}(d)$. Haxell's proof of Theorem 1.4.6, as well as Loh and Sudakov's argument [60] showing $m(d) \leq d + o(d)$, were both originally formulated for a more general independent transversal problem, and hence their arguments give the same upper bounds on $\ell(d)$ as well.

We summarize the discussion above with the following inequalities:

$$\begin{aligned} d+2 \leq m(d) &\leq \ell(d) \leq 2d \\ d+2 \leq m(d) &\leq \ell(d) \leq d + o(d). \end{aligned} \tag{3.2}$$

In [3], Aharoni, Berger, Chudnovsky, Havet, and Jiang considered the value $m_{\mathcal{F}}(d)$ for the class $\mathcal{F}$ of forests. These authors obtained a lower bound for $m_{\mathcal{F}}(d)$ from a construction and obtained an upper bound for $m_{\mathcal{F}}(d)$ by using a creative application of the Lovász Local Lemma that resembles an earlier method used by Bernshteyn, Kostochka, and Zhu [14, Section 4.2], which involves giving each vertex in the problem a random color inventory and then attempting to greedily give each vertex a color from its inventory. Since the method for obtaining an upper bound on $m_{\mathcal{F}}(d)$ also applies to the cooperative list coloring problem with no changes, we have the following result from [3]:

$$\log_2 \log_2 d \leq m_{\mathcal{F}}(d) \leq \ell_{\mathcal{F}}(d) \leq (1 + o(1)) \log_{4/3} d. \tag{3.3}$$

In this section, we will use Theorem 3.2.3 to extend the upper bound in (3.3) to all graphs of bounded degeneracy, at the expense of a constant factor. We present this result in Theorem 3.3.1 below. Additionally, we will construct a family of forests which we can use to prove that $m_{\mathcal{F}}(d) \geq (1 + o(1)) \frac{\log d}{\log \log d}$, improving the lower bound in (3.3) significantly. One interesting feature of our construction is that each graph in our family is a forest of stars. Hence, we write $\mathcal{S}$ for the class of of star forests, and since $\mathcal{S} \subseteq \mathcal{F}$, we observe that $m_{\mathcal{S}}(d) \leq m_{\mathcal{F}}(d)$. With $\mathcal{S}$ defined, we remark that our construction actually implies the stronger lower bound $m_{\mathcal{S}}(d) \geq (1 + o(1)) \frac{\log d}{\log \log d}$. With a lower bound for $m_{\mathcal{S}}(d)$ established, it is also natural to ask for an upper bound on $m_{\mathcal{S}}(d)$. We will prove two results that both imply, as a corollary, that $m_{\mathcal{S}}(d) \leq \ell_{\mathcal{S}}(d) \leq (1 + o(1)) \frac{\log d}{\log \log d}$, and hence, we will conclude that both $m_{\mathcal{S}}(d)$ and $\ell_{\mathcal{S}}(d)$ are of the form $(1 + o(1)) \frac{\log d}{\log \log d}$.

3.3.2 Graphs of bounded degeneracy

In this subsection, we will prove that the upper bound in (3.3) can be extended to all graphs of bounded degeneracy at the expense of a common factor.

Theorem 3.3.1. *Let $\mathcal{G}$ be a family of m graphs on a common vertex set V. Suppose each graph $G \in \mathcal{G}$ is at most k-degenerate and of maximum degree d. Then, whenever*

$$m \geq 13(1 + k\log(kd)),$$

$\mathcal{G}$ has a cooperative coloring.

Proof. If $d \leq 28$, then as $13(1+\log(d)) \geq 2d$, then (3.2) gives us the result. Hence, we assume that $d > 28$. Furthermore, if $k = 1$, then the corollary holds by (3.3), since the $1 + o(1)$ coefficient in this theorem is less than 3 for $d > 10$. Hence, we assume that $k \geq 2$. Additionally, if $d \leq 70$, then as $13(1 + 2\log(2d)) \geq 2d$, (3.2) again gives us the result.

By Observation 1.4.5, the graph $G = \bigcup_{H\in\mathcal{G}} H$ may be edge-colored in such a way that the cooperative coloring problem on $\mathcal{G}$ is equivalent to the adapted coloring problem on G. Observe that the maximum degree of G is at most md, and G has an orientation of its edges so that every vertex has an out-degree of at most mk.

Furthermore, by Theorem 3.2.3, G contains an adapted coloring as long as

$$m \geq 2\sqrt{mk[1 + \log((mk+1)md)]},$$

or stronger, as long as $m \geq 4k[1 + \log(2m^2kd)]$. It is enough to prove the corollary just for

$$m = \lceil 13k(1 + \log(kd)) \rceil,$$

which is at most $\frac{1}{\sqrt{2}}k^{1.125}d^{1.125}$ for $k \geq 2$ and $d > 70$. Hence, the corollary holds as long as

$$m \geq 4k[1 + \log(k^{3.25}d^{3.25})],$$

which holds whenever

$$m \geq 13k(1 + \log(kd)).$$

This completes the proof. ∎

3.3.3 A lower bound for $m_{\mathcal{S}}(d)$

In this section, we will give a construction that shows that $m_{\mathcal{F}}(d) \geq m_{\mathcal{S}}(d) \geq (1+o(1))\frac{\log d}{\log\log d}$. For ease of presentation, we will work in the setting of adapted colorings, which is equivalent to the cooperative coloring setting by Observation 1.4.5.

Theorem 3.3.2. $m_{\mathcal{S}}(d) \geq (1 + o(1))\frac{\log d}{\log\log d}$.

Proof. For each value $t \geq 1$, we will construct a graph G_t whose edges are colored with $\{1, \ldots, t\}$ by some function φ_t and whose monochromatic subgraphs are star forests. We will show that

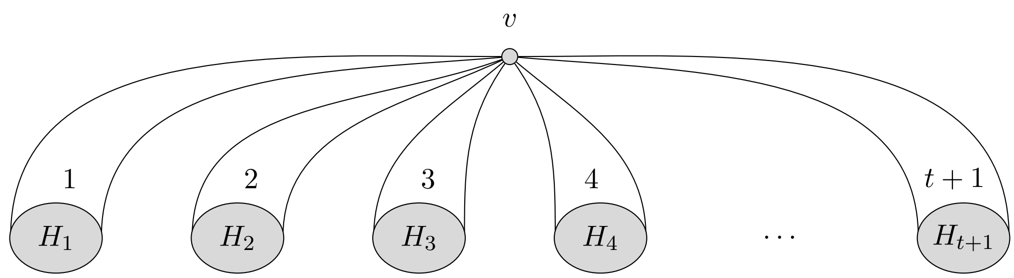

Figure 3.1: The figure shows the construction of (G_{t+1}, φ_{t+1}) from (G_t, φ_t). First, we make $t+1$ copies $H_1, \dots, H_{t+1}$ of G_t, and we obtain an edge-coloring of each H_i from φ_t by shifting the colors so that no edge of H_i uses the color i. Then, we add a universal vertex v that is joined to each vertex in each H_i by an edge of color i. These colored edges are denoted by the numbers above each H_i. In any cooperative coloring of this new graph using the set $\{1, \dots, t+1\}$, some vertex of each H_i must be colored i, and hence there is no available color at v.

(G_t, φ_t) does not have an adapted coloring with the colors $\{1, \dots, t\}$. Then, we will translate the edge-colored graph (G_t, φ_t) into a graph family $\mathcal{G}_t$ that proves our lower bound.

We will construct the edge-colored graphs (G_t, φ_t) recursively. First, we let (G_1, φ_1) be a K_2 whose edge is colored with the color 1. Now, suppose we have constructed G_t along with an edge-coloring $\varphi_t : E(G_t) \to \{1, \dots, t\}$, and suppose that (G_t, φ_t) does not have an adapted coloring with the color set $\{1, \dots, t\}$. For $1 \le i \le t+1$, we define a shift function $\psi_i : \{1, \dots, t\} \to \{1, \dots, t+1\}$ so that

$$\psi_i(x) = \begin{cases} x & 1 \le x \le i-1 \\ x+1 & i \le x \le t. \end{cases}$$

Now, we construct (G_{t+1}, φ_{t+1}) first by creating $t+1$ disjoint copies $H_1, \dots, H_{t+1}$ of G_t, where each H_i is edge-colored with the function $\psi_i \circ \varphi_t$. Observe that $(H_i, \psi_i \circ \varphi_t)$ is isomorphic to (G_t, φ_t) as an edge-colored graph, and hence $(H_i, \psi_i \circ \varphi_t)$ does not have an adapted coloring with the colors $\{1, \dots, i-1, i+1, \dots, t+1\}$. Therefore, in any adapted coloring of $(H_i, \psi_i \circ \varphi_t)$ using the color set $\{1, \dots, t+1\}$, some vertex must be colored i. Now, we construct (G_{t+1}, φ_{t+1}) by first taking our $t+1$ disjoint edge-colored copies $(H_i, \psi_i \circ \varphi_t)$ of G_t and adding a single new vertex v, and then adding an edge of color i joining v and each vertex of H_i, for $1 \le i \le t+1$. We call this new graph G_{t+1}, and we call its edge-coloring φ_{t+1}. We sketch the construction of (G_{t+1}, ϕ_{t+1}) from (G_t, ϕ_t) in Figure 3.1.

Observe that by construction, all monochromatic subgraphs of (G_{t+1}, φ_{t+1}) are star forests. Furthermore, for each value $1 \le i \le t+1$, some vertex of H_i must be colored with i, and hence no color from the set $\{1, \dots, t+1\}$ is available at v. Therefore, (G_{t+1}, φ_{t+1}) has no adapted coloring using the set $\{1, \dots, t+1\}$.

Now, we compute the maximum degree of each monochromatic subgraph of G_t. We write $V_t = |V(G_t)|$, and we write Δ_t for the maximum number of edges of a single color incident to a vertex in (G_t, φ_t). It is easy to see that $\Delta_1 = 1$, $V_1 = 2$, and that the following recursion holds for $t \geq 2$:

$$\begin{aligned}\Delta_t &= V_{t-1} \\ V_t &= tV_{t-1} + 1.\end{aligned}$$

Solving this recurrence, we see that

$$\begin{aligned}V_t &= V_1 t^{\underline{t-1}} + t^{\underline{t-2}} + \cdots + t^{\underline{2}} + t^{\underline{1}} + 1 = (e + o(1))t! \\ \Delta_t &= (e + o(1))(t-1)!,\end{aligned}$$

where $t^{\underline{k}} = t!/(t-k)!$ is the falling factorial.

Now, consider a value d, and choose t so that $\Delta_t \leq d < \Delta_{t+1}$. We construct (G_t, φ_t) as above, and we obtain a graph family $\mathcal{G}_t = \{G_1, \dots G_t\}$ on the universal vertex set $V(G_t)$ by letting each $G_i \in \mathcal{G}_t$ have an edge set consisting of those edges of color i in (G_t, φ_t). Observe that each graph in $\mathcal{G}_t$ is a star forest of maximum degree at most d. Furthermore, since (G_t, φ_t) has no adapted coloring using the color set $\{1, \dots, t\}$, it follows that $\mathcal{G}_t$ has no cooperative coloring. Since $d \leq (e + o(1))t!$, it follows that $t \geq (1 + o(1))\frac{\log d}{\log \log d}$. Hence, $m_{\mathcal{S}}(d) \geq (1 + o(1))\frac{\log d}{\log \log d}$, completing the proof. □

3.3.4 A partition lemma and an upper bound on $\ell_{\mathcal{S}}(d)$

In this section, we aim to show that $\ell_{\mathcal{S}}(d) \leq (1 + o(1))\frac{\log d}{\log \log d}$. In order to prove this upper bound, we establish a partition lemma, which essentially shows that if $\mathcal{H}$ is a graph class whose graphs can be vertex-partitioned into members of classes $\mathcal{A}$ and $\mathcal{B}$ for which $\ell_{\mathcal{A}}(d)$ and $\ell_{\mathcal{B}}(d)$ are not too large, then $\ell_{\mathcal{H}}(d)$ is also not too large. While proving the partition lemma, it is essential that we work in the setting of cooperative list colorings rather than the setting of cooperative colorings.

While we can use our partition lemma to prove the upper bound $\ell_{\mathcal{S}}(d) \leq (1 + o(1))\frac{\log d}{\log \log d}$ directly, we will see that the lemma gives us stronger results that imply this upper bound on $\ell_{\mathcal{H}}(d)$ as a corollary. We will prove two results that both show an upper bound on $\ell_{\mathcal{H}}(d)$ for some graph class $\mathcal{H}$ based on certain forest structures in the graphs of $\mathcal{H}$, and both of these results will imply that $\ell_{\mathcal{S}}(d) \leq (1 + o(1))\frac{\log d}{\log \log d}$.

Our partition lemma is as follows.

Lemma 3.3.3. *Let $\mathcal{H}$, $\mathcal{A}$, and $\mathcal{B}$ be graph classes, and let $t = t(d)$ be a function of d. Suppose that*

- *Each graph $G \in \mathcal{H}$ of maximum degree at most d can be vertex-partitioned into sets A and B so that $G[A] \in \mathcal{A}$ and $G[B] \in \mathcal{B}$, and so that each vertex in A has at most t neighbors in B,*
- *$\ell_{\mathcal{A}}(d) = o(\log d)$,*
- *$\ell_{\mathcal{B}}(d)t = o(\log d)$.*

Then,

$$\ell_{\mathcal{H}}(d) \le (1+o(1))\frac{\log d}{\log\log d - \log(\ell_{\mathcal{B}}(d)t)} + \ell_{\mathcal{A}}.$$

It may help the reader first to visualize $\mathcal{A} = \mathcal{B}$ as the class of edgeless graphs and to visualize $\mathcal{H} = \mathcal{S}$ as the class of star forests. In this special case, for each star forest $G \in \mathcal{H}$, we may let A denote the leaf set of G and let B denote the set consisting of the centers of the star components of G. In this special case, $\ell_{\mathcal{A}}(d) = \ell_{\mathcal{B}}(d) = t = 1$, so the lemma immediately implies that $\ell_{\mathcal{S}}(d) \le (1+o(1))\frac{\log d}{\log\log d}$.

Proof. We fix a value d, and we consider a family $\mathcal{G} = \{G_1, \dots, G_k\}$ of graphs from $\mathcal{H}$ of maximum degree at most d whose vertex sets are subsets of a universal vertex set V. We will write $\ell_{\mathcal{A}} = \ell_{\mathcal{A}}(d)$ and $\ell_{\mathcal{B}} = \ell_{\mathcal{B}}(d)$. We assume without loss of generality that each vertex $v \in V$ belongs to exactly ℓ graphs in $\mathcal{G}$. We will show that for each $\gamma > 0$, if $\ell = (1+\gamma)\frac{\log d}{\log\log d - \log(\ell_{\mathcal{B}} t)} + \ell_{\mathcal{A}}$, then when d is sufficiently large, $\mathcal{G}$ has a cooperative list coloring.

We let $\varepsilon > 0$ be a sufficiently small constant (which is at most 1). For each graph $G_i \in \mathcal{G}$, we suppose that $V(G_i)$ can be partitioned into sets A_i and B_i satisfying the properties of A and B in the lemma's hypothesis. Note that if every vertex of V belongs to at most $\varepsilon(\ell - \ell_{\mathcal{A}})$ sets B_i, then every vertex of V must belong to at least $\ell - \varepsilon(\ell - \ell_{\mathcal{A}}) \ge \ell_{\mathcal{A}}$ sets A_i, and hence a cooperative list coloring on V can be found by taking independent subsets of the graphs $G_i[A_i]$. Therefore, we assume that for some nonempty set $U \subseteq V$ of vertices, each vertex $u \in U$ belongs to more than $\varepsilon(\ell - \ell_{\mathcal{A}})$ sets B_i.

Before we proceed to the next step of our proof, we need to show that $\ell_{\mathcal{B}} < \varepsilon(\ell - \ell_{\mathcal{A}})$. To show this, we use the third condition of the lemma to write $\ell_{\mathcal{B}} t = \log d / f$, for some unbounded function f for which $\inf_{x \in [d,\infty]} f(x)$ is increasing with respect to d, which is possible by the third condition of the lemma. Then, we observe that when d is large, $\log d \cdot \frac{\log f}{f} < \varepsilon \log d$, which implies

$$\ell_{\mathcal{B}} t = \log d / f < \frac{\varepsilon \log d}{\log f} = \frac{\varepsilon \log d}{\log\log d - \log(\log d / f)} = \frac{\varepsilon \log d}{\log\log d - \log(\ell_{\mathcal{B}} t)} < \varepsilon(\ell - \ell_{\mathcal{A}}),$$

which is even stronger than what we needed to show.

Now, for each vertex $u \in U$, we write $\mathbf{B}_u$ for the family of all sets B_i containing u, for $1 \le i \le k$. Then, we choose a family $\mathbf{B}'_u$ of exactly $\ell_{\mathcal{B}}$ sets B_i uniformly at random (without replacement) from $\mathbf{B}_u$, and we write $C_u = \{i : B_i \in \mathbf{B}'_u\}$. This is possible due to the fact that $|\mathbf{B}_u| \ge \varepsilon(\ell - \ell_{\mathcal{A}})$ and the inequality that we have just shown. We assign each vertex u a color from C_u so that $\mathcal{G}[U]$ receives a cooperative list coloring, where $\mathcal{G}[U] = \{G[U \cap V(G)] : G \in \mathcal{G}\}$. Note that this is possible, since $|C_u| = \ell_{\mathcal{B}}$ for each vertex $u \in U$, and since $u \in G_i[B_i]$ for each $i \in C_u$. After this assignment, if a vertex $v \in V$ has a neighbor $u \in U$ via a graph G_j and u is assigned the color j, we then say that j is *unavailable* at v. If $v \in A_j$ and the color j is not unavailable at v, then we say that j is *available* at v. Observe that if each uncolored vertex $v \in V$ has at least $\ell_{\mathcal{A}}$ available colors, then we may extend our cooperative list coloring on $\mathcal{G}[U]$ to a cooperative list coloring on $\mathcal{G}$. Therefore, for each vertex $v \in V \setminus U$, we define a bad event X_v, which is the event that fewer than $\ell_{\mathcal{A}}$ colors are available at v. The bad event X_v depends on at most $t\ell$ neighbors of v, each of which has at most $d\ell$ neighbors. Therefore, X_v is dependent with at most $\ell^2 td + t\ell < 2\ell^2 td$ other bad events. We will use the Lovász Local Lemma (Lemma 1.7.2) to show that with positive probability, no bad event occurs and that we can hence find a cooperative coloring of $\mathcal{G}$.

Now, consider a vertex $v \in V \setminus U$. Suppose that $v \in A_j$ for some value j. Recall that v has at most t neighbors $u \in B_j$ via G_j, and each such neighbor u belonging to U is colored from a randomly chosen set C_u of $\ell_{\mathcal{B}}$ potential colors. The probability that a given vertex $u \in U \cap N_{G_j}(v)$ is assigned the color j is at most the probability that $j \in C_u$, which is at most $\frac{\ell_{\mathcal{B}}}{\varepsilon(\ell - \ell_{\mathcal{A}})}$. Therefore, the probability that j is unavailable at v is at most $\frac{\ell_{\mathcal{B}} t}{\varepsilon(\ell - \ell_{\mathcal{A}})}$. Note that this argument remains true even if it is given that some other set of colors has already been made unavailable at v. Therefore, since v belongs to at least $\ell - \varepsilon(\ell - \ell_{\mathcal{A}})$ sets A_i, $\Pr(X_v)$ is bounded above by the probability that more than

$$\ell - \varepsilon(\ell - \ell_{\mathcal{A}}) - \ell_{\mathcal{A}} = (1 - \varepsilon)(\ell - \ell_{\mathcal{A}})$$

colors are made unavailable at v, which is at most

$$\binom{\ell}{(1-\varepsilon)(\ell - \ell_{\mathcal{A}})} \left(\frac{\ell_{\mathcal{B}} t}{\varepsilon(\ell - \ell_{\mathcal{A}})} \right)^{(1-\varepsilon)(\ell - \ell_{\mathcal{A}})} < 2^{\ell} \left(\frac{\ell_{\mathcal{B}} t}{\varepsilon(\ell - \ell_{\mathcal{A}})} \right)^{(1-\varepsilon)(\ell - \ell_{\mathcal{A}})}.$$

Since each bad event X_v is dependent with fewer than $2\ell^2 td$ other bad events, the Local Lemma (Lemma 1.7.2) tells us that all bad events are avoided with positive probability as long as

$$2^{\ell} \left(\frac{\ell_{\mathcal{B}} t}{\varepsilon(\ell - \ell_{\mathcal{A}})} \right)^{(1-\varepsilon)(\ell - \ell_{\mathcal{A}})} \cdot 2\ell^2 td \cdot e \le 1.$$

Equivalently, by taking the natural logarithm on both sides, no bad event occurs with positive probability as long as

$$\ell \log 2 + (1-\varepsilon)(\ell - \ell_{\mathcal{A}})(\log(\ell_{\mathcal{B}} t) - \log \varepsilon - \log(\ell - \ell_{\mathcal{A}})) + \log 2 + 2\log \ell + \log t + \log d + 1 \le 0.$$

This inequality can be written more simply as follows:

$$(1 - \varepsilon + o(1))(\ell - \ell_{\mathcal{A}})(\log(\ell_{\mathcal{B}} t) - \log(\ell - \ell_{\mathcal{A}})) + (1 + o(1)) \log d \le 0.$$

We claim that this inequality holds when d is sufficiently large and ε is sufficiently small. Recall that $\ell = \frac{(1+\gamma)\log d}{\log\log d - \log(\ell_{\mathcal{B}} t)} + \ell_{\mathcal{A}}$. When we substitute this value for ℓ and assume d is large, we can first write the inequality as

$$(1 - \varepsilon + o(1)) \left(\frac{(1+\gamma)\log d}{\log\log d - \log(\ell_{\mathcal{B}} t)} \right) (\log(\ell_{\mathcal{B}} t) - \log\log d) + (1 + o(1)) \log d \le 0,$$

or more simply,

$$-(1-\varepsilon+o(1))(1+\gamma)\log d+(1+o(1))\log d\le 0,$$

which holds when ε is sufficiently small and d is sufficiently large. Therefore, with positive probability, our random procedure allows us to complete a cooperative list coloring of $\mathcal{G}$. Since $\gamma > 0$ can be arbitrarily small, this completes the proof. $\square$

As mentioned before, we can use Lemma 3.3.3 directly to prove that $\ell_{\mathcal{S}}(d) \le (1+o(1))\frac{\log d}{\log\log d}$, which shows that the lower bound in Theorem 3.3.2 is best possible up to the $o(1)$ function. We will see that Lemma 3.3.3 also implies much stronger results, and we will prove two such results that both imply this upper bound on $\ell_{\mathcal{S}}(d)$ as a corollary.

For the first of our results, we will need some definitions. Given a rooted tree T with a root r, the *height* of a vertex v in T is the distance from v to r, and the height of T is the maximum height achieved over all vertices $v \in V(T)$. Given integers $q \ge 1$ and $h \ge 1$, a *q-ary tree of height h* is a rooted tree in which every vertex of height at most $h-1$ has exactly q children. Given an integer $k \ge 1$, we write $\log^{(k)} d = \underbrace{\log\log\ldots\log}_{k \text{ times}} d$. Then, we have the following result.

Theorem 3.3.4. *Let $q \ge 2$ and $h \ge 1$ be fixed integers. If $\mathcal{H}$ is a family of graphs with no q-ary tree of height h as a subgraph, then*

$$\ell_{\mathcal{H}}(d) \le (1+o_{q,h}(1))\frac{\log d}{\log^{(h)} d}+O_q(1).$$

Proof. We will prove the theorem by induction on h. When $h = 1$, then our hypothesis implies that each graph of $\mathcal{H}$ has maximum degree $q-1$. Hence, by (3.2), it holds that $\ell_{\mathcal{H}}(d) \le 2q-2$, which is certainly of the form $O_q(1)$. Hence, the theorem holds when $h = 1$.

Now, suppose that $h \ge 2$ and that the graphs of $\mathcal{H}$ contain no q-ary tree of height h as a subgraph. We write $t = 2q^h$. We consider a graph $G \in \mathcal{H}$, and we let $A \subseteq V$ be the set of all vertices $v \in V$ for which $\deg_G(v) < t$. Now, we claim that $G \setminus A$ has no q-ary tree subgraph of height $h-1$. Indeed, suppose that $G \setminus A$ contains a q-ary tree T of height $h-1$ as a subgraph. Since no vertex of T belongs to A, this implies that every vertex of T must have degree at least k in G. However, since

$$t = 2q^h > (q^{h-1}-1)q + 2q^{h-1} > (q^{h-1}-1)q + |V(T)|,$$

we hence can greedily choose a set N_x of q neighbors in $N_G(x)\cap(V(G)\setminus V(T))$ for each of the q^{h-1} leaves $x \in V(T)$ in such a way that the sets N_x are pairwise disjoint. Then, by taking the union of T and the sets N_x, we have a q-ary tree of height h in G, a contradiction. Thus, we conclude that $G \setminus A$ has no q-ary tree of height $h-1$.

Now, for each G, we define the set A as described above, and we let $B = V(G) \setminus A$. By construction, each vertex of A as at most t neighbors in B via the graph G. Furthermore, $G[A]$ belongs to the family $\mathcal{A}$ of graphs of maximum degree t, which satisfies $\ell_{\mathcal{A}}(d) \leq 2t$ by Theorem 1.4.6, and $G[B]$ belongs to the family $\mathcal{B}$ of graphs with no subgraph isomorphic to a q-ary tree of height h. By the induction hypothesis, it holds that $\ell_{\mathcal{B}}(d) \leq (1+o_{q,h}(1))\frac{\log d}{\log^{(h-1)} d} + O_q(1)$. Therefore, we can apply Lemma 3.3.3.

By applying Lemma 3.3.3 and recalling that k and $\ell_{\mathcal{A}}(d)$ are constants depending on q and h, we conclude that

$$\ell_{\mathcal{H}}(d) \leq (1 + o_{q,h}(1))\frac{\log d}{\log\log d - \log(\ell_{\mathcal{B}})}.$$

As $h \geq 2$, it holds that $\log \ell_{\mathcal{B}}(d) = \log\log d - \log^{(h)} d + O_{q,h}(1)$, and thus the theorem is proven. $\square$

To use Theorem 3.3.4 to prove that $\ell_{\mathcal{S}}(d) \leq (1 + o(1))\frac{\log d}{\log\log d}$, consider a binary tree of height 2, which has 7 vertices and 4 leaves. Since no star forest contains this binary tree as a subgraph, the upper bound on $\ell_{\mathcal{S}}(d)$ follows from Theorem 3.3.4 with $q = h = 2$.

Next, we show that if $\mathcal{H}$ is a graph class whose graphs have a certain quotient of bounded treedepth, then $\ell_{\mathcal{H}}(d)$ can be bounded above. For this next theorem, we will need some more definitions. If G is a graph and $U_1, \dots, U_k$ is a partition of $V(G)$, then the *quotient graph* $G/(U_1, \dots, U_k)$ is the graph on k vertices obtained by contracting each part U_i to a single vertex and deleting all resulting loops and parallel edges.

Given a rooted tree T with a root r, we define the *closure* of T as the graph on $V(T)$ in which two vertices $u, v \in V(T)$ are adjacent if and only if u and v form an ancestor-descendant pair. Given a rooted forest F, in which each tree component has a root, the closure of F is the union of the closures of the components of F. For a graph G, if there exists a rooted tree T of height $h-1$ such that G is a subgraph of the closure of T, then we say that the *treedepth* of G is at most h. The reason for this "off-by-one error" is that if T has height $h-1$, then the longest path in T with the root as an endpoint contains exactly h vertices.

With these definitions in place, we are ready for our second theorem implying that $\ell_{\mathcal{S}}(d) \leq (1 + o(1))\frac{\log d}{\log\log d}$.

Theorem 3.3.5. *Let $0 < \varepsilon < \frac{1}{2}$ be a fixed value. Let $\mathcal{H}$ be a graph class for which each graph $G \in \mathcal{H}$ has a partition into parts $U_1, \dots, U_k$ of size at most $t = (\log d)^{\varepsilon}$, so that each component of the quotient graph $G/(U_1, \dots, U_k)$ has treedepth at most h. Then, $\ell_{\mathcal{H}}(d) \leq \frac{h-1+o_h(1)}{1-2\varepsilon} \cdot \frac{\log d}{\log\log d}$.*

Proof. We prove the theorem by induction on h. When $h = 1$, for each graph $G \in \mathcal{H}$, the quotient graph $G/(U_1, \dots, U_k)$ is an independent set, so each component of G has at most t vertices. Therefore, $\ell_{\mathcal{H}}(d) < 2t = o\left(\frac{\log d}{\log\log d}\right)$ by (3.2).

Now, suppose that $h \geq 2$. Consider a graph $G \in \mathcal{H}$. Let F be a rooted forest subgraph of $G/(U_1, \dots, U_k)$ in which each component has height at most $h-1$ and so that the closure of F contains $G/(U_1, \dots, U_k)$. We partition $V(G)$ into parts A and B so that B contains the sets U_i

corresponding to the roots of F and A contains all other vertices of G. Observe that each component of $G[B]$ contains at most t vertices, and each component K of $G[A]$ can be partitioned using the sets U_i so that the quotient graph of K with respect to this partition has treedepth at most $h-1$. Finally, observe that a vertex $v \in A$ is adjacent to a given vertex $u \in B$ only if v belongs to a set U_i, U_j is the root ancestor of U_i in F, and $u \in U_j$. Hence, each vertex $v \in A$ has at most $|U_j| \leq t$ neighbors in B.

Now, we apply Lemma 3.3.3 to $\mathcal{H}$. We let $\mathcal{A}$ be the graph class defined to satisfy the same conditions of $\mathcal{H}$ except with h replaced by $h-1$, and we let $\mathcal{B}$ be the class of graphs whose components each have at most t vertices. By the induction hypothesis, $\ell_{\mathcal{A}}(d) \leq \frac{(h-2+o_h(1))}{1-2\varepsilon} \cdot \frac{\log d}{\log \log d}$, and $\ell_{\mathcal{B}}(d) < 2t$ by (3.2). Since $2t^2 = o(\log d)$, all of the hypotheses Lemma 3.3.3 are satisfied, and we can apply the lemma to $\mathcal{H}$.

By applying Lemma 3.3.3 and using the induction hypothesis, we see that

$$\ell_{\mathcal{H}}(d) \leq (1+o(1)) \frac{\log d}{\log \log d - 2\log t} + \frac{h-2+o(1)}{1-2\varepsilon} \cdot \frac{\log d}{\log \log d} = \frac{h-1+o(1)}{1-2\varepsilon} \cdot \frac{\log d}{\log \log d}.$$

Hence, the theorem is proven. $\square$

In order to use Theorem 3.3.5 to prove that $\ell_{\mathcal{S}}(d) \leq (1+o(1)) \frac{\log d}{\log \log d}$, we observe that if G is a star forest, then every component of G has treedepth at most 2, so we can apply Theorem 3.3.5 with $h = 2$ and obtain the upper bound.

3.3.5 Conclusion

By combining (3.3) and Theorem 3.3.2, we obtain the following inequality:

$$(1+o(1)) \frac{\log d}{\log \log d} \leq m_{\mathcal{S}}(d) \leq m_{\mathcal{F}}(d) \leq \ell_{\mathcal{F}}(d) \leq (1+o(1)) \log_{4/3} d.$$

While this inequality is certainly much tighter than (3.3), the correct asymptotic growth rates for $m_{\mathcal{F}}(d)$ and $\ell_{\mathcal{F}}(d)$ remain open. While we do not have a conjecture for the correct growth rates of these quantities, we remark that if $m_{\mathcal{F}}(d) = \Theta(\log d)$, then Theorem 3.3.4 gives a strong necessary condition for forest families that demonstrate this growth rate. Namely, suppose that $\{\mathcal{G}_d\}_{d \geq 1}$ is a sequence of forest families such that $|\mathcal{G}_d| = \Theta(\log d)$, the forests of $\mathcal{G}_d$ have maximum degree at most d, and $\mathcal{G}_d$ has no cooperative coloring. Then, Theorem 3.3.4 implies that for each finite tree T, T must appear as a subgraph of infinitely many forests from the families in $\{\mathcal{G}_d\}_{d \geq 1}$.

Chapter 4

The hat guessing game

4.1 Introduction

Recall that the hat guessing game is a graph coloring problem defined as follows. We have a graph G, a set $S = \{1, \dots, k\}$ of colors, and a family $\Gamma = \{\Gamma_v\}_{v \in V(G)}$ of functions, where each function is a mapping $\Gamma_v : S^{N(v)} \to S$. We say that the *hat guessing number* of G is less than k if, for every family $\Gamma = \{\Gamma_v\}_{v \in V(G)}$ of functions, there always exists a coloring $\phi : V(G) \to S$ so that no vertex v satisfies $\phi(v) = \Gamma_v(\phi(N(v)))$. We write $\mathrm{HG}(G)$ for the hat guessing number of G. In Chapter 1, we show that this problem can also be described as a game in which players try to guess the colors of their hats.

In this chapter, we will establish an upper bound on the hat guessing number of outerplanar graphs (Theorem 1.5.2), as well as an upper bound on a larger class of planar graphs, which we call *layered planar graphs* (Theorem 1.5.3). The chapter will be organized as follows. In Section 4.2, we will introduce some important tools that we will need for our two main theorems. In Section 4.3, we will show that every outerplanar graph has a vertex partition satisfying certain key properties, and then with the help of the tools introduced in Section 4.2, we will prove Theorem 1.5.2. In Section 4.4, we will use a similar strategy to extend our methods beyond outerplanar graphs and prove Theorem 1.5.3. Finally, in Section 4.5, we will show that if an upper bound can be obtained for a certain stronger version of the hat guessing number on planar graphs, then an upper bound on the hat guessing number can be obtained for all graphs of bounded genus.

4.2 Multiple guesses, vertex partitions, and edge density

In this section, we will outline three key tools that we will use to prove Theorem 1.5.2 and 1.5.3. These tools use a modified version of the hat guessing game in which each player attempts to guess his hat color s times (without hearing the guesses of the other players). Given a graph G and an integer $s \geq 1$, if k is the maximum integer for which the players on G, when assigned hats from the color set $\{1, \dots, k\}$, have a strategy that guarantees at least one correct hat color guess when each player is allowed to guess s times, then we write $\mathrm{HG}_s(G) = k$.

The first of our tools follows from a simple application of Lovász's Local Lemma [34]; see [38] for more details.

Lemma 4.2.1. *Let $s \geq 1$ be an integer. If G is a graph of maximum degree Δ, then* $\mathrm{HG}_s(G) < (\Delta + 1)es$.

Our next tool tells us that if the vertices of a graph can be partitioned into sets satisfying certain conditions, then the hat guessing number of the graph is bounded. This tool was introduced by Bosek et al. [18].

Lemma 4.2.2 ([18])**.** *Let $s \geq 1$ be an integer. Let G be a graph, and let $V(G) = A \cup B$ be a partition of the vertices of G. If each vertex in A has at most d neighbors in B, then* $\mathrm{HG}_s(G) \leq \mathrm{HG}_{s'}(G[A])$, *where $s' = s(\mathrm{HG}_s(G[B]) + 1)^d$.*

By using the same approach originally used in [18], we can prove the following more general version of Lemma 4.2.2. Note that Lemma 4.2.2 is obtained from the following lemma by setting $k = 2$, $V_1 = A$, and $V_2 = B$.

Lemma 4.2.3. *Let $s \geq 1$ be an integer. Let G be a graph with a vertex partition $V(G) = V_1 \cup \dots \cup V_k$, and let $\ell_1, \dots, \ell_k$ be positive integers. Assume that for each pair i, j satisfying $1 \leq i < j \leq k$, each vertex $v \in V_i$ has at most $d_{i,j}$ neighbors in V_j. For $1 \leq i \leq k - 1$, define*

$$s_i = s\ell_{i+1}^{d_{i,i+1}} \cdot \ell_{i+2}^{d_{i,i+2}} \cdot \dots \cdot \ell_k^{d_{i,k}},$$

and define $s_k = s$. If, for each value $1 \leq i \leq k$, it holds that

$$\mathrm{HG}_{s_i}(G[V_i]) < \ell_i,$$

then $\mathrm{HG}_s(G) < \max\{\ell_1, \dots, \ell_k\}$.

Proof. For $1 \leq i \leq k$, we fix a list of exactly ℓ_i colors at each vertex in V_i, and we consider the game in which each player makes s guesses. We will prove the following statement:

> For each value $1 \leq i \leq k$, if the hat colors on $V_1 \cup \dots \cup V_{i-1}$ are already fixed, then there exists a hat assignment on V_i for which no vertex in V_i correctly guesses its hat color, regardless of how $V_{i+1} \cup \dots \cup V_k$ is colored.

By iteratively applying this statement for each value $1 \leq i \leq k$, we obtain a winning hat assignment on G that uses at most $\max\{\ell_1, \dots, \ell_k\}$ colors, proving the lemma.

To prove this statement, consider a value $1 \leq i \leq k$, and assume that hat colors are fixed on $V_1 \cup \dots \cup V_{i-1}$. With these hat colors fixed, every vertex in V_i has a hat guessing function depending only on $G[V_i \cup \cdots \cup V_k]$. Furthermore, for each vertex $v \in V_i$, every possible coloring of $N(v) \cap (V_{i+1} \cup \cdots \cup V_k)$ gives v a unique guessing function depending only on $G[V_i]$, and there are $\ell_{i+1}^{d_{i,i+1}} \ldots \ell_k^{d_{i,k}}$ possible colorings of $N(v) \cap (V_{i+1} \cup \cdots \cup V_k)$. Therefore, with color lists fixed at every vertex of $V_{i+1} \cup \cdots \cup V_k$, for each hat assignment on $G[V_i]$, v will guess from a total of at most s_i possible guesses. (Note that when $i = k$, the set $V_{i+1} \cup \cdots \cup V_k$ is empty, so v guesses from a total of $s_k = s$ guesses.) By our assumption, we may assign each vertex of V_i a hat from its set of ℓ_i colors in such a way that no vertex of $G[V_i]$ guesses its hat color correctly, even with s_i guesses. We give $G[V_i]$ such a hat assignment, and since each vertex $v \in V_i$ guesses from a set of at most s_i colors, no vertex of V_i guesses its hat color correctly. This completes the proof. $\square$

Finally, we will define a third tool that we will need for Theorems 1.5.2 and 1.5.3. Our last tool relies heavily on theory related to a Turán-type edge density problem. We will need some definitions. First, an *r-partite r-uniform hypergraph* $\mathcal{H}$ is defined as a set V of vertices and a collection E of r-tuples from V, satisfying the following property: V can be partitioned into r parts $V_1, \ldots, V_r$ so that every r-tuple in E intersects each part V_i at exactly one vertex. We often use the term *r-partite r-graph* to refer to an r-partite r-uniform hypergraph, and we often call the r-tuples in E *edges*. We say that an r-partite r-graph is *balanced* if $|V_1| = \cdots = |V_r|$. We say that an r-partite r-graph $\mathcal{K}$ is *complete* if it contains every possible edge e satisfying $|e \cap V_i| = 1$ for each vertex part V_i, and if $\mathcal{K}$ is also balanced and has $r\ell$ vertices, then we write $\mathcal{K} = K_\ell^{(r)}$. Next, for integers $r \geq 1$, $\ell \geq r$, and $n \geq \ell$, we define the quantity $\mathcal{E}^{(r)}(n, \ell)$ to be the maximum number of edges in a balanced r-uniform r-graph with rn vertices and with no complete $K_\ell^{(r)}$ subgraph. Since a balanced r-uniform r-graph with rn vertices and n^r edges certainly must contain such a subgraph, we see that $\mathcal{E}^{(r)}(n, \ell)$ is well-defined and less than n^r.

We give several examples of the quantity $\mathcal{E}^{(r)}(n, \ell)$. When $r = 1$, an r-partite r-graph is simply a collection of vertices in which some of these vertices are also called edges, and a $K_\ell^{(1)}$ is simply a collection of ℓ of these "edges." Any 1-partite 1-graph with at least ℓ edges clearly must contain a $K_\ell^{(1)}$, so for all $1 \leq \ell \leq n$, $\mathcal{E}^{(1)}(n, \ell) = \ell - 1$. When $r = 2$, the quantity $\mathcal{E}^{(2)}(n, \ell)$ describes the maximum number of edges in a balanced bipartite graph on $2n$ vertices containing no copy of $K_{\ell,\ell}$. The question of determining the precise value of $\mathcal{E}^{(2)}(n, \ell)$ is a special case of a classic problem of Zarankievicz, which asks how many edges a bipartite graph on $m + n$ vertices can have without containing a copy of $K_{s,t}$. This problem of Zarankiewicz has a long history and has led to many beautiful results; see [42] for an extensive survey of this area of combinatorics.

For our final tool, we will need the following theorem of Erdős [33], which gives an upper bound for $\mathcal{E}^{(r)}(n, \ell)$. The original result of Erdős assumes that n is sufficiently large, but we will need a result that holds even for small n, so we present a slightly modified form of Erdős's original result.

Lemma 4.2.4. *Let $r \geq 2$, let $\mathcal{H}$ be a balanced r-partite r-graph with rn vertices, and let $2 \leq \ell \leq n$. If $|E(\mathcal{H})| \geq 3n^{r - \frac{1}{\ell^{r-1}}}$, then $\mathcal{H}$ contains a copy of the r-graph $K_\ell^{(r)}$.*

Proof. The proof of the lemma is very similar to the original proof of Erdős, and we defer the proof to the appendix. □

Before we introduce our last tool for proving Theorem 4.3.5, we will need some more definitions. For a graph G and a vertex subset $U \subseteq V(G)$, we write $N(U)$ for the set of vertices with at least one neighbor in U. Given a graph G with a vertex partition $V_1, \dots V_t$, we define the *quotient graph* $G/(V_1, \dots, V_t)$ as the graph obtained from G by contracting each part V_i and deleting loops and parallel edges. In other words, $G/(V_1, \dots, V_t)$ has t vertices $v_1, \dots, v_t$, and v_i is adjacent to v_j if and only if some edge in G joins a vertex of V_i to a vertex of V_j.

We are now ready for our last main tool for proving Theorem 4.3.5. The following result, which is useful for bounding the hat guessing number of outerplanar and layered planar graphs, is also interesting in its own right. Our proof of this result uses key ideas from the proof of Butler et al. [25] that $\mathrm{HG}(T) = 2$ for every tree T on at least two vertices.

Theorem 4.2.5. *Let $r, s \geq 1$ be integers. Let G be a graph, and let $V_1, \dots, V_t$ be a partition of $V(G)$ such that the quotient graph $G/(V_1, \dots, V_t)$ is a tree. Suppose that for each distinct pair V_i, V_j, it holds that $|N(V_i) \cap V_j| \leq r$. If $\mathrm{HG}_s(G[V_i]) < \ell$ for all $1 \leq i \leq t$, then $\mathrm{HG}_s(G) \leq (3\ell)^{r\ell^{r-1}}$ when $r \geq 2$, and $\mathrm{HG}_s(G) \leq \ell(\ell-1)$ when $r = 1$.*

Proof. We would like to assume that for each V_i, every neighboring set V_j satisfies $|N(V_i) \cap V_j| = r$. This can be achieved by adding isolated vertices to each neighboring set V_j and then adding edges between these new vertices and vertices of V_i. These extra vertices will not cause any of our hypotheses to be violated, nor will they decrease the hat guessing number of G.

We let $k = (3\ell)^{r\ell^{r-1}} + 1$ when $r \geq 2$, and we let $k = \ell(\ell-1) + 1$ when $r = 1$. We aim to show that $\mathrm{HG}_s(G) < k$. We first make the following claim.

Claim 4.2.6. $\ell^r \, \mathcal{E}^{(r)}(k, \ell) < k^r$*.*

Proof. When $r = 1$, the claim asserts that $\ell(\ell-1) < k$, which is clearly true. When $r \geq 2$, Lemma 4.2.4 states that $\mathcal{E}^{(r)}(k, \ell) < 3k^{r - \frac{1}{\ell^{r-1}}}$. Then, $\ell^r \, \mathcal{E}^{(r)}(k, \ell) < 3\ell^r k^{-\frac{1}{\ell^{r-1}}} k^r < k^r$. □

Now, we fix a guessing strategy $\Gamma = \{\Gamma_v\}_{v \in V(G)}$ on G. We prove the following stronger statement.

> Let $1 \leq i \leq t$. If every vertex in V_i has a list of ℓ colors and every other vertex in G has a list of k colors, then the adversary has a winning hat assignment.

We induct on t. When $t = 1$, the statement holds from the fact that $\mathrm{HG}_s(G[V_1]) = \mathrm{HG}_s(G) < \ell$. Now, suppose $t > 1$, and let i be some value satisfying $1 \leq i \leq t$. We must show that if every vertex in V_i has a list of ℓ possible hat colors and every other vertex of G has a list of k possible hat colors, then the adversary has a winning hat assignment.

In each neighboring set V_j of V_i, there exists a set $U_j \subseteq V_j$ of exactly r vertices that have neighbors in V_i, and there also exists a set $W_j \subseteq V_i$ of exactly r vertices that are neighbors of U_j. We write C_j for the component of $G \setminus V_i$ containing V_j. If a hat assignment α on W_j is fixed, then Γ determines a unique hat guessing strategy on C_j. Furthermore, by the induction hypothesis, if each vertex of C_j has the color list $\{1, \dots, k\}$, then with α fixed, the adversary has a winning hat assignment on C_j. We let $B_{\alpha,j}$ be the set of hat assignments on C_j that cause the adversary to win the restricted game on C_j when the players use the hat guessing strategy determined by α. Then, we let $A_{\alpha,j}$ be the set of distinct colorings of U_j that can be obtained by restricting an assignment from $B_{\alpha,j}$ to U_j. We see from the induction hypothesis that $A_{\alpha,j}$ is nonempty. Now, we make the following claim:

Claim 4.2.7. *If α is a fixed hat assignment on W_j, then $A_{\alpha,j}$ contains at least $r^k - \mathcal{E}^{(r)}(k, \ell)$ distinct colorings.*

Proof. Suppose that $A_{\alpha,j}$ contains at most $r^k - \mathcal{E}^{(r)}(k, \ell) - 1$ distinct colorings. We construct a balanced r-partite r-graph $\mathcal{H}$ on kr vertices as follows. We write $U_j = \{u_1, \dots, u_r\}$. Then, we let the kr vertices of $\mathcal{H}$ be indexed by (u_p, q), where $u_p \in U_j$ and $1 \leq q \leq k$. Finally, for each hat assignment in $A_{\alpha,j}$ in which each vertex $u_p \in U_j$ is given a hat of some color γ_p, we add an edge to $\mathcal{H}$ of the form $\{(u_1, \gamma_1), (u_2, \gamma_2), \dots, (u_r, \gamma_r)\}$. Now, since $A_{\alpha,j}$ has at most $r^k - \mathcal{E}(k, r) - 1$ edges, it follows that the complement r-graph $\overline{\mathcal{H}}$ contains at least $\mathcal{E}^{(r)}(k, \ell) + 1$ edges, and hence $\overline{\mathcal{H}}$ contains a copy of $K_\ell^{(r)}$. In other words, there exist sets $L_1 \subseteq \{1, \dots, k\}, \dots, L_r \subseteq \{1, \dots, k\}$, each of size ℓ, such that $A_{\alpha,j}$ contains no hat assignment in which each vertex u_p is assigned a hat with a color from L_p. It then follows that when each vertex $u_p \in U_j$ is given the color list L_p and every other vertex of C_j is given a list of k colors, the adversary has no winning hat assignment on C_j using these lists. However, this contradicts the induction hypothesis applied to C_j and with V_j instead of V_i. Thus, the claim holds. $\square$

Now, for each component C_j, we compute $A_{\alpha,j}$ for each of the ℓ^r hat assignments α on W_j using the pre-assigned lists of size ℓ. Since $\ell^r \, \mathcal{E}^{(r)}(k, \ell) < k^r$, for each j, the intersection $\bigcap_\alpha A_{\alpha,j}$ is nonempty by the pigeonhole principle, where α is taken over the ℓ^r hat assignments on W_j. Hence, for each j, we can choose an assignment A_j from this intersection, and we use A_j to assign hats to the vertices in U_j.

Next, with hats already assigned to each U_j, the vertices in V_i have a fixed guessing strategy that depends only on the hat assignment at V_i. Since $\mathrm{HG}_s(G[V_i]) < \ell$, we can assign each vertex in V_i a hat in such a way that no vertex guesses its hat color correctly.

Finally, with colors assigned to V_i, we argue that we can complete a hat assignment on each component C_j in such a way that no vertex in C_j guesses its hat color correctly. Since V_i has already been colored, a coloring α on W_j is fixed, and hence the adversary can give a winning assignment to C_j if and only if some assignment $B_j \in B_{\alpha,j}$ extends the already fixed coloring A_j at U_j. However, since $A_j \in A_{\alpha,j}$, the set of restricted assignments of $B_{\alpha,j}$, such a winning assignment B_j must exist by definition. Therefore, we use B_j to color C_j, and hence no vertex in C_j will guess its hat color correctly. By repeating this process for each component C_j, we assign hats to all remaining vertices in such a way that no vertex will guess its hat color correctly. □

4.3 Outerplanar graphs

In this section, we prove Theorem 1.5.2, showing that the hat guessing number of outerplanar graphs is bounded. We need some definitions and lemmas.

Definition 4.3.1. We define a *petal graph* G to be a graph obtained from a (possibly empty) path P by adding a vertex v adjacent to every vertex of P. We say that v is the *stem* of G. Then, we define a *petunia* to be a graph in which every block is a subgraph of a petal graph.

We note that a petal graph is an example of a *fan graph*, which is constructed from a path and a coclique joined by a complete bipartite graph. We use the term *petunia* rather than *flower* so as not to be confused with other uses of the word *flower* in combinatorics (e.g. [11]). In the following lemma, we show that petunias admit a vertex partition satisfying the conditions described in Theorem 4.2.5.

Lemma 4.3.2. *If G is a petunia, then $V(G)$ can be partitioned into forests $F_1, \dots, F_t$ such that the quotient graph $G/(F_1, \dots, F_t)$ is a forest and such that for each distinct pair i, j, $|N(F_i) \cap V(F_j)| \leq 3$.*

Proof. We add edges to G until every block of G is a petal graph. Then, we will color $V(G)$ red and blue, and we will let each maximal connected monochromatic subgraph of G give the vertex set of a tree F_i. After removing our extra edges from G, the subgraphs $F_1, \dots, F_t$ will make a family of forests satisfying the conditions of the lemma.

We give a general procedure for how to color the vertices of a block H of G with red and blue. We let H consist of a stem v and a path P. If v is colored with either color, then we color the path P formed by the non-stem vertices of H so that P alternates between red and blue. If a vertex $w \in V(P)$ is colored with either color, then extend the coloring of w to the entire path P formed by the non-stem vertices of H so that P alternates between red and blue. Then, we color v with either color. Then, to color G, we begin by coloring a vertex of each component of G with an arbitrary color, and then we extend the coloring using the procedure we have described. After G is colored, we observe that each maximal connected monochromatic subgraph of G intersects each block of G either in a star or at a single vertex, and these single vertices form an independent set. Therefore, each maximal connected monochromatic subgraph of G is a tree (which may span several blocks), and these trees give a partition $F_1, \dots, F_t$ of $V(G)$.

We argue that for a pair i, j, the tree F_i has at most three neighbors in F_j. If F_i has at most one neighbor in F_j, then we are done. Otherwise, choose vertices $u, v \in V(F_i)$ so that u has a neighbor

$x \in V(F_j)$ and v has a neighbor $y \in V(F_j)$ distinct from x. Since F_i is a tree, there exists a path P (possibly of length 0) in F_i from u to v, and similarly, there exists a path Q in F_j from x to y. Then, $V(P) \cup V(Q)$ gives the vertex set of a cycle C in G, and C must belong to a single block H of G. It then follows that every vertex in $N(F_i) \cap V(F_j)$ belongs to the block H. Thus it is easy to see from our construction of $F_1, \dots, F_t$ that F_i has at most three neighbors in F_j.

Finally, we argue that the quotient graph $G/(F_1, \dots, F_t)$ is a forest. Suppose that this quotient graph contains a cycle C. We assume without loss of generality that F_1 belongs to C, with neighbors F_2 and F_3. Using a similar argument to that above, there exists a cycle C' in G that visits F_i, then F_{i+1}, and then later visits F_{i+2} without once again visiting F_i (with $i \in \{1, 2, 3\}$ and addition "wrapping around"). Since C' is two-connected, C' must belong to a single block H of G. However, again, because H is partitioned by the trees F_i into a star and single independent vertices, the cycle C' cannot satisfy these properties. Therefore, the quotient graph $G/(F_1, \dots, F_t)$ is a forest. □

Lemma 4.3.2 shows that the hypotheses of Theorem 4.2.5 hold for petunias with $r = 3$. Therefore, we can obtain an upper bound on the hat guessing number of petunias as follows.

Theorem 4.3.3. *If G is a petunia, then* $\mathrm{HG}_s(G) \le (3s^2 + 3s + 3)^{3(s^2+s+1)^2}$.

Proof. We partition G into forests F_i as described in Lemma 4.3.2. By Theorem 4.2.5 (and also by a result of Bosek et al. [18]), each forest F_i satisfies $\mathrm{HG}_s(F_i) \le (s + 1)s$. Then, by applying Theorem 4.2.5 to each component of G with $r = 3$ and $\ell = (s + 1)s + 1$, we obtain the result. □

In order to prove Theorem 1.5.2, we will need one more lemma. In the following lemma, we show that using the notion of petunias, we can find a useful vertex decomposition of any outerplanar graph.

Lemma 4.3.4. *If G is an outerplanar graph, then $V(G)$ can be partitioned into two sets A and B such that $G[A]$ is a petunia, B is an independent set, and each vertex of A is adjacent to at most three vertices of B.*

Proof. If $|V(G)| \le 1$, then we let $A = V(G)$, and we are done. Otherwise, we assume $|V(G)| \ge 2$. Since the class of petunias is closed under taking subgraphs, we may add edges to G until G is a maximal outerplanar graph (or equivalently an outerplanar triangulation whenever $|V(G)| \ge 3$), and doing so will not make our task any easier.

We prove the following stronger claim:

> Let G be a maximal outerplanar graph on at least two vertices, and let uv be an edge of G oriented from u to v. Then $V(G)$ can be partitioned into two sets A and B such that the following hold:
>
> 1. $G[A]$ is a petunia containing uv;
> 2. B is an independent set;

3. u is not adjacent to any vertex of B;

4. v is adjacent to at most two vertices of B;

5. Every other vertex of G is adjacent to at most three vertices of B.

We prove the statement by induction on $|V(G)|$. If $|V(G)| = 2$, then again letting $A = V(G)$ satisfies the claim. Now, suppose that $|V(G)| \geq 3$, and let $uv \in E(G)$. We fix an outerplanar drawing of G. Since G is an outerplanar triangulation, it follows that G is 2-connected, and hence G contains a Hamiltonian cycle C separating the outer face of G from all other faces.

We begin to construct our sets A and B as follows. We first add all vertices in $N[u]$ to A. Note that $N[u]$ induces a petunia in G. Now, let $t = \deg(u)$, and write w_1 and w_t for the neighbors of u via C, and assume without loss of generality that the neighbors of u, in clockwise order, are $w_1, w_2, \dots, w_t$. We observe that any given component K of $G \setminus N[u]$ is separated from the rest of G by two vertices of the form w_i, w_{i+1}, for some $1 \leq i \leq t-1$. We show an example of such a component K in Figure 4.1.

Now, if $N[u]$ contains all vertices of G, then it is easy to check that we are done. Otherwise, let K be some component of $G \setminus N[u]$, and let K be separated from the rest of G by w_i, w_{i+1}. Since G is an outerplanar triangulation, uw_i and uw_{i+1} lie on some triangle T in the interior of C, so w_i and w_{i+1} are adjacent. Furthermore, since w_i and w_{i+1} separate some component K from the rest of G, it must follow that the edge w_iw_{i+1} does not belong to C, and hence T shares the edge w_iw_{i+1} with a second triangle T' in the interior of C. The triangle T' includes the vertices w_i and w_{i+1}, along with a third vertex x_i. We add x_i to B, and we add all neighbors of x_i to A.

Now, let the neighbors of x_i, in clockwise order, be $y_1, \dots, y_s$, with $y_1 = w_i$ and $y_s = w_{i+1}$. Let y_j and y_{j+1} be neighbors of x_i along C. Since G is an outerplanar triangulation, it follows that G contains the edge $y_\alpha y_{\alpha+1}$ for $1 \leq \alpha \leq j-1$, as well as for $j+1 \leq \alpha \leq s-1$. Now, for $1 \leq \alpha \leq j-1$, we orient the edge $y_\alpha y_{\alpha+1}$ from y_α to $y_{\alpha+1}$. Then, for $j+1 \leq \alpha \leq s-1$, we orient the edge $y_\alpha y_{\alpha+1}$ from $y_{\alpha+1}$ to y_α. Finally, for each value $\alpha \in [s-1] \setminus j$, we apply the induction hypothesis to the outerplanar subgraph of G that is either 2-connected or isomorphic to K_2 whose outer facial walk is given by the edge $y_\alpha y_{\alpha+1}$ along with the path from y_α to $y_{\alpha+1}$ along C. By induction, all vertices of G are partitioned into the sets A and B.

We claim that all criteria of the induction statement are satisfied. First, we must show that $G[A]$ is a petunia containing uv. By construction, A contains u and v. Also, clearly $N[u]$ is a petunia, as G is outerplanar. Furthermore, each vertex y_α described in the process above must be a cut-vertex in $G[A]$, so $G[A]$ remains a petunia even after adding vertices using the induction hypothesis. Second, clearly no pair x_i, x_j is adjacent, so B is initially an independent set. Furthermore, as all neighbors of each x_i are added to A, B remains an independent set even after applying the induction hypothesis. Third, by construction, u is not adjacent to any vertex of B. Fourth, if $v = w_i$, then as G is outerplanar, v is initially adjacent to at most two vertices of B, namely x_i and x_{i-1}. Furthermore, each vertex w_i is the tail of an arc in all outerplanar graphs containing w_i for which

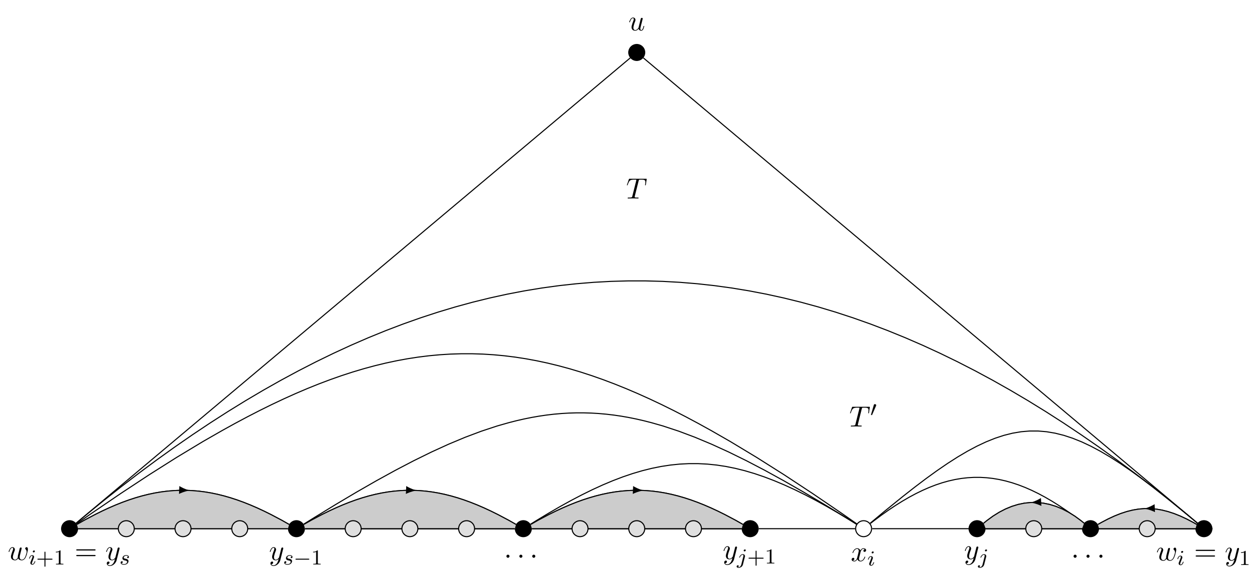

Figure 4.1: The figure shows part of the outerplanar graph G from the proof of Lemma 4.3.4. The black vertices in the figure belong to the set A, and the white vertex belongs to the set B. Each shaded region represents some outerplanar subgraph of G. We may partition $V(G)$ as described in Lemma 4.3.4 by applying the induction hypothesis to each of the outerplanar graphs shown as a shaded region in the figure.

the induction hypothesis is applied, and thus w_i does not gain any neighbors in B after applying induction because of criterion (3).

Finally, we argue that each vertex $z \in A$ is adjacent to at most three vertices of B. If z is of the form w_i or u, then (5) holds for z. Otherwise, if z is of the form y_α in the process described above, then z belongs to at most two outerplanar graphs H, H' for which the induction hypothesis is called, and z belongs to an arc of both H and H'. Furthermore, z is the head of at most one of these arcs. Therefore, by criteria (3) and (4), z gains at most two neighbors in B during induction. As we have assumed that z is not of the form u or w_i, it follows that z is adjacent to at most one vertex of the form x_i. Therefore, z has at most three neighbors in B. Finally, if none of the above holds, then z belongs to an outerplanar graph H on which induction is applied, and z is separated from all vertices x_i by some edge $y_\alpha y_{\alpha+1}$. Therefore, by criterion (5), z has at most three neighbors in B. Thus the induction statement holds, and the proof is complete. □

With Lemma 4.3.4 proven, we are ready to apply Lemma 4.2.2 and Theorem 4.3.3 to obtain an upper bound for the hat guessing number of outerplanar graphs. Letting $s = 1$ in the following theorem immediately implies Theorem 1.5.2.

Theorem 4.3.5. *If G is an outerplanar graph, then*

$$\mathrm{HG}_s(G) \le (3(s+1)^6 + 3(s+1)^3 + 3)^{3((s+1)^6+(s+1)^3+1)^2}.$$

Proof. We partition G using Lemma 4.3.4 so that $G[A]$ is a petunia, B is an independent set, and each vertex of A has at most three neighbors in B. Since $\mathrm{HG}_s(G[B]) = s$, we follow Lemma 4.2.2 and set $d = 3$ and $s' = (s+1)^3$. Then, according to Lemma 4.2.2 and Theorem 4.3.3,

$$\mathrm{HG}_s(G) \le \mathrm{HG}_{(s+1)^3}(G[A]) \le (3(s+1)^6 + 3(s+1)^3 + 3)^{3((s+1)^6+(s+1)^3+1)^2}.$$

□

4.4 Layered planar graphs

In this section, we compute an upper bound for the hat guessing number of layered planar graphs. Formally, we define layered planar graphs as follows.

Definition 4.4.1. Consider a planar graph H obtained from the following process. We begin with a 2-connected outerplanar graph G_1 embedded in the plane so that the unbounded face is incident to all vertices of G_1. Then, we choose some integer $\tau \ge 1$, and for each $2 \le i \le \tau$, we draw a 2-connected outerplanar G_i inside some interior face of G_{i-1} so that in the drawing of G_i, the unbounded face contains all vertices of G_i. Then, we add some set of edges between G_{i-1} and G_i in such a way that does not introduce a crossing. If G is a subgraph of a graph H constructed in this way, then we say that G is a *layered planar graph.*

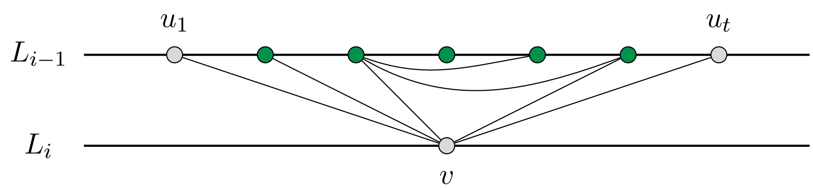

Figure 4.2: The figure shows a vertex $v \in L_i$ with u_1 as its counterclockwise-most parent and with u_t as its clockwise-most parent. We use green to color all vertices in L_{i-1} on the clockwise side of vu_1 and on the counterclockwise side of vu_t.

We obtain the following upper bound for the hat guessing number of layered planar graphs. Letting $s = 1$ in the following theorem immediately implies Theorem 1.5.3.

Theorem 4.4.2. *If G is a layered planar graph, then* $\log_2 \log_2 \log_2 \log_2 \mathrm{HG}_s(G) < 2^{149} s^{35}$.

Proof. We fix a drawing of G in the plane. We partition the vertices of G into *levels* L_i as follows. First, we let L_1 denote the set of vertices on the outer face of G. Then, for $i \geq 1$, we let L_{i+1} denote the set of vertices on the outer face of $G \setminus (L_1 \cup \cdots \cup L_i)$. Since G is a layered planar graph, we may assume that G_i is a 2-connected outerplanar graph for each level L_i, and that every edge of G either has both endpoints in some L_i or has one endpoint in some L_i and the other endpoint in L_{i+1}. If a vertex $v \in L_i$ has a neighbor u, then we say that u is a *parent* of v if $u \in L_{i-1}$, a *sibling* of v if $u \in L_i$, and a *child* of v if $u \in L_{i+1}$.

Now, we will begin to partition the vertices of G into color classes. Initially, we let every vertex of G be colored `blank`. Then, for each vertex v in each level L_i, let $u_1, \ldots, u_t$ be the parents of v in clockwise order. We let K_v denote the (possibly empty) subgraph of G_{i-1} that is separated from the rest of G_{i-1} by u_1 and u_t and that can be reached by travelling from v to u_1 and then turning right. In other words, K_v is on the "clockwise side" of the arc vu_1 and the "counterclockwise side" of vu_t, and if K_v is nonempty, it contains the vertices $u_2, \ldots, u_{t-1}$. We color every vertex of K_v green, as shown in Figure 4.2. Observe that by planarity, since G_i and G_{i-1} are 2-connected, each vertex of K_v can only have v as a child.

Next, for every vertex $v \in V(G)$, if v has at least three children, then we color v red. Since each green vertex has only one child, no vertex will be colored both green and red. Then, if a red vertex v has at least one red child, then we use pink to recolor the clockwise-most and counterclockwise-most red child of v, as shown in Figure 4.3. Finally, we use blue to color all remaining `blank` vertices in a level L_i with i even, and we use indigo to color all remaining `blank` vertices in a level L_i with i odd.

Now, we make a series of claims about our coloring of G. The ultimate goal of these claims will be to show that we can apply Lemma 4.2.3 to obtain an upper bound for $\mathrm{HG}_s(G)$.

Claim 4.4.3. *The green vertices of G induce an outerplanar graph, and every green vertex has at most five neighbors in a color other than green.*

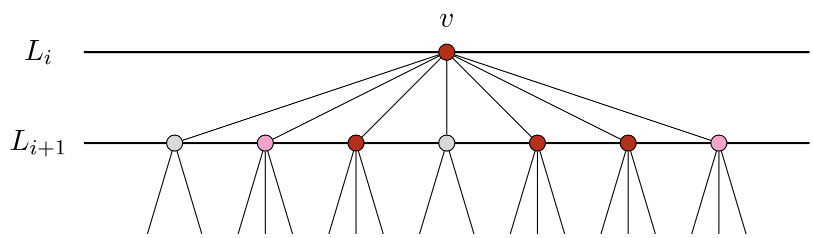

Figure 4.3: The figure shows a red vertex $v \in L_i$ with several red children. We use pink to recolor the clockwise-most and counterclockwise-most red children of v.

Proof. We first show that each green vertex has at most five non-green neighbors. Let v be a green vertex. Since all parents of v except for the clockwise-most and counterclockwise-most parent are colored green, v has at most two non-green parents. If $v \in L_i$, then by construction, v belongs to a connected green subgraph in G_i that is separated from the rest of G_i by two non-green vertices. Therefore, v also has at most two non-green siblings. Finally we have observed previously that v has at most one child. Therefore, v has at most five non-green neighbors.

Now, we show that the green vertices of G induce an outerplanar graph. Since each G_i is an outerplanar graph, clearly the green vertices in any single level L_i induce an outerplanar graph. Furthermore, by construction, for any two distinct green vertices $u, v \in L_i$, no green parent of u is equal to or adjacent to a green parent of v, and hence no ancestor of u in the green induced subgraph is equal to or adjacent to an ancestor of v in the green induced subgraph. Therefore, the green induced subgraph of G is a graph in which each block is an outerplanar graph contained in some level L_i, and hence the green induced subgraph of G is outerplanar. □

Claim 4.4.4. *The blue vertices of G induce an outerplanar graph, and every blue vertex has at most six neighbors in indigo, red, or pink.*

Proof. Clearly the blue vertices induce an outerplanar graph, as they induce a subgraph of the disjoint union of the outerplanar graphs G_i for even values i.

Now, let $v \in L_i$ be a blue vertex. As v is not red or pink, v has at most two indigo, red, or pink children. As all but at most two parents of v are colored green, v has at most two indigo, red, or pink parents. We also observe that by construction, v has no indigo sibling.

Finally, we argue that v has at most two red or pink siblings. If G_{i+1} is empty, then clearly v does not have a red or pink sibling, so we assume that G_{i+1} is a 2-connected outerplanar graph. Since G_i is an outerplanar graph, G_i has a Hamiltonian cycle C_i such that $E(C_i)$ and the interior of $E(C_i)$ contain all edges of G_i. Suppose that v has at least three red or pink siblings. Assume that when starting outside C_i and then visiting the edges incident to v in clockwise order, we visit edges incident to three red or pink siblings u_1, u_2, u_3, in order. Observe that since G_i is outerplanar, the edge vu_2 separates the interior of C_i into two regions, one containing u_1 and one containing

u_3. However, since G_{i+1} is connected, the edge vu_2 separates one of u_1, u_3 from G_{i+1}, contradicting the assumption that both of these vertices have neighbors in G_{i+1}. Therefore, v has at most two red or pink siblings. □

Claim 4.4.5. *The indigo vertices of G induce an outerplanar graph, and every indigo vertex has at most six neighbors in red or pink.*

Proof. The proof is similar to that of Claim 4.4.4. □

Claim 4.4.6. *The red vertices in G induce a petunia, and every red vertex has at most six pink neighbors.*

Proof. In our proof, we will often use the fact that for each value $1 \leq i \leq \tau$, the subgraph of G_i induced by those vertices with at least one child is a subgraph of a cycle. This fact follows easily from planarity.

To show that the red vertices in G induce a petunia, we first claim that if two distinct red vertices u, v in a common level L_i have respective red children u' and v', then u' and v' are not joined by a path of red vertices in L_{i+1}. Indeed, u has two pink children $w, w' \in L_{i+1}$ with respective children $x, x' \in L_{i+2}$, and the edges wx and $w'x'$ along with G_{i+2} separate u' from v'. Thus, the claim holds, and by a similar argument, u' and v' cannot be equal.

Next, consider the subgraph G' of G induced by the red vertices of G, and consider a 2-connected subgraph H of G'. If all vertices of H belong to a single level L_i, then H is a subgraph of a cycle and hence a petal graph. Otherwise, by the previous observation, for each vertex $v \in V(G')$, v separates the descendants of v in G' from all other vertices in G'. Therefore, if two vertices of H belong to a level L_i, then as H is 2-connected, no vertex of H can belong to L_{i+1}. Therefore, it must follow that H contains exactly one vertex in some level L_i and that all other vertices of H belong to L_{i+1}. As seen before, the red and pink vertices in L_{i+1} induce a subgraph of a cycle, and hence the red vertices of H in L_{i+1} induce a subgraph of a path. Therefore, H is a petal graph, and G' is a petunia.

Now, let v be a red vertex. As argued before, v has at most two non-green parents. By the same argument used in Claim 4.4.4, v has at most two pink siblings. Finally, if $v \in G_i$, then since G_{i+1} is 2-connected and outerplanar, v has at most two pink children. □

Claim 4.4.7. *The pink vertices in G induce a graph of maximum degree* 6*.*

Proof. The proof is similar to that of Claim 4.4.6. □

Now, we are ready to apply Lemma 4.2.3 and obtain an upper bound for the hat guessing number of planar graphs. Following Lemma 4.2.3, we let V_1 denote the green vertices of G, V_2 the blue vertices, V_3 the indigo vertices, V_4 the red vertices, and V_5 the pink vertices. Then, we define the following values:

$$
\begin{aligned}
\ell_5 &= 20s \\
s_4 &= s\ell_5^6 = (20)^6 s^7 \\
\ell_4 &= (3s_4^2 + 3s_4 + 3)^{3(s_4^2+s_4+1)^2} + 1 < 2^{2^{138} s^{30}} \\
s_3 &= s\ell_4^6 < 2^{2^{141} s^{35}} \\
\ell_3 &= (3(s_3+1)^6 + 3(s_3+1)^3 + 3)^{3((s_3+1)^6+(s_3+1)^3+1)^2} + 1 < 2^{2^{2^{145} s^{35}}} \\
s_2 &= s\ell_3^6 < 2^{2^{2^{146} s^{35}}} \\
\ell_2 &= (3(s_2+1)^6 + 3(s_2+1)^3 + 3)^{3((s_2+1)^6+(s_2+1)^3+1)^2} + 1 < 2^{2^{2^{2^{147} s^{35}}}} \\
s_1 &= s\ell_2^5 < 2^{2^{2^{2^{148} s^{35}}}} \\
\ell_1 &= (3(s_1+1)^6 + 3(s_1+1)^3 + 3)^{3((s_1+1)^6+(s_1+1)^3+1)^2} + 1 < 2^{2^{2^{2^{2^{149} s^{35}}}}}
\end{aligned}
$$

We verify these estimates in the appendix. Since the upper bound given by this method is probably too large, we make no real effort to optimize our estimates.

Now, we must check that all of the hypotheses of Lemma 4.2.3 hold. It is easy to check from our claims that we have given appropriate definitions to each value s_i. (In fact, we may overestimate the values of s_1, s_2, s_3, but this is fine.) Then, we show that each ℓ_i is large enough as follows.

As the pink vertices induce a subgraph of maximum degree at most 6, it follows from Lemma 4.2.1 that $\mathrm{HG}_s(G[V_5]) < 20s$. As $G[V_4]$ is a petunia, $\mathrm{HG}_{s_4}(G[V_4]) < \ell_4$ by Theorem 4.3.3. Finally, as $G[V_i]$ is outerplanar for $i \in \{1,2,3\}$, $\mathrm{HG}_{s_i}(G[V_i]) < \ell_i$ for $i \in \{1,2,3\}$ by Theorem 4.3.5. Therefore, $\mathrm{HG}_s(G) < \ell_1$, and the proof is complete. $\square$

4.5 Graphs of bounded genus

While it is still unknown whether the hat guessing number of planar graphs is bounded, it is straightforward to show that if $\mathrm{HG}_s(H)$ is bounded for every planar graph H, then $\mathrm{HG}_s(G)$ is also bounded for every graph G of bounded genus. We will use the following lemma of Mohar and Thomassen, which follows from first principles of algebraic topology. For a graph G embedded on a surface S, we say that a cycle C in G is *separating* if $S \setminus C$ has at least two connected components, or equivalently, if C is zero-homologous.

Lemma 4.5.1 ([63]). *Let G be a graph embedded on a surface, and let x and y be two distinct vertices of G. If P_1, P_2, and P_3 are distinct internally disjoint paths with endpoints x and y, and if $P_1 \cup P_2$ and $P_1 \cup P_3$ are both separating cycles, then $P_2 \cup P_3$ is also a separating cycle.*

This lemma will allow us to use a straightforward inductive argument to prove the following theorem.

Theorem 4.5.2. *If f is a function such that every planar graph H satisfies* $\mathrm{HG}_s(H) < f(s)$*, then every graph G of genus g satisfies* $\mathrm{HG}_s(G) < f(3^{2(6^g-1)}s^{6^g})$.

Proof. If $g = 0$, then G is planar, and there is nothing to prove. Otherwise, we assume that $g \geq 1$. Since G has no planar embedding, we must be able to find some non-separating cycle C in G. We choose C to be a shortest non-separating cycle.

Now, we claim that every vertex $v \in V(G) \setminus V(C)$ has at most five neighbors in C. If $|V(C)| \leq 5$, then this claim clearly holds. Otherwise, suppose that C is of length at least six. If v has at least six neighbors in C, then we must be able to choose two neighbors $x, y \in V(C)$ of v that are at a distance of at least three along C. We define the path $P_1 = (x, v, y)$. We also define the path P_2 to be a shortest path from x to y along C, and we define P_3 to be the path with edge set $E(C) \setminus E(P_2)$. We observe that the length of $P_1 \cup P_2$ is at most $\frac{1}{2}|V(C)| + 2 < |V(C)|$, and since P_3 has at most $|V(C)| - 3$ edges, the path $P_1 \cup P_3$ has length at most $|V(C)| - 1$. Since C is a shortest nonseparating cycle, it follows that $P_1 \cup P_2$ and $P_1 \cup P_3$ are both separating cycles. However, then Lemma 4.5.1 implies that $P_2 \cup P_3 = C$ is a separating cycle, a contradiction. Thus our claim holds.

Now, we apply Lemma 4.2.2. We let $A = V(G) \setminus V(C)$ and let $B = V(C)$. Observe that since C is a nonseparating cycle, $G[A]$ has genus at most $g - 1$. Following Lemma 4.2.2, we set $d = 5$. Furthermore, since $G[B]$ has maximum degree 2, it follows from Lemma 4.2.1 that $\mathrm{HG}_s(G[B]) < 3es < 9s$, so we let $s' = (9s)^5 s = 3^{10}s^6$. Then, according to Lemma 4.2.2, $\mathrm{HG}_s(G) \leq \mathrm{HG}_{s'}(G[A])$. By the induction hypothesis,

$$\begin{aligned} HG_{s'}(G[A]) &< f(3^{2(6^{g-1}-1)}s'^{6^{g-1}}) \\ &= f(3^{2\cdot 6^{g-1}-2}(3^{10}s^6)^{6^{g-1}}) \\ &= f(3^{12\cdot 6^{g-1}-2}s^{6^g}) \\ &= f(3^{2(6^g-1)}s^{6^g}). \end{aligned}$$

This completes the proof. □

4.6 Conclusion

While we are not able to prove an upper bound for the hat guessing number of planar graphs, one principle that is clear from our method is that by bounding $\mathrm{HG}_s(G)$ for graphs G in some small graph class, it is often possible to use such a bound along with some vertex partitioning method to obtain an upper bound for the hat guessing number of a larger graph class. Indeed, in order to obtain our upper bound for the hat guessing number of layered planar graphs, we started with an upper bound on $\mathrm{HG}_s(F)$ for forests F, and then we extended this result to an upper bound for petunias, and then we extended this result to an upper bound for outerplanar graphs and finally for layered planar graphs. We hope that our upper bound in Theorem 4.4.2, along with some clever observations, will be enough to bound the hat guessing number of all planar graphs and give an affirmative answer to Question 1.5.1.

4.7 Appendix

Proof of Lemma 4.2.4. We induct on r. When $r = 2$, we must show that if $\mathcal{H}$ is a balanced bipartite graph on $2n$ vertices containing at least $3n^{2-\frac{1}{\ell}}$ edges, then $\mathcal{H}$ contains a copy of $K_{\ell,\ell}$. By a classical theorem of Kővári, Sós, and Turán [58], $\mathcal{H}$ contains a copy of $K_{\ell,\ell}$ as long as $|E(\mathcal{H})| \geq (\ell-1)^{1/\ell}(n-\ell+1)n^{1-1/\ell} + (\ell-1)n$. To show that $3n^{2-\frac{1}{\ell}}$ is greater than this lower bound, we begin with the following inequality, which can easily be verified graphically:

$$\frac{3}{2}\ell^{1-\frac{1}{\ell}} - \ell + 1 > 0.$$

Now, since $(\ell-1)^{1/\ell} < \frac{3}{2}$ for all ℓ, and since $n \geq \ell$, we have

$$(3 - (\ell-1)^{1/\ell})n^{1-\frac{1}{\ell}} - \ell + 1 > 0.$$

Next, since $n > 0$, we have

$$3n^{2-\frac{1}{\ell}} - (\ell-1)^{1/\ell}n^{2-\frac{1}{\ell}} - (\ell-1)n > -(\ell-1)^{\frac{\ell+1}{\ell}}n^{1-\frac{1}{\ell}}.$$

Rearranging this equation gives us

$$3n^{2-\frac{1}{\ell}} > (\ell-1)^{1/\ell}(n-\ell+1)n^{1-\frac{1}{\ell}} - (\ell-1)n,$$

which is exactly what we need to finish the base case. Next, suppose that $r \geq 3$. We will need to borrow a lemma from the original proof of Erdős.

Lemma 4.7.1 ([33]). *Let $S = \{y_1, \dots, y_N\}$ be a set of N elements, and let $A_1, \dots, A_n$ be subsets of S. Let $w > 0$, and assume that $\sum_{i=1}^{n} |A_i| \geq \frac{nN}{w}$. If $n \geq 2\ell^2 w^\ell$, then there exist ℓ distinct sets $A_{i_1}, \dots, A_{i_\ell}$ such that $|A_{i_1} \cap \cdots \cap A_{i_\ell}| \geq \frac{N}{2w^\ell}$.*

Now, suppose we have a balanced r-partite r-graph $\mathcal{H}$ with rn vertices and $t \geq 3n^{r-\frac{1}{\ell^{r-1}}}$ edges. We choose one of the r partite sets of $\mathcal{H}$ and name its vertices $x_1, \dots, x_n$. Next, we set $N = n^{r-1}$, and we let $y_1, \dots, y_N$ denote the set of $(r-1)$-tuples of vertices that can be obtained by choosing exactly one vertex from each partite set of $\mathcal{H}$ outside of $\{x_1, \dots, x_r\}$. Then, for each x_i, we let A_i

contain those y_j for which $x_i \cup y_j \in E(\mathcal{H})$. We have

$$\sum_{i=1}^{n} |A_i| = t \geq 3n^{r-\frac{1}{\ell^{r-1}}}.$$

We set $w = \frac{1}{3}n^{\frac{1}{\ell^{r-1}}}$, and then it is easy to verify that $t \geq \frac{nN}{w}$ and that $n \geq 2\ell^2 w^\ell$, so the hypotheses of Lemma 4.7.1 hold. Hence, we may choose ℓ vertices $x_{i_1}, \ldots, x_{i_\ell}$ whose neighborhoods intersect in at least

$$\frac{N}{2w^\ell} = \frac{3^\ell}{2}n^{r-\frac{1}{\ell^{r-2}}} > 3n^{r-\frac{1}{\ell^{r-2}}}$$

$(r-1)$-tuples. Then, by the induction hypothesis, we may find a copy of $K_\ell^{(r-1)}$ among these $(r-1)$-tuples, and this $K_\ell^{(r-1)}$ along with the vertices $x_{i_1}, \ldots, x_{i_\ell}$ form a copy of $K_\ell^{(r)}$. This completes the proof. □

Proof of estimates in Theorem 4.4.2. Recall that $\ell_5 = 20s$ and $s_4 = (20)^6 s^7$. We will use the inequalities

$$(3s^2+3s+3)^{3(s^2+s+1)^2}+1 \quad < \quad 2^{(3s)^5} \tag{4.1}$$

$$(3(s+1)^6+3(s+1)^3+3)^{3((s+1)^6+(s+1)^3+1)^2}+1 \quad < \quad 2^{(3s)^{13}} \tag{4.2}$$

for $s \geq 1$. From (4.1), we see that

$$\log_2 \ell_4 < (3s_4)^5 = 3^5 2^{60} 5^{30} s^{35} < 2^{138} s^{35}.$$

Then,

$$\log_2 s_3 = \log s + 6\log_2 \ell_4 < 2^{141} s^{35}.$$

Then, using (4.2), $\log_2 \ell_3 < (3s_3)^{13}$, and so

$$\log_2 \log_2 \ell_3 < 13\log_2 3 + 13\log_2 s_3 < 2^{145} s^{35}.$$

The remaining bounds can be proven similarly using (4.2). □

Chapter 5

The graph coloring game

5.1 Introduction

Recall that the graph coloring game is a game in which Alice and Bob take turns coloring the vertices of a graph G with colors from the set $\{1, \ldots, k\}$, with Alice moving first. Alice's goal is to complete a proper coloring of G, and Bob's goal is to prevent Alice from doing so. The *game chromatic number* of G, written $\chi_g(G)$, is the minimum value k for which Alice has a winning strategy in the graph coloring game played on G with the color set $\{1, \ldots, k\}$.

Recall also that the *graph marking game* is a game in which Alice and Bob take turns marking the vertices of a graph G with a black pen. The *score* of an uncolored vertex v at some state of the game is equal to the number of neighbors of v that have already been marked. (A marked vertex can be said to have score 0.) We say that the *game coloring number* of G, written $\mathrm{col}_g(G)$, is the minimum value t for which Alice has a strategy in the graph marking game that limits the score of every vertex $v \in V(G)$ to $t - 1$ throughout the entire game. A greedy coloring argument shows that $\chi_g(G) \leq \mathrm{col}_g(G)$.

This chapter will be organized as follows. In Section 5.2, we prove that a properly colored graph whose bicolored subgraphs have bounded game coloring number must have a bounded game chromatic number, and we list a number of corollaries. Then, in Section 5.3, we apply the method of Section 5.2 to calculate upper bounds on the game chromatic numbers of certain graph products, namely the Cartesian product and the strong product of two graphs. In doing so, we will answer a question of Zhu [82]. Finally, in Section 5.4, we pose some questions.

5.2 Bounding χ_g with the game coloring number of bicolored subgraphs

In this section, we will show that the game chromatic number of a properly colored graph G may be bounded by a function of the number of colors used to color G and the game coloring numbers of the bicolored subgraphs of G. Dinski and Zhu [29] show that for a graph G, $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$, where $\chi_a(G)$ is the acyclic chromatic number of G. We will follow the ideas of Dinski and Zhu to prove a more general upper bound on $\chi_g(G)$ in terms of the game coloring numbers of the bicolored subgraphs of G with respect to some proper coloring.

We consider a slight variation of the graph marking game, which we name the *Bob marking game*. In the Bob marking game on a graph G, Alice and Bob play on G by the same rules as those in the graph marking game, but Alice marks with a red pen, and Bob marks with a blue pen. In the Bob marking game, we let Bob move first. When a play of the game is finished, for each vertex $v \in V(G)$, we define the score of v as the number of neighbors of v marked in blue at the time v was marked. In other words, only the neighbors of v marked by Bob contribute to the score of v. Then, for a play of the Bob marking game on G, we say that the score of the play is equal to the maximum score over all vertices of $V(G)$, plus one. We say that the value $\text{Bob}(G)$ is equal to the minimum integer t for which Alice has a strategy to limit the score of a play of the Bob marking game on G to t. Defining $\text{col}_g^B(G)$ to be the lowest score achievable by Alice in the traditional marking game on G with optimal play when Bob moves first, it is clear that $\text{Bob}(G) \leq \text{col}_g^B(G)$. Furthermore, Zhu [82] shows that $\text{col}_g^B(G) \leq \text{col}_g(G) + 1$, so it follows that $\text{Bob}(G) \leq \text{col}_g(G) + 1$.

It is worth giving an example of a graph G for which $\text{Bob}(G) < \text{col}_g^B(G)$ in order to show that these parameters are indeed different. Bodlaender [15] shows that there exists forests F for which $\chi_g(F) = \text{col}_g^B(F) = 4$. In contrast, we will prove that $\text{Bob}(F) \leq 3$ holds for every forest F using the following strategy for Alice, which is used implicitly by Dinski and Zhu [29]. At a given state of the Bob marking game on F, let F' denote the subgraph of F that is obtained by removing from F the vertices marked in red by Alice, as well as the edges whose endpoints are both marked in blue by Bob. We will show that at the end of each of Alice's turns, she can ensure that at most one vertex from each component of F' is marked in blue by Bob. Alice can certainly ensure that this condition holds at the end of her first turn. Now, suppose that the condition holds at the end of Alice's ith turn. On Bob's $(i+1)$th turn, Bob chooses a component K of F' and marks a vertex $v \in K$ blue. If v is the only blue vertex in K, then Alice marks an arbitrary vertex, and the condition is satisfied again at the end of Alice's $(i+1)$th turn. Otherwise, there exists a single other blue vertex $w \in K$. If w is a neighbor of v, then the edge vw is removed from F'. Then, Alice marks an arbitrary vertex, and the condition holds again at the end of Alice's $(i+1)$th turn. On the other hand, if w is not a neighbor of v, then there exists a unique path P in K connecting v and w with at least one internal vertex. Alice marks an internal vertex u of P, and then since u is removed from F', the condition again holds at the end of Alice's $(i+1)$th turn. Now, if $\text{Bob}(F) \geq 4$, then at some point in the game, an unmarked vertex u must have three blue neighbors, and u along with these three blue neighbors belong to a single component of F'. However, Alice's strategy ensures that at any point in the game, a component of F' contains at most two blue vertices, giving us a contradiction. Therefore, $\text{Bob}(F) \leq 3$.

Now, consider a graph G for which $\text{Bob}(G) \leq t$. In a play of the Bob marking game on G, Alice has a strategy in which every vertex $v \in V(G)$ is marked before the number of blue marked neighbors of v exceeds $t-1$. We say that Alice's strategy on G with respect to the bound $\mathrm{Bob}(G) \leq t$ is *reactive* if for each vertex v, if v ever has $t-1$ blue marked neighbors after Bob's move, then Alice marks v immediately. For example, the strategy for forests F described above is reactive with respect to the bound $\mathrm{Bob}(F) \leq 3$, because if Bob ever marks two neighbors of an unmarked vertex u, then Alice will immediately mark u. If Alice plays a strategy on G to limit the score of each vertex to $t-1$, then the only way that Alice's strategy would not be reactive would be if Alice were to allow a vertex to remain unmarked when all of its neighbors were marked, with exactly $t-1$ neighbors marked in blue. Indeed, if an unmarked vertex v has $t-1$ blue marked neighbors and at least one unmarked neighbor on Bob's turn, then Bob can achieve a score of $t+1$ on G by marking an additional neighbor of v, so in any successful strategy, Alice would need to mark v to prevent its score from increasing. Most strategies that we consider for a graph G that give a bound of the form $\mathrm{Bob}(G) \leq t$ will be reactive, as it is not usually convenient to try to ensure that all neighbors of an unmarked vertex v are marked, and it is usually easier for Alice just to mark a vertex v in order to prevent its score from increasing.

The following theorem generalizes the method of Dinski and Zhu [29] originally used to prove that for any graph G, $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$. The method of Dinski and Zhu considers an acyclically colored graph G, and using the acyclical coloring of G, these authors devise a winning strategy for Alice in the graph coloring game on G. Using the strategy above, Dinski and Zhu implicitly show that $\mathrm{Bob}(F) \leq 3$ holds for every forest F, and they essentially use the fact that every bicolored subgraph H of G satisfies $\mathrm{Bob}(H) \leq 3$ to devise their strategy. The following theorem, however, shows that in order to bound the game chromatic number of a properly colored graph G, it is enough simply to ensure that $\mathrm{Bob}(H)$ is bounded for every bicolored subgraph of G. We use the term *k-coloring* to refer to a proper graph coloring using k colors.

Theorem 5.2.1. *Let G be a graph with a k-coloring ϕ, and suppose that every bicolored subgraph H of G with respect to ϕ satisfies* $\mathrm{Bob}(H) \leq t$. *If Alice has a reactive strategy with respect to each graph H and the bound* $\mathrm{Bob}(H) \leq t$, *then*

$$\chi_g(G) \leq k((k-1)(t-2)+2).$$

Proof. Let ϕ be a proper coloring of G using k colors that satisfies the assumptions of the theorem. In order to show that $\chi_g(G) \leq k((k-1)(t-2)+2)$, we must show that Alice has a winning strategy in the graph coloring game using $k((k-1)(t-2)+2)$ colors. We will define a set C of $k((k-1)(t-2)+2)$ values with which Alice and Bob will play the graph coloring game, and to avoid confusion, we will refer to the values in C as *shades*, rather than colors. That is, on each turn, we will let Alice or Bob assign a shade from C to a vertex of G that has not already been assigned a shade. On the other hand, we will refer to the k values in the image of ϕ as *colors*. We will partition C into k parts of size $(k-1)(t-2)+2$, and we will say that for each color c used by ϕ, C contains $(k-1)(t-2)+2$ shades of c.

For two colors c and d, let $G_{c,d} \subseteq G$ be the subgraph of G induced by the vertices of $V(G)$ that are colored with c and d by ϕ. Let $S_{c,d}$ be a reactive strategy of the marking game on $G_{c,d}$ by which Alice can limit the score of any vertex of $G_{c,d}$ to $t-1$ in the Bob marking game. We will describe Alice's strategy for the coloring game on G. In Alice's strategy, Alice will always color some vertex $v \in V(G)$ with a shade of $\phi(v)$. We will sometimes allow Alice to choose an arbitrary vertex v to assign a shade of $\phi(v)$, and in this case, we say that Alice plays an *idle move*.

As Alice plays the game, Alice will in fact consider $\binom{k}{2}$ different Bob marking games played on the graphs $G_{c,d}$, one for each color pair $c, d \in \phi(V(G))$. Each time Bob makes a move, Alice will consider Bob's move to be a move in a Bob marking game on one of the subgraphs $G_{c,d}$. Alice will calculate a response to Bob's move in the Bob marking game on $G_{c,d}$ using the strategy $S_{c,d}$, and based on Alice's response in the Bob marking game on $G_{c,d}$, Alice will respond to Bob's move in the coloring game on G.

Alice's strategy is as follows. Alice begins the game with an idle move. On each of Bob's turns, if Bob chooses a vertex $v \in V(G)$ and colors v with a shade of $\phi(v)$, then Alice responds by playing an idle move. If Bob colors a vertex v with a shade c that is not one of the shades of $\phi(v)$, then Alice considers Bob's move as if it were a move in the Bob marking game on $G_{c,\phi(v)}$. Alice then uses $S_{c,\phi(v)}$ to choose a vertex $w \in V(G)$ to mark in response to Bob's move in the Bob marking game on $G_{c,\phi(v)}$. Then, in the coloring game on G, Alice colors w with any available shade of $\phi(w)$. If w has already been colored, then Alice plays an idle move. Alice repeats this process for each of Bob's moves.

We now show that Alice's strategy always succeeds in producing a proper coloring of G. Suppose that on some turn, Alice attempts to color a vertex v with a shade of $\phi(v)$. For any neighbor w of v that is colored with a shade of $\phi(v)$, w must have been colored by Bob. Equivalently, w must have been marked by Bob in the Bob marking game on $G_{\phi(v),\phi(w)}$. However, Alice has used the strategy $S_{\phi(v),\phi(w)}$ to ensure that Bob does not mark more than $t-1$ neighbors of an unmarked vertex in the Bob marking game on $G_{\phi(v),\phi(w)}$. Therefore, for each color $c \in \phi(V(G))$ that appears in the neighborhood of v, at most $t-1$ vertices $w \in N(v)$ with $\phi(w) = c$ have been colored by Bob with a shade of $\phi(v)$. Furthermore, as the strategy $S_{\phi(v),\phi(w)}$ is reactive, there exists at most one color c^* for which $t-1$ vertices $w \in N(v)$ with $\phi(w) = c^*$ have been colored with a shade of $\phi(v)$, and this color c^* must satisfy $c^* = \phi(w^*)$, where $w^* \in N(v)$ is the vertex that has just been colored by Bob with a shade of $\phi(v)$ on the last move. For all other colors $c \in \phi(V(G))$, at most $t-2$ neighbors $w \in N(v)$ with $\phi(w) = c$ have been colored by Bob with a shade of $\phi(v)$. This implies that the total number of shades of $\phi(v)$ that appear in the neighborhood of v is at most $(t-2)(k-1)+1$. As Alice has $(t-2)(k-1)+2$ shades of $\phi(v)$ to use, Alice thus has an available shade of $\phi(v)$ to use at v. Hence, Alice's strategy succeeds at every move.

As Alice always succeeds in coloring a vertex of G with a shade from C on her turn, the only way that G would not be properly colored would be if Bob were unable to color any vertex of G on some turn, in which case the coloring game would end prematurely with Alice losing. However, Bob may always "pretend to be Alice" and successfully color a vertex of $V(G)$ with an idle move using the previous argument. Therefore, Bob also always has a legal move, and hence G is properly colored. $\square$

We make several observations about Theorem 5.2.1 and its proof. First, we have defined the graph coloring game with Alice moving first, but it is easy to see that the strategy of Theorem 5.2.1 works regardless of which player moves first. Second, the upper bound on $\chi_g(G)$ from Theorem 5.2.1 also holds for any subgraph of G, as removing edges from G does not make the strategy any more difficult for Alice, and if a vertex of G that Alice wishes to color is not present in some subgraph, then Alice may simply play an idle move. Third, while the Bob marking game is not a standard part of the literature, the inequality $\mathrm{Bob}(H) \leq \mathrm{col}_g^B(H) \leq \mathrm{col}_g(H) + 1$ implies that we can replace the condition $\mathrm{Bob}(H) \leq t$ of Theorem 5.2.1 with a bound using more standard parameters. Finally, if $\mathrm{Bob}(H) \leq t$ holds for every bicolored subgraph H of G, but Alice does not necessarily have a reactive strategy with respect to these bounds, then a very similar argument gives the following upper bound, which is only slightly worse than the bound in Theorem 5.2.1.

Corollary 5.2.2. *Let G be a graph with a k-coloring ϕ, and suppose that every two-colored subgraph H of G with respect to ϕ satisfies* $\mathrm{Bob}(H) \leq t$. *Then*

$$\chi_g(G) \leq k((k-1)(t-1)+1).$$

We note that the strategy of Zhu [82] used to bound the game chromatic number of graph Cartesian products bears some resemblance to the strategy of Theorem 5.2.1, as Zhu explicitly devises a single graph coloring strategy by combining many graph marking strategies on smaller subgraphs. However, the strategy of Zhu still relies on acyclic colorings, so the strategy of Theorem 5.2.1 is the first strategy, to the best of our knowledge, that uses more general bicolored subgraphs.

Theorem 5.2.1 has a number of corollaries. First, the upper bound of Dinski and Zhu [29] follows immediately.

Corollary 5.2.3. *For every graph G, $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$.*

Proof. We have shown previously that $\mathrm{Bob}(F) \leq 3$ holds for every forest F, and furthermore, that Alice has a strategy that is reactive with respect to this bound. In an acyclic coloring on G, every bicolored subgraph on G is a forest, so letting $k = \chi_a(G)$ and $t = 3$ in Theorem 5.2.1 yields the result. $\square$

Additionally, a number of similar upper bounds follow.

Corollary 5.2.4. *Let G be a graph with a proper k-coloring in which every bicolored subgraph has treewidth at most w. Then $\chi_g(G) \leq k(3w(k-1)+2)$.*

Proof. Zhu [81] shows that for a graph H of treewidth at most w, $\mathrm{col}_g^B(H) \leq 3w + 2$, and furthermore, the strategy for Alice that Zhu gives is reactive with respect to this bound. Therefore, letting $t = 3w + 2$ in Theorem 5.2.1 yields the result. $\square$

Corollary 5.2.5. *Let G be a graph with a proper k-coloring in which every bicolored subgraph is planar. Then $\chi_g(G) \leq k(15k - 13)$.*

Proof. Zhu [83] shows that for a planar graph H, $\mathrm{col}_g^B(H) \leq 17$, and furthermore, the strategy for Alice that Zhu gives is reactive with respect to this bound. Therefore, letting $t = 17$ in Theorem 5.2.1 yields the result. $\square$

Corollary 5.2.6. *Let G be a graph with a proper k-coloring in which every bicolored subgraph is of genus at most g. Then $\chi_g(G) \leq k((k-1)\lfloor \frac{1}{2}(3 + \sqrt{1 + 48g} + 19)\rfloor + 2)$.*

Proof. Zhu [81] shows that for a graph H of genus at most g, $\mathrm{col}_g^B(H) \leq \lfloor \frac{1}{2}(3+\sqrt{1+48g}+23)\rfloor$, and furthermore, the strategy for Alice that Zhu gives is reactive with respect to this bound. Therefore, letting $t = \lfloor \frac{1}{2}(3 + \sqrt{1 + 48g} + 23)\rfloor$ in Theorem 5.2.1 yields the result. $\square$

It is natural to ask whether these upper bounds for the game chromatic number of a graph obtained using the method of Theorem 5.2.1 are optimal. Giving an overall answer to this question is difficult, as graph colorings in which bicolored subgraphs have bounded game coloring number have not yet received any attention. It is known, however, that Corollary 5.2.3 often does not give a tight upper bound. For instance, using the fact that a planar graph has an acyclic chromatic number of at most 5 [16], Corollary 5.2.3 implies that a planar graph has a game chromatic number of at most 30, a result shown in [29], but as stated, a different method of Zhu [83] shows that the game chromatic number of a planar graph is in fact at most 17.

5.3 The Cartesian product and strong product of graphs

In this section, we will show that Theorem 5.2.1 may be used to calculate an upper bound on certain graph products, namely the Cartesian product of two graphs and the strong product of two graphs. We first consider the Cartesian products of two graphs, which we define as follows. Given two graphs G_1 and G_2, the *Cartesian product* of G_1 and G_2, written $G_1 \square G_2$, is defined as the graph on the vertex set $V(G_1) \times V(G_2)$ in which two vertices (u, v) and (u', v') are adjacent if and only if either $u = u'$ and $v \sim v'$ in G_2, or $v = v'$ and $u \sim u'$ in G_1, where $\sim$ represents adjacency. An example of the Cartesian product of two graphs is shown in Figure 5.1. In [82], Zhu calculates an upper bound on the game chromatic number of the Cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2, but Zhu's upper bound relies on the acyclic chromatic number of one of the graphs and the game coloring number of a modified form of the other graph. Using Theorem 5.2.1, however, we may show that $\chi_g(G_1 \square G_2)$ may be bounded above only by $\mathrm{col}_g(G_1)$ and $\mathrm{col}_g(G_2)$. Recall that

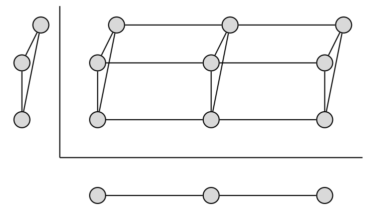

Figure 5.1: The figure shows a K_3, a path of length 2, and the Cartesian product of these two graphs.

for a graph G, we define $\mathrm{col}_g^B(G)$ to be the lowest score achievable by Alice in the graph marking game on G with optimal play when Bob moves first.

Theorem 5.3.1. *Let G_1 and G_2 be graphs. Let $k = \chi(G_1)\chi(G_2)$, and let $t = \max\{\mathrm{col}_g^B(G_1), \mathrm{col}_g^B(G_2)\}$. Then*

$$\chi_g(G_1 \square G_2) \leq k((k-1)(t-1)+1).$$

Proof. Let $\phi_1 : E(G_1) \to \{1, 2, \ldots, \chi(G_1)\}$ be a proper coloring of G_1, and let $\phi_2 : E(G_2) \to \{1, 2, \ldots, \chi(G_2)\}$ be a proper coloring of G_2. For each pair $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, we may color the corresponding vertex $(v_1, v_2) \in V(G_1 \square G_2)$ with the color $(\phi_1(v_1), \phi_2(v_2))$, which gives us a proper coloring

$$\phi : E(G_1 \square G_2) \to \{1, 2, \ldots, \chi(G_1)\} \times \{1, 2, \ldots, \chi(G_2)\}$$

using k colors.

We claim that each connected bicolored subgraph H of $G_1 \square G_2$ under ϕ satisfies $\mathrm{col}_g^B(H) \leq t$. Indeed, let $H \subseteq G_1 \square G_2$ be a connected bicolored subgraph with respect to ϕ. If H is a single vertex, then $\mathrm{col}_g^B(H) = 1$; otherwise, H has at least one edge e. We assume without loss of generality that e has endpoints $(u, v_1), (u, v_2)$, where $u \in V(G_1)$, and $v_1, v_2 \in V(G_2)$, and hence that H is colored with the colors $(\phi_1(u), \phi_2(v_1))$ and $(\phi_1(u), \phi_2(v_2))$. If every vertex of H is of the form (u, v) for some $v \in V(G_2)$, then H is isomorphic to a subgraph of G_2, and hence $\mathrm{col}_g^B(H) \leq t$. Otherwise, as H is connected, H must contain a vertex of the form (u', v), where $u' \in V(G_1)$ is a neighbor of u in G_1, and $v \in V(G_2)$ is any vertex in G_2. However, as u and u' are neighbors, $\phi_1(u) \neq \phi_1(u')$, so $\phi(u', v)$ cannot be one of $(\phi_1(u), \phi_2(v_1))$ and $(\phi_1(u), \phi_2(v_2))$, a contradiction to the assumption that H is bicolored. Therefore, H is isomorphic to a subgraph of G_2, and $\mathrm{Bob}(H) \leq \mathrm{col}_g^B(H) \leq t$. The same upper bound holds even if H is not connected, as $\mathrm{col}_g^B(H)$ is equal to the maximum value $\mathrm{col}_g^B(H')$ over all components H' of H. As Alice does not necessarily have a reactive strategy with respect to the game coloring numbers of G_1 and G_2, we apply Corollary 5.2.2 with our values k and t, and we obtain an upper bound of $\chi_g(G_1 \square G_2) \leq k((k-1)(t-1)+1)$, which completes the proof. $\square$

We note that given r graphs $G_1, \ldots, G_r$, we may use the same method to obtain the upper bound $\chi_g(G_1 \square \cdots \square G_r) \leq k((k-1)(t-1)+1)$, where we let $k = \chi(G_1) \cdots \chi(G_r)$ and $t = \max\{\mathrm{col}_g^B(G_1), \ldots, \mathrm{col}_g^B(G_r)\}$.

Corollary 5.3.2. *Let G_1, G_2 be graphs, and let $t = \max\{\mathrm{col}_g(G_1), \mathrm{col}_g(G_2)\}$. Then*

$$\chi_g(G_1 \square G_2) \leq t^2((t^2-1)t+1) = t^5 - t^3 + t^2.$$

Proof. The bound follows directly from Theorem 5.3.1 after applying the inequalities $\chi(G_i) \leq \chi_g(G_i) \leq \mathrm{col}_g(G_i) \leq t$ and $\mathrm{col}_g^B(G_i) \leq t+1$ for $i = 1, 2$. $\square$

Corollary 5.3.2 answers a question of Zhu [82] asking if $\chi_g(G_1 \square G_2)$ is bounded whenever $\mathrm{col}_g(G_1)$ and $\mathrm{col}_g(G_2)$ are bounded. Zhu asks this question for the graph coloring game in which Bob moves first, but the original strategy from Theorem 5.2.1 works the same regardless of which player moves first. The upper bounds of Theorem 5.3.1 and Corollary 5.3.2 are often far from tight, however. For example, Theorem 5.3.1 tells us that the game chromatic number of the Cartesian product of two planar graphs is at most $16(15 \cdot 16 + 1) = 3856$, but using a different method, Zhu [82] obtains a sharper upper bound of 105. Furthermore, in the following example, we show two graphs G_1 and G_2 for which the Cartesian product $G_1 \square G_2$ has a game chromatic number equal to the trivial lower bound of $\chi(G_1 \square G_2) = \max\{\chi(G_1), \chi(G_2)\}$, which is far from the upper bound given in Theorem 5.3.1.

For an even integer $n \geq 2$, let G_1 be the union of a complete graph K_n and a single isolated vertex, and let G_2 be the union of an edge K_2 and a single isolated vertex. We illustrate G_1, G_2, and their Cartesian product in Figure 5.2. $G_1 \square G_2$ has four components: a K_n component, a $K_n \square K_2$ component, a single vertex component, and a K_2 component. We observe that $\chi(G_1 \square G_2) = n$, and we will show that $\chi_g(G_1 \square G_2) = n$ by giving a strategy using n colors with which Alice may win the graph coloring game on $G_1 \square G_2$. In comparison, the upper bound for $\chi_g(G_1 \square G_2)$ given by Theorem 5.3.1 is $4n^3 - 2n^2 + 2n$, which is far from optimal.

Alice's strategy is as follows. On the first move, Alice colors the isolated vertex of $G_1 \square G_2$ with any color. Then, whenever Bob colors a vertex in a component C of $G_1 \square G_2$, Alice colors a vertex of C on the next move. As each component of $G_1 \square G_2$ of size at least 2 has an even number of vertices, Alice will always be able to respond to Bob by coloring a vertex in the same component that Bob just colored, provided that each uncolored vertex still has a legal color. Therefore, in order to show that $\chi_g(G_1 \square G_2) = n$, it suffices to show that Alice wins the coloring game with n colors on each component of $G_1 \square G_2$ of size at least 2 when Bob moves first.

It is clear that Alice wins the coloring game on K_2 and K_n with n colors when Bob moves first; thus, we will only explicitly describe Alice's strategy for winning the coloring game on $K_n \square K_2$. Let $K_n \square K_2$ have $2n$ vertices $u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}$, so that $u_i \sim u_j$ and $v_i \sim v_j$ for each pair $0 \leq i < j \leq n-1$, and so that $u_i \sim v_i$ for each $0 \leq i \leq n-1$. Alice will play as follows. Whenever

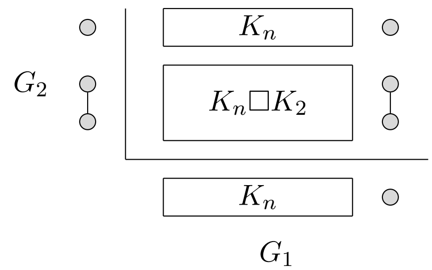

Figure 5.2: The figure shows two graphs G_1 and G_2 along with their Cartesian product $G_1 \square G_2$. In this example, $\chi_g(G_1 \square G_2) = \chi(G_1 \square G_2)$, showing that the upper bound in Theorem 5.3.1 may be far from optimal.

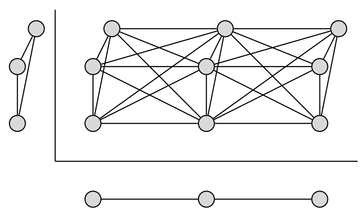

Figure 5.3: The figure shows a K_3, a path of length 2, and the strong product of these two graphs.

Bob colors a vertex u_i with a color c, Alice will respond by coloring v_{i+1} with c, and whenever Bob colors a vertex v_i with a color c, Alice will respond by coloring u_{i-1} with c, with addition calculated modulo n. It is easy to check that after each of Alice's turns, the partial coloring on $u_0, \dots, u_{n-1}$ is equal to the partial coloring at $v_0, \dots, v_{n-1}$, but "shifted down" by one. Therefore, Alice's strategy always gives her a legal move, and together Alice and Bob will complete a proper coloring of $K_n \square K_2$ using n colors. Therefore, $\chi_g(G_1 \square G_2) = \chi(G_1 \square G_2) = n$, which is much smaller than the upper bound we would obtain from Theorem 5.3.1.

Next, Theorem 5.2.1 allows us to establish the following result about the strong product of two graphs. Given two graphs G_1 and G_2, the *strong product* of G_1 and G_2, written $G_1 \boxtimes G_2$, is defined as the graph on the vertex set $V(G_1) \times V(G_2)$ in which two vertices (u, v) and (u', v') are adjacent if and only if both of the following hold:

- $u = u'$, or $u \sim u'$ in G_1;
- $v = v'$, or $v \sim v'$ in G_2.

An example of the strong product of two graphs is illustrated in Figure 5.3. Furthermore, given a graph G, the *square* of G, written G^2, is defined as the graph on $V(G)$ in which two distinct vertices $u, v \in V(G)$ are adjacent in G^2 if and only if u and v are at a distance of at most 2 in G. With these definitions in place, we have the following result.

Theorem 5.3.3. *Let G_1 and G_2 be graphs, let $t = \mathrm{col}_g(G_1)$, and let $k = \chi(G_1)\chi(G_2^2)$. Then*

$$\chi_g(G_1 \boxtimes G_2) \leq k((k-1)t+1).$$

Proof. Let $\phi_1 : V(G_1) \to \{1, 2, \ldots, \chi(G_1)\}$ be a proper coloring of G_1, and let $\phi_2 : V(G_2) \to \{1, 2, \ldots, \chi(G_2^2)\}$ be a proper coloring of G_2^2. As in Theorem 5.3.1, we define a proper coloring

$$\phi : E(G_1 \boxtimes G_2) \to \{1, 2, \ldots, \chi(G_1)\} \times \{1, 2, \ldots, \chi(G_2^2)\}$$

using k colors by coloring each vertex $(u, v) \in G_1 \boxtimes G_2$ such that $\phi(u, v) = (\phi_1(u), \phi_2(v))$.

If G_1 is an independent set, then $G_1 \boxtimes G_2$ consists of copies of G_2, so $\chi_g(G_1 \boxtimes G_2) \leq \Delta(G_2)+1 \leq \chi(G_2^2)$, as G_2^2 has a clique of size $\Delta(G_2)+1$. Hence, the theorem holds in this case, and we thus assume that G_1 has at least one edge, and hence that $t \geq 2$.

Consider a connected bicolored subgraph H of $G_1 \boxtimes G_2$ with respect to ϕ. We aim to show that $\mathrm{col}_g(H) \leq t$. If H contains no edge, then $\mathrm{col}_g(H) = 1$. If H contains an edge of the form $(u_1, v)(u_2, v)$ for vertices $u_1, u_2 \in V(G_1)$ and $v \in V(G_2)$, then by the argument of Theorem 5.3.1, H is isomorphic to a subgraph of G_1, and hence $\mathrm{col}_g(H) \leq t$. Similarly, if H contains an edge of the form $(u, v_1)(u, v_2)$ for vertices $u \in V(G_1)$ and $v_1, v_2 \in V(G_2)$, then by the argument of Theorem 5.3.1, H is isomorphic to a subgraph of G_2. However, as ϕ_2 is a proper coloring of G_2^2, v_2 is the only neighbor of v_1 in G_2 with color $\phi_2(v_2)$, and v_1 is the only neighbor of v_2 in G_2 with color $\phi_2(v_1)$. Hence, H must be isomorphic to K_2, and $\mathrm{col}_g(H) = 2 \leq t$.

Finally, suppose H contains an edge of the form $(u_1, v_1)(u_2, v_2)$ for two adjacent vertices $u_1, u_2 \in V(G_1)$ and two adjacent vertices $v_1, v_2 \in V(G_2)$. Again, as ϕ_2 is a proper coloring of G_2^2, v_1 and v_2 must be the only vertices of G_2 that appear as the second entry in an element of $V(H)$. Furthermore, as ϕ_1 is a proper coloring of G_1, every edge of H must be of the form $(u, v_1)(u', v_2)$, where $u, u' \in V(G_1)$ may be any distinct pair of adjacent vertices in G_1. We recall that H is colored with two colors and hence that H is bipartite. Therefore, H is isomorphic to the subgraph $G' \subseteq G_1$ induced by the vertices $u \in V(G_1)$ that appear in some pair $(u, v_i) \in V(H)$, where $i \in \{1, 2\}$, and we see that the index i of the pair (u, v_i) in which a vertex $u \in V(G')$ appears indicates to which partite set of G' the vertex u belongs. Hence, $\mathrm{col}_g(H) \leq t$, and furthermore, $\mathrm{col}_g^B(H) \leq t+1$.

In each case, the bound $\mathrm{col}_g^B(H) \leq t+1$ holds even when H is not connected, as the value of $\mathrm{col}_g^B(H)$ is equal to the maximum value $\mathrm{col}_g^B(H')$ over all components H' of H, and $\mathrm{col}_g^B(H') \leq \mathrm{col}_g(H')+1 \leq t+1$. Hence, we have a proper coloring ϕ of G using k colors in which $\mathrm{col}_g^B(H) \leq t+1$ holds for each bicolored subgraph H of G. Then, the result follows from Corollary 5.2.2. ☐

Theorem 5.3.3 has the following corollary, which shows that a the strong product of a graph with bounded game coloring number and a second graph of bounded degree must have bounded game chromatic number.

Corollary 5.3.4. *Let G be a graph, and let G' be a graph of maximum degree Δ. Then*

$$\chi_g(G \boxtimes G') \leq \chi(G)^2(\Delta^2+1)^2 \operatorname{col}_g(G) \leq (\Delta^2+1)^2 \operatorname{col}_g(G)^3.$$

Proof. The chromatic number of the square of G' is at most $\Delta^2 + 1$, so the result follows from Theorem 5.3.3 by letting $t = \operatorname{col}_g(G)$, using the fact that $k \leq \chi(G)(\Delta^2 + 1)$, and noting that the upper bound of Theorem 5.3.3 is at most $k^2 t$. □

Corollary 5.3.4 tells us, for instance, that the strong product of any graph G with a cubic graph has a game chromatic number of at most $100 \operatorname{col}_g(G)^3$, and that the strong product of a planar graph G with a graph of maximum degree Δ has a game chromatic number of at most $4^2 \cdot 17(\Delta^2+1)^2 = 272(\Delta^2+1)^2$. However, Theorem 5.3.3 and Corollary 5.3.4 are likely far from best possible. Furthermore, if we consider two complete graphs K_m and K_n, we see that $\chi_g(K_m \boxtimes K_n) = \chi(K_m \boxtimes K_n) = mn$, so it is possible for the strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 to have a game chromatic number equal to the trivial lower bound of $\chi(G_1 \boxtimes G_2)$, which is far from the upper bound of Theorem 5.3.3.

5.4 Conclusion

We have shown in Corollary 5.3.2 that if two graphs G_1 and G_2 have their game coloring numbers bounded by a constant, then $\chi_g(G_1 \square G_2)$ is also bounded by a constant. It seems natural to try to strengthen this result by asking whether $\operatorname{col}_g(G_1 \square G_2)$ is also bounded by a constant; however, Bartnicki et al. [12] have shown $\operatorname{col}_g(G_1 \square G_2)$ is unbounded when $G_1 = G_2 = K_{1,n}$, while $\operatorname{col}_g(K_{1,n}) \leq 4$. Similarly, it is natural to ask whether the bounds on $\operatorname{col}_g(G_1)$ and $\operatorname{col}_g(G_2)$ in the hypothesis can be replaced by bounds on $\chi_g(G_1)$ and $\chi_g(G_2)$. This fails as well, however, as Barnicki et al. [12] show that while $\chi_g(K_{n,n}) \leq 3$ for all n, there exist values k and m for each integer t such that $\chi_g(K_{k,k} \square K_{m,m}) > t$.

On the other hand, we have shown in Corollary 5.3.4 that given a graph G_1 of bounded game coloring number and a graph G_2 of bounded degree, $\chi_g(G_1 \boxtimes G_2)$ is bounded by a constant. However, the following question remains open, which could strengthen Corollary 5.3.2 and Corollary 5.3.4.

Question 5.4.1. *Let G_1 and G_2 be graphs, and suppose that $\operatorname{col}_g(G_1)$ and $\operatorname{col}_g(G_2)$ are both bounded by a constant. Is it true that $\chi_g(G_1 \boxtimes G_2)$ is bounded by a constant?*

Bibliography

[1] Dimitris Achlioptas and Amin Coja-Oghlan. Algorithmic barriers from phase transitions. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 793–802, 2008.

[2] Ron Aharoni, Eli Berger, Maria Chudnovsky, Frédéric Havet, and Zilin Jiang. Cooperative colorings of trees and bipartite graphs. *Electron. J. Combin.*, 27(1):Paper 1.41, 2020.

[3] Ron Aharoni, Eli Berger, Maria Chudnovsky, Frédéric Havet, and Zilin Jiang. Cooperative colorings of trees and of bipartite graphs. *Electron. J. Combin.*, 27(1):Paper No. 1.41, 9, 2020.

[4] Ron Aharoni, Ron Holzman, David Howard, and Philipp Sprüssel. Cooperative colorings and independent systems of representatives. *Electron. J. Combin.*, 22(2):Paper 2.27, 14, 2015.

[5] Sameer Al-Dahidi, Osama Ayadi, Jehad Adeeb, and Mohamed Louzazni. Assessment of artificial neural networks learning algorithms and training datasets for solar photovoltaic power production prediction. *Frontiers in Energy Research*, 7(130):1–18, 2019.

[6] Noga Alon. Restricted colorings of graphs. In *Surveys in combinatorics, 1993 (Keele)*, volume 187 of *London Math. Soc. Lecture Note Ser.*, pages 1–33. Cambridge Univ. Press, Cambridge, 1993.

[7] Noga Alon. Degrees and choice numbers. *Random Structures Algorithms*, 16(4):364–368, 2000.

[8] Noga Alon, Omri Ben-Eliezer, Chong Shangguan, and Itzhak Tamo. The hat guessing number of graphs. *J. Combin. Theory Ser. B*, 144:119–149, 2020.

[9] Noga Alon, Stijn Cambie, and Ross J. Kang. Asymmetric list sizes in bipartite graphs. *Ann. Comb.*, 25(4):913–933, 2021.

[10] Noga Alon and Michael Krivelevich. The choice number of random bipartite graphs. *Ann. Comb.*, 2(4):291–297, 1998.

[11] C. Avart, P. Komjáth, T. Ł uczak, and V. Rödl. Colorful flowers. *Topology Appl.*, 156(7):1386–1395, 2009.

[12] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, and I. Peterin. Game chromatic number of Cartesian product graphs. *Electron. J. Combin.*, 15(1):Research Paper 72, 13, 2008.

[13] Anton Bernshteyn. The local cut lemma. *European J. Combin.*, 63:95–114, 2017.

[14] Anton Bernshteyn, Alexandr Kostochka, and Xuding Zhu. Fractional DP-colorings of sparse graphs. *J. Graph Theory*, 93(2):203–221, 2020.

[15] Hans L. Bodlaender. On the complexity of some coloring games. *Internat. J. Found. Comput. Sci.*, 2(2):133–147, 1991.

[16] O. V. Borodin. On acyclic colorings of planar graphs. *Discrete Math.*, 25(3):211–236, 1979.

[17] O. V. Borodin and A. V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *J. Combinatorial Theory Ser. B*, 23(2-3):247–250, 1977.

[18] Bartłomiej Bosek, Andrzej Dudek, Michał Farnik, Jarosław Grytczuk, and Przemysław Mazur. Hat chromatic number of graphs. *Discrete Math.*, 344(12):Paper No. 112620, 10, 2021.

[19] N. Bowler, J. Erde, F. Lehner, and M. Pitz. Bounding the cop number of a graph by its genus. *Acta Math. Univ. Comenian. (N.S.)*, 88(3):507–510, 2019.

[20] Peter Bradshaw. Cooperative colorings of forests, 2022.

[21] Peter Bradshaw. Graph colorings with restricted bicolored subgraphs: II. The graph coloring game. *J. Graph Theory*, 100(2):371–383, 2022.

[22] Peter Bradshaw. On the hat guessing number of a planar graph class. *J. Combin. Theory Ser. B*, 156:174–193, 2022.

[23] Peter Bradshaw and Tomáš Masařík. Single-conflict colorings of degenerate graphs. 2021.

[24] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[25] Steve Butler, Mohammad T. Hajiaghayi, Robert D. Kleinberg, and Tom Leighton. Hat guessing games. *SIAM J. Discrete Math.*, 22(2):592–605, 2008.

[26] Gregory J. Chaitin, Marc A. Auslander, Ashok K. Chandra, John Cocke, Martin E. Hopkins, and Peter W. Markstein. Register allocation via coloring. *Computer Languages*, 6(1):47–57, 1981.

[27] Ilkyoo Choi. *Extremal problems on variations of graph colorings*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[28] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2018. Paperback edition of [MR3644391].

[29] Thomas Dinski and Xuding Zhu. A bound for the game chromatic number of graphs. *Discrete Math.*, 196(1-3):109–115, 1999.

[30] David R. Ditzel and H. R. McLellan. Register allocation for free: The c machine stack cache. *SIGPLAN Not.*, 17(4):48–56, mar 1982.

[31] Zdeněk Dvořák, Louis Esperet, Ross J. Kang, and Kenta Ozeki. Single-conflict colouring. *J. Graph Theory*, 97(1):148–160, 2021.

[32] Zdeněk Dvořák and Luke Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *J. Combin. Theory Ser. B*, 129:38–54, 2018.

[33] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel J. Math.*, 2:183–190, 1964.

[34] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, Colloq. Math. Soc. János Bolyai, Vol. 10, pages 609–627. North-Holland, Amsterdam, 1975.

[35] Paul Erdős, András Gyárfás, and Tomasz Ł uczak. Independent transversals in sparse partite hypergraphs. *Combin. Probab. Comput.*, 3(3):293–296, 1994.

[36] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157. Utilitas Math., Winnipeg, Man., 1980.

[37] U. Faigle, U. Kern, H. Kierstead, and W. T. Trotter. On the game chromatic number of some classes of graphs. *Ars Combin.*, 35:143–150, 1993.

[38] Michał Farnik. *A hat guessing game.* PhD thesis, Jagiellonian University, 2015.

[39] Uriel Feige. You can leave your hat on (if you guess its color). Technical report, Weizmann Institute, 2004.

[40] Pierre Fraigniaud, Marc Heinrich, and Adrian Kosowski. Local conflict coloring. In *57th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2016*, pages 625–634. IEEE Computer Soc., Los Alamitos, CA, 2016.

[41] A. M. Frieze and T. Ł uczak. On the independence and chromatic numbers of random regular graphs. *J. Combin. Theory Ser. B*, 54(1):123–132, 1992.

[42] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdös centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013.

[43] Maximilien Gadouleau and Nicholas Georgiou. New constructions and bounds for Winkler's hat game. *SIAM J. Discrete Math.*, 29(2):823–834, 2015.

[44] Stefan Glock and Benny Sudakov. An average degree condition for independent transversals. *J. Combin. Theory Ser. B*, 154:370–391, 2022.

[45] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[46] Penny E. Haxell. A note on vertex list colouring. *Combin. Probab. Comput.*, 10(4):345–347, 2001.

[47] Xiaoyu He, Yuzu Ido, and Benjamin Przybocki. Hat guessing on books and windmills. *Electron. J. Combin.*, 29(1):Paper No. 1.12, 19, 2022.

[48] Xiaoyu He and Ray Li. Hat guessing numbers of degenerate graphs. *Electron. J. Combin.*, 27(3):Paper No. 3.58, 8, 2020.

[49] Pavol Hell and Xuding Zhu. On the adaptable chromatic number of graphs. *European J. Combin.*, 29(4):912–921, 2008.

[50] Anders Johansson. Asymptotic choice number for triangle free graphs. 1996.

[51] Gwenaël Joret, Marcin Kamiński, and Dirk Oliver Theis. The cops and robber game on graphs with forbidden (induced) subgraphs. *Contrib. Discrete Math.*, 5(2):40–51, 2010.

[52] John Jumper, Richard Evans, Alexander Pritzel, Tim Green, Michael Figurnov, Olaf Ronneberger, Kathryn Tunyasuvunakool, Russ Bates, Augustin Žídek, Anna Potapenko, Alex Bridgland, Clemens Meyer, Simon A. A. Kohl, Andrew J. Ballard, Andrew Cowie, Bernardino Romera-Paredes, Stanislav Nikolov, Rishub Jain, Jonas Adler, Trevor Back, Stig Petersen, David Reiman, Ellen Clancy, Michal Zielinski, Martin Steinegger, Michalina Pacholska, Tamas Berghammer, Sebastian Bodenstein, David Silver, Oriol Vinyals, Andrew W. Senior, Koray Kavukcuoglu, Pushmeet Kohli, and Demis Hassabis. Highly accurate protein structure prediction with alphafold. *Nature*, 596:583–589, 2021.

[53] Ross J. Kang and Tom Kelly. Colorings, transversals, and local sparsity. *Random Structures Algorithms*, 61(1):173–192, 2022.

[54] Charlotte Knierim, Anders Martinsson, and Raphael Steiner. Hat guessing numbers of strongly degenerate graphs, 2021.

[55] K. P. Kokhas and A. S. Latyshev. Cliques and constructors in the "hats" game. I. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 488(Kombinatorika i Teoriya Grafov. XI):66–96, 2019.

[56] K. P. Kokhas and A. S. Latyshev. The Hats game. The power of constructors. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 498(Teoriya Predstavleniĭ, Dinamicheski Sistemy, Kombinatornye Metody. XXXI):26–37, 2020.

[57] Alexandr V. Kostochka and Xuding Zhu. Adapted list coloring of graphs and hypergraphs. *SIAM J. Discrete Math.*, 22(1):398–408, 2008.

[58] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloq. Math.*, 3:50–57, 1954.

[59] Marek Kubale, editor. *Graph colorings*, volume 352 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 2004.

[60] Po-Shen Loh and Benny Sudakov. Independent transversals in locally sparse graphs. *J. Combin. Theory Ser. B*, 97(6):904–918, 2007.

[61] D Marx. Graph coloring problems and their applications in scheduling. *Periodica Polytechnica, Electrical Engineering*, 48, 10 2003.

[62] Antonio Mauttone, Gonzalo Mercadante, María Rabaza, and Fernanda Toledo. Bicycle network design: model and solution algorithm. *Transportation Research Procedia*, 27:969–976, 2017.

[63] Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces.* Johns Hopkins series in the mathematical sciences. Johns Hopkins University Press, 2001.

[64] Michael Molloy. The adaptable chromatic number and the chromatic number. *J. Graph Theory*, 84(1):53–56, 2017.

[65] Michael Molloy. The list chromatic number of graphs with small clique number. *J. Combin. Theory Ser. B*, 134:264–284, 2019.

[66] Michael Molloy and Bruce Reed. *Graph colouring and the probabilistic method*, volume 23 of *Algorithms and Combinatorics.* Springer-Verlag, Berlin, 2002.

[67] Robin A. Moser and Gábor Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010.

[68] B. Reed. ω, Δ, and χ. *J. Graph Theory*, 27(4):177–212, 1998.

[69] Bruce Reed. A strengthening of Brooks' theorem. *J. Combin. Theory Ser. B*, 76(2):136–149, 1999.

[70] David Saxton and Andrew Thomason. Hypergraph containers. *Invent. Math.*, 201(3):925–992, 2015.

[71] James B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46(1):83–87, 1983.

[72] James B. Shearer. A note on the independence number of triangle-free graphs. II. *J. Combin. Theory Ser. B*, 53(2):300–307, 1991.

[73] Witold Szczechla. The three colour hat guessing game on cycle graphs. *Electron. J. Combin.*, 24(1):Paper No. 1.37, 19, 2017.

[74] Terence Tao. Moser's entropy compression argument, 2009. https://terrytao.wordpress.com/2009/08/05/mosers-entropy-compression-argument/.

[75] Zsolt Tuza and Xuding Zhu. Colouring games. In *Topics in chromatic graph theory*, volume 156 of *Encyclopedia Math. Appl.*, pages 304–326. Cambridge Univ. Press, Cambridge, 2015.

[76] Ian Wanless and David Wood. Exponentially many hypergraph colourings, 08 2020.

[77] P. Winkler. Games people don't play. In David Wolfe and Tom Rodgers, editors, *Puzzlers' Tribute: A Feast for the Mind*, chapter 10, pages 301–313. A K Peters, Baltimore, 2002.

[78] Jiaojiao Wu and Xuding Zhu. Lower bounds for the game colouring number of partial k-trees and planar graphs. *Discrete Math.*, 308(12):2637–2642, 2008.

[79] Lenka Zdeborová and Florent Krzakala. Phase transitions in the coloring of random graphs. *Phys. Rev. E (3)*, 76(3):031131, 29, 2007.

[80] Xuding Zhu. The game coloring number of planar graphs. *J. Combin. Theory Ser. B*, 75(2):245–258, 1999.

[81] Xuding Zhu. The game coloring number of pseudo partial k-trees. *Discrete Math.*, 215(1-3):245–262, 2000.

[82] Xuding Zhu. Game coloring the Cartesian product of graphs. *J. Graph Theory*, 59(4):261–278, 2008.

[83] Xuding Zhu. Refined activation strategy for the marking game. *J. Combin. Theory Ser. B*, 98(1):1–18, 2008.