The Teacher–Tool–Mathematics Ensemble: Before, During and After Implementing *TouchTimes*

by

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Abstract

Using the theoretical constructs of double instrumental genesis, instrumental distance, didactical landmarks and instrumental orchestration, I examined case studies of four elementary school teachers in British Columbia, Canada, who implemented *TouchTimes* (hereafter, TT) as a pedagogical tool for teaching mathematics. TT comprises a pair of novel, multi-touch iPad applications that provide embodied and relational experiences of multiplication through two different dynamic multiplicative models. In interviews, these teachers shared their personal experiences learning about this relatively new digital application themselves, the obstacles they encountered and their professional experiences integrating TT into their instructional repertoires as a tool for student learning.

The aim of this research was to study the teacher, the digital tool – in this case, TT – and the mathematical concept of multiplication as an ensemble, as it played out in the classroom. Rather than examining each of the parts separately, the ensemble views each piece in relation to the whole, rather than individually, and became my way of honouring the complexity and the nuances that emerged from the information and experiences shared. I examined how TT, and its way of materialising the mathematical concept of multiplication, affected the personal and professional instrumental geneses of the teachers and, conversely, how the teachers' accommodated TT within their previous ways of thinking about, and teaching, multiplication. Furthermore, I studied how the teachers' experiences of double instrumental genesis evolved during their use of TT as a pedagogical instrument.

My analysis indicates that the implementation of TT, and its way of presenting multiplication, was multi-faceted and complex, that the personal and professional instrumental geneses that teachers undergo may be closely intertwined and that, when speaking of TT, they clearly differentiate between ways of teaching with it and how students may learn using this technology. In keeping the teacher–TT–mathematics ensemble in mind, a few noteworthy implications of the research include: (1) the mutually interactive nature of personal and professional instrumental genesis; (2) an acknowledgement of the agency of the tool; (3) the importance of sociocultural influences on the teachers' processes of double instrumental genesis; (4) the identification of three new instrumental orchestrations that emerged from this research.

Keywords: double instrumental genesis; elementary school teachers; instrumental distance; multiplication; touchscreen technology; *TouchTimes*

Dedication

It always seems impossible until it's done (Nelson Mandela)

For all the family, friends, teachers and mentors – some of whom were stuck with me and others who chose me – who have left indelible marks upon my journey to this point. Words can not effectively convey just how important "my people" have been in helping me through the ups and downs along the way. Though many of these people will never read this page, they will always be in my heart. Thank you so much for the education, the encouragement and the support, as well as the love and the laughter.

Acknowledgements

"It's not about the goal or the dream you have. It's about who you become *on your way to that goal" (Rachel Hollis)*

I would like to acknowledge the invaluable contribution made by each of the four teachers, whose pseudonyms are Amy, Kate, Leah and Rachel, and thank them for agreeing to participate in this research project. I deeply appreciate the time each spent with me, sharing their experiences and self-reflections, and for generously providing their thoughts about their own learning experiences and teaching practices for analysis and scrutiny. The contributions made by each of these highly professional and skilled teachers to my work have been invaluable, and form the basis of this dissertation.

The guidance, encouragement and support that I have received from so many throughout the six years of my doctoral studies has been an invaluable part of developing my confidence and my skillset as a researcher, writer and presenter.

Next, I would like to thank Dr. Nathalie Sinclair for taking me to 'Mathland' and for introducing me to the purposeful use of digital technology as a tool for teaching mathematics. As a senior supervisor, she has been a rock. I always knew that Nathalie was there if I needed advice, assistance or direction and I would also like to recognise the patience that she demonstrated as I slowly found my way. Not only did she include me in various research projects, she also allowed me the space to "do my own thing". A big part of finding my feet occurred as a result of my desire to attend international conferences. Nathalie encouraged this, both by providing timely constructive feedback while co-writing my initial papers and by filling out the necessary paperwork so that I could access the funding to pay for it. Writing all of those conference papers was pivotal in building the skills that eventually led to the completion of this dissertation. Truly a role model to look up to, the high calibre of Nathalie's work, as well as her thoughtfulness and care towards her students are all traits that I strive to emulate. Nathalie, thank you for all that you have shared with me that has enabled my successful attainment of this goal.

During the first graduate course that I took with Dr. David Pimm, he stated that, "When the student is ready, the teacher will appear" and that, "Schooling is exchanging time for experiences". Since first writing these bits of wisdom in my notes back in 2013, they have proven to be true. When David first asked me if I had considered pursuing a Ph.D., I thought the idea was crazy. By asking the question though, he planted the seed that eventually did lead to my pursuit of a Ph.D. With David's care and attention, my ability to analyse data and write about my findings has continued to grow stronger and my belief in my own abilities has taken root. His eye for detail, explicit feedback, positivity and gift for providing a nudge at just the right moment have made him one of the "teachers" that this student was ready for. Thank you, David, for your kindness and care, your belief in my potential, as well as your high standards, which have made the quality of my work so much better.

I would also like to express my appreciation to my defence members, Dr. Sean Chorney and Dr. Annie Savard for their time and care in preparing for my thesis examination. I have had the privilege to work with Sean during more than one research project and have learned so much from him. As a former elementary teacher herself, Annie provides a perspective not commonly found in mathematics education and is one that I highly value. Thank you for your encouragement and positivity, both have made a difference as I grow into this new role.

My thanks also go to Dr. Mariam Haspekian for providing thoughtful feedback about my earlier work using her theoretical ideas. I was fortunate to meet her via Zoom in February 2022 during a conference, and not only did she attend my research presentation, she also kindly suggested a personal Zoom call to further discuss my research. Mariam's willingness to engage with and question my ideas, while providing suggestions for further exploration and development, is something truly special for a beginning researcher, and I am grateful to have had this opportunity.

Throughout the past six years, there are a number of fellow doctoral students who were an important part of my journey. I would like to express my appreciation to all those who shared their own work and provided feedback for mine – a few of the regulars including, Sheree Rodney, Jason Forde, Victoria Guyevskey and Canan Gunes. In particular, I would like to thank Annette Rouleau, Sam Riley, Ofer Marmur and Ronald Boersen for being pillars of support and encouragement while I climbed what felt like Everest. I am so glad that I had you all in my corner. I have benefited so much from your advice in so many areas of my life, and am a better person for our interactions.

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List of Acronyms

BC	British Columbia
CERME	Congress of the European Society for Research in Mathematics Education
DEME	Digital Experiences in Mathematics Education
SFU	Simon Fraser University
TC	*TouchCounts*
TT	*TouchTimes*
YERME	Young European Researchers in Mathematics Education

Chapter 1. Introduction

> *Individual events mean nothing by themselves, but are part of a larger totality and can only be understood within the whole. (Matt Haig, 2021)*

I will begin with the story of how I arrived at the point of sharing my research in the form of this dissertation. This story commences with my memories of learning mathematics as an elementary school student, then shifts to my thoughts of teaching multiplication as an elementary school teacher and, finally, outlines the transition experienced as my roles have shifted from teacher to some sort of student–teacher–researcher hybrid. My initial foray into graduate studies involved becoming a student again, then implementing what I had learned about teaching mathematics better into my pedagogical practice and then, finally, drawing to a close with engaging in research within my classroom. However, the story then continues with my decision to pursue doctoral studies and I will outline the opportunities and experiences that occurred during this pursuit and describe how they have influenced and shaped the research that is written about here. Lastly, I will conclude the chapter by providing a brief overview of what to expect in the chapters that follow. Now that that I have outlined the progression that this chapter will follow, let me return to where the story first begins.

1.1. Learning multiplication

Perhaps it is not a coincidence that my most vibrant memories of elementary school mathematics involve learning the multiplication facts, and the feelings associated with these memories are powerful, having stayed with me, even decades later. There is nothing about my mathematics classes that stands out in my memory prior to this point, but I will never forget the feelings of frustration connected to the timed drills, star charts and around-the-world games that were used in the early 1980s intended to help us memorise our times tables.

I can clearly visualise the star chart on the wall of my third-grade classroom, and I vividly recall the three gold stars beside my name that I had earned for getting all of the questions on the zero-, one- and two-times table drills correct. I loved those gold stars in the way that only a very keen, conscientious child who was already well on the path to becoming an overachiever could. When I hit the three-times tables though, I stalled. Writing and rewriting that particular timed drill over and over again without success, and the gold stars stopped coming my way for a very long time. I watched most of the other students continue to get stars on the chart and I was somehow trapped at three, likely getting the same questions incorrect each time.

Alongside the math drills came ‘around the world’. A simple game where two students stand side-by-side and the teacher displays a multiplication flashcard that the two players compete to answer first. Whoever says the correct answer aloud first wins that round and moves on to the next student. The game continues in this manner until all students have a chance to participate. The last person standing is the winner. My competitive spirit enjoyed playing games and loved to win them even more. Initially, I was excited to play this game, but that excitement soon fizzled out, as each time it was my turn, I was still figuring out the answer while my fellow competitor was already saying it aloud. Playing a game only remains fun as long as there is a chance actually to engage in, play and possibly win once in a while. It was almost always the same student who won this game each time we played it in class.

1.2. Teaching multiplication

Despite these vivid memories of my own frustration with learning multiplication facts as a young student, when I began my teaching career, I also used star charts, created my own versions of multiplication timed drills and even played around the world with my fourth-grade students. How could I possibly inflict these types of activities on my students after my own negative experiences with them? It seems that I just did what my own teachers had done because I did not know any better and, also, this was what all the other more experienced fourth-grade teachers in my school were doing.

Although it quickly became apparent that these strategies were not working, particularly with struggling students, I still continued to use them for many years. All the while, buying into the idea that students just needed more practice to learn their facts. At this point, I had not yet learned the correlation between high-quality teaching and learning. I was still using a mindset of, ‘I taught it, I’m not sure why they haven’t yet learned it’. As I began to adopt a mindset of, ‘My students haven’t yet learned this skill, so how will I adjust my teaching to help them succeed?', the quality of the learning in my classroom began to improve, and my approach to teaching multiplication began to change. My entire approach to teaching mathematics would soon be impacted after attending a professional development session at the Mighty Peace Teachers' Convention in March of 2011.

1.3. An unforeseen opportunity

On Friday, March 4th, 2011, I attended an eighty-minute professional development session on mathematics problem solving given by Dr. Peter Liljedahl during the Mighty Peace Teachers' Convention in Grande Prairie, Alberta. Prior to this session, I had never even heard of Peter Liljedahl nor his work and little did I know that this moment would become a turning point in my life.

The room was packed. There were teachers lining the walls and sitting on the floor. We were given two different problems to work on in small groups. I remember feeling nervous about working in a group, because math has never been my strongest subject, and I was worried that I would look stupid and have nothing to contribute. Suffering in silence had been my mathematics survival tactic for many years, and I had not been asked actually to do any mathematics in a classroom setting since completing my undergraduate degree fifteen years earlier. I was already in the room though and, at that point, there was no turning back. I was so relieved when the first problem given to us was something that I could use in my classroom and, although there was not an obvious answer, it was also clear that that was the whole point of the problem. I took a deep breath and tried to dig into the problem with my group.

When the second problem was given to us, it involved planning a celebrity's schedule for a day. The goal was for the celebrity to raise money for a charity of our choice (or something along those lines) and there were various options for doing this that involved driving the celebrity to different functions. Keeping in mind that, for each function, the time commitment and payment value varied, and the time needed to drive to each location was also different. I was hooked. I did not want to stop! For those who have had a positive relationship with mathematics, this may be a familiar feeling, but for me, this was noteworthy. Later, I went home and continued to work on the problem – again, this was very unusual behaviour for me and also an indicator of the quality of the problem. I remember thinking to myself, “If I had learned math like this, my experiences would have been different”. The way that I was engaged with mathematics that day inspired me to want to learn more.

During Peter’s session, we learned that Simon Fraser University (hereafter, referred to as SFU) was considering Grande Prairie as a possible site for a master’s degree in numeracy cohort. Anyone interested in hearing more information about this opportunity was invited to attend an information session that afternoon at 4pm. It was a Friday and I had planned to attend back-to-back professional development sessions and to then take my lunch break at the end of the day for an early start to the weekend. I came back at 4pm to learn more. In the months that followed, I applied to be part of this master’s cohort, despite my previously negative experiences related to mathematics as a learner and all the baggage that entailed. Once I was accepted into the program, it was then that my fear of mathematics rose again, almost swamping me.

Showing up for the first night of class in the master’s cohort took all of my courage. I dug deep. I carried all of my insecurities related to mathematics with me into that class. This was THE subject in school that had left me feeling defeated. The only subject (outside of physics) that no matter how hard I worked and how many help sessions I went to after school, I still did not understand. I felt successful in every other subject that I took from Kindergarten through to grade twelve, except math (and physics 10, as it was based on a LOT of math). This meant that I had a lot of emotional baggage and few tools in my mathematics problem-solving toolbox when I embarked upon the master’s journey.

I will never forget the fear I felt that night. That feeling of just wanting to shrink into yourself and be invisible. I only knew one of the other fifteen students, and in my mind, it seemed like everyone else knew more math than I did. I sat there like a deer in the headlights, hoping that Dr. Rina Zazkis, our professor, would not ask me any questions. And I took as many notes as I could throughout class, so that later, at home, I could try to figure out all that I did not understand during class. More than once Rina would tell us, “Just put your pencils down”. But I could not. If I wanted to figure it out, I needed the notes, all the hints that emerged and sometimes even an answer, so that I could try to understand it all later. I was very much an island. At that point, I had nobody that I could ask for help. I was very much on my own, and at the mercy of whatever mathematics Rina challenged us with.

I had nothing but my courage, my quick note-taking skills and my dogged determination to figure out what I did not understand during class, in the week that followed. Despite having to battle through my own fight-or-flight response each time a new mathematics problem was given, I stayed, endured and learned. And, somehow, I made it through successfully. Though there is a part of me that wishes I could take that course again, knowing that I would start out so much further ahead now that I have accumulated a few tools in my mathematics problem-solving toolbox.

From my perspective, there were three important building blocks that comprised the master's program. The first concerned the mathematics, which was ever present throughout each course. The second involved the teaching of mathematics, and the final piece was related to the research.

Dr. Peter Liljedahl led the second course, bringing with him the teaching component that allowed us to experience, as learners, his Building Thinking Classrooms (Liljedahl, 2020) concept. Our cohort was now working in small groups on vertical, non-permanent surfaces and no longer was I alone with the mathematics: for the first time, I was now part of a team. Usually one of the members of my group would have an idea or strategy to get us started and I could then ask questions as we worked to ensure that I understood the process that led our group to the solution. Although I still did not always understand the whole process that led to the solution, I was making significant progress in my learning and my confidence was slowly starting to increase. With the support of my group, I was no longer working from a space of fight or flight, which freed me from feeling so isolated, particularly when I did not understand a problem or have any idea where to start.

These experiences inspired me to implement this type of approach into my own mathematics teaching. I wanted to provide my students with the opportunity to engage with mathematics in a way that was interesting, included problem solving and allowed them to work collaboratively with others on vertical, non-permanent surfaces. I also began to approach my planning differently. How could I first provide students with an experience related to what I wanted them to learn, prior to teaching it? How could I build my teaching from these experiences? I absolutely loved the difference I was seeing in regard to student engagement, attitude, learning and an improved classroom community. I was hooked and wanted to learn more.

Throughout the next class, Dr. David Pimm brought academia to us, in the form of a carefully selected 'taste' of academia that was more relevant to practicing elementary school teachers. We were introduced to readings related to mathematics education and began to learn more about research itself. The articles and chapters we were assigned to read were often about ideas that I was unfamiliar with, and often involved approaches that I had never considered using in my mathematics teaching, even if I was already utilising those teaching strategies in other subject areas. We were also encouraged to go beyond our perspective as teachers and to question and think about the content of the readings from a researcher's perspective. Some of these questions included: what is the phenomenon of interest? How did the researcher 'look' for the data? What data did the experiences generate? What claims did the author(s) make? How did they support these claims?

We also had to begin thinking about how we would engage in research within our own classrooms and, for me, it was coming up with an idea to research that proved to be most difficult. Fortunately, though, inspiration arrived during the next course, when Dr. Nathalie Sinclair introduced us to various technology options for learning mathematics. One of the assigned readings was about a program for use with *The Geometer's Sketchpad* called Shape Makers (Battista, 2008), that provided students a dynamic way to learn about quadrilateral or triangular shapes. Coincidentally, quadrilaterals happened to be part of the mathematics curriculum for the grade level I was teaching. After reading the Battista chapter, I wanted to try to using *Shape Makers* as a teaching tool with my students and set out to create this as my research project.

In hindsight, there are two things of importance that emerged from these particular experiences that were pivotal in my journey towards academia. The first was prompted by Nathalie's introduction of digital technology as an interactive, hands-on tool for student learning. The second occurred after I completed collecting the data and writing up my project experiences. I'll never forget the conversation that began with David asking, "Have you considered pursuing a Ph.D.?" To which I replied, "No! I'm not that good at math." And his response, "But you can write". A conversation which was to become another pivotal moment in the journey leading me towards this dissertation.

I continued teaching elementary school for another two years before leaving to pursue doctoral studies in mathematics education. It would be unjust not to acknowledge the emotional impact that leaving teaching to further my education had. All aspects of my life were affected in some way by this decision, though perhaps exacerbated by my age – I am in my forties – and having to change provinces in Canada. It took a couple years to find my feet again after facing situation after situation where I had little idea what to do or how even to begin doing it. This was in sharp contrast to my competence as a teacher. My transition from schoolteacher towards academic researcher was far from smooth, with the biggest jolt occurring during my first course. I thought I wanted more experiences (like those of my master's degree) to connect research to practice and engage with research through my own teaching, but, in hindsight, I now recognise how naïve this was. Somehow, I had not realised that the Ph.D. process was going to involve turning me into an academic.

The nature of the readings assigned were very different from those of my previous course-based graduate studies. I was not used to reading lengthy academic papers and found the content challenging. For instance, I was not prepared for the difficulty of theoretical readings, and my limited experience of theory only permitted connections with my mathematics experiences as a teacher or a student. It took an inordinately long time to read each assigned piece, as I wrote extensive notes in preparation for class discussion with others who had completed thesis-based master's degrees and could fit these theories into existing 'theory' folders. This was both exhausting and time-consuming and, just as I was beginning to grasp a theory, we would study a new one, and the whole process would start again. Additionally, if the writings were dense, packed with unfamiliar information or mathematical examples that were difficult for me to access, it added another layer of complexity.

Writing is an ability that I have always felt confident about and was part of how I justified my pursuit of a Ph.D., despite feeling less confident about my broad competence in mathematics. However, my struggles with academic writing began while analysing some data using the theory of instrumental genesis and writing the results using the PME template during my first assignment as a doctoral student. The expectation to write in an academic style to which I had only just been introduced, coupled with the concern of not knowing how to write in this style, effectively paralysed my pen. I tried turning to published writing for exemplars, but these finished, polished pieces left me floundering with where to begin with my own writing, which was nowhere near the sophisticated level of published work. Only a month into my studies, my self-confidence was taking a beating, much of it self-inflicted. If I could not write, then did I belong?

Having emerged from an educational model of practise, practise, practise and demonstrate understanding by completing a test, it has been an entirely new challenge to persevere in learning about academic writing by engaging in the process itself. This learning-through-doing has proven, at times, an excruciatingly difficult way to learn and there appear to be no shortcuts to avoid this painful process. As a result of feedback from my supervisors, as well as my peers, constructive feedback from reviewers and the opportunity for co-writing situations, I began to get a better sense of how to engage with academic writing in order to share my ideas and analysis in a format acceptable in the field.

I have shared this part of my story in order to highlight the importance of confidence in my growth as a researcher. The impact of experiences that I describe in the sections that follow often influenced my confidence. Without a clear appreciation of how low my confidence was when I began my doctoral studies, it would be difficult to understand why each new writing project I engaged in was so important. And why I repeatedly mention the impact of these activities on my growing confidence in a journey towards seeing myself as a researcher.

1.4. *TouchTimes*: Researching multiplication

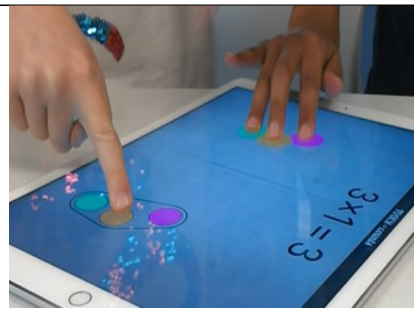
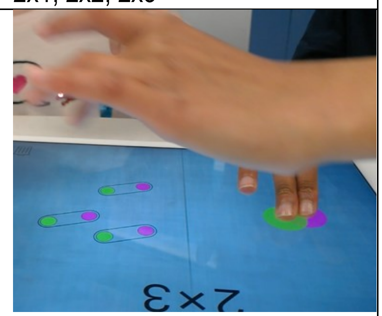
When I arrived at Simon Fraser University in the fall of 2016, I had absolutely no idea what my phenomena of interest would be, other than I was interested in elementary school mathematics teaching and learning. However, during the beginning of my third year of studies I was invited by Dr. Nathalie Sinclair and Dr. Sean Chorney to be a research assistant in a research project they were investigating that involved a new iPad application called *TouchTimes* (hereafter, called TT). The project focus involved working collaboratively with elementary teachers on the development of tasks for use with TT, and to study the effects of introducing multiplication using this digital technology on student learning. With the project focus on student learning, it left a lot of space for me to learn more about teachers, and how they implemented TT in their classrooms. This allowed for a slightly different angle within a project that was already set up, while also providing an opportunity that fit well with my area of interest, while narrowing my research focus towards TT specifically.

As my attention and energy were increasingly directed towards becoming an academic, this shift is also apparent in my story, which will now become one about theory and constructs. Though what is shared next are heavily related to my journey towards academia, the mathematics itself is perhaps the underlying reason that I felt compelled to engage in this research. My own difficulties and frustration as a learner trying to memorise the multiplication facts, and my desire to improve teaching by increasing ways of thinking about multiplication, are both themes that remain close to my heart and continue to motivate me. The academic aspect of my journey, though it will take me away from these themes, is a necessary path to travel, in order eventually to develop my research questions and study the teaching of multiplication with technology.

1.4.1. Writing as the way to my research questions

Although the focus of my research was still rather broad, I began to use the data that was collected as part of the TT research project to write conference papers. In my role of research assistant, I was asked to analyse five minutes of a 25-minute video recording of two third-grade children engaging with TT for the first time. In order to capture the three different, yet interacting, modalities of speech, gestures and touchscreen technology, I created an orchestral transcription (see Table 1.1). This form of transcription, delineated into ten-second increments, emerged as a method to capture what the children were saying, while at the same time noting what their hands were doing on the left or right side of the iPad screen and showing, visually and with a written description, what was displayed by TT.

Table 1.1 An orchestral transcription

Voices	Jacy		Laughs
	Kyra	Wait, now I'll make a three. And then	[laughs] They're dancing! [Laughs]
	Interviewer		
Hands	Left Screen	3-finger simultaneous touch (RH)	3 sequential finger touches
	Right Screen	Pointer finger touch (RH)	Hold two fingers
iPad Screen	Top of Screen	3 3x1=3	2x1, 2x2, 2x3
			
	Left Screen	3 pips	
	Right Screen	1 pod	1, 2, 3 pods

Source: Bakos & Sinclair (2019)

In order to capture the progression of what could be seen and heard in the video I was analysing, the left to right progression of an orchestral score came to mind. Trying to replicate this idea, I created a table where the top three rows were designated for each of the three voices, which provided a way to sequence visually the speaking in a manner that would effectively show overlapping voices. Given the importance of the left and right side of the screen in TT, the next two rows were used to capture what the children's hands were doing on each side of the iPad screen. My difficulty in clearly describing what the hands were doing, in a way that someone without access to the video would understand, quickly became apparent. As a way of resolving this issue, I created an additional row in the table to include screenshots from the video that illustrated what the users' hands were doing. While doing this, I realised that what was happening within the digital technology itself was also important. Therefore, additional rows were created in the table to reflect what could be seen on the top, left and right sides of the iPad screen. I felt empowered by the time the orchestral transcription was complete and, after putting so much time and effort into capturing the sequencing of the three different modalities, I wanted to find a way to utilise it in some manner. And I did so by co-writing the conference paper, Pips (times) pods: Dancing towards multiplicative thinking (Bakos & Sinclair, 2019) for the Congress of the European Society for Research in Mathematics

Education (hereafter, referred to as CERME). The acceptance of this paper at an international level conference contributed to building my confidence and was an important part of my growth as an early academic.

Using the same video of the two girls engaging with TT, with David's support, I submitted an abstract proposal for a special issue of Digital Experiences in Mathematics Education (DEME). The theme of the issue addressed the intertwined contributions of physical and digital tools for the teaching and learning of mathematics, and it seemed to me that, when using TT, the children's fingers are themselves physical tools needed to manipulate the technology. When invited to write the article proposed, David and I examined how the girls' fingers were mutually interactive with each other, as well as with TT, and how the ways the two entities would combine seemed to challenge physical and digital tools as separate entities (Bakos & Pimm, 2020). With David's assistance, I was able to engage more deeply with these ideas, while also being part of co-writing a lengthier piece of work in a genre I had not written before. This was another notable step forward, as it was one more challenge faced and overcome, and another contribution towards building my growing self-confidence as an academic writer.

1.5. My research questions come into focus

In the fall of 2019, I found myself reading about the theoretical notion of instrumental orchestration (Drijvers et al., 2010), and was inspired by this article to use this notion to examine data from the TT project. For my first solo international conference paper, I used this construct to examine the instrumental orchestrations being used by primary school teachers who were implementing TT in their classrooms (Bakos, 2021). Without the strength of experience that my co-writers brought to my previous work, my data analysis was not pushed to the same depth as other writing, but despite this, I felt more confident in my abilities and could feel some forward momentum starting to build. Although I will go into more detail in chapter three regarding how I arrived at the theory that will be utilised in this dissertation, let me briefly mention the three pieces of writing that were essential to bringing my research questions into focus.

The first was a chapter that I was invited to write after submitting a proposal to the second edition of *The Mathematics Teacher in the Digital Era: An International Perspective on Technology Focused Professional Development* (Clark-Wilson, Robutti &

Sinclair, 2022). When I came up with the idea for this proposal, I was hoping that the chapter could be something that I would write in parallel to part of my thesis. While writing that chapter, the word *ensemble* was first used in reference to the technology, the teaching and the mathematics. When one of the reviewers questioned the meaning of this term within this context, it forced me to think more deeply about it and define it in a way that made sense to me, as well as to the readers. An ensemble in this context "views each part in relation to the whole, rather than separately" (Bakos, 2022a). The ensemble idea became an important way of viewing and thinking about the technology, the teaching and the mathematics. I did not want to examine them as separate entities: rather, I wanted to honour all three aspects as a complex entity of intra-actions (Barad, 2007) influencing each other and working inseparably. Once I had more clearly defined ensemble within the context of my research, I then went back to my data analysis to ensure that I had consistently approached the teaching, the mathematics and TT using this idea. Another pivotal suggestion provided by one of the chapter's reviewers was to consider the possible usefulness of other theoretical approaches, such as double instrumental genesis. Though I will go into more detail about this in Chapter 3, suffice it to say, that it was at this point that I was first introduced to this construct.

The next two pieces of writing that propelled my thinking forward were an article written for a DEME special issue and a conference paper for CERME. I began by writing an abstract proposal for the DEME special issue on transitions within, across and beyond digital experiences for the teaching and learning of mathematics. For the first time, I was going to be utilising the data that I had collected for my doctoral research and I could easily see connections between the ideas shared by the teachers I interviewed and the theme of transitions. However, in the time interval between submitting the proposal and being invited to write the article, I wrote a conference paper for CERME. Given that I was already thinking about this idea of transitions, I began to examine my data in search of illustrative examples of transitions related to the teacher–TT–multiplication entity (Bakos, 2022b).

After writing the CERME paper, I found out that my abstract proposal was accepted and I was invited to write and submit an article for the DEME special issue (Bakos, 2022c). My initial engagement in the CERME paper with this idea of transitions provided a solid foundation of ideas that I could now expand and engage with more deeply for the article. It was a natural extension that allowed my ideas to develop further without the space constraints of an eight-page conference paper. For both of these writing endeavours, I was using my own data and writing more independently, both a reflection of my growth as a writing and increasing confidence as a researcher. I finally possessed both the ability, and the confidence in that ability, to begin writing my dissertation.

1.6. Overview

Now that I have shared the journey that brought me to the point of writing this dissertation, I will provide an overview of what can be expected in the chapters that follow.

Chapter 2 contains a literature review that provides the context necessary for better understanding two of the key underlying problems that are applicable to my research. The first relates to multiplication and the way it is traditionally introduced to students and why this is problematic for future mathematics learning. The second involves the complexities and challenges experienced by mathematics teachers when implementing digital technology as a tool for teaching and learning. Bridging these two key issues, I provide some background on the links between using fingers to do mathematics and number sense before describing the emergence of touchscreen technologies and the opportunities these create for elementary school mathematics. This chapter lays the foundation for why my study is both important and relevant to the current state of mathematics education.

Throughout Chapter 3, the progression of theoretical considerations that have shaped the development of my research questions will be shared. These theoretical influences are an essential part of my approach to the research questions and my interpretation of the data. This chapter closes with the presentation of my three questions.

I begin the fourth chapter by describing *TouchTimes*, and its two microworld applications, Grasplify and Zaplify. I then share the context of the larger TT teacher–researcher group, as well as the teacher interviews completed as part of my research. After sharing the context of the study, I will describe the methods used for data collection and analysis.

The first of four chapters that will examine specific examples shared by the four elementary school teachers who were part of this research study, chapter 5 highlights the teachers' initial impressions of TT. While Chapter 6 presents in-depth examples of the instrumental orchestrations utilised by two of the elementary teachers who were implementing TT with their students. Chapter 7 considers the purpose described by each teacher for the use of TT as a pedagogical tool for teaching multiplication. In Chapter 8, I examine the sociocultural influences experienced by the four teachers throughout their processes of double instrumental genesis. Finally, in Chapter 9, I return to my research questions to consider the possible implications of this study and potential avenues for further research.

Chapter 2. Literature Review

> *“The more that you read, the more things you will know. The more that you learn, the more places you’ll go.” (Dr. Seuss)*

In this chapter, I will provide an overview of related literature, in order to situate the ensemble components of technology, teacher and mathematics that form the backbone of this dissertation within the landscape of existing mathematics education research. In so doing, the necessary context will be established for understanding better why my research is both important and relevant to the current state of mathematics education and how it addresses an area in need of further study.

My approach to reviewing the pertinent literature is to highlight research from the field that is related to three different themes associated with the components of the teacher–digital technology–mathematics ensemble. In examining the first theme (sections 2.1–2.3), I will provide some background, specifically on the mathematics that constitutes multiplication and how multiplicative thinking is different from additive thinking. I will then describe the importance of ensuring student development of multiplicative thinking, before finally explaining ways that multiplication is commonly introduced to elementary school students in many countries and why this approach can be problematic for future mathematics learning. In so doing, the focus of the first theme will be on the ensemble components of mathematics and teacher.

In order to address the second theme (sections 2.4 and 2.5), I will conduct a brief examination of research findings that highlight the links between using fingers to do mathematics and number sense. I will then connect these ideas with the technology and mathematics components of the ensemble, and examine the emergence of touchscreen technologies and the opportunities these create for elementary school mathematics.

Finally, my examination of the third theme (section 2.6) brings together all three aspects of the ensemble and looks at some of the complexities and the challenges experienced by mathematics teachers when implementing digital technology as a tool for teaching and learning.

2.1. Multiplication and multiplicative thinking

In their work on the procedural vision of teaching multiplication, Polotskaia and Savard (2021) pointed out that multiplication is often seen as simply an operation to perform and that basic facts are considered by some to be the most important thing for students to know. Thus, the importance of basic facts, multiplication included, is still being debated in Canada.[1]

The research related to multiplication is also diverse in focus, and the views of multiplication that have been adopted by researchers since the early 1980s have also varied. These views have ranged from considering multiplication as an arithmetic operation to a focus on multiplicative structures (e.g. Vergnaud, 1983), classifications of situations modelled by multiplication (e.g. Izsák & Beckmann, 2019; Greer, 1992, 1994) and/or an examination of multiplicative relationships (e.g. Beckmann & Izsák, 2015; Davydov, 1992). Additionally, there are researchers who have examined and compared multiplicative models and their visual representations (e.g. Kosko, 2020; Maffia & Mariotti, 2018; Vest, 1971) and there are those who have studied both students' and teachers' interactions with visual representations of multiplication (e.g. Anghileri, 1989; Bolden et al., 2015; Lunney Borden et al., 2021).

Therefore, given the breadth of research on multiplication, my goal throughout this chapter is to highlight the literature that is pertinent to the elementary teacher–TouchTimes–mathematics ensemble that is the focus of my own research. In so doing, I will begin by defining multiplication, then outline relevant research related both to multiplicative structures and to multiplicative relationships and, finally, I will examine multiplicative thinking and its importance for student progress in mathematics. As Clark and Kamii (1996) pointed out, "If we want to teach multiplication, we must first understand the nature of multiplicative thinking" (p. 50).

Multiplication is an arithmetic operation used for finding the product of two or more factors, whereas multiplicative thinking is a cognitive process used in mathematical problem contexts that necessitate the processing of multiplication through the application

[1] Given that the basis of the British Columbia mathematics curriculum rests on teaching curricular competencies and understandings, rather than on the memorisation of basic facts, the approach used in this dissertation is in line with this curriculum.

of different methods or approaches (Singh, 2012): “Multiplicative thinkers are those who have understood the concept of multiplication and are able to apply the concepts and solve problems relationally” (p. 2389). In what follows, I will outline in more detail some of the research that is relevant to multiplication and its mathematical structure, as well as clarify the differences between additive and multiplicative thinking.

Since Vergnaud (1983) defined the conceptual field of multiplicative structures as “sets of problems involving arithmetical operations and notions of the […] multiplicative type (such as multiplication, division, fraction, ratio, similarity)” (p. 128), considerable attention has been given to the comparison of quantities using multiplicative thinking (e.g. Clark & Kamii, 1996; Schwartz, 1988; Steffe, 1988). Vergnaud noted that multiplication problems consist of a four-term relationship and, in describing some of the differences between additive and multiplicative structures, he acknowledged that, “multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects” (p. 128).

Piaget (1983/1987) observed that, although multiplication appears in principle to be nothing more than repeated addition, it is actually much more complex, and that multiplication is “synthesized as a simultaneous composition instead of being carried out successively” (p. 78). Drawing from Piaget’s work, Clark and Kamii (1996) illustrated and described how the levels of abstraction required for additive thinking and for multiplicative thinking are distinctly different. Thinking additively involves tracking one quantity at a time, as depicted in Figure 2.1, where each unit is made up of three ones which are combined successively.

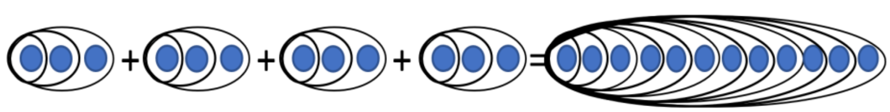


Figure 2.1 Representation of additive reasoning (3 + 3 + 3 + 3)

In contrast, multiplicative thinking requires tracking two quantities through a co-ordination of composite units. The first co-ordination occurs when the three ones become a single unit of three, and the second one does when the four units of three become a single product of twelve (see Figure 2.2). In this many-to-one process, the groups are combined simultaneously. Downton and Sullivan (2017) argue that, “the co-ordination of composite units is the core of multiplication, and that young children’s (8-

year-olds) concept of multiplication is based on the meaning they give to the composite units they construct" (p. 306). To think multiplicatively involves thinking simultaneously about units of one and units of more than one that form the number of groups.

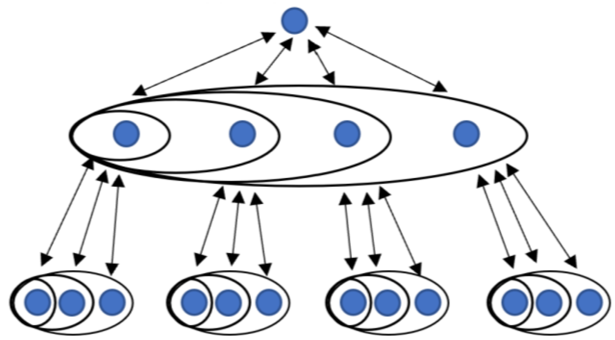


Figure 2.2 Representation of multiplicative reasoning for 4 × 3 (Clark & Kamii, 1996)

As children transition from additive to multiplicative reasoning, a reconceptualisation of the notion of unit is required. Hiebert and Behr (1988) note that there is an inherent difficulty in this transition due to the fact that, "A change in the nature of the unit is a change in the most basic entity of arithmetic" (p. 2). This shift from the construction of single entities as units in addition to the construction of composite units composed of multiple entities in multiplication is described by Steffe (1988) as constructing a composite unit and, by Lamon (1994), as unitising. This idea is fundamental to thinking multiplicatively and necessitates a change in what counts as number. For example, five individual objects can now be counted as a single unit of five.

The essence of these ideas was captured nearly three decades earlier by Dienes and Golding (1966), who, in describing multiplication, stated that:

> It is important to realise that in this operation we have gone beyond the idea of addition. It is true that the same answer can be obtained to the problem by an addition of the three terms as by multiplication by three. Just because the answer is the same does not mean that the operation is the same. Multiplication involves a new kind of variable, namely the multiplier, which counts *sets*. The multiplier is a property of sets of sets. The multiplicand is a property of sets. (p. 34; *emphasis in the original*)

Also significant in the transition from additive to multiplicative situations is a change in the *nature* of the referents, a change in what numbers are about (Schwartz, 1988).

Whereas additive situations involve *extensive* quantities, or actual amounts, multiplicative situations almost always require the manipulation of *intensive* quantities (relationships between quantities) which cannot be directly counted, nor measured. For example, calculating price per kilogram is an intensive quantity.

In order to provide another way of understanding multiplication, Confrey (1994) advocated for a construct called ‘splitting’, which is defined as “an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram” (p. 292). The focus of splitting is on the one-to-many action and she suggests that splitting should be introduced as a complement to other strategies related to counting (such as repeated addition), in order for students to build multiple concepts of multiplication simultaneously. Essentially, this approach begins with a single unit and acts upon it with successive n-splits, as opposed to beginning with composite units of singletons. Confrey and Smith (1995) showed evidence that, in the minds of children, this splitting action, which results in creating equal parts or copies of an original, develops naturally as a way to structure problem situations involving halving, doubling, sharing, similarity and magnification. Siemen and colleagues (2011) argue that splitting is “inherently tied to multiplicative operations of replicating, magnifying and shrinking” (p. 357).

In a shift towards a relational paradigm, Davydov’s (1992) work focused attention on multiplicative relationships and the relationships *between* quantities. He stated that multiplication is “genetically tied to counting (measuring)” (p. 12) and the two factors “indicate the relationship of the units themselves and the relationship of the object to the unit with which the count was actually carried out” (p. 12). He argued that the identification and construction of the multiplier and the multiplicand are necessary, and that the simultaneous co-ordination of these factors are what signifies a multiplicative response. However, as Boulet (1998) pointed out, “simply by considering the more general sense of *measure* rather than the particular sense of *count*, Davydov’s definition becomes foundational in the domain of all real numbers, no longer confined to the set of natural numbers” (p. 17; *emphasis in the original*). Kosko (2020) suggests that, “careful attention is needed in how unit is conveyed, but also the particularly operands involved” (p. 16)

Davydov (2008) described a *multiplicative comparison relationship* as having three interrelated elements: a measured quantity, the unit of measurement and the number resulting from the measurement. In further developing this three-fold idea of relationship, and in an effort to improve primary school students' understanding of both simple and complex multiplicative situations, Polotskaia and Savard (2021) have proposed a typology of three basic multiplicative relationships with accompanying graphical representations: multiplicative composition, multiplicative comparison and Cartesian product. This typology is helpful in orienting us to different multiplicative situations, two of which – multiplicative composition and Cartesian product – are important elements in the TT design, which will be discussed in more detail in Chapter 4.

2.2. Introducing multiplication through repeated addition

Encouraged by curriculum documents and embraced in teaching practice, the action of repeated addition is commonly used in the primary grades (K–3) as the initial (and, for some, the sole) means for introducing and working with multiplication (Askew, 2018). Consequently, it tends to become firmly entrenched as the dominant perception of multiplicative situations, both for students and for primary teachers. In studying third-grade teachers' practice, Kosko (2019) also found that numeric-only, groups-of or array models were predominantly used for teaching multiplication. According to Davis and Renert (2013), although it appears to be very important to provide young learners with varied approaches in early mathematics learning to topics such a multiplication, teachers often select approaches that fit particular aspects of a concept (such as multiplication-as-repeated-addition). These dominant approaches may be so well rehearsed that they preclude teachers from considering other approaches that could nevertheless also be beneficial for student learning.

Repeated addition, hopping along a number line, skip counting and equal grouping (see Figures 2.3 a–d) are all closely related and detail prevalent methods for introducing multiplication by many elementary mathematics teachers. Equal grouping, using the expression '*a* groups of *b* objects' can often be found in primary school textbooks as a way of representing multiplication pictorially. The products are represented by the sum total of all the elements in the groups or the cardinality of the set (Davis & Renert, 2013).

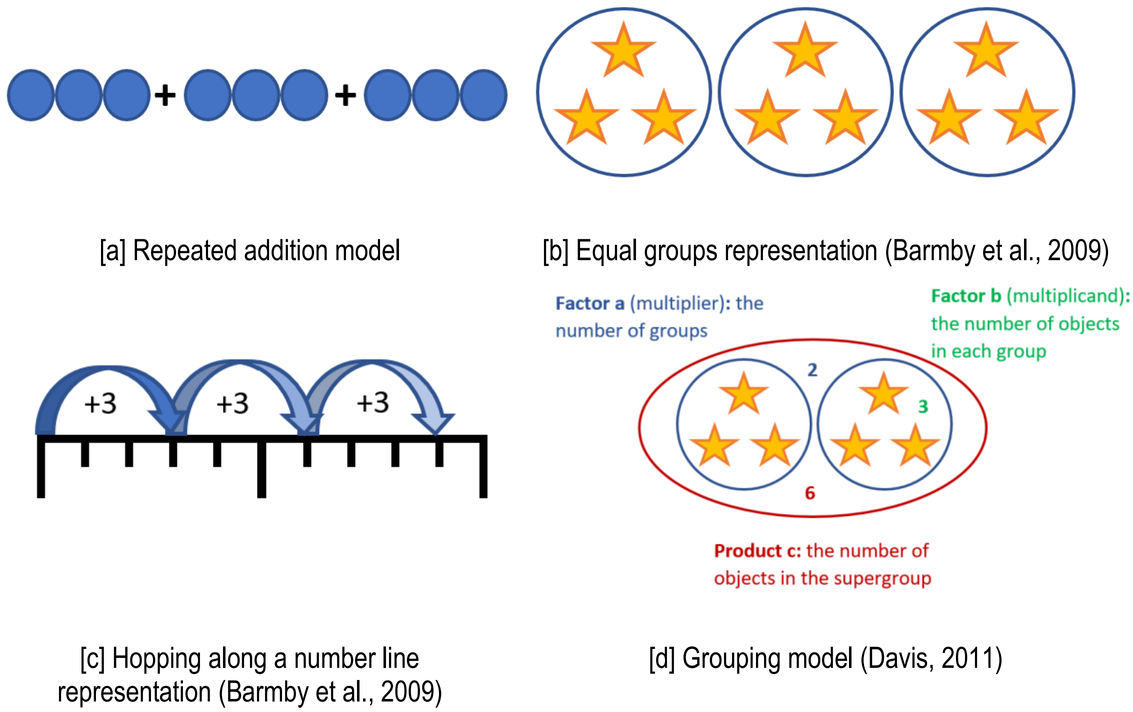


Figure 2.3 Multiplicative models

Each of these representations involves thinking of one quantity at a time, encourages thinking related to equal groups and prompts addition as the method of calculation (Barmby et al., 2009). Although some of the visual representations, such as the number line, provide support with the calculation, none of these models easily illustrates the characteristics of the commutative or distributive properties. In fact, if the numbers were swapped in the diagrams, the representations would be different, making it less obvious how the commutative law applies.

Using the process of repeated addition, the multiplication of $a \times b$ can be solved by adding a, b times or alternately by adding b, a times (2 × 3 can be seen as either 2 + 2 + 2 or 3 + 3). Fischbein and colleagues (1985) found that, "*each fundamental operation of arithmetic generally remains linked to an implicit, unconscious and primitive intuitive model*" (p. 4; *emphasis in the original*). Repeated addition is considered one of these models and is frequently chosen to introduce children to multiplication, because it appears to provide the most natural way for children to understand this new concept. However, learners commonly associate certain effects with specific operations which may lead to overgeneralisations, such as the belief that multiplication always makes bigger (Bell et al., 1981).

In a study of grade 2 and 3 students, Mulligan and Mitchelmore (1997) found that students would progress from direct counting to repeated addition and then multiplicative operations. They noted that students showed a strong preference for the repeated addition model, using strategies such as skip counting, rhythmic counting or additive calculation, which occurred because students were able to recognise the equal-sized group structure: “The intuitive model employed to solve a particular problem therefore does not reflect any specific problem feature, but rather the mathematical structure that the student is able to impose on it” (p. 327). In investigating two student’s multiplicative understanding, Larsson and colleagues (2017) found that, instead of connecting calculations to models of multiplication, both students relied on equal groups or repeated addition which limited their use of commutativity.

There is, however, research that indicates that young learners possess some ability to reason multiplicatively prior to formal instruction (Bakker et al., 2014), which Confrey (1994) also observed as evident in children’s ability to imagine repeated splits before exposure to repeated addition. In the case of young learners in Australia, research findings by Cheeseman and colleagues (2020) suggest that there is value in encouraging mathematical visualisation of multiplicative thinking and in utilising drawing as a form of mathematical communication for these students. In South Africa, Askew and colleagues (2019) show that the use of context-based problems and array images helped young learners create pictures or diagrams that supported their understanding of multiplicative structures. Despite this, repeated addition models of multiplication are still prevalent in elementary school mathematics.

2.3. Why repeated addition models can become problematic

Researchers have long highlighted the challenges experienced by students when transitioning from additive to multiplicative reasoning and have identified significant limitations when multiplication is characterised as repeated addition (e.g. Greer, 1992; Nunes & Bryant, 1996; Simon et al., 2018; Thompson & Saldanha, 2003).

Bechtel and Dixon (1967) cautioned that the role of specific topics in mathematics, such as multiplication, must be considered in relation to the overall structure of the system, in order to ensure that the concepts are presented in a mathematically sound manner:

“Failure to do this often leads to confusion for a student who must relearn or radically alter a previously learning principle” (p. 373) and that, when considering multiplication as repeated addition from a mathematical viewpoint, “such a presentation becomes a deterrent to an understanding of a basic mathematical principle” (p. 373).

In the case of multiplication, many elementary teachers commonly introduce it using the repeated addition model, which, “tacitly affects the meaning and use of multiplication” (Fischbein et al., 1985, p. 5). Findings show that repeated addition models are incomplete and, even though they eventually come into conflict with formal concepts of multiplication, by that point, “*The initial didactical models seem to become so deeply rooted in the learner’s mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct*” (p. 16; *emphasis in the original*).

Multiplication is not viewed as commutative in this model; rather, one factor represents the multiplicand, which must be a whole number, and the other is the multiplier, which can be any positive quantity. Therefore, in the repeated addition model, multiplication always makes bigger because the multiplicand is always a whole number, a fact which becomes problematic later when students begin to multiply fractions or decimals. They encounter difficulty as a result of the “constraints of the underlying tacit model. The constraints may force the choice of an inadequate operation” (p. 4). As the first interpretation of multiplication that students learn, the repeated addition model tends to become entrenched and resistant to change, continuing to influence students’ mathematical reasoning, even after they have progressed to higher level mathematics where this model is no longer sufficient.

Elementary mathematics teachers are therefore faced with a dilemma:

> On the one hand, if one continues to introduce the operations of multiplication and division through the models described above, one will create—as our findings demonstrate—strong, resistant, and, at the same time, incomplete models that soon will come to conflict with the formal concepts of multiplication and division. On the other hand, if one tries to avoid building the ideas related to arithmetical operations on a foundation that is behaviourally and intuitively meaningful, one certainly will violate the most elementary principles of psychology and didactics. (Fischbein et al., 1985, p. 14)

In their work on concept study with teachers, Davis and Renert (2009) exposed the complexity of multiplication across grade levels, and the different metaphors, images, analogies and applications that emerged when teachers engaged more deeply with the question ‘What is multiplication?’. The number of interpretations they were able to generate about a seemingly ‘basic’ mathematical concept surprised the teachers, and by delving more deeply into multiplication, they became more aware of “the dynamic, embodied, and enacted dimensions of mathematics […] And thus, an opening is created for teachers to consciously participate in bringing about a more authentic mathematics for teaching” (p. 43).

While acknowledging the value of repeated addition when introducing multiplication, Jacob and Willis (2001) suggest that, “to maintain that interpretation of multiplication is ultimately disabling because it does not provide children with important multiplicative structures” (p. 306). Lay (1963) pointed out that zero and one merit special attention as multipliers and that young students’ difficulties with multiplication first appear as stumbling blocks in relation to zero and one facts when multiplication is defined as repeated addition. Steffe (1988), too, argued that repeated addition is an inadequate model for many situations requiring multiplication, primarily because of its foundation in the counting of single rather than composite units.

Though I will discuss Davydov’s (1992) model of multiplication in more depth in Chapter 4 when examining the design of TT, it is worth noting that his double change-in-units approach to multiplication does not correspond well with repeated addition. Grounded in measurement and ratio, the multiplicand (or unit of measure) must be established first, prior to creating the multiplier (or unit quantity). This process highlights the relationship between the number of units and the quantity of each unit which is established first when creating composite units.

Approaches to multiplication that are grounded in repeated addition become problematic when students begin to engage with mathematics that requires a direct capacity to think multiplicatively (e.g. Siemon et al., 2005). Consequently, multiplication may prove a crucial turning point in student learning, possibly a turnstile for mathematical competence. Rather than relying so heavily on repeated addition, Siemon and colleagues (2011) advocate a greater emphasis in the early years of schooling on sharing and splitting as an approach for developing multiplication. The importance of multiplicative reasoning for elementary students' mathematical learning for early and later schooling is well documented (e.g. Askew, 2018; Hackenberg & Tellema, 2009; Norton et al., 2015).

In order to reason multiplicatively, students must demonstrate the ability to work flexibly and efficiently with "the concepts, strategies and representations of multiplication (and division) as they occur in a wide range of contexts" (Siemon et al., 2005, p. 2), concepts that include direct and indirect proportion. Among other things, such reasoning involves learners viewing situations of comparison in a multiplicative rather than an additive sense. As students progress to larger whole numbers – and then to decimals, fractions, percentages, ratios and proportions – multiplicative reasoning becomes key to a large number of mathematical situations found in upper elementary and middle school (Brown et al., 2010).

Extensive research has documented difficulties students have with employing multiplicative reasoning in middle school and beyond, which Brown and colleagues (2010) claim has not improved since the 1970s. Some argue that the underlying cause for student difficulties is an over-reliance on additive structures when multiplicative reasoning is required (e.g. Askew, 2018). Sun and colleagues (2019) assert that, "conceptual understanding of multiplication, or simply put, multiplicative reasoning is the missing foundation, which is less often discussed than multiplication models in the international literature" (p. 5). Limited student experience with different multiplicative situations is also proposed as a contributing factor to this problem (e.g. Downton & Sullivan, 2017). Furthermore, Askew (2018) contends that the lack of development of multiplicative reasoning in the primary grades is "a consequence of predominant approaches to teaching multiplication limiting access to opportunities through which thinking functionally can emerge" (p. 1).

Now that some background related to the mathematics and the teacher parts of the ensemble has been provided, in the next section, I will shift my focus towards the technology and its related mathematics. I will begin by highlighting the links between number sense and using fingers to do mathematics, and then follow up by connecting these ideas to the emergence of touchscreen technologies and the opportunities these have created for elementary school mathematics.

2.4. Using fingers to do mathematics

The role of finger usage in young children's development of number sense has received considerable attention, both in psychology and in mathematics education research (e.g. Baroody, 1987; Fuson & Kwon, 1992; Butterworth, 2005). In his work with finger-math, Gattegno (1974) highlighted the value of fingers as tools that students carry with them for learning mathematics, and demonstrated how classes of equivalent sets of fingers can exemplify cardinal numbers. Additionally, he pointed out that complementary subsets can be created by using folded and unfolded fingers, and that the union of these two subsets will always result in the whole set of fingers. Through carefully orchestrated games, fingers could also be used as a way of naming any numeral in the ordinary system and to write numerals in a table now referred to as a Gattegno chart.

Neuroscientific research suggests that there is a functional and beneficial relationship between early numerical representations involving and invoking fingers, and the development of later mental representations of number (e.g. Domahs et al., 2008). There has been increasing research on the relationship between using fingers to do math and basic numerical performance in children (Noël, 2005; Penner-Wilger et al., 2007). In a synthesis of constructivist mathematics education and embodied neurocognitive research, Soylu and colleagues (2018) contend that, "features of numerical cognition may be grounded in the finger sensorimotor system" (p. 127), which extend beyond childhood and into adulthood.

With the technological development of multi-touch devices, there emerge new opportunities for research to examine the interactions among fingers, touchscreen technology and the mathematics afforded by these applications.

2.5. The emergence of touchscreen devices

Although various forms of digital technology have been implemented in British Columbia schools since the early 1980s, it was the 2010 release of Apple's iPad, with its touchscreen interface, that has provided an alternative to the keyboard and mouse input mechanisms, which were difficult for some users. This is particularly relevant for those children in Kindergarten to grade three, whose fine motor skills are still developing and, therefore, have less dexterity for manoeuvring a mouse or using a keyboard efficiently.

In comparison, touchscreen devices are more intuitive and user-friendly, and, through simple tactile actions such as pressing, tapping, swiping and pinching, users are able to interact with the technology directly through screen contact with their fingertips. Their ease of use, alongside the increasing availability of iPads in schools, offer elementary teachers access to new pedagogical resources and alternative methods which show promise in supporting early mathematics learning (see, for instance, Calder and Murphy, 2018, or Sinclair and Coles, 2017). Santi and Baccaglini-Frank (2015) summarise that, "[microworlds] activate the use of gestures, natural language, symbolic language, touching, tapping, dragging, artifacts, etc. that enhance the student's sensory motor experience in space and time" (pp. 226–227).

Consequently, there is a growing body of mathematics education research that seeks to build an understanding of how the affordances of touchscreen devices, and the digital applications used with them, can facilitate mathematics learning (e.g. Attard, 2015; Barkai et al., 2022; Sedaghatjou & Rodney, 2018). Given the requisite screen–finger contact for the use of multi-touch devices, considerable attention has been given to the role of the fingers and their associated movements when engaging with mathematics using this form of digital technology (e.g. Ferrara & Savioli, 2018; Sinclair & Coles, 2017). Baccaglini-Frank and Maracci (2015) examined the affordances of multi-touch devices and their potential influence on developing young children's mathematical abilities in Northern Italy. They observed links between the multi-touch devices and the strategies used by young children to represent numbers through their fingers which can potentially foster their development of number sense.

The research findings about using fingers to do mathematics outlined in the previous section, which have highlighted the significance of the relationship between finger-based interactions and number sense, have also influenced the development of new learning technologies in mathematics, such as *TouchCounts* (TC)[2] (Sinclair & Jackiw, 2011) and *TouchTimes* (TT) (Jackiw and Sinclair, 2019). Through the use of multi-touch affordances, both of these interactive applications have provided new and different ways for children to encounter, engage with, investigate and process mathematical ideas such as counting, numbers and arithmetic. By engaging 'directly' through finger touch,

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[2] For a more detailed description of how *TouchCounts* operates, see Jackiw and Sinclair (2017) or Sinclair and Heyd-Metzuyanim (2014).

children can summon and make ‘objects’ appear on an iPad screen, and, through the dynamic actions of their fingertips, it is possible to interact with mathematical phenomena in an open-ended and exploratory way.

2.5.1. TouchCounts

The *TouchCounts* design was “intended to offer an expressive environment in which learners could create and relate mathematical objects directly with their fingers and hands” (Sinclair & Heyd-Metzuyanim, 2014, p. 84). Research related to the use of TC in particular has shown that there may well be a strong connection between the ways that children use their fingers when interacting with the iPad application and the ways they come to think about numbers (Sinclair & Sedaghatjou, 2013). The linked interaction between fingers and eyes – between the tangible and the visible – enables new ways of thinking about number (Sinclair & de Freitas, 2014), which may include the development of symbol-symbol awareness, which characterises ordinality (Sinclair & Coles, 2017) and the development of ‘gestural subitising’ may be assisted through the task of placing several or all available fingers simultaneously on the screen (Sinclair & Pimm, 2015b).

Sedaghatjou and Campbell (2017) found that the multimodal affordances of TC, which include visual and auditory feedback in response to the user’s touch(es) on the iPad screen, were beneficial to a young boy’s development of cardinality. Tangibility was also found to be of particular significance in Sinclair and Pimm’s (2015b) examination of the mutually interactive visible, audible and tangible effects of TC on young children’s coming to terms with counting and early arithmetic. However, the input actions used by the children were noted to be “different from those [gestures] discussed in the mathematics education literature in two ways: they involve contact with a screen and they perform an action” (Sinclair & Pimm, 2014, p. 210).

Calder and Campbell (2016) have pointed out that, “the visual and dynamic elements of engaging mathematical thinking through digital technologies reposition both the types of knowledge and understanding required and the ways in which learning emerges, which simultaneously shape the learning experience in a range of interrelated ways” (p. 50). After using TC in a classroom-based intervention with first-grade students, Ferrara and Savioli (2018) described seeing the children’s “mathematical doing not only in terms of creatively touching numbers, but also in terms of imagining and feeling quantities” (p.

243). This is an interesting blend of finger use, mathematics and technology that can also include aesthetic and affective elements.

2.5.2. TouchTimes

Whereas the focus of TC is on counting, addition and subtraction, *TouchTimes* grew out of the findings of TC and extends the operations to multiplication. Also, a multi-touch digital application, TT is composed of two microworlds, Grasplify and Zaplify. Its design enables children to create tactile, pictorial and symbolic representations of multiplicative situations that adjust simultaneously in response to their fingertips on the screen (and will be described in greater detail in Chapter 4).

Most of the existing research on TT is teacher-focused. However, there have been some research results related to student learning. In her research on the reciprocal use of pencil-and-paper and the Zaplify microworld as a duo of artefacts, Güneş (2021b) found that a five-year-old's experiences with each artefact was enriched as a result of reciprocal use and that each artefact also mediated the relationships necessary for multiplicative thinking.

While observing two second-grade students (who had yet to learn about multiplication in school) experimenting with Grasplify, Chorney and Sinclair (2019) noted that the pair expressed multiplication in simultaneous ways that involved the co-ordination of three quantities. When examining the use of Grasplify by two third-grade students, Chorney and colleagues (2019) suggested that the use of two hands in the application is a co-ordination of two quantities, which is relevant to multiplicative reasoning. Further analysis by Güneş (2021a) revealed that, when asked to create a quantity in a specific way using Grasplify, it was through recurrent interactions with the digital technology that the two third-grade students were able to solve a unitising task.

2.5.3. Other multi-touch applications

Although many of the digital applications that have been developed to target mathematics can be categorised as 'edutainment' (Larkin, 2016), there are some multi-touch applications designed for young learners that have potential for improving student learning of mathematics (e.g. Baccaglini-Frank & Maracci, 2015; Chorney & Sinclair, 2018; Hewitt & Jones, 2022).

Designed to support early numeracy, Fingu is a virtual manipulative game which aims to assist the development of proficiency with number and arithmetic competency in young children. This multi-touch application engages children in subitising and the identification of part–whole relationships (Barendregt et al., 2012; Holgersson et al., 2016). In a case study of a pre-school child using Fingu, Tucker and Johnson (2020) observed that conceptually congruent gestures supported the development of embodied early number sense. In their analysis of the educational potential offered by the software applications Fingu and Ladybug Count, Baccaglini-Frank and Maracci (2015) noted that both multi-touch digital applications can potentially foster the development of important aspects of number sense in preschool children.

Using a web-based variation of *The Geometer's Sketchpad* (Jackiw, 1991, 2001) called Web Sketchpad, Chorney and Sinclair (2018) engaged first-grade students with open and exploratory geometry sketches. Using their fingertips on the iPad screen, students would drag or move objects in order to discover how each object's reflected image moved in response. Findings indicated that students expressed the concept of symmetry in numerous ways which were intricately connected through gestures and body motions during their engagement with the geometry sketch.

In a case study of a fifth-grade student using the digital iPad application *Stick and Split*, Hewitt and Jones (2022) found indications of a gradual shift from repetitive use of additive thinking towards the use of multiplicative thinking in later, more demanding, tasks. This, they argued, is due to the implicit feedback inherent in the computer application.

In the previous sections, I have reviewed some prior research findings related to multiplication and multi-touch digital technology, and I will now turn my focus to the third theme, which examines research related to the mathematics teacher implementing technology.

2.6. Teaching with technology

The pedagogical practices of the teacher, which may include the use of technology as a teaching or learning tool for mathematics, is one of the most significant influences on student engagement (Hayes, Mills, Christie & Lingard, 2006). With the increasing prevalence of digital technology in schools, mathematics education researchers have a continued interest in how teachers choose to implement these resources for the teaching of mathematics (e.g. Monaghan, 2004; Thomas & Palmer, 2014) and their effect on student learning (e.g. Calder & Murphy, 2018; Sinclair & Baccaglini-Frank, 2015).

As Monaghan and Trouche (2016) point out, “many of the actions of doing mathematics involve selecting, using and creating tools” (p. 4), and, therefore, students require exposure to, instruction with and experience of using a diverse range of tools, including digital technology. There is a variety of digital technologies which have been specifically designed to provide valuable mathematics learning experiences. However, Ruthven (2014) found that it is the relevant expertise of the teacher that determines whether or not these technologies are successfully integrated into everyday teaching practice. For many teachers, not only is it daunting to integrate technology into their practice, it is also a complex and challenging endeavour to leverage skilfully these digital tools for student learning (Trigueros et al., 2014), which Sinclair and colleagues (2020) also found to be true when primary teachers began using the iPad touchscreen application *TouchTimes* with their students. As Guin and Trouche (1999) explain, it is the role of the teacher to draw attention to and help students make appropriate connections between the digital technology and the mathematical knowledge they evoke.

Although digital technology is providing new resources and means that show promise in supporting the mathematical learning of young children (e.g. Sinclair & Baccaglini-Frank, 2015), in schools, however, there is often a mismatch between the developers’ intended purpose of technology devices and how they are implemented and used by teachers (e.g. Clark-Wilson et al., 2016). Contributing to this mismatch are obstacles and constraints, such as the availability (or lack of) resources (both technology resources for student use and professional development or pedagogical resources for teacher use) and the impact of teacher confidence and attitude towards technology and its use in the classroom (Thomas & Palmer, 2014).

Many elementary schools are investing in mobile technologies such as iPads; however, their integration and use in classrooms has often occurred without accompanying professional development support for teachers. Consequently, the ability of teachers to use these forms of technology effectively to enhance teaching and learning has proven to be challenging (Attard & Curry, 2012). Furthermore, Larkin and Milford (2018) have found that many of the apps downloaded onto these devices for classroom use are chosen, “without a strong conceptual, pedagogical, or methodological underpinning” (p. 12).

Attard and Curry (2012) observed that teacher inexperience with choosing iPad applications was problematic during their research on the influence of the introduction of iPads on student engagement with mathematics in an Australian primary classroom. Rather than opting for applications that would facilitate rich mathematical learning, the use of games that were essentially rote drill and practice of basic facts was prevalent. Given that the majority of ‘educational apps’ available in the iTunes App Store were found by Highfield and Goodwin (2013) to be reflective of ‘drill-and-practice’ techniques, this is, perhaps, not surprising.

Mishra and Koehler (2006) argue that introducing technology into education is not sufficient. In surveying secondary mathematics teachers in Australia, Goos and Bennison (2008) found that access to technology and professional development were both influential factors in teacher use of digital technology in their classrooms. In order to transition from traditional mathematics teaching to that which includes digital technology, it is necessary for teachers to commit to learning how to use the technology as a pedagogical tool effectively (Pierce & Ball, 2009).

Clark-Wilson (2010) found that teachers’ mathematical ideas shaped how they used digital tools, but that their teaching was also shaped by their increasing familiarity with the tool itself. In her analysis of a model–task dyad using TT’s Zaplify and pencil-and-paper, Güneş (2021c) found that though the end product of the task may be the same, it develops from different processes that mediate very different meanings of multiplication. This is an aspect that may be difficult for teachers when planning for and implementing new technology. Indeed, as Noss and Hoyles (1996) point out, “tools wrap up some of the mathematical ontology of the environment and form part of the web of ideas and actions embedded in it” (p. 227).

2.7. Summary

In this chapter, I examined three themes of mathematics education literature that are relevant to the teacher–technology–mathematics ensemble that is the focus of this dissertation. I began with the mathematics and teaching aspects of the ensemble and highlighted research on multiplication and multiplicative thinking, and examined how multiplication is commonly introduced to many elementary students, while explaining why this can be problematic. I then addressed the technology and mathematics components of the ensemble, and conducted a brief overview of research findings that draw attention to the links between using fingers to do mathematics and number sense before describing the emergence of touchscreen technologies and the opportunities these create for elementary school mathematics. Finally, I brought together all three aspects of the ensemble and outlined some of the complexities and the challenges experienced by mathematics teachers when using digital technology as a pedagogical tool.

A more detailed examination of technology in the context to mathematics teaching will occur in the theoretical considerations chapter that follows.

Chapter 3. Theoretical Considerations

> *"*Technology can *shape teaching and learning mathematics*, while reciprocally *being shaped* by its use." (Maschietto & Trouche, 2010; *emphasis in the original*)

Throughout this chapter, I will describe, in detail, the theoretical constructs that were essential in guiding the development of my research questions, as well as my data analysis. I will begin by returning to the story of the journey that brought me to this point and, more specifically, I will recount how I arrived at the theory used for the research shared in this dissertation. Then, beginning with the theory of instrumentation (Vérillon & Rabardel, 1995), and the adoption of some if its key ideas into mathematics education as part of the instrumental approach (Artigue, 2002; Guin et al., 2005; Guin & Trouche, 1999), I will then describe the theoretical constructs of double instrumental genesis (Haspekian, 2007), instrumental distance (Haspekian, 2005), didactical landmarks (Haspekian, 2017) and instrumental orchestration (Guin & Trouche, 2002). Finally, I will bring the chapter to a close by sharing my three research questions.

3.1. Moving towards the theory

My introduction to theoretical frameworks and their use occurred during my first doctoral level course. Our assigned reading for the initial class was an article written by Luc Trouche (2004), and was my first exposure, not only to theory in general, but also to the theory of instrumental genesis, specifically, and its ideas about humans and tools. This was also the theory that I used or, perhaps more accurately, attempted to use, while writing my first doctoral paper, though I would not think much about, or utilise it again, during the next four years. Despite this, the instrumental approach to tool use, and the process of instrumental genesis, would become the underlying theoretical framework to provide the foundation for much of the theory that I draw from for this dissertation, the final written work of my doctoral journey.

My return to instrumental genesis occurred as a result of an article written by Paul Drijvers and colleagues (2010) that used the instrumental approach to tool use and the notion of instrumental orchestration to examine teacher behaviour when using digital technology. Through analysis that was both theory- and data-driven, six different orchestration types were identified and described in the article and, while reading these ideas, I thought to myself, "*I* could do something like this!" At the time of reading, I needed to write a conference paper for the annual SFU Mathematics Education Doctoral Students Conference, and though the article's focus was on a group of secondary mathematics teachers and their implementation of a Java applet, I had a clear vision of how I could apply many of the ideas introduced in the article to my own data.

Inspired, I began my analysis by attempting to imitate how the various orchestrations were identified by Drijvers and colleagues, using my own data of an elementary school teacher implementing the touchscreen technology, *TouchTimes*. I was hoping that the elementary school context, and the use of a different form of digital technology by the teacher, would provide enough differences from the original article, still to prove interesting. Although not necessarily adding anything new to the field about the theory, or the way it was used, this copy-cat approach enabled me to engage with and develop a better understanding of instrumental orchestration, and how to use and apply these theoretical ideas to my own work.

Shortly after completing this conference paper, I was invited to submit a chapter for a book that was examining, from an international perspective, teachers' use of digital technology in the mathematics classroom. Wanting to continue my work with the instrumental approach, and to push further using the construct of instrumental orchestration, I began to examine the video recordings from the larger *TouchTimes* project, in the hope of identifying common *sequences* of instrumental orchestrations used by the teachers. I created case studies of three different elementary teachers who were implementing this new digital technology with students, and wanted to identify and highlight the specific orchestrations each were using while teaching with TT.

During the review process, after submitting the chapter, it was suggested that it may be helpful to describe other theoretical frameworks that have been used to analyse the use of applications for teaching mathematics in primary schools. This reviewer then directed my attention towards double instrumental genesis. And though I was unfamiliar with this notion prior to this recommendation, as I began to read and learn more about it, it seemed that an important piece of theory that would be helpful for examining my data had fortuitously fallen into my lap. With these new ideas, I could finally see just what needed to be done in order to move forward with writing my dissertation.

3.2. Situating the theoretical foundation

I begin by tracing the emergence of and outlining some of the theoretical elements of the instrumental approach to tool use (Artigue, 2002; Guin et al., 2005; Guin & Trouche, 1999). Following this, I will introduce double instrumental genesis (Haspekian, 2007), instrumental distance (Haspekian, 2005) and didactical landmarks (Haspekian, 2017). I will then summarise the construct of instrumental orchestration (Guin & Trouche, 2002), highlighting some of its conceptual tools and results, as well as outlining some of the additional orchestrations that have emerged from more recent research. Finally, I will briefly consider sociocultural aspects of instrumental orchestration, as well as acknowledging the influence of the artefact itself, before bringing the chapter to a close with my research questions.

3.2.1. The instrumental approach

The *theory of instrumentation* on the human use of tools is a psychological and sociocultural framework created in cognitive ergonomics (Vérillon & Rabardel, 1995). The *instrumental approach* is a theoretical construct that was developed for analysing technology-mediated learning in mathematics education (Artigue, 2002; Guin et al., 2005; Guin & Trouche, 1999), and adopted two key ideas from the theory of instrumentation. The first was the distinction made by Vérillon and Rabardel between an *artefact*, a human-made material object, and an *instrument*, defined as a psychological construct. And the second significant idea involves *instrumental genesis*, the process in which a human transforms the artefact into an instrument through its use. An instrument does not exist in itself, rather, an artefact must first be adapted to and appropriated into the user's activity before the psychological construct that is an instrument, develops.

The interactions that occur during the process of instrumental genesis involve complex, simultaneous, two-way processes of instrumentation and instrumentalisation. During *instrumentation*, the artefact itself influences or shapes the actions or thinking of its user whilst the user integrates and shapes the artefact into a functional instrument for specific purposes through the process of *instrumentalisation* (Rabardel, 1995/2002). The artefact is not passive, rather, it makes its mark on the user during instrumentation, and “far from ‘investing the world with his vision’, the computer user is ‘mastered by his tools’” (Noss & Hoyles, 1996, p. 58). Whereas, during instrumentalisation, there is a differentiation process directed towards the artefact that is unique to each individual user (Trouche, 2004) and may include different stages of experience, such as, a discovery phase, a phase of personalisation, or even a phase where the artefact is transformed in ways that may or may not be intended by the designer. Ultimately, during instrumentalisation, “the user’s conception of the instrument is formed through use” (p. 295).

3.2.2. Double instrumental genesis

In the case of teachers, the transformation of an artefact into an instrument for pedagogical purposes involves even more complexity. Haspekian (2007) observed that mathematics teachers, prior to implementing digital technology for pedagogical purposes, undergo dual experiences of personal and professional instrumental geneses a process she referred to as double instrumental genesis. During personal instrumental genesis, the artefact becomes a personal, working instrument for mathematical activity; a process which may be similar for teachers and students. In addition to this, however, in order to appropriate and utilise the artefact effectively as a didactical instrument, the teacher must also undergo a professional instrumental genesis.

“The teacher’s professional genesis with the tool is much more complicated as it includes the pupils’ instrumental genesis” (Haspekian, 2014, p. 254). It is necessary for the teacher to understand, predict and plan in advance, for the process of instrumental genesis that students will experience while engaging with the digital technology, which creates an additional layer of complexity in the teacher’s professional genesis.

> The instrument that is created as a result of this process of professional genesis (for instance the ‘spreadsheet as a tool to teach algebra’) is different from the instrument built through a personal genesis (the spreadsheet as a tool of personal work of calculation, plotting, data treatment, etc.). From the same artefact, two instrumental geneses (that may have interferences/interactions on each other) lead to two different instruments. (p. 254)

A teacher’s personal and professional instrumental geneses are not necessarily independent of each other, which Haspekian (2014) observed during one teacher’s implementation of spreadsheets over two years. She noted that the two processes “interacted in a relational sense” (p. 256), which sometimes caused *interferences*. In the examples given, these interferences between the personal and professional instrumental geneses of the teacher were at times related to the purpose of using a spreadsheet, for personal calculation or as a didactic instrument, interferences which can also influence the instrumental genesis of the students.

Additionally, when adopting digital technology into their pedagogical practice, teachers may experience what Clark-Wilson (2010) refers to as a *hiccup*, or “the perturbation experienced by teachers during lessons stimulated by their use of the technology, which illuminates discontinuities with teachers’ knowledge” (p. 2). Furthermore, the integration of technology may also displace other previously used ways of teaching, and consequently, the teacher must then develop and implement new teaching methods for use with the digital technology (instrumentation). During the process of professional instrumental genesis, while teachers are transforming a digital tool into a new didactical instrument, they may also experience a phenomenon called instrumental distance (Haspekian, 2005), which will be explained further in the next sub-section.

3.2.3. Instrumental distance

While researching teacher integration of spreadsheets for teaching algebra in France, Haspekian (2005) noted deviations, or gaps, which were caused by the impact of introducing something new (the use of spreadsheets) into previous ways of teaching mathematics (algebra). This gap, referred to as *instrumental distance*, provides a means of focusing attention on how the digital tool affects the mathematical concepts and conceptualisations (instrumentation). It is also useful for examining the changes that occur with the introduction of a digital tool, thus, providing a way of explaining teacher resistances related to the difficulties of integrating technology into their pedagogical practice. The instrumental distance must be large enough that the advantages of using the technology are apparent for the teacher, yet small enough not to discourage the teacher from making the extra effort required for incorporating something new into their pedagogical practice.

Haspekian (2011, 2014) found that, over time, there was a tendency for teachers to minimise the instrumental distance, and one way this was done was by using spreadsheets where the curriculum specifically mandated their use for mathematics.

When presenting a complex technological tool, familiar content that students had already been introduced to was used. Once students gained more familiarity with the digital technology, then teachers would use it to develop new mathematical knowledge. In order to highlight similarities in the mathematics, the teacher was also more intentional in alternating between the use of spreadsheet technology and the use of paper-and-pencil activities. This was referred to as the old/new game and contributed to reduce the distance between the familiar paper-pencil environment and the new technology situation, while creating a greater proximity to the mathematics within both environments.

In more recent work, Haspekian and Nijimbéré (2016) examined the introduction of 'algorithmique'[3] at the high school level in France and found that teachers used two possible approaches to this new domain: one that focused on the mathematics but minimised the technological context, and the other which was oriented towards the computer programming but was more distant from the mathematics. Though the focus was on teaching a new domain, rather than on the integration of a new digital technology to teach mathematics, the phenomena of distance reduction was also observed, leading to an extension of instrumental distance to *distance from the usual mathematics practices*.

With the introduction of a new French mathematics curriculum in 2015, computer science, coding, programming, robotics and 'algorithmique' are now required teaching beginning in primary school. Consequently, this provided new opportunities for the examination of instrumental distance experienced by primary school teachers using Scratch programming software or Bee-Bot robots for the first time when teaching 'algorithmique' and computer science (Haspekian, 2017, 2018; Haspekian & Gélis, 2021). Emerging from this research was the importance of didactic landmarks for teachers, an idea which will be explained further in the sub-section that follows.

[3] The word 'algorithmique' in French curriculum is related to the overlapping use of algorithms in mathematics and in computer programming. When Haspekian (2017) translated 'algorithmique', a noun, into English, the word chosen was 'algorithmic', which is an adjective and does not clearly convey the meaning of this term. Therefore, I have chosen here to use the French word in reference to this idea.

3.2.4. Didactical landmarks

Haspekian (2017) describes the function of *didactical landmarks*[4] as reference markers referred to by teachers during the development of their teaching practice using digital technology. The theoretical tools used for examining didactical reference markers are drawn from the didactical and ergonomic framework of the *Double Approach* (Robert & Rogalski, 2002), which describe teachers' practices as being complex, coherent and stable objects that "result from singular (personal) recompositions from knowledge, representations, experiences, and individual history according to belonging to a profession" (p. 508; *my translation*).

These practices consist of five components: cognitive, mediative, institutional, social and personal. The didactical perspective of the framework relates to the teacher's activity, and is comprised of the cognitive component, which involves the mathematical goals and intentions of the teacher (e.g. task selection, classroom management, choice of tools) and the mediative component, which involves the interactions that occur during a lesson between students, or between the teacher and students. The ergonomic aspect of the framework includes the institutional, social and personal components and recognises that, "teachers must be seen as professionals having craft knowledge, beliefs and previous experience whilst working in given institutional and social conditions" (Abboud-Blanchard, 2014, p. 302). The institutional component relates to the constraints of the institution, which may include curriculum, institutional guidelines, teacher training, scheduling, etc. The social component refers to the work conditions and environment of a school, resource availability, classroom culture, and teaching colleagues as individuals and as a community. The personal component includes such things as individual conceptions of knowledge, teaching processes, mathematics learning, willingness to take risks, comfort with technology, professional experience, etc.

It is worth noting that both Robert and Rogalski's Double Approach (2002) and Haspekian's (2007) notion double instrumental genesis emerged from France, where there is a strong tradition of research in didactics of mathematics. Though both of these

[4] I will refer to didactical landmarks as teacher reference points or reference markers throughout this thesis, as landmarks to me represent physical objects or features of land. I feel that the prior experiences or background knowledge that teachers are accessing in Haspekian's didactical landmarks are better referred to as reference points or reference markers.

theoretical constructions include the word ‘double’, each does so in relation to different ideas. The ‘double’ in Haspekian’s double instrumental genesis is used in reference to teachers’ dual experiences of personal and professional instrumental geneses when integrating digital technology into their teaching practice. Whereas, the ‘double’ in Robert and Rogalski’s Double Approach relates to the teacher’s concern for their students’ mathematics learning and by the various constraints of the teaching profession.

> The interconnections among practices from the perspective of their relationship with the mathematics-learning path of the students concerned and from the point of view of the teaching profession. In other words, we are concerned not only with the teaching procedures implemented by the teacher, which are analyzed in terms of the potential learnings of students, but also with the teacher as actor, whose activity is examined in the context of a particular professional field. (Robert & Rogalski, 2002, p. 505)

Haspekian (2017) and Haspekian and colleagues (2021, 2022) have used Robert and Rogalski’s (2002) Double Approach to focus on the personal component and to consider the other four components (cognitive, mediative, institutional and social) from the perspective of the teacher. Their research has shown how the personal component is comprised of knowledge from the cognitive, mediative, institutional and social components (Haspekian, 2017). If the distance is large between a teacher’s new and former practices, or if the teacher’s previous reference markers are largely disrupted without the consideration of new ones, then the distance will be problematic and the creation of new reference markers is necessary. However, teachers may be able to move away from former practices more easily and quickly if they are experienced and therefore already possess various reference marks and an increased ability to adapt and adjust old reference marks to create new ones (Haspekian & Gélis, 2021). Although institutional and social components were found to be more favourable for the integration of information and communication technology by teachers, barriers to it were mostly found at the cognitive and mediative levels (Haspekian & Fluckiger, 2022).

As previously described in earlier sections of this chapter, technological integration involves the appropriation of an artefact by the mathematics teacher, who then utilises this to support the implementation of the artefact for learning by their students. With the increasing availability of technological resources for mathematics teaching, there has been more attention from researchers into how teachers plan for and make decisions within an instrumentalised classroom. One of the conceptualisations used for this study has been that of instrumental orchestrations, which will be described in more detail in the subsection that follows.

3.2.5. Instrumental orchestration

First introduced by Guin and Trouche (2002), the term *instrumental orchestration* is used "to point out the necessity (for a given institution – a teacher in her/his class, for example) of *external steering* of students' instrumental genesis" (Trouche, 2004, p. 296; *emphasis in the original*). It entails, "the teacher's intentional and systematic organisation and use of the various artefacts available in a – in this case computerised – learning environment in a given mathematical task situation" (Drijvers et al., 2010, pp. 214–215).

There are three elements involved in an instrumental orchestration that teachers must take into consideration: a didactical configuration, an exploitation mode and the didactical performance. The *didactical configuration* involves the arrangement of the classroom environment and the artefacts in it (Trouche, 2004). Technological tools, as well as the tasks students are assigned to work on, may both be considered artefacts. An *exploitation mode* involves the choices made by the teacher to exploit the didactical configuration to the benefit of their teaching intentions. These decisions may include how tools or tasks are introduced and engaged with, student groupings (e.g. partner work), the role of the artefact and the techniques used by the students. Extending beyond the didactical configuration and exploitation mode, both of which can be planned for in advance, Drijvers and colleagues (2010) added didactical performance as a third component of instrumental orchestration, that occurs spontaneously during a lesson.

> A *didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals. (p. 215)

This includes how teachers respond to situations and/or students 'in the moment', as well as how lessons are adjusted 'on the fly' while teaching. This includes resolving unanticipated issues that arise related to the technology, the mathematics, or the students themselves.

After Trouche's (2004) introduction of the Sherpa-student orchestration, the collection of instrumental orchestrations was expanded considerably by the work of Drivers and colleagues (2010, 2013). Outlined in Table 3.1, there are eight instrumental orchestration types that can be used for whole-class, teacher-led instruction, and an additional five orchestrations for use with individual learners. The teacher may choose to use either teacher-centred or individual orchestrations to encourage student participation with the purpose of guiding the collective instrumental genesis of the entire class, or for assisting the individual instrumental genesis during one-to-one interactions with students.

Table 3.1 An overview of instrumental orchestrations

Orchestration type	Teacher's didactical intention	Whole-class	Individual
Technical-demo	To demonstrate techniques for tool use	✓	✓
Guide-and-explain	To explain or ask closed-type questions based on what can be seen on a computer screen	✓	✓
Explain-the-screen	To explain the mathematical content guided by what happens on the computer screen	✓	
Link-screen-board or Link-screen-paper	To connect the mathematical ideas and/or representations of the technology to the conventional ways of recording the mathematics	✓	✓
Discuss-the-screen	To discuss what is happening on the screen	✓	✓
Spot-and-show	To share student work samples for class discussion and/or teaching purposes	✓	
Sherpa-at-work	To have a student use the technology to present work and/or carry out teacher-directed actions	✓	
Board-instruction	To provide instruction on the board without use of, or reference to, technology	✓	
Technical-support	To provide technical support		✓

Source: Drijvers and colleagues (2013)

3.2.6. Extending the repertoire of orchestrations

Since Drijvers and colleagues (2010) identification of these different orchestration types, there has been continued interest by mathematics education researchers in the identification of orchestrations used by teachers while teaching mathematics with technology to students of various ages.

While exploring the potential synergy of two different theoretical frameworks: the concept of Technological Pedagogical Content Knowledge (Mishra & Koehler, 2006) for analysing teachers' knowledge, and the notion of instrumental orchestration (Trouche, 2004) to analyse teacher's actions, Tabach (2011) noted a *not-use-tech* orchestration. She argued that if the teacher intentionally chooses *not* to use technology that is available, that this too, is an orchestration type.

In their earlier research, Drijvers and colleagues (2010) focused on identifying orchestrations that occurred within technology-rich, whole-class teaching situations in eighth-grade mathematics classrooms. While examining how a twelfth-grade mathematics teacher adjusted his practice when using technological resources in his teaching, Drijvers (2012) observed the teacher repeatedly using a *work-and-walk-by* orchestration in order to assist students who were working independently. Tabach (2013), however, during her research in fifth- and sixth-grade classrooms, noticed that teachers would use a *monitor-and-guide* orchestration to observe student progress *and* provide in-person or electronic feedback to guide the independent learning of students. Additionally, in a situation where laptop computers on carts were being used and distributed to students, she also observed that teachers would sometimes provide instructions related to the technology that was not yet physically present, in what she called, *discuss-tech-without-it* orchestrations.

While studying technology use in a Kindergarten classroom in France, Gueudet, Bueno-Ravel and Poisard (2014) observed the use of two new orchestration types to manage the diversity of individual learner needs: *autonomous-use* and *supported-use*. During *autonomous-use*, the children were capable of independently using the technology, therefore allowing the teacher to monitor their progress from a distance. However, when a child required support from the teacher, either with using the technology itself, or with the mathematical ideas, the teacher engaged in a *supported-use* orchestration to support the learner. In the context of another French Kindergarten, Besnier (2018) described the use of teacher-created concrete manipulative replicas of the digital technology. These were displayed on the board for students to see and were used by the teacher to demonstrate or explain the software in, what Besnier described as, a *manipulatives-and-software-duo* orchestration.

When examining the implementation of touchscreen technology in elementary school classrooms, I described three new orchestrations: *compare-successive-screens*, *document-screen-on-paper* and *discuss-the-finger-and-the-screen* (Bakos, 2022a). During the *compare-successive-screens* orchestration, the teacher would refer to a mathematical representation on an iPad screen that could be seen by everyone, and then asked questions that required students to recall the representation from the previous screen for comparison purposes. Students were also sometimes asked to reproduce or record on paper, the various elements of the mathematical representations created on an iPad screen using the *TouchTimes* application, in what was called a *document-screen-on-paper* orchestration. The final orchestration observed was, *discuss-the-finger-and-screen* which involved the teacher, or a student, using their finger to physically point at something specific related to the touchscreen application, or the images created on it.

With the onset of the COVID-19 pandemic, many universities transitioned to virtual teaching for undergraduate and graduate courses. Orozco-Santiago and colleagues (2022) identified four new instrumental orchestrations that are based on experiences teaching university mathematics synchronously online. Unable to use the work-and-walk-by orchestration in a virtual environment, the teacher instead chose to have a student to share their screen with the whole-class, in a *student-shared-screen* orchestration. Once the student screen was shared for all to see, the teacher could then choose which orchestrations, such as, spot-and-show or Sherpa-at-work, best served his didactical purposes. A *share-board* orchestration was used when the teacher or students would write on either a blank PowerPoint slide, or a digital ‘whiteboard’, to present, explain or solve mathematical problems. When students wanted to share their paper-and-pencil work, they would take a photo of their notebook and upload the photo for other classmates to view, in a *shared-resource* orchestration. Although it is not explicitly explained in their paper, a *board-instruction-tech* orchestration is the final one mentioned for use in a virtual teaching environment.

Although there has been some research related to instrumental orchestration at the Kindergarten level (Besnier, 2018; Gueudet et al., 2014), the secondary-school level (Drijvers et al., 2010; Trouche, 2004) and even recent work at the tertiary level (Orozco-Santiago et al., 2022), there are few instances of instrumental orchestration studies in the elementary school grades. In fact, Joubert (2013) noted that out of over one hundred papers submitted to a digital technology conference on 'orchestrating learning' in mathematics education, there were only five papers about technology use in primary schools. Additionally, much of the prior work with instrumental orchestration has focused on the teacher use of desktop software, interactive whiteboards or virtual environments, and there has been little attention paid to the orchestrations used by teachers during the implementation of touchscreen technologies.

3.2.7. Sociocultural considerations

Although instrumental orchestration may appear to be a one-way, teacher-directed process, Hoyles, Noss and Kent (2004) contend that a sociocultural aspect is present throughout, where, ''instrumentation could be regarded as part of the process of developing participation within a community of practice, a process by which individual understanding and behaviour develops from and contributes to the collective activity" (p. 317). The technological medium, they argue, can be considered a *boundary object* (Star, 2010; Star & Griesemer, 1989); an object that can be shared, even if used differently by students and teachers. The technological medium, as a boundary object, can provide common ways to think and talk about ideas, which is useful for shared meaning-creation. While "students construct and interpret procedures and teachers (re)interpret their constructions" (Hoyles et al., 2004, p. 321), there is an expression of both the students' and the teacher's views of the boundary object. In a collaboration between researchers and teachers, Sinclair and colleagues (2020) found that the digital technology *TouchTimes* could be considered a boundary object for studying the pedagogical and/or mathematical ideas shared during group meetings.

In describing the possible co-construction of knowledge using a technological tool, Hoyles, Noss and Kent go further and contend that the influence of the tool is, "not only a cognitive tool but also a genuine mediator of social interaction through which shared expressions can be constructed" (p. 318). Additionally, they argued that the tool itself plays an important role in shaping the user's evolving (instrumented) mathematical knowledge, and that the mathematical ideas being developed and expressed are influenced by user interaction with the tool *and* interactions with the community. Which takes us back to a focus on the artefact itself, which will be discussed further in the next subsection.

3.2.8. Coming back to the tool

As Hoyles and Noss (2003) stated, “Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures” (p. 341). Ruthven (2014) also points out that the organisation of classroom activity has been the primary, and almost exclusive, focus of instrumental orchestration research, with little attention given to the adaptation of the tool itself. Instead, the focus has been on the teacher’s role in planning, co-ordinating and adapting the learning tasks, the digital tool and the classroom interactions through didactical configuration choices, modes of exploitation, and their didactical performance. Underlying this, is an inherent assumption that the teacher’s actions are both prompted and shaped by the presence of the tool. However, it has only been recently that the agential role of the digital tool itself has been acknowledged explicitly by researchers (e.g. Chorney, 2017), though this has generally been done in passing.

For example, it may be seen as a recognition of the agency of the tool when Gueudet, Bueno-Ravel and Poisard (2014) noted that their data clearly indicated that the orchestration choices made by the Kindergarten teachers in their study were influenced by several of the software’s features. Though this notion was not expanded upon further, it was explicitly mentioned. In another example, Carlsen and colleagues (2016) drew on Pickering (1995) when describing the teacher, the digital tool, the child and the mathematics as agents, interacting in what they refer to as “distributed agency”. They were critical of Drijvers and colleagues (2010) view of the use orchestration types, generally, as *teachers’ choices*, arguing that this phrasing “obscures the influence/agency of digital tools in understanding teachers’ use of digital technology in mathematics classrooms” (Carlsen et al., 2016, p. 15). Their findings illustrated how the orchestration choices made by the Kindergarten teachers observed during their research were “interrelated with the affordances of the environment” (p. 15) and they argued for the consideration of the agency and mediation of digital tools in classroom activity.

3.3. Research questions

Unlike spreadsheets, which were not created for purpose of education, *TouchTimes* is a digital technology that *was* specifically designed as a learning tool for school mathematics. When the four teachers who are part of this research study were first introduced to TT, it was newly developed and not yet publicly available. As such, none of them had had any previous exposure to, or experience with using this new technology. This allowed for a unique opportunity to examine the process of double instrumental genesis that the four teachers would undergo as they each engaged in their own personal instrumental genesis when first using TT as learners themselves, as well as the progression of their professional instrumental genesis. I was interested in learning more about how these teachers learned to use TT themselves, their impressions of its way of multiplying and their thoughts about using it with students. I was also curious to learn how their personal instrumental genesis would interact with their professional instrumental genesis and wondered if there would be interferences between these two processes. Once the teachers began to implement TT in their classrooms, what would be the effects of instrumental distance and would the use of their previous didactical reference markers be apparent? Would the instrumental orchestrations used in elementary school mathematics be similar to what previous mathematics education researchers have noted in the contexts of Kindergarten, middle-school or secondary mathematics classrooms?

In previous writing, I used an *ensemble* approach to study the teacher, the digital tool and the mathematical concept of multiplication (Bakos, 2022a, 2022c). Rather than examine each of these parts individually, I wanted to study them in relation to the whole, or in what I referred to as an ensemble. This approach seemed to better capture the complexity, as well as the nuances, of the interactions amongst the teacher–tool–mathematics and, therefore, I have chosen to continue utilising this way of thinking about my research. However, in order to effectively engage with this ensemble, I have narrowed my research focus to the following three research questions.

1. In what ways does *TouchTimes*, and its way of presenting the mathematical concept of multiplication, affect teachers as they progress through the process of double instrumental genesis (instrumentation)?

2. How do teachers accommodate the instrumental distance between *TouchTimes* and their previous ways of thinking about, and teaching, multiplication (instrumentalisation)?

3. How do teachers' experiences of double instrumental genesis evolve while utilising *TouchTimes* as a pedagogical instrument?

In order to engage with, and respond to these questions, I draw upon the data gathered during the broader teacher–researcher project and from interviewing these four teachers about their experiences implementing TT as a pedagogical tool for teaching multiplication to their students. A more detailed description of the research context and the methods used for gathering and analysing data will be provided in the methods chapter that follows.

Chapter 4. Methodological Considerations

> *There are events and records of events. We can only re-enter the event through the record of the event, which is not the event itself. (David Pimm, taken from personal class notes)*

In this chapter, I will begin by describing the iPad application *TouchTimes* (Jackiw & Sinclair, 2019), its central design features and the concerns that motivated its development. I will then provide an overview of the research context, including the TT teacher–researcher group, the classroom observations and the semi-structured interviews which were part of my doctoral research. Next, I will share some background information about the teachers who participated in my research and, finally, I will articulate my data-analysis process.

4.1. TouchTimes: Motivation and design

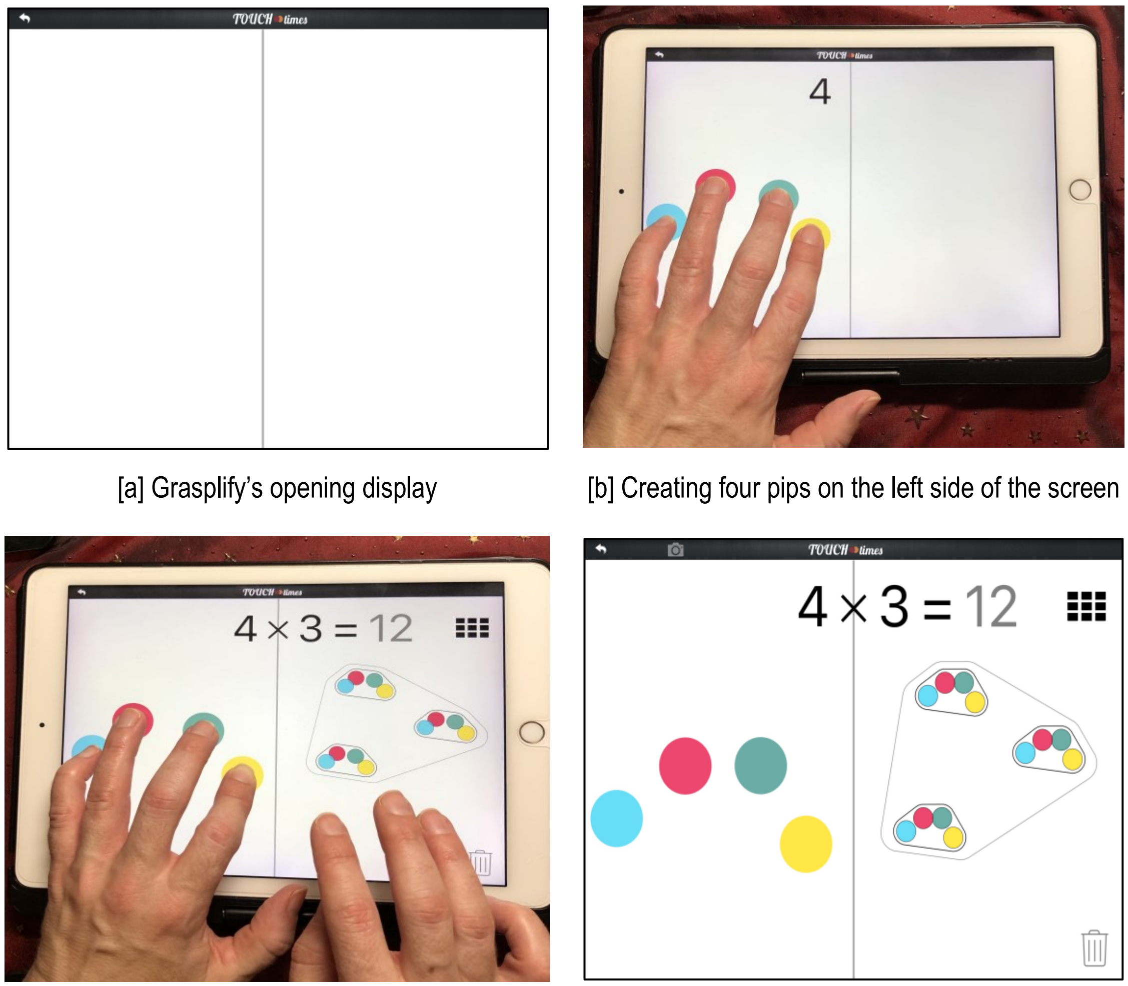
A multi-touch iPad application, TT provides young learners with multiplicative, rather than additive, experiences of multiplication using an embodied co-ordination of units approach. Inside TT, there are two complementary microworlds, *Grasplify* and *Zaplify*, each of which highlight different properties of multiplication in visually distinct ways. In Grasplify, the user's hands separately take on the roles of multiplier and multiplicand, emphasising the function of each in determining the product, whereas Zaplify's design draws attention to the commutative property of multiplication and the symmetry of the multiplicative factors. As children engage with *TouchTimes* to create, adjust and transform digital representations of multiplicative situations, they receive immediate haptic, visual and symbolic feedback.

4.1.1. Grasplify: Function and design[5]

Outside of the mid-screen vertical line that divides the display in half (see Figure 4.1a), the Grasplify opening screen appears blank until finger contact is made on the surface of the iPad. When the user touches the screen, coloured discs (referred to as 'pips') appear

[5] For a more detailed account of Grasplify, see Bakos and Pimm (2020).

at each point of contact, as if summoned by the user's specific fingertips (see Figure 4.1b). Pips require continuous screen contact to remain visible, and if a pip-making finger is lifted from the screen, the pip (and all pips associated with it) ceases to exist and vanishes from view. Grasplify resets when all pip-fingers are removed from the screen.



[a] Grasplify's opening display

[b] Creating four pips on the left side of the screen

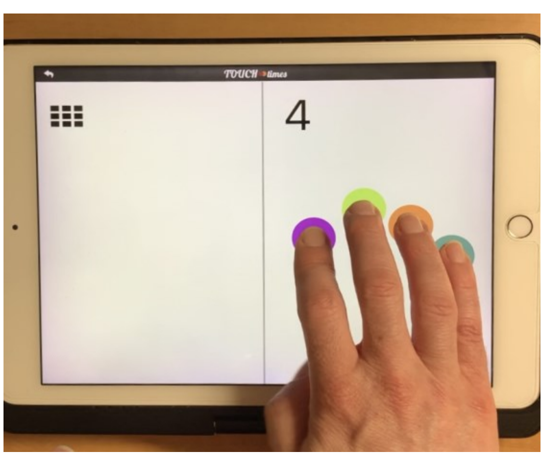
[c] Creating pods on the right side of the screen

[d] Two composite units

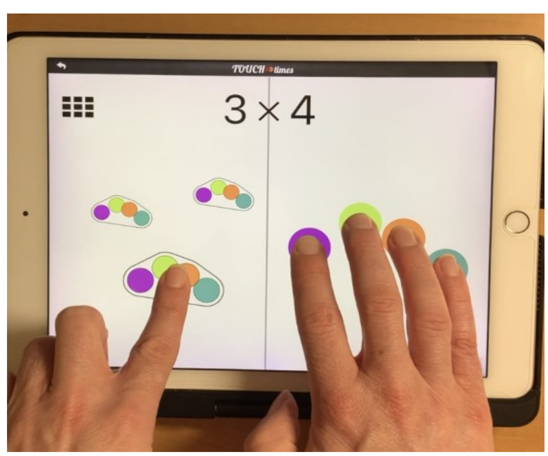
Figure 4.1 Grasplify display and pip- and pod-creation

The numeral that corresponds to how many pips made by the user instantly appears on the same side of the screen (see Figure 4.1b) and automatically adjusts in response to the creation or removal of pips. Once pips have been established, it is then possible to create 'pods', or enclosed groups of pips, with the touch of a finger on the other side (see Figure 4.1c). Unlike pips, pods do not require continuous screen contact in order to remain on display. Each pod reflects the colour, shape formation and number of pips that the user is actively maintaining on the opposite side of the screen. After a momentary delay, the entire group of pods is encircled by a ‘lasso’ to form the product visibly, which then appears alongside the equals sign to complete the numerical expression (see Figure 4.1c). The composition of the pods, essentially the pips within them, and the numerical expression at the top of the screen, alter instantly to reflect the creation or removal of pips on the pip-creation side of the screen, and its effect on the product.

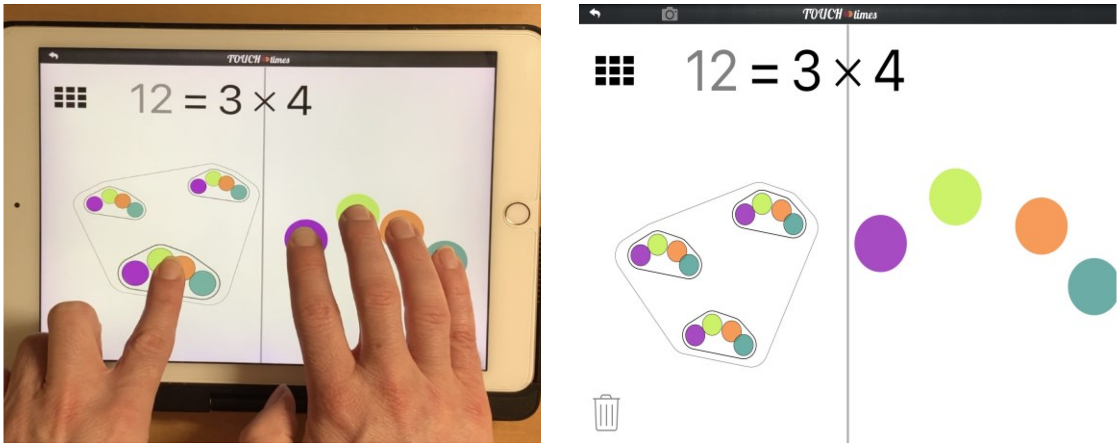
Designed to be symmetric, pip- and pod-creation can occur on either side of the screen, as can be seen by the progression of visuals shown in Figures 4.1 and 4.2. The pips are always produced first (see Figures 4.1b and 4.2a) and the pods are then created on whichever side of the vertical line that is opposite to where the pips were established (see Figures 4.1c and 4.2b). With the creation of the first pod, the numerical expression at the top of the screen adjusts to display the number of pods, and the ‘×’ symbol becomes visible between the two numerals representing the pips and the pods (see Figure 4.2c). The appearance of each part of the numerical expression always corresponds in time, order and direction to the initial establishment of the pips (see Figures 4.1b and 4.2a), followed by the pods (see Figures 4.1c and 4.2b) and, finally, the product (see Figures 4.1c and Figure 4.2c). The symbolic mathematics also visually coincides with the left/right arrangement of pips/pods and will be displayed as either pips × pods = product (Figure 4.1d) or product = pods × pips (Figure 4.2d). It is worth being aware that the symbols appearing at the top in Figure 4.2 go right to left in time, against the expectation of the writing direction in English.



[a] Establishing pips on the right side of the screen



[b] Creating the pods on the left side of the screen



[d] The full numerical expression [d] Two composite units

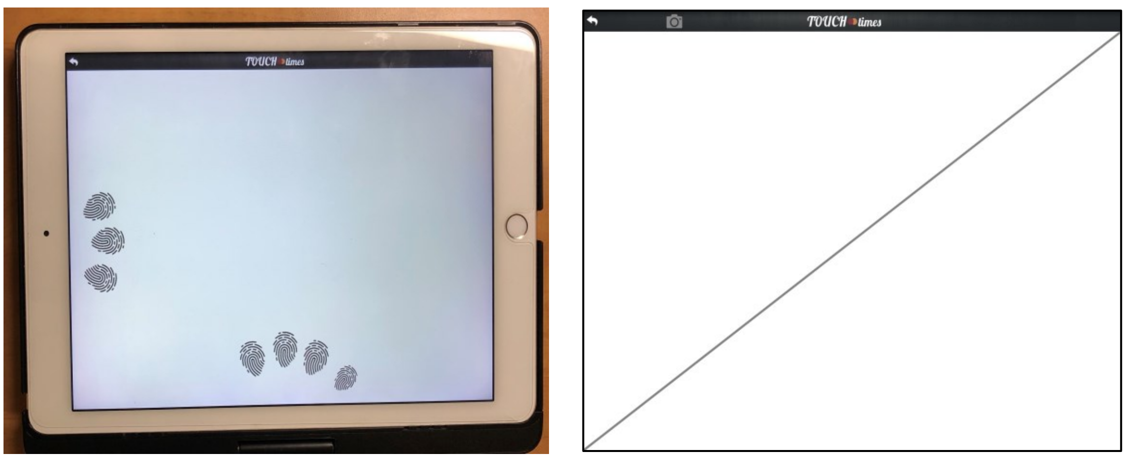
Figure 4.2 Multiplication using Grasplify

Grounded in approaches to mathematics that are based on measurement and ratio, Davydov's (1992) double change-in-units approach describes multiplication as involving "a count of a [larger] unit for which a relationship to another, smaller unit, is already established" (p. 12). When creating composite units, this requires the unit of measure (the multiplicand) be established before the unit quantity (the multiplier). These ideas were influential in the design of Grasplify, where the pods (the multiplier) are a quantity dependent on the establishment of pips (the multiplicand) and serve as the unit of measurement for the final product. The co-ordination of Davydov's double change-in-unit process is embodied by this microworld, where the first unitising is seen when the pips are established (in Figures 4.1b and 4.2a, there are four pips), the second unitising takes place as the pods (a unit of units) are created (in Figures 4.1c and 4.2b, there are three pods), and the product is reflected by the encircling of the pods by the lasso (in Figures 4.1d and 4.2d, there are two composite units), which is when the third, and final, unitisation occurs.

This multiplicand × multiplier ordering is intentionally incorporated into Grasplify, though it is the opposite of what elementary teachers in British Columbia (BC) usually find in textbooks and teaching resources (and, potentially, their own student school experience). This ordering of factors, where the multiplicand precedes the multiplier in time, is reflective of the Davydovian approach to multiplication, where the *unit quantity* must be established prior to determining how many units. The chronological ordering of the appearance of each factor is important and is therefore asymmetric.

4.1.2. Zaplify: Function and design[6]

Zaplify's opening screen briefly displays images of fingerprints on the bottom and left edges of the screen, which function as visual guides for where users should make screen contact (see Figure 4.3a). Visible only for a few seconds, the fingerprints fade away and a diagonal line that stretches from the bottom left to the top right corner becomes visible (see Figure 4.3b). As soon as screen contact is made, the diagonal line fades away but will reappear if all fingers are removed from the screen, effectively resetting Zaplify when in unlocked mode. There is also an option to use Zaplify in a locked mode, which will be described in more detail shortly.



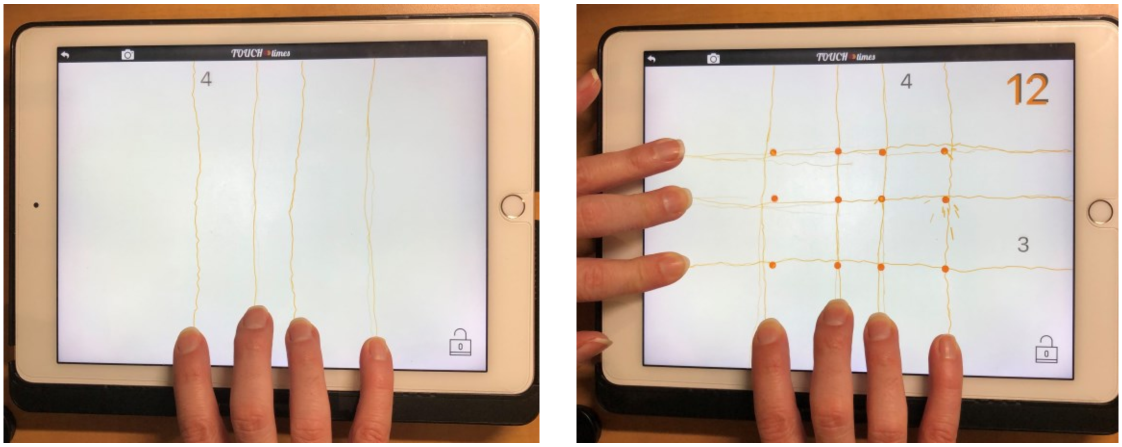
[a] Zaplify's screen with fingerprints [b] Zaplify's screen with its diagonal line

Figure 4.3 Zaplify's opening screen

When screen contact is made by the user's fingertips on the left edge of the upper triangle, horizontal lightning bolts are created, and vertical lightning bolts will appear beneath the user's fingertips when placed along the bottom edge of the lower right triangle (see Figure 4.4a). The two multiplicative factors are represented by the number of horizontal and vertical lightning bolts created, and the numerals associated with each move, either back-and-forth across the top of the screen (for vertical lightning bolts) or up-and-down the right edge of the screen (for horizontal lightning bolts). Orange sparks automatically appear where the vertical and horizontal lightning bolts intersect and are

[6] For a more detailed description of Zaplify, see Güneş (2021c).

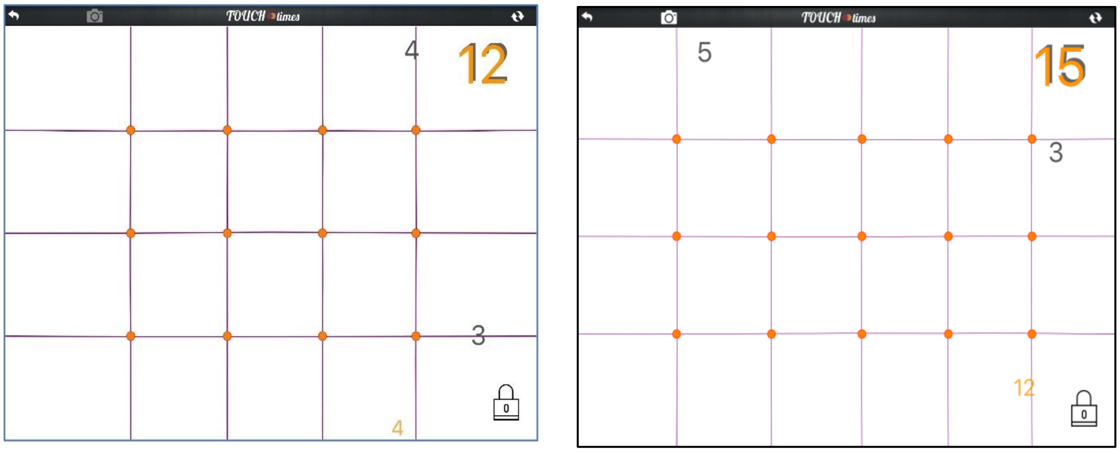
the multiplicative product formed by the two factors. The product is also displayed symbolically by orange numerals in the top right corner of the screen (see Figure 4.4b).



[a] Four vertical lightning bolts [b] Intersections and orange sparks

Figure 4.4 Multiplication using Zaplify

As previously mentioned, there is an option to engage a locked mode in Zaplify, which will automatically transform all existing horizontal and vertical lightning bolts into grid lines, with orange discs marking each point of intersection (see Figure 4.5a). While in locked mode, continuous screen contact is unnecessary for lines and their intersections to remain visible. This allows for the creation of larger products and makes it easier for children to follow what happens after each touch of the screen, which also helps to increase their awareness of the multiplicative relationship between the two factors and the resulting product. Zaplify functions to distribute both factors simultaneously, and changes to one, or both, factors are immediately displayed in the product. If there are three horizontal lines distributed across four vertical lines, for example, then these four vertical lines can also be viewed as being distributed over the three horizontal lines (see Figure 4.5b). If one factor is changed, the product alters as the change is distributed over the other factor. As can be seen in Figure 4.5, when $4 \times 3 = 12$ is changed with the creation of another vertical line to $5 \times 3 = 15$, the product increases by three. With the newly created vertical line, Zaplify highlights the additional intersection points with the creation of three orange discs at each horizontal line.



[a] Distribution of both factors in locked mode

[b] increasing the vertical factor by 1

Figure 4.5 Functional and relational aspects of Zaplify

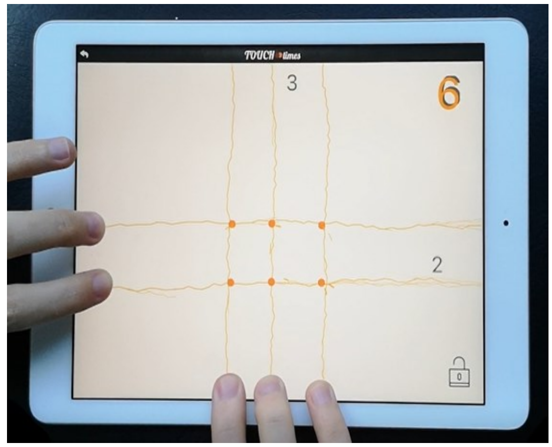
Vergnaud's (1983) work on the conceptual field of multiplication, and its focus on doubling, tripling, etc., was influential in the functional and relational characteristics of Zaplify's design. The binary nature of the multiplicative relationship is embodied by Zaplify in the orientation of the user's hand placement on the screen. The horizontal and vertical orientation of the lightning bolts (or lines) that represent the factors and the orange sparks or discs at each intersection work together to embody and visually highlight the properties of multiplication. This model can be seen as a rectangular array, where $a \times b$ is interpreted as b rows with a elements in each row – which Maffia and Mariotti (2018) claim is consistent with the Cartesian product (see Figure 4.6a).

> Cartesian product of sets A and B is defined as the sets of all ordered pairs where the first elements are chosen from set A and the second elements are chosen from set B. (Vest, 1971, pp. 222–223)

When in locked mode, the Cartesian product is represented by the grid-line intersections and their orange discs, thereby emphasising both the symmetry and the commutative property of multiplication (see Figure 4.6a and b).

	0	1	2	3
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)

[a] Double-entry table



[b] Zaplify crossing lines

Figure 4.6 Cartesian product representations

4.2. Sources of data

There are three data sets that will be utilised as part of my research, all of which have emerged in some way from the larger, multi-phase project led by Dr. Nathalie Sinclair and Dr. Sean Chorney, which Canan Güneş (another doctoral student) and I were invited to be part of in the role of research assistants. As mentioned in Chapter 1, the project focus involved the collaborative development of tasks for use with *TouchTimes*, and to study the effects of using this digital technology on student thinking about multiplication.

In the first phase of the project, there were ten teachers, working with students from 5–13 years of age, in eight different schools throughout Metro Vancouver, who volunteered to participate in this collaboration. Nine of these teachers had completed a numeracy-focused master's degree through SFU at some point previously, and the tenth was alerted to the project by a colleague who was also a former graduate of the master's program (though not a project participant). The teachers had chosen to join the project because of an interest in using technology to teach mathematics.

During the initial stage of the project, the research team and the teacher participants met as a group on five separate occasions throughout the 2018–2019 school year. Four of the five meetings were hosted after school by three different teacher participants in their classrooms and were approximately 60-minutes long. Substitute teachers were provided for the afternoon of the final meeting, which took place on campus at SFU and was between 1.5–2 hours in duration. As indicated in Table 4.1, attendance at each of the meetings varied, as each teacher (all identified by pseudonyms here) was not always able to be present.

Table 4.1 ***TouchTimes* project meetings**

Teachers[7]	Teacher–Researcher Meeting Dates				
	10/24/2018	12/6/2018	1/31/2019	3/7/2019	4/25/2019
Amy	✓				✓
Andy	✓				✓
Chloe				✓	✓
Jasmine				✓	✓
Kate[8]					
Leah	✓	✓	✓		✓
Monica		✓			
Rachel		✓			✓
Stella	✓				✓
Susan	✓	✓	✓	✓	
Tina	✓	✓			

It was during the first of these meetings that the teachers were introduced to TT and were given an opportunity to use the digital technology to engage with Grasplify tasks shared by the research team. When TT was initially introduced to teachers in the project, Grasplify was the only microworld available and Zaplify was developed shortly afterwards. After the teachers had a chance to explore Grasplify for themselves, the research that inspired the development of Grasplify was explained, and, as the group discussed its different features, some of the underlying mathematical intentions behind the design of that microworld were also shared. In subsequent meetings, more tasks were developed and provided to the teachers, who would then try these out with their students. This led to further discussion about the purpose of TT and, after experimenting with using Grasplify and its associated tasks in their classrooms, the teachers would provide feedback about refinements or suggested improvements for the technology itself.

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[7] I later interviewed Amy, Kate, Leah and Rachel as part of the third data set, which will be described in more detail later in this section.

[8] Given the location of Kate's school district, she was unable to attend the teacher–researcher group meetings, though she did provide feedback about the task booklet and shared her experiences using TT with students.

Other topics of conversation during these meetings included: individual teachers sharing which tasks they had tried; descriptions of teaching strategies that worked well when implementing TT; questions the teachers had for either the research team or other teachers in the group; discussions about ways to overcome various challenges that the teachers were experiencing. A few of these issues were related to implementing digital technology, while others had to do with the way that TT presents multiplication. This provided opportunities for the teachers to share strategies and resources they found effective for teaching multiplication and also for explicit links to be made between the tasks and the provincially mandated curriculum.[9]

All of these meetings were video-recorded by a single camera that was placed to capture the whole group. When there was a larger number of teachers in attendance, there was sometimes a second camera used, in order to record discussion between pairs or small groups of teachers. Each of these video-recordings was later transcribed for analysis as part of the larger TT research project.

The video-recordings and their corresponding transcripts form the first source of data and, although the meetings themselves are not the primary focus of my research, they do contain some data that is relevant to my research questions. It was during these meetings that TT was first experienced and, consequently, there are reactions to, and thoughts shared with, the larger group that provide insight into the teachers' personal and/or professional instrumental geneses. These moments are useful as a way of examining how three of the teachers whom I later interviewed, initially spoke about TT and their experiences of using it, in comparison to how they described it two or three years later, while at a more developed stage of double instrumental genesis.

The second source of data emerged during the 2019–2020 school year as part of the next phase of the TT teacher–researcher project. The research team was invited into the classrooms of five of the participating teachers (see Table 4.2) to observe mathematics lessons where Grasplify was being used with students. Depending on availability, between two and four members of the team would video-record, observe and interact with students throughout a total of twelve 60–90-minute mathematics lessons in the various classrooms. During teacher-led, whole-class sections of each lesson, a single

[9] For a more detailed breakdown of what occurred during the first three TT teacher–researcher meetings, see Sinclair and colleagues (2020).

video camera was set up either in the corner or in the centre of the classroom. However, when individual or pairs of students began exploring their assigned task, an additional camera was used by one member of the research team to capture this activity.

Table 4.2 Classroom observations

Teachers	Grade Level of Students	Dates of Classroom Observations
Amy	3–4	2/5/2020
	3–4	2/10/2020
Erin/Jenn[10]	4–5	11/15/2019
	4–5	11/28/2019
	4–5	11/29/2019
Leah	3	10/29/2019
	3	11/20/2019
	3	11/21/2019
Rachel	3–4	12/13/2019
	3–4	12/17/2019
	3–4	3/10/2020
	3–4	3/12/2020

The video-recordings from the classroom observations in Leah and Rachel's classrooms that show the teacher-led, whole-class aspects of their lessons will form the basis of data analysed in Chapter 6. I was in attendance for six of the seven classroom observations that occurred in these two classrooms, which is why I have chosen to examine these videos more closely. After viewing all of the video-recordings, I have excluded those recorded during Jenn's and Erin's lessons and in Amy's classroom from the analysis, primarily because these were focused on student use of TT and little was recorded of the teacher-led portion of the lessons.

The third data set involves semi-structured interviews of four different teachers involved in the TT project. Although these interviews were intended to take place in 2020, with the onset of the COVID-19 pandemic in mid-March of that year, the interviews were delayed until the summer of 2021.

Between May and August of 2021, I performed, via Zoom, six 60–80-minute, semi-structured interviews as part of my doctoral research (see Table 4.3). Each teacher was

[10] Erin and Jenn brought their two classes together for a team-teaching situation during the three classroom observation dates.

first interviewed individually and then, later, based on their availability, I paired each up (Leah/Kate and Rachel/Amy) for an additional interview (one further interview per pair).

The idea to interview pairs of teachers together was suggested by another doctoral student in one of the bi-weekly writing seminar sessions led by Dr. Nathalie Sinclair and Dr. David Pimm. This suggestion brought to mind how the teachers would often clarify or expand further the ideas shared within the TT teacher–researcher group setting. I hoped to encourage this elaboration and clarification of experiences and ideas in the interview setting by pairing the teachers up so they would not only be sharing their thoughts and ideas with me, but with each other as well.

Table 4.3 Teacher interviews

Teachers	Dates of Interviews	Type of Interview
Amy	6/1/2021	Individual Interview
	6/8/2021	Interviewed with Rachel
Kate	7/9/2021	Individual Interview
	8/20/2021	Interviewed with Leah
Leah	7/15/2021	Individual Interview
	8/20/2021	Interviewed with Kate
Rachel	5/31/2021	Individual Interview
	6/8/2021	Interviewed with Rachel

During the interviews, I asked each teacher to share what they recalled of their initial exposure to TT and its way of presenting multiplication, as well as any memories of learning to use it themselves. We also discussed their experiences of implementing TT as a pedagogical tool for teaching multiplication, their observations related to the mathematics and how students interacted with multiplicative ideas when using this digital technology and the mathematics it presents The teachers also shared any thoughts or concerns about TT as a technological tool, either of its models of multiplication, or any other observations about how students used the technology or what they learned about multiplication. I will provide more detail about the process of analysis used with this data, later in this chapter.

4.3. The research participants

All four teachers interviewed had been implementing TT as a teaching tool for multiplication with grade 3 or 3–4 students (8–9-year-olds), which is when multiplication is first introduced to students in the BC mathematics curriculum. I will now share the educational background and teaching experience of Amy, Kate, Leah and Rachel, along with details related to the context in which they were teaching, during the classroom observations or when interviewed.

Amy's educational background includes fulfilling the requirements of the French-module of SFU's Professional Development Program to earn a Bachelor of Education degree and later completing the master's degree in numeracy program offered by SFU, which is where I first met her. I had been asked to share my experiences researching my own practice and the process of engagement used when I wrote up my experiences for my course assignment, with Amy's cohort, which was in the early stages of thinking about their own research projects. After beginning her career as a resource teacher for French-immersion Kindergarten to grade 7, Amy has since been a homeroom teacher for grades 2, 3–4 and 4, all in French-immersion settings.

As a previous student in the SFU master's program, who is currently teaching in Metro Vancouver, Amy received information that described the TT project and invited interested teachers to join. When asked why she had chosen to become part of the project, Amy explained that she is "always looking for different ways to teach math without worksheets" and that she wanted to find "different ways to make math engaging and have kids think more deeply about problem solving". Her desire to engage students with mathematics in ways that were creative and open-ended motivated her to volunteer to be part of the TT teacher–researcher collaboration.

Leah is an elementary generalist French-immersion teacher, with a master's degree from the University of British Columbia in Literacy. She has had over twenty years of classroom teaching experience in grades two through seven French-immersion in Metro Vancouver. She became part of the larger TT teacher–researcher project after a colleague shared the project information with her, who was teaching grade three that year. The project, with its focus on multiplication, was of interest to her. Having already completed a master's degree, she wanted to continue to grow as a professional, and chose to do this by becoming involved in what she called 'manageable projects'. By joining the teacher–researcher collaboration, Leah could engage in professional development in a way that fulfilled her desire to grow as a teacher, but the time and energy required to participate was something she considered to be manageable.

Rachel completed the Professional Development Program at the University of British Columbia and is also a graduate of the two-year master's degree in numeracy program offered by SFU. She has been teaching French-immersion for nearly ten years in the Metro Vancouver area and is an elementary generalist who has taught a variety of grade levels that include grades 1, 1–2, 3–4, 4–5, 6 and 7–8. While completing her master's degree, Rachel became interested in using digital technology for teaching mathematics and, as a former master's student still teaching in the Metro Vancouver area, she received the invitation to join a research collaboration project for TT. She explained how she "was interested in the opportunity of continuing [being involved in research using touchscreen technology] and being part of a group of teachers who was going to be talking about it [TT] and looking at it". For Rachel, the chance to engage in discussions about teaching mathematics through the use of digital technology with other practicing teachers was what motivated her decision to become part of the project.

Although Kate was unable to attend the TT teacher–researcher meetings, her contributions included piloting the resources and assessment tools developed as part of the project and providing detailed feedback about them. Her educational background includes a Bachelor of Science degree with a major in mathematics, prior to enrolling in the Professional Development Program to earn a Bachelor of Education degree and she has also completed graduate studies in mathematics education. Kate has had 20 years of teaching experience in what she described as varying capacities, which include being a former secondary mathematics teacher (grades 8–12).

When interviewed, Kate was in her fifth year as a math lead teacher for her school district in the Lower Mainland region of BC. In this role, she creates and shares resources, such as pre- and post-assessment instruments for use in Kindergarten–grade six that align with the BC mathematics curriculum, supports teachers through professional development training sessions, goes into classrooms to perform demonstration lessons or will assist teachers with lesson planning, preparation and implementation of high-quality mathematics pedagogy. In her effort to improve mathematics teaching, Kate was working in a third-grade classroom to model the implementation of TT as a digital tool for teaching multiplication. Given her more recent experiences using TT (in the spring of 2021), her background and training as a secondary mathematics teacher and her willingness to be interviewed, the inclusion of Kate, as part of my interview data, provided a singular opportunity. I was curious to examine her responses to the interview questions and her shared experiences with TT, with those of the three other teachers who were elementary generalists who had been using TT over the course of 2–3 years.

4.4. Data analysis

As previously mentioned, these four teachers were participants in the larger TT teacher–researcher project and, therefore, there was relevant data available from all three data sets. In the case of Kate, given the difficulty of attending the teacher–researcher meetings, there was only the semi-structured interview data from which to draw. During both data collection and its analysis, I was cognisant of three underlying factors of importance in this research: (1) the elementary school context; (2) the touchscreen aspect of the digital technology being studied; (3) the novelty (for both teachers and students) of the two multiplicative models used by TT.

There were two overarching themes for the data analysis which will be examined in detail throughout Chapters 5 through 8. The first is related to the process of double instrumental genesis that each teacher experienced in relation to TT, which applies to Chapters 5, 7 and 8. The second theme involves the instrumental orchestrations used by the teachers when implementing TT with their students, which will be the focus of Chapter 6. Within these overarching themes, there were two very different perspectives, directly tied to the instrumental approach, that were utilised: instrumentation and the impact of the digital technology and its way of materialising multiplication on the teachers, and instrumentalisation and the ways that the teachers accommodate TT into their previously existing ways of teaching and thinking about multiplication.

In order better to understand the early processes of personal and professional instrumental geneses experienced by each teacher, Chapter 5 highlights the four teachers' initial impressions of TT *before* using it with students. I will examine both the influence of TT and its presentation of mathematical ideas on the teachers, as well as how they each accommodate this digital tool and its multiplicative models.

The data analysed was drawn from comments Amy, Leah and Rachel shared during the TT teacher–researcher meetings, from video-recordings of their teaching during our classroom observations and from the six semi-structured interviews of all four teachers.

I began by studying the six interview transcripts, which I transcribed in their entirety. With my research questions in mind, I analysed the interview transcripts with two specific aims: (1) I wanted to capture instances of transition experienced by the teachers during their personal and professional instrumental geneses of TT that were provoked by the digital technology or by the way it presents the mathematics; (2) I wanted to identify specific examples of how the teachers adopted and integrated TT and its multiplicative models into their pedagogical practice. I hoped to understand better the effects of TT on each teacher throughout their individual experience of double instrumental genesis, as well as how they each dealt with instrumental distance and the effects of this on their professional instrumental genesis.

In order to get a better sense of how Amy, Leah and Rachel's experiences of double instrumental genesis evolved, I went back to the original teacher–researcher meeting video-recordings and transcripts. It had been a few years since I had last watched them, and this time I was paying attention to specific instances of personal or professional instrumental genesis in its early stages. I also examined the classroom observation videos for any instances that demonstrated a progression in each teacher's double instrumental genesis process.

The focus of Chapter 6 is on the teacher–tool–mathematics ensemble *during* the implementation of TT with students. I present in-depth examples of the instrumental orchestrations utilised by Leah and Rachel while implementing Grasplify in their classrooms and identify didactical reference markers that were used by each teacher to help them negotiate the instrumental distance between TT and former ways of teaching multiplication.

Prior to writing Chapter 6, I re-watched in considerable detail the nine video-recorded lessons taken during the classroom observations that took place in Amy, Leah and Rachel's classrooms as part of phase two of the TT teacher–researcher project. My first step was to identify the instrumental orchestration types used by each teacher during the whole-class, teacher-led portions of their lessons, while also watching for any new orchestrations used by the teacher(s) externally to guide the instrumental genesis of the students. During our classroom observations, when the students were working independently, the cameras were focused on the students, and therefore there was little data available about specific teacher orchestrations used with individual or pairs of students during that time.

Once the instrumental orchestrations were identified, I examined these looking for sequences of orchestrations used repeatedly by individual teachers, or for sequences common to all three teachers. For any newly identified orchestrations used by a teacher, I would look to see if any of these orchestrations were also being used in either of the other two classrooms. Furthermore, I wanted to understand better how each teacher's professional instrumental genesis evolved while utilising TT as a tool for teaching multiplication.

While examining these video-recordings, I was also watching for any 'hiccups' (Clark-Wilson, 2010) or "perturbations experienced by the teachers during the lesson, triggered by the use of the technology that seemed to illuminate discontinuities in their knowledge and offer opportunities for the teachers' epistemological development within the domain of the study" (p. 138). I was particularly mindful of the teacher–tool–mathematics ensemble and how the tool–mathematics interacted with each other while also affecting the teacher (instrumentation) and, conversely, how the teacher accommodated TT–multiplication (instrumentalisation). The episodes that are shared in Chapter 6 were chosen to provide a picture of the instrumental orchestrations used by Leah and Rachel while teaching with TT, to highlight instances of instrumental distance and to note any didactical reference markers that were explicitly referenced by the teachers.

Chapters 7 and 8 are both concerned with the thoughts and experiences shared by each of the teachers *after* they had implemented TT in their classrooms. Chapter 7 considers the purpose described by each teacher for the use of this digital technology as a pedagogical tool for teaching multiplication. The final of the four chapters that examine the data, Chapter 8 highlights the sociocultural influences experienced by the four teachers throughout their processes of double instrumental genesis. The data analysed throughout Chapters 7 and 8 was drawn from the ideas shared by Amy, Leah, Kate and Rachel during the six semi-structured interviews and were chosen because of what MacLure (2010) describes as "glow", an idea which I will proceed to describe in more detail in the chapter that follows.

Chapter 5. Grasplify: Teachers' First Impressions

> *One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children. (Stephen I. Brown, 1981)*

This is the first of four chapters where I will detail specific examples shared by Amy, Kate, Leah and Rachel regarding their experiences using *TouchTimes.* Each of these teachers involved in this research study shared their thoughts about implementing a new digital tool in their grade 3 or 3–4 classrooms, and described how the dynamic, multiplicative models of TT influenced their ways of thinking about and teaching multiplication. In this chapter, I will be focusing on some of the teachers' initial impressions of Grasplify and, in the chapters that follow, I will examine the instrumental orchestrations used during classroom implementation of this digital technology as a pedagogical tool, the purpose that each teacher shared for using TT and the sociocultural influences during the process of double instrumental genesis engaged in by each of the four teachers.

5.1. Data that 'Glows'

As stated in the epigraph shared above, the richness of a single incident or experience by, in this case, a single teacher has much to convey. I was an active participant in each of the TT teacher–researcher group meetings, the classroom observations and the semi-structured interviews. I acknowledge that my role as researcher was not impartial and, in choosing the data examined throughout this chapter, I want to honour the experiences that were particularly powerful individually for the teachers themselves who chose to share them with me.

Learning to use new technology and then integrating it into teaching practice is a complex process (Monaghan, 2004), one that is different for each teacher. The instances that will be analysed and interpreted throughout this chapter, and those that follow, though descriptive in nature, are guided by the constructs of double instrumental genesis and its associated processes of personal and professional geneses, instrumental distance and each teacher's didactical reference markers. These theoretical ideas not only guide how I tell the story that unfolds throughout this dissertation, but they also facilitate my understanding of why the process of double instrumental genesis is so complex. One way better to understand this complexity experienced Amy, Kate, Leah and Rachel, is by examining the instrumental distance and attending to the didactical reference markers that are unique to each individual teacher.

Though there are experiences shared in the teacher transcripts that the theory helps guide my attention towards, and illuminates as interesting; there are other ideas that captured my attention that are unrelated to the theory, but may be influenced by my own experiences as an elementary school teacher and by my own understanding of multiplication. When examining the teacher interview transcripts, I started my data analysis by grouping similar ideas that were shared by more than one of the teachers together, as a way of trying to identify common themes. I found that I was able to group some of these ideas, which were similar in some way, into categories, some of which included: initial impressions of the digital technology, difficulties projecting TT for all students to see, capturing student learning using whiteboards or drawings, the value each teacher saw in using the technology. I discovered, however, that the individuality of each teacher was stronger than the categories themselves. As a result, my interest was primarily drawn to those instances, often shared by a single teacher, that stood out in some way in the category that it seemed to best fit (or that it did not quite fit). These were the experiences shared that I *sensed* were particularly powerful for that specific teacher.

Initially, I struggled with the idea of sharing data that was chosen on the basis of this *sense*, rather than some sort of 'impartial' coding scheme. I was concerned about how to effectively describe my way of choosing the examples that are written about in chapters five through eight, until I read about MacLure's (2010) argument "for 'exemplary' practices, in which theory proliferates from examples" (p. 277). She goes further to acknowledge that there is a type of 'glow' or an intensity that emanates from qualitative data.

> There is, I suggest, another potentiality associated with data, beyond and beside their capacity for mute surrender to the colonialist administrations of social science. This potentiality can be felt on occasions where something—perhaps a comment in an interview, a fragment of a field note, an anecdote, an object, or a strange facial expression—seems to reach out from the inert corpus (corpse) of the data, to grasp us. These moments

> confound the industrious, mechanical search for meanings, patterns, codes, or themes; but at the same time, they exert a kind of fascination, and have a capacity to animate further thought. (MacLure, 2013, p. 228)

MacLure's work freed me from this feeling that I had to make my data 'fit' into common themes, and allowed me to confidently embrace the potentiality of those instances that exemplified the process of double instrumental genesis experienced by Amy, Kate, Leah and Rachel that were capturing my attention. I will now begin by detailing Leah and Rachel's initial impressions of TT, which only consisted of the Grasplify microworld when they were first introduced to it.

5.2. Initial impressions of *TouchTimes*

The ideas that emerged as powerful were sometimes those that came up repeatedly in either the TT teacher–researcher group meetings or in the individual and/or paired interviews. This was particularly evident in Leah's response to the multiplicand × multiplier = product ordering displayed by Grasplify and in Rachel's acknowledgement that she did not initially find TT to be intuitive to use. Mentioned during their individual interviews nearly three years later, each of these instances remained with Leah and Rachel long after their first exposure to this digital technology and exemplify very different initial impressions of TT.

Although Amy mentioned that she remembered finding TT very user friendly and straightforward, which she liked, she did not share much beyond that. Therefore, her initial impressions are not included here. During her paired interview with Leah, Kate mentioned finding two design aspects of TT, the blank opening screen and the lack of permanence of the screen, surprising when she initially used it. Given the nature of the paired interviews, when Kate discussed these ideas, it prompted Leah to share related experiences that had not previously emerged in her individual interview.

When the teachers were first introduced to TT, it was expected that they would initially undergo personal instrumental genesis when learning how to use this digital technology themselves, *prior to* beginning the process of professional instrumental genesis. It is not uncommon for teachers to explore digital technology first themselves, in order to understand how it works as an instrument for mathematics before determining how best to use it as a pedagogical tool to promote student learning. Haspekian (2014) noted that there were sometimes interferences that occurred between a teacher's personal and professional instrumental geneses. I also found this to be the case with TT, and the episodes that follow were shared by Leah, Kate and Rachel and I have chosen them to illustrate the influence of their professional instrumental genesis during their initial exposure to Grasplify.

5.3. Leah's reaction to Grasplify: "It's backwards"

Multiplication is first introduced in the British Columbia mathematics curriculum in grade three and, as a third-grade teacher, Leah joined the research group hoping to learn alternative methods for teaching multiplication. In the description that follows, her initial response to Grasplify's arrangement of pips and pods, and its corresponding numerical expression, which she firmly stated was "backwards" (in a tone that implied backwards was wrong), was immediate and would prove to be persistent. Each time the teacher–researcher group met over the next year, and in the interviews conducted later, Leah mentioned her difficulty with this. In the description that follows, some background context will be provided and Leah's initial response will be examined in more detail.

Prior to introducing Grasplify at the first TT teacher–researcher meeting, Dr. Sean Chorney shared four different ways of thinking about multiplication: multiplication as repeated addition, as two quantities producing an array, as splitting or partitioning, and as a change in unit. The double change in units' approach to multiplication is an essential part of the Grasplify design, involving two processes which are depicted in Figure 5.1 (based on a figure included in Boulet, 1998, p. 13). At the bottom of the diagram, there are four units, each composed of three discs encircled into single units. Then, during the many-to-one operation depicted in the middle section of the diagram, the four units of three discs are encapsulated into a single unit of twelve. This involves a second change in unit.

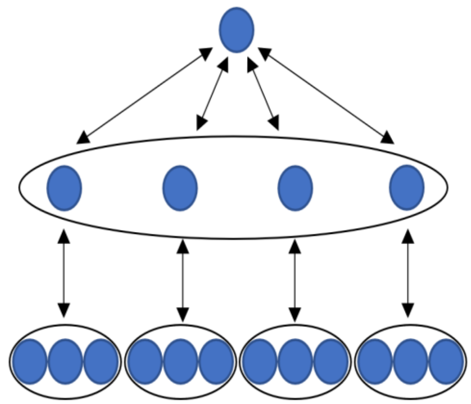


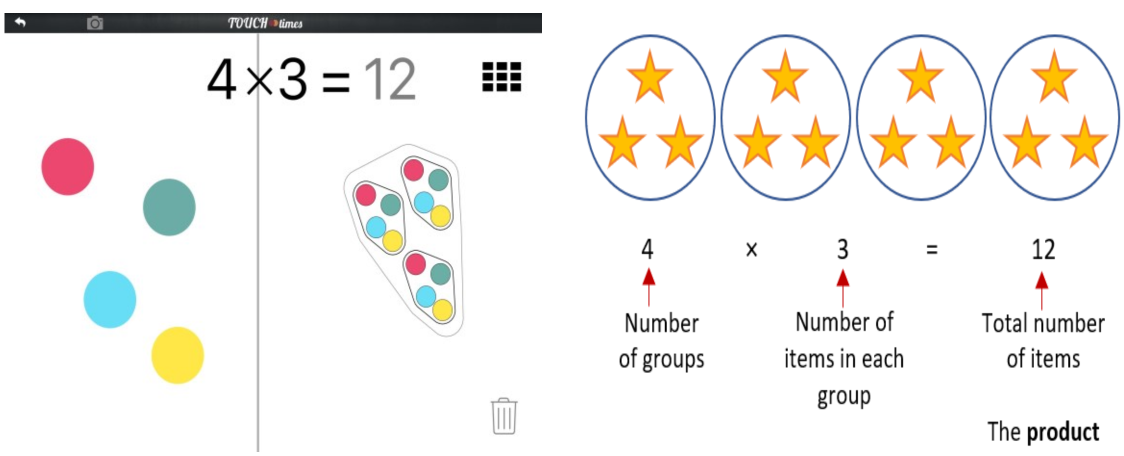
Figure 5.1 Multiplication as a change in unit

As Sean reviewed this way of thinking about multiplication, Leah asked, “So the units meaning, we look at unit as group?”. Once it was confirmed that a unit could be a group, Leah stated, “So, groups-of”. Although TT as an instrument had not yet been introduced, Leah’s way of thinking about multiplication as involving groups-of first became apparent in this moment. The instrumental distance between Grasplify’s multiplicative model and Leah’s groups-of model was large from the outset. The groups-of model was a didactical reference marker used by Leah as a way of explaining the meaning of numerical multiplication expressions to her students and one that she herself had internalised. For example, 3 × 4 would be explained by Leah to her students as three groups-of four. The effects of this distance between Grasplify’s way of representing multiplication and how Leah personally conceived, and the way she taught, multiplication created conflict within the mathematics–technology–teacher ensemble in ways that were not yet apparent at this point in the first TT teacher–researcher project.

Although Sean immediately returned to the idea of unit, clarifying that if five is thought of as the unit and we want four of those units, that would be twenty, the groups-of idea persisted. Another teacher in the group looked towards Leah and confirmed, “Four groups-of”. At this point, Leah explained that she would use the wording *groups-of* and that: “I’ve never thought of it as one unit, but that [group] is one unit. That group.” Her previous reference marker for thinking of multiplication as groups-of had been disrupted and this was her initial expression of considering a new way of thinking about the mathematics.

Shortly thereafter, the iPads containing the Grasplify application were distributed amongst the teachers and they were given time to explore the application for themselves. This exploration is part of the videorecording made of this meeting and Leah could be overheard saying to Amy, who was sitting beside her, that she made three groups-of five on her iPad but the numerical expression displayed by Grasplify was 5 × 3. Amy responded affirmatively in a way that clearly expressed her acceptance of this. Leah then turned to Susan, who was sitting on her other side, and expressed that even though multiplication is commutative, “I always picture the first number as this [pointing to 5 in the 5 × 3 displayed at the top of the Grasplify screen] groups-of and this would be marked wrong in the textbook” [indicating the visual pod-representation displayed by Grasplify]. Although she had initially created 5 × 3, Leah had removed a pip-making finger from the screen (whether this was intentional or not, is unknown), changing the numerical expression and its associated multiplicative model to 4 x 3, which she then referred to in the conversation that followed. The whole group stopped and listened as Leah explained that the 4 × 3 now displayed on her iPad screen *should* show a visual representation of four groups-of three, rather than the three groups-of four created by Grasplify (see Figure 5.2).

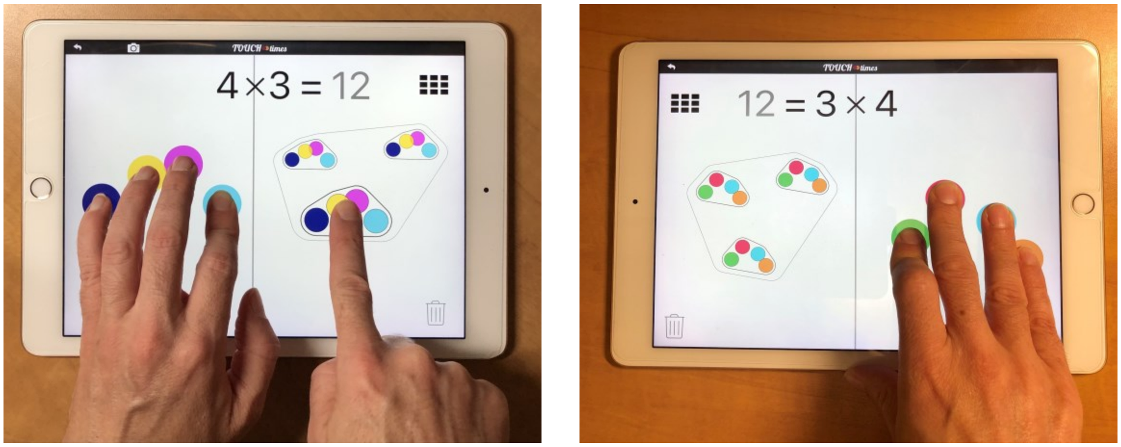
> “It’s backwards. So we’re saying, like when you look at Math Makes Sense […] *that* [pointing at 4 × 3] wouldn’t be four groups-of three, *that* [points again at 4 × 3] would be three groups-of four that makes *this* [pointing at the pod creation on the right screen in Figure 5.2a]. […] Whereas our textbooks say, four groups-of three [referring to the numerical expression 4 × 3].”



[a] Grasplify display of 4 × 3 [b] Sample textbook explanation of 4 × 3

Figure 5.2 Visual representations of 4 × 3

Canan, a fellow research assistant who was involved in the project, pointed out that, if Leah placed her pip-creating hand on the right side of the screen, the resulting equation would be consistent with the groups-of textbook model that was being referred to (see Figure 5.3b).



[a] Pip-creation on the left screen [b] Pip-creation on the right screen

Figure 5.3 Grasplify's symmetric design

Leah created pips on the right screen and, ignoring that the product came first (in other words, on the left in the notation), confirmed that this ordering did fit the three groups-of four model used in the textbook. Sean intervened again, explaining further that the first hand is always the multiplicand or the unit, and that the groups are established after, when the user touches the other side of the screen. And the notation was generated right to left. Leah continued to experiment with which side to create the pips on first and, after making 4 × 3 by establishing pips on the left screen, she stated, “It’s the opposite”, and then started over by making the pips on the right screen which showed the groups-of ordering in the textbook. Canan acknowledged this, “So, in the textbook, I think they fix the place of the multiplicand and multiplier, but this [Grasplify] makes it like both ways”. To which Leah responded, “Which it should be”, and explained how she has always questioned, “Why does it matter where the kids put the four and the three in a matrice[11] [array]? It shouldn’t matter because it’s commutative”.

Another didactical reference marker emerged for the first time related to Leah’s personal knowledge of the mathematical properties of multiplication. Though she referred to the commutative property, and her understanding that the ordering of the factors would not affect the product, she was unable to make sense of how pip-creation on the left and

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[11] As a French Immersion teacher, Leah would often refer to mathematical terms in French, as she did here.

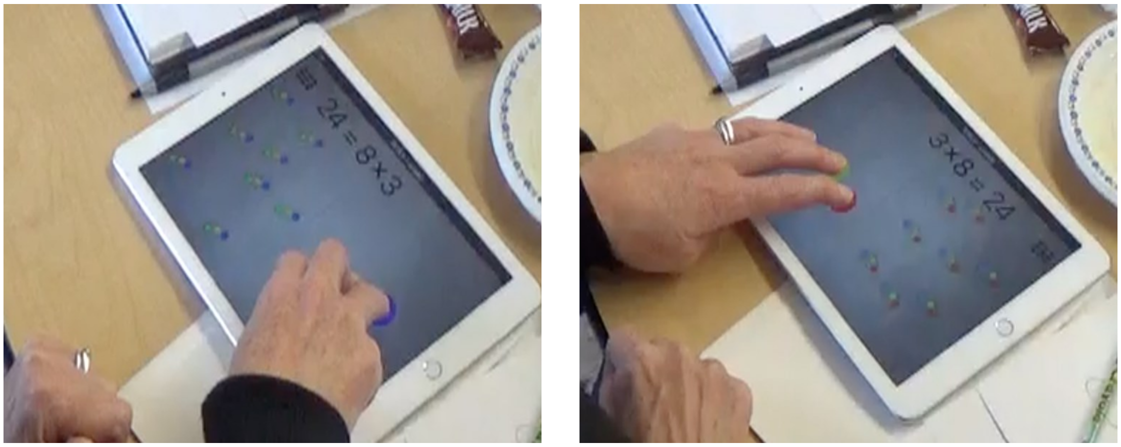
right sides of the iPad screen occur in Grasplify. Drawing from her knowledge of multiplicative models defined by a textbook, she had difficulty reconciling why Grasplify's model is consistent with the groups-of model in the textbook when the pips are created on the right screen, but is "backwards" when the pips are created on the left side of the screen. Despite Sean's repeated explanations of the differing roles of the multiplicand and the multiplier – an essential part of the formation of the multiplicative model in Grasplify – there was an interference between Leah's personal understanding of multiplication and the model used by the technology.

This emerged when the teachers were asked to draw some of the different ways that the pips and pods could make a product of 24. Leah continued to question the ordering in the following exchange:

> Leah: When I go like this, it's inappropriate still. I'm still starting with this hand [places three pip-creating fingers on the right screen]. It's eight three times. [See Figure 5.4a]
>
> Sean: But it's not eight three times because you're still, you're looking at eight times three as an order and it's not an order. It's what you typed in. What you touched down first is your multiplicand, it grew the unit.
>
> Leah: But in both times, in both times I touched down the same amount, three, but?
>
> Sean: Yeah, but you always will get the, the first hand you touch [the pip-creating hand] will be the one that is grouped.
>
> Leah: But it's the same both times.
>
> Sean: Do it again.
>
> Leah: So, watch. Three [creates three pips on the left screen], so that's my first touch is three.
>
> Sean: That's your unit.
>
> Leah: Then one, two, three, four, five, six, seven, eight [touches the right screen to create eight sequential pods]. [See Figure 5.4b]
>
> Sean: Yeah, so you get eight groups-of three because eight groups-of three because you touched three first.
>
> Leah: Yeah, but watch this. Three first times [creates three pips on the right screen]. One, two, three, four, five, six, seven, eight [touches the left screen to create eight sequential pods]. [See Figure 5.4a]

Sean: You still get eight groups-of three because you touched three first.

Leah: It’s the same thing, but this is flipped [refers to 8 × 3 displayed at the top of the screen] because of the sides? Makes sense. Okay.



[a] Pip-creation on the right screen: Product = multiplier × multiplicand

[b] Pip-creation on the left screen: multiplicand × multiplier = product

Figure 5.4 The creation of the multiplier is key

Though she indicated that ‘it’ (the ordering) makes sense and earlier had made a comment related to commutativity, at the start of the second TT teacher–researcher group meeting, Leah again brought up her discomfort with Grasplify’s ordering of the terms in the equation and expressed that she required reassurance. “It’s the opposite way that the app is looking at it than some of us are used to teaching it.” Between meetings she consulted with her teacher colleagues and double checked her teaching resources, all of which confirmed what she described as the backwardness of the app, “because when I’m teaching it, I’m always teaching that multiplication means groups-of, not this five times”. She again referred to commutativity and then provided an example, “if I am doing three times four, I would expect three groups-of four to show up but four groups-of three is showing up for three times four”. Her way of thinking about and teaching multiplication was primarily based on thinking of 3 × 4, in terms of repeated addition, as three groups-of four. Leah finished by sharing that she was trying to understand that, “you can look at it the other way […] and that this is an opportunity to teach that”.

In returning to the textbook and speaking with other teachers, Leah was confirming that her didactical reference markers for multiplication were correct. In Grasplify, though, multiplication involves units of units, and therefore 3 × 4 is a unit of three, taken four times. The implications of establishing a unit of measure (the multiplicand) prior to creating the number of groups was not an idea that Leah was associating with multiplication. Her conceptualisation of multiplication was primarily based on groups-of and commutativity. Though the multiplier and multiplicand relationship is relative to the idea of groups-of, Leah's understanding of groups-of was firmly attached to the ordering of the numerical expression without a conscious acknowledgement that in this way of thinking, the first number in the multiplicative expression is always the multiplier.

Given the intertwined nature of the teacher–tool–mathematics ensemble, the interference created by the multiplicand × multiplier ordering used by Grasplify on Leah's own understanding of multiplication was a significant disruption to her double instrumental genesis. In order to reconcile this disturbance, Leah relied upon the commutative property. Though the product for $a \times b$ and $b \times a$ is the same, the emphasis in Grasplify is not on the product itself. It is designed to draw the user's attention to the creation of the pip(s) and how these 'spread' throughout the pod-units. The multiplicand and multiplier are assigned different roles (and involve different hands) and are not interchangeable – and therefore commutativity is not relevant in this context, nor functional to assist with Leah's personal genesis of Grasplify.

Though Leah's concern with the ordering of the numerical expression was connected to her way of thinking about multiplication, which influenced her own personal instrumental genesis, it was even more impactful in relation to her professional instrumental genesis and *how* she explained multiplication to her students. The instrumental distance between the multiplicative ideas displayed by Grasplify, and the ordering of the numerical expression to reflect the "groups-of" depiction that Leah believed to be correct and previously used as a teaching method, was an issue not easily dismissed. She had used the textbook as a reference tool for explaining mathematical concepts to her students for more than two decades and the textbook described 4 × 3 = 12 as 4 groups-of 3, resulting in a product of twelve. When explaining multiplication to her students, the '×' symbol was replaced by the term "groups-of". Grasplify's depiction of 4 × 3 = 12 as "four, three times" created a disturbance in Leah's process of double instrumental genesis. The gap between her prior ways of thinking about and teaching multiplication did not match the multiplicative representation in Grasplify and initially the instrumental distance was large.

I will return to Leah's implementation of Grasplify in her classroom and how she came to terms with the "backwardness" of the multiplicative expression in chapter 8 when I examine the impact of socio-cultural influences on the teachers' processes of double instrumental genesis.

5.4. Kate and Leah: The impermanence of *TouchTimes*

Kate too mentioned in passing that two times five, which would normally be thought of as two groups of five, was the opposite somehow in Grasplify. Though with a strong mathematical background, Kate's didactical reference markers enabled Grasplify's representation of multiplication to be easily accommodated within the knowledge she already possessed without a disruption. She explained that, "I already knew that that was possible, but it was just a different way of thinking about it". Though Kate herself understood this about multiplication, she did not mention if this was an aspect of Grasplify that she would incorporate into her teaching to ensure that her student would also benefit from this understanding of multiplication. This aspect of her professional instrumental genesis is unknown based on the information that was shared during the interviews.

When questioned about her initial impressions of TT, Kate recalled two things that struck her, the first involved what she called, "the initial blank canvas", and the second related to the impermanence of both microworlds. She described how most digital applications she used with students would invite the user in, whereas with TT, the opening screen was blank, which she found unusual. For Kate this was really intriguing, though she remembered thinking, "Okay, what do I do with it?". Once she placed her fingertips on the screen, what next caught her attention was the impermanence of the objects created on the screen, "I was more surprised by if I move my, if I lift up my hand, everything disappears. That was more of the surprise, like figuring out how I could make it work so that some things stay". At this point, Kate shifted to share her frustration with this lack of permanence as a teacher. When she removed her fingers, everything would disappear and therefore she was left with taking screen shots of TT to project as part of her teaching. Both of these responses were prompted by the digital technology itself, and initially began as part of Kate's personal instrumental genesis, though later became part of her professional instrumental genesis.

This lack of permanence being surprising emerged during Kate's paired interview with Leah and prompted Leah to also share some of her thoughts related to that aspect of TT which were closely related to her professional instrumental genesis. Leah mentioned how she initially disliked this lack of permanency but then found it to be useful in her teaching. "I didn't like that at first, but then I actually went to loving that because what that forced when I was using it, and then when the kids went to use it, it actually forced the kids to think." Leah had observed that when students removed their fingers when using Grasplify and everything on the screen "goes away", that it had forced her students to think back to what they had just created on the screen. She remembered giving a task where students were using Grasplify to find the answer but then they had to draw on a mini-whiteboard how Grasplify showed the answer they had found. She watched as students "let go" and while they completed their drawings, they had to think back to what they had created on their iPad screens, and how, "You could hear the conversations of why and how many pips are there, how many pods were there". This lack of permanence was positive for Leah because, "It adds an element of concentration. A bit of planning. Yeah, planning, purposefulness, planning, you're thinking about why. There's more of a meta-cognitive aspect to it. It's not just playing, which I mean playing is great. It's playing but it's a *thoughtful* play".

Leah's reaction to the lack of permanence of Grasplify was related to her professional instrumental genesis and how she would use the digital technology to benefit student learning. Initially she did not like this feature of TT, however, when she saw how this aspect of the design could prompt student conversation and more purposeful planning on the part of her students, which she viewed as more meta-cognitive, Leah then embraced this as a positive element of the technology. Her didactical reference markers relating to the benefits of play and student engagement in conversation about their learning were stronger than the disruption caused by a design aspect of the digital technology that initially caused a disturbance for Leah.

Kate extended this idea of play, explaining that many children associate iPads with games or play, though in the case of TT, it's the nature of the application itself that is playful. She noted that it is very different from math drill type applications that give prompts or hints and returned to the idea of the open space of TT, noting that for students the fact that nothing happens until they do something, "adds to the mystery". For Kate, the play experience was related to students' curiosity when faced with a blank canvas, prompting questions such as: “What is this thing? What’s going on here?”. Although the opening screen being a blank canvas was something she noted during her own personal instrumental genesis, it was during her professional instrumental genesis that Kate saw this as having the potential to engage students’ innate curiosity while using an iPad, which she felt they viewed as fun rather than work.

5.5. Rachel’s reaction: “It’s not intuitive”

As part of an action research project completed during her master’s degree a few years prior to the start of the *TouchTimes* project, Rachel had extensive prior experience integrating the touchscreen application *TouchCounts* (Sinclair & Jackiw, 2011) into her mathematics teaching practice in her first-grade classroom. When interviewed in 2021, she explained to me how she enjoyed engaging with research in her classroom and using *TouchCounts* with her students. Her continued interest in implementing new forms of digital technology as teaching tools for mathematics, alongside the opportunity to be part of a cohort of teachers who shared this interest, were what prompted Rachel to join the TT research group.

When asked about her initial impressions of TT, Rachel described how she had missed the first TT teacher–researcher meeting and, as a result, she did not quite understand some of the tasks given during the second meeting. She compared her prior experiences with *TouchCounts* with her initial experience with Grasplify. “*TouchCounts* I get. […] And I know it’s also for a lower grade level, but still, multiplication at the grade two to five level shouldn’t be that complicated. It wasn’t intuitive.” Rachel described TT as more difficult to use initially: “I found it harder to use and harder to think of things on my own. […] I was able to think of other things I could do with *TouchCounts*, whereas with *TouchTimes*, I still really needed the teacher guide to assist me”. The instrumental distance for Rachel was large at this point; her personal instrumental genesis was influenced by not quite understanding some the tasks shared at the second meeting, having missed the first one, and she did not feel comfortable implementing or utilising Grasplify without further support.

She also described sharing *TouchCounts* with a group of pre-service teachers and their responses, which were: “I don’t get it. What do I do? How is this helpful? I found the same thing with *TouchTimes* and it's like I don't know how to come up with a task for this, so I need help with that". Rachel explained that the resources that had since been developed as part of the TT research project, which included two teacher task booklets, student task cards and the inclusion of short videos demonstrating the tasks, would make it easier for those teachers first using TT. "I think showing people like here's what you do. Here's a little bit of what you say, here's why you say it, will be helpful."

Rachel's didactical reference markers were heavily based on her experiences using *TouchCounts*, which did not prove helpful in understanding Grasplify and the tasks used with it. Being able to access tasks that were already developed provided her with a place to start and helped narrow the instrumental distance enough for her to begin implementing Grasplify in her classroom. Without didactical references to refer to when using Grasplify, Rachel was left relying on the teacher task resources for TT that were in the early states of development in order for professional instrumental genesis to occur. Her progression through double instrumental genesis involved a close intertwining of her personal and professional instrumental geneses, which will be examined further in relation to the socio-cultural influences of the TT teacher–researcher group on the ensemble of Rachel–Grasplify–multiplication in chapter 8.

5.6. Summary

In the analyses above, I have attempted to highlight the impact of Leah, Kate and Rachel's initial impressions of TT on their early engagement in the process of double instrumental genesis. Given that TT was a new digital technology that the teachers had no prior experience using, it was expected that a phase of personal instrumental genesis (learning to use it as a personal tool for mathematics) would occur first, prior to the teachers entering the process of professional instrumental genesis (how they would use it as a teaching instrument). In this chapter we begin to see that even when in the early stages of their own personal instrumental genesis, these teachers would often quickly transition to thinking about how TT would be used as a teaching tool with students. This shift from personal to professional instrumental genesis was not necessarily sequential, rather it was multi-faceted and complex, and though there are already hints of that emerging here, this will become much more apparent in the chapters that follow.

Conscious of the two-way interaction of instrumentation and instrumentalisation in the process of double instrumental genesis, I wanted to ensure that my research questions also included an examination of how the digital technology influenced the teachers, as well as how the teachers' integrated TT into their teaching practice. In keeping the teacher–TT–mathematics ensemble in mind, here are a few notable effects that the technology and its presentation of the mathematics had on the teachers that were examined in this chapter.

Leah's resistance to the multiplicand × multiplier = product ordering in Grasplify is an example of an effect that the mathematics as it was presented by the digital technology, had on the user. This opposite ordering created instrumental distance from the groups-of approach previously used by Leah when teaching multiplication and began a process that would provoke Leah to rethink what multiplication means. Kate's reaction to the "blank canvas" of the opening screen and the lack of permanence of the objects created on the TT screen influenced how she thought of TT as a teaching tool. Leah's didactical reference markers relating to the power of play and student discussion were beneficial in narrowing the instrumental gap in relation to the lack of permanence in the Grasplify design. The instrumental gap between Rachel's previous experiences teaching with *TouchCounts* and her introduction to Grasplify, resulted in her description of TT not being intuitive for her.

In the next chapter, I will examine, in detail, some of the instrumental orchestrations used by Leah and Rachel during their implementation of Grasplify with their grade 3 and 3–4 students.

Chapter 6. Teaching with *TouchTimes*

> *The conductor (the teacher) leads the musicians (his/her students) to use artefacts and transform them into musical instruments (didactic instruments), which will allow them to perform the musical sheets (solve the mathematical situations). (Lucena, Gitirana & Trouche, 2022, p. 190)*

In this chapter, the focus will be on the two teachers'[12] arrangements of the classroom environment and the artefacts within it (the didactical configuration), in order intentionally to guide students' instrumental genesis for the benefit of their teaching intentions (instrumental orchestrations) (Guin & Trouche, 2002; Trouche, 2004). I will share detailed examples and some general-level observations of the instrumental orchestration types and the didactical configurations that were used by Leah and Rachel when implementing TT as a pedagogical tool for teaching multiplication to their students.

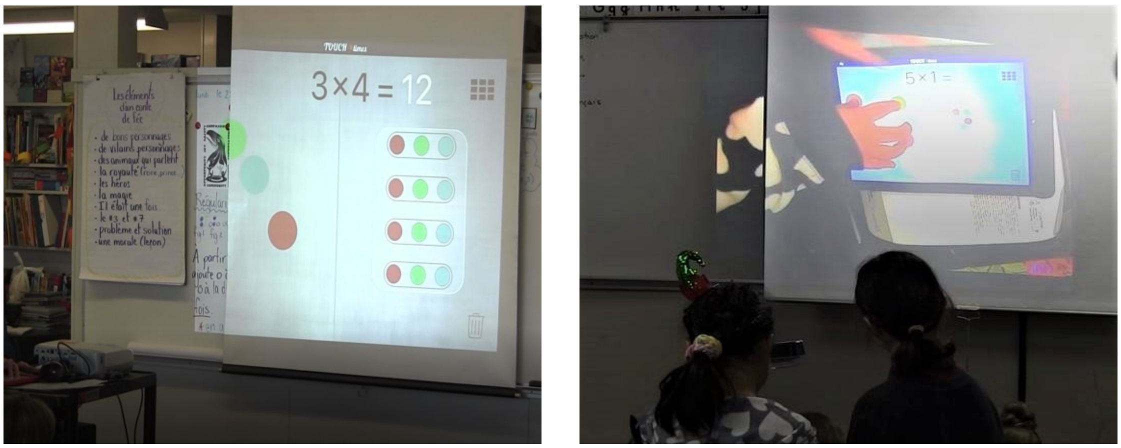
While pointing out instances of the teacher–digital technology–mathematics ensemble, I will also highlight the development of each teacher's double instrumental genesis and the influence of lesson 'hiccups' when using Grasplify. Additionally, I will identify and describe three new orchestration types that were observed being utilised both by Leah and by Rachel when teaching with TT. Finally, as part of the ensemble approach which recognises the interacting roles of the teacher–tool–mathematics, I will turn my attention away from the agency of the teacher, in order also to acknowledge the influential role of the digital tool. In the final sections of this chapter, I will describe episodes which illustrate the strong agential role played by Grasplify itself.

6.1. Introducing Grasplify to students

Leah and Rachel each had access to a sufficient number of iPads with TT downloaded onto them either for individual or for pairs of students to use. They could also plug an iPad into a projector in order to display an enlarged image of the screen on the

[12] As previously mentioned in Chapter 4, only the video-recordings from Leah's and Rachel's classrooms show the teacher-led, whole-class aspects of their lessons, which is why they are the focus of this chapter. The video-camera was primarily focused on student use of Grasplify in Amy's classroom and there was not an opportunity for the researchers to video-record any of Kate's TT lessons.

whiteboard for all students to see (see Figure 6.1a). Although Rachel also had access to a document camera which had the advantage of showing the user's fingers as they manoeuvred on the iPad screen, the projection did not display the colours in the pips or the pods very clearly, which was seen as detrimental and, therefore, was only used once (see Figure 6.1b).



[a] Grasplify screen projection on the wall [b] Document camera projection of Grasplify

Figure 6.1 Projecting Grasplify for all to see

While present both in Leah's and in Rachel's classrooms, the TT research team observed (and video-recorded) a similar sequence of orchestrations used by both teachers when introducing tasks using Grasplify. For both teachers, this started with a *Technical-demo*[13] in order to demonstrate for students certain techniques for using TT that were relevant for the task to be assigned. For example, after connecting her iPad to a projector for the class to see (see Figure 6.2a), Leah used Grasplify to model creating 1 × 3 = 3 (a Technical-demo) and then asked, "What is the product? Who can tell us what the product is? That's a new word for us." In the BC mathematics curriculum, multiplication is first introduced in grade three and, with this question, Leah was introducing students to unfamiliar mathematical vocabulary (a component of the mathematics register). This closed-type question is an instance of the *Guide-and-explain* orchestration, used as a way of explaining what can be seen on the iPad screen. After a student responded that the product was three, Leah also went to the screen to point out physically that the white-coloured 3 in the numerical expression was indeed the product.

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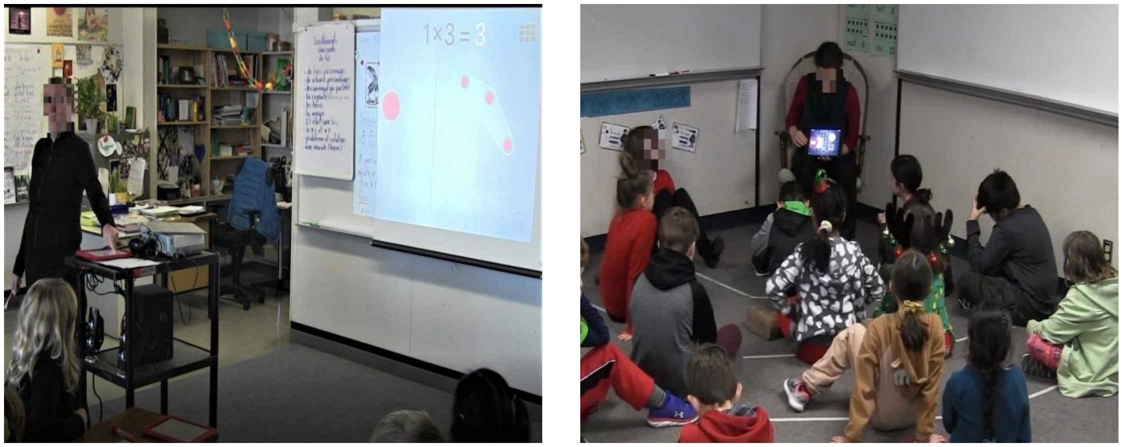
[13] Refer to Table 3.1 for an overview of instrumental orchestrations, including the Technical-demo orchestration.



[a] Leah’s Technical-demo projection [b] Rachel’s Technical-demo

Figure 6.2 Technical-demo orchestrations of Grasplify

When beginning her lessons with TT, Rachel preferred to gather her students together as a group, seated on the carpet in front of her (see Figure 6.2b). She would often hold up an iPad in front of her, screen facing the class, so that students could watch both her finger actions on the screen and the effects of her finger actions on the screen itself. She would then proceed with a Technical-demo to set up the task that she wanted students to engage with. For certain tasks, Rachel would also use a *Board-instruction* orchestration to provide both verbally and visually information that students could reference on the board that was related to the task. For example, when students were challenged to skip count by three using Grasplify, she recorded the target products (3, 6, 9, 12, 15, 18, …) on the board for all students to refer to. In this way, she would differentiate her instruction to provide support for students who might need it.

Students in both classrooms were given an extended opportunity for free exploration of the digital technology for much of the first lesson. Leah used this as an opportunity to engage her class in recording, and later sharing, what they noticed and wondered about TT. And the lesson in Rachel’s grade 3–4 classroom concluded with a member of the research team assigning a drawing activity related to students’ explorations of Grasplify. In the lessons that followed, the teachers utilised the affordances of the digital technology more purposefully, engaging students with tasks designed to focus their attention on the mathematics enabled by Grasplify.

In the section that follows, I will provide a full elaboration of the sequence of orchestrations utilised by Rachel during the second TT lesson which was video-recorded and observed by the research team.

6.2. Orchestrations used by Rachel

Rachel began by asking the children, who were gathered on the carpet in front of her chair, about the product if they were counting by two. As soon as she used the word product, however, Rachel paused to ask students what colour the product is in Grasplify. Students had been introduced to this term in prior lessons about multiplication, which allowed her to review this terminology while connecting it to TT. When the question was asked, Rachel was not displaying the Grasplify screen for students to see, yet one of the students immediately responded, "*Blanc* [white]".[14] Without the technology visible for students to refer to, Rachel drew upon the mental images that students had of Grasplify through this *Explain-the-screen* orchestration (Bakos, 2022a), thereby linking the colour of the product displayed on the screen to specific mathematical vocabulary. Students were then challenged to make the product increase by twos using TT.[15]

After assigning students to work in pairs, Rachel moved throughout the room in a *Monitor-and-guide* orchestration, attending to the pairs' progress, answering questions, providing further assistance or instruction, as necessary, and watching for student work examples for the up-coming class discussion. The students were brought back together on the carpet once most pairs had successfully achieved the task. Rachel asked a pair of students to share and, in so doing, she engaged in *Spot-and-show* orchestrations as a way of highlighting student work to prompt discussion and for teaching purposes. Two students were chosen and, once at the front of the class, became *Sherpas-at-work* (see

[1] Grasplify has two screen display options. The default display is a grey background and, at the top of the screen, the two factors that form the numerical expression are written in black, while the product is displayed in white. There is also a second display option that can be chosen in the settings to make the screen easier to see when it is projected or during screen shots (as I have done here). This optional setting involves a white background display and, although both factors in the numerical expression remain black, the product is grey.

[15] For a short video description of this task, and the mathematical differences between these two ways of skip counting, please go to: https://youtu.be/NQBhXdBjD1E

Figure 6.3). They held up an iPad so that their classmates could see the screen and proceeded to demonstrate and explain the process they used to skip count up by twos.

Figure 6.3 Sherpas-at-work

While the two girls were sharing their work, Rachel sometimes intervened either to clarify what was happening in a *Discuss-the-screen* orchestration or, guided by what was being demonstrated on the screen, to provide additional information about the mathematical content in an Explain-the-screen orchestration. When doing this, she simultaneously interacted with the pair who were sharing, as well as with the whole class, in order to draw attention to the mathematics. In the grade-three mathematics curriculum, teachers are directed to: “provide opportunities for concrete and pictorial representations of multiplication” and “connect multiplication to skip-counting” (Province of British Columbia, 2022, Grade 3 mathematics content section).

In Grasplify, students can skip count in two different ways. In the first, they create however many pips they are skip counting by (unitising) and then sequentially tap the screen to make pods; an action that is reflective of repeated addition. In the second, they begin with a single pip and then create however many pods they are skip counting by. With the placement of each additional pip-creating finger on the screen, the composition of every pod alters. In the transcript that follows, Rachel draws student attention towards covarying[16] and spreading,[17] two key ideas that underly multiplication and are enabled by the design of Grasplify. The touchscreen display in TT makes it possible for students to create or remove pips and immediately see the results of these actions in a digital–pictorial representation. [The transcript below was translated from French.]

Rachel: Show us *one* way to count by two. [Student 1 places two pip-creating fingers on the screen. Then student 2 creates two pods.] Wait, describe what you're doing. [Students remove fingers and start again.]

Student 2: So, go up two. [Student 1 creates two pips.]

Rachel: Those are two pips. [Points to left screen where the pips are being created.]

Student 2: Pips.

Rachel: Pips. [Nodding.]

Student 2: And then we're going to add pods here. [Student 2 adds two pods to create 2 × 2 = 4.]

Rachel: Pods. [Nodding.]

Student 2: We just add one and then… [Student 1 places another pip-finger on the screen, creating 3 × 2 = 6. Then sequentially adds one, two, three more pip-fingers from her other hand; 4 × 2 = 8, 5 × 2 = 10, 6 × 2 = 12.]

Rachel: You're adding to the pips or the pods? [Student 2 adds one, two, three more pips; 7 × 2 = 14, 8 × 2 = 16, 9 × 2 = 18.]

Student 2: Pips. And it makes it go by two. [Rachel assists by placing two pip-creating fingers on the screen. 10 × 2 = 20, 11 × 2 = 22.]

Rachel: So, how many pods are there?

Student 2: Two.

Rachel: And how many pips in each pod? How many?

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[16] Multiplication is a varying of two quantities; so when one factor varies, the other factor covaries in response. In Grasplify, altering the pips affects the composition of every pod, as well as, consequently, the product.

[17] Spreading can be seen when the pips are increased and/or decreased and that variation can be seen to spread across all other units (pods). This idea emphasises scaling.

Student 2:	Eleven.
Rachel:	Have we seen what the girls did?
Students:	[Some respond yes, and others respond no.]
Rachel:	Where do you add? Did you add to the pods or to the pips?
Student 2:	The pips.
Rachel:	The pips. Okay, thanks. Who found another way to make the product go up by two?

As the two girls demonstrated and explained their method of skip counting for their classmates, Rachel used her fingers to help the pair create the number of pips they needed. At the same time, she engaged in a Discuss-the-screen orchestration, in order to clarify how many pips and how many pods were visible on the screen and to draw student attention to the number of pips visible in each pod. When she questioned how they made the product increase by two – by pips or by pods – Rachel was highlighting the mathematical content guided by what was happening on the screen in an Explain-the-screen orchestration. By explicitly pointing out that the pair had skip counted up by two each time by changing the number of pips, she was leading the two girls (and the class) beyond the technique used to count by twos using Grasplify and towards a more detailed mathematical explanation related to multiplication. In this way, Rachel was building the collective instrumental genesis of the whole class, as well as the pair.

The elements of the teacher–tool–mathematics ensemble were working in concert while mutually influencing one another. The manner in which Grasplify visually displayed the immediate effects of covarying the pips through the expanding (or removal) of pips across each of the pods, provided a way for Rachel to introduce multiplicative ideas in an accessible way for her students. Her language of instruction involved an interweaving of pips and pods, singular terms that are part of this particular digital application, with skip counting, which is part of the prescribed BC mathematics curriculum for her grade level. The ensemble is also visible in the way that the teacher's fingers on the TT screen became part of the multiplicative situation. As part of the same lesson, students had explored skip counting backwards from twenty. In the class debrief, a pair came to front, created 5 × 4 = 20 on their iPad screen and were asked by Rachel: "Can you explain why you can count down by five or four? If you take away one finger right now, will it count down by five or by four?". In this instance, there was a merging of the digital tool and the mathematics where it is difficult to isolate where the technological tool stops and the mathematic begins.

Though skip counting had been taught to students earlier in the school year, it was more nuanced and complex during Rachel's implementation of Grasplify, becoming a new mathematical procedure. In this way, the mathematical concept of multiplication had changed and, with the option to approach it in different ways, Rachel's teaching pedagogy had also changed to include the TT-inflected concept. The design of Grasplify, which affords a dynamic model of multiplication that is able to change with the touch (or removal) of a finger, has mathematical implications which can also influence the didactical choices made by the teacher. Within this microworld, there is the possibility for the product to increase by a specified amount in two different ways, thus creating a pedagogical opportunity, which I will examine in more detail in the section that follows. Rachel's ability to recognise and use this affordance of the digital technology to support the instrumental genesis and mathematical learning of her students is an indication of her growing professional instrumental genesis of TT as a tool for teaching multiplication.

6.3. Three New Orchestrations

In this section, I will examine three new instrumental orchestrations that I identified in the video analysis of Rachel's and Leah's classrooms. The first orchestration is best exemplified by a continuation of Rachel's lesson that was introduced in the previous section. Two new students had just shared with the class that a different way of skip counting up by twos was to create successive pods.

Rachel:	How many pods are there now? [Pointing toward the right side of the screen where the pods are 2 × 12 = 24.]
Student 3:	Twelve.
Rachel:	Yes, twelve. And how many pips are there?
Student 4:	Two.
Rachel:	Look at this image. [2 × 12 = 24.] How is their image different from the image of the first pair? [12 × 2 = 24.] How is this image different?
Student 5:	Because they add the pods and not the pips.

As the pair demonstrated and explained their method for skip counting up by two on their iPad for their fellow classmates to see, Rachel again used a Discuss-the-screen orchestration when intentionally questioning to highlight the number of pips and the number of pods visible on the final screen. After pointing to the screen (see Figure 6.4), she asked the class to identify how the current visible image was different from the one shared by the previous pair of students. In referring to a visible screen *and* to students' mental images of a previous screen image, Rachel was using a variation of the Discuss-the-screen orchestration that required students to make a mental, sequential comparison. The use of this type of orchestration was observed both in Rachel's and in Leah's teaching as a way to encourage student comparison of different visual effects on Grasplify that were related to the mathematics. I shall refer to it as a *Compare-successive-screens* orchestration (Bakos, 2022a).

Figure 6.4 Discuss-the-screen *orchestration*

Although the Compare-successive-screens orchestration was used by both teachers when teaching with TT, Leah's implementation of it was slightly different from Rachel's. Leah would project images of pip and pod configurations for all to see and then ask students to predict what would happen next. A few examples of the type of prediction questions used by Leah include: "If I put one more pip-finger down, make a prediction about how the sentence at the top will change. What is that going to look like? What are the numbers at the top going to say?". By asking students to compare the current configuration that was visible on the screen with an imagined future screen configuration, Leah was using a Compare-successive-screens orchestration. In addition to this, by focusing student attention on the relationships between the pips and the pods and the numerical expression displayed by Grasplify, she was also engaging in a *Link-screen-board* orchestration, though, in this case, she was not required to write the mathematics on the board being that it was already visible at the top of the iPad screen.

Given the dynamic nature of TT, the emergence of the Compare-successive-screens orchestrations seemed to be a natural way for Leah and Rachel to exploit the didactical configuration to the benefit of their teaching intentions. This is particularly apparent in the way that Rachel highlighted the mathematical differences between the two ways of skip counting in Grasplify using this type of orchestration. In both the prediction and skip counting examples shared above, the mathematics is briefly visible in the *action* of altering the pips and the pods, which is difficult to portray in a static image. In order to draw attention to what was happening in Grasplify and its related mathematics, both teachers asked their students to compare what could presently be seen on the screen, either with a previously viewed screen image or with a possible future screen image.

By highlighting how the multiplicand (the pips) impacted the multiplier (the pods), Rachel and Leah used the Compare-successive-screens orchestration as an exploitation mode, a process which effectively exemplifies the teacher–TT–multiplication ensemble. Rachel's purpose was to develop student awareness that there are two possible ways for multiplication to occur using Grasplify. The first pair of students who shared their strategy for skip counting up by twos started by creating a single pip and then two 1-pods. As they sequentially placed one more pip-finger onto the screen (e.g. a second, third, fourth, fifth, etc.), each newly-created pip 'spread' across both pods until the pair of students stopped at 12 × 2 = 24. In this case, having two pods meant that, with the creation of one more pip, since *each* pod also received an additional pip, the outcome was increasing the product by two each time. The skip-counting strategy shared by the second pair of students is reminiscent of repeated addition and involved the establishment of two pips followed by the sequential creation of more pods, in order eventually to arrive at 2 × 12 = 24.

By using the Comparing-successive-screens orchestration, Rachel (in the example above) and Leah (with her 'what if?' predictive questions) both focused student attention on the effects of the addition (or removal) of finger–placements on the product and also on the symbolic mathematics. This new orchestration emerged spontaneously during each teacher's didactical performance, an indication that Grasplify was becoming a mathematics–teaching instrument used by these teachers to target their teaching intentions.

Although similar to the *Link-screen-paper* orchestration, which involves connecting the mathematical ideas represented by the technology to the conventional ways of recording the mathematics on paper, I first observed the *Document-screen-on-paper* in Leah's classroom. Students were initially shown a configuration of 1 × 3 = 3 (involving one pip and three pods) and were asked to draw on their mini-whiteboards what configuration would be visible on the Grasplify screen were they to double the product. This new orchestration is distinct since it required the children to recreate, through drawing, various elements of the screen that are mathematically important and specific to TT. As can be seen in the two photos that comprise Figure 6.5, in the Document-screen-on-paper orchestration, student drawings included such things as the vertical line found mid-screen on Grasplify, the numerical expression, some of the screen icons and even the use of different colours for the pips and the replication of these colours in the pods.

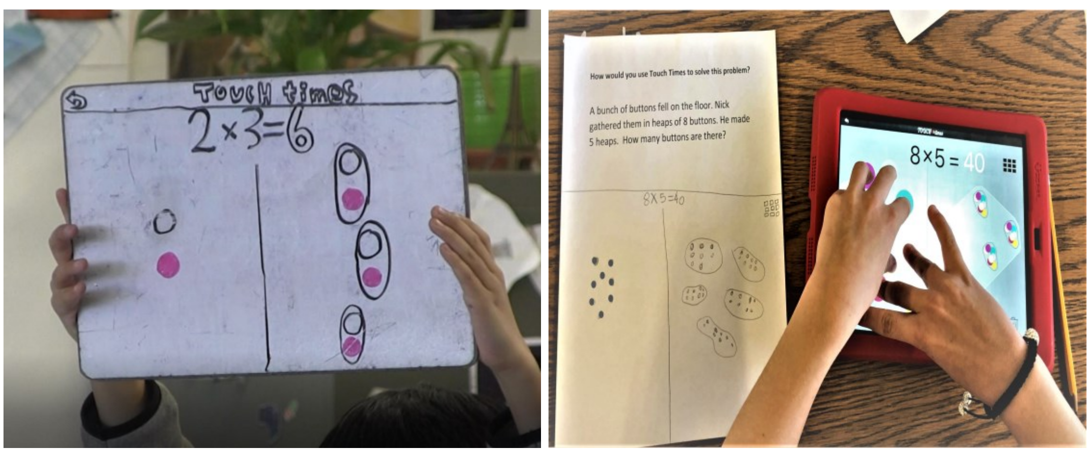


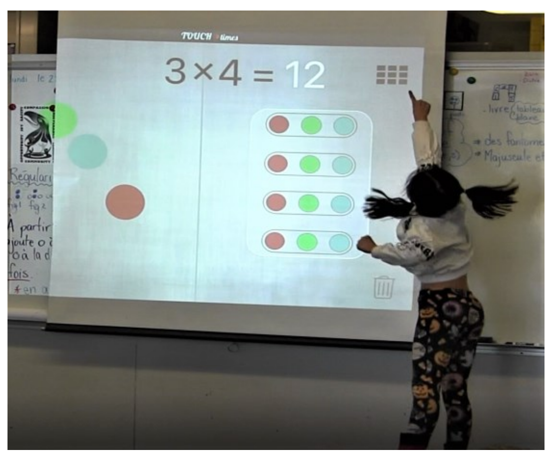
Figure 6.5 Examples of the *Document-screen-on-paper* orchestration

The dynamic and relational aspects of Grasplify enable an opportunity for students to immediately see, and feel through their fingertips, the effects of creating and changing multiplicative situations. This is a multi-sensory experience of multiplication that is very different from the static images encountered on worksheets or in textbooks. Though in TT, as soon as users remove their fingertips from the screen, the images vanish. Commonly used in elementary school classrooms, drawing is an effective way for students to capture and/or express ideas. Through the use of the Document-screen-on-paper orchestration, Leah was encouraging her students to pay closer attention to the details of the multiplicative images on the Grasplify screen, so these could be replicated in their drawings. Details such as the colour, shape and composition of the pips within the pods (the multiplier) are significant in relation to the pips (the multiplicand).

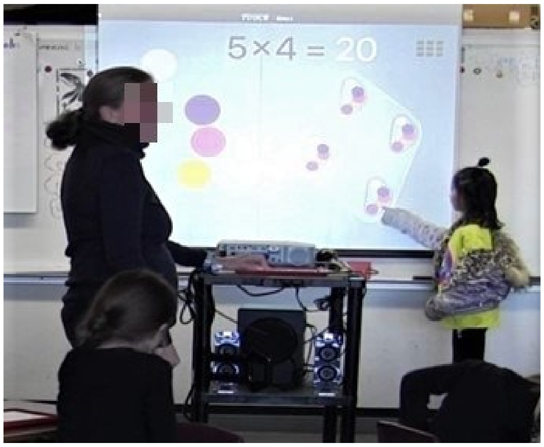
Specific to the touchscreen aspect of TT, the third new orchestration that was observed was a result of a lesson 'hiccup' (Clark-Wilson, 2010) during Leah's first use of the digital technology with her class. She was projecting the iPad screen for the whole class to view when an unforeseen problem emerged. Though the children could see the effects of Leah's finger-manipulations on the projected screen image, they were unable to see the fingers themselves. In the early stages of her professional instrumental genesis of TT, this was Leah's first lesson 'hiccup'. She needed to find other ways to share with students *where* her fingers were touching the screen, so they could better understand how her finger-to-screen touches were related to the results being seen on the projected image.

Leah addressed this issue using what I call a *Discuss-the-finger-and-screen* orchestration. There were multiple variations of this orchestration which included Leah physically pointing to the projected image on the screen herself (see Figure 6.6a), asking a student to come to the front to point out something specific on the screen (see Figures 6.6b and 6.6c) and occasionally she would hold an iPad up vertically, so that a student could demonstrate on the screen while explaining an idea to the class (see Figure 6.6d).

[a] Leah pointing to TT icon



[b] A student pointing to the array button



[c] A student pointing to a yellow pip within a pod

[d] *Sherpa-at-work* on an iPad

Figure 6.6 *Discuss-the-finger-and-screen* orchestration

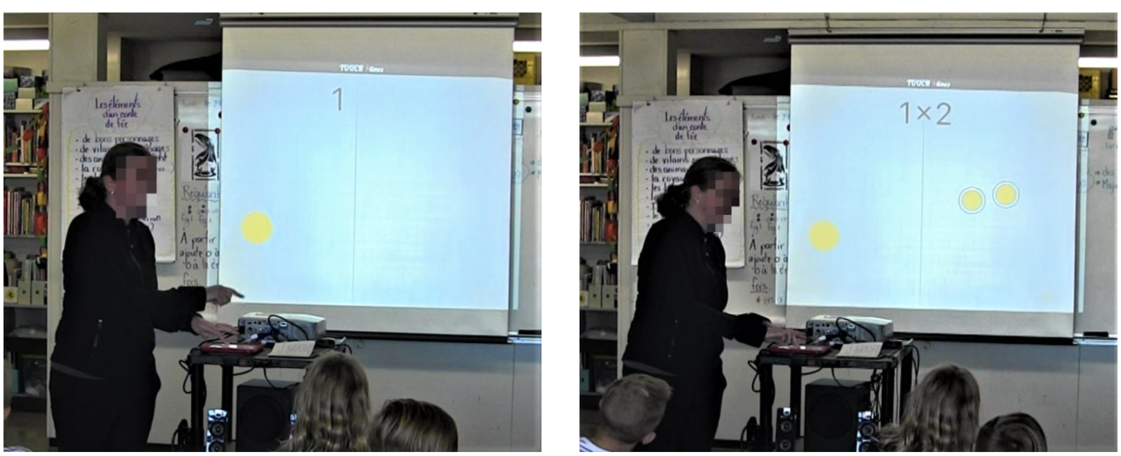
Grasplify requires continuous finger–screen contact to function. Consequently, the emergence of the Discuss-the-finger-and-screen orchestration is a natural extension of this requirement. It is only through finger–screen contact that the mathematics afforded by this touchscreen application can be materialised by the user. Therefore, the importance of students being able to see the user's fingers manipulating Grasplify is an example of the digital tool's influence on the actions of a teacher. Furthermore, the significance of the fingers, as a way of expressing multiplication and attending to the process of producing a multiplicative expression, increases beyond merely considering the result of multiplication in a static image (as in Figure 6.6c, for example).

Both the importance of the user's fingers and the mathematics that is enabled through finger–screen contact are highlighted by the Discuss-the-finger-and-screen orchestration. An orchestration emerges specifically because TT is a *touchscreen* technology that embodies multiplication in a dynamic way through the user's fingertips. The touchscreen aspect of the digital technology, the dynamic elements of the multiplicative models offered by TT and the pedagogical choices made by the teacher to highlight the multiplicative ideas enabled by TT are all intertwined. It is again possible to witness the emergence of an ensemble that is technologically, mathematically and pedagogically distinctive from traditional approaches to multiplication such as static models, repeated addition and/or paper-and-pencil worksheets.

6.4. Exploring the Tool–Teacher Relation

So far in this chapter, I have highlighted sequences of orchestrations used by Leah and Rachel while teaching with Grasplify and have identified three new orchestrations while emphasising that the teacher, the mathematics and the technology do not function independently of each other, but rather they function as an ensemble. In this section, my focus turns to episodes that illustrate the strong agential role played by the digital tool.

When Leah first introduced Grasplify to her students, she used a teacher-centred, Technical-demo orchestration to show the class which button they needed to press in order to enter the Grasplify microworld. A student could be overheard asking in the video-recording of this lesson, “the light blue?”, in reference to which of the two coloured buttons in TT that Leah meant. She then demonstrated, using exaggerated hand gestures over the iPad screen, how to create a pip (see Figure 6.7a) and some pods (see Figure 6.7b), while asking: “What do you notice happens on this side? [Figure 6.7a] What do you notice happens on that side? [Figure 6.7b]”. Though the Grasplify screen was being projected, which enabled students to see the effects of Leah’s finger touches, as mentioned earlier in this chapter, students were not able to see where Leah’s fingers were touching the screen. She instructed students to play with and explore TT, while reminding them to pay attention to what they noticed happening at the top of the screen when they lifted, moved or added a finger.



[a] Creating a pip [b] Creating two pods

Figure 6.7 Leah demonstrating Grasplify

As the class began to explore Grasplify, one student asked Leah a question. At this point, Leah stopped the class and had the student share her question with everyone, which was, “I was wondering if you can do […] because there’s always like, one times something, but I was wondering if you can do zero times something?”. Leah did not directly address the student’s question: rather, she used it as an example of a ‘wondering’ that students could engage with while exploring TT, and then instructed students to write down any of their ‘wonderings’ on their mini-whiteboards to share later.

In this instance, there were a number of elements that resulted in Leah sharing this particular student ‘wondering’ with the class as a whole. Her purpose for the lesson was clear when she stated, “I want you to play with what happens when”. The issue of multiplication by zero was not part of Leah’s intention for that lesson, and she was likely surprised when asked if it were possible to obtain zero on the pip-making side of the screen.[18] This moment exemplified a lesson ‘hiccup’ that was prompted by Grasplify which provided Leah with an opportunity to address multiplying by zero with the class.

In the expression *a x b,* where both *a* and *b* can be any number, the mathematics meets the physicality of Grasplify, where it is not possible to create a unit of 0 pips by touching the screen zero times (though in theory, you could see the answer is always zero, because nothing appears on the screen). Though multiplication by zero is not commonly addressed in approaches to multiplication that rely on repeated addition, Leah continued with her originally intended purpose for the lesson, which was for students to familiarise themselves with the technology. The student’s question was used instead as an opportunity to encourage the act of ‘wondering’, which was consistent with Leah’s approach to teaching through the Curricular Competencies aspect of the BC curriculum, which will be examined in more detail in Chapter 7. By not engaging in the mathematics discussion made available by this question, however, Grasplify was inadvertently established as a whole-number multiplying machine. This student question exemplifies a clear instance where the mathematical characteristics of this microworld provided an opportunity for the teacher to build relational understanding about multiplying by zero. An opening created by the design of the technology, rather than instruction driven solely by the teacher.

[18] TT works with natural numbers.

6.5. If the Order Matters, Then Language Matters Too

Throughout Leah's professional instrumental genesis, she was engaged in a process of instrumentalisation where she "instrumentalised the tool in order to service didactic objectives" (Haspekian, 2014, p. 253). As she continued to use Grasplify with her students, becoming more familiar with its design, Leah's own thinking about what multiplication is and the language she used in reference to the multiplicative models represented by the digital technology began to undergo a transformation.

Previously described in Chapter 5, where I detailed how Leah was first introduced to Grasplify in 2018 as part of the TT research project, her response to the multiplicative model created using the digital application was that it was "backwards". Her way of thinking about multiplication was grounded in her way of teaching it, which was primarily based on thinking of 3 × 5, in terms of repeated addition, that is as three groups-of five. In Grasplify, multiplication involves units of units, and therefore in this example it would be a unit of three, taken five times. Given the interconnected nature of teacher–technology–mathematics, the multiplicand × multiplier model used by Grasplify created a 'hiccup' for Leah. Although this 'hiccup' was not experienced while teaching a lesson using the technology, it was a perturbation triggered by the design of Grasplify that brought to light a discontinuity in Leah's mathematical understanding of multiplication.

Throughout the TT teacher–researcher meetings that occurred during the first year of the project, and even when interviewed three years after her first experience with Grasplify, Leah continued to return to her initial difficulty with the multiplicand × multiplier ordering. Her discomfort with the ordering of the numerical expression in relation to the multiplicative model displayed by Grasplify significantly affected her personal instrumental genesis of the technology, though this did not prove to be an obstacle to her professional instrumental genesis.

In the fall of 2019, during one of the research team's observational visits to her third-grade classroom, Leah's effort to adjust her thinking about multiplication was made visible as the result of a lesson 'hiccup'. This issue arose spontaneously in relation to the terminology she expressed while teaching with Grasplify projected onto a screen for the class to see.

Leah: If we think of a pod of whales, is there more than one? It's a group, right? So, the pods, if you think of a pea, with a pod, a pod of peas with all the pods. There's more than one in it, it's a *group of* things. Oh, I don't want to use that word. [Laughs and looks towards the research team.]

Researcher: It's okay.

Leah: It's a collection of things. It's more than one. Right?

Student: Why don't you want to say […] groups-of?

Leah: [Leah looks to the research team.] Before Madame started *TouchTimes*, I was stuck on calling multiplication as being groups-of. But we don't always want to be thinking about multiplication that way. And *TouchTimes* helps us start thinking about it in a different way, so I'm trying to avoid that language that I'm using and change the way we look at it. I'm learning that as I go along.

This moment illustrates how the digital technology had influenced Leah's thinking about multiplication, while offering an example of how her personal instrumental genesis had interfered with her professional instrumental genesis. It was the mismatch between Grasplify's multiplicative model and its corresponding numerical expression and her way of thinking about and teaching multiplication that was based on a 'groups-of' model that initially drew her attention to this issue. Multiplication was no longer limited to 'groups-of' for Leah and, in reconciling her mathematical thinking about this, it also altered the language that she used in reference to multiplication when discussing it with her students. Rather than verbally describing 3 × 4 as three groups-of four, Leah now spoke of it using language such as three, four times.

It is striking that, in spite of Leah's personal difficulty reconciling the multiplicand × multiplier model of Grasplify with those of repeated addition, she was still able to instrumentalise the digital technology to serve her didactic objectives and she proceeded to integrate it into her pedagogical repertoire as a method for teaching multiplication. However, it is unlikely that Leah's personal and professional instrumental geneses would have evolved as they did, had she not been part of this research project. When confronted with a mathematical model that did not conform to her ways of thinking, speaking and visualising multiplication, would Leah have persisted with her use of Grasplify? If the technological tool *and* the mathematical concept *both* create dissonance for the teacher, is that an obstacle to the development of the teacher–tool–mathematics ensemble?

6.6. Summary

Throughout this chapter, I have continued to use an ensemble approach when examining the ways in which the teacher–tool–mathematics mutually influence each other when elementary teachers are implementing Grasplify as a tool for teaching multiplication. Although many of the instrumental orchestrations noted by Drijvers and colleagues (2010, 2013) were also observed in these elementary school mathematics lessons, research on orchestrations used by elementary teachers and/or those used with touchscreen technology remained limited, and this work contributes to this area.

Though the scope of my study of instrumental orchestrations here was limited to two teachers, both of whom were voluntary participants in a research project involving the integration of TT, the comments that follow are not necessarily limited to these specific cases, nor are the three newly named orchestrations. The orchestrations that emerged are closely related to the dynamic, relational design of TT and as a consequence, the touchscreen technology of this application lends itself well to these types of orchestrations. I have described below how the teachers implemented Grasplify as part of their pedagogical practice for teaching multiplication and have also detailed an example of how the digital technology influenced the thinking and altered the language used by the teacher when teaching multiplication using Grasplify.

This chapter addressed my first research question and examined in more depth the effects of TT and its ways of potentially materialising multiplication on teachers as they progressed through the process of double instrumental genesis. It is common practice for many elementary school teachers to introduce students to multiplication through repeated addition and, like Leah, many of these teachers become firmly attached to thinking about and teaching multiplication using a 'groups-of' model. The design of Grasplify, however, necessitates that the user creates the multiplicand prior to the multiplier, and the resulting numerical expression, which reflects the number of pips and pods that were created by the user, does not match this 'groups-of' model. In this way, TT, and its way of presenting multiplication, shapes how teachers talk about the mathematics with their students, also seen when they refer to ‘pips’ and ‘pods’ in relation to the multiplicative model created in the application.

Leah became very intentional in her use of language such as ‘three, five times’, which matched the multiplicative model visible on the iPad screen. This (for her) new way of describing multiplication corresponds to the mathematics of Grasplify and exemplifies the interacting nature of the teacher–tool–mathematics ensemble. The way Leah talked about multiplication with her students when using Grasplify may be very different from how she spoke of it without the presence of different visual representations of multiplication to teach with, something which has not been suggested in the literature (e.g. Anghileri, 1989; Davis & Renert, 2013, Kosko, 2018).

Additionally, whole-class orchestrations were used by both teachers to draw student attention to the effects of adding or removing pip-fingers on the pip-configuration within the pods, and also to link these pip and pod units to the numerical expression displayed by Grasplify. In so doing, the *Compare-successive-screens, Document-screen-on-paper* and *Discuss-the-finger-and-screen* emerged as new orchestrations. The *Compare-successive screens* and *Discuss-the-finger-and-screen* orchestrations both occurred spontaneously while the teachers interacted with their students and the mathematical ideas being shared as part of whole-class discussion. The *Compare-successive-screens* orchestration was used to highlight mathematical ideas that emerged from Grasplify which both teachers deemed important. This included focusing student attention on what happened to the multiplicative model and/or the numerical expression when the unit is changed – an idea that was new for teachers when teaching multiplication.

Asking students to record mathematical images and ideas through drawings is a pedagogical practice commonly used by elementary teachers, and therefore the *Document-screen-on-paper* orchestration asks students to re-present the mathematics created by their fingertips using Grasplify in their notebooks for future reference. Asking students to draw the multiplicative models directs their attention towards the relationship between the pips and the pods, which is reflected by the number, colour and shape of the pip-configuration within the pods. The *Discuss-the-finger-and-screen* orchestration is very particular to the touchscreen aspect of TT, where the purpose of the finger movements is not merely instrumental (to push buttons), but is also conceptual (to create relationships). Both Leah and Rachel wanted to connect the finger actions on the screen when using Grasplify with drawings that brought attention to the mathematics.

Both Leah's and Rachel's way of teaching using Grasplify involved assigning students (often in pairs) a mathematical task, providing time for them to explore the task and then bringing everyone together for sharing and whole-class discussion. When analysing the instrumental orchestrations used by the two teachers when teaching with TT, their approach to using whole-class orchestrations was seamlessly integrated within student sharing and whole-class discussion. This style of teaching, where elementary students are situated in leadership roles for demonstrating, discussing and sharing their findings with their classmates, while the teacher assists through the use of questioning to draw out student thinking or by highlighting and/or extending the mathematical ideas further is not uncommon in elementary classrooms. When in this coaching role, the didactical performance of the teacher is responsive to the ideas that emerge from the children, the mathematics that is the focus of the lesson and the teaching opportunities that emerge either from Grasplify or from the mathematical task assigned.

Though many of the instrumental orchestrations used by Leah and Rachel are similar, the pedagogical approach to the way in which they were implemented differed from those described in secondary mathematics classrooms (as documented by Drijvers et al., 2010, 2013, for example). It also differed from the teacher-led approach used in Kindergarten in a case study described by Carlsen and colleagues (2016). In that situation, the teacher mediated basic interactions between the child and the interactive whiteboard technology, whether they be technical or mathematical, by assuming the role of assistant, mediator or teacher in response to individual student needs. Besnier (2018) also studied instrumental orchestrations in a Kindergarten context and noted the teacher's emphasis "on verbalisation in mathematics and the idea that peer-to-peer exchanges promote learning" (p. 261). Although this aspect is similar to the approach used by Leah and Rachel in their elementary school classrooms, Besnier does not elaborate on the teacher–student interactions, while instead focusing on the Kindergarten teacher's *Manipulatives-and-software-duo* orchestration where the children manipulated labels on the board to mimic the actions of the software.

In the earlier-discussed ensemble, there was a synergistic influence amongst and between all three components of teacher, technological tool (TT) and mathematics (multiplication) which included occasions in which Grasplify was observed exerting agency. The first instance of this occurred in a non-intentional way as a result of a student question about multiplying by zero. This is worth noting, as the tool prompted a novel set of actions and the features of the technology had an unintended influence on the practice of the teacher. The challenge of unseen fingers on the projection screen and the mathematics vanishing with the removal of fingers from the iPad are both difficulties entangled within the teacher–tool–mathematics ensemble. The solutions created by teachers to overcome these problems are indicative of the growing professional genesis of these professionals in their implementation of TT as a teaching tool for mathematics.

Additionally, as a result of the presence of Grasplify, there were new classroom orchestrations initiated by the teachers, some of which were related to the nature of the body(finger)–Grasplify interaction. The very design of TT creates certain ways of seeing and embodying multiplication, which can influence the way teachers talk about multiplication with students, while also encouraging new attention on the fingers as tools for expressing and engaging physically with multiplicative relationships.

Now that I have examined the instrumental orchestrations commonly used by Leah and Rachel while implementing Grasplify in their classrooms, my attention shifts in the next chapter to the thoughts and experiences shared by Amy, Kate, Leah and Rachel *after* they had each used TT as a pedagogical tool for teaching multiplication with their students.

Chapter 7. After *TouchTimes*: Teacher Reflections

> *The moment causes you to notice, but the reflection is what really changes you. (David Pimm, taken from personal class notes)*

My examination of teachers' personal and professional instrumental geneses is sequenced to highlight the progression of experiences before, during and after using TT. The process of double instrumental genesis was initiated during the teachers' first exposure to and experiences using TT themselves, *before* using it with students, which was the topic of Chapter 5. As Leah and Rachel proceeded to implement this digital technology in their classrooms, some of the instrumental orchestrations that were utilised *during* teaching with Grasplify were detailed in the preceding chapter. Which brings me to the focus of this chapter, which will be on the reflections shared by the four teachers *after* they had each implemented TT as a pedagogical tool for teaching multiplication to their students.

During the interviews, Amy, Kate, Leah and Rachel shared their insights and reflections regarding their purpose for implementing and using this digital technology with their students. While Kate used TT to build foundational understanding through mathematical intention, Leah and Amy chose to use Grasplify to build student understanding through learning processes and Amy and Rachel both described how Grasplify was only one model of many that they used to teach multiplication. Using the theoretical constructs of double instrumental genesis, instrumental distance and didactical reference markers, I will again interweave my analysis and interpretation of Amy, Kate, Leah and Rachel's thoughts and experiences throughout this chapter. Although the data was shared from the perspective of each teacher during the individual and paired interviews, I am conscious of examining it in a way that honours the intra-action[19] of the teacher–technology–mathematics ensemble.

[19] Intra-action recognises a dynamism of forces, in this case of teacher–*TouchTimes*–mathematics, working inseparably in a relationship of continuous exchange or influence (Barad, 2007).

7.1. Purpose: Why use *TouchTimes* in school?

Between the start of the 2018 school year and mid-March 2020, Amy, Leah and Rachel implemented and utilised Grasplify with two subsequent years of students. During this time, their descriptions of how they were using TT as a tool for teaching multiplication were becoming more detailed and personalised, reflecting the on-going evolution of their professional instrumental genesis. During their June–August 2021 interviews, however, all three teachers shared, almost apologetically, that during the COVID-19 pandemic they did not use TT. They described how exhausting teaching was in those circumstances, how much longer additional handwashing took with elementary school children and how they just did not possess the energy to sanitise a class set of iPads before and after student use (if their school district even allowed shared touchscreen devices to be used at that time).

Kate was the only teacher interviewed who had recently used TT with a class of students and her approach to using it was in sharp contrast to the other three teachers' shared experiences. There was something about this difference that captured my attention and I found myself trying to pinpoint exactly what it was that was different about Kate's use of TT. In order to do this, I went back to the individual teacher interview transcripts and examined more closely how the other three teachers described using TT. Though Amy, Leah and Rachel spoke about and used Grasplify in a similar way to each other, what they shared was very different from what Kate said. This difference was directly connected to Kate's purpose for implementing TT, which was to leverage it as a pedagogical tool for mathematics. This idea was so powerful and such a continuous thread throughout Kate's individual interview that, in the paired interviews that followed – with Amy and Rachel, then with Leah and Kate – I asked the pairs to clarify their purposes for using TT with their students.

I will begin by describing in detail what Kate shared regarding her intent for using TT as a tool for teaching multiplication. This will then be followed by the value of exploration and discovery shared by Leah and Amy, and the purpose for integrating this digital tool into their pedagogical repertoire shared by Amy and Rachel.

7.2. Kate: Building foundational understanding through mathematical intention

Grasplify and Zaplify, the two different microworlds that compose *TouchTimes*, present different visual models of multiplication and their corresponding mathematical symbols. The creation of these distinctive multiplicative models also involves very distinctive kinaesthetic experiences for users in regard to their hand positioning. Each microworld not only *looks* different, each one provides differing sensory experiences during the creation of each multiplicative model. The ability to access two different mathematical representations within *TouchTimes* permits teachers an opportunity to provide students with experiences of more than one model of multiplication within the same digital resource. The role of the teacher, however, is crucial in ensuring that this mathematics–technology opportunity benefits learners and one teacher shared how she encouraged this through purposeful planning and questioning.

Of the four teachers interviewed, Kate was the only teacher in this study who utilised both Grasplify and Zaplify with her students. Her use of both multiplicative models in her teaching enables an examination not only of the gap between her former ways of teaching multiplication and *TouchTimes*, but also of the distance between these two different, yet related, digital tools. I refer to the distance between Grasplify and Zaplify as *intra-instrumental distance* (Bakos, 2022b).

In her role as a mentor teacher, Kate did not have her own classroom, but in the spring of 2021, she had the opportunity to use TT with a class of third-grade students in her school district. As a result of her recent implementation of this digital resource, what she shared when interviewed in July–August 2021 was very focused on how she had used TT in her teaching. Another characteristic that is singular to Kate in this quartet of teachers interviewed is her specialised training as a secondary mathematics teacher. Perhaps as a result of this, she was very purposeful when planning her instruction to focus specifically on the numerical expressions created in conjunction with the multiplicative models in TT. Her didactical reference markers were very firmly situated in the content learning standards of the mathematics curriculum.

Though Kate acknowledged that students found TT more engaging and that “kids are more open to learning when they’re doing something fun”, she was also very clear that discovery and exploration were teaching methods that she was cautious of, unless they were purposeful. “If I just let them explore, they’re just going to keep playing. So, I had to find ways to connect it back to the mathematics.” This influenced how Kate approached using TT and she described continually questioning herself about what she was trying to do, what she wanted students to learn and where she wanted students to be “having very clear intentions. They [students] have to be made aware that it’s multiplication. That will not happen on its own.”

In the teacher–tool–mathematics ensemble, the mathematics took on a leading role for Kate, whose primary purpose for using the digital technology was very clearly linked to the symbolic mathematics. With a very strong background in mathematics, she possessed a number of didactical markers related to multiplication and, therefore, the models and numerical expressions presented by the software did not provoke any changes in her understanding of this topic. Consequently, the focus in this sub-section will be on how Kate ensured that her students connected “what they were seeing with the actual equations” and that they possessed an understanding of multiplication that would transition across both multiplicative models afforded by TT.

Kate was very intentional in using the intra-instrumental distance between both microworlds to provide students with embodied experiences of the different multiplicative relationships highlighted by each of the models. By having her students go back and forth between the multiplicative models of Grasplify and Zaplify, her intention was that students gain a deeper understanding of the mathematics that would allow them to transition between multiplicative models while being able to recognise “that multiplication is the common theme”. She believed that, when using TT, it “has to be a conscious effort on behalf of the teacher” to ensure that students can “make connections between the two different worlds, make connections between the symbols, the equations, the representations, because, at the end of the day, I wanted them to know how to multiply and what multiplication was, so I kept coming back to that one idea”.

Ultimately, for Kate, it always came back to the mathematics itself and providing ‘mathematically rich’ experiences. She did not see the technological tool and the mathematics as separate entities. Rather, her intent was to use the dynamic affordances of TT as a way for her students to visualise, conceptualise and experiment with multiplication and “to connect what they were seeing with the actual equations. Like so that they were getting a sense that what they were doing was multiplication." She described how seeing the dynamic nature of TT caused her to question, "Are they paying any attention to the symbols? No. How can I get them to?" She wanted her students to see and feel how their bodies, in real time, were affecting the numerical expression displayed by Grasplify and the factors and product shown in Zaplify.

Kate was mindful that the numerical expression was displayed by Grasplify the entire time it was being used. Though students did not appear initially to notice it, the equation was something that she felt students were comfortable with because they were simply used to it being there. Kate described subtly drawing student attention to the numerical expression by asking students, "deep-thinking questions about how the symbolic connects to what they're doing with their body". She also consciously directed attention towards the effects of adding or lifting fingers from the screen on the equation, noting that, "they started looking for that and it was like a whole different way of experiencing multiplication". Kate's professional instrumental genesis was very much driven by what mathematical opportunities she saw as being offered by the digital technology. The influence of the numerical expression being available but unnoticed by students, prompted her action as a teacher to draw student attention towards how their fingertips were affecting the equation displayed at the top of the screen.

> For me, it was it was a having the equation there was like a subordination piece, but also like an extra tool for me to use to ground what they're doing in the mathematics. That part's really important for me. […] What is this? What are you actually doing? And then you can look at how things change together, so it's not just the two hands now, it's the two hands *and* the equation. It's almost like another relationship. You can see how things change together.

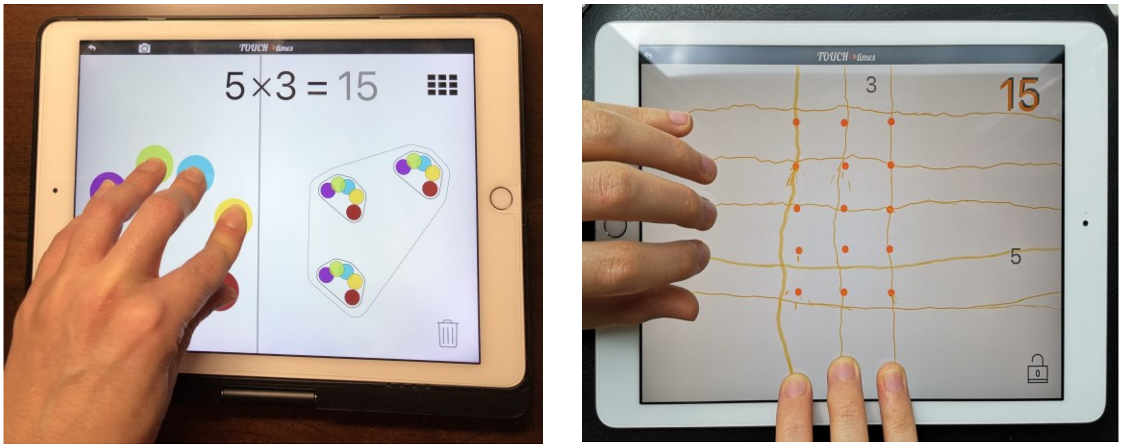
Unlike Grasplify, which displays the numerical expression that corresponds to the multiplicative model that has been created by the user, Zaplify only displays the applicable floating factors and the product. This was frustrating for Kate, who felt that, "it would have been easier for them to see the change and what they do with their fingers and easier for them to make the connection to multiplication" if Zaplify also displayed the numerical equation in its entirety. She found that she "had to work harder to connect it [Zaplify] with the mathematics" and that her students just saw it as "the lightning video game". Even though Kate did not like this aspect of Zaplify, she felt that, by going back and forth between both microworlds, her students were "looking at the concept and trying to play with both", which she believed to be beneficial for their mathematical understanding. She also noted that, “one of the things I liked the most about the Grasplify array model is that it corresponds more to Zaplify”. The advantage of this was that it allowed students to see the mathematical connections between the two models more clearly. Because Kate wanted to keep the intra-instrumental distance narrow, these differences between the two microworlds were viewed as obstacles for student learning, rather than pedagogical opportunities.

Class discussion was not a teaching technique that Kate found effective with the class that she was teaching and, as a result, her choice of teaching methods was somewhat mediated by student behaviour and classroom management. Therefore, after students had engaged with TT for the first few classes, she felt that the student activity part of the lessons provided in the TT Task Booklet (Bakos et al., 2021) she was using to guide her teaching were not enough for this particular group. Students needed more independent practice time to work on the concepts and to “figure it out and play with it and write down what they found. So, less talk [whole-class discussion], more do.” Kate would determine what idea she wanted students to be attending to in a lesson and would then develop an independent practice sheet that targeted that idea.

> Whatever was on those pages is what I was trying to get their attention on. I was very intentional about like everything I did with them. I was very intentional about what do I want them to learn? What do I want them to notice? Whatever it was that I asked them to do [on the worksheet], that was what I wanted them to be attending to.

Once students had finished their assigned exploration task, they would engage in these activity sheets.

The teacher–TT–mathematics ensemble functioning as a single unit is evident in Kate’s explicit efforts to draw student attention to the similarities and differences of the two multiplicative models created in TT. After displaying screenshots of the same multiplicative expression both in Grasplify and in Zaplify (see Figure 7.1), students were asked to compare and contrast them. She wanted her students to notice that multiplication was the common theme. Through intentional questioning and the use of independent practice sheets, Kate used the intra-instrumental distance between TT’s two microworlds to transition students across Grasplify and Zaplify’s different multiplicative models, in order to encourage deeper student understanding of what multiplication is.



[a] Grasplify model [b] Zaplify model

Figure 7.1 ***TouchTimes*' multiplicative models**

Drawing activities were another medium that Kate would have her students engage in during independent work and these were directly related to the tasks assigned during that lesson. On one activity sheet created by Kate, there was a screenshot of a multiplicative model created in one microworld, and students were asked to draw how the other microworld would display the corresponding multiplicative model (see Figure 7.2). In another question, students were asked what equation the Zaplify screen showed (a screenshot of Zaplify was provided) and if there were one or two possible equations for that screenshot.

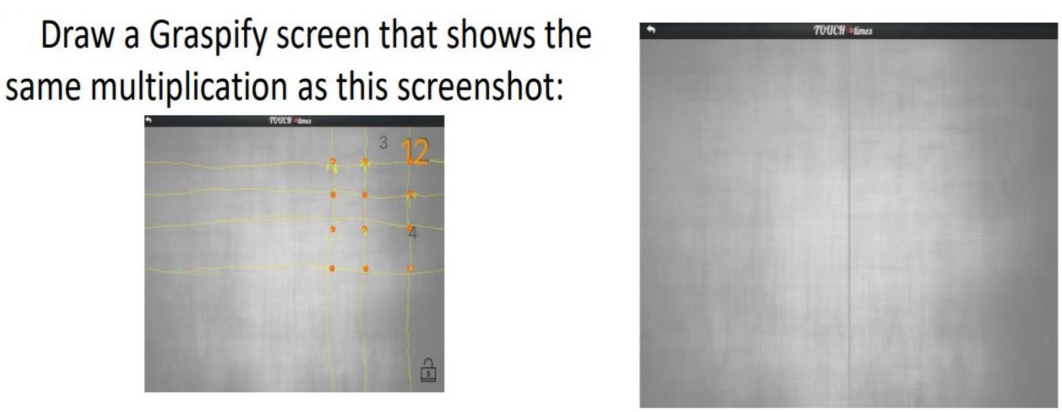


Figure 7.2 **Transitioning across multiplicative models**

After engaging the class in a doubling task using Grasplify, students were given a screenshot and asked to "draw on the picture to show how you could double the amount shown by changing the number of pips. Write what the new equation would be." (Figure

7.3). By having students complete these activities which often included drawing and/or writing the corresponding multiplication equation, Kate was working to scaffold student understanding of multiplication beyond the digital technology by directing attention to the connections between the multiplicative models that compose TT. Her intentional use of these carefully created independent practice sheets was another means used by her to narrow the instrumental distance between what she considered most important mathematically for students to understand about multiplication and the learning opportunities afforded by TT.

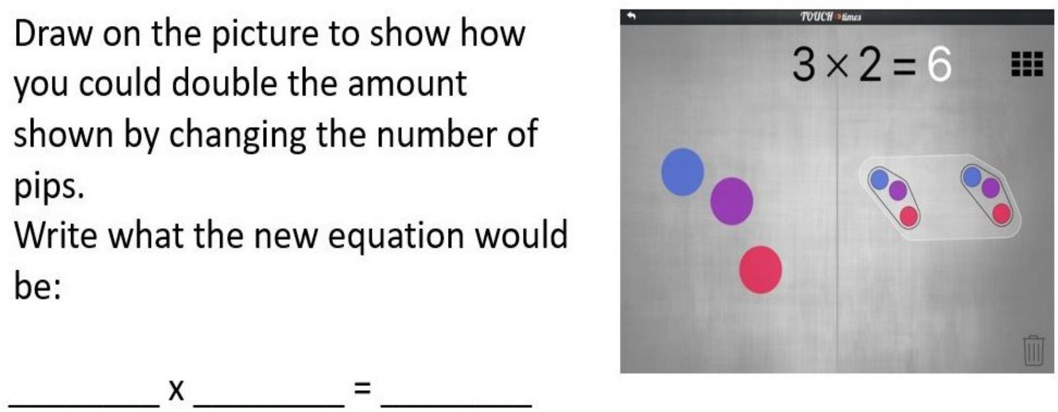


Figure 7.3 Drawing attention to doubling

The prioritisation of students making connections between the different multiplicative representations that could be created in Grasplify and Zaplify and, in particular, their corresponding numerical expressions was an essential characteristic of Kate's professional instrumental genesis of TT. Her didactical reference markers were grounded in multiplication being the symbolic equation. As a result, her pedagogical goal was to focus student attention on connecting the multiplicative models in TT to the symbolic mathematics, which she justified with the requirements of the content learning standards of the mathematics curriculum.

She was very conscious of keeping the instrumental distance (between the technology and her belief that multiplication is the symbolic equation) very narrow. Students were engaged in activities focused on the representational models of multiplication and the symbolic mathematics, and, in consequence from no equation being present in Zaplify, that proved more difficult for her to achieve. Rather than using the digital technology as a way of developing student understanding of the meaning of multiplication, TT was utilised as a pedagogical tool to direct student attention towards the symbolic mathematics that Kate prioritised as most important.

The ability of students to transfer their mathematical understanding of multiplication created using TT into other contexts was important to Kate, who was concerned that TT would not be a resource used by their next teacher. She wanted to ensure that her students would be able to “go from this experience and carry it over and it [multiplication] all makes perfect sense”. Kate’s willingness to engage in the process of professional instrumental genesis was clearly hinged on her ability to “bring it all together and connect it [TT] to the curriculum”. However, the aspect of the curriculum she was referring to was specifically the content learning standards (knowledge), which was done at the expense of the curricular competencies (skills, strategies and processes). Kate believed that, when first learning multiplication, what is most important is for students “to understand the different representations and how to go between them”. In meeting that purpose, Kate worked to take advantage of the intra-instrumental distance between Grasplify and Zaplify, while maintaining a small gap between what she believed was most important mathematically for children to learn about multiplication and the affordances of integrating TT into her pedagogical practice.

7.3. Leah and Amy: Building understanding through learning processes

Kate’s very focused use of TT as a way of addressing curricular content learning standards was in sharp contrast to the value that Leah and Amy placed on the learning processes engaged in by students while using TT. In order to understand this better, it is worthwhile to examine the design of the current British Columbia curriculum. This concept-based, competence-driven curriculum rests on a foundation of literacy, numeracy and the core competences, which “are sets of intellectual, personal and social and emotional proficiencies that all students need in order to engage in deep, lifelong learning” (Province of British Columbia, 2022, Core Competencies section). These core competencies effectively underpin the skills, strategies and processes outlined by the subject-specific curricular competencies that students are meant to develop over time. In mathematics, the specific skills, strategies and processes that students are expected to engage in fall into four categories: reasoning and analysing; understanding and solving; communicating and representing; connecting and reflecting.

Leah, a very experienced teacher, explained that the curricular competence aspect of the mathematics curriculum was the primary resource she used to guide her teaching. Therefore, the open-ended design of TT, which Leah viewed as “more of a competency app”, enabled her to provide her students with opportunities to explore mathematical concepts through play, problem solving and visualisation, while also encouraging the use of mathematical vocabulary when students are talking about TT tasks with a partner or when contributing to whole-class discussion. The curricular competencies themselves provided the didactical reference markers used by Leah when teaching mathematics and being able to see opportunities to use TT as a way of addressing the curricular competencies provided a way for her to bridge the instrumental distance between TT and other ways of teaching multiplication.

Leah described true learning as involving curiosity and playing with intention. By engaging students in what she called, “mathematical play”, they were encouraged to take risks and guess and check without constraints, which Leah believed was beneficial for learning. The children were given a short time to explore TT without limitations at the start of each lesson before being given a task to do or a question to answer. Prior to modelling an idea for her students herself, Leah would provide them with an opportunity to discover the answer through play. She used a student-centred approach based on what the children noticed or wondered. After bringing the whole class together, students would share their discoveries with the group, while Leah led them toward the intended mathematical ideas through careful questioning. When using this discover and share approach, Leah noted that, “I wanted them to believe that they had discovered it”. Although the underlying mathematics is what the tasks and/or questions were based on, what was most important for Leah were the processes with which the students engaged, in order to ‘discover’ the mathematical ideas for themselves. Despite the fact that her personal genesis of TT was still developing, Leah’s ability to use Grasplify as a tool for exploration and discovery enabled her to appropriate it for her mathematics teaching.

After teaching for eight years, Amy no longer relied on the textbook to guide her mathematics teaching, preferring to create many of the activities that she used with students herself. Like Leah, she also shared how, being more familiar with and comfortable using Grasplify in her second year, she gave her students more freedom to explore and make their own observations. In comparison with other technology, which required a lot of groundwork to teach students how to use it, which Amy felt was taking away from teaching the intended concept, TT was seen as user-friendly, so students could “just get down to business and start exploring multiplication with it”. Preferring for students to explore activities with partners when using Grasplify, Amy would circulate amongst them while they were working, in order to respond to questions, encourage them to keep thinking or to provide them with the next level of challenge. The benefits of student exploration are described in detail by Amy:

> Over time I’ve learned that it's actually really valuable to kind of let them ask their questions, let them make their own predictions and kind of let them guide their own explorations. They make the best observations when they're just kind of doing their own thing and they're given the freedom to choose what they do with the app and then everyone can build their own idea of how the app works and how it can support them in their multiplication journey and then they're much more receptive when you give them tasks to do. I think exploration’s a huge thing, a big benefit for them.

After students had time to explore the task given, Amy would gather them together on the carpet at the front of her classroom and have student pairs share their discoveries. Having students explain their ideas to each other, and to the class as a whole, was important to Amy, who saw value in letting her “kids be the teachers” while she took on a coaching role to guide their explorations. She would use the ideas shared by the students as a starting point for deeper discussion about the mathematical concepts that were the focus of the task given, or as a “jumping point” (as she described it) for the next activity that she would engage student with.

This style of teaching/learning is more commonly used in other subject areas, such as social studies and, as an elementary generalist teacher, Amy possessed didactical reference markers that may have facilitated her use of Grasplify to encourage student exploration and discovery. These didactical markers helped to narrow the instrumental distance between the digital technology and Amy’s former ways of teaching multiplication while making the process of professional instrumental genesis smoother for her.

A similar approach was used by Leah, who explained that, when students were sharing their discoveries with the class, she was watching for specific mathematical ideas to emerge. For example, when students were showing all the different ways to make 24 using Grasplify, she described how she hoped that one child would mention two times twelve and another would answer twelve times two. This would allow the conversation naturally to emerge where she could ask students: “Do we count that one? Is it the same thing? How would it look if we drew it?” These moments provided an opportunity for explicit teaching, which Leah described as being more valuable for students when the initial ideas emerged from them. When a student noticed something mathematically relevant, she felt that her role as a teacher was “to make those noticings explicit for the others, either through the child, or ourselves”.

Leah’s lessons would sometimes involve ‘what if?’ questions, where students were asked to make predictions about ‘what would happen if’ scenarios. Student would draw their predictions on individual mini-whiteboards at their desks. For instance, while projecting an image of a Grasplify scenario for all to see, Leah asked, “What would happen if I placed another pip-finger down? Show me on your whiteboards.” Leah described using these drawings as a way of gauging student learning. She felt that these drawings and her observations while students engaged in multiplicative experiences by “playing with” Grasplify provided her with insight into student learning. This information enhanced Leah’s ability to build student understanding in a more individualised way through intentional questioning which she used to redirect attention towards the mathematics that she wanted the student(s) to notice.

The ability to watch what the children were doing while they were “playing with” and learning a concept was valuable to Leah. She wanted students to develop deep understanding and was looking to see if they were “really understanding” the concept. She noted that, even if student exploration was not in the direction she had originally intended, she still had a better understanding of where that child was coming from and could then better assist them. Leah felt that, when her students were using Grasplify, their engagement with the digital technology allowed her “to actually see *how* kids were thinking about multiplication”, in ways that were not previously possible when she expected students simply to mimic her demonstrations of “how to do” multiplication.

Part of Leah’s professional instrumental genesis of Grasplify involved her ability to assess individual students’ mathematical awareness. She wanted to know if students were understanding the concept of multiplication as experienced in Grasplify. While observing and interacting with individual students as they completed their ‘what if’ drawings or engaged in TT tasks, Leah could intervene and assist students on-the-spot while they worked, or during whole-class discussion. Based on what she observed in a lesson, Leah would use her didactical reference markers to plan for and scaffold the next day's lesson, in order to make it meaningful for student learning.

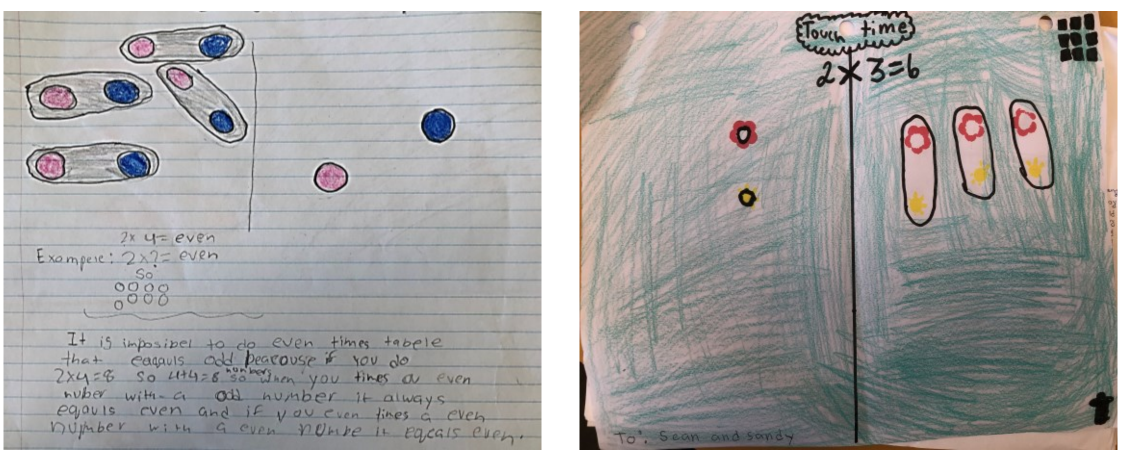
Her didactical markers were based on the knowledge she possessed about how students learn multiplication and what difficulties they commonly encounter during that process. As she became more confident using Grasplify, and her professional instrumental genesis continued to develop, Leah was able to bridge the instrumental distance by seeing TT as a tool that better enabled her to gauge student understanding of multiplication.

The importance of the visual aspect of Grasplify was also brought up by Leah in relation to its usefulness in prompting mathematical conversations. The non-permanent nature of the pips prompted more "oohs and ahs" which led to students sharing their "wonderings and noticings". When students came to her, "Oh, look at what it did, Madame! Look at what it did, Madame!", Leah described how this allowed an opportunity for her to ask the student a relevant question. If students were not seeing the mathematics which she wanted them to, she could redirect their attention through questioning. There was a benefit in using Grasplify because it allowed for conversations within the discoveries, and engaging students in mathematical explanations and conversations was something that Leah valued, believing that it led to better student understanding. Again, she possessed didactical reference markers for addressing student misconceptions. Once she began to see that TT provided an opportunity for the types of conversations and/or questions that led to student understanding, she was better able to leverage the technology as a pedagogical tool.

Amy also described the importance of students being able to explain their learning and internalise their understanding in their own ways. She found it interesting, even though all students were using the same application, how "they made sense of that app in their own way and drew it in their own way". She shared a Grasplify activity where she put three fingers down on a projected iPad and asked, "Can you find an equation that can make an even number?" Students were given time to explore this question and, when they had an answer, they were asked to draw it out and try to explain their understanding.

One student drew a number line and then used skip counting to explain his answer. Amy shared to me, “I never thought about transferring it to a number line and I was really impressed that he did. […] I loved just seeing how the kids were able to internalise it their own way.” Providing opportunities for students to develop the ability to explore, notice, draw diagrams and explain their thinking were processes that Amy valued, believing that these enabled students to explore deeper multiplication concepts. “Kids were able to, instead of seeing multiplication as just a really surface kind of basic function, where you just fill it out in a worksheet, they were able to really dive in deeper.”

After a Grasplify lesson, Amy asked her students to answer questions such as: what surprised you? What did you learn? For her, “Math doesn't have to be all about numbers. You can be like explaining your thoughts.” Sometimes she would have her class complete drawings of what they had noticed during their explorations. It was important to Amy that students had opportunities to express their understanding in various ways (see Figure 7.4a), giving an example of the student who used stamp markers to express her understanding of the pips and pods in Grasplify (see Figure 7.4b). These drawings were also useful as a formative assessment, enabling Amy to “see what they noticed. Like did they notice the colours? What were they able to pick up on? What did they attend to?”



[a] Student drawing and written reflection of personal understanding

[b] Student stamp marker drawing reflecting pips and pods

Figure 7.4 Ways students shared their understanding

In Leah’s classroom, she noticed that the dynamic, non-permanent and faster aspects of Grasplify allowed for more conversations and, consequently, more shifts in student thinking. For instance, she could ask a student, “Can you do six, four times on here? What would that look like?” and the student could try it, get an immediate answer and then conversations, starting with questions such as, “Why did you decide to do it this way? Would it matter if I change the order of the numbers?”, could be triggered right away. Whereas, when students were drawing arrays on paper, the process was much slower and Leah felt that these kinds of interactions were either not possible or did not have the same impact. As she continued to use Grasplify with her students, the instrumental gap continued to narrow as she noticed affordances of the digital technology that benefited student learning in ways that were not possible using pencil-and-paper.

Leah also described how her exposure to TT had influenced her way of thinking about multiplication as a teacher, in a way that stuck with her when introducing multiplication to her students the following school year. Her teaching approach had become more focused on trying to build the concept of what multiplication means, rather than simply knowing 6 × 4 = 24. Developing students’ ability to explain why 6 × 4 = 24, as well as for them to understand what that numerical expression means, was now Leah’s teaching goal. Like Amy, as Leah became more comfortable and familiar with using TT, her professional instrumental genesis was becoming stronger and she could more consciously taking advantage of the open-ended nature of TT.

After beginning the lesson with a pre-determined task, she was not only open to, but was also watching for, other opportunities to emerge as a result of student discoveries that would enable her teaching “to go in a direction that works for you and your students”. Leah was also clear, however, that if her mathematical goal was not met in that day’s lesson, that she would plan to come back to that idea in a future lesson. She would set students up to return to these ideas again, by asking them to identify and communicate: “What did we notice today? What did we discover today? And what do we wonder about for next time?” She was no longer completing a single, pre-determined task using TT in a lesson, Leah was looking beyond TT to the mathematical idea that she wanted students to understand. Her ability to do this seamlessly is an indication of the growing depth of her professional instrumental genesis and reflects the dynamic relationship amongst the teacher–technology–mathematics ensemble.

Since my interview with Leah occurred after Kate’s, I was curious to know her thoughts about linking what was occurring in Grasplify back to the numerical expression. Leah’s husband, who is a high school mathematics teacher, happened to be nearby when we were discussing this topic and Leah asked him for his opinion. He shared the importance of students seeing, “That it's [TT] not just a game or that it's not just a new way of seeing it [multiplication], but that it also links back to the pure math”. Leah’s response was: “Whereas for me that's not what teaching math is”. She then referred back to the curricular competencies that she used to guide her teaching:

> What do I want them to get out of this? It's not that they know that six times four is twenty-four. It's what does that mean and what does that look like? And how can you show that in different ways? It's the process, not the product. That's the most important thing to me because in those early years they need to understand the process and have flexible thinking with number.

Leah’s prioritisation of student learning processes made linking the process of multiplication very purposefully back to the numerical expression or ‘pure math’ less important for her. This was very different from Kate’s perspective and, when Leah shared her use of the curricular competencies to guide her teaching and her use of Grasplify during her paired interview with Kate, Kate pointed this out. She explained that, while Leah was seeing TT as all about the curricular competencies, she was seeing it as, “Okay, I have to connect this to the content for them [students] as well”.

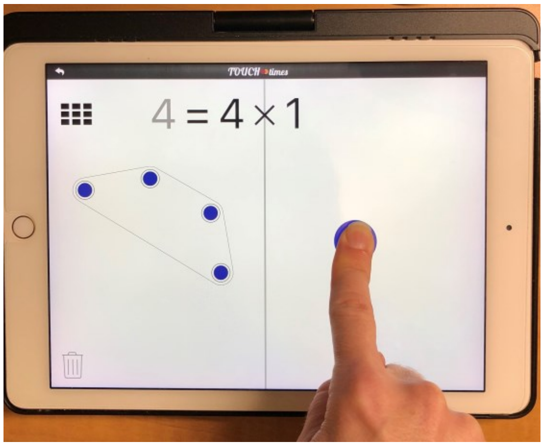
As experienced elementary school teachers, both Amy and Leah possessed a number of didactical reference markers related to what third-grade students should know and understand about multiplication. These enabled them more easily to bridge the instrumental distance between other methods of teaching multiplication and those afforded by Grasplify. Their process of professional instrumental genesis was such that Amy and Leah were able to use Grasplify in a manner that prioritised student exploration and discovery while tailoring their teaching to the needs of their students and the mathematical ideas that naturally emerged from the tasks given. The teacher–tool–mathematics ensemble was working in a way that allowed each teacher to utilise the digital technology to build student skills, strategies and processes related to multiplicative thinking, which they believed was most important in mathematics.

7.4. Amy and Rachel: “It’s not your *only* model, it’s *one* model”

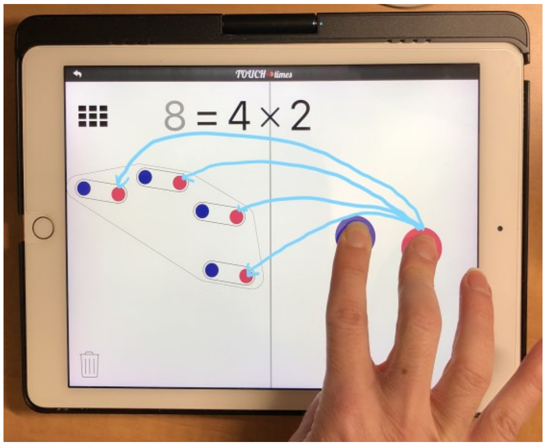
In this section, I will analyse how Amy and Rachel accommodated the instrumental distance between Grasplify and their previous ways of teaching multiplication, as well as how their experiences of double instrumental genesis evolved as they used the digital technology in their classrooms. During their paired interview, these two would often elaborate upon, clarify or extend each other's ideas and, as a result, the experiences shared were more nuanced and detailed. Their main purpose for using TT and the manner in which they approached their teaching were very similar and, for those reasons, I am choosing to write about Amy and Rachel together. In what follows, I will examine how their purpose for using TT was based on what the digital technology offered them as a teaching tool that was different from the approaches previously used by both teachers for introducing multiplication.

When asked to describe the methods previously used for introducing and teaching multiplication to their students, Amy and Rachel seemed naturally to juxtapose their prior teaching methods with their use of Grasplify. Rachel started by explaining how she would often introduce a multiplication problem to her students and have them physically engage in recreating the situation. She provided an example that involved having some baskets and picking up leaves and putting them into the baskets. Rachel would take her students outside where they would collect the leaves and then use them to make little groups-of leaves. After this, she would have them draw their baskets and the leaves that would be put into each basket. Additionally, building arrays with manipulatives and introducing area models with base-ten blocks, then having her students colour these area model representations using graph paper, were all ways that Rachel approached teaching multiplication. Amy then described how she formerly introduced multiplication by writing a numerical expression on the board, 3 x 2, for instance, and would then model, using physical manipulatives and/or drawings, three groups-of two. She also used number-line drawings to demonstrate visually skip counting by three, two times, as well as manipulatives to model building arrays, and introduced area models when teaching multiplication to her students.

Throughout Amy's explanation of these teaching methods, Rachel nodded her head in agreement, then clarified that when introducing multiplication using Grasplify that, "It's almost like the opposite". She went on to share that normally she would write three times two on the board and would then draw three little groups-of two or build an array. She pointed out, however, that she demonstrated and explained multiplication, that previously this started with the numerical expression. She contrasted this with an example of how she used Grasplify, where her students were challenged to skip count forward by fours to twenty by *only* changing the pips. Skip counting by changing the pips embodies a multiplicative rather than an additive approach. Although there is more than one way to skip count to twenty using Grasplify, by limiting students only to adjust the pips, they must understand the effects of the pips within the pods. In order successfully to skip count by four this way, students must establish the correct number of pips and pods at the beginning that will enable this increase of four to occur each time (see Figure 7.5a).



[a] The initial set-up required



[b] The "spreading effect" of an additional pip-finger within the pre-established pods

Figure 7.5 Skip counting by four by only changing the pips

When she assigned this task, Rachel explained that she was, "not necessarily looking to show a model for that equation, […] the equation is there, but you're trying to get at concepts that might be harder to get by just drawing something". Her purpose in using this task was to direct student attention towards the effects of creating additional pips, and how each new pip would 'spread' across each pod (as shown in Figure 7.5b, blue lines added by me), thereby increasing the product by four each time. Rachel's teaching goal in this example is directly influenced by the dynamic, kinaesthetic and visual experience of multiplicative growth through spreading afforded by Grasplify's design. There is a closely intertwined and mutually interactive relationship reflected by the ensemble of Grasplify, the mathematics it enables and how Rachel implemented and used the digital technology as a way of teaching multiplication as growth.

Though Amy and Rachel each shared their thoughts and experiences implementing Grasplify as a pedagogical tool for teaching multiplication throughout their individual interviews, it was when they were interviewed together that their purpose for using it became much more apparent. Given that their paired interview occurred shortly after I interviewed Kate, I was curious about their beliefs related to the importance of the numerical expression. Kate's strong epistemological beliefs – regarding the importance of linking the numerical expression to student actions and the corresponding multiplicative models when using TT – were striking. Yet this focus on the symbolic mathematics had not emerged as essential in the previous interviews with the other teachers, nor was it at the forefront of the lessons observed in Leah, Amy and Rachel's classrooms. Therefore, I was curious about this and asked Amy and Rachel to share their thoughts regarding linking the equation to what their students were physically creating in Grasplify and to clarify further their teaching intention(s) when using this digital tool to teach mathematics.

The dynamic nature of the pips and pods representation of Grasplify was acknowledged both by Amy and by Rachel as being beneficial for building student understanding of multiplication. However, when asked about the importance of the symbolic mathematics in this context, they were quick to point out that TT was not the *only* method they used for teaching multiplication; rather, Grasplify was merely *one* model that they taught amongst others. Their purpose in using the digital technology was to engage students in the multiplicative process through embodied experiences. Neither Rachel nor Amy felt that they had to choose between using their traditional methods of teaching multiplication or those offered by this digital application. Instead, their intention was to use both options in complementary ways, in order for students to build deeper conceptual understanding of multiplication.

For Amy, a focus on the equation seemed to signify finding the answer and then moving on, without providing an opportunity for deeper understanding and exploration of concepts. She explained that, even when using an array model, it's "just stuck" in that one configuration that corresponds to the equation, and then the students simply go on to the next question. In contrast, Amy felt that the constant changes and movement within Grasplify's multiplicative models encouraged more open-ended thinking and whole-class discussion led by student discoveries. When asked about the role of the numerical expression in TT, she clarified that, for her, when using the digital technology, "the emphasis is on the exploration because the answer is already provided by TT and, therefore, that isn't where the focus is". This was consistent with her valuing student-led exploration guided by their own questions, predictions and discoveries.

In contrast to the other models she used for teaching multiplication, TT provided a visual representation of multiplication that immediately changed in response to student actions, something not easily demonstrated with manipulatives or drawings. By having students explore tasks using Grasplify, Amy felt that students were provided with a better picture of what was happening, which would lead to a deeper understanding of multiplication and the type of thinking in which she wanted her students to engage. She stated that:

> With the app, you're not using the equation to teach multiplication, you're just kind of exploring multiplication. The equation is just secondary. The equation's there to *support* your understanding of multiplication, instead of the equation *being* the understanding of multiplication.

Amy went on to clarify, however, that when her intention *was* to focus on equations and getting the right answer, that she would chose to use other models with her students.

The didactical reference markers that Amy associated with the symbolic mathematics of multiplication involved worksheets, getting the right answer and moving on to the next question. It was a combination of her dislike of this type of approach to mathematics and her preference for more open-ended exploration and student-led, whole-class discussion, which she could do when using Grasplify with her students, that helped make Amy's professional instrumental genesis a positive process. Her ability to use the digital technology in ways that encouraged students to engage in the learning processes that she most valued helped Amy to bridge the instrumental distance more easily.

Rachel concurred with Amy's comments about using other multiplicative models, explaining that she exposes her students to various models, which include arrays, area models and groups-of to show repeated addition, but when using these models, that she was, "building a model to represent an equation [… whereas] the equation in *TouchTimes* is secondary". Though Grasplify was one model amongst many, she recognised that it offered different affordances than drawing or building multiplicative scenarios out of concrete materials. What was most compelling about Grasplify, for Rachel, was the impact of being able to add or remove fingers from the screen and then to see the results of these actions instantly, in ways that were "perfect each time". She pointed out that the consistent colours and immediate responses in Grasplify were not possible to replicate with drawings or physical manipulatives. Rachel's didactical markers were visible in her reference to using manipulatives or drawings to illustrate these ideas for students, and the gap between these strategies and those offered by Grasplify was narrow but favourable for using the technology.

Returning to her previous example of the skip counting by fours in Grasplify, Rachel reiterated that she wanted students to explore what effects the sequential and/or simultaneous placement and removal of their fingertips had on the pips within the pods. Then, during the whole class sharing and discussion that followed the task, she would use intentional questioning to guide student attention toward the numerical expression displayed at the top of the screen. In this way, Rachel would highlight how the multiplier and the multiplicand were affected by each pip- and pod-creating (or -removing) action and the subsequent effect on the product as well. She compared starting with the equation first to how she used TT and would draw attention to the equation later:

> With an equation you're like, here's the equation and now I'm going to represent it with area or with a drawing or with an array. Whereas *TouchTimes* is like I want to try and show how, when I put more fingers down, this is what happens to my number, and I want to see where I get to. And then, at the end, I'm like, Oh cool. Look at my equation. This is what I built.

Rachel also noted that multiplication is a big concept for grade four and five students who are used to thinking of multiplication only as repeated addition. Therefore, prior to using Grasplify in her teaching, she looked to see what the application offered and noted that:

> *TouchTimes* is really good at showing that kind of growth. You know something is getting bigger by a certain amount as opposed to maybe other models where kids are just drawing groups-of something and they're thinking more of it as repeated addition. […] I would choose to use that app, I guess, to get at the concepts of something growing. That's harder.

Rachel saw this concept of something "growing" as being "at the heart of multiplication […] beyond additive thinking" and that this concept was much easier to see in Grasplify. As part of her double instrumental genesis, Rachel needed to become familiar with what Grasplify could offer and, once she had a better understanding of this, then she could think about "why am I using this app?" in the context of her teaching. She was very conscious that TT provided immediate visual feedback and embodied experiences that could "enhance students' comprehension of multiplication in ways that are different, you know harder, to get at through pencil-and-paper or even manipulatives".

Rachel was also interested in better understanding what her students were attending to as a result of the tasks they engaged in using Grasplify and would ask them to complete drawings as a way of depicting their learning. She described using these to help her determine, “What did they understand and what should I work on next time?”, and how she would use this information to plan her next lesson with TT. The colour, arrangement and number of the pips within the pods are essential features of Grasplify’s multiplicative model, which were characteristics that Rachel would watch for in student drawings. In order to inform her teaching practice, she wanted to know: “Okay, they really didn’t pick up on, you know, the colours or whatever it may be. And so next time, I’m going to focus on that because I didn’t see that in any of their drawings.”

As her professional instrumental genesis continued to develop, Rachel began to think about the strategies that she wanted to teach to her students and whether she could use Grasplify to do so. One of these strategies was decomposition, or breaking down a seven into a five and a two, in order to make it easier to multiply. Decomposition was a strategy that she would normally teach for two-digit multiplication, which she would soon introduce to her students. This prompted her to think about, “How can I use this [TT] to work on these strategies that I’m teaching to my students? […] I wanted to be able to work on certain strategies that were going to help my students multiply larger numbers because that's where we were going.” The instrumental gap continued to narrow for Rachel as she began to think about how to utilise the affordances of the technology to approach strategies, such as decomposition, that she traditionally taught as part of multiplication. In this sense, there was a merging of the didactical reference markers that she previously possessed with new ways of approaching them using this technology. This did not involve the creation of a new reference marker: rather, it involved an update which incorporated the specific technology into what was already there.

As a result of their professional instrumental genesis, Amy and Rachel’s method of teaching multiplication using Grasplify was a complete inversion of how they formerly taught this concept. The dynamic embodied and relational experiences provided by the digital technology of TT enabled an instructional approach that was “the opposite way” from the operational method they had traditionally used for introducing multiplication. Rather than starting with the symbolic mathematics and then explaining its meaning using diagrams or models created with manipulatives, Grasplify was used as a tool to expose their students to the process of multiplication. Rachel, in particular, was very intentional in using the opportunities that arose during whole-group discussion to link student actions on the pips and pods with the numerical expression found at the top of the screen. Students could first experience the process of multiplication through their hands and, with the use of purposeful questioning, Rachel would create a bridge between students' Grasplify experiences and the symbolic mathematics commonly associated with multiplication.

Neither Amy nor Rachel were discouraged by the instrumental distance between their prior methods of teaching multiplication and those that included TT, instead they embraced this gap and saw it as beneficial. The experiences of multiplication that were enabled by Grasplify were different in ways that both teachers believed to be advantageous for student understanding. Rachel went further to acknowledge that the digital technology was also influential in broadening her own perspective about multiplication: "*TouchTimes* was definitely good for expanding both mine and the students' kind of horizons as to what multiplicative thinking is". In this case, the teacher in the teacher–tool–mathematics ensemble was affected by the digital tool's presentation of the mathematics, which also became part of her professional instrumental genesis. *TouchTimes* itself, and its way of presenting the mathematics, created opportunities for Rachel to develop new didactical reference markers that focused on multiplicative versus additive approaches to teaching multiplication.

7.5. Summary

Throughout this chapter, I have highlighted the insights and reflections shared by Amy, Kate, Leah and Rachel during their interviews related to their purpose for implementing and using TT with their students as a way of learning about multiplication. Each teacher's growing professional instrumental genesis was reflected by their increasing responsiveness to the individual learning needs of their students and their ability to recognise and take advantage of the teaching opportunities that emerged while using TT. That said, the process of double instrumental genesis is multi-faceted and complex.

When implementing unfamiliar digital technology into their mathematics teaching, teachers must be familiar with how the technology functions, as well as how to leverage it effectively as a tool for teaching. Doing this involves differentiating between ways of teaching with TT, which are teacher-centred, from how students are expected to learn using TT, which is student-centred. When examining Amy, Kate, Leah and Rachel's purposes for using this digital technology, what was common was sometimes related to themselves as teachers, while at other times the purpose was connected to the learning they wanted students to engage in. Occasionally, the purpose was found in the overlapping space where student learning was gauged by the teacher in order to address individual student learning needs or to plan instruction.

My first research question focused on the ways that TT presents multiplication and how this affected the teachers as they progressed through the process of double instrumental genesis. Kate was very conscious that TT was an iPad application, a type of digital tool that she felt many students associated with play, rather than learning. As a result, she was very intentional in her teaching and what she wanted to draw student attention to in regard to the mathematics. The dynamic nature of the application was influential in her professional instrumental genesis, as Kate saw this affordance of the technology as a way with which for students to experiment, visualise and conceptualise the process of multiplication in regard to the symbolic mathematics. Meanwhile, her role as a teacher was to ensure that students connected these actions back to the numerical expression in Grasplify and the factors and product displayed in Zaplify.

As described in Chapters 5 and 6, the influence of TT's design had a significant impact on Leah's process of double instrumental genesis: however, as detailed in this chapter, the effects were more general. The open-ended design of Grasplify allowed both Leah and Amy to focus on learning processes when using the digital technology to build student understanding of multiplication. The curricular competencies aspect of the mathematics curriculum is a foundational aspect to how Leah approaches her teaching and her engagement in the process of professional instrumental genesis was influenced by her ability to use Grasplify in ways that allowed for student exploration, mathematical play and student discussions. Amy also valued student exploration, something that the open-ended nature of Grasplify enabled her to use a teaching strategy.

Rachel and Amy were conscious of exposing students to different learning experiences with a variety of visual representations of multiplication. They viewed the dynamic aspect of Grasplify's design as an opportunity for students to experience multiplication in a relational way where they could create new, visually multiplicative meanings of multiplication using their bodies. Building multiplicative situations to develop this type of understanding, however, was detailed by Rachel as difficult to demonstrate using physical manipulatives or static drawings. She viewed Grasplify's way of demonstrating multiplicative growth as significant, even noting that it expanded her own way of thinking about and also teaching multiplication.

Much of this chapter focused on my second research question, which was how teachers accommodated the instrumental distance between TT and their previous ways of teaching multiplication. Kate was very purposeful in her intent to ensure student understanding of the two different multiplicative models that are part of TT and to connect these models with the commonly used symbolic mathematics. The teacher–TT–mathematics ensemble was visible as Grasplify and Zaplify provided opportunities for students to make connections between two different multiplicative models, something for which Kate intentionally planned. When she observed that her students could successfully do this, the intra-instrumental distance between Grasplify and Zaplify's two different, though related, multiplicative models narrowed.

The instrumental distance continued to narrow both for Leah and for Amy as they became more familiar with using Grasplify and as they witnessed the mathematics learning that their students were experiencing as a result of using it. The curricular competencies, such as exploration through play, problem solving and visualisation were didactical reference markers that Leah already possessed. As she began to see ways of addressing these competencies through the use of the digital technology, the instrumental distance between Grasplify and her former ways of teaching multiplication began to narrow. As Amy became more comfortable with using Grasplify, she began to provide more student freedom for them to explore and to ask their own questions, so that they could build their own understanding. Similar to Leah, Amy already had didactical reference markers for this type of teaching and, being able to teach in this way, made it easier for her to adopt an unfamiliar digital technology into her mathematics teaching.

Rachel and Amy described previously teaching multiplication by writing a numerical expression on the board and then explaining what the numbers meant through drawings or the use of physical manipulatives. When using Grasplify, however, there was a complete inversion of this process. With the digital technology, students were given tasks that focused on the process of multiplication first, and it was only later that either teacher drew student attention to the numerical expression. Although this is a significant shift in how they taught multiplication, in this instance, Rachel and Amy saw this difference as advantageous and embraced the technology as a result. Although they both saw the dynamic, relational affordances of TT as beneficial for student learning, they were also clear that Grasplify was *only one* model that they used to teach multiplication. The instrumental distance between Grasplify and their previous teaching methods was not a deterrent either for Amy or for Rachel: instead, it was an additional pedagogical tool that offered something different for teaching multiplication.

My third research question was related to the evolution of the teachers' experiences of double instrumental genesis while using TT as a teaching tool. Although each teacher's purpose for using TT may have been slightly different, as part of their professional instrumental genesis, Amy, Kate, Leah and Rachel not only used the technology as an instrument for teaching mathematics, but also had their students' complete drawings to depict their understanding. A type of formative assessment, these drawings were used to gauge student understanding, in order to address student learning needs either on the spot or to inform the next lesson's plan.

The integration of these types of drawing activities with the use of the technology is another indicator that the instrumental gap between prior ways of teaching multiplication and those enabled by the use of TT was closing. All four teachers had experience using a static medium to assess student learning: these didactical reference points made it possible to narrow the instrumental distance between accustomed ways of teaching and/or assessing learning and the dynamic, embodied tasks that are part of TT. This digital technology was becoming increasingly incorporated with other teaching techniques, such as making predictions, recording 'wonderings' and drawing.

In the next chapter, I will examine the sociocultural influences experienced by the Amy, Kate, Leah and Rachel throughout their processes of double instrumental genesis of TT.

Chapter 8. Sociocultural Influences

> *The true direction of the development of thinking is not from the individual to the socialized, but from the social to the individual. (Vygotsky, 1934/1962, p. 20)*

Though my research questions were initially focused on how TT and its way of materialising the mathematics affected the teachers (instrumentation) and how the teachers accommodated the instrumental distance between TT and their previous ways of thinking about, and teaching, multiplication (instrumentalisation), an unexpected factor that was influential for all four teachers also emerged in the data. It was during the individual and paired interviews, when I was trying to get a sense of how each teacher's thinking about multiplication and/or about TT had evolved as a result of their personal process of instrumental genesis, namely that the infuence of others was described by each, though in different ways, as a critial factor.

From the time of their initial exposure to TT to the point of being interviewed, each teacher had implemented and used this digital technology as a pedagogical instrument for teaching multiplication. Amy, Kate, Leah and Rachel all described the impact of other people as being significant during their adoption and integration of TT into their professional practice. Throughout this chapter, the last of the four data chapters, I will examine the personal interactions that were described by each teacher as being valuable, in order to understand better how the influence of others affected each teacher's process of double instrumental genesis.

When examining the influence of others on each of the teachers, I am doing so from a sociocultural viewpoint, one that acknowledges the critical role of the social context in cognition and social development (Vygotsky, 1978). From the perspective of mathematics education, the term 'sociocultural' often refers to a view of thought, learning and knowledge-creation as inherently social, cultural and situated (made rather than given). Schleppegrell (2010) explains that, "Sociocultural perspectives focus on discursive practices and the social engagement of students. They draw on Vygotskyan frameworks that stress the interaction between language and cognition and highlight the social dimension of learning and the role of communication and participation" (p. 76).

Though this quotation refers to the social engagement of ‘students’, I see the role of communication and participation in the TT teacher–researcher group as being a significant factor in Amy’s, Leah’s and Rachel’s[20] process of double instrumental genesis. This is an idea which will be examined in more depth in the section that follows. After that, my focus will shift to the impact of the interactions between the teachers and their students. In particular, Kate and Leah both described how the students themselves proved to be an influential factor in the evolution of their personal and professional instrumental geneses.

8.1. The influence of the *TouchTimes* teacher–research group

Amy, Rachel and Leah all mentioned how valuable it was to be involved in the TT teacher–researcher project, and to have a chance to interact with other teachers who were also implementing this digital technology as part of their mathematics teaching. Though Amy did not go into great detail about this in her interviews, she did express during her paired interview with Rachel how beneficial attending the teacher–researcher group meetings was for her, and that, “it was a really good place were everyone was able to share their ideas and learn from each other, which is what I really like doing”. She found it helpful to hear other teachers’ ideas and the opportunity to share with others the activities she had created for use with Grasplify was something Amy highly valued.

The influence of the TT teacher–researcher project was described both by Rachel and by Leah as being fundamental in helping them overcome their initial difficulties with understanding and implementing Grasplify as a pedagogical tool for teaching multiplication. I will first examine the experiences shared by Rachel, before returning to Leah and her difficulty with the multiplicand × multiplier ordering that was introduced in Chapter 5.

[20] Given the location of Kate’s school district, she was unable to attend the teacher–researcher group meetings and therefore this did not emerge as part of her process of double instrumental genesis.

8.2. Rachel: Overcoming challenges and rethinking teaching multiplication

Initially finding TT less intuitive to use and harder to think of tasks by herself than she had previously found with *TouchCounts* (Sinclair and Jackiw, 2011), Rachel described her reliance on the TT task ideas handout that was developed during the project and on the experiences and ideas that were shared at the research group meetings by the other teachers. During these meetings, those who had used TT in their classrooms would share their experiences with the larger group, describing which tasks they had used, as well as how they had implemented them, what worked well, any challenges encountered while trying to teach using this digital application and what their students liked, had difficulty with or were learning while using TT. Being able to listen to what was shared by the other teachers in the project during small- and large-group discussions was beneficial for Rachel in overcoming her initial difficulty with how to implement Grasplify: "Just understanding how it [Grasplify] worked and seeing what other people were doing […,] it was just nice to get a sense of what people were doing in their classrooms and what I might be able to try".

The instrumental distance between Rachel's prior experiences with using *TouchCounts* and her introduction to TT was large. It seems that in order to make the gap small enough to feel confident to try implementing TT in her classroom, Rachel relied on the shared experiences of other teachers in the group and required the support of pre-developed tasks for use with the technology. It was almost as if the shared experiences of other teachers enabled Rachel to create 'shadows' of her own didactical reference markers, in order to rely on during her initial implementation of Grasplify until she could solidify these through her own teaching experiences.

During the first TT teacher–researcher group meeting, Dr. Sean Chorney explained that, "there tends to be a real focus on additive thinking in presenting multiplication and in doing multiplication" and that one of the goals of the project was to improve multiplicative thinking in students. The research team also shared with teachers that, if students are only able to think about multiplication in terms of repeated addition, this creates problems for them when they reach middle-school mathematics, which requires an ability to think multiplicatively. Grasplify's design and the tasks that were developed for use with it have been developed to expose students to ways of thinking about multiplication that are more multiplicative in nature. As a result, Grasplify's way of presenting multiplication and the importance of highlighting these multiplicative ideas continued to be a topic of discussion within the TT teacher–researcher group meetings.

During her interview, Rachel explicitly described how being part of the TT project helped her develop a deeper understanding of the way that Grasplify presents multiplication and how this influenced her thinking about teaching multiplication. She explained, "I think it was like coming to the sessions and listening and having the discussions [in response to questions prompted by Sean]" that impacted her thinking about the properties of multiplication and the importance of developing her students' ability to reason multiplicatively. "I think it definitely did get me thinking about the properties of multiplication more than I had if I weren't using it." She went on to add that, when members of the research team were in her classroom to observe her teaching with Grasplify, their presence made her pause and think, "Oh, I should think really deeply about this [the properties of multiplication]".

As a result of being part of the teacher–researcher group, Rachel was becoming more intentional in addressing these ideas in her own teaching, while becoming increasingly aware of wanting to encourage the development of multiplicative thinking in her students. She explained that, "If I were just using it [TT], not being part of a cohort of teachers who are meeting and talking about it and really thinking about it, would it affect me as much? Probably not, but it did."

Listening to and engaging in these group discussions was clearly significant to Rachel's thinking about multiplication and her growing recognition of the importance of developing multiplicative thinking in her students. Although it was the mathematical model presented by Grasplify that created opportunities for these conversations, it was the influence of the conversations within the group that caused Rachel to begin thinking about and teaching multiplication differently using Grasplify.

> For me, using *TouchTimes* or listening to Nathalie, or being part of those discussions, it was like, "Oh, we really need to be showing students that multiplication is more than just this". So, in terms of impacting my teaching, whether using the app or not, I guess it made me think about, like the words I'm using or what I'm saying, or how I'm showing multiplication or teaching it or what, you know, whatever the activity is that the students are doing, that I need to be aware of how I'm representing multiplication.

Both Rachel's personal and professional instrumental genesis were significantly impacted by her engagement with the TT teacher–researcher group. Although she did not initially find Grasplify to be an intuitive tool for teaching mathematics, she was able to overcome this difficulty through her participation with the project and the shared experiences and ideas of the other participating teachers. The discussions and interactions in this group also influenced her own thinking about multiplication and, as a result, her approach to teaching it using Grasplify was also changing.

The effects of Rachel's interactions with the researchers and the teachers who were part of the project affected the entire teacher–tool–mathematics ensemble in a positive manner. The task ideas shared by the researchers and the discussion amongst the group about TT and the way it materialised multiplication influenced how Rachel implemented Grasplify with her students and also drew her attention to the mathematical affordances of the tool. She was now being more intentional in her teaching about different ways of representing multiplication and was working to provide her students with opportunities to think multiplicatively through the tasks she implemented with Grasplify.

Her personal instrumental genesis and understanding of Grasplify and its way of representing the mathematics were becoming deeper which could be seen in the way that Rachel was beginning to leverage this understanding into her teaching. Her beliefs about what students needed to learn about and understand regarding multiplication were changing as she learned more about what was important for students to understand about multiplication and how TT could be used as a pedagogical tool to achieve those aims. The sociocultural influence of the group was a significant part of Rachel's ability first to construct 'shadow' didactical reference markers, which enabled her to bridge the instrumental distance between her prior ways of teaching multiplication and those using Grasplify. Once she had bridged this distance, Rachel was then able to build her own didactical reference markers based on her teaching experiences which effectively helped her to begin narrowing the instrumental gap.

8.3. Leah: Context, commutativity and relationship

As mentioned previously in Chapters 5, 6 and 7, the multiplicand × multiplier = product ordering of Grasplify caused a significant disturbance for Leah and there were two very different sociocultural influences that proved to be beneficial to her process of double instrumental genesis. The first, which will be addressed in this section, was the TT teacher–researcher group and the second, which will be examined in the section that follows, was the influence of Leah's students themselves.

The opportunity to share questions and to discuss possible answers to these questions with other teachers in the group was a significant factor in Leah's process of double instrumental genesis. She explicitly stated, "I think that teachers being able to work with this together is also really beneficial. […] The ideas that come out of it are pretty big, especially when somebody is new to it [TT] too." Leah also described herself as being very self-reflective and could remember thinking to herself at one of the teacher–researcher meetings, "Okay, there's something missing [in reference to her difficulty with Grasplify's representation], these are people who really know a lot of math and number. I'd better think about my thinking. It bothered me that there was another way that I hadn't thought about." The presence of 'knowledgeable others' made Leah pause and question her own understanding of multiplication.

During the second TT teacher–researcher project meeting, Leah expressed to the group a need for further reassurance, stating that she was still confused about "the opposite way that the app is looking at it [multiplication] than some of us are used to teaching it". In the following excerpts, another teacher in the group, Monica (a pseudonym), noted the importance of context in relation to the numerical expression and its multiplicative model.

> For me, it's context, right? So, that's when you want to make sure that the groups-of works. Like if you've got, there's a difference between if you're looking at cars with four wheels and you know they [students] know how many wheels. There's a difference between two times four and four times two in that case.

Monica went on to question whether or not this idea that 3 × 4 had to be three groups-of four was a "real rule". Then, a few minutes later, she provided an example of a multiplicative situation where the groups-of scenario would be influenced by which factors would be easier to visualise, rather than their position in the numerical expression.

> The other thing I was going to say too is if you have a hundred times three, are you going to expect like you're doing a hundred groups of three? Or are you going to visualise three groups of a hundred?

This explanation and the two examples that were provided by Monica in November of 2018 made an impact on Leah, who shared the essence of the cars and wheels example with me during an interview nearly three years later. She explained how helpful Monica's example was and that, "it [the order] does matter, sometimes. So, if you're talking about something that has a set amount of things, like cars have four wheels. Groups-of matters in that language." Leah already had didactical reference points for thinking about groups-of and this example, and its acknowledgement that context sometimes did matter when creating a multiplicative model, somehow seemed to help her accept that, in most multiplicative situations, "It doesn't really matter because it all equals the same thing".

The instrumental gap was becoming narrower as a result of this discussion with another teacher, who provided an option to think of multiplicative models in relation to context. Though this notion of visual representations being dependent upon context was not explicitly connected to the ordering of the numerical expression (which is what created the disturbance for Leah, initially), it was still effective in transforming her thinking. Perhaps what was helpful for Leah was the acknowledgement that, unless a certain context demands that a multiplicative model be represented in a specific way (as in the cars and wheels example), mathematically the ordering of the factors is not important.

In reconciling for herself that the multiplicand × multiplier = product ordering of Grasplify should not matter, Leah also referred to another didactical reference point, which was her knowledge of commutativity. She explained how this property of multiplication was helpful as a way of making sense of the irrelevance of the ordering of the factors. Though her understanding of commutativity enabled Leah to bridge the instrumental distance between Grasplify's way of modeling multiplication and her previously held groups-of model, commutativity is not applicable in the context of Grasplify.

Though the product is the same for $a \times b$ and $b \times a$, the consequence of the commutative property, the emphasis in Grasplify is not actually on the product itself: rather, it is on the procedural establishment of the pip(s), then the pods and the subsequent 'spreading' of the pips across each of the pod-units. In the context of Grasplify, the multiplicand (represented by the pips) and the multiplier (by the pods) have differing roles and are not interchangeable. Despite commutativity not being relevant to Grasplify, having the knowledge of this property was helpful in permitting Leah to disregard her previous belief that the only correct way to represent 3 × 4 was as three groups-of four.

When interviewed three years later from her starting with TT, Leah explained that, "I was so stuck on this groups-of thing and then I started thinking about, well, what does multiplication mean? So, it really changed my thinking about what it [multiplication] means." She went on to describe further the transition that she had experienced, which I believe nicely reflects the intertwined nature of her double instrumental genesis. "If I really believe in […] how this [application] works, and what multiplicative thinking means, it doesn't matter what happens next. It's what happens in their [her third-grade students'] thinking."

Leah also spoke about how, as a result of the car and wheels example and the subsequent discussions within the teacher–researcher group about how to describe the multiplicative model in Grasplify, she now used different ways of speaking about multiplication with her students.

> So, to me, you can do the three cars with four wheels, three groups of four. I still do use that language sometimes, but then you can teach, "Well, on *TouchTimes,* it's actually thinking this way: three, five times." And so, when you're teaching that, you are thinking about it in that way. And this way, it actually deepens the understanding, even more. So, the conflict I had was actually a gift.

The instrumental distance between Grasplify's way of representing multiplication and the groups-of approach previously used by Leah when introducing multiplication in her class provoked her to question her own understanding of what multiplication means. Her previous ways of teaching multiplication had come to influence her own thinking about multiplication. When confronted with Grasplify's model, which was not consistent with her understanding of multiplication, Leah's personal instrumental genesis was challenged. The influence of other members of the teacher–researcher group was instrumental in Leah's ability to overcome this challenge. As a result of her personal instrumental genesis of Grasplify becoming stronger, her professional instrumental genesis strengthened as well. She began to pose the questions that she had asked herself in reconciling her understanding of multiplication to her students.

> I'd say, "Well, what does this multiplication [draws an '×' sign with her hand in the air] mean? Well, can it mean this, this many times? Or this many groups of? Can it mean both? And why can it mean both? And I got them [her students] to start thinking about that and showing that. So, for me, it means both. It can mean three times five, it can mean three groups-of five or it can mean three, five times. It means both. Depending on the context, depending on the wording.

Leah's reference to context and the language used to describe a multiplication, both ideas which emerged as topics discussed during project meetings, illustrate how Leah's process of double instrumental genesis was positively impacted by her interactions with other members of the TT teacher–researcher group. While she was initially rigid about the 'correct' way for a numerical expression to be depicted in a 'groups-of' scenario, Leah valued flexible student thinking and meaning-making in her approach to teaching mathematics. If she had been rigid both in her presentation of concepts *and* about student thinking, Leah would not have experienced the transition in thinking about multiplication that she did.

Like Rachel, Leah also mentioned the impact of having the research team observe in her classroom, though what she described as being beneficial were the short conversations with the team following her teaching. These occurred immediately after the lesson when the students were outside for recess and the research team was packing up to leave. Leah would often share what she had noticed or thought went well in the lesson and the research team would share their observations in response to Leah's comments.

In one of these post-TT lesson conversations, Canan, one of the members of the research team, pointed out that a few students chose to use colour in their drawings of Grasplify, even though Leah had not directed them to do so. Leah explained that she had never discussed the colours with her students. Thinking about how Leah's preferred way of teaching by encouraging student noticings and discoveries, I responded that perhaps having a few students share their drawings, some with colour and some without, may provide an opportunity to draw attention to the relevance of the colour in Grasplify. At this point, Leah described how valuable she found these conversations with the research team. Here is a description that is best shared in her own words:

> This is why this needs to be done like this [in reference to the group coming in to observe her teaching]. Like, I think you guys coming in, because you're, you think about these things because you're thinking about the app all the time. And then having these conversations after,

> in the context of what happened, like what I just got out of this is like whoof, mind blowing. [She laughs.] Like it's making me think about the whole thing in a whole different way […] I'm thinking about the colours, I'm thinking about the pips and the pods, and the importance, like, you know how I got over the groups-of? And I got like, really over that. Now I'm like, it's *really* about the relationship.

Moving beyond her initial challenge with the multiplicand × multiplier ordering of Grasplify occurred in multiple steps for Leah. Her first step occurred when another teacher explained the importance of context and the next step happened when Leah used her knowledge of commutativity to reconcile that $a \times b$ can be a groups-of b or b groups-of a.

Though there is another significant factor, which will be addressed in the next section, the comment above is an important step forward in Leah's understanding of Grasplify's way of materialising multiplication. A step that occurred during a spontaneous conversation between the researchers and Leah after she had taught a lesson using Grasplify to her students. The influence of others throughout this process had a significant impact on Leah's understanding of multiplication and her ability to use TT effectively as a pedagogical tool for teaching mathematics to her students, both of which were now more in alignment with her valuing of flexible thinking and personalised approaches to mathematics.

8.4. The influence of the students

A second sociocultural aspect, that was explicitly mentioned by three of the four teachers as being influential to their evaluation of TT as a pedagogical tool, was from the students themselves. This assessment of the effectiveness of the digital technology is an important aspect of the teacher's double instrumental genesis, informing their choice to continue using this digital tool for teaching multiplication or to abandon it in favour of other teaching strategies or tools.

Although the influence of Amy's students on her thinking about TT did not emerge during her interview, Kate, Leah and Rachel all described different ways that student comments, actions and learning were significant to their evaluation of TT. This aspect of evaluating the effects of their teaching and the digital tool chosen to address their mathematics teaching goals is a crucial consideration when examining the evolution of the double instrumental genesis process the teachers were engaged in.

In the next sections of this chapter, I will return to Leah's story and how her students were instrumental to her acceptance of the multiplicand × multiplier ordering of Grasplify that caused her discomfort for so long. After that, I will examine how Rachel's and Kate's processes of double instrumental genesis were impacted by the students they were teaching.

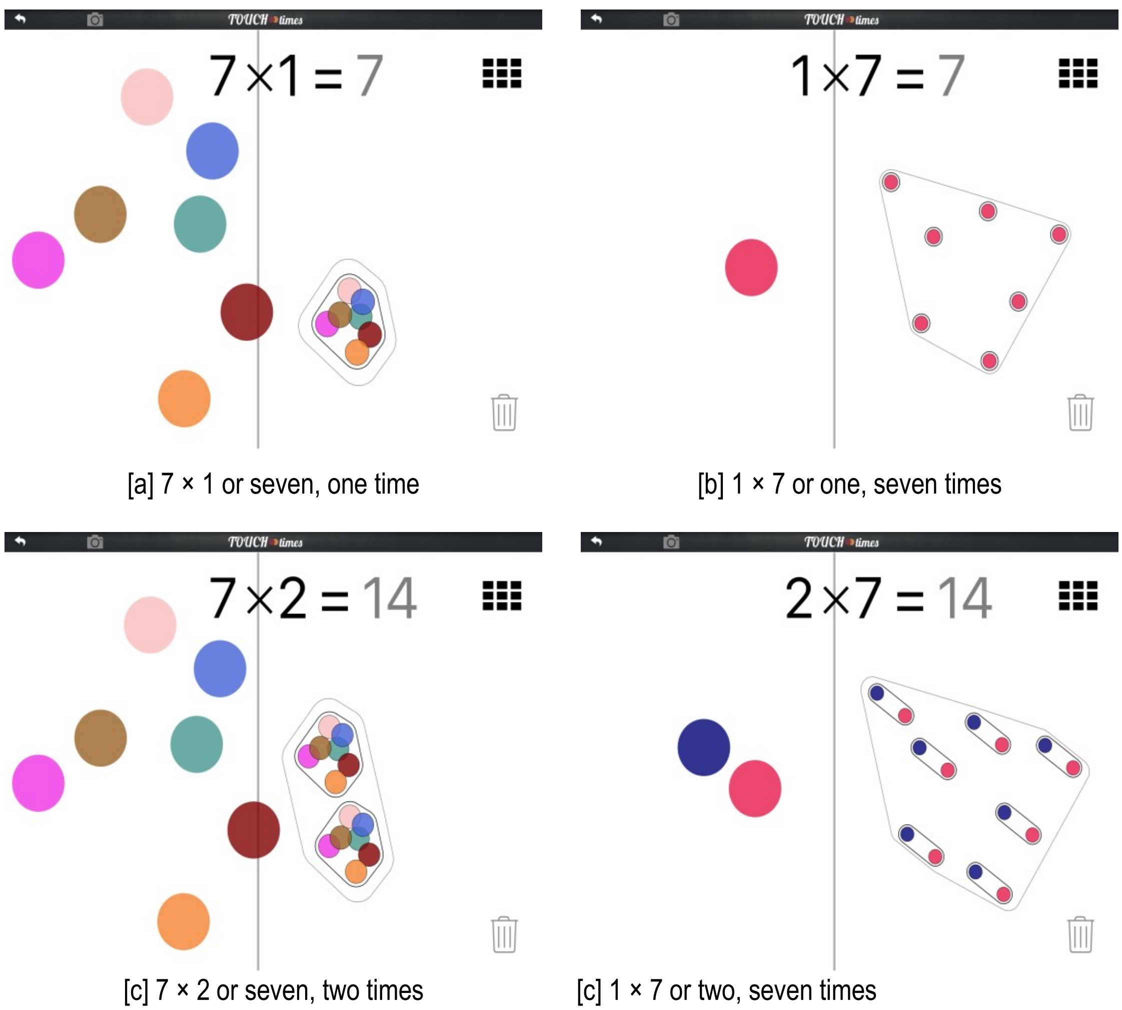
8.5. Leah: "The kids are okay with it"

As Leah began using Grasplify in her classroom, the successful instrumental orchestration by her students of the digital tool and the mathematics it presents was deeply compelling for her. She was very forthright in sharing that the children's reactions to Grasplify were quite influential for her: "one thing that affected me is the conversations I had with the kids as a teacher". Unhindered by the rigid view of multiplication models that must reflect a multiplier × multiplicand ordering that was so troublesome for Leah herself, her students easily used the language of 'three, five times' or 'five groups-of three' for 3 × 5 when using Grasplify. She observed that her students "didn't know any different, and so they were understanding it [the ordering] the way it was, and it didn't matter".

Given that Leah's didactical reference markers were based upon her previous ways of referring to multiplication through an $a \times b$ numerical expression which was explained as *a* groups-of *b*, she was surprised when her students used the term "groups-of", even though she had been very intentional about not formally teaching her students this expression. "Interestingly enough […] they still talk about groups-of, even though the sentence is the other way around. So, they'll say 'oh, that's three groups-of two', even though it [the numerical expression] says two, three times."

After observing her students speaking about multiplication in terms of groups-of, in a way that was based on the visual model depicted rather than the order of the numerical expression, Leah began to question her own thinking about multiplication. She described herself as being curious about why the ordering in Grasplify was so problematic for herself, but was not for her students. Therefore, in her lessons, she began to explore the words used to describe the multiplicative scenarios created using Grasplify more intentionally with her students. Leah shared an example of how she approached this:

> So, I think it was like one times seven I put on the board and then I said, “Okay, make that.” [using Grasplify] Yeah and then I said, “Is seven times one the same thing on *TouchTimes*?” [see Figure 8.1a] And of course, because of commutativité, how I always say it in French, commutative, commutative property. They all said, “Well, it’s the same thing”. And I said, “But does it look like the same thing?” So, then I put one times seven on the board. “Make that on your *TouchTimes*. [see Figure 8.1b] What does that mean on *TouchTimes*? Talk to me in *TouchTimes* language.” It was funny how a lot of kids went, “Well, it's one group of seven”. Finally, one kid pipes up, “No, it’s not. It’s one, seven times." So, it was so fascinating because it made them *look* at that. […] Some of them got confused by that, but we changed it with two. If you change it with that, you’ll see. There you go. There’s two, seven times. [see Figure 8.1c]



[a] 7 × 1 or seven, one time

[b] 1 × 7 or one, seven times

[c] 7 × 2 or seven, two times

[c] 1 × 7 or two, seven times

Figure 8.1 What does 7 × 1 mean on Grasplify?

In one of the video-recordings taken while the research team was observing a lesson in Leah's classroom, she explained to a member of the research team that her students were "okay with the order". Curious to know more, I shared with Leah the video-clip containing this moment in a later interview and asked her to elaborate further on this comment. She explained:

> That's not the old-fashioned group-of. […] This whole groups-of thing, the way the order of the sentence is. Which still, people I've showed it to can't get past it. But the kids are okay with it. It's working for them, this five times. So why are we so stuck on it? Right? That's what I meant by it. See, the kids are okay with this, they're learning, they're understanding it. I was amazed that that child did it exactly how *TouchTimes* would do it without even a flinch. […] She does two, three times. Not two groups of three. Whereas she's able to look at this and go, "Oh look, I have two, three times". Exactly like it would be up there on *TouchTimes* and make that connection quickly without prompting.

Although Leah's personal instrumental genesis of Grasplify was significantly impacted by her discomfort with the way the application modelled the numerical expression created by the user, this did not prove to be an obstacle for her students, nor for the development of her professional instrumental genesis.

There are two things worthy of note about this. The first is that, despite Leah's own personal instrumental genesis being challenged, she proceeded to implement Grasplify in her classroom and was able to instrumentalise the digital technology to serve her didactical intentions. As a result, her professional instrumental genesis developed much more quickly – and in a stronger way – than her personal instrumental genesis initially did. The second is the impact on Leah when her class did not have the same difficulty with the ordering that she had experienced herself. When her students seemed to understand and were able to talk easily about the multiplicative models created in Grasplify, it caused Leah to ask question, "So why are we [elementary teachers] so stuck on it?".

Possibly because TT is a digital tool designed specifically for teaching mathematics, Leah's experience of double instrumental genesis seemed to begin with little attention paid to her personal genesis of TT. Instead, she immediately focused on how Grasplify was backwards to her previous ways of thinking about *and* teaching multiplication. As a member of the teacher–researcher group, however, Leah committed to implementing TT in her classroom, which created an opportunity for the development of her professional instrumental genesis.

After witnessing her student success using the digital technology, Leah then recognised a need to revisit her own thinking and understanding of multiplication. And, in so doing, she was able to reconcile for herself the relevance of the multiplicand × multiplier = product ordering and to develop further her own personal instrumental genesis of TT. The process of double instrumental genesis for Leah involved a unique interaction of TT–students–teacher–mathematics which impacted her understanding of multiplication, as well as her acceptance of this particular digital technology as a pedagogical tool for teaching mathematics that she had come to value.

8.6. Rachel and Kate: Convinced by student success

Similar to Leah, Rachel's processes of personal and professional instrumental geneses were both notably influenced using Grasplify with her students. During our individual interview, Rachel recalled that her students had clearly enjoyed using the digital technology and were engaged for much longer than she had expected them to be. Noting that it was only part-way through the school year when she had implementation of Grasplify and, although her fourth-grade students could engage in their learning for longer periods of time, the length of her third-grade students' attention span was still developing.

Although Rachel herself had initially found Grasplify to be less intuitive to use, during her observations of student pairs engaged in the tasks and in her evaluation of the drawings that her students completed to illustrate their learning, she had observed that this was not the case for her students:

> A lot of them got it kind of faster than I thought they would too because I, you know, I'd been to the meetings and was thinking like this app isn't like quite as intuitive as *TouchCounts*. Let's see how this goes and their understanding, I think, was better than I thought it was going to be, kind of right away. And a lot of them had a lot of success with the tasks we gave them. […] They got there quicker than I thought they would.

Despite TT not being intuitive for her to use initially, Rachel continued with her implementation of it with her students. As she noticed the success that students were having with Grasplify and its accompanying tasks, it motivated her to continue using the application to draw student attention to the multiplicative ideas of covarying and spreading. In so doing, Rachel was further developing her professional instrumental genesis while also learning more about Grasplify from her students during their use of it. The development of her personal process of instrumental genesis was similar to Leah's in that it developed in a deeper way as a result of using TT as a teaching tool with her students.

Using TT as a teaching tool and later witnessing her students' ability to recognise and transfer the multiplicative models both from Grasplify and from Zaplify to other representations was influential in Kate's evaluation of the technology. She described projecting onto the whiteboard at the front of the room some practice-page examples taken from the textbook resource used by her school district after the class had completed a variety of Grasplify and Zaplify tasks over the course of a couple weeks. She was curious to know if students would be able to see the various representations that were visually different from but mathematically similar to those they had created in the two microworlds that compose TT.

Kate described her purpose as "helping students make and see connections between TT and the content and also connections with the other resources [mathematics textbooks] they were using". She described how she found it "really powerful" to witness how her students connected the example images of different multiplicative representations with either Grasplify or Zaplify. "Switching back and forth between the models connected to the different representations. So, when they see an array model, they're connecting it to Zaplify or when they're seeing the groups-of, they're seeing it as pips and pods [Grasplify]."

Kate considered this skill to be especially beneficial for students, believing it necessary to transition them beyond the dynamic, embodied multiplicative experiences enabled by TT into "the more formal symbolic type of math" more commonly taught by teachers. Though she described her purpose as "helping students make and see connections between TT and the content and, also connections with the other resources [mathematics textbooks] they were using", these connections were very much focused on students' ability to recognise the symbolic mathematics in the static drawings of multiplicative models that were used in the teacher resource that Kate both had had experience with and trusted.

> What was the coolest was, at the end of the whole thing, they [students] could see what multiplication meant through both lenses. Like they understood that the pips and the pods were actually the same as the grids. Like it was the same result, so that they saw that these are all different ways of representing multiplication and I know that representation is one of the competencies.

When Kate determined that her students were able to do this successfully then she began to see the digital technology as a more valuable asset to her teaching which narrowed the instrumental gap and further developed her double instrumental genesis of the tool. What was most convincing for Kate was the ability of her students to recognise the different representations of multiplication outside of TT.

8.7. Summary

Throughout this chapter, I have tried to draw attention to various sociocultural factors that were influential in further developing Amy's, Kate's, Leah's and Rachel's processes of double instrumental genesis. The teachers described how other members of the TT teacher–researcher group and/or the success of their own students influenced their ability to utilise TT as a tool for teaching mathematics as well as increased their willingness to accept the technology as a viable pedagogical resource.

Amy, Leah and Rachel all mentioned in some way the benefits of being able to engage with others (both teachers and researchers) who were part of the TT project. Not only was Rachel able to overcome her initial difficulties with how to use Grasplify as a teaching tool, but she also talked of an increasing recognition of the importance of developing multiplicative thinking in her students. These results were facilitated by her interactions with the other teachers who were sharing their experiences using TT and being involved with the research team.

Leah's ability to move past the multiplicand × multiplier ordering in Grasplify was due to the influence of another teacher in the TT teacher–researcher group and as a result of her students not having the same difficulties with this that she had had. The influence of others can not be underestimated in the development of Leah's double instrumental genesis and the narrowing of the instrumental gap between her former ways of thinking about and teaching multiplication, and the relational and embodied experiences of TT.

The processes personal and professional instrumental geneses engaged in by Leah and Rachel were far from linear: rather, the two processes were very much intertwined. Their personal comfort with, understanding of and confidence in using the application improved because of their interactions with other teachers in the project and as a result of their students' successful use of the technology. The instrumental distance continued to narrow for Leah and Rachel, as well as for Kate as they each witnessed the mathematical understanding that their students were demonstrating after using TT. The impact of sociocultural influences was a significant factor in each teacher coming to recognise and appreciate the opportunities for interacting with multiplication that this tool could offer for student learning.

Hoyles, Noss and Kent (2004) argued that there are sociocultural aspects present through instrumental orchestrations, that the technological medium can offer, as a boundary object, an opportunity for students and teachers to think about and discuss ideas in a shared meaning-creation process. This was apparent during the lessons that were observed in Leah's and Rachel's classrooms. However, there were also sociocultural aspects which were influential to Kate's, Leah's and Rachel's processes of double instrumental genesis. The influence of the students' successful engagement with the embodied and relational models of multiplication provided by TT was significant in contributing to the progression of each teachers' professional and personal processes of instrumental genesis.

Previous research by Sinclair and colleagues (2020) has also found that TT acted as a boundary object when teachers and researchers come together to share pedagogical and/or mathematical ideas. This finding was also confirmed by the experiences described by Amy, Leah and Rachel during their individual and paired interviews. TT and the mathematics it presented acted as a mediator in the TT teacher–researcher project and, in so doing, provided an additional opportunity for the development of each teachers double instrumental genesis of the digital technology.

In returning to my first research question, namely how TT affected the teachers' process of double instrumental genesis, although TT and its way of presenting multiplication clearly did influence and shape the actions and thinking of the teachers, I would argue that this process of instrumentation was not solely individual. In other words, it did not occur exclusively between the digital tool and the individual teacher using it. Rather, there was a complex intra-action (Barad, 2007) occurring within TT, the individual teacher and the collective TT teacher–researcher group. The reverberations of the effects of this digital tool affected individual teachers, as well as the group as a whole. This is particularly apparent in the multiplicand × multiplier situation that caused such difficulty for Leah and consequently emerged within the larger group as well.

When examining my second research question about how teachers instrumentalise or accommodate the instrumental distance between TT and their former ways of teacher multiplication, sociocultural aspects emerged as influential here too. The opportunity to share successes, challenges and questions related to TT, the mathematics it presents and implementing it with students with other members of the research project was explicitly stated as beneficial by each of Amy, Leah and Rachel. The instrumental gap was narrowing as they built 'shadow' didactical reference markers based on the shared experiences of other teacher prior to building their own didactical reference markers through their own teaching with TT.

The influence of the students themselves was also a significant part of Kate's, Leah's and Rachel's evaluation of TT as a pedagogical tool that was effective and worth continued use or as something to be abandoned. When these teachers saw that their students were understanding the multiplicative models afforded by the technology, this impacted their professional instrumental genesis as they came to accept TT as a valuable tool for teaching mathematics. Additionally, in the case of Leah, when the students were not having the same mathematical difficulties that she had experienced, it caused her to rethink her definitions of multiplication and how to teach them, causing growth in her personal instrumental genesis of the tool.

In the next, and final, chapter of this dissertation, I will return to my research questions, the results of my data analysis, their implications and suggestions for possible future avenues of research.

Chapter 9. Before, During and After *TouchTimes*: Bringing it all Together

We write to make sense of it all. (Wallace Stegner)

In this chapter, the final one of this dissertation, I will summarise the key findings from Chapters 5 through 8 while explaining how this data contributes to responding to the research questions that were initially posed in Chapter 3. After addressing each of the research questions in turn, I will then highlight the implications of the findings, describe the limitations of this study and, finally, by identifying directions for future research, I will bring the chapter (and this thesis) to a close.

9.1. Returning to the research questions

While engaging with the data collected as a significant part of this research, I continued to consider the teacher, the digital tool and the mathematical concept of multiplication as an ensemble (Bakos, 2022a, 2022c), rather than studying each of the parts individually and separately. In so doing, I was trying to honour the complexity, as well as the nuances that emerged during the interactions amongst all three components which form the whole. I will now return to the three research questions, which are listed once more below, and will summarise my responses to them, found within the data examined throughout Chapters 5, 6, 7 and 8.

1. In what ways does *TouchTimes*, and its way of presenting the mathematical concept of multiplication, affect teachers as they progress through the process of double instrumental genesis (instrumentation)?
2. How do teachers accommodate their instrumental distance between *TouchTimes* and their previous ways of thinking about, and teaching, multiplication (instrumentalisation)?
3. How do teachers' experiences of double instrumental genesis evolve while utilising *TouchTimes* as a pedagogical instrument?

Although I initially attempted to address each of these questions separately, I found that, in some situations, this created an artificial separation between instrumentation and instrumentalisation. In fact, the three questions seemed to merge into their own form of ensemble. Even though various aspects of TT did affect the teachers with whom I engaged, their response to the technology is also reflected in how they adopted the tool. These are not two entirely separate actions: rather, they occur nearly simultaneously at times in an intermingled sort of way. Therefore, I will address the themes that emerged in response to all three research questions in a more holistic way, while trying to highlight aspects of each question that are particularly relevant in the instances shared.

While Amy, Kate, Leah and Rachel shared how different aspects of TT and its way of materialising multiplication impacted them, as they progressed through the process of double instrumental genesis, they also described how they responded to these aspects of the technology. In some instances, the teachers' responses were illustrative of how they accommodated each of their instrumental distances between TT and their previous ways of thinking about, and teaching, multiplication, thereby contributing to research question one and aspects of research question two. In other circumstances, the focus of Amy, Kate, Leah and Rachel was not on how the digital technology itself influenced them, but in how they chose to utilise the technology as a pedagogical tool for mathematics. Often, this resulted in co-mingled aspects of research questions two and three being addressed at the same time.

9.2. How *TouchTimes* affected teachers: Responses to research question one

Recognising that the responses to my three research questions have not emerged in clear-cut distinctions or consistently co-mingled responses to all three questions at once, I will begin by describing how different aspects of the TT design affected Amy, Kate, Leah and Rachel in the sub-sections that follow. To do this, I will focus on all responses that pertain to research question one, many of which comprised the focus of Chapter 5. When relevant, I will also draw elements from Chapters 6, 7 and 8 in following through to its conclusion any response(s) that also address(es) research questions two and/or three. There are some instances where the digital technology created a disturbance for the teacher, but the teacher was able successfully to accommodate the instrumental distance involved during their process of double instrumental genesis. In these cases, I will follow the example through in responding to research questions one, two and three.

9.2.1. A “blank canvas” that was “not intuitive”

One of the two issues related to the design of TT that affected Kate’s personal instrumental genesis – and later influenced her professional instrumental genesis – was what she described as “the initial blank canvas” of TT’s opening screen. When first presented with TT’s open-ended blank canvas, Kate was unsure what to do with it or how to use the technology effectively to teach mathematics. Consequently, she relied on the TT task booklet to guide her teaching and to provide task ideas for use with her students.

Part of Kate’s professional instrumental genesis of TT involved thinking about how students would receive the open space of the digital application during their first experiences with it. She felt that the nature of TT is playful and that students would view its blank canvas as a mystery to be solved. Kate could imagine them wanting to know, “What is this thing? What’s going on here?” and that their curiosity would be sparked when nothing happened on the screen until they made physical screen contact with some of their fingertips. Although this design aspect of TT was noted by Kate as part of her personal instrumental genesis, her thoughts quickly turned towards the potential of the blank screen for engaging students’ innate curiosity in what they would view as something fun on the iPad, rather than schoolwork.

For Rachel, there was a large instrumental gap between *TouchCounts* (Sinclair & Jackiw, 2011), another mathematics iPad application with which she had previous teaching experience, and *TouchTimes*. When first introduced to TT, she did not find it to be intuitive to use and also had difficulty thinking of ways to use it with her students. After missing the first TT teacher–researcher group meeting, Rachel described not understanding some of the tasks given at the second meeting, which negatively affected her personal instrumental genesis of the tool. Lacking didactical reference markers for TT, she relied on the ideas shared by other teachers in the teacher–researcher group meetings and the pre-developed tasks shared with the group, in order to feel comfortable enough to implement Grasplify in her classroom.

9.2.2. The impermanence of the multiplicative models in *TouchTimes*

Kate also shared her frustration that the multiplicative models created in TT would only remain visible while the user maintained continuous screen contact. Unaware of the screenshot button that is an option in the TT settings, she found it cumbersome to capture the multiplicative images that she created in TT using the traditional two-handed technique necessary for taking a screenshot on an iPad. Two people were needed in order to make this happen and it interrupted the flow of her teaching. This influenced how Kate approached demonstrations using TT and the manner she used, which included worksheets that she created to provide visual images of the multiplicative models for students to refer to.

Leah also disliked this lack of permanence when she first experimented with Grasplify, but after observing how this design feature forced her students to recall what had been previously created on their screen, and how they had created it, she saw it as beneficial for student learning. The pedagogical value of the impermanence of the user created multiplicative models in Grasplify was embraced by Leah and this aspect of her professional instrumental genesis allowed for her to move past her initial dislike for it during her personal instrumental genesis.

The comments that she overheard between students while they completed drawings of Grasplify images after they had removed their fingers from the iPad screen was something that Leah felt was purposeful and thoughtful, and she encouraged students to think about the relationship between the pips and the pods. Although this design aspect of the digital technology was initially disliked by Leah, her didactical reference markers, which prioritised the benefits of play and student verbal and/or written (or drawn) expressions of their ‘noticings and wonderings’, enabled her to change her position to benefit her professional instrumental genesis.

9.2.3. Grasplify’s multiplicative model

Although Kate mentioned in passing that Grasplify was somehow the opposite of the groups-of model of multiplication, this did not create a disturbance for her. With a strong background in mathematics, Kate possessed a number of didactical reference markers in relation to multiplication, which enabled her easily to accommodate Grasplify's multiplicative model within the mathematical knowledge she already had.

Leah, however, was significantly challenged by the multiplicand × multiplier = product ordering of Grasplify. During the initial TT teacher–researcher group meeting, when she was first exploring the application, Leah turned towards Amy, the teacher sitting closest to her, to express her concerns about this, but Amy was unbothered by Grasplify's ordering.

From the outset, Leah's instrumental distance was large between Grasplify's multiplicative representation and the model that she had internalised and been using for many years to teach multiplication. Her didactical reference markers for explaining the meaning of numerical multiplication expressions were based on $a \times b$ being a groups-of b, as it was described in the textbook resource used by her school district. Therefore, the arrangement of pips and pods, and their corresponding numerical expression as it was created in Grasplify, was "backwards" to the groups-of model that Leah used for thinking about, and teaching, multiplication. The digital technology and its way of materialising multiplication conflicted with Leah's previous reference markers for multiplication and created an interference within the teacher–technology–mathematics ensemble. This caused a significant disruption to her process of double instrumental genesis and ultimately provoked Leah to reconsider what multiplication means.

During her personal instrumental genesis, while trying to reconcile why the multiplicative model in Grasplify seemed consistent with her groups-of model when pip-creation occurred on the right side of the screen, even though it was not when pip-creation occurred on the left side of the screen, Leah drew from another of her didactical reference markers. According to the commutative property of multiplication, the order of the factors does not impact the product. Leah used this knowledge to convince herself that the ordering of the numerical expression and how it corresponds to the multiplicative model in Grasplify should not matter. Though, in Grasplify, it does matter.

The formation of the multiplicative model is based on the differing, and not the interchangeable, roles of the multiplicand and the multiplier. The unit of measure (the multiplicand) must be established first in Grasplify, prior to creating the number of groups (the multiplier), therefore making commutativity irrelevant in this context. This unit of units concept, however, was not one that Leah associated with multiplication. Rather, her didactical reference markers were based on groups-of and commutativity, which proved to be disruptive to her personal instrumental genesis of Grasplify.

Leah's didactical reference markers were even more impactful during her professional instrumental genesis, given that the way the mathematics was presented in Grasplify interfered with how she explained multiplication to her students. Though the instrumental distance was large, ultimately Leah did not let it become an obstacle that prevented growth of her professional instrumental genesis.

While she implemented this technology in her classroom, Leah became more familiar with its design, and the language she used to describe Grasplify's multiplicative model began to reflect the transformation in her understanding of what multiplication means that she was undergoing. She was no longer limited to describing 3 × 4 only in terms of three groups-of four; in the context of Grasplify, she now referred to the multiplicative model as a depiction of three, four times. This expression corresponded to the mathematics of Grasplify and nicely illustrated the interactive nature of the teacher–technology–mathematics ensemble.

Ultimately, the explanation shared by another teacher in the TT teacher–researcher group and her students' lack of difficulty with Grasplify's ordering were the two factors that proved to be most influential in Leah's ability to accept the multiplicand × multiplier ordering in Grasplify. This narrowing of the instrumental gap for Leah occurred as a result of interacting with others in the teacher–researcher group, as part of the process of implementing the technology as a pedagogical tool for teaching mathematics to her students and because of her observation that her students were not experiencing the same difficulty with the ordering that she had. Leah's success with instrumentalising Grasplify to serve her didactical objectives is reflective of the development of her professional instrumental genesis of the tool. Her successful use of Grasplify with students prompted Leah to question her own thinking and understanding of multiplication.

When interviewed, Leah described how the influence of other teachers in the TT teacher–researcher group, the impact of having researchers observe her teach using Grasplify and the conversations we had about the lesson afterwards and the effect of her students' success with the technology and the mathematics it presented all caused her to really think about the colours and the pips and pods in Grasplify. Multiplication, for her, was no longer solely about groups-of; instead, "it's really about the relationship". Leah's process of double instrumental genesis was not straightforward; rather, it involved a complex interaction between and amongst multiple factors: TT itself, Leah's prior ways of thinking about and teaching multiplication, her students, the teacher–researcher group and the mathematics itself.

Leah's process of double instrumental genesis went beyond using TT as a tool for the personal calculation of multiplication (personal instrumental genesis) or using TT as a tool to teach multiplication (professional instrumental genesis). During the process of instrumentation, TT prompted Leah to engage in a process far deeper and more complex than simply learning how to use a technological tool to do the personal work of mathematics – it provoked a re-examination of her personal understanding of what multiplication means.

9.2.4. Using the affordances of Grasplify with intention: One model amongst many

Amy and Rachel described using TT as a teaching tool because it offered experiences of multiplication that were different from those they commonly presented to students. Both teachers were conscious of introducing students to multiple representations of multiplication using various methods for building student understanding, and had been doing so prior to implementing TT. However, Amy and Rachel considered the dynamic, embodied and relational features of the pips and pods representation of Grasplify as an opportunity for learners to create and manipulate multiplicative models using their fingertips. However, they were also very clear in pointing out that Grasplify was not the only approach they used; namely, that it was only one model amongst others that they used for teaching multiplication.

During their interview together, Amy and Rachel described a number of ways that they engaged their students in learning about multiplication. Amy explained how she traditionally began by writing a multiplication equation on the board and then used drawings and/or physical manipulatives to model an explanation of what these numbers represented. Rachel added that she too began this way, usually modelling groups-of objects or building an array to represent a numerical expression, and then proceeding to providing students with a multiplication problem situation to recreate physically. Additional methods they described using when teaching multiplication included building arrays with manipulatives, introducing area models with base-ten blocks, having students colour area model representations using graph paper and using number-line drawings to demonstrate skip counting visually.

Rachel contrasted this with how she introduced multiplication using TT, noting that she initially directed student attention towards the creation and manipulation of the multiplicative model when using the digital technology, and the equation itself was secondary. Both Amy and Rachel described how they used the Grasplify tasks to focus students on the process of multiplication, noting that it was only after students had developed an understanding of this process that they would draw attention to the numerical expression. This was the opposite to the operational approach previously used by the two teachers when introducing multiplication to their classes. The provision of the numerical expression, which included the product, removed the necessity for students to find the answer when using Grasplify and, therefore, changed the focus of Amy's and Rachel's teaching.

Amy explained that TT provided dynamic multiplicative models that immediately adjusted in response to the user's fingertips, something which provided students with a better visual of what was happening in regard to the mathematics. Rachel agreed, noting that the consistent pip colours and instantaneous arrangement of pips within the pods made it possible for students visually to experience multiplicative growth that occurred as a result of their actions. These affordances of the technology had mathematical implications that influenced the pedagogical choices of the teachers.

In the video-recordings of Rachel's teaching using TT, the mutual influence of the different components that form the teacher–tool–mathematics ensemble was evident. During a lesson where student pairs were challenged to skip count in Grasplify by only changing the pips, they created and could see the visual effects of expanding (or removing) pips and how these spread (or disappeared) across each of the pods instantaneously. The colour, composition and shape of the pips within the pods immediately adjusted to reflect any changes made by the pip-creating hand and, consequently, so did the numerical expression.

As a result of these design features of Grasplify, Rachel was able to highlight multiplicative ideas in a way that was accessible for her grade 3–4 students, an opportunity of which she took advantage. Students were first given a task that allowed them to experience the process of multiplication through their fingertips and, with the use of purposeful questioning, Rachel would later create a bridge between students' Grasplify experiences and the symbolic mathematics commonly associated with multiplication. As her professional instrumental genesis continued to develop, she became more intentional about linking student actions on the pips and pods with the numerical expression at the top of the screen during whole class discussion.

The growing professional instrumental genesis of both Amy and Rachel was visible in slightly different ways. As Amy became more familiar with Grasplify, she began to create and implement her own tasks with students – tailoring the tasks to fit within classroom structures that she already had in place. In Rachel's case, she began to think about the multiplication strategies that she usually taught, such as decomposition, and considered how Grasplify could be used for modelling these strategies. In this sense, both teachers were merging the didactical reference markers that they already had with new ones related to TT, a process that ultimately caused a narrowing of the instrumental gap.

Another factor that significantly contributed to Rachel's growing double instrumental genesis was her participation in the TT teacher–researcher group meetings. She explicitly pointed out that the ideas shared and discussions held amongst members of the project were beneficial in furthering her own thinking about multiplication and how to implement the technology, which, consequently, positively influenced her teaching. As a result of participating in the group, she developed a better understanding of the way that Grasplify presented multiplication and was increasingly aware of the properties of multiplication and the importance of developing students' ability to reason multiplicatively. Rachel described how Grasplify afforded her students a visual and physical experience of multiplicative growth that she was unable to provide through physical manipulatives or static drawings. The mathematical model presented by Grasplify, and the influence of the discussions within the teacher–researcher group, expanded Rachel's way of thinking about multiplication herself and also changed her pedagogical intent when teaching multiplication using the digital technology.

An important aspect of Rachel's double instrumental genesis involved becoming familiar with what Grasplify could offer and, once she had a better understanding of this, then she could think about how best to implement the digital application in the context of her teaching. She was becoming more intentional in highlighting the multiplicative aspects of the digital technology, in order to expose her students to ideas about multiplication that went beyond an additive approach. Rachel's ability to use the affordances of Grasplify to support her students' instrumental genesis and mathematical learning provides an example of how the teacher in the teacher–tool–mathematics ensemble was affected by the mathematics enabled by the digital tool and is also a reflection of her growing professional instrumental genesis. Grasplify itself, and its way of presenting multiplication, created opportunities for Rachel to form new didactical reference markers that focused on multiplicative versus additive approaches to teaching multiplication.

As a result of the embodied, relational models of multiplication enabled by Grasplify, Rachel and Amy shifted their focus of instruction towards developing student understanding of what multiplication means, instead of concentrating on finding the right answer. Though this represented a significant shift in their approach to teaching multiplication, neither teacher was deterred by the instrumental distance between the digital technology and their former ways of teaching multiplication. Rather, Amy and Rachel embraced this difference as advantageous, and chose to use both options in complementary ways to benefit student learning.

They considered TT to be an additional pedagogical tool that could be utilised to help students build a deeper conceptual understanding of multiplication. Amy and Rachel's willingness to integrate the digital technology into their teaching was influenced by the fact that they already possessed didactical reference markers for creating visual models of multiplication for students using manipulatives or drawings, Therefore, the instrumental gap was narrow and enfolding the technology within these pre-existing markers was more easily accomplished.

9.2.5. New instrumental orchestrations

Much of Chapter 6 examined the effects of TT and its ways of materialising multiplication on Leah and Rachel as they progressed through the process of double instrumental genesis. The emergence of new instrumental orchestrations that are closely related to the dynamic, relational design of TT and/or its touchscreen aspect are indications of Leah's and Rachel's increasing professional instrumental genesis with the tool. During whole-class orchestrations using Grasplify, both teachers were becoming more purposeful in focusing student attention onto the multiplicative effects of changing the number of pip-fingers on the configuration of pips within the pods. Eventually, they also used intentional questioning to point out the connection between changes made to the pip and pod units and what happened to the numerical expression displayed at the top of the screen. While the two teachers were using TT purposefully to draw attention to the mathematics that they wanted students to notice, three new orchestrations naturally emerged, which I refer to as: Compare-successive-screens, Document-screen-on-paper and Discuss-the-finger-and-screen orchestrations (Bakos, 2022a).

The Compare-successive-screens orchestration was observed being used both by Rachel and by Leah to engage students in comparing previously seen (or predicted) visual images on Grasplify with currently visible images. When utilising this orchestration, the teachers were drawing student attention towards visible changes on the screen that were mathematically important. The changes could be related to the multiplicative model and/or the numerical expression when one of the units is changed – an idea enabled by Grasplify that was new for teachers when teaching multiplication.

The Document-screen-on-paper orchestration was utilised by Amy, Kate, Leah and Rachel while teaching using TT and they each asked their students to re-present the mathematics created by their fingertips in their notebooks (or as part of a worksheet) for future reference. By drawing the multiplicative models that they had created in TT, the students' attention was directed towards the relationship between the number, colour and shape of the pips and the pip-configuration in the pods in Grasplify and (with Kate's students) the intersections of vertical and horizontal lighting bolts (or lines) in Zaplify.

Unlike the Compare-successive-screens and Document-screen-on-paper orchestrations, which emerged naturally as part of each teacher's way of teaching, the Discuss-the-finger-and-screen orchestration first appeared as a result of a lesson 'hiccup' (Clark-Wilson, 2010). When Leah's students could only see the effects of their teacher's finger touches in Grasplify, but were unable to see her actual fingers themselves, it proved to be problematic. The Discuss-the-finger-and-screen orchestration emerged as a solution to this problem and sometimes involved Leah or a student volunteer physically pointing at the projected image on the screen and, occasionally, Leah would hold an iPad up so that the class could see her fingers or a student's fingers manipulate the pips and pods on the iPad screen.

The mathematics in TT can only be materialised through finger–screen contact: therefore, the necessity of students being able to see the user's finger manipulations is another instance of the influence of TT on the actions of a teacher. As noted in Chapter 6, the Discuss-the-finger-and-screen orchestration is specific to the touchscreen aspect of TT, where the use of the fingers is not merely to push buttons; rather, they are used to create relationships. The Discuss-the-finger-and-screen orchestration was used both by Leah and by Rachel to draw attention to the importance of the user's finger manipulations, as well as to the mathematics that is enabled through the finger–screen contact of the user.

9.3. Teacher accommodations: Shifting the focus to research questions two and three

As I carried out in the previous section, where responses to research question one were summarised, I will now continue a similar process with responses to my second and third research questions. Much of this section returns to the insights and reflections shared in the interviews by the four teachers regarding their purposes for implementing TT with their students that I examined throughout Chapter 7. There were times when the responses to research question two also pertained to the evolution of each teacher's double instrumental genesis, which was the focus of question three. When this occurred, I chose to address the responses to both questions in this section.

I will now highlight how Amy, Kate and Leah accommodated their instrumental distance between TT and their previous ways of teaching multiplication, and also follow through with any related ideas from Chapter 8. Rachel's primary purpose for using Grasplify related to the opportunities it provided for teaching about multiplicative growth. This was described in detail earlier in this chapter and, therefore, she will not be mentioned in this particular section.

As the teachers become more familiar with the digital technology and how it functioned, they began to leverage it more effectively as a tool for teaching multiplication. The increasing ability of the teachers to recognise and use the teaching opportunities that arose while using TT to benefit their students' mathematics learning is a clear indicator of their growing professional instrumental genesis. This process, however, is multi-faceted and complex, because teachers must take into consideration their teaching goal, the digital technology, the mathematics, the students and their own role in bringing each of these together to ensure student learning.

When Amy, Kate, Leah and Rachel described their purpose(s) in using TT, these sometimes involved what they considered valuable as teachers, while, at other times, they were related to the type of learning they wanted students to engage in or the mathematics they wanted students to learn.

9.3.1. Kate: Building mathematical understanding with intention

Of the four teachers interviewed as part of this study, Kate was the only one who implemented both Grasplify and Zaplify. Consequently, this enabled an examination of the instrumental distance between TT and the previous ways used by Kate to teach multiplication, as well as providing an opportunity to analyse the intra-instrumental distance (Bakos, 2022b) between these two related, but different, microworld applications.

Conscious that students often associate iPads with play, Kate was very intentional in her teaching with TT to ensure that discovery and exploration tasks were what she described as "mathematically rich experiences". Perhaps due to her training and experience as a secondary mathematics teacher, Kate's instructional aim was to ensure that students understood that what they were creating in TT was multiplication.

The mathematical opportunities afforded by Grasplify and Zaplify were what motivated Kate's professional instrumental genesis. In the ensemble of teacher–mathematics–digital technology, although she prioritised the mathematics, Kate did not necessarily view the technology and the mathematics as separate entities. Her intent was to take advantage of the embodied, dynamic nature of TT to provide students with different ways to visualise, conceptualise and experiment with multiplication. She considered the embodied, dynamic nature of TT to be beneficial for student learning, as long as students noticed the effects of their fingertips on the symbolic mathematics. In this way, Kate ensured that the instrumental distance between TT's multiplicative models and the symbolic mathematics that she prioritised was intentionally kept narrow.

When she noticed that students were not paying attention to the numerical expression in Grasplify, or to the floating factors and the product displayed in Zaplify, Kate used intentional questioning to direct student attention towards what the relational effects of adding or removing fingers from the screen had on the multiplicative model(s), and the corresponding symbolic mathematics as well. Zaplify's display of the floating factors and the product, rather than a full numerical expression that corresponded to the multiplicative model created by the user, was disliked by Kate. In keeping the intra-instrumental distance narrow, she considered these differences between the two microworlds to be obstacles to student learning, rather than pedagogical opportunities.

The teacher–TT–mathematics ensemble functioning as a single unit is evident in Kate's explicit efforts to highlight the similarities and differences of the two multiplicative representations in Grasplify and Zaplify. She provided opportunities for students to go back and forth between the different multiplicative relationships embodied by each of the models. She also displayed screenshots of the same multiplicative expression in both applications and had her students compare and contrast the two models, then subsequently engaged them in drawing activities and independent practice sheets.

Kate's intention was for students to develop an understanding of multiplication that was strong enough to extend across both of TT's multiplicative models, and she endeavoured to do this by taking advantage of the intra-instrumental distance between the two microworlds. The activities that were independently developed by Kate, in order to ensure that students experienced the related, but different, multiplicative relationships offered by each microworld, are indicative of her increasing professional instrumental genesis of TT. Kate was no longer relying on the TT task booklet of ideas (Bakos et al., 2021); instead, she was creating her own activities that addressed the mathematics that she believed was most important. In so doing, she was able to keep the instrumental distance narrow between what she considered most important mathematically for students to understand about multiplication and the learning opportunities afforded by TT.

Kate's didactical reference markers were grounded in multiplication being the symbolic equation which became an essential characteristic of her professional instrumental genesis of TT. As a result, her pedagogical aim was for students to make connections between the multiplicative models in TT and their corresponding numerical expressions, which she justified with the requirements of the content learning standards of the mathematics curriculum. However, this focus on the curriculum content (knowledge) was done at the expense of the curricular competencies (skills, strategies and processes). Instead of using TT as a way of developing student understanding of the meaning of multiplication, Kate utilised it to direct student attention towards the symbolic mathematics that she prioritised as most important. And, in so doing, she was very intentional about keeping the instrumental gap narrow between TT and her belief that multiplication is the symbolic equation.

9.3.2. Leah and Amy: Valuing learning processes

Grasplify's open-ended design was embraced by both Leah and Amy who used the digital technology to build student understanding of multiplication by focusing on learning processes. Leah relied on the curricular competencies component of the mathematics curriculum to guide her teaching, and thus provided many of the didactical reference markers she used when teaching mathematics. After challenging students with a Grasplify task, she would give them exploration time to discover the intended mathematical ideas on their own. Later, while students explained their discoveries to the whole class, if the specific mathematical idea(s) that Leah wanted students to learn did not naturally emerge from the students themselves, then she would either guide student discussion in that direction or explicitly teach the concept. What was most important for Leah were the processes that students engaged in while 'discovering' for themselves, the mathematical ideas that the assigned task was based on.

Being able to use TT in ways that engaged students in 'mathematical play', where curiosity, exploration and risk-taking were encouraged and learning discoveries were explained by students had a significant impact on Leah's process of professional instrumental genesis. Despite being in the early stages of her own personal instrumental genesis of TT, Leah's ability to implement Grasplify in ways that addressed the curricular competencies enabled her to appropriate it as a mathematics teaching tool and allowed her to bridge the instrumental distance between TT and her previous ways of teaching multiplication.

In her second year of using Grasplify, Amy was more familiar with the open-ended nature of the digital technology and, therefore, felt more comfortable allowing her students the freedom to explore and make their own observations. While student pairs were working, Amy would circulate amongst them to answer questions, encourage their thinking or to extend their learning with a new challenge. She observed the value of students asking and trying to find the answers to their own questions through exploration and how engaging in learning this way made them more receptive to the types of exploratory tasks she assigned them later.

The teaching approach used by Amy with Grasplify was very similar to that used by Leah. After providing time for students to engage with the task given, she would then bring the whole class together for sharing and discussion. The ability of students to explain their learning and internalise their understanding in their own ways was valued by Amy. While the student pairs shared their discoveries, she took on a coaching role where she would use the ideas shared by students to enter deeper mathematical discussions related to the task given or use them as a 'jumping point' for the next activity. Amy's didactical reference markers for encouraging student exploration and discovery helped to facilitate her use of Grasplify, making the process of professional instrumental genesis smoother for her while simultaneously narrowing the instrumental distance between the digital technology and her prior ways of teaching multiplication.

Like Amy, as Leah became more familiar with using Grasplify, her professional instrumental genesis was developing and she was more purposefully taking advantage of the open-ended nature of TT. As she became more conscious of her own ways of thinking about multiplication, Leah's teaching approach became more focused on student understanding of, and their ability to explain, what the concept of multiplication means. She shared how valuable it was for her to observe and interact with students while they were using Grasplify or when they were drawing their individual predictions to the 'what-would-happen-if' scenarios she gave them.

These were opportunities to gauge student learning and provided Leah with insight into how students were thinking about multiplication. She could then intervene to address student misconceptions immediately while they worked, or during whole-class discussion, and by using intentional questioning she would redirect attention towards the specific mathematics that she wanted the student(s) to notice. Leah felt that this encouraged a deeper mathematical understanding than she had observed in her previous teaching when students simply mimicked her demonstrations of 'how to do' multiplication. Leah's perception of TT as a tool that allowed her insight into how students were understanding multiplication was beneficial in helping her bridge the instrumental distance.

Leah's didactical reference markers were grounded in her experiences of how students learn multiplication and the common difficulties encountered while so doing. Therefore, she was also able to use her student observations to intervene immediately or to plan for the next day's lesson, leveraging the affordances of the technology to scaffold student learning for deeper understanding. Leah's perception of TT as a tool that allowed her to assess the mathematical awareness of her students within her lessons was also beneficial in helping her bridge the instrumental distance and reflects the dynamic relationship amongst the teacher–technology–mathematics ensemble. Her increasing ability to assess and intervene 'on the fly' during her didactical performance is an indication of the growing depth of Leah's professional instrumental genesis.

In Amy's classroom, after giving student pairs time to explore a Grasplify task, she sometimes asked them to depict their learning through drawings, which they later referred to while explaining their understanding. Wanting her students to explore deeper multiplication concepts, Amy engaged her students in processes such as exploring, noticing, drawing and explaining their thinking to others. She felt it was important that students be able to express their understanding in various ways. Like Leah, Amy also used these drawings for formative assessment. She wanted to know what students noticed, what they picked up on and what they were attending to so that she could plan her next lesson to address any gaps in understanding that emerged.

Leah also noted how Grasplify's dynamic, non-permanent aspects – and the speed in which a student could try something and immediately see the results – allowed for more fruitful teacher–student interactions and, consequently, more shifts in student thinking. Compared with when she previously had students draw arrays on paper, the process was much slower and ideas could not be easily checked the way they can be using

Grasplify and, consequently, Leah felt that her interactions with students did not have the same impact. As her professional instrumental genesis continued to develop, and she noticed affordances of TT that benefited student learning in ways that were not possible using pencil-and-paper, the instrumental gap for Leah continued to narrow.

As Amy and Leah became more familiar with using Grasplify as a pedagogical tool and, as they observed the mathematics learning that occurred while students were using it, the instrumental distance continued to narrow for both teachers. Their ability to use Grasplify in ways that were consistent with their established styles of teaching, that prioritised student exploration, discovery and whole class discussion, while tailoring their teaching to student needs and the mathematical ideas that naturally emerged from the tasks given, made it easier for Amy and Leah to incorporate TT into their mathematics teaching. The teacher–tool–mathematics ensemble was working in ways that enabled each teacher to utilise the digital technology not only to build student understanding of multiplication, but also further to develop student learning processes, which they believed was most important in mathematics and was significant to their professional instrumental genesis.

9.4. Evolving processes of double instrumental genesis: Responses to research question three

Though this question has been addressed throughout the previous two sections, there are a few instances that reflect the evolution of the teachers' processes of double instrumental genesis that were not part of the responses to research questions one or two. I will briefly summarise some of the highlights of Amy's, Kate's, Leah's and Rachel's experiences of personal and professional instrumental genesis while implementing TT as a pedagogical tool for teaching multiplication and will finish by describing how all four teachers began to integrate formative assessment into their lessons using TT.

9.4.1. Amy: Student-led exploration and the creation of tasks

The information shared by Amy did not provide much of a glimpse into her personal instrumental genesis. When first exposed to TT, she was sitting beside Leah, who questioned the multiplicand × multiplier = product ordering of Grasplify; this was something with which Amy did not have difficulty. The experiences that were shared by

Amy in her interviews were highly reflective of her process of professional instrumental genesis throughout her implementation of Grasplify.

As Amy became more familiar with Grasplify and the mathematical ideas it presented, she described two factors that I see as reflective of her growing professional instrumental genesis. The first related to her increasing willingness to allow her students to lead their own explorations with Grasplify. She encouraged them to ask their own questions, make predictions, search for the answers to their questions and to share what they learned with their peers. In doing this, Amy was no longer explicitly directing student experiences; instead, she was using the student ideas that emerged within whole-group sharing to guide and direct class discussion toward the mathematical concepts that she wanted students to learn.

While Amy did encourage student ownership of their explorations, she usually prompted these explorations by providing students with some sort of task. Initially, she used the task ideas provided within the TT teacher–researcher group meetings, but as Amy gained experience using Grasplify with students, she also began to develop her own tasks for use with it. For Amy, part of her growing professional instrumental genesis included the opportunity to share her own experiences of implementing Grasplify with the other teachers, and to hear and learn about the ideas shared by others in the project group.

The increasing depth and breadth of Amy's professional instrumental genesis can be seen in her ability to predict the mathematical ideas that would likely to emerge from the tasks with which she used to prompt student explorations, some of which she created herself. Amy's comfort with letting students take control and ownership of their learning, and her flexibility in using the ideas shared by students in class discussion to highlight the mathematical ideas on which she wished to focus attention, are also indicators of her growing professional instrumental genesis of TT.

9.4.2. Kate: *TouchTimes* as a viable tool for teaching

Similar to Amy, the experiences shared by Kate were much more focused on her experiences as a teacher implementing TT with students and, therefore, she shared less information about her own process of personal instrumental genesis. Although Kate described being surprised initially by TT's blank opening screen, as well as by the disappearance of objects on the screen when she removed her fingers from the iPad screen, neither of these impacted her personal instrumental genesis. She also noted that the multiplicand × multiplier ordering displayed in Grasplify was a different way of thinking about multiplication, though she already knew this was possible.

When using TT to teach multiplication, Kate's professional instrumental genesis was highly influenced by her use of the digital technology to highlight the mathematical ideas she valued, which were primarily based on the symbolic mathematics which she prioritised as most important. This was also demonstrated by her creation of worksheets for students to complete that drew attention to the numerical expression. When students were able to recognise and identify the corresponding multiplicative model from Grasplify or Zaplify in the images from another teaching resource that were projected by Kate, she was then more willing to accept the usefulness and value of the digital technology for teaching mathematics. The success of her students to transfer their learning from TT to static images of models of multiplication was significant in Kate's process of double instrumental genesis, seeming to allow her to acknowledge the technology as a viable tool for teaching.

9.4.3. Leah: Intertwining personal and professional instrumental geneses

Given that the evolution of Leah's process of double instrumental genesis has already been examined in detail in previous sections of this chapter, this summary will be brief. When confronted by the way that Grasplify modelled the numerical expression created by the user, which was not consistent with her own understanding of multiplication, Leah's personal instrumental genesis was disrupted. Despite her personal discomfort with the multiplicand × multiplier ordering in Grasplify, the development of her professional instrumental genesis continued as she proceeded to implement it and was able to instrumentalise the digital technology to serve her didactical intentions. Consequently, the development of Leah's professional instrumental genesis was faster and stronger than that of her personal instrumental genesis.

Participation in the TT teacher–researcher group provided opportunities to ask questions and to discuss their possible answers with other teachers and/or members of the research team. This was beneficial for Leah, who described the impact of another teacher's explanation of the importance of context in some, but not all, situations. With this in mind, Leah then used her knowledge of commutativity to reconcile that $a \times b$ can be *a* groups-of *b* or *b* groups-of *a*. Though not applicable in the context of Grasplify, Leah found the commutative property of multiplication helpful as a way of making sense of the irrelevance of the ordering of the factors.

Leah also became more intentional in the language she used to describe the visual model depicted by Grasplify and, when her students understood and had no difficulty describing the multiplicative model depicted rather than the order of the numerical expression, she revisited her own thinking and understanding of multiplication. And, in so doing, her personal instrumental genesis of TT began to grow stronger. The process of double instrumental genesis was not linear for Leah. She did not first adopt TT as a personal, working instrument for mathematical activity and then appropriate and utilise it effectively as a didactical instrument. Rather, Leah's professional instrumental genesis of TT developed prior to, and therefore influenced, the progression of her personal instrumental genesis.

The evolution of Leah's process of double instrumental genesis is exemplified by her ability to move beyond Grasplify being 'backwards' to her previous ways of thinking about and teaching multiplication. This process was facilitated by her participation in the TT teacher–researcher group and the success of her students when using Grasplify. There was a singular interaction of TT–students–teacher–mathematics which impacted Leah's understanding of multiplication, as well as her acceptance of and ability to use TT effectively as a pedagogical tool for teaching mathematics to her students – both of which are now consistent with the flexible thinking and personalised approaches to mathematics that Leah most values.

9.4.4. Rachel: The impact of the TT teacher–researcher group

Detailed descriptions of Rachel's experiences with TT and subsequent progression through double instrumental genesis have already been examined in earlier sections of this chapter. Therefore, in this sub-section, I will focus on the highlights of her journey.

Perhaps most beneficial to Rachel's processes of personal and professional instrumental genesis was her engagement with the TT teacher–researcher group. Though she did not initially find the implementation of Grasplify as a pedagogical tool to be intuitive, she described how the provision of task ideas during the project meetings and listening to the shared experiences and ideas of the other participating teachers helped her to overcome this difficulty.

Furthermore, the group discussions and Rachel's interactions with the research team were influential to the entire teacher–tool–mathematics ensemble, providing a deeper understanding of the way that Grasplify materialises multiplication, while highlighting the importance of developing students' ability to reason multiplicatively. As her personal instrumental genesis and knowledge of the digital technology and its mathematical affordances increased, her own thinking about multiplication expanded and, subsequently, her teaching approach also changed. She became more intentional about engaging students with TT tasks that provided experiences of multiplication that were multiplicative, rather than additive. Rachel's growing professional instrumental genesis was demonstrated by her increasing ability to leverage her understanding of the digital tool and the mathematical experiences it affords into her teaching.

Rachel, like Leah, continued with her implementation of TT despite her own initial challenges with the technology and also, similar to Leah, was the influence of Rachel's students on her processes of personal and professional instrumental genesis. The children's success with Grasplify and its accompanying tasks motivated her use of the technology to draw student attention to the multiplicative ideas of covarying and spreading. Thereby, Rachel was further developing her professional instrumental genesis while also learning more about Grasplify from her students during their use of it.

This is a nice example of the complex back-and-forth intra-action (Barad, 2007) between Rachel's personal process of instrumental genesis and that of her professional instrumental genesis (and vice-versa). Her personal comfort with, understanding of and confidence in using Grasplify improved due to her interactions with those in the TT teacher–researcher group, and as a result of witnessing her students' successful use of the technology to better understand multiplication.

9.4.5. Drawings as formative assessment

Amy's, Kate's, Leah's and Rachel's growing professional instrumental genesis was demonstrated by their independently motivated shift in focus from teaching using the digital technology to assessing student understanding of TT and its way of materialising multiplication. All four teachers asked their students to depict their learning through drawings, which they then used to gauge student understanding (or misunderstanding) and to note which mathematical ideas students were attending to when using the digital technology. The teachers' described various ways that these formative assessments were utilised. Some of which included: (1) to inform their teaching practice; (2) to tailor their planning for the next lesson; (3) to address individual student or whole class learning needs either on the spot or to inform the direction of whole-class discussion. The inclusion of formative assessment into their lessons with TT was another indicator of their growing double instrumental genesis.

9.5. Implications of the findings

In keeping the teacher–TT–mathematics ensemble in mind, here are a few noteworthy implications of the research presented throughout this dissertation. The first relates to the mutually interactive nature of personal and professional instrumental genesis, the second involves an acknowledgement of the agency of the tool, the third concerns the importance of sociocultural influences on the teachers' processes of double instrumental genesis and, finally, three new instrumental orchestrations that emerged from this research.

When first introduced to TT, I expected that the teachers would initially undergo a phase of personal instrumental genesis while learning to use it themselves and, once they possessed an understanding of the digital tool and how it presented multiplication, that they would then transition into a period of professional instrumental genesis while considering how to use TT effectively as a tool for teaching. This, however, was not the case.

Perhaps because TT was specifically designed for teaching multiplication to elementary students, when the teachers first engaged with it, the technology and the mathematics it presented *appeared* to be fairly straightforward and, therefore, the transition into thinking about how TT could be implemented as a pedagogical instrument for teaching multiplication occurred quickly. Integrating digital technologies often involves unexpected complexities and, in the case of TT, the complexity is nested within the teacher–tool–mathematics ensemble. The way that TT materialises multiplication using multiplicative, as well as additive, models required the teachers to possess a depth of mathematical understanding of multiplication, in order effectively to leverage the affordances of the tool to ensure student learning.

The process of double instrumental genesis did not occur in a linear, sequential fashion: rather, it was muti-faceted and complex. The teachers' personal and professional instrumental geneses were mutually interactive; growth in one area influenced the development of the other. In her spreadsheet integration research, Haspekian (2014) found interferences between the two modes of instrumental genesis that were related to the purpose of the technology. In the case of TT, there were initially interferences between some of the teachers' personal and professional instrumental geneses which were related to the technology's presentation of the mathematics itself. The mutually interactive nature of the two modes of instrumental genesis, however, eventually facilitated a progression by the teachers past any interferences as part of their processes of double instrumental genesis.

In much of the literature on instrumental genesis, and on instrumental orchestrations in particular, the focus is on how the teacher organises, implements, adapts and manages the task, the technology and the classroom interactions. Although there is an underlying assumption that the technological tool affects the teacher, it is the teacher that is studied with little attention paid to the role the tool plays in shaping teacher practice. By examining the data in a more holistic way that honours the synergistic influence amongst and between all three ensemble components of teacher, technological tool (TT) and mathematics (multiplication), there were occasions in which TT was observed exerting agency which influenced the practices of the teachers.

Sociocultural factors also proved to be beneficial to the development of the teachers' personal and professional instrumental genesis. Being part of a larger teacher–researcher project meetings where information about TT design, the mathematics it presents and ideas for implementing it as a pedagogical tool were shared and discussed amongst the group was significant in facilitating the teachers' processes of double instrumental genesis. Having the research team in to observe in their classrooms when Leah and Rachel were teaching with Grasplify and the conversations that occurred during this process were also described as being powerful.

The students' noticeable engagement and success with TT and their understanding of the multiplicative models it presented was another sociocultural factor that increased the teachers' willingness to accept the technology as a viable pedagogical resource and further developed their professional instrumental genesis. In the case of Leah, this also caused her to rethink her own understanding of multiplication, which also positively influenced her personal instrumental genesis. These sociocultural effects were significant for each teacher's growing recognition and appreciation of the mathematical experiences that the affordances of TT could offer for student learning.

It was during classroom observations of Leah and Rachel teaching with Grasplify that three new orchestrations emerged from the elementary school context. While Rachel and Leah were focusing student attention on the effects of adding or removing pip-fingers to the configuration of the pods and linking these unitisations to the product displayed in the numerical expression, the new orchestrations – Compare-successive-screens, Document-screen-on-paper and Discuss-the-finger-and-screen – were observed (Bakos, 2022a).

The Compare-successive-screens and Discuss-the-finger-and-screen orchestrations emerged spontaneously as the teachers interacted with their students and the ideas being shared during whole-class discussion. The Compare-successive-screens orchestration was linked to the importance of highlighting mathematical ideas that emerged from Grasplify, such as what happens when the unit is changed, which proved to be a new idea for these teachers when teaching multiplication. The Document-screen-on-paper allows children to re-present the mathematics created with their fingertips using TT into their notebooks for future reference and directs attention to the relationship between the pip-side and the pod-side of the screen.

Given the touchscreen nature of this technology, there was particular attention paid to the users' fingers as a means to express and engage physically with multiplicative relations. This, therefore, underlines the importance of studying the implementation of different digital technologies by teachers at all levels of schooling, as these factors may lead to specific orchestrations.

9.6. Limitations of the study

In this section, I will detail some of the methodological limitations of this study and examine the potential constraints that arose from situations relating to me as the researcher. Given that this research focused on the experiences of only four elementary school teachers, the sample size was very small. Thus, the results described throughout Chapters 5 to 8 are specific to these individual teachers and the situations in which they taught mathematics using TT and, therefore, generalising or transferring the results of this study must be done with caution.

Although there is existing research on the use of touchscreen technology for teaching mathematics to young children, when this research began there was no pre-existing research on the newly developed touchscreen application *TouchTimes*. Throughout the course of gathering my data and writing this dissertation, members of the TT teacher–researcher group, including myself, have analysed and written about findings related to TT (e.g. Chorney et al., 2019; Güneş, 2021a; Sinclair et al., 2020).

The teacher interview data that I collected as part of this research was self-reported and, as such, there may be potential sources of bias, for instance the teachers, either consciously or unconsciously, providing answers to questions that they believe are most acceptable to the researcher or the teachers experiencing selective memory about teaching events that occurred some time ago. As a result of the COVID pandemic, the teacher interviews were conducted more than a year after three of the four teachers had used TT with their students. Therefore, their experiences teaching with the digital tool were not recent and their answers were based on what they were able to recall about teaching with TT in previous school years with a different class of students than the one they had most recently taught.

The four teachers who formed the basis of the data collection for this dissertation were chosen as a result of their involvement with the TT teacher–researcher project and their willingness to share their experiences via interviews. There were also video-recordings of mathematics lessons using TT in Amy's, Leah's and Rachel's classrooms that I was able to access and analyse as one of the research assistants involved in the larger project (some of which I had also been present).

There is researcher bias that is inescapable given the qualitative nature of my research and my desire to honour the experiences that the teachers chose to share that were particularly powerful in some way for them individually. As a former elementary teacher myself, what I perceived as a powerful experience could have been influenced by my own experiences implementing digital technology into my own mathematics teaching. My interactions with the teachers during the interviews was sometimes that of a fellow teacher, while at other times as a researcher in my own right and, occasionally, as part of a research team led by more knowledgeable others. In assuming different roles throughout the interviews, at times, this likely influenced the responses given by the teachers.

When first analysing my data, I began by trying to group the ideas taken from the teacher responses to the interviews into common categories. These categories were not pre-established, but instead emerged from the data I had. However, when I found myself drawn to the instances shared that I sensed were particularly powerful for that specific teacher, I chose not to use a coding scheme or other methods of data analysis that are perhaps more impartial. Rather, I embraced the data that 'glowed' (MacLure, 2013) for me. In so doing, there may have been other instances shared that provided insight into the teachers' experiences implementing TT that I chose not to include in this dissertation because I found them less compelling in some way. I am conscious that every method emphasises and de-emphasises things and that is also the case in choosing data that glows.

9.7. Future directions

I have been fortunate for opportunities to discuss smaller 'snapshots' of this research through conference papers written during the journey that brought me to writing this dissertation. An area of interest for future research deals with the embodied aspect of TT as a touchscreen application. How do teachers' personal instrumental genesis benefit from their kinaesthetic experiences of multiplication when using TT? As their professional instrumental genesis grows, how do teachers leverage this affordance of the digital technology when planning their teaching with it?

Of the four teachers that formed the basis of my research, only Kate had utilised the Zaplify microworld with her students and there is more data needed regarding teacher implementation of *both* Grasplify and Zaplify. How do they structure their teaching using two different multiplicative models within the TT application? Leah and Rachel were both explicit about how influential their use of Grasplify was to their own ways of thinking about multiplication and how this expanded their approach to teaching multiplication with their students. What are the effects of Zaplify on teachers' thinking about multiplication and does it influence approaches to teaching it?

There are two areas that I began to examine as part of my research that are often overlooked in mathematics education research that focuses on digital technology. The first is the agency of the technology itself and, although I tried to address this in my research here, there is much more research that can be done in relation to the effects of the tool on the teacher and/or the mathematics. The second involves an examination of sociocultural influences on teachers when implementing technology. This emerged as being a significant part of these teachers' processes of double instrumental genesis of TT, yet it has not previously emerged in the research using this theoretical notion. More research is needed to understand better the role of sociocultural influences on elementary teachers and their mathematics teaching. Though this emerged in relation to implementing digital technology, I am curious to explore further what mathematics education research already exists in a more general sense about sociocultural factors and their influence on elementary teachers in relation to their mathematics teaching.

Additionally, the teachers that formed the basis of my research had been teaching from 9–24 years, and therefore had experience in teaching multiplication and some of whom also had experience using digital technology as part of their mathematics teaching. It would be worthwhile exploring what the experiences of novice teachers, or those without prior experience implementing digital technology in their mathematics teaching, would be. This would perhaps provide valuable information about what supports teachers need in order to effectively implement technological resources into their mathematics teaching.

9.8. Final words

As I described back in Chapter 1, the journey that brought me to this point began when I was a student of mathematics, continued when I became a teacher of mathematics and underwent a significant upheaval when I began graduate studies and started to learn more about mathematics education research and, ultimately, began the transition that led to becoming a mathematics education researcher. I bring the experiences of each of these roles into my new one.

It took what seemed to be a very long time (at least to me) to arrive at my research focus and I did not take a direct route in gathering the data and writing this dissertation. Rather, I began with writing a four-page paper describing research that I had yet to decide upon, in order to apply for the Young European Researchers in Mathematics Education (YERME) summer school in Montpellier, France in the summer of 2018. From there, my topic gradually came into focus with each successive piece of writing completed for a conference paper or an article. My increasing confidence and my growth as a writer, a researcher and a presenter have been positively influenced by each of the smaller steps I have taken to arrive at the conclusion of writing this thesis. A task that seemed insurmountable not so very long ago.

I consider the personal growth experienced throughout this process to be invaluable and it is my hope that my work will be able to contribute in some way to improving mathematics teaching at the elementary school level.

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