A Complete Framework for Modelling Defined Benefit Pension Plans

by

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Abstract

This report proposes a complete framework to model the operation of a defined benefit pension plan, which contains a Canadian economic scenario generator, a stochastic mortality model, an administrative cost model, and an asset optimization procedure. We suggest the use of economic capital-based measures and expected utility-based measures to quantify the solvency and welfare of the plan. The economic capital-based measure is based on the value-at-risk and expected shortfall measures over three-year and 50-year horizons. Members' expected utility is compared through certainty equivalent consumptions. Using simulated results from the framework, we find a feedback loop in the asset allocation, the valuation rate, and the funded ratio: the funded ratio influences the asset allocation, and these asset weights affect the valuation rate used to discount the actuarial liability which, in turn, impacts the funded ratio.

Keywords: Defined Benefit Pension Plan; Economic Scenario Generator; Asset Optimization; Pension Plan Valuation; Stochastic Simulation; Economic Capital; Expected Utility.

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Table of contents

List of tables

List of figures

Chapter 1

Introduction

Most occupational pension plans can be characterized as defined benefit (DB) pension plans or defined contribution (DC) pension plans.[1] DB plans provide members with pension payments or a lump sum retirement benefit, which are determined by a formula based on the member's salary and years of service. In DC plans, on the other hand, members pay contributions into a fund, and the fund earns investment income. The members receive the fund value after retirement. They can use the fund value to buy an annuity, or they can continue investing and withdraw savings from the fund.

This report focuses on the operation of a DB plan. In the real world, the operation of this type of pension plan depends on the outcomes of many random variables. The interest rate, inflation rate, and asset returns are always changing, which impact the pension plan's actuarial liability and fund value. Longevity risk is an increasing concern nowadays as populations tend to live longer, and the pension fund surplus could be negatively impacted if the pension payments last longer than expected. Administration costs change with fund size—which is random—because economies of scale exist in large plans. Last but not least, asset allocation plays a critical role in pension plan operation. The asset allocation impacts the fund return, and further impacts the surplus of the fund and the contribution rate that members and sponsors need to pay every year.

To account for all the random pieces, we propose a general framework to model the operation of a DB plan. This complete framework allows for simulating the operation of the plan in the real world by using realistic stochastic models for the key assumptions, which include (1) an economic scenario generator (ESG) that generates future inflation rates, interest rates, and rates of return on various assets based on Canadian economy, (2) a stochastic mortality model considering future mortality improvements, (3) an administrative cost model reflecting economies of scale, and (4) an asset portfolio optimization procedure based on utility maximization.

[1] Note that nowadays, there are some other hybrid plans between DB and DC plans, such as cash balance plans, target benefit plans, underpin plans, collective DC plans, and notional DC plans.

An ESG is used to generate the joint future behaviour of the Canadian economy; it extends the model proposed by Bégin (2021). In the ESG, the monetary policy is the primary driver. Regime variables are constructed based on the monetary policy. The price inflation, wage inflation, and the dividend yield are modelled by autoregressive models with regime-dependent long-run mean levels. A generalized autoregressive conditional heteroskedasticity (GARCH) model is applied to the short rate, stock index returns, total private equity returns, investment grade corporate bond yields, and high yield corporate bond yields to capture the changes in volatility over time.

A stochastic mortality model is used to project plan membership. Death is modelled as a random variable generated from a Bernoulli distribution. The parameter of the Bernoulli distribution is the mortality rate in the CPM 2014 Male Mortality Table with the CPM Improvement Scale B provided by Canadian Institute of Actuaries (CIA, 2014). This model considers mortality improvement trends in the future.

The administrative costs of the pension plan are modelled by a traditional cost function proposed by Bikker et al. (2012) that reflects economies of scale. The function considers the number of members, service quality, pension plan complexity, pension fund type, number of pension plans offered, and the country.

We apply a utility-based method to form optimal portfolios, similar to the method proposed by Warren (2019). The utility function is based on the funded ratio and is parameterized for relatively high risk aversion in relation to deficits. It also attaches a relatively modest value to surpluses. The optimal allocation is found by maximizing the expected utility of the funded ratio.

The main objective of this report is to quantify solvency and welfare arising from DB plan operation by using a complete framework; these measures are useful to assess the performance of the plan in a straightforward way. Through the simulated realizations of economic, financial, and mortality variables, we get distributions of pension plan quantities such as the portfolio weights, the funded ratio, the contribution rate, and so on. The information contained in these distributions is then summarized into one-dimensional metrics.

One way to determine solvency and welfare is to understand the extent of plan gains and losses and compute risk measures of the present value of the profit stream—similar to the process used by banks and insurers to understand their risk. This yields so-called economic capital metrics, generally speaking. Economic capital is the capital that shareholders should invest in the company to limit the probability of default up to a certain level (i.e., the one used to determine the risk measure). It is a key part of the supervisory approach in banking and insurance. Basel (2009) recommended an economic capital framework for banks and supervisors in the supervisory review process of Basel II. The latter framework requires to calculate economic capital, which is measured as the amount of capital that a bank needs in order to absorb unexpected losses. In insurance, economic capital is part of the Solvency II regime in Europe for capital requirement. Also, economic capital can be used to satisfy the regulatory requirement of Own Risk and Solvency Assessments (ORSA) in Canada and US to project future capital needs.

In the pension context, Porteous et al. (2012) applied the Solvency II framework to compute economic capital under a value-at-risk (VaR) measure for a DB plan. Andrews et al. (2019) updated the work of Porteous et al. (2012) and calculated both VaR and expected shortfall (ES) measures. Andrews et al. (2022) further analyzed the impact of the choice of time horizon and asset allocation on economic capital of DB plans.

The expected utility framework is another approach to measure welfare. Expected utility theory is a classical decision-making approach, which can be used for welfare comparison. von Neumann and Morgenstern (1947) first stated the axioms supporting expected utility theory. They proved that, under certain conditions, individuals are guaranteed to have a real-valued utility function and that their preferences are consistent with maximizing expected utility. In the pension context, expected utility theory has been used to determine the optimal funding strategy. For example, Josa-Fombellida et al. (2018) optimized the asset portfolio by maximizing the expected utility of the fund surplus over a finite planning horizon in a DB plan, and Cairns et al. (2006) maximized the expected utility of the members' final replacement ratio to find the optimal asset allocation in DC plans.

Following the work of Andrews et al. (2022), we apply economic capital-based measures to evaluate the solvency and welfare of the plan, which summarize the distributions of the funded ratio, the fund return, and the asset allocation into a single number. The economic capital-based measures include the VaR and ES measures at confidence levels of 50%, 90%, and 99.5% over a three-year and a 50-year horizon. In addition, we apply an expected utility-based measure to calculate the welfare of the plan members, which summarizes the distribution of the contribution rate, the salary and the wage inflation. We further compute certainty equivalent consumptions (CECs) for each member to compare the expected utility in a more meaningful way.

The pension plan operation is projected through Monte Carlo simulation. Nested stochastic projections are used for optimizing asset allocation, requiring ESG scenarios (i.e., *inner loop*) that are based on current economic conditions throughout each path (i.e., *outer loop*). Outer loop scenarios are first generated to project the asset returns through the projection period. Inner loop scenarios are then generated at each node along the outer loop paths, using the economic conditions generated by the outer loops as the starting point. This nested simulation exercise has a high computational cost; we therefore perform the simulations by using the heterogeneous computer cluster Cedar, part of the Digital Research Alliance of Canada, running multiple parallel jobs simultaneously.

We obtain several key results by using the proposed framework with an initial funded ratio of 1, an initial valuation rate of 6%, and a reference funded ratio of 0.9. We find a U-shape relationship between the initial funded ratio and the optimal total stock index weight when we only allow for the investment in the total stock index and the investment grade bond portfolio in our fund. The optimal total stock index weight is lowest when the initial funded ratio is near 1.1. When the initial funded ratio deviates from 1.1, the optimal total stock index weight increases. This kind of U-shape relationship is consistent with the findings of Warren (2019).

Because of this U-shape relationship, we also observe possible oscillations in the evolution of the asset allocation, the valuation rate, and the funded ratio in early years. These three quantities impact each other in the framework: the funded ratio influences the asset allocation, these asset weights affect the valuation rate used to discount the actuarial liability which, in turn, impacts the funded ratio. The oscillations occur in the first ten years because the initial asset allocation, the valuation rate, and the funded ratio have not reached their steady states yet. In the long term, the distributions of the optimal asset allocation weights become stable with a significant allocation to the investment grade bond portfolio. Under the optimal asset allocation, the distributions of the funded ratio and the contribution rate are stable in the long term as well.

Based on the solvency and welfare metrics introduced above, we find that the economic capital results are worse at high confidence levels for the 50-year horizon than those for the three-year horizon. We also consider some alternative inputs to our model and find that the CECs are impacted in a different way than the economic capital measures in some cases. When we increase the reference funded ratio, increase the initial valuation rate, or remove private equity from the available assets, the economic capital measures change at the 90$^{\text{th}}$ and 99.5$^{\text{th}}$ confidence levels. The CECs also change slightly. On the other hand, when we decrease the initial valuation rate or change the smoothing factor—the percentage of the funding shortfall used to calculate the adjustment to the contributions—both the economic capital results and CECs are significantly impacted.

This report is organized as follows. Chapter 2 introduces a realistic ESG model and explains some results about the economic variables. In Chapter 3, we describe the complete framework for simulating the operation of a DB plan. The solvency and welfare metrics are discussed and applied to the framework in Chapter 4. In Chapter 5, we summarize the key findings from the simulation in the base case using the assumptions described in Chapter 3, and Chapter 6 verifies the robustness of the base case results. Concluding remarks are provided in Chapter 7.

Chapter 2

Economic and financial framework

A realistic ESG is an important component needed to study pension plan operation because assets and liabilities depend on future inflation rates, interest rates, and the rates of return on various assets. An ESG generates a range of realistic outcomes for economic variables over the long run, which helps us better understand the pension plan dynamics.

The first comprehensive ESG used in the actuarial literature is the Wilkie (1986) model. It uses a cascade structure to model four related variables: the inflation rate, the dividend yield, the long-term interest rate, and equity returns. The inflation rate is modelled by an autoregressive process, and it is the primary driver for the other variables. The dividend yield is a function of the inflation rate described by an autoregressive process. The long-term interest rate and equity returns both consider the dividend yield and the inflation rate in their respective models.

Over the years, various extensions of the Wilkie model have been proposed. For instance, Wilkie (1995) added models for a wage index, short-term interest rates, property prices, index-linked stock yields, and currency exchange rates. Huber (1997) further reviewed and critiqued the assumption of the Wilkie model. Sahin et al. (2008) revisited the Wilkie model and suggested most of the model parameters are not stable. Zhang et al. (2018) recently updated the Wilkie model for the US economy.

More complex ESGs were proposed in recent years as well. Ahlgrim et al. (2005) used a two-factor Hull and White (1994) model for interest rates and applied a regime-switching process to equity returns. Bégin (2021) proposed an ESG based on observable regime-switching dynamics and observable dynamic variances.

We use an economic and financial framework for the purpose of modelling the future joint behaviour of the Canadian economy that extends the ESG proposed by Bégin (2021). Our implementation includes a model for monetary policy, price inflation, wage inflation, short rate, forward rates, dividend yield, stock index returns, total private equity returns, investment grade corporate bond yields, and high yield corporate bond yields.

Our ESG is based on a cascade structure. The monetary policy is the primary driver and impacts all the other economic and financial variables. Regime variables are constructed upon the monetary policy. The price inflation, wage inflation, and the dividend yield are modelled by autoregressive models with regime-dependent long-run mean levels. Besides, a GARCH model is added to the short rate, stock index returns, total private equity returns, investment grade corporate bond yields, and the high yield corporate bond yields to capture the changes in volatility over time. Each variable is modelled on a monthly basis. The following sections introduce each variable's model, how to convert variables to annual rates, how to simulate series, and the forecasting results.

2.1 Monetary policy

Monetary policy is modelled by a discrete-time Markov chain with three states: a tightening or upward stage (u), a status quo stage (s), and an accommodating or downward stage (d). The observed states of the Markov chain at integer times t, m_t, are inferred from a reference rate fixed by the central bank:

$$m_t = \begin{cases} \mathrm{u} & \text{if } \exists\, t' \in [t-3, t] \text{ and } t'' \in [t, t+3] \text{ such that } R_t - R_{t'} > 0 \text{ and } R_{t''} - R_t > 0 \\ \mathrm{d} & \text{if } \exists\, t' \in [t-3, t] \text{ and } t'' \in [t, t+3] \text{ such that } R_t - R_{t'} < 0 \text{ and } R_{t''} - R_t < 0 \\ \mathrm{s} & \text{otherwise} \end{cases},$$

where R_t is the policy rate at time t. The transition matrix of this Markov chain is associated with the transition probabilities from one state to another:

$$\Pi = \begin{bmatrix} p_{\mathrm{uu}} & p_{\mathrm{us}} & 0 \\ p_{\mathrm{su}} & p_{\mathrm{ss}} & p_{\mathrm{sd}} \\ 0 & p_{\mathrm{ds}} & p_{\mathrm{dd}} \end{bmatrix} = \begin{bmatrix} p_{\mathrm{uu}} & 1 - p_{\mathrm{uu}} & 0 \\ p_{\mathrm{su}} & 1 - p_{\mathrm{su}} - p_{\mathrm{sd}} & p_{\mathrm{sd}} \\ 0 & 1 - p_{\mathrm{dd}} & p_{\mathrm{dd}} \end{bmatrix},$$

where $0 \leq p_{\mathrm{uu}}, p_{\mathrm{su}}, p_{\mathrm{sd}}, p_{\mathrm{dd}} \leq 1$. The "=" sign holds because the summation of row probabilities equals 1. Therefore, m_t given m_{t-1} can be generated as follows:

$$m_t = \begin{cases} \mathrm{u} & \text{if } U \leq p_{m_{t-1}\mathrm{u}} \\ \mathrm{d} & \text{if } p_{m_{t-1}\mathrm{u}} < U \leq p_{m_{t-1}\mathrm{u}} + p_{m_{t-1}\mathrm{s}} \\ \mathrm{s} & \text{if } p_{m_{t-1}\mathrm{u}} + p_{m_{t-1}\mathrm{s}} \leq U \end{cases},$$

where U is a uniform random variable generated over (0,1).

2.2 Price inflation

Price inflation impacts the projected administrative and investment costs in the pension plan. It is modelled by an autoregressive model of order one, AR(1), with a regime-dependent long-run mean level:

$$q_t = \mu_{q,m_t} + a_q(q_{t-1} - \mu_{q,m_t}) + \sigma_q \varepsilon_{q,t}, \quad \varepsilon_{q,t} \sim \mathcal{N}(0,1),$$

where μ_{q,m_t} is the regime-dependent long-run mean level of the price inflation, a_q is the parameter that governs the strength of the mean reversion, and σ_q is the standard deviation of price inflation. As usual for an AR(1) model, $|a_q| < 1$.

2.3 Wage inflation

Wage inflation is considered in our ESG since it drives future salary increases. Similar to price inflation, wage inflation is modelled by an AR(1) model with a regime-dependent long-run mean level as follows:

$$w_t = \mu_{w,m_t} + a_w(w_{t-1} - \mu_{w,m_t}) + \sigma_w \varepsilon_{w,t}, \quad \varepsilon_{w,t} \sim \mathcal{N}(0,1),$$

where μ_{w,m_t}, a_w, and σ_w are defined similarly to the price inflation parameters.

Besides, we adopt a correlation between $\epsilon_{q,t}$ and $\epsilon_{w,t}$ to capture the underlying relationship between price inflation and wage inflation innovations; that is, $\mathrm{Corr}(\epsilon_{q,t}, \epsilon_{w,t}) = \rho_{q,w}$.

2.4 Short rate

The risk-free interest rates are important in our ESG since the liability discount rate and the rates of return on risk-free government bonds are constructed based on them. The risk-free interest rates are modelled as a short rate model with a term structure component constructed on top. We illustrate the short rate model and term structure model in this section and the next section, respectively.

A transformation function is applied to the short rates in advance to allow for negative interest rates. In fact, there are different views on whether to allow for negative interest rates in the short rate model. For instance, the Vasicek (1977) model allowed for negative rates while Cox et al. (1985) proposed a model with strictly positive rates. Similar issue exists in the ESG literature. Wilkie (1986, 1995) applied a logarithmic transform to the real interest rates, forcing them to remain positive. On the other hand, negative rates were allowed in the model proposed by Ahlgrim et al. (2005). Bégin (2021) combined these two views, by relying on the transformation function proposed by Engle et al. (2017). Specifically, a linear transform is applied to higher short rates and a logarithmic transform is applied to lower short rates. Then the transformed short rates are modelled by an AR(1) model with a regime-dependent long-run average and conditional heteroscedasticity. The transformation function is:

$$\tilde{r}_t \equiv T_r(r_t) = \begin{cases} r_t & \text{if } r_t > \bar{r} \\ c_{r,0} + c_{r,1}\log(r_t - c_r) & \text{if } r_t \leq \bar{r} \end{cases},$$

where r_t is the short rate at time t, $\bar{r}$ is the threshold for the transformation, c_r is the minimum short rate assumed by the model, $c_{r,0} = \bar{r} - (\bar{r} - c_r)\log(\bar{r} - c_r)$, and $c_{r,1} = \bar{r} - c_r$. In this case, short rates can be negative when $c_r < 0$. In the present report, we let $c_r = 0$

and $\bar{r} = 0.005$. The transformed short rate dynamics are given by:

$$\tilde{r}_t = \mu_{r,m_t} + a_r(\tilde{r}_{t-1} - \mu_{r,m_t}) + \sigma_{r,t}\varepsilon_{r,t}, \quad \varepsilon_{r,t} \sim \mathcal{N}(0,1),$$
$$\sigma^2_{r,t+1} = \sigma^2_r + \alpha_r\big((\sigma_{r,t}\varepsilon_{r,t} - \sigma_{r,t}\gamma_r)^2 - \sigma^2_r(1+\gamma^2_r)\big) + \beta_r(\sigma^2_{r,t} - \sigma^2_r),$$

where μ_{r,m_t} and a_r are interpreted similarly to μ_{q,m_t} and a_q, respectively. Moreover, $\sigma^2_{r,t}$ is the time-t conditional variance of the transformed short rate and is modelled by a GARCH process, where σ^2_r is the unconditional variance, and α_r, γ_r and β_r are the reaction, asymmetry, and persistence parameters, respectively. As usual for a GARCH model, $\beta_r + \alpha_r(1+\gamma^2_r) < 1$, and $\alpha_r, \beta_r > 0$.

To capture the relationship between price inflation, wage inflation and the transformed short rate, we assume correlations between $\epsilon_{q,t}$, $\epsilon_{w,t}$, and $\epsilon_{r,t}$; that is $\mathrm{Corr}(\epsilon_{q,t}, \epsilon_{r,t}) = \rho_{q,r}$, and $\mathrm{Corr}(\epsilon_{w,t}, \epsilon_{r,t}) = \rho_{w,r}$.

Finally, we convert $\tilde{r}_t$ into r_t:

$$r_t = \begin{cases} \tilde{r}_t & \text{if } \tilde{r}_t > 0.005 \\ e^{(\tilde{r}_t - c_{r,0})/c_{r,1}} & \text{if } \tilde{r}_t \leq 0.005 \end{cases}.$$

2.5 Forward rates

Forward rates are used to model the rest of the term structure. Let $f_{i,t}$ be the forward rate observed at time t for a contract starting at $t + \tau_{i-1}$ and ending at maturity $t + \tau_i$:

$$f_{i,t} = \frac{\tau_i r_{i,t} - \tau_{i-1} r_{i-1,t}}{\tau_i - \tau_{i-1}}, \quad i \in \{1, 2, ..., 7\}, \tag{2.1}$$

where $r_{i,t}$ is the yield of the i^{th} zero-coupon bond with maturity of τ_i years. Zero-coupon bonds with eight different maturities are used here. The yield of the bond with the shortest maturity is assumed to be the short rate, which implies $r_{0,t} = r_t$, and τ_0 is 0.25 years. The other tenors considered are one, two, three, five, seven, 10, and 20 years, denoted by maturities from τ_1 to τ_7 years.

A similar transformation to what is applied to the short rate is used for forward rates:

$$\tilde{f}_{i,t} \equiv T_f(f_{i,t}) = \begin{cases} f_{i,t} & \text{if } f_{i,t} > \bar{f} \\ c_{f,0} + c_{f,1}\log(f_{i,t} - c_f) & \text{if } f_{i,t} \leq \bar{f} \end{cases},$$

where $\bar{f}$ is the threshold for the transformation, c_f is the minimum forward rate assumed by the model, $c_{f,0} = \bar{f} - (\bar{f} - c_f)\log(\bar{f} - c_f)$, and $c_{f,1} = \bar{f} - c_f$. Here we let $\bar{f} = 0.005$, and $c_f = 0$.

Litterman and Scheinkman (1991) argued that the yield curve's total variation was explained by a level component, a slope component, and a curvature component. Building on this idea, Bégin (2021) used the slope and the curvature as observable factors to explain the term structure, because the level component is already considered as a part of short rate model. This report follows a similar logic. We let the spread over the transformed short rate $\tilde{r}_t$ be generated by the observable slope and curvature:

$$\tilde{\boldsymbol{f}}_t - \mathbf{1}_7 \tilde{r}_t = \boldsymbol{\mu}_f + \boldsymbol{A}_f \boldsymbol{F}_t + \boldsymbol{\Sigma}_f \boldsymbol{\varepsilon}_{f,t}, \quad \boldsymbol{\varepsilon}_{f,t} \sim \mathcal{N}_7(\mathbf{0}_7, \boldsymbol{I}_7),$$

where $\tilde{\boldsymbol{f}}_t = [\tilde{f}_{1,t} \quad \tilde{f}_{2,t} \quad ... \quad \tilde{f}_{7,t}]^\top$, and $\mathbf{1}_7$ is a seven-dimensional vector of ones. The vector $\boldsymbol{\mu}_f = [\mu_{f_1} \quad \mu_{f_2} \quad ... \quad \mu_{f_7}]^\top$, is a seven-dimensional vector of average spread levels. The vector $\boldsymbol{\varepsilon}_{f,t} = [\varepsilon_{f_1,t} \quad \varepsilon_{f_2,t} \quad ... \quad \varepsilon_{f_7,t}]^\top$, is a seven-dimensional vector of error terms. The matrix $\boldsymbol{\Sigma}_f$ is a 7×7 diagonal matrix that contains the standard deviations of the measurement errors:

$$\boldsymbol{\Sigma}_f = \begin{bmatrix} \sigma_{f_1} & 0 & \cdots & 0 \\ 0 & \sigma_{f_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{f_7} \end{bmatrix}.$$

Note that the error terms are independent. The distribution $\mathcal{N}_7(\mathbf{0}_7, \boldsymbol{I}_7)$ is a seven-dimensional multivariate normal distribution with mean $\mathbf{0}_7$ and variance $\boldsymbol{I}_7$, which is the 7×7 identity matrix. The matrix $\boldsymbol{A}_f$ is a 7×2 matrix given by:

$$\boldsymbol{A}_f = \begin{bmatrix} a_{f_1,1} & a_{f_1,2} \\ a_{f_2,1} & a_{f_2,2} \\ \vdots & \vdots \\ a_{f_7,1} & a_{f_7,2} \end{bmatrix}.$$

The time-t values of the observable slope factor $F_{1,t}$ and the observable curvature factor $F_{2,t}$ are calculated as below:

$$\begin{aligned} F_{1,t} &\equiv \tilde{f}_{7,t} - \tilde{f}_{1,t}, \\ F_{2,t} &\equiv \tilde{f}_{1,t} + \tilde{f}_{7,t} - 2\tilde{f}_{3,t}. \end{aligned}$$

Then, we model these two observable factors through a two-dimensional autoregressive model:

$$\boldsymbol{F}_t = \boldsymbol{\mu}_F + \boldsymbol{A}_F(\boldsymbol{F}_{t-1} - \boldsymbol{\mu}_F) + \boldsymbol{\Sigma}_F \boldsymbol{\varepsilon}_{F,t}, \quad \boldsymbol{\varepsilon}_{F,t} \sim \mathcal{N}_2(\mathbf{0}_2, \boldsymbol{I}_2),$$

where $\boldsymbol{F}_t = [F_{1,t} \ F_{2,t}]^\top$ and $\boldsymbol{\mu}_F = [\mu_{F_1} \ \mu_{F_2}]^\top$, which contains the long-run mean parameters. Matrices $\boldsymbol{A}_F$ and $\boldsymbol{\Sigma}_F$ are 2×2 diagonal matrices containing the autoregressive parameters and the variance parameters:

$$\boldsymbol{A}_F = \begin{bmatrix} A_{F_1} & 0 \\ 0 & A_{F_2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_F = \begin{bmatrix} \sigma_{F_1} & 0 \\ 0 & \sigma_{F_2} \end{bmatrix}.$$

Furthermore, we convert $\tilde{f}_{i,t}$ into $f_{i,t}$ using the following transformation:

$$f_{i,t} = \begin{cases} \tilde{f}_{i,t} & \text{if } \tilde{f}_{i,t} > 0.005 \\ e^{(\tilde{f}_{i,t} - c_{f,0})/c_{f,1}} & \text{if } \tilde{f}_{i,t} \leq 0.005 \end{cases}.$$

Finally, we obtain the risk-free interest rate term structure by inverting Equation (2.1):

$$r_{i,t} = \frac{f_{i,t} \times (\tau_i - \tau_{i-1}) + \tau_{i-1} r_{i-1,t}}{\tau_i}, \quad i \in \{1, 2, ..., 7\}.$$

2.6 Dividend yield

Similar to the inflation rate, the logarithm of the dividend yield is modelled by an AR(1) model with a regime-dependent long-run mean:

$$\log(d_t) = \log(\mu_{d,m_t}) + a_d\big(\log(d_{t-1}) - \log(\mu_{d,m_t})\big) + \sigma_d \varepsilon_{d,t}, \quad \varepsilon_{d,t} \sim \mathcal{N}(0,1),$$

where μ_{d,m_t}, a_d, and σ_d have similar definitions to the inflation rate parameters.

As with the price inflation, wage inflation and the short rate, we assume correlations between $\varepsilon_{q,t}$, $\varepsilon_{w,t}$, $\varepsilon_{r,t}$, and $\varepsilon_{d,t}$ as well; that is, $\text{Corr}(\varepsilon_{q,t}, \varepsilon_{d,t}) = \rho_{q,d}$, $\text{Corr}(\varepsilon_{w,t}, \varepsilon_{d,t}) = \rho_{w,d}$, and $\text{Corr}(\varepsilon_{r,t}, \varepsilon_{d,t}) = \rho_{r,d}$.

2.7 Stock index returns

The (ex-dividend) stock index returns are modelled by a regime-switching model using observable monetary regimes to capture the changing nature of the average return. A GARCH structure is added to capture the changing nature of volatility:

$$\begin{aligned} y_t &= \frac{r_t}{12} + \mu_{y,m_t} + \sigma_{y,t}\varepsilon_{y,t}, \quad \varepsilon_{y,t} \sim \mathcal{N}(0,1), \\ \sigma^2_{y,t+1} &= \sigma^2_y + \alpha_y\big((\sigma_{y,t}\varepsilon_{y,t} - \sigma_{y,t}\gamma_y)^2 - \sigma^2_y(1+\gamma^2_y)\big) + \beta_y(\sigma^2_{y,t} - \sigma^2_y), \end{aligned}$$

where μ_{y,m_t} is the regime-dependent average monthly excess return, and $\sigma^2_{y,t}$ is the time-t conditional variance of the monthly return. The conditional variance is modelled by a GARCH model, where σ^2_y, α_y, γ_y and β_y have similar explanations and restrictions as the parameters of the short rate.

2.8 Total private equity returns

Similar to (ex-dividend) stock index returns, total private equity returns are modelled by a regime-switching model with a GARCH structure for the variance:

$$p_t = \frac{r_t}{12} + \mu_{p,m_t} + \sigma_{p,t}\varepsilon_{p,t}, \quad \varepsilon_{p,t} \sim \mathcal{N}(0,1),$$

$$\sigma_{p,t+1}^2 = \sigma_p^2 + \alpha_p\big((\sigma_{p,t}\varepsilon_{p,t} - \sigma_{p,t}\gamma_p)^2 - \sigma_p^2(1+\gamma_p^2)\big) + \beta_p(\sigma_{p,t}^2 - \sigma_p^2),$$

where μ_{p,m_t} and $\sigma_{p,t}^2$ are analogous to μ_{y,m_t} and $\sigma_{y,t}^2$. The parameters σ_p^2, α_p, γ_p, and β_p are interpreted similarly to those of the short rate.

We also assume a correlation between $\varepsilon_{y,t}$ and $\varepsilon_{p,t}$ to reflect the relationship between the stock index returns and total private equity returns; that is, $\mathrm{Corr}(\varepsilon_{y,t}, \varepsilon_{p,t}) = \rho_{y,p}$.

2.9 Investment grade corporate bond yield

To study the dynamics of the investment grade corporate bond yield, we model its spread over the long-term zero-coupon bond yield. The yield of a seven-year zero-coupon bond, $r_{5,t}$, is assumed to be the approximate average yield of the long-term zero-coupon bond. Similar to forward rates, we transform the spread as:

$$\tilde{i}_t \equiv T_i(i_t - r_{5,t}) = \begin{cases} i_t - r_{5,t} & \text{if } i_t - r_{5,t} > \bar{i} - \bar{r}_5 \\ c_{i,0} + c_{i,1}\log(i_t - r_{5,t} - c_i) & \text{if } i_t - r_{5,t} \leq \bar{i} - \bar{r}_5 \end{cases},$$

where $c_{i,0}$, $c_{i,1}$, and c_i have a similar definition to those used for transformed forward rates, $\bar{i} - \bar{r}_5 = 0.005$, and $c_i = 0$.

The transformed excess yield $\tilde{i}_t$ is modelled with a regime-dependent long-run mean and conditional heteroscedasticity:

$$\tilde{i}_t = \mu_{i,m_t} + a_i(\tilde{i}_{t-1} - \mu_{i,m_t}) + \sigma_{i,t}\varepsilon_{i,t}, \quad \varepsilon_{i,t} \sim \mathcal{N}(0,1),$$
$$\sigma_{i,t+1}^2 = \sigma_i^2 + \alpha_i\big((\sigma_{i,t}\varepsilon_{i,t} - \sigma_{i,t}\gamma_i)^2 - \sigma_i^2(1+\gamma_i^2)\big) + \beta_i(\sigma_{i,t}^2 - \sigma_i^2),$$

where μ_{i,m_t} is the regime-dependent average excess yield, $\sigma_{i,t}^2$, σ_i^2, α_i, γ_i, and β_i have similar meanings to the parameters used in the short rate.

Finally, we convert $\tilde{i}_t$ to i_t, we apply the following transformation:

$$i_t = \begin{cases} \tilde{i}_t + r_{5,t} & \text{if } \tilde{i}_t > 0.005 \\ e^{(\tilde{i}_t - c_{i,0})/c_{i,1}} + r_{5,t} & \text{if } \tilde{i}_t \leq 0.005 \end{cases}.$$

As the relationship between stock index returns and total private equity returns, we introduce correlations between $\varepsilon_{y,t}$, $\varepsilon_{p,t}$, and $\varepsilon_{i,t}$; that is, $\mathrm{Corr}(\varepsilon_{y,t}, \varepsilon_{i,t}) = \rho_{y,i}$, and $\mathrm{Corr}(\varepsilon_{p,t}, \varepsilon_{i,t}) = \rho_{p,i}$.

2.10 High yield corporate bond yield

The high yield corporate bond is modelled in the same way as the investment grade corporate bond. A similar transform is applied for the spread over the investment grade corporate bond yield:

$$\tilde{h}_t \equiv T_h(h_t - i_t) = \begin{cases} h_t - i_t & \text{if } h_t - i_t > \bar{h} - \bar{i} \\ c_{h,0} + c_{h,1}\log(h_t - i_t - c_h) & \text{if } h_t - i_t \le \bar{h} - \bar{i} \end{cases},$$

where $c_{h,0}$, $c_{h,1}$, and c_h are interpreted similarly to the parameters used for transformed forward rates, $\bar{h} - \bar{i} = 0.005$, and $c_h = 0$.

Identically, the transformed spread $\tilde{h}_t$ is modelled based on a regime-dependent long-run average and conditional heteroscedasticity:

$$\tilde{h}_t = \mu_{h,m_t} + a_h(\tilde{h}_{t-1} - \mu_{h,m_t}) + \sigma_{h,t}\varepsilon_{h,t}, \quad \varepsilon_{h,t} \sim \mathcal{N}(0,1),$$
$$\sigma^2_{h,t+1} = \sigma^2_h + \alpha_h\big((\sigma_{h,t}\varepsilon_{h,t} - \sigma_{h,t}\gamma_h)^2 - \sigma^2_h(1+\gamma^2_h)\big) + \beta_h(\sigma^2_{h,t} - \sigma^2_h),$$

where the AR(1) parameters and GARCH parameters are interpreted as those in the investment grade corporate bond yield.

The high yield corporate bond yield h_t is then converted from $\tilde{h}_t$ as follows:

$$h_t = \begin{cases} \tilde{h}_t + i_t & \text{if } \tilde{h}_t > 0.005 \\ e^{(\tilde{h}_t - c_{h,0})/c_{h,1}} + i_t & \text{if } \tilde{h}_t \le 0.005 \end{cases}.$$

Correlations between $\varepsilon_{y,t}$, $\varepsilon_{p,t}$, $\varepsilon_{i,t}$, and $\varepsilon_{h,t}$ are considered in the same way to catch the underlying relationships; that is $\mathrm{Corr}(\varepsilon_{y,t}, \varepsilon_{h,t}) = \rho_{y,h}$, $\mathrm{Corr}(\varepsilon_{p,t}, \varepsilon_{h,t}) = \rho_{p,h}$, and $\mathrm{Corr}(\varepsilon_{i,t}, \varepsilon_{h,t}) = \rho_{i,h}$.

2.11 Conversion to annual rates

Recall that our ESG is modelled on a monthly basis, while our pension plan is projected yearly. In order to use inflation rates and interest rates in our pension plan operation, we convert them into annual rates. We also construct the annual rates of return for cash, the long-term risk-free government bond portfolio, the total stock index, the total private equity, the investment grade, and the high yield corporate bond portfolio based on the economic variables generated from our ESG. These asset returns are then used to project our pension fund performance. This section shows how to convert monthly rates into annual rates and how to derive various asset returns.

All the rates modelled from our ESG are continuously compounded rate, so we add all the monthly rates in a year to get the annual rates. We convert the monthly price inflation q_t and monthly wage inflation w_t into annual rates Q_i and W_i:

$$Q_i = \sum_{t=T+12(i-1)}^{T+12i-1} q_t,$$
$$W_i = \sum_{t=T+12(i-1)}^{T+12i-1} w_t,$$

where i refers to the i^{th} year after the observed sample, which finishes at time T. The conversion will happen after simulation, which we will discuss later in Section 2.12.

The risk-free interest rates are annualized rates even though they are modelled on a monthly basis. So we pick the risk free interest rate, $r_{k,t}$, at the end of each year, to obtain the annual risk-free interest rate $R_{k,i}$:

$$R_{k,i} = r_{k,T+12i}, \quad k \in \{1, 2, ..., 7\}.$$

We derive the year-i return of cash $R_i^{(C)}$ based on the short rate, which is calculated by summing up monthly short rates, $\frac{r_t}{12}$, in the year:

$$R_i^{(C)} = \sum_{t=T+12(i-1)}^{T+12i-1} \frac{r_t}{12}.$$

We assume that the long-term risk-free government bond portfolio comprises zero-coupon bonds with increasing maturities from one to ten years. The year-i return of the long-term risk-free government bond portfolio $R_i^{(RF)}$ is computed as follows:

$$R_i^{(RF)} = \log \left(\frac{\sum\limits_{\tau=1}^{10} \exp(-r_{\tau-1,T+12i}(\tau-1))}{\sum\limits_{\tau=1}^{10} \exp(-r_{\tau,T+12(i-1)}\tau)} \right).$$

The denominator in the logarithm above calculates the price of the portfolio at year $i-1$, and the numerator computes the price of the same portfolio at year i. Thus, it represents the annual return of the portfolio at year i.

The total stock index returns are constructed based on the (ex-dividend) stock index returns and dividend yield. So, the year-i total stock index return $R_i^{(S)}$ is

$$R_i^{(S)} = \sum_{t=T+12(i-1)}^{T+12i-1} \log \left(e^{y_t} + \frac{d_t}{12} \right).$$

Similar to the inflation rate, the year-i total private equity return $R_i^{(P)}$ is

$$R_i^{(P)} = \sum_{t=T+12(i-1)}^{T+12i-1} p_t,$$

meaning that the annual rate $R_i^{(P)}$ is the summation of the monthly rate p_t in the i^{th} year after the last observed time T.

We assume the investment grade corporate bond portfolio consists of investment grade corporate bonds with maturities from one to ten years, and the investment grade corporate bond yield modelled from our ESG, i_t, is the average yield. The annual return of investment grade corporate bond portfolio $R_i^{(IG)}$ is

$$R_i^{(IG)} = \log\left(\frac{\sum_{\tau=1}^{10}\exp(-i_{T+12i}(\tau-1))}{\sum_{\tau=1}^{10}\exp(-i_{T+12(i-1)}\tau)}\right).$$

Similar to $R_i^{(RF)}$, the denominator in the logarithm function represents the price of the portfolio at year $i-1$, and the numerator indicates the price of the same portfolio at year i.

We obtain the year-i return of the high yield corporate bond portfolio $R_i^{(\mathrm{HY})}$ in a similar way:

$$R_i^{(\mathrm{HY})} = \log\left(\frac{\sum_{\tau=1}^{10}\exp(-h_{T+12i}(\tau-1))}{\sum_{\tau=1}^{10}\exp(-h_{T+12(i-1)}\tau)}\right).$$

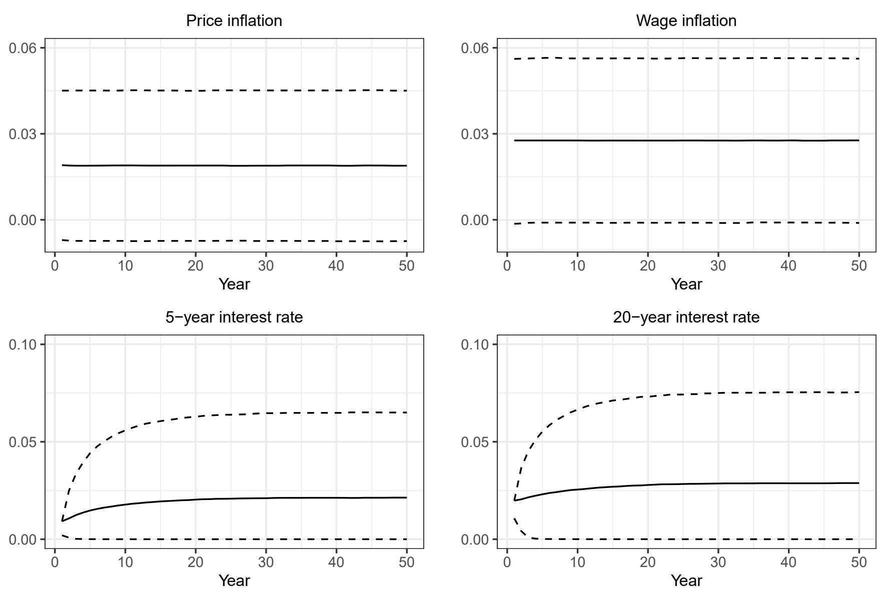


Figure 2.1: **The 5$^{\mathbf{th}}$, 50$^{\mathbf{th}}$, and 95$^{\mathbf{th}}$ percentiles of price inflation, wage inflation, 5-year interest rate, and 20-year interest rate.**

2.12 Simulation and forecast

First, we introduce the data used to fit the ESG. The time period of the data is from February 1998 to June 2021. The Bank of Canada policy rate, obtained from Bloomberg, is selected as the policy rate in the monetary policy model. The price inflation is based on non-seasonally adjusted monthly consumer price index, which is extracted from Statistics Canada. The total hourly remuneration for Canada extracted from Bloomberg is used for the wage inflation model. The three-month zero-coupon bond risk-free yields from the Bank of Canada's website is used for the short rate model, and the one-, two-, three-, five-, seven-, 10-, and 20-year zero-coupon risk-free yields from the Bank of Canada's website are used for the forward rates model. The dividend yield is constructed from dividends paid out on the stocks and level of the S&P/TSX Composite index, which is obtained from Compustat. The stock index returns are based on the S&P/TSX Composite index from Compustat. The total private equity returns are based on Thomson Reuters private equity buyout index in Canadian dollars extracted from Bloomberg. At last, the investment grade and the high yield corporate bond yield are the yield on S&P Canada investment grade corporate bond index and S&P Canada high yield corporate bond index from Refinitiv Eikon, respectively.

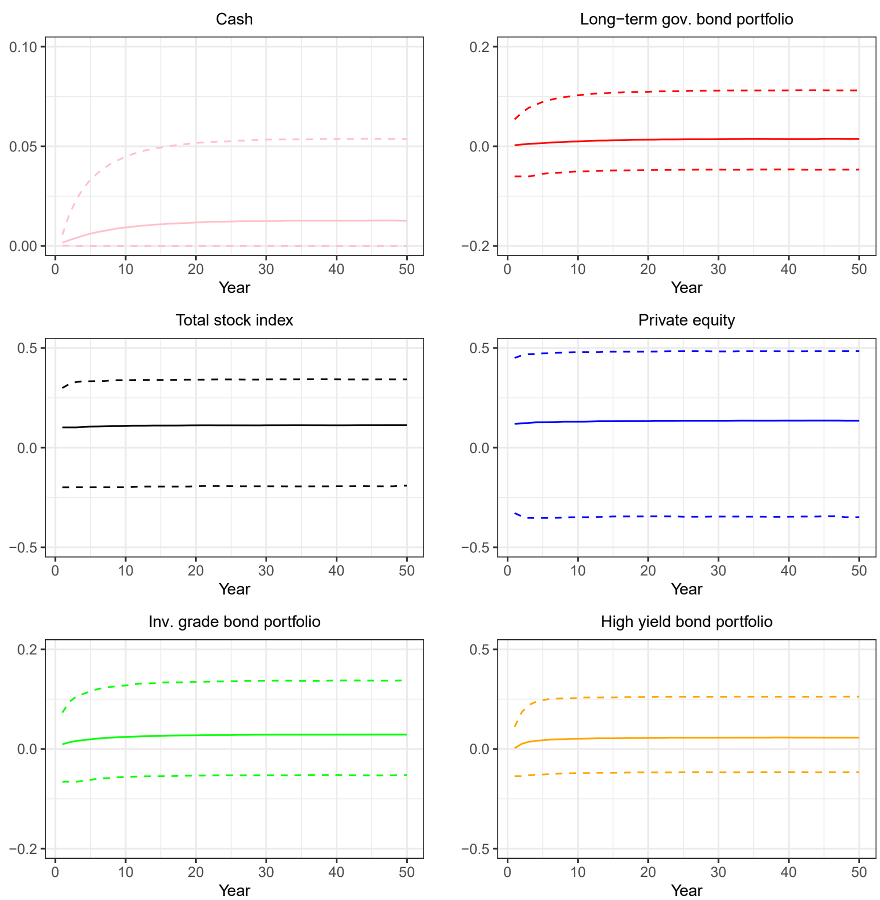


Figure 2.2: **The 5$^{\text{th}}$, 50$^{\text{th}}$, and 95$^{\text{th}}$ percentiles of annual returns for cash, the long-term government bond portfolio, the total stock index, the total private equity, the investment grade, and the high yield corporate bond portfolio.**

The parameters are estimated via Bayesian inference. Parameter samples are generated from the posterior distribution based on non-informative priors and Markov chain Monte Carlo (MCMC) scheme with the adaptive Metropolis algorithm of Haario et al. (2001). Bégin (2021) illustrates details of the estimation methodology.[2]

We forecast the economic series from July 1, 2021 by using the ESG. We pick 10,000 sets of parameters from the MCMC sample, and generate 10,000 sets of inflation rates, interest rates, and asset returns (one path for each parameter set). Figure 2.1 illustrates the 90% confidence intervals of price inflation, wage inflation, five-, and 20-year interest rates. Price and wage inflation are stable over the years. The interest rates increase in the first several years, and become stable later.

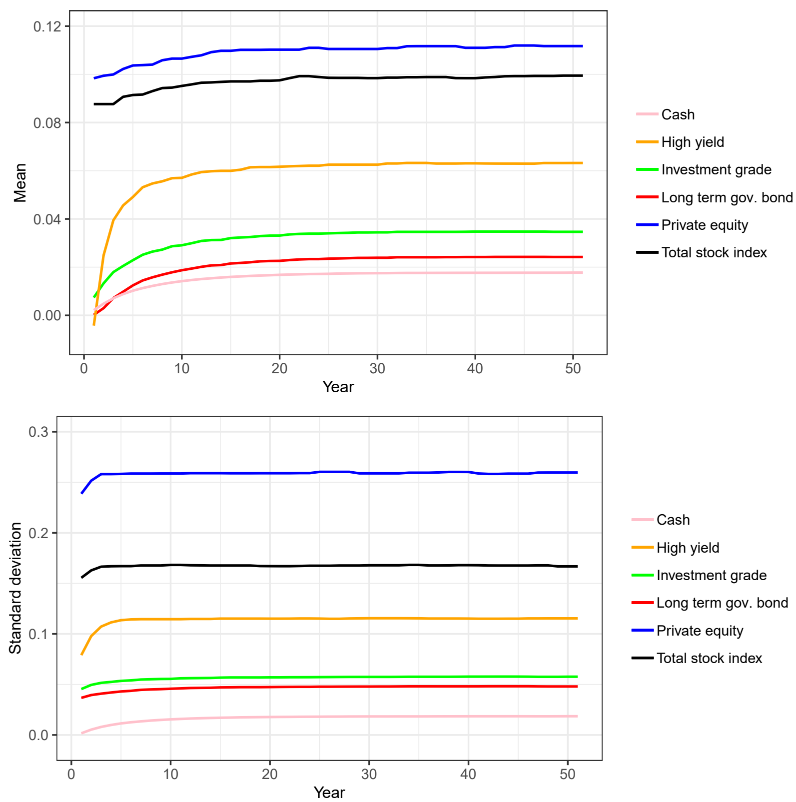


Figure 2.3: **The mean and standard deviation of annual returns for cash, the long-term government bond portfolio, the total stock index, the total private equity, the investment grade, and the high yield corporate bond portfolio.**

Figure 2.2 shows the 5^{th}, 50^{th}, and 95^{th} percentiles of annual returns for cash, the long-term government bond portfolio, the total stock index, the private equity, the investment grade, and the high yield corporate bond portfolio. We find that the returns for cash, the long

[2]The estimation of the parameters and the generation of the parameter samples were not part of this project.

term government bond portfolio, the investment grade, and the high yield corporate bond portfolio start low at the beginning and increase over the time until they are stable. For the total stock index returns and the total private equity returns, they are stable throughout the projection. The private equity has the highest median return as well as the widest range in return. On the opposite, cash has the lowest median return and the narrowest distribution. Similar conclusion can be reached with Figure 2.3, which shows the mean and standard deviation term structure of these asset returns. We find that cash has the lowest mean and standard deviation of return among six assets. Mean and variance of long-term government bond portfolio return is higher than that of cash, but lower than that of investment grade bond portfolio. High yield bond portfolio has higher mean and variance than investment grade bond portfolio, but lower than total stock index. Private equity has the highest mean and highest variance of return.

Chapter 3

Pension plan operation

This chapter describes the operation of an open-group DB plan over the course of 50 years. We describe the plan provisions and the models for the plan membership, the administrative costs, and the salaries. We outline key features of the pension plan valuation process, including the assumptions and methods used as well as the main outputs produced. Finally, we describe the process for selecting the optimal investment strategy of the pension fund.

3.1 Plan provisions

The pension plan provisions, which are chosen by the sponsor when setting up the plan, are summarized below:

- Benefit: 2% of final salary for each year of service.
- Normal retirement age: 65 years old.
- Frequency of pension payments: annually in advance.
- Pre-retirement death and termination benefits: none.

3.2 Membership model

In our hypothetical open pension plan, we aim to construct a stable membership. For simplicity, we assume all members in our plan are male. Mortality and longevity modelling plays a critical role in pension plan operation since the pension fund surplus will be impaired if pension payments last longer than expected. In the actuarial literature, various mortality models were introduced to cope with mortality and longevity risk. Lee and Carter (1992) proposed the well-known Lee-Carter model which contained a mortality reduction factor. Renshaw and Haberman (2006) extended the Lee-Carter model to include a cohort effect in the mortality reduction factor. Cairns et al. (2006) introduced a two-factor model considering a slower mortality improvement rate for higher ages.

We take a different approach: we use a deterministic table to project expected mortality and then add randomness to account for idiosyncratic mortality experience. Expected mortality follows the CPM 2014 Male Mortality Table with CPM Improvement Scale B provided by Canadian Institute of Actuaries (CIA, 2014). CPM 2014 is a table for base Canadian pensioners' mortality for the year 2014 which was developed by CIA using the combined experience of Canadian public and private sector plans in calendar years 1999 to 2008. CPM Improvement Scale B considers mortality improvement trends in the future, based on the observed trends in Canadian mortality experience since 1967. The scale provides improvement rates by age that decrease linearly in the years 2012–2030 and ultimate rates for all years after 2030. Therefore, the expected mortality rates for years past 2014 are

$$q_x^y = q_x^{2014} \prod_{t=2015}^{y} (1 - I_x^t),$$

where q_x^y is the probability that a person aged x at the beginning of calendar year y will die before reaching the end of the calendar year, and I_x^y is the improvement rate in mortality for persons aged x at the start of calendar year $y-1$ to those aged x at the start of calendar year y. The expected survival rates are then calculated as follows:

$${}_tp_x^y = \prod_{j=0}^{t-1} p_{x+j}^{y+j} = \prod_{j=0}^{t-1} (1 - q_{x+j}^{y+j}),$$

where ${}_tp_x^y$ is the probability that a person aged x at the beginning of calendar year y survives t more years.

Individual deaths are then modelled as Bernoulli random variables. Thus, the random variable $\mathcal{D}(x, y)$, which is the number of deaths from a group of n members aged x in calendar year y, has a binomial distribution with parameters n and q_x^y:

$$\mathcal{D}(x, y) \sim \mathrm{Bin}(n, q_x^y).$$

Equipped with a mortality model, we are ready to project plan membership. At the start of the projection (time 0), the plan contains members from age 25 to age 115. We assume there are 100 members aged 25. The number of members at each subsequence age decreases following the pattern of expected survival rates. The number of members aged x at time 0 is $\lfloor 100\, {}_{x-25}p_{25}^{2021} \rceil$, $x \in \{25, 26, ..., 115\}$, where $\lfloor \cdot \rceil$ represents rounding to the closest integer.[3] Note that 115 is the age at which the mortality table ends.

Starting at time 1, 100 new employees aged 25 are added to the pension plan each year. Figure 3.1 shows the evolution of plan membership. We assume that active members do not withdraw from the plan before retirement, so that post-retirement mortality is the

[3]We use the mortality rates applicable to year 2021 for all ages for this purpose.

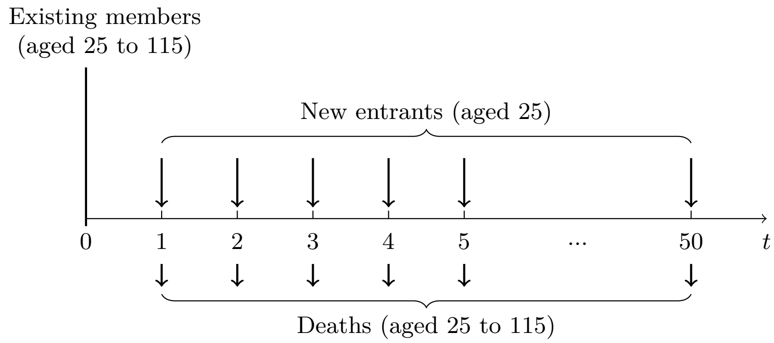


Figure 3.1: **Structure of the membership.**

only decrement. The membership will be close to stable over the projection period because members who die will be replaced by the new members aged 25. If each member had the same expected mortality, the expected number of members in the plan would be unchanged. However, since expected mortality is improving each year, the members who enter later will have lower expected mortality than those who enter earlier. As a result, the number of members in the plan will be approximately stable with slight increases.

Let e be the time when a member is first valued under our pension plan projections (the so-called "starting time"), and let x be the starting age of the member at time e. Specifically, for members already in the plan at time 0, $e = 0$, and x is the age of these members at time 0. For new entrants, e and x are the time and age when they enter the plan. We can classify plan members into different groups according to the combination of starting time and starting age. Let $\mathcal{L}(x, e, j)$ be the population in force at integer time $e + j$ among the group of members with starting age x and starting time e. We define $\mathcal{L}(x, e, j)$ to be zero when the members have not yet entered the plan; that is, $j < 0$. For $j = 0$,

$$\mathcal{L}(x,e,0) = \begin{cases} 100 & \text{if } e > 0,\ x = 25 \\ \lfloor 100\, {}_{x-25}p_{25}^{2021} \rceil & \text{if } e = 0,\ x \in \{25, 26, ..., 115\} \\ 0 & \text{otherwise} \end{cases}.$$

Otherwise, $\mathcal{L}(x, e, j)$ follows a binomial distribution:

$$\mathcal{L}(x,e,j) \sim \text{Bin}(\mathcal{L}(x,e,j-1), 1 - q_{x+j-1}^{2021+e+j-1}), \quad j \geq 1.$$

The quantity $q_{x+j-1}^{2021+e+j-1}$ is the expected mortality rate applicable in year $2021 + e + j - 1$ for a member aged $x + j - 1$ at that time.

Let $\mathcal{L}^r(x, e, j)$ denote the number of retired members at time $e + j$ among the group of members with starting age x and starting time e:

$$\mathcal{L}^r(x,e,j) = \begin{cases} \mathcal{L}(x,e,j) & \text{if } x + j \geq \text{NRA} \\ 0 & \text{otherwise} \end{cases},$$

where NRA is the normal retirement age. Let L_t denote the population of the plan at time t:

$$L_t = \sum_{x=25}^{115} \mathcal{L}(x, 0, t) + \sum_{e=1}^{50} \mathcal{L}(25, e, t-e),$$

where the first summation represents the number of members in the plan at the beginning of the projection who still in force at time t, and the second summation represents the number of new entrants who are still inforce at time t. Figure 3.2 shows the distribution of the population under 10,000 simulations, which is stable and increases slightly over the time due to mortality improvements.

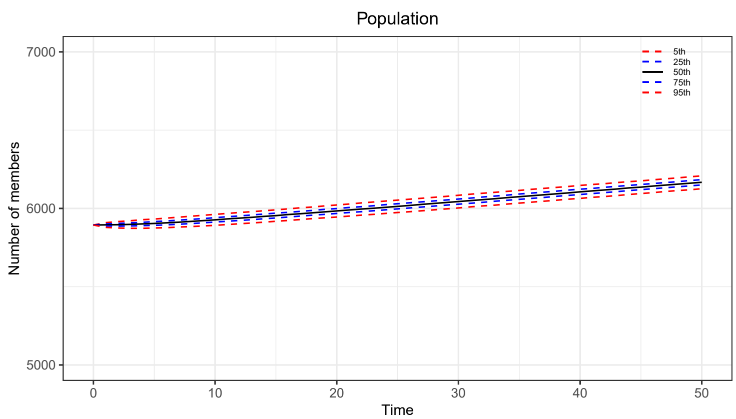


Figure 3.2: **The 5$^{\text{th}}$, 25$^{\text{th}}$, 50$^{\text{th}}$, 75$^{\text{th}}$ and 95$^{\text{th}}$ percentiles of the population based on 10,000 simulations.**

Similary, let L_t^r denote the retired population of the plan at time t:

$$L_t^r = \sum_{x=25}^{115} \mathcal{L}^r(x, 0, t) + \sum_{e=1}^{50} \mathcal{L}^r(25, e, t-e).$$

We assume all members start working at age 25 and will not withdraw from the plan, so the members of the same age have the same service at each valuation. If the members have not retired yet, the service is the number of years that have passed since age 25. If the members have retired, the service is 40 years. Let $s(x, e, j)$ denote the years of service of a member with starting age x and starting time e at time $e + j$:

$$s(x, e, j) = \begin{cases} x + j - 25 & \text{if } x + j < \text{NRA} \\ 40 & \text{otherwise} \end{cases} .$$

3.3 Administrative cost model

The administrative costs of a pension plan include all of a pension fund's operating expenses except investment costs. Past research on administrative costs suggests economies of scale exist in pension funds (e.g., Mitchell and Andrews, 1981; James et al., 2001; Bikker and De Dreu, 2009). Bikker et al. (2012) found the administrative costs vary with pension fund size, pension plan complexity, service quality, and plan maturity (proxied by percentage of active members) in Australia, Netherlands, Canada, and United States. More recently, Alserda et al. (2018) showed economies of scale for the vast majority of pension fund sizes in the Netherlands.

In our pension plan projections, we use a traditional cost function proposed by Bikker et al. (2012) to explain administrative costs. The function considers the number of members, service quality, pension plan complexity, pension fund types, number of pension plans offered and the country. Let AC_t denote the administrative cost of the pension plan at time t. We generalize the administrative cost function as follows:

$$\log\left(AC_t\, f\right) = \alpha + \beta\, \log L_t + \lambda\, \frac{L_t^r}{L_t},$$

where f is the foreign exchange rate at which one Canadian dollar will be exchanged for Euros since the administrative cost function was measured in Euros in Bikker et al. (2012). The parameter α is a constant that contains the intercept and the factors of service quality, pension scheme complexity, pension fund types, the number of pension plans offered, and the country. The variable L_t is the number of participants at time t, which is the main driver of administrative costs. The parameter β measures economies of scale. If β is less than one, economies of scale exist. The values of the parameters are shown in Table 3.1.

Parameters	Value
α	5.1935
- Intercept	7.035
- Service quality	-0.0063
- Complexity	-0.0152
- Single pension plan offered	-0.129
- Corporate pension fund	-0.092
- Canadian pension fund	-1.599
β	0.945
λ	-0.003
f	1.45

Table 3.1: **Administrative cost model parameters.**

Figure 3.3 illustrates administrative costs per member as the number of members increases. We find that administrative costs per member decrease as the number of members increases. Economies of scale exist in the model because β is less than one.

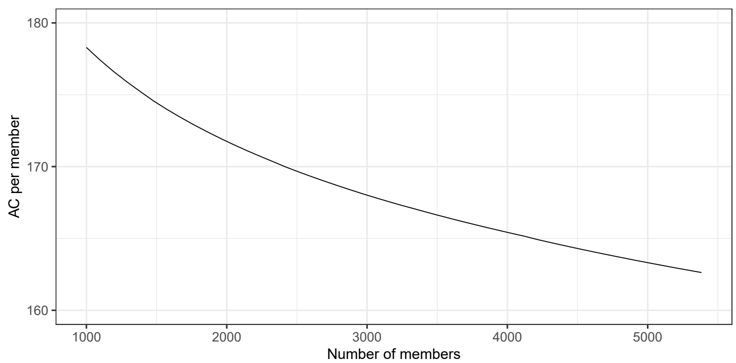


Figure 3.3: **Administrative costs per member**

3.4 Salary model

We assume that members begin employment at age 25 with a starting salary of \$40,000 at time 0. Their salary increases with wage inflation and merit increases of 0.5% each year. The salary becomes zero when members retire from the plan. The salary of a member with starting age x and starting time e at time $e + j$, $S(x, e, j)$, is given by

$$S(x,e,j) = \begin{cases} 40000\,(1+0.5\%)^{(x+j-25)} & \text{if } e+j=0 \text{ and } x+j<\text{NRA} \\ 40000\,(1+0.5\%)^{(x+j-25)} \prod_{z=1}^{j} e^{W_{e+z}} & \text{if } e+j>0 \text{ and } x+j<\text{NRA} \\ 0 & \text{otherwise} \end{cases},$$

where W_{e+z} is the wage inflation at time $e + z$ derived from our ESG.

Hence, the total payroll at time t, S_t, can be defined as:

$$S_t = \sum_{x=25}^{NRA-1} S(x,0,t)\,\mathcal{L}(x,0,t) + \sum_{e=1}^{50} S(25,e,t-e)\,\mathcal{L}(25,e,t-e).$$

3.5 Pension plan valuation

In a DB plan, the employee's pension benefit is set in advance. The sponsor and the employees contribute to the pension fund every year based on the current funding level and the cost of future benefits. In our projections, we value the plan every year to determine the appropriate contribution level.

3.5.1 Valuation assumptions

First, we illustrate the assumptions used in the pension plan valuation, which include the various salary assumptions, the valuation rate, and the mortality rate.

We take the actuary's assumed salary scale to match the model described in Section 3.4. Members start working at age 25, with a starting salary of \$40,000. The salary grows with wage inflation (from the ESG) and merit increases of 0.5% every year.

The valuation rate is set based on the expected long-term return of the asset portfolio using the asset allocation from the previous year. Suppose the pension fund invests in four assets: a long-term government bond portfolio, the stock index, private equity, and an investment grade bond portfolio, respectively. Let $\pi_{i,t}, i \in \{1,2,3,4\}$, denote the proportion of these four assets at time t. The valuation rate at time t, g_t, is defined by:

$$g_t = \pi_{1,t-1}\,\bar{R}^{(RF)} + \pi_{2,t-1}\,\bar{R}^{(S)} + \pi_{3,t-1}\,\bar{R}^{(P)} + \pi_{4,t-1}\,\bar{R}^{(IG)}$$

where $\sum_{i=1}^{n} \pi_{i,t} = 1$, and $g_0 = 6\%$. The initial valuation rate, g_0, is set based on the expected long-term return of the pension fund. The factors $\bar{R}^{(RF)}, \bar{R}^{(S)}, \bar{R}^{(P)}$, and $\bar{R}^{(IG)}$ are the expected long-term returns of the long-term government bond portfolio, the stock index, private equity and the investment grade bond portfolio, respectively. Let v_t denote the corresponding discount factor, and then

$$v_t = \frac{1}{1+g_t}.$$

The assumed mortality rates in each valuation match the expected mortality under the model described in Section 3.2. That is, rates follow the CPM 2014 Male Mortality Table with CPM Improvement Scale B.

3.5.2 Valuation method

We use the projected unit credit (PUC) method to value the actuarial liability and normal cost. Under the PUC method, expected future salary increases are considered in the benefits. The projected pension payments are based on projected final salary, which is defined as the salary in the last year before retirement. Let $FS(x,e)$ denote the final salary of members with starting age x and starting time e:

$$FS(x,e) = \begin{cases} S(x,e,\mathrm{NRA}-1-x) & \text{if } x < \mathrm{NRA} \\ 40000\,(1+0.5\%)^{(\mathrm{NRA}-1-25)}\,e^{-(x-\mathrm{NRA}+1)\bar{W}} & \text{if } x \geq \mathrm{NRA} \end{cases},$$

where $\bar{W}$ is the long-term wage inflation rate generated by our ESG. For members who have already retired at time 0, we assume the final salary is based on the current starting salary with merit increases for 39 years adjusted by wage inflation.

The pension plan is valued as a closed group. In other words, we only consider existing members at the valuation date, and do not consider the future new entrants. The actuarial liability of the plan is calculated as the sum of the expected present values of all the accrued benefits as of the valuation date. The normal cost is the difference between the expected present value of the actuarial liability in the next year and the current actuarial liability. The actuarial liability and normal cost are the same for members of the same group who have the same starting age and starting time.

Let $AL(x, e, j)$ be the actuarial liability of a member in the group with starting age x and starting time e, given the member is still alive at time $e + j$, then

$$AL(x, e, j) = \frac{\sum_{z=\max(j,\mathrm{NRA}-x)}^{115-x} 0.02\, s(x, e, j)\, FS(x, e)\, {}_z p_x^{2021+e}\, v_{e+j}^{z-j}}{{}_j p_x^{2021+e}}.$$

Note that the years of service, $s(x, e, j)$, are based on the member's past service since we consider only the accrued benefit in the valuation.

The actuarial liability of the pension plan is the sum of the actuarial liabilities in all the groups. Let AL_t be the actuarial liability of the pension plan at time t given as follows:

$$AL_t = \sum_{x=25}^{115} AL(x, 0, t)\, \mathcal{L}(x, 0, t) + \sum_{e=1}^{50} AL(25, e, t-e)\, \mathcal{L}(25, e, t-e).$$

Let $NC(x, e, j)$ be the normal cost of a member in the group with starting age x and starting time e, at time $e + j$. So

$$NC(x, e, j) = \begin{cases} p_{x+j}^{2021+e+j}\, \widetilde{AL}(x, e, j+1)\, v_{e+j} - AL(x, e, j) & \text{if } x + j < \mathrm{NRA} \\ 0 & \text{if } x + j \geq \mathrm{NRA} \end{cases},$$

where $\widetilde{AL}(x, e, j+1)$ indicates the actuarial liability in the next year, but discounted at the current valuation rate; that is,

$$\widetilde{AL}(x, e, j+1) = \frac{\sum_{z=\max(j+1,\mathrm{NRA}-x)}^{115-x} 0.02\, s(x, e, j+1)\, FS(x, e)\, {}_z p_x^{2021+e}\, v_{e+j}^{z-j-1}}{{}_{j+1} p_x^{2021+e}}.$$

The normal cost of the plan is calculated by adding all the normal costs in all groups. Let NC_t be the normal cost of the pension plan at time t:

$$NC_t = \sum_{x=25}^{115} NC(x, 0, t)\, \mathcal{L}(x, 0, t) + \sum_{e=1}^{50} NC(25, e, t-e)\, \mathcal{L}(25, e, t-e).$$

3.5.3 Funded ratio and contribution requirement

The funded ratio of the pension plan is the ratio of the fund value to the actuarial liability. Let FR_t and F_t be the funded ratio and fund asset value of the pension plan at time t, so

$$FR_t = \frac{F_t}{AL_t}.$$

The plan is said to have a surplus if the funded ratio is larger than one.

Contributions are added to the fund at the beginning of each year to make the funding level be sufficient to pay future pension payments. Let C_t be the contribution to the pension plan at time t. Then

$$C_t = NC_t + \kappa\,(AL_t - F_t),$$

where we set $\kappa = 0.2$. The difference between the actuarial liability and the fund value is the funding shortfall of the plan. Therefore, the contribution equals the normal cost plus an adjustment which is 20% of the funding shortfall.[4] We add the adjustment to smooth the contribution through the projection period.

3.6 The pension fund

3.6.1 Recursive formula for the fund value

In the pension fund, the initial injection equals the actuarial liability at time 0; that is, AL_0. The fund is invested in the financial market by the sponsor. Contributions are added to the fund, and pension payments and administrative costs are deducted from the fund at the beginning of the year. Let $B(x, e, j)$ be the pension payment for a member with starting age x and starting time e at time $e + j$, then

$$B(x, e, j) = \begin{cases} 0.02\, s(x, e, j)\, FS(x, e) & \text{if } x + j \geq \text{NRA} \\ 0 & \text{otherwise} \end{cases} .$$

Let B_t denote the total pension payments made from the pension fund at time t, which is defined as the sum of the pension payments in each group:

$$B_t = \sum_{x=25}^{115} B(x, 0, t)\, \mathcal{L}^r(x, 0, t) + \sum_{e=1}^{50} B(25, e, t-e)\, \mathcal{L}^r(25, e, t-e).$$

Let R_t be the fund return at time t. The recursive formula for the fund value F_t at integer time t is then:

$$F_t = \begin{cases} AL_0, & \text{at } t = 0 \\ (F_{t-1} + C_{t-1} - B_{t-1} - AC_{t-1})\, e^{R_t}, & \text{at } t > 0 \end{cases} . \tag{3.1}$$

[4]In Canada, this adjustment is called a "special payment" when positive. Note that the adjustment is negative when the plan has a surplus; this corresponds to a contribution reduction relative to the normal cost. We do not place any constraints on C_t, so it is possible to have $C_t < 0$, which corresponds to contribution refunds or withdrawals.

3.6.2 The sponsor's asset allocation

The pension fund is invested in the financial market in order to receive enough return to pay for the future pension payments. There will be a deficit if the fund value is lower than the actuarial liability, so the investment strategy is an important concern for the fund sponsor.

Warren (2019) suggested to use a reference-dependent utility function to measure the sponsor's preference over different funded ratios and find the optimal asset allocation of the DB pension fund by maximizing the expected utility in three years' time. The utility function was based on the prospect theory of Kahneman and Tversky (1979) and Tversky and Kahneman (1992), where the utility was determined based on gains and losses relative to a reference point, with losses being penalized more than gains.

We apply the utility function used by Warren (2019) to our pension plan. The utility function is a ratio-based reference-dependent utility function, and is parameterized for relatively high risk aversion in relation to deficits, while attaching a relatively modest value to surpluses. Let U_t be the utility of the funded ratio at time t:

$$U_t = \begin{cases} \gamma\left(\left(\frac{FR_t}{FR_t^*}\right)^{\alpha} - 1\right), & \text{if } FR_t > FR_t^* \\ \lambda\left(\left(\frac{FR_t}{FR_t^*}\right)^{\beta} - 1\right), & \text{if } FR_t < FR_t^* \\ 0, & \text{if } FR_t = FR_t^* \end{cases}, \tag{3.2}$$

where

$$\begin{aligned} FR_t^* &= 0.9, \\ \gamma &= 1, \\ \lambda &= 4.5, \\ \alpha &= 0.11, \\ \beta &= 0.88. \end{aligned}$$

The variable FR_t^* is the reference funded ratio at time t, which is set to be 0.9. In effect, we assume that the sponsor can bear small losses. Parameters α and β are the curvature parameters on surpluses and deficits, respectively, and parameters γ and λ are corresponding weighting parameters. Figure 3.4 shows the utility of the sponsor over funded ratios from 0.5 to 2. There is a high penalty when the fund fails to meet the target, reflecting the fact that the sponsor prefers to avoid deficits more than gaining surpluses. The parameters are same as those used in Warren (2019) except for the parameter α. The lower α we chose decreases the curvature beyond the reference funded ratio.

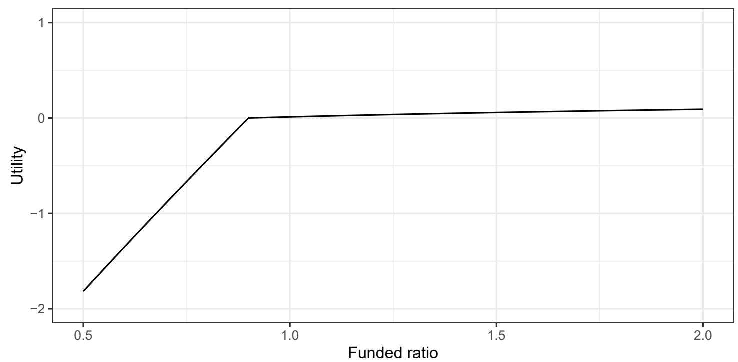


Figure 3.4: **The sponsor's utility over different funded ratios.**

3.6.3 Optimal asset allocation

The pension fund is assumed to invest in four assets: a long-term government bond portfolio, the stock index, private equity, and an investment grade bond portfolio. At time t, we aim to find the optimal allocation among these four assets by maximizing the expected utility of the sponsor three years hence; that is, at time $t+3$. In order to avoid too much volatility in the asset portfolio, we perform the optimization once every three years.

Recall that $\pi_{i,t}$, for $i \in \{1,2,3,4\}$, denotes the proportion of the long-term government bond portfolio, stock index, private equity, and the investment grade bond portfolio we invest in at time t, respectively. The fund return at time t is then defined by

$$R_t = \pi_{1,t-1}\, R_t^{(RF)} + \pi_{2,t-1}\, R_t^{(S)} + \pi_{3,t-1}\, R_t^{(P)} + \pi_{4,t-1}\, R_t^{(IG)},$$

where $\sum\limits_{i=1}^{4} \pi_{i,t} = 1$. The variables $R_t^{(RF)}, R_t^{(S)}, R_t^{(P)}$, and $R_t^{(IG)}$ are the time-t returns of the long-term government bond portfolio, stock index, private equity, and the investment grade bond portfolio generated from our ESG, respectively.

To perform the optimization that updates the asset allocation at time t, we use nested stochastic projections. First, we simulate 10,000 outer loop paths for the asset returns through the period of projection. Along each of these paths, we perform the optimization every three years. The points where the optimization is done are referred to as "node". At each node, we simulate 10,000 distinct paths of asset returns for three years using the ESG; we call these the inner loop. We calculate the utility for each inner loop path at the end of the third year based on the node's state, and estimate the expected utility by taking the average of the utilities across the 10,000 distinct inner loop paths emanating from each node. The optimal asset allocation at time t along a single outer loop path, $\pi_{i,t}$, can be defined as follows:

$$\arg\max_{\pi_{i,t}} E[U_{t+3}] = \arg\max_{\pi_{i,t}} \frac{\sum\limits_{p=1}^{10000} U_{p,t+3}}{10000},$$

where $U_{p,t+3}$ is the utility at time $t+3$ along inner loop path p. Note that in the inner loop simulation, the fund value and actuarial liability are calculated based on the valuation rate at time t and the mortality rate assumed in the valuation; that is, the valuation rate and the mortality rate are deterministic. Specifically, the valuation rate is based on the asset allocation at time $t-1$. The optimal asset allocation along each outer loop path is found by using the *NlcOptim* package in R, which is created for the optimization of nonlinear, constrained objectives.

Chapter 4

Solvency and welfare metrics

The plan sponsor and members are two different groups of stakeholders with divergent needs and concerns. To evaluate the performance of a given pension plan, we need some solvency and welfare metrics relevant to each group of stakeholders mentioned above (i.e., the plan sponsor and members, respectively). In this chapter, we introduce an economic capital-based measure for the plan sponsor, and an expected utility-based measure for the plan members.

4.1 Economic capital-based measure

Economic capital is a risk measure related to capital. Porteous and Tapadar (2005) defined economic capital as the amount of capital required to support the risks a financial service firm is running on an economic or market value basis over a specified time horizon with a prescribed probability. As stated in Porteous and Tapadar (2005), a financial service firm can use economic capital in its operations in many ways, such as checking its capital adequacy, validating its regulatory capital requirements, defining its risk appetite, forecasting economic capital requirements for business planning, and deriving actual rates of return on economic capital as a risk adjusted performance measure.

In banking, economic capital calculation is involved in the second pillars of Basel II and Basel III. Basel II is a supervisory approach which has three pillars; those are, (1) the minimum capital requirements, (2) supervisory review, and (3) market discipline. Basel III further strengthens the capital adequacy requirement of Basel II. The second pillar—supervisory review—is intended to ensure that banks have adequate capital to support all their business's risks, and encourages banks to develop and use better risk management techniques to monitor and manage their risks (see Basel, 2006, for more details). Basel (2009) recommends economic capital frameworks for banks and supervisors in the supervisory review process. As mentioned in Basel (2009), economic capital is measured as an amount of capital that a bank needs to absorb unexpected losses over a certain time horizon at a given confidence level. The risk measure, time horizon, and confidence level of economic capital measures are not prescribed by regulation. Basel (2009) points out that the VaR and ES are the most widely used risk measures. The choice of confidence level is related to banks' target rating, and most banks use a time horizon of one year for economic capital calculation according to a survey done in 2007 (IFRI Foundation and CRO Forum, 2007).

Closer to our pension content, in insurance, the concept of economic capital is involved in the first pillar of Solvency II in Europe. Solvency II is a supervisory approach similar to Basel II with three pillars: (1) quantitative capital requirements, (2) qualitative supervisory review, and (3) market discipline (Asadi and Al Janabi, 2020). The quantitative capital requirements include solvency capital requirements and minimum capital requirements. The solvency capital requirement is the amount of capital that can cover economic capital based on a one-year VaR measure at the 99.5^{th} confidence level, which is valued on a market consistent basis. The minimum capital requirement is lower than the solvency capital requirement, which is measured at the 85^{th} confidence level. It is the trigger amount of the ultimate supervisory intervention (see Solvency II, 2009, for more details). In Canada and the US, economic capital can be used in fulfilling the regulatory requirement of insurers' ORSA. ORSA requires insurers to assess their risk exposure and project future capital needs under normal and stressed environments. Insurers can use their own model, which could include economic capital models, to perform this assessment (Farr et al., 2016).

4.1.1 Economic capital in the pension context

The concept of economic capital has been applied to pension plans in the literature by a few researchers. Porteous et al. (2012) proposed a definition of economic capital for DB plans and applied the Solvency II framework to compute a VaR measure for a UK DB plan. Andrews et al. (2019) updated the work of Porteous et al. (2012) and compared the economic capital of UK DB plans with that of US pension plans by using both the VaR and ES measures for run-off horizons. Andrews et al. (2022) further analyzed the impact of the choice of time horizon and asset allocation on the economic capital of these plans.

We use the following definition of economic capital as stated by Andrews et al. (2022):

Definition 1. The economic capital of a pension scheme is the proportion by which its existing assets would need to be augmented in order to meet net benefit obligations with a prescribed degree of confidence. A pension scheme's net benefit obligations are all obligations in respect of current scheme members, including future service, and net of future contributions to the scheme (Andrews et al., 2022).

In other words, economic capital is the amount of additional assets needed to ensure the solvency of the pension fund, which is consistent with the definition of economic capital in finance and insurance.

4.1.2 Calculation method

We follow the risk measure framework proposed by Andrews et al. (2022) to calculate the economic capital. We start by calculating the present value of "future profit" (PVFP) of the pension plan over a horizon of T years. Let P_t denote the additional surplus emerging in the pension fund at integer time t, and let X_t denote the net cash flow of the pension plan at time t. Let $I_{t-1,t}$ denote the accumulation factor from time $t-1$ to time t, based on the annual random return on the fund assets, R_t, such that

$$I_{t-1,t} = e^{R_t}. \tag{4.1}$$

Then,

$$P_t = \begin{cases} F_0 - AL_0 & \text{at } t = 0 \\ (AL_{t-1} - X_{t-1})\, I_{t-1,t} - AL_t & \text{at } t > 0 \end{cases}, \tag{4.2}$$

where

$$X_t = B_t + AC_t - C_t.$$

The PVFP of the pension plan over a horizon of T years, $V_{0,T}$, is defined by

$$V_{0,T} = \sum_{t=0}^{T} P_t\, D_{0,t}, \tag{4.3}$$

where $D_{0,t}$ is the discount factor from time 0 to time t. That is,

$$D_{0,t} = \prod_{s=1}^{t} I_{s-1,s}^{-1} = \prod_{s=1}^{t} e^{-R_s}, \text{ for } s > 0 \text{ and} \tag{4.4}$$

$$D_{0,0} = 1. \tag{4.5}$$

From Equations (4.1) and (4.4), we can derive that

$$\begin{aligned} I_{t-1,t}\, D_{0,t} &= I_{t-1,t} \prod_{s=1}^{t} I_{s-1,s}^{-1} \\ &= \prod_{s=1}^{t-1} I_{s-1,s}^{-1} \\ &= D_{0,t-1}. \end{aligned} \tag{4.6}$$

Based on Equations (4.5) and (4.6), we can then combine Equations (4.2) and (4.3) to find that the actuarial liabilities over a horizon of $T-1$ years cancel out:

$$V_{0,T} = F_0 - \sum_{t=0}^{T-1} X_t\, D_{0,t} - AL_T\, D_{0,T}. \tag{4.7}$$

Note that when $V_{0,T}$ is negative, the absolute value of $V_{0,T}$ is the amount of excess assets needed by the pension plan. To let $V_{0,T}$ be comparable across pension plans with different asset and liability scales, we denote the standardized PVFP by $V^*_{0,T}$:

$$V^*_{0,T} = \frac{V_{0,T}}{F_0}. \tag{4.8}$$

So, $V^*_{0,T}$ can be interpreted as the proportion by which the initial fund assets, F_0, must be augmented in order to meet net benefit obligations.

Because $V^*_{0,T}$ is a random variable that depends on future projections, we adopt two common risk measures to understand the economic capital risk of the pension plans under study: VaR and ES.

VaR can be interpreted as the worst loss over a target horizon that will not be exceeded with a given confidence level (Jorion, 2006). As mentioned in Linsmeier and Pearson (2000), VaR began to be used in late 1980s to measure the risks of trading portfolios. In 1994, J.P. Morgan established a market standard for VaR, which promoted its growth. VaR is now widely used by financial institutions, nonfinancial corporations, institutional investors, and regulators.

Let VaR_p be the VaR of $V^*_{0,T}$ at confidence level $(1-p)$. Then,

$$P[V^*_{0,T} \leq \mathrm{VaR}_p] = p,$$

where VaR_p indicates the proportion of additional initial assets required at time 0 for the pension plan to meet its net benefit obligations at a confidence level $(1-p)$.

ES is defined as the conditional expectation of losses beyond the VaR level. ES is proposed as an alternative risk measure to VaR as it considers tail risk beyond VaR, and it is a coherent measure, unlike VaR (Yamai and Yoshiba, 2005).

Let ES of $V^*_{0,T}$, ES_p, be the expected value of all standardized losses that are worse than or the same as VaR_p for a given confidence level $(1-p)$. Then,

$$\mathrm{ES}_p = E[V^*_{0,T} | V^*_{0,T} \leq \mathrm{VaR}_p].$$

4.2 Expected utility-based measure

In the expected utility framework, a utility function reflects the preference of individuals and assigns a value to a payoff or consumption amount. Individuals' preferences are typically consistent with maximizing their expected utility. Expected utility theory facilitates our understanding of risk and uncertainty. Trowbridge (1989) pointed out that utility theory can be seen as the philosophical basis for risk management in insurance.

4.2.1 Consumption function

Before calculating our utility-based measure, we first need to define the amount of annual consumption for plan members. We assume that the sponsor and active members each bear half of the annual pension fund contributions. The overall contribution rate of the pension plan at time t, CR_t, is defined by the ratio of the dollar contributions divided as the total payroll:

$$CR_t = \frac{C_t}{S_t}.$$

The time-t contribution rate of the members is therefore given by $\frac{1}{2}CR_t$. The dollar contribution for each active member is obtained by multiplying the contribution rate and their salary in the given year. Let $c(x, e, j)$ be the dollar contribution of active members with starting age x and starting time e at time $e + j$:

$$c(x, e, j) = \frac{1}{2}\, CR_{e+j}\, S(x, e, j).$$

For active members, we further define their consumption in a given year as the difference between their salary and contribution at that time. For retired members, their consumption is the pension payment they receive. Let $\text{Consumption}(x, e, j)$ denote the consumption at time $e + j$ of a member with starting age x and starting time e, so

$$\text{Consumption}(x, e, j) = \begin{cases} S(x, e, j) - c(x, e, j) & \text{if } x + j < \text{NRA} \\ B(x, e, j) & \text{if } x + j \geq \text{NRA} \end{cases},$$

where NRA is the normal retirement age. The above consumption is in nominal terms because it is measured in dollars and does not consider the impact of price inflation. We can deflate consumption by price inflation to make consumption comparable (i.e., expressed in real terms). If $\text{Consumption}'(x, e, j)$ is the inflation-adjusted consumption, then

$$\text{Consumption}'(x, e, j) = \text{Consumption}(x, e, j)\, e^{-\sum_{i=0}^{e+j} Q_i},$$

where $Q_0 = 0$ by construction. Recall that variable Q_i is the i^{th} year price inflation.

4.2.2 Expected discounted utility

For each member, the expected discounted utility is calculated as the average value of the total discounted utility coming from our simulation. Recall that we simulate 10,000 outer loop paths that represent 10,000 possible future paths of pension plan outcomes, including salaries, contributions, and benefit payments. We calculate the total discounted utility along each of these paths by adding up the discounted utility of consumption in each future year along a given path, considering members' subjective time preference and survival probability over the specific period.

Let $u(\cdot)$ be the members' utility function, and let $U(x, e)$ denote the total discounted utility of consumption over the period $[0, T]$ for a member with starting age x and starting time e. Then the expected discounted utility for such a member is given by the following equation:

$$E[U(x,e)] = E\left[\sum_{t=0}^{T} \beta^t \, u(\text{Consumption}'(x,e,t-e)) \, {}_{t-e}\hat{p}_x^{2021+e}\right],$$

where

$${}_{t-e}\hat{p}_x^{2021+e} = \frac{\mathcal{L}(x,e,t-e)}{\mathcal{L}(x,e,0)}.$$

Parameter β represents members' subjective time preference for the utility and is set to $e^{-0.02}$ in this report. The variable ${}_{t-e}\hat{p}_x^{2021+e}$ stands for the realized member's survival probability.

We choose a power utility function with constant relative risk aversion; that is,

$$u(\chi) = \frac{\chi^{1-\gamma}}{1-\gamma}, \quad \gamma \geq 0, \ \gamma \neq 1, \tag{4.9}$$

where χ is consumption and γ represents the constant relative risk aversion level. Here, we let $\gamma = 5$.

4.2.3 Certainty equivalent consumption

To compare the expected discounted utility across different pension plans in a more meaningful way, we adopt the concept of CEC. CEC is defined by the condition that the member is indifferent between receiving a stream of variable consumption or fixed amount of CEC over the period (Denuit et al., 1999). The CEC for a member with starting age x and starting time e, $\text{CEC}(x, e)$, is obtained by solving the following equation:

$$E[U(x,e)] = \sum_{t=0}^{T} \beta^t \, u(\text{CEC}(x,e)) \, {}_{t-e}p_x^{2021+e}, \tag{4.10}$$

where the variable ${}_{t-e}p_x^{2021+e}$ is the expected survival probability derived from the mortality table used in Section 3.2.

By combining Equations (4.9) and (4.10), we obtain a closed-form expression for the $\text{CEC}(x, e)$:

$$\begin{aligned} E[U(x,e)] &= \sum_{t=0}^{T} \beta^t \, \frac{(\text{CEC}(x,e))^{1-\gamma}}{1-\gamma} \, {}_{t-e}p_x^{2021+e} \\ &= \frac{\text{CEC}(x,e)^{1-\gamma}}{1-\gamma} \sum_{t=0}^{T} \beta^t \, {}_{t-e}p_x^{2021+e} \end{aligned}$$

$$\Rightarrow \text{CEC}(x,e) = \left((1-\gamma) \frac{E[U(x,e)]}{\sum_{t=0}^{T} \beta^t \, {}_{t-e}p_x^{2021+e}} \right)^{\frac{1}{1-\gamma}}.$$

Chapter 5

Simulation study: Base case

We conduct a simulation study to evaluate the performance of our stylized pension plan based on the economic and financial framework presented in Chapter 2. This chapter presents the key findings from the simulated results regarding the evolution of the pension plan and the sponsor's and members' solvency and welfare metrics introduced in Chapter 4.

5.1 Simulated results

In the simulation study, as mentioned in Section 3.6.3, we generate 10,000 outer loop paths of the asset return, price inflation, and wage inflation scenarios based on the ESG. In addition, we generate 10,000 inner loop paths every three years along each outer loop scenario to obtain the optimal asset allocation.

5.1.1 Initial funded ratio and asset allocation

To understand the asset allocation dynamics, we first study the relationship between the initial funded ratio and the asset allocation. To simplify, we only allow for investment in a risky asset (i.e., the total stock index), and a less risky asset (i.e, the investment grade bond portfolio) in our fund for now. The other assumptions and features remain the same. Figure 5.1 plots the optimal weight of the total stock index under different initial funded ratios based on a single path at time 0. The plot is U-shaped; the optimal total stock index weight is lowest when the initial funded ratio is near 1.13. At this initial funded ratio, investing more assets in the investment grade bond portfolio should allow the plan to reach a decent funded ratio three years hence with a low probability of losses (or funded ratio less than 0.9). When the initial funded ratio is lower than 1.13, we need to invest more in the risky assets to increase the probability of gains over the next three years. On the other hand, when the initial funded ratio is higher than 1.13, the fund is less likely to fall below the target in the next three years, so the plan can invest more in the total stock index to increase its utility.

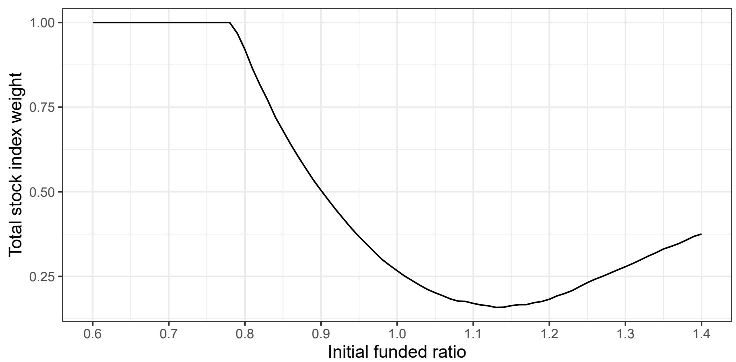


Figure 5.1: **Total stock index weight for different initial funded ratios.**
Notes: We assume the fund is invested only in the total stock index and the investment grade bond portfolio. The other assumptions and features remain the same.

5.1.2 Evolution of the asset allocation, valuation rate, and funded ratio

To study the evolution of our pension plan, we plot funnels of doubt for the asset allocation, the valuation rate, the funded ratio, the normal cost rate, and the contribution rate in Figure 5.2. Panels A to D of Figure 5.2 show the distribution of asset weights over the 50 year horizon. Given an initial funded ratio of 1, an initial valuation rate of 6%, and a reference funded ratio of 0.9, the initial optimal asset allocation has a high proportion in the investment grade bond portfolio because more weight in the less risky asset increases the probability that the funded ratio exceeds the target. The variability of the asset allocation distribution becomes larger starting in year 3 because the optimal asset allocation changes as a function of the various economic scenarios simulated at that time, unlike the time-0 allocation that depends only on current (known) values. Asset allocations change sharply after year 3; this is caused by a feedback effect between the asset allocation, the valuation rate, and the funded ratio, which will be discussed in detail in the next section.

In the long run, most of the assets are allocated to the investment grade bond portfolio, followed by the total stock index and private equity. Few assets are allocated to the long-term government bond portfolio, all in all. Because the plan sponsor is risk averse, more assets are allocated to the less risky assets to help to secure a safe fund in the long run. Recall from Figure 2.2 that the investment grade bond portfolio has an expected return that is higher than that of the long-term government bond portfolio; their 5^{th} percentile are similar, however. The investment grade bond portfolio can help the plan sponsor seek higher returns without increasing risk too much. So, the investment grade bond portfolio is preferred to the long-term government bond portfolio, generally speaking. The total stock index has higher weight than private equity, which is consistent with the fact that private equity has a higher average return but is more volatile than the total stock index.

Panel E of Figure 5.2 shows the evolution of the distribution of the valuation rate. As mentioned in Section 3.5.1, the valuation rate is set to the weighted average of the expected

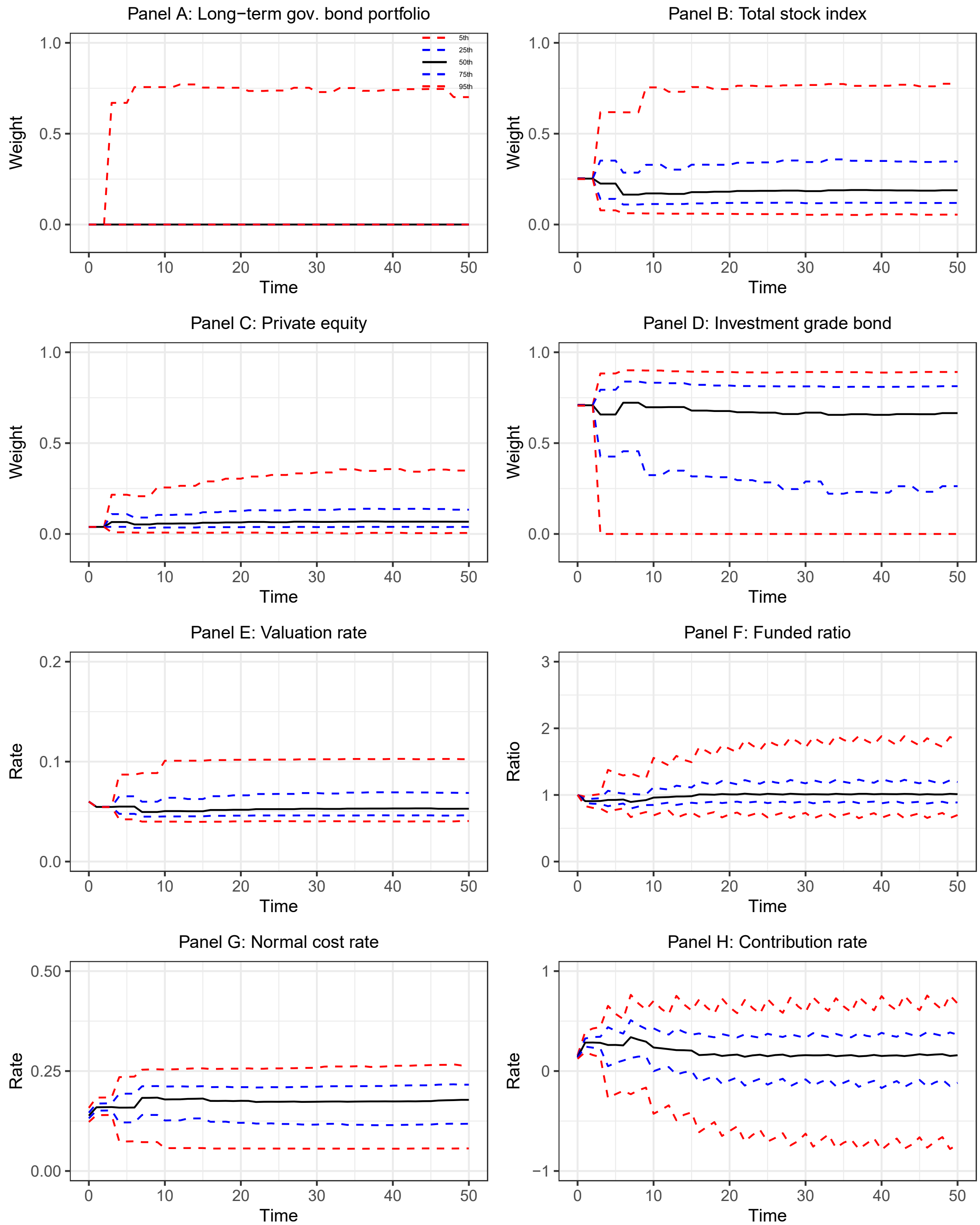


Figure 5.2: **The 5$^{\text{th}}$, 25$^{\text{th}}$, 50$^{\text{th}}$, 75$^{\text{th}}$ and 95$^{\text{th}}$ percentiles of the asset allocation, the valuation rate, the funded ratio, the normal cost rate, and the contribution rate over the projection period.**

Notes: The red dashed lines show the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles, the blue dashed lines show the 25$^{\text{th}}$ and 75$^{\text{th}}$ percentiles, and the black solid line shows the median of the asset allocation, the valuation rate, the funded ratio, the normal cost rate, and the contribution rate.

long-term asset returns based on the previous year's asset weights. So, the evolution of the valuation rate is consistent with the evolution of the asset weights. The distribution of the valuation rate changes significantly in the first ten years and then becomes stable in the long run.

Panel F of Figure 5.2 shows the evolution of the distribution of the funded ratio. The median of the funded ratio increases gradually over the projection period despite the first several years exhibiting an oscillatory behaviour. As time passes, the median funded ratio will be around one and reaches a stable state, together with the asset allocation and valuation rate.

The distribution of the funded ratio has low dispersion in the first three years. The funded ratio's dispersion only comes from the uncertainty in asset returns during this period because the asset weights remain constant in the simulation over the first three years. After year 3, the asset allocation varies for each scenario based on the different valuation rates and funded ratios, so the dispersion of the funded ratio becomes larger.

5.1.3 Oscillations and feedback in the asset allocation, valuation rate, and funded ratio

The feedback loop in the asset allocation, the valuation rate, and the funded ratio mentioned earlier can be described as follows. The funded ratio influences the asset allocation in the optimization. The asset weights affect the valuation rate used to discount the actuarial liability which, in turn, impacts the funded ratio. The feedback loop occurs at the beginning of our pension plan's operation; during this time, the asset allocation, the valuation rate, and the funded ratio transition together from their initial values to their steady states.

We can further understand the feedback mechanism by looking at the outcomes of a single scenario. Figure 5.3 shows the asset allocation, the valuation rates, and the funded ratio from a single path of the simulation. Based on the initial settings, the investment grade bond portfolio has a higher weight. At time 1, the valuation rate decreases based on the previous year's asset allocation. The lower valuation rate increases the actuarial liability, which reduces the funded ratio. The funded ratio remains at a low level in the next two years because the portfolio will not be optimized during that time, so the asset allocation and valuation rate will remain unchanged. At time 3, a higher weight is allocated to the risky asset according to Figure 5.1 because the funded ratio is lower than the time-0 level. So, the total stock index and private equity weights increase, and the investment grade bond portfolio weight decreases. Then, the feedback occurs again, but the valuation rate, the funded ratio, and the asset weights change in the opposite direction. In general, as shown in Panels A to F of Figure 5.2, after about three of these feedback loops, the valuation rate, the asset allocation, and the funded ratio reach their steady states, and their medians stay stable in the long run.

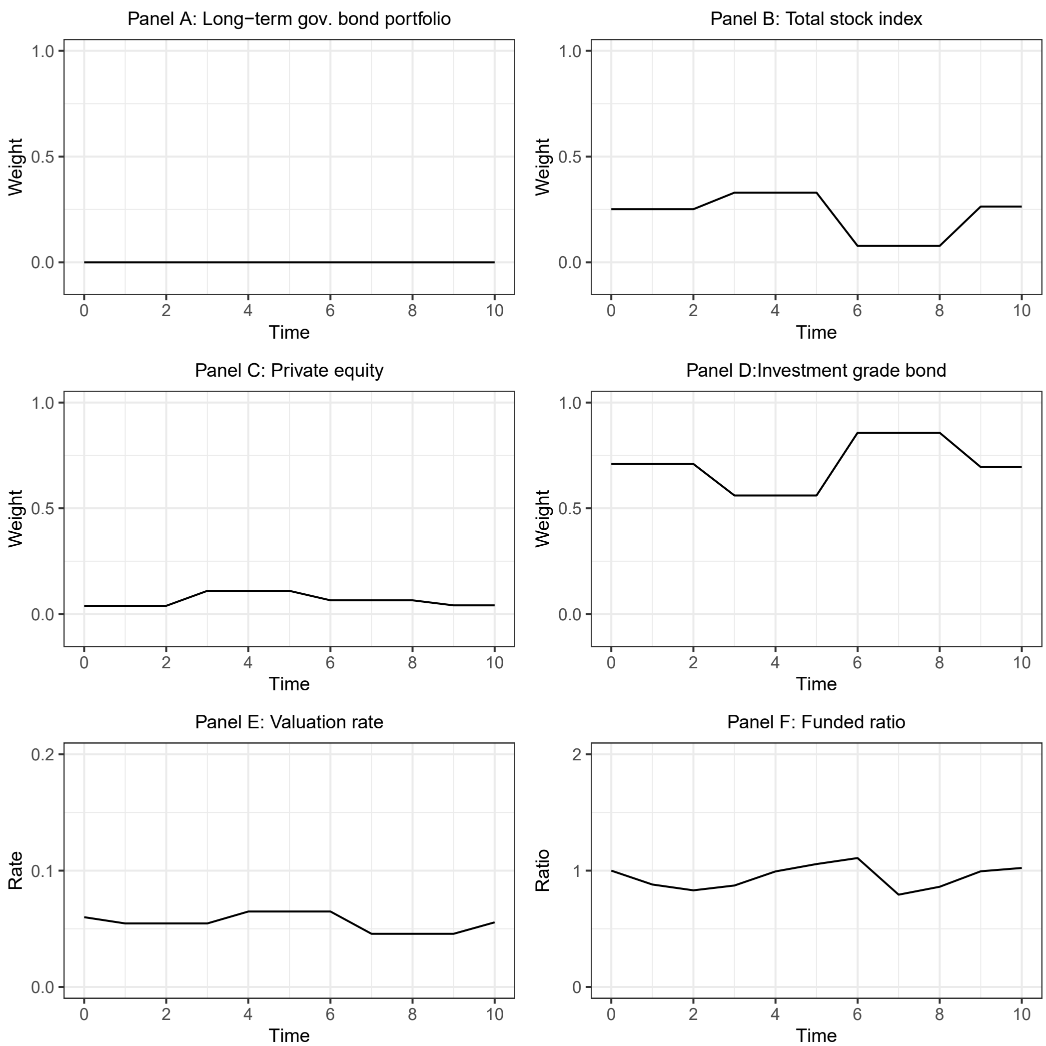


Figure 5.3: **The asset allocation, the valuation rate, and the funded ratio in first ten years of a single path.**

5.1.4 Evolution of normal cost rate and contribution rate

Panels G and H of Figure 5.2 show the distributions of the normal cost rate and the contribution rate. Similar to the asset allocation, the valuation rate, and the funded ratio, the normal cost rate and the contribution rate both change sharply in the first ten years.

Changes in the normal cost rate are caused by changes in the valuation rate. Similar to the contribution rate, the normal cost rate is the ratio of the normal cost to the total salary. The normal cost measures the difference between the expected present value of next year's actuarial liability and the current actuarial liability. If the valuation rate decreases, the actuarial liability increases which, in turn, increases the normal cost (and vice versa). After ten years, the normal cost rate becomes stable because the valuation rate also becomes stable.

The contribution rate is related to the normal cost rate and the funded ratio because the contribution is the sum of the normal cost and an adjustment for any funding shortfalls. If there is a deficit in the fund, our smoothing mechanism increases the contributions. So, in the first ten years, we see the contribution rates behaving similar to the normal cost rates but are higher than the normal cost rates. After ten years, the contribution rate decreases because the funded ratio increases.

5.2 Simulated results of solvency and welfare metrics

Table 5.1 shows results for the economic capital-based measures, which include the probability of running a deficit at years 3 and 50, and the economic capital based on VaR and ES measures at the 50$^{\text{th}}$, 90$^{\text{th}}$, and 99.5$^{\text{th}}$ confidence levels for three-year and 50-year horizons. The economic capital measures for a three-year horizon are much worse than those for the 50-year horizon at the 50$^{\text{th}}$ and 90$^{\text{th}}$ confidence level because of the generally low funded ratio at year 3. This is consistent with the 97% probability of running a deficit at year 3 and the 39% probability of running a deficit at year 50. The economic capital measures at the 99.5$^{\text{th}}$ confidence level are better at the three-year horizon than those at the 50-year horizon because the dispersion of funded ratio is higher after 50 years than after three years. In other words, we expect more extreme funded ratios in the long run.

Table 5.1: **Economic capital-based measure results for the base case.**

		50$^{\text{th}}$ confidence level		90$^{\text{th}}$ confidence level		99.5$^{\text{th}}$ confidence level	
Base case	$P[V_{0,T}^{*} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Three-year horizon	0.92	−0.09	−0.16	−0.20	−0.27	−0.38	−0.43
50-year horizon	0.48	0.00	−0.07	−0.10	−0.24	−0.55	−1.47

Notes: This table reports the probability of the initial fund assets being insufficient, as well as the VaR and ES of $V_{0,T}^{*}$ at the 50$^{\text{th}}$, 90$^{\text{th}}$, and 99.5$^{\text{th}}$ confidence levels, for both three-year and 50-year horizons.

Figure 5.4 shows the CECs for the existing members at time 0 across different ages, which are calculated based on their (remaining) whole life consumption. The CECs decrease with age: younger members work longer and enjoy more years of wage inflation (net of price inflation), meaning their final salary and pension payments should be higher in real terms because wage inflation tends to be higher than price inflation. The younger members are therefore expected to have higher consumption when they retire, which contributes to their higher CEC values. For the retired members, the older (retired) members have lower pension payments because they retired earlier and their final salary was lower at that time. So, the CECs decrease with age for retired members.

Table 5.2 reports the CECs of members aged 25, 45, 65, and 85 for the base case. The bottom row of the table shows the average consumption of these members in their remaining lifetime. These values are calculated based on the member's median contribution rate in the long term, which is 15%, the long-term average price inflation rate, and the long-term average wage inflation rate. In general, the CEC values make sense because they are close to the average consumption levels. In fact, a CEC of \$38,281 for a member aged 25 means that the member's expected discounted utility of receiving a stream of salary (net of contribution) or pension payment is equivalent to that of receiving a fixed amount of \$38,281 in real terms in each year during his remaining lifetime.

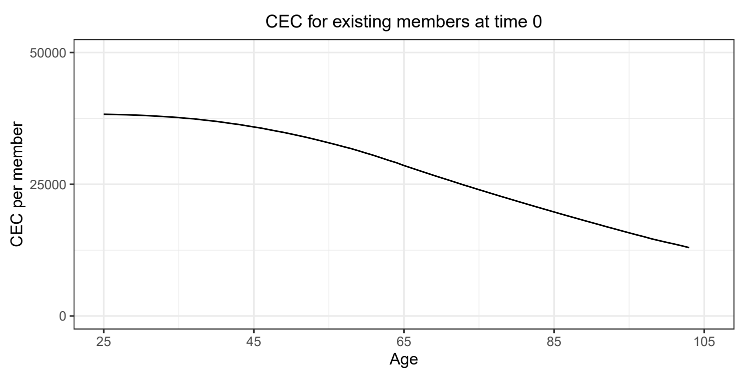


Figure 5.4: **Certainty equivalent consumptions of existing members at time 0 for different ages.**

Table 5.2: **Certainty equivalent consumption and average consumption for members of different ages.**

Age	25	45	65	85
Certainty equivalent consumption	38,281	35,872	28,552	19,745
Average consumption	40,071	32,971	23,742	16,228

Notes: The average consumption is based on the member's (remaining) whole life consumption and calculated by applying the member's contribution rate of 15%, the long-term average price inflation rate, and the long-term average wage inflation rate.

Chapter 6

Simulation study: Robustness tests

To test the robustness of our pension plan modelling, we change some of the assumptions and redo the simulations. This chapter describes the different robustness tests and shows the results. When analyzing the results, we focus on the changes in the solvency and welfare metrics; that is, the economic capital-based measures and the CECs.

6.1 Factors impacting the solvency and welfare metrics

First, to analyze the changes in solvency and welfare metrics more concisely, we list the factors that might influence solvency and welfare metrics in our robustness tests.

Economic capital depends on the funded ratio, fund returns, and asset allocation. Recall from Equation (4.7), the PVFP of the pension plan is expressed as

$$V_{0,T} = F_0 - \sum_{t=0}^{T-1} X_t \, D_{0,t} - AL_T \, D_{0,T}. \tag{6.1}$$

The first two terms on the right side of the equation are the present value of the end-of-horizon fund value. We can derive this result by multiplying the accumulation factor from time 0 to time T, $I_{0,T}$, with these two terms:

$$\begin{aligned} \left(F_0 - \sum_{t=0}^{T-1} X_t \, D_{0,t}\right) I_{0,T} &= \left(F_0 - X_0 \, D_{0,0} - \sum_{t=1}^{T-1} X_t \, D_{0,t}\right) \prod_{t=1}^{T} I_{t-1,t} \\ &= \left(I_{0,1} \, (F_0 - X_0 \, D_{0,0}) - I_{0,1} \sum_{t=1}^{T-1} X_t \, D_{0,t}\right) \prod_{t=2}^{T} I_{t-1,t}. \end{aligned} \tag{6.2}$$

Recall from Equation (4.4) that we can derive the following relationship:

$$I_{s,t} \, D_{s,T} = I_{s,t} \prod_{z=s+1}^{T} I_{z-1,z}^{-1}$$

$$
\begin{aligned}
&= \prod_{z=t+1}^{T} I_{z-1,z}^{-1} \\
&= D_{t,T}. \qquad (6.3)
\end{aligned}
$$

Using Equations (6.3) and (3.1), and the fact that $D_{t,t} = 1$ in Equation (6.2), then we get

$$
\begin{aligned}
&\left(I_{0,1}\ (F_0 - X_0\, D_{0,0}) - I_{0,1} \sum_{t=1}^{T-1} X_t\, D_{0,t} \right) \prod_{t=2}^{T} I_{t-1,t} \\
&= \left(I_{0,1}\ (F_0 - X_0) - I_{0,1} \sum_{t=1}^{T-1} X_t\, D_{0,t} \right) \prod_{t=2}^{T} I_{t-1,t} \\
&= \left(F_1 - \sum_{t=1}^{T-1} X_t\, D_{1,t} \right) \prod_{t=2}^{T} I_{t-1,t} \\
&= \left(F_1 - X_1\, D_{1,1} - \sum_{t=2}^{T-1} X_t\, D_{1,t} \right) \prod_{t=2}^{T} I_{t-1,t} \\
&= \left(I_{1,2}\ (F_1 - X_1) - I_{1,2} \sum_{t=2}^{T-1} X_t\, D_{1,t} \right) \prod_{t=3}^{T} I_{t-1,t} \\
&= \left(F_2 - \sum_{t=2}^{T-1} X_t\, D_{2,t} \right) \prod_{t=3}^{T} I_{t-1,t} \\
&= \quad \cdots \\
&= (F_{T-1} - X_{T-1})\ I_{T-1,T} \\
&= F_T.
\end{aligned}
$$

So, the standardized PVFP can be expressed as

$$
\begin{aligned}
V_{0,T}^{*} &= \frac{V_{0,T}}{F_0} \\
&= \frac{F_T / I_{0,T} - AL_T\, D_{0,T}}{F_0} \\
&= \frac{F_T - AL_T}{F_0}\, D_{0,T} \\
&= (FR_T - 1)\ \frac{AL_T}{F_0}\, D_{0,T}. \qquad (6.4)
\end{aligned}
$$

Because the initial fund value is known, the standardized PVFP is determined by the funded ratio at time T, the actuarial liability at time T, and the discount factor between time 0 and time T. With a higher funded ratio at the end of the horizon, the standardized PVFP increases, which can improve the economic capital results. The actuarial liability at time T, AL_T, is impacted by the valuation rate at that time, which depends on the asset allocation. The discount factor, $D_{0,T}$, is based on the fund return. When the funded ratio is less than

1, the standardized PVFP and economic capital results are improved as the AL_T and $D_{0,T}$ decrease.

The CEC is determined by the expected discounted utility of consumption. High expected discounted utility leads to high CECs. For active members, the consumption is related to the contribution rate. A low contribution rate increases the consumption, and further increases the expected discounted utility. Since the power utility function is concave, the utility increases more slowly as the consumption increases. So, an increase from a lower consumption level leads to a more substantial increase in utility than the same increase from a higher consumption level. In other words, a decrease in the contribution rate at a high percentile—e.g., the 95^{th} percentile—increases utility more than the same decrease in the contribution rate at a low percentile—e.g., the 5^{th} percentile.

For retired members, consumption equals the pension payment, so, their CECs change with the pension payment. In our robustness tests, the CECs for a given cohort of retired members do not change because none of our tests change the assumptions regarding salary and years of service. So, the payments received after retirement will not change from one robustness test to the next.

The description of each case and the analysis of the changes in solvency and welfare metrics are provided in the following sections. Because the asset allocation determines the fund return and greatly impacts the funded ratio and contribution rate, which further influence the solvency and welfare metrics, we start by analyzing the changes to the asset allocation process in the following sections. Afterwards, we consider changes to the plan membership and plan design.

6.2 Change in the asset optimization frequency

6.2.1 Description

In this robustness test, we change the frequency of the asset optimization from every three years to the following:

Case 1. Optimize asset mix every year.

Case 2. Optimize asset mix every six years.

Case 3. Optimize asset mix only once, at time 0; that is, have a static asset mix.

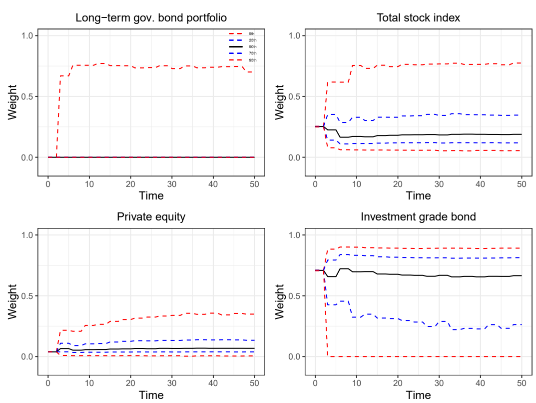
In the base case, the asset mix is optimized every three years. If we optimize the portfolio more frequently, the fund performs better, but it also becomes more volatile. If we optimize the portfolio less frequently, the fund is more stable but performs worse.

6.2.2 Impact on asset allocation

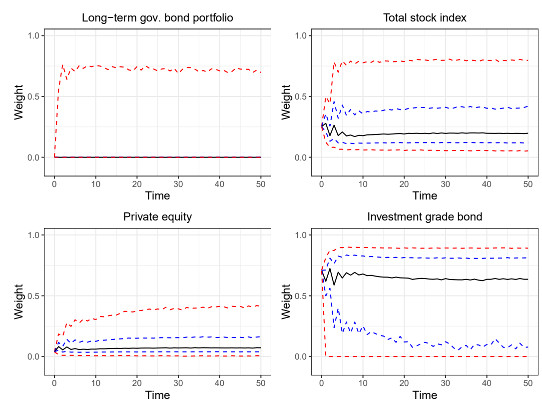
Figure 6.1 shows the asset allocation results under the base case and different optimization frequencies. When we increase the optimization frequency, the median asset weights are more volatile in the first 10 years because the optimization occurs more frequently, and the fund has not yet reached its steady state value. The total stock index and private equity weights increase and the investment grade bond portfolio weight decrease slightly. The range of the total stock index and private equity weights increases when the asset allocation is changed frequently.

By contract, when we decrease the asset optimization frequency, the median asset weights become more stable in the first 10 years. The median asset allocation stays nearly unchanged in the long term. The range of total stock index and private equity weights also tends to decreases in this case.

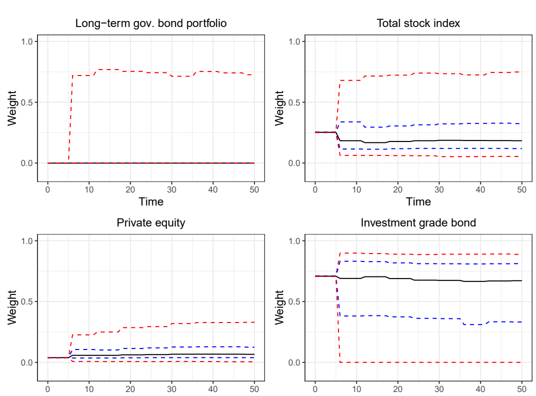
Under static optimization, the asset allocation stays constant at the initial allocation. The total stock index and investment grade bond portfolio weights are higher than those of the base case.



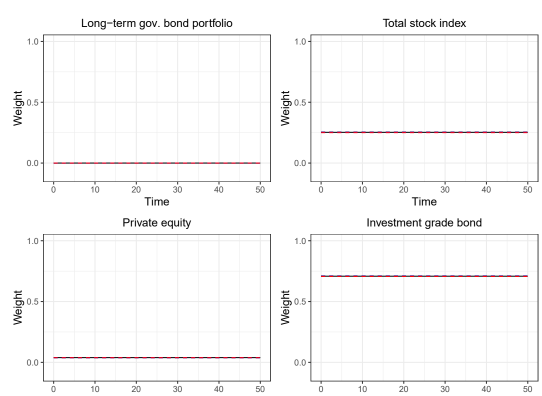
(a) Base case



(b) Increased optimization frequency



(c) Decreased optimization frequency



(d) Static optimization

Figure 6.1: **Asset allocations for different asset optimization frequencies.**

6.2.3 Impact on economic capital

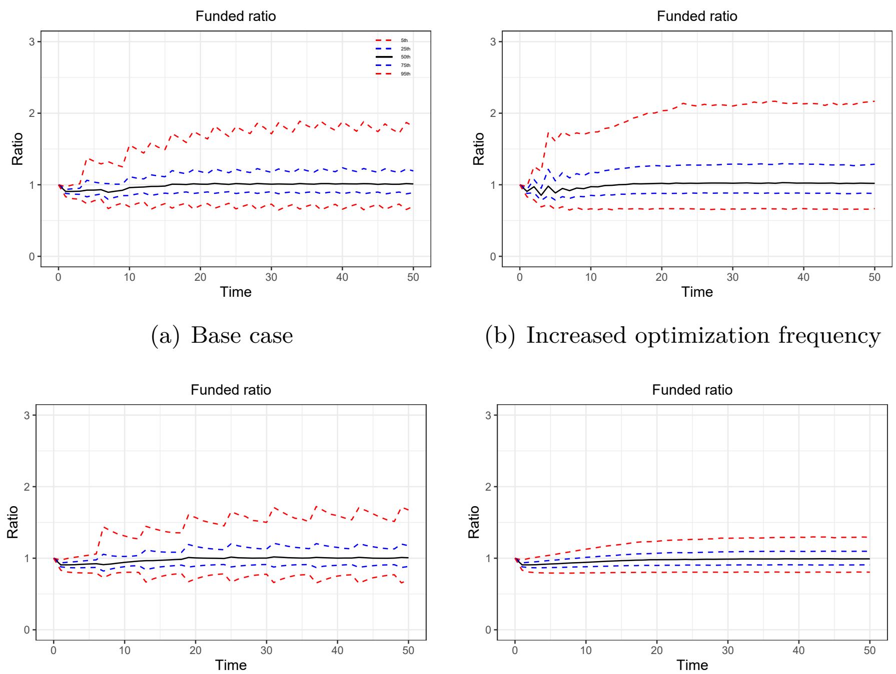
Table 6.1 reports economic capital results for different optimization frequencies. We find that, when we increase the optimization frequency, the economic capital measures at the three-year horizon deteriorate. Panel (b) of Figure 6.2 shows the funded ratio distributions when we increase the optimization frequency. The median funded ratio at year 3 is lower than that of the base case, which also worsens the economic capital at year 3. The lower funded ratio is mainly caused by the oscillations of asset allocation in the first several years. For the 50-year horizon, economic capital is generally improved except for the VaR at the 99.5^{th} confidence level. The VaR is worse because the higher total stock index and private equity weights leads to lower fund returns in the left tail. However, because of more frequent optimization, the asset allocation can be adjusted quickly when the funded ratio is low, so the extreme scenarios are improved, leading to better ES at the 99.5^{th} confidence level.

Table 6.1: **Economic capital results for different asset optimization frequencies.**

Panel A: Three-year horizon							
		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V_{0,3}^* \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 1	0.82	-0.16	-0.28	-0.35	-0.44	-0.60	-0.66
Case 2	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 3	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Panel B: 50-year horizon							
		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V_{0,50}^* \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.48	0.00	-0.07	-0.10	-0.24	-0.55	-1.47
Case 1	0.46	0.00	-0.07	-0.09	-0.24	-0.63	-1.30
Case 2	0.49	0.00	-0.08	-0.10	-0.26	-0.58	-1.63
Case 3	0.52	0.00	-0.08	-0.12	-0.24	-0.51	-0.67

As we decrease the optimization frequency in Case 2, economic capital is unchanged for the three-year horizon because the asset allocation is the same as in the base case at year 3. Economic capital measures for the 50-year horizon deteriorate, however. Because the asset allocation is changed only every six years in this case, it cannot adjust immediately when the funded ratio is low, making the economic capital worse at high confidence levels.

Economic capital measures under static optimization yield similar results to the base case. Yet, we see an improvement at the 99.5^{th} confidence level for a 50-year horizon when compared to the base case. Panel (d) of Figure 6.2 shows the funded ratio results under static optimization. The volatility in the funded ratio decreased and the tail of the funded ratio is improved, which increases the economic capital overall.



(a) Base case

(b) Increased optimization frequency

(c) Decreased optimization frequency

(d) Static optimization

Figure 6.2: **Funded ratios for different asset optimization frequencies.**

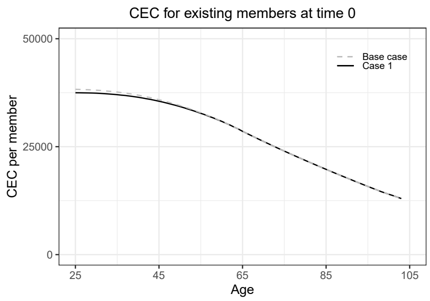
6.2.4 Impact on certainty equivalent consumptions

Figure 6.3 compares CECs for Case 1, 2, and 3, against the base case. The dashed line in the figure represents the base case CEC, and the solid line represents the CECs of the robustness cases.

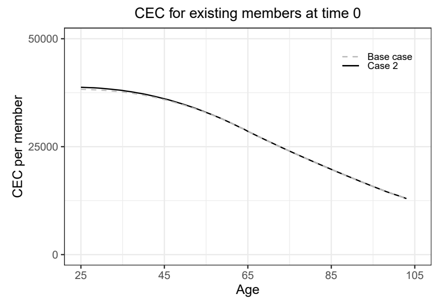
For active members, the CECs decrease when we increase the optimization frequency. The funded ratio allows for a wider range of values because of higher allocations to the total stock index and private equity, which translates to wider range for the contribution rate. The contribution rate becomes higher in the right tail of the distribution, making utility much worse. Therefore, the expected utility and CECs decrease.

On the other hand, the CECs increase when we decrease the optimization frequency. The funded ratio falls in a narrower range, on account of lower volatility of fund return due to less allocation to the total stock index and private equity. This also results in a less volatile contribution rate. Compared to the base case, contribution rates are lower in the right tail of the distribution, yielding higher utility overall. Therefore, the expected utility and CECs also increase.

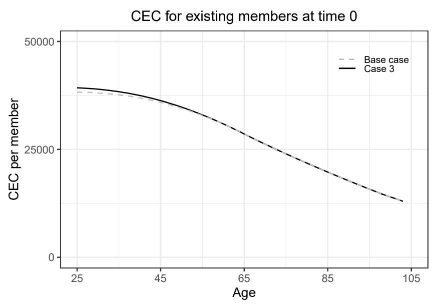
The CECs increase under the static optimization. The funded ratio has a much narrower range than that of the base case because of the constant asset allocation, leading to a much narrower range for the contribution rates. Similar to Case 2, the expected utility and CECs increase because contribution rates are lower in the right tail of the distribution.



(a) Increased optimization frequency



(b) Decreased optimization frequency



(c) Static optimization

Figure 6.3: **Certainty equivalent consumptions for different asset optimization frequencies.**

6.3 Change in the reference funded ratio in the asset optimization

6.3.1 Description

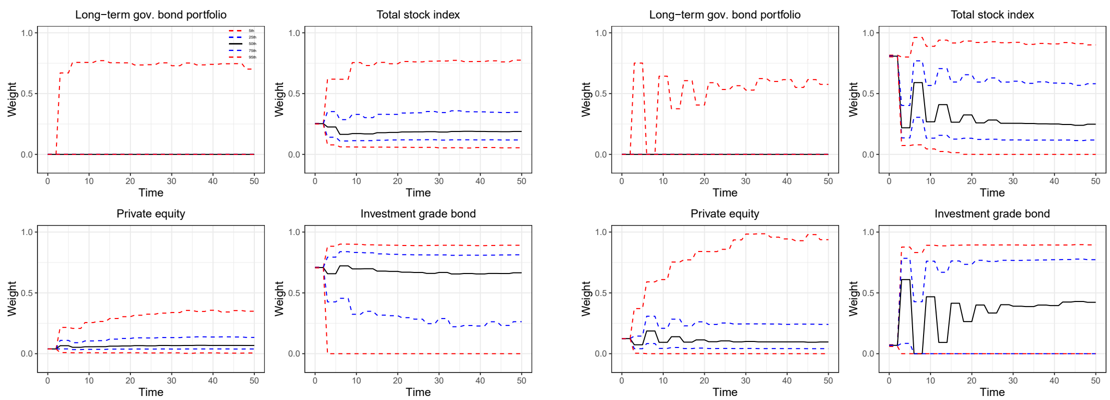
In this robustness test, we change the reference funded ratio in the utility function of Equation (3.2) from 0.9 to 1:

Case 4. Increase the reference funded ratio from 0.9 to 1.

We set the reference funded ratio to 0.9 in the base case because we assumed that the sponsor can bear small losses. We increase the reference funded ratio to one to test the performance of the pension plan when the sponsor is more risk averse and does not want to bear small losses.

6.3.2 Impact on asset allocation

Figure 6.4 shows the base case asset allocation results and the results when increasing the reference funded ratio. Based on the higher reference funded ratio of 1, the plan allocates a higher proportion of its assets to the total stock index and private equity to increase the funded ratio and the corresponding utility. The median asset weights become more volatile, with wider oscillations in the first 30 years because the initial asset allocations are far from the asset weights reached in the steady state. This is because the feedback in the asset allocation, valuation rate, and funded ratio becomes stronger in this case.



(a) Base case (b) Increased reference funded ratio

Figure 6.4: **Asset allocations when increasing the reference funded ratio.**

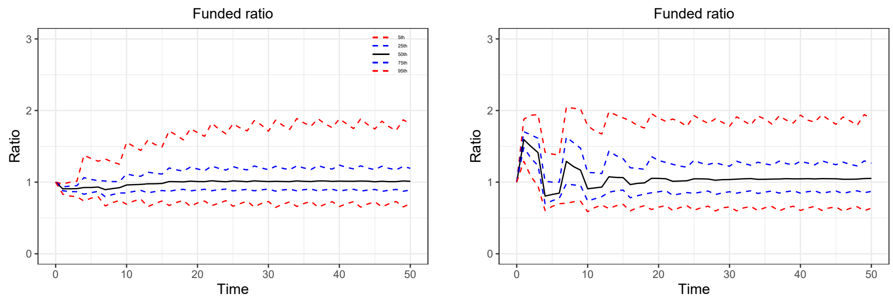
6.3.3 Impact on economic capital

From Table 6.2, we find that the Case 4 economic capital measures are improved for the three-year horizon at the 50$^{\text{th}}$ and 90$^{\text{th}}$ confidence level, but are worse at the 99.5$^{\text{th}}$ confidence level. Figure 6.5 shows the funded ratio results when increasing the reference funded ratio. Compared to the base case, the median funded ratio at year 3 is higher because of the higher total stock index and private equity weights in the asset portfolio. However, the significant allocation to the total stock index and private equity also increases the volatility of the funded ratio. The funded ratio is lower at high confidence levels, meaning that the economic capital is lower at the 99.5$^{\text{th}}$ confidence level.

Table 6.2: **Economic capital results when increasing the reference funded ratio.**

Panel A: Three-year horizon							
		50$^{\text{th}}$ confidence level		90$^{\text{th}}$ confidence level		99.5$^{\text{th}}$ confidence level	
Cases	$P[V^*_{0,3} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 4	0.07	0.22	0.11	0.04	-0.09	-0.38	-0.54
Panel B: 50-year horizon							
		50$^{\text{th}}$ confidence level		90$^{\text{th}}$ confidence level		99.5$^{\text{th}}$ confidence level	
Cases	$P[V^*_{0,50} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.48	0.00	-0.07	-0.10	-0.24	-0.55	-1.47
Case 4	0.42	0.00	-0.07	-0.04	-0.30	-1.19	-2.98

Similarly, for the 50-year horizon, the probability of insufficient initial fund assets decreases, but the economic capital measures deteriorate at the 90^{th} confidence level for ES and the 99.5^{th} confidence level for both VaR and ES. As mentioned above, more investments in the total stock index and private equity improve the median funded ratio but also increases the volatility of the funded ratio, which makes the economic capital worse at high confidence levels.



(a) Base case

(b) Increased reference funded ratio

Figure 6.5: **Funded ratios when increasing the reference funded ratio.**

6.3.4 Impact on certainty equivalent consumptions

As shown in Figure 6.6, the CECs increase slightly for active members when increasing the reference funded ratio. This result is explained by higher funded ratios, which lead to lower contribution rates in Case 4. The lower contribution rate increases consumption. Therefore, the expected utility increases, which contributes to the higher CECs.

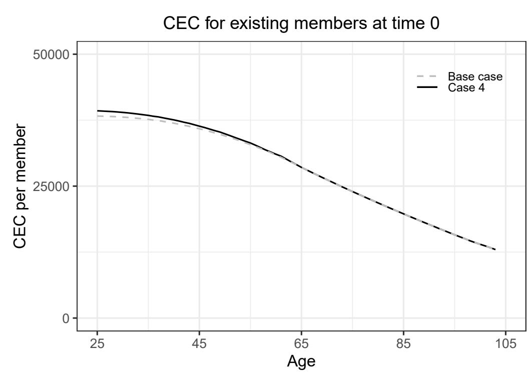


Figure 6.6: **Certainty equivalent consumptions when increasing the reference funded ratio.**

6.4 Change in the initial valuation rate

6.4.1 Description

In this robustness test, we change the initial valuation rate of 6% to:

Case 5. An initial valuation rate of 9%.

Case 6. An initial valuation rate of 3%.

In the base case, the initial valuation rate of 6% is an approximation to the long-term weighted average return on the fund assets. When the initial valuation rate deviates from the long-term weighted average return, the asset weights and funded ratio might be affected. Therefore, we increase and decrease the initial valuation rate, respectively.

6.4.2 Impact on asset allocation

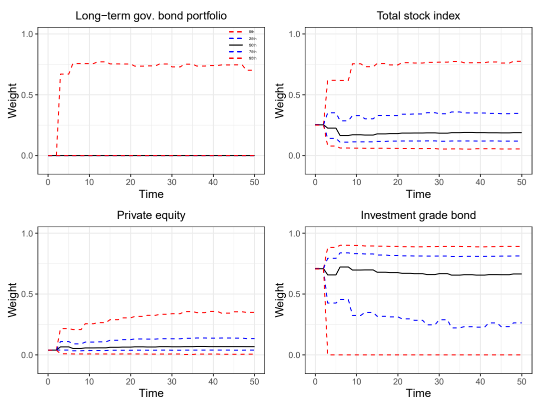
Figure 6.7 shows the asset allocations for different initial valuation rates. In general, both Cases 5 and 6 have similar long term asset allocations when compared to the base case. When the initial valuation rate is set to 9%, the total stock index and private equity weights are higher at the beginning to let the fund return match as much as possible the valuation rate. In later years, the asset weights are generally closer to those of the base case.

Similarly, when we decrease the initial valuation rate to 3%, the initial asset allocation tends to be highly concentrated in the investment grade bond portfolio. Asset weights then approach to those in the base case in the long run.

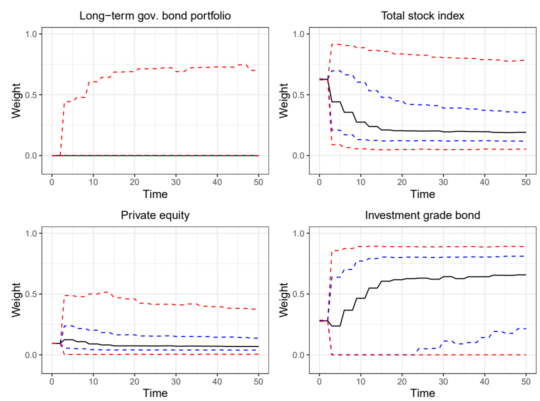
6.4.3 Impact on economic capital

From Table 6.3, we find that when we increase the initial valuation rate, the economic capital for three-year horizon deteriorates for confidence levels higher than the 50^{th} percentile. Panel (b) of Figure 6.8 shows the distribution of funded ratios when the initial valuation rate is set to 9%. At year 3, the volatility of the funded ratio is higher than that of the base case because more assets are invested in the total stock index and private equity. So, the economic capital measures tend to become worse at high confidence levels. For the 50-year horizon, the economic capital is worse for levels higher than the 90^{th} confidence level. The funded ratio is similar to the base case in the long term, while the higher total stock index and private equity weights in the early years increase the volatility of the fund return. The fund return is lower at high confidence levels. According to Equation (6.4), lower fund returns increase the discount factor, which decrease the economic capital measures at high confidence levels.

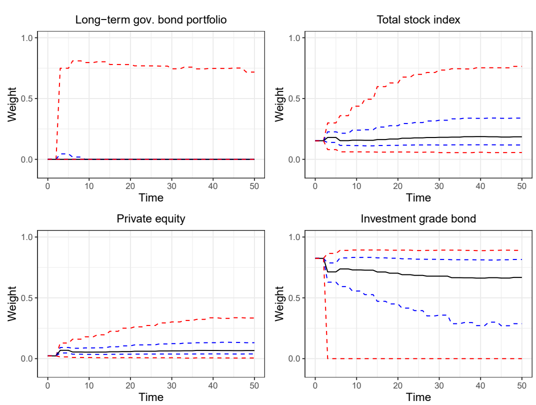
When we decrease the initial valuation rate to 3%, the economic capital metrics are improved for both the three-year and 50-year horizons. Panel (c) of Figure 6.8 shows the distribution of the funded ratios when the initial valuation rate is decreased. The funded



(a) Base case



(b) Increased initial valuation rate of 9%



(c) Decreased initial valuation rate of 3%

Figure 6.7: **Asset allocations when changing the initial valuation rate.**

Table 6.3: **Economic capital results when changing the initial valuation rate.**

Panel A: Three-year horizon

		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V^*_{0,3} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 5	0.72	-0.08	-0.23	-0.32	-0.47	-0.82	-0.99
Case 6	0.01	0.13	0.10	0.07	0.05	0.00	-0.03

Panel B: 50-year horizon

		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V^*_{0,50} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.48	0.00	-0.07	-0.10	-0.24	-0.55	-1.47
Case 5	0.48	0.00	-0.09	-0.09	-0.33	-1.10	-2.78
Case 6	0.47	0.00	-0.05	-0.07	-0.16	-0.37	-0.70

ratio at year 3 tends to be greater than 1 because the median valuation rate increases based on the initial asset allocation which, in turn, decreases the actuarial liabilities of the plan. The high funded ratio implies increases in the economic capital for the three-year horizon. The economic capital measures are also improved for the 50-year horizon because of the high fund returns in the early years, which decrease the discount factor.

6.4.4 Impact on certainty equivalent consumptions

Panels (a) and (b) of Figure 6.9 compare the CECs obtained with initial valuation rates of 9% and 3%, respectively, to that of the base case. For active members, the CECs increase when the initial valuation rate is 3%, and decrease when the initial valuation rate is 9%.

For an initial valuation rate of 9%, the CECs decrease slightly because of the low funded ratio in the early years that increases the contribution rate. Comparing Panels (a) and (b) of Figure 6.8, we find that the funded ratio is lower than that of the base case when the initial valuation rate is 9% from year 4 to 20. Because the valuation rate decreases as the total stock index and private equity weights decrease in the early years, the actuarial liabilities increase, which leads to a lower funded ratio in the early years.

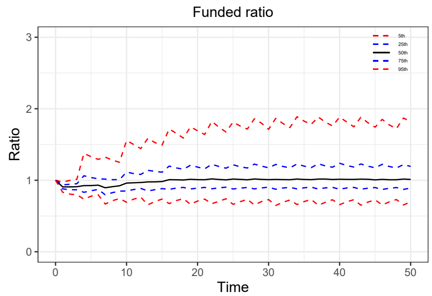
For an initial valuation rate of 3%, the CECs increase because of the lower contribution rate in the first several years as a result of the high funded ratio.

6.5 Change in the available asset classes

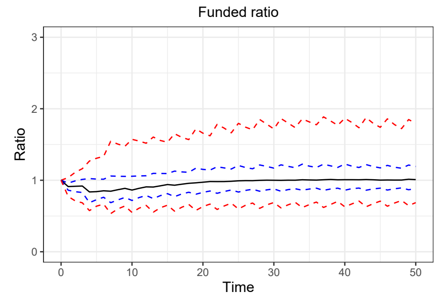
6.5.1 Description

We remove some of the asset classes from the portfolio in three additional robustness tests, which are described as follows:

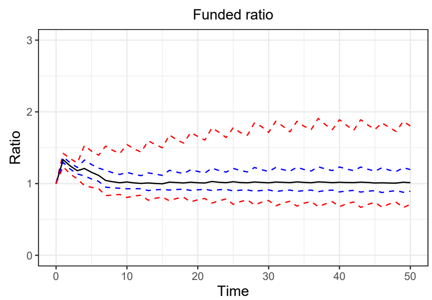
Case 7. Remove the long-term government bond portfolio from available assets.



(a) Base case

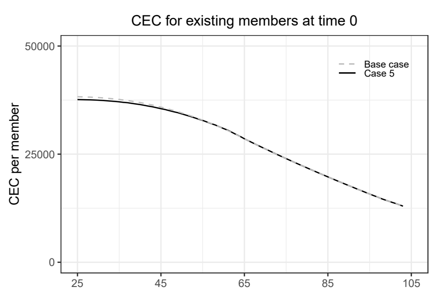


(b) Increased initial valuation rate of 9%

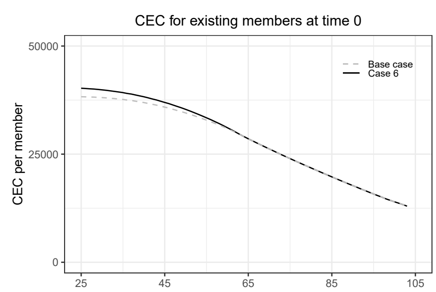


(c) Decreased initial valuation rate of 3%

Figure 6.8: **Funded ratios when changing the initial valuation rate.**



(a) Increased initial valuation rate of 9%



(b) Decreased initial valuation rate of 3%

Figure 6.9: **Certainty equivalent consumptions when changing the initial valuation rate.**

Case 8. Remove private equity from available assets.

Case 9. Remove both the long-term government bond portfolio and private equity from available assets.

In the base case, we invest in four asset classes which include a long-term government bond portfolio, the total stock index, private equity, and an investment grade bond portfolio. The number and types of assets available might affect the fund return, and hence the solvency and welfare metrics.

6.5.2 Impact on asset allocation

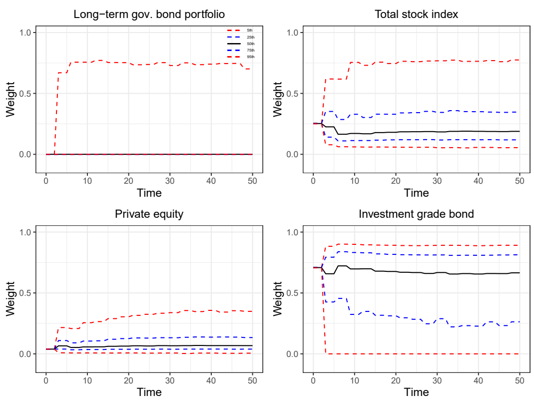
Figure 6.10 compares the asset allocation after removing some asset classes from the investment universe to that of the base case. The investment grade bond portfolio weight increases when we remove the long-term government bond portfolio. Both the investment grade bond portfolio weight and the total stock index weight increase when we remove private equity or when we remove both the long-term government bond portfolio and private equity.

6.5.3 Impact on economic capital

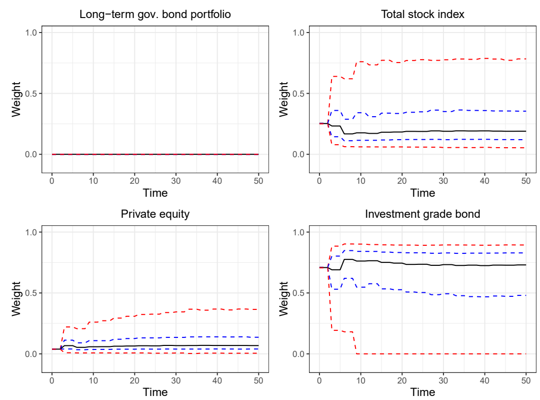
From Table 6.4, we find that the economic capital measures deteriorate when we reduce the number of asset classes in the portfolio because it leads to less potential for diversification. If we have more asset classes, the weak performance of one asset can be balanced out by the good performance of another asset, which reduces the risk of having extremely low funded ratios. Because we have fewer asset classes in these robustness cases, we observe extremely low funded ratios, which deteriorate the economic capital results.

Table 6.4: **Economic capital results when changing asset classes.**

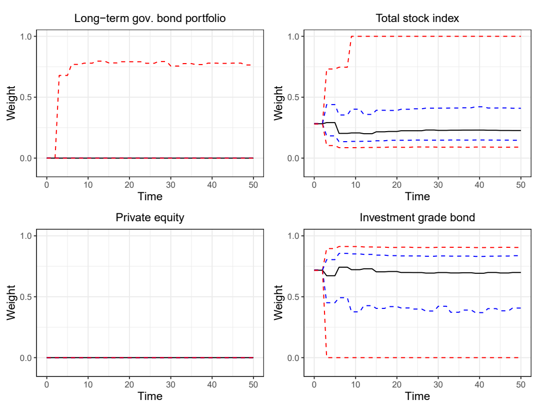
Panel A: Three-year horizon							
		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V^*_{0,3} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 7	0.92	-0.10	-0.17	-0.20	-0.27	-0.38	-0.43
Case 8	0.94	-0.11	-0.18	-0.22	-0.28	-0.39	-0.46
Case 9	0.94	-0.11	-0.18	-0.22	-0.28	-0.40	-0.46
Panel B: 50-year horizon							
		50^{th} confidence level		90^{th} confidence level		99.5^{th} confidence level	
Cases	$P[V^*_{0,50} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.48	0.00	-0.07	-0.10	-0.24	-0.55	-1.47
Case 7	0.49	0.00	-0.07	-0.09	-0.24	-0.56	-1.56
Case 8	0.49	0.00	-0.10	-0.12	-0.31	-0.83	-1.88
Case 9	0.50	0.00	-0.09	-0.11	-0.30	-0.83	-1.89



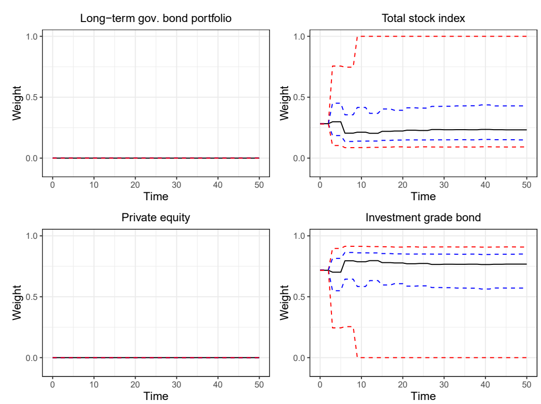
(a) Base case



(b) Remove long-term gov. bond portfolio



(c) Remove private equity



(d) Remove long-term gov. bond portfolio and private equity

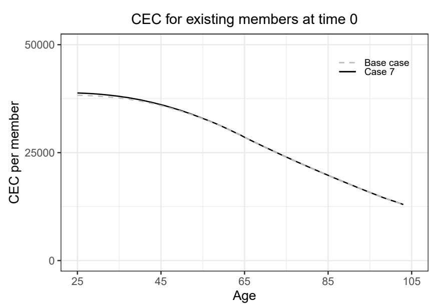
Figure 6.10: **Asset allocations when changing asset classes.**

6.5.4 Impact on certainty equivalent consumptions

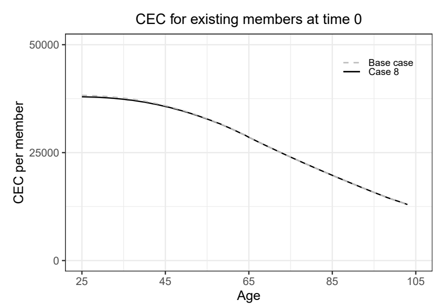
Panel (a) of Figure 6.10 compares the CECs obtained by removing the long-term government bond portfolio to those of the base case. For active members, the CECs increase because the long-term government bond portfolio has the lowest average return among the four available assets, so the funded ratio increases when we remove this asset, on average, which causes the CECs to increase.

Panel (b) of Figure 6.10 shows decreasing CECs when removing private equity. Private equity has the highest average return among the four available assets, so the funded ratio decreases when we remove private equity, on average, which causes the CECs to decrease.

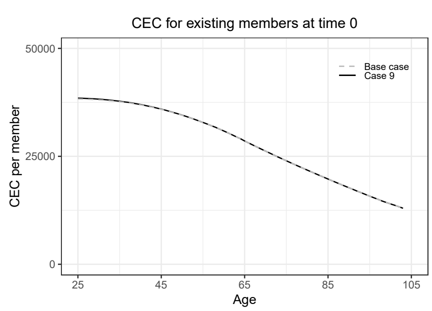
Panel (c) of Figure 6.11 reports the CECs obtained by removing both the long-term government bond portfolio and private equity. Because it is hard to see the difference of CECs in the Panel (c), we further list the CECs of the members aged 25, 45, 65, and 85 for the Cases 7, 8, and 9, as well as the base case in Table 6.5. We find the CECs slightly increase for active members in this case. The change is not obvious as the positive effect on CECs from removing the long-term government bond portfolio almost offsets the negative impact of removing private equity.



(a) Remove long-term gov. bond portfolio



(b) Remove private equity



(c) Remove long-term gov. bond portfolio and private equity

Figure 6.11: **Certainty equivalent consumptions when changing asset classes.**

Table 6.5: **Certainty equivalent consumptions when changing asset classes.**

Age	25	45	65	85
Base case	38,281	35,872	28,552	19,745
Case 7	38,815	36,099	28,552	19,745
Case 8	37,931	35,684	28,552	19,745
Case 9	38,478	35,928	28,552	19,745

6.6 Membership size

6.6.1 Description

We increase the number of plan members in these robustness tests, which are summarized via the following two cases:

Case 10. Increase the population by 100%.

Case 11. Increase the population by 300%.

The plan population might impact the administrative costs because of economies of scale. So, we increase the size of the membership to test the effect of economies of scale on the pension plan operation.

6.6.2 Impact

The economic capital and CEC results are nearly unchanged when increasing the population by 100% and 300%. This is because the administrative costs are very small when compared with the fund value. In fact, the administrative cost is around 0.1% of initial fund assets. So, the potential for economies of scale has little impact on the plan.

6.7 Change in the smoothing factor κ

6.7.1 Description

We change the value of the smoothing factor κ in two robustness tests as follows.

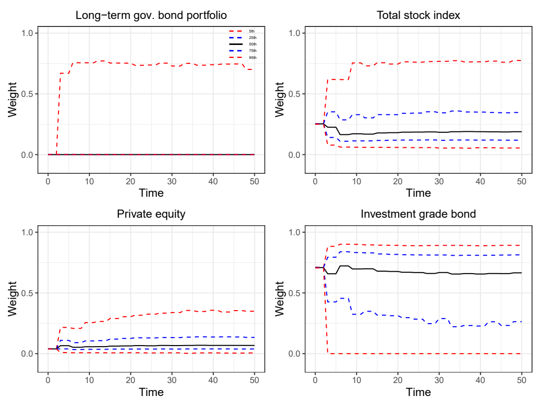
Case 12. Change κ from 0.2 to 0.1.

Case 13. Change κ from 0.2 to 0.5.

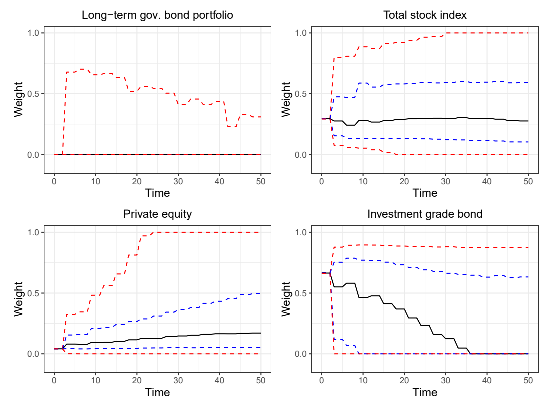
Different smoothing factors κ result in different levels of adjustments in the contributions, which further impacts the fund level. Surpluses and deficits are spread out over 5 years in the base case, 10 years in Case 12, and 2 years in Case 13.

6.7.2 Impact on asset allocation

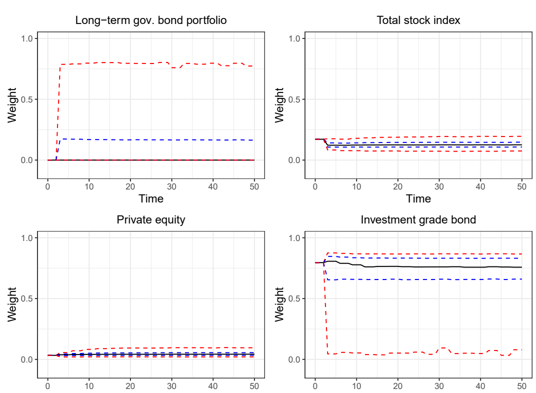
Figure 6.12 shows the asset allocations when the smoothing factor κ is changed. When κ decreases to 0.1, the allocation to the total stock index and private equity increases. Panel (b) of Figure 6.13 shows the evolution of funded ratios when κ equals 0.1. The funded ratio increases quickly in later years. The adjustment in the contribution decreases because κ is small. When the fund is in surplus in later years, the contribution increases as the adjustment decreases, so the fund value and funded ratio increase. The high funded ratio further increases the total stock index and private equity weights since, when the funded ratio is high, more investment in these assets can help the fund achieve higher return without much risk of falling into a deficit.



(a) Base case



(b) $\kappa = 0.1$



(c) $\kappa = 0.5$

Figure 6.12: **Asset allocations when changing the smoothing factor κ.**

When κ is set to 0.5, the allocation to the total stock index and private equity decreases. Panel (c) of Figure 6.13 shows the evolution of the funded ratio distribution when κ equals 0.5. For similar reasons, the funded ratio becomes closer to 1. The adjustment in the contribution increases because κ is large, leading to a decrease in contributions. So, the fund value and funded ratio also decreases, which further decreases the total stock index and private equity weights.

6.7.3 Impact on economic capital

From Table 6.6, we find that, when κ equals 0.1, the economic capital measures deteriorate at the 99.5$^{\text{th}}$ confidence level for the three-year horizon and deteriorate for levels above the 50$^{\text{th}}$ confidence level for the 50-year horizon. As shown in Panel (b) of Figure 6.13, the funded ratio is lower at high confidence levels than the funded ratio in the base case because more investments in the total stock index and private equity increase the dispersion of the funded ratio. The small smoothing factor also impacts the improvement in the funded ratios when the plan is in deficit. So, the left tail of the funded ratio distribution becomes fatter, which worsen the economic capital measures at high confidence levels.

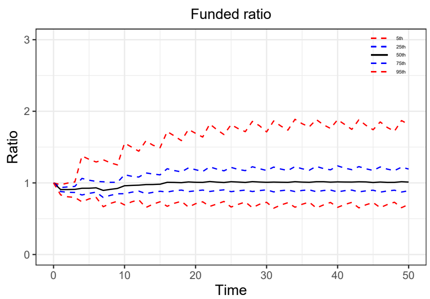
When κ equals 0.5, the economic capital measures for the three-year horizon are improved because a higher κ increases the contribution in year 3, which increases the funded ratio. For the 50-year horizon, the economic capital measures are also improved for levels above the 50$^{\text{th}}$ confidence level. As shown in Panel (c) of Figure 6.13, the funded ratio dispersion decreases because of higher allocations to the investment grade bond portfolio. The funded ratio also increases when the plan is in deficit because of the large smoothing factor. This implies a thinner left tail in the funded ratio distribution, which improves the economic capital measures at higher confidence levels.

Table 6.6: **Economic capital results when changing the smoothing factor $\boldsymbol{\kappa}$.**

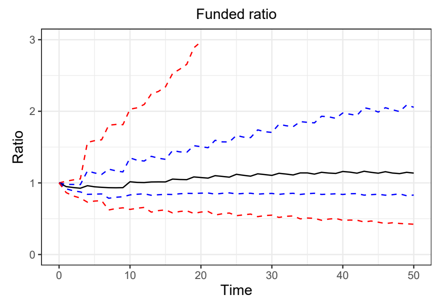
Panel A: Three-year horizon							
		50$^{\text{th}}$ confidence level		90$^{\text{th}}$ confidence level		99.5$^{\text{th}}$ confidence level	
Cases	$P[V^*_{0,3} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.92	-0.09	-0.16	-0.20	-0.27	-0.38	-0.43
Case 12	0.84	-0.07	-0.15	-0.20	-0.27	-0.40	-0.47
Case 13	0.93	-0.08	-0.13	-0.17	-0.21	-0.29	-0.33
Panel B: 50-year horizon							
		50$^{\text{th}}$ confidence level		90$^{\text{th}}$ confidence level		99.5$^{\text{th}}$ confidence level	
Cases	$P[V^*_{0,50} \leq 0]$	VaR	ES	VaR	ES	VaR	ES
Base case	0.48	0.00	-0.07	-0.10	-0.24	-0.55	-1.47
Case 12	0.40	0.01	-0.15	-0.10	-0.65	-2.56	-5.95
Case 13	0.51	0.00	-0.06	-0.10	-0.18	-0.37	-0.47

6.7.4 Impact on certainty equivalent consumptions

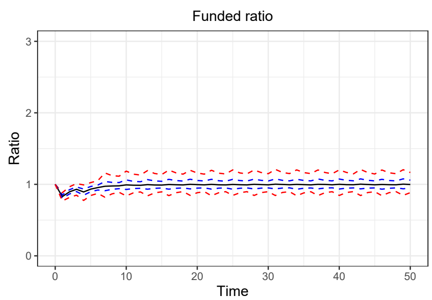
Figure 6.14 reports the CECs for the case when κ is set to 0.1 and 0.5. The CECs of active members increase when κ equals 0.1 and decrease when κ equals 0.5.



(a) Base case



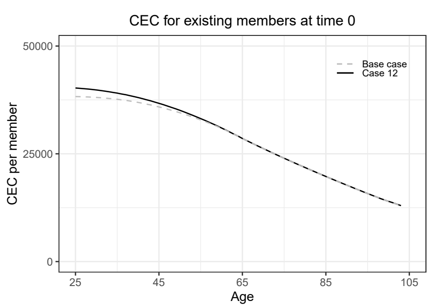
(b) $\kappa = 0.1$



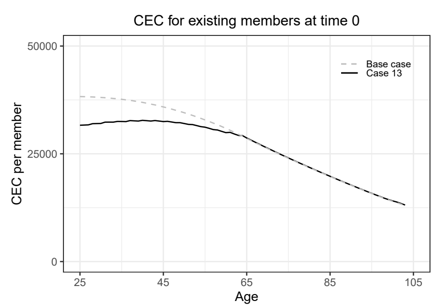
(c) $\kappa = 0.5$

Figure 6.13: **Funded ratios when changing the smoothing factor κ.**

When κ equals 0.1, the funded ratio is higher than that of the base case in the long term, and the contribution rate is lower, thus improving the active members' CECs. When κ equals 0.5, on the other hand, the funded ratio decreases significantly at the 75^{th} and 95^{th} percentiles, leading to a higher contribution rate, which ultimately decreases CECs.



(a) $\kappa = 0.1$



(b) $\kappa = 0.5$

Figure 6.14: **Certainty equivalent consumptions when changing the smoothing factor κ.**

Chapter 7

Concluding remarks

This report proposed a complete framework to model the operation of a DB plan, accounting for all random pieces. The framework included a realistic ESG, a stochastic mortality model, a dynamic administrative cost model, and an asset portfolio optimization procedure. The ESG was used to generate economic variables for the Canadian economy. The deaths were modelled through Bernoulli distributions, where the parameters were taken from the CIA mortality table that considered mortality improvements over time. The administrative costs were modelled by a traditional cost function that reflected economies of scale. The optimal asset allocation was based on maximizing the expected utility of the funded ratio, where the utility function put a high penalty on deficits and assigned a relatively modest value to surpluses.

To quantify solvency and welfare under the DB plan and condense our simulation results into one-dimensional metrics, we applied two types of measures—economic capital-based measures and expected utility-based measures—to the pension plan. The economic capital-based measures included the VaR and ES measures at 50^{th}, 90^{th}, and 99.5^{th} confidence levels over three-year and 50-year horizons. In addition, the CECs were calculated for comparing members' expected utility of consumption.

From the simulations, we observed a U-shape relationship between the initial funded ratio and the optimal total stock index weight, leading to oscillations in the evolution of the asset allocation, the valuation rate, and the funded ratio in early years. In the long term, the distributions of all quantities of interest were stable.

In terms of the solvency and welfare metrics results, the economic capital measures were worse at high confidence levels for the 50-year horizon than those obtained for a three-year horizon. We also found that when increasing the reference funded ratio, increasing the initial valuation rate, and excluding private equity from the asset portfolio, the economic capital measures changed at the 90^{th} and 99.5^{th} confidence levels while the CECs only changed slightly. The economic capital results and CECs were both impacted by lowering the initial valuation rate and by changing the smoothing factor.

The solvency and welfare metrics tend to be more meaningful when compared across pension plans with different designs. For example, when a pension plan merges, the pension plan's features such as the benefit, the plan's membership, and the fund's relative size may change. As a result, the pension plan's operation and profit will be impacted after the merger which influence the stakeholders' welfare. To quantify the impact of mergers on the stakeholders, we can compute and compare the measures before and after the merger by using the framework and metrics introduced in this report.

In future research, the proposed framework can be extended. In all our tests, the membership is stable over time. We could change the membership according to that of an existing pension plan and include deferred vested members who are no longer accruing benefits but have not started drawing a pension. This extension would allow us to study the operation of a real pension plan as well as its solvency and welfare. Another idea for future research would be to consider alternative utility functions, such as a power utility function based on fund surplus in the asset portfolio optimization procedure. Another avenue for future research is to consider adding more asset classes to our economic scenario generator and making them available to the pension plan.

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