Mathematics Lecturing:
Gestural Practices in Contexts

by

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Abstract

The purpose of this research is to examine the actions of an instructor in the undergraduate mathematics classroom over a full term of lecturing on abstract algebra. A micro-ethnographic, natural history approach was adopted, guided by the *Natural History of an Interview* project, and especially influenced by the application of such a methodology by Jürgen Streeck. The analytic framework owes much to George Herbert Mead's focus on the act, on the necessity of social interactions for the emergence of meaning, mind, and the self, and on the critical importance of gestures in meaningful interaction; together with Gregory Bateson's focus on metacommunication and the creation of contexts in interaction; synthesized with Streeck's analysis of gestures as a human praxis engaging directly and actively with a material world.

Thirty-five lectures were video-recorded, transcribed and summarized in multiple ways. Contexts, and the gestural practices achieving mathematical ends that occurred within them, were analyzed. Lecturing was found to be segmented, using a constellation of bodily resources, into local contexts: *stanzas* and *lines*. Six varieties of gestural practice were found. *Manipulating the object*: interacting, using the hands, with a physical, textual, or imagined object. *Looking at side-by-side*: moving back and forth between two pieces of writing, handling each in turn. *Regarding as*: expressing with the body and hands the manner in which some writing is to be considered. *Deducing that*: touching multiple pieces of writing to figure out what ought to be written next. *Communicating about*: stepping back from the writing action, gesturing and speaking about writing to come or that was just finished. *Correcting self and others*: occasions when the lecturer interrupts themselves, or an interactant fixes an ongoing mathematical action. The structure and function of these gestural practices, alone or in combination, were studied. Three broad mathematical situations were considered: the three lectures on isomorphisms; the appearances of the mathematical object D_4, namely the group of symmetries of the square; the appearances of the notion of well-definedness.

Keywords: interaction; gesture; writing; practices; lecturing; well-definedness

To my wife, Tara, and my children, Liam and Zoe. I love you.

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To my son Liam and my daughter Zoe: I want to tell you how much you have helped me along the way just by being who you are. Liam, when I close my eyes I can see you as you were when you were nearly three years old, and we were exploring together the campus at Simon Fraser University. When I asked you which way we should go next you

pointed confidently with your index finger: your gesture is emblazoned in my imagination. Zoe, you have told me so many riddles, and you have transformed so many conversations into occasions filled with laughter: your impeccable timing has made me smile time and again when I relive those talks in my mind.

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Chapter 1.

Introduction

1.1. Folk theorems, taking metaphors seriously, and making new mathematics from old: the impact of mathematical lecturing

I attended the lectures of Dr. Edward Barbeau at the University of Toronto in the fall of 1988 and the spring of 1989. The title of the course was Analysis I and the subject matter was the standard material from first-year calculus: limits, continuity, intermediate value theorem, extreme value theorem, mean value theorem, Taylor's theorem, definitions of the derivative and the Riemann integral, sequences and series. The textbook was Spivak's *Calculus*. I learned a great deal in that course. In particular, I vividly remember two incidents from his lectures. I will describe them and try to say what they have meant for me over the years.

One day Barbeau wrote a result on the board, some result that I do not now remember. He stepped back a few paces while nearly colliding with his desk, ignored the near-trip, and faced the class with a big grin. Flinging his chalk from hand to hand uninterruptedly, he excitedly announced: "This is an example of a Folk Theorem. Everybody knows it, everyone's heard it from someone, you learn it from your supervisor or a fellow student. But it's not in the written literature anywhere." He raised his eyebrows up and down a few times as if enjoying a great joke. I remember thinking that this was very interesting, and it also puzzled me.

At the time I literally did not realize that such a thing was possible. I thought that all the results that mathematicians knew about, and could prove, were results that were somewhere written down 'in the literature'. I had a naive view of mathematics, naive in multiple ways. I don't know that it is possible for me to list all the ways, but I will talk about two of them.

First, I underestimated the extent to which the 'same result' could be written in innumerable different ways, with the degree to which these ways were different also varying. This meant that I did not realize that each person exposed to some result might themselves modify in some tiny or not so tiny manner the justification of the result, or the statement of the result itself, so that a new truth could emerge that might never reach the level of being considered different enough, or novel enough, in order to be publishable in the literature. Such variations might emerge in live conversations, or in live lectures, and thus be spread by word of mouth. My view of the nature of mathematical results was too rigid. I thought results were large in number, but somehow rigidly only themselves as they were exactly written, and unalterable.

Second, (and this naivety is related to the first), I did not really understand how new mathematics was developed. Although I did not articulate it consciously in this way, I thought it was developed mostly by reading someone else's written argument (and secondarily listening to someone else's spoken argument), absorbing it exactly as it was written or spoken, and collecting together enough such exact absorptions, so as to 'know enough mathematics' to be able to advance the frontier of mathematics by some small amount. Barbeau's remark helped spark in me a new way of thinking about the creation of new mathematics. It began to seem that mathematics developed much more often in the following manner. A person would be exposed to an argument, either in written form or in spoken form (but perhaps much more often in spoken form); this argument had likely already been modified often in many ways by a community of people. Upon encountering the argument, such a person might or would attempt any modifications possible that occurred to them to try, and they would realize from the constraints of the mathematical situation which modifications could not work, and then they might occasionally realize that some modification could work.

Even then, when some individual has noticed a modification that works which does not appear in the written argument that they have been reading, or in the spoken argument that they have been listening to, I was wrong in thinking that the appropriate next step would be to search the literature in order to see if the result were new. I learned in this moment from Barbeau that probably the appropriate thing to do would be to speak to someone who is knowledgeable in that area to see if they are familiar with the result. Either the result has been written down somewhere, in which case they will very probably either know immediately where the result can be located, or they know where to begin looking. Or the result is 'well-known' to them, and some others, but is actually not written down anywhere in the literature in that precise form, constituting a folk theorem much as Barbeau described. Or perhaps the result would be new to them, and thus might very well be new to the mathematical community as a whole. I began to understand that communicating mathematics to another person, live, often by writing things down and speaking about them, perhaps at a blackboard or whiteboard, or huddled over some paper, is an enormously important part of the spread and growth and development of new mathematical results, of new mathematics.

So far, I have talked about what I think such a remark meant for the development of my sense of how mathematics evolves. More personally, I also attribute to this remark a hastening of my own realization, which took a few more years to rise fully to articulation, that my own deepest intellectual curiosity lies not in mathematics, but in how it is that people learn mathematics, how it is that they develop new mathematics, and how it is that they communicate this mathematics to each other. For many years I fought against this inclination, utterly swallowing the line repeated by many mathematicians, that any work 'about' mathematics that is not actually about mathematics itself, but is instead about (in their view) second-order concerns (philosophy of, history of, communication of, etc. of mathematics), is of less importance.

I will not quote anyone here, but it is all too easy to find such pontifications. I believed them. I thought less of myself for always being so interested in what I privately thought of as 'the rhetoric of mathematics' (and later, 'the rhetoric of physics') – by which I meant, more or less, 'the study of how it is in practice that mathematicians and physicists actually speak and write to each other'. I wished to take their aims utterly seriously, but to put in pride of analytical place all of the things that they knew were important but were not the principal focus of their efforts: their exact way of defining concepts and making and justifying claims, yes; but also their informal comments, their pictures and diagrams, their prefaces to textbooks, the way they designed their courses for each other, their talk of beauty and elegance and complications and ugliness and messiness, their secret motivations that they never revealed in papers but only mentioned in response to a question at the end of a seminar, and so on. It took me too long to discover that people whose work I respected and admired had taken as their subject precisely what I thought of as a private and un-confessable interest. I was slow to realize that perhaps the mathematicians and physicists who denigrated this activity had their own private reasons for casting aspersions on the subject.

By being in the audience for one of Barbeau's lectures, I took in one of his comments and it helped me realize something important about the oral nature of much mathematical activity, and something important about myself and what I hoped to spend my intellectual life concerned with. What I have not really got across very well is how emotionally charged the original moments were for me. I knew already that this incident was fundamental long before I understood why. I realize now, too, that I remembered in detail many of Barbeau's body movements, and this too provided a seed for the concerns and focus of this work.

For the second incident from Barbeau's lectures, I do remember the result he was explaining. It was the intermediate value theorem. The statement of the theorem was already on the board. He drew a slowly waving line on the board, representing the surface of the ocean, and a cartoon sketch of a dolphin underneath the water–air interface. Again, the charming grin of someone excitedly interested as if for the first time: "This is obvious, right? The dolphin has to cross the surface of the water to get into the air, right? So, what is there to prove?" Pause, pacing, chalk-flinging, pacing stops suddenly, chalk-flinging ceases. "But just a second. What is this interface anyway? We look more closely – there's turbulence and foam. There are splashing droplets in the air. There are air bubbles in the water. There are impurities. The water and air are really made of molecules."

I remember how soundly his point struck home. You have a result, you have the intuition that it is obvious, you find a metaphor that makes the intuition clear. Then, instead of leaning back with your hands behind your head, you lean into the metaphor and take it seriously. Everything starts to unravel. The feeling of certainty fades. You realize that we ought to prove this result, and we had better check to see if our hypothesis is, in fact, strong enough. Perhaps we ought to be considering alternative conclusions that might be weaker, because suddenly this is a result that might not be obviously true after all.

This process of hitting upon a metaphor, and then staring into it for real, seems to have the power to give you a sense of where to look for pitfalls, and how to chase away the potential unravelling. Many times since then I have gone through a similar process: seeking an intuitive understanding of some mathematical result or object or situation, realizing simultaneously that said intuition must be bolstered and interleaved with rigorous reasoning that can withstand skeptical inquiries. I really do think Barbeau was the first to make this process utterly clear and visible for me. He instantiated it so vividly and memorably that he changed how I thought about mathematics.

I have described these incidents in some detail for several reasons: to set the stage for this thesis, the subject of which is mathematics lecturing at the undergraduate level; to pay respectful tribute to the generosity of spirit and the pedagogical efforts of my former teacher; to say something about myself that will help make clear why I chose this subject for my thesis, and why I find it compelling and important. But most of all I hope that these moments were described in enough detail so that they spark in the reader some memories of their own. I am confident that the reader has attended more than a few mathematics lectures, and that at least one such transformative moment has occurred for them. I suppose I want the reader to share with me some inner suspicion that witnessing a mathematics lecture can be a very powerful experience indeed.

It is important to interpose here two caveats. One: I do not believe, and I am not suggesting, that the sole, or even the most important, virtue of mathematics lectures is the potential for such occasions as described above. I attended many lectures from which I learned many things that never struck me at the time with the force that these moments did, but nevertheless they left their mark in hundreds of ways, large and small, on how I think and feel about mathematics. I do not believe that only the starkly memorable incidents are the ones that ‘count’. Two: I also do not believe that just because my life was changed by two choices that Barbeau made while lecturing that that means mathematics lectures cannot also be, and very often so, unpleasant or unprofitable experiences. They can be boring or stultifying or confusing. They can contain long stretches during which multiple students are following nothing.

As true as it is that some moments in mathematics lectures have been crucial to me, it is also true that I skipped a lot of lectures as an undergraduate, and that I was lost in a lot of the lectures I did attend, doodling and daydreaming. I also remember moments of frustration, writing down line after line of notes one symbol at a time, irritated that the ‘how come we’re doing this?’ question was going to go completely unaddressed for another long lecture, my dull pencil and the dried sweat on the page working together to drive me into a silent state of disconnection. I also realize now, to my embarrassment, just how active some of my former lecturers were in various very interesting areas of mathematics. I did not have the presence of mind to realize how much I could learn from them if only I were to ask the right question, or if only I were to watch carefully enough, with sufficient curiosity, what they were doing moment by moment in the classroom.

I am now on the other side of this divide. I have been lecturing to undergraduate classrooms for nineteen years. I taught high school for five years. I have taught introductory calculus, for the physical sciences and for the life sciences, pre-calculus, linear algebra, intermediate calculus, differential equations, geometry, abstract algebra, real analysis. At the high school level, I taught physics, mathematics, and theory of knowledge. I have given many poor lectures. Even in lectures that I think are mostly good, there are always moments or sequences of moments that I think did not come off as well as they might have. When a lecture is going well, or when a student tells you later that they got something from one of them, it is a very good feeling indeed.

I am fascinated with this human interaction. When lecturing, I am speaking and writing and moving my body and moving my hands. This is pretty much a complete list of what I can be felt by a student to be doing. I am joined in this activity by thousands of people in the world, all lecturing to hundreds of thousands of undergraduate mathematics students. These students will go on to be just about any conceivable sort of person in society. We instructors or lecturers are all doing *something*, and there seems to be large amounts of analyzable commonality in what we do. The goal of this thesis is to determine some of the reliable and checkable things that can be truthfully said about undergraduate mathematics lecturing: an activity that is restricted enough in some respects that one can hope it can be understood in some detail, and an activity that is common enough and significant enough in the world that a better understanding of it will be of fundamental importance to the world. It seems clear that any attempt to improve this activity so that more students become capable of learning, communicating and developing more mathematics more often and more naturally *must* begin with a thorough-going description and analysis of the act of mathematics lecturing itself.

1.2. Road to this research topic

1.2.1. Initial Steps

When I started the doctoral program, my plan was to analyze a particular mathematics text: *The Gamma Function* by Emil Artin (1931/1964). Whenever I had seen that book referred to by other mathematicians, they always wrote in admiring terms: “this elegant work”, “this beautiful monograph”. I wanted to know why. I admired it myself, but what in the writing brought forth this loving and grateful response?

Looking now at the preface of the 1964 edition, Edwin Hewitt refers to it as “Artin’s little classic”, “read with joy and fascination by many thousands of mathematicians and students of mathematics” (Hewitt, 1964, p. v). A little later he adds: “His undergraduate lectures in the calculus, for example, were filled with elegant constructions and theorems which, alas, Artin never had time to put into printed form” (p. v). This resonates well with what my eventual research interest turned out to be. At the time, I was asking questions like: what exactly is it about this book that seems to strike so many varied mathematicians in such a nearly uniformly positive way? What is it in the writing of the book that gives rise to these emotional reactions, these responses filled with aesthetic language?

Years earlier I had fallen in love with reading literary criticism, and had gravitated towards critics like M. H. Abrams and William Empson, who to me seemed like powerful combinations of sensitive readers and clear thinkers. They read closely and deeply, they sought to understand how it was the writers they read achieved the effects they thought they saw, and they aspired to gather together their observations to make rich, complex claims about writing more generally: Abrams (1953) on the writing from the Romantic period; Empson (1930) concerning the uses writers make of ambiguities.

One book that stayed with me was Barthes’ (1970/1975) *S/Z*. What I loved about it was the ambition, as it seemed to me, to contain a single story (*Sarrasine*, by Balzac); to analyze it patiently and minutely from a variety of standpoints and perspectives; to somehow swallow it whole and come out the other side with a penetrating and thought-provoking analysis of a rich network of possible meanings of the story. No part of the story was too small, too humble, too minute to escape Barthes’ interest. In fact, sometimes a small detail, when considered carefully enough, and put together with earlier discussion, helped throw a new light on the whole work. I thought: “I’d like to do that to Artin’s ‘little classic’”. I was bolstered by the notion that, like the Balzac story, this was a text of manageable size, and that, again like the Balzac story, this was a text held in high esteem by other practitioners of the art-craft in question (writing for Balzac, writing mathematics for Artin).

I thought I might be positioned to do the job. I understood the moves that Artin was making (I could follow the proofs, I understood the mathematics), while also being interested in examining how and why Artin wrote it in this particular way and not in other ways. I was interested in the rhetoric of this mathematical text. I dreamed I might be able to say something about how Artin made something so beautiful, something about what this beauty might be said to be made of or subsist in, and something about the manner or form in which this beauty might be seen or felt to manifest itself.

I said something of this to my supervisor. I dived into the readings of my various courses. I slowly educated myself on the voluminous rich work that was being done and had been done in the mathematics education literature on language, communication, textbooks. In conversations with my supervisor, she suggested a golden opportunity: a professor that she knew in the mathematics department was going to be teaching a course where one of his aims was not only to use a 'standard' textbook for his group theory course, but also a second textbook that was much more deliberately visual in nature. Knowing of my interest in textbooks, and the choices made in their construction by their authors, and knowing that I had also expressed interest in other 'beautiful' books – like *Geometry and the Imagination* (Hilbert & Cohn-Vossen, 1932/1952), based on Hilbert's lectures in the winter of 1921, which contains page after page of lovely thought-enhancing figures and diagrams – she offered this chance to watch how this professor might use these two textbooks in teaching a course.

I loved the notion that there was a more vital component to this possible study. Artin's little classic is finite and closed, and there is a certain interest in being able to 'contain' and 'swallow it whole', as I had initially thought Barthes had done. But Barthes' project went further than this. His long creatively analytical work serves also to show the *open-endedness* of Balzac's story: the contradictions that are not resolvable, but which generate endless discussion. Barthes' writing was precise, but in the service of being fecund. After reading Barthes, there was more to read in Balzac, and more to write about.

Balzac's story fights back and is living. It was not really true that I would end up 'containing' Artin's book (writing this out now makes it obvious how absurd this part of the original aim was). Would it not make sense, and perhaps be even more interesting, to take the animatedness and the vitality seriously from the beginning, and to watch and record a mathematician in real time writing and drawing and erasing the sorts of inscriptions that I had originally intended to analyze in their so-called frozen form? This dovetailed with what I had learned from Barbeau, that the making of new mathematics was likely to be occurring all the time, and in live public conversations with excited people engaged in action, long before some of it curled up and nestled into an elegant book. All the more reason to study this making close up.

1.2.2. Beginnings and endings of bits of speech

I began to attend the lectures, and I videotaped them. After a few lectures, I began to transcribe what was said in the lectures. It was very hard to decide, I found, where commas should go, where periods should go. By contrast with the difficulty in placing commas or periods, it was very easy to hit the ‘return’ key whenever there seemed to be a bit of a pause that was slightly longer than any inter-word silences, so I did that.

Often, the lecturer (I will refer to him henceforth as J) self-corrected himself: he would interrupt himself mid-word, or just after a word, and then begin again his stream of speech, picking up a word back, or a handful of words back. I had decided by then that I was not going to ‘clean up’ the transcript and omit his self-corrections, as I thought they had value. They might indicate a preferred and dis-preferred stream of speech, and in comparing the two the reader of the transcript would learn something about the various axes of value that J possessed, and how they were revealed in practice.

Then I noticed there was a significant amount of patterned repetition. He might speak of a reflection in the vertical axis, and then immediately speak of a reflection in the horizontal axis, in bits of speech that were symmetric. He might speak of the multiplication of two elements in one order, then immediately afterwards speak of the multiplication of these two elements in the opposite order, again in symmetric phrases, echoing many of the same words of the previous bit of speech. Additive notation mentioned for one group, multiplicative notation mentioned immediately after for another group. Similarities indicated, contrasts drawn, very often in as stark and as explicit terms as possible, with every word matching except for a “just as” or a “not” respectively. It was impossible to ignore that the lines that were being created in the transcript (the lines between pressing the ‘return key’) possessed structural features that the lecturer was emphasizing, and that the little pauses were demarcating beginnings and endings of some sort of unit.

Similarly, after 5 or 8 or 10 or 19 (but never as uninterruptedly long as 40 or 50 of these lines), there would appear a transitional mark that impacted me more deeply. I was noticing that, in my transcript, I was pressing the ‘return’ key twice at these moments. These more strongly marked endings and more strongly marked beginnings were, at first, simply registered by me more or less unconsciously. If pressed to explain why I was separating the next bit of transcription with an interleaved blank line, I would probably not have been able to articulate a reply until I watched the videotape again. I ‘felt’ there was a break. It was ‘obvious’, or ‘clear’ that there was some separation, and indeed, an intended separation. A lecture seemed to end up having one of these breaks about every half a minute on average.

This was the beginning of my conviction that the lecturer was breaking up his lecture into pieces within which he was witnessably accomplishing a ‘doing’, a ‘step’, some *quantum* of mathematical action. At the time, I saw it as him breaking up the string of words he was unspooling; later, I saw it as the deliberate markings of boundaries between two periods of time within which different specific actions were embarked upon, undertaken, and successfully completed.

1.2.3. Pointing

It is difficult now to remember just what exactly began this focus. It seems now like I have never not noticed mathematics lecturers pointing, but I know that can’t be right. One way or another it became hard to ignore the fact that within virtually every short passage of time in which the lecturer was trying to attain some goal, to achieve some announced aim, the lecturer was pointing at some thing or multiple things on the board, touching things on the board, sometimes holding them for short periods of time. If he was a step or two from the board, he would often close the gap and touch a term directly in order to say something about it.

At first, I focused my attention on a single lecture. This felt at the time like a proof-of-concept, pilot study. Something about this focus appealed to the empiricist in me. There was nothing ambiguous or vague or debatable about his acts of pointing.

Anyone else looking at the video could not deny that at that moment the lecturer took a step to the board and touched that particular word or symbol. Whatever attempt a viewer could make of figuring out what the lecturer was communicating would probably have to come to grips with these acts of pointing if their concern to understand the lecturer's actions were serious. After all, why would a lecturer move their body, lift their hand, select a single item from hundreds on the board and touch it, and expect a viewer to ignore it in their analysis of what was happening? And if they repeat variations on these actions dozens of times per lecture, how likely could it be that they have chosen repeatedly to do something unnecessary or pointless in the efforts they make to communicate the subject matter that they love?

These acts of pointing were frequent indeed. There were certainly patches of time where the lecturer was, for example, facing and addressing the class, and not pointing at all (and still gesturing; this became important later). But predominantly the mode seemed to be that if he were near the board, and was in the middle of writing some section of something, that pointing was mandatory, and happened often. There was no shortage of occasions, too, where if he happened to be distant from the board, then suddenly I would see him close the gap in order to sweep a group of terms, to hold an equation. It seemed almost as if he were compelled to do so, as if the board were a magnet and he an iron filing.

Having made a count of these acts in a lecture, I found that acts of pointing occur on average once per seven seconds. Speaking more informally, if one randomly selected an instant in a lecture, and watched the next 20 seconds, it would be rare to not see an act of pointing. It seemed very unlikely that a lecturer would so frequently make use of a communicative act if it were not playing a key role in their actions (Hare & Sinclair, 2015).

Speaking from my own experience as a lecturer, and from my experience in talking with fellow lecturers, or reading the self-reports of mathematics lecturers, it is fair to say that a major factor in teaching mathematics at the undergraduate level is the sheer amount of content that, in principle, might be developed in a term, and the still huge subset of this content that somehow 'ought' to be covered. Many is the hallway conversation that laments how it would be nice to 'get to Baire's theorem' or what have you. In addition, if the course one is teaching is a pre-requisite for a later course, there is a strong pressure to cover all the topics that the instructor for the later course can assume the students have already seen.

From this perspective, any time that is being used to do anything in a mathematics lecture is already under the microscope. Any time doing anything has an opportunity cost to it, the cost of what could have been otherwise done during that time interval. If a lecturer takes a second to point at something, and takes another second to point at something else, and repeats this thousands of times in a course, it seemed to me that there was something significant going on here that would be impossible to ignore. After all, why bother? Why not just write what there is to write, and say whatever there is to say, and never point or hold or touch anything on the board at all? It is difficult to recreate now the mindset I had then, and these are not meant to mirror identically my present considerations. Nevertheless, I still appreciate the force of them.

This was the beginning of my conviction that, in order to properly appreciate what this lecturer was doing, I could not, even for a moment, ignore what he was doing with his hands. I saw the gesturing of his hands as a first among three comparable equals: speaking, writing, *gesturing*.

1.3. Outline of Thesis

In Chapter 2, I undertake a review of relevant literature. Within the field of tertiary mathematics education, there is a growing number of researchers interested in teaching and teachers. I concentrate on some empirical studies performed on those teachers who predominantly use the lecturing mode by scholars such as Keith Weber and Tim Fukawa-Connelly, and others. In addition, I review the work of Elena Nardi, who with a different methodology (primarily interview-based) sought to discover some of the beliefs and practices that university teachers have adopted, and why they have done so.

Outside of mathematics education, I review three notable attempts to come to grips with teaching advanced mathematics at the board: the work of Christian Greiffenhagen, using an ethnomethodological framework, the work of Michael Barany and Donald MacKenzie, situated within the science and technology studies program, and the work of Natasha Artemeva and Janna Fox, adapting a rhetorical genre approach. These are sharply observant studies, filled with rich and perceptive accounts of the practical realities of teaching undergraduate mathematics, and both put writing on the board, and the myriad actions accompanying this writing, at centre stage.

This thesis takes a naturalistic micro-ethnographic approach to its subject matter. In Chapter 3, I discuss the influence on this work of George Herbert Mead's writings on mind, meaning, and gesturing, as well his method or approach to arriving at his ideas. I review some of the work of Gregory Bateson, including the seminal collaboration which gave birth to *The Natural History of an Interview*, and examine its importance for body- and gesture-researchers, as well as video-researchers. I then take a close look at two books in anthropology by Jürgen Streeck that form the basis for my theoretical frame. In *Gesturecraft: The Manu-facture of Meaning*, Streeck (2009) gives a useful typology of what he calls "gesture ecologies", and he uses these as a basis to organize his analysis and descriptions of various body movements and gestures that he considers critical in human interaction and communication. In *Self-Making Man: A Day of Action, Life, and Language*, Streeck (2017) takes an entire videotaped day in the working life of an owner of an auto-mechanic shop as his data for an extended and fascinating analysis of this man's speech, motion, and gesturing. I summarize the view of gestures that emerges from Streeck's work (and the work of those who make similar theoretical assumptions).

In Chapter 4, I set out my methodology. In Chapter 5, I identify and analyze six families of gesturing practices: *manipulating objects*; *looking at side-by-side*; *regarding as*; *deducing that*; *commenting about*; *correcting self and others*. In Chapter 6, I explore how these practices come together to unfold the mathematical meaning of an example of a mathematical object: the dihedral group D_4, which is the symmetry group of the square. In Chapter 7, I explore how these practices co-ordinate to elaborate the mathematical meaning of an example of a mathematical concept: the notion of well-definedness, as particularized in the context of quotienting by normal subgroups of a group. In Chapter 8, I respond to my research questions. I conclude with a self-examination of how this research has altered my own lecturing.

Chapter 2.

Literature Review: Undergraduate Mathematics Lecturing

2.1. Research in Undergraduate Mathematics Lecturing

The sub-discipline of mathematics education that concerns itself with research on undergraduate mathematics education has experienced a long period of growth that within the last thirty years has greatly accelerated. In 1991, the first special session on research in undergraduate mathematics education (RUME) was held at the AMS–MAA annual meeting; in 1996, the first RUME conference was held; in 2000, a special interest group of the MAA was formed, devoted to RUME. In 2015, the International Journal of Research in Undergraduate Mathematics Education (IJRUME) was launched; in 2016, the first INDRUM conference was held (International Network for Didactic Research in University Mathematics), and it continues on a biennial basis. In the opening lecture to this conference, Artigue (2016) reviews two earlier key works attempting to synthesize the state of the art of this research in recent history: the work of the Advanced Mathematical Thinking working group of PME (begun in 1986) culminating in the book of the same name (Tall, 1991); and the ICMI Study conference on the teaching and learning of mathematics at university level (held in Singapore in 1998), including eleven working groups, culminating in the book (Holton, 2001). More recent surveys of this literature have also appeared (Artigue, Batanero, & Kent, 2007; Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2017; Rasmussen & Wawro, 2017).

The study of mathematics teaching practices at the university level (Speer, Smith, & Horvath, 2010; Nardi & Rasmussen, 2020) is one of the growing areas of interest in this field. A recent survey of 126 abstract algebra instructors in the United States (Johnson, Keller, & Fukawa-Connelly, 2018) revealed that 79 reported that they lectured for at least half the time in every class, confirming the widely held anecdotal view that lecturing is the most commonly found mode of instruction at the undergraduate level. At the same time, the authors found a more complicated picture of lecturers vs non-lecturers than what anecdotes might suggest: about half of the lecturers reported sometimes conducting whole-class discussions, asking students to work on problems individually or having students give presentations in class; more than a third of the non-lecturers reported lecturing more than a quarter of the class time in every class.

A key study for this research is that of Nardi (2008), because of the insights it contains into the perspectives and beliefs of mathematicians regarding mathematics and pedagogy. Nardi unearths what it is they think they are trying to achieve while lecturing, and why. She conducted eleven half-day focused group interviews with twenty pure and applied mathematicians from different parts of the UK. A week before the interviews she distributed data samples consisting of students' written work and interview transcripts. Nardi's book is structured as a dialogue between two characters, M (the mathematician, a composite of all the participants), and RME (the researcher in mathematics education, a stand-in for Nardi herself and her co-researcher Paola Iannone); Nardi edited, paraphrased and assembled the dialogue from her transcripts of the interviews. The result is a rich and rewarding text filled with specific observations and comments on a truly wide range of topics and themes of teaching, speaking, writing, reading, and in all senses doing mathematics.

Many of the concerns expressed by M resonate with concerns that J shares; often, J addresses the issue explicitly. Themes within mathematical argumentation and reasoning such as how much detail and rigour is required in a proof depending on the context (p. 47), the nature of arguments where there is no choice about what to do (p. 63), the importance of being able to take an abstract concept and work out what it means in an example (p. 53) all resonate with J's actions; indeed, they often show up in his own comments about what he has done or will do. M frequently expresses concerns about the mathematical writing of their students, noting the lack of syntactic structure ("no commas, no beginning of sentence, no full stops", p. 119), and decrying the absence of those words and sentences which would have served the purpose "of emphasis, of clarification, of explanation, of unpacking the information within the symbols" (p. 151). For J, too, these issues are paramount. Mathematical seeing, and the importance of the viewpoint and perspective that the lecturer or mathematician is taking towards the mathematics they are currently writing or talking about, is a third crucially important theme that M returns to repeatedly: "We are not just communicating facts, we are saying that this is one way you can view it and that is another way you can view it, let's put these together somehow. And it's not an easy job, believe me!" (p. 218). This also accurately predicts an important ingredient to J's mathematical lecturing.

It is obviously true that a person's self-aware description of their own actions, though useful, cannot constitute the final word on what those actions are. One of the aims of this research is to closely study a lecturer's actions over a whole course to glean what happens in practice. One of the suppositions is that while a study of a single ten-minute incident in a lecture can be of great significance, it might give a slanted view, depending on how the incident was selected in the first place. By way of contrast, it would be hard for a lecturer to hide or fake or in some other way disguise for the length of an entire course how it is they mathematize in interaction with students in a classroom. In addition, there can be rarer kinds of behaviour that one can only unearth by witnessing an entire course, so that from the new background of what is analyzed to be regularly occurring, and the structures and items of behaviour that have already been adduced, these exceptional behaviours emerge in the researcher's awareness together with an improved picture of what is interesting about the behaviour and why it is infrequent. I turn now to look at some research that focuses on observation of mathematics lectures with a view to indicating features of this practice.

An early and influential example of research on the practice of mathematical lecturing is Weber (2004). Weber met weekly with the instructor of an introductory real analysis course and he observed and took field notes for the 75-minute classes that met twice a week over the 15-week semester; some of the classes were video-recorded and transcribed. One of Weber's explicit aims was: "To describe in detail the teaching styles of this professor using traditional instruction in the advanced mathematical classroom" (p. 116). He found three teaching styles, which he termed *logico-structural*, *procedural*, and *semantic*.

In the logico-structural style, the instructor drew hardly any diagrams. He would write the elements of the hypothesis at the top of the board, the conclusion at the bottom of the board, and on the side would write out the definitions of the terms appearing in the hypothesis. Without talking about the semantic meaning of the concepts or the proof, he would justify implications either coming down from the top of the board or moving up from the bottom, by unpacking the definitions, until the chain of inferences had met somewhere in the middle. At this point the instructor would pass through the proof from the beginning to the end linearly, explaining why every step was justified. He would not discuss the meaning of the concepts involved beyond using the definitions of the terms.

In the procedural style, the instructor would write out the skeleton of a proof on the board, consisting of pieces of writing that would remain in the final complete proof, and which revealed the structure of the proof. Then, using the side board for his rough work, he would think aloud about how he might complete the gaps that existed in his incomplete argument; also, he would make more general comments about tactics and heuristics. Only when all the gaps had been filled would he then discuss why the argument was logically sound; the meaning of the proof would not be discussed.

In the semantic style, the instructor would draw a diagram and use intuitive language to try to get across the idea that the definition he was introducing was intended to capture; if it were a proof he was presenting in this style, he would hand out a written version of the complete proof, and ask students not to take notes but to try to understand his work at the board. This work would include drawing pictures with the aim of communicating why the result being proved was a reasonable thing to expect to be true.

Subsequently, Weber and collaborators have examined various aspects of lecturing using a variety of methods. By interviewing nine mathematicians about their pedagogical practices in the classroom with respect to proving, Weber (2012) found that they proved results in lectures not to convince the students of their truth, but to illuminate methods of proving or to help the students see why the result is true. He also found that, although the instructors believed that students did not properly recognize how challenging and involved a process it is to read a proof and understand it, they often could not report specific pedagogical moves they made in the classroom to help students improve their ability to do so.

In Lew, Fukawa-Connelly, Mejia-Ramos, and Weber (2016), a lecture on real analysis was video-taped, the instructor was interviewed about his lecture, and the researchers watched the lecture, in order to determine what they took to be the main mathematical ideas. Six students were then interviewed concerning their interpretation and understanding of the lecture. The findings were that although the students indicated some understanding, they were not able to grasp the main ideas of the lecture. The authors postulate three reasons for this: the use of colloquial terms like ‘small’ in a context where the true meaning was something more like ‘arbitrarily small’; the students had a readiness to view proofs as a sequence of calculations, rather than the unfolding of one or more key ideas or methods; informal but important comments were only spoken by the instructor but not written.

A later study (Fukawa-Connelly, Weber, & Mejia-Ramos, 2017) examining 11 video-recorded mathematics lectures and photographs of the lecture notes of 96 students found that though informal content was common, most of it was presented orally and that typically students did not record it. The questioning practices of lecturers of advanced mathematics courses were examined in Paoletti, Krupnik, Papadopoulos, Olsen, Fukawa-Connelly, and Weber (2018). Olsen, Lew and Weber (2020) examined the metaphors frequently occurring in advanced mathematics lectures.

A very useful study by Artemeva and Fox (2011) categorized and analyzed data (video-recorded lectures, observational notes, interviews and written artifacts) collected from the lectures of 50 mathematics instructors of undergraduate mathematics from seven countries. Framed by rhetorical genre theory and activity theory, the authors determined and discussed a few of the typical and recurrent elements of the genre of undergraduate mathematics lecturing. They found across all the local contexts a "highly complex and underexplored pedagogical genre" (p. 355) which they called *chalk talk*.

Features and categories of this genre that they highlighted were: running commentary (a continuous spoken accompaniment to their writing accounting for every part of it), metacommentary (occasions when the teachers would switch from accounting for the writing to talking about what they were writing), board choreography (erasing parts of the board no longer needed, dividing the board into sections, enclosing significant writing in boxes), the use of lecture notes or scripts (many participants acknowledged that their preparation for lectures included creating or reviewing such notes, many used them while teaching), discursive signaling ("shifts in *action* during the enactment of chalk talk", p. 362, using expressions indicating logical relationships, or expressions inviting student responses and so on), and the beauty of mathematics (most of the participants commented on the beauty of mathematics in their interviews, and therefore the opportunity they have while lecturing to help exhibit it to students).

In Fox and Artemeva (2011), the phenomena of interest are the positions of the instructor with respect to the board and the students, and the focus of the instructor's gaze, that they found to recur across the lectures of the 50 mathematics instructors in their various local contexts: facing the board with intermittent looking at lecture notes; half-turn away from board, focus still on board (or notes); three-quarter turn away from board and toward class, intermittent focus on board; full turn away from board facing class. They associate these configurations with the functions that they see them serving. Facing the board, the instructor can write while voicing a “meaningful reenactment of established mathematical practices” (Cobb, 2000, p. 20). The half-turn gives the instructor an opportunity to check or confirm their most recent activity. The three-quarter turn features meta-commentary from the instructor, a fuller stop to the ongoing writing, and an opportunity to further explicate the recent actions. The full turn invites student questions.

The authors comment on the complexity of lecture pedagogy:

> we witnessed a remarkable lecture in which the class sat in rapt attention, all eyes on the professor for nearly two hours. Not a word was spoken by any of the students in the class during the lecture. Rather, they wrote, observed, consulted their notes and textbook, while concentrating fully on the professor's performance in the front of the room. At the end of the lecture, the class erupted in talk. Four students rushed up to the professor to speak with him, to question, to extend their understandings, and express their appreciation. Other students in the class began to discuss the lecture themselves. (Fox & Artemeva, 2011, p. 97)

They note that from one viewpoint, such a lecture was lacking in student interaction; they argue “this is a highly reductive view” (p. 97).

Interviewing undergraduate mathematics students about their experiences in the classroom, Rodd (2003) was struck by similar aspects of what a mathematics lecture can be like for students, which she regarded as under-represented in the literature: “students are kept ‘rapt’ by the mathematical ideas presented and are 'lifted' to reflect and review afterwards” (p. 18). Where Fox and Artemeva spoke of the “cinematic art of teaching university mathematics” (the title of their article), Rodd’s theoretical perspective is informed by work on the processes by which texts of plays become live performances in front of audiences, quoting from Mudford (2000): “A performance continues in the mind after the performance is over; and we ask ourselves what we have witnessed” (p. 8). Rodd is alive to the importance of inspiration, awe and wonder experienced in the mathematics classroom, and she invites us to reconsider the mathematics lecture as a site where students can enter the mathematics community and develop their own identity as mathematicians. Further research on undergraduate mathematics lectures (Marmur, 2019; Marmur & Koichu, 2021) isolates and illustrates a theoretical construct named “Key Memorable Events”. These are events occurring in the mathematics classroom that are memorable and meaningful to students and associated with strong emotions, positive or negative; occasions that combine affective and cognitive significance for a student.

Some research on the practice of mathematics lecturing heightens the focus on materiality within the interaction (Barany & MacKenzie, 2011; Greiffenhagen, 2014), observing and analyzing in detail the inter-dependencies of material media, notably chalk and the blackboard, with mathematical thinking and writing. Both articles contain multiple examples of lecturer-with-blackboard behaviours; Barany and MacKenzie study the mathematics seminar and Greiffenhagen the graduate mathematics classroom. A selection from the Barany and MacKenzie article includes: pointing and tapping boards at places where assumptions or conditions have been written at the moment the ongoing explanation requires them (p. 13); blocking expressions with the hand to momentarily remove them from consideration in the current argument (p. 13); re-invoking an earlier written argument by tracing it top to bottom with the hand (p. 13); taking care with the positioning of expressions so as to separate individual parts or to group them for the purposes of their argument (p. 13).

Greiffenhagen highlights the erasing of asides and ‘side proofs’ which have been used and are no longer needed, and the reluctance to erase portions of the main argument (p. 518); separating the board into three separate regions, the first reserved for writing results from the previous lecture, the second for recording the main steps of the developing proof; the third for ‘side proofs’ (p. 516); drawing boxes around certain expressions to visually foreground that they go together, or to indicate key links in the chain of an argument (p. 513). In both papers the verb ‘unfolding’ is used in key moments to describe what the lecturer is doing at the board.

Barany and MacKenzie also note that there is continuity between the writing–thinking practices of the mathematician working publicly in the seminar and the writing–thinking practices of the mathematician working privately in an office. A similar observation is explored in Greiffenhagen and Sharrock (2011), who conclude that, “we have tried to show that ‘working behind the scenes’ in mathematics does not involve being exposed to a kind of reasoning that conflicts with the kind that appears ‘up front’” (p. 32).

Greiffenhagen (2008) contains description and analysis of short scenes of mathematics lecturing that are closely related to some of the work in this thesis. Going beyond merely noting that pointing and tapping are frequent in mathematics lectures, he shows that the manner in which the lecturer organizes a sequence of movements of his hand and pen expresses a meaningful approach to what is, for the lecturer, going on in the current mathematical move. For example, in one sequence, with two items on the board currently being engaged with, the instructor initially points to the first item and withdraws the pen, then, later, he points to the first item and keeping the pen close to the board he moves to point to the second item as well. Greiffenhagen interprets this sequence as a gesturally achieved mathematical distinction between treating these two items either as one quantity (in the first case), or as two.

Another key observation in this paper is the instructor’s frequent use of verbal boundary markers serving to close and open episodes: the example is “right”, which the instructor uses to indicate the beginning of a new activity. Greiffenhagen calls these “formal markers”, citing Turner (1972, p. 369). In the next chapter, when I discuss the work of Bateson relevant to this research, I will return to such moments when the instructor punctuates the ongoing interaction.

2.2. Summary

In this chapter, I have collected some of the research on university mathematics lecturing that is marked by a serious reckoning with the aspects, themes and structural features of the dynamic unfolding process or activity of mathematics teaching. This prior work has brought out the importance of mathematical writing, including a sequential analysis of how this writing emerges in practice, the importance of the board on which the writing occurs, the positioning and orientation of the instructor with respect to this space, the physical interaction of the instructor’s hands with the board, and the ongoing complex relationship between what the instructor says, what the instructor gestures, and what the instructor writes.

In the next chapter, I develop the perspective on gestures and the body that will be adopted in this research.

Chapter 3.

Analytic Framework: Interaction, Contexts and Practices, Gestures and the Body

3.1. Introduction

In this chapter I focus primarily on the work of three people, with a view to indicating how my approach to the data and phenomena of this research was shaped by their ideas, methods, distinctions and observations. I have tried to isolate my attention on the most central figures, those who have most directly and strongly influenced how I saw, and what I thought and wrote about what I saw, as I watched J's lectures and tried to understand what he was doing and how he was doing it, moment by moment.

These three people are George Herbert Mead, Gregory Bateson, and Jürgen Streeck. Though they have each written extensively on a wide variety of topics, they nevertheless share enough with each other that I can consistently weave a theoretical view from my selections from their writings. It will be obvious that I am not trying to comprehensively summarize their 'positions' in some wider landscape, even if it were possible to do so. Mead and Bateson wrote for many years and cannot be pigeonholed into lockstep agreement with their former selves, while Streeck is still writing and growing his understanding of the topics he cares about with each new work, sensitively adjusting his former stances, while at the same time clearly deepening some existing, continuing understanding of these phenomena that concern him.

At the same time, none of these thinkers ever suffered a 'break' in their theoretical approach, as might be said of Wittgenstein between his two major works, to take a familiar example. Each thinker can, I think, be seen to be continually developing and enriching their elaboration of an ongoing intellectual commitment to certain fundamental axioms or views of the world, the people in it, and the nature of communication and meaning. Capturing these intellectual commitments in as few words as possible is worthwhile, even though it may do considerable violence to their unique, rich, detailed, closely analyzed and, above all, creative and fertile contributions. Mead argues that social interaction is the precondition for the development of mind. Bateson argues that context and actions reflexively co-create each other. Streeck emphasizes the making body-in-the-world as essential to understanding and appreciating human gestures and how they make meaning.

Bateson cites Mead at critical junctures in his writing, Streeck cites both at critical junctures of his. It seems to me possible to form a coherent and consistent theoretical framework in the following way: model my framework on that of Streeck, since the work he has done rings true to me; note that Streeck early on cites Bateson and Mead as influential predecessors shaping his perspective, and explains why; read Bateson and Mead closely myself; if I happen to find insights and observations that seem important for my purposes, and which Streeck has chosen so far not to highlight in his written work, incorporate them into my framework. This is what I have done. I treat the three authors in historical order.

3.2. George Herbert Mead: interaction; conversation of gestures; the role of the other; I and the me; the hand

Mead wrote and lectured extensively for many years, devoting his energies to philosophy. He is considered, together with John Dewey, William James and Charles Sanders Peirce, as one of the classical pragmatists. He published many articles but no books in his lifetime. His most famous work, *Mind, Self, and Society* (Mead, 1934), was posthumously published, and was constructed by an editor from notes taken by students, and a professional stenographer, from Mead's lectures. Huebner (2014) undertook an extensive and detailed archival investigation, succeeding in assembling "a substantial and unique dataset of correspondence and notes from which to reconstruct the process of creating this text" (p. 116). Based in part on this work, Huebner and Joas published a revised edition of Mead's book (Mead, 2015) which includes a lengthy and extremely useful appendix that reveals paragraph by paragraph the sources that the original editor, Charles Morris, used to construct the original edition. Here is one notable example: the original subtitle of the book, *From the Standpoint of a Social Behaviorist*, was Morris' own invention; Mead "never used the phrases 'social behaviorist' or 'social behaviorism' in any extant lecture manuscripts or published writings" (Huebner, 2014, pp. 130–1).

In what follows, I try to briefly flesh out a few of the notions that will be useful for me in this work: Mead's consistent commitment to empiricism and naturalism while avoiding positivism and reductionism; his emphasis on the importance of temporality in understanding the 'world that is there', a 'process philosopher' before the term became popular; what he means by the 'conversation of gestures', and 'significant symbols'; his story of the development of the mind and the self out of the interactions of a growing child with others, in particular in play and in games; and what he means by the 'I', the 'me', and the 'generalized other'.

Although Mead considered himself a philosopher, his reputation first grew in the domain of sociology; some of his ideas, as altered by his student Herbert Blumer, developed into the field known as symbolic interactionism (Blumer, 1969). Over time, his influence has grown in multiple fields, with ever-more careful readings performed by each generation, finding anew Mead's original voice and his productive and engaging ideas (Burke & Skowronski, 2013; Carreira da Silva, 2007; Joas, 1980/1997; Joas & Huebner, 2016; Miller, 1973; Natanson, 1956).

Mead seems eternally modern. One reason, perhaps, is his long and searching interest in the history of science (for careful discussions of Mead's writings on science, see Carreira da Silva, 2008; Puddephatt, 2008). Indeed, he has some reason to be thought of as one of the great historians of science, reading the works of scientists from his past to his own contemporaries. His reading of Charles Darwin, for example, is thoughtful, and it is clear that though Mead does not do philosophy in order to reduce it to science, he believes it important that whatever he philosophizes about ought at least to be consistent with scientific findings, and more importantly, consistent with the method by which scientists actually research the world.

What is most exciting about Mead's approach is that he develops a story of the development of mind that is consistent with the story of how it is that scientists advance their research. In other words, rather than a simplistic history of science, that assumes along with an unreflective scientist that the way science progresses is, for example, to break things down into parts, and to posit abstractions out of which all experimental phenomena are now somehow better understood, Mead correctly begins and stays with what actually happens: "The research scientist starts from a specific problem that he finds as an exception to what has been regarded as a law" (Mead, 1936, p. 264). Mead goes on to emphasize that such problems are not only stumbled upon but actively sought: “There is no phase of the world as we know it in which a problem may not arise, and the scientist is anxious to find such a problem” (p. 265). Mead also sees the creativity in the ‘I’ in the finding of solutions to present problems, bumping us out of habit and routine. In later chapters it will be observed that gestures of the hands share this double role of enacting conventionalized or patterned movements as well as adapting such movements, or improvising hybrid combinations of movements, for the perhaps never to be repeated needs of the present moment and contextual conditions.

This is what I mean by his empiricism, his naturalism. He kept up to date with the findings of psychology; he wrote about relativity soon after it was developed. His interest in scientists was professional: he cared about the details of their work – the new laws, the new discoveries – but he also cared about the ways that scientists as a group periodically radically changed their perspective on the workings of the world, the attitude they took to it. His lifelong friendship with Dewey meant that the two shared and communicated their ideas with each other. Dewey speculated that many of the ideas Mead developed over his lifetime stemmed and flowed from what he referred to as Mead’s “haunting question” (Dewey, 1932, p. xxxvii). Dewey described the Mead of his first acquaintance as being concerned with the problem that consciousness is personal and private. Dewey notes that at the time a widely accepted ‘solution’ was idealism: “Mind as consciousness was at once the very stuff of the universe and the structural forms of this stuff; human consciousness […] was at most but a variant, faithful or errant, of the universal mind” (p. xxxvi).

Dewey characterizes Mead as not only not accepting such a solution, but instead as observing that it did not even answer the problem of real interest to Mead. Consider the private and personal thoughts of a Galileo, for instance – hesitantly entertaining skepticisms about erstwhile commonly held beliefs about the world, and cautiously raising new hypotheses. How can these thoughts later become objects shared by a community of scientists, belonging to all? Mead’s lifelong development of his understanding of the fundamental role of temporality is here as well: how can the new (in this case, a new set of ideas that Galileo wrote about and spread) emerge from the old. In *The Philosophy of the Present* (1932) (Dewey contributed the prefatory remarks) Mead explores these questions in ways far beyond my needs in this thesis. So far, I am taking from Mead the following: a shared commitment to observe exactly what is the case and not to posit abstractions-as-dogma, but rather to posit as few as possible abstractions-as-tentative-postulates. This I take to be a rough description of naturalism and empiricism. Dewey, writing in the months after Mead's death, admired Mead's manner of perceiving the world: "The power of observing common elements, which are ignored just because they are common, characterized the mind of George Mead." (p. xxxviii). As Mead himself puts it: "Science does not attempt to set up a dogma, as I have already insisted; and, of course, science cannot tolerate any other person's setting up a dogma." (Mead, 1934, p. 274).

For Mead, gesture is the isolatable beginning of the act that another person can use to anticipate the rest of the act. In his running example of the dog fight, Mead analyzes how one dog responds to another (Mead, 2015, pp. 14–15, pp. 42–3, p. 63). The very beginning of one movement by the first dog – a snarl, a feint – is for the second dog a *gesture* that indicates to the second dog what the next movements of the first dog will likely be. When the second dog responds with a movement of their own – backing up a step, circling around, snarling louder – the first dog interprets these gestures as the very beginnings of acts that would follow. The interaction here is what Mead refers to as a *conversation of gestures*. Sometimes Mead uses the word *attitude* to connote the readiness of an animal to behave in a certain way.

Mead argues that human interaction evolved from such conversations of gestures, and still contains these, but with the additional feature of what he calls *significant symbols*, or language: these are *vocal gestures*. People respond to vocal gestures as the beginnings of the acts that would follow. Mead suggests that in *play*, children take one role and then the other. First playing at being the mother, then the child, when playing with a doll, for example; or first playing at being a member of the police and then the criminal. This is what Mead calls taking on the *role of the other* or the *attitude of the other*. He argues that it is through such play that people learn how to interpret the meaning of vocal gestures.

Mead also emphasizes the importance of *games*. Here children learn how to take on the attitude of all the rest of the people who are involved in the game. For Mead this is the birth of the ability to take on the role of what he calls the *generalized other*. Mind, for Mead, emerges from *social interactions*. He describes thinking as a conversation involving significant symbols that is an interiorized version of the adoption of one role and then the other during play and games and the other social interactions of the thinker's history. He gives the name *me* to the interiorized generalized other, and the name *I* to the part of the self that responds to the contribution, the vocal gesture, of the *me*.

Mead's fundamental insistence, repeated in a variety of forms in his lectures and writings, is: "the unit of existence is the act, not the moment" (Mead, 1938, p. 65). In this research, J's mathematical acts are consistently the dominant concern.

3.3. Gregory Bateson: visual ethnography, frames, metacommunication, natural history

Gregory Bateson was an interdisciplinary thinker, whose work significantly impacted different fields such as psychiatry, anthropology, linguistics, semiotics, and cybernetics (Bateson, 1972, 1979). His writing is characterized by observation-driven constructions of abstract concepts about systems and processes, and the thoughtful and measured application of them to fundamental questions concerning the natural interactions that occur between people in the world.

Bateson began his career as an anthropologist. Together with the anthropologist Margaret Mead (in the first portion of this section, "Mead" refers to "Margaret Mead"), he lived in Bali for two years shortly after they married, and they conducted a pioneering study in ethnography, notable in part for the important role that photography and film played in their method and analysis. As they explain in the preface of Bateson and Mead (1942), they had each, separately, previously published work that had been critically received, but which shared the goal of communicating unambiguously and clearly to a scientific community, insights about facets of culture that up to then only artists had captured.

Mead's approach had been to describe "those intangible aspects of culture which had been vaguely referred to as its *ethos*" (p. xi); Bateson's approach had been to attempt to demonstrate that categories like *ethos* were not collections of specific bits of behaviour, but instead abstractions that could be applied to all behaviour. Mead's method was criticized for being too synthetic and Bateson's for being too analytic. In this work, they attempted an "experimental innovation" (p. xi): they placed on the same page multiple jointly germane photographs of the Balinese "moving, standing, eating, sleeping, dancing, and going into trance" (p. xii). Mead wrote a long introduction to the Balinese culture as a whole; Bateson wrote short accompaniments to the pages of the photographs. This way the reader could look at each photograph on the page while also attending to the abstracting concept that related them; they hoped this would avoid, on the one hand, overly fictionalizing the cultural behaviour and, on the other hand, presenting insufficiently few actual items of behaviour.

The book is counted as an early instance of visual anthropology. Bateson and Mead were married in Singapore just before they arrived in Bali. Their book marries their methods of analysis by using the hundreds of photographs it contains as the interface between the two. They examined together thousands of pictures, placing and arranging them together this way and that way, and from this side-by-side looking they abducted many of the themes they develop in their work. In my research, there was also a period where I watched scenes from the course of lectures in sequence, having grouped them together for one reason or another, seeking patterns and commonalities, or seeking to find where the patterns did not fit.

Further, the book itself contains numerous images concentrating on the positioning and configuration of the hands of the Balinese. Mead and Bateson observe that the Balinese form hand postures that serve to accentuate the sensory and exploratory functions of each fingertip, and also that the Balinese seem to emphasize their left hand in their dances and when making their art. Related to this observation, they note that the Balinese habitually run their fingers over their own bodies seeking irregularities and protuberances of the surface, and continually readjusting their costumes and their hair. There is a particular focus on self-touch of the mouth with the hands. These are a selection of a few examples of close attention to the details of how the Balinese actually use their hands as they go about their day-to-day life, interacting with objects of the world and with themselves.

Bateson regularly attended the Macy conferences on Cybernetics (1946–1953), and on Group Processes (1954–1960), which gathered together thinkers in multiple disciplines in order to explore questions related to the overarching theme of understanding the human mind. The format of the conferences was that of conversation: the goal was to spark new ideas from the frank and direct interaction of members of disciplines that bordered each other. One such conversation (Bateson, 1956) has become extraordinarily influential in most disciplines interested in communication of some kind. Bateson launches the discussion, and for the first few pages the conversation contains short comments or questions from others interspersed with rather longer responses from Bateson. After this, however, many of the others now speak for extended paragraphs just as Bateson did. In addition, it more frequently happens that multiple speakers contribute their perspective in a row before Bateson speaks again. I count twenty-four speakers, including Bateson. Disagreements are registered immediately and clearly, multiple fascinating extended examples are shared, definitions are hazarded and rejected, on and on for about a hundred journal pages.

The discussion touches on several themes. First, Bateson introduces the topic by observing that in his experience, when most people talk about play, they are inclined to say what it is *not*. His examples are “it is not real” (Bateson, 1956, p. 145) and “it is not serious” (p. 145); much later Erik Erikson, the developmental psychologist, will add “it is not work” (p. 201). Bateson says that he would like the group to first consider the word *not*. He draws a circle on the board, calls it the class of chairs, and asks the group to quickly, without too much reflection, offer up instances of the class of not-chairs. There is an ellipsis in the transcript at this point, followed by Bateson announcing the list of suggestions he received (tables, dogs, people, autos). Now he suggests one of his own: “tomorrow”. He asks if this example makes the group uncomfortable. He goes on to distinguish between the class of proper not-chairs (with members like “tables”) and the class of improper not-chairs (with members like “tomorrow”), and he says this is his way of getting at the meaning of the *not* in statements like “play is not serious”. He suggests that play, like when a child might play at being an archbishop, might be a way in which people learn how to structure categories of not-objects more generally.

Twelve pages into the discussion Bateson announces a new beginning: “Let me try an approach from another angle” (p. 157). He now begins what is likely the most famous example from this discussion. He says he went to the zoo to see if he could find an observational basis for determining when an animal sends a signal while also revealing some degree of awareness that they were sending a signal. He watches otters play. The messages at one level are, Bateson says, the bite, the scratch, the flip. Apart from a few occasional moments of uncertainty, Bateson reports that the majority of the time the otters are somehow also labelling the class of these bites and scratches and flips as “play”, which he refers to as a message that is one logical type higher.

I end the close look at this discussion here. This is an early and clear presentation by Bateson of two themes that were important for him and other communication theorists: frames and meta-communication. The movements and actions of the otters were meaningful in a particular way for the otters because they occurred within a particular frame or context, that of ‘play’; and this continuously successful achievement of the shared frame was a temporally and sequentially continuously successful act of meta-communication: what was being communicated was *about* the ongoing communication itself.

In this thesis, both of these themes play important roles. In interaction with the students in his class, J begins, sustains, and ends contexts within which the movements of his body and hands are meaningful, in part because they occur within those contexts. In addition, the movements of his body and hands, accompanied by his speech and creation of objects of textual writing, must necessarily succeed in certain goals so that a local context can be ended. J touches terms, words and gaps in the writing, points, traces, sweeps, and holds pieces of the already written, while continually sending the message ‘this is justifying this step of the proof’ and simultaneously the message ‘this is proving this result’. When the justification of the step is finished, the smaller context is over, but the larger one continues; when the proof is over, the large context is finished, and new large context may be begun; for example, it might be time to initiate a context within which the meta-communicative message is ‘this is defining’. I take up these considerations in Chapter 4.

In the academic year 1955–1956 a project called *The Natural History of an Interview* (I sometimes use the acronym *NHI*) was initiated at the Center for Advanced Study in the Behavioral Sciences by a group of scholars: Frieda Fromm-Reichman and Henry Brosin, both psychiatrists; Norman McQuown and Charles Hockett, both linguists; Ray Birdwhistell, an anthropologist keenly interested in communication with the body, and Bateson, who joined the collaboration at the beginning of 1956 (for detailed and authoritative histories of this influential project see Leeds-Hurwitz, 1987 and Leeds-Hurwitz & Kendon, 2021).

The project was born in conversations between Fromm-Reichman and McQuown. Fromm-Reichman was aware that, during therapy sessions, there were moments when she intuitively understood or grasped something meaningful about her patient (Fromm-Reichman, 1955), and there were also occasions when her patients arrived at significant realizations about themselves; she sought to examine and understand how these moments came about. Part of her goal was to be able to teach other therapists, including her own students, what it was she was doing in those episodes where the interaction between her and her patient seemed to be generative of insights and growth. In addition, according to Birdwhistell, she "was losing her hearing, and knew she needed to *see* with more control" (Leeds-Hurwitz, 1987, p. 5).

Having made some audio recordings of her sessions, she invited McQuown to analyze them with her together. It rapidly became clear that analyzing the speech alone would not be enough, and so they reached out to others at the Center; when they realized that the analysis of body movements would be essential, they invited Birdwhistell. In turn, Birdwhistell, having worked with Bateson before, suggested inviting him to the group, knowing that Bateson had made films of families with at least one member who had attended psychotherapy, and that some of these might serve as the data to be analyzed.

Bateson did bring film, and the group viewed films of a number of episodes: an interview of 'Doris' by Bateson with her son 'Billy' playing in the room; a conversation between Bateson and Doris' husband Larry while Doris cooked; a scene where they all ate dinner; Billy playing outside; Larry giving Billy a bath; a party at the house. The group analyzed in very close detail only a few scenes from Bateson's interview of Doris; watching all of these films was considered to be crucially important in setting the right background for accurate interpretation of the scenes selected for analysis. McQuown broke down the method the team used to analyze the data into the following steps (McQuown, 1971a, p. 5). First was "'soaking' (multiple viewing-listening)". Second was "scene selection and intensive study". Third was matching (in time) body movements with speech. Fourth was "uncovering the interaction profile".

Transcription was not explicitly mentioned in McQuown's list but it was a key part of their analysis, was the most time-consuming, and required the most space when the project was written up. In fact, the *Natural History of an Interview* was never published, and was available for many years only on microfilm (McQuown, 1971b). It is 982 pages long. Audio of the scenes to be analyzed were transcribed by McQuown from tapes with the sound alone; soundless video of the scenes was transcribed by Birdwhistell; the records were matched by frame numbers of the video. The manner of transcription and the analysis of the scenes formed an interconnected complex: an existing portion of the transcript would help them notice structural units or behavioural items in the data, which they would seek in other portions of the scenes, which would suggest changes in what to capture in the transcription and how to capture it, and so on.

Bateson contributed the chapter titled *Communication* that served as the theoretical framework for the project (Bateson, 1971). In it he cites Mead as the grounds for their treatment of interactions, and he cites the Gestalt psychologists as their analytical background for punctuating an interaction into local contexts. In the next section I discuss how Streeck adapts from Bateson, and Mead, an approach to studying interaction which is characterized by close attention to the hands and the body of the participants, and to the materials of the local environment, as the interactants make meaning in contexts they themselves define.

3.4. Jürgen Streeck: the craft of gesturing; making the self

In this section, I review those aspects of the work of Jürgen Streeck directly related to this research, concentrating on two book-length treatments of his long-term research on gestures, the body, and communicating in interaction with others, using materials and tools and other objects in the world (Streeck, 2009, 2017).

Streeck (2009) notes that much work on gesture has centred on a particular ecology, by which he means, “a distinct pattern of alignment between human actors, their gestures, and the world” (p. 7): the ecology in question is the *experimental* situation of a subject in a lab talking about actions or events that they lived through. By contrast, Streeck is interested in studying gesture as it occurs in everyday life, closely analyzing video-recordings of *naturally occurring interactions*: examples include a late-night host talking with a guest; two friends sitting in a living room at home, chatting; a rice-farmer discussing fertilizer with her nephew while both are standing on a hill near her farmhouse; an auto-mechanic interacting with a customer. Streeck (2009) identifies six *gesture ecologies* which he names as follows: *making sense of the world at hand*;

disclosing the world in sight; *depiction*; *thinking by hand*: *gesture as conceptual action*; *displaying communicative action*; *ordering and mediating transactions*.

The last two ecologies are concerned with the communicative process that the gesturer is involved in, and Streeck uses the term *pragmatic mode* to refer to both. He describes *displaying communicative action* as involving “the use of the hands in the embodiment of communicative action” (p. 10), observing that it incorporates gesturing that: reveals or heralds the sort of communicative act now occurring or about to occur; indicates features of the structure of the speech currently emitted by the gesturer; presents “the stance that the person takes towards an utterance or the content expressed” (p. 10). This last consideration is of especial importance for this work. I will later show (in Chapter 5) that the lecturer, J, is frequently concerned with indicating, with his hands and body, his attitude towards what he is saying, the way or manner or angle with which he views his current or recent or upcoming utterances.

The other ecology in the pragmatic mode (*ordering and mediating transactions*) is “other-centered” and “addresses *other* interaction participants, whose actions it is intended to regulate” (p. 10). This conversational ecology occurs more rarely in my data corpus; when it does, it can sometimes be significant. Several notable occasions when J seeks the response of an individual student of his choosing from the class as a whole are discussed in Chapter 7. I show in Chapter 4 that there is a good deal of student involvement in the course. In Chapter 7 I analyze one incident where a back and forth between J and a student goes on for many turns.

Streeck sometimes uses the term *conceptual mode* to distinguish the ecology *thinking by hand* from the two ecologies discussed so far under the umbrella *pragmatic mode*. The conceptual mode is similar to the pragmatic mode in that both sorts of gesturing occur in a manner that we might refer to as ‘under the radar’. Neither the gesturer, nor the other speakers in the interaction, seem to be paying attention directly to the gestures. The gesturer is not looking at their hands while they gesture. The contrast to the pragmatic mode is that *thinking by hand* refers to occasions when a speaker’s hands are giving form to the content of their spoken words. A useful term subsuming the gestures from all three of these ecologies is *gesticulation*, which appears in Streeck (2009), but features more prominently in Streeck (2017).

When the speaker or other members of the interaction direct their attention to the hands of the speaker, “turn away from the world” (Streeck, 2009, p. 9) and “by their bodily orientation and positioning in space, mark off the space between them as territory of their interaction” (p. 9), this is the gesture ecology called *depiction*.

Streeck describes and analyzes a few subtypes of depiction. One method of gestural depiction is *mimetic gesturing* – “the performance of gestures to depict physical acts or behaviors” (p. 144). He distinguishes two sorts of mimetic gesturing: one where the gestural depiction of some real-life act serves to depict an object or some other implement involved in the action, and another where it is the act itself that is focus of the communication. He calls the first *handling* and it is the most frequently occurring kind of depiction in his corpus: here, the gesturer configures their hands and moves them as if they are holding an object and doing something with it, like lowering or raising it, picking it up or putting it down, in the cases of more generic types of object; or an enormous variety of more specific actions tailored to the particular affordances of the object evoked by the gestures.

The second he calls *mimesis*. One of his examples of *mimesis* is an interaction participant re-enacting a moment in her experience of a car crash, when her forehead was hit by the rear-view mirror: “She slowly brings her left hand, the palm facing up, before her face, freezes the motion, and intently looks at her hand. [...] Then she lightly hits her forehead with the bottom of her hand” (p. 147). Streeck notes that the duration of mimesis can vary, from a few moments being enacted, to longer pantomimes or caricatures, on to fuller, near-theatrical depictions of a former self or another self in some dramatic situation.

The distinction between these two kinds of mimetic gesturing is not useful often enough in my analysis to maintain. J frequently flips over, rotates, reflects, and otherwise manipulates the air in the way one would manipulate a physical object in that space: both the object itself and the transformation it undergoes are the focus. In Chapter 5 when I identify and analyze such a practice I will use the term *manipulating the object*.

The two remaining ecologies are defined by the relationship between the hand of the gesturer and the world around them. There are the parts of the world that are near at hand that can be touched and felt, and there are the more distant parts that can be pointed at. In this work I will use Streeck's term *spotlighting* to refer to these.

Streeck discusses sub-varieties of each. One such class is *gestures of orientation*: by conforming their hand into various configurations while pointing near or touching something, gesturers can elevate a figure from a ground while simultaneously *construing* this figure in a particular manner: a car mechanic moves a hand back and forth with his index finger extended towards a group of tires, gathering them into a collective. A second class is *tracings*: the car mechanic uses his finger to trace along a head-gasket, saying "we have bad leak here" (p. 70); the customer repeats the tracing with their own finger. A third class has to do with the *decomposition and reassembling* of some object at hand, highlighting some part of the multi-part object, and some aspect of this part with respect to the whole.

Streeck (2017) studies a full day of the actions of a car mechanic, Hussein, who he recorded on video interacting with his employees and his customers during a regular workday. Streeck is interested in his *practices* – his "*methods for doing things*, for performing social actions" (p. 7) – and he notes the following features of practices: they are "embodied", they include practices for "instrumental, including solitary, actions" as well as "social, communicative acts", they may comprise "the acts of only one or a combination of body parts", they are "skilled, methodical" (p. 8), they can be "applied to new circumstances", they are "adaptive", they are "nested". He also observes that the term *practice* is "scalable", applicable to short and long time scales.

In this thesis I use a micro-ethnographic, natural historical methodology, as adapted from the way Streeck uses such a method. I use his terminology of spotlighting, depiction, and gesticulation. I too am interested in investigating the practices of an expert; in my case, the practices of a mathematics lecturer.

3.5. Research Questions

In this final section I state the three research questions of this study.

What are the features, components, structures and functions of the kinds of practices by which this lecturer, within the local hierarchical context of the ongoing interaction, moment-to-moment, sequentially, publicly, and accountably, creates the next *pieces of mathematical writing, while speaking, moving his body, and, most importantly, gesturing with his hands?*

What aspects of these gestural practices emerge prominently, and how do they co-operate, during those occasions in the course when a mathematical object (the dihedral group of order eight indicated as D_4) is at the centre of the ongoing lecturing interaction?

What aspects of these gestural practices emerge prominently, and how do they co-operate, during those occasions in the course when a mathematical notion (well-definedness) is at the centre of the ongoing lecturing interaction?

Chapter 4.

Methodology: generating and using data documents from video

4.1. Introduction

A micro-ethnographic, natural historical approach to the mathematical actions J took while teaching means that even scenes that only last for a few minutes will contain a wealth of notable and observable behaviour that matter for the analysis: an individual action like touching the board, or turning to face the class, or shrugging, and other such examples, might each only take a second; moreover, two or more of them can, and often do, happen simultaneously. The most significant challenge in performing a micro-ethnographic, natural history analysis of the mathematical actions that J took while teaching a course of group theory lectures is developing tools which allow the researcher to notice, name, record, select, prioritize, and cross-reference these highly numerous observed actions. In this chapter I detail the process by which I developed these tools to make clear how it was I used them once they were complete, and to explain how some of my eventual understanding of J's actions was borne out of this process.

I will refer to these tools as *data documents*. There are six of them. The first is the *Speech data document*, which I will refer to as the *S transcript*. To a first approximation, it can be described as a record of all the speech that occurred in the group theory classroom during the course. A deeper explanation of the making of the *S transcript* discloses the fundamental role of a certain unit of interaction – a certain time duration of a local context defined by J's own actions – in organizing and separating behaviour into smaller interaction periods. In addition, because the ending of the local context is an indication that J's aim or purpose for that context has reached a resolution – often achievement, sometimes partial; occasionally failure – the meaning of J's gestures, movements, speech and writing, and the speech of the students, within the context, are more readily determined and interpretable. A heap of ten thousand actions in a lecture is recognized as a structured hierarchy of segmented contexts within which a more manageable number of actions are combining to form a clearer interaction sequence.

The second is the *Writing data document*, which I will refer to as the *W transcript*. To a first approximation, it can be described as a record of all the writing that occurred on the whiteboard in this group theory course. It is obvious that, in all of J's lectures, he has the goal of writing on the board some subset of the notes he has pre-developed for this course. In addition, it quickly becomes clear in watching J teach that a large fraction of this writing only appears on the board, or stays on the board, as a result of J arguing for, or justifying, or motivating this writing. In other words, one primary goal of many local contexts is to be *allowed to write* the next piece of writing. Therefore, although the focus of this research is not directly on mathematical writing, in section 4.3 I consider in some detail an example of mathematical writing which has been contributed to by many members of the mathematical community, in order to draw out those features of mathematical writing that the *W transcript* reflects, and that also help shape, J's gestural behaviour.

The third is the *Pictures data document*, which is an alternative perspective on all the writing in the course, and which I therefore also discuss in section 4.3. It consists of a series of snapshots from the videos of all the lectures. The snapshots are taken at those moments when full boards have been completed, and when J is not blocking any of the writing by his position.

In section 4.4, I discuss the *Episodes data document*. This is a high-level summary of each lecture, organized in a way that reflects the structures in J's speaking and J's writing that were captured in the *S transcript* and *W transcript*. This document therefore served to co-ordinate these two, as well as remind me of the short list of major achievements of each lecture. The *Episodes* data document can be thought of as a more formal version of what a lecturer might say was covered in class that day to a student who was absent.

In section 4.5, I detail the creation of the *Student Contributions data document*, which captures all the occasions when students speak, a concise description of who said it, what they said, and why they said it, whether it was correct or convincing, and how it was J responded. Many of J's actions and choices are only intelligible, of course, by tracking the other interactants in the mathematical conversation. It will be seen that the number of words spoken by a student in an interaction is not a reliable guide to the impact that the engagement has on the ensuing interaction. Some of the major scenes to be analyzed in the following chapters involve important contributions from individual students.

Finally, in section 4.6, I treat the *Gestures and the Body data document*, which came in two forms, the *original* and a *compressed* version. This is a record of the movements of the hands and the movements of the body that occurred in all the lectures, except for six of them. I explain what guided my selection of which gestures or body motions to record in the original data document, and I discuss the criteria by which I compressed it into a more manageable, searchable form.

I used all these data documents together. Watching a scene repeatedly is, of course, a necessary component of arriving at a sensible analysis of what J is doing with his hands and why he is doing it. Informed by the example of the *Natural History of an Interview*, and also by Streeck's work, the best way that I knew how to arrive at a collection of scenes whose analysis would reveal what is typical, what is representative, of the mathematical actions of a lecturer, and also what is unusual, what is anomalous, in the mathematical actions of a lecturer, was to go through this process where I created these documents as rigorously, carefully, and consistently as I could, and then wield these documents as tools to find, collect, compare, and contrast scenes from everywhere in the course.

4.2. Features of Mathematical Speaking: creating the S transcript

In sub-section 4.2.1 I explain how J segments his lectures into local contexts. The rest of this work will take this as a foundational observation. In sub-section 4.2.2 I demonstrate that such a segmentation is unlikely to be an imagined construct of the observer. I consider a continuous time duration (six and a half minutes) in a single lecture, starting from the beginning of the class, and I detail how J marks off this time into smaller units.

4.2.1. Susan Staats: mathematical speaking in units of stanzas and lines

When I came to transcribe the videos that I recorded, two structures naturally emerged: the *line* and the *stanza*. I have adopted these names from Staats (2008, 2018, 2021). She gave an example of classroom discourse that had been transcribed using a prose format, with sentences and paragraphs, and she then re-transcribed it in poetic form, using lines and stanzas. This form made many patterns immediately visible, led to interesting conjectures about relationships between the discourse and the structure of the mathematics that was being discussed, and allowed her to analyze key features of the discourse that would have been harder to spot in the original form. Her insightful paper had a dramatic effect on my appreciation and understanding of what I was seeing in the videos. Staats' research immediately helped me see the pauses, and the patterned repetitions, that I discussed in sub-section 1.2.2, as instances of the structures that she described and analyzed.

Three core principles I adopted for the *S transcript* were *readability*, *consistency*, and *minimalism*. These principles emerged from the circumstances of my research: I was creating a transcript of a large quantity of speech, I would be using this transcript very frequently, and I wanted to always have full confidence in this transcript whenever I would use it. By *readability* I simply mean I wished to create a document that I could read quickly and easily. By c*onsistency* I mean that for any *type* of choice that I made I would determine to make the same choice in the same way throughout the transcript. By *minimalism*, I mean that I wished to minimize the number of types of decisions I needed to make in capturing the speech in the classroom in transcript form. Any new kind of decision would have to be applied throughout the whole corpus, and if I was not confident in its *consistent* and reliable application, then I preferred not to hazard it.

Here is an example of the application of these principles: in addition to recording faithfully the line and stanza structures in transcripts, Staats also advocated for the use of successively greater indentation in order to bring out in the transcript yet other features of speech in mathematics classrooms. It is clear that by doing so she was able to obtain further insights. I decided against attempting such indentations myself.

To do so would first go against minimalism, because it would require making decisions which seemed to me to be less clear-cut than the decisions as to when a line ended and when a new line began. If I were to incorporate in the transcript judgments that I found difficult to make, this would dilute my trust that the transcript was accurately reflecting the structure that I was hearing and seeing in the lectures. Minimalism as a principle might be reinterpreted as a cautious, conservative approach to transcript-making. I only included segmentations that I felt certain or highly confident about.

Attempting such indentations would also make the transcript less readable. The eye can flit back to the beginning of a line very quickly and easily when all the beginnings fall on the same vertical axis. But to ask the eye to flit back to a different spot each time slows down my reading far too much. This would not be an issue if my corpus were much smaller. I needed to be able to read any portion of this long transcript quickly and methodically – *effortlessly*.

A criticism may be that the line and stanza structure I have introduced already slows readability. I believe that it does, to some extent, slow the speed with which I can read the transcript. I can read ordinary paragraph text faster than the transcript I have made of the videos. There are, however, three advantages that outweigh the slight loss in speed. First, it is not such a bad thing to make the familiar a little strange in this way: it helps to jog me into a slightly ‘as if for the first time’ perspective. Second, because of the mirroring of the structure with the vocal delivery, what I read in my head matches the rhythm of the spoken language better than a sentence transcription. This has the important consequence that I can mimic J far better with the aid of the Staats-influenced transcript than with a prose version. Third, it is easier to choose to read small selections of this transcript. There are more ‘hooks’ to begin with and ‘hooks’ to leave off with. Paragraphs of text can be very long, and it can be hard to figure out if there is something in the middle that might be interesting. By contrast, I can begin reading from any line in the *S transcript* and I am quickly engrossed in the current unfolding mathematical scene.

Staats refers in her work to Dell Hymes and Dennis Tedlock, linguistic anthropologists, who devoted their careers to the collection, publication, and analysis of oral narratives from what at the time they called the American Indian people, now called the Native American people. Hymes and Tedlock often come paired in ethnopoetics literature that refers to at least one of them because of the perception of later researchers that they each stand for separate and distinguishable paradigmatic methodological choices that can be made in the textual transcriptions of taped oral narratives.

The received wisdom is that Tedlock (1983) was mostly guided by *pauses* in his transcription of text, and divided his lines at these pauses, whereas Hymes (1981) was guided by certain repeated words, *particles*, which to him announced the beginning of a new line or verse. Tedlock emphasizes the *performance* of the narrative, while Hymes emphasizes the *structure of meaning* of the narrative. Both are committed to demonstrating that what had been previously thought of as prose narrative is better understood as having poetic form. The contrast between them can be sometimes over-simplified; Hymes (2003) observes: “Dennis has sometimes attended to particles as relevant, and I have never attended to particles alone” (p. 37). Indeed, Bright (1979) found that in independently applying a Hymesian approach and a Tedlockian approach he found a great deal of consistency in his resulting transcript.

This literature was useful to me because the story I have told so far (“introduce a line-break at a pause”) became a little more complicated as I transcribed more and more videos. I was noticing that there were small pauses in the middle of the lines I was making, and I was noticing that sometimes there were hardly any pauses at all at the ends of other lines I was making. Instead, I seemed to be preferring to let a clause complete itself as a line. When as an experiment I attempted to ruthlessly follow only pauses, I sometimes got lines that were only three or four words long. A jagged and unruly text was beginning to emerge, one that contrasted badly with the transcript up that point. I analyzed my earlier intuitive judgments, and they seemed to indicate, for example, that most lines ought to include a single verb. Many of the alleged lines as dictated by the pause rule, however, did not have a verb at all. When I shifted my focus from pauses to moments when J took a brief breath, I found much more harmony with my earlier intuitive judgments.

These reflections, coupled with reading and studying *The Natural History of an Interview*, helped me to return to the videos with a conscious attention to J’s body, eyes, voice, and hands, when seeking markers that punctuate the ongoing interaction into a hierarchy of units dissected into smaller units. I realized that some of my former intuitive judgments could, in retrospect, be understood as tacit (Polanyi, 1958; 1966) recognitions of a conversation of gestures that I could anticipate. Like everyone else, I have a lifetime of experience in changing topics – in starting afresh – and in recognizing when others are doing so.

I paid close attention to the variety of resources J uses to indicate the end of a local piece of action, a stanza, and the beginning of a new one. These resources include gaze direction, changes in gaze direction, and the speed of such changes; movement of the head; sudden stops or starts in body motion; orientation of the body, changes in body orientation, and the speed of such changes; picking up or putting down objects; as well as the various speech resources of intonation, pitch, volume, pace, tone, and silence that I had more consciously been alert to earlier. Even this list does not really capture the over-determinedness and the detailedness with which J gestures with his body and hands in a synchronous and well-timed way. In the next sub-section, I describe the sorts of doings that marked for J the beginning and ending of stanzas, with a view to obtaining a closer understanding of this important phenomenon.

4.2.2. Stanza Transitions in Lecture 7

In this sub-section, I undertake a micro-ethnographic moment-to-moment examination of J in the short period of time during which he ends one stanza and begins a new one. I will demonstrate that, with a multitude of means, J can show that he has finished his dealings with what had previously been of importance to him – he has concluded some action – and he can show that he is readying himself and the class to initiate a new action, which might very well flow logically or thematically from the previous action, but which is distinguishable enough and separate enough to mark it out as a new start.

In the opening six and a half minutes of Lecture 7, J determines fourteen such interfaces – *punctuations* in the language of the *Natural History of an Interview* – and so there are fourteen stanzas in this time period. On the one hand, this example is deliberately intended to *not* be special; the point is that if I were asked about some other time period in the course, then the corresponding demonstration of its segmentation into stanzas would differ from this one, to be sure, in the details, but would exhibit enough similarities to be recognizable as another collection of applications of segmenting resources.

On the other hand, since I do have to select *some* period of time to analyze in this way, I chose this one for the following reasons. First, it begins with the start of the lecture, which is certainly an important kind of transition: between not-lecturing and lecturing. Second, during this period J introduces a definition that plays an important role throughout the rest of the course: the definition of a subgroup. Third, I picked a duration of time long enough to indicate clearly a good *variety* of the actions J performs to segment the interaction into stanzas, and long enough to already witness *repeats* of certain types of these actions, thus revealing which segmenting resources are more commonly used. The final reason requires some preliminary explanation.

A segmentation in interaction can, to a first approximation, be thought of as a two-dimensional quantity: one dimension is its *location in time*, and the other dimension is *how pronounced* the segmentation is. To use a metaphor, if the segmentations are notches in a piece of wood, then the location of the notch, and the depth of the groove of the notch, are the two dimensions. To mark a segmentation in this course, I was therefore responding to two questions: *when* does the segmentation occur, and *what units* does it separate.

Such judgments were occasionally more challenging to make. In some interactions, although it was clear that a stanza transition had occurred, it was challenging to determine which one of two or three successive transitions between lines truly marked the stanza transition. In other interactions, although it was clear that a transition had occurred which was more pronounced than a transition between two lines, it was challenging to determine if the segmentation was pronounced enough to constitute a division between two stanzas. A fourth reason I selected this time period to analyze here is that it includes an instance of each of these two types of difficulty in judgment. Other time periods of comparable length in the course commonly include neither type, or only an instance of one type.

Some of the sentences below will begin straight with the verb and omit the subject; I hope in this way to draw more attention to the behaviour that accomplishes the segmentation of the interaction. The portion of the *S transcript* that corresponds to these fourteen stanzas appears in Appendix A. The headings in the next paragraphs each mark an interface between two successive stanzas.

Before Class Starts – Stanza 1. Standing behind desk facing class, head down. Looks at watch (on left hand). Puts this hand up to his face, to his lips, concealing his mouth for a few seconds. Brings hand down, looks at watch again. As he does so he takes one step so that he is by the transparency projector in position to read it and touch it. Says “Ok folks” – the intonation pattern is falling then rising (twenty-four of the lectures begin with the word “Ok”). He adjusts the transparency in a very minor way, then speaks his first line over a slowly quietening murmur of student conversation.

Stanza 1 – Stanza 2. He had been facing the class two steps away from the transparency. He looks back to the transparency, he puts his hand on his chin and rubs it, his voice lowers a little in tone and volume. There is a brief pause as he steps back to the transparency. Once there he starts speaking again.

Stanza 2 – Stanza 3. He had wandered a couple of steps away from the transparency – he walks back and touches briefly a term when he says “power”, then finishes his last line with falling tone. There is a very fast touch at “infinite”, the final word. Then with very little pause, he begins with his normal standard higher pitch tone, starting with “And”. He sweeps with his finger the first line of the next paragraph of the transparency. There are three paragraphs (each one sentence long) on the transparency; he has devoted one stanza to each of these. In sub-section 4.3.1, I will discuss segmentation in written mathematical text: it is by no means solely responsible for the segmentation of mathematics lectures into stanzas; however, it is critical to understand it in order to understand the dynamics of the interaction between the two kinds of segmentations.

Stanza 3 – Stanza 4. Falling tone on “into cycles”. Hands make a little circle in the air before he claps them together just before “ok”. Then a vigorous “So”, picks up notes from desk, removes pen top, turns and walks to empty board. At top left he says “so a subgroup” as he begins the first writing of the day ‘A subgroup’. The clap and the “So” seem to mark this segmentation more deeply than the three segmentations so far. In section 4.4, I develop the terminology of *episodes*. In that language, stanzas 1 to 3 form an episode (reviewing highlights from the previous lecture), stanzas 4 to 7 form an episode (defining the concept of a subgroup), and stanzas 8 to 31 form an episode (giving four examples of subgroups).

Stanza 4 – Stanza 5. Finishes sentence, the writing-with-reading of which has occupied the entirety of stanza 4. Turns to face class. Takes a few steps towards his desk, away from board, towards students. At the end of this little walk, he looks down at his notes just as he finishes his spoken phrase. The pitch of his voice lowers in the last two words. When he stops talking, he turns towards the board again, and begins walking towards the expected next portion of the board. When his pen touches the board, he starts speaking again, starting with "And".

Stanza 5 – Stanza 6. Finishes his spoken phrase, lowering pitch in his standard way. He continues to write for a few seconds in silence until his written sentence is finished. Backs away (parallel to board, ends up between desk and board) while looking at his notes, comes to a standing halt, looks at the whole board of what he has written so far. Next, he does a false start of writing: he says the phrase "and we also", takes step to board, indeed places pen on the board in position to write; instead says "so", backs off and takes a couple of steps back, turns to orient towards class with head down, places his notes back on his desk. This is often a signal that the newly commenced stanza will be largely or entirely spoken, as he almost always carries his notes when he is writing. After the notes are down, he puts the top back on his pen, a second such signal. Indeed, the next stanza is entirely spoken with body square to the class, standing behind desk, taking a step or two from side to side.

Stanza 6 – Stanza 7. He takes one step to his right (parallel to board), so that he is standing directly behind his notes that are on his desk. The step is precisely timed with a strong emphasis on the "course" of "of course we want to understand". His finishing phrase is said a little more slowly, slightly spacing out the words, as he is looking down at the notes on his desk. He picks up the notes, with head down looking at them he takes a step back, lifts head up and swivels to look at board. He takes the top off his pen, moves to the next writing point. His pen sways around in the air above the point in question, vacillating. A second false start of writing (the pen never quite makes actual contact with the board): he steps back, turns, asks the class for what he will write next.

Stanza 7 – Stanza 8. Finishes writing sentence, underlines 'trivial subgroup'. Backs away two steps, puts top on pen. Starts new talk with "well" as he looks at his desk for a pen of a different colour. Steps parallel to board and a half step away from board so that he is beside his desk facing them. Turns to go write on the board. He draws a vertical division line and begins a new board (top left of a new blank region – in sub-section 4.3.2, I discuss some aspects of J's writing that have to do with the space on the whiteboard).

Stanza 8 – Stanza 9. Finishes writing sentence. Looks down at his notes during the last few words of Stanza 8, falling tone on last word. Begins diagram as he says "So". He is facing the board squarely throughout.

Stanza 9 – Stanza 10. Finishes the diagram, turns to face the class and asks them a question ("What's that a Cayley diagram for?"). Looks at his notes as he waits for a reply, then puts pen in position, facing board, draws a bullet-point, all with unhurried and deliberate movements, displaying expectation that a student will give the label he will write in, but no student is saying anything. He turns to look at the class and after looking at them for a half second his body suddenly jolts. Startled awake to the discrepancy between what he had, in fact, written versus what he had been interacting with as if he had written, he immediately looks again at his notes and says out loud that he has made an error. He walks to his desk to get a different pen (the pen he needs to fix the mistake).

Stanza 10 – Stanza 11. Finishes the diagram again, drawing the last few arrows with the red pen (indicating a different action than the one with the black pen that he had previously written and, in this stanza, erased). Turns to face the class and asks them the same question as at the end of the previous stanza. The intonation of the entire question is very similar to the previous one: he is doing this over, as if this is a reset. As he asks he walks to his desk, putting the top on his red pen, for a moment trying to replace the red pen on a shelf on the board, but realizes mid-move that there is no adequate shelf there.

This little body-error occurs several times in the course. More precisely, there is a little shelf, but it is intended for the dry erasers, not for his pens. When in the course he forgets and tries this, the pen falls to the ground. Here, he catches himself before trying. He puts the red pen back on his desk, retrieves his standard default black pen, returns to his position on the board. As he returns, a student speaks precisely the next bit of writing 'Z_6' which fits the slot in the question that J had left. Questions-with-slots are a common technique as will be seen again later, as indeed are pieces of writing with slots or holes. J duly writes this after he agrees, “That’s a Cayley diagram for Z_6”.

Stanza 11 – Stanza 12. Takes a step to the board and touches in turn three nodes in succession, precisely timing the touches so that they coincide with the words “zero”, “two”, and “four” in his closing phrase “the nodes zero two and four form a subgroup”. At the word “a” he turns to look at class, then at “subgroup” he takes a step directly towards the class. He then takes two more steps while scratching the side of his nose and saying nothing. Then, as he stops in front of them, he starts a new stanza with a higher pitch tone: “if you say why is that?”. Then he turns again to move to the board. A little later in this stanza he will begin touching the diagram again while making an argument to justify the closing phrase of the previous stanza (and to answer the question that opened this stanza).

This stanza transition is a shallower one than most. The link with the last stanza is strong – there is an unfolding of the action that has evolved very naturally and easily from the action that had gone on before, even though there has observably been a punctuation. This transition comes close, then, to the edge of the complex boundary between those sorts of action segmentations which constitute transitions between stanzas, and those other, far fewer, sorts of action segmentations which are *not quite so pronounced*, and which seem most comfortably viewed as sub-stanza segmentations that are *more pronounced* than the transition between lines. The number of these were so few that I did not see the need to introduce a unit size between the line and the stanza. Such a unit size may be of value in research involving a smaller data-set.

Stanza 12 – Stanza 13. He is standing close to the class, facing them squarely. Uses a falling tone on the last two words of “keeps the associative operation”. On “keeps” he turns to move to the board, and he arrives as he finishes the phrase. The first phrase of the new stanza mentions the second property of a subgroup that we need to check – that the identity is in the subgroup – and as he speaks it he turns to face the class again while at the board.

Stanza 13 – Stanza 14. Here the falling tone occurs during the last words of the second last line of the stanza. The last line is uttered brightly and begins with “so”. There is a short pause at the end of the line, which he uses to step away from the board and towards the class again. Then he says “we need to have inverses”. The intonation on this line (low pitch steady for first four words, last word uttered at a much higher pitch) indicates a continuation of an ongoing list. This is now the third property of subgroups that is being talked about, the first two being associativity and containing the identity of the group. The last few stanzas have seen a lot of movement between the position at the board, and the position two steps from the board.

A suggestion could be made that the last line of stanza 13 ought to be interpreted as the first line of stanza 14: after all, “so” is often an opening particle, and the falling tone just before this word often indicates an end to a stanza. On the other hand, there is no hesitation or pause accompanying the falling tone, no looking at notes or some other sudden change in direction of gaze, no coincident turning or re-orientation of the body, and this final line (“so we better have zero”) summarizes the action of stanza 13. In addition, there are all the observable indications listed above in support of the interpretation that I have given.

This is an example of the class of boundary decisions where it is the *location in time* of the segmentation that is more challenging to determine than is normally the case. The previous class of boundary decisions involved a potential cleavage of a stanza into two (or a few) roughly equal sized chunks (smaller stanzas). Here we are dealing with what we might think of as the dual of that situation: we are considering the possibility of shaving off a line (or a few lines) from the beginning or end of a stanza and attaching it to the neighbouring stanza instead.

Most boundary decisions are easy to make and provoke hardly any inner doubt or hesitation – just as, most of the time, in natural conversations, we hardly notice our almost infallible precisely-timed understandings of similar transitions – we only notice our rare mistakes. If debatable stanza transitions are rare, then two in close succession ought to be roughly an order of magnitude rarer. It follows that the most frequently occurring case of the relatively rare debatable instance of stanza-fracturing ought to be (and indeed is) deciding between a single stanza or splitting it into two (and not, for example, into three or more).

Of the three thousand stanzas that constitute the course, about fifty made me think hard about whether these were genuinely two stanzas, or instead were just one stanza with a sub-stanza break that felt stronger than the break at the end of a line. The other kind of boundary decision, where it was not so clear where to demarcate the interface between two stanzas, where I might be off by a little bit – and the metric for this little bit is the unit of next smaller size, the line – occurred about this often as well.

J can make deliberate use of, and play with, both dimensions of segmentation. For example, J can be witnessed to genuinely close a stanza, but then be seen to reverse course and genuinely continue for one or two or a few more lines before concluding again. On such occasions, I determined that he thought better of finishing, that he realized he was not quite done nailing down whatever he wanted nailed down in this chunk of the action.

Similarly, J can harness the other of our two sorts of boundary decisions, the degree to which a segmentation is pronounced. Occasionally, J will seem to be drawing a rather short stanza to a close, during which a rather small chunk of action has gotten done, perhaps the first of a clearly pre-marked out list of two or three things to do. However, rather than closing this stanza devoted to that first small chunk, or that first of the list of two or three, he visibly changes course and instead opts to meld the performance of all these duties into a single larger stanza. By having enacted only a few moves from his repertoire of stanza-closers and refrained from enacting many of the clusters of others, and by relatively quickly and adroitly enacting enough of the moves from his repertoire of standard default stanza-developers, he can be seen to change his mind about ending a stanza.

In both types of occasions, J deliberately sails close to a definitive segmentation, either roughly near the ends of the unit in question, or near the middle of the unit (note: this is where I get the duality intuition from). I call it deliberate because as he is doing so he is witnessably constructing said segmentation. However, he abandons the segmenting and then just as publicly constructs the continuation of the unit. The sewing together is clear, and I trust my boundary decisions in these cases.

Panning out a little more widely, it is worth observing here that any ambiguity in interpretation of one of J’s actions can be, for that for very reason, used by J for his purposes in either that context, or some other context, if the potential ambiguity in question affords him an opportunity to make in that moment a particular meaningful action. This is exactly analogous to how, in ordinary conversations, any conceivable ambiguities can be repurposed by speakers in order to make puns, hide or disguise intentions, satisfy two or more aims simultaneously, attain greater emphatic force, or do any other of the things people wish to do to achieve their particular, local, contextual ends.

I report here that I experienced again and again an extraordinary feeling of decisiveness while analyzing the action of these lectures. While analyzing the video corpus, I experienced I-cannot-know-how-many thousands of moments and instants of internal “yeses”. Many of them were simply conscious little ‘clicks’, perhaps barely conscious, felt to be unerringly accurate, or just about unerring. Perhaps even more of these internal yes moments occurred so immediately and naturally that they required many viewings and long analytical effort to even bring to consciousness what it is in the action that are the proximate causes of these agreements and understandings. I say an internal yes to the decision that a stanza of action has just ended. I say an internal yes to the decision that he has touched or pointed to a particular bit of writing on the board. As I watch his body perform segmentations, and enact spotlighting gestures, I feel inside of me the making of these movements, and I sense and perceive inside of me the constructing of these actions. This natural and seemingly effortless agreement is an outstanding and notable feature of witnessing lecturing. Such segmentations and gestures can occasionally be ambiguous, and J can exploit such ambiguities for refining and fine-tuning his mathematical actions.

In this sub-section I have detailed how J begins and ends stanzas, and therefore how I determined the places in my transcript of his speech where such divisions are marked. This is the most important analytic feature captured by the *S transcript*. The reliability and consistency of this punctuation of the ongoing interaction is a fundamental feature of J’s mathematics lecturing.

4.3. Features of Mathematical Writing: creating the W transcript, creating the Pictures documents

My aim in this section is to explain the choices I made in how to capture what J wrote on the board in two data documents that I will call the *W transcript* and the *Pictures* data document. First, I choose a prominent example of a large piece of mathematical writing, collaborated on by many members of the mathematical community, in order to identify some features of written mathematics which deserve a central place in this analysis of mathematics lecturing. After this, I use what I have noted about these features to discuss how the *W transcript* was made. Then I will discuss the method by which an alternative record of all the writing that occurred in the course was constructed: a collection of still images from the videos making up the *Pictures* data document. I will also discuss how I used the *W transcript* and the *Pictures* data document in this research.

4.3.1. The Stacks Project: mathematical community as co-authors; environments in mathematical writing

Aise Johan de Jong, the originator and maintainer of *The Stacks Project* (de Jong, 2022a), describes it in this way: “The Stacks project is an ever-growing open-source textbook and reference work on algebraic stacks and the algebraic geometry needed to define them.” (de Jong, 2022b). He goes on to note some facts about it. It is an advanced text aimed at graduate students and researchers. It has the goal of starting algebraic geometry from the ground up, containing all that is needed to make sense of all the definitions and results in that area, building up to those objects known as algebraic stacks. He welcomes any contributors to participate; he will read the edits and new submissions of writing, and he will either accept or modify or reject. He observes that *The Stacks Project* is meant to be “read online”, which has the consequence that chapters can be as long as they want them to be. Perhaps the most important fact is the last one: that he will use “tags” to permanently identify and name “items”. What are these “items”?

Mathematical environments

Currently, there are about 21,000 tags; they permanently identify and name 21,000 items (the total when you read this can be found by going to the *Statistics* page at *The Stacks Project*). Here is de Jong's list of all the types of items: part, chapter, section, subsection, definition, example, exercise, lemma, proposition, remark, remarks, situation, theorem. This comprehensive list of categories of items in this long work is short. The first four are hierarchical divisions that could belong to any text. The rest are mathematical *parts* which I will refer to as mathematical *environments*, for reasons I will make clear shortly.

Looking these over they do not surprise. More granular reactions might include the following questions, to which I respond with the answers. Why no environment called "proof"? Because each of the environments called "lemma", "proposition" and "theorem" already include automatically within them the proof that establishes that result. What is the "situation" environment? It refers to occasions when a list of a few hypotheses is made; then the next few lemmas or theorems can just begin with the sentence "In Situation such and so" rather than repeating this same list of hypotheses over and over. Why no environment called "corollary"? Because of a style decision made by de Jong which he describes as item (9) in a document titled *List of style comments*: "Instead of a "corollary", just use "lemma" environment since likely the result will be used to prove the next bigger theorem anyway." (de Jong, 2022c).

A typical section of a typical chapter of *The Stacks Project* consists of a nearly uninterrupted sequence of such environments, one after the other. Importantly, it is *nearly* uninterrupted: at the beginnings of sections, and occasionally in between two successive environments, there are one or more sentences that set up or project or anticipate what the next environment or environments will treat and why. I will call these interstitial sentences *text-mortar*.

Here are two examples of sections taken at random. Section 13.7 is titled *Adjoints for exact functors* and its opening text-mortar sentence is: "Results on adjoint functors between triangulated categories." (The Stacks Project Authors, 2022a). This sentence is succeeded by Lemma 13.7.1 (with its proof), followed immediately by Lemma 13.7.2 (with its proof), completing the section. Section 9.25 is called *Artin-Schreier extensions* (The Stacks Project Authors, 2022b). The first six text-mortar sentences set up the machinery to state a particular result; the seventh sentence announces that a converse to this result is possible; this converse is Lemma 9.25.1 (statement and proof), which concludes the section.

Mathematical writing style: precise segmenting, identifying and isolating results, modularity.

I now examine more closely the *List of style comments* document (de Jong, 2022c) with a view to identifying as clearly as possible the features of mathematical writing that *The Stacks Project* exemplifies.

Here is the first sentence: “These will be changed over time, but having some here now will hopefully encourage a consistent LaTeX style.” I will be pointing to something here that is obvious to members of the mathematical community, and to researchers in mathematics education, but I will prefer to underline the obvious in order to emphasize its importance. de Jong simply assumes as a matter of course that *all* the contributing writers will submit their writing in LaTeX, an extensive set of macros written by Leslie Lamport (1985) atop the ground-breaking and game-changing mathematical *typesetting system* called TeX built by Donald Knuth (1984). I see the universal adoption by the mathematical community of this manner of typesetting their mathematics as further evidence of the commonality, the sharedness of approach-to-writing, that mathematicians have together. This is one of the many reasons why Mead is the appropriate theoretical figure here, and not, for example, Descartes. One of the things LaTeX makes easy is the building of what Knuth called *environments* (theorems, examples, and so on). This is where I adopt this term from.

Item (10) in the *List of style comments* reads: “Directly following each lemma, proposition, or theorem is the proof of said lemma, proposition, or theorem. No nested proofs please” (de Jong, 2022c). If one follows this style, one can always be certain what result is being argued for at any given point in the project. Presumably de Jong would prefer not to have a result stated and proved inside of a lemma because then two unwanted things could happen. One, the reader might be confused at some moments as to whether the current argument is trying to justify the lemma or the result-buried-in-the-lemma. Two, this buried result will not later be easily found, used or precisely referred to, because it has not been isolated and identified with its own name and tag. Both dangers are to be avoided. Therefore, de Jong encourages a style of writing mathematics that prioritizes *clear segmentation*; *identifying* any items presently inside larger items which can stand alone; *isolating* these items in their own environment. This style increases the likelihood that a community of mathematicians will collectively write trustworthy mathematics that contributors can safely add to in a justifiable and checkable manner.

Item (18) reads: “State all hypotheses in each lemma, proposition, theorem. This makes it easier for readers to see if a given lemma, proposition, or theorem applies to their particular problem.” (de Jong, 2022c). I read this as a wish for each environment to be meaningful even if read in complete isolation from its surrounding context (assuming the ability to unpack each of the terms in the environment, which may require many logically preceding environments). There ought to be no tacitly understood hypotheses. This push for what we might call *modularity* is another key feature of the written mathematics that is made and accepted by the mathematical community. It is not hard to imagine the reasons for this drive to make an environment contain within it as much as what is needed to read it correctly. If written mathematics routinely consisted of long running textual conventions where, for example, the letter *G* always meant a group, then someone unfamiliar with this convention might believe that a certain result did not require the hypothesis that that set *G* was actually a group.

Of course, there *are* long running textual conventions all over the place in mathematics, and this raises the question of how a lecturer will deal with this. Indeed, sometimes mathematics lecturers do rely on a background context of “we’ve been discussing groups for weeks, so of course occasionally I might forget to write ‘is a group’ in some result involving the letter *G*”. But we see here de Jong expressly require the elimination of such invisible conventions.

Other items in the style document also push in the direction of ending one environment *completely* and beginning a new one, so that the locations of endings and beginnings are in no way smeared or smudged, but are instead marked with precision. Item (17) reads: “Do not have a sentence of the type “This follows from the following” just before a lemma, proposition, or theorem. Every sentence ends with a period.” (de Jong, 2022c). Other items also push in the direction of *isolatability*, *ease of direct reference*. Item (21) reads: “Put any definition that will be used outside the section it is in, in its own definition environment. Temporary definitions may be made in the text.” Interestingly we find here de Jong's word, “text”, for the text-mortar occasionally found in the interstitial places joining environment-bricks, which I discussed above. This manner of reference that I have just used (“I discussed above”) is singled out as one to avoid by de Jong, again for reasons of modularity – to be able to move environments around if need be – in item (13): “Never refer to “the lemma above” (or proposition, etc.).”. Instead, de Jong advises referring to the lemma by its label.

Another key item that promotes *isolatability* is item (19): “Keep proofs short; less than 1 page in pdf or dvi. You can always achieve this by splitting out the proof in lemmas etc.”. Although de Jong does not explain why he thinks this is crucial, I think one can surmise the rationale. Short proofs that prove particular claims are easier to understand. Keeping track of the fact that in a long proof one has successfully proved, say, seven of the eleven statements needed in order to establish the result is a taxing demand on the prover. An additional benefit is that one can later refer to these isolated proven lemmas, regarding them as individual building blocks that one can *trust* and *use* to make theorems out of.

Although de Jong never uses the word “trust” in this document, it seems to me that it runs throughout as a tacitly understood goal. He encourages a writing style that contributors can use that will lead to a growing document all of whose results are trustworthy and can be safely used by graduate students and researchers for their purposes. Indeed, one of the aims of *The Stacks Project* is to empower readers to contribute new results to the mathematical literature based on having read parts of it which helped them learn some construction, or definition, or method of proving.

Improving mathematical writing, catching and fixing mistakes.

There is one more aspect of *The Stacks Project* to emphasize before I turn to the transcript I made of J’s writing on the board in his course of lectures. When mathematicians write, they sometimes, like everybody, mean to say one thing, but write another; or they write what they mean to say, but they are mistaken as to what they are claiming; or they think they have written something with a single meaning, but they have allowed two or more meanings: in short, they make mistakes of all possible varieties, despite all of their best efforts.

Beneath each section of *The Stacks Project* there is a space for Comments, open to anyone to make. Comments point out mistakes of all kinds: typos, misspellings, gaps in arguments, notational errors, and so on. Some comments detail what is missing in the section in its current state, perhaps listing what other results they expected to see there, or advising a shift in the level of abstraction or generality, so that a few results can be subsumed into one. Comments contain questions about sentences in proofs that the writer does not understand. One by one eventually de Jong reads through all the comments. Often he replies with “Thanks! Fixed here.” and points to a link to a file that highlights the change that has now been made. Sometimes the question is more difficult to resolve, requiring a more radical revision of the paragraph, or more rarely the environment, or even more rarely, the section. Sometimes de Jong disagrees that there is a problem at all.

It is in these corrections that we truly begin to see the power of isolating items, cracking items into smaller ones, segmenting the project, making it modular. The commenter can identify and locate the problem with great specificity. The ripple effect, or domino effect, of the consequences of the problem is easily traced. Each tagged item that depends on the problematic item is immediately identifiable. The structure of this mathematical work is nearly exactly that of a tree. If the features above were less insisted upon, many more of these comments would have upended the entire project, unravelling it because of the interlocked, interconnected nature of the whole block of text. As it is, one can look at the *Recent Comments* page of *The Stacks Project* and scroll through in reverse chronological order all the (at-present) 7,400 or so comments. None of these comments vitiated the entire project. None forced the revision of some macroscopic portion of the whole work. Instead, the changes affected only the branches growing from the position of that item in the tree.

I want to make explicit one aspect of segmentation and modularity that follows from the discussion so far, and that is the *hierarchical* nature of the segmentation. The writing consists entirely of sentences, put together into paragraphs, that themselves form the constituents of the environments, and the environments together with the text-mortar make up the sections, and the sections one after another form the chapters, which form the parts, which form the whole. Each of these super-containers provides a layer of protection preventing an error or bit of trouble in some sub-container from affecting externally located containers. To repeat, then: the ability to precisely refer to each tagged environment makes it possible to track exactly when any particular result, if shown to be problematic, can affect any later environment. This way de Jong can immediately trace the effects of any problem.

The *Comments* section of *The Stacks Project* I will see as analogous to student contributions in the lectures – an analogy only, as student contributions come in richer forms, and come intermingled with J's lecturing, rather than isolated at the bottom of some written section. The text that occasionally appears in between environments I will see as analogous to comments J makes before or after some portion of time that he spends writing on the board. The tags or environments themselves will have much more direct analogues which I treat in the next section.

Widespread agreement in the mathematical community on desirable features of mathematical writing.

It seems to me that I could have made some of the above observations about essential features of mathematical writing in other ways. Edwina Michener (1978a, 1978b) wrote in detail about the classic environments of mathematics: her triad of definitions, examples and theorems, and her rich discussion of their interrelations already capture both the importance of these divisions to mathematics, as well as how one might help students navigate these divisions. She explains with close attention to the specifics of a mathematical–pedagogical situation how to help students appreciate for themselves how to construct examples to elucidate definitions, how to isolate from a theorem or its proof a definition that might be of use in another mathematical context, how to see what a theorem is telling us by means of using a particular example of a mathematical object that satisfies the hypothesis of a theorem, and more. Maybe it would have been enough to point to her work, and then claim that no analysis of lectures would have much value if it did not confront the problem of how it is a lecturer tries to get these environments on the board in a manner that meets certain of their mathematical values.

Alternatively, I could have first observed from reading some mathematics course syllabi from well-known universities that certain textbooks seem to recur year after year with great regularity, the names of which I could simply list and expect a random member of the mathematical community to recognize them. Here are some sample names of widely and regularly used textbooks of various undergraduate mathematics courses. Analysis: Rudin (1976). Topology: Kelley (1955), Munkres (1974). Abstract algebra: Dummit and Foote (2004), Herstein (1964). Differential geometry: Do Carmo (1976); and so on. The point here is not that there are not other excellent texts; this introductory material is gone over in many such books. The point is what professionals in other fields might find to be a considerable amount of agreement.

Of course, mathematicians will have their favorite texts, and they might not like what another lecturer might like, and after a generation a text might go out of fashion; but predominantly it is the agreement that is the rule, the agreement that respects that such and such a text is seen by the community as one of the *standard* texts. Then I could look at these texts and show by random sampling of pages how they consist of setting up sharply begun and sharply ended “pieces” that they label with the sorts of environmental names as shown above. Or alternatively I could take a random sample from a series of textbooks, say the Springer series *Graduate Textbooks in Mathematics*, and again take a random sampling of pages.

Perhaps I could have chosen the most famous earlier example of a collective group of mathematicians co-authoring a lengthy foundational text together, the Bourbaki group. I would have found in their multi-volume work ample evidence again of the necessity for lemmas and definitions and examples; the need to split lengthy proofs into short ones; the obligation to label environments in order to precisely refer to them later: always the segmentation, the hierarchy, the modularity, the consistent drive for sentences that form paragraphs, the self-standing structure of the statements of results so that they can be read alone. Lucid expressions of the value of these elements of mathematical writing can be found in much of the literature devoted to advice to mathematicians on how to write well (Gillman, 1987; Higham, 1998; Poonen, 2020; Steenrod et al., 1973).

Conclusion

Let me take a step back. I have chosen *The Stacks Project* to serve as an exemplary instance. More than 500 mathematicians, led by de Jong, jointly writing an online textbook that currently consists of at least 7,400 pages which is almost entirely correct on a sentence-by-sentence level is simply an astonishing achievement. There must be something extremely powerful about the features of mathematical writing I have highlighted using this exemplary instance: clear and definitive segmenting into environments, stating explicitly within a result environment all the hypotheses required, identifying stand-alone results and useful definitions and isolating them in their own environments, cracking containers into smaller containers if possible (“splitting out the proof into lemmas”). There must also be a high degree of shared understanding among these mathematicians not just of the content of the mathematics that is being written about, but also of the way this material is being presented. I find this example convincing with respect to these two conclusions.

4.3.2. Creating the W transcript

Informed by the observations made in the previous sub-section, in this sub-section I will explain the choices I faced in constructing the *W transcript*, and I will justify the decisions I made. There is one major choice that I must discuss here first.

Would my *W transcript* itself *look like* what J wrote? Or would it be a document that records the *structure* of what J wrote? If I chose the first approach, my document would roughly look like an electronic version of what a student might capture in their lecture notes, embodying a WYSIWYG (What You See Is What You Get) philosophy. I would likely be able to *read* such a document very easily. If I chose the second approach, the readability of the document would suffer. On the other hand, I would be able to perform computer-assisted *searches* for character strings in the document.

I chose the second approach. The *W transcript* afforded me the enormously powerful ability to instantly locate all instances of a type of writing in the whole course. As discussed in section 4.2, mathematicians have already unanimously agreed on a typesetting system that captures the structure of mathematical writing: LaTeX. I therefore use a LaTeX-inspired symbolism all over the place in the *W transcript*.

In Appendix B, I exhibit a portion of this transcript: all the writing by J on the whiteboard in Lecture 24 of the course. I analyze a scene from this lecture in section 7.4. The writing at this stage in the course is about as sophisticated and complex as the course will get.

In the rest of this sub-section, I discuss mathematical *environments*, mathematical *symbols* and *expressions*, and the *visual appearance* of the writing in J's classroom.

Mathematical environments on the whiteboard

Here is the complete list of mathematical *environments* that I used in the *W transcript*, together with the number of times they appeared in the course: *diagram* (103), *example* (60), *theorem* (56), *definition* (50), *proof* (46), *remark* (40), *fact* (37), *exercise* (35), *solution* (33), *corollary* (23), *notation* (7), *question* (2), *special case* (2), *lemma* (1), *preliminary note* (1).

I now explain how I determined the beginnings and endings of the mathematical environments that J makes on the whiteboard, and how I determined what kind of environment it was.

It was always the case that J titled his theorems using the numbering from the textbook he was using (Gallian, 2013): for example, ‘Theorem 6.2. (Properties of isomorphisms acting on elements)'. Therefore, it was obvious for me when to begin a *theorem* environment. Any result that was given a theorem title I called a theorem no matter what. Similarly, he began every *corollary* environment with the word 'Corollary'. The *question*, *special case*, *lemma*, and *preliminary note* environments are also named after the title J used on those occasions. I ended all these environments right after the statement of the result.

J always began his proofs with 'Proof.', followed often by a new line. He always ended his proofs with a little black box (sometimes referred to in the literature as the Halmos symbol, which has seemingly replaced the earlier QED convention). Therefore, it was always clear where to begin and end *proof* environments in the *W transcript*. I opted not to distinguish between proof environments where he proves theorems and those where he proves corollaries. He always began his *exercise* environments by giving a title using the numbering from the textbook he was using (for example, 'Ex. 5.7'). I ended the exercise environment immediately after the exercise statement.

He never began an *example* environment with the word 'Example' or any other such title. Instead, he would begin a new left-justified paragraph, and I would realize, almost always very easily, that he was doing an example. He never started his solution with any word like 'Solution'. Instead, in the line below the last line of the exercise statement he would begin the argument that consisted of the solution. So just as with the example environment, it was my decision to label explicitly the *solution* environment, as opposed to following his titling decision with a label of my own. His solutions ended with a period and no other symbol.

The *remark* and *fact* environments were again my names and were not titled as such by J. This is the loosest distinction in all the environments I considered. There is no real clear dividing line between a remark and fact. I preferred to keep a rough division between those sentences consisting of a true observation that was to some extent informal (remark), and those sentences consisting of a relatively formal expression of a true mathematical statement, perhaps together with its brief justification (fact). If I judged that some statement could have easily been a theorem statement, accompanied by a proof, but was being stated here as something the students already knew from a previous course, it was a fact. Or if the result was so easy, and the proof of it was so short, it was a fact. If I deemed some statement to be a written version of some helpful spoken comment, it was a remark. For example, Rem8 is ‘Associativity means parentheses are permitted everywhere but required nowhere.’. Fac8 is ‘Corollary 4 shows that Z_n has exactly *phi*(*n*) generators.’.

J always underlined the word that was being defined inside a definition. I began the *definition* environment at the start of that sentence. If J defined a word in one sentence and another word in the next sentence, I created two successive definition environments. The *notation* environments were again my invention. They could have all been definition environments, but what J was doing in them was defining notation, so I thought it was worth carving this small category out. I discuss the *diagram* environment briefly when I take up considerations of visual appearance of the writing.

I begin and end all environments (the diagram environments excepted) with \begin{} and \end{}. Inside the curly brackets, I place a short identifier: for example, \begin{T38} and \end{T38} means that the material between these two delimiters is the 38th theorem that appeared on the board in the course. Sometimes in one lecture J wrote a theorem statement on the board but did not prove it or did not finish proving it, and then in the next lecture he wrote that same theorem statement again. I needed to decide whether or not to uptick my counter for theorems. I chose to do so. So, the number after the ‘T’ represents the number of times a theorem statement appeared on the board, not the number of different theorems in the course.

What I have briefly discussed here is an example of a *boundary case* decision. There are some criteria for determining how worrisome or disturbing a particular boundary case decision might be. A first criterion is the *degree of difficulty* in determining which cases are, in fact, boundary cases. In this instance, the boundary cases were easy to identify. The second criterion is *how many* edge cases there are. Here there were three. These two criteria applied together implied that the stakes for this boundary case decision were very low. If I simply kept track of which theorem environments constituted my three edge cases, then I could simply reverse my decision later if I wanted to.

In resolving this particular boundary case dilemma, I was guided by the following rationale: the theorem statement he writes today at the beginning of class in order to review the last lecture is a new action and deserves a new name. Although this boundary case turned out to be entirely unproblematic, other boundary case decisions could be more troublesome (edge cases harder to diagnose, more edge cases found in total). I made such decisions using a similar approach to the one I explained here: by privileging the occasions of writing, the *act of writing*, over the textual material itself, as part of the theoretical vantage point of focusing on mathematical acts by J.

Mathematical symbols and expressions.

I turn now to the text that appears inside these mathematical environments. As I encountered no difficulties transcribing any of the mathematical prose, I will concentrate here on mathematical symbols and expressions. I used LaTeX notation whenever reasonably possible. A dollar sign is used to open and close formulas, to distinguish them from ordinary text: for example, ‘$ G = D_3 $’ is used to transcribe the obvious equation it contains. Many other symbols have their usual LaTeX meanings.

First, the *delimiters*: ‘\langle’ and ‘\rangle’ are the left and right angle brackets surrounding a letter in order to denote the cyclic group generated by that letter; ‘\{‘ and ‘\}’ are the left and right curly brackets for set notation.

Second, *subscripts* and *superscripts*: subscripts are denoted using the underscore character ‘_’, superscripts using the caret ‘^’.

Third, *binary operations*: ‘cap’ means intersection; ‘cup’ means union; ‘\cdot’ means a dot positioned between two symbols indicating multiplication; ‘\times’ means Cartesian product symbol; ‘\circleplus’ is the notation representing the external product of groups.

Fourth, *binary relations*: ‘=’, ‘<’ and ‘>’ are self-explanatory; ‘\neq’, ‘\leq’, ‘\geq’ mean “not equal to”, “less than or equal to”, “greater than or equal to” respectively; ‘\equiv’ means “equivalent to”; ‘\iso’ means “isomorphic to” while ‘notiso’ means “not isomorphic to”; ‘\in’ means “element of” and ‘\notin’ is obvious; ‘\subset’ is obvious; ‘\subgp’ means “subgroup of”, ‘\nsubgroup’ means “normal subgroup of”, ‘\propsubgp’ means “proper subgroup of”. There are a few more but these are the vast majority.

Fifth, *logical symbols*: ‘\iff’ is the double-implication symbol, ‘\implies’ is the implication symbol, ‘\backwardsimplies’ is the reverse-implication symbol, ‘\therefore’ is the three-dotted symbol meaning “therefore”; ‘\box’ is the Halmos symbol placed at the end of a proof; ‘\contradiction’ is his symbol representing the moment in a proof where they have arrived at a contradiction.

Visual appearance of the writing on the whiteboard.

I now discuss how and to what extent I captured in the *W transcript* the *appearance* on the whiteboard of all this writing. How exactly was the writing spatially positioned, and how much of this did I record?

I will first introduce the term *board*. Most boards are characterized as follows: a fresh beginning at the top of the whiteboard, starting at a leftmost position on that line, followed by a sequence of horizontal lines that continue underneath that beginning, until J runs out of space, most often because he has reached the bottom of the whiteboard, but sometimes because he will collide with already existing writing. A handful of exceptional boards differed from this description of the prototypical board. There were 249 boards made by J in the 35 lectures, for an average of about seven a lecture.

I kept track in the *W transcript* when a new board was begun. In Lecture 24, for example (see Appendix B), there were seven boards. These were boards 168 through 174 of the whole course.

J typically makes three boards on the whiteboard, at which point it is full and some board must be erased. He never erases the board he has just finished writing. The whiteboard has a vertical dividing metal separator. He customarily begins the new writing of the day by starting immediately to the right of this separator, at the top. He writes rightwards for a certain amount of space, then proceeds downwards as discussed above. He then begins the second board at the top and to the right of the rightmost character in the first line of the first board. Very occasionally he feels the need to draw a line separating two boards (seven times in the course).

Once the second board is complete, he usually goes all the way to the left half of the whiteboard and begins a new board there. It is more rarely the case that this left half gets split into two boards the way we have seen the right half of the whiteboard usually is. After this typical beginning some variants can occur. He might erase the first board and repeat the process I have just described. Because the screen for his transparency projector, when it is rolled down, covers the left half of the whiteboard, and because he typically begins classes by using this projector, he might not use this half of the whiteboard at all for long sections of a class. Occasionally he gets into the trouble where some writing on the whiteboard that he wants to touch or handle will be blocked by the screen he wants to pull down. Another projector screen, which is connected to his computer, covers part of the right half of the whiteboard when it is rolled down. Sometimes, if he is planning to use his computer, he preferentially writes on the left half of the whiteboard. Sometimes he needs to begin a new board but does not want to erase any of the existing boards. On these occasions, he might erase the bottom half or portion of an existing board, keeping only the part of it that he still needed to be visible, and begin his fresh board beneath it. I called this a new board.

In Appendix C, Appendix D, and Appendix E, I include three images of the whiteboard at different moments in the course. In Appendix C is shown, from left to right, boards 4, 3 and 2 of Lecture 7. This means that after he drew the two diagrams underneath the sentence “The marked nodes form a subgroup”, he erased board 1 of the day and drew the two diagrams that can be seen in the middle of the whiteboard, and then moved all the way to the left to start board 4. Note the rare vertical line separator. In Appendix D is shown, ordered from left to right, boards 5, 6 and 7 of Lecture 8. In Appendix E is shown, on the right half of the whiteboard, boards 7 and 8 of Lecture 33. Note that the statement of the Lemma was the first piece of writing from an earlier portion of the lecture (board 3 in fact), and board 7 begins with the words ‘Theorem 24.3 (Sylow’s First Theorem)’.

I turn now to some examples of writing on the board that are spatially arranged in a particular form, or whose appearance or location is exceptional in some respect.

There were many occasions when J arranged multiple lines so that a few symbols, for example, several equal signs, were *aligned* vertically. I inserted a ‘&’ on either side of each of the mathematical symbols that were aligned. This is a modification of the LaTeX approach to alignment.

J often wrote material in a *two-column* format within a single board. For example, sometimes proofs might contain a series of lines connected by double implication symbols, or implication symbols, or equal signs. In a second column a little to the right of one or more of these lines, he might write a short phrase providing the justification for the step that resulted in that line. Here is an example: see Appendix D. On the ninth line of board 5, beside the equation '*abg* = *agb*', there appears the justification 'since $b \in Z(G)$ so $bg = gb$'. In the transcript, this entire line appears as follows: '$ a b g = a g b $ \scb since $ b \in Z(G) $ so $ b g = g b $ \sce'. The '\scb' and the '\sce' indicate a second column beginning and a second column ending.

There were some occasions when J presented material in a *list* format (usually bullets, sometimes dashes), and some occasions when he employed a *tabular* format (ordinary tables; I considered group multiplication tables to be diagrams). For lists, I just used notation like '\bullet' and '\dash'. For tables, I again used a modification of the LaTeX approach to indicating alignment. I wrote a new row of the table on a new line, and I separated the terms that appeared in the table by the '&' symbol.

Frequently J drew a *brace* next to some portion of writing, and then wrote some new text that served to *label* or *annotate* the previous writing. He displayed considerable freedom in doing so, and in the *W transcript* I have examples of overbraces, underbraces, and sidebraces. Sometimes he did not bother with the brace, and simply wrote some text above or below some earlier writing, or he drew a short line segment starting at some point in his writing and at the other end would write some comment or formula; I gave these names like overlabels and underlabels.

A diagram is a cardinal example of writing on the board where visual appearance and spatial arrangement are critically important features. In the *W transcript*, I limited myself to recording any labels, headings or legends attached to the diagram, and I would tag the environment with a name like 'Dia82'.

These are the main features of the visual appearance of the writing on the whiteboard which I kept track of in the *W transcript*.

The W transcript: powerful tool, dark temptation.

In the opening of this sub-section, I asserted how powerful a tool the *W transcript* is: it affords the researcher the ability to search it for any character string whatsoever. Having now reviewed the notation I employed to name environments, symbols and expressions, and visual arrangements, I can be more explicit here about this power. I could collect all the occasions when J annotated or labelled. I could collect all the occasions when J aligned his mathematical expressions. If I wanted to see J proving, solving, drawing, remarking, equating, subscripting, implying, sentence-ending, second-columning, I could find all such instances.

Speaking more abstractly, if I had succeeded in turning some verb connoting a mathematical action by J into a notation, then I could search the *W transcript* for that notated verb, and then turn to the videos to watch all that action. I could also turn this around. I could consider every mathematical item in the *W transcript* – for example, the symbol standing for ‘normal subgroup of’ – as the residue of a mathematical interaction in a local context where J and the class succeeded in justifying or motivating the placement of that item in that location. Then, every such searchable item in the *W transcript* would be an instance of the family of occasions in the course where these successful actions are present.

I could return to the videos to watch these eleven or seventeen or however many slivers of film containing only these sorts of actions one after another. If all the individual frames of video of the course are imagined as stacked one on top of the other, searching for strings in the *W transcript* corresponds to precisely isolating all those and only those thin slices that correspond to occasions when the same kind of writing on the whiteboard gets made. The *W transcript* allows for a *cross-sectional anatomization* of the mathematical actions in the course.

There is a dark counterpart to this power, or at least I experienced it as dark. Over and over again I faced the following temptation. Since the whole course contains such a great many actions, and I can observe, for example, his proving practices that take place on dozens of occasions, perhaps I ought to do something *quantitative* with this material. I was never able to do so without in a few short weeks completely swamping or overwhelming whatever else I wanted to say about the phenomena of interest in this work. The reader will doubtlessly find various remnants of this formerly seductive obsession of counting of items.

Guided by the experience of the researchers engaged in *The Natural History of an Interview* project, I judged that an excellent way to benefit from noticing that some type of behaviour happened often was to use this observation to help me scan through the instances of this type in the corpus, so that I could determine whether or not this practice was likely to be appreciated or regarded by later mathematics education researchers as notable, interesting, arresting, or in some form attracted them to compare it to their own observations of other lecturers or themselves. I resolved to find intriguing yet characteristic members of a collection of instances which captured most clearly and compellingly whatever it was about the collection that impressed me as essential for an understanding of mathematics lecturing.

This choice is, I think, akin to the choice that Streeck (2017) makes: “What micro-ethnographic research and the limited space of a book allow are analyses of exemplary moments and interactions whose representativeness can only be established by future studies of other cases.” (p. 13). I am a Meadian member of a mathematical education community recognizing that some instance of a practice stands for many others; when I write up such an instance as a detailed-enough analytical treatment, reading it might well bring forth a sensation of recognition in others in the community.

One of the origins of the dark temptation to count items is the reduction of information that results from translating the videos of J actively writing on the whiteboard into the notationally restricted *W transcript*. In the next sub-section, I describe an alternative device for accessing all the writing in the course which in some respects enjoys considerable advantages over the *W transcript*. This is the *Pictures* data document.

4.3.3. Creating and using the Pictures data document.

The various data documents or data files that I am describing in this chapter are all ways of capturing aspects of the original live events. Since the only movements of the camera I allowed myself were rare horizontal adjustments, in case J ever decided to use more of the whiteboard in some direction than he normally did, the mostly static camera was not subject to choices on my part. Such choices might be common, and welcome, in researching an alternative educational situation: zooming in here, panning over to a different portion of the room, walking around some space to film this or that. In such cases we would believe that the videographer was playing a not-to-be-ignored role in the capturing of ‘what happened’. Indeed, even my static camera is not an absolutely objective eye. However, of all the ways in which I captured the lectures and turned them into data, the video recordings were the least invasive.

But these videos are too many hours long (about 28 hours and 19 minutes in total) to be able to watch them all in a row and somehow contain them in one's mind (see Appendix F for the duration of each lecture video, as well as the number of stanzas in each). I did watch all the videos in a row one day at eight times speed (no audio possible at this speed) to get an overview, a look at the whole; trusting that within this viewing of the whole some patterns would emerge to me. This was a one-off. Mostly I would watch videos on different days. I was searching for representative significant mathematical practices, and when I watched an individual lecture on one day, I knew that I would find and notice practices that resonated with me as being similar to practices I had noticed in watching other videos on other days. But the actual passing of time between my viewing of the videos meant that such direct comparisons suffered from the asymmetry of the newly watched practice being very fresh, and the other being stale.

However, I am taking from Streeck, Bateson, and others an analytic focus on what is happening moment-to-moment, and on how such actions are sequenced. My theoretical framework requires concentration on the details of his gestures, movement of his body, speech, and writing. In this way I aim to build from the ground up a sturdy, convincing analysis of key components of mathematical lecturing practices. A predominantly *micro* analytic focus is being applied to a *macro* data-set. The primary purpose of fashioning multiple data documents, and employing them in interaction with each other, is so that an intelligent and careful combination of these vantage points will unearth, name, and determine the structures and functions of, representative exemplary practices which might only last a few seconds, and which might occur at very different temporal locations in the course. I felt a strong double need for local analysis as well as a global perspective.

What are the options for getting this global view? One is to read the *S transcript* from beginning to end. I have never managed to achieve this in a single go. It is faster, yes, to read everything he says in a lecture, than to watch the whole lecture. I timed myself and found that I could read a lecture from the *S transcript* in about 20 minutes (the length of the standard lecture was about 50 minutes). But 700 or so minutes is nearly 12 hours, not to mention that the pace of my reading would slow down as my concentration faltered. So that was out. Nevertheless, reading a few lectures at a sitting, was very useful! Another option was to scan all of the *W transcript* in a sitting. This was achievable, and I did so on a couple of occasions.

But the best method for achieving a global view of this course of lectures was by using the *Pictures* data document. I now describe the method by which I created it. For each lecture, I watched the videos until certain moments when one or more boards were completed, and just before one (or more) of them were going to be erased. I also tried to select, from these few seconds, an instant when J was not blocking any of the writing on the visible boards. At this instant I took a snapshot of the video and noted the timestamp. From each lecture I made about five or so such pictures (3 such pictures appear in Appendix D). Sometimes the new picture only had one new board on it, and still had in it visible some of the boards of the previous picture. From the 35 lectures I made 157 such images. The title for each image told me the lecture and the board numbers of that lecture which were visible in the image. These pictures proved extremely useful in several ways.

First, on a large computer screen, these images were for the most part quite easy to read, and were a fast way of performing the transcription of J's writing. It was faster to transcribe a static image than it was to transcribe the video of J unspooling the writing bit by bit as he lectures. Sometimes there were individual words or symbols that were difficult to read in this manner (the light from the window often made parts of the left half of the whiteboard difficult to read), and for these occasions I needed to watch the video, immerse myself in the context that J was in, hear his speech, and thus figure out what the writing was. In the opening lectures I was new to video-recording, and I was afraid to move the camera. So, in some of the first few lectures there were a few minutes where the right edge of the camera prevented me from later viewing the last three or four words of each line of the rightmost board, for example. However, I could listen to J talk and determine what he wrote. I also took notes during the lectures myself, and could cross-check those as well.

Second, this folder of pictures proved to be the best way for me to contain in myself what he did in the whole course, to own inside of me the structure and content of the whole course. I would open all the pictures of the writing in full screen mode on my computer, and with a press of the left or right arrow, move my way through the course as quickly as I wanted. In other words, not only could I scan unidirectionally from beginning to end faster using these pictures; but I could also find moments faster this way than I could by scanning through the *W transcript*. For example, suppose I was watching a lecture and noticed a particularly nice example of J comparing two portions of writing side-by-side. I might remember there was another example of such a thing at another time in the course, and I might want to watch that scene right away to compare the instances of this practice.

I do not know if this is the case for other researchers, but it surely mattered to me how quickly I could find that other scene. And the answer is not to have gone through the whole course and noted down on a page 'all the side-by-side scenes' for two reasons: one, maybe this observation was not going to lead anywhere in the end so it would be strange to think that I had already made such a list before I realized that there was a chance this might be interesting; and two, suddenly one confronts all the considerable problems with *boundary cases*, and with trying to decide what truly constitutes side-by-side comparisons. Is comparing two words enough? Should I restrict myself to two boards, or two diagrams? So, the answer did need to be that, if I ever found some phenomenon that I thought 'showed up' in important ways, I needed to be able to find other phenomena which I wanted to consider in relation to it, and I needed to be able to look at those instances relatively quickly after I found that first instance.

The *Pictures* data document served this need best. I could reliably find within a minute or so the other occasion that I had been reminded of. I could spot, from the picture of the writing on the whiteboard, enough to trigger in me the recognition required to find analogous or contrasting examples elsewhere in the course of the behaviour I was currently exploring; invariably the *Pictures* data document was the one which most quickly afforded me the ability to retrieve that 'elsewhere' example. The better my recognition was, the faster I could find the other occasion. What I mean, of course, is I could locate the other lecture and the board within it where that writing occurred, and then go back to the video, or to any of the other data documents, to dig deeper in analyzing the scene as it unfolded. The *Pictures* data document was the tool that afforded me the fastest *transportation* from one location in the temporal sequence of actions in the lectures to another. In my experience, even a gain in speed of a factor of two is utterly transformative in analyzing a macro data-set employing a microanalytic framework.

Flipping through the pictures was fast indeed. This tool was also reliably the fastest way for me to suddenly *change scales*. If I were analyzing a five second sequence where J is touching some terms in an equation in order to justify some claim, I could jump to the Pictures data document, and at a glance take in the much wider written context – ‘oh yes, that was the lecture where he defined permutations, and introduced various notations for it’. Then, if I stared at the images of the boards of that lecture, I could see which other equations were likely to have been touched by J in related ways; then I could return to the video, move quickly to those moments, and dive back down to the smaller time scale, the shorter unit of writing, the shorter unit of speech.

I relied on the pictures to capture for me any details about the *visual appearance* of the writing on the whiteboard that I did not bother to capture in notation on the *W transcript*. For example, I could gaze at diagrams to my heart’s content using the *Pictures* data document.

In sections 4.2 and 4.3, I have discussed the creation of the *S transcript*, which is the record of all of what J and the students said during the lectures, organized into units called lines and stanzas, the creation of the *W transcript*, which is the record of all of what J wrote on the whiteboard in the course, divided into boards and environments, and the creation of the *Pictures* data documents, which allowed for the rapid examination of all the writing in the course. In the next three sections, I examine the three remaining types of data documents I created to help me navigate the mathematical interactions in the lectures, and locate commonalities and connections among them: the *Episodes* data document, the *Student Contributions* data document, and the *Gestures and the Body* data documents.

4.4. Creating the Episodes data document

We have seen that the action of the lectures is naturally divided into smaller portions within which some local bit of business is accomplished, with numerous resources available to J to mark the beginnings and endings of these portions that have been termed *stanzas*. We have also seen that one of the accomplishments of a complete lecture is a number of *boards* of writing, divided into segments of their own, called *environments*. In both cases, the *S transcript*, and the *W transcript*, the whole consists entirely of contiguous portions. The *S transcript* is nothing but stanzas (with no spoken word in between one stanza and the next), and the *W transcript* is nothing but environments (with no written word in between one environment and the next).

A frequent determiner of the 'local bit of business' that I mentioned above is the goal to complete the local writing environment that is presently happening. However, it usually takes many stanzas to complete an example, or a proof. A larger unit size than the stanza is naturally called for. This is what I will call *episodes*. An example of a lecture divided into episodes is given in Appendix G (Lecture 25). The *Episodes* data document consists of all the lectures of the course divided, in this way, lecture by lecture, into episodes.

The *Episodes* data document is a *condensation* of each lecture of the course. Its most valuable feature is to capture in each page the ten to twenty major 'doings' of a lecture. By glancing at a page, it is possible in a matter of seconds to recall what this lecture was about. In addition, it forms an important and highly useful *connection* between the *S transcript* and *W transcript*.

On the *Episodes* data document I noted the time of the beginnings and endings of episodes in order to facilitate quick location in the videos whenever needed; I kept track of which stanzas of the *S transcript* corresponded to the episode; I kept track of what environment(s) of the *W transcript* corresponded to the episode. In addition, I subdivided the episodes into the smaller actions that J accomplishes in shorter multi-stanza units, and I wrote headings for the episodes and sub-headings for the sub-episodes that captured as pithily as I could what J did in that unit. For example, the episode titled 'Exercise 9.11' starts at 04:38, and ends at 20:50; it is 29 stanzas long, lasting from stanza 25.06 to 25.34 inclusive; it corresponds to the 25th exercise of the course (Exe25) and its solution (Sol23; by this point in the course two exercises were stated but not solved by J). This episode is divided into 11 sub-episodes; the heading for the episode is the statement in the exercise that they are trying to prove, and a couple of examples of sub-headings are 'draws Cayley diagram of *G*' and 'visualizes *G/H*'.

In practice, I frequently made use of the *Episodes* data document in the following way. Suppose I was in the *S transcript*, reading a certain stanza. This stanza would have a label (say 25.24). I could rapidly locate this stanza in the *Episodes* data document for Lecture 25. Then, since the *Episodes* data document contained also the label for the environment that was being written in this episode (here Exe25), I could search my *W transcript* for 'Exe25' and locate that portion of the transcript immediately. This was one of the ways that the *Episodes* data document served to *co-ordinate* the complete record of the speech in the course and the complete record of the writing in the course.

An additional important function of the *Episodes* data document is that it brings to the fore those stanzas, or series of stanzas, where J is not writing and is making spoken comments. For example, we see that the beginning of Lecture 25 consists of an episode called ‘Reviews’. The first five stanzas consist of going over, or recapping, material from the previous lecture, with the aid of a transparency. I could have, but in the end did not, write down in a document all the transparencies in the course: they consisted of subsets of the *W transcript* anyway, and I took pictures of all these transparencies so that I could view them whenever I needed them. In other words, I decided I never needed to perform a computer-aided search for words in transparencies but relied on looking at them in picture form if I ever needed to.

In Lecture 25 there were no other comments that came after the ending of one written environment and before the start of the next one, but this occurred frequently in the course: I called these episodes Comments. Inside of episodes there were often comments that came after the ending of one portion of a written environment and before the start of the next portion, and less often right in the middle of a written sentence when he decided to break off for some reason. In this lecture, stanza 7 is such an interstitial comment, coming after he has finished writing the statement of the exercise, and before he has begun writing its solution. Similarly, stanzas 24 through 27 are a multi-stanza comment on strategy coming after he has finished drawing a diagram, and before he has begun writing some results that the diagram has helped them realize.

I worked hard in making the *S transcript* to ensure that I captured exactly what was said, and that I found the naturally occurring divisions between beginnings and endings of locally accomplished mathematical actions. I also worked hard in making the *W transcript* to capture exactly what J wrote, separated into the environments discussed in section 4.2. In making the *Episodes* data document, I did *not* demand a similar ruthless consistency from myself, or in my judgments. For example, if a comment stanza at the tail end of an ‘episode’ might otherwise be seen as a stand-alone episode of its own, I did not try hard to determine rules or consistent criteria by which to determine such, and I did not watch these moments over and over in order to deepen my ability to discern or observe what were the salient criteria that would, if applied throughout my viewing of the course, give a consistent approach to the episode unit. The only thing that mattered was whether the comment in question seemed to be part of the action of the episode that was being completed, or whether it seemed to be part of the action of the new episode that was starting next, or whether it felt like a comment that did not belong to either one.

With this document, then, when it came to determining the positioning and labelling of comments, there were some instances where more judgment on my part was being used. J ended and began his writing environments in ways I had little to no choice how to register. J marked his openings and closings of stanzas with such an overdetermining of resources that I almost always had no choice in capturing these segmentations. The episode unit is also unquestionably real, but its edges are slightly fuzzier. J did not determine the entirety of the *Episodes* data document for me. Where I opted to see a student question that J answers as a little episode of its own, another researcher might deem the exchange to be the tail end of the previous episode. Perhaps it is natural that the larger a unit of action, the more likely there is to be some difficulty in determining the boundaries of that action. It matters to me to register the degree to which I imposed the segmentations or J imposed them on me, and I have tried to be honest about these differences.

Setting the Comments episodes aside for the moment, is there otherwise a one-to-one mapping between *environments* and *episodes*? Certainly, Theorem episodes almost always correspond to theorem environment plus proof environment. Definition episodes are baggier: they might contain a couple of definition environments perhaps, or they might also contain some nearby relevant remarks. I incorporated diagram environments into the episode they belong in, and so on. Example episodes often include not only example environments but nearby remark environments. Otherwise, I was usually content to name episodes by the same name as the name of the written environment that the episode accomplishes as its major goal.

The benefits of the *Episodes* data document, then, were as follows. I could break down an episode into meaningful chunks of stanzas that I could label in as few words as I could, and thus name the major mathematical actions that J was involved in on the time scale of multiple minutes, rather than the typical stanza time scale of half a minute. I could move back and forth between the *S transcript* and *W transcript* fluidly and easily. I could quickly and reliably identify regions of time when J was speaking but was in between writing environments, or in between large sub-units of these environments.

4.5. Creating the Student Contributions data document

While the vast majority of the spoken words in the course are spoken by J, there are, of course, many words spoken by students. In this section, I describe how I constructed the data document called *Student Contributions*. By the phrase *student contribution*, I will mean the words a student speaks during the period of time they are speaking uninterruptedly.

The starting point was the already-created *S transcript* of all the spoken words in the course. Each of the student contributions had been signalled in that transcript by being contained inside square brackets, for example: [show that it's like well-defined]. In the *S transcript* I did not keep track of which student made which contribution. I systematically went through the *S transcript*, searching for square brackets, then watched the video of this instance in order to determine who spoke. I organized the information by stanza.

Creating the *Student Contributions* data document is a perfect example of the phenomenon that recurred with the development of every tool I discuss in this chapter. The completed tool was highly useful and valuable; the process of developing the tool was exceedingly useful and valuable to me as well, because I was necessarily immersed in an extraordinary number of lecturing occasions which I viewed in a certain light as I had particular goals to fulfil. Here, for example, developing the *Student Contributions* data document required the viewing of every interaction in the course that involved a student. What I learned from this experience far exceeded the information I kept track of which I will describe below. At the same time, demanding a systematic approach to what information I would record helped me to maintain a consistent quality of attention to the scenes that I watched.

In Appendix H (Student Contributions: Lecture 9) and Appendix I (Student Contributions: Lecture 28) I give two examples of entire lectures from the *Student Contributions* data document. I chose them for a few reasons. They are generic; I have avoided a few lectures that were extremely atypical in some respect. They give an indication of classroom behaviour from earlier in the course as contrasted with later behaviour. I can use them to instantiate the observations I want to make in this section.

Finally, each lecture contains at least one scene with a student contribution that is of great significance. In Lecture 9, a student question prompts J to realize that he must reword a mathematical justification he had given some time earlier: I discuss this moment in section 5.7. In Lecture 28, a student surprises J in a different way. J had a few minutes earlier indicated a specific mathematical assertion to be tempting but untrue; the student creates and explains a compelling counterexample showing the assertion is indeed false. I analyze this scene in section 6.6. Student contributions vary in their impact on the ongoing action. Some student contributions, such as the two mentioned here, alter the nature of the mathematical interaction in the room dramatically. Such occasions occur regularly in the course.

Student contributions are frequent. For example, there were 93 stanzas in Lecture 9, 16 of which contained student contributions; there were 90 stanzas in Lecture 28, 36 of which contained student contributions. These are significant fractions. It would be hard to properly understand the nature of the stanza unit, and the actions that go on during it, if no data document were created on student contributions at all.

All entries in *Student Contributions* include at a minimum the following information: the stanza number, the number of separate student contributions in that stanza, the list of students who spoke which contribution in sequence, a brief reason for why the student said what they said was said (response to question from J, self-initiated by student, response to other student, and so on), and a brief description of what the student said – that is to say, what the main topic or focus was of the contribution.

Some of this information was easy to track. It was possible, with only a handful of exceptions, to determine what a student said. It was almost always possible for me to determine which student had said what. Although the camera was trained always on the board, and the students were invariably out of frame, their voices were easily recognizable to me. There were a few occasions when I could not assign a name to a student speaker because of overlap of talk. There were also a few occasions when a student spoke so softly, or answered with such a short reply (perhaps one word), that although I could determine what was said, I could not figure out who said it. For these occasions I would write “Somebody”. There are a few such examples in Lecture 9, and there were none in Lecture 28. Most lectures had none.

Overlap of talk was rare. Interruption of J by a student was very rare. Interruption by one student of another student who had the floor was also very rare. The typical stanza that included student contributions could be routinely divided, then, into separate speeches from J and from one or more students. Student contributions constituted a well-defined category of talk; it was very easy to keep track of these separate student speeches.

I next describe the principles I used in determining what to capture from a student contribution.

The first principle was that this data document was not intended to be a replacement for the information in the *S transcript*. I would not be solely relying on this data document as my record of what happened in a stanza that contained a student contribution. Therefore, I could afford to write a brief enough description that captured the kernel of what was at stake in the stanza. This would remind me of what was going on here on this occasion: then I could go find that stanza in the *S transcript* to read fully what was said there by J and the students involved. I smoothed this transition of data documents by using the same notation for stanzas in each document: a search for ‘28.94’ would take me to stanza 94 of Lecture 28 in both documents, for example. If this stanza would turn out to be important enough to analyze more fully, then of course I could take the next step up and look at the video itself.

Sometimes this brief description was accomplished by means of a very short quotation of what a student said (see, for example, 9.88 and 28.14); sometimes a very short quotation of what J said (see, for example, 9.76 and 28.10). These short quotations usually involved the key words that I remembered as my own shorthand for the scene in question. It was remarkable how often such a short quotation could bring the scene to mind for me; remarkable too how often one particular short phrase from all the speech in the scene recommended itself as the obvious choice. Often the phrase itself occurred so rarely in the lectures that they might even constitute the sole instance. I sought the *pithiest* description possible that would allow me later to read the description and retrieve the event and its local context as quickly and decisively as possible.

The second principle was that I had no intention of tagging student contributions with keywords concerning the content of their speech with a view to, for example, sorting out common themes of student contributions – for example, which mathematical topics came up how many times, and so on – fascinating as such a study might be.

I resolved to keep track of the *precipitating factor* for the student contribution. In as many entries as possible I tried to make clear *who had caused* the student contribution: was it *J* who had asked a question to which the student is responding, or was it the *student* who had spoken on their own initiative. I did this because of two reasons: one, I wanted to know the role J was playing in making such contributions happen; two, this information was vital in helping to understand the mechanism of formation of openings and closings of the stanza unit. It is a bonus that this *bit* of information (J or student) was almost always wholly unambiguous to determine, and therefore could be leaned on reliably.

J routinely organized the path of a solution to an exercise, or a proof of a result, by asking specific questions at specific moments in order to elicit an idea that might help them move forward. For some examples, see 28.28, 28.29, 28.30, 28.31, 28.35 in Appendix I. I hope these examples indicate how frequent such a move is from J. As the course proceeds, it becomes a more and more frequent mechanism by which J organizes the unfolding action. I now describe three variations of this dynamic that J and the students enact, using the lens of the stanza segmentation.

A frequently occurring type of stanza is one in which J has summarized a little bit of what is going on mathematically, and by slowing down his speech, altering the tone of voice, incorporating pauses, and slowing the movement of his body, concludes with an offered question to the class (and ending that stanza). When the response comes, and he moves again, perhaps to write on the board what the response allows him to now write with justification, speaking with higher pitch, volume, speed, and/or certainty, he in so doing marks the beginning of a new stanza.

A second frequently occurring type of stanza is one in which J wraps up the business at hand in the usual way, perhaps by finishing a short line of active inquiry with some of the closing markers he has at his disposal. Then with fresh vigour, perhaps with an opening particle like “Ok”, and other such opening markers, he asks a question, often in a single line. Then the student response comes in the second line, and in the rest of the stanza J either uses this correct response to advance a discrete amount further or to settle this issue J has raised, or J uses this incorrect response to give further prompts and more detailed and specific hints in order to get the response he wants. The contrast with the previous variety is that in this variety the stanzas begin with the question and contain the student response immediately following, whereas in the previous variety the stanzas end with the question and the next stanza opens with the student response.

Rarer, but still regularly occurring, is the obvious third variety suggested by these two varieties: the stanza whose second or third last lines contains the question, and whose last line contains the student response. The instances of this category tend to involve simpler questions to which the student response is more or less inevitable. In this course there was a particular student, Bart (pseudonym), who mostly took on the role of answering-the-obvious-questions.

Note that in all three of these varieties the timing of the invitation by J and the response by the student is structurally related to the stanza transition time. In the first type the transition occurs between the invitation and the response; in the second the transition occurs before the invitation; in the third the transition occurs after the response. I made use of this observation in the following way. By recording in the *Student Contributions* data document all the occasions when J was the precipitating factor in a student contribution, I had thereby assembled a large collection of occasions when there was a good chance that a stanza transition would occur in one of the three positions just listed. In other words, I had a sort of divining rod for occasions highly likely to be marked as a stanza transition. I then watched all of these scenes. In this way I was able to examine many dozens of stanza transitions, I could watch J mark all these transitions, I could watch bunches of them in quick succession, and I could therefore learn so much more about how he accomplishes this punctuation of his lecturing into stanza-sized pieces.

What I have described in the preceding paragraphs is another consequence of crafting these different tools: the construction or application of one tool affords the ability to reinforce, strengthen, or augment the observations and conclusions stemming from the construction or application of another tool. In making the *S transcript* I had already divided all the lectures into stanzas. But now I could apply the method I have just described to more deeply understand the nature of those divisions. I could also here and there occasionally fine-tune or clean up portions of the *S transcript* itself.

Often a student spoke not because J had asked the class a question, but because they themselves wanted to ask a question, or indeed to make a statement of their own. It was vital to succinctly note that it was a student question or offering for some of the same reasons as outlined earlier. Very frequently the student question would mark the beginning of a new stanza. There are a few reasonable explanations why such behaviour ought to be expected. Often the student could not have been unaware that the topic or focus of their question was not on the current main line of what J was presently concerned with. Also, the student could likely not have been uninfluenced by the social pressure not to interrupt J while he was in mid-speech.

The combination of these two effects lead regularly to situations where J has just concluded a stanza and a student uses what might have been even a very a brief pause in order to insert their question about what had transpired ten or thirty seconds earlier, or even minutes earlier. Alternatively, they might jump into this brief silent interval in order to ask a question that concerns the strategy behind what they are doing, or why they are not doing something else, or why J did not write something else, and so on.

Regardless of whether the focus or topic of the student question is something that had transpired earlier, or whether it in some other way concerns something that was not being paid attention to currently by the class-as-led-by-J, many student questions are inserted right at the junction of the end of a stanza, and therefore appear at the beginning of new stanzas. This observation itself counts towards a justification for the existence of and importance of the stanza unit: students themselves clearly show by their actions that they organize their actions (here speech) subject to, with respect to, taking into account, the stanza. Whether they could articulate this or not, the stanza was meaningful to them, and they made choices based on this understanding.

I hope the preceding paragraphs have given sufficient justification for the principle of recording who prompted the student contribution: whether it was a response to a question or offer from J, or whether it was self-started. I turn now to the next principle governing the method of creation of the *Student Contributions* data document: that I would keep track of the *rightness* and *wrongness* of arguments, and related to this, whether the speakers involved – J and the students – *understand* what is being said by any of the others in the interaction, and whether or not they are *convinced* by what is being said. For example, in 9.43, a student is initially convinced by an answer from J, then changes his mind and argues against him. In 28.21, a student correctly answers a question. In 28.89, a student concedes a point. In 28.56, a student stammers out an answer: here, the student is not really convinced by the argument they are themselves currently attempting.

The students are taking an upper-year undergraduate mathematics course on group theory, where what might be expected to be regularly at stake is producing convincing arguments for the truth of statements of results concerning precisely and unambiguously defined conceptual quantities. Moreover, student contributions in this course regularly concern every aspect of the preceding sentence: student responses involve the attempted production of arguments, or the naming of the global strategy of an argument, or the answer to the detail of a computation within the argument, and so on; student questions involve why a justification was needed, why some justification is missing, whether or not some expression has been precisely referred to or defined properly, and so on. For these reasons it was essential to keep track of *correctness* and *convincingness* considerations in this data document.

The last principle was *comprehensiveness*. I kept track of *all* student contributions, even 'irrelevant' ones, like when a student makes a joke.

To summarize, in this data-document I kept track of: all stanzas where students made contributions; how many separate contributions were made in each stanza; which student made each contribution; whether it was J who prompted the contribution or whether it was initiated by the student; a succinct description of the content of the stanza so that I could instantly recall what the comment was about; an indication of whether the comment was correct, or whether the student was convinced or convincing, or found J unconvincing or incorrect.

4.6. Creating the Gestures and the Body data documents

It may seem incredible to the reader that it has taken this long to get to the documents that are of primary interest in this research. The families of gestural practices that I wish to discuss in the remainder of this work were largely found by the process by which I created the *Gestures and the Body* data documents. So why invest so much time in creating the previously described data documents, and why devote so much space to describing them?

First, I consider it an important finding that lecturing, as performed by J, consists of these stanzas of actions; and second, it is obvious that J intends to get on the board all of the writing that he does succeed in writing on the board. It was therefore inevitable that I pass through the process of creating the *S* and *W transcripts*. Third, I will throughout the rest of the thesis take as background, or for granted, that all of the gestures to be discussed are happening within a local context of a stanza. Before I go on to treat the *Gestures and the Body* documents I ought to answer a question that may have occurred to the reader already.

I talked a great deal about 'what J writes on the board', and my attempts to capture this in picture form, and in transcript form. But my analytic approach centres on the *act*, on the actual physical material way that this writing gets on the board. It is past time, then, to confront the temporal manner in which the writing appears on the board, and not be content only with the accumulated result of, say, the ten minutes of writing that ended up creating a board that is still visible when he moves on to begin the next one. What about how fast he writes this part or that? Erasings of a single word? Is it true that he always advances his writing by a steady left-to-right accumulation of characters and words, broken only by making new lines or finishing one board and starting a new one? What about capturing the information concerning what gets written in a single unbroken writing effort?

The *W transcript* is keeping no information about the pauses between periods of writing; it does not reveal that J wrote the first clause of a sentence, left the board after marking the comma, returned to write the next three words, stopped again to make a quick comment, and then completed the writing of that sentence. A physicist might say that the *W transcript* is keeping only the information about the final state of a board, and keeping no information at all about the path that the system (J plus students plus classroom) took to get to these states. What should I do with this information?

In practice, I tried a few alternative approaches. For a few lectures, I tried to cut the *W transcript* into numbered pieces. I used the terms *writing streak* and *not-writing streak* to refer to time intervals of uninterrupted writing and uninterrupted not-writing respectively. I had envisioned a document where the *W transcript* had been cut into parts where, say, the seventh part corresponded to what J wrote in his seventh writing streak. This turned out to create documents whose complexity seemed to be disproportionate to the information I wanted to spot or the use I wanted to make of it.

For example, it is a feature of J's lecturing that he, in fact, regularly writes material that comes later in the *W transcript* before he writes the earlier material. This *non-linear writing* comes largely in the following three forms: writing the *headings first* of some piece of writing, followed by the material inside of those headings; leaving a short *gap* in his writing (a word or symbol) for various purposes, including asking students to offer to him what the missing piece is that fills the gap; and the most common type, writing forwards from the *beginning* of some proof and also writing backwards from the *end*, and alternating in whatever manner he sees as easiest to follow, so that in the end he joins the chain of reasoning up. A person who walked in after the proof was complete would have no idea that it had been written nonlinearly. Sometimes there occur complicated combinations of these types.

In addition, there are occasions of *temporary writing*, where he writes something on the board, and it is clear even as he is writing it that he intends to erase it almost immediately, because the purpose it is serving is to briefly explain something, or to indicate a competing notation he will not use, and he wants to clear the board to continue the writing he intends to keep.

For these reasons, although large portions of the lectures proved easy to demarcate on the *W transcript* using the writing streaks approach (e.g. 'this portion is writing streak number 14, and it consisted of this contiguous set of 8 words'), a small fraction of the *W transcript* would take nearly as much labor as all of the rest, and the difficulties of notation proved very challenging indeed.

Another attempt I made, then, was to only keep track of a time series of writing streaks and not-writing streaks (that is to say, keeping track of all the time-stamps of the moments he began writing or stopped writing). I did this for a few lectures, and it gave me a sense of how many such streaks there were, and the distribution of time intervals that such streaks come in. Statements of theorems and exercises tended to be written in one streak and regularly emerged as the longest streaks. Definitions also tended to be written in one streak. Long not-writing streaks tended to be digressions, or stanza-long, sometimes multiple-stanza-long comments which pre-viewed or post-viewed episodes. Long intermittent series of writing and not-writing streaks corresponded to proofs.

I tried marking these time stamps in the *S transcript*; and to give myself a break in complexity, decided to round off to the nearest stanza. In this notational effort, stanzas came in three varieties: either a stanza had no writing at all, or it was all writing, or it had at least one switch from one kind of streak to the other.

I will not display any of these attempts. I think the effort to make all of these was worthwhile. But, in the end, I did not find some system of keeping track of writing streaks in a transcribed form that I found useful to note patterns and regularities with, that in conjunction with my other data documents would serve the purpose of finding related practices, or of understanding a particular instance of a practice in more detail.

Thus far, I have talked about creating data documents that were comprehensive: that covered the whole course. I am saying I chose not to do this with data concerning writing streaks. It is true, however, that in any individual instance of a practice that I analyzed – some stanza or sequence of stanzas that I wanted to understand in detail – I did keep track of when J started writing and not writing. The distinction, which is an important one, is this: writing streaks were one of the features I understood must be kept prominent in the *analysis* of a scene whose importance had *already been unearthed* through the data documents I am discussing in this chapter, but they were not something that needed to be transcribed in full in order to become a *tool* for *unearthing* important scenes.

The next observation is that although it is possible, in principle, to gesture and write at the same time, in actual practice J never does this in the entire course. There are a few occasions when he pauses from writing for a fraction of a second to point or touch the board, but this is as close as he comes to simultaneous writing and gesturing. I will not include ‘writing’ or ‘inscribing’ as a part of the verb ‘gesturing’, although I understand that one could well use a superset word that includes both and that this would get at the fact that there is no real dividing line between these verbs: he writes in the air with his finger, he traces a line underneath a term with his finger to emphasize it, he writes an actual line with his pen – these are clearly all very closely related. In his book Streeck (2017) emphasizes the continuity between Hussein's instrumental and communicative gestures, where the slightest adjustment in how Hussein handles a car part serves to indicate something to his employee. Nevertheless, I will separate 'writing' from 'gesturing'.

Similarly, Mead's approach to gestures includes vocal gestures (that is to say, speech) as a sub-type, stemming from his general approach of tracing incipient acts back deeper and deeper into the body, the nervous system, and the brain those physical movements that project and set in motion the more publicly visible actions, like a movement of the larynx, tongue and mouth that constitute a spoken word, and like a movement of the hand that constitute what I will call a 'gesture'. Nevertheless, I will separate 'speaking' from 'gesturing'.

So, all of the work I tried to do in order to transcribe writing streaks I will now reinterpret in the following way: all the gestures and body movements that I captured in the *Gestures and the Body* documents occurred within not-writing streaks. I made it a principle that if I were to write about some gesturing occurring in some scene, I would determine what writing was to come and what writing had just occurred in the neighbouring writing streaks on either side of this not-writing streak.

I created the original *Gestures and the Body* document in the following way. I first of all kept in mind Streeck's sorting of gestures into spotlighting, depicting, and gesticulating. Secondly, I kept in mind Streeck's exemplar representative gesturing practices that fell within these divisions and that I discussed in the previous chapter. Thirdly, I read closely a section of Streeck (2017) where he goes through a minute-long sequence and takes up the gestures that happen one by one. Fourth, again modeled from Streeck (2017), I knew that shrugs, stances and poses, mimicry and pantomime, body movements and facial expressions of all kinds could feature prominently in gesturally significant episodes. From this background, I aimed to go through the lectures, and steer tightly to the minimal end. Did I notice a spotlighting gesture, a depicting gesture, a gesticulation? Did I notice a movement of J's hands that I interpreted as a meaningful act? Did J move his body in a way that departed from various standard default background movements that I already understood? If so, what was the briefest, most succinct one or few word phrase that would capture these observations?

Mid-way through the first lecture that I was examining in this way I realized (again) that spotlighting and gesticulating happen far too frequently to try to capture all of them. I felt a bit like how I feel when reading a proof that contains all the details – the main steps together with every last little step – there is too much clutter, and it is hard to notice anything that stands out. I needed to rely on such intuition and judgment as I could offer – as a person who lectures in mathematics, including the subject matter of J's course, who does not understand the mathematics as deeply as J does, but who can follow what he is doing – in order to be somewhat selective of the instances of spotlighting and gesticulating that I captured. I captured a good number, but not all, of the instances that had begun to seem similar to each other, so that these would stand for the others that I would pass over, and I captured all of the instances that stood out from this background as being in some way idiosyncratic or special.

I did this for almost all the lectures. I decided not to do Lectures 9 to 12, and Lectures 21 and 22. I had organized my work by the chapters of Gallian that J was covering, and I had roughly sorted these chapters in, quite frankly, the order I was curious about. These omitted lectures correspond to the cyclic groups chapter and the direct products chapter. The direct products chapter I put last because the results in that chapter seemed less interesting to me than those in the other chapters; the cyclic groups chapter was about as interesting as other chapters, and I intended to cover it. However, long before I got to these two chapters I had begun to feel that I had seen the sorts of things that I was going to see, and that there was a strong diminishing returns aspect to continuing.

More precisely, I mean this: with every new lecture that I viewed and added to the *Gestures and the Body* data document, I was improving the quality and utility of that data document as a tool for succeeding in two self-reinforcing aims: one, unearthing scenes of exemplary interest, and two, understanding what was typically and repeatedly happening throughout the course at many hundreds of occasions of mathematical actions, so that I knew what *exemplary interest* meant. The amount of improvement in quality and utility of this data document, after ten lectures had been captured, was already smaller than it had been near the beginning; after twenty, smaller still; at length I decided that the improvements were now smaller than what I could accomplish by using this tool as it was now rather than continuing to sharpen it. Later in my research, there came moments when I needed to analyze a few scenes from these omitted lectures: then I examined those scenes for the gestures and body movements they contained.

Finally, I took this original *Gestures and the Body* document, which by virtue of the manner of its creation had variation in the detail with which various gestures were described or captured, and I compressed it into a document that allowed only a single line to any given time stamp. I did not try to design a code that I rigidly followed. Occasionally, some short phrase would become very useful; once I happened upon such a phrase I made the effort to use the same phrase for all similar instances. This way I could later find very quickly lots of relevant instances of gesturing practices that I considered similar. In Appendix J, I show the portion of the compressed *Gestures and the Body* document corresponding to Lecture 24.

The original and the compressed *Gestures and the Body* documents, whose construction I have detailed in this section, were the main source for the representative instances of gesturing practices that appear in the remainder of the thesis.

4.7. Summary

Birdwhistell had this to say about methodology:

> From the most technical point of view there are four cardinal steps in the development of valid and reliable social behavioural data: (a) learning to observe; (b) learning to record the component events and relevant context of that which is observed; (c) the organization, preservation, and preparation for analysis of stored data; (d) the development of relevant and efficient methods for the review and analysis of such data. (Birdwhistell 1967, p. 554)

In this chapter, I have described the process by which I developed and used six tools which helped me learn how to do the actions Birdwhistell lists here. These tools addressed the whole course: a transcript of all the spoken words; a transcript of all the writing on the whiteboard; a collection of still images that together exhibited all the writing on the whiteboard; a document of all the episodes; a record of all student contributions. Inspired from their development, and the use of them in conjunction with each other, together with targeted and repeated viewings of the videos of the lectures in the course, a final tool was generated by passing through all but six lectures and recording the gestures and the body movements integral to the ongoing mathematical action.

In the next chapter I discuss the families of gestural practices that J engages in while lecturing in the undergraduate mathematics classroom.

Chapter 5.

Gestural Practices

5.1. Introduction

I assume now that I have performed all of the following kinds of work that I discussed in the previous chapters.

I leaned on key terms from Mead: the I and the me; the social interaction as fundamental; the beginning of acts as crucial, that gestures – and their interpretive significance to others – stand for the acts that would have followed the beginning of the act; that communication for humans in their entire development from childhood has consisted of myriad such social dialogues and conversations of gestures and talk and body movement, where we have all learned to anticipate later actions from the earliest action, and can in this way take on the attitude and the role of the other.

I leaned on Bateson for the importance of beginnings and endings of scenes and frames, the constant reminder to look at what was happening before me with a minimum of preconceived assumptions as to what was important or what the right language might be to talk about it with, and the urge to make accurate micro-ethnographic observations coupled with meaningful macro-analysis into abstractions which would not straitjacket talk about these phenomena, but instead allow for richer and closer attention to the observations themselves.

I leaned on Streeck for many things, but I will highlight four of them here. First, for his language of kinds of gestures (spotlighting, depiction, gesticulation) and his analysis of them. Second, for his conception of practices. Third for his emphasis on simultaneity and timing (that what follows and what precedes some talk combined with gesture is important for the analysis, as well as where we are in the ongoing interaction). But most of all for how it was he accomplished what seemed to me very similar goals to mine: to study a single mechanic for many hours and say something meaningful about the methods by which this mechanic makes sense of his world for the people he interacts with.

The result is six *families* of gestural practices that emerged from this method of researching this video record of this course of lectures. They show up repeatedly in the lectures: any given random three minutes in succession would almost certainly contain at least one or more instances of one or more of these practices. In the following paragraphs, I answer some natural questions about this manner of sorting and arranging J's gestural practices. Can a short sequence of gestures in a single scene belong simultaneously to *more than one* family? Or if not simultaneously, can short sequences of gestures be seen as belonging first to one family and then to another in rapid succession? To what degree is the categorization *complete*? How *strict* and how *elaborate* is the classification scheme?

Boundaries between the gestural practices are not sharp in two ways: instances of different families can occur within short time intervals, and instances can belong to more than one family at once. It is possible to interleave gestural practices smoothly and easily, moving the hands in a way that forms a part of one practice, and then in the next few seconds move the hands in another way that forms a part of another practice. Gestural practices are productive and generative: finishing an action not only gives or provides something that was needed, it also often raises a new problem or need which the next action begins to fulfil. This next action may very well be achieved by movements of the hands that are part of a different family of gestural practices.

Considered, then, as a function of *time*, the current gestural practice can change rapidly; such relatively sudden transitions from one family to another are by no means uncommon. It is also possible for the same hand movements during an interaction to be considered as an instance of more than one family of gestural practices. This possibility is the exception and not the rule. Most of the time any given gestural movement is naturally viewed as a member of a single gestural practice family. Abstractly, overlaps between families of gestural practices are non-empty, but sparsely inhabited.

This typology is not intended to be comprehensive. I identified those families that I considered to be elemental in importance, and which were comparably important to each other. Further practices, which occur more rarely, may well be of future interest to researchers in mathematics teaching. No other practice that I could identify or define occurred nearly as often.

This classification scheme is not defined in a precise top-down manner. As someone with mathematical leanings, I yearned for three yes/no distinctions whose answers would sort, for example, eight gestural practices into their respective categories. The Aristotelian approach seems to die very hard; nevertheless, try as I might, I could not formulate such a set of distinctions that did not immediately result in too many counterexamples. The natural historical approach is different; perhaps an appropriate analogy is the species taxonomy of the animal kingdom. Which approach to the classification of species is best, and even the definition of the species concept, remains a highly contested and complex issue to this day, after centuries of observation and analysis (for a discussion of the philosophical problems of classification in modern science with careful reference to this example, see Mayr & Bock, 2002). It would be a mistake to expect that the families I discuss and exemplify in this chapter, generated as they were from careful micro-ethnographic observation of a very large data-set of actions performed by a mathematics lecturer, would be sortable into a two-by-two-by-two division of boxes.

Finally, there were occasions when I faced the choice of splitting one of the six families into several smaller sub-families, on the basis of some further distinctiveness that I could observe and define in these sub-families. On each of these occasions, I chose simply to keep the larger family for two reasons. First, the sub-families were less robust as categories. Second, I knew I could discuss such sub-varieties when they became relevant, and indicate their features, while stopping short of giving the sub-variety its own name and identity.

In choosing the instances that exemplify these practices I decided in this chapter to stick to the topic of *isomorphisms*. This allows for certain advantages: a homogeneity of theme; a shortening of prefatory material explaining what topic J is talking about whenever I want to discuss some practice; a greater ability to sense what is distinctive about the practices when the topic concerned is being held invariant. This choice affords me an invaluable opportunity to indicate how J treats the fundamental concept of isomorphism by discussing and analyzing these various episodes through the lens of introducing these six families of gestural practices. After all, all of these practices show up throughout the course, so there is a wealth of examples to choose from, even when I restrict attention to a single chapter of the course. On the other hand, there were phenomenally nice instances of some of these practices that happened to occur at other times and so fell to the sword.

A word about my conventions. I use single quotation marks (‘ ’) to surround text that J wrote on the board. I use double quotation marks (“ ”) to surround text that J or a student speaks. I use a forward slash (/) to separate two spoken lines (in other words, to indicate the segmentation between one *line* and the next *line* as recorded in the *S transcript*). I use a hyphen (-) to indicate those moments when J self-interrupts. I use italics for mathematical symbols. There are a few mathematics symbols that are not easily typeset by Microsoft Word. For these I use a backslash followed by a word that indicates what symbol I mean. For example, ‘*H* \normalsubgroup *G*’ means that J wrote that expression on the board, only he used the appropriate mathematical symbol instead. In J’s speech I record numbers using words and not numerals. For example, when he refers to the written symbol ‘R_{90}’ in his speech I will write this here as “*R* ninety”.

5.2. Manipulating Objects

The first family of gestural practices can be recognized as a compilation of occasions when J’s hands are engaged in handling an object: moving it, rotating it, flipping it, and in all such ways transforming it; or moving his hands over an object in a manner that is context-specific or specific to the features of the object that currently concern him. This gestural practice is called *manipulating an object*. J can be manipulating a *physical* object; or J can be manipulating a *pretend* object, moving his hands in the air *as if* he were manipulating a physical object; or, lastly, J can be manipulating a *textual* object, moving his hands near and on already-written markings on the board.

J begins his teaching of the chapter on isomorphisms immediately after the week-long reading break. Most lectures in the course begin with a transparency-assisted review of the material from the previous lecture; more rarely a lecture will start a new chapter, which occasions a different sort of beginning. This lecture is unusual in that J must begin from a dead stop. It is similar in that sense to the first lecture of the course. The mid-term occurred just before the break, timed with the end of chapter 5, and now he must re-launch the course.

He does so by announcing the theme of this new chapter, and then reminds the students that there were moments earlier in the course that foreshadowed the concept of isomorphism of groups. First, there were those occasions when they had noted that finite cyclic groups of order n are "structurally identical" to Z_n (the additive group of the integers modulo n) and that infinite cyclic groups "behave like" Z, the integers. I will discuss a couple of such occasions later in this chapter. A second previous occasion where the concept of isomorphism was implicitly present is what he reminds them of next.

He fishes for and finds in his notes on his desk a square whose corners are marked with the numbers 1, 2, 3, and 4, saying, "if you think about our games with this" – on "this" he holds the square up at head height, flipping it back and forth – "which is how we started this whole business". Here he is referring to Lecture 2, which I analyze in some detail in the next chapter. "We talked about the group of symmetries of the square", he says, punctuating moments of tonal emphasis with a forward movement of the square in the direction of the students. Then he systematically flips the square over two different axes of reflection, saying "we talked about this kind of- this kind of map".

Now he warms to his present theme: "and then we said that structurally" – he holds his right hand in a forward grasping motion – "that group of symmetries / a vertical reflection" – with his right forefinger he twirls a couple of horizontal circles to indicate a flip over a vertical axis – "a rotation" – he brings his right hand down to grip the square together with his left hand that had held it in order to physically rotate the square by ninety degrees – "that's structurally identical to the group of permutations / of the vertices of the square". On "vertices", he points at the square with his right index finger.

In his next few lines, which I will not quote, he contrasts D_4, understood as the group of symmetries of the square, with D_4 thought of as a subgroup of the permutation group on four symbols, S_4, by handling the square in two contrasting ways. While talking about D_4 in the first sense, he continues to flip and rotate the square in front of him while looking at the students; while talking about D_4 in the second sense, he holds the square with his left hand and uses his right index finger to circle around the corners while pointing at it, indicating how the vertices were being permuted under some action of D_4. He goes on to briefly mention one example of such a permutation "one goes to two two goes to one" and with his left index finger he motions in the air a horizontal line that starts to his left at 1 and ends a little to his right at 2, and then back again.

In this scene, we see J physically *manipulating an object* (the cardboard square), in order to perform the transformations from one configuration to another that constitute elements of the group that is being discussed. The manipulations are of two kinds: one where the object gets moved around, and another where J's hands and, in particular, fingers are moving around adjacent to, and touching, the object, in order to spotlight a different aspect of transforming the object. We also see how smoothly J pivots from moving the physical object itself in order to meaningfully communicate "reflection", to twirling a pretend invisible replica of the object with his finger in order again to meaningfully communicate "reflection".

It is noteworthy that this second gesture occurs even as the cardboard square is still being held in his other hand, readily available to flip. A moment such as this one induced me to gather under one gestural practice not only the occasions when he holds an honest-to-goodness prop in his hands (a square, a tetrahedron, an icosahedron, a textbook, a piece of paper, a coffee mug) but also the occasions when his hands manipulate the air in front of him, as if there were such a material object in the space being manipulated. J's choice here indicates that he sees this second option as of comparable value in acting out a reflection.

Streeck's analysis of the car mechanic, Hussein, also linked the spotlighting gestures he might use when physically touching an engine part in front of a customer with those gestures he used when away from the engine (either in the air, as J did here, or by seizing opportunistically at an available object at hand that could stand for the engine). Lecturing proceeds so swiftly, accelerates and slows and digresses so rapidly, that J might not wait to find a physical object to manipulate, even when it is already in the other hand.

The power of some gestures to accompany, and help enable, a faster mathematical action emerges even more explicitly in a later vignette two lectures later, which also involves a *manipulating an object* gestural practice. J has written the definition of automorphism on the board, and he has also written his first example of this concept: complex conjugation on the set of complex numbers (considered as a group under multiplication). He touches the equation on the board that defines complex conjugation, and says, “and if we want we can go through all the properties”. In a sing-song voice, he lists one property after another that ought to be checked, followed immediately by a deep voice bored-tone rejoinder of “yes”. His body stays fixed, and he bows after each sentence a little as if he is a mechanical robot compelled to go through the motions.

Suddenly, he says “but”, and he begins to walk back over to the writing on the board – “visually” – he holds his left hand out with palm up, holding for presentation the statement that complex conjugation is an automorphism (Figure 1) – “for goodness sake” – he turns his left hand over 180 degrees (Figure 2) – “isn’t it obvious?”. After a beat he lets his left hand fall to his thigh with an audible slap, and he exhales audibly with a smile.

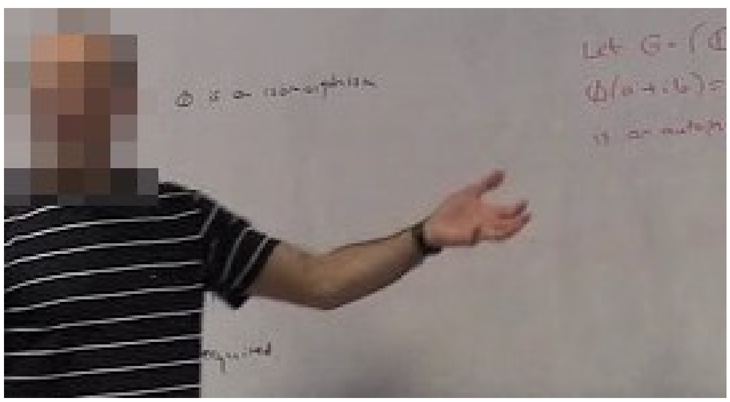


Figure 1. Manipulating an object – before.

Figure 2. Manipulating an object – after.

Then he brings his left arm up again, hand in the position it was a few seconds before (in the flipped over position), and then undoes his previous turning-over gesture, so that his hand is now palm-up the way it was initially: “turns the complex plane upside down". Looking at the class the whole time, his hands now at his sides, he moves them both upwards towards the class palms-up saying "of course it's structure preserving / do we really need to write anything down"; he is emphatic on the word "course".

Here, J explicitly tags his gesture as capturing enough of what matters of the mathematical argument as to be sufficient to carry the day. Hearing himself say this he goes on to joke that he can now anticipate that students will write on their exams "it's obvious" rather than writing out their arguments. Still, he does not write any more justification on the board, saying only that the students can check for themselves "algebraically" what his ten-second one-way flip and then back-again flip has performed "visually". The cheerful impatience on display here has to do with considering it to be a routine task to write out of the details of an algebraic verification of a result when those details follow immediately and inevitably from the turning-over, there and back again, of a phantasmal complex plane in the air. J's robotic pantomime of the bored checker of algebraic properties helps draw an even more marked contrast to the rapid visual approach executed decisively with two flips of his hands.

Mapping gestures abound in this course. In the two examples thus far in this section – twirling a finger to reflect a hypothetical square about an axis, flipping over a hand to reflect a hypothetical plane about an axis – the mapping is from a group to itself. In other situations, the maps gestured about are from one group to another. In the second of his three lectures on the isomorphism chapter, J has reached the point where he is about to state two theorems on isomorphic groups. The slogan of the first theorem is 'the properties of isomorphisms acting on elements' and the slogan of the second is 'the properties of isomorphisms acting on groups'. The hypothesis for each theorem is the same: 'Suppose *phi* is an isomorphism from *G* to *G* bar'. In each theorem there is a list of statements; an example in the first theorem is '*phi* maps the identity of *G* to the identity of *G* bar' and an example in the second is '*G* is cyclic $\Leftrightarrow$ *G* bar is cyclic'. J is making a few comments before writing out the statement of the first theorem concerning these two theorems. (Here and throughout *G* bar means the symbol *G* with a short horizontal line drawn above it).

He begins with what the first theorem will be about – "so there's a bunch of statements to do with element properties" – and then he seizes on an example: "the order of an element in the original group". On "original", he takes both hands (one of which is holding his notes, the other holding his marker) and moves them together to point to a space on the left side of his body. Then, saying “does this compared with the element- / the order of the element in *G* bar”, he moves both his hands together in a semi-circular *carrying* gesture over to the right side of his body, repeating the movement a second time so that his hands arrive at the second space at the moment he says the word “*G* bar”.

Next he takes an example of the second theorem – ‘*G* is cyclic ⇔ *G* bar is cyclic’ – and when he first says the word “cyclic”, both his hands are roughly holding some space on his left side, and then by the time he says the word cyclic a second time to refer to the codomain “*G* bar” he has completed the same semi-circle in the air and landed with his hands holding some space on his right side. He repeats this carrying gesture a few times: each time his hands move through an uninterrupted flowing path from a source space to a target space. His hands hold imaginary material in one region, while he utters words naming properties like “order” and “cyclic”; then, while maintaining the same shape and conformation, as if they are persistently holding some cargo, his hands sweep through the air in a semi-circle to arrive at a second region; finally, having arrived, his hands still hold imaginary material, while he repeats his earlier words, “order” or “cyclic”. This sequence of movements of the hands is a public display of what is shared, what is in common, what is the *same*, about these groups. It is one of many examples of gestural actions which simultaneously show that there are *two* groups in the local context (that may not be equal to each other) *and* that there is really only *one* group *qua* group in this context which has been realized in two different ways.

In the numerous contexts in which mappings are being lectured on, and in which J moves his hands and body in order to meaningfully act mathematically, he sometimes makes other choices. For example, it is often his finger that traces a path in the air from the source set to the target set. Sometimes he uses his hand to trace a path from the domain to the codomain so that he can *then* use his finger to indicate a path from a particular element in the domain to its image in the codomain. It is interesting to observe that in the very occasion when J is keen to distinguish two theorems he uses exactly the same hand movement to perform the example statements of each theorem – this is less surprising under the interpretation that his hands are moving between two regions that are “structurally the same”.

So far what J has been *manipulating* has either been a *physical* object, or it has been a *pretend* physical object (gesturing a transformation from that object to itself, or from this object to a second object). I turn now to the third sort of the gestural practice of *manipulating an object*: occasions when the object that is being manipulated is a *textual* object on the board. I examine two scenes; in the first the textual object is a large blank space bordered at the top and bottom by two mathematical expressions he intends to show are equal; in the second the textual object is an individual mathematical equation.

In Lecture 18 J is midway through an exercise where he must show that if *H* is any proper subgroup of *Q* (the rationals), then *H* cannot be isomorphic to *Q*. As part of this exercise he wishes to establish the following claim: that if *phi* were such an isomorphism, then it would perforce have to satisfy the identity *phi(x) = x phi(1)* for all *x* in *Q*. He has written a nonlinear begin-and-end bit of writing, which begins with '*phi(x) = phi(m 1/n)*' (because he can assume *x = m/n* for some *m*, *n* integers), followed one line below by '*= m phi(1/n)*'; then there is a large vertical blank space, ended by '*= x phi(1)*' at the bottom. He walks over to the blank spot after the second line, pauses there with his marker hovering over the writing position but does not write anything, saying "ok uhhhhhh".

Then he takes one deliberate step back from the writing position, turns to the class, and points at the second line: "well I don't know what to do with that". He swivels to face the class fully, then swivels back to the board, and says "so I'll come up from the bottom", using his right index finger to sweep up the blank space starting from the bottom line. He begins to write a line that is immediately above the bottom line, but before he finishes writing it (and therefore also before he can mathematically justify why it implies the bottom line) a student pipes up with an offer, to which J replies "which-which way are you- / you helping me here or here". On "which way" J points with his finger at the beginning of the gap, then sweeps down to the bottom of the gap, then back to the top; on the first "here" he has moved a step to the board and points closely at the top of the gap and on the second "here" he points closely at the bottom of the gap. The student says "down" and J moves into position to write what the student will say.

Here, as on many other such occasions, a piece of the board (here a block of equations aligned vertically, with a sizable gap between the start and the finish) is *manipulated* with the hands to highlight features of interest, in this case, features which might help J determine which next written line will be easiest to justify. Should they *attach* a line immediately below the beginning of the gap, and justify why this line is implied by the previous line that is already there? Or should they *attach* a line immediately above the end of the gap, and justify why this line implies the next line that is already there? In each option there is an accompanying mathematical justification that will support the inference; once the inference is convincingly argued for, J can alter the shape and form of the textual object on the board. He can make the gap smaller by building more writing either at the top edge or the bottom edge of this empty space. His hands roam over each edge, indicating the choice of where to build, sweeping downwards and upwards. Each such sweep encourages the students to imagine or determine the logical implication which will allow another line.

To students who have come to expect that written proofs are textual objects that must be built from the leftmost topmost textual element to the rightmost bottommost textual element, this sequence of lecturing actions, including these instances of this gestural practice – handling the edges of a piece of writing with a one-dimensional hole in it, deliberating over which edge is at this moment more amenable, more accessible to immediate action – will not be the expected, dull actions that one has been habituated to and can easily ignore.

What has been observed here about a textual slot in a written proof can and does occur in the course for any conceivable piece of mathematical writing. The last example of *manipulating* a textual object in this section concerns a single step in a proof where the previous line is an equation 'a x a^{-1} = b x b^{-1}' and J proposes "let's do a rearrangement of that" as he sweeps his right index finger underneath this equation back and forth twice. Saying "I'm going to multiply on the left by *b* inverse" he sweeps his finger along the board underneath the equation to the right and then writes 'b^{-1} a x'. He pauses for a second, says "then I'm going to multiply on the right by *a*", and he uses his finger now to start at the rightmost spot of the equation and sweeps over to the left. He touches the 'a^{-1}' term briefly to say "so that will kill that". Then he slides his right finger to the right above the equation to slide it into the '*b*' term, saying "and I'll get- / the *b* inverse killed that", now touching the '*b*' term. Then he finishes writing the new line.

Suppose we were to watch a ten second film of a plumber handling a short tube open at both ends, showing where liquid will flow in from the left opening with a trajectory directed to the right, spotlighting how it might be that liquid could flow in from the right opening headed to the left, tapping each end to indicate that a stopper could be placed there. Such a film would look very similar to this bit of mathematical action. Instead of a tube there is a previously written equation. Part of what students experience in J's lectures is witnessing him act on, with his hands, accompanied by his speech, the pieces of writing he has already made, often in order to build a next piece.

In the next two chapters there will be many appearances of the *manipulating an object* gestural practice. In Chapter 6 I will show that two kinds of diagrams – Cayley diagrams and group multiplication tables – are significant examples of textual objects that J manipulates. In Chapter 7 I will discuss the major modifications J makes to the mapping gestures analyzed here when he treats the concept of a one-to-one mapping, as well as what it means to confirm that a proposed mapping is indeed a function.

5.3. Looking at Side-by-Side: the same and different, better and worse

The family of gestural practices discussed in this section are often the easiest to identify in the lectures. They invariably involve two pieces of writing that are simultaneously visible. J walks back and forth between the two textual objects. He might touch or point at one and then the other, and then back again several times in succession. He might move his hands over one in a particular way, handling it in a context-specific way, or specific to that object, then move over to the other object, and handle it the same way, or in some slightly different way. The actions he can take, or does take, are seen to be similar, or are witnessed to have important differences. Often the two textual objects are immediately adjacent to each other. Often it has taken J considerable time and planning to get to the moment where such a gestural practice can begin. Sometimes the gestural practice begins even while J is still making the two textual objects. This is the *looking at side-by-side* family of gestural practices. In this sub-section I illustrate this family by discussing two scenes that occur very near each other in the same lecture.

Nine minutes into his first lecture on isomorphisms (Lecture 16), J has already written on the board the definition of 'isomorphism', along with the definition of what it means for two groups to be isomorphic. Reading along from his notes he says: "ok so the idea is that *G* and *G* bar are structurally identical / but possibly notationally different".

He walks to the board announcing his intention to “look at this idea from the point of view of Cayley diagrams”. He writes the letter ‘*G*’ and above it draws an oval with two dots in it. He labels the top dot ‘*b*’ – “let’s have *b* here” – then draws an arrow from the top dot to the other dot and labels it ‘*a*’ – “and here is the action *a*”. Then he says “and when *a* acts on *b* we get *ab*”, at which point he labels the bottom dot ‘*ab*’.

Next he moves a little to his right, and a similar bit of drawing takes place. Saying “now over here” – he draws a second oval – “in *G* bar” – he labels the oval underneath just as he had labeled the first oval ‘*G*’ – “let’s suppose we start off here” – he draws a top dot in an analogous position in the second oval – “with *phi* of *b*” – he labels the top dot – “and we act on that” – he draws an analogous arrow to a second dot – “with action *phi* of *a*” – he labels the arrow, and turns around to face the class. He looks them over and pauses. The stage is set. It is obvious that the second diagram, which shares so many commonalities with the first diagram, has one missing piece that needs to be filled with writing. It is impossible not to feel some degree of curiosity, expectation and suspense.

Saying “now”, J turns to the board again and touches and holds (Figure 3) the second dot in the second oval: “by rights”, he says in a convincing tone, “what you should write here is *phi* of *a phi* of *b*”, looking at the class while still holding the dot.

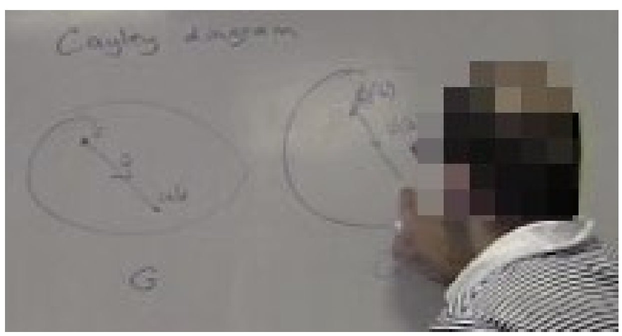


Figure 3. Looking at side-by-side.

He turns back and explicates with his hands to make certain they see the justification of what he has just said: “the effect of applying this action” – he touches the arrow labelled ‘*phi(a)*’ – “to this” – he turns, holding the top dot labelled ‘*phi(b)*’, staring at the class – “group element”. A couple of seconds of silence ensues as he waits. Then he releases his hold on the diagram – “but the whole point is” – and he walks over to the definition of isomorphism on the board where the condition of preserving the group structure (‘*phi(ab) = phi(a)phi(b)*’) sits and touches it as he says “is” – “if you have an isomorphism" – here J touches three locations in quick succession: the equal sign in the equation, the line of words above it ('1-1 correspondence between *G* and *G* bar'), the leftmost part of the equation – then he walks back to his diagram in order to complete it by labelling the bottom dot with '*phi(ab)*' – "that is *phi* of *a b*". While the actions in this paragraph are all part of the *looking at side-by-side* gestural practice, the sequence where he touches the arrow and the top dot is an example of *manipulating an object*, and the sequence where he touches three locations in order to determine how to finish the second diagram is an example of *deducing that*, the gestural practice family that is discussed in section 5.5.

J shows here, by the very act of making the mathematical expression that will complete his second diagram, that one of the defining properties of an isomorphism is *exactly* what is needed for him to complete it in this way ('*phi*(*ab*)'). He has in addition shown that, if it were not for this condition, he would have been forced to complete the diagram in a different way ('*phi*(*a*)*phi*(*b*)'). Now he needs to explain why the first way is so special.

A few spoken lines later, J is reaching his punchline: "so if we have an isomorphism *phi* / then this picture" – he taps audibly the first diagram with his marker precisely on "this", then shuffles backwards a half-step so that he is beside the second diagram – "and this picture" – he taps the second diagram audibly exactly on "this" – "are the same picture" he concludes, with a high pitch that I would interpret as a tone indicating a surprised or shocked realization (his pitch becomes higher and his voice almost breaks). He says "you've just stuck *phi* of in front of everything" while his hand moves his pen in a dash of writing a left and right parenthesis in the air. Audible touches of the board in the course are not common but also not rare; they usually accompany occasions where J is acting in a mathematical context in a definitive, unmistakably correct way, or perhaps one where he has no choice. Here the symmetric sounds of the vigorous strikes on each diagram propel the conviction that both diagrams are the same apart from the extra '*phi*( )' labelling in the second diagram.

Now J begins a very deliberate pattern of three pairs of touches, with the third pair of touches distinctly marked as different from the first two pairs. His pinky touches the top dot of the diagram of *G* while he says "*b*"; then he moves the pinky over to the diagram of *G* bar touching the corresponding dot and says '*phi* of *b*'. His pinky touches the labelled arrow in *G* while he says "*a*"; then he moves the pinky over to the diagram of *G* bar touching the corresponding arrow and says '*phi* of *a*'. These two pairs of touches and the accompanying speech take two seconds in total. His face was aimed at the board, and from behind it is clear that his eye gaze goes from the first diagram to the second, back to the first, and then back to the second. But now that his pinky returns to the diagram of *G* to touch the bottom dot, he turns to face the class and his speech slows: "*ab*" he says deliberately, then he moves his pinky to touch the corresponding dot in the diagram of *G* bar without looking and says "*phi* of *ab*" with a smile. "It's the same picture" he concludes, shaking his marker up and down repeatedly to emphasize the syllables: "you've just relabelled the nodes and the actions". He stops talking for a few seconds while looking at them all, before quietly saying "ok" with a nod.

Once J has got his diagrams in place, he will patiently elaborate with his hands to systematically reinforce and deepen the quick impression of the eyes. Back and forth he goes, on this occasion from a generic picture of one group to the other, carefully touching the points that play analogous roles in each picture. His pacing and timing are also measured and controlled; rhythmic and straightforward for the first two pairs of touches; a studied pause in which his interaction with the students purposefully increases before the culminating finish of the third pair of touches. I am reminded while watching this scene of the classic three-part structure of many standard jokes: an X, a Y and a Z do something or see something; X says this, and Y says that, and then – pause for effect and look at your audience – Z says *that*.

There has been a running theme, in the course, of visualizing the behaviour of groups not only with Cayley diagrams but also with their multiplication tables (which he refers to as Cayley tables). A few spoken lines later he begins a new diagram underneath what he has just drawn: when he is finished, on the left is a square with a single columnar rectangle running vertically and a single row rectangle running horizontally, with the column labelled '*b*' at the top and the row labelled '*a*' at the left, with '*ab*' written in the small square where the two rectangles intersect. This Cayley table is labelled '*G*' and sits directly beneath the Cayley diagram of *G* that he drew earlier. Next to it, directly below its Cayley diagram counterpart, he has drawn another such square with two intersecting rectangles, with a top label of '*phi(b)*' and a left label of '*phi(a)*'. He performs this drawing quite quickly and efficiently and his tone is less theatrical and more mechanical. He now says "and again you would expect to see *phi* of *a phi* of *b*" –

his right hand pointing first to the left label and then casually pointing to the centre of the square (he does not bother to point directly at the label at the top of the column) – "but you actually see *phi* of *a b*" – he writes this label in the open space that had so far been left unlabelled, the corresponding little square where the column and row rectangles intersect.

Notice the nearly exact repetition of the moment of suspense in the drawing of the Cayley diagrams: everything in both diagrams has been constructed except the final piece of the second Cayley table. Exactly like last time, two options for filling in this hole are presented, and like last time, the first option is spoken aloud, while the second option, which is special, and only occurs because *phi* is an isomorphism, is chosen and therefore written.

Placing his open palm over the second diagram he says "you get this table" and moves his open palm so it rests just over the first diagram saying "from this one". Then he finishes by saying "by applying *phi* everywhere in sight" and he flings his hand vertically upward over the whole diagram, almost dismissively, facing the class as he does so. He stares at his notes, nods, and says "good" concluding the stanza.

The two scenes analyzed here involved two diagrams positioned side-by-side. Many instances of the *looking at side-by-side* family do involve diagrams, and there is no doubt something about diagrams that dovetails very well with this gestural practice. However, there are also many occasions in the course when what is being *looked at side-by-side*, and then systematically and deliberately touched, pointed at, and handled in a patterned way, is two portions of mathematical text that are simultaneously accessible to J's hands. Moreover, while the two examples above showcase what is the *same* about two textual objects, it is just as common for the patterned pointing to spotlight noteworthy remark-able *differences* or *distinctions*.

Here is one example. When introducing the notions of the centre of a group, and the centralizer of an element of a group, J takes pains to write on the board the proofs that each of these forms a subgroup so that both proofs are visible simultaneously. He also tells the students that he worked on the proofs to make them look as similar as possible. J says he did this precisely to be able to point to those portions of the proofs which are different, so he could disclose, in a focused and concentrated manner, exactly how these concepts, though named so similarly, differ from each other in significant ways.

A final lesson from our pair of examples here also has other echoes in the course. The students are not only able to glance back and forth at the two side-by-side Cayley diagrams as J touches on one and the other repeatedly; once the two side-by-side Cayley tables join them on the board directly beneath, the students can glance up and down to compare and contrast these two different *methods* by which these visual representations reveal the crucial role of the '*phi*(*ab*) = *phi*(*a*)*phi*(*b*)' condition. In short, we have a pair of pairs. This doubling of doublings, or in some cases a chain of doubles (A side-by-side with B; now B side-by-side with C), occurs at a handful of important moments in the course. Indeed, there is another such example later in this same lecture. Although he does not make use of this opportunity here, on some of these other occasions J touches and handles one pair of textual objects, and then the other pair of textual objects, moving back and forth repeatedly, in order to reveal how one pair affords one perspective or reveals one distinction while the other pair affords another or reveals another that is similar in some respects, different in others. Such second-level comparisons, analogies between analogies, can be a powerful step towards abstraction.

While it might not be surprising that mathematical material concerning isomorphic groups might very well include scenes where two bits of writing side-by-side are compared and somehow found to share group-theoretic features, the gestural practice of *looking at side-by-side* is by no means restricted to the chapter on isomorphisms but indeed is a phenomenon of interest that occurs regularly throughout the course. Some of the other families are populated with far more instances, and these instances can vary widely in the scale of their individual impact on the ongoing mathematical interaction they are expressed in. In the *looking at side-by-side* family, however, the instances nearly all have a powerful impact.

5.4. Regarding As: how a mathematical object or environment is to be viewed and used

At the beginning of Lecture 17, as J is reviewing the previous lecture, he gets to the part where they had considered an illustrative example of Cayley's theorem (that every finite group is isomorphic to a group of permutations). He is working with a transparency, and on it is written the Cayley table of the group *U(12)*, whose elements are 1, 5, 7 and 11 (because these are the positive integers less than 12 that are relatively prime to 12 – the group operation is multiplication modulo 12). He holds the title of the Cayley table as he says "the illustration was we were looking at a Cayley- / the Cayley table for *U* of *12*", then he moves his finger down to circle the table once briefly as he says "and it looks like this". He sweeps his index finger across a row of the table, saying "and we interpreted each row"; then he runs his finger up and down the labels of each row, saying "as being the action of this element"; then he runs his finger across the labels at the tops of the columns, saying "on all the other elements".

Now he does an example. He says "so the action of 5" while holding the symbol '5' with two fingers, one from each hand: this symbol '5' is the label on the second row of the table – the entries of this row are the result of multiplying 5 by the column headings 1, 5, 7, 11 in turn. He continues "is to permute 1 and 5", and he twins his index and middle finger together, and twirls them around by 180 degrees so that his index finger which used to point to the 1 has now flipped around to point to the 5 and the middle finger has done the reverse. Then he repeats this turning around gesture with his two fingers as he says "to switch them round". Then he moves his hand over a little to the right so it is hovering over the column headings '7' and '11', and repeats this same gesture twice saying "and to switch 7 and 11".

Beside the table he has already written 'the action of 5 is T_5 = (1 5)(7 11)' (which expresses the action he has just talked about in what he calls the cycle notation for writing permutations). He sweeps the beginning of this writing, says "so the action of 5", then smoothly moves his finger to be underneath the 'T_5' term, says "which is", then moves his finger to touch the 1 – "1 goes to 5 and 5 goes to 1" – his finger moves rigidly from each number he names to the next one like he is moving the number from the spot it was in to the other spot – "7 goes to 11 and 11 goes to 7" – he does the same here with his finger underneath these two numbers in the permutation. Two lines later he is concluding by saying "and then we think about composing these actions" – he circles his finger around the table – "by composing these permutations" – he similarly circles his finger around the permutations he has written next to the table.

This is a classic instance of the *regarding as* gestural practice. The *manner with which* he twirls his two fingers in order to "interpret" a row of the Cayley table, and *the* *way* he moves his finger in little precise sideways shifts as he speaks out loud the permutation, simultaneously performing and saying the permutation: these are the actions of a person who is showing that they are considering a mathematical situation from a particular *point of view*. The word “interpret” is a tell; others include “see this as”, “represent this as”, “view this as”. It is a small word, but it is the word *as* that is the surest spoken signal that the *regarding as* gestural practice is at work. This is why I have included it in the naming of this practice.

A row of a Cayley table can be thought about in many ways; to translate into the theoretical terms of this thesis, a row of a Cayley table can be a bit of writing that J visibly gestures with or near, while speaking and perhaps writing something new, in many ways. If he wanted to *regard* this Cayley table in the way they had done most regularly thus far in the course (as indicating the results of multiplication of elements), he would have touched the heading at the top of the column, touched the heading at the left of the row, said those labels out loud perhaps, then touched the spot in the table where that column and row intersected and named the element that resulted from multiplying those elements of the group (as we saw him do in the previous section when he was *looking at* two Cayley tables *side-by-side*). But to consider the entire row as being a permutation of the elements of the group, and to name and refer to this permutation by the label at the end of the row; this is a different and novel perspective on this row. The existence of this interpretation is in fact the central idea of Cayley’s theorem. In some sense it is the only ‘idea’ in the theorem.

In this scene J manifests this interpretation by isolating two numbers in the row with two fingers and flipping his fingers in order to show the numbers being switched. He has another go at this interpretation a little later, and this time opts to slide one finger from one position to the other and back. J is showing – like Hussein does when explaining what is going wrong with a car to customer, bringing components of the engine to life using what Streeck (2017) calls *action figures* – how each row can be thought of as a movement of the original list of elements (ordered in the column headings) into new positions in that list (in other words, a permutation). J is making visible the invisible movement that occurred between the static display of the row of column headings (the original list of numbers) and the static display of this row of the Cayley table (the permuted list of numbers). Once this little movie has been witnessed, the view of the two lists is altered, and all the rows are regarded differently.

In this first scene the *regarding as* gestural practice was accomplished by means of the *manipulating an object* gestural practice: the *way* in which J handled the Cayley table is how he demonstrated the permutation interpretation of the rows of the table; the *way* he twirled his fingers, permuting a phantom pair of symbols in the air, reinforced the permutation interpretation. But instances of the *regarding as* family by no means require instances of the *manipulating an object* family in order to be enacted, as I will now show.

Let us return to Lecture 16, the first lecture on isomorphisms, to the moment when J has just finished writing the portion of the definition of isomorphism that consists of the preserving group structure condition: '*phi*(*ab*)=*phi*(*a*)*phi*(*b*)'. He walks away from the board, saying "we really need to understand what we mean by that". He switches markers from black to red, goes back to the board, touches the symbol '*G*' and says "we start off in a group *G*". He positions his pen so that it is exactly between the '*a*' and '*b*' inside the expression '*phi(ab)*', saying as he does so "so when we multiply"; having written nothing he moves a little to his right to touch the membership relation 'a, $b \in G$' saying "elements *a* and *b* in *G*". He turns to face the class, then back to the board, touching the '*a*' and '*b*' in '*phi(ab)*' saying "this *a* and *b*", then he draws a line from the spot in between the letters down to below the equation and writes a label on the line 'group operation in *G*'. In the next two spoken lines he reminds them that they used to use a symbol for this operation, like a star, but now they do not write anything at all, and he refers to it as "the invisible operation that combines *a* and *b*". Having finished the label, he takes two steps back, faces the class and pauses.

Then he starts again, with a higher pitch and the emphatic tone of marking a distinction "now *phi* of *a* and *phi* of *b*" – while touching these two terms on the right side of the condition – "are in the image" – now his index finger is wandering around the text of the definition searching for the symbol he wants to touch. He walks over to where the term '*G* bar' is and then just before touching it his hands drop and he says "sorry". What next ensues is a minute long digression on terminology. He decides that "codomain" is the term he wants (and not "image") and he returns to the board, and holds the term '*G* bar'. He turns to face the class, then audibly taps it repeatedly, saying "that's where the *phi*- the images live / they live in *G* bar". He then holds the term '*G* bar' for a few seconds of silence until he says "ok".

He releases his hold, says “so” and then his right index finger touches the ‘*phi(a)*’ term – “*phi* of *a* lives in *G* bar” – and then touches the ‘*phi(b)*’ term – “*phi* of *b* lives in *G* bar” – then he draws a red line down from the space in between these terms, saying “so that invisible group operation”, where on “that” he holds the end of the line, the spot in between the terms, for the rest of that phrase – “is actually the group operation in *G* bar” – where he uses the time it takes to say this to write ‘group operation in *G* bar’ as a label on the line.

Here all the deliberate and precise work of touching terms is in the aid of making definite the nature of the binary operation that is occurring on each side of this equation. It is noteworthy that what concerns him on this occasion is an *absence*: there is literally no marking on the board at all between the ‘*a*’ and the ‘*b*’ on the left hand side, or in between the ‘*phi(a)*’ and ‘*phi(b)*’ on the right hand side. J has spotlighted two specific locations inside a mathematical expression. As can so often happen in mathematics, the very frequency with which some operation occurs can generate a notational convention where *nothing at all* is written in a place where one might expect a symbol, or some bit of writing, to appear, which would denote the operation. The default practice is to write nothing in the place where the binary operation symbol would occur; adjacency of terms is itself the operation. J is keen here, as well as later, to explicitly touch these invisible operations and talk about *how* he is *regarding* them – since indeed adjacency of terms means different things on the different sides of the equation. On the left hand side of the equation, adjacency of terms stands for the binary operation in the group *G*; on the right hand side, adjacency of terms stands for the binary operation in the group *G* bar.

Also of interest in this incident is the fact that J commits to the board in writing the interpretation attested to by his *regarding as* gestural practice. In this sense the incident is atypical. Usually when J touches three or four bits of writing in succession as he shapes a few spoken lines around the manner in which he is presently considering the mathematical situation at hand, and which often gives strong clues about what sort of mathematical action they ought to take next, none of this gets written down. Here they are considering a critically important definition. They will encounter precisely this condition in every subsequent moment of the course when they work or make arguments involving isomorphisms. Perhaps J has decided, then, that it is worth the few seconds it takes to make these annotative labels on the board. On many subsequent occasions J does indeed perform a *regarding as* gestural practice concerning the invisible binary operation without writing anything down.

5.4.1. What aspect of what is known about a mathematical object is currently most useful?

In each of the first two scenes the *regarding as* gestural practice developed an interpretation that might otherwise have been *missed*; in addition, the concern on each occasion was how to currently and appropriately view *multiple* mathematical objects that were in interaction. I turn now to instances of the *regarding as* family which appear in contexts where attention is currently focussed on a *single* mathematical object; moreover, the business at hand is to *select* the manner with which to view the object from a collection of relatively accessible alternatives, or to *recognize* there is only the one possible manner available.

In Lecture 17 J is engaged in proving Cayley's theorem. He is in a local context where his goal is to show that if the permutations labelled T_g and T_h are equal then this must imply that g equals h. He wrote the first equality, followed by an implication symbol; he left a sizable horizontal blank space; he followed this with another implication symbol, and finally the second equality. He asks the class for ideas. No students reply immediately so he walks over to the transparency in order to touch the equation that defines the permutation T_g, which is '$T_g(x) = gx$'. Pointing at it he prompts them "I want to end up with g equals h / how do I do that" and he sweeps the equation again.

Bart suggests "apply them to the same thing" and J repeats this phrase. Then J gesticulates keep-going circles with his hands, saying "can you give me a nice simple thing I should apply them to". Then he shrugs both hands high in the air (Figure 4), continuing "since I know nothing about G". He pauses expectantly.

Figure 4. Regarding as – first shrug.

Bart offers "*x*" but immediately J shrugs again, saying "which *x*" (Figure 5).

Figure 5. Regarding as – second shrug.

Then J shrugs a third time, in a sort of search-me-I'm-innocent tableau, and repeats: "I know nothing about *G*". Bart persists as the penny hasn't dropped – "some *x* in *G*" – to which J replies with a quick wave of his right hand "yeah ok but *G* is an arbitrary group". His left hand is held back as he continues his I-don't-know stance. He swings his right arm forward again saying "I've got no idea which *x* to pick".

Throughout this exchange, whenever he answers Bart, J does little shakes of his head to emphasize that he does not know, cannot know, what *x* to pick. His gaze stays on Bart during the entire interaction. Now it comes: “oh the identity”, Bart says, and J’s entire body relaxes. His head falls to the side, he smiles, he takes a step – his two feet had been planted the entire time – “yeah good”. He points towards Bart quickly: “apply it to *e* the identity”. He moves quickly over to the portion of the board where he will now write what Bart has suggested, saying “and then we’re away”.

This is another example of the *regarding as* gestural practice. The scene looks very different from the last two: no objects are being manipulated, and no board items are being spotlighted. However, J is using unmistakably obvious movements of his body to insist on the student’s manner of perceiving *G*: that the only element the student can rely on to be in there is the identity, and that no other element can be talked about because nothing else is known about the group. J ruthlessly and immediately shakes off every response that is other than the phrase the student finally emits, continuously pantomiming the part of the person who knows nothing about the group. It is obvious that this is a ‘moment’ for J. There isn’t anything casual here, there aren’t multiple goals being attended to. He simply repeats the seemingly gnomic utterance “I know nothing about *G*”, as if this itself is the only hint needed in order to yield the right next action. J is *regarding* this group in a particular way – that because he knows nothing about it, the only element he can be certain will be in the group is the identity – and he is making a *display* of it, his elaborate shrugs underscoring how to recognize a mathematical situation where their very lack of knowledge about some object can be turned to their advantage. He and the students can thus carry with them into the future a methodical tool that can always be used at times like this (try the identity).

Just as in the *side-by-side* example of isomorphic groups we see a precisely timed payoff: the visible slackening of his temporary pantomime back into his ordinary normal easeful self comes immediately after the student regards the mathematical object in the same way as J. The student’s action triggers the end of J’s pantomime as if the two were connected by a string. This is one of the many moments of strong interaction between J and a student. No one is hugging, no one is striking a blow. But close observation of the sequence of actions between student and J often reveals tight interplay like that experienced here. This scene reminds me of a parent stubbornly preventing a conversation with their child from ending until they have heard the child say out loud that no matter what the situation is, no matter how late it is, or where they are, or what is going on, that they will call home. No other answer is permitted, and only when the right answer comes will the parent relax. Similar such occasions of regarding a mathematical object, in the currently active context, from the point of view of a property that it at the very least must enjoy, recur regularly throughout the course.

On other occasions what is at stake in considering some mathematical object is what it is about the structure or properties of that object that they will *use at that time*. J might touch a symbol that stands for a *subgroup*, but what he is using at that moment, the way he wants to view that subgroup, is only that it is a *subset* of the larger group. So he wants to regard this mathematical object in such a way as to temporarily *forget* some of the structure or properties that it *has*. On another occasion it might be that a certain *subgroup* is in fact *not a normal* subgroup, and that it is this *lack* that is of current interest. So an object can be considered from the point of view of emphasizing *extra* structure or properties that it could have had but which it does *not have*. A few minutes later he refers again to the equation '$T_g = T_h$' by touching it and saying "now T_g and T_h are equal as functions". Here both terms are, simply, functions; but J knows that equations between symbols can mean all sorts of things depending on the symbols. When the symbols are functions, then there is a specific kind of work that must be done to justify that equation, which requires the definition of what the word *function* means. A necessary preliminary step to invoking that definition is to regard the situation in the first place as one that requires this invocation. So an object can sometimes be regarded as the *exact* object it *already is*, no more, no less. In practice, this act of regarding an object for what it already is – what might be called *reminding* – can be tougher to do than this description makes it sound; it occurs in every lecture multiple times.

5.4.2. Failing to regard as

Occasions of gestural practice that do not come off, misfire, fail, or otherwise stand out from the background sea of occasions by going awry, are tremendously useful, as the *NHI* team observed, for further understanding of the gestural practice itself. I consider here a scene where for a time J cannot achieve a particular *regarding as* gestural practice.

Near the beginning of Lecture 18, immediately after his review of the previous lecture, J embarks on an exercise. They must show that Q (the set of all rational numbers) is not isomorphic to a proper subgroup H of Q (we saw him manipulate a gap in his writing in this exercise in section 5.2). He has written the membership relation '$a \in Q$ \setminus H' (since if H is a proper subgroup, there must be some element that is in Q and not in H). He has written a claim on the board that he says they will prove later, and he touches it repeatedly to say that they will assume for now that they have this result "in hand". He announces brightly "now let's see how we finish off". He says "we can choose *a* that lives in *Q* but not in *H*" – he touches '*Q*' and '*H*' at exactly the moments you would expect – "and now?" he asks while looking at his notes. He turns to the class and no one answers – "how does that kill it?" he asks again.

No one speaks. J turns to the board with a finger outstretched, and utters a frequently occurring formulaic phrase: "so what are we trying to do". His finger retracts into his balled fist as he slowly continues "we're trying to"; then his voice loses half its volume, he looks at the membership relation again, moves his finger to point at it as he repeats "what are we trying to do". His finger stops short of touching the board, recoils, comes back away from the board now pointing aimlessly upwards, his eyes now gazing down at his notes, his voice trailing off. Some seconds pass in silence, his finger goes back into a loose fist, he steps away from the board a little awkwardly, and he starts again: "*phi*".

Having said this word J moves again with some vigor to the board with his finger outstretched to touch the symbol *phi* on the board (they had begun the proof by assuming towards a contradiction that there was an isomorphism *phi* between *Q* and *H*). He taps it and repeats out loud what they had already assumed "so it maps *Q* to all of *H* right?". By this point he has ambled away from the board, and he takes a few hurried steps back to the membership relation with his finger outstretched. In fact, since all this has been occurring at the leftmost part of the whiteboard, and he has therefore been at the far left of my video, he had actually temporarily walked off screen. What I really see in the video is his finger first, then the rest of his supporting arm, then the rest of J hurrying behind, only to stop short just as he reaches the board with the finger repelled from touching the relation at the last second – "all of-" he mutters indistinctly, his face grimaces, he looks down at his notes and his left hand with the finger that refused to touch the board falls to his side like a stone, making an audible slap against his thigh. He stares at his notes for a few seconds.

J is unsure. J is lost. This is rare for J. J has not made an error, is not caught in some path that he must retrace his steps out of. He just does not currently know how to view this membership relation from the right vantage point that will launch him and the class onto the next mathematical actions. The little stammer of attempts to connect with the writing on the board that yield to little repulsions away from the board, the body hiccups when he reached for the relation because for a moment he felt able to proceed, followed by the recoil when the feeling passed like some kind of temporary disturbance leaving nothing behind: I interpret this jerky dance as an ongoing, although ultimately short-lived, failure to regard this relation in the way that J's former self regarded it when he was writing those notes in the first place. The J that confidently wrote those notes could touch and point to the writings on the page in the right order and with the right emphasis so as to make the next written lines inevitable, perhaps even trivial or obvious. But the J in this room cannot so touch the board. He stares at his notes – at what his former self considered evident enough to need no further commentary.

There is a struggle here between two incompatible action-sequences; the very beginnings of these action-sequences are seen to alternate. One action-sequence is the touching of the membership relation simultaneous with the utterance of words that indicate a certain *manner of viewing* the relation, followed by subsequent touches that will confirm and unfold the consequences of this view, with the sequence being conducted in a whole-hearted manner throughout. The other action-sequence is to be frozen and touch nothing. The first cannot presently be achieved and hence it can barely be begun; the second is intolerable and can hardly be continued.

In Ruesch and Bateson (1951), Bateson discusses a version of the liar paradox: a person who says "I am lying". He elaborates a mechanical model for what he calls this "oscillating or paradoxical system": the buzzer doorbell (p. 194). When the buzzer is pressed a circuit closes, forming a magnetic field, which lifts the contact arm breaking the circuit. But with the circuit broken there is now no magnetic field, so the contact arm falls, and the circuit closes again, taking the doorbell back to the beginning. Bateson goes on to explain how this models the liar paradox. I invoke it here because the doorbell serves as a good model of J's actions in this sequence. J is frozen and cannot touch anything because he does not see what he ought to touch subsequent to a first touch; that is to say, he cannot meaningfully touch the board. He cannot currently succeed in his attempted *regarding as* gestural practice. But he must proceed with this exercise, and to do so requires meaningfully touching the board, so he reaches to touch something, perhaps something that will spark a new touch. When he has no view of what to touch after, he recoils, takes his hand away from his writing, and he is back to the beginning.

One of the reasons I am so grateful to J is that he is never, not even once in an entire course of lectures, a mathematical phony. J is never going to copy on the board what he previously wrote in his notes at a time when he presently cannot make the meaning of the text visible in the classroom with his hands, body, speech, and writing. Perhaps it has never even occurred to J to fake such a moment. It's unclear to me to what extent such unwavering intellectual honesty continuously operates in mathematical lecturing in all universities in the world. Perhaps if I were studying a different lecturer I would have introduced a seventh gestural practice, *faking it*, or maybe just *hand-waving*, which would consist of the gestures of the hands and movements of the body that occur as a lecturer pretends to justify a step in an argument when in those moments they themselves do not understand the reason the step is valid. I am lucky to be researching someone whose commitment to honesty in public meaning-making is total. It is possible that this is the deepest, and most important lesson the students of this course ever get from J.

J talks a little to himself "maps Q to H right". It looks like he is picking up a bit of steam. Now he nods firmly so his whole upper body moves down and up – "it doesn't map anything" – he turns to look directly at the class, his finger leaps up from his side and plants onto the board precisely underneath the 'a': "there's nothing-" he says, "there's no way of getting to this a" as he taps the board confidently. He turns back towards the board and unexpectedly begins drawing a diagram which captures exactly the point of view he had not been able to achieve a few seconds before. He draws a first circle which he labels Q; an arrow to a second circle which is also Q; a smaller circle sitting inside the second one which he labels H; a dot outside H but inside Q that he labels a. He says "if we could find an element in Q" – he draws a little dot in the first circle – "which maps to that a" – he draws a line connecting this dot to the dot labelled a – "then it's game over ok?".

This diagram is as concrete a response as could be wished to his original question "how does that kill it". It is not that the diagram shows the killing; it is that it shows the strategy for the next few minutes of writing which will give the required contradiction: they will not be able to find such a dot in the first circle. He created this diagram out of the crucible of his fifteen second experience of losing the plot. And why did he lose the plot? After all, he knew that the next lines of his notes required him to make use of, take stock of, somehow jump off from, the membership relation he kept being drawn towards to touch. He just could not get the right angle on the connection between this membership relation, the supposed isomorphism *phi*, and the goal of forcing something to go wrong. The diagram captures how this ‘*a*’ is to be *regarded*: as an element you cannot get to by applying *phi* to elements of Q. This manner of considering ‘*a*’ unlocks the writing of the next few lines, converting them at once from a sequence of equations that a dishonest mathematics lecturer could mindlessly copy from their notes onto the board, into a sequence of equations J can mindfully and meaningfully gesture into being. There is something magical about this.

A hypothetical version of J, taken back in time before his uncertainty, who had already drawn this very diagram on the board, would have experienced no hesitation. He would have likely done then what this J was able to do just now. Subtle forms of *regarding as*, like this instance here, can be substantially supported by a diagram that includes only the required features.

Instances of the *regarding as* gestural practice family appear in local contexts where J is concerned with how some piece of writing ought to be viewed. He can *manipulate an object* in a fashion which affords him and the students a particular perspective on a mathematical situation, and he can point to, touch, and hold items on the board in deliberate and ordered sequences in order to see their present mathematical interaction in a certain way. Situations containing multiple mathematical objects may require an interpretive angle that is unexpected or new to the students, or one that is easily overlooked. Situations where a single object is momentarily foregrounded may require perspectival decisions as to how much of the structure of the object is currently of interest. Some occasions when a *regarding as* gestural practice is required are subtle and challenging. The previous writing is all there and yet the secret of how to look at it correctly seems hidden in plain sight. A diagram can serve as an additional piece of writing that unlocks the correct viewpoint.

5.5. Deducing That: spotlighting old writing to justify new writing

In this section I treat the *deducing that* gestural practice: it is marked by J pointing to, touching, holding, sweeping, or otherwise spotlighting some individual portions of writing that already appears on the board, which supports the making of a new piece of writing. The order can be reversed: J can write the new item first and then subsequently perform the spotlightings which justify it. Sometimes he does both. Often there is an obviously visible location on the board where the new piece of writing must go, and often it is at the *leading edge* of his writing: the location just to the right of the last thing he has written.

In Lecture 17, his second lecture on isomorphisms, after doing a detailed illustrative example in the previous class aimed at motivating the "principle" behind Cayley's theorem, J is now in the middle of proving it. He says "now we've got to define a *phi* that maps us from *G* to *G* bar", where *G* is a group and *G* bar is the collection of all mappings from *G* to *G* of the form '$T_g(x) = gx$' (so there is one such mapping for each element *g* in *G*).

He stands in front of the class to *comment about* the current situation, distinguishing the "working mathematician" and the "theoretical mathematician". The theoretical mathematician apparently pulls mappings out of thin air and then says "check it out", the mapping does what we want it to. By contrast, "working mathematicians say / I've got no idea what's going on / can you give me an example please". By this point J is already rifling through his transparencies on his desk and has found the transparency he had used at the start of the class to review the illustrative example from last time. This was the transparency I discussed in the last section, displaying a Cayley table for *U(12)*, accompanied next to it by the definition of four permutations, one for each row of the table. Below on the transparency he had also drawn the Cayley table for these four permutations, and at the time he had performed multiple touchings of both tables to indicate how both tables are identical except for the replacement of a number like 5 with a permutation labelled T_5.

Walking again over to the board to indicate with his open hand the place where he is about to define '*phi*' he says "as a working mathematician / I don't know what *phi* is meant to be" and with a puzzled expression he steps back to the transparency saying

“I’m lost in symbols”. He asks out loud “how did we get from *G* to *G* bar?” and while placing a finger directly on the first Cayley table under the element 5 he answers his own question: “oh that was easy”. He says “we mapped little *g*”, he then moves his finger over to the corresponding T_5 term in the other table so it is held underneath it and then continues “to *T* little *g* / to *T* subscript *g* of course ok?”. Then he removes the transparency, walks over to the leading edge of his writing on the board, points at it one more time, saying “define a mapping *phi* from *G* to *G* bar”, and then simultaneously speaks and writes the definition “*phi* of little *g* is *T* subscript *g*” (‘$phi(g) = T_g$’).

In this short sequence we see many of the common features of the *deducing that* gestural practice. First, the existence of a promised specific *next* piece of writing whose exact form requires some mathematical work to sculpt. Second, the spotlighting of another piece of writing forged *earlier*. Third, a *directionality*: the second piece of writing is touched and handled so that J may walk back and justifiably, confidently, write the very next unit of what is to be written.

Despite the ordinary connotation of the phrase *deducing that*, the reader can convincingly argue that in this instance J isn’t logically deducing anything. He is using an example that he treats in a generic way. Note the touching of the specific element 5 but saying “little *g*”. Here his spotlighting gesture helps him *regard* this mathematical symbol *as* any old element in any old group rather than the element 5 in the group *U(12)*. This serves to motivate what he has very good reason to believe will be a good definition of the mapping he seeks in his theorem: it worked so beautifully in his example, and further, although he does not say this explicitly, nothing in his example seemed to depend on the specific choice of *U(12)* as his group example. So why name this practice *deducing that* when deducing has a specific meaning in mathematics roughly synonymous, say, with ‘logically inferring’ or ‘rigorously establishing’?

One possibility I considered was to name the practice something slightly more general: *determining that*. There is clearly a little more room in *determining* than in *deducing*. Another possibility I considered was to give a short list of verbs to cover the main varieties of such directional gesturing: *deducing that* / *generalizing to* / *specifying that*. A third possibility, that I remain strongly sympathetic to, is to move to a more abstract level: the *from–to* gesturing practice.

I will continue to use *deducing that* for the following reasons. First, the vast majority of occasions of the *deducing that* gestural practice are, in fact, *logical implications* of one step to the next in a mathematical argument. Second, even though it is clear that the following summative clauses are different – "because of this earlier writing I am compelled to write as follows", "because of this earlier writing, it seems like a very good idea indeed to write as follows", and even "because of this earlier writing, I am certainly allowed to write as follows" – in living experience, when J performs the mathematical actions of which these clauses are the summary, the similarities in the actions, particularly the movements of the hands, prove stronger and more robust than the differences, and therefore seem to me more important to recognize. Names are important. I do not want to lose sight of the features of this gestural practice that lie at its core: J has written up to some current leading edge, and in order to permit himself to *accountably* write beyond this edge, he must go over to touch or spotlight some other, pre-existing and pre-accounted for, piece of writing.

We saw in this example that the need to touch the previous writing is strong enough to compel J to seek and find a transparency that had the bit of writing from the previous class that he wanted to handle. Far more frequently, of course, the writing is still there on the board from where he wrote it some minutes earlier. For example, a few minutes later J has by now shown that this mapping *phi* is indeed one-to-one and he now wants to show that *phi* is onto. In fact, he had first written the three headings '*phi* is 1 – 1', '*phi* is onto', '*phi* preserves group structure', and he is at the point where he filled in the argument justifying the first heading. He starts a new stanza by audibly writing a colon after 'onto', and reads the heading out loud, followed by "it means that" and he scratches the side of his head, takes a little step back, and says again "it means that what". He hazards "if we pick a *T*", he looks at his notes, and asks himself out loud "what notation am I using here?". He takes two more steps back and says "well actually let's slow down".

These moments may remind us of when J was lost in the previous section but there is a difference. J is moving and weaving physically, his hand has not dropped, he is looking mostly at the board, his tone is upbeat and hopeful: "*phi* is onto" he says again, and he moves decisively towards the definition of *phi* on the board, saying "ok let's take a look at *phi*". His right finger moves towards the codomain of the mapping, *G* bar, hesitates briefly, then continues onwards to touch the term. He holds it and says "so

I want to know that I can get to every *G* bar-", *correcting* himself to "every element of *G* bar". He says "ok", then looking back at the board he begins to move, saying "now what are the elements of *G* bar" and he goes to touch the spot two lines up where *G* bar is defined as a set (set-builder notation: set of all T_g such that *g* is in *G*). He holds his finger underneath the symbol 'T_g', turns his head to look at the class while still holding the term, and says "they are the set of the *T g*'s", pauses for a few seconds, then adds "for which *g* is in *G*", touching each part of the '$g \in G$' membership relation in turn. His face scrunches up a little awkwardly, and he says "so by definition absolutely / can I get to a *T*-", then his right arm extends out to the students and he starts a different sort of clause "you- you pick me a *g* in *G* / can I get to *T g* ? absolutely" and he touches the T_g term again. He finishes by saying, with his eyebrows raised and with a grin, "apply *phi* to", short pause for effect, "*g*", while touching the '*g*' symbol. Still smiling and blinking his eyes as if something a bit weird has occurred he lifts his right arm in a warding-off gesture, waving it in small circles with his palm facing the class: "now that one- the less you say the easier it is". Then he goes back to where he was supposed to write a justification for why *phi* is onto, and he writes 'by defn of *G* bar'.

After this J looks back at the lines he had been spotlighting parts of just a few moments before and he steps towards them again. He holds the 'T_g' term in the definition of the mapping *phi* ('$phi(g) = T_g$'), says "can I get to a particular T_g"; touches the '*g*' in the membership relation in the set that defines what *G* bar is, says "absolutely"; touches the '*phi*' and the '*g*' in quick succession in the definition of the mapping *phi*, says "apply *phi* to little *g*".

I want to highlight first here that J goes back to *try* his hand *again* at a sequence of touchings that he already attempted. The new sequence is similar to the old one but the differences are important: the new sequence is shorter (less things are touched), faster (less time, less spoken lines), and more concentrated (only the things that need to be touched are touched). It also seems to me that the *right* terms are being touched. Suppose another instructor were to paraphrase the argument here in a rhetorical question and answer format; in brackets I will note whether or not J touches any of the key ingredients that come up in the paraphrase. You want to get to every element of the codomain? Well, you'd better get to every element T_g (J touches this term). And what is T_g? It is the map that is associated to *g* (J touches *g* in the very set that defines the collection of all the T_g's). And can you get to every such T_g? Yes you can (J touches the map *phi*, and touches the element *g* that *phi* acts on that takes it to T_g). There is nothing that is touched that could be trimmed, and no key ingredient is left untouched.

J did not achieve this pinnacle of *deducing that* gesturing in his first go-through. It is notable that J went back for another attempt, even though his first go-through really had nothing wrong with it. Still, he went back and tightened it all up; in the second pass there were no pauses, and no hunting for where to touch, and no hesitating as to whether to touch. Very often in the course when J runs through fifteen seconds worth of touching and holding and pointing at a handful of terms in order to convince everyone, including himself, that he can justifiably write the next little chunk (or be permitted to), he will in fact go through it *twice*. The second flurry is hardly ever identical to the first; there is usually some alteration in the timing or tempo, some variation in the order of what gets touched, some change in what gets spotlighted at all, some shift in where his voice grows in emphasis, some compression (less often, dilation) in the spoken accompaniment. Such changes often reveal axes of valuation for J; he self-improves by adjustments in gesture and speech, and the choice of adjustments can't help but showcase one or another aspect of his system of values in mathematical lecturing.

Second, although most occasions of *deducing that* conclude with the writing of some mathematical statement or equation or symbol (this is the vast majority), it is interesting and unusual here that what he writes is simply a short phrase "by defn of *G* bar". This phrase is operating as a shorthand, as a stand-in or a reminder, of the actual mathematical argument that the classroom just experienced twice; once before the writing of the phrase, and once after. The number of such phrases in written mathematics that fill in for complex patterned spotlighting-flurries, as witnessed here, is, arguably, immense. Indeed, innumerable such occasions give rise to no remnant writing whatsoever.

It is also remarkable that J cannot bring himself to write out some lines that copy out on the board the sorts of things he was saying and the sorts of terms he was pointing at. J is a dedicated and meticulous lecturer, always keen to improve his written notes, careful to number and name results and refer to them accurately and consistently. He is committed to an ideal of clarity and is evidently willing to work hard behind the scenes and on the classroom stage to fudge nothing and to write out exactly what he proves (or be clear about what he wants the students to prove for themselves). Yet even J has his limits: “the less you say the easier it is”. On this occasion, and others, it will turn out that the written record is not a reliable indexical reference to the behaviour required from a mathematician to convincingly advance forward in this step in front of an audience of potentially skeptical listeners (again, recalling Mead’s *I–me*, including himself). Perhaps the argument is too tedious to write out. Perhaps it would be too confusing to read, misleading the reader into thinking the argument is harder than it really is. Perhaps it would lengthen the proof in such a way as to make it harder for the reader to discern what are the important steps. There are many reasons why mathematical lecturing contains many examples of complex multi-touch sequences, which are necessary to perform, but which might leave only the kind of deceptively transparent residue found here: “by defn of *G* bar”. Unpacking exactly these sorts of expression is what Nardi’s composite mathematician M suggested students typically do not realize they must do when reading a proof.

Examples of the *deducing that* gestural practice are truly legion. If section lengths of this chapter were proportional to the amount of time in the course where J is engaged in the gestural practice of that section, then the *deducing that* section would be, by far, the longest. Variations of this directional gestural practice are seemingly without limit.

The *deducing that* gestural practice is the one that most strains the ability to be described, moment-to-moment, in writing. The greatest strength of this practice is the sheer *speed* with which one or more specific concepts or objects can be handled in a specific order, and with specific tempo and emphasis – in other words, mathematically arguing in a specific context for a specific conclusion. This very strength works against the ability to write out what goes on as the practice ensues. Where it takes me two lines to remind the reader of what term in what nested hierarchy of mathematical containers is being touched (the g in the membership relation ‘$g \in G$’, which is the condition in the set ‘$\{T_g \mid g \in G\}$’, which is the set that G bar is defined as), J has already touched this term in half a second and moved on.

The gestural practices of the last two sections frequently emerge together in the same scene, working in concert. J can hold a term so that he can *regard* it *as*, for example, an element in some set; then this feature of his knowledge of that term can allow him to *deduce* some fact about some other object of his current interest. In the next two chapters we will encounter the *regarding as* gestural practice as a regular helpmeet of the *deducing that* gestural practice. The co-operation between the two families of gestural practices also extends in the reverse direction. Repeated instances of *deducing that* involving a particular kind of mathematical item lead over time to associating this quality to that item: they are the kinds of items which participate in such kinds of deductions, and therefore this is an available manner in which they can be viewed.

5.6. Commenting About: chapters, lectures, environments, words and symbols

In the last section we saw that the deducing that gestural practice typically has a strong *forward* impulse, propelling some argument along to the next link in the chain. By contrast, one of the features of the *commenting about* gestural practice is that there is a lull in the forward logical movement during this period of time. J and his students may be in the middle of an example, or exercise, or proof of a theorem, but during the *commenting about* phase the usual drive to advance a discrete step towards the finish line of the local written environment has been provisionally suspended. Instead, J metaphorically takes a step back from the ongoing action. In practice, J almost always literally takes at least one step back from the board.

The final part of the proof of Cayley's theorem is to check that the mapping *phi* that J defined "preserves the group structure"; so J has written on the board the beginning and ending of a horizontal sequence of equations. The opening term is '$T_{gh}(x)$' followed by an equal sign, then a longish gap, then the closing term '$T_g(x)T_h(x)$' together with the statement that these equations must be true 'for all x in G'. Saying "if I want to get these a little bit closer" he uses his two hands to shove inwards on both sides of this piece of writing (compressing the gap with his hands twice as if it were an accordion). Beginning with the opening term '$T_{gh}(x)$' he says "that's easy" – he looks at the transparency and points to it from a metre away – "that's g h times x" – and once he writes this after the opening term he walks over to the transparency to touch the definition of T_g on the transparency. Then he walks back to the gap in his writing, touches the terms on either side of the gap, saying "can I get these any closer". So far what I have described has fallen under the *manipulating an object* gestural practice. Twenty seconds later he has completed the gap and moves a few feet away from the board to look at the class.

Then he goes back to the board, traces his fingers underneath the series of equations starting from the left and begins a new stanza – "so you know if you go forwards and you get to about here" – he stops his fingers underneath the second term in the sequence, looks at the class with a pained expression on his face (Figure 6) – "you may think oh what do I do now" – then he reaches over to the end of the sequence of equations on the right and begins to sweep to the left (Figure 7) – "well don't be afraid to work backwards". He finishes this advice with the following phrase: "and just let the algebra do the work for you".

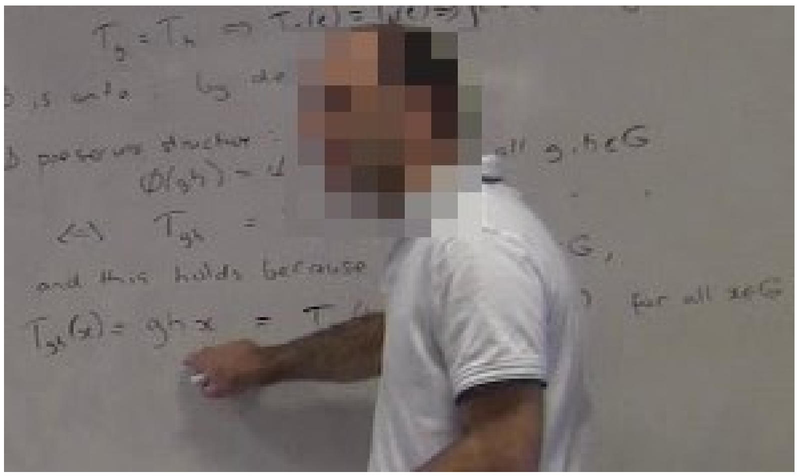


Figure 6. Commenting About – before.

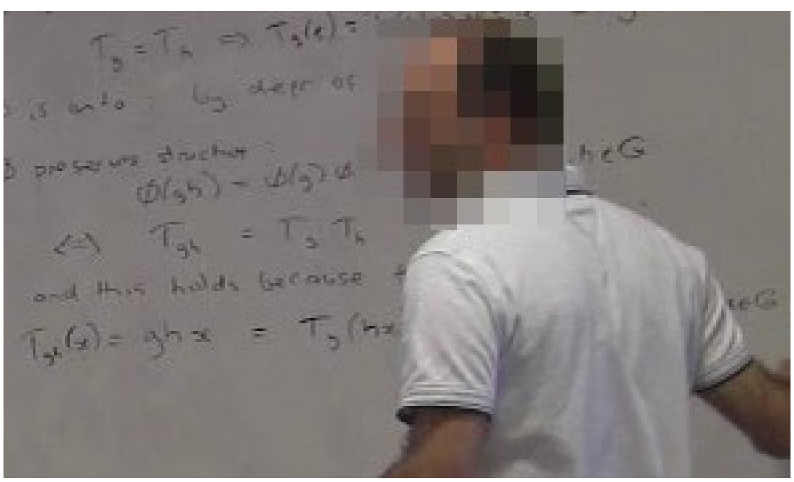


Figure 7. Commenting About – after.

He puts a little filled-in square to indicate that the proof is complete, begins to walk over to a board, presumably to erase the writing on it as there is no more room anywhere on the board, and along the way he is struck and stops walking and asks the class "do you know what yellow pages is?". J has heard his closing phrase, and after a few lines where he jokes that maybe the students have not experienced them firsthand, he warms to his theme: “there was an advert for the yellow pages when I was a kid”. He reaches out his right hand, extending his index and middle finger, and begins to sing “let your fingers do the walking”; as he does so he pantomimes a walking being where his two fingers are the legs (Figure 8).

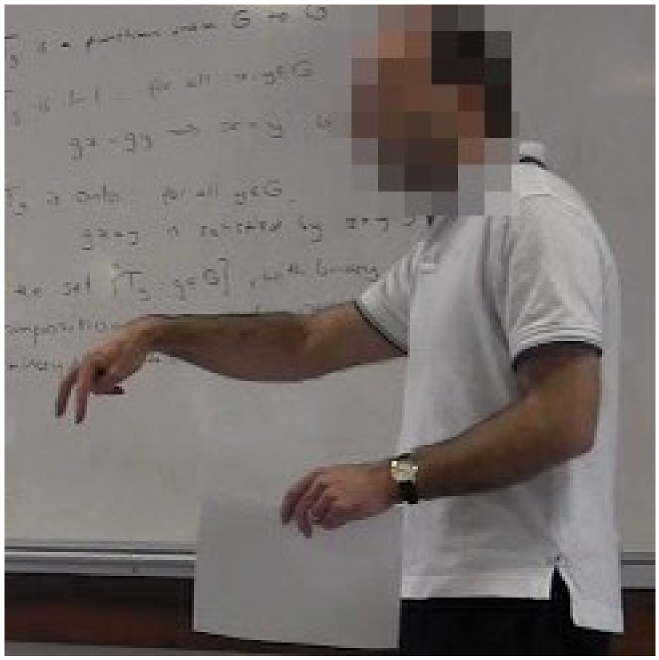


Figure 8. Commenting About – pantomime.

After a few words he himself begins to half walk half dance, rhythmically to the beat of his song, his fingers still tripping along: “yellow pages”. He cuts off his singing, holds the notes he is carrying in his left hand out as a flat surface and begins to explain “they had a little thing walking along the yellow pages”, and his fingers tap their way along the page. He walks back over to the writing on the board, his body facing halfway between the board and the students, facing them quizzically while pointing with his thumb to the writing on his right: “let your” – he pauses and looks at the board, then finishes in a questioning rapid tone “your fingers do the thinking? / I don’t know”. Then he looks at the class and with a finishing tone, with more conviction, points one last time at the writing and concludes “let the algebra do the / let the algebra do the walking there we go / that’s what I was trying to say”.

This example of the *commenting about* gestural practice (I take it to begin with J tracing his fingers and saying “so you know”) exhibits an important feature of most members of the family: the purpose is some *meta-commentary* about what has just been accomplished. Here the meta-commentary consists of rather high-level but practical advice about how to do mathematics. This is where the *about* comes from in the name of this gestural family.

There are two sorts of hand movements in this scene. There are spotlighting gestures that highlight the writing they have just done and the order in which they did it, accompanied by words starkly different in kind from the words accompanying the *deducing that* phase that came a little earlier, when they were still building this writing. During this *commenting about* scene they are *looking back* (Polya, 2004) at the writing that has just been performed, and J is retouching this writing to announce that the approach to the mathematical actions the students have just experienced can be profitably adopted in many similar occasions (strategic knowledge: Weber, 2001), even occasions where they may fear doing so. There are also depictions, here a very vivid and concrete example, pantomiming the fingers walking along his writing, and trusting them to do the thinking.

The observation earlier that the forward movement of the course – in terms of coverage, in terms of the logical development of the architecture of introductory group theory – is arrested during the *commenting about* phase has the following double consequence: J is free to hop back on the logical train whenever he wants, whether it is ten seconds later or ten minutes later. In addition, within this window of freedom from the usual burden of advancing-the-writing-justifiably, he has the time to move his body in dynamically interesting ways, exploring a wider range of stances and postures, and committing to the performance for enough time to make it become real.

Indeed these are two important characteristics of the family of *commenting about* gestural practices. One: members of the family occupy *widely varying time scales*. Some instances of *commenting about* last for a few lines in a stanza; many take about one stanza, or two; some occupy five or ten stanzas or more. No other gestural practice exhibits this degree of *elasticity* in their duration in time. Two: the bulk of the *mimicry* and *pantomime* behaviours that occur in the course occur within instances of the *commenting about* gestural practice. We have already seen J's body emerging in important ways in the *regarding as* practice, and to this extent many of those moments enact relationships between J and the mathematical expressions or concepts on the board: still, on those occasions there is a swiftness in resolving the interaction, as the momentum of the ongoing proof or example sweeps him on to the next step, where a different mode of gestural practice may well be evoked in the ongoing action. But in *commenting about* phases, J can linger in the enactment of these relationships for as long as he cares to, and he can dance along to a song he has just made up: visibly enjoying this instance of a mathematical moment when the algebra itself can be enough to “tell” them what to think.

In the next scene we witness again the regularly occurring feature of *commenting about*, which is that some spotlighting of writing serves as a springboard to move some distance from the board (Artemeva & Fox, 2011), gesticulate, and make comments about the mathematics that has been written. The *commenting about* gestural practice *stands apart* from the writing. No such statement can ever be one hundred percent true. Sometimes a comment about some writing might be so short that he does not really take a step away from the board, although often on those occasions he leans back and turns to steadily face the class. Some comments are considered by him to be of such interest that he goes on to write some brief version of them on the board. Sometimes he is midway through some commenting, breaks off to do some further writing, then steps back to continue commenting on the same theme. J may step back towards the board to spotlight this or that while the *commenting about* continues, but he is, ordinarily, not stepping back in to write a newly authorized, freshly justified, next piece of writing. As a general rule, *commenting about* involves a stepping away from the ongoing action at the board.

It is still Lecture 17, and J has finished writing the statement of Theorem 6.2, whose tagline is ‘Properties of Isomorphisms’. Standing in front of his desk facing the class, he has spoken for the last thirty seconds about how he will not prove this theorem, and that it would be useful for the students to justify the seven assertions in the theorem for themselves. He looks down again at his notes and begins a new stanza by turning around and pointing in the direction of the last two assertions that are a few feet away – “six and seven are very very useful” – he turns to face the class and uses this hand to rhythmically beat along with his words “if we wanna show that two groups are not isomorphic” – he finishes his beats with a long sharp downward movement exactly on the word “not”. A few moments later, while saying “if I wanna show that some group / is not isomorphic to another one”, he walks over and touches the word ‘elements’ in assertion seven, and continues “well if I can show they have different numbers of elements of order three / I’m done”, and he opens his palm face up to accompany “done”.

There is a pause and then he walks back over to the board just as he did before, saying “if I can show they have different numbers of solutions to” – he touches two parts of assertion six, the start of it, and the equation ‘$x^n = a$’ – “x to the five is” – he is walking back to his desk and he *waves* his left hand casually in front of his body as if to say *it doesn’t matter* which choice he makes in his next words, hits his thigh with his hand, shakes his head – “the identity” in an upward inflected tone – “then I’m done”. He goes on to say he could have picked any element in the group, while repeating his hand movement that raises up in a casual wave and then comes back down to hit his thigh, it did not have to be the identity but could be “anything I want”. He looks back down at his notes, begins to turn around, and he starts a new stanza by saying in a louder tone “So”.

I think a fair capturing of part of what J accomplishes in this scene is that he explains how and where the present bit of writing will likely show up in his students’ *future* mathematical actions. J can assert this explanation with confidence since it emerges directly from his own experience in using these results. The phrasing “if we wanna...well if I can show...then I’m done” does not include the word “should” once: does this scene constitute advice? Is it some attempt by J to shape the mathematical value system of his students? Whether we use the word advice, or not, it is clear that what J describes as the thing he or they will likely do is going to be interpreted as something that would be better to do than some other unspoken uncommented-upon alternative. As in the first scene, some writing has occurred, and now J has moved at least one level up (Bateson, 1956). He has written two assertions – he did not need to deduce them, so it is not the mechanics of that justification that serves as the seed of his commenting – and it is the utility in practice of these assertions that he makes as his one-level-up theme.

There is, then, a further freedom involving *time* that is afforded to the *commenting about* gestural practice. The first freedom was the *time away from* advancing-the-writing: J and the students temporarily step off this moving walkway during the *commenting about*. This second freedom concerns *when in time* the actions are taking place that J is talking and gesturing about. Stepping away from the writing physically, moving one level up to gesture and talk about the writing, coincides with commentary that involves the actions of the students at other times in their lives. The suggestion to “let the algebra do the walking” might very well be a phrase that a student will recall in some mathematical moment years or decades in the future, or, more likely, is one ingredient among many in a complex history that leads to that student being ready to let the algebra do the walking years or decades in the future. The suggestion that assertions “six and seven are very very useful” in showing that two groups are not isomorphic – the other five assertions are not highlighted in this manner, so he has made a distinction among these seven all-true assertions – might be something a student remembers and puts into their actions when doing the homework later that day. Both suggestions might even be something a student repeats to their own students, or their own children, someday. The *commenting about* gestural practice has a tremendous potential for *making leaps forward in time*.

Here is a third example of such a leap forward in time made within a *commenting about* phase. In the next lecture J has defined the group *Aut(G)*, which is the group of automorphisms of a group *G*, and has also defined the group *Inn(G)*, which is the group of inner automorphisms of a group *G*. On the board he has written one of his relatively rare written remarks, contrasting the ease with which *Inn(G)* can always be calculated, with the difficulty in calculating *Aut(G)* in general. He walks away from the board and begins a two-and-a-half minute scene entirely at his desk. His theme is a “technique” that he recommends if his students are ever researching some mathematical situation where they are looking for “a combinatorial object...some code / some set of points in a finite geometry / you wanna find some incidence structure...”; in short, any time where they might say to themselves: “I wonder if there’s a blah blah blah”. He mentions a student he worked with who used this “method” and got some good results that they published together. He names it “the method of prescribed automorphism group”; two lines later he refers to it more informally as “a fancy word for wishful thinking”.

The method is to pretend that the object being sought is mapped to itself under a particular automorphism or group of automorphisms. There is a stanza where he depicts a hypothetical example with his hands: they cycle around in the air as he talks about a cyclic group mapping some elements of the sought-after object to other elements; he uses both his hands to form a globular sort of object that he moves around, *manipulating it*, saying “it will decompose into these cycles”. Then he hits the punchline “what you’ve done is / you’ve reduced the search space hugely” and he throws his hands inwards from a wide distance to a smaller distance, a gesture he repeats three more times as he says “not just by a factor of two / but like square root or cube root”. Each time his hands start far apart up near his shoulders and then rush together to meet in a small ball in front of his body. He explains that because they have assumed more structure, the object they are looking for is more special and more easily described. He points his fingers randomly at a few locations to indicate some elements of the object that they might know about, and which they could use to build the rest of the object by availing themselves of the assumed automorphism group. He declaims in an excited, exultant tone: “and you go looking for that object / and if you’re lucky you find it”.

This scene has no spotlighting of existing writing at all, and indeed it is clear that these comments are not about the present course: it is unlikely that he intends to assess them on this material, no later results will depend on these observations. The scene is rich in gesticulation (in particular, a *reduction of complexity* gesture) and depiction (acting out in the air between him and the students a complex hypothetical action of an automorphism group). The springboard is the present topic of automorphism group, and again, the payoff of these comments can only be in the *future*, perhaps quite far in the future indeed.

As in the first two scenes we here see J recommending the *value* of some mathematical concept or practice. I will briefly indicate how subtle and complex the potential influences or effects of the *commenting about* gestural practice can be. I have chosen these three examples to indicate just how creatively vast is the universe of choices for J to comment about: “algebra”; two properties of isomorphisms; the notion of an automorphism group. But also varied are the potential contexts within which J’s recommended attitudes would emerge.

Suppose we introduced three axes along which to locate the potential value of J’s suggestions: how many sorts of contexts are the suggestions applicable to; how frequently do contexts come up to which these suggestions apply; how challenging is it to mathematically act successfully in the contexts that the suggestions are applicable to. Now we can list our three instances of the *commenting about* gestural practice studied in this section, together with the potential future contexts they may influence, and then an indication of a positioning on these axes. First scene: applicable to any occasion when a student is trying to join up a series of relatively straightforward equations or implications or double implications. Wide variety of sorts of contexts; commonly occurring contexts; relatively straightforward to mathematically act successfully. Second scene: applicable any time a student is trying to show two groups are not isomorphic. Narrow range of sorts of contexts; not as commonly occurring; medium difficulty. Third scene: any time a student is researching new mathematical terrain and wishes to establish the existence of some particular object. Wide range of sorts of contexts; commonly occurring if the students end up doing mathematical creative work; medium to high difficulty. The *commenting about* gestural practice is truly protean.

We have seen a few examples where *commenting about* involves mathematical actions that will occur in the future (farther in the future than the next piece of writing). A particularly important kind of such future-oriented *commenting about* occasions is one where the future actions will actually occur later in the course. The concept of isomorphism itself is explicitly foreshadowed on many occasions before the class arrives at the isomorphism chapter. A lovely and short instance of this occurs in Lecture 8. J has just defined what it means for a group to be cyclic, and he has remarked on the board that "Every cyclic group is abelian" – followed by the short equational justification of this remark: '$a^j a^k = a^{j+k} = a^{k+j} = a^k a^j$'. After writing this J goes back to his desk, looks down at his notes on the desk without looking at the students, and says in a low tone "so sure of course every cyclic group is abelian".

Then he suddenly picks up the notes and starts moving to the board and he has a frowning puzzled face on, starting a new stanza – "But let's think about that a little bit more" – and by this point he has reached the equations and he puts his fingers underneath the first terms. He sighs and turns to look at the class – "multiplying elements" – he looks at the board and places his index and pinky finger under the 'a^k' and 'a^j' terms respectively, holds them there and looks back at the class – "of a cyclic group" – he turns to the board, lifts his hand maintaining the index and pinky finger outstretched configuration – "is like" – he places these two fingers underneath the '$j + k$' exponent, positioning his fingers so that there is one finger underneath each letter, holds them there and looks back at the class – "adding their exponents". He pauses and stares at them in silence, nodding slightly, his hand still straight out at his side holding the symbols in the exponent.

Later in this scene he expounds in more detail what he means by “is like”, with more spotlighting of these terms, and he notes that “we haven’t got there yet” in terms of being able to say precisely what it means to say that finite cyclic groups behave like the integers mod *n*. The critical gesture is the one where he carries his hand from touching the two terms that multiply in the cyclic group to touching the two exponents that are adding in the integers; if this thesis were concerned with collecting *isomorphism gestures*, this would be a major example. One of the great strengths of the *commenting about* gestural practice is that J can move his hands in certain ways early in a course, preceding some formal precise definition say, and then he can move those hands in the same way later in the course after the precise definition! It was the success of mathematicians in the past to give an unambiguous verbal/symbolic definition of the concept of *isomorphism*. One of the ways they determined whether or not they had succeeded is if the definition didn’t do any violence to the gestural practices they were using in their mathematical arguments that involved this concept – whether the concept was being tacitly invoked, whether it showed up only implicitly in the writing, or whether it showed up explicitly but with a definition that would now be looked at as lacking in precision or definiteness of reference.

The freedom in *commenting about* to leap to another time is not limited to leaping to the future; J can also *leap to the past*. At the very end of his lecturing on the chapter on isomorphisms J does not begin chapter 7 (on Lagrange’s theorem) immediately. Instead he says: “let’s just- just push pause / and have a think about how we got here”. What follows is a remarkable review of the course thus far: one stanza each on chapters 1, 2 and 3; two stanzas on chapter 4 and three stanzas on chapter 5; one stanza on chapter 6. It takes about three and a half minutes. The factor of compression between the original lecturing time devoted to a chapter to the current summarizing time of that chapter is so large (roughly two orders of magnitude of stanzas) that one might correctly expect that the hands would play a lesser role. There is no spotlighting. There is one depicting gesture: his hands move from a wide position to a small one when he talks about what a subgroup is. There are a few key gesticulations: J numbering with his fingers key observations from that chapter; J using various gestures of emphasis to accompany key summarizing statements of the main conclusions or results from that chapter.

Other such reviews of already-covered material, such *remember when we* moments, have a much smaller factor of compression, and J will often recreate with his hands and body a significant fraction of the sorts of movements he actually made during the occasion they are bringing to mind. These reviews occur regularly. They create fascinating cross-pollination experiences: sometimes J can foreshadow a later event, and then during the later event, after it has ended, remind them that he told them this was coming.

J can also leap backward in time beyond the confines of the course. He recalls moments in his own undergraduate career, pantomiming the actions of his own group theory lecturer as he spoke about admiring Abel because unlike Einsteinian and Newtonian (adjectives named after a person), the adjective *abelian* "has passed into the language". In one fascinating portion of a lecture J admits that the way he is about to present Cayley's theorem is not the way he has always presented it.

He tells the students a story about a former student in J's group theory course, who had witnessed J's former self lecture on the proof of Cayley's theorem. This student suggested to J that it seemed like part of that proof of Cayley's theorem was really just a statement about groups in general: that if one had a one-to-one correspondence between a group *G*, and another *set G* bar with a binary operation on it, that this alone would guarantee that *G* bar was a group which was isomorphic to *G*. J did not at first believe the student (he acts out his reaction – shaking his head "I don't think so"). The "persistent" student emails with more details (J acts out his earlier self's reading of the email – hand on chin frowning, pause, "hmm looks good to me"). J then checked with a friend who had been teaching group theory from the same textbook (Gallian) for longer than J had; J's friend not only agreed but confirmed that he too had at some point realized this improvement was possible and that he now teaches his class the above weakened definition of isomorphism.

Later J has written this observation on the board and invites the class to check it themselves. Then he pulls out a printout of the email his friend had written to him. He sets up the scene again, acting out his own puzzled email where he says to his friend "am I missing something here? it looks like he's right" – quizzical facial expression, right hand palm up shaking it slightly, shoulders hunched up. Then he reads aloud his friend's email. He points to the writing on the board when his friend begins talking about various parts of Cayley's theorem. He puts the paper down when he has finished reading and looks at the class: "it's the eighth edition!" he says incredulously, eyes wide, both hands indicating the textbook on his desk, "eighth edition of Gallian" he says again, repeating the gesture of both hands presenting the book to the class. He is not sure if no one has told Gallian, or if someone has and Gallian does not care: but he finishes by pointing at the observation on the board and says "I think it's a point well worth making".

This *commenting about* scene is truly powerful. In it, J is re-enacting former mathematical interactions that he clearly believes are of great value for the students in his present class to experience. To me, the quoting from his friend's letter indicates not only the great respect J has for his friend, and his admiration for exactly how the friend phrased both the mathematics involved and his own perspective on the mathematical situation more generally, but it serves as an emphatic instance of the tight and close community of mathematicians that J belongs to. When having found, with the help of his student, an unexpected new understanding of material he had worked with for a long time, and had taught, he shared this experience with a fellow traveller. On this occasion, his peer had also encountered the same surprise. Even though this undergraduate mathematics textbook has been read by so many for so long, not all the secrets have been found. What J does not need to say explicitly, for his story already accomplishes it, is that perhaps they too, the students in his present class, can do what this other student had done; can be like that other student.

In the very first lecture of the course, when J has not even begun writing anything on the board yet, one of his expressed hopes is that the students will not read the textbook like a newspaper, but instead read it with some kind of fighting-back "belligerent" attitude. He acts out for a few moments the character of a fighter: "oh yeah? says who?". Now, many weeks later, the students see a real-life example of someone who was in exactly the position that they are now, only a few years earlier, who earns J's admiring words here in Lecture 16: "amazing what you can learn if you take an appropriately skeptical attitude". *Commenting about* lends itself to such play-acting, such pantomimes: the imperative to write a bit more mathematics is in a brief intermission; the imperative to strive to write better mathematics in the future continues to be in ascendance.

5.7. Correcting Self and Others: in advance, in the moment, or after

I want to revisit the episode in Lecture 16 where J has drawn two Cayley diagrams *side-by-side* that I discussed earlier in section 5.3. Immediately after finishing the second diagram he touches the first diagram with his pen and says “and what that means is / remember the Cayley- a Cayley diagram / captures everything we need to know about the group”. I want to focus attention on the self-correction that happens when he stops abruptly after saying “Cayley” (the stop being indicated by the hyphen) and immediately restarts but switches out the word “the” for “a”. This incident is a specimen of the *correcting self and others* family of gestural practices. Here he is *self-correcting*: he interrupts himself, bringing his flowing speech to a sharp sudden stop; he restarts quickly at a location a few words earlier in his speech, and he smoothly continues. Occasions like these occur often in the course.

As with many of these other incidents, there is a mathematical mistake that he has fallen into in his rapid fluent speech and that he just as quickly climbs out of. Here the mistake is assuming uniqueness of a Cayley diagram representation of a group when, as he made very clear in the first lecture of the course, and repeated since then, there are many possible Cayley diagrams associated to any group (you can pick different generators, and you can also redraw it so that the directed edges and the nodes have the same graph structure but the positions of the nodes might be very different). This distinction between *uniqueness* and *multiplicity* is a fundamental one in a variety of contexts in mathematics and shows up repeatedly for J in this group theory course.

Importantly, there is *no warning* whatsoever that the break is coming. There isn't anything that I can see in his body or hands that makes me anticipate that a self-interruption and subsequent self-improvement of his words is about to occur. It is possible that a Birdwhistell might see something in the frames of the video just preceding the moment in question. But up to ignoring actions on the time scale of hundredths of a second, the self-corrections do not come pre-telegraphed.

In this scene his self-interruption is rather mild. On other occasions there can be an accompanying facial expression: perhaps a fleeting grimace, some such pained reaction. Sometimes the correction itself, and whatever mathematical conceptions are in conflict, becomes the theme of his ongoing talk, and J segues into a *commenting about* gestural practice.

The *correcting self and others* gestural practice shares with *deducing that* the feature that exactness, correctness, precision, and intolerance of ambiguity are dominant concerns. For *deducing that* it is this term and no other that he must touch on the board at this time before touching this specific other term to carry the local inference to the finish line; for *correcting self or others* it is this sequence of words and no others that he or the student must say. A concise formula for *deducing that* is: *from this, that*; for *correcting self and others* it is *not that, this*.

This gestural practice is essential to investigate and analyze if one wished to compile a large collection of conceptual or mathematical errors that are easy or natural to fall into. All of J's self-corrections are fascinating because they are so readily seen to be tempting and interesting errors. J has been doing mathematics alone and with colleagues and students for long enough that he no longer makes obvious gaffes. Those have clearly been ironed out patiently over time. Any slips that remain (which he so often catches as soon as he hears himself say it) usually have pedagogical value. A student in J's course could do worse than to assign themselves the task in a lecture to only write down what they remember of moments when J self-fixed something he just said, and ask themselves "What was wrong about the discarded words? What was right about the new ones?".

5.7.1. Self-corrections, the me and I, the line, and the breath

Mead (1934) included vocal gestures as part of what he meant by gestures; in this work I have largely centred attention on the hands, along with movements of the body, while not excluding the eyes or what J says or how his voice sounds when he says it. In the case of this gestural practice, I might have chosen, perhaps, to more narrowly define the family so that it only included moments in the action where he corrects a mistake of his own, or that of a student, by using his hands to spotlight or depict or gesticulate, or by moving his body notably. I believe the category is analytically more robust, however, when *all* self-correcting incidents are included, even those incidents where the role of the hands or the body is slighter than usual. I have two grounds for this belief: one, what

Mead deems to be an inseparable continuity between body movements and vocal cord movements; two, Mead's analytic construct of the *me* and *I*.

As soon as the movement of the body is taken seriously – to and from the board, away and towards the class, the stops and starts of ending a context and beginning a new one, and so on – it feels artificial and awkward to analytically split apart sudden and deliberate changes of body movement from sudden and deliberate changes of vocal cord movement. What had been before that very second a continuous rumbling flowing vocal cord movement has surprisingly halted, generating an unexpected little gap of silence, after which their vibrations ring out anew an instant later. The slightly jarring discrete little skip backwards in time of the new speech is experienced by the listener as a mild jolt or push.

The second important reason for a broad view of the category is that one could not ask for a more concrete example of Mead's *me* and *I*. The *me* of J that hears his slightly earlier *I* saying "the Cayley" is very quickly followed by a new *I* that corrects this to "a Cayley" even before J can reach the word "diagram".

Empirically I found the great majority of moments of *self-correction* occurred with J hearing something within the line he is presently uttering that his *me* takes issue with; a minority of moments of self-correction occurred with J's *me* hearing something problematic in the previous line, usually a word coming late in that line. The regularity of these self-emendations makes it very plausible that such a *me* is listening to a slightly former *I* during every *line* of these lectures. The remaining moments of self-correction did not come about from hearing something in his own words that he did not agree with, but occurred when he discovered that he was in some sort of mathematical trouble, which then led him to realize he had taken a wrong turn some time earlier.

If one adopts Mead's view of communication and meaning, and if one follows Staats in attending to the separation between successive *lines* of ongoing talk, then the evidence from this research on moments of self-correction supports the hypothesis that one purpose of line-segmenting is to break the flow of speech into a unit size that does not exceed the capacity of the *me* of that speaker to respond to what the *I* has just said. That is to say, the behavioural correlate of the length of the *line* is the length of time within which the present J–*me* can hear the present J–*I*. If J did not pause every six to twelve words (or so) in that slight way, and if he did not adjust the pace and rhythm of his ongoing talk so that it came out in bunches of single clauses or perhaps two clauses, then he could not hear himself, and in Mead's theoretical terms, he could not think properly. If it were instead true that J was able to hear himself over the space of twenty words, or thirty words, I would have found more evidence of self-correction at those length scales.

This hypothesis is strengthened by the following considerations. Recall that these are *mathematical* mistakes. J is highly motivated to hear himself say them because he is highly motivated to catch them and fix them. He is also exceptionally strong at catching the rare errors he makes – it is extremely rare for him to make a verbal slip that he does not himself catch. These facts make it even more likely that I would have found more evidence of J, in twenty-eight hours of talking, catching spoken mistakes that occurred three or five lines ago if the true unit length of 'present J–*me* hearing the present J–*I*' was longer than a single line.

There is a connection between self-corrections and the breath. In Chapter 4 I observed that the ends of J's lines are frequently moments when he takes a little breath, though there is no rigid one-to-one relationship between his breaths and the ends of the lines. When I try to enact J's lectures, the ends of his lines are frequently very natural places for me to take a quick little breath. I can report as an observer that moments when J interrupts himself to correct himself induce in me a short cessation of my regular pattern of breathing. I experience some degree of briefly heightened tension. It would be an exaggeration to say that they take my breath away; but the sudden stop in J's words – the air not coming out like it was – followed by the sudden restart – releasing air again – generates some sympathetic interruption in my breath. I have tried to articulate more sensitively here what I captured earlier in short words like 'jarring' and 'jolt'.

5.7.2. Correcting a student

In the next two scenes J is not correcting himself but is instead correcting a student. J has so much richly felt experience with mathematics in general, and so much experience with teaching mathematics, and he has seen so many student errors, that he can regularly warn students about such errors. A few minutes later in the same lecture J is in the middle of a solution to an exercise. He has written three lines on the board, where the first line implies the second line and the second implies the third. He stops for a moment and says "please notice"; he plants a finger directly on the implication symbol and continues "that I use this symbol quite carefully". He looks at the class the whole time and his finger keeps holding the implication symbol.

He clarifies by saying that what he means by this symbol, and what he suggests that they mean by it, is: "if this is true" – he touches the first line – "then this is true" – he touches the second line – "and then this is true" – he touches and holds the third line. Then he says that "a lot of students would mean by this" – indicating the three lines with his palm – "they would say" – he touches the first line – "we know that this is true" – touches the next line – "therefore this" – touches the last line – "therefore this". He repeats by saying "they say you know / now I've got this now I've got this" and he moves his hand down from line one to line two and from line two to line three each time he says "got this".

To some extent this is a *commenting about* scene: it consists of explicit discussion using spotlighting of an aspect of mathematical writing that he commends to them for their future mathematical work. To some extent this is a *regarding as* scene. J contrasts two perspectives on, for example, the second written line: the right perspective is that it is implied by the previous line; the wrong perspective is that it is necessarily true. But it is also a *correcting others* scene where the commenting takes the following specific form: *here is the wrong way and now here is the right way*. Whereas a self-correction achieves this structure unintentionally (presumably J would prefer just to have said "a Cayley diagram" the first time in the opening scene of this sub-section), here this structure is created deliberately. J's correction of former students doubles as correcting hypothetical actions that his current students might take in the future. He temporarily writes on the board the options students have for indicating "now I've got this": three dots for "therefore" or the word "so". This completes his correction: if his students indeed want to mean "now I've got this" he has written the notation that connotes this.

It is worth thinking about the *correcting* and the *looking at side-by-side* gestural practices together. There is a reasonable analogy to be drawn, as well as a distinction. In *side-by-side* scenes J positions in *space* two diagrams or pieces of writing adjacent to each other, one *next* to the other; in correcting scenes J positions in time two bits of speech adjacent to each other, one *after* the other. In *side-by-side* scenes both textual objects are simultaneously visible; in *correcting* scenes both bits of speech are so closely contiguous in time that, with some effort, they can be considered together. In *side-by-side* J compares and contrasts the two; in *correcting* J selects the second as superior and discards the first.

J sometimes engineers moments when he can entice a student into answering one of his questions in a way that exhibits a tempting error. The next scene occurs a little later in the same exercise. The exercise requires them to show that the set of all positive real numbers under multiplication forms a group that is isomorphic to the set of all real numbers where the binary operation is addition. They have defined a mapping *phi* from the positive reals to the reals (the logarithm function), shown it to be onto and one-to-one, and now wish to show that it preserves the group structure. J has written '*phi(xy)* = ' and invites the class to fill in the empty slot.

For a moment there are no takers, and J, with his arm outstretched and his palm up, gestures broadly to the preserving group structure condition – '*phi*(*ab*) = *phi*(*a*)*phi*(*b*)' – that is still on the board from when he wrote it near the beginning of the lecture. Now Bart takes the hint and says "*phi* of *x phi* of *y*". J is mock-exultant: "splendid someone fell into the manhole" – he waves his arm high in the air almost triumphantly – "excellent" – broad smile. The student speaks up again: "*phi* of *x* plus *phi* of *y*" – "good" says J quickly, and as he often does with a student response that he accepts, repeats it verbatim: "*phi* of *x* plus *phi* of *y*". He points at the student quickly while still smiling and says admiringly "and got himself out immediately".

J now goes back to the condition that I talked about at the beginning of the *regarding as* section, and spotlights those invisible operations and the comments he wrote beneath them all over again. He rereads this condition out loud, touching it term by term as he moves through it, but now replacing the first operation with "times" as he knows the first group is the positive real numbers, and replacing the second operation with "plus" as he knows the second group is the set of real numbers. This last rereading is another example of *regarding as*. Bart shows here that it is not only J who can fall into a mathematical trap but then quickly reword their spoken expression and thus smoothly extricate themselves. All that was needed was for Bart to insert the single word "plus" at the right spot in his previous speech.

5.7.3. J is corrected by a student

The last two scenes that I discuss in this section are occasions when J is corrected by a student; put another way, these are occasions when J receives a correction. They share some characteristics with self-corrections, particularly after the moment when J has figured out how to address the error that a student has prompted him into recognizing. New features appear in the two stages that precede this moment. The first stage begins with the student contribution and ends with the moment when J has properly understood what the student means. Then the second stage begins, and it ends at the moment when J figures out what point in the recent past of their mathematical action he needs to return to, and what different action they must choose at that point.

With two minutes to go in this lecture J has begun the proof of Cayley's theorem. He says out loud that he knows he cannot complete it today, and that he intends to simply make a start. He has written one sentence that begins 'For every *g* in *G* bar define $T_g : G \to G$ bar by', and he is continuing to add to it when he is interrupted by a question: "where is *g*?" which J, as usual, repeats exactly. The student adds with a rising tone "in *G* bar?". For a while J cannot figure out which *g* the student is talking about, and he touches three *g*'s on the board, one after the other, each of which is not what the student is referring to. The whole time J is scanning what he has written, hunting for what to touch that will get the student to confirm J is on the right spot.

Finally, J says with a note of alarm "oh for each little *g* in-" and suddenly stops talking. His scanning sweeping finger had reached and touched the '*g*' and '$\in$' symbols exactly at the moments when he said "*g*" and "in" in his sentence; his finger had stopped before touching the (incorrect) *G* bar term at exactly the moment his voice stopped. This is a good example of symmetric behaviour in vocal cord movement and hand movement at a moment of avoiding error or recognizing error. The occurrence of such twinned movements constitutes a further justification for including in the *correcting self and others* family those occasions which are predominantly a vocal gesture phenomenon.

J looks down at his notes glumly, and his hand, whose finger had been so close to touching the *G* bar term, falls to his side loudly hitting his thigh. Slightly more accurately, it looks like he has released all tension in his arm, and then makes it fall faster than it would under gravity so that it collides with his thigh. He stands motionless looking at his notes, then turns his head slightly towards the student, shifting his eyes from the notes in his hand to make eye contact, saying “that’s not good”. He speaks the correct formula: “for each little *g* in *G*”, in a surely-that’s-what-I-want-here tone. He looks at the student for a few moments, still motionless.

Then he finally breaks into movement. He shakes his head, says “I’m sorry”, moves to the desk, grabs the eraser, goes to the board, erases the bar on the *G* bar term. He says “that’s a mistake in my notes you’re quite right”, puts his notes down on the desk, he picks up a pen, and corrects his notes right there and then. As he does so he makes a kind of “aargh” sound of frustration, followed by “well spotted”.

When the correction flows from student to J, J’s response shows up in at least the following important ways: first, his hands no longer confidently spring towards terms and spring away, but show *hesitation*, dithering, uncertainty; second, there is often a *collapse of the hand* from a near-board position held with strength down rapidly to his side where they stay, often with an audible slap with his thigh; third, his whole body *ices up* into a stance; fourth, a *sudden rebirth of all movement* – the body free to walk and turn, the hands and fingers ready and eager to point and write.

Here the mistake was minor, a sort of typo, although serious enough in its potential consequences. It could have been confusing indeed to a student trying to understand a proof they have never encountered before. Occasionally the mistake is more subtle. In Lecture 9 an error is made that can ultimately be traced to the fact that in set notation, when elements are listed within the curly brackets, any repetitions of elements are to be ignored. For example, the set {1, 2, 2, 3} contains three elements and is identical to the set {1, 2, 3}. In a context where J is considering the set of all elements of the form a^i, where a is an element of a group and i runs from 0 to $n – 1$, at some moment in the justification of an argument J traces his fingers through this set and says it has n elements. A few minutes later a student asks J how he knew that there were n elements (after all, perhaps some of these ostensibly different elements are the same). Because the mathematical context has some other moving parts it takes J ten or twenty seconds to figure out that the student was right, and he should not have just assumed that. Remarkably quickly after this J also realizes that, importantly, there is another statement on the board that guarantees that no two of those powers of *a* are identical. It had, in fact, been necessary to *appeal* to this statement at precisely the moment that he had asserted that this set had *n* elements. He admits “I said a bad proof” and then systematically takes up his previous argument, this time saying it right.

While I am not discussing this example in detail, I mention it to note that corrections need not operate only on the time scale of a few seconds. Occasionally, when prompted by some component of the mathematical interaction, whether it be a student, or a later snarling difficulty in a step of a mathematical argument, J might be forced to cast aside a minute, say, or longer, of earlier mathematical gesturing and talking, and do it over, but this time really succeed in his mathematical activity.

I close this discussion of the *correcting self and others* gestural practice with a description of how to draw a useful diagram of *self-correction* moments. Such a diagram would consist of a horizontal line beginning at the far left, headed to the right until a point marked *A*, continuing further a short distance to a point marked *B* where the horizontal line would end. There would be a dotted line beginning at *B* that would form a semicircle ending at *A*. After touching *A* the dotted line would continue forward horizontally to the right, parallel to and beneath the earlier segment *AB*.

The point marked *A* represents the place in the word flow that J returns to, and *B* marks the last word he utters before he breaks off his flow in order to correct himself. The dotted line does not represent speech at all; it may be interesting to speculate exactly what it does represent. The new line emanating from *A* is the new word flow. The diagram makes tangible, I think, the key features of such self-corrections: *cutting off* (as if with scissors) and *regluing* (as if with tape) the ongoing flow of spoken words.

5.8. Conclusion

Often J handles or manipulates a physical object. Relatedly, he often depictively gestures with his hands as if he is manipulating a physical object. Also, he often handles or manipulates a textual object on the board as if it were a physical object. I call all these the gestural practice *manipulating objects*.

When J has created two pieces of writing that are on the board simultaneously, or is in the process of creating two such pieces of writing, and handles one or both of them in sequence, going back and forth from one to the other, I call this practice *looking at side-by-side*.

When J is holding or touching a single piece of writing on the board and showcasing with his speech the manner in which he wishes to view this object, and/or with his next gestures or movements acts out the manner in which this object is being used, I say this practice is *regarding as*. His attitude towards the writing in question, the point of view he is taking, or the perspective he takes on the object, especially as determined by the gestures immediately preceding or anteceding this holding or touching, are the primary indicators of the practice.

When J is spotlighting a piece of writing or pieces of writing during an action where he is forced to continue with a certain bit of writing, or strongly encouraged or guided to so, or must conclude that such and such, or deduces that some bit of writing is justified or will be justified, this practice I call *deducing that*.

When J is not writing for a little while, when a longish period of writing is going to be begun or has now ended, when he is talking about what has happened in retrospect or what will be happening, when he is in some sense a step back from the writing-action and is providing some kind of meta-frame to a unit of a period of action, these are the indicators of the practice I call *commenting about*.

Finally, when J stops himself or others, when he catches himself, says “sorry”, says “no”; when he must begin anew, begin again at a moment before the breakage of the flow, and continue in a way that is distinct from how he had just been continuing, when he gestures and talks about why this way is the correct way and the other way was the mistaken way, I call this practice *correcting self and others*.

The next two chapters will develop further understanding of the nature of these practices as they interact during scenes in the undergraduate mathematics classroom. The dihedral group with eight elements, D_4, is the fundamental concrete example of a group that J returns to repeatedly as the course unfolds. How instances of the families of gestural practices co-operate to create meaning on occasions when J and the students are concerned with some aspect of this group is the focus of Chapter 6. In Chapter 7 I examine the manner in which these families of gestural practices emerge and interact on occasions when J and the students are engaged with proposed definitions of mappings in mathematical contexts when it may be that such definitions may fail. They must check that the mapping is well-defined; the concepts of normality, quotient groups, and equivalence classes are closely involved.

Chapter 6.

D_4: the symmetry group of the square

6.1. Introduction

A square can be mapped to its own space under the action of various transformations. It can be rotated, counterclockwise say, by 90 degrees, or 180 degrees, or 270 degrees. It can be reflected in a vertical axis that passes through the midpoints of the top and bottom sides. Similarly, it can be reflected in a horizontal axis that passes through the midpoints of the left and right sides. It can also be reflected in each of the two diagonal axes that pass through pairs of opposite corners. Or it can be left alone. These eight transformations are the eight symmetries of the square. The set of these eight transformations, together with the binary operation of composition of the mappings, forms a group, called the dihedral group of order 8, written as D_4.

In this course it is the central running example of a group. The so-called Klein 4-group, $Z_2 \times Z_2$, also shows up regularly in examples, but D_4, and more generally, D_n, are the most frequently occurring example of a group. The course carefully and comprehensively covers the first ten chapters of Gallian, and then chooses selections from chapters 11 and 24. Each of the first ten chapters includes at least one, and often two or more, episodes where D_4 or more generally D_n, is the central mathematical object being considered and thought about. In particular, the second lecture is entirely devoted to D_4, and introduces all the features of a group before the formal definition appears midway through the third lecture.

It is not possible to consider all these occasions here. All the gestural practices discussed in the previous chapter appear in most of the occasions. There are often striking instances indicating notable variations of a gestural practice that must be passed over. I have tried to be guided by the following criteria for inclusion: to treat at least some of the episodes in the crucial introductory Lecture 2, and to pick at least a few episodes that occur at various stages in the course in order to show some of the time development of J's approach to D_4. J's attitudes towards D_4, and the aspects of D_4 and his mathematical self that emerge in their interactions, undergo an exciting evolution as the course goes on, especially as new concepts are introduced, such as Cayley diagrams and Cayley tables, subgroups, permutations, and homomorphisms. I focused on portions of those episodes where there was a particularly high concentration of moments which are dominated by communicative actions that seemed to me well-understood by the gestural practices discussed before.

This chapter and the next I see as a pair. In this chapter the central character is an example of a group; this is the fundamental mathematical concept of this course. Some in the mathematical community might refer to D_4 as a mathematical object, and they might even call it something like 'a concrete example of a group'. Whether or not it is a good example, or one that is useful in understanding group theory, is a subjective question. Certainly one defense of its use is that it is a nonabelian group, and a second is that it is a relatively uncomplicated group. Therefore, D_4 can serve as an example that is not going to be misleading by having properties that are not shared by groups in general (abelian groups are too special); at the same time, it will not be confusing by being unwieldy in size or difficult to grasp intuitively.

In the next chapter, the central character is the notion of well-definedness: proposed definitions may fail in their purpose, and one must explicitly check that they do not fail. This chapter is about how J acts with respect to a mathematical object which is an instance of a mathematical concept; the next is about how J acts with respect to the regular situation of ensuring that a putative definition of a mathematical object is above reproach. It will transpire that this very act of ensuring leads directly to the birth and understanding of important related mathematical concepts such as normal subgroups and quotient groups. In the research phase of this dissertation, I saw the pairing more simply still: the material generated for this chapter was about mathematical meaning-making via gestural practices concerning a mathematical *object*, and the material for the next was about mathematical meaning-making via gestural practices concerning a mathematical *concept*.

6.2. Manipulating a square to determine its symmetries: points of view in conflict and distinguishing possibilities from certainties

Lecture 2 of the course revolves entirely around the group D_4. In this section and the next I discuss several important scenes in this lecture. One of the aims of this lecture is to construct the Cayley diagram of D_4. In this section we follow the action up to the moment when J and the class have convinced themselves that they have found all the symmetries of the square.

6.2.1. Improvised drawing, improvised gesture

At the very end of Lecture 1 J previewed what he would look at next time. He held up a square with the vertices numbered, and he asked the students to see if they could work out the Cayley diagram of the group of symmetries of the square. In that lecture J himself had worked out the Cayley diagram of the group of symmetries of the rectangle.

Now at the very start of Lecture 2 he looks at his watch – a routine move for J to mark the end of any informal chatter and the beginning of the lecture proper – and he holds up the square again, asking “so did anyone try to do the Cayley diagram for the square?”. Thomas (pseudonym) answers “yeah”. After checking that Thomas found eight nodes, J asks to see his picture and walks towards him.

As it happens Thomas has not brought his picture with him. Nevertheless, he agrees to draw it quickly right then and there. This occurs off-camera. A few seconds go by while Thomas draws. At one moment he says “it’s just like counter-rotating”, and J agrees. Ten seconds of silence later J says “would you like to come up and teach the class instead of me / excellent”.

There are no hand gestures visible from J. I include this incident because a student demonstrates what J models in every moment of his lecturing. It would be one thing if Thomas had worked out the Cayley diagram at home on a piece of paper and then simply presented this bit of already-done writing now. It is quite another for Thomas to walk around being able to draw this diagram whenever the occasion demands. Somehow whatever Thomas worked out at home transformed him into someone who could manufacture this diagram on command.

J’s admiration is clear in his tone. Two minutes later J begins to make an observation concerning his own experience constructing this Cayley diagram, but then interrupts himself out loud and decides instead to put the question to Thomas: “what you just showed me / was that exactly what you wrote down the first time you started-”. Thomas cuts in with a long “no” and J answers with “excellent good”. J goes on to remark that although Thomas' first stab at the diagram may have looked "crazy", he was then able to redraw it to make it look "neat and tidy": at this point J impulsively walks back over to Thomas, saying "if I may". This is a unique instance in the course: a student's writing being held up and shown to the class.

In between these scenes, when J initially refers to Thomas after approving of his live drawing of the diagram, he gets his name wrong. There is some laughter, some back and forth dialogue with the class, and it is revealed that the two students whose names he mixed up in fact have switched seats since the first lecture. J uses his pen to point back and forth along a line joining the students, saying "it's like one of those blue arrows / you just keep switching with this reversible action". In the first lecture J had drawn such blue arrows to represent a reflection of a rectangle with respect to a horizontal axis.

I realize there is nothing very much happening in the group theory course at this moment, at least not in the traditional sense. It's a joke, I know. I include the example to indicate just how fluid, flexible, and spontaneously improvisational gestures can be in the mathematics classroom, just as we would expect them to be from past work on gestures in all sorts of interactions. In an inspired instant J has, with a small movement of his hands, turned Thomas and Bart into living nodes of a Cayley diagram in the classroom, and performed the Thomas-goes-to-Bart and Bart-goes-to-Thomas transformation. Everybody laughs. I draw two conclusions here. One, any attempt to sort gestures into tidy, distinct, non-overlapping categories is likely to be foolhardy, for how could any classification anticipate creative movements of the hands like this one? Two, if even such an idiosyncratic, never-to-be-repeated gesture can be understood immediately by the students, it would be equally foolhardy to be too skeptical about the degree to which students can share interpretations with J of what the movements of his hands mean.

6.2.2. Points of view in conflict

J now begins his own live attempt to draw the Cayley diagram for D_4. This will take him some fifteen minutes involving a great many steps.

He writes 'Square mapping to its own space' as a heading and then he walks away from the board to his desk to face the class, holding the square and looking at it.

He touches each corner of the square in turn – “I’ve labelled the four corners”. He touches the corner marked ‘1’ – “there’s a blue one” – he flips the cardboard over and touches the same corner – “on the other side I’ve got a red one”. He jokes that he drew the number ‘1’ on the red side as if it were a mirror image, but he says “the point is that’s one / whether we’re looking at it on either side”. He says he wants to know if he is looking at the front or the back, and he flips the square to show each face a few times.

This prompts a question from Peter (pseudonym): “can you say that the front is one two three four / and the back is five six seven eight”. J says a long “uh” and rewords slightly while touching the corners of the back of the square as he stares at it: “could I label the back as five six seven eight?”. Just as he finishes saying this he adds “I don’t want to do that”, then he immediately corrects himself to say “I can label- I can do what I want” and he makes friendly eye contact with Peter. He ever so slightly puts his palm down, in an assertive gesture, as if he is reassuring someone that what they suspected was impossible was in fact quite doable. This hand suddenly turns into a single pointing finger as he suddenly says “but I don’t want to do that”. Soon he is holding the square with one hand only by corner number one, in a kind of pinching position. He begins to wave the square around in all sorts of ways while keeping hold of just that corner, saying “I want to consider this as corner number one / whether I’m looking at the square / under all solid motions of the square”. He continues to flip the square, move it around, as he finds other words to say much the same thing. He tries another attempt a few moments later, performing a single flip while pointing at corner number one: “if this is corner number one / I still want it to be corner number one” – he flips it over again – “I just need to know that I flipped over”.

J is *manipulating* this object in specific ways, spotlighting a specific corner before and after transformations, so that he can act out how it is he has chosen to *regard* the corners of the square. The tension is between what is the *same* and what is *different*. Two different colours are being used. Peter is suggesting that since the red ‘1’ and the blue ‘1’ are two distinguishably different labels, then why not call them ‘1’ and ‘5’, since presumably (he thinks) they are being regarded as *different*. For J, who is pushing back on Peter’s offer, the red ‘1’ and the blue ‘1’ are to be regarded as labelling the *same* corner. J’s deliberately random movements of the square while clinging to corner numbered ‘1’ – play-acting an island of stability inside a sea of tossing and turning the square around – deliberately *counters* Peter’s manner of viewing the square. Every random time the square gets flipped over, J is visibly *not caring*, because this will not matter to him; he knows that throughout this crazy process the corner he is tightly holding stays the same. Peter, on the other hand, will have to care every time the square changes which face it is displaying towards him; one moment he will see corner '1', and a moment later he will see corner '5'. This deliberately haphazard manipulation of the object exaggerates the difference between their respective interpretations, and sharpens exactly what *distinguishes* J's manner of viewing the labels of that corner from Peter's manner.

What is happening here in the early minutes of the second lecture is an encounter with an *equivalence class*, though the interactants do not discuss it explicitly in these terms. Red '1' and blue '1' are members of the same class (set), and the name of this set is "corner number one"; similarly for red '2' and blue '2', and the other corners. The power of the concept of equivalence class resides in exactly this ability to consider *different* objects as being, in some sense, the *same* – because they are living in the same class – in a way that allows for consistent and precise mathematical reasoning. It is intriguing to see that the issue that is at the heart of the concerns in Chapter 7 are already here in this conversation. To be clear on the nature of the difference in interpretations, it is not that Peter wants or needs to regard that corner as being two corners named '1' and '5', say; it is that Peter thinks that J is *already* regarding that corner as having two names anyway, so he does not understand why J is acting as though swapping out J's red '1' for Peter's '1' and J's blue '1' for Peter's '5' would change anything about the situation. After all, the labels are already different!

J does realize here that he must justify why he used two different colours for the label of "corner number one" when he at the same time wishes to consider both labels as labelling the same corner. He says that he wants to "keep track" of which face he is looking at. There is more that J does in this scene to try to elicit from Peter some reaction of agreement, but Peter withholds it, and ultimately J must press on.

Here we have seen that opposing views on a mathematical object, when each view has grounds for legitimacy, can lead quickly to issues and concepts of deep mathematical importance. Judging two objects to be the same, or different, is a fundamental ingredient in one's point of view on those objects. In this incident the clashing views that seem difficult to reconcile have already been found by mathematicians to have at least one resolution: the notion of equivalence class. This notion, as it is presently understood, is quite modern: in a lovely paper Asghari (2019) discusses the complex history of the name and the concept, and notes that disagreements continue among historians of mathematics as to who to trace it back to: perhaps Dedekind, or Frege, or Cantor.

It is likely that as I write there are many animated conversations at boards taking place between mathematicians where each believes their own point of view to be legitimate and the other's point of view to be subtly incorrect or misguided. Perhaps such contested instances of the *regarding as* gestural practice will give birth to new mathematical concepts. What J does here, which is to manipulate an object so that it *heightens the contrast* between the two perspectives, is an important type of instance of the *regarding as* family: exaggerating an aspect of an object's mathematical properties that aligns with one perspective but which causes some mathematical trouble for another perspective.

6.2.3. Manipulating the square to determine all the elements of D_4

J now begins determining all the elements of D_4.

He writes 'Initial position' followed by a drawing of his square, labelling the corners of the top edge with the numbers '1' and '2', and labelling the corners of the bottom edge '3', '4'. Then he writes 'can map to any one of'. He leaves a big space underneath, and on the right he puts a sideways brace that he labels 'Rotations'. Finally, he quickly draws four squares in a row on this line. In the next twenty seconds he performs a number of rotations of the square in his hand: first by 90 degrees counterclockwise, then rotating further another 90 degrees counterclockwise, then again another 90, and again one last time. He then repeats each of these *manipulations* directly in front of the board, and each time he does so, he records on the board, inside one of his four blank squares, what the square in his hands now looks like. He labels each of the configurations with a name: R_{90}, R_{180}, R_{270}, and since he left a gap at the beginning of the row, he goes back to fill in "the result of the doing nothing action", which he labels R_0.

By visibly performing the manipulations of the square that return the square to itself – in other words, by acting out a symmetry of the square in front of the class – and by annotating the square with enough labels so that he can determine the new configuration of the square, he is beginning to record, element by element, the members of a set which later will be named D_4. Each little three second action sequence of the path that his-hand-and-the-square took, in the air, from their starting position to their final position, is frozen onto the board in the form of a snapshot of the final position of the square alone.

J next draws four axes on the far right of the board: a horizontal axis, a vertical axis, and two diagonal axes. After each axis that he draws, he flips the square in his hands over said axis – to be accurate, over an axis parallel to the one drawn on the board, but translated about a foot away from the board. It is exactly the ease and freedom of performing this (geometric) translation of a written line-segment from a writing surface to the air that is at the heart of so much of the *manipulating an object* gestural practice. It is clear from occasion after occasion in this course that this (geometric) translation just as frequently occurs the other way around: from air to board.

He draws another sideways brace beneath the first one, labels it ‘Reflections’, draws another four (currently unlabeled and empty) squares below the first four. He faces the class and waves his hands at these squares: “right now see if you say / how do I know what order to do them in / well I don’t that’s the whole point”. He is repeating the observation he made with Thomas’ help earlier in the lecture. He waves his hands in little circles: “I’ve got to play around...it’s not like I’m getting the right order right now”.

What is the distinction J is keen to make here by first waving carelessly at all eight squares, vaguely referring to all of them, then waving his hands as he says “play around”? When J is lecturing on mathematics, it is very often the case that the order in which J has written something mattered. An individual step has justified a next step, which has allowed for a succeeding step, and so on, in that order. Moreover, J knows that his students will have experienced such moments many times during other mathematics courses taught by fellow members of his community. Therefore, J waves his hands to make clear that the students should *not regard* what he will write next as being designedly ordered. The *waving* of his hands is a frequent gesticulation for J during mathematical occasions where he *delimits* a meaning: *I'm saying this, but not more than this*; or *thus far and no farther*.

There will come a moment later in the lecture when this diagram of eight configurations will be complete, and it will sit next to a second diagram. This second diagram will be a genuine Cayley diagram: it will consist of these same eight configurations, symbolized with letters instead of depicted as pictures, and the configurations will be connected with lines or arrows which represent the actions of two generators of the group. At that time the students will witness a wealth of patterned touches of the Cayley diagram, highlighting various aspects of the group structure. I suggest that this little sequence of careless waving at the half-finished diagram of eight configurations which he is presently making displays that he is *regarding* what he is building as simply an unstructured list, a *set*, and *not more* than this, like a *group*. J is cautioning his students to *not regard* what he is presently creating on the board as possessing more structure than it really does.

In the next forty seconds, J dutifully performs each of the said reflections, looks carefully at the result on the square, and then records the result on the board. It is only after he has acted out all four reflections, live, that he compares the result on the board with what he wrote down before the lecture in the notes in his hands. He has now finished eight configurations of the square.

It is difficult to overstate how fundamental a role the *manipulating the object* gestural practice is playing in this scene. The subject of group theory was born out of the consideration of transformations of geometric objects that leave the object invariant, as well as the composition of these symmetries (Wussing, 1969/1984). Equally, this is one natural explanation for why J is beginning his course by treating in detail this concrete example. The elements of any group they will consider in the future can be similarly regarded as a snapshot of the resulting state of some mathematical object, perhaps a very complicated object, that has undergone a particular *transformation* from an agreed upon initial state of the object, or in the intuitive and evocative language that J uses in the first few lectures, a particular *action*. Group theory is the study of such actions and their composition.

6.2.4. When has a mathematical question been resolved? When can they move on?

Has J finished finding all the symmetries of the square? Has he found all possible snapshots of configurations of the square that can result from applying a symmetry to the starting position? When I listed eight symmetries of the square in the first paragraph of this chapter, did I list all the symmetries of the square?

J puts his notes on the desk, turns to the class suddenly with a puzzled expression on his face: “am I done?”. He spotlights the configurations on the board, noting that they have established that there are at least eight configurations of the square achievable by moving the square around so that it maps to its own space – “but maybe I- there’s some more” – he swings the square up sharply in a search-me who-knows gesture – “maybe there’s sixteen positions”.

Moments like this one go beyond the hands and reach out to his full body. His stance, his bearing, his facial expressions, unite in a committed pantomime of *perhaps it is the case that*. J takes on the role of an *other*, some open-minded, agreeable, equable other, who believes that the possibility that there might be more configurations has not yet been ruled out. There are easily thirty little play-acting performances like this in the course which distill some *attitude*, in the sense of Mead; making as plainly visible as possible some *readiness to act* in certain ways to the ongoing mathematical situation. Some such scenes are nearly tableaus: J frozen in some pose at the board like a painting on an urn. His attitude towards their present mathematical exploration of the square and its symmetries is *we’re not done yet*.

J follows through on his attitude in the next few minutes. First he calls on Thomas: “your diagram had eight [yes] positions / are you sure” – J emphasizes the last word pulling both hands up and striking down forcefully – “that there’s only eight”. Thomas responds with a long sigh. J stays silent, then points to Peter: “you’re sure Peter why are you sure”. Peter says that the corners numbered one two and three are fixed in relation to each other; J says “good start”. J then indicates John (pseudonym) with a hand. John gives the longest speech a student ever gives in the course: sixty-eight uninterrupted words.

He argues that if you use “brute force”, you can apply every one of the transformations they have considered, to every one of their eight configurations, and they will always end up back in one of those eight. J says a long “uh”, moving a little towards the diagram of the eight configurations on the board – the vocal gesture could be called hesitation – then he turns back with a quick grimace, saying “well maybe”, and he comes to a stop to look at John. Suddenly he moves again, walking over to a space on the board beside the eight configurations, both of his hands outstretched in front of him, taking a number of steps, saying “but maybe there’s another eight / that sit over here that I haven’t thought of”. J circles his hands in the air saying that maybe all those new transformations will take all those new configurations to themselves. He returns to his original spot and repeats his request for a “simple argument” – he holds the square up for them to look at. He moves the square back and forth in rhythm to the beat of his next words “give me a simple argument / why there are exactly eight positions that I can map to”. He immediately cuts off and says “sorry”. He *self-corrects*, with a quick roll of his eyes, “why there are no more than eight”. He uses one hand with palm facing the board to spotlight the configurations they found: “certainly there are at least eight”.

Two more students make their attempts. To the first one J replies “that’s warm”; he interrupts the second one quite early while saying that one aspect of what was said was good. J tries to sum up that “they’re getting warm”. Then he switches gears entirely, now deliberately enlisting the students in committing themselves to declaring their attitude publicly. He stands in front of them, both hands moving together, palm-down: “who thinks” – hands go up to his head dramatically then down again swiftly – “that what has been said so far” – now his left hand makes a pointing gesture with the index finger, the other is still holding the square, and J begins to perform sweeping motions, as if clearing the air in front of him, starting in the middle, and ending out to each side – “is a one hundred percent” – he repeats the neat horizontal clean sweep outwards on the next word – “convincing argument / that there are no more than eight positions” – he repeats it again when he hits the word “no”. The gesture is repeated a few more times in the next few lines as he repeats a variant on this question, reaching its most emphatic version on the word “nailed” in: “who thinks...we’ve collectively nailed this question”. He looks around the class and no one raises their hand – “I’d agree” he says, “it’s warm”.

He proposes his own argument, and when he is finished, he flips both hands over so that the palms face down, pushes his hands downwards and says “done”. Then he pokes one finger in the sky shyly and says “who votes for a hundred percent now”. In a few moments he will be surprised to see that there is a holdout: Peter. I will return to Peter’s objection, and J’s treatment of it, in the next section. When this is over, J sums up by play-acting, during a *commenting about* scene, what students could do in the future every time they are fashioning a mathematical argument. He holds his chin with his right hand – “is that a bit flaky?” – holds his hand there – “can I improve it?”. Later he looks strained or pained and holds his hand to his temple – “is there something missing there?”. He encourages them to keep doing this “to the point where you say / that’s it” – here his right hand sweeps out and to the side, palm down “it’s nailed” – repeats the sweep out – and, intriguingly, finishes with “there’s no further discussion to be had”. It is no longer possible to convincingly play the *role* of the skeptical *other*. The talking-and-gesturing is over, the argument carried the day, and the matter has been completely settled.

In this entire scene, J contrasts two mathematical *attitudes* with large expressive, emotionally resonant, movements of his hands and accompanying pantomime with his face and body. The first is the attitude in which uncertainty has persisted, doubt has lingered, questions remain, puzzles subsist: hand stroking the chin, hand on his temple, inhibited body movement, vocal tone rising into high pitch questions. The second is the attitude in which certainty has arrived and all questions have been resolved: huge vigorous clearing motions of the hands, repeated often, palms-down pushes to the ground, emphatic vocal tone. J’s *done*, *nailed*, *one hundred percent sure* gesticulations and confident downward tone are correspondingly forceful and unhesitating. These movements gain even more in power because they live in sequential comparison to his *quizzical*, *can I improve that*, *is that flaky* gestures, puzzled facial expressions and upward rising tone.

6.3. Assembling the Cayley diagram for D_4

It is time for J to draw the Cayley diagram for D_4. He throws both hands towards the eight configurations on the board, saying “so those are all the positions”, and walks to the board to start writing. First there is a written remark describing what he is about to do: ‘Compose mappings to form a Cayley diagram for the group of symmetries of the square’. Such written remarks get rarer as the course goes on. Then he draws four nodes in a diamond shape. Each node is a large enough circle so that inside them he can write the labels for the rotations: 'R_0', 'R_{90}', 'R_{180}', 'R_{270}'. Once he is done he starts from the top, proceeds clockwise, and rhymes off each label as he points to it: "*R* zero *R* ninety *R* one eighty *R* two seventy". He picks up his red pen and draws four arrows, from the north node to the east node, and continuing clockwise; next to the diagram he begins a little legend, drawing a red arrow – "my red action here is" – then both saying and writing "rotation through ninety degrees".

He stops for a moment to look at the diagram so far, and looks at the class, then points to the diagram, doing a twirl of his fingers as if following the arrows of the diagram, saying and *self-correcting* twice "and notice that the red arrow- the red line- the red action / now has an arrow on it". He starts a new spoken line and gets only two words out – "last time" – before suddenly jolting himself around almost 360 degrees in order to find a different Cayley diagram at the very far left of the board that he himself had drawn before class had started. This diagram has been unlooked at, unspotlighted, and completely ignored for the entire lecture so far. It is a Cayley diagram for the group Z_2 x Z_2 (the Klein 4-group) which J had constructed in Lecture 1. J will not officially name it this until he covers direct products much later in chapter 8. Up until then he calls it the light switch group, since it represents the transformations one can apply to a pair of light switches: both up, left up right down, right up left down, both down. With his left index finger pointing the way, J walks over to touch it, continuing "when we did the light switch group". By now J is holding this diagram with his left hand placed in the middle of it and he is facing the class.

In the ensuing scene we have a *looking at side-by-side* gestural practice, prepared for long in advance, explicated in more detail with the aid of *manipulating an object*. The essential contrast will come in two spotlightings: tracing a *line* in the Cayley diagram for the light switch group; touching an *arrow* in the Cayley diagram for D_4.

He traces one of the lines that joins two nodes, saying "these were bidirectional". He touches one node – "if we started here and applied the red action" – he traces the red line to tap the node at the end of this line – "we got here". Then he says much the same thing while retracing his finger-steps. Now he walks back over to the Cayley diagram he is engaged in constructing in the present lecture – "but here" – he changes direction to go to his desk to pick up the square and rotates it by a quarter revolution counterclockwise – "if I rotate by *R* ninety from this position I get to here" – he stops and looks at the class briefly. He continues "if I rotate by *R* ninety from this position / I don't come back to where I started" – he rotates the square counterclockwise – "I get to somewhere new". He concludes by touching the arrow on his new diagram, saying "so I need an arrow".

J continues with his diagram, drawing the four remaining nodes, adding the second action to his legend ('reflection about a vertical axis'). Because a reflection is its own inverse, he uses a line and not an arrow to represent it in the diagram. He will show that all the configurations – all the nodes of his Cayley diagram – can be reached by travelling along some sequence of the two actions he has chosen as generators: rotation counterclockwise by 90 degrees (R_{90}), and reflection across a vertical axis (V). He accomplishes this with another instance of the *looking at side-by-side* gestural practice, coupled with *regarding as*.

He points with his pen to the eight configurations on the board from earlier, and notes that he labelled the four reflections by naming the axes they were reflections about – he goes over to his diagram of the four axes and touches it. But now in his Cayley diagram, although he has labelled the four remaining nodes in the same way, he explains how he and the class will *regard* the transformations as he points to the diagram he has just finished: "now when I wanna write a Cayley diagram / I'm just gonna think about reflection about a single axis the vertical one" – "gonna think" is a decent paraphrase of *regarding as*. So for example they are "gonna think" of the action 'H' – reflection across a horizontal axis – as the sequence of actions $V\ R_{90}\ R_{90}$. Their convention is to apply the actions right to left. J checks with his square two nodes of his diagram to see if indeed composing the arrows correctly leads to the nodes as he has labelled them. He then spends ten minutes touching the Cayley diagram in many sorts of ways as he lists a few features of this Cayley diagram that are true in general for Cayley diagrams, and which are informal versions of the axioms of a group. In the section on subgroups I will look at a few examples of J handling a Cayley diagram so I will pass over this episode here.

When this material is over J goes to his desk, picks up a new page of his notes, and says "now". The next few lines are quite interesting, in the sense that J is attempting to say two nearly contradictory things at once. On the one hand he seeks to praise the Cayley diagram and on the other hand he seeks to point out its limitation. He starts off with the insult, his brow furrowed “this doesn’t show-“, but he breaks off and puts his hand up, palm facing the class, literally in the *stop* or *halt* gesture. He continues with the opposite, conciliatory tack: “ok in one sense this shows the whole story”; having said this he looks over at the diagram, walks toward it and places his palm on it: “that’s everything that we need to know about the group / and visually that’s telling us a load of information”. He takes his hand away and walks away from the board with a loud “but” – he has returned, as he telegraphed initially, to the downside – “there’s still a lot of work”.

The *self-interruptions* in this sequence share features with self-corrections. J is not interrupting himself in order to unequivocally replace his earlier speech with a correct version that he happily continues from then on; thus this instance does not really belong in the *correcting self and others* family. Nevertheless, he stops himself in order to pursue the opposite tack, and then stops himself again to pursue the original tack.

A few lines later, keen to demonstrate this limitation of a Cayley diagram (“lot of work”) he sets himself the task of computing a product of two actions using the Cayley diagram, in the case when neither of these actions is a generator. Now he acts out the role of a person who must perform this calculation but is unhappy to do so: he puts his notes down, sighs in pretended exasperation “ohhhh”, puts both his hands to his head – it is a short performance of someone who is required to do something complicated and tedious.

Ironically, the calculation only takes him a few seconds – less time, perhaps, than his performance of near-despair had forecasted – and J’s tone shifts back to the praise side again: “I can work out the answer to any such question from the diagram / so that’s one reason why this is useful”. Then he shifts tack again, immediately qualifying his approval with a let-me-level-with-you look at the class, nodding with eyes a little narrowed: “but at the same time / if I want to answer all such questions / it could get a little bit tricky ok.”.

There are not many occasions in the course when J needs to negotiate a *regarding as* that is as balanced, delicate, and conflicted as this one. The only one that is comparable is the nearly paradoxical first stanza of his introduction to permutation groups, when he foreshadows Cayley’s theorem. He says of permutation groups that studying them “tells us everything there is to know about groups / except that it doesn’t /

um in some sense it really does". While decisions as to the validity of an argument are one hundred percent or they are not, estimations along various axes of value – how readily can one extract some mathematical information about an object from a certain presentation of it, for example – can contain contradictory forces.

Here in the second lecture, J is making these comments about the Cayley diagram in order to motivate a second representation of a group, the group multiplication table (J always uses the phrase "Cayley table"). He begins to construct a Cayley table for D_4 on the board, but he only completes the column headings and the row headings; he does not fill in any entries. Instead, he displays a transparency of the completed Cayley table.

Having done so he switches off the projector and he begins a *side-by-side* comparison of the Cayley table to the Cayley diagram, returning to his earlier *regarding as* theme, but now contrasting the two representations. He holds the Cayley table on that board and says it is "complete". He means that the product of each two elements of the group is displayed in the table (ironically it literally is not complete because he opted not to take the time to fill in the sixty-four entries, but the hypothetical one is). Then he walks over to touch the Cayley diagram and says he could make that one "complete as well", but this would require drawing in all the arrows for all the actions using enough colored pens. The resulting diagram would be very cluttered, In J's pun it would be "complete-ly useless / cause I won't see what's going on".

In the next section I discuss J's handling a Cayley table in some detail (which occurs in Lecture 3) so I will skip over the episode in this lecture where he performs some similar but less systematic handling. I also skip over an exercise he solves, except to mention one curiosity, germane to this Cayley diagram versus Cayley table discussion. There comes a moment when to solve the exercise he needs to calculate the squares of elements of D_4. He pulls the screen down to use again the transparency of the Cayley table of D_4; but in doing so the screen now covers the Cayley diagram of D_4 that is written on the board. When J realizes this he freezes, turns his head to the class, and says "huh" – pause – "interesting". For two seconds he looks genuinely stymied and flummoxed. He soon smoothly continues, deciding not to move the projector, and opts for the Cayley diagram on the board. It is a compelling example of the ongoing need J has to have the writing that he needs to touch visible; and if *side-by-side* becomes front-over-back, then the action lurches to a stop.

The final incident I will look at in this lecture occurs near the end. J says he will not write down the definition of the general dihedral group today, but that he will do one more example. He proposes that they consider the group of symmetries mapping the regular pentagon to itself. He points to the Cayley diagram of D_4 on the board with his marker – "well if you just take a look at that picture" – he starts to write the title 'A Cayley diagram for D_5' – "we should be able to just see or intuit or deduce / or whatever word you wanna use" – he has finished the title and walks away from the board – "we should be able to" – he points at the empty space below his title – "spot infer guess" – he chuckles and begins waving both hands in wide circles as if attempting to stop his compulsively growing list of alternatives – "those will be equivalent for the purposes of the rest of this lecture" – he moves now to touch the empty space, his eyes closing in conclusion – "we should be able to write down the Cayley diagram for D_5 / just by seeing how this" – he touches the definition of D_5 – "relates to this" – he points to D_4.

This is a quintessential moment of the *deducing that* gestural practice, and as a bonus it contains an explicit articulation of the key directional quality ("we should be able to *write down*" X "just by seeing how *this* relates to *this*"). In addition, the wealth of analogous verbs that J lists is consistent with the wide view taken in this thesis that the *deducing that* gestural practice is best thought of as all those practices where from some *this* (or collection of *thises*) a mathematical actor is now able to write down a particular sought-after *that* (whether it be by seeing, or intuiting, or deducing, or spotting, or inferring, or guessing, or the like). In this scene it is a student, Peter, who given J's stanza-length setup, delivers the directive of what to write in his customary casual but correct manner: "it looks like that but with more nodes". J accepts "sure that'll do" and he proceeds to draw it.

6.4. Manipulating a Cayley table to explain the definition of a group

In the previous two sections, I discussed how J previewed the properties that groups satisfy by building a Cayley diagram for D_4. In this section I will analyze how J handles the Cayley table for D_4 after he has just introduced the formal definition of a group.

Early in Lecture 3 J writes the definitions of *binary operation* and *group* on the whiteboard. J says “let’s check with something we’re already familiar with”, and he puts up a transparency of the Cayley table of D_4 on the projector. In the next few minutes he will be working his way sentence by sentence through the writing on the board by *manipulating* and handling his textual object, the Cayley table for D_4.

His definition of binary operation is ‘A binary operation on a set G is a function from G x G to G (closure)’, and his written definition of a group begins ‘A group is a set G with a binary operation for which’. Saying “the set is these eight elements” he runs his index finger along the row at the top of the Cayley table; his emphasis is on the word “set”. They are *regarding* these eight elements that his finger has grazed as the ‘set’ being referred to in the fifth word of his definition of a group.

Then saying “with a binary operation / it takes pairs of elements” he runs his index finger along the row at the top and the column at the left. Each sweep brushes past a potential element to be selected. He has reached the ninth and tenth word (‘binary operation’) of his definition of a group, and to unpack this term he must use his definition of that term, which requires him to discuss a map from ordered pairs of elements of his set. He continues with “so we said first this one then this one”, where he repeats the sweeping spotlighting gesture, timing it so the first “this” coincides with a line through the top row, and the second “this” coincides with a line through the left column. The emphasis on “first” and “then” is more pronounced. They are *regarding* the eight elements in the top row as possible choices of the first entry in an ordered pair, and the eight elements in the left column as possible choices for the second entry.

During this scene J is engaged in *reading and embodying* an abstract definition. He reads it in strict linear order with one exception: when he encounters a term with its own definition he jumps to a different location, where this term is defined, and reads there in linear order from the beginning. Every time he encounters a mathematical object or item in the definition, he at that moment handles his Cayley table in a suitable manner: a single sweep can indicate the set of elements; an ordered sequence of two sweeps is required to indicate an ordered pair of such elements.

He has reached the words ‘to G’ in the definition of a binary operation. He says “the binary operation spits out some element”, and he circles all sixty-four elements in the table with his finger. He next says “in other words all of the elements in this eight by eight table” – he traces his finger again along the top row – “are one of these eight elements we see on the top”. The circling of all the elements is the moment when he is regarding each of them as potential outputs of the binary operation; the gesture corresponds to the ‘to’. The tracing of his finger along the top row corresponds to the ‘*G*’.

Then he traces the left column up and down a few times followed by a last sweep of the top row, saying “these eight elements repeat these eight elements”; the gestures are simultaneous with the two “these”s. Now he is no longer advancing his reading. These movements repeat the tracings he performed earlier of the top row and the left column. He accomplishes two things. First, emphasis of this earlier manipulation. Second, a subtle adjustment to the impression left by the tracing of his finger along the top row which at the time corresponded to the ‘*G*’. The left column *also* corresponds to ‘*G*’, and his present movements serve to clarify this. The stanza is marked as ending with the hand leaving the transparency, the pitch of his voice falling, and the spoken words “so that’s the closure property”. He has reached the final word of his written definition of binary operation.

Now he moves further in his definition of a group. His definition states the three properties that a set with a binary operation must satisfy: associativity, existence of an identity, and existence of an inverse for every element of the group. The new stanza is marked as beginning with the hand returning to the transparency, the pitch higher, and the spoken words “Ok a set and a binary operation”. He continues “so a set” – here with his right hand vertical to the transparency, the edge of his hand traces the top row yet again; “and a table” – and here his right hand sweeps down to pass over the sixty-four elements of the table.

He next says “now what are the properties of the table? / well associativity that’s not gonna be a problem”, and for this portion, and the next couple of lines, his right hand has spread fingers and from above the transparency he pushes his hand down towards it repeatedly, indicating the table as a whole. However his speech soon runs into a wall: “the way that we’ve represented this set / because we’re just talking about buh buh buh buh buh / is that gonna be a problem?”. For the next few lines his voice is less sure and quieter, and his hands are off the projector, and he asks and answers as follows: “am I guaranteeing associativity off the table / no". He slows down at "table", pauses very briefly, and then the "no" is already back at ordinary pace and rhythm.

From this moment on his confusion is over and he can already begin explaining why he stopped and *corrected* himself: "no I'm not / sorry take that back / when I made- / ok I was- I confused something there". It is the first of two highly instructive instances of the *correcting self* gestural practice that occur in the scenes in this section. He goes on to explain his error.

In making the Cayley table for D_4 in Lecture 2 he had relied on the Cayley diagram for D_4 that he had built by manipulating the square. These symmetries of the square – reflections, rotations – are examples of transformations, or mappings; and composition of transformations is automatically associative. Therefore, the Cayley table for D_4 that he built in that way would be guaranteed to inherit this associativity.

But if instead he is, like now, only looking at this table itself, *with no knowledge of how it came to be written*, he has no way of guaranteeing that compositions are associative. The error came in his *attitude* towards the table. Was it an object that he built in the way I just described? Or was it an object that has been handed to him, whose origin and method of construction is unknown? He has recognized that he must *regard* this object not as one that he himself generated in the previous lecture, but as a putative Cayley table constructed by a self in some other process, for which the associativity property must be checked. He says "but if I throw all of that away" – he gestures throwing away with his hand – "and just write eight different symbols here / and eight different symbols here" – he sweeps his hand along the top row and left column – "I can't assume" – he places his hand down on the table and leaves it there – "I was about to fall into a trap of my own making / I can't assume that this table will come out to be associative". When J *self-corrects* because of a short-lived *failure to regard* the mathematical situation properly, it usually precipitates *commenting about*, and there is a longer lasting impact. *Self-correcting* in a *deducing that* occasion is commonly, though not always, a low stakes affair; adjusted and tidied up very quickly.

In the next stanza he attempts to check one example of associativity. Picking a random element by touching it in the top row, then picking some random other element to multiply it on the left by (and touching it in the left column), he traces to the element which lies at the intersection of that row and column. But he realizes out loud that he will never remember these elements if he does not write them down, so he writes down on the transparency itself an equation expressing associativity of the multiplication of three elements: 'D (H R_{270}) = (D H) R_{270}'.

In the following stanza he computes these two products to show that they are equal. One might call the entire choreography of touches here something like *touches-computes*. He uses both of his hands. The right hand, still holding the transparency pen, is responsible for touching the element R_{270} from the top row; his left hand touches the required element H from the left column, and then both hands move together to find the element D' which is their product. He now finds and touches that element D' in the top row with his right hand which holds a pen, finds and touches the element D from the left column with his left hand index finger, moves the pen down the column until it reaches the row held by his left hand, announces the result "R_{180}". He goes on to similarly *touch-compute* the other product. The entire touch-computation (four multiplications) takes seventeen seconds, including writing down the results.

Throughout the course J often *touches-computes* with Cayley diagrams and Cayley tables. They constitute an important class of instances of the *manipulating the object* gestural practice. Instead of moving a square in space with his hands, performing one transformation after another with his hands in order to multiply elements of D_4, J can find products in D_4 by operating a Cayley table much as someone might use the grid markings on a map to find an island, or a player might find a location when playing Battleship; or he can find products by touching the arrows and lines of a Cayley diagram in a particular sequence in order to arrive at the right destination. Later in this Chapter we will see occasions when the way J handles a Cayley diagram *does not have* a counterpart in manipulating the square; handling a textual object can allow mathematical actions not possible to achieve with manipulating a physical or pretend object.

Next in the sequence of properties of a group is the existence of "an identity element / that's R zero in this case" and J holds this term in the top row of the table with his right hand in a palm-up orientation. He goes on: "with the property that if you do R zero" – he has shifted his hand so that it is his pen that is touching this term – "and then anything else" – here he runs his marker down the entire column labelled at the top by R_0 – "you get that anything else back" – he runs his pen down the column again. He now repeats his explanation: “in this table this means look down the column *R* zero” – right index finger touching this term at the top – “and this” – runs his finger down the column – “has gotta be identical to this” – runs his finger down the left column (which lists the elements of the group), timing it to coincide with the word “identical”. Both sweep down motions with the finger are nearly identical.

His next words are “also the other way round” and he draws a little circle with his index finger in the air, clearly apparent on the projector. Now he runs his finger along the top row to indicate those elements – “take any element of the group”; touches the term R_0 in the left column and holds it – “then do *R* zero” – he sweeps his finger along the first row, which is identical to the top row which lists the elements of the group – “and then you get exactly that member of the group back”. Just as before, when he performed the explanatory gestures twice, he goes through the showing process again. He rests his pen on the transparency so that it acts as an underline of the top two rows of the Cayley table, and by sweeping his finger along it he spotlights how these two rows are identical. He then shifts the pen so that it is vertical, aligning it so that it separates or marks out the two leftmost columns, and by sweeping his finger along those columns he shows that they are identical.

These were two instances of the *looking at side-by-side* gestural practice. J has carefully shown, with his table, that multiplying any element by the identity on the right, or on the left, gives back that element. He has looked at the top row side-by-side with the row indicating the result of multiplying on the left by the identity and they are clearly the same. He has looked at the left column side-by-side with the column indicating the result of multiplying on the right by the identity, and they are also clearly the same.

J is now ready to discuss inverses. He opens a new stanza by stating in a few words what they have just done, and then announcing what they will now examine, his pitch high again: “so that’s the identity part / for each element of the group there is an inverse element”. He uses his pen to touch a specific column label on the Cayley table, saying “let’s pick an element *H*”. J continues “if I do *H* first” – the pen still fixed on *H*, the finger on his other hand beginning to sweep up and down the labels of the rows – “there’s gotta be some” – sweeping up and down multiple times – “group element down here” – his finger now stops at the bottom – “which I can then do so that I get” – suddenly his finger jumps from the bottom to touch the identity which labels the first column, timing it to coincide exactly with his next words, then leaps back to where it was – “the identity element which we decided was *R* zero”.

He pauses for a second then *self-corrects* himself: “sorry an identity which we decided was *R* zero”. He has swapped the word “the” for “an”, and he emphasizes “an”. With his pen and finger fixed in place he says “I just assumed there was only one but we haven’t proved that”. The finger on the projector suddenly points ‘in real life’ to the writing on the board as J says “take a look”. He begins to walk away from the projector and towards his desk to pick up a red marker – “because I know what’s coming / it’s so easy for me to anticipate what’s coming up” – smiling, he goes to the board and draws a circle around the word ‘an’ of ‘an identity’ in the statement of the definition of a group. He turns around and enunciates the word “an” with a wide mouth in his next words: “there’s an identity”. He goes to the transparency to trace the first column of the table underneath the label R_0 saying “*R* zero is an identity”, again emphasizing the article very deliberately – “maybe there’s more than one we don’t know at this point”.

One line later he walks again to the definition of a group on the board, his arm outstretched in front of him, pointing for all five strides until he touches the circled word ‘an’ again – saying “all that says is that there’s an identity” – emphasizing “an”. Then he goes on to touch the word ‘an’ of ‘an inverse’ in the same definition, saying “an inverse”, emphasizing “an”. He touches the equations that the inverse *b* of an element *a* must satisfy (‘$ab = ba = e$’), saying “satisfying this” – then he walks over to the line where he defined the identity and touches the symbol ‘*e*’ there – “for the *e* that we identified here”.

At this moment J takes his hand away from the board, faces the class, and swings his arm up high in the air, palm facing the class, waving it up carelessly twice, timing it with each “maybe”: “maybe there’s an *f* maybe there’s something else”. This *waving* gesture is again a *delimiting* gesture: J knows there is at least one inverse, but he does not know more than that.

This is the second important instance of *self-correction* in this scene, and it again involves the *regarding as* gestural practice. A very little word which escaped his lips suggested a point of view on a mathematical object that was not yet warranted. When J is in this classroom, lecturing at this particular stage in the logical development of this course, his point of view on mathematical objects is not identical to his point of view on those same objects when he is, for example, in his office. In his office, in the hallway, *of course* a group has a unique identity, and *of course* every element in a group has a unique inverse. To be discussing a group and to refer to "the" identity element in that group is to make no mathematical error whatsoever. However, 'discussing a group' is not a complete description of the present situation. The present context is J discussing a group when all he and the class have justified to be known about groups is that they have "an" identity. The *self-correction* in this scene shows that the *regarding as* gestural practice is constituted in part by what stage of the course J and the student are at. Regarding objects while lecturing is regarding them under a constraint: the constraint of what has been established to be true in the course thus far. J must negotiate two selves: his ordinary self, who knows very well how many identities a group contains; and his lecturing self, who at this moment must not know.

Now J can return to where he had been before he uttered "the" instead of "an". He says "ok", walks back to the transparency, says "so we've identified R zero is an identity" and he touches this term labelling the first column of the Cayley table. With his right index finger he holds H again – "so given H what can we do" – he traces up and down with his left index finger the labels of the rows again – "so that we get R zero back" – now this finger has moved to trace up and down the column beneath the H that he is still holding – "well look down the column" – the moving left finger finds 'R_0' and the finger on H moves to also touch R_0 – "there's R zero" – now the right finger holds this R_0 in the table and his left finger sweeps quickly left to touch the label on that row (which is also H, since two successive reflections across a horizontal axis does indeed give the identity) and then it sweeps quickly back to the join the right finger – "so do H again" – he flicks his right finger up and down – "and we'll get back to R zero". J has touch-computed an inverse of a group element by handling and touching a Cayley table in a systematic patterned manner.

After doing another example of finding an inverse to a group element using the Cayley table he moves to the board and touches each of the three conditions for a group in turn: holding the identity condition he says "this I found that was straightforward"; holding the inverse condition and then tapping it firmly he says "this I have to do some checking" (emphasis on "some"); finally he touches the associativity condition, turns to look at the students with a pained expression on his face saying "this" – he pauses, tilts his head and shakes it – "a lot of checking".

He had used a finishing tone and walks back to the transparency. But then he stops just before he gets there and starts again “ok we almost never actually check” – he turns to walk back to the associativity condition, his arm outstretched pointing multiple steps before he gets there – “associativity” – and he touches the condition. Now he turns to the class, continuing his *commenting about*: “we almost always” – pauses briefly, and he gesticulates a twice-repeated big circular inwards motion with both hands at chest level while saying “inherit associativity” – now his right arm and hand perform a careless scattering to the right motion as he says “from somewhere else”.

The waving, shaking motion again indicates a vagueness, a delimiting of knowledge. J is visibly *not caring* about precisely where the associativity is coming from. Such deliberately careless hand movements contrast sharply with, for example, the five-stride-walk-while-pointing, which ended with him touching the single word “associative” on the board. J elaborates: “oh we already know” – he flings his right hand at shoulder level back and to the right, as if swatting away some fly – “function composition is associative” – right hand still waving away and to the right, arm fully extended, and then as he says the next words, the right arm shifts so that his hand now sweeps from the left over to his right, like he is pushing some small object to his left at eye level – “so that will carry over”.

In this scene J read step by step through the written definition of a group by means of manipulating a textual object, the Cayley table. There were two key *self-corrections* that revealed subtle components of the *regarding as* gestural practice. In both occasions, what J actually knows caused him the trouble; he knows more than he is supposed to know, given the frame or context.

6.5. Subgroups of D_4

At the start of Lecture 7 J writes the definition of a subgroup H of a group G. He includes a sentence that explains the notation for saying that H is a subgroup of G, as well as the notation for expressing that H is a proper subgroup of G. Finally, he writes down the so-called trivial subgroup of G. He then proceeds to consider four examples of subgroups, using Cayley diagrams as his vehicle. The first was the cyclic group Z_6. Now he begins to draw a Cayley diagram for D_3. Just like his diagram for D_5 in Lecture 2, he does not bother to label the nodes, so the circles that represent the nodes are smaller. This diagram does not have a legend. So much for relatively minor variations on the theme of drawing a Cayley diagram: now comes the first serious change. He marks in bold two of the open circle nodes, adding “and I say that that’s a subgroup”.

He barely takes a breath and launches immediately into a justification – “well again if I do it visually” – he has taken a step to the board, and he quickly did a once-over of the diagram with his index finger as he said “visually” – “from here to here” – he touches the top node then the other node then back to the first one – “that’s blue followed by red”. He touches the blue line and then the red arrow that take him from the top node to the other one, then holds this node while looking at the class, asking them “where can I go if I’m only allowed to do blue followed by red?”. He looks at the diagram and moves his index finger to touch the blue line emanating from the node he was holding, then the red arrow that takes him back to the top node – “blue followed by red takes me back to where I started”. J holds his hand high and palm open and then shrugs “and there’s nothing else to do”, and then pauses for a long moment staring at the students with eyebrows up, as if to say he has no options.

Now he turns to the diagram again and double-points at the two nodes with his index and pinky finger simultaneously – “so if I look at these two” – he turns his head slightly squinting his eyes and hits a high pitch on his next words – “then I almost have a subgroup”. He closes his eyes and flickers them (he never repeats this expression in the course), blinking rapidly, with his lips wide, and then he raises an index finger to the air saying “what one extra thing do I need to mark on my diagram / to make sure that these nodes” – he touches the two nodes individually again – “really are a subgroup”. A few seconds later Bart answers: “the identity”. J agrees (repeating the phrase) and he labels the top node as R_0. A moment later he touches a node different from the two he marked initially, saying “if I marked that one as *R* zero then no”, shaking his head as he continues to hold it. He puts a palm to push towards the diagram “apart from that we’re good”, looks at his notes, and walks away. The diagram definitely shows a subgroup and can be pressed home and dismissed.

J erases all the writing about the definition of a subgroup, draws the Cayley diagram of D_4, saying as he finishes “this is our friend *D* four”. He marks four nodes, including the top one which he labels R_0. He touches the top node, says “I can do blue followed by red” and he touches the line and the arrow in turn to arrive at the second node, exactly as he had done with D_3; now he applies the same two actions from the second node to arrive at the top node – "blue followed by red gets me home", again repeating the actions he had done with D_3. He double-points to these two nodes with his index and pinky – "so these two alone would be a subgroup" – he breaks off, stops looking at them, keeps his index and pinky in the same pointing configuration, turns around to walk over to his previous diagram of D_3, reaching out his arm fully to point near the corresponding subgroup he marked on D_3 – "just like those two were".

It seems important to me that he makes eye contact with D_3 at the beginning of "just" but already at "those" he looks back at D_4 even as his arm stretches to point at D_3 behind him to his right. It is hard to capture in words how quick and fluid this motion is of carrying his index and pinky in the same formation, swinging his arm around expertly to reach and point inches away from the two nodes in a diagram five feet away. Later in the course he will stop using the phrase "just like", and he will start using phrases like "isomorphic to". This is a very good example of a spotlighting gesture that *carries* structural and mathematical significance. J goes on to touch his way around the Cayley diagram of D_4 to show that indeed the four nodes form a subgroup.

Five lectures later J has embarked on drawing the subgroup lattice of D_4: a diagram that shows the containment relationships of all the subgroups of D_4. It has not taken long for the course to progress from an example discussing and justifying one subgroup in detail to a multi-minute episode where all of them are found. I will just zoom in to one ten second interval when J has the Cayley diagram of D_4 on the transparency, and with his two hands, he shows all three subgroups of D_4 that have four elements. One of them is the one we have just discussed. He uses a thumb and index finger of one hand, and a thumb and index finger of the other, and plants them simultaneously and holds them – "so these four form a subgroup" – lifts them, finds it more convenient for the next subgroup to use index and middle finger of each hand – "aaaaaaand these four form a subgroup", his tone indicating that he is finished the job of finding such subgroups; then in a very quick gesture he touches the four outer nodes saying "actually so do these" but he had already found this subgroup (isomorphic to Z_4) earlier in the episode.

Once the class have in hand the concept of a subgroup J can now begin defining types of subgroups that make sense for any group. In Lecture 8, as has already been anticipated in Chapter 5, J writes on the board the definition of the centre of a group, *Z(G)*, writes the proof that it is a subgroup of *G*, and then he defines the centralizer of an element *a* of a group, *C(a)* (but he has not yet proved that this forms a subgroup). After each definition he has written a parenthetical comment. After the definition of *Z(G)* it is: 'the subset of *G* whose elements commute with every element of *G*'. After the definition of *C(a)* it is: 'the subset of *G* whose elements commute with this fixed *a*'.

Now he stands away from the board and says "so the centralizer" – he emphasizes this last word, and he walks to the board to touch the symbol *a* in the term *C(a)* in the definition – "is with respect to a particular element". He explains that the notation is implicitly suggesting that "that centralizer could change / according to which element" – he touches the symbol again – "you make as the argument". His tone falls and he pauses. He looks over at the previous board, begins walking to it, and says "whereas this" and he combines, as we have so often seen him do, pointing and walking until he arrives at the *G* of *Z(G)* in its definition, now saying "is a property of the entire group", and he turns to look at them meaningfully, his hand pointing in mid-air in front of his face at the definition.

He takes another crack at it, sweeping his index finger lightly under the first words in the parenthetical comment after the *Z(G)* definition: "ok so this is all the elements-" – he *self-corrects* nimbly "the subset of all elements" – he makes two clockwise turns with his right hand in the air, an exchanging gesture – "which commute with every other element of the group" – he is already looking over at the centralizer definition, starts to walk towards it – "this is the subset" – he has reached the word 'elements' in the parenthetical comment there, sweeps a line with his index finger under the rest of the comment – "whose elements all commute" – he turns to face them, makes a particular hand formation with his right hand, all fingers touching the thumb – "with a specific" – on this word he moves that hand down and up three times, as if he is planting some seed in some particular location, and only when he is finished does he finish his line – "element".

J acknowledges in his next line that it is easy to mix these up, and as he moves the transparency projector into position he adds "and a good way to distinguish / is to get a nice example going". He turns the projector on and, indeed, it is the Cayley diagram for D_4. It looks much like it did on Lecture 2. There is a small but important variation. Recall that the four nodes written in the internal cycle were the four reflections, named after the axis about which the reflection occurs. These labels are still there, but next to the nodes for H, D and D', J has also written what these group elements are in terms of a rotation and V. So next to D he has written $R_{90}V$, next to H he has written $R_{180}V$, next to D' he has written $R_{270}V$. He says "we're going to see the distinction between these two" – he points first at the $Z(G)$ writing, then the $C(a)$ writing, then points to the transparency, saying "by doing an example on D four". In this *commenting about* scene, he makes explicit his advice "if you find two definitions that look really similar" – he waves his left hand back and forth – "and you're trying to work out how to distinguish them" – more waving back and forth – "having an example is a good way".

J invites the students to tell him elements that live in $Z(D_4)$. The first reply gets recorded immediately – the identity. The next offer is "all the rotations" which J echoes exactly, then says "ok", and begins walking over to the Cayley diagram (I suspect within two steps the student's heart sank because if they were right J would have probably recorded it instantly and not bothered to check or confirm it. I have no evidence for this.) J takes an example of a rotation "let's take R_{90}" – he holds it on the Cayley diagram – "does R ninety commute with everything". He confirms that R_{90} is the red action, performs red followed by blue starting at the top node (R_0), then performs the other order, and arrives at a different location. Now a student offers the only other element of the centre: "R_{180}".

After a few comments listing the various ways they now know of arriving at $Z(D_4)$ – "off a Cayley diagram", "off a Cayley table", "geometrically" (he manipulates an imaginary square) – he begins his look at the centralizer of the element V in D_4. He writes '$C(V)$ = ', and while he says "so that's everything that commutes" – higher pitch for the next words – "just with V" – he opens up a set parenthesis, and lists a first element of $C(V)$, which is R_0. He quickly sweeps the $Z(D_4)$ equations he has just written, saying "not with everything else"; spotlighting here an equation to serve as a contrast to the present equation. He immediately repeats in exactly the same higher pitch tone "just with V" and he holds his palm up in a *stop* or *no-more* gesture. He turns back to the board and also writes the term R_{180}, saying "we'd better put these two elements in" – he sweeps the writing in the line above again back and forth a few times – "these are meant to commute with everything / so they certainly commute with V" – on "certainly" he gives a confident push downwards with his open hand.

J invites the students to name other elements of the group that commute with V. Peter names "V" and J enthusiastically repeats it and records it. Peter next offers "all the flippy things" and J says "all the flippy things / not all the flippy things / just one more flippy thing". Soon someone says "H", which J agrees with, though he writes $R_{180}V$ for a reason he explains later (that it is more obvious when written this way that $C(V)$ is closed under multiplication). In the lectures J will accept an informal reference to a referent that has been precisely indicated, as we see here. In fact he is happy here to join in with the short-lived lexical improvisation. However, he will almost always correct an ambiguous or imprecise reference (of course here he corrects a mistake, but he does not correct "flippy thing" to "reflection").

A useful contrasting moment occurs many lectures later, in Lecture 22, while J is solving an exercise: they must show that two groups, $D_3 \times D_4$ and $D_{12} \times Z_2$, are not isomorphic. They will do this by following the advice J gave earlier (which I discussed in Chapter 5): by showing that each of these groups have a different number of elements of order 12. J has just written 'the n rotations of D_n (forming a subgroup isomorphic to Z_n)' and breaks off his writing and steps back a few steps, beginning a *commenting about* scene with the phrase "notice in passing how / we've sort of been talking about this stuff all along / but it's only now" – he takes a step to the board and with his finger sweeps the writing he has just done – "that we have the language- / only since we did isomorphisms" – he moves closer to point at the phrase in brackets – "that we could rattle off a phrase like" – and then he proceeds to read exactly what he has just written. He scrunches up his face, puts on a pained expression, saying "until then we had to sort of use these" – he holds his hand up and swivels it back and forth – "weasel words about how it looked like or behaved similarly to". His comment done he goes immediately back to writing.

The swivelling gesture simultaneous to the phrase "weasel words" creates a powerful expression of slipperiness, mutability; by contrast, every lecture contains many dozens of examples of gestures of specificity and precision – every single time he touches a particular term or symbol on the board, even when walking from many feet away, he arrives with an accuracy of the tip of his finger. J can respond to Peter's "flippy things" without skipping a beat because he knows exactly what Peter meant; vague language like "behaves similarly to", though useful at the time, can now from their present superior vantage point be safely disparaged.

Also important in this scene, occurring multiple times in the course, is a *comment* by J that such and such that they are presently concerned with in their present mathematical situation is in fact a such and such that they have encountered before, either before in this very course, or before in his students' education, but under another name, or in some other fashion occurring implicitly or disguised. Here this feature, comment as *revelation* or *recognition*, is subsumed within a comment where we also add a value judgment, that what has been recognized from before is now being written about and spoken about in mathematically precise terms, and with terms that are applicable to all sorts of other such situations as well.

But sometimes a recognition comment appears without such a reference to an improvement in precision. When J defines quotient groups (he calls them factor groups) one of the examples he goes through is the real numbers modulo 2 *pi*. In doing so he reveals that when his students had been learning about trigonometry, they had been talking about quotient groups without knowing it. There was nothing vague or incorrect about their previous work with, for example, evaluating the sine function. J's comments there do not include any devaluing gestures or phrases like "weasel words". Nevertheless, there is still an expressed favoring of the current manner of regarding the mathematical situation: "really" the students were working with quotient groups "and didn't know it". A veil has been lifted, multiple apparently different kinds of mathematics have been unified or seen to be manifestations of the same concept, and this more enveloping viewpoint has been acquired in this course.

So far the examples in this section have consisted of situations involving subgroups of D_4, or D_3, or D_n. In the following final example it is D_4 that is the subgroup of a larger group: S_4. The episode occurs in Lecture 13, shortly after J has defined, in short succession, what a permutation is, what a permutation group is, what the symmetric group is, and then performed a short example of multiplying two permutations. Here too J is about to reveal that, although they did not talk about it this way before, and although they were not consciously aware at the time (these are my words), they have in fact looked at permutation groups already. J is at his desk, rifling through various pages of his notes trying to find the next page that he needs, and starts up after a silence with a surprised tone: "now we've actually got / a very natural example of a permutation group" – he times this with a remarkable rabbit-out-of-a-hat move where he lifts the old cardboard square that has not been seen for some time, holds it aloft with one hand and puts his other hand behind his back in a quite formal stance (a rare pose) – “to look at / our friend the square”.

He reminds them that D_4 is the group of symmetries of this object, and he now points to various corners, saying “and if we just think about where- / we’ve got four numbered vertices”. He rotates the square, and flips it around a vertical axis, in an offhand manner (as if he could have selected any transformation to make his argument here, shaking his head a little, slight shrug of shoulder) and notes that “we can just say well / where do the four vertices map to” – holding his other hand palm up in a well-obviously gesture. He has embarked on a *regarding as* scene, where D_4 will be considered, in his hands, spoken words, and writing, as a group of permutations of the numbers 1 to 4. Attention is oriented towards the 4 numbers at the vertices and how a transformation permutes them, and the geometric viewpoint of the numbers as standing for the corners of the square is no longer foregrounded. He reminds them that the order of D_4 is 8, and he walks over to touch the writing on the board that states that the order of S_n is n!, so the order of S_4 is 24. He sums up “so D_4 is a subgroup of S_4” and turns to the board to begin writing.

I said “attention is oriented” and I want to pinpoint as precisely as I can when this happens and how I think I can tell that it is happening. J reaches a point where he has written R_{90} and V on the board and he wants to write each of these “in this permutation language”. He has already redrawn the square on the board (the only other time he does so is in Lecture 2, as we saw in section 6.2). The permutation notation he is currently using is a two-row array, where the first row is the numbers 1, 2, 3, 4 and the second row will be these 4 numbers permuted. He fills in the top rows. He begins with V. He holds his right hand directly in front of his drawing of the square, palm facing the square, and as he says “V is vertical reflection” he flips his hand over the vertical axis so his hand is flat against the board but palm facing him, and then undoes this flip. Then he touches the two top numbers in the corners, saying “so one and two get switched” and he fills in ‘2 1’ in the first two slots of the second row of V. Then he uses his index and pinky to point just in front of the 3 and 4 of his square, flips his hand over and back again, saying “and three and four get switched” and he writes these in too. Notably, he does not manipulate the physical square to determine where the numbers go, even though he held the square just moments ago.

Now he turns to R_{90} and he puts his hand on his diagram of the square, and carefully rotates his hand counterclockwise by ninety degrees. His voice slowing, he touches the 2 and then touches the 1, saying "so two goes to one", writing a 1 under the 2 in his permutation. Also noteworthy, I think, is the touching of a number followed by the number it is going to; different from rotating a physical square and copying down what the whole square now looks like. This individual number touching continues for the rest of the permutation. He touches 1 and then 3, says "one goes to three", writes 3 under the 1; touches 3 and then 4, says "three goes to four", writes 4 under the 3; touches 4 and then 2, says in a finishing tone "and four goes to two", writes 2 under the 4. The flipping and rotating gestures in the air above the diagram, and the tightly co-ordinated rhythm of touching saying and writing a source number followed by its target number, together manifest as an intricate little *regarding as* scene showing the generators of D_4 to be permutations on four symbols or letters (here 1 2 3 and 4).

J's observation that the order of S_4 is larger than the order of D_4 might remind us of the moment in Lecture 2 when J invites his students to come up with a convincing argument that the order of D_4 is not larger than 8. Recall that we left them in the moment when J has given his argument and conducts for a second time a vote to determine which students are one hundred percent convinced. "Who votes for a hundred percent now" he says, putting his own hand up, and then after a couple of seconds, encouraging more votes by using that hand in a scurrying upwards gesture, nodding eagerly. "Not bad" he says at last; then his face changes into a smile and he says in a surprised voice "someone's not convinced? / anyone not convinced?". Peter speaks up: "that only works because they're fixed in relation to each other".

It takes a few lines for J to figure out what the problem is: first he acknowledges that maybe "I skipped something verbally in my description" – J touches a hand to adjust his hair in a very rare gesture that I speculate is a rare occasion of nervousness or embarrassment of some kind – "there's only one- well have I" – after he self-interrupts he puts his right index finger on his lips and looks away from the class parallel to the board. This gaze direction – not at the board, not at the class, not at his notes, not at some object he is holding, but to the middle distance somewhere along a channel parallel to the board and the class – is rare, and occurs at moments when J is considering something he had not expected. J holds the square and moves it a bit, continues to demur – “I don’t buy that” – then suddenly his vocal inflection changes entirely: “ohhhh”.

He leans over the square and his right index finger begins to move towards a corner and touches it – “I see” J declares confidently. He has been holding corner number 1 of the square with his left hand the whole time; he moves his right index finger over the other 3 vertices touching each one in a circle, then rotating his finger above these corners three times – “if I were to move the two the three and the four around amongst themselves / that could create more”. Somewhere in here Peter makes a sound of agreement as of someone who feels he has been heard. J says “I agree with you” and he gestures his hand towards Peter – “there’s something about saying / ok now I’m with you” – he gestures again towards Peter, same hand but now it has the square in it.

As an observer who has also taught this course, out of many many moments that I admired J’s lecturing, these moments here are a series that I deeply respect. Not only did he have the presence of mind to hear and then act out precisely, in seconds, even when skeptical, Peter’s objection to his argument (an argument that seconds earlier J was so sure was completely convincing he took a vote he was visibly surprised to find was not unanimous); he is now able to chisel a little jewel out of Peter’s observation: “once we’ve fixed position- / the position of number one / and we’ve fixed the orientation / we have fixed the positions of two three and four”. Peter says “yeah”, J quickly points at Peter and says “ok good I like that”.

No one uses the word permutation here, and no one out loud recollects this moment in Lecture 13 when S_4 is observed to have three times as many elements as D_4 does. It is exactly these words from J, prompted by Peter’s withholding of being convinced, that nail down why no other elements of S_4 are symmetries of the square. The connection was apparent to me because of what J did when he segued from a stance of finger on lips pondering, and a defiant stance saying “I don’t buy that” to a sudden and emotionally resonant “ohhhh” of the penny dropping: what he did was he moved his right index finger from the 2 to the 3 to the 4, circling his finger around these labelled vertices, visibly and obviously permuting these numbers.

Not only do I admire how J rose to this occasion, I admire Peter’s actions as well. He refused the call to agree with the authority figure who had just announced “done” after concluding their argument; to Peter they were not done. He did not hide his state of unconvincedness from the others but stated it publicly. Finally, he shaped his contribution into the form of addressing the gap that he saw as being ignored, or not addressed, or not considered – "that only works because they're fixed in relation to each other". A more mathematically sophisticated or experienced Peter might have used particular nouns instead of "that" (J's argument) and "they're" (the vertices). But I suggest it is hard to beat the pithiness and precision of "fixed in relation to each other".

This scene is a joint creation of J and Peter. I understand that J can write on the board and Peter cannot, and that J gets to say orders of magnitude more words than Peter can, and most importantly, J can touch any piece of writing on the board or transparency that he wants to and Peter cannot. These are stark differences between J and his students in this course. And yet the power of interjection and interruption is not to be underestimated. Peter does not have to get up and physically force J's hands to move a certain way; he can simply refuse to be convinced and name as best he can the lack. By this limited means, he can influence J to move his hands within a mathematical context in ways that are unexpected to J, and which are in fact witnessably and evidently new to J. It is possible that J's gestural practices concerning this aspect of D_4 will never be the same as they were before Peter refused to put his hand up and say that he was convinced when he was not.

This incident helped me grasp that though my camera was trained on J, I was in reality analyzing a person deeply enmeshed in an interaction network that was social through and through. Social in the sense I am drawing attention to here – the powerful influence radiating from the students, sharpening almost into shared control or shared dominance in a scene like this one; but also in the sense that runs through every single example I have analyzed in this thesis – the powerful influence radiating from other mathematicians, who gestured and argued with each other and with themselves, seeking to convince, and who when they occasionally felt that they had succeeded, wrote their arguments down. People like J re-inhabit and re-create, in their own person, inevitably making it new, the movements and touchings of these predecessors, moment by moment publicly re-convincing themselves in front of their skeptical interjecting successors.

6.6. D_4: spontaneous and unplanned

In this section I analyze a few examples where J interacts with the D_4 group in contexts marked by spontaneity and the unplanned.

6.6.1. Spontaneously drawing a diagram: spotlighting temporary writing to enable regarding as

In Lecture 19 J has written the statement of a theorem on the board and has asked the students to read the proof of it in Gallian and to submit their own rewritten proof for their assignment: "re-present it in your own way / hopefully improving it or making it easier to digest".

The theorem statement is that if G is a group whose order is $2p$, where p is an odd prime, then G is either isomorphic to Z_{2p} or isomorphic to D_p. He then writes the start of a Corollary: '$S_3 \approx$ ' and then leaving the right hand side blank he invites the students to tell him what to write next. There is a single slot which needs to be filled correctly with a single item: the entire sequence that follows constitutes a *deducing that* gestural practice. In order for their deduction to succeed, some other gestural practices contribute along the way, as we will see.

He gets the answer "D_3" and he writes this, repeats it and asks "because?"; a student gives the right response "nonabelian". Holding his right index finger underneath S_3, J reminds them what this group is, and that it has six elements – "six is twice three / three is an odd prime" – and now he looks back at the theorem above it and traces his finger underneath the two options in turn. He holds his finger underneath Z_{2p} – "this is abelian and it's cyclic" – he moves his finger underneath D_3 – "this is neither abelian nor cyclic-" – suddenly he *self-interrupts*, and jolts his finger underneath S_3 instead – "ah sorry this / is neither abelian nor cyclic".

So far we have a fairly routine *regarding as* gestural practice. He is holding terms in a theorem that depend on a *variable* (p is an odd prime) and interpreting them in a current situation where the variable has been *specified* (now p is 3). In addition, he is holding terms that are currently of interest and saying what it is about them that is relevant or necessary *at this moment*: first, that because the group S_3 has order six it satisfies the hypothesis of the theorem; second, viewing it from the vantage point of whether or not it is commutative or whether or not it is generated by powers of a single element.

There was also a *correcting self* occasion. His *deducing that* gestural practice went momentarily awry: he touches D_3 too soon. He correctly touched the two options in the theorem first, thus establishing there are only two possible choices for what S_3 is isomorphic to. He correctly touched Z_{2p} next, regarding it as an abelian and cyclic group. He *ought* to touch S_3 next, regard it as neither abelian nor cyclic, and thus rule out one of the two options (the Z_6 option). *Then* he ought to touch D_3, the only remaining option. Instead he touched D_3 one move early. J instantly catches this slip and fixes it. This occasion may serve as a good demonstration of the sort of tightly patterned and ordered sequence of spotlightings which are so characteristic of the *deducing that* gestural practice. The order of touches matters.

Now J goes a little further. He has kept his gaze on the class for a few seconds of silence after finishing the above, and he suddenly turns back to the board and begins drawing a diagram. This diagram lives on the board for just under twelve seconds. He draws a small-scale fast sketch of the Cayley diagram for D_3 in two seconds. No little circles for the nodes here, and certainly no labels telling us which node stands for which element of D_3. All he does is draw the lines, and where they join at corners; this is where the nodes must be. He sweeps his marker just once over it and says “so that’s not cyclic”. Then in a fluid motion J moves his pen to a position just beside this Cayley diagram saying “cause cyclic would be in a big ring with six things” – here he just draws an oval shape in half a second and sweeps his marker over it once. This oval shape is meant to stand for the Cayley diagram for Z_6. It exists for less than a second. He grabs an eraser and erases all this *temporary writing*.

One of the features of gestural practices emphasized by Streeck, and confirmed again here, is not only the spontaneity and freedom that I have noted before, but also an ease with which boundaries between ostensibly ‘different’ behavioural choices are erased. J can at any moment advantageously hybridize the options and negotiate a method to quickly capture what is needed, or what is best, from each of the options. J could easily have swung his marker around in the air to indicate the Cayley diagram for Z_6, perhaps including six little stops in the vertical circle to quickly suggest the nodes. This would have been an instance of *manipulating the object*, in this case, a pretend object. However, J cannot as easily draw in the air the Cayley diagram for D_3. It is not impossible to so, but it would be harder. Somewhere between these two sorts of Cayley diagrams some sort of line is crossed: drawing the second diagram in the air would overly burden, perhaps, the short-term visual memory.

J opts for a lightning-fast sketch on the board. This rapidly constructed textual object marks enough of the features of the mathematical situation to allow him to spotlight the feature that is in contrast between the two groups: one is cyclic and abelian, the other is neither. He performed the temporary writing to afford him the opportunity to compare the two diagrams *side-by-side*. The complex interface between pure depiction, in the service of *manipulating the object*, and diagramming, in the service of *looking at side-by-side*, is here reckoned with. J chooses the *looking at side-by-side* gestural practice and draws just what he needs in order to achieve his goals by means of that practice.

This is another occasion of J *trying it again*. It may be that J in that moment believed that there was something more he could add to the explanation he had just given using the gestural practice of *regarding as*. There was something about what he was privileging in his spoken words about how to view D_3 – it is not abelian, not cyclic – that he could literally point to visibly on the board if he were to quickly draw the Cayley diagrams, and he opted to do this. By far the bulk of J's gestural practices work with the writing that he has already done. Indeed, the bulk of this writing was planned in advance; J prepares the terrain for his spotlighting needs. But J will also, as we have seen, search for and find transparencies of old writing to project when he needs to spotlight them at some moment in the ongoing action. Similarly, as here, J will opportunistically write new things – diagrams, definitions of terms – on-the-fly so that he can spotlight them.

6.6.2. Thomas explains his counterexample; J eventually understands

In the last scene J drew a spur of the moment diagram. In the next scene a student, Thomas, surprises J with an observation that takes the class six stanzas to get to the bottom of.

It is a moment in Lecture 28 when J has stated a number of properties of homomorphisms. The common hypothesis to all the statements is 'suppose *phi* is a homomorphism from *G* to *G* bar, and let *H* \subgroup *G*'. Property 4 states that '*H* \normalsubgroup *G* ⇒ *phi*(*H*) \normalsubgroup *phi*(*G*)'.

After writing this result, J spotlighted the *phi*(*G*) term and *commented about* it that the students might have expected to see the term '*G* bar' here, but that in fact such a result was not true ("you can very easily write the wrong thing here"). J says he came up with a good example to show why this tempting-to-write result is not true, but that he is saving it for later.

Here, for clarity, is the statement of this *tempting but untrue result*: 'suppose *phi* is a homomorphism from *G* to *G* bar, and let *H* \subgroup *G*. Then *H* \normalsubgroup *G* ⇒ *phi*(*H*) \normalsubgroup *G* bar'.

Nearly three minutes later, just as J has finished writing the statement of the 7th property, J calls on Thomas who has his hand up, and Thomas says: "there's like a trivial counterexample for four". He is about to explain it but J interrupts because he does not know what Thomas is referring to. In a few seconds J is touching property 4, saying "this?" and Thomas clarifies "well yeah not- not as stated". J writes '*G* bar' above the *phi*(*G*) term and says "to having the *G* bar?", and Thomas agrees. In other words, Thomas believes he has a trivial counterexample to the *tempting but untrue result*.

Thomas begins to explain. It is interesting to me that after repeated viewings I now easily understand what Thomas is trying to say; I remember that when I watched this unfold in real life I could not at first get a hold of where he was going. Here is what he says (again, a rare long speech for a student in this course): "if you take just any subgroup that you know is not normal / and you treat that as *G* / you treat that as your whole group / it's obviously normal in its image / uh the whole group to itself is normal right". J has been standing still for all of this, scratching the back of his neck; on "right" he begins walking to his desk and puts his notes down (the notes he almost always is carrying as he writes on the board) and Thomas continues a little further "so *G*- *G* is normal in *G*". Putting the notes on his desk is the moment J has decided to pursue Thomas' offer of what the class will do next.

At this point J walks back to the board saying "ok let's be specific / so we start off with a subgroup that's not normal" and Thomas cuts in "not normal in some larger group that you're going to go to / that's gonna be your *G* bar eventually". J says "ok think of a subgroup that's not normal". In a few seconds he and the class have opted for D_3 as their group, with a subgroup {R_0, F} that they already saw before was not normal in D_3. I will discuss the scene where they learned this in Chapter 7.

J reconfirms with his two hands starting at the same place, and then heading off in different directions, that this subgroup is not normal (in the next chapter we will see this *scattering* gesture is a characteristic *non-normality* depictive gesture). J begins to write on the board somewhere, roughly near property 4. He is not writing in the space immediately after the last thing he has written, nor is he filling in some gap or space in writing he has already done, thus marking this writing as different than the ordinary writing of the course (ordinary writing either advances linearly, or fills in some deliberately nonlinear portions). This writing *floats* in a bubble of its own.

He writes '<*F*> \notnormalsubgroup D_3', and underneath D_3 he writes '*G* bar' saying "that's gonna be the *G* bar?" and Thomas confirms. Underneath '<*F*>' he writes '*phi*(*H*)' and confirms that in his words. Thomas adds more: "uh that's gonna be in fact your whole group as well / it's gonna be *phi* of *G*". J writes '<*F*> \normalsubgroup <*F*>', and says slowly "*F* is normal in *F* trivially"; Thomas says "yes".

J does not catch, or perhaps does not care, that he should have said "cyclic group generated by *F*"; there is a lot going on already that J is attending to. Thomas is leading him, or J is prompting Thomas to lead him, bit by bit through the assembling of the pieces required for the counterexample. In this scene J and Thomas are co-responsible for the writing and the mathematical argument that occurs. At this stage, we can distinguish Thomas and J by the fact that J can write no more than the next little statement, and his spotlighting is limited to touching a single term, saying exactly what is written, and achieving a bare bones *regarding as*: namely, regarding what has been written as the portion of the statement of property 4 that has been captured thus far by the growing counterexample. There is no confident change in the manner of how to view any of this writing, because J does not know how else to view any of this; there is no spotlighting two or three terms in a row to *deduce that* because J does not know what to deduce or how.

These observations approach tautologies the more one believes or trusts the analytical approach, and the empirical support for this approach, of this research. After all, when J reveals at any time in the course what he knows or understands, he does so via smooth, fluid, confident gesturing – well-timed and evocative depictions, precisely patterned and deliberately sequenced spotlightings of specific portions of existing writing, a nearly continuous flow of tightly synchronized gesticulations – interacting together, and therefore exemplifying, one or more of the six gestural practices I have described. J is hardly doing any of those things here. He is doing, as I said, a minimum level *regarding as*: first eliciting, and then writing, short pieces of mathematical text that Thomas agrees with, and then touching the terms to at least name, and label, what pieces of the statement of property 4 that they correspond to. This is the most that J can currently do. It could very well be that J will not succeed in anything significantly better than this. Perhaps J's public attempt to understand the counterexample will just collapse, and J will say something like "ok let's talk about that further after class".

An alternative interpretation would be to not consider J's gestural actions here as an example of *regarding as*, on the grounds that J is not able, yet, to *regard as a whole* the mathematical situation that Thomas can regard as a whole. It is clear from J's questions, and the speed with which Thomas can reply, that Thomas has a view, a perspective, on the mathematical situation at hand, which allows him to answer J while also giving short accompanying explanations. It is also clear from J's questions that before Thomas answers them, J does not know what the answer will be. According to this interpretation, this classroom incident is unfolding as if Thomas is looking at, and describing, a scene that J cannot see.

I consider this interpretation to be too sweeping. While it is true that J's actions – his choice of questions, his touches, his shrugs, his decisions as to what to write – reveal a person who does not know before Thomas speaks what the answer will be nor what he will himself write in a few seconds, there are aspects to J's gestural behaviour that reveal his role as an active and knowing co-creator of this mathematical counterexample. First, it was J who moved to an unused part of the board and announced the local proximate goal of the next few stanzas: to instantiate Thomas' counterexample in a specific context. Second, J visibly redisplays to the class a characteristic feature of a subgroup that is not normal, manipulating a pretend coset in the air with his hands. Third, J structures his questions by carving out, one by one, the elements of the hypothesis and conclusion of property 4; fourth, J writes in an organized succinct manner the answers he gets from Thomas: to accomplish these two tasks he touches terms ('*G*', '*H*', '*G* bar') while saying out loud and writing down what they correspond to in their specific context. J does not have the view that Thomas has, but he shows that he does have a global view of what it will take for this counterexample to be accepted as valid and he shows that does have a restricted local view of the developing counterexample that grows with each of Thomas' replies.

J looks at the statement of the theorem and says "and now *phi*" – this is the next ingredient that Thomas must have an answer for. J writes *phi* on the board, writes nothing more, not even the colon, and asks "now what's *phi*? *phi*" he says in a tone like there will be some words that will come out of his mouth soon, but nothing comes. Thomas cuts in on cue: "uh *phi* of anything just maps to what it is / except in D_3". J has already said "but" halfway through this, but Thomas was determined to finish and overrides him. J goes on "hang on if *G* is *H* / then *phi* of" – again he is speechless and his shoulders sag and his hands curl back to his body.

Other voices now emerge. Peter says "*phi* is the identity". This does not get picked up. J writes on the board ': $<F> \rightarrow$ ' and turns again to look at Thomas, who says "to D_3". J repeats it, but unlike the tone he uses when he repeats a student offering on all the other occasions when J is expectant, there is no authority in his tone, and he is not energized and bustling with movement. He listlessly copies 'D_3' and then his hands fall to his sides. Now two students start talking simultaneously: it is the students who on this occasion are quicker to figure out what is going on with Thomas' example. Bart is trying to explain, Thomas is saying "yeah" to what another student has said; but J is shrugging his shoulders: "I guess". Even a non-mathematical bystander would say that his heart was not in it.

Bart now repeats what Peter had said earlier, only louder and with a bit more insistence "just use the identity map" (after all, J has yet to write down what *phi* does to elements of $<F>$, so the definition of his map is not yet complete). J looks at Bart for a few seconds, Peter tries to say that he agrees and that this is what he had said, J steps to the board and says "so *phi* is the identity / so *phi* of *x* is *x*". He writes this clearly and quickly on the board and his tone is the most assured it has been since he started acting as Thomas' scribe. Bart is convinced: "and that works / that's the counterexample".

J says “so H is normal in G / but *phi* of H” – he touches <F> – “which is F” – short pause – “is not normal in” – he touches the G bar term but does not say out loud “D_3”. This is only two touches. It is a restrained and rather inhibited example of *deducing that*. Touching G bar but not saying out loud “D_3”; not immediately attempting again the sequence of touches in order to find a faster, better, cleaner way of spotlighting-and-saying; these are indications that his hold on the argument is not yet mature and complete. Nevertheless, this is the moment that marks the dawning of his understanding. He turns to look at Thomas with a not-bad-at-all facial expression – corners of the mouth turned down, nodding, eyebrows up – a picture of surprised approval, his head and upper body leaning back away from Thomas.

Peter breaks the silence with “I don’t know how trivial that was” and the class laughs. J folds his arms and takes two steps back from the board, looking at what he has written. J makes a joke as well: “is that what you were doing when I was trying to teach number 7”. He puts his right hand under his chin (the Thinker pose), takes a step back to the board, and says “let’s see”. Now we will see J enter a *consolidation* phase, a period where he visibly, over a period of about a minute, tightens, organizes, and adapts both the writing and his gesturing and succeeds in more convincingly persuading himself and the class of the truth of this counterexample. He makes more explicit and evident the inescapability of the logic.

First he writes the words ‘*phi* identity’ underneath the map he wrote earlier and he says in a quiet tone that contains a bit of wonder “so *phi* is the identity”. Then he turns to the class and says “you’ve sort of artificially” – on “artificially” he uses both hands and pulls them backwards and outwards from a smaller sphere to a much larger one – “inflated G bar”. Thomas immediately jumps in – “yes” – even while J is continuing to say “you’re just gonna put some dummy elements” – his shoulders shrug on “dummy” – “that you’re never gonna go to on purpose”.

When I watch this scene, I trace the moment of J’s *complete* understanding to the very instant he depicts a deliberate expansion with his hands. It is moments like this one that re-compel me to watch the hands of someone who is acting mathematically. That half-second motion of J’s hands jumping outwards crystallizes Thomas’ counterexample. It is difficult to believe that J would from this moment on his life ever be able to *forget* this counterexample. It seems to me that students in this class, witnessing such a radical act of compression, enjoy a remarkable and rare experience. I read into Thomas' quick and unhesitating "yes" that even he could admire how J had boiled down his own intuition or reasoning into what some mathematicians might call the *key idea*. The whole sentence is gold ("dummy", "on purpose"), but the nucleus to me, of that mysterious mathematical act, understanding, is the simultaneous "artificially inflated" and the hands widening on purpose.

6.7. Summary

In this chapter I have analyzed several scenes where J employs one or more gestural practices to interpret and explain aspects of the dihedral group of order 8, D_4. He constructs the Cayley diagram of this group by *manipulating an object*, a square, and visibly performing all of its symmetries: this precedes his formal definition of the concept of a group. After this definition he *handles* a textual object, the Cayley table of D_4, and unpacks each part of the definition bit by bit by *regarding* the features of his table in particular ways using appropriate touches and movements of his fingers and hands. Later he *touches-computes* with both the Cayley diagram and the Cayley table, performing calculations of products of group elements. He *holds structure*, simultaneously touching multiple nodes on a Cayley diagram to regard what he is touching as a subgroup. Looking at a Cayley table and a Cayley diagram *side-by-side* he *comments about* which is useful in which circumstances. Some of J's depictive gestures, especially when *manipulating* a phantom object, as well as some of J's temporary writings, especially diagrams, prove useful when clarifying a situation he previously could not *regard* in the appropriate way.

Chapter 7.

Well-definedness: cosets, normal subgroups, quotient groups, and mappings

7.1. Introduction

The term “well-defined” appears often in mathematical texts. It can appear in contexts like the running narrative or in sections titled “remarks”; more commonly it appears in the portions of the text that involve the bulk of the logical and conceptual development of that text: in proofs or examples or solutions to exercises. Thus far, much the same could be said about a term like “group”.

But there is a difference between these terms. Suppose a person encounters the word “group” in a mathematical text and wants to know what it means. If the text is sufficiently introductory or elementary, the chances are very good that the word “group” appears in the index, along with plenty of subsidiary terms listed underneath it, and also that the word “group” is defined precisely somewhere in the text itself. If the text is not elementary then the person can go find an introductory book; or they can perform an internet search and find a definition very quickly. The definitions may look and read a little different from one another, but some good and worthwhile effort will convince that person that they all specify the same thing. Most terms in mathematical texts are like the term “group” in this respect.

I claim that the experience of a person who encounters the word “well-defined” in a mathematical text will be distinctly different. This is not a claim I can prove but I will try to make a plausible case. Suppose that person is reading the kind of text whose very goal is to capture as many of the important concepts and objects of mathematics as possible, and to define and explain them succinctly and correctly, while also treating the theorems and results they show up in. Let me take the example of the *Princeton Companion to Mathematics* (Gowers, Bower-Green, & Leader, 2008), which surely aims at something like what I just described, as well as many further aims. It is a remarkable book. One can open to virtually any random page and read well-chosen carefully authored treatments of all sorts of mathematics. The index and the table of contents is a who's who of concepts and objects and results.

Now let us imagine that our reader encounters the word "well-defined" in this text. It is easy to do: the word appears in multiple articles. The reader, let us say, has already had reason to turn to the index to learn or remind themselves of what a tangent space is, what the Yang-Baxter equations are, what a minimal polynomial is. There, in the index, they found these among hundreds and hundreds of such entries. So they look for the word "well-defined" in the index. But it is not there. They look under "functions" and "mapping"; no luck. Isn't this a bit odd?

A rejoinder might be: ok, but maybe the occasions when "well-defined" is used are informal ones; maybe this is not really a mathematical notion and so it is understandable that this term is not in the index. But a look through the actual occasions of its usage reveals that the term "well-defined" is bang in the middle of the heart of the mathematics being written about: angles between intersecting curves on an abstract Riemannian surface; free groups and group presentations; theory of distributions; trace class operators, and so on. Again and again there pops up this word which the reader must understand if they are to make sense of what is going on.

So the reader does what most of us do most of the time when confronting words not in our personal lexicon, which is to work out, from a sufficient number of examples of the use of that word in context, what this word must mean. By way of contrast most students do not figure out the meaning of the word "group" by teasing it out from the sentences it is involved in, nor do their teachers present the subject in this way (it is probable that mathematicians who are trying to answer research questions of their own do in fact sometimes read proofs with a view to see what some object is actually doing regardless of its present definition, perhaps to come up with a new widened or weakened or altered definition). The term "well-defined" sits in an extraordinary class of mathematical terms which mathematicians use in sentences they are utterly confident in the truth of, and which by contrast with the far larger ordinary class of mathematical terms, is not so easily traced to a precise definition.

Let me back off a little so that the claim is not understood too broadly. The main editor, Tim Gowers, also authored the seventy-six page introduction, whose sections are titled “What is Mathematics About?”, “The Language and Grammar of Mathematics”, “Some Fundamental Mathematical Definitions”, and “The General Goals of Mathematical Research”. Gowers has made a choice in his introduction to explain the idea of well-definedness without using this term explicitly. On page 25 he has just noted that two expressions for the same rational number, *a/b* and *c/d,* are “genuinely different, but we think of them as denoting the same object”. He then observes: “If we do this, we must be careful whenever we define functions and binary operations”. Then he does an example where he says “suppose we tried to define a binary operation” on the set of rational numbers by the “natural-looking formula” $a/b + c/d = (a + b)/(c + d)$. Now he points out that this definition has “a very serious flaw”. He shows that 1/2 + 1/3 doesn't equal 2/4 + 1/3 under this addition. His conclusion is that “although the formula defines a perfectly good binary operation on the set of *expressions* of the form *a/b*, it does not make any sense as a binary operation on the set of *rational numbers*”.

How was Gowers able to explain so much about what the term “well-defined” means without using it? The way Gowers has set up this example, he can use the phrase “binary operation” every time in an entirely legitimate fashion. For example, the first time he uses it, he is clear that they are *trying to define* a binary operation. They are attempting something, and then he shows right away that the attempt fails. Another way to write the example would have been to start out by saying “define the binary operation on *Q* by”. Then he might have noticed that they had better check that if they choose two different expressions denoting the same rational number that the binary operation will give the same output both times: he could say “we had better check that this binary operation is well-defined”. If Gowers did this, he would be temporarily abusing the language: he would be using the phrase “binary operation” before he was even sure that what he had defined *actually was* a binary operation. So Gowers, in this context, opted for the language of “tried to define” and then observing that the attempt fails.

However, in many instances of the term “well-defined” in mathematics texts, the attempted definition actually succeeds. Knowing this, the authors of these texts don't bother to write (I paraphrase) “let’s try to define” followed by “oh good the attempt succeeded”. Instead, the dominant social convention among mathematicians at present is to define the function or binary operation first, even though there is some checking that must be done. When they perform the checking, they call this “checking that the mapping is well-defined”.

So another peculiarity of the term “well-defined” is that a rewriting of the local contexts it appears in can often eliminate its usage. Instead of “we define” followed by “it is well-defined” (assert first, abusing the language for a short period, justify immediately afterwards), one can justify first and then define after. It is interesting that logically it seems that the term is not necessary, but in practice it appears to be so useful that it is a universally understood term among the mathematical community.

Such a choice to temporarily commit an abuse of language is not restricted only to the term “well-defined”. Two sorts of analogous occasions occur in the course (as they do more widely in mathematics practice). First, J temporarily refers to some objects as “subgroups” before he has done the checking that they indeed are “subgroups” (he explicitly comments on this and warns students to look out for this in the mathematics writing that they read). Second, J consistently examines the notation *G/H* whenever he encounters it to check that *H* is normal (so that the quotient group, which has implicitly been announced as such, can actually exist).

Why did Gowers avoid the use of the term “well-defined” in his introduction to the Companion? I don’t know. What I can say is that he wrote a long and detailed blog post on exactly the topic of “well-definedness” (Gowers, 2009). He even treats as one of his examples exactly the same proposed binary operation on the set of rational numbers. But there are key differences. In an attempt to make clear the current epistemological status of the alleged function in question, he suggests that they might call it a “gunction”. He includes, with the luxury of space on his blog, plenty of discussion surrounding the specific features of mathematical occasions when he, or the students he is lecturing (for whom he initially wrote the post), must check if some mapping is well-defined.

Binary operations on a set *A* are really just functions from the Cartesian product of *A* with itself to *A*. So “well-defined” with reference to binary operations is really a special case of “well-defined” with reference to functions (or synonymously, mappings). There are two ways that a proposed function *f* can fail to be well-defined.

First, as above: that *f* applied to some element of the domain gives *two* (or more) elements in the codomain. When this occurs what is being defined is termed a relation (an older term that still survives in informal text or speech is “multifunction”). Many occasions of the use of “well-defined” could be eliminated by saying “define the relation” followed by the checking that shows “in fact our relation is a function”. I say “could be”; in practice I do not remember a single occasion that I have encountered it.

The second way that a proposed function *f* can fail to be well-defined is as follows: that *f* applied to some element of the domain gives *no* element of the codomain. An informal term for such a near-miss to being a function, not universally accepted at present, is “partial function”.

Both problems occur in the course, but it is the first one that by far predominates. The reason for this is that it is natural in a group theory course to consider the set of cosets of a subgroup of a group. It is then natural to hope to define a binary operation on the set of cosets. The condition required to do so (the condition required to make the binary operation well-defined) is precisely the condition for a subgroup to be normal. One of the central advances that Galois made to mathematics was to recognize the importance of this definition and to state it clearly.

This chapter centres on J’s movements of his body and his hands, in local contexts, interacting with writing on the board and the students in the class, within occasions when the word “well-defined” gets spoken out loud. I made a list of such occasions (84), reviewed multiple times the episodes they occurred in, weeded out occasions of lesser interest, and selected occasions to analyze thoroughly that captured phenomena that occurred also in the occasions I did not select. The mathematical concepts of normal subgroup, coset, quotient group and mapping emerge in full vitality from J’s mathematical action inside of episodes involving the six kinds of gestural practices introduced in Chapter 5.

7.2. “you can’t go on until you’ve shown that the definition is clean”

It is Lecture 11 and J is midway through proving what Gallian refers to as the Fundamental Theorem of Cyclic Groups. This theorem gives a complete description of all the subgroups of a cyclic group of order *n*. Having begun the proof by saying that the result follows from three Claims which he states up front, he finishes proving Claim 1. He says “ok so far so good” and walks back to the board to write ‘Proof of Claim 2’ while commenting “this is the interesting one”. Claim 2 states that each subgroup *H* of $<a>$

(the cyclic group generated by a single element *a*) is itself a cyclic group, and can be written in the form $<a^m>$, where *m* is some positive integer. Walking away from the board he continues to remark that "this is the one non-obvious bit of the theorem".

As so often at times like these in the course, he explicitly contrasts the interesting and non-obvious part of a proof from the part that is, in his words, mechanical or trivial or obvious or straightforward. Stepping towards the board he sweeps his right hand holding the marker over the region of the board containing his proof of Claim 1 saying "I mean this you can see your way through" – he sweeps his hand over the statement of Claim 1 – "once you've got the claim". Then he stops, looks at the class, points at the place where his proof of Claim 2 will go and continues "but this one" – his tone now a question, delivered with high pitch – "what is the right *m*?". J walks to where Claim 2 sits on the board and he reads it all aloud in a questioning tone while jabbing his marker over the text; then he does a puzzled face and says "hmmm". Telling them that they will "explicitly" (palm-up gesture with his right hand, shaking it for emphasis) find such an *m*, he turns his eyes to his notes and ends the stanza by walking to where he will begin the proof of Claim 2, saying "this part I would say is not obvious", his right hand pushing with flat palm twice in front of him in a gesture of *caution*.

In this *commenting about* sequence he has highlighted this portion of the proof as more difficult, foregrounding it as a place where they will need to work and think harder, forewarning the students that a more significant, less already-telegraphed mathematical action will be required of them in the moments to come.

In the next stanza he writes and reads the first sentence of the proof of claim 2. He pauses after reading the written clause 'and let *m* be', interleaves the single line comment "no I don't think it's obvious either", and continues reading while writing the definition of this integer: "let *m* be the smallest positive integer for which *a* to the *m* lies in *H*". The definitional sentence now completed he backs away a few steps from the board and stares at the class for a few seconds in silence, looking the students over: "ok", he says finally, "who wants to hit the alarm bell before I go on".

Something needs to happen here. Something needs to be done. He waits for the students to provide for him what he is expecting. This is the very first occasion of the course where this specific kind of expectation has occurred, so the form and shape of this expectation is of a different nature than those where the acts and responses of previous occasions have already helped mold the character of the current situation. His look of expectancy, his unmistakable attitude of not continuing to write unless and until students provide for him the mathematical action of the next sentence, is not then an action from within the social occasions of the course, but is rather an action from within the social occasions of mathematical occasions more generally, perhaps earlier courses where students may have encountered such definitions before, but certainly from mathematical occasions that are encountered with regularity in their futures and in J's mathematical past.

The first offer, from Peter, is whether or not they need to write down some words that justify or confirm that every element of a cyclic group generated by *a* is of the form *a* to the *m*. J shrugs, says "sure" in a careless tone, raises his hands in a gesture that indicates that nothing of any importance really attaches to the suggested action, and then goes on with a few lines which, though they do not correct a mathematical error, serve to correct and adjust the future behaviour or set of responses of his students: "you're right / but I'm thinking at this stage / we already know that that's the case". Such a justification is no longer required at this point in the course, and putting his right hand flat and pushing down, he concludes this phrase on a downward tone. Taking a step to the board again and pointing his right index finger to the definitional sentence he returns to his question to end this stanza "but there's something else I need to say before I go on?". What the student had suggested was not the problem.

The next offer from a student, John, states the problem exactly: "you're using well-ordering but you need to show it's non-empty". In other words, the sentence defines *m* as the smallest positive integer that satisfies some condition, but they need to establish that there indeed are some positive integers that satisfy the condition (otherwise their definition of *m* will amount to defining it as the smallest positive integer belonging to the empty set). As the student finishes J smiles. J is a positive teacher, whose general disposition is upbeat and energetic, who is not sour or frowny or moody. Still, even against this steadily warm backdrop, the smile registers as warmer still, a moment of gratitude and respect; "very good" he says, and then rewords the offer: "how do we know that's well-defined / suppose there are no positive integers for which *a* to the *m* is in *H*".

J is now right at the board, first underlining the containment relation with his right hand, then finally holding his hand on the board underneath it as he finishes. His words while holding the containment relation indicate the manner in which he, and the student who spoke, are *regarding* this relation: *as* an *implicit* question asking whether or not there are in fact *any* such powers of the element *a* that indeed sit in this subgroup. The relation sits inside a larger sentence, and it is important at this moment to see this relation as one that requires existential justification; otherwise, the definition will fall apart and will not serve to specify a number at all. J walks away from the board, amplifying his comments: “we’ll see a few times in this course / where we get stuck on / oh this is well-defined / this definition makes sense because / and you can’t go on until you’ve shown that the definition is clean”. Indeed, J had stopped in his tracks when he had finished writing the definitional sentence in question, his attitude performing “can’t go on until”.

Now J returns to the board, holds the relation again, saying “we’ve gotta exhibit a positive integer for which *a* to the *m* is in *H*” (Figure 9).

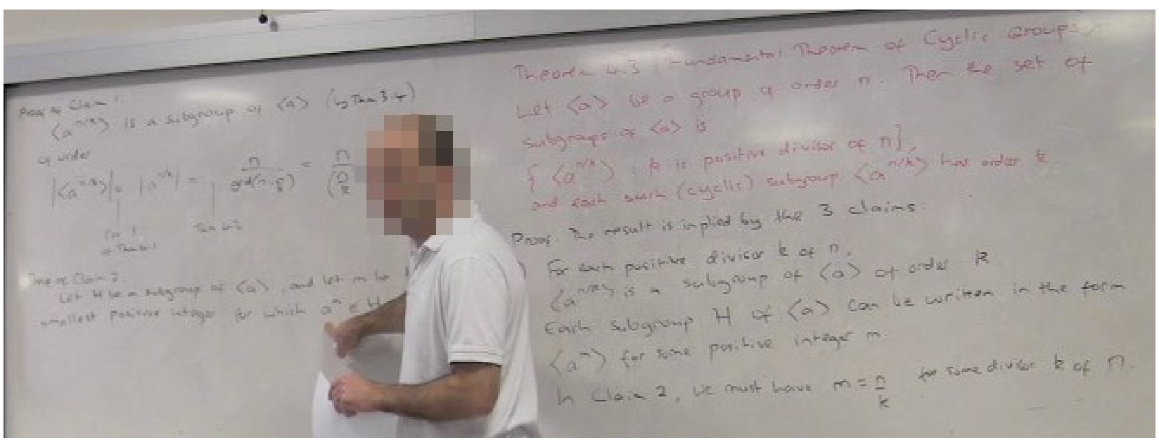


Figure 9. Well-definedness – first moment in the course.

In a few lines he has reworded this as a question: “*a* to the what power is definitely in *H*?”. The first student suggestion, from Peter, is “0” – *a* to the 0 is the identity, and the identity has often been the right answer to such “definitely” hints as to what elements are guaranteed to be in some group or subgroup. J steps towards the written word ‘positive’, touches it, and *corrects* the student “that’s not a positive integer”.

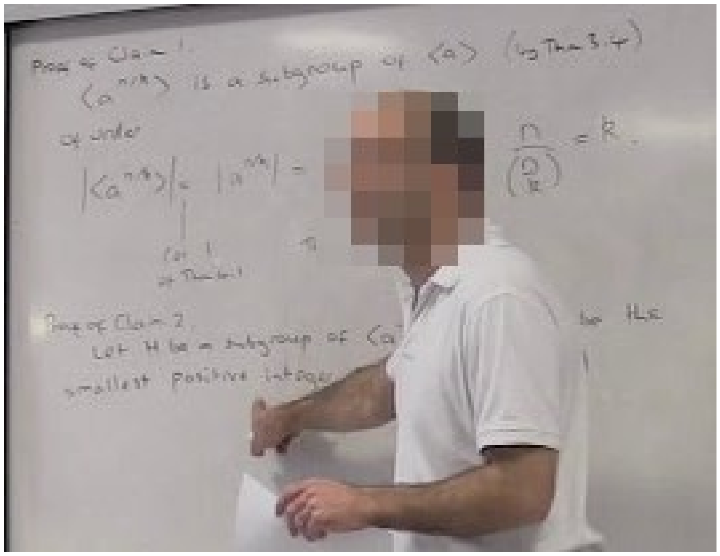
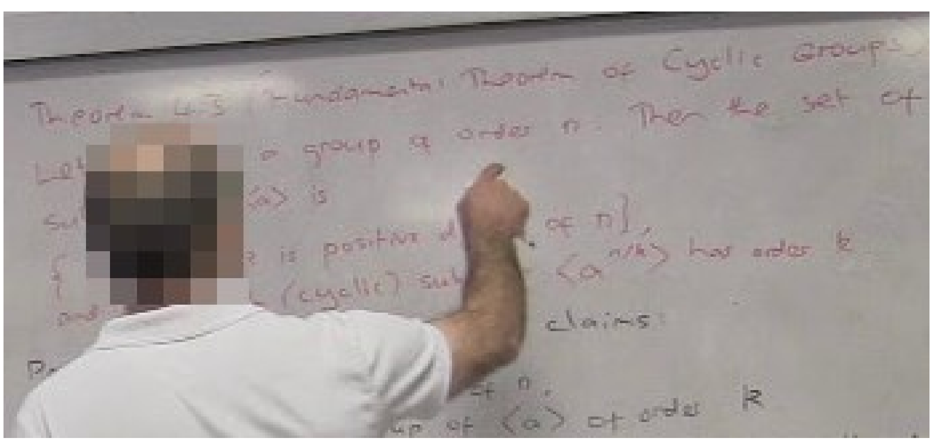


Figure 10. Correcting – not a positive integer.

While he is doing so another student, Thomas, gives the answer “*n*”, which is the order of the cyclic group generated by *a* (and therefore *a* to the *n* is the identity, which must be in the subgroup). J doesn’t react, so Thomas says more distinctly “*a* to the *n*”; J repeats these words emphatically, and he points to the ground on “*n*”. He pauses, and then turns to the board – “right because” – his right index finger is extended out and he hunts with it until he finds exactly where the symbol *n* appears in the hypothesis of the theorem.



Deducing that – the element *a* has order *n*.

When he succeeds, he underlines the phrase ‘has order *n*’ with his finger, and then underlines ‘*n*’ with his marker. Only now does he turn to face the class with a smile and concludes his spoken line: “*a* has order *n* / that means that *a* to the *n* is the identity”. He ends the stanza with a repetition of the necessity of writing this justification of the definition: “so we do need to say that”, and then he writes in parenthesis: ‘well-defined since $a^n = e \in H$’.

This is the first occasion that ‘well-defined’ appears on the board, the first occasion “well-defined” appears in his speech, and the first time mathematical actions involving well-definedness have transpired in the course. Apart from these firsts (and their consequences in how the action unfolds) there is one other respect in which the occasion is unusual when compared with the later ones: all the later ones occur when a mapping is defined and what must be checked is that the mapping is well-defined (in other words, actually defines a function). What this episode shares with the later ones is the stopping of an action in order to check something, and more specifically, a stopping of an action where the need to stop is easy to forget or miss. The *commenting* here combines evaluations of non-obviousness with warnings of easy-to-make mistakes as well as emphasis on the necessity to notice and write the required justification. J regards a relation, and regards the sentence it is in, in a manner that results in a *readiness to act* in a particular mathematical way: to justify that his proposed definition of a single object succeeds in its purpose and does select a single object.

7.3. “[well we defined it weirdly] ooh I have to get some candy”

Eight lectures later, J is midway through the proof of the Orbit-Stabilizer theorem, which states that for each element i of a set S, the order of a finite group G of permutations of S is equal to the product of the order of the orbit of this element (under the action of the group) and the order of the subgroup of G which stabilizes (leaves invariant) this element. He has written a claim and already shown how the claim would imply the conclusion of the theorem. The claim is that there exists a one-to-one correspondence T from the left cosets of the stabilizer (of i) subgroup to the elements of the orbit, and he says while walking away from the end of that writing “and now we gotta try and prove that”. As he erases the board he needs next, he comments, much as he did on the first well-definedness occasion, about the mathematical doings to come: “there’s only one thing that requires any thought in this proof”. He even adds later in this *commenting* stanza: “the entire proof is like turning the handle with one exception / there’s one place where we need to think”.

He writes 'To prove the claim, define *T*' and then walks back over to the claim, touching it word by word with the page of notes in his left hand as he reads along. Using his right index finger he touches the term '$stab_G(i)$' and asks himself and the class to regard the term from the point of view of the following question: "what is a typical element of a left coset of *stab G* of *i*?". Consideration of a typical element occurs frequently in the course, and any conventional choices of notating such typical elements tend to be systematically repeated. It is not different here: because permutations have often been denoted by Greek letters like *alpha* or *beta* or *phi*, he writes the mapping *T* as acting on "*phi stab G* of *i*". In more detail, what he actually does is write in the air in front of them a "typical left coset *a H*" where *H* was the standard letter denoting a subgroup, and *a* was a standard letter for an element of a group *G*, and so *aH* was their standard choice for left coset. So that now, as he writes $stab_G$ of *i*, he says "here's the *H* that's the *stab*". When he has finished writing what *T* is acting on, he emphasizes how this term is to be seen by writing a brace beneath it and titling it 'left coset'.

A similar procedure ensues in order to determine what *T* maps this element to. He walks back over to the claim, holds the term '$orb_G(i)$' saying with a mock-puzzled face to the class "which element do you think it maps it to?". He pantomimes the faux-difficulty of the question, and in seconds a student answers "*phi* of *i*" to which he gesticulates with his right hand a little twist near his ear indicating an of-course-how-simple attitude, adding "*phi* of *i* what else could it map it to". Walking back, he writes this down, then again puts braces underneath the term, titling it '$\in orb_G(i)$'. Both of these last achievements, the writing of the element that *T* acts on and the element *T* sends it to, are accomplished in what he calls a "no choice" manner from the statement of the claim itself. While these are not mathematical deductions, there is an inevitability to them: once you know the domain of the mapping, and the codomain, there is really only the one mapping one could write down, and the gestural practice in play is *deducing that*.

A few moments later, standing at the board and facing the class, he sweeps empty space below this mapping, saying "now you wanna leave three lines / for some technicality". Beneath this empty space he writes two headings '*T* is 1-1:' and '*T* is onto:'. Then he proves *T* is onto and fills in the work in the heading, including the white square which he will shade into a black square when he is finished the proof (normally he only draws this square when he has in fact arrived at the end of the proof, and he shades it black immediately). He announces: "so that will be the end of the proof / now what am I faffing about”, walking back to sweep up and down the empty space he left before with his marker (Figure 11): “what’s this missing proof- / missing part of the proof doing?”.

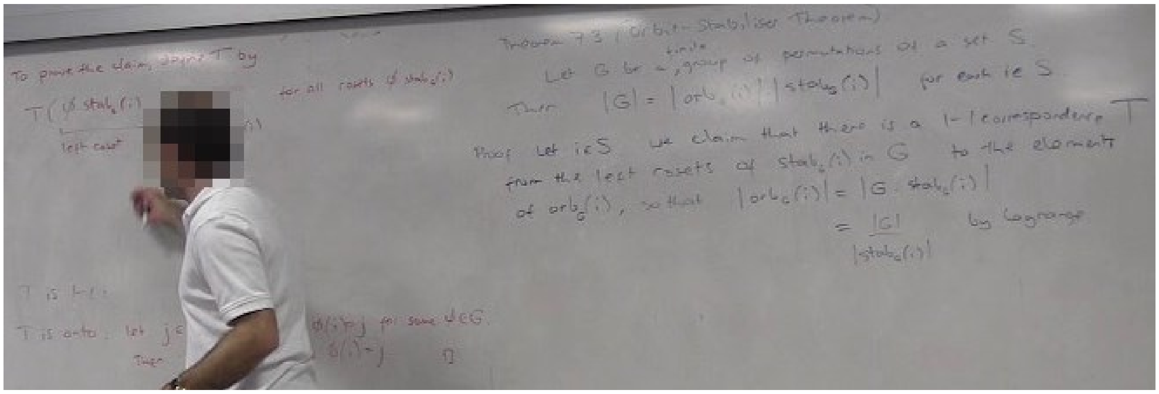


Figure 11. Manipulating a textual object – the missing part of a proof.

He backs away a few steps from the board, looks at the board again, and sweeps up and down in the air with his open right hand, saying: “why don’t we just show it’s one one and onto and go home?” (when J says “one one” this is one of the ways he says *one-to-one*, probably because the written notation ‘1-1’ can be read as “one one”). John jumps in to do the only bit of thinking required in this proof: “show that it’s like well-defined”.

J takes in a deep breath and says ‘ooh’. He gesticulates a strong fist emphasis striking down and says “I have to get some candy” – his whole demeanour alive with praise – “good excellent gotta show it’s well-defined”. He paces away from the board and pursues further, moving his right hand towards John to encourage him to continue “tell me more so what does that mean”. John does, and with some more questioning from J as to how it was that John realized that this proposed function might not actually have a unique output, John explains “well we defined it weirdly”, which J agrees is an excellent clue.

Now J takes the reins again. Touching the left coset that *T* is acting on and saying “we said well pick a coset” he then uses both hands in front of his body to cradle a sphere or a ball region (Figure 12).

Figure 12. Manipulating a phantom object – picking a coset.

He *manipulates* this pseudo-body and reminds the students "there are different ways of calling the same coset". He says "I could have some *psi stab G* of *i* that could mean the same coset", and again collects his two hands together as if holding a solid region. He goes on "but let's suppose that *phi* and *psi* send me somewhere different". Here his right hand begins to move on *phi*, as if it is responsible for this naming of the coset, and his left hand shakes in response to the word *psi*, as if it is performing this alternative naming of the coset, and finally on "different" his two hands turn into two index fingers pointing in two different directions. The two hands that had been together have suddenly *scattered* into moving towards and pointing along two different vector directions. His back has been to the board through all these lines, but suddenly on "different" his head turns back to the board as if he had forgotten something there.

In the next moment he is walking to the board and begins to draw there a textual instantiation of the pseudo-objects he had been *manipulating* a moment before ("let's see what the picture / what's the problem"). He draws a circle, and labels it with the equation '$phi\ stab_G(i) = psi\ stab_G(i)$'; he underlines this equation with his index finger, saying out loud the manner in which he *regards* this "that's the same coset with two coset representatives". Touching the mapping he had defined above he says "now *T* tells me", and from this he figures out "in one case if you use the first name" (he touches the left hand side of the equation) "I'm gonna map you to *phi* of *i*" (speaking in the voice of the mapping *T*). Then he says "but if you use the second name for the same object" (he touches and holds the right hand side of the equation), "I'm gonna map you to *psi* of *i*". These are two examples of *deducing that*; again, these aren't logical inferences, but they are moments of writing where he is compelled to write what he writes; this is a case in the class of cases where what he must write is dictated by a definition. Walking away from his diagram, which shows that *T*, a supposed function, takes the same input to two different outputs, he concludes “that would not be good” (recall that the diagram was premised on the supposition that “*phi* and *psi* send me somewhere different”).

Now he paces back to his drawing, and using his right index and pinky, he simultaneously touches both sides of the equation (‘*phi* $stab_G(i)$ = *psi* $stab_G(i)$’), says “if this happens” (Figure 13), then he transfers his hand with the same index and pinky finger positions fixed in place and now simultaneously touches the two outputs on the diagram marked ‘*phi*(*i*)’ and ‘*psi*(*i*)’, and says while looking at the class “these are the same” (Figure 14). He holds this pose for a few seconds in silence, nodding.

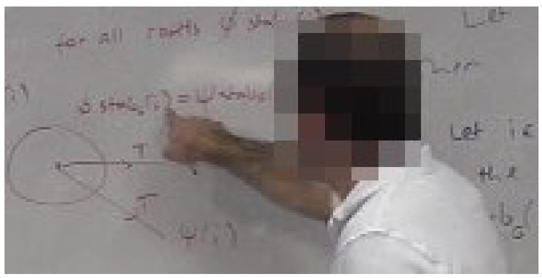


Figure 13. Carrying the equality relation – before.

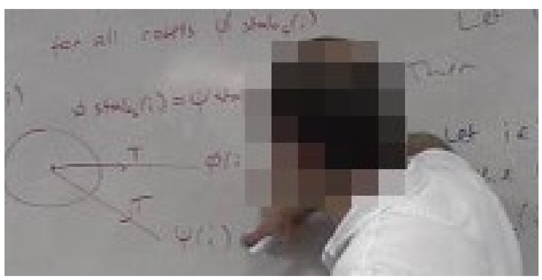


Figure 14. Carrying the equality relation – after.

Moments like this one appear to me to be highly concentrated. Moving his right hand from the equation that gives two names for the same object, and transferring it rigidly to the two differently named outputs that he intends to show are equal to each other, takes only a few seconds, and the words accompanying the motion are few. He has *carried* a mathematical relation with his hands. The occasion has been prepared for during the previous few minutes, but there is a charge and a payoff that is so compelling to me that it, and occasions like it, lie at the heart of my interest in gestural practices in mathematics lecturing.

He breaks off suddenly, shifts from a motionless stance to one of high movement, pointing at John, pacing sideways to the board, and asking "how do you diagnose John when to check that a function is well-defined / because I sometimes overlook it". The student pauses before answering, then laughs a bit embarrassedly before saying "it was a weird definition". There are no gestures here, but I include this description as the exchange foreshadows later occasions where this same topic crops up again: how is one to realize or remember in mid-proof or mid mathematical work to begin this mathematical action? How will one know to do this? J agrees with the student "so look out for weird definitions well done".

He writes '*T* is well-defined:' as a heading to his empty space (the diagram was a float to the side) and then goes back to his diagram to repeat his touchings so as to re-launch the writing which will prove the implication he needs. While he is there he adds something new. Holding the two outputs in his right hand index and pinky, he leans down with his left hand holding his notes and touches the board where the heading '*T* is 1-1' is written. He says "what I'm going to say there is / reverse the above argument". He retaps the two outputs saying "what one to one says is that if these are equal" – he simultaneously touches the two outputs '*phi*(*i*)' and '*psi*(*i*)' – "then these are equal" – he simultaneously touches both sides of the coset equation. In these seconds, with a few movements of his hand atop a textual object, the meaning of showing a mapping is well-defined and the meaning of showing a mapping is one-to-one have been publicly and plainly visible.

7.4. "why are normal subgroups interesting"

J begins Lecture 24 by reviewing on the transparency the definition of normal subgroup and the statement of the normal subgroup test. When he has finished this, his transition to the first writing of the day consists of saying "ok" – with an accompanying strike of his hand – "the question is why are normal subgroups interesting" – and he walks to the board, where he writes another question, which he acknowledges is different: 'When do the left cosets of a subgroup H form a group under the operation $(aH)(bH)= abH$'. Then he begins an example and draws the Cayley diagram of D_3. Just as in the example he created in response to Thomas' prodding that I discussed in section 6.5, he lets H be the subgroup consisting of two elements, the identity R_0 and a reflection F across a vertical axis (that goes through the top vertex of the equilateral triangle, and through the midpoint of the base; recall that D_3 is the group of symmetries of the equilateral triangle).

Having set the stage, he touches individually the three terms *aH*, *bH* and *abH*, restating the written question out loud. He touches the two nodes R_0 and *F* simultaneously with his right thumb and index finger, saying "*H* is the subgroup comprising these two elements". We will see later, especially in section 7.5, other ways that J negotiates discussing a single set which contains multiple elements: J has various ways of using his hands to *construe* (in Streeck's language) *one* or *many*. Here he can achieve both construals simultaneously, as we have seen on other occasions: the simultaneity of the touch construes the collection as a single entity and the multiply pronged nature of the touch construes the entity as consisting of exactly the individually touched members.

J next writes the equation '$FH = R_0H$'. In the ensuing sequence he will *manipulate* the Cayley diagram to show that both sides of this equation are just the subgroup *H*. He first holds the two nodes with his left thumb (on *F*) and middle finger (on R_0)– "this is *H*" – looks at the class while doing so, looks back at the board – "so if I multiply by *R* zero" – he lifts his hand slightly and then taps the two nodes lightly twice – "that just gives me back *H*". His pitch rises again on the next phrase – "if I take *H*" – he taps the two nodes twice lightly on "take *H*" – "and apply *F*" – he looks at the class – "which is the blue action" – he looks back at the board and simultaneously flips his hand over in a little somersault so that his thumb is now next to R_0 and his middle finger is next to *F* – "then I just do this" – he undoes the flip and then repeats this somersault-and-return, then takes his hand away and with his palm flicks out towards the equation '$FH = R_0H$' and says "and I still have the same set". Note that he performed the action of *F* on *H* and then redid the performance again for good measure. J gesticulates his hand in the air while shrugging his shoulders and says "so that's true".

J had opened the stanza with the comment "well now just follow these calculations for a moment"; his *shrug* now, as ever in the course, takes something away, reduces some value. Here the shrug is not taking away the truth value of his actions. but instead taking away something else from how it is we are *regarding* our present state in this mathematical context. The equation is true, but at the moment J is communicating to the class that he knows that it is not clear yet *why* such a result is interesting or why exactly J is doing these “calculations”. The second line of the stanza was “you may not see where we’re going”. The shrug says that J knows that his present actions have not yet been publicly *motivated*.

Now J begins a second calculation (it will be the first of a pair). He writes ‘$(R_0H)(R_{120}H)$’ and then he touches the coset-multiplication condition ‘$(aH)(bH)=abH$’ while saying “if we’re following that rule”. Then he says “then that should be” and goes on to write ‘$= R_0R_{120}$’; midway through writing it he *self-corrects* to say “well that is by definition” and breaks off the writing to touch again the coset-multiplication condition to clarify “the question is not can we do this / but when do we get a group”. J heard himself say “if” and realized that there is no “if” about it; they *are* following that rule and seeing what happens.

He finishes writing the equation, which now looks like ‘$(R_0H)(R_{120}H) = R_0R_{120}H$’. He determines the multiplication by R_0 out loud and he uses the diagram to *touch-compute* the product of R_{120} and H. The result of his calculation is the set $\{R_{120}, R_{120}F\}$. J performs a third calculation in a similar way (“another calculation that I will pluck out of thin air”): his initial expression is ‘$(FH)(R_{120}H)$’ and after a chain of equal signs his final expression is the set $\{R_{240}, R_{240}F\}$. This calculation also involved a touch-computation performed on the Cayley diagram of D_3. Since such computations were treated in Chapter 6 I pass over them here. He concludes by writing ‘$\neq (R_0H)\ (R_{120}H)$’.

Finally, he steps away from the board, looks at the class, raises his volume and the pitch of his tone, starting a new stanza: “now where am I going with this”. J takes about ten seconds to get going properly on the touches he wants to make: “take a look at this” he says – he holds ‘$R_0H = FH$’ for a long pause, looks at his notes – “aaaand this” – his finger hovers in the air uncertain, finally lands on the expression ‘$(FH)(R_{120}H)$’, which he traces underneath with his finger, and then more quietly murmuring he also traces under the last expression he wrote ‘$(R_0H)\ (R_{120}H)$’. He has only looked at the board and his notes throughout this somewhat hesitant period. This attempt at *deducing that* was a misfire. From another point of view, it was also a trial run: like a free-throw shooter practicing a shot with an imaginary ball before their real attempt.

Now he starts again more confidently. He traces under the terms of each side of the equation ‘$R_0H = FH$’ saying “*R* zero *H* and *F H* are the same coset” while looking at the class. His index finger swings downwards as he says “but when I operate with *FH*” – his finger finds the expression ‘(FH)($R_{120}H$)’ and he touches underneath the first bracket – “on the left” – now he touches underneath the second bracket repeatedly – “if I operate on this coset” – he briefly touches the *FH* in the equation referred to above and then holds his finger underneath the *FH* bracket in the present expression – “on the left with *FH*” – his hand jumps back up to the equation and his index finger touches the equality between R_0H and *FH* – “I don’t get the same thing” – he moves and his finger now searches for where to touch in the expression ‘(R_0H) ($R_{120}H$)’ – “as if I operate on the left” – now his finger his touching the board underneath the $R_{120}H$ coset – “of *R* one twenty *H* with this” – and on “this” he touches his finger underneath the R_0H coset and looks at the class in a concluding tone.

He has been holding his notes in his left hand throughout this rather complicated sequence of spotlightings which is, as anticipated above, best thought of as an instance of the *deducing that* gestural practice. It has all the hallmarks of such a practice: tight coordination of hands with individual terms along with a directionality of attention, driving from the initial equality of the cosets R_0H and *FH*, to the destination of the unequal results of multiplying the coset $R_{120}H$ on the left by each of these identical cosets. For the time being it appears that, on the other hand, there is in this case some sort of exception to another feature of *deducing that* practices: normally there is some sort of upshot to the spotlighting of previous writing, resulting in some new writing which the *deducing that* gestural practice has paved the way for. In actual fact it will turn out that J only remembers later, midway through the statement of the theorem that this example is preparation for, that he “forgot to write the tagline there” and he proceeds to fill in the missing comment at that time. So on this occasion it is not an equation or some other mathematical expression that the *deducing that* gestural practice makes necessary or highly plausible; it is instead an important written comment.

J puts his notes between his legs, needing both hands for what is to come next, and says “so”. As so often in the course J takes another go at what he has just done. Let me try to be as precise as I can. J is not mechanically ‘doing it again in the identical way’. Just like those other occasions, the second attempt looks and sounds and is different from the first. But there is obviously something that is ‘the same’ about it. I suppose the best way I can put it is that whatever it is that is experienced in a repeated gestural scene that feels ‘shared’, any features that can be seen to be ‘in common’, it is from these analytical pieces that researchers might profitably build meanings for terms like 'the lecturer's goal was' or 'the lecturer was aiming to' and so on. The second observation about such repetitions is how often it seems to me that the second gestural scene is a compression or a *distillation* of the first one. There are repetitions on all time scales, and I am not claiming that, for example, the second and third somersault flip of his hand is a compression of the first one; some repetitions really are just near-copies of the first gesture. But on a longer time scale, as for example the scene I am looking at here, where his first run-through took about twenty seconds (preceded by the indecisive trial run taking about ten seconds), the second run-through often features remarkable improvements.

First of all, improvement in the signal to noise ratio. He systematically *removes unneeded* motion: trimming all unnecessary wasteful motion. Second, improvement in increased precision of *synchronicity* of touches with vocal gestures. Third, improvement in *efficiency*: he *compresses* his practice, he uses less words. Fourth, improvement in *patterning* and *symmetries*: better parallelism of spoken lines, more consistent rhythm. Fifth, improvement in control over the moments of *emphasis* of particular words. This improvement is clearly correlated with the improvement in synchronicity of hand and voice, and the increased control over rhythm and parallelism. Sixth, he *fixes any errors* in his ordering and sequencing of gestures. And finally, incorporations of new forms of spotlightings and depictions and gesticulations that give an alternative experience of 'the same' calculation/explanation/justification/mathematical action that just occurred. Let's see how some of this shows up in this scene.

J touches two expressions simultaneously: his left index finger touches and holds underneath the $R_{120}H$ coset that appears in the expression '$(FH)(R_{120}H)$' and at the same moment his right index finger touches and holds underneath the $R_{120}H$ coset that appears in the expression '$(R_0H)(R_{120}H)$'; as he does so he says "these guys are the same". Coordinating both hands to move together, he lifts both hands in concert off the board, moves them both slightly to the left in sync, and lands back on the board to touch and hold the 'FH' coset with his left index finger and the 'R_0H' coset with his right index finger; when they land he says "and these guys are the same". His tone on both phrases is very similar and builds in an expectation that there will be a third phrase that will serve as the conclusion. His left hand changes its shape so that his left thumb and first two fingers simultaneously touch underneath both cosets in the expression '$(FH)(R_{120}H)$' and he says "but that product" – he moves that left hand, keeping the conformation intact, and touches underneath the expression '$(R_0H)(R_{120}H)$' – "comes out different". He releases his hand as he looks at the class. The action of this paragraph takes five seconds. There are three spoken lines, six words in each. The first two lines are identical up to exchanging the opening word "so" for "and". There is one simultaneous spotlighting for line 1, an analogous spotlighting for line 2 (the hands moving five cm to the left is the counterpart of the "so" becoming an "and"), and the two clauses of line 3 have one touch each. In these five seconds there is nothing but a few unhurried, unerring, and marked-as-separate meaningful actions.

J now says out loud the comment referred to above that he (for now) forgets to write down. He looks at the class – "so what that tells me is" – he indicates some of the writing in the question that launched the example – "I'm never gonna get a group under this operation" – he leans in to point more closely at the coset-multiplication condition – "because that multiplication" – he now points to the last writing he has done that he just showcased – "is not well-defined". He nods, holds up a hand palm-up to point to a spot in the classroom, moves his head to the side to see better, cannot find the student he is looking for: "where's John when I need him / so John is the keeper" – J holds his hand out in a mid-grasping pose – "of non-well-definedness in this class ok" – J turns to the board and with his right index finger sweeps up and down repeatedly the calculations he has just written – "and so he of course would be jumping up and down / saying no no no that's an illegal operation".

I will jump now to a few minutes later. J has written the statement of a theorem whose label is 'Factor Groups': it states that 'H \normalsubgroup $G \Leftrightarrow$ the set of left cosets $G/H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = abH$'. He has also just finished writing the comment underneath his motivating example that he forgot to write earlier. As he walks back over to begin the proof of this theorem, he says "we're gonna see that the only way / that this operation could fail to be a group" – on "fail" he raises his left arm high up, index finger raised, and brings it down emphatically, while his pitch on "fail" is very high, on "group" he reaches the condition '$(aH)(bH) = abH$' and holds it – "is if the operation is not well-defined" – he turns to touch the question that started his motivating example, and he ends in a concluding tone. He shakes his hand back and forth palm out in a negation gesture – "everything else is gonna work just fine"; a few words later all his fingers on his left hand are touching his thumb and he moves this hand up and down as he says “the only place that can fail is not well-defined” – on “not well-defined” he quickly alters the shape of his hand so his index finger is pointing at his example.

In this *regarding as* sequence, J telegraphs in advance a mathematical fact that is not emblazoned in the statement of the theorem itself, but which is an observation about the proof. He comments elsewhere in the course that he much prefers not “fishing out” results that are buried in a proof, and instead wants all results in theorem statements, or lemma statements; this is a rare counterexample. J here explicitly regards what he referred to earlier in the lecture as “this weird condition” (the condition for a subgroup to be normal) as *equivalent* to the demand that the obvious binary operation one might define on the set of left cosets of that subgroup be well-defined.

The adjective “weird” here has some disparaging connotations. Elsewhere J repeats this devaluation of the algebraic definition of normality. The present occasion is an opportunity he has carefully created in order to substitute for this manner of viewing normality (‘$aH = Ha$’ for any a in G) another one entirely; substitute is the most neutral verb I can think of when the episode is in my memory. While watching his efforts over the long minutes of writing out the question, drawing a Cayley diagram for D_3, picking a subgroup that is not normal, computing the two products of cosets, touching and spotlighting the calculation he has just performed in order to boil down the essence of the problem that can arise in multiplying cosets, J is more easily interpreted as *revealing* or *disclosing* the underlying reason for the normality condition, as if the algebraic definition, though clearly useful, were somehow disguising or hiding what is *really* going on. What is *really* going on is that the normality condition on a subgroup is exactly the condition which ensures that the obvious binary operation of multiplication on the set of left cosets of that subgroup is well-defined.

A final observation to close this section. A few minutes later J arrives at the moment in the proof of the forward implication when he will use the normality condition. J takes a few seconds to depict with his hands how the normality condition affords them a certain specific power in moving from one expression to another: “we can always take a group element through the H” – he takes his left hand, holds it at shoulder height, touches his index finger and middle finger to his thumb, and begins to move his hand horizontally back and forth multiple times as if running through a tube about two feet long

– “ok either way / move it past the *H* in either direction” – then he stops looking at the class and walks toward the part of the board where he can immediately use this power. Even a blackbox inscrutable algebraic equation (‘$aH = Ha$’) can nevertheless be transformed into a depictive gesture that gets at the core of how this bit of algebra gets used in practice in the middle of proofs. J isn’t always reaching for the conceptual essence of this or that, unlocking secret mechanisms with sequenced touchings of complex well-selected writing; he is at home showing with his hands how to go from one expression to another with peculiarly specific and appropriate gestures that treat writing as an object. This is a short *manipulating the object* scene, where the object is the normality condition.

7.5. “if I look at a group / how do I know I’ve got a normal subgroup”

In this section we will see J pursue his quest to understand normal subgroups in public together with his class. Whereas the previous section centred on J posing and addressing the question “why do we care about normal subgroups” (as he puts it in his review of the previous lecture at the start of Lecture 25), this section centres on J highlighting how it is they can recognize, instantly, from a Cayley table, whether a particular subgroup is normal or not.

To properly appreciate J’s interest and intensity in addressing this question, it is necessary to make contact with a fascinating portion of the very first lecture of the course. J is sitting on a table, performing the same two twists on a Rubik’s cube repeatedly, while making his introductory comments to the course. He recalls to mind his own undergraduate course in group theory, where they began with the definition of a group, and “off we go / theorem proof theorem proof lemma theorem proof proposition”. Every second or two he looks back at the cube he is twisting; both hands are engaged. J admits that although he could manipulate the symbols and he “kinda got the hang of it kind of” (crinkly eyes puzzlement), at the same time he “didn’t really have very much idea of what it was all about”.

He says that years later, when he was doing research for his doctorate, he came across “some thing some object” – his right hand leaves the cube and he begins waving it in small circles in a region in front of his face “and I said you know you kind of have these things" – he moves his hand to make circles a little bit over to his left now – "and you move them around like this" – now his left arm joins the movement, moving towards his right hand, left hand still holding the cube – "and you paste them together like that" – he had been looking at the class throughout but now returns his gaze to the cube, both hands return, and he goes back to his systematic two twist transformations of the cube.

Eventually, J says, someone tells him that what he had described "has something to do with factor groups". J replays his younger self's incredulity: "factor groups? you're kidding me / I did them in my undergrad"; disbelief on his face. His right hand leaves the cube again and palm-up holds some invisible object from the bottom "I'd been staring" – hand back to cube – "at this object" – hand back out palm-up on "object"; and he confesses that he "had no idea" that what he had been staring at was the *same* object he had studied when he was an undergrad. J concludes: "something really really wrong happened there". This story of his own early steps in the mathematical subject he is about to explore with them foreshadows the efforts J will make to lower the chances that his students will share the same fate of studying an object closely without developing the ability to recognize that object when it is operating and acting within some mathematical context.

Finally, I observe that in this episode J's hands are occupied closely indeed: he is launching the course with this set-piece of showing that the Rubik's cube will return to its original state after he has performed the same two twists a certain number of times – this will fall very sadly flat if at some point he makes a mistake and twists the wrong way at some point. And yet, even at risk of making such a movement error, J moves his hands at the precise moment he talks about the object in his research that excited his interest and which did not spark any moment where J could say "I know you". Here again is an incident that reinforced my own growing sense that I could learn the most about what makes J's lecturing tick by concentrating my attention on his hands. So that the reader can relax, J does finish all his twisting without mishap. He and the students had already been engaged in questions about what they thought might happen, and a student had suggested and tried to justify that the cube would return to its starting point: when J gets there the ta-da moment is satisfying.

I pick up the action in the middle of Lecture 25. His hand is in the same palm-up gesture (one difference – he has a marker in it), and he is saying "if I look at a group /

how do I know that I've got a normal subgroup / ok what's all this stuff mean". J continues his *commenting about* gestural practice with another significant palm-up grasping gesture exactly on the word "look": "here's a great place where we're gonna just learn / to look at a visual representation of a group" – he raises his right hand like a detective in the moment of identifying the guilty party and points his right index finger dramatically – "and say aha!" – he flicks his index finger forward then to the left and right while saying "that subgroup is or is not normal". By this point the reader will correctly guess exactly which word is associated to each pointing of the finger ("that"; "is" "not"). Two common locations in time that a *commenting about* gestural practice appears in are just before J begins writing in some episode (or a larger unit, like a new chapter) and just after J has finished writing in some such context; perhaps useful terms for these are *previews* and *postviews*.

A few minutes later J is midway through drawing a Cayley table. He has assumed that he has a group G and a normal subgroup H; he has listed the elements of H explicitly ($\{h_1, \ldots, h_r\}$); he has written that he will arrange the Cayley table for G so that the elements in the same coset of H appear consecutively. His diagram has two sorts of labels for the columns; this is an unusual feature for his Cayley tables. Normally the students would expect column labels that list the elements of the group. Here the labels nearest the table are indeed the elements: 'ah_1, ah_2, . . . , ah_r, bh_1, bh_2, . . . , bh_r, ch_1, ch_2, . . . , ch_r'. But there is a second layer of labelling: there is a brace collecting all the elements that begin with a, and this brace is labelled 'aH', representing the coset aH which consists of the elements ah_1 through to ah_r. Then another brace 'bH' and a third 'cH'. J sweeps his hand across the element labels, *commenting* "remember the Cayley tables are unique / up to the order in which I write the elements". He continues to sweep his hand back and forth – "I'm saying organize your group elements" – his hand changes shape into a partial grasp formation, holds it under the 'aH' brace – "block them into cosets". The grasp gesture is one of J's ways of indicating a *one*; here a *one* that is also a *many* that is being *regarded* at that moment as a *single* set.

J completes, in the manner in which he started, his labelling of the columns and also of the rows of his Cayley table, and he has drawn vertical and horizontal lines to separate the coset braces on the columns and rows. He walks to the board and touches the square formed by the wide column labelled 'bH' and the wide row labelled 'aH' and says "so now what will I see here". His index finger has touched the board on "here"; the individual finger is another singleness gesture; here a *one* that is only a *one*. To reword the distinction: the palm-up grasp gesture foregrounds a *one* but leaves room for a present reminder of the *many*, whereas the individual finger touch foregrounds a *one* and, although with some extra work one can remember there is a *many*, the gesture does not express this. J takes one step back from the board, and then steps to the board again, and with the same index finger now traces a square outline just within the borders of the empty square in his Cayley table – "how can I describe the set of *r* squared entries that live here". This outlining gesture indicates the *one* "here" while also indicating the "*r* squared" *many* within. So far we have three variants of gesture where the meaning of "many elements of a single set" emerges from the actions of the hands.

J continues to prompt "they're all the elements of" and he touches the empty square with his finger again. No one is answering and he traces a square again "just this block here". After someone's wrong guess Thomas supplies the answer "*abH*" which J repeats, and he writes 'elements of *abH*' in the empty square, and walks away. J now enters a post-hoc or retrospective *deducing that* gestural period. He walks back to the board – "remember the whole point is that" – he slows his speech a little while he hunts for the right thing to touch – "because we're normal" – he touches the condition '*H* \normalsubgroup *G*' – "coset *aH*" – he holds his hand palm-up underneath the coset brace row label – "times coset *bH*" – he moves his hand to be palm-up underneath the coset brace column label – "is" – his tone is emphatic and his hand moves to be palm-up under his writing inside the block where that row and column intersect – "coset *abH* and that's well-defined".

J had stepped away from the board and was looking at the class, and now suddenly turns back – "I'm not gonna have" – his finger somewhat erratically selects one of the element row labels within the *aH* coset brace – "something falling in here" – he moves that finger to tap audibly inside the intersection block – "and something else" – he very vaguely points at the list of element row labels and then flings his finger off to point farther to the right than the block – "falling outside". This last movement is an early instance of a group of some gestures that I will refer to as *scattering* gestures, which are depictions which emerge in contexts of non-normality. Above I called this post-hoc or retrospective *deducing that* simply because the culmination writing which is so often a hallmark of *deducing that* gestural scenes was on this occasion supplied by a student and did not get step by step built at the board by J; J builds it step by step on the board anyway.

This is perhaps a useful moment to point out a very common feature of J's lecturing, which is the tight coupling between sudden sharp jerky changes of direction of J's body (as here), and tone of voice, and what observers in the audience would comfortably call "J changed his mind about stopping". J's gesturing could very easily have ended with the word "well-defined"; his tone lowers in exactly the way it does on hundreds of occasions when stanzas end, he has turned his back on the writing he had been touching. It is a claim of this work that the very empirical evidence which first plausibly, and upon so many self-reinforcing repetitions, convincingly, correlates certain spoken words, tones of voice, volume, eye gaze changes, and other such criteria discussed in Chapter 4, with beginnings and ends of episodes, and therefore to marked changes in what it is J is talking about – this very evidence makes it possible to point to moments when such an ending is registered, only for J to suddenly re-begin and continue as if he had not ended. Such moments can be with good confidence labelled as false endings: occasions when J changed his mind about finishing. In mathematics the temptation that is seen here that J succumbs to (J has shown what *is* the case; he is tempted and succumbs to showing what it is that *cannot* happen, what it is that is forced to *not be* the case) is almost always an available option.

In the next little while J writes the corresponding version of what he has just done for the two squares beneath the one he has done. They happen much more quickly. First J says "here I will see all the elements of" – he touches coset brace row label '*bH*' and says "*b*", he touches coset brace column label '*bH*' and says "squared" (!), he finishes writing 'elements of b^2H' and says "*H*". The repetition for the coset *cbH* is nearly exact. This makes his concluding "and so on" quite convincing.

A minute later J has drawn on the board a new Cayley table, this time for the quotient group *G/H*. It is smaller in scale, and the column and row labels are now only the coset names: *aH*, *bH*, *cH*. Now he begins a period of *looking at side-by-side*. He taps the term *abH* which he has written in the block where the row labelled *aH* and the column labelled *bH* intersect – "it's a single element" – he walks over to his original Cayley table for *G* and traces an outline around the corresponding block – "that was *r* squared elements" – he returns to the new Cayley table – "and I shrink them all down to this".

The *looking at side-by-side* gestural practice affords J the opportunity to slow down and separate the hand movements required to distinguish the *many* elements inside the *one* coset *abH*. He can spotlight the block where *r* squared elements live in the Cayley table for *G* and then walk over to touch the block where the one element lives in the Cayley table for *G/H*. There is a clean separation that *looking at side-by-side* provides J, affording maximum contrast while still maintaining so much similarity in overall appearance. The coset labels are the same, the grid or checkerboard appearance looks the same. But the individual element labels are gone. The blocks where rows and columns intersect are named; but whereas before J wrote 'elements of *abH*' now he has written simply '*abH*'. The preparatory writing, this precondition of the two *side-by-side* tables, now allows J to move back and forth between the tables to tease out exactly what is going on in a factor group – something J missed in the version of this course that J himself took when he was a student.

J is at his desk, sighs, and says "the point is this" and moves to the original Cayley table. With his open palm left hand he points to the coset row label, then the coset column label, then the block where they intersect – "this multiplication is well-defined". He points quickly over to the other board where the new Cayley table is – "what we do when we form the factor group" – he begins circling the 'elements of *abH*' block over and over – "is I don't care what mess is going on inside here" – looks at the class on "mess", still circling again and again – "this could be something very very complicated" – last two tracings are more like squares – "going on with the *r* squared elements of this table".

Now J suddenly takes his hand from the board, brings it in front of his face, holds it briefly palm-up all fingers splayed, and then closes all his fingers into a fist while at the same moment making a sound effect "whisshhhtt" – "I just collapse that" – he pumps his fist once, begins to walk over to the new Cayley table – "down to a single element" – he touches the column label '*bH*' – "and all I care about" – he touches the row label '*aH*' and then the column label '*bH*' – "is how *a* and *b* interact" – he touches the '*abH*' written in the block where that row and column intersect, then holds this term with his arm outstretched behind his back as he looks at the class and pauses.

After the pause he adds more quietly “ok those coset reps” – he walks back over to the old Cayley table and places his hand over the corresponding block – “the internal structure” – he takes his hand off the block and begins a vigorous clearing motion, repeating this sweeping away gesture multiple times – “gets smudged gets thrown away gets ignored” – he closes his hand and stops moving it on “ignored” and he pauses again. Now both his hands form a shape like they are holding the top half of a large sphere, and then J moves them backwards and outwards as if the sphere were inflating – “and I reveal” – repeats the gesture to coincide with the next two words – “the superstructure” that’s going on. This *superstructure* gesture is repeated a few times in the course. So are various gestures that express *mess* and *complicated*, especially when J does not “care” about the mess: one such occurs a little later, which consists of J poking around in the air with his finger, turning it in little circles, saying “if we ignore the complication of the internal stuff”.

Stepping away from the board, and marking a transition from a *looking at side-by-side* period to a *commenting about* period, J waves his left hand in a dismissive gesture – “so it’s not just that I can make” – he plunges both hands together to hold briefly a small sphere – “small groups” – he pulls them both out wider – “inside larger groups”. The contrast between the superstructure quotient group depiction and the subgroup (substructure if you like, J never uses this term) depiction has occurred within a few seconds.

7.6. “[how do you see that?] ... I don’t know”

Two themes are present in this section: the connection between what it means for a function to be well-defined and what it means for this function to be one-to-one, and the troubling question of how it is an apprentice mathematician can know when they must show that a function is well-defined.

In Lecture 29 J has reached an important milestone in an undergraduate algebra course on group theory: he has written on the board the statement of the First Isomorphism Theorem. This important theorem requires the concept of quotient groups in order to be stated, and its proof showcases the deep connections between all of the concepts I have discussed in this chapter as well as the central concept discussed in

Chapter 5: normal subgroups, quotient groups, well-defined mappings, and isomorphisms.

J begins by tracing his finger underneath the term '*G/Ker phi*' and says "why am I allowed to talk about that". This marks the sixth time in the course that J has gone through this little action sequence – spotlighting some symbols that stand for a quotient group, and then justifying that the notation is ok by stating that the subgroup he is quotienting by is in fact a normal subgroup. So it is here: "I am allowed to talk about that / because the kernel of *phi* is normal in *G*". In the next episode I discuss in this section, an interesting sort of counterexample occurs to this recurring, one might say, habitual reaction to seeing the notation for a quotient group.

J then writes the word "Define" and then says out loud that he will define a mapping "out of thin air" (it turns out he will ask the students and they provide it) and he writes the letter '*psi*'. He turns to the class and says he will show that this mapping is one-to-one, onto, and preserves group structure; then he turns back to the board to continue writing the domain of the mapping. As he writes he continues speaking as if he had not just concluded (had he forgotten for one second? is this a self-correction? it is not possible for me to be sure) "and that it is" – while his hand is still holding the marker in the writing position he leans back, swivels his head slightly, winks.

No one says anything yet, so he swivels more, abandons the writing position and holds his index finger up to hold it on the side of his nose. Of the repertoire of gestures J makes this is among the small class that seem more culturally specific to J's past – from my own experience in seeing this gesture I have taken it to mean something like 'you know what I cannot say right now', almost like a translation of a wink into a hand movement. One student offers "a homomorphism", J waggles his fingers a little and nods quickly as if to say 'no but go on' – it feels like a small game of Charades has broken out – another student offers "isomorphism", J just continues shaking his fingers and nodding, and then John says "well-defined" and J leans back to point at John and says "thank you" and J smiles.

A little while later J has written the definition of the mapping on the board and he stops, sighs, and steps away from the board *commenting* that once before in the lectures he had proven that a mapping was well-defined and then wrote that to show it was one-to-one they just needed to "reverse the above argument / and someone which was probably you Peter said" – J points to Peter in the sort of full hand gesture used when performing an introduction of one person to a group – "couldn't you just have done both at the same time" – J begins nodding – "ok fine you win / I'll do them both at the same time" and he turns back to the board to begin writing. In a response that can make me smile even after many many viewings, Peter replies in a pretended whining tone "now it will be confusing". The class, including J, laugh. A minute or so later J has written on the board the double implication '*g Ker phi = h Ker phi ⇔ phi(g) = phi(h)*'.

Now J begins his deliberate, patterned, ordered and economical touchings. He asks in the tone of one who will answer his own question "so what do we need for well-defined" – he is at the board, looking closely at his writing, orienting himself – "we need to say" – he positions his right index finger under the '*g Ker phi*' term that is sitting inside the definition of the mapping '*psi(g Ker phi) = phi(g)*' – "that if there are two different names for this coset" – he turns his look away from the class towards the board and moves his hand so that his index finger and pinky finger simultaneously touch the equality of the cosets '*g Ker phi = h Ker phi*' – "then this function *psi*" – he touches the '*psi*' – "sends them both to the same place" – he touches the right side of the equation defining the mapping, then he moves his hand and taps once audibly underneath each of '*phi(g)*' and '*phi(h)*' – "good it did".

I wrote but erased a sentence about this *deducing that* gestural practice, using the words "orchestrated" and "choreographed"; these terms carry with them too much connotation of planning in advance. But with a little forgiveness of the terms, it is astonishing how systematic and methodical the short dance of touchings is in this sequence. If I take seriously such terms, then it seems that some combination of J and his already written mathematics is calling the specific tune that J then moves to.

The loveliness of this episode grows with what occurs next. The very same written mathematics now equally well serves the different purpose of showing that the mapping is one-to-one. J pauses only for a second, shrugging, before launching into it – "what do we need for one to one" – he touches the right hand side of the definition of the mapping – "if the images of two objects are the same" – he touches underneath the equality '*phi(g) = phi(h)*' and then immediately retouches the spot his finger had just left – "then the original objects" – he touches the '*g Ker phi*' term in the expression '*psi(g Ker* *phi)*' and then touches the equality '*g Ker phi = h Ker phi*' – "are the same". This takes four seconds; the movements and spotlightings of his one-to-one *deducing that* gesturing look like a sped-up *time-reversal* of his well-defined *deducing that* gesturing.

J points to Peter again who has a question: "does this mean one to one and well-defined are like the same thing". J slowly sighs, looks at the board, says "no", turns back with a grin, "in general not". He says that it depends on the mapping, and that here they could wrap them both up at the same time, and then he moves on with his proof. Although I cannot see this in the video, because of J's later actions, I know that at some point after this moment, J wished he had said more.

I know this because of the next episode I will analyze. It is from a moment in the third last lecture of the course, Lecture 33. The statements have become more sophisticated, the results they contain are more powerful, and the proofs of those statements have become, on the whole, longer, and pull in material from multiple different parts of the course. Here the theorem he is trying to prove is that the number of conjugates of an element *a* of a group *G* is equal to the index of the centralizer of *a* in *G*.

Within his proof he has defined a mapping from the set of left cosets of the centralizer of *a* to the conjugacy class of *a*. He writes "Check *T* is" and says "we need to check that"; he pauses expectantly leaving his thought unfinished, turns around, looks at the class, then takes a deliberate step to his left and looks more intently at a particular student (John) and lays his hand out towards him in a be-my-guest gesture; this movement prompts Peter to say "well-defined". J says "thank you", goes to the board, swivels his upper body around to add "ventriloquist". I detail these moments, these interactions between J and his running-joke of expecting John to immediately spot when it is time for a mapping to be checked as to whether it is well-defined, because I think they point towards two things.

First, that there is something different or unusual about this mathematical move. No other concept or calculation or object in the course is associated in this way to any particular student in the course. For example, it is clear to me from many viewings that Thomas earned the right to have become associated with surprising counterexamples; that Bart time and again provides or fills whatever gap J has just left in a hinting prompting sentence, regularly responding with helpful and on the mark answers.

However, it is very hard to imagine that if Thomas or Bart had answered some question about the centre of a group that J would have called that person "the keeper of the centre", or to look for that student anytime the centre of a group came up. Rather I suspect (I can't prove this) that the act of realizing that there might be a problem with the definition of some mapping is a little peculiar, and perhaps distributed rarely. A problematic mapping is perhaps not obvious to spot, and that there is something to be done when encountering one is hidden from common awareness. When someone did so spot, it is more natural for J to suspect that they will continue to have the knack for doing so. Second, as an instance of how easy and natural it is for the common pursuit of mathematical understanding to breed interpersonal connections and bonds. J points at John, but Peter has seen this play out before and can figure out what J expects John to have seen and he can say it before John does. Everyone can laugh at J's conceit that John has spoken through Peter.

A minute later J has finished the proof. In fact, a portion of the proof has really been left to the students (he finishes writing the sentence 'Check *T* is well-defined, 1-1, and onto.' but does not do this checking himself – as I said, we are now in the late stages of the course, and J is choosier about what he writes and can delegate more to the students to fill in on their own). J enters a *commenting about* gestural scene. He recalls that "one of you asked last time / I was wittering on about well-defined" and he restates Peter's question from the last scene. J looks off in a channel parallel to the board, saying "I should have answered that better I realized afterwards / well-defined is if you like the converse of one one". He is looking at the class now, steps forward to put his notes down so that both his hands are free – "well-defined is". What he now begins is a powerful depictive gestural sequence which forms a no-writing counterpart of his sequence of patterned spotlightings I analyzed above, as well as a no-drawing counterpart of the diagram he drew and touched back when he proved the orbit-stabilizer theorem.

He raises both hands in the air to just above his eye level, hands loosely in fists, hands about a foot apart – "if you take two things with the same name" – he shakes them a little in rhythm to his next words – "and you apply the map" – he now moves both hands vertically downwards to about waist level and brings the hands so that they are together, not separated by a horizontal gap anymore – "you end up in the same place" –

slight upturn in pitch on “place” (he is not finished). He will now *repeat* or *try again* this sequence.

He pauses, lifts both hands up again higher than they were at the start, farther apart than they were, loosely in fists – “so two things” – shakes his fists lightly – “with the same name” – sharp drop vertically to the low waist position again, this time fists stay separated horizontally – “end up equal to each other” – he looks at his hands briefly – “one one is the images are equal to each other” – he brings his hands up vertically and then at the peak moves them horizontally together to touch – “the things you started off with are equal to each other”. He nods and takes his right hand and sweeps it up and down along the vertical path his two hands had just travelled along – “so if you like one is the reverse process of the other”. J points at Peter a line later as he remembers it was Peter who asked the question.

In the last section we encountered a variety of gestures or sequences of gestures where J negotiated different approaches to a coset, delicately shading how much awareness and attention was addressed to the *many* members of the coset, and how much to the *one* coset. In this scene we instead are witnessing a struggle or a contest between a *two* and a *one*. This is because in this context if J can show that applying the map to any *two* representatives of a coset will give the same output, then the map will take any of the *many* representatives of the coset to the same output. Therefore *any two* is all that is needed as a proxy for *the many*, and it explains the difference in hand and finger gestures.

J faces the same tension that he faced last time, however, in that his hands must somehow move in such a way as to mean both “two names” and “for the same object”. One of his solutions in the previous section when he needed to mean both “*r* squared elements” and “same coset” was to do two different gestures in sequence: first tracing a square around the collection of elements, then touching the centre of the square. The meaning of *many* followed quickly by *one* is a decent attempt at achieving a simultaneity of meaning using gestures that are linearly ordered in time. In this *manipulating the object* scene J resolves the problem in two different ways.

The first way is the two hands in fists high in the air: the *twoness* is obvious, there are *two* fists. The *oneness* is less explicit: both hands are at the *same* height, they look like mirror images. Perhaps the most telling feature that there is a *one* here is when he repeats the well-defined sequence and shakes the hands at the *same time* when he says “same name”. In this first solution to the problem I see a mixture of meaning, with the *twoness* more prominent.

J’s second solution occurs at the end of his one-to-one depiction: his *two* hands move vertically indicating the *two* nominally different starting points in the domain (the *twoness* predominates) but then he moves the hands horizontally so that they meet together (now the *oneness* predominates). So here, like his first-trace-then-touch solution that I discussed in the last section, he opts to divide into two successive moments the two gestures that mean two different things: first, the *two* nominally different points x and y, say, which satisfied $f(x) = f(y)$; second, identifying x and y as just *one* point.

J is not done with this theorem. He points at the proof, turns to the class with a scrunched up expression on his face and asks “so why is that proof a little bit different than what you might have expected”. J does a little performance of how a student might have reacted – puzzled, slightly unhappy look – “kind of a little bit strange”. There is a suggestion from a student, but it is not what J is looking for. He sweeps the proof up and down and then waves his hand at it a few times saying “there’s a lot of words in this proof no?”. In this *commenting about* gestural phase, J is acting out everything except for the reason for his reaction, hoping that a student will look at the proof in the same way that J is, which would mean they would share that sense of their expectations being upended. Bart comes up with it in his usual succinct way: “normal subgroup”.

J says “right”, and he writes temporarily on the board ‘$G/C(a)$’, asking “why did I not just say G mod C of a”. Bart replies that $C(a)$ is not normal in G and J exclaims “ah!” sharply and raises his right hand high in the air. J now reports that when he first read the proof he was tempted to write this and then said “whoa whoa whoa whoa” to himself. Like many a *commenting about*, this one doubles as a cautionary remark (warding off potential *correcting self* sequences) as well as marking the proof of yet another theorem as having some “oddity” about it that makes it different, special.

J’s individuated reactions, complete with emotional intensity, to the details of the written proofs he encounters, are shared and expressed by J as though these might very well have been or would be how his students would respond to these mathematical situations. It is a satisfying coda to his knee-jerk reaction to always check that the '*H*' in some supposed quotient group '*G/H*' is actually normal. So strong and so habitual is this impulse that he is temporarily impatient with a proof that doesn't simply use a quotient group when he thinks that it could: then he realizes that it can't. Finally, as I have noted before, recall that this habitual checking that alleged quotient groups actually are quotient groups is an analogue of checking if a mapping is well-defined.

In the final episode I wish to analyze, the question of when to check if a mapping is well-defined is brought explicitly to the forefront by Peter. Of all the interactions between students and J in this course this one stands out as containing the most tension. J has written the statement of an exercise on the board 'Why is the correspondence x \mapsto 3x from Z_{12} to Z_{10} not a homomorphism?'; he walks back to his desk and says "certainly looks like one to me".

About a minute goes by as two suggestions are made which do not work. Peter thinks that this map does not send the identity to the identity, then he realizes with J that the identity in both groups is 0. Bart begins an approach that suggests the map does not preserve the group structure, but begins to sputter out, and he begins explaining why his suggestion is incorrect after a few phrases. J is silent through Bart's speech and soon tilts his head downwards, starts stroking his chin and squinting his eyes. Finally he gestures his hand towards John saying "ok John for historical reasons / I have a feeling you may be best placed to resolve this / just because you've been the keeper of such things before" – at which point Bart breaks in "is it not well-defined". J lets out a celebratory "yeah!" and jokes that now he can "just name the person and you know what the answer is". Bart, like we saw Peter do above, can do the John-move when he is prompted.

J writes out the three lines that show that the correspondence is not well-defined, and then walks over to an earlier piece of writing (the defining condition of a homomorphism: '*phi(ab) = phi(a)phi(b)*') and touching it with a smile says "don't even go that far". Now Peter asks "how do you see that?". J is standing at his desk putting the top back on his marker, audibly exhales, replies "how do you see that it needs to be well-defined?". J pauses, looks at Peter with his head tilted and his two hands up in a do-I-have-this-right pose. Peter answers "I understand that things need to be well-defined" and J drops the hands and says “right”. Peter continues “when you’re looking at something for the first time”. J is about to speak, breaks off, then lifts both hands up and bows his head in a what-would-you-have-me-do pose, and says “that’s why I’m giving it as an example”. He repeats the two hands palm-up gesture, smiling, and there is some laughter. The sequence in this scene contains several moments where the stakes are raised. The atmosphere is charged.

Peter replies “ok so there’s one example” (repeating J’s own word but reversing the value judgment placed on it), and J says “right” and he is back to the serious gaze at Peter to see what is coming next. Peter says “how would you see-” and John interrupts “if it looks like that”. Weeks earlier it was John who was interacting with J concerning this very issue, as I discussed in section 7.3. J had asked John “how do you diagnose John / when you need to check that a function’s well-defined”. John’s answer at the time was “uh cause there’s this weird definition”. Here, on this occasion, his interjection is half-joke and half-repetition of his earlier view. At the time J accepted this answer and moved on, but Peter has no intention of moving on.

J laughs, and putting on a voice he says “if it looks at me in a funny way”; then he drops the act and says in a sincere high-pitch tone “I don’t know I mean I’ve said / sometimes these things are very hard to spot”. Now J tilts his head back to look at the ceiling, takes a deep breath, audibly exhales, and starts anew: “ok again do I give the real answer or the politically correct answer”. Already we are clearly in territory that is unusual. J’s answers to questions in the course are routinely delivered confidently. There may be alternative ways of gesturing his way through an explanation, but by *self-correcting*, and by repeating with improvement, J consistently responds to questions from students, or to moments of choice in proofs, with purposeful singly-directional work. There usually aren’t two different answers from which J must select what he feels comfortable doing.

J starts “I think-” – his shoulders sag suddenly and he loses his steam briefly, then he raises his hands again, shrugging his shoulders – “I guess in principle you should think” – and now for the first time during this scene a single hand comes out to gesture. He makes a grasping shape with it, and continues “is this well-defined” – now he makes sweeping motions down and to his left to emphasize the words “every” and “any” that follow – “every time anyone shows you a mapping”. These sweeps are reminiscent of his *one hundred percent*, *nailed* gestures that I discussed in section 6.2. He waves his hands up in the air one more time "that's the politically correct answer" – waves them up again when he hits the word "always" – "you should always check that your mapping is well-defined / as a matter of principle" – on "principle" he turns both hands into fists and stiffens his body for a moment.

J's next speech is too long to quote as closely. He notes that most of the time mappings are not checked because "it's tedious" and "most mappings are" well-defined. He moves both hands now in a leisurely side-to-side palm-up motion, saying "I guess it's brought to our attention when it's very easy to trip up / and write something down that's not well-defined I guess". More shoulder movements upwards, more loose and vague upward motions of the hands and then letting them drop.

I agree with, and admire, J's entire answer. It is true that in principle one should always check if a proposed mapping is well-defined, and it is equally true that in practice one usually performs this checking only when a mapping has been defined in a way that arouses a suspicion. In addition, it is true that within the course in general, and even during this episode, J has given very useful information as to what sorts of definitions of mappings arouse, as he puts it, this "sense of danger". J is exceptionally articulate even at moments like this when he is facing pointed unanticipated questions.

At the same time, J's body and J's hands are not doing what J's body and hands do when he is being convincing; his use of "I guess" three times in one stanza is the spoken counterpart of his movements. J is being asked about how Peter ought to act when faced with a part of the mathematical landscape that J has encountered many times before, and I believe him when he says "I don't know". There is no stanza's worth of words that can replace the experience required to be able to tell whether or not a proposed mapping is obviously well-defined or not. What J can say is what he does say: suggest that his students get into the habit of checking all mappings, and give some rules of thumb indicating those situations where the checking is more likely to be needed.

J closes his speech with a return to his "in principle" advice, which is to check every mapping, "otherwise we have no idea whether what we're talking about makes any sense" – J looks down, picks up his notes. Peter keeps at it: "so should we do that every time". J mentions cosets as a site for possible problems with mappings, and he mentions that this chapter on homomorphisms contains a few examples.

Peter tries one last time to pin J down, while also fine-tuning the focus: "I guess my question is when we're doing a proof" – J has taken one step from the desk, his notes in his left hand, marker in the right – "the only time we mention this is when it goes wrong". J suppresses a grin, points downward, and says "in this chapter when we talk about homomorphisms / you'll see that I keep writing / note that it's well-defined / even if I just say brackets check". Peter says "ok". I think it is fair to say that Peter pressed the issue as far as he could and that this is the most complete response he can draw from J on this day. Looking at the episode again I realize that some of the tension I see comes from me recognizing that J wants to move forward from the moment he picks up his notes, and Peter does not obey the directive that this gesture has come to mean in this classroom. The rarity of this moment is a forceful reminder of how easily the students have understood the meaning of J's movements and gestures, and at the same time this emphasizes how important it was to Peter to settle this issue.

I sympathize with Peter and I admire his courage. If he can ask what a conjugacy class is, and how he is supposed to use it in some problem, and J will answer instantly by writing it down and then touching this part of it and that part of it, drawing a temporary diagram next to it if need be, why can't J just tell him exactly when he needs to check if a mapping is well-defined, and exactly when he needs to incorporate this checking in his written proof? Where has the precision gone from a J who will stop himself from saying "the" and replace it with the word "an"? Why is this meticulous, clear, confident and unambiguous mathematical actor now shrugging and raising his hands and saying "I guess"? From the perspective of the mathematics education researcher, it is Peter who is the keeper of well-definedness in this class. He is the gadfly who provokes a scrupulously honest and estimable response that seems to be a somewhat discomfiting experience for J to deliver. He asks the questions that spur J on to beautiful gestural depictions of the relationship between well-definedness and injectivity.

7.7. Summary

Checking that a mapping is well-defined is a mathematical action of an unusual type: it can be difficult to recognize that one is in a situation that requires this action. J pantomimes an attitude of refusing to continue unless some particular action is taken, and by doing so prompts students to realize they must perform this checking. Associating a student with this recognition, later students can perform this role of noticing they should check for well-definedness simply by J referring to that student. J regards the normality condition for a subgroup as the solution to a problem: that multiplication of cosets would otherwise not be well-defined. He uses a selection of hand-configurations to *handle* a Cayley table to discuss and explain the properties of cosets. He looks at two Cayley tables *side-by-side* and walks back and forth between them to explain the quotient group concept. A *collapsing* gesture, an open hand with fingers splayed, moving downwards to a closed fist, and a *superstructure* gesture, moving his hands outwards suddenly from a smaller sphere to a larger one, are both used in *manipulating* a phantom *object* occasions, as he discusses and demonstrates the quotient group concept. A *scattering* gesture, with fingers moving in random directions away from a center, is used when he comments about non-normality.

Chapter 8.

Summary of the argument: in conclusion

In this final chapter, I review my method, and I respond to the three research questions. I close by discussing how this research has impacted my own mathematics lecturing.

8.1. Review of Methodology

In the course of developing a data document consisting of all the spoken words in the course – the *S transcript* – I observed that J's lecturing is methodically broken into segments of time: *stanzas*. These stanzas constitute local contexts within which some individual specific action, or a short sequence of related actions, takes place. A typical fifty-minute lecture is broken into eighty to a hundred stanzas. J's spoken words in stanzas naturally break into smaller units termed *lines*. These are roughly of length six to ten words, usually contain a single verb, usually unfold a single clause or perhaps two in quick succession, and usually are separated by a short breath. When J's gestures and body movements were analyzed, the fact that they occurred within a particular stanza radically reduced the scope of interpretation of their meaning. This data document allowed for searches of any words or phrases throughout the course of lectures.

It was observed that a large proportion of the actions that were carried out in stanzas took place so that a new item of mathematical writing could be achieved on the board, or that an item of mathematical writing that had just been performed could be justified. A second data document was thus developed, which consisted of all the writing on the whiteboard, recorded in a format that captured the organization, structure, and some features of the visual arrangement of the writing: the *W transcript*.

An examination of a major collective work by a large community of mathematicians resulted in an articulation of a series of observations about the nature of mathematical writing. Lessons learned there were systematically and consistently applied to the construction of the *W transcript*. The resulting data document was rich in information concerning a wide variety of individual items located in a small list of types of segmented, hierarchically organized, environments, together with some information concerning the visual appearance of this writing on the whiteboard. This document was searchable, making it possible to filter or select video from the lectures related to any of those individual aspects or features of J's writing which had been recorded.

The fastest method of scanning all the writing appearing on the board in the course was achieved by a collection of still images from the videos: the *Pictures data document*. Since the writing on the board is almost invariably a proxy for what was going on in the classroom at the time the writing was made, I made use of the *Pictures data document* whenever I recalled another incident in the course which I wished to compare to an incident I was currently investigating; this way I could transport as quickly as possible to viewing this scene. The *W transcript* was most suited for extracting detailed information and the *Pictures data document* excelled in providing a global view.

Coordination between the *S transcript* and the *W transcript* was most easily achieved by means of the *Episodes data document*, which demarcated the lectures into a unit larger than a stanza: the *episode*. An individual lecture broken up into episodes afforded a top-level summary of the main achievements of the lecture. Episodes were often the time interval during which an environment of writing was built: mismatches between the two units afforded an easy and efficient tool for locating stanzas in which J is making comments and is not writing.

Interactions between J and the students occur regularly and are of great importance. During these interactions, J is pushed or pulled in directions he may not have planned to go, or must give explanations or answer questions that are unexpected to him. A student contribution is any uninterrupted speech a student makes. They are frequent. The *Student Contributions* data document recorded the essential features of all these contributions: what stanza they occurred during, who spoke, were they instigating the speech or responding to J, what was the topic, and were they convincing or correct.

The creation of the above five data documents thoroughly achieved the 'soaking' phase that the *Natural History of an Interview* team of researchers described. When I created the *Gestures and the Body* data documents – an original version, and then a compressed one – my observation and recording of his gestures and body movements in some scene was influenced by my recognition of the local context of action, the current stanza; it was guided by the precise structure of what had been written up to that point, and what would be written in the moments to come; it was shaped by my knowledge of the overarching theme of the current episode; it was informed by what the student involvement had been, was, or soon would be; and it was highly directed by my knowledge of related incidents in the course that the creation and study of all the previous data documents had afforded me.

8.2. Response to Research Question 1

In this section I respond to my first research question:

What are the features, components, structures and functions of the kinds of practices by which this lecturer, within the local hierarchical context of the ongoing interaction, moment-to-moment, sequentially, publicly, and accountably, creates the next pieces of mathematical writing, while speaking, moving his body, and, most importantly, gesturing with his hands?

I identified six families of gestural practices: *manipulating the object*, *looking at side-by-side*, *regarding as*, *deducing that*, *commenting about*, and *correcting self and others*.

The *manipulating the object* gestural practice emerges on occasions when J uses his hands to interact with an object: it may be a *physical* object, a *pretend* object, or a *textual* object. These are scenes where J is *handling* the object, searching for and discovering its possibilities, feeling it over. If it is a physical object, he is moving it around in space, turning it, flipping it over; or tracing his fingers on it in specific ways to gather meaning about it. If it is a pretend object he can move his hands as if he is working to adjust or arrange it in one configuration or another. If it is a textual object he can run his hands over this or that feature, showing what might be done to alter or change the structure of that textual object, or select and reveal structure within it.

The second family of gestural practices emerges in scenes when two textual objects are simultaneously visible and touchable on the board. J walks back and forth between the two pieces of writing, and touches parts of one and then parts of the other. He can spotlight individual items in each or he can manipulate each object in a more complex way. Specific features of one or the other object are highlighted and foregrounded. Contrasting behaviour is accentuated, especially when every feature of the two objects is the same except for one difference. The second object can be almost complete and by walking back and forth from the first object to the second J can complete the second object in a desired way. This is the *looking at side-by-side* gestural practice.

J is often interested in how he ought best to view a mathematical object or environment that is of current concern. He may spotlight an item of writing on the board and say about it what it is about its structure that he presently needs for his purposes. He may manipulate a textual object involving many items in ways to indicate the manner in which he views the mathematical relationship between these items. The *regarding as* gestural practice involves J, an object, and the perspective or angle that J is taking on the object at that time. On the rare occasions when J fails to regard an object or situation in the appropriate way, his hands might seek to touch the board and recoil just before doing so. When he solves the problem of how to view the situation, J may draw a diagram to encapsulate the resolution.

In the *deducing that* gestural practice, J justifies a new portion of writing by touching in an ordered, patterned way some sequence of items of visible previous writing. Either the touches occur first, or they occur after, or both before and after: they are necessary in order to make acceptable of the new item of writing. Most often these spotlightings have served the purpose of providing an air-tight logical justification of an inferential step. But the family includes all instances where a new item of writing has been prepared for, or has been determined, by J handling or touching visible writing. Sometimes he will pull out transparencies of writing from earlier in the course simply to view it, and then touch the right terms in the right order, so that he can conclude that some statement is true, which he then writes.

When J steps back from his ongoing writing, and offers remarks that are one level up from the logical flow of his material, he is *commenting about*. This gestural practice affords J the ability to make leaps forward and backward in time. He may spotlight individual items of writing in order to make observations about whether such a step was obvious or difficult and how to notice in the future whether such a step will be needed or likely. He may be inspired to recall a mathematical story from a previous time he taught the course, or an experience from when he was a student. He may evaluate in all sorts of ways the writing they are doing, and why they are doing it. He may foreshadow later rigorous definitions with current intuitive language, or may recall their earlier struggles with a theorem because of how limited their tools were back then, to contrast with their present strengths.

J has freedom in how long such comments last, and during this time he sometimes pantomimes or mimics the actions of another person, or of his former self, or enacts some mathematical object. Many occasions of the *manipulating the object* gestural practice, that involve a phantom object, occur during *commenting about* scenes. Writing rarely occurs during a *commenting about*.

The sixth and final family of gestural practices is *correcting self and others*. Sometimes J will hear himself say something that is, in some sense, inferior to what he would prefer to say. It might be logically incorrect, it might be ambiguous, it might be any word or clause where he would prefer so much to replace it that he interrupts his ongoing speech sharply, with no warning, and starts up his speech again with the new, superior version of what he had been saying. These occasions sometimes turn into *commenting about* scenes when the mistake is one that J considers mathematically interesting: perhaps a mistake that allows for an opportunity to explain some mathematical situation more closely, more precisely.

The *regarding as* gestural practice co-ordinates easily with many of the other families of practices. A common component of a *deducing that* scene is at least one occasion when J is *regarding* one of the mathematical objects from a certain, locally advantageous, point of view. Frequent appearances of mathematical objects in similar *deducing that* sequences increase the likelihood that in future mathematical actions J will regard that object from the point of view that made the previous deducing that sequences more amenable. When J *looks at* two textual objects *side-by-side*, sometimes he holds one to regard it as having certain capabilities and a certain value, then holds the other to regard it as having different capabilities and different values; each may be useful in a different context. *Commenting about* is a natural co-operating gestural practice with *regarding as*: the manner in which J is considering some mathematical object now can be recommended for other occasions, or he may caution them to regard it differently in some other circumstances. Frequently, the best way for J to express his *point of view* on an object is to *manipulate* it, or nearby objects, in appropriate ways.

The *looking at side-by-side* and *correcting self and others* gestural practices have similarities: the first requires adjacent pieces of writing; the second typically involves contiguous portions of speech. In *looking at side-by-side* J is equally concerned with both textual objects; in *correcting* J privileges the second portion of speech as being indisputably preferred in some way.

More relationships between the gestural practices emerge in responses to the next two research questions.

8.3. Response to Research Question 2

In this section I respond to my second research question:

What aspects of these gestural practices emerge prominently, and how do they co-operate, during those occasions in the course when a mathematical object (the dihedral group of order eight indicated as D_4) is at the centre of the ongoing lecturing interaction?

The scenes which first introduce the dihedral group of order 8, D_4, were dominated by two gestural practices: *manipulating the object* and *regarding as*. J carefully rotates a cardboard square whose corners are labelled into four different configurations, and he reflects it across each of four axes to find another four configurations. These physical *movements* of a material object – the visible movements of the square through some path in space that returns it to exactly the same region in space it had originally occupied – are precisely the *elements* of the group D_4.

When J and Peter discuss their differing perspectives on the labels of the corners of the square, J manipulates the square in a deliberately uncaring manner, twisting it suddenly and unexpectedly this way and that to heighten the contrast between the way J regards the corner labels and the way Peter does. The *deducing that* and *correcting self and others* gestural practices are typically connected to the consideration of questions that have right or wrong answers and arguments about what is or is not *true*. By contrast, when J enacts the *regarding as* practice, though rigour and correctness are never absent from concern, the focus is more *pragmatic*: “I don’t want that” J says, to Peter’s question, “I can do what I want”. His labelling of the corners is not dictated to him. He has chosen them because of how he wants to view the square, and what information he wants to keep track of. Convincing another of one's own manner of viewing a mathematical object has challenges that are not identical to convincing another of the inevitability of the next item of mathematical writing through the *deducing that* gestural practice. Harmonizing two points of view in mathematical conflict can require new concepts: such as the notion of an equivalence class.

The *regarding as* gestural practice is not limited to individual items, like the labels of a square, but can emerge in situations where a large field of concern is being viewed. When J wants to determine if what they have done so far has convincingly established that they have found all the symmetries of the square, he uses all the resources of his body to pantomime the two contrasting *attitudes* one can take in such situation. Either they are done, in which case he makes vigorous sweeping *one hundred percent nailed* gestures, repeating them at will, and he makes pushing down gestures, settling the situation firmly in one place. Or they are not done, in which case he strokes the chin, puts his hands on his temples, mimics puzzlement, doubt, and skepticism.

The second attitude is to be thoroughly embodied until they can no longer seriously entertain any doubts. Then the first attitude can emerge. In this second lecture, J will brook no sitting on the fence between the two attitudes. Clearly, this is an important performance for him. In his *commenting about* gestural practice, which recapitulates many of these movements, he explicitly recommends this approach for every future occasion in which the students are constructing a mathematical argument.

A short instance of the *manipulating the object* gestural practice, where J briefly moves his hand from pointing to one student to another and back as if they were elements in a group, exemplifies the lightning-fast creativity of gestural practices. The immediate laughter in the room is a clear sign that J's gestural practices, even when they are arguably complex and unusual, can be shared and understood in the moment.

J commonly makes a second attempt to gesture with his hands in the accomplishment of some aim after he has made a first attempt: he *tries it again*. He has *deduced*, with a sequence of touches, that it must be that the permutation group S_3 is isomorphic to D_3. This required him to *regard* two groups in a way that foregrounded whether they were abelian, or cyclic. To explicate again, he sketches two Cayley diagrams quickly s*ide-by-side* and, by spotlighting one of them again, he regards one as abelian and cyclic, and the other not.

An informative instance of the *deducing that* gestural practice was an occasion when J asked his students to tell him how to draw the Cayley diagram for D_5; on the board already was the Cayley diagram for D_4. A student replied "it looks like that with more nodes". The student can determine the form of this object from the visible example; he helps to create a second textual object which, together with the first, can be *looked at side-by-side*.

In another occasion of *looking at side-by-side*, J walked back and forth between the Cayley diagram of Z_2 x Z_2 and the Cayley diagram of D_4. In the Cayley diagram of the first group, every generator is its own inverse; in the intuitive language early in the course, every action is reversible. On the other hand, in the Cayley diagram of the second group, one of the generators does not satisfy this condition. J has made visible each property in his textual object, and by spotlighting one after the other in quick succession he can accentuate the distinction. *Looking at side-by-side* can also allow J to make observations about potential future mathematical actions, and, for example, make *comments about* them to compare their value. J contrasted the Cayley diagram and Cayley table of D_4 from the vantage point of which one affords the most efficient computation of the multiplication of elements of the group.

J *manipulated* a textual object, the Cayley table of D_4, as his first mathematical actions after writing the definition of a group on the board. Each successive component of the definition was accompanied by a gestural manipulation of his table: *reading and embodying* the abstract definition. This is a subfamily of manipulating the object. He spotlights the top row and the left column of the table to *regard them* each as a *set* of elements. He orders his spotlighting, top row followed by left column, when considering an ordered pair formed by an element chosen from each. The action of the identity is revealed by a *looking at side-by-side* gestural practice, spotlighting two rows, and two columns, as being identical. He also handles the object to compute products: he *touches-computes*. As the name indicates, this is a subfamily of both *manipulating the object* and *deducing that*.

While *reading and embodying* the portion of the group definition corresponding to the property of associativity, he *self-corrects* when he hears himself say that associativity is “not gonna be a problem”. He must regard his Cayley table as one that has just appeared before him, not as one whose history of construction he is familiar with, and which therefore gives him extra information he should not have.

He *self-corrects* when he hears himself say “the identity”; he must regard a group at this stage in the course as possibly having more than one. A *waving* gesture serves as a *delimiting* gesture: J knows this much, but not more than this, about some mathematical object or situation.

J also handles Cayley diagrams of D_4. Sometimes he touches-computes: he performs a calculation using his fingers and hands to step along the actions of the generators of the group. Other times he simultaneously touches multiple nodes in his textual object at the same instant with his hands. These multiple positions commonly form a mathematical substructure of D_4: J can *hold structure* (here a subgroup). As the name indicates, this is a subfamily of both *manipulating the object* and *regarding as*. With two such diagrams on the whiteboard, and *looking at them side-by-side*, J can *carry a structure*: he can hold a structure in one object, lock his hand in that conformation, move to the other textual object, and place his hand down on the second diagram to hold an isomorphic structure.

J *comments about* how effective it can be to use a concrete example, like D_4, in order to distinguish between two similar-looking definitions of distinct mathematical concepts. He sets down the definition of the centre of a group and the centralizer of an element of a group, and also writes the proofs that each of these is a subgroup of the group on the board: now he can *look at* these *side-by-side*. He touches-computes with a Cayley diagram of D_4 to determine the elements of each subgroup in this example: the differences between the two concepts are foregrounded.

By *manipulating an object* in two distinct ways, J can switch from one manner of viewing it to another. When manipulating a square, J circles his fingers around three of the corner labels and slides his fingers from one corner to another when he regards the symmetries of the square as performing permutations of the labels of the corners. These are visibly very different ways of interacting with the square with his hands when compared to when he rotated and flipped the square to perform the symmetries of the square. In this way J can switch from *regarding D_4* as a group of *geometric transformations* to *regarding D_4* as a group of *permutations* – a subgroup of the permutation group S_4.

A student who was not convinced that J had established that they had found all the elements of D_4 prompted J to improve his argument. One component of this improvement was a manipulation of the square that highlighted those elements of the permutation group S_4 that were not permitted as elements of D_4 because, as the student said, the labels in the corners are "fixed in relation to each other". The interaction between J and the student, like many such interactions that continue for multiple stanzas, concerned a distinction between two manners of regarding a mathematical object.

The *regarding as* gestural practice can be limited and local, as is apparent in a scene where J is attempting to understand a student's mathematical observation. On this occasion, he exhibited a *readiness to act* in the following way: one, to instantiate a general result in a specific situation; two, to take a mathematical result and analyze it piece by piece. We have seen that many instances of the *regarding as* gestural practice concern touching a general object and seeing it as a specific instance of that object. We have also seen that many instances of the *deducing that* gestural practice concern touching, in a particular order, individual pieces of a larger portion of writing. When J's limited and local *regarding as* gestural practice is accompanied by these strategies developed from previous occasions of the *regarding as* and *deducing that* gestural practices, a global view of the mathematical situation can be progressively obtained.

J can at first operate locally, viewing terms and expression appropriately, understanding what it is about them that makes the current logical step acceptable. He does not yet regard the example as a whole, and he cannot perform any portion of the *deducing that* gestural practice. He assembles bits of writing in an organized way, but he cannot touch them in sequence, and connect them in a deduction. When enough writing is in place he begins to understand: he can touch two terms in a row appropriately. J soon reveals his full understanding with a powerful depictive gesture that constitutes a significant instance of the *manipulating the object* practice. By inflating the distance between his hands, he encapsulates and demonstrates the key idea of the student's example.

8.4. Response to Research Question 3

In this section I respond to my third research question:

What aspects of these gestural practices emerge prominently, and how do they co-operate, during those occasions in the course when a mathematical notion (well-definedness) is at the centre of the ongoing lecturing interaction?

On occasions when J explains that a map they have defined may, in fact, fail to be a function, and that they need to check that it is well-defined, he sometimes employs a *looking at side-by-side* gestural practice. He will deliberately leave a gap in the writing of his proof, and preferentially write out all of the proof except for this portion: he has created a textual object of a particular form. Then he *manipulates* his textual object by spotlighting the two components: the part that has been completed and the part that is missing. While doing so, he *comments about* the contrast, explaining that the part they have done was mechanical or straightforward; whereas the part he has saved for last is non-obvious, and is a place where they have to think.

At other times, again when considering a potentially problematic definition of a mapping, but when he has not constructed his proof in this way, J will make a point of his *attitude*. He will stop, he will wait, he will freeze in a stance, awaiting instructions from the student as what he must now do. He exaggerates his readiness to act in a certain way: that they will not continue further with their mathematical argument until they have confirmed that their definition is “clean”. He insists on this manner of *regarding* the present situation.

Two further instances of *deducing that* were notable. One instance revealed that some moments in proofs, like the moment when a mapping is defined, are, in fact, occasions of *no choice*. Given the situation they are in, there is really only the one mapping that can be written down: so, although a passive reader of such a proof in a textbook might think this definition required a decision or some creativity, J shows with his ordered touches of previously written that it was inevitable. Sometimes the very next bit of writing is dictated by a definition: this is a common type of instances of the *deducing that* family.

There were three significant occasions when J performed a series of gestures to show that a mapping was well-defined, and then reversed the order in which he performed these in order to show that this mapping was one-to-one. The occasions were different. Once he used large expressive depictive gestures that *manipulated* a phantom mapping in the air from an element with two names to potentially two target elements which in fact coincide, then attempted a reversal of these depictions. Another time he took a series of double implications that were already written, and using the *deducing that* gestural practice he touched the terms in an order that justified the implications in the forward direction. Then he engaged in a reversal of these touches. On the third occasion he wrote a series of implications, used the *deducing that* practice to confirm the truth of those statements by means of a carrying gesture from an equation to a diagram, as well as a sequence of spotlightings; he then time-reversed the gestures, justifying his ability to write a series of implications in the reverse direction. J made visible with the hands the close relationship between the concepts of well-definedness of a mapping and one-to-oneness of a mapping.

J's ability to *touch-compute* with, and *hold structure* in, a textual object revealed itself in further, more sophisticated mathematical situations. He could touch-compute with a Cayley diagram to determine the product of a subgroup with an element: he could determine a *coset*. In addition, he could also multiply cosets together.

J *tries again* with many of his gestural practices: those occasions, when he tries again a *deducing that* sequence of gestures, are of particular note. His second attempt often shows many improvements, and also helps to define for the viewer what J considers an improvement in his gesturing. He commonly removes any unneeded movements, improves the synchronicity of his hand and voice, compresses his actions into a shorter time, improves the consistency of any parallel movements or parallel lines of speech, improves control of any moments of emphasis, and fixes errors. Sometimes he incorporates new gestures that accomplish the deduction through a distinct manner of regarding some object in the mathematical situation.

At a critical moment in the course, J *deduces that* two expressions which one might have hoped would equal each other, do not equal each other: in that context, this meant that multiplication of cosets with respect to that subgroup was not a well-defined binary operation. Nevertheless, one might wish to multiply cosets. The resolution of this tension is the definition of a condition on a subgroup which would guarantee that this multiplication would be well-defined. The desired condition is the normality condition; it is the definition of what it means for a subgroup to be a normal subgroup.

There are elements of the *self-correcting* gestural practice operating here. It is not a mathematical error that is being fixed. It is that a wished-for outcome has been demonstrated to not occur in a situation. To correct this, a new definition is needed. Harmonizing two manners of *regarding* an object can lead to the definition of a new concept; trying to fix a detailed *deducing that* which ended in a failure to achieve a wished-for outcome can also help extract the definition of a new property. J regards the normality condition from this perspective: that it permits the desired multiplication of cosets. He contrasts this manner of regarding the condition with regarding it as an algebraic condition. J also *manipulates* the normality condition by pulling at terms and passing them through the symbol standing for the normal subgroup.

J writes the Cayley table for a group *G*, where he has organized the top row, and left column, by cosets of a normal subgroup *H*. He uses his hands to *regard* the coset as a *many* and also a *one*. He sweeps his finger along a list of *many* elements of the coset, then with a palm-up grasp gesture under a label that covers all of them he talks about the *one* coset. He traces an outline around a region where all the *many* products of two lists of elements will go: the whole region is a *single* coset, and he touches this region with a single finger. He draws a Cayley table for the quotient group *G/H* next to his first table. Then in a powerful *looking at side-by-side* gestural practice, he walks back and forth between the two diagrams, showing how each region of *many* elements in the first diagram transforms into a *single* element in the second diagram.

Using a depictive gesture of a hand with fingers splayed moving downwards into a lower position where the hand makes a fist, a *collapsing* gesture, he demonstrates how the *many* become a *one*. This is one of his two quintessential *quotient group* gestures. The other is a two-gesture sequence. First, quick and random poking movements with his fingers in a *mess* or *complications* gesture: a *manipulating the* *object* gestural practice where the phantom object is the normal subgroup *H*, whose internal structure might be complex. Second, a widening movement of his hands to a larger sphere, a *manipulating the object* gestural practice where the phantom object is the quotient group *G/H*, whose structure might be simpler: a *superstructure* gesture. To contrast with the situation with a subgroup that is not normal, he uses *scattering* gestures, his fingers moving into many locations, to depict how in that situation the products of elements do not so nicely organize themselves into regions as they do here. This is his typical *non-normality* gesture.

One student, John, distinguished themselves in the course by their ability to recognize when J and the students were in a situation where they needed to check that a proposed definition was well-defined. Later, two students demonstrated they could take on the role of the *other*, in this case the role of John: suggesting a mapping should be checked to see if it is well-defined after J hinted by using John's name.

An interaction between a student and J revealed the complexity of J's manner of *regarding* proposed mappings in the mathematics he reads: in principle all such mappings ought to be immediately tested, and yet this is tedious; in practice only a subset of these occasions are more often the ones where mappings ought to be checked, and it is difficult to give hard and fast rules which define that subset of occasions.

8.5. How has my lecturing practice been changed?

In this final section, I mention some of the ways I have changed as a lecturer as a result of this research.

Writing clearly on the board has become a higher priority. I have always tried to write in a way that was legible and neat. However, when I wanted to proceed more quickly, or when my focus while lecturing was taken up with some aspect of the mathematical situation, this principle would be sacrificed. Doing this research made it clearer than ever before that I was touching and handling my previous writing on innumerable occasions. I was no longer comfortable pointing at a term or an expression that was not as clearly written as it could be.

I began to be very aware of the following tension: rushing through the writing of some expression to save seconds; returning to this expression multiple times to use it. I became conscious of how often I returned to hold expressions either to make comments about when we might use it later, or to regard it in some way to allow another deduction. It was more enjoyable to hold an item that I had written mindfully, even if more slowly. Over time, the act of noticing which terms and expressions I returned to more often also had an impact on how it was I prepared my lecture notes.

I also found that I was no longer satisfied with touching some item on the board if it was really an item *inside* or *near* that item that I was using in my mathematical actions at that moment. In other words, I tried within every lecture to achieve a tighter coordination between what it was I was specifically pointing to on the board and what I presently was doing in that mathematical step. In doing so, I became aware of dozens of occasions of slightly blurred touches: resting my hand under a relation when in fact I was engaged in regarding the term on one side of that relation in a particular way.

This practice, mundane or humble as it may sound, improved my understanding of many mathematical arguments significantly. It trained me to find the best item to touch; the best moment to touch it; the best amount of time to hold it for. I believe that even though such evaluations are necessarily subjective, the act of striving to attain them forced a world of interesting distinctions for me that helped me appreciate and notice the particularities of different logical arguments.

Appreciating the power of the looking at side-by-side gestural practice made me more conscious about deliberately creating occasions when I had on the board, visible, two pieces of writing that I wished to compare or contrast. While it is somewhat effective to do an example on the board, and to recall out loud an earlier example that is no longer visible, that was similar or different, it is much more powerful to arrange my delivery so that both examples are on the board at the same time. I also learned to be specific about my touches in such a situation. I feel more free to walk back and forth and dwell in the side-by-side practice for much longer than I used to.

My conversations with my colleagues about teaching have become more targeted. I have discovered that what I often really need or seek from colleagues are their points of view or manners of regarding certain results or objects. So now, when I talk with them about what I covered in class, I talk immediately about those occasions in proofs where I want to learn from them how they regard the main ingredients in the hypothesis.

I have learned, from watching and studying J, the importance of making a second attempt at an explanation or justification. I simply made it a habit to try again at a deducing that to see what I might do better or differently the second time. Even the attempt to do so proves exhilarating. I so often feel a tighter control over the mathematical step, even steps that I thought I already understood.

Towards the end of his book, Streeck (2017) quotes a passage from some reflections by Bateson (1972). These lines are equally appropriate here:

> Whenever we pride ourselves upon finding a newer, stricter way of thought or exposition; whenever we start insisting too hard on 'operationalism' or symbolic logic or any other of these very essential tramlines, we lose something of the ability to think new thoughts. And equally, of course, whenever we rebel against the sterile rigidity of formal thought and exposition and let our ideas run wild, we likewise lose. As I see it, the advances in scientific thought come from a combination of loose and strict thinking, and this combination is the most precious tool of science. (Bateson 1972, p. 75)

In this research, I have tried to steer a path through these two extremes as well. I developed, as carefully and consistently as I could, tools to analyze, micro-ethnographically, a large data-set. In using the tools, I stayed alive to analogies or intuitions that might help me disclose patterns in behaviour, which others might recognize or be astonished by, in a series of mathematics lectures that I loved watching.

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Appendix A.

Stanza Transitions in Lecture 7.

07.01

```
ok folks                                                    =04:14
so last time we defined the order of a group
group might be finite or infinite
and the order is the number of elements in the group.       =04:24
```

07.02

```
the order of an—                                            =04:24
of a single element of a group
also written with bars
but we're now talking about just one element
is the smallest positive integer which gives you back the identity
uh when you raise the group element to that power
or if there isn't such an integer then it's infinite.       =04:41
```

07.03

```
and I said the cycle graph                                  =04:42
shows the decomposition of the group elements of finite order
into cycles that intersect in the identity
and then afterwards I thought well
you know what I mean
and I know what I mean
but we're not decomposing the elements
we're not splitting the elements up
we're decomposing the set of group elements
so I think probably a better word is partition
we're partitioning the set of elements
of a group of finite order into cycles ok.                  =05:08
```

07.04

```
so let's just— like we haven't previewed this enough        =05:09
let's actually have the definition of subgroup
so a subgroup $H$ of a group $G$
is a subset of $G$
a subset of $G$
```

that is itself a group under the binary operation of G
so keep the same operation
take a subset
and you still want to be a group
a group that lives inside a group. =06:00

07.05

and we will write H less than or equal to G =06:02
that's the sign for less than or equal to
but that when these are groups
that means H is a subgroup of G
and we will write H strictly less than G
this is just a convention
this is not universal
that's what Gallian's doing
H less than G means that H is a proper subgroup of G
that's not the sort of proper as in
Jonathan's a proper English gentleman
that is- which has been said before I can't imagine why
that means- proper subgroup means it's a subgroup
and it's not the whole of G. =06:54

07.06

and we also =06:59
so first of all why do we care about subgroups
well this whole abstract algebra lark is about structure
in fact come to think of it all of mathematics is about structure
mathematics back of the envelope definition
mathematics is the study of pattern
ok what we are about
we are in the business of studying structure
and so if we've got this thing called a group
and we have another group living inside it
that's structurally interesting
of course we want to understand when we have a subgroup
how many subgroups have we got
what are the properties
if I know that the big group does this
what does the little group do
of course we want to understand the structure of the subgroups.=07:41

07.07

so what's always a subgroup of G =07:42
[G] G is always a subgroup of G very good
what's always a proper subgroup of G
[the identity] the identity element only
so just the set consisting of the el- of e
is called the trivial subgroup of G. =08:07

07.08

well we went to all that hassle =08:10
to get our visual intuition going
so we may as well keep going
now that we've got those pictures in our- in our minds
let's take a look at what some subgroups look like
in terms of a Cayley diagram
so I'll draw a Cayley diagram
and then I'll mark the nodes that form a subgroup. =08:36

07.09

| so here 0 1 2 3 4 5 and 0 =08:36
what's that a Cayley diagram for? =08:51

07.10

[um] oh ahem =08:54
that's not a Cayley diagram for anything the way I've drawn it
good point let's hold that thought
silly me well it's not just the colour
the point is I didn't put an arrow on
so we've got no idea what that was
let's try that again ok
once more ok
what's that a Cayley diagram for? =09:16

07.11

[$Z6$] that's a Cayley diagram for $Z6$ =09:16
and if I mark oops

that's a zero not a node
right if I mark this one and this one and this one
then those- the nodes 0 2 and 4 form a subgroup. =09:39

07.12

if you say well why is that =09:40
well let's have a think about it quickly
we're gonna spend-
we're gonna do this all algebraically in a minute
but what does that mean to be a subgroup
if the big group is associative
of course the subset is associative
so we never have to check that
if we're in a group
of course every subset keeps the associative operation. =09:57

07.13

we have to worry ok =09:58
do we still have the identity
well we better make sure we do
so I couldn't mark 1 3 and 5 and hope that that's a subgroup
cause every group has an identity
and we already decided by example ahem
that we were not going to uh-
that we were not gonna try and change the identity
when we went to a subset
so we better have 0. =10:18

07.14

we need to make sure we have inverses =10:18
and now we have to be careful with the language
when we had the group definition
we talked about a binary operation
and I just said that means you go from G cross G to G
now if we're in a subgroup
we - I'm saying we use the same binary operation
but now we have to explicitly check
that we don't go from H cross H to something outside H
in other words we have to check closed explicitly
we have to be sure that the group- the subgroup is closed. =10:48

Appendix B.

W transcript: Lecture 24.

```
168 --- 24  1--7

\begin{Que2}

When do the left cosets of a subgroup $ H $ form a group under the
operation $ (a H) (b H) = a b H $?

\end{Que2}

\begin{Exa46}

Consider $ G = D_3 $ and $ H = \langle F \rangle = \{ R_0 , F \} $ .

We have
Dia80
\labelb $ R_0 $ $ R_120 $ $ R_240 $ $ F $ $ R_120 F $ $ R_240 F $
\labele
 \scb $ R_0 H = F H $ \scb ( 1 )  \sce
but $ (R_0 H) (R_120 H) = R_0 R_120 H $
$ = R_120 H $
$ = \{ R_120 , R_120 F \} $ \scb ( 2 ) \sce
and
$ (F H) (R_120 H) = F R_120 H $
$ \{ R_240 , R_240 F \} $
$ \neq (R_0 H) (R_120 H) $ \scb ( 3 ) by ( 2 ) \sce \sce

( 1 ) and ( 3 ) show that left coset multiplication is not well-
defined for this choice of $ H $, so the left cosets cannot form a
group under this operation .

\end{Exa46}
```

169 --- 24 2--7

\begin{T38}

Theorem 9.2 (Factor Groups) .

Let $ G $ be a group . Then $ H \nsubgp G $ \iff the set of left
cosets $ G / H = \{ a H : a \in G \} $ is a group under the operation
$ (a H) (b H) = a b H $.

\end{T38}

\begin{P30}

Proof .

170 --- 24 3--7

By Chapter 7 Lemma, $ a' = a h_1 $ and $ b' = b h_1 $ for some $ h_1 ,
h_2 \in H $.

Then
$ (a' H) (b' H) = a' b' H $
$ = a h_1 b h_1 H $ \scb substitute for $ a' , b' $ \sce
$ = a h_1 b H $ \scb $ h_1 H = H $ \sce
$ = a h_1 H b $ \scb $ b H = H b $ since $ H \nsubgp G $ \sce
$ = a H b $ \scb $ h_1 H = H $ \sce
$ = a b H $ \scb $ H b = b H $ \sce
$ (a H) (b H) $

171 --- 24 4--7

$ G / H $ is a group .

Identity is $ e H = H $

Inverse of $ a H $ is $ a^{-1} H $

Associativity $ a H (b H c H) = a H b c H $
$ = a b c H $
$ = a b H c H $
$ = (a H b H) c H $

\backwardsimplies

Let $ b \in G $ and $ h \in H $.

We show that $ b h b^{-1} \in H $, so that $ H \nsubgp G $ by Normal Subgroup Test .

$ b^{-1} H = (e H)(b^{-1} H) $ \scb $ G / H $ is a group \sce
$ = (h H)(b^{-1} H) $ \scb $ e H = h H $ \sce

$ = h b^{-1} H $
\therefore $ b^{-1}^{-1} h b^{-1} \in H $
\therefore $ b h b^{-1} \in H $ \box .

\end{P30}

172 --- 24 5--7

\begin{D36}

When $ H \nsubgp G $, we call $ G / H $ the _ factor group _ (_ quotient group _) _ of $ G $ by $ H $ _ .

\end{D36}

\begin{Fac29, Fac30}

We have
$ | G / H | = | G : H | $, and when $ G $ is finite
$ | G / H | = | G | / | H | $ (Lagrange) .

\end{Fac29, Fac30}

173 --- 24 6--7

\begin{Fac31, Fac32}

Let $ H \nsubgp G $.

If $ G $ is cyclic then $ G / H $ is cyclic (Ex . 9.12) .

If $ G $ is abelian then $ G / H $ is abelian (Ex . 9.13) .

\end{Fac31, Fac32}

\begin{Exa47}

\floatb $ Z_7 $ $ 3 + 7 Z \verticalarrow 3 $ \floate

Let $ n \in \mathbb{N} $, and $ n Z = \{ n i : i \in Z \} \subgp Z $.

$ Z $ is abelian, so $ n Z \nsubgp Z $.

A coset of $ n Z $ in $ Z $ is $ j + n Z = \{ j +n i : i \in Z \} $
= set of integers integers congruent to $ j $ modulo $ n $.

\floatb
$ a H = b H $
\iff $ a^{-1} b H \in H $
\iff $ (a H) R (b H) $
\iff
\floate

Factor group $ Z / n Z $ is
$ \{ n Z , 1 + n Z , 2 + n Z , \ldots , n - 1 + n Z \} $
and $ Z / n Z \iso Z_n $.

Forming $ Z / n Z $ collapses the coset $ j + n Z $ to a single
element $ j $.

\end{Exa47}

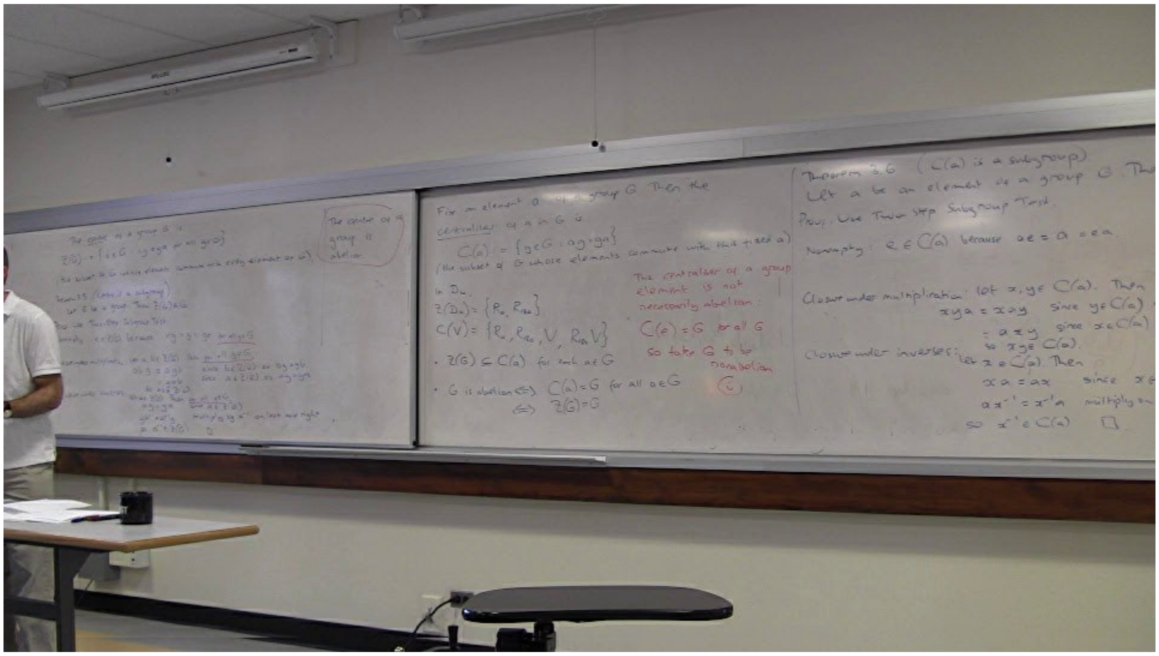
```
174 --- 24  7--7

\begin{Exa48}

Let $ G = (\mathbb{R} , + ) $ and $ H = \langle 2 \pi \rangle $ .

$ G $ is abelian , so $ H \nsubgp G $ .

A coset of $ H $ in $ G $ is
$ \theta + H = \{ \theta + 2 n \pi : n \in Z \} $
= set of angles coterminal to $ \theta $
and factor group $ G / H $ is $ \{ \theta + H : 0 \leq \theta < 2 pi
\} $ .

Forming $ G / H $ collapses the coset $ \theta + H $ to $ \theta $ .

\end{Exa48}

\begin{Exa49}

$ ( \mathbb{Q} , + ) $ is abelian, so $ Z \nsubgp \mathbb{Q} $ .

A coset of $ Z $ in $ \mathbb{Q} $ is $ q + Z = \{ q + n : n \in Z \}
$ .

Factor group \boxb $ \mathbb{Q} / Z $ \boxe is $ \{ q + Z : 0 \leq q <
1 \} $ : an infinite group, each of whose elements has finite order .

\end{Exa48}
```

Appendix C.

Pictures data document: Lecture 7, boards 4, 3, 2.

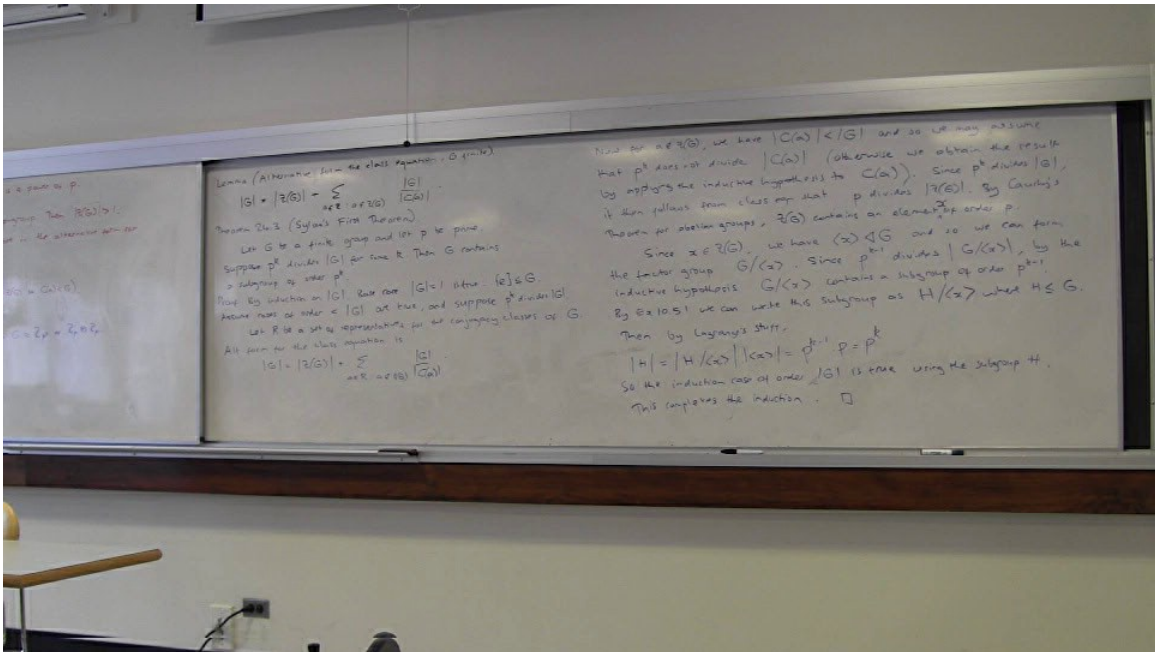
Appendix D.

Pictures data document: Lecture 8, boards 5, 6, 7.



Appendix E.

Pictures data document: Lecture 33, boards 7, 8.



Appendix F.

Lectures: dates, durations, stanzas, correspondence with chapters of the textbook.

Video	Date (2014)	Total Time	Start Time	End Time	Stanzas
1	Jan 6	47:33	02:52	50:25	74
2	Jan 8	53:00	00:50	53:50	92
3	Jan 10	52:40	02:38	55:18	82
4	Jan 13	52:00	02:26	54:26	93
5	Jan 15	49:28	03:58	53:26	90
6	Jan 17	27:47	00:26	28:13	54
7	Jan 20	50:22	04:14	54:36	103
8	Jan 22	51:01	02:40	53:41	96
9	Jan 24	50:06	04:55	55:01	93
10	Jan 27	50:51	02:07	52:58	104
11	Jan 29	52:13	01:40	53:53	104
12	Jan 31	49:53	00:00	49:53	102
13	Feb 3	51:48	03:21	55:09	85
14	Feb 5	49:49	01:27	51:16	102
15	Feb 7	53:41	00:02	53:43	95
16	Feb 19	50:24	01:11	51:35	93
17	Feb 21	49:10	00:53	50:03	89
18	Feb 24	50:00	00:30	50:30	92

19	Mar 3	51:08	00:52	52:00	98
20	Mar 5	26:50	00:19 00:11	08:30 18:50	38
21	Mar 7	50:48	01:12	52:00	76
22	Mar 10	49:55	00:30	50:25	80
23	Mar 12	50:47	00:44	51:31	74
24	Mar 14	49:34	00:01	49:35	86
25	Mar 17	50:04	02:16	52:20	104
26	Mar 19	50:02	01:48	51:50	83
27	Mar 21	51:01	00:27	51:28	73
28	Mar 24	49:15	00:32	49:47	90
29	Mar 26	51:46	01:20	53:06	90
30	Mar 28	34:12	00:18	34:30	59
31	Mar 31	50:34	01:02	51:36	90
32	Apr 2	51:03	00:11	51:14	82
33	Apr 4	51:39	00:00	51:39	91
34	Apr 7	48:54	00:16	49:10	84
35	Apr 9	39:46	00:47	40:33	75
		28:19:04			3006

Average time length of stanza is 33.9 seconds.

Notes:

1. Classes were Monday, Wednesday, Friday.
   Each class was about 50 minutes long.

2. There are 4 videos that are shorter than the regular length:
   Videos 6 (27:47), 20 (26:50), 30 (34:12), 35 (39:46).
   Battery died in Videos 6 and 30.
   Fire alarm happened within Video 20.
   Course evaluations were conducted after the end of Video 35.

3. There was one midterm, on Monday Feb 17.
   Reading week was Monday Feb 10 - Friday Feb 14.

4. There were 2 classes that were unfilmed - Wednesday Feb 26 and Friday Feb 28. These occurred between Videos 18 and 19. I was attending a conference.

5. The course followed the textbook "Contemporary Abstract Algebra" by Joseph Gallian.

Chapter	Title	Videos
1	Introduction to Groups	1 - 3 [19:44]
2	Groups	3 [19:44] - 6
3	Finite Groups; Subgroups	7 - 9 [13:24]
4	Cyclic Groups	9 [13:24] - 12
5	Permutation Groups	13 - 15
6	Isomorphisms	16 - 18
7	Cosets and Lagrange's Theorem	19 - 20
8	External Direct Products	21 - 23 [28:02]
9	Normal Subgroups and Factor Groups	23 [28:02] - 27 [14:05]
10	Group Homomorphisms	27 [14:05] - 31 [35:48]
11	Fundamental Theorem of Finite Abelian Groups	31 [35:48] - 32
24	Sylow Theorems	33 - 35

Appendix G.

Episodes data document: Lecture 25.

```
Video 25 - Mar 17                 50:13


Reviews.
  1. definition of normal subgroup.
  2. normal subgroup test.
  3. Theorem 9.2 (factor groups), size of factor group.
                                     02:16 - 04:38    25.01 - 25.05
Exercise 9.11.  H isomorphic to K need not imply that G/H is
 isomorphic to G/K.
  [Exe25 Sol23]
  statement.
                                     04:38 - 05:18    25.06 - 25.06
  commentary.
                                     05:18 - 06:03    25.07 - 25.07
  solution.
    comments that he must show H and K are normal in G.
                                     06:03 - 07:48    25.08 - 25.10
    shows this.
                                     07:48 - 08:24    25.11 - 25.12
        shows H and K are isomorphic.
                                     08:24 - 10:03    25.13 - 25.15
        shows G/H is not isomorphic to G/K.
          draws Cayley diagram of G.
                                     10:03 - 13:20    25.16 - 25.19
          visualizes G/H.
                                     13:20 - 15:00    25.20 - 25.22
          visualizes G/K.
                                     15:00 - 15:47    25.23 - 25.23
          comments on strategy.
                                     15:47 - 18:03    25.24 - 25.27
          shows G/H is isomorphic to Z4.
                                     18:03 - 19:19    25.28 - 25.29
          shows G/K is isomorphic to Z2 x Z2.
                                     19:19 - 20:50    25.30 - 25.34
Student Question. how many cyclic groups are there of order n.
                                     20:50 - 21:34    25.35 - 25.35
Illustration (of definition). normal subgroups. visualization.
  [Exa50]
  motivation.
                                     21:34 - 22:36    25.36 - 25.37
  Cayley tables: G (arranged nicely) and G/H.
  side by side comparison.
                                     22:36 - 30:23    25.38 - 25.54
```

```
Example. A4 and 12 tetrahedrons.
  [Exa51]
  Cayley table for A4. (transparencies).
                                    30:23 - 32:21    25.55 - 25.60
  shows that a certain subgroup H is normal in A4.
                                    33:08 - 35:13    25.61 - 25.66


   punchline. A4/H is isomorphic to Z3.
                                    35:13 - 37:44    25.67 - 25.73
 Example. Group explorer.
   1. Cayley table for A4. similar visuals to what he did on
       the board. but also counterexamples (organizing A4 by
       subgroups that are not normal).
                                    37:44 - 41:28    25.74 - 25.83
   2. Cayley table for D4.
                                    41:28 - 43:18    25.84 - 25.87
   3. Cayley table for S4.
                                    43:18 - 44:28    25.88 - 25.89
   4. Cayley diagram. intro.
                                    44:28 - 45:28    25.90 - 25.92
   5. Cayley diagram for A4. normal subgroup.
                                    45:28 - 49:12    25.93 - 25.98
   6. Cayley diagram for A4. subgroup supposedly not normal.
       gets into problems.
                                    49:12 - 52:29    25.99 - 25.104
```

Appendix H.

Student Contributions: Lecture 9.

Student Contributions: Lecture 9

09.17	John. About the centralizer and center of a group. Not a question, but a statement.	2
09.24	They answer questions about the generators of Z.	3
09.42	John asks about the definition of a to the 0. Reveals soon that he has a followup that is his real question: about whether he could replace a justification that J used with a justification of his own that was different.	3
09.43	At start John begins to agree, but then breaks off, in a 'hold on a sec' spirit, renewing his belief with a further back-to-you query. J rebuts at greater length.	1
09.44	Peter. That one direction of the double implication is always true, and they are checking the other part. Here it is a question of the form "Statement, right?".	1
09.47	Somebody answers his question about which of the two set-containments is easy to prove.	1
09.50	Somebody answers his question about technique to use: the division algorithm.	1
09.59	Somebody answers his question about which technique to use: the division algorithm. He thinks momentarily they are wrong, looks at his notes and realizes they are right.	1
09.62	They eventually nail down what to conclude and why it is true (answer hinges upon the definition of order of an element).	7
09.66	Peter (with assist from John). Why did J say at first that they didn't need to use the division algorithm. P P J P.	4
09.76	John asks about when did they prove that the n elements they wrote down on the board were all distinct. J realizes he didn't say a proper proof and starts saying one right then and there.	5

09.77	John says "ok:.	1
09.86	Peter. J pulls it out of him. This whole episode is spontaneous and unplanned. J is improvising. Peter had been talking to John about an infinite order group that had a finite order element.	2
09.88	John offers "symmetry group of a circle".	1
09.90	Peter offers "rotation by 1 radian" and Bart justifies why this would have infinite order.	2
09.91	Thomas. Makes joke about how the boring thing J will show is probably what he would have thought of.	1
Total.		36

Appendix I.

Student Contributions: Lecture 28.

Student Contributions - Lecture 28

28.03	John makes joke "for being late".	1
28.04	Peter recognizes J's reference "mission impossible". Off-topic comment.	1
28.06	Thomas makes one word responses to J's jokes about how much toffee he will give them.	2
28.07	John answers J's "what do I write as the next line".	2
28.10	Bart slowly unfurls, with J's pulling, an answer to "how does that determine phi of 1", together with a justification.	3
28.12	Peter asks "what's our group operation". He slowly opens up the problem of $Z15$ vs U of 15.	4
28.13	It transpires that this question came up in the tutorial. Many students jump in. Ron P P B B P B x5 P P.	13
28.14	Peter answers a bit self-righteously "I know that exists" when J speculates "you don't think that 7 inverse mod 15 exists".	1
28.15	Peter asks again because he still doesn't have clarity "can we do that". J gives long, not completely satisfying answer. J starts using the word "interpret".	1
28.16	Bart makes joke comment. Peter still doesn't understand. J promises to give better answer another time. B P B.	3
28.21	Peter correctly answers J's "what can we do with that".	2
28.23	Thomas correctly fills in the gap in the equation "phi of what equals 3". J says there are other ways, but this one is easiest.	1
28.28	Peter correctly answers J's "what's a good way to get some conditions going".	1

28.29	Peter correctly gives an upper bound for J's "how many possibilities are there for phi of 1".	2
28.30	Peter and Bart bat around an idea a bit desultorily. J retries his question "don't think too hard". Bart correctly answers "Lagrange". P P B P B B.	6
28.31	Kevin correctly answers J's "what's the order of 1".	1
28.35	Thomas gives all 3 correct answers to J's promptings about the orders of various elements.	3
28.38	Peter and Thomas give correct answers "0 of those", "uh none" to J's "how many of those are onto". As very frequently, his followup is "because__". John supplies the answer to this. P T A T.	4
28.40	Peter says "well-defined" in answer to J's "what do I need to check about phi n". J is impressed.	1
28.46	Thomas talks at length out loud to say in his own words his explanation for why J did what he just did. J says "exactly".	3
28.49	Peter asks one of his "so we don't need to say anything" sort of questions. He believes a justification is missing, and wants to know why it is missing. He was expecting a "because" statement that never came. J says this justification is logically not needed. Other students start to pipe up to help explain, including a lengthy one from Thomas. Eventually they all agree that, yes, the result requires commutativity, but that it's obvious, so it's not worth writing (or even saying). P P B T P T B P.	8
28.50	John makes lengthy unprompted comment, attempting to "generalize" the strategy of what had been done a bit earlier. He times this just before J starts a new thing. J says "sounds reasonable" and adds nothing else. He wants to move on. J J J P J.	5
28.55	Thomas suggests the conclusion to the result J is writing after J asks "what do you think". Then J wants to take a vote as to whether this is right or not. Peter makes joke comment "I gotta think about it". J's intention here is to point out that what one might have naturally expected to be the case is not true.	2

28.56 Bart eventually stammers out the correct conclusion. J says "it's gonna take us a little while / till we get to the example that demonstrates that". B B B P B P. 6

28.57 Peter comments, then asks question, revealing that he thought G bar was the same as phi of G. He is quickly set straight by Bart and J. P P B P P. 5

28.58 Bart asks if "phi G is a subgroup of G bar". John answers. J agrees. Then Peter begins to make a comment in the nature of a remark, then John thinks of an objection to the remark, and the two sort of trail off talking to each other. J cuts in to begin the next stanza. B J B P J P. 6

28.66 Thomas, in a lovely and important stanza, comes up with a counterexample to the result they first began in 28.55. One of his contributions is quite long. It takes J a few stanzas for him to figure out that Thomas is right. 5

28.67 Thomas continues his explanation. J wants to be "specific". Ironically, this is one of those occasions where being specific gives you no advantage. J does temporary writing. T T B B B T x 4. 9

28.68 Thomas continues his explanation. He is impeded by so many different objects really being identified in his "trivial" example. Peter contributes a suggestion, but Thomas points out, to Peter's agreement, that that won't work. T P T P T T. 6

28.69 Bart essentially takes over the explanation now that he gets it, and he is the first to state the crucial words "identity mapping". Quite a bit of overlapping talk, the most in the course.
J B J B T P B P B T P B x3. 14

28.70 J nods and agrees. Peter makes joke about "I don't know how trivial that was". J names the strategy: "you've sort of artificially inflated G bar". Thomas says "yeah". 2

28.74 Peter asks a question about notation and name. "why do we call it K bar". Bart answers, but it's mostly J.
P B P P. 4

28.76 Peter wonders why they don't "just" do something. J says they will, but they have to go the other way as well, which requires some more justification. 4

28.87	Peter offers what they should do in order to begin a certain proof. John supplies a bit more detail. J accepts. P J P.	3
28.88	J asks "how am I gonna get there". Peter and Bart eventually combine to give what J wants. P T B P B.	5
28.89	J wonders if Peter prefers a different method, Peter concedes "I guess that's a better way of dealing with it".	1
Total.		140

Appendix J.

Gestures and the Body: Lecture 24.

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Gestures and the Body - Lecture 24

24-00:00    transparency touches_reads
24-00:56    quick_sudden_body shock 'suspense'
24-01:48    first_writing_of_the_day
24-04:12    taps condition_multiplication_cosets
24-04:14    pass
24-04:17    grab_subgroup
24-04:40    flips reflections
24-05:00    condition_multiplication_cosets YES
24-05:13    pass
24-05:25    touches_performs_computation
24-05:53    condition_multiplication_cosets
24-06:30    compares_contrasts not_equal
24-06:54    hands_slow wavering uncertain body_motion confidence
24-07:10    well_defined star
24-07:27    same simultaneous
24-07:29    same simultaneous
24-07:31    spotlight not_equal YES
24-08:04    shrugs well_defined
24-08:07    palm_up 'what went wrong here'
            episode_factor_group_well_defined_example
24-08:59    spotlight how_to_say
24-10:05    spotlight why_important
24-10:13    spotlight implication_symbol
24-10:21    waves factor_group_gesture
24-10:27    spotlight type group
24-10:32    depicts backwards_implication_symbol
24-10:37    shrugs his_vs_Gallian
24-10:50    shakes deliberately_imprecise vague
24-10:52    shakes 'fail'
24-12:19    the_only_way_gesture singleness_gesture
24-12:27    push_away 'failure of group structure'
24-14:35    writes_question_mark diagram
24-16:40    taps
24-17:13    nonlinear_writing_begin_end
24-17:25    nonlinear_writing_begin_end
24-18:02    sweeps_gap
24-18:07    swivels left_coset_right_coset commutativity
24-18:09    action_figure move_term_through_normal_subgroup YES
24-18:23    back_and_forth 'get rid'
24-18:35    covers conceals hides
24-18:57    spotlight move_term_through_normal_subgroup
24-19:13    pacman_gesture 'gobble_up' YES
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24-21:25	notint
24-21:43	nonlinear_writing_begin_end
24-23:02	spotlight condition_normal_subgroup YES
24-23:44	holds_up_transparency YES
24-23:52	spotlight neuro
24-24:08	sweeps compares_contrasts no_choice_vs_not_easy
24-24:52	walks_to_touch single_word group 'allowed'
24-26:00	this_proof_breaks_his_rules
24-27:02	shakes factor_group_gesture superstructure_gesture star
24-27:08	shakes factor_group_gesture superstructure_gesture star personifies_subgroup
24-27:16	shakes 'internal complications'
24-27:13	factor_group_gesture superstructure_gesture
24-27:27	factor_group_gesture superstructure_gesture
24-27:30	shakes 'poke around'
24-28:41	spotlight cautions warns 'might be infinite'
24-29:38	temporary_writing warning_notation
24-31:07	wags_finger invites_students_to_recognize_object factor_group
24-34:00	spotlight_two intentional_relation subgroup
24-37:28	spotlight notation_of_others looks_off_distance coset_mod precise
24-37:33	shrugs
24-37:44	goodbutpass
24-38:14	undergrab 'really doing'
24-38:48	coset_reduced_representative_gesture sound_effect collapse_gesture YES
24-38:55	shakes uncertain temporary_writing writes_to_spotlight
24-39:07	writes_mapping_vertically unique_instance
24-39:20	many_to_1 fist singleness_gesture 'factoring out'
24-39:22	churns 'get rid' factor_group_gesture superstructure_gesture
24-39:24	fist 'just focus on' factor_group_gesture superstructure_gesture
24-39:46	absorbing_gesture melds gathers amalgamates star YES
24-41:00	distant_point next_writing
24-40:38	sweeps_gap
24-41:32	rotations_gesture
24-41:40	waves shakes 'you've been doing this since before'
24-41:39	waves temporal
24-41:47	sweeps up_down
24-43:08	spotlight_sequence restriction_on_domain typical_element
24-43:22	receiving_gesture
24-43:34	many_to_1 fist collapse_gesture
24-43:43	repeats_spotlight quotation absorbing_gesture many_to_1 star
24-43:50	nods

24-44:45	chops 'way back when'
24-45:50	series_examples spotlight corresponds_to analogous star
24-46:02	palm_up grasp complete_gesture
24-46:22	spotlight_condition
24-47:30	boxes symbol callback
24-48:55	temporary_writing to_understand_student