On the calculation of risk measures for variable annuities with guaranteed benefits

by

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Abstract

With the development of the life insurance industry, different types of life insurance products, in addition to the traditional ones, are being developed. A common and well-known life insurance product is the variable annuity with different types of guaranteed benefit riders, which provides policyholders a high rate of investment return with downside risk protections. Two typical distortion risk measures, VaR (value at risk) and CTE (conditional tail expectation), are widely used to manage insurers' future liabilities to avoid the potential of insolvency. In this project, we consider variable annuities with certain types of guaranteed benefits and various asset price processes, and focus on the calculation of the two risk measures of insurers' net and gross liabilities at the maturity date. Specifically, we consider two types of guaranteed benefit riders, the guaranteed minimum death benefit (GMDB) and the guaranteed minimum maturity benefit (GMMB), and assume that the logarithm of underlying asset returns follows a Cauchy or a skew-normal distribution. Analytical expressions of VaR and CTE for insurers' future liabilities are obtained, and numerical calculation algorithms are proposed. Comparisons of the calculated risk measure results with that under the normal distribution are also presented.

Keywords: Analytical expressions; Risk measures; Variable annuity; Cauchy distribution; Skew-normal distribution.

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Chapter 1

Introduction

1.1 Overview

In recent years, many types of equity-linked life insurance and annuity products have been developed. One common type of equity-linked life insurance product is the variable annuity contract with or without guaranteed benefits. In the U.K. and most of the European countries, variable annuity contracts (commonly used terminology in the U.S.) are called equity-linked policies and, in Canada, they are known as segregated fund policies. In general, variable annuities have a benefit linked to the performance of an investment fund. There are many types of guaranteed benefits associated with variable annuity contracts; we call them rider (or riders if multiple types of guaranteed benefits are applied).

For a variable annuity with guaranteed benefit riders, the policyholder pays the insurer a single premium at the beginning of the policy term or periodic premiums during the policy term. The insurer invests the initial premium and follow-up premiums in an asset or assets (normally a risky asset such as a stock or any other financial derivatives) in a separate account at the time that the premiums are being collected. By the end of the term, the policyholder receives a value that is the accumulated value of the premiums, if no guaranteed benefit riders are carried. In the case that the variable annuity carries a guaranteed minimum maturity benefit (GMMB) rider, the policyholder receives a value at the maturity date, which is the maximum value between the accumulated value of the premiums and the guaranteed level of the benefit set at the beginning of the contract. In the case that the variable annuity contract carries a guaranteed minimum death benefit (GMDB), the policyholder receives a value at the time of the policyholder's death, which is the maximum value between the accumulated value of the premiums and the roll-up guaranteed benefit level set at the beginning of the contract. Throughout the term, the management fees and rider charges are deducted from the separate fund.

Variable annuity products become increasingly popular around the world because they can be used as a long term investment plan for retirement. They are tax-deferred investment vehicles; that is, policyholders do not need to pay income tax on any investment gains during the policy term until they withdraw their money as income. Variable annuities are not absolutely guaranteed with respect to their investment growth; while they allow for huge gains, they also face potential losses. Variable annuities with guaranteed benefits provide policyholders protections against inflation risk and market volatility; for example, a variable annuity with a guaranteed lifetime withdrawal benefit provides the policyholder a guaranteed income for life even if the market remains unstable or drops precipitously. Hence, variable annuities with guaranteed benefits become one of the ideal choices for investors to receive higher expected investment returns with downside financial market protection.

From the insurers' point of view, it is essential to manage and monitor the fund performance of the separate account during the policy term. Two well-known risk measures, the value at risk (VaR) and the conditional tail expectations (CTE), have been used as management tools by insurance and banking industries over the past few decades. The VaR is a quantile of the loss distribution at a specific significance level, while the CTE gives the conditional expected loss given that the losses exceed the VaR at a given significance level. Both risk measures provide references when monitoring and mitigating losses in the worst-case scenarios. Moreover, the CTE is superior to the VaR in the sense that the CTE is a coherent risk measure and is more robust to sampling errors (Hardy, 2006).

In this project, we are interested in valuing variable annuity contracts with either a GMMB and a GMDB rider. Among other related references, Feng and Volkmer (2012) propose an analytical calculation method for computing VaRs and CTEs for insurers' liabilities of variable annuities with either of the above mentioned two riders. One of the crucial assumptions for their study is that the prices of the underlying asset follow a geometric Brownian motion (GBM). Due to advanced properties of the GBM, the analytical expressions for calculating the risk measures such as VaR and CTE can be obtained. As pointed out in Feng and Volkmer (2012), Monte Carlo simulation method is one of the most commonly used methods by the insurers. However, the Monte Carlo simulation method could be extremely demanding and time consuming. In particular, this approach may be costly and challenging in practice for small sized companies. A literature review on related topics to those discussed above is provided in Chapter 2.

1.2 Motivation

Stylized facts of stock (asset) returns show that continuously compounded returns (returns henceforth) may not be Gaussian (i.e., normally distributed), and the empirical distribution of these returns may be skewed (i.e., not symmetry) and heavy-tailed (see, for example, Cont, 2001). In this case, the GBM model for the stock prices or the normal distribution for returns may not be appropriate for modeling stock returns. Eling (2014) considers several skewed stock returns distributions. Motivated by the study presented in Feng and Volkmer (2012), and considering the empirical characteristics of returns, we propose two alternative distributional models for modeling asset returns in variable annuities with guaranteed benefits. The return models that we consider in this project are the Cauchy distribution and the skew-normal distribution, which could capture the skewness and the heavy tails presented in the data. We follow similar techniques as in Feng and Volkmer (2012) to derive analytical expressions of VaR and CTE for gross liabilities of variable annuities with either GMMB or GMDB rider. We present calculation algorithms based on Monte Carlo simulation for calculating both VaR and CTE risk measures for net liabilities of variable annuities with either rider. The S&P 500 stock index historical data are fitted to the Cauchy and skew-normal models, and numerical values of risk measures under these asset price models are presented. The results for the normal model are also included for comparison purposes.

1.3 Outline

The remainder of this project report is organized as follows. Chapter 2 provides a literature review on the modeling of stock returns, related studies on the pricing and valuation of variable annuities with guaranteed benefits, and risk measure calculation methods. Chapter 3 presents details of three asset models and introduces the concept of future liabilities for a variable annuity with GMMB or GMDB rider. Analytical expressions for gross liabilities are derived, and calculation algorithms for net liabilities are presented under the three asset models. Chapter 4 shows the statistical analysis on the S&P 500 returns data, and the numerical risk measure results under the three fitted asset models. The conclusion of this project and possible further research on related topics are provided in Chapter 5.

Chapter 2

Literature review

In this chapter, we provide a literature review on three topics. We first review the modeling techniques for asset prices, mainly focusing on the application of the lognormal model or the geometric Brownian motion, and other alternative distribution models. We then review recent research and publications on the pricing and valuation of variable annuities with various types of guaranteed riders. Lastly, we review studies related to the risk measures and corresponding theoretical analysis on the risk measures.

2.1 Models for asset prices

One of the most widespread models used for modeling asset prices is the geometric Brownian motion (GBM), a continuous-time stochastic process. The GBM can be used to describe the dynamics of stock prices; in this case, the logarithm of the asset gross returns follows a Brownian motion (also called Wiener process) with drift. It is also known as a lognormal model because, in this case, the logarithm of the returns follows a normal distribution at any fixed time point. Lidén (2018) fits the normal and Cauchy distributions to stock returns data and shows that the Cauchy distribution fits better. Simulated future stock price paths are also obtained by using Monte Carlo simulation. Gerber and Shiu (2003) fit the GBM to the asset and liability processes of a pension fund and calculate the expected discounted value of the payments to be made by the sponsor and that of the refunds to the sponsor. Reddy and Clinton (2016) present a comparison of the actual stock prices movements and simulated stock prices paths by using the lognormal model. According to their study, there are above 50% chances that stock prices simulated by the lognormal model have the same directional movements as the actual ones.

Many stylized empirical facts and statistical issues of the asset returns are discussed in Cont (2001). One of the most significant stylized facts is that the empirical distribution of the stock returns exhibits fat tails, which excludes the normal distribution for modeling purposes. Aggarwal et al. (1989) study Tokyo stock market data from 1965 to 1984, in which significant skewness and kurtosis are observed from empirical distribution of the stock returns. Thereafter, many studies in financial engineering and actuarial science fields considered other desirable distributions as alternatives to the normal distribution. For example, Eling (2014) fits the stock returns data to some skewed distribution models such as skew-normal and skew-student t distributions; the analysis shows that such skewed distribution models are promising for modeling returns. Choi and Yoon (2020) present model comparison study on several stock returns data by using twelve different distributions, including fat-tail distributions such as the Cauchy distribution, and skewed distributions such as the skew-normal and skew-student t distributions. More recently, Mahdizadeh and Zamanzade (2019) fit the Cauchy distribution to the stock returns data and proposes six new goodness-of-fit tests to show that fat-tail distributions like the Cauchy fit data better than the normal distribution.

2.2 Variable annuity with guaranteed benefits

A variable annuity, also called an equity-linked insurance contract, is a life insurance product that has been common worldwide since 1960s. Hardy (2003) provides a comprehensive guide and detailed information on life insurance products with investment guarantees, including their modeling and risk management. Major benefit riders introduced in this book are guaranteed minimum maturity benefit (GMMB), guaranteed minimum death benefit (GMDB), guaranteed minimum accumulation benefit (GMAB), guaranteed minimum surrender benefit (GMSB), and guaranteed minimum income/withdrawal benefit (GMIB/GMWB). These guaranteed benefit riders are designed to provide policyholders with downside risk protection when markets are in turmoil. There are many studies on different aspects of variable annuities such as pricing and valuation of variable annuities with guaranteed benefit riders from the insurers' point of view; see Feng et al. (2022) for a survey on variable annuity pricing, valuation, and risk management.

When pricing variable annuities over long time periods, the dynamics of the fund values with guaranteed benefit riders are typically based on the randomness of risk factors such as the underlying asset returns, interest rates, and mortality rates. Peng et al. (2012) present pricing formulas for variable annuities with a GMWB rider by assuming the stochastic interest rate model of Vasicek (Vasicek, 1977); lower and upper bounds of the pricing functions are obtained, and the pricing behaviour when model parameters are changed is also analyzed. Dai et al. (2015) present a valuation method to price variable annuities with a guaranteed lifelong withdrawal benefit (GLWB) rider, which is the lifelong version of the GMWB. Mortality risk plays a significant role in valuing such variable annuities with long policy terms. A valuation method, called the three-dimensional tree, is used to analyze the impact of pricing GLMB with different policy provisions. Barigou and Delong (2022) present a pricing formula for variable annuities with both GMDB and GMMB riders by expressing the underlying stock price as the solution of a backward stochastic differential equation with jumps and obtain numerical results by using the back-propagation neural network. Bacinello et al. (2011) present a unifying approach for the valuation of variable annuities with guaranteed benefit riders. The contract values are computed and compared under different valuation approaches using the ordinary and least squares Monte Carlo simulation methods. Huang et al. (2022) develop a computationally efficient approach to value variable annuities with GMAB and GMMB riders, where different stochastic processes of risk factors include lapse rates. Their valuation method is more efficient than the ordinary Monte Carlo simulation method. Gerber et al. (2012) present closed-form formulas for the expected discounted value of the death benefit payment at the time of death for variable annuities with equity-linked death benefits. The discounted density approach is used, and the time-of-death distribution is under the assumption of constant force of mortality. Liang et al. (2016) extend the study by Gerber et al. (2012) by assuming that the time-until-death distribution follows the piecewise constant force of mortality assumption.

2.3 Risk measures and calculation methods

As discussed in the introduction, managing and monitoring the fund performance is one of the most important parts for equity-linked life insurance providers. Risk management plays a vital role in managing the fund performance. Balbás et al. (2009) study the properties of distortion risk measures that need to be satisfied to avoid inconsistent investment portfolio decisions. Note that both VaR and CTE are distortion risk measures. Artzner et al. (1999) present a set of four properties for risk measures (monotonicity, subadditivity, positive homogeneity and translation invariance) and call a risk measure satisfying these four properties coherent. It is well-known that VaR is not a coherent risk measure because it fails to satisfy the subadditivity property[1]. Wirch and Hardy (2003) present a necessary and sufficient condition for a risk measure to be coherent; that is, the risk measure can be expressed in terms of a concave distortion function. To determine which risk measures are more adequate and stable, Cont et al. (2010) provide a robustness and sensitivity analysis of risk measure procedures and illustrates that using CTE leads to a less robust risk measurement procedure than VaR based on their proposed estimators and criteria.

Generally, there are two types of methods for evaluating the prices or risk measures of variable annuity with guaranteed benefits. One is to derive the analytical closed-form expressions for certain guaranteed riders as in Feng and Volkmer (2012), Peng et al. (2012), Barigou and Delong (2022), Huang et al. (2022), Gerber et al. (2012), and Liang et al. (2016). The another method is based on numerical algorithms as in Bacinello et al. (2011) and Dai et al. (2015). In this project, we calculate the risk measures based on analytical

[1]Let ρ: $X \to \mathbb{R}$ represent the risk measure functional, where X is a random variable. The risk measure ρ satisfies the subadditivity property if, for any two random losses X_1 and X_2, $\rho(X-1+X_2) \leq \rho(X_1)+\rho(X_2)$.

expressions for gross liabilities of the variable annuity with either a GMMB rider or a GMDB rider and use Monte Carlo simulation to calculate the two risk measures based on their corresponding net liabilities.

Chapter 3

Future liabilities and Risk measures

In this chapter, we first introduce two types of insurers' future liabilities for variable annuities with either a GMMB or a GMDB rider. The two types of future liabilities are the gross liability and net liability. We then consider two risk measures, VaR and CTE, and use them to evaluate risks with respect to their future liabilities with guaranteed benefit riders under three equity models (normal, Cauchy, and skew-normal).

We first introduce the notation we use in this study.

- G — the guaranteed level of variable annuity. It represents the lowest value that the policyholder will receive at the end of the policy term.
- T — the maturity date of the policy, which is when the policyholder can cash out the policy benefits.
- F_t — the projected future value of fund in the separate account at time t (≥ 0), where F_0 is the initial fund value at the beginning of the policy term. For simplicity, we assume that the policyholder pays only one single premium F_0 (purchase amount) at issue; that is, no additional purchase payments or withdrawal is allowed.
- S_t — the projected future value of underlying asset at time t (≥ 0). In this project, we assume that the initial fund value F_0 is fully invested in one asset.
- m — the annualized rate of mortality and expenses fees that is deducted from the separate account, which is being taken continuously during the term since no withdrawal is allowed.
- m_e — the annualized rate of charges allocated to the GMMB rider.
- m_d — the annualized rate of charges allocated to the GMDB rider.
- ${}_eL_g^0$ — the present value of insurers' gross liability for GMMB rider at time 0.

- ${}_eL_n^0$ — the present value of insurers' net liability for GMMB rider at time 0.
- ${}_dL_g^0$ — the present value of insurers' gross liability for GMDB rider at time 0.
- ${}_dL_n^0$ — the present value of insurers' net liability for GMDB rider at time 0.
- ${}_tp_x$ — the survival probability for a life age x who will survive t years.
- μ_{x+t} — the force of mortality for a life age $x+t$.
- τ_x — the future lifetime for a life age x.

The rest of this chapter is organized as follows. Two risk measures for gross and net liabilities are introduced in details in Section 3.1. Section 3.2 introduces three models for the underlying equity, and then presents analytical expressions of risk measures for gross liabilities under the three different models. Section 3.3 provides general analytical expressions and numerical algorithms for computing and approximating net liability risk measures.

3.1 Risk measures and future liabilities

As in Feng and Volkmer (2012), we study the two most popular and widely applied risk measures: VaR and CTE. VaR is a quantile risk measure that gives the quantile of a random variable at some specific significance level α $(0 \leq \alpha \leq 1)$. Given the significance level α, VaR represents the loss amount that will not be exceeded with probability α. It is helpful for investors to measure their potential losses and manage their risk capital during the investment period. CTE is the conditional expectation of a random variable given that the random variable is greater than its VaR at a given significance level. CTE helps investors to estimate the expected value of losses given that the losses exceed the given VaR. CTE is a coherent risk measure, while VaR is not because it is not subadditive.

3.1.1 GMMB future liabilities

A variable annuity with a GMMB rider provides a policyholder with a guaranteed minimum maturity benefit at the end of policy term. The payment varies based on the investment performance of the fund, especially the fund value at the maturity date. Since the variable annuity with a GMMB rider has a fixed term policy, the future underlying asset returns is the only source of randomness that impacts the insurer's future gross liabilities.

Let ${}_eL_g^0$ be the insurer's gross liability for the GMMB rider at time 0 given by

$$ {}_eL_g^0 = e^{-rT}(G - F_T)_+ I_{\{\tau_x > T\}}, \tag{3.1} $$

where r is the risk-free force of interest. Function $(x)_+$ returns the maximum value of $\{x, 0\}$, and $I_{\{x\}}$ is the indicator function which equals 1 if the statement x is true or equals 0 otherwise. Note that in this project, T is measured in years.

Before we formulate the net liabilities, we introduce an additional quantity called the management expense at time t and denoted by M_t. The management expense represents the management fee that is charged continuously by the insurer during the policy term. Following Feng and Volkmer (2012), we assume that the amount of management expense at time t is proportional to F_t, the value of the fund at time t; it is given by

$$M_t = m_x F_t, \qquad 0 \leq t \leq T, \tag{3.2}$$

where m_x=m_e for the GMMB case, and m_x=m_d for the GMDB case. When we formulate the present value of gross liabilities at time 0, the only cash flow we consider is the future maturity benefit payoff to the policyholder. However, insurer collects management fees during the entire policy term, which results in a negative cash flow from the insurer's losses point of view. Hence, the two sources of randomness for the present value of the net liability are the underlying asset price and the policyholder's future lifetime. The present value of insurer's net liability for the GMMB rider at time 0 is then given by

$${}_eL_n^0 = e^{-rT}(G - F_T)_+ I_{\{\tau_x > T\}} - \int_0^{\tau_x \wedge T} e^{-rs} M_s ds, \tag{3.3}$$

where the first component of Equation (3.3) is the present value of the gross liability at time 0, and the second component is the collected management expenses during the policy term. We notice that if a policyholder survives beyond the policy maturity date, the integral in Equation (3.3) is bounded at T, representing that the management expense is fully deducted by the insurer from policyholder's variable annuity separate account.

3.1.2 GMDB future liabilities

A variable annuity with a GMDB rider provides a policyholder with a guaranteed minimum death benefit at death. The payment to the policyholder is determined by the roll-up guaranteed amount and the separate account fund value at the time of the policyholder's death. Unlike the GMMB rider, the GMDB rider sets a fixed amount G at time 0, and G is accumulated with a fixed roll-up rate during the survival period of the policyholder, which produces a variable guaranteed amount depending on the time of death. In general, the roll-up rate used to accumulate the guaranteed amount is less than the risk-free force of interest. We denote the roll-up rate by δ and assume $0 \leq \delta \leq r$. The gross liability of GMDB rider at time 0 is given by

$${}_dL_g^0 = e^{-r\tau_x}(e^{\delta\tau_x} G - F_{\tau_x})_+ I_{\{\tau_x \leq T\}}. \tag{3.4}$$

Similar to the net liability defined for the GMMB rider case, the net liability for GMDB rider is given by

$$ {}_dL_n^0 = e^{-r\tau_x}(e^{\delta\tau_x}G - F_{\tau_x})_+ I_{\{\tau_x \le T\}} - \int_0^{\tau_x \wedge T} e^{-rs} M_s ds. \tag{3.5}$$

3.1.3 Two risk measures for guaranteed riders

We discuss the two risk measures, VaR and CTE, for both the GMMB and GMDB riders in this subsection. The quantile risk measure with a given significance level α, denoted by V_α, is define as

$$ \mathrm{V}_\alpha \equiv \inf\{x : \mathbb{P}[L^0 \le x] \ge \alpha\}, \tag{3.6}$$

where L^0 is a general form of insurer's loss, representing the net present value of insurer's future liability at time 0 in this project. Typical values for α are 95% or 99% (Hardy, 2006). The value of V_α estimates the amount that with probability α, the present value of insurer's future liability will not be exceeded.

The conditional tail expectation risk measure with a given significance level α, denoted by CTE_α, is defined as

$$ \mathrm{CTE}_\alpha \equiv \mathbb{E}[L^0 | L^0 > \mathrm{V}_\alpha]. \tag{3.7}$$

Typical values of α for CTE_α are 90%, 95%, or 99% (Hardy, 2006). The value of CTE_α estimates the amount that represents the average amount of insurer's future liabilities when they exceed V_α.

For insurance companies, it is essential to analyze both gross and net liabilities. Gross liabilities give an insurer a good sense to manage liability risks because gross liabilities do not include any future negative cash flow (management fees), while the net liability includes both the future positive cash flow (benefit payout) and the future negative cash flow (management fees). The latter helps the insurer manage both liability risks and asset risks. In this project, we calculate the two risk measures for gross liabilities by using the analytical formulas we derive, and we estimate the two risk measures for net liabilities based on Monte Carlo simulation.

3.2 Analytical results for gross liabilities

We have introduced the definitions of gross liabilities, net liabilities, and two risk measures for variable annuities with either a GMMB rider or a GMDB rider in Section 3.1. In this section, we first present three models for the asset price process. We then review the analytical results for gross liability risk measures based on the normal model used in Feng and Volkmer (2012) and provide analytical expressions for the gross liability risk measures based on the Cauchy and skew-normal models considered in this project. In the last section of this chapter, we provide algorithms for calculating risk measures of net liabilities by using the Cauchy model as an example.

3.2.1 Underlying equity models

As in Feng and Volkmer (2012), we assume that the account value (market value) of a variable annuity at time t, F_t, is described by

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \le t \le T, \tag{3.8}$$

where S_t is the market value of the underlying asset at time t, F_0 is the initial payment (premium) paid at inception, S_0 is the initial value of the asset, and m is the annualized rate of mortality and expenses fees being deducted continuously. Note that this account value does not take the benefit guarantees into consideration.

The lognormal model for equity returns is assumed in Feng and Volkmer (2012); it is also known as a geometric Brownian motion model. That is, the underlying asset price process $\{S_t\}_{t\ge 0}$ on a probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$ is given by

$$S_t = S_0 e^{\mu t + \sigma B_t}, \qquad t > 0, \tag{3.9}$$

where $\{B_t\}_{t\ge 0}$ is a standard Brownian motion.

In this project, we consider two additional models for equity returns: log-Cauchy and log-skew-normal models. The lognormal model studied in Feng and Volkmer (2012) is included in the following for the purpose of numerical comparisons presented in Chapter 4. We denote these three models as normal, Cauchy, and skew-normal in this report. We now describe these three models.

- Normal model

  Assume that the logarithm of S_t/S_{t-1} for $t \in \mathbb{N}^+$ follows a normal distribution with mean μ and standard deviation σ (>0). This means that the returns of the underlying equity fund from time $t-1$ to time t follows a normal distribution, namely,

  $$\ln\left(\frac{S_t}{S_{t-1}}\right) \sim \text{Norm}(\mu, \sigma), \qquad t \in \mathbb{N}^+,$$

  or

  $$\frac{S_t}{S_{t-1}} \sim \text{Lognorm}(\mu, \sigma), \qquad t \in \mathbb{N}^+.$$

  The normal distribution is symmetric. The probability density function (pdf) of the normal distribution with parameters μ and σ is given by

  $$\phi(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty, \tag{3.10}$$

and its cumulative distribution function (cdf) is given by

$$\Phi(x;\mu,\sigma) = \frac{1}{2}\left[1+\text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \qquad -\infty < x < \infty, \tag{3.11}$$

where $\text{erf}(x)$ is the error function given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Assume that the returns in the underlying equity fund from time $t-1$ to time t, for $t \in \mathbb{N}^+$, are identical and independent distributed (i.i.d.). We now first write $\ln(S_t/S_0)$ as

$$\ln\left(\frac{S_t}{S_0}\right) = \underbrace{\ln\left(\frac{S_t}{S_{t-1}}\right) + \ln\left(\frac{S_{t-1}}{S_{t-2}}\right) + \cdots + \ln\left(\frac{S_1}{S_0}\right)}_{\text{independent and identically distributed}}. \tag{3.12}$$

Notice that the terms on the right-hand-side of (3.12) are i.i.d. and follow $\text{Norm}(\mu,\sigma)$. Applying the properties of the normal distribution, we then have that

$$\ln\left(\frac{S_t}{S_0}\right) \sim \text{Norm}\left(\mu t, \sigma\sqrt{t}\right), \qquad t > 0,$$

or

$$\frac{S_t}{S_0} \sim \text{Lognorm}\left(\mu t, \sigma\sqrt{t}\right), \qquad t > 0.$$

- Cauchy model

Assume that the logarithm of S_t/S_{t-1} for $t \in \mathbb{N}^+$ follows a Cauchy distribution with location parameter λ and scale parameter θ (> 0). This means that the returns of the underlying equity fund from time $t-1$ to time t follows a Cauchy distribution, namely,

$$\ln\left(\frac{S_t}{S_{t-1}}\right) \sim \text{Cauchy}(\lambda,\theta), \qquad t \in \mathbb{N}^+,$$

or

$$\frac{S_t}{S_{t-1}} \sim \text{Log-Cauchy}(\lambda,\theta), \qquad t \in \mathbb{N}^+.$$

The pdf and cdf of the Cauchy distribution with parameters λ and θ are given by

$$f_C(x;\lambda,\theta) = \frac{1}{\pi\theta}\left[\frac{\theta^2}{(x-\lambda)^2+\theta^2}\right], \qquad -\infty < x < \infty, \tag{3.13}$$

$$F_C(x;\lambda,\theta) = \frac{1}{\pi}\arctan\left(\frac{x-\lambda}{\theta}\right) + \frac{1}{2}, \qquad -\infty < x < \infty, \tag{3.14}$$

respectively. When $\lambda = 0$ and $\theta = 1$, it is called the standard Cauchy distribution.

Note that the Cauchy distribution is also symmetric similar to the normal one, but it does not have a mean, a variance or higher moments. The latter characteristic implies that the Cauchy distribution has a fat tail.

Assume that $\ln(S_1/S_0), \ln(S_2/S_1), \ldots,$ are i.i.d. and follow Cauchy(λ, θ). By applying the properties of the Cauchy distribution, we can similarly get that

$$\ln\left(\frac{S_t}{S_0}\right) \sim \text{Cauchy}(\lambda t, \theta t), \qquad t > 0,$$

or

$$\frac{S_t}{S_0} \sim \text{Log-Cauchy}(\lambda t, \theta t), \qquad t > 0.$$

Similar to (3.9), in this case we can write the underlying asset price process $\{S_t\}_{t\geq 0}$ as

$$S_t = S_0 e^{\lambda t + \theta C_t}, \qquad t > 0,$$

where $\{C_t\}_{t\geq 0}$ is a Cauchy process with location parameter 0 and scale parameter t; that is, for a fixed t, $C_t \sim$ Cauchy$(0, t)$.

- Skew-normal model

  Assume that the logarithm of S_t/S_{t-1} for $t \in \mathbb{N}^+$ follows a skew-normal distribution with location parameter η, scale parameter κ (> 0), and shape parameter $\omega \in \mathbb{R}^+$. This means that the returns in the underlying equity fund from time $t-1$ to time t follows a skew-normal distribution, namely,

$$\ln\left(\frac{S_t}{S_{t-1}}\right) \sim \text{skew-norm}(\eta, \kappa, \omega), \qquad t \in \mathbb{N}^+,$$

or

$$\frac{S_t}{S_{t-1}} \sim \text{Log-skew-norm}(\eta, \kappa, \omega), \qquad t \in \mathbb{N}^+.$$

The pdf and cdf of the skew-normal distribution with parameters η, κ and ω are given by

$$g_S(x; \eta, \kappa, \omega) = \frac{2}{\kappa}\phi\left(\frac{x-\eta}{\kappa}; 0, 1\right)\Phi\left(\omega\left(\frac{x-\eta}{\kappa}\right); 0, 1\right), \qquad -\infty < x < \infty, \quad (3.15)$$

$$G_S(x; \eta, \kappa, \omega) = \Phi\left(\frac{x-\eta}{\kappa}; 0, 1\right) - 2T\left(\frac{x-\eta}{\kappa}, \omega\right), \qquad -\infty < x < \infty, \quad (3.16)$$

respectively, where $T(h, \omega)$ is Owen's T function given by

$$T(h, \omega) = \frac{1}{2\pi}\int_0^{\omega} \frac{e^{-\frac{1}{2}h^2(1+x^2)}}{1+x^2}dx, \qquad -\infty < h < \infty,\ -\infty < \omega < \infty.$$

When $\eta = 0$ and $\kappa = 1$, it is called the standard skew-normal distribution.

Notice that the skew-normal distribution is an asymmetric distribution. The skew-normal distribution becomes normal distribution when $\omega = 0$, and it becomes half normal distribution when $\omega = -\infty$ or ∞.

Assume that $\ln(S_1/S_0), \ln(S_2/S_1), \ldots,$ are i.i.d. and follow skew-norm(η, κ, ω) distribution. By applying the properties of the skew-normal distribution, we can similarly get

$$\ln\left(\frac{S_t}{S_0}\right) \sim \text{skew-norm}(\eta t, \kappa\sqrt{t}, \omega), \qquad t > 0,$$

or

$$\frac{S_t}{S_0} \sim \text{Log-skew-norm}(\eta t, \kappa\sqrt{t}, \omega), \qquad t > 0.$$

Similar to (3.9), the underlying asset price process $\{S_t\}_{t\geq 0}$ can be written as

$$S_t = S_0 e^{\eta t + \kappa M_t}, \qquad t > 0,$$

where $\{M_t\}_{t\geq 0}$ is a skew-normal process with location parameter 0, scale parameter $\sqrt{t}$, and shape parameter ω; that is, for a fixed t, $M_t \sim$ skew-norm$(0, \sqrt{t}, \omega)$.

3.2.2 Risk measures for variable annuities with a GMMB rider

Before we proceed to the analytical results, we first determine the probability that positive liabilities occur. The insurer only considers the situation where there is a chance for positive future liabilities because negative future liability represents a profit. Considering the gross liability for a variable annuity with a GMMB rider, the probability that no guarantee payment will be made at maturity is given by

$$\xi_e = 1 - \mathbb{P}\left[G \geq F_T, \tau_x > T\right].$$

When calculating VaR and CTE risk measures, the significance level α should be chosen to be larger than ξ_e in the GMMB rider situation. By using (3.8) and assuming that the future lifetime of the policyholder and the account value are independent, we have

$$\xi_e = 1 - \mathbb{P}\left[G \geq F_T, \tau_x > T\right] = 1 - {}_Tp_x \mathbb{P}\left[e^{-mT}\frac{S_T}{S_0} \leq \frac{G}{F_0}\right].$$

Because $\ln(S_T/S_0) \sim \text{Norm}(\mu T, \sigma\sqrt{T})$, in the normal model case, we can easily get

$$e^{-mT}\frac{S_T}{S_0} \sim \text{Lognorm}((\mu - m)T, \sigma\sqrt{T}).$$

We then obtain

$$\xi_e = 1 - {}_Tp_x \Phi\left(\ln\left(\frac{G}{F_0}\right); (\mu - m)T, \sigma\sqrt{T}\right). \tag{3.17}$$

Similarly, in the Cauchy model case, the value of ξ_e is given by

$$\xi_e = 1 - {}_Tp_x F_C\left(\ln\left(\frac{G}{F_0}\right); (\lambda - m)T, \theta T\right), \tag{3.18}$$

where F_C is the cdf of the Cauchy distribution with corresponding parameters in the brackets and, in the skew-normal model case, the value of ξ_e is given by

$$\xi_e = 1 - {}_Tp_x G_S\left(\ln\left(\frac{G}{F_0}\right); (\eta - m)T, \kappa\sqrt{T}, \omega\right), \tag{3.19}$$

where G_S is the cdf of the skew-normal distribution with corresponding parameters in the brackets.

We now present the analytical results for the VaR and CTE on the gross liabilities, respectively, in the following two propositions for the variable annuities with a GMMB rider and for the three equity return models.

Proposition 3.1. For the three equity return models described in Section 3.2.1, we have the following results for the value at risk V_α, given that $\alpha > \xi_e$, and for GMMB gross liabilities. Note that ξ_e for corresponding normal, Cauchy, and skew-normal models are respectively given in (3.17), (3.18) and (3.19).

(1) Under the normal model, we have

$$\mathrm{V}_\alpha = e^{-rT}G - F_0 \exp\{(\mu - r - m)T + \sigma\sqrt{T}z_\beta\},$$

where z_β is the $100\beta\%$ percentile of the standard normal distribution with β = (1 - $\alpha)/{}_Tp_x$.

(2) Under the Cauchy model, we have

$$\mathrm{V}_\alpha = e^{-rT}G - F_0 \exp\{(\lambda - r - m)T + \theta T c_\beta\}, \tag{3.20}$$

where c_β is the $100\beta\%$ percentile of the standard Cauchy distribution with β = (1-$\alpha)/{}_Tp_x$.

(3) Under the skew-normal model, we have

$$\mathrm{V}_\alpha = e^{-rT}G - F_0 \exp\{(\eta - r - m)T + \kappa\sqrt{T}s_\beta\},$$

where s_β is the $100\beta\%$ percentile of the standard skew-normal distribution with β = (1-$\alpha)/{}_Tp_x$.

Proof. See Appendix A.1. □

Proposition 3.2. For the three equity return models described in Section 3.2.1, we have the following results for the conditional tail expectation CTE_α, given that $\alpha > \xi_e$, and for GMMB gross liabilities. Note that ξ_e for corresponding normal, Cauchy, and skew-normal models are respectively given in (3.17), (3.18) and (3.19).

(1) Under the normal model, we have

$$\text{CTE}_\alpha = e^{-rT}G - {}_Tp_x \frac{F_0}{1-\alpha} \exp\{(\mu - r - m)T + \sigma^2 T/2\}\Phi(Z_\beta; \sigma\sqrt{T}, 1),$$

where z_β is the $100\beta\%$ percentile of the standard normal distribution with β = (1 - $\alpha)/{}_Tp_x$, and Φ is the cdf of the standard normal random variable.

(2) Under the Cauchy model, we have

$$\text{CTE}_\alpha = e^{-rT}G - {}_Tp_x \frac{F_0}{1-\alpha} \int_{-\infty}^{\ln(a)} e^y \cdot f_C(y; (\lambda - r - m)T, \theta T)dy, \tag{3.21}$$

where $a = (e^{-rT}G - \text{V}_\alpha)/F_0$, and f_C is the pdf of the Cauchy distribution with location parameter $(\lambda - r - m)T$ and scale parameter θT.

(3) Under the skew-normal model, we have

$$\text{CTE}_\alpha = e^{-rT}G - {}_Tp_x \frac{F_0}{1-\alpha} \int_{-\infty}^{\ln(a)} e^y \cdot g_S(y; (\eta - r - m)T, \kappa\sqrt{T}, \omega)dy,$$

where $a = (e^{-rT}G - \text{V}_\alpha)/F_0$, and g_S is the pdf of the skew-normal distribution with location parameter $(\eta - r - m)T$, scale parameter $\kappa\sqrt{T}$, and shape parameter ω.

Proof. See Appendix A.2. □

3.2.3 Risk measures for variable annuities with a GMDB rider

We now derive analytical expressions for the V_α and CTE_α on gross liabilities with a GMDB rider. The gross liability in this case is given by (3.4). We first find an expression for ξ_d, which is the probability that the guarantee payment will not be paid before the maturity. Note that ξ_d is different from ξ_e in the sense that the guarantee payment increases at a rate δ and the payment is made at the time of death of the policyholder. The rate δ is called the roll-up rate which is normally set to be less than the risk-free rate r; that is, $0 < \delta < r$.

Assuming that the future lifetime of the policyholder and the account value are independent and conditioning on the time of death, we have

$$\xi_d = 1 - \mathbb{P}\left[e^{\delta\tau_x}G \geq F_{\tau_x}, \tau_x \leq T\right] = 1 - \int_0^T {}_tp_x\mu_{x+t}\mathbb{P}\left[e^{\delta t}G > F_t\right]dt,$$

in which ${}_tp_x\mu_{x+t}$ is the density function of the future lifetime of (x). Using (3.8), we can further write ξ_d as

$$\xi_d = 1 - \int_0^T {}_tp_x\mu_{x+t}\mathbb{P}\left[e^{-(m+\delta)}\frac{S_t}{S_0} < \frac{G}{F_0}\right] dt.$$

Because $\ln(S_t/S_0) \sim \text{Norm}(\mu t, \sigma\sqrt{t})$ in the normal model case, we can easily get

$$e^{-(m+\delta)t}\frac{S_t}{S_0} \sim \text{Lognorm}((\mu - m - \delta)t, \sigma\sqrt{t}).$$

We then obtain

$$\xi_d = 1 - \int_0^T {}_tp_x\mu_{x+t}\Phi\left(\ln\left(\frac{G}{F_0}\right); (\mu - m - \delta)t, \sigma\sqrt{t}\right) dt. \quad (3.22)$$

Similarly, in the Cauchy model case, the value of ε_d is given by

$$\xi_d = 1 - \int_0^T {}_tp_x\mu_{x+t}F_C\left(\ln\left(\frac{G}{F_0}\right); (\lambda - m - \delta)t, \theta t\right) dt, \quad (3.23)$$

where F_C is the cdf of the Cauchy distribution with corresponding parameters in the brackets. In the skew-normal model case, the value of ξ_d is given by

$$\xi_d = 1 - \int_0^T {}_tp_x\mu_{x+t}G_S\left(\ln\left(\frac{G}{F_0}\right); (\eta - m - \delta)t, \kappa\sqrt{t}, \omega\right) dt, \quad (3.24)$$

where G_S is the cdf of the skew-normal distribution with corresponding parameters in the brackets.

We now follow a similar idea to that used for GMMB rider for deriving analytical expressions of two risk measures for the GMDB rider. The following two propositions present V_α and CTE_α on the gross liabilities for the variable annuities with a GMDB rider under three equity return models.

Proposition 3.3. For the three equity return models described in Section 3.2.1, we have the following results for the value at risk V_α, given that $\alpha > \xi_d$, and for GMDB gross liabilities. Note that ξ_d for the normal, Cauchy, and skew-normal models are respectively given in (3.22), (3.23) and (3.24).

(1) Under the normal model, we have

$$1 - \alpha = \int_0^T {}_tp_x\mu_{x+t}\Phi\left(\ln\frac{e^{(\delta-r)t}G - \text{V}_\alpha}{F_0}; (\mu - r - m)t, \delta\sqrt{t}\right)dt,$$

where Φ is the cdf of the normal distribution with corresponding parameters given in the brackets.

(2) Under the Cauchy model, we have

$$1-\alpha=\int_0^T {}_tp_x\mu_{x+t}F_C\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\lambda-r-m)t,\theta t\right)dt, \tag{3.25}$$

where F_C is the cdf of the Cauchy distribution with corresponding parameters given in the brackets.

(3) Under the skew-normal model, we have

$$1-\alpha=\int_0^T {}_tp_x\mu_{x+t}G_S\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\eta-r-m)t,\kappa\sqrt{t},\omega\right)dt,$$

where G_S is the cdf of the skew-normal distribution with corresponding parameters given in the brackets.

Proof. See Appendix A.3. □

Proposition 3.4. For the three equity return models described in Section 3.2.1, we have the following results for the conditional tail expectation CTE_α, given that $\alpha > \xi_d$, and for GMDB gross liabilities. Note that ξ_d for the normal, Cauchy, and skew-normal models are respectively given in (3.22), (3.23) and (3.24).

(1) Under the normal model, we have

$$\begin{aligned}\mathrm{CTE}_\alpha=&\frac{G}{1-\alpha}\int_0^T e^{(\delta-r)t}{}_tp_x\mu_{x+t}\Phi\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\mu-r-m)t,\sigma\sqrt{t}\right)dt\\&-\frac{F_0}{1-\alpha}\int_0^T e^{(\mu-r-m)t+\sigma^2t/2}{}_tp_x\mu_{x+t}\\&\qquad\times\Phi\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\mu-r-m+\sigma^2)t,\sigma\sqrt{t}\right)dt,\end{aligned}$$

where Φ is the cdf of the normal distribution with corresponding parameters in the brackets.

(2) Under the Cauchy model, we have

$$\begin{aligned}\mathrm{CTE}_\alpha=&\frac{G}{1-\alpha}\int_0^T e^{(\delta-r)t}{}_tp_x\mu_{x+t}F_C\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\lambda-r-m)t,\theta t\right)dt\\&-\frac{F_0}{1-\alpha}\int_0^T {}_tp_x\mu_{x+t}\times\int_{-\infty}^{\ln(c_t)}e^y\cdot f_C(y;(\lambda-r-m)t,\theta t)dydt,\end{aligned} \tag{3.26}$$

where $c_t=\left(e^{(\delta-r)t}G-\mathrm{V}_\alpha\right)/F_0$, and f_C and F_C are the pdf and cdf of the Cauchy distribution with location parameter $(\lambda-r-m)t$ and scale parameter θt, respectively.

(3) Under the skew-normal model, we have

$$\mathrm{CTE}_\alpha = \frac{G}{1-\alpha}\int_0^T e^{(\delta-r)t}{}_tp_x\mu_{x+t}G_S\left(\ln\frac{e^{(\delta-r)t}G-\mathrm{V}_\alpha}{F_0};(\eta-r-m)t,\kappa\sqrt{t},\omega\right)dt$$
$$-\frac{F_0}{1-\alpha}\int_0^T {}_tp_x\mu_{x+t}\times\int_{-\infty}^{\ln(c_t)} e^y\cdot g_S(y;(\eta-r-m)t,\kappa\sqrt{t},\omega)dydt,$$

where $c_t = \left(e^{(\delta-r)t}G-\mathrm{V}_\alpha\right)/F_0$, and g_S and G_S are the pdf and cdf of the skew-normal distribution with location parameter $(\eta-r-m)t$, scale parameter $\kappa\sqrt{t}$, and shape parameter ω, respectively.

Proof. See Appendix A.4. □

3.2.4 Calculation notes for gross liabilities

It is worth noting that, in real life applications, mortality rates and corresponding assumptions vary for different regions, insurance companies, and different groups of policyholders. This will result in different numerical results when calculating risk measures using the analytical expressions for gross liabilities. In this section, we present the calculation based on two most commonly used mortality assumptions, the uniform distributed death (UDD) assumption and the constant force of mortality assumption for modeling mortality rates for fractional ages. We use the Cauchy model as an example to provide some notes for computing the VaR of a GMDB gross liability.

Note that, in Chapter 4, the parameters for the normal, Cauchy, and skew-normal models are estimated based on the weekly historical data. Since all the financial rates used for this project are assumed on the annual basis, we need to properly convert the corresponding parameters and financial rates based on an annual time basis that we choose for simulation and approximation purposes. In the rest of this report, we assume the data used for estimating the model parameters have the same time unit (increment) as the unit we choose for approximating some functions presented below.

1: Under the UDD assumption, ${}_sq_x = s\cdot q_x$ for $0\le s\le 1$, where ${}_sq_x$ is the probabilities that a life age x dies before age $x+s$. In this case, ${}_sp_x\mu_{x+s} = q_x$, for $0\le s\le 1$. Assume a constant time increment of 1 unit with in total n units in a year (so that there are nT time units in T years), and that the death benefit is payable at the end of each time unit. Under the Cauchy model, Equation (3.25) can be approximated by

$$1-\alpha \approx \sum_{k=1}^{nT} {}_{k-1}p_xq_{x+k-1}\cdot F_C\left(\ln\frac{e^{(\delta-r)\frac{k}{n}}G-\mathrm{V}_\alpha}{F_0};\left(\lambda-\frac{r+m}{n}\right)k,\theta k\right),$$

where F_C is the cdf of the Cauchy distribution.

2: Under the constant force of mortality assumption, we have ${}_sp_x\mu_{x+s} = \mu_x e^{-\mu_x s}$ for $0 \le s \le 1$. Then under Cauchy model, Equation (3.25) can be approximated by

$$1-\alpha \approx \sum_{k=1}^{nT} \int_{k-1}^{k} {}_{k-1}p_x \mu_{x+k-1} e^{-\mu_{x+k-1}\cdot t} \cdot F_C \left(\ln \frac{e^{(\delta-r)\frac{k}{n}} G - \mathrm{V}_\alpha}{F_0}; \left(\lambda - \frac{r+m}{n} \right) k, \theta k \right) dt,$$

where F_C is the cdf of the Cauchy distribution.

When $n = 1$, all the above calculations are on an annual basis, assuming that the annual mortality rates are available from the mortality table.

3.3 Calculation algorithms for net liabilities

We have derived analytical results of risk measures for an insurer's gross liabilities under the three equity models. In this section, we present the corresponding probabilities that no guarantee payment will be made at maturity. Then, the algorithms for general simulation of an insurer's future net liabilities with GMMB and GMDB riders by taking the Cauchy model for the underlying equity model are shown as an example.

3.3.1 Risk measures algorithms for the GMMB rider

We first determine ξ_e based on the GMMB net liability based on (3.3). In this case, ξ_e is the probability of non-positive liabilities, and it can be expressed by

$$\xi_e = 1 - \mathbb{P}\left[{}_eL_n^0 > 0, \tau_x > T \right].$$

Using (3.2), (3.3) and (3.8), and assuming that the future lifetime of the policyholder and the account value are independent, we can write $\mathbb{P}\left[{}_eL_n^0 > 0, \tau_x > T \right]$ as

$$\begin{aligned}
\mathbb{P}\left[{}_eL_n^0 > 0, \tau_x > T \right] &= \mathbb{P}\left[e^{-rT}(G - F_T)_+ I_{\{\tau_x > T\}} - \int_0^{\tau_x \wedge T} e^{-rs} M_s ds > 0, \tau_x > T \right] \\
&= {}_Tp_x \mathbb{P}\left[e^{-rT}(G - F_T) - \int_0^T e^{-rs} M_s ds > 0 \right] \\
&= {}_Tp_x \mathbb{P}\left[e^{-(r+m)T} \frac{S_T}{S_0} + m_e \int_0^T e^{-(r+m)s} \frac{S_s}{S_0} ds < \frac{e^{-rT} G}{F_0} \right].
\end{aligned}$$

Let

$$P(T, x) = \mathbb{P}\left[e^{-(r+m)T} \frac{S_T}{S_0} + m_e \int_0^T e^{-(r+m)s} \frac{S_s}{S_0} ds < x \right]. \tag{3.27}$$

Then, ξ_e can be expressed as

$$\xi_e = 1 - {}_Tp_x \cdot P\left(T, \frac{e^{-rT} G}{F_0} \right).$$

Now, given that $\alpha > \xi_e$, the expression for V_α based on net liabilities of a GMMB rider, ${}_eL_n^0$, can be obtained from the following equation:

$$\begin{aligned}1-\alpha &= {}_Tp_x \cdot \mathbb{P}\left[e^{-rT}(G-F_T) - \int_0^T e^{-rs}M_s ds > \mathrm{V}_\alpha\right] \\ &= {}_Tp_x \cdot \mathbb{P}\left[e^{-(r+m)T}\frac{S_T}{S_0} + m_e \int_0^T e^{-(r+m)s}\frac{S_s}{S_0}ds < \frac{e^{-rT}G-\mathrm{V}_\alpha}{F_0}\right] \\ &= {}_Tp_x \cdot P\left(T, \frac{e^{-rT}G-\mathrm{V}_\alpha}{F_0}\right), \end{aligned} \tag{3.28}$$

where function P is given in (3.27) and at a significance level of α.

Note that under the normal model, an explicit expression of $P(T,x)$ is presented in Feng and Volkmer (2012) (see Equation (3.5) in Proposition 3.3). However, when S_T/S_0 follows a Cauchy or skew-normal model, the explicit expression for $P(T,x)$ is not available. We propose the Monte Carlo simulation algorithm below for computing (i.e., approximating) ξ_e and V_α.

Let

$$e^{Y_s} = e^{-(r+m)s}\cdot\frac{S_s}{S_0}, \qquad 0 \le s \le T. \tag{3.29}$$

Note that the integral $\int_0^T e^{Y_s}ds$ is defined path by path. For a fixed sample path of $\{Y_s\}_{s\ge 0}$, the integral $\int_0^T e^{Y_s}ds$ is a continuous function, so the integral can be calculated or approximated over the interval $[0,T]$. Assume a constant time increment of a unit with in total n units in a year (so that these are nT time units in T complete years), and then for a fixed sample path of $\{Y_s\}_{s\ge 0}$, we have

$$\int_0^T e^{Y_s}ds \approx \sum_{k=1}^{nT} e^{Y_k}. \tag{3.30}$$

This integral approximation approximates $P(T,x)$ from Equation (3.27) by

$$\tilde{P}(nT,x) = \mathbb{P}\left[e^{Y_{nT}} + m_e\sum_{k=1}^{nT} e^{Y_k} < x\right], \tag{3.31}$$

where $Y_1, Y_2, \ldots, Y_{nT}$ are random variables at corresponding times $1, 2, \ldots, nT$.

We use the Cauchy model as an example to present the calculation algorithm for approximating ξ_e, V_α, and CTE_α. In this case, for $k = 1, 2, \ldots, nT$,

$$e^{Y_k} = e^{-(r+m)\frac{k}{n}}\frac{S_k}{S_0} \sim \text{Log-Cauchy}\left(\left(\lambda - \frac{r+m}{n}\right)k, \theta k\right).$$

For notation simplicity, we further let Q be the summation of the random variables in (3.31); that is,

$$Q \equiv e^{Y_{nT}} + m_e \cdot \sum_{k=1}^{nT} e^{Y_k}.$$

Algorithm 1: approximate ξ_e

Based on approximation formula (3.31), ξ_e can be approximated by

$$\xi_e \approx 1 - {}_{nT}p_x \cdot \tilde{P}\left(nT, \frac{e^{-rT}G}{F_0}\right). \tag{3.32}$$

We present below the detailed steps for calculating ξ_e.

Step 1: Simulate N sets of $\{e^{Y_k}\}_{k=1}^{nT}$ and calculate corresponding N realizations of Q, $Q_1, Q_2, \ldots, Q_N$, based on the Cauchy model.

Step 2: Calculate $\tilde{P}\left(nT, \frac{e^{-rT}G}{F_0}\right)$ using the following empirical probability formula:

$$\tilde{P}\left(nT, \frac{e^{-rT}G}{F_0}\right) = \frac{1}{N}\sum_{l=1}^{N} I_{\left\{Q_l < \frac{e^{-rT}G}{F_0}\right\}}.$$

Step 3: Compute ξ_e by using (3.32).

Algorithm 2: approximate V_α for the GMMB rider

Using (3.31), the expression of V_α defined by (3.28) for the GMMB net liabilities can be approximated from the following equation:

$$\frac{1-\alpha}{{}_{nT}p_x} = \tilde{P}\left(nT, \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right).$$

Let $\zeta_\alpha = 100(1-\alpha)/{}_{nT}p_x$. We present below the detailed steps.

Step 1: Simulate N sets of $\{e^{Y_k}\}_{k=1}^{nT}$ and calculate corresponding N realizations of Q, $Q_1, Q_2, \ldots, Q_N$, based on the Cauchy model.

Step 2: Order simulated $Q_1, Q_2, \ldots, Q_N$ to $Q_{(1)}, Q_{(2)}, \ldots, Q_{(N)}$ such that $Q_{(1)} \leq Q_{(2)} \leq \cdots \leq Q_{(N)}$, where $Q_{(j)}$ is the j-th smallest simulated values of Q, and determine the ζ_α%-quantile of simulated values of Q, which we denote it as $Q^{(1)}_{(\lfloor \zeta_\alpha \% N \rfloor)}$, where $\lfloor \cdot \rfloor$ is the floor function.

Step 3: Calculate V_α by equaling $Q^{(1)}_{(\lfloor \zeta_\alpha \% N \rfloor)} = (e^{-rT}G - \mathrm{V}_\alpha)/F_0$, which yields $\mathrm{V}^{(1)}_\alpha = e^{-rT}G - F_0 Q^{(1)}_{(\lfloor \zeta_\alpha \% N \rfloor)}$.

Step 4: Repeat Steps 1–3 for a total of $M-1$ times to get $\mathrm{V}^{(2)}_\alpha, \mathrm{V}^{(3)}_\alpha, \ldots, \mathrm{V}^{(M)}_\alpha$.

Step 5: Compute the estimated V_α by

$$\widehat{\mathrm{V}}_\alpha = \frac{1}{M}\sum_{j=1}^{M} \mathrm{V}_\alpha^{(j)}.$$

We now present the steps for calculating the conditional tail expectation CTE_α for the GMMB net liabilities. Given that $\alpha > \xi_e$, and by Equation (3.3), the expression of CTE for net liabilities of GMMB, ${}_eL_n^0$, is given by

$$\mathrm{CTE}_\alpha = \frac{{}_Tp_x}{1-\alpha}\mathbb{E}\left[\left(e^{-rT}G - e^{-rT}F_T - \int_0^T e^{-rs}M_s ds\right) \times I_{\left\{e^{-rT}G - e^{-rT}F_T - \int_0^T e^{-rs}M_s ds > \mathrm{V}_\alpha\right\}}\right],$$

at a significance level α.

By using (3.8) and the definition of e^{Y_s} given in (3.29), CTE_α can be written as

$$\begin{aligned}\mathrm{CTE}_\alpha = {} & \frac{{}_Tp_x}{1-\alpha}e^{-rT}G \cdot \mathbb{P}\left[e^{Y_T} + m_e\int_0^T e^{Y_s}ds < \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right] \\ & - \frac{{}_Tp_x}{1-\alpha}F_0 \cdot \mathbb{E}\left[\left(e^{Y_T} + m_e\int_0^T e^{Y_s}ds\right) I_{\left\{e^{Y_T} + m_e\int_0^T e^{Y_s}ds < \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right\}}\right].\end{aligned}$$

We further let

$$Z\left(T,x\right) = \mathbb{E}\left[\left(e^{Y_T} + m_e\int_0^T e^{Y_s}ds\right) I_{\left\{e^{Y_T} + m_e\int_0^T e^{Y_s}ds < x\right\}}\right]. \tag{3.33}$$

Then, using (3.28), we can obtain the following expression for CTE_α:

$$\mathrm{CTE}_\alpha = e^{-rT}G - {}_Tp_x\frac{F_0}{1-\alpha}Z\left(T, \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right).$$

Similar to $P(T,x)$, an explicit expression of $Z(T,x)$ is also presented in Feng and Volkmer (2012) for the normal model(see Equation (3.7) in Proposition 3.4). When S_T/S_0 follows a Cauchy or a skew-normal model, a similar Monte Carlo simulation algorithm can be used to approximate CTE_α. This is presented again for the Cauchy model.

Algorithm 3: approximate CTE_α for the GMMB rider

Using the same integral approximation described in (3.30), we can approximate $Z(T,x)$ by

$$\tilde{Z}\left(nT,x\right) = \mathbb{E}\left[\left(e^{Y_{nT}} + m_e\sum_{k=1}^{nT} e^{Y_k}\right) I_{\left\{e^{Y_{nT}} + m_e\sum_{k=1}^{nT} e^{Y_k} < \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right\}}\right], \tag{3.34}$$

and CTE_α can be approximated by

$$\mathrm{CTE}_\alpha \approx e^{-rT}G - {}_{nT}p_x \frac{F_0}{1-\alpha}\tilde{Z}\left(nT, \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right). \tag{3.35}$$

Below are the detailed steps.

Step 1: Simulate N sets of $\{e^{Y_k}\}_{k=1}^{nT}$ and calculate corresponding N realizations of $Q, Q_1, Q_2, \ldots, Q_N$, based on the Cauchy model.

Step 2: Follow Steps 2–3 in **Algorithm 2** to obtain $\mathrm{V}_\alpha^{(1)}$.

Step 3: Calculate $\tilde{Z}(nT, e^{-rT}G - \mathrm{V}_\alpha^{(1)}/F_0)$ given by (3.34) using the following empirical formula:

$$\tilde{Z}\left(nT, \frac{e^{-rT}G - \mathrm{V}_\alpha^{(1)}}{F_0}\right) = \frac{1}{N}\sum_{l=1}^{N} Q_l I_{\left\{Q_l < \frac{e^{-rT}G - \mathrm{V}_\alpha^{(1)}}{F_0}\right\}},$$

and $\mathrm{CTE}_\alpha^{(1)}$ using (3.35).

Step 4: Repeat Steps 1–3 for a total of $M-1$ times to get $\mathrm{CTE}_\alpha^{(2)}$, $\mathrm{CTE}_\alpha^{(3)}$, …, $\mathrm{CTE}_\alpha^{(M)}$.

Step 5: Compute the estimated CTE_α by

$$\widehat{\mathrm{CTE}}_\alpha = \frac{1}{M}\sum_{j=1}^{M} \mathrm{CTE}_\alpha^{(j)}.$$

3.3.2 Risk measures algorithms for the GMDB rider

We now determine ξ_d based on the GMDB net liability as given in Equation (3.5). In this case, ξ_d is the probability of non-positive liabilities, and it can be expressed as

$$\xi_d = 1 - \mathbb{P}\left[{}_dL_n^0 > 0, \tau_x < T\right].$$

Denote $K_x = \lceil \tau_x \rceil$ as the complete future lifetime of (x) including the year of death, where $\lceil \cdot \rceil$ is the integer ceiling function. Assume that the guaranteed death benefit is payable at the end of the year of death, and that the investment account is accumulated and the rider charges ξ_d are deducted continuously until the end of the year of death. By using (3.2), (3.5), and (3.8), and assuming that the future lifetime of the policyholder and the account value are independent, we can write $\mathbb{P}\left[{}_dL_n^0 > 0, \tau_x < T\right]$ as

$$\begin{aligned}\mathbb{P}\left[{}_dL_n^0 > 0, \tau_x < T\right] &= \mathbb{P}\left[e^{-r\tau_x}(e^{\delta\tau_x}G - F_{\tau_x})_+ I_{\{\tau_x \le T\}} - \int_0^{\tau_x \wedge T} e^{-rs}M_s ds > 0, \tau_x \le T\right] \\ &= \sum_{k=1}^{T} \mathbb{P}\left[e^{-rk}(e^{\delta k}G - F_k) - \int_0^k e^{-rs}M_s ds > 0\right]\mathbb{P}\left[K_x = t\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{T} \mathbb{P}\left[e^{-(r+m)k}\frac{S_k}{S_0} + m_d \int_0^k e^{-(r+m)s}\frac{S_s}{S_0}ds < \frac{e^{(\delta-r)k}G}{F_0}\right] {}_{k-1}p_x q_{x+k-1} \\
&= \sum_{k=1}^{T} P\left(k, \frac{e^{(\delta-r)k}G}{F_0}\right) {}_{k-1}p_x q_{x+k-1},
\end{aligned}
$$

where function P is given in Equation (3.27) and m_e is replaced by m_d. Then, ξ_d for the GMDB net liabilities can be expressed as

$$
\xi_d = 1 - \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} P\left(k, \frac{e^{(\delta-r)k}G}{F_0}\right).
$$

Similarly, given that $\alpha > \xi_d$, V_α for net liabilities of GMDB, ${}_dL_n^0$, can be obtained from the following equation:

$$
\begin{aligned}
1-\alpha &= \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} \mathbb{P}\left[e^{-rk}(e^{\delta k}G - F_k) - \int_0^k e^{-rs}M_s ds > \mathrm{V}_\alpha\right] \\
&= \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} \mathbb{P}\left[e^{-(r+m)k}\frac{S_k}{S_0} + m_d \int_0^k e^{-(r+m)s}\frac{S_s}{S_0}ds < \frac{e^{(\delta-r)k}G - \mathrm{V}_\alpha}{F_0}\right] \\
&= \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} P\left(k, \frac{e^{(\delta-r)k}G - \mathrm{V}_\alpha}{F_0}\right), \qquad (3.36)
\end{aligned}
$$

at a significance level of α.

Note that similar to the GMMB net liabilities, the explicit expression for $P(k,x)$ is not available when S_k/S_0 follows a Cauchy or a skew-normal model. The same approximation ideas presented in Section 3.3.1 for the GMMB net liability case can be applied here. We again use the Cauchy model as an example to present calculation algorithms for approximating ξ_e, V_α, and CTE_α.

For $k = 1, 2, \ldots, T$, $P(k,x)$ can be approximated by

$$
\tilde{P}(nk, x) = \mathbb{P}\left[e^{-(r+m)k}\frac{S_{nk}}{S_0} + m_d \sum_{l=1}^{nk} e^{Y_l} < x\right]. \qquad (3.37)
$$

Recall that, for $l = 1, 2, \ldots, nT$,

$$
e^{Y_l} = e^{-(r+m)\frac{l}{n}}\frac{S_l}{S_0} \sim \text{Log-Cauchy}\left(\left(\lambda - \frac{r+m}{n}\right)l, \theta l\right).
$$

For notation simplicity, we further let Q'_k be the summation of the random variables in (3.37); that is,

$$
Q'_k = e^{-(r+m)k}\frac{S_{nk}}{S_0} + m_d \sum_{l=1}^{nk} e^{Y_l} = e^{Y_{nk}} + m_d \sum_{l=1}^{nk} e^{Y_l}.
$$

Algorithm 4: approximate ξ_d

Based on the approximation formula of Equation (3.37), ξ_d can be approximated by

$$\xi_d \approx 1 - \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} \tilde{P}\left(nk, \frac{e^{(\delta-r)k}G}{F_0}\right). \tag{3.38}$$

Below are the detailed steps.

Step 1: Simulate N sets of $\{e^{Y_l}\}_{l=1}^{nT}$ and calculate corresponding N sets of realizations of Q'_k, for $k = 1, 2, \ldots, T$, denoted as $Q'_{(k,1)}, \ldots, Q'_{(k,N)}$, based on the Cauchy model.

Step 2: For each k, calculate $\tilde{P}\left(nk, \frac{e^{(\delta-r)k}G}{F_0}\right)$ using the following empirical probability formula:

$$\tilde{P}\left(nk, \frac{e^{(\delta-r)k}G}{F_0}\right) = \frac{1}{N}\sum_{l=1}^{N} I_{\left\{Q_{(k,l)}<\frac{e^{(\delta-r)k}G}{F_0}\right\}}.$$

Step 3: Compute the approximated ξ_d by using (3.38).

Algorithm 5: approximate V_α for the GMDB rider

Using (3.36), V_α for the GMDB net liabilities can be approximated using the following equation:

$$1 - \alpha = \sum_{k=1}^{T} {}_{k-1}p_x q_{x+k-1} \tilde{P}\left(nk, \frac{e^{(\delta-r)k}G - \mathrm{V}_\alpha}{F_0}\right). \tag{3.39}$$

Below are the detailed steps.

Step 1: Simulate N sets of $\{e^{Y_l}\}_{l=1}^{nT}$ and calculate corresponding N sets of realizations of Q'_k, for $k = 1, 2, \ldots, T$, denoted as $Q'_{(k,1)}, \ldots, Q'_{(k,N)}$, based on the Cauchy model.

Step 2: Generate a vector of $\{\mathrm{V}_\alpha\}$ from 0 to 1 with increment of 0.0001. Note that $\{\mathrm{V}_\alpha\}$ is a vector that consists of 10001 elements.

Step 3: For each value in $\{\mathrm{V}_\alpha\}$ and each k, calculate $\tilde{P}\left(nk, \frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right)$ using the following empirical probability formula:

$$\tilde{P}\left(nk, \frac{e^{(\delta-r)k}G - \mathrm{V}_\alpha}{F_0}\right) = \frac{1}{N}\sum_{l=1}^{N} I_{\left\{Q_{(k,l)}<\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right\}}.$$

Then, calculate the right hand side of (3.39) for each value in $\{\mathrm{V}_\alpha\}$ based on the N sets of realizations of $\{Q'_k\}$.

Step 4: Find the value of $\mathrm{V}_\alpha^{(1)}$ from $\{\mathrm{V}_\alpha\}$ such that Equation (3.39) holds.

Step 5: Repeat Steps 1–4 for a total of $M-1$ times to get $\mathrm{V}_\alpha^{(2)}, \mathrm{V}_\alpha^{(3)}, \ldots, \mathrm{V}_\alpha^{(M)}$.

Step 6: Compute the estimated V_α by

$$\widehat{\mathrm{V}}_\alpha = \frac{1}{M}\sum_{j=1}^{M} \mathrm{V}_\alpha^{(j)}.$$

We now present the steps for calculating the conditional tail expectation CTE_α for the GMDB net liabilities. Given that $\alpha > \xi_d$, and by using Equation (3.5), CTE for the net liabilities of GMDB, ${}_dL_n^0$, is given by

$$\mathrm{CTE}_\alpha = \frac{1}{1-\alpha}\sum_{t=1}^{T}\mathbb{E}\left[\left(e^{-rk}\left(e^{\delta k}G - F_k\right) - m_d\int_0^k e^{-rs}F_s ds\right)\right.$$
$$\left.\times I_{\left\{e^{-rk}\left(e^{\delta k}G-F_k\right)-m_d\int_0^k e^{-rs}F_s ds > \mathrm{V}_\alpha\right\}}\right]\mathbb{P}[K_x = k],$$

at a significance level of α.

Note that

$$I_{\left\{e^{-rk}\left(e^{\delta k}G-F_k\right)-m_d\int_0^k e^{-rs}F_s ds > \mathrm{V}_\alpha\right\}} = I_{\left\{e^{Y_k}+m_d\int_0^k e^{Y_s}ds < \frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right\}}.$$

By using (3.8) and the definition of e^{Y_s} given in (3.29), and (3.27), and (3.33), CTE_α can be written as

$$\begin{aligned}
\mathrm{CTE}_\alpha = \frac{1}{1-\alpha}\sum_{k=1}^{T}&\left(e^{(\delta-r)k}G\cdot\mathbb{P}\left[e^{Y_k}+m_d\int_0^k e^{Y_s}ds < \frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right]\right.\\
&\left.-F_0\cdot\mathbb{E}\left[\left(e^{Y_k}+m_d\int_0^k e^{Y_s}ds\right)I_{\left\{e^{Y_k}+m_d\int_0^k e^{Y_s}ds<\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right\}}\right]\right)\,{}_{k-1}p_x q_{x+k-1}\\
= \frac{1}{1-\alpha}\sum_{k=1}^{T}&\left(e^{(\delta-r)k}G\cdot P\left(k,\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right)\right.\\
&\left.-F_0\cdot Z\left(k,\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right)\right)\,{}_{k-1}p_x q_{x+k-1}.
\end{aligned}$$

Function $Z(k,x)$ can be approximated by using (3.34). Similarly, we use the Monte Carlo simulation to approximate CTE_α for the Cauchy model.

Algorithm 6: approximate CTE_α for the GMDB rider

Function $Z(k,x)$ can be approximated by

$$\tilde{Z}(nk,x) = \mathbb{E}\left[\left(e^{Y_{nk}}+m_d\sum_{l=1}^{nk}e^{Y_l}\right)I_{\left\{e^{Y_{nk}}+m_d\sum_{l=1}^{nk}e^{Y_l}<x\right\}}\right]. \tag{3.40}$$

Using the same approximation for $P(k,x)$ as in (3.37), CTE_α can be approximated by

$$\mathrm{CTE}_\alpha \approx \frac{1}{1-\alpha}\sum_{k=1}^{T}\left(e^{(\delta-r)k}G\cdot\tilde{P}\left(nk,\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right)\right.$$
$$\left. -F_0\cdot\tilde{Z}\left(nk,\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha}{F_0}\right)\right){}_{k-1}p_x q_{x+k-1}. \tag{3.41}$$

Below are the detailed steps.

Step 1: Simulate N sets of $\{e^{Y_l}\}_{l=1}^{nT}$ and calculate corresponding N sets of realizations of Q'_k, for $k=1,2,\ldots,T$, denoted as $Q'_{(k,1)},\ldots,Q'_{(k,N)}$, based on the Cauchy model.

Step 2: Follow Steps 2–4 in **Algorithm 5** to obtain $\mathrm{V}_\alpha^{(1)}$.

Step 3: For each k, calculate $\tilde{P}\left(nk,\left(e^{(\delta-r)k}G-\mathrm{V}_\alpha\right)/F_0\right)$ using the value calculated in Step 3 in **Algorithm 5** and calculate the value of $\tilde{Z}\left(nk,\left(e^{(\delta-r)k}G-\mathrm{V}_\alpha^{(1)}\right)/F_0\right)$ using the following empirical formula:

$$\tilde{Z}\left(nk,\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha^{(1)}}{F_0}\right)=\frac{1}{N}\sum_{l=1}^{N}Q'_{(k,l)}I_{\left\{Q'_{(k,l)}<\frac{e^{(\delta-r)k}G-\mathrm{V}_\alpha^{(1)}}{F_0}\right\}},$$

and then calculate $\mathrm{CTE}_\alpha^{(1)}$ using (3.41).

Step 4: Repeat Steps 1–3 for a total of $M-1$ times to get $\mathrm{CTE}_\alpha^{(2)}$, $\mathrm{CTE}_\alpha^{(3)}$, …, $\mathrm{CTE}_\alpha^{(M)}$.

Step 5: Compute the estimated CTE_α by

$$\widehat{\mathrm{CTE}}_\alpha=\frac{1}{M}\sum_{j=1}^{M}\mathrm{CTE}_\alpha^{(j)}.$$

Chapter 4

Numerical illustrations

4.1 Data and Models

In this section, we first introduce the data and perform a preliminary data analysis by looking at the histograms, time series plots, and autocorrelation function plot of data. In Section 4.1.2, we present the maximum likelihood estimation method for estimating the model parameters for normal, Cauchy, and skew-normal models, and then determine the better-fit distributions based on some model selection criteria. In Section 4.1.3, we provide a graphical comparison of theoretical and the empirical distributions and examine simulated projections.

4.1.1 Data

In this project, we use the S&P 500 weekly stock index prices[1] over the past two decades, between the week of February 6^{th} 2000 and the week of January 26^{th} 2020, as our historical data. We calculate the returns by taking the logarithm of the ratio of two consecutive stock index prices. The time series plot of historical weekly stock index prices are shown in Figure 4.1.

The historical weekly returns are plotted in Figure 4.2 and the relevant statistics of this data are shown in Table 4.1. From Table 4.1, we see that the skewness and kurtosis of historical returns data are -0.8928359 and 10.42228, respectively. The empirical skewness of a data is a measure of asymmetry of the empirical distribution. The data is symmetrically distributed if its empirical skewness has a value of 0, and the empirical distribution is left-skewed (right-skewed) if its empirical skewness has a negative (positive) value. The empirical kurtosis of a data is a measure that assesses whether the data are heavy-tailed or light-tailed relative to a normal distribution. The data is normally distributed if its empirical excess kurtosis has a value of 0, and the data has heavier (lighter) tails than normal if its empirical excess kurtosis has a positive (negative) value. Based on the skewness and excess kurtosis

[1] https://ca.finance.yahoo.com/

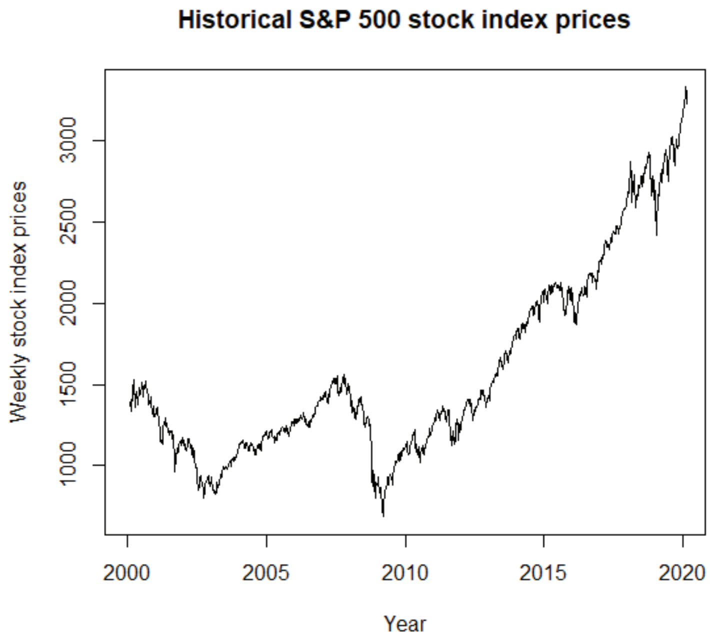

Figure 4.1: Time series plot of S&P 500 weekly prices.

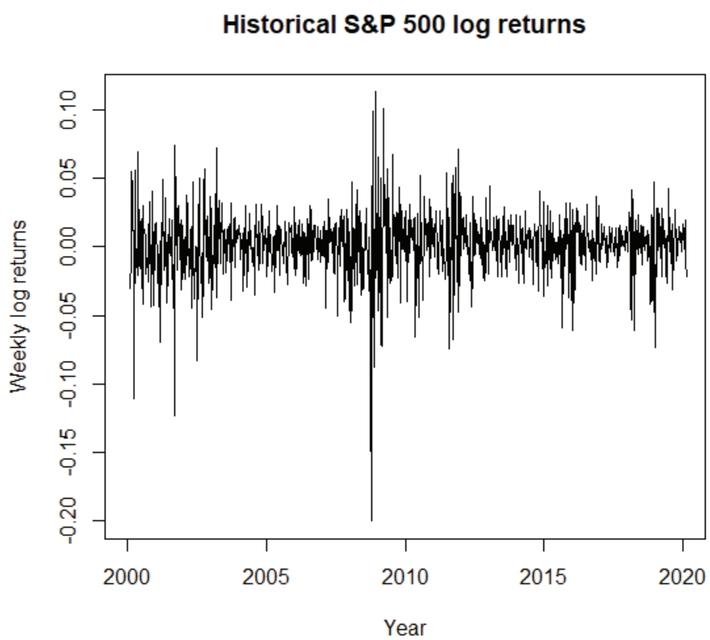

Figure 4.2: Time series plot of S&P 500 weekly returns.

obtained from our historical returns data, we can conclude that the empirical distribution of historical returns is left-skewed and has heavy tails. This can also be observed from the histogram of historical weekly returns based on the stock index prices shown in Figure 4.3.

Table 4.1: Statistics for historical S&P500 weekly returns.

Min	Max	Median	Mean	Variance	Skewness	Kurtosis
-0.2008375	0.113559	0.0020705	0.0008098	0.0005717	-0.8928359	10.42228

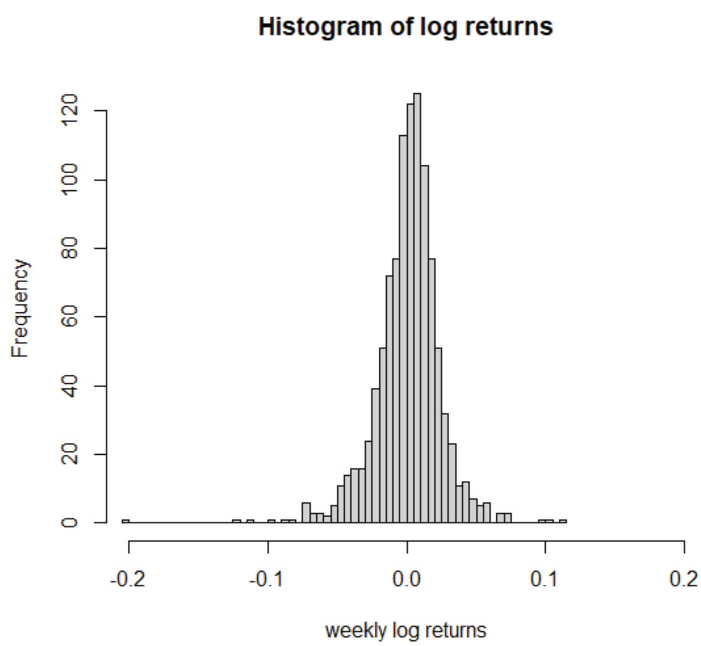

Figure 4.3: Histogram of S&P 500 weekly returns.

In this study, we assume that the returns are independent and identically distributed. To test the independence assumption of our returns data, we plot the autocorrelation function (ACF) of our historical weekly data in Figure 4.4. In time series studies, the autocorrelation function measures the correlation of a time series with itself after lagging. It can be observed from Figure 4.4 that the data have correlation of 1 at lag 0. This means that the data is perfectly correlated with respect to itself. The dashed blue lines represent a confidence interval of zero correlations. As we can see from Figure 4.4 and for any positive lag levels, the values of sample autocorrelation function are all within the dashed blue lines, implying that the historical lagged returns are not correlated.

In addition to the ACF plot, both Ljung-Box and Box-Pierce tests are also commonly used to verify the independence assumption of time series data. The Ljung-Box test, proposed by Ljung and Box (1978), examines whether a time series contains autocorrelation. The Box-Pierce test, proposed by Box and Pierce (1970), is a simplified version of Ljung-Box test. Both tests set up the null hypothesis in the same way; the null hypothesis assumes that the time series data are independently distributed. We perform both tests for our returns data at a lag level of 1 and their p values are obtained using the *stats*[2] package in R. The p values are 0.02677 and 0.02699, respectively. Based on these p values, we fail to reject the null hypothesis of both Ljung-Box and Box-Pierce tests at 1% significance level. Hence, the independence assumption should hold for our historical returns data.

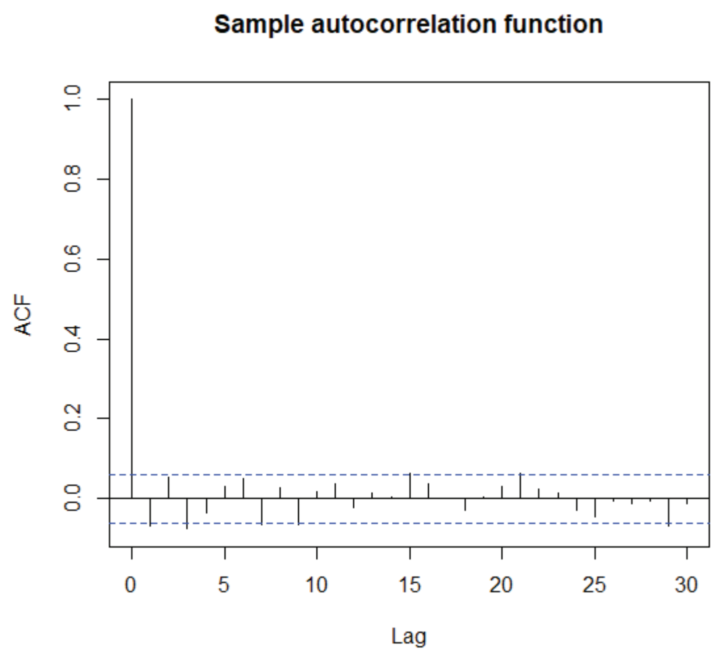

Figure 4.4: Sample ACF of S&P 500 weekly returns.

4.1.2 Models and estimations

Recall that the asset price at time t ($0 \leq t \leq T$) is denoted as S_t, and the logarithm of the quotient of two consecutive stock prices at time $t-1$ and t, also known as equity return at time t, is denoted by $\ln(S_t/S_{t-1})$.

Let $X_i = \ln(S_i/S_{i-1})$, $i = 0,1,\ldots,n$. We first consider the lognormal model. In this case, the X_is are assumed to be independent and normally distributed with mean μ and standard deviation σ. The pdf and cdf of this normal distribution are given by (3.10) and (3.11), respectively.

In this project, we use the maximum likelihood estimation (MLE) method to estimate the parameters for the three distributions we consider. In the normal distribution case, the explicit expressions of the MLE of the parameters can be easily obtained by solving a

[2] https://www.rdocumentation.org/packages/stats/versions/3.6.2

system of equations, called estimating equations. The likelihood function based on a sample of observations $x_1, x_2, \ldots, x_n$, $L(\mu, \sigma^2; x_1, x_2, \ldots, x_n)$, is given by

$$L(\mu, \sigma^2; x_1, x_2, \ldots, x_n) \propto \frac{1}{\sigma^{2n}} \exp\left(-\frac{\sum\limits_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right),$$

and the log-likelihood function can be expressed as

$$\ell(\mu, \sigma^2; x_1, x_2, \ldots, x_n) = -\frac{n}{2}\ln(\sigma^2) - \frac{\sum\limits_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}.$$

By taking the first derivative of the log-likelihood function with respect to parameter μ and σ^2, respectively, and setting them equal to 0, we have the following system of estimating equations:

$$\begin{cases} -2\sum\limits_{i=1}^{n} x_i - n\mu = 0, \\ -\frac{n}{\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^4} = 0. \end{cases}$$

By solving this system of equations, we get

$$\begin{cases} \hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}, \\ \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^2}{n}. \end{cases}$$

We now consider the Cauchy model. In this case, the X_is are assumed to be independent and to follow a Cauchy distribution with location parameters λ and scale parameter θ. The pdf and cdf of this Cauchy distribution are given by (3.13) and (3.14), respectively.

The likelihood function based on a sample of observations $x_1, x_2, \ldots, x_n$ is given by

$$L(\lambda, \theta; x_1, x_2, \ldots, x_n) \propto \frac{1}{\theta^n} \prod_{i=1}^{n}\left[\frac{\theta^2}{(x_i-\lambda)^2+\theta^2}\right],$$

and the log-likelihood function can be expressed as

$$\ell(\lambda, \theta; x_1, x_2, \ldots, x_n) = -n\ln(\theta) + \sum_{i=1}^{n}\ln\left[\frac{\theta^2}{(x_i-\lambda)^2+\theta^2}\right].$$

By taking the first derivative of the log-likelihood function with respect to parameters λ and θ, respectively, and setting them equal to 0, we have the following system of estimating equations:

$$\begin{cases} \sum_{i=1}^{n} \frac{2(x_i-\lambda)}{\theta^2+(x_i-\lambda)^2} = 0, \\ \sum_{i=1}^{n} \frac{2(x_i-\lambda)^2}{\theta(\theta^2+(x_i-\lambda)^2)} - \frac{n}{\theta} = 0. \end{cases}$$

The MLE of λ and θ can be obtained by solving the above system of equations numerically.

According to Figure 4.3, the empirical distribution of the returns shows that there are more observations falling into the range $(-0.1, 0)$ rather than range $(0, 0.1)$. This implies that a distribution with an additional parameter modeling skewness, such as the skew-normal distribution, might be a better fit distribution to this specific historical returns data.

We now consider the skew-normal model. In this case, X_i's are assumed to be independent and to follow a skew-normal distributed with location parameter η, scale parameter κ, and shape parameter ω. The pdf and cdf of this skew-normal distribution are given by (3.15) and (3.16), respectively.

The likelihood function based on a sample of observations $x_1, x_2, \ldots, x_n$ is given by

$$L(\eta, \kappa, \omega; x_1, x_2, \ldots, x_n) \propto \left(\frac{1}{\kappa}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\eta}{\kappa}\right)^2\right) \prod_{i=1}^{n} \Phi\left(\omega \cdot \frac{x_i-\eta}{\kappa}; 0, 1\right),$$

and the log-likelihood function can be expressed as

$$\ell(\eta, \kappa, \omega; x_1, x_2, \ldots, x_n) = -n\ln(\kappa) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\eta}{\kappa}\right)^2 + \sum_{i=1}^{n} \ln\left(\Phi\left(\omega \cdot \frac{x_i-\eta}{\kappa}; 0, 1\right)\right).$$

By taking the first derivative of the log likelihood function with respect to parameters η, κ, and ω, respectively, and setting them equal to 0, we have the following system of estimating equations:

$$\begin{cases} \sum_{i=1}^{n} \frac{x_i-\eta}{\kappa^2} - \frac{\omega}{\kappa}\sum_{i=1}^{n} \frac{\phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)}{\Phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)} = 0 \\ -\frac{n}{\kappa} + \sum_{i=1}^{n} \frac{(x_i-\eta)^2}{\kappa^3} - \frac{\omega}{\kappa}\sum_{i=1}^{n} \frac{\phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)}{\Phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)} \frac{x_i-\eta}{\kappa} = 0 \\ \sum_{i=1}^{n} \frac{\phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)}{\Phi\left(\omega\cdot\frac{x_i-\eta}{\kappa};0,1\right)} \cdot \frac{x_i-\eta}{\kappa} = 0. \end{cases}$$

The MLE of η, κ, and ω can be obtained by solving the above system of equations numerically. A well-known method for solving the complex system of equations is called the Newton-Raphson method, which has been used by most well developed packages in R for estimating the MLE of parameters for the fitted distributions. In this project, we estimate the parameters by using R packages *univariateML*[3] and *sn*[4].

[3] `https://www.rdocumentation.org/packages/univariateML/versions/1.1.1`

[4] `https://www.rdocumentation.org/packages/sn/versions/2.0.2`

Table 4.2: The estimated parameters and maximum log-likelihood.

Model	Parameters			Log-likelihood
Normal	$\mu = 0.00080985$	$\sigma = 0.02390084$		2412.129
Cauchy	$\lambda = 0.00284392$	$\theta = 0.01103033$		2422.943
Skew-normal	$\eta = 0.00048935$	$\kappa = 0.02363083$	$\omega =$ -0.26741422	2431.782

Table 4.2 shows the values of the estimated parameters for the three models. We notice that both Cauchy and skew-normal models have larger log-likelihood values at the estimated parameters than that of the normal model case. Based on this criterion, both Cauchy and skew-normal models fit our historical data better than the normal one. To determine which model fits the data better out of models with different numbers of parameters, we use the Akaike information criterion (Akaike, 1973) and Bayesian information criterion (Schwarz, 1978).

Definition 4.1. The Akaike information criterion (AIC) is a measure of goodness of fit defined as

$$\text{AIC} = 2k - 2\ell(\hat{\theta}),$$

where k is the number of estimated parameters, $\hat{\theta}$ represents a set of estimated parameters in the model, and ℓ is the log-likelihood function.

Definition 4.2. The Bayesian information criterion (BIC) is a measure of goodness of fit defined as

$$\text{BIC} = k \cdot \ln(n) - 2\ell(\hat{\theta}),$$

where n is the number of observations.

Both AIC and BIC can help to compare the goodness-of-fit for models with different numbers of parameters. A smaller AIC or BIC indicates a better-fit model within all the candidate models. We present the AIC and BIC values for the three models in Table 4.3.

Table 4.3: The AIC and BIC values of models.

Model	Number of parameters	Log-likelihood	AIC	BIC
Normal	2	2412.129	-4820.257	-4810.360
Cauchy	2	2422.943	-4841.885	-4831.987
Skew-normal	3	2431.782	**-4857.564**	**-4842.717**

According to the AIC and BIC values in Table 4.3, the skew-normal model has the lowest AIC and BIC values. Hence, we can conclude that the skew-normal distribution is the best-fit model for our S&P 500 historical returns data.

4.1.3 Graphical analysis of fitted models

In this subsection, we use several graphical tools to help understand the relationship between the fitted theoretical distributions and the empirical distribution. The useful plots include density plots, cdf plots, Q-Q plots, and P-P plots. The density plot shows the density functions of the fitted distributions along with the empirical histogram of the data; the cdf plot shows the cumulative distribution function of the fitted distributions and the empirical distribution function. The Q-Q plot compares the quantiles of the fitted distribution with the empirical distribution, and the P-P plot compares the theoretical probabilities to the empirical probabilities. These plots for our data are displayed in Figures 4.5 to 4.7.

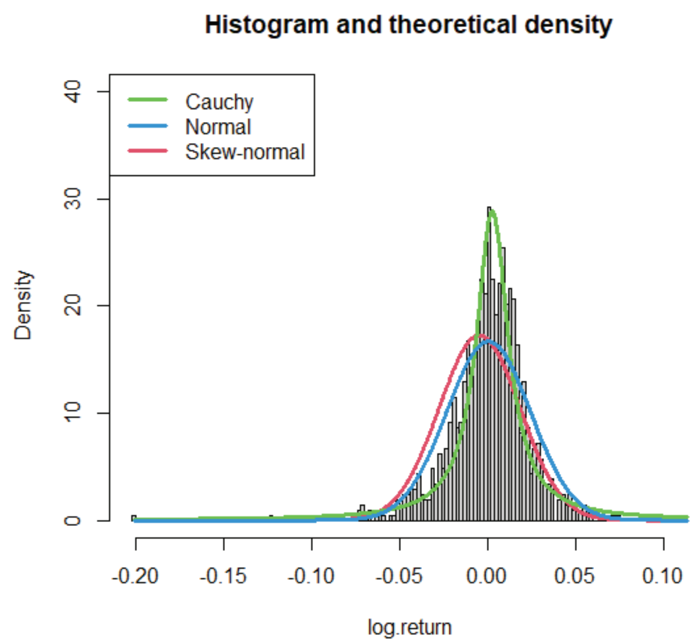

Figure 4.5: Density plot of theoretical distributions and histogram of empirical distribution.

In Figure 4.5, we observe that the histogram shows a large portion of the data sitting around 0, and relatively heavy-tailed. Visually, the Cauchy distribution fits the data well in the middle and is able to capture the peak of the empirical distribution. The skew-normal distribution shows a slightly negative (left) skew and fits the data well in the right part of the empirical distribution. In Figure 4.6, we observe a similar distributional behaviour of Cauchy and skew-normal distributions with respect to the empirical one. We also notice that the cdf curve of the Cauchy distribution lays above that of the empirical distribution and the other two theoretical distributions on the left tail side and lays below on the right tail side. This implies that the Cauchy distribution over-estimates the data at the tails, while it fits the middle part perfectly.

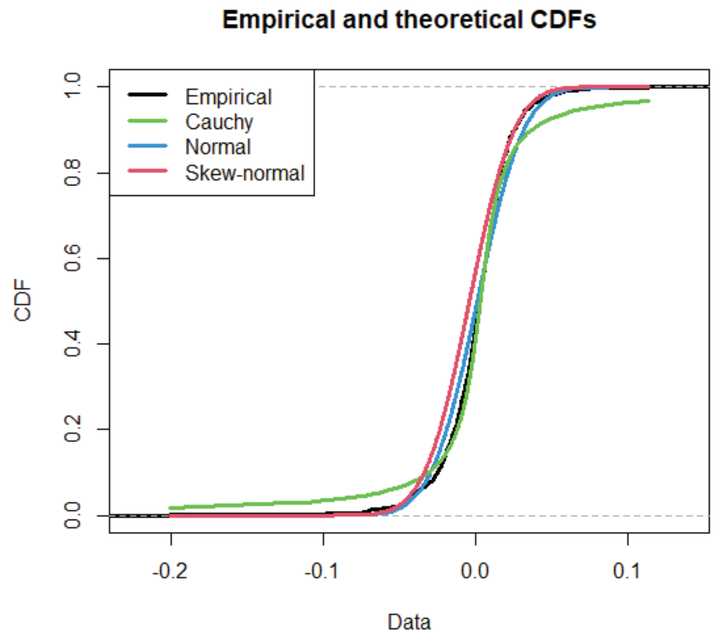

Figure 4.6: CDF plot of theoretical and empirical distributions.

In Figure 4.7, we show the quantiles of the fitted three distributions against the empirical distribution on the left, and the cumulative probabilities of the fitted three distributions against the empirical distribution on the right. We observe from both the Q-Q plots and P-P plots that the Cauchy distribution shows a good fit for the values around 0 in the middle part of the empirical distribution. The skew-normal Q-Q plot illustrates again that the empirical distribution is negatively skewed.

In Figure 4.8, we display S&P 500 projections for the next 10 years based on the fitted normal, Cauchy, and skew-normal models on the left, and corresponding return projections for the next 10 years on the right. Note that the lower and upper quantiles of price projections showed in the figure are 25% and 75% for the Cauchy model, and 5% and 95% for the normal and skew-normal models. The projections of the stock prices under the Cauchy model show an extremely wide price range compared to that under the normal and skew-normal models. Moreover, the extremes of the projected prices under the Cauchy model become more extreme to the upside. The reason for this phenomenon is that the Cauchy distribution is heavy-tailed. As expected, the prediction of returns under the Cauchy and normal models are symmetric, while that under the skew-normal model shows negative skewness. Because of the negative skewness of our fitted skew-normal model, the prediction shows a downward trend of stock prices. In addition, the median of projected returns is below 0 (under the green horizontal line) in the skew-normal model, which demonstrates the downward trend of its projected stock prices.

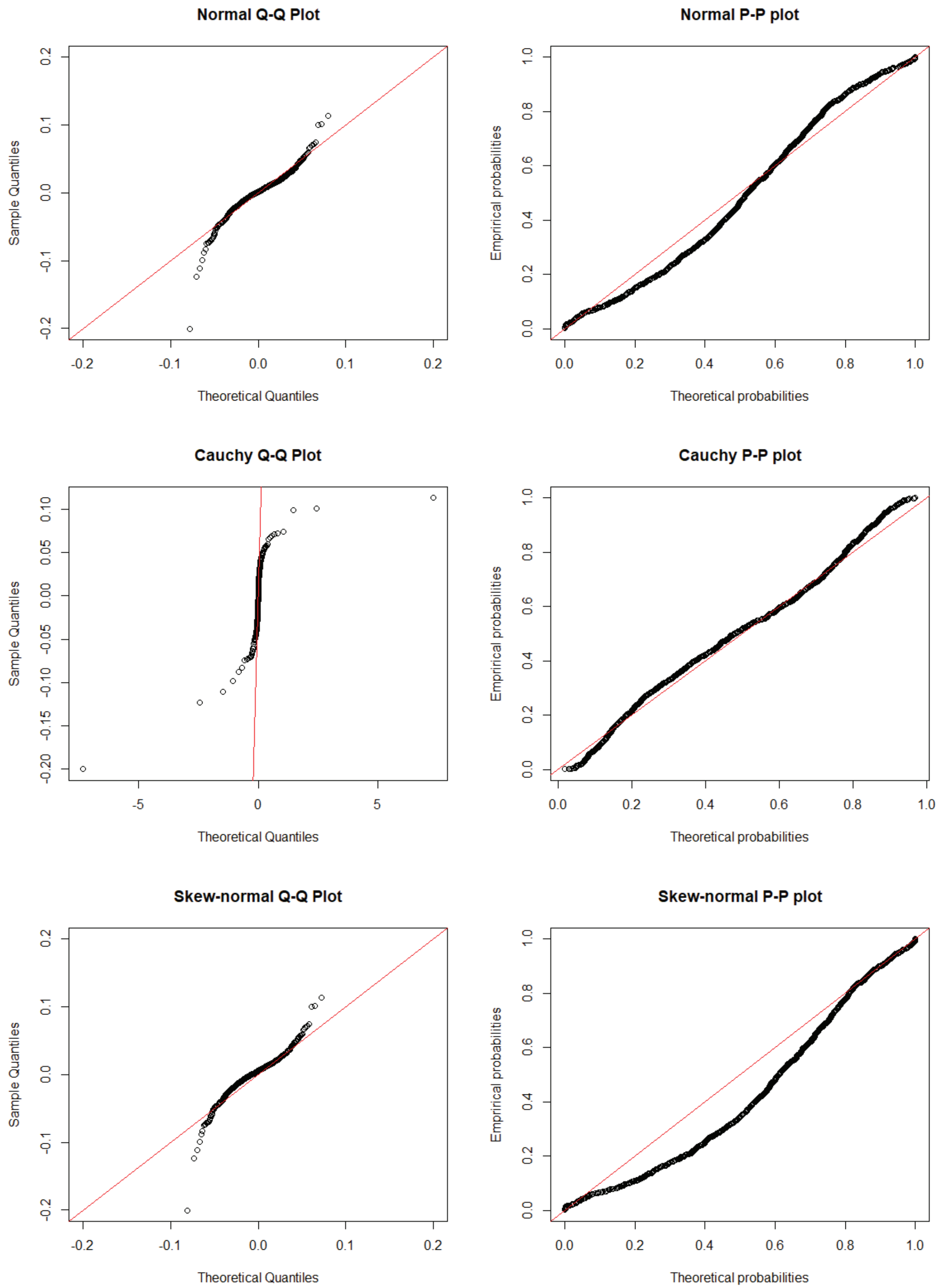

Figure 4.7: Q-Q plots and P-P plots.

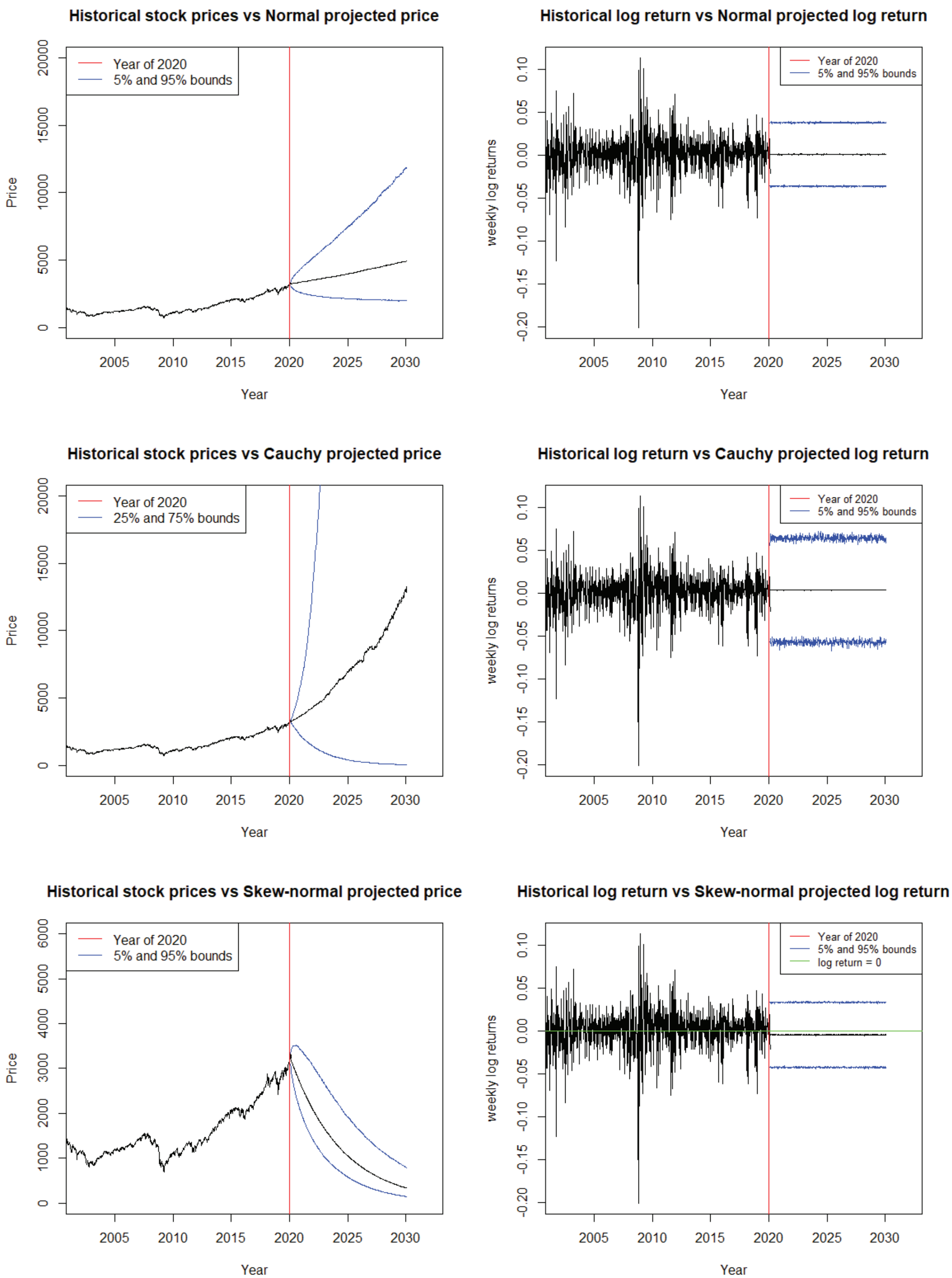

Figure 4.8: Projections versus historical data.

4.2 Risk measure results and analysis

In this section, we calculate and compare the VaR and CTE for the insurer's future liabilities (gross and net) with respect to GMMB and GMDB riders based on three fitted return models. For example purposes, the variable annuity product is set up for a male policyholder with age 65 at the time of purchase, and the policy term of this variable annuity is 10 years (i.e., $T = 10$). The guaranteed levels of GMMB and GMDB riders relative to F_0 are set to be 75%, 100%, and 120%, respectively. According to Segregated Funds Task Force of the CIA (2002), a report published by the Canadian Institute of Actuaries, the most common recommended risk measures to determine risk capital are CTEs at 80% and 95% levels. In this project, we calculate VaRs and CTEs at levels of 80%, 90%, and 95%, respectively. The mortality rates being used in our numerical calculations are from a life table provided by the U.S. Social Security Administration (Bell and Miller, 2005).

For the sake of efficiency in calculations, we assume that all the guaranteed payments are payable at the end of year during the policy term, and thus the approximation is on the annual basis (i.e., $n = 1$). The risk-free force of interest as well as the roll-up rate are set arbitrarily within a reasonable range for both GMMB and GMDB riders. We simulate $N = 1000$ sets of $\{e^{Y_k}\}_{k=1}^{nT}$, and then repeat the experiments $M = 100$ times to get the approximated values of ξ_e, ξ_d, V_α and CTE_α. Because the MLE parameters estimated for the models considered in this study are based on weekly data, we need to convert the MLE parameters to the annual basis for all fitted distribution models. For example, under the normal model, the return from $(t-1)$-th week to t-th week is normally distributed as

$$\ln\left(\frac{S_t}{S_{t-1}}\right) \sim \mathrm{Norm}\left(\mu, \sigma\right),$$

where t is measured in weeks. Converting it to the annual basis, we then have

$$\ln\left(\frac{S_{52t}}{S_{52(t-1)}}\right) \sim \mathrm{Norm}\left(52\mu, \sigma\sqrt{52}\right),$$

or equivalently,

$$\ln\left(\frac{S_t}{S_{t-1}}\right) \sim \mathrm{Norm}\left(52\mu, \sigma\sqrt{52}\right),$$

where t is measured in years.

4.2.1 Probabilities of non-positive liabilities

Before we proceed to calculate all the risk measure based on the analytical expressions and approximation methods, we need to calculate the corresponding probabilities of non-positive liabilities ξ_e and ξ_d for both gross and net liabilities. Recall that, in Sections 3.2 and 3.3, either the analytical expressions or numerical approximation methods are provided to calculate these probabilities. Table 4.4 shows all the probabilities of non-positive liabilities under different combinations of the guaranteed levels, models, and liability types. As explained earlier, the level α is assumed to be greater than the corresponding ξ_e or ξ_d in this study; accordingly, the risk measure results are displayed as 0* when α is less than the corresponding ξ_e or ξ_d values. In latter case, no guaranteed payment will be made at the date of maturity. For instance, in Table 4.4, the ξ_e of the normal model at 75% guaranteed level for the net GMMB liability has a value of 0.9146. This means that the VaR values at the risk levels of 80% and 90% would have a value of 0* in the case of net GMMB liability under the normal model showed in Table 4.5.

Table 4.4: Probabilities of non-positive liabilities.

			Models		
Liability		Guaranteed level (%)	Normal	Cauchy	Skew-normal
Gross GMMB	ξ_e	75	0.90008	0.68963	0.79947
		100	0.78965	0.67834	0.64652
		120	0.69758	0.67105	0.54301
Net GMMB	ξ_e	75	0.91460	0.72445	0.70070
		100	0.81079	0.71046	0.53523
		120	0.71898	0.70332	0.43672
Gross GMDB	ξ_d	75	0.91279	0.89814	0.87379
		100	0.84244	0.88893	0.80634
		120	0.80333	0.88273	0.77834
Net GMDB	ξ_d	75	0.91827	0.90310	0.85421
		100	0.84713	0.89361	0.79375
		120	0.80615	0.88668	0.77078

4.2.2 Risk measure results for GMMB

Recall that the GMMB rider provides the policyholder with the largest value between the guaranteed amount and the separate account fund value at maturity date. We calculate the risk measures for the insurer's future gross liabilities based on the propositions presented in Section 3.2.2, and the risk measures for the net liabilities based on the algorithms presented in Section 3.3.1. Below is the valuation basis used for calculations for the GMMB rider:

- Parameters of corresponding models are showed in Table 4.2;
- Risk-free force of interest $r = 0.04$;
- GMMB rider annual rate of management fees and charges of $m = 0.01$;
- GMMB rider annual rate of management expenses of $m_e = 0.0035$;
- Initial premium paid for the variable annuity (F_0) with a GMMB rider is \$100.

Table 4.5 shows the calculated risk measure results relative to the initial fund value F_0 for the GMMB gross liabilities with different predetermined guaranteed levels and different risk levels for normal, Cauchy, and skew-normal underlying equity models. From the insurer's points of view, only the positive liabilities are meaningful in real life applications. The values with an asterisk mark in Table 4.5 imply negative risk measures for gross liabilities. For example, with the guaranteed level of 75% and the normal model, the insurer has a risk capital of 0% of the initial fund, which indicates that no capital is exposed to risk at levels of 80% and 90%. In this case, such products with corresponding risk levels and guaranteed levels could be profitable.

Table 4.5: The numerical results of GMMB gross liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}/F_0$	0*	0.48871	0.00056
	$CTE_{80\%}/F_0$	0.01617	0.50137	0.12680
	$V_{90\%}/F_0$	0*	0.50273	0.11422
	$CTE_{90\%}/F_0$	0.11071	0.50274	0.19775
	$V_{95\%}/F_0$	0.09601	0.50274	0.18644
	$CTE_{95\%}/F_0$	0.17584	0.50274	0.29438
100	$V_{80\%}/F_0$	0.01490	0.65629	0.16814
	$CTE_{80\%}/F_0$	0.18375	0.66895	0.29438
	$V_{90\%}/F_0$	0.16744	0.67031	0.28179
	$CTE_{90\%}/F_0$	0.27829	0.67032	0.36533
	$V_{95\%}/F_0$	0.26359	0.67032	0.35402
	$CTE_{95\%}/F_0$	0.34342	0.67032	0.41457
120	$V_{80\%}/F_0$	0.14896	0.79035	0.30221
	$CTE_{80\%}/F_0$	0.31781	0.80301	0.42844
	$V_{90\%}/F_0$	0.30150	0.80437	0.41586
	$CTE_{90\%}/F_0$	0.41235	0.80438	0.49940
	$V_{95\%}/F_0$	0.39765	0.80438	0.48808
	$CTE_{95\%}/F_0$	0.47749	0.80438	0.54863

Based on the values showed in Table 4.5, we notice that all the risk measures relative to the initial fund value F_0 for the Cauchy model are significantly greater than that of the normal and skew-normal models. For the guaranteed level at 120% of the initial premium, around 80% of the insurer's capital is exposed. This reminds that the Cauchy model should be used with a great caution. Because the Cauchy distribution features fat-tails compared to the normal and skew-normal distributions, it is more likely to incur enormous future losses if the Cauchy distribution is used.

Table 4.6 shows the calculated risk measure results relative to F_0 for GMMB net liabilities with different predetermined guaranteed levels and different risk levels for normal, Cauchy, and skew-normal models. Because the net liability is the gross liability net of the margin offset, we expect the risk measure values in Table 4.6 to be all slightly less than the corresponding ones showed in Table 4.5. However, this holds only for the normal and Cauchy models. By comparing the risk measure values in both Tables 4.5 and 4.6 for the skew-normal distribution, we notice that the net liability risk measure values are larger than those for the gross liability. This is because the fitted skew-normal distribution is negatively (left) skewed, which would cause the simulated future paths of underlying equity prices have a downward trend.

Table 4.6: The numerical results of GMMB net liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}/F_0$	0*	0.39717	0.08511
	$CTE_{80\%}/F_0$	0*	0.47745	0.18810
	$V_{90\%}/F_0$	0*	0.48843	0.17772
	$CTE_{90\%}/F_0$	0.08851	0.49754	0.24529
	$V_{95\%}/F_0$	0.07433	0.49820	0.23647
	$CTE_{95\%}/F_0$	0.15524	0.50115	0.28471
100	$V_{80\%}/F_0$	0*	0.56594	0.25342
	$CTE_{80\%}/F_0$	0.15964	0.64497	0.35547
	$V_{90\%}/F_0$	0.14247	0.65607	0.34543
	$CTE_{90\%}/F_0$	0.25645	0.66513	0.41288
	$V_{95\%}/F_0$	0.24306	0.66580	0.40355
	$CTE_{95\%}/F_0$	0.32256	0.66874	0.45279
120	$V_{80\%}/F_0$	0.12151	0.70024	0.38716
	$CTE_{80\%}/F_0$	0.29329	0.77935	0.48946
	$V_{90\%}/F_0$	0.27647	0.79002	0.47897
	$CTE_{90\%}/F_0$	0.39037	0.79918	0.54687
	$V_{95\%}/F_0$	0.37543	0.79983	0.53804
	$CTE_{95\%}/F_0$	0.45694	0.80281	0.58747

4.2.3 Risk measure results for GMDB

The GMDB rider provides the policyholder a death benefit during the policy term. The death benefit is equal to the larger value between the guaranteed amount accumulated at a roll-up rate and the separate account fund value at the time of death of the policyholder. In this numerical example, we assume that the death benefit is payable at the end of the year of death, and we calculate the risk measures for the insurer's future gross liabilities based on the propositions presented in Section 3.2.3, and the risk measures for the net liabilities based on the algorithms presented in Section 3.3.2. Below is the valuation basis used for calculations for the GMDB rider:

- Parameters of corresponding models are showed in Table 4.2;
- Risk-free force of interest $r = 0.07$;
- Roll-up rate $\delta = 0.06$;
- GMDB rider annual rate of management fees and charges of $m = 0.01$;
- GMDB rider annual rate of management expenses of $m_d = 0.0035$;
- Initial premium paid for the variable annuity (F_0) with a GMMB rider is \$100.

The calculated risk measure results relative to the initial fund value F_0 for GMDB gross and net liabilities are listed in Tables 4.7 and 4.8, respectively. As we can see from both tables, risk measure values for GMDB rider are generally stable when assuming that the underlying equity returns follow a normal or a skew-normal distribution. However, the risk measure values relative to F_0 show significant differences for the Cauchy model. For instance, the 90% VaR (0.04410) and 95% VaR (0.67486) differ significantly when the guaranteed level is set at 75%. Notice that for a guaranteed level of 120% and with a risk level of 95% for the Cauchy model, the insurer encounters a risk capital of 100%, which means the entire initial premium could be exposed at a 95% level.

Table 4.7: The numerical results of GMDB gross liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}/F_0$	0*	0*	0*
	$CTE_{80\%}/F_0$	0.07688	0.28756	0.12849
	$V_{90\%}/F_0$	0*	0.04410	0.07820
	$CTE_{90\%}/F_0$	0.15377	0.57472	0.24670
	$V_{95\%}/F_0$	0.13211	0.67486	0.23262
	$CTE_{95\%}/F_0$	0.25971	0.69691	0.33961
100	$V_{80\%}/F_0$	0*	0*	0*
	$CTE_{80\%}/F_0$	0.22120	0.41295	0.31879
	$V_{90\%}/F_0$	0.19462	0.28353	0.31363
	$CTE_{90\%}/F_0$	0.38472	0.80971	0.47942
	$V_{95\%}/F_0$	0.36603	0.90429	0.46554
	$CTE_{95\%}/F_0$	0.49172	0.92942	0.57084
120	$V_{80\%}/F_0$	0*	0*	0.15881
	$CTE_{80\%}/F_0$	0.38964	0.52064	0.50645
	$V_{90\%}/F_0$	0.38380	0.47505	0.50205
	$CTE_{90\%}/F_0$	0.57177	0.99770	0.66570
	$V_{95\%}/F_0$	0.55325	1^+	0.65196
	$CTE_{95\%}/F_0$	0.67747	1^+	0.75592

From the insurer's point of view, the risk measures for net liabilities are more meaningful than that for gross liabilities. Table 4.8 shows the calculated risk measure values relative to F_0 for net GMDB liabilities. The values calculated for net GMDB liabilities are smaller than those for the gross GMDB liabilities for the normal and Cauchy models only. We also observe that, with either a GMMB rider or a GMDB rider, the Cauchy model returns the largest risk measure relative to the initial fund value F_0 due to the fat tails of the Cauchy distribution, while the skew-normal model returns larger results than normal model. The latter is because the estimated location parameter of the skew-normal model is smaller than the estimated location parameter of the normal model. In addition, the estimated negative shape parameter corresponding to the negative (left) skewness for the skew-normal model implies a fat left tail in distribution compared to the normal model, which results in larger risk measure values in this case.

Table 4.8: The numerical results of GMDB net liabilities.

		Models		
Guaranteed level (%)	Risk Measures	Normal	Cauchy	Skew-normal
75	$V_{80\%}/F_0$	0*	0*	0*
	$CTE_{80\%}/F_0$	0.06966	0.26435	0.16508
	$V_{90\%}/F_0$	0*	0.00753	0.13942
	$CTE_{90\%}/F_0$	0.13890	0.53101	0.29654
	$V_{95\%}/F_0$	0.11369	0.63524	0.28510
	$CTE_{95\%}/F_0$	0.24150	0.68515	0.38128
100	$V_{80\%}/F_0$	0*	0*	0.03587
	$CTE_{80\%}/F_0$	0.20713	0.38692	0.37423
	$V_{90\%}/F_0$	0.17880	0.16539	0.37521
	$CTE_{90\%}/F_0$	0.36872	0.76163	0.52977
	$V_{95\%}/F_0$	0.34861	0.86678	0.51769
	$CTE_{95\%}/F_0$	0.47483	0.91904	0.61174
120	$V_{80\%}/F_0$	0*	0*	0.22600
	$CTE_{80\%}/F_0$	0.37253	0.48659	0.56205
	$V_{90\%}/F_0$	0.36891	0.36146	0.56039
	$CTE_{90\%}/F_0$	0.55630	0.95021	0.71322
	$V_{95\%}/F_0$	0.53731	1^+	0.70269
	$CTE_{95\%}/F_0$	0.66154	1^+	0.79588

Moreover, in Tables 4.5 to 4.8, the values with the asterisk mark can be viewed as being left-truncated. In this case, the risk measure results relative to F_0 are less than 0 (0% of the initial premium) and thus are truncated at 0 because the insurers do not face risks in terms of management on the future liabilities. On the other hand, the results with superscript (+) mean that the calculated risk measures are all greater than F_0, the initial premium. It is worth noting that such situations occur only in the GMDB case because the guaranteed amounts are allowed to accumulate along the policy term.

Chapter 5

Conclusion

In this project, we studied variable annuities with two types of guaranteed benefits: the GMMB and GMDB riders. We assumed that the returns of the underlying asset for the considered variable annuities follow Cauchy or skew-normal distributions. Two typical risk measures, the VaR and the CTE, are calculated for the insurer's future gross and net liabilities with either of the two guaranteed benefit riders. In an illustration, we fitted our proposed asset return models to the historical S&P 500 weekly returns data. We then compared the calculated risk measure results under the fitted asset return models for both insurer's gross and net liabilities with one of the guaranteed benefit riders.

Our main findings of this study are as follows. First, we found that our newly proposed Cauchy and skew-normal models can fit the returns data better than the normal model under the maximum likelihood estimation. While the Cauchy model can capture the peak of the empirical distribution of the returns, the skew-normal model is suitable for left or right skewed returns data. Second, we observed that the risk measure results relative to F_0 under the two proposed models are all greater than that under the normal model. This finding implies that the normal model used in Feng and Volkmer (2012) may under-estimate the financial risk faced by the insurer in the context of variable annuities with guaranteed benefit riders. Third, we found in our example that the VaR results vary greatly over different risk levels for all three models, compared to the changes of the corresponding CTE results. This suggests that CTE risk measures are not too sensitive to the changes of risk levels in our study. Our last finding of this study is on the approximation methods: Monte Carlo simulation to approximate VaR and CTE values is feasible and efficient in our context.

Although the Cauchy model fits the returns data better than the normal model according to the criteria we use, the Cauchy model should be used with caution. The reason is that while it captures well the peak of the data, it fits tails relatively poorly comparing to the other two normal models, which may over-estimate the risks at the right tail. The skew-normal model we studied can capture the skewness of the returns data as shown from our data application compared to the other two symmetric distributions. In the negative (left) skewed case, the insurers may under-estimate their profitability of the variable annuity contracts. Insurers need to be aware of the positive and negative aspects of using the distributions we study in this project.

This study can be extended in different ways. We could consider other distributional models (for example, the skew t distribution) to model the underlying asset returns, and we may also consider mixture models which are able to capture the peak, skewness, and heavy tails of the equity returns. We may also consider a variable annuity with both GMMB and GMDB riders and study the risk measures of the insurer's liabilities in this case. In addition, we may use alternative risk measures to calculate the risk capitals of insurers' future liabilities. For example, the weighted value-at-risk proposed by Cont et al. (2010), also called the range value-at-risk (RVaR), is the truncation version of the CTE. The RVaR is suitable in dealing with the fat-tail distributions and infinite tail expectations (Bairakdar et al., 2020).

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Appendix A

Proofs

The proof for the results presented in Propositions 3.1 to 3.4 for the normal model are given in Feng and Volkmer (2012). Here, we prove only the results for the Cauchy model. (i.e., case (2) in each of Propositions 3.1 to 3.4). The results under the skew-normal model can be proven similarly, so they are omitted.

A.1 Proof of Proposition 3.1

Proof. Recall that the gross liability of a GMMB rider defined by (3.1) is

$$ {}_eL_g^0 = e^{-rT}(G - F_T)_+ I(\tau_x > T). $$

From Equation (3.6) and because the future lifetime of the policyholder is independent of the market value of the underlying equity fund, V_α can be determined by

$$ 1 - \alpha = {}_Tp_x \mathbb{P}[e^{-rT}(G - F_T) > \mathrm{V}_\alpha], $$

when $\alpha > \xi_e$.

Using (3.8), we have

$$ \begin{aligned} 1 - \alpha &= {}_Tp_x \mathbb{P}\left[e^{-rT}\left(G - F_0\frac{S_T}{S_0}e^{-mT}\right) > \mathrm{V}_\alpha\right] \\ &= {}_Tp_x \mathbb{P}\left[e^{-rT}G - F_0\frac{S_T}{S_0}e^{-(r+m)T} > \mathrm{V}_\alpha\right] \\ &= {}_Tp_x \mathbb{P}\left[\frac{S_T}{S_0}e^{-(r+m)T} < \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right], \end{aligned} $$

or

$$ \frac{1-\alpha}{{}_Tp_x} = \mathbb{P}\left[\frac{S_T}{S_0}e^{-(r+m)T} < \frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right]. \tag{A.1} $$

Since $S_T/S_0 \sim$ Log-Cauchy$(\lambda T, \theta T)$, we can easily get that

$$\frac{S_T}{S_0} e^{-(r+m)T} \sim \text{Log-Cauchy}((\lambda - r - m)T, \theta T),$$

or

$$\ln\left(\frac{S_T}{S_0} e^{-(r+m)T}\right) \sim \text{Cauchy}\left((\lambda - r - m)T, \theta T\right).$$

Let $\beta = (1\text{-}\alpha)/{}_Tp_x$. Then, we have

$$\ln\left(\frac{e^{-rT}G - \mathrm{V}_\alpha}{F_0}\right) = (\lambda - r - m)T + \theta T c_\beta,$$

where c_β is the $100\beta\%$ percentile of the standard Cauchy distribution. This gives

$$\mathrm{V}_\alpha = e^{-rT}G - F_0 \exp\left\{(\lambda - r - m)T + \theta T c_\beta\right\},$$

which proves (3.20). □

A.2 Proof of Proposition 3.2

Proof. From Equation (3.7) and because the future lifetime of the policyholder is independent of F_T, the CTE_α for gross liability of a GMMB rider is given by

$$\mathrm{CTE}_\alpha = \frac{{}_Tp_x}{1-\alpha} \mathbb{E}\left[e^{-rT}(G - F_T) I_{\{e^{-rT}(G - F_T) > \mathrm{V}_\alpha\}}\right],$$

when $\alpha > \xi_e$.

Using (3.8) and letting $Y = \ln(S_T/S_0 \cdot e^{-(r+m)T})$ and $a = (e^{-rT}G - \mathrm{V}_\alpha)/F_0$, we have

$$\begin{aligned}\mathrm{CTE}_\alpha &= \frac{{}_Tp_x}{1-\alpha} \mathbb{E}\left[\left(e^{-rT}G - F_0 \frac{S_T}{S_0} e^{-(r+m)T}\right) I_{\left\{e^{-rT}G - F_0 \frac{S_T}{S_0} e^{-(r+m)T} > \mathrm{V}_\alpha\right\}}\right] \\ &= \frac{{}_Tp_x}{1-\alpha} \mathbb{E}\left[\left(e^{-rT}G - F_0 e^Y\right) I_{\{e^Y < a\}}\right].\end{aligned}$$

Because $Y \sim \text{Cauchy}\left((\lambda - r - m)T, \theta T\right)$, and using (A.1), we further obtain

$$\begin{aligned}\mathrm{CTE}_\alpha &= \frac{{}_Tp_x}{1-\alpha}\left[\int_{-\infty}^{\ln(a)} \left(e^{-rT}G - F_0 e^y\right) f_C(y; (\lambda - r - m)T, \theta T) dy\right] \\ &= e^{-rT}G - \frac{{}_Tp_x F_0}{1-\alpha} \int_{-\infty}^{\ln(a)} e^y f_C(y; (\lambda - r - m)T, \theta T) dy,\end{aligned}$$

where f_C is the pdf of the Cauchy distribution with location parameter $(\lambda - r - m)T$ and scale parameter θT. This proves (3.21). □

A.3 Proof of Proposition 3.3

Proof. Recall that the gross liability of a GMDB rider defined by (3.4) is

$$ {}_{d}L_g^0 = e^{-r\tau_x}(e^{\delta\tau_x}G - F_{\tau_x})_+ I(\tau_x \le T). $$

From Equation (3.6) and by conditioning on the future lifetime of the policyholder τ_x, V_α can be determined by

$$ \begin{aligned} 1-\alpha &= \int_0^T \mathbb{P}\left[{}_{d}L_g^0 > \mathrm{V}_\alpha\right] f_{\tau_x}(t)dt, \\ &= \int_0^T \mathbb{P}\left[e^{-rt}\left(e^{\delta t}G - F_t\right) > \mathrm{V}_\alpha\right] f_{\tau_x}(t)dt, \end{aligned} $$

when $\alpha > \xi_d$, and where $f_{\tau_x}(t) = {}_tp_x\mu_{x+t}$.

Using (3.8), we have

$$ \begin{aligned} 1-\alpha &= \int_0^T \mathbb{P}\left[e^{-rt}\left(e^{\delta t}G - F_0\frac{S_t}{S_0}e^{-mt}\right) > \mathrm{V}_\alpha\right] {}_tp_x\mu_{x+t}dt \\ &= \int_0^T \mathbb{P}\left[\frac{S_t}{S_0}e^{-(r+m)t} < \frac{e^{(\delta-r)t}G - \mathrm{V}_\alpha}{F_0}\right] {}_tp_x\mu_{x+t}dt. \end{aligned} $$

Because $S_t/S_0e^{-(r+m)t} \sim$ Log-Cauchy$((\lambda - r - m)t, \theta t)$, V_α can then be determined by

$$ 1-\alpha = \int_0^T F_C\left(\ln\left(\frac{e^{(\delta-r)t}G - \mathrm{V}_\alpha}{F_0}\right); (\lambda - r - m)t, \theta t\right) {}_tp_x\mu_{x+t}dt, $$

where F_C is the cdf of the Cauchy distribution. This proves (3.25). □

A.4 Proof of Proposition 3.4

Proof. From Equation (3.7) and by conditioning on the future lifetime of policyholder τ_x, the CTE_α for gross liability of a GMDB rider is given by

$$ \begin{aligned} \mathrm{CTE}_\alpha &= \mathbb{E}\left[{}_{d}L_g^0 | {}_{d}L_g^0 > \mathrm{V}_\alpha\right] \\ &= \frac{1}{1-\alpha}\int_0^T \mathbb{E}\left[e^{-rt}\left(e^{\delta t}G - F_t\right) I_{\left\{e^{-rt}\left(e^{\delta t}G - F_t\right) > \mathrm{V}_\alpha\right\}}\right] {}_tp_x\mu_{x+t}dt, \end{aligned} $$

when $\alpha > \xi_d$.

Using (3.8), and letting $Y_t = \ln(S_t/S_0 \cdot e^{-(r+m)t})$ and $c_t = (e^{(\delta-r)t}G - \mathrm{V}_\alpha)/F_0)$, we have

$$ \begin{aligned} \mathrm{CTE}_\alpha &= \frac{1}{1-\alpha}\int_0^T \mathbb{E}\left[\left(e^{(\delta-r)t}G - F_0\frac{S_t}{S_0}e^{-(r+m)t}\right) I_{\left\{e^{(\delta-r)t}G - F_0\frac{S_t}{S_0}e^{-(r+m)t} > \mathrm{V}_\alpha\right\}}\right] {}_tp_x\mu_{x+t}dt \\ &= \frac{1}{1-\alpha}\int_0^T \mathbb{E}\left[\left(e^{(\delta-r)t}G - F_0e^{Y_t}\right) I_{\{e^{Y_t} < c_t\}}\right] {}_tp_x\mu_{x+t}dt. \end{aligned} $$

Because $Y_t \sim \text{Cauchy}\left((\lambda - r - m)t, \theta t\right)$, we have

$$\begin{aligned}
\text{CTE}_\alpha &= \frac{1}{1-\alpha}\int_0^T {}_tp_x\mu_{x+t}\left\{\int_{-\infty}^{\ln(c_t)} e^{(\delta-r)t}Gf_C(y)dy - F_0\int_{-\infty}^{\ln(c_t)} e^y f_C(y)dy\right\}dt \\
&= \frac{G}{1-\alpha}\int_0^T e^{(\delta-r)t}{}_tp_x\mu_{x+t}\cdot F_C\left(\ln(c_t); (\lambda-r-m)t, \theta t\right)dt \\
&\quad - \frac{F_0}{1-\alpha}\int_0^T {}_tp_x\mu_{x+t}\int_{-\infty}^{\ln(c_t)} e^y\cdot f_C(y; (\lambda-r-m)t, \theta t)dydt,
\end{aligned}$$

where f_C and F_C are the pdf and cdf of the Cauchy distribution with location parameter $(\lambda - r - m)t$ and scale parameter θt. This proves (3.26). □