Generation and propagation of Lebesgue norms for the homogeneous Boltzmann equation without angular cutoff

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Matt Spragge

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Declaration of Committee

Name: **Matt Spragge**

Degree: **Master of Science**

Thesis title: **Generation and propagation of Lebesgue norms for the homogeneous Boltzmann equation without angular cutoff**

Committee: **Chair:** Ralf Wittenberg
Associate Professor, Mathematics

Weiran Sun
Supervisor
Associate Professor, Mathematics

Nilima Nigam
Committee Member
Professor, Mathematics

Razvan Fetecau
Examiner
Professor, Mathematics

Abstract

We consider kinetic systems comprised of a large number of interacting particles and discuss one specific approach to modelling such an object from a physical point of view. The subsequent kinetic model obtained through this process is a nonlinear integro-partial differential equation; namely, the Boltzmann equation. We focus, in particular, on the spatially homogeneous Boltzmann equation for soft potentials (HBESP) and without angular cutoff and use techniques adapted from the hard potential theory together with classical ideas to study the instantaneous appearance (generation) and propagation of L^p-norms. By considering solutions over a fixed interval of time and with sufficient assumptions regarding their L^1 and L^2 moments, we are able to demonstrate that L^p-norms of solutions to the non-cutoff (HBESP) are both generated and propagated in time.

Keywords: kinetic theory; nonlinear Boltzmann equation; analysis of partial differential equations; integro-differential equations; nonlocal analysis; well-posedness

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Chapter 1

Introduction

1.1 Introduction to kinetic models and equations

The aim of kinetic theory, in a general sense, is to describe the dynamics of systems comprised of a large number of interacting particles, such as gases or plasmas. The most obvious approach to this problem is to solve the dynamical system which details the precise position and velocity of every particle in the system. Of course, this approach is also highly intractable, since one would ultimately require knowledge of the system's initial state and in most practical cases, due to the typically very large number of constituent particles as well as their size relative to measurement devices, being able to accurately determine both the position and velocity of each particle at an initial time is not physically realistic. However, even though we are physically limited in our ability to measure such quantities, we will afford ourselves the mathematical convenience of denoting the total state of a system of N particles at a given time by a vector

$$z_N \coloneqq (x_1, x_2, ..., x_N, v_1, v_2, ..., v_N),$$

where $x_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ correspond to the position and velocity of the i^{th} particle respectively. We then refer to the $6N$-dimensional space of all possible states z_N as the *phase space* for the system. Now, to account for the uncertainty in any simultaneous measurement of a particle's position and velocity, we modify our approach to the problem by instead modelling the system with a probability density function, $f(t, z_N)$, such that $f(t, z_N)dz_N$ represents the probability that the N particles occupy the volume element dz_N of the phase space at time t. While this kinetic model is more physically appropriate, it still depends on far too many variables for the problem to be computationally feasible. It is then the central idea of kinetic theory that for a system comprised of only one particle species, one may consider as a kinetic model for the entire system, the *one-particle marginal density*, $f^{(1)}(t, x, v)$, that corresponds to the probability of finding a single representative particle at the point (x, v) in the reduced *one-particle phase space* at time t. Then, if we had a closed evolution equation for $f^{(1)}$ (or which is sometimes referred to as a *kinetic equation*) that would describe how $f^{(1)}$ changes in time, this would potentially address the issue of computational feasibility as $f^{(1)}(t, x, v)$ depends only on seven variables. As we shall see below, it is in fact possible, under certain assumptions, to derive (at least formally) such an expression.

Let us consider, for the moment, a system whose particles are identical and follow the laws of classical mechanics but do not interact. Then, the time evolution of a given one-particle phase point $(x(0), v)$ is given by

$$(x(t), v) = T_t(x(0), v) \coloneqq (x(0) + tv, v).$$

In this case, the probability that a particle is in the state $(x(0), v)$ at time $t = 0$ must be the same as the probability that a particle is in the state $T_t(x(0), v)$ at some later time t; that is, for any given initial region, Ω_0, of the one-particle phase space, we have

$$\int_{\Omega_0} f^{(1)}(0, x(0), v) dx(0)\, dv = \int_{\Omega} f^{(1)}(t, x(t), v) dx(t)\, dv,$$

where $\Omega = T_t(\Omega_0) = \{(x(t), v) = T_t(x(0), v) \in \mathbb{R}^{3\times 3} : (x(0), v) \in \Omega_0\}$. Then, for any time t, the transformation T_t has unit Jacobian determinant since

$$|DT_t| = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1,$$

and hence

$$\begin{aligned} \int_{\Omega} f^{(1)}(t, x(t), v) dx(t)\, dv &= \int_{\Omega_0} f^{(1)}(t, x(0) + tv, v) |DT_t| dx(0)\, dv \\ &= \int_{\Omega_0} f^{(1)}(t, x(0) + tv, v) dx(0)\, dv. \end{aligned}$$

Therefore, since Ω_0 is any arbitrary region in the phase space, we may conclude from the above integrals that

$$f^{(1)}(t, x, v) = f^{(1)}(0, x(0), v).$$

By differentiating this expression, we then obtain the evolution equation

$$f_t^{(1)} + \frac{\partial(x(0) + tv)}{\partial t} \cdot \nabla_x f^{(1)} = f_t^{(1)} + v \cdot \nabla_x f^{(1)} = 0.$$

Of course, for any real kinetic system this equation is incomplete as we have yet to include the effect of particle interactions on $f^{(1)}$. It turns out that there are various ways in which one may rectify this omission and, in fact, this is the primary reason why there is no unique kinetic equation for any given system. In the proceeding section, we shall provide one such method for dealing with particle interactions.

1.2 Boltzmann's collision operator

We begin by first outlining the various postulates regarding the interactions between particles that will allow us mathematically describe their impact on the density function $f^{(1)}$, through which we may also introduce the relevant terminology and notation.

It is firstly assumed that the way in which constituent particles interact is only through *elastic binary collisions*; that is, collisions that conserve momentum and energy and occur at a given position and time between two particles only. A relevant remark to make about this assumption is that it inherently imposes a restriction on the types of systems we may consider since they must be dilute enough for us to be able to neglect collisions simultaneously occurring between three or more particles. We also note that our elasticity condition allows us to accurately describe the geometry of such collisions and, in fact, there are two common ways for us to do this.

Indeed, if we let v' and v'_* denote the velocities of two particles before collision, referred to as *pre-collisional velocities*, and similarly let v and v_* denote the *post-collisional velocities*, then by conservation of momentum and kinetic energy we have

$$v' + v'_* = v + v_* \tag{1.1}$$

and

$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \tag{1.2}$$

By rewriting (1.1) as

$$v' - v = v_* - v'_*,$$

it is clear that the vectors $v' - v$ and $v_* - v'_*$ lie in the same plane. Moreover, by (1.1), we have that

$$|v' + v'_*|^2 = |v + v_*|^2$$

and so together with (1.2) we find that

$$|v'|^2 + |v'_*|^2 + 2\langle v', v'_* \rangle = |v|^2 + |v_*|^2 + 2\langle v, v_* \rangle = |v'|^2 + |v'_*|^2 + 2\langle v, v_* \rangle,$$

which implies $\langle v', v'_* \rangle = \langle v, v_* \rangle$. With this in mind, we may apply (1.2) again to obtain

$$\begin{aligned} |v' - v'_*|^2 &= |v'|^2 + |v'_*|^2 - 2\langle v', v'_* \rangle \\ &= |v|^2 + |v_*|^2 - 2\langle v, v_* \rangle \\ &= |v - v_*|^2. \end{aligned}$$

Geometrically, this corresponds to the lengths of the diagonals between the vectors $v' - v$ and $v_* - v'_*$ being equal, and since these vectors are parallel we may conclude that they form opposite sides of a rectangle in some plane in the particle phase space; see Figure 1.1.

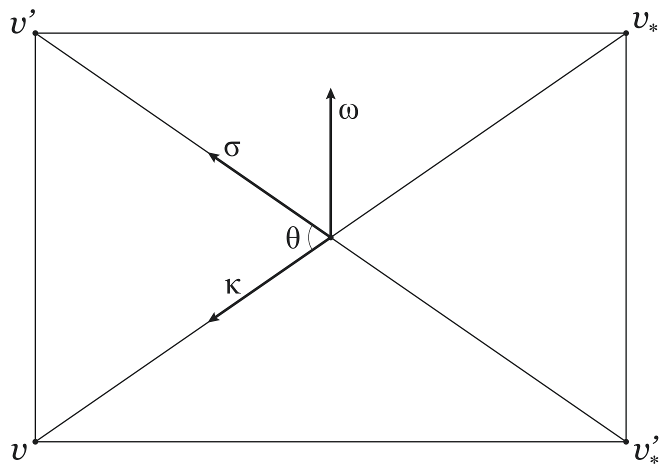

Figure 1.1: Geometry of an elastic binary collision

Defining now the spherical variable $\sigma \in \mathbb{S}^2$ by $\sigma \coloneqq \frac{v'-v'_*}{|v'-v'_*|}$ and referring to the geometry of binary collisions described above or Figure 1.1, we see that we may relate the pre-collisional velocities to the post-collisional velocities through the following formulas known as the *σ-representation* of the pre-collisional velocities

$$\begin{aligned} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \\ v'_* &= \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma. \end{aligned} \tag{1.3}$$

Note also, that we may write the post-collisional velocities in terms of the pre-collisional velocities in a very similar manner:

$$\begin{aligned} v &= \frac{v'+v'_*}{2} + \frac{v'-v'_*}{2}\kappa, \\ v_* &= \frac{v'+v'_*}{2} - \frac{v'-v'_*}{2}\kappa, \end{aligned}$$

where $\kappa \in \mathbb{S}^2$, similarly to σ, is defined by $\kappa \coloneqq \frac{v-v_*}{|v-v_*|}$. With this notation, the *deviation angle* between pre- and post-collisional velocities, θ, must satisfy

$$\cos\theta = \langle \kappa, \sigma \rangle.$$

For convenience, if we let x and x_* denote the center positions of the two particles involved in the collision, we may alternatively use the spherical variable $\omega \in \mathbb{S}^2$, defined by $\omega \coloneqq \frac{x_*-x}{|x_*-x|}$, to relate the pre- and post-collisional variables. The easiest way to see this relationship is by first noticing that, at the time of collision, ω must bisect the angle formed between the relative velocities $(v_* - v)$ and $-(v'_* - v')$ and can therefore be expressed as

$$\omega = \frac{(v_* - v) - (v'_* - v')}{|(v_* - v) - (v'_* - v')|} = \frac{(v' - v) + (v_* - v'_*)}{|(v' - v) + (v_* - v'_*)|},$$

but since $(v' - v)$ and $(v_* - v'_*)$ are parallel (see Figure 1.1) we may more simply write

$$\omega = \frac{v' - v}{|v' - v|}.$$

Thus, referring again to Figure 1.1, we get the *ω-representation*:

$$\begin{aligned} v' &= v - \langle v - v_*, \omega\rangle\omega, \\ v'_* &= v_* + \langle v - v_*, \omega\rangle\omega, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} v &= v' - \langle v' - v'_*, \omega\rangle\omega, \\ v_* &= v'_* + \langle v' - v'_*, \omega\rangle\omega. \end{aligned} \tag{1.5}$$

Now that we have made precise what we mean by a collision, we may begin to discuss their impact on $f^{(1)}$. Recall that $f^{(1)}(t,x,v)dx\,dv$ gives the probability that, at time t, there is a particle in the volume element $dx\,dv$ of the one-particle phase space. So, if our goal is to determine how $f^{(1)}(t,x,v)$ evolves in time, we must therefore keep track of the number of collisions involving particles with state (x,v) over time. Consequently, our updated kinetic equation should take the form

$$f_t^{(1)}(t,x,v) + v\cdot\nabla_x f^{(1)}(t,x,v) = G(t,x,v) - L(t,x,v), \tag{1.6}$$

where the *gain term* $G dx\,dv\,dt$ is the probability that a particle enters the volume element $dx\,dv$ during the time interval dt due to collision and the *loss term* $L dx\,dv\,dt$ similarly denotes the probability that a particle leaves the volume element $dx\,dv$ in the time interval dt due to collision. Continuing with our consideration of a system of N identical particles, we may then write

$$G = (N-1)g, \quad \text{and} \quad L = (N-1)\ell,$$

where $g dx\,dv\,dt$ and $\ell dx\,dv\,dt$ represent the probability of a collision between any two given fixed particles, say the *starless* particle (whose phase point we denote with starless variables (x,v)) and the *starred* particle (whose phase point is denoted with starred variables (x_*, v_*)).

Let us focus, for now, on computing the loss term and for simplicity we will proceed with the assumption that the constituent particles are *hard spheres* with radii $\frac{r}{2}$, where by hard we mean the particles are perfectly rigid so that the joint probability of there being two particles such that $|x - x_*| < r$ is zero. Now, let us take the starless particle as our frame of reference. If we treat the starless particle as having twice its actual radius and

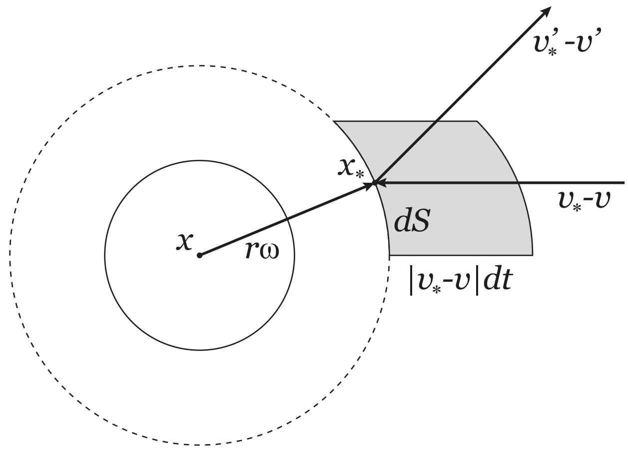

Figure 1.2: ω-representation of a binary collision relative to one particle

the starred particle as being just its center point with relative velocity $(v_* - v)$, then the cylinder whose base is given by the surface area element of the extended starless particle, $dS = r^2 d\omega$ (where $d\omega$ denotes the surface area element about ω of the unit sphere), and side given by $(v_* - v)$ with length $|v_* - v|dt$ (see Figure 1.2), contains all possible starred particles with relative velocity $(v_* - v)$ that will collide with dS within the time interval dt (or have already collided with dS within dt time depending on the sign of $\langle v_* - v, \omega\rangle$). Then, since this cylinder has volume $|\langle v_* - v, \omega\rangle| dt\, r^2 d\omega$ and the probability of finding two particles in any volume element $dx\, dx_*\, dv\, dv_*$ of the two-particle phase space at time t is given by the two-particle marginal density $f^{(2)}(t, x, x_*, v, v_*) dx\, dx_*\, dv\, dv_*$, we find that the probability of the starred particle colliding with dS of the starless particle within dt time is $f^{(2)}(t, x, x + r\omega, v, v_*) dx\, dv\, dv_*\, |\langle v_* - v, \omega\rangle| dt\, r^2 d\omega$. The total probability that the starred and starless particles will collide in the fixed volume of the phase space $dx\, dv$ within the time interval dt is then

$$\ell dx\, dv\, dt = r^2 dx\, dv\, dt \int_{\mathbb{R}^3} \int_{\mathbb{S}^2_-} f^{(2)}(t, x, x + r\omega, v, v_*) |\langle v_* - v, \omega\rangle| d\omega\, dv_*,$$

where $\mathbb{S}^2_-$ is the hemisphere of $\mathbb{S}^2$ such that $\langle v_* - v, \omega\rangle < 0$ since this corresponds to the case where the particles are moving towards each other. Therefore, we may write

$$L = (N-1) r^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2_-} f^{(2)}(t, x, x + r\omega, v, v_*) |\langle v_* - v, \omega\rangle| d\omega\, dv_*.$$

As for the gain term, since the volume $|\langle v_* - v, \omega\rangle| dt\, r^2 d\omega$ also contains all possible particles with relative velocity $-(v_* - v)$ that have already collided with the starless particle within dt time, we may compute G in the same way as for L only by integrating over the hemisphere, $\mathbb{S}^2_+$, of $\mathbb{S}^2$ that corresponds to $\langle v_* - v, \omega\rangle > 0$:

$$G = (N-1)r^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} f^{(2)}(t,x,x+r\omega,v,v_*)|\langle v_* - v,\omega\rangle| d\omega\, dv_*.$$

If we further postulate that the pre-collisional velocities of any particles which are about to collide are independent, an assumption which physically corresponds to a sufficient level of molecular chaos (often referred to as Boltzmann's chaos assumption), then we may rewrite the loss term as

$$L = (N-1)r^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_-} f^{(1)}(t,x,v)f^{(1)}(t,x+r\omega,v_*)|\langle v_* - v,\omega\rangle| d\omega\, dv_*.$$

We cannot, however, simply do the same for the gain term since the velocities appearing in the joint density are understood as post-collisional and should be treated as dependent. To deal with this, we recognize that

$$f^{(2)}(t,x,x+r\omega,v,v_*) = f^{(2)}(t,x,x+r\omega,v',v'_*),$$

in which case we may write

$$\begin{aligned} G &= (N-1)r^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} f^{(2)}(t,x,x+r\omega,v',v'_*)|\langle v_* - v,\omega\rangle| d\omega\, dv_* \\ &= (N-1)r^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} f^{(1)}(t,x,v')f^{(1)}(t,x+r\omega,v'_*)|\langle v_* - v,\omega\rangle| d\omega\, dv_*. \end{aligned}$$

Moreover, since the ω-representation for v' and v'_*, as well as $|\langle v_* - v,\omega\rangle|$, are even functions in ω, we see that under the change of variable $\omega \to -\omega$ we have

$$G = (N-1)r^2 \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_-} f^{(1)}(t,x,v')f^{(1)}(t,x-r\omega,v'_*)|\langle v_* - v,\omega\rangle| d\omega\, dv_*.$$

By taking the so-called *Boltzmann-Grad limit*, where we let $N \to \infty$ and $r \to 0$ in such a way that Nr^2 tends to a constant C, then $(N-1)r^2 \simeq Nr^2 = C$ and we no longer distinguish between x and $x \pm r\omega$. Therefore, in the limit we may express the right hand side of (1.6) as follows:

$$G(t,x,v) - L(t,x,v) = C \int_{\mathbb{R}^3}\int_{\mathbb{S}^2_-} \left(f'^{(1)} f_*'^{(1)} - f^{(1)} {f_*}^{(1)}\right) |\langle v_* - v,\omega\rangle| d\omega\, dv_*, \tag{1.7}$$

where we have used the abbreviations: $f'^{(1)} := f^{(1)}(t,x,v')$, $f_*'^{(1)} := f^{(1)}(t,x,v'_*)$, $f^{(1)} := f^{(1)}(t,x,v)$, and ${f_*}^{(1)} := f^{(1)}(t,x,v_*)$. Note that for the remainder of this thesis we will employ this notation more broadly so that for any function, say h, of velocity v we may write for example:

$$h'_* := h(v'_*).$$

We may also observe that in the Boltzmann-Grad limit, while the domain of integration changes from $\mathbb{S}^2_-$ to $\mathbb{S}^2_+$, the integrand in (1.7) is invariant under the previously performed change of variable $\omega \to -\omega$. Therefore, rather than integrating over the hemisphere $\mathbb{S}^2_-$, we may instead simply integrate over the whole sphere:

$$G(t,x,v) - L(t,x,v) = \frac{C}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left(f'^{(1)} f'^{(1)}_* - f^{(1)} f_*^{(1)} \right) |\langle v_* - v, \omega\rangle| d\omega \, dv_*, \tag{1.8}$$

the right hand side of which we call *Boltzmann's collision operator* (with respect to ω) for hard spheres and is viewed as an operator acting on $f^{(1)}$ at the point (t,x,v). Finally, by replacing the right hand side of (1.6) with Boltzmann's collision operator, we then arrive at the *Boltzmann equation* for a system of hard spheres

$$f_t^{(1)} + v\cdot\nabla_x f^{(1)} = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left(f'^{(1)} f'^{(1)}_* - f^{(1)} f_*^{(1)} \right) \left(\frac{C}{2} |\langle v_* - v, \omega\rangle| \right) d\omega \, dv_*. \tag{1.9}$$

Now that we have a kinetic equation which is closed in terms of $f^{(1)}$, for brevity we shall begin writing f to represent the one-particle marginal probability density function in place of $f^{(1)}$.

While the formal derivation leading to the Boltzmann equation for hard spheres (1.9) is a useful exercise which highlights how the structure of the collision operator directly connects to the postulates assumed about the system, it is not of the particular form that we shall consider moving forward. For one, we should like to consider the Boltzmann equation with respect to σ rather than ω and, in general, we will not restrict ourselves to the case of hard spherical particles and extend the range of possible interactions to inverse-power law potentials. Indeed, one typically expresses Boltzmann's collision operator acting on two functions g and h as follows:

$$Q(g,h)(t,x,v) \coloneqq \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left(h' g'_* - h g_* \right) B(v - v_*, \sigma) d\sigma \, dv_*, \tag{1.10}$$

and the Boltzmann equation is then given by

$$f_t + v\cdot\nabla_x f = Q(f,f). \tag{1.11}$$

Furthermore, if we consider densities, f, which are constant in the spatial variable x, the kinetic equation for the system is given by the *spatially homogeneous* (or just *homogeneous*) Boltzmann equation:

$$f_t = Q(f,f), \tag{1.12}$$

and it is this equation that is the central object of this thesis. We note also that in the homogeneous setting, we may consider the further reduced phase space for one particle where we neglect the spatial states, in which case the one-particle phase space is written

$\mathbb{R}^3_v$ (though we will generally tend not to include the subscript and simply write $\mathbb{R}^3$), and the density $f = f(t, v)$ is taken over $[0, T] \times \mathbb{R}^3_v$.

The function B is called *Boltzmann's collision kernel* and for inverse power law potentials of the form r^{1-k} for $k > 2$, it can be separated as the product of the *kinetic collision kernel*, $|v - v_*|^\gamma$ for $\gamma = \frac{k-5}{k-1} \in (-3, 1)$, and an *angular collision kernel* denoted by $b(\langle \kappa, \sigma \rangle)$, or equivalently $b(\cos\theta)$; that is,

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta). \tag{1.13}$$

Since B depends only on the magnitude of the relative velocity and the cosine of the deviation angle, we may occasionally abuse notation by writing $B(|v - v_*|, \cos\theta)$. The literature for the Boltzmann equation is often divided according to the sign of the parameter γ appearing in the kinetic collision kernel and we typically refer to the cases when: $\gamma > 0$ as *hard potentials*, $\gamma = 0$ as *Maxwellian potentials*, and $\gamma < 0$ as *soft potentials*. The precise form of the angular kinetic kernel is not of particular use to us and, in fact, an explicit definition of $b(\cos\theta)$ does not exist in general. For our purposes, this fact bears no significant relevance, though it is however important to note that we may treat this function as being symmetric with respect to σ and as a locally smoothing function that satisfies:

$$\sin\theta b(\cos\theta) \approx \frac{b_0}{\theta^{1+2s}}, \quad \theta \approx 0, \tag{1.14}$$

where, as noted for example in [18], $s = \frac{1}{k-1} \in (0, 1)$ and b_0 is a constant. In particular this shows that b has a nonintegrable singularity as $\theta \to 0$ and is consequent not integrable over $\mathbb{S}^2$ since

$$\begin{aligned}
\int_{\mathbb{S}^2} b(\cos\theta) d\sigma &= \left|\mathbb{S}^1\right| \int_0^\pi b(\cos\theta)\sin\theta d\theta \\
&= \left|\mathbb{S}^1\right| \left(\int_0^\epsilon \frac{b_0}{\theta^{1+2s}} d\theta + \int_\epsilon^\pi b(\cos\theta)\sin\theta d\theta \right) \\
&= \left|\mathbb{S}^1\right| \left(\frac{b_0}{2s} \cdot \lim_{\delta\to 0} \left(-\theta^{-2s} \Big|_\delta^\epsilon + \int_\epsilon^\pi b(\cos\theta)\sin\theta d\theta \right) \right).
\end{aligned}$$

Grad's angular cutoff is the additional postulate that

$$\int_{\mathbb{S}^2} b(\cos\theta) d\sigma < \infty.$$

It is not hard to imagine that Grad's angular cutoff considerably changes how one may mathematically approach the Boltzmann equation and for this reason it is important to draw the distinction between the Boltzmann equation with and without angular cutoff. As the title of this thesis suggests, it is indeed the latter that we shall be discussing and therefore the nonintegrability of the angular collision kernel poses an issue that we will have to address. As we will see, this is dealt with primarily through the so-called *Cancellation Lemma* which we will introduce in Section 1.5.

1.3 The weak formulation of Boltzmann's collision operator

We provide now the weak formulation of $Q(f, f)$, as it will play a vital role in our discussion moving forward. One of the key tools for this derivation is the so-called *pre-post-collisional change of variables*:

$$\mathcal{K}' : \mathbb{R}^{3\times 3} \times \mathbb{S}^2 \longrightarrow \mathbb{R}^{3\times 3} \times \mathbb{S}^2;\ (v, v_*, \sigma) \longmapsto (v', v'_*, \kappa) \tag{1.15}$$

and we claim that this transformation has unit Jacobian determinant. To see this, we first let F denote the entire integrand of the collision operator $Q(f, f)$; that is,

$$F \coloneqq (f' f'_* - f f_*)|v - v_*|^\gamma b(\cos\theta).$$

Then, similarly to observation made in the previous section regarding the invariance of the integrand of (1.7) under the change of variable $\omega \mapsto -\omega$, we note, due to the symmetry of the angular collision kernel with respect to σ, that the same is true for F under $\sigma \mapsto -\sigma$. Therefore, it remains valid to write the collision operator (with respect to σ) as

$$Q(f, f) = 2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2_+} F d\sigma\, dv_*,$$

where, similarly to before, $\mathbb{S}^2_+$ denotes the hemisphere where $\langle \kappa, \sigma\rangle \geq 0$. Integrating the above expression over $v \in \mathbb{R}^3$ then gives

$$\int_{\mathbb{R}^3} Q(f, f) dv = 2\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2_+} F d\sigma\, dv_*\, dv.$$

Our goal now is to decompose $\mathcal{K}'$ into a series of transformations each of which may be more easily applied to the integral above and, since we are currently interested in the claim that $|\det D\mathcal{K}'(v, v_*, \sigma)| = 1$, we perform these transformations while focusing purely on how they impact the differential elements and integration ranges. To this end, let us first consider the map $T_1 : (v, v_*, \sigma) \mapsto (v, v_*, \omega)$. To formally describe this map we notice, referring to Figure 1.1, that in spherical coordinates we have

$$\sigma = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \text{for } \theta \in [0, \pi/2], \tag{1.16}$$

and

$$\omega = \left(\sin\left(\frac{\theta+\pi}{2}\right)\cos\phi, \sin\left(\frac{\theta+\pi}{2}\right)\sin\phi, \cos\left(\frac{\theta+\pi}{2}\right)\right), \quad \text{for } \theta \in [0, \pi/2],$$

and therefore $T_1(v, v_*, \sigma)$ may be characterized by the map $\theta \mapsto \frac{\theta+\pi}{2}$. Moreover, we may similarly characterize the inverse transformation $T_1^{-1}(v, v_*, \omega)$ by the map $\xi \mapsto 2\xi - \pi$, where ξ is the angle between ω and κ. An important subtlety here is that, while v and v_* do not explicitly appear in the characterizations for $T_1(v, v_*, \sigma)$ or $T_1^{-1}(v, v_*, \omega)$, the parameterization of σ and ω implicitly depends on them since the azimuthal angles θ and ξ are taken with respect to $\kappa = \frac{v-v_*}{|v-v_*|}$. In particular this means that $T_1^{-1}(v', v'_*, \omega) = (v', v'_*, \kappa)$, since in the case the fixed direction is σ which implies $T_1^{-1}(v', v'_*, \omega)$ corresponds to the map

$$\xi = \frac{\pi - \theta}{2} \mapsto 2\left(\frac{\pi - \theta}{2}\right) - \pi = -\theta, \tag{1.17}$$

which is precisely the (signed) angle between σ and κ.

Now, if we denote by $u \otimes v$ the outer product of vectors u and v, by introducing the map $T_2 : (v, v_*, \omega) \mapsto (v', v'_*, \omega)$ defined by

$$T_2(v, v_*, \omega) = \begin{pmatrix} I - \omega \otimes \omega & \omega \otimes \omega & 0 \\ \omega \otimes \omega & I - \omega \otimes \omega & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} v \\ v_* \\ \omega \end{pmatrix},$$

we see that $\mathcal{K}' = T_1^{-1} \circ T_2 \circ T_1$. Furthermore, since T_2 is linear, its Jacobian determinant is simply the determinant of the matrix above. Now, since (1.5) and (1.4) imply that the matrix

$$\begin{pmatrix} I - \omega \otimes \omega & \omega \otimes \omega \\ \omega \otimes \omega & I - \omega \otimes \omega \end{pmatrix}$$

is an involution, it must have unit determinant and thus so must T_2.

Therefore, by first switching to spherical coordinates and applying the change of variable $T_1(v, v_*, \sigma)$ we get

$$\begin{aligned} 2\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2_+} F d\sigma\, dv_*\, dv &= 2\left|\mathbb{S}^1\right| \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^{\frac{\pi}{2}} F|\sin\theta| d\theta\, dv_*\, dv \\ &= 4\left|\mathbb{S}^1\right| \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} F|\sin\theta| d\xi\, dv_*\, dv \\ &= 4\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}_{1/4}} F\left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\right| d\omega\, dv_*\, dv, \end{aligned}$$

where $\mathbb{S}_{1/4}$ denotes the quarter of the unit sphere corresponding to $\xi \in [\frac{\pi}{2}, \frac{3\pi}{4}]$. Then, by applying $T_2(v, v_*, \omega)$ followed by $T_1^{-1}(v', v_*', \omega)$ we obtain

$$\begin{aligned} 2\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2_+} F d\sigma\, dv_*\, dv &= 4\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}_{1/4}} F\left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\right| d\omega\, dv_*'\, dv' \\ &= 4\left|\mathbb{S}^1\right|\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} F\left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\sin\xi\right| d\xi\, dv_*'\, dv' \\ &= 2\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2_-} F\left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\frac{\sin\xi}{\sin(2\xi-\pi)}\right| d\kappa\, dv_*'\, dv_*, \end{aligned}$$

where, as in (1.17), $\xi = \frac{\pi-\theta}{2}$. Hence, we may compute

$$\left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\frac{\sin\xi}{\sin(2\xi-\pi)}\right| = \left|\frac{\sin\theta}{\sin\left(\frac{\theta+\pi}{2}\right)}\frac{\sin\left(\frac{\pi-\theta}{2}\right)}{\sin(-\theta)}\right| = 1,$$

and therefore, again since F is invariant under the mapping $\kappa \mapsto -\kappa$, we see altogether that $|\det \mathcal{K}'(v, v_*, \sigma)| = 1$, so

$$\int_{\mathbb{R}^3} Q(f,f) dv = 2\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2_+} F d\kappa\, dv_*'\, dv' = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} F d\kappa\, dv_*'\, dv',$$

and hence we see the validity of the claim.

Now, consider an arbitrary continuous function, φ, of v. Then, by the pre-post-collisional change of variable (1.3), we have

$$\begin{aligned} \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \varphi f' f_*' |v-v_*|^\gamma b(\langle\kappa,\sigma\rangle) d\sigma\, dv_*\, dv = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \varphi\left(\frac{v'+v_*'}{2} + \frac{|v'-v_*'|}{2}\kappa\right) f' f_*' \\ \cdot |v'-v_*'|^\gamma b(\langle\kappa,\sigma\rangle) d\kappa\, dv_*'\, dv'. \end{aligned}$$

By relabelling (v', v_*', κ) as (v, v_*, σ), we may then rewrite the right hand side above by

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \varphi\left(\frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\right) f f_* |v-v_*|^\gamma b(\langle\sigma,\kappa\rangle) d\sigma\, dv_*\, dv,$$

which, by the σ-representation (1.3), is the same as

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \varphi' f f_* |v-v_*|^\gamma b(\langle\kappa,\sigma\rangle) d\sigma\, dv_*\, dv.$$

With this in mind, by multiplying $Q(f,f)$ by φ and integrating over all $v \in \mathbb{R}^3$, we formally obtain the weak formulation of Boltzmann's collision operator:

$$\int_{\mathbb{R}^3} Q(f,f)\varphi dv = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} (\varphi' - \varphi) f f_* |v - v_*|^\gamma b(\langle \kappa, \sigma\rangle) d\sigma\, dv_*\, dv. \tag{1.18}$$

An interesting observation to make here is that the transformation $\mathcal{K}^* : (v, v_*, \sigma) \mapsto (v_*, v, -\sigma)$ also has unit Jacobian determinant since it is also involutory, and making this change of variable on the weak formulation (1.18) yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q(f,f)\varphi dv \\
&\qquad = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \left(\varphi\left(\frac{v+v_*}{2} - \frac{|v-v_*|}{2}(-\sigma)\right) - \varphi\right) f f_* |v-v_*|^\gamma b(\langle\kappa, -\sigma\rangle) d(-\sigma)\, dv\, dv_* \\
&\qquad = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} \left(\varphi\left(\frac{v_*+v}{2} - \frac{|v_*-v|}{2}\sigma\right) - \varphi_*\right) f_* f |v_*-v|^\gamma b(\langle\kappa, \sigma\rangle) d\sigma\, dv_*\, dv \\
&\qquad = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} (\varphi'_* - \varphi_*) f f_* |v-v_*|^\gamma b(\langle\kappa,\sigma\rangle) d\sigma\, dv_*\, dv.
\end{aligned}$$

Combining this result with (1.18) then gives yet another weak formulation:

$$\int_{\mathbb{R}^3} Q(f,f)\varphi dv = \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} (\varphi' + \varphi'_* - \varphi - \varphi_*) f f_* |v-v_*|^\gamma b(\langle\kappa,\sigma\rangle) d\sigma\, dv_*\, dv. \tag{1.19}$$

In this form we may now more easily verify that the object:

$$\int_{\mathbb{R}^3} Q(f,f)\varphi dv$$

is well-defined when φ is sufficiently smooth. Indeed, following the discussion in [17], since $\cos\theta = \langle\kappa,\sigma\rangle$ we see that for fixed velocities v and v_*, $\sigma \to \kappa$ as $\theta \to 0$. Thus, recalling the σ-representation of the pre-collisional velocities (1.3), taking $\theta \to 0$ gives

$$v' \longrightarrow \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\kappa = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\frac{v-v_*}{|v-v_*|} = v$$

and hence, if φ is at least in C^2, the linear approximation of φ' about $\theta = 0$ can be expressed as

$$\varphi' \approx \varphi'\Big|_{\sigma=\kappa} + \left\langle \nabla\varphi'\Big|_{\sigma=\kappa}, \sigma - \kappa\right\rangle = \varphi + \frac{|v-v_*|}{2}\langle\nabla\varphi, \sigma-\kappa\rangle.$$

Moreover, with the same reasoning we may also write

$$\varphi'_* \approx \varphi_* - \frac{|v-v_*|}{2}\langle\nabla\varphi_*, \sigma-\kappa\rangle, \quad \theta \approx 0.$$

Furthermore, there exists some $\kappa^{\perp}$ such that $\langle \kappa^{\perp}, \kappa \rangle = 0$ and

$$\nabla\varphi - \nabla\varphi_* = \langle \nabla\varphi - \nabla\varphi_*, \kappa \rangle \kappa + c\kappa^{\perp}$$

for some constant c and hence

$$\begin{aligned}\langle \nabla\varphi - \nabla\varphi_*, \sigma - \kappa \rangle &= \langle \nabla\varphi - \nabla\varphi_*, \kappa \rangle \langle \kappa, \sigma \rangle + c\langle \kappa^{\perp}, \sigma \rangle - \langle \nabla\varphi - \nabla\varphi_*, \kappa \rangle \\ &= \langle \nabla\varphi - \nabla\varphi_*, \kappa \rangle (\langle \kappa, \sigma \rangle - 1) + c\langle \kappa^{\perp}, \sigma \rangle.\end{aligned}$$

Additionally, similarly to (1.16), since

$$\sigma = (\cos\phi \sin\theta) e_1 + (\sin\phi \sin\theta) e_2 + (\cos\theta)\kappa$$

with the spherical coordinates taken about the κ-axis, $\langle \kappa^{\perp}, \sigma \rangle$ must be a linear combination of $\cos\phi$ and $\sin\phi$ and therefore

$$\begin{aligned}\int_0^{2\pi} \varphi' + \varphi'_* - \varphi - \varphi_* d\phi &\approx \int_0^{2\pi} \frac{|v - v_*|}{2} \langle \nabla\varphi - \nabla\varphi_*, \sigma - \kappa \rangle d\phi \\ &= \int_0^{2\pi} \frac{|v - v_*|}{2} \langle \nabla\varphi - \nabla\varphi_*, \kappa \rangle (\langle \kappa, \sigma \rangle - 1) + c\langle \kappa^{\perp}, \sigma \rangle d\phi \\ &= \pi \langle \nabla\varphi - \nabla\varphi_*, v - v_* \rangle (\cos\theta - 1) \\ &= O(\theta^2 |v - v_*|^2)\end{aligned}$$

where the last equality follows from the Taylor expansion of $\cos\theta$ about $\theta = 0$ and the Mean Value Theorem. Thus, from the approximation above and (1.14), we find that

$$\int_0^{2\pi} (\varphi' - \varphi) \sin\theta b(\cos\theta) d\phi \approx \frac{\widetilde{b}_0 |v - v_*|^2}{\theta^{2s-1}}, \quad \theta \approx 0$$

and hence, since $(2s - 1) > -1$, we may conclude that the right hand side of (1.19) is finite when

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* |v - v_*|^{\gamma+2} dv_* \, dv < \infty.$$

While we will not make use explicitly of (1.19) in the main chapters of this thesis, there is another important and quite immediate consequence of this weak formulation that is worth mentioning. By not expanding the time derivative on the left hand side of the Boltzmann equation (1.11) (the expansion that originally led to the PDE at the end of section 2.1) and integrating over the whole one-particle phase space, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi dx \, dv = \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q(f, f) \varphi dx \, dv. \tag{1.20}$$

Thus, by (1.19), when φ satisfies

$$\varphi' + \varphi'_* = \varphi + \varphi_*, \quad \forall (v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \varphi dx\, dv = 0.$$

In particular, due to our elastic collision assumption, this demonstrates that we have the following conservation laws for the Boltzmann equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx\, dv = 0, \tag{1.21}$$

which correspond to conservation of total mass, momentum and kinetic energy respectively.

In a similar manner, applying the pre-post-collisional change of variable (1.3) to (1.19) and relabelling gives

$$\int_{\mathbb{R}^3} Q(f,f) \varphi dv = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{S}^2} (\varphi + \varphi_* - \varphi' - \varphi'_*) f' f'_* B(v - v_*, \sigma) d\sigma\, dv_*\, dv,$$

which we may add to (1.19) to obtain

$$\int_{\mathbb{R}^3} Q(f,f) \varphi dv = -\frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{S}^2} (\varphi' + \varphi'_* - \varphi - \varphi_*)(f' f'_* - f f_*) B(v - v_*, \sigma) d\sigma\, dv_*\, dv. \tag{1.22}$$

Now, since the map $(x, y) \mapsto (\log x - \log y)(x - y) \geq 0$, if we take $\varphi = \log f$, then from (1.22) we see that

$$\int_{\mathbb{R}^3} Q(f,f) \log f dv = -D(f) \leq 0,$$

where $D(f)$ is the *entropy dissipation functional*:

$$D(f) \coloneqq \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{S}^2} (\log(f' f'_*) - \log(f f_*))(f' f'_* - f f_*) B(v - v_*, \sigma) d\sigma\, dv_*\, dv.$$

Then from (1.20) we obtain the famous result originally stated in *Boltzmann's H Theorem*:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f) = -\int_{\mathbb{R}^3} D(f) dx \leq 0, \tag{1.23}$$

where H is known as Boltzmann's H functional (or *entropy*) and is defined as

$$H(f) \coloneqq \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \log f dx\, dv.$$

Hence, Boltzmann's H Theorem states, in particular, that the Boltzmann entropy is non-increasing in time.

1.4 Introduction of main results

We now discuss the central problem that is dealt with in this thesis; that is, the *generation* and *propagation* of L^p-norms of solutions to the homogeneous Boltzmann equation (1.12). By *generation* of L^p-norms we mean the following: if $f(t,v)$ solves (1.12) with suitable initial data f_0 such that $f_0 \notin L^p(\mathbb{R}^3_v)$, then $\|f(t)\|_{L^p(\mathbb{R}^3_v)} < \infty$ for any $t > 0$. Whereas *propagation* of L^p-norms refers to the case where $f_0 \in L^p(\mathbb{R}^3_v)$ implies $\|f(t)\|_{L^p(\mathbb{R}^3_v)} < \infty$ for all $t > 0$. This problem has been extensively studied under Grad's angular cutoff assumption for hard and Maxwellian potentials ($\gamma \in (0,1)$, and $\gamma = 0$ respectively). We first note that in this case, existence and uniqueness of solutions to the Boltzmann equation (1.11) was established in [19] and later, existence and uniqueness of solutions which conserve mass, momentum, and energy was proven in [13] in the spatially homogeneous setting. Then for hard potentials, it was shown that L^1 moments of these solutions are generated in time, a result which was then extended in [5] where it was shown that both generation and propagation of *exponential* L^1 moments hold for solutions to (1.12), where we generally refer to L^p-norms with a particular polynomial or exponential weight (see (1.27) below for the polynomial case) as L^p moments or exponential L^p moments respectively. Together these papers both imply the generation of L^1-norms due to the fact that they are bounded by L^1 moments (either polynomial or exponential) and that it is assumed that the initial data has finite second moment (which corresponds to the assumption that the system begins with finite mass and energy). A similar result for L^p moments was demonstrated in [10, 14] under the assumption that the initial data has sufficiently many L^p moments. Propagation of L^∞-norms was also studied and proved in [11, 6] for $\gamma > 0$. Moreover, it is known from [6, 15, 8, 20] that the results mentioned above for $p \in [1,\infty]$ remain true when $\gamma = 0$. In fact, with the added assumption that the angular collision kernel b is finite, [8] proves the propagation of a uniform Maxwellian upper bound. As a final remark regarding the cutoff theory, we note that various results related to the generation and propagation of higher L^p moments (both polynomial and exponential) have also been obtained in [7, 20, 10, 14, 5, 6] for $\gamma > 0$, and in [19, 20, 6, 15] for $\gamma = 0$.

In the non-cutoff setting, weak solutions were shown to exist for (1.12) for $\gamma > -1$ and suitable initial conditions. This result was extended in [17] to show that the same is true for all soft potentials $\gamma < 0$ and later it was proven in [12] that, with regular enough initial data, there exists a unique classical solution for the whole range $\gamma \in (-3,1)$. Now, while the problem of L^p generation and propagation has been studied far less in the non-cutoff case, due to the development of new tools, as in [2, 3] for example, the community has seen recent progress within this context. In particular, it was proved in [16] for $\gamma > 0$ and in [15] for $\gamma = 0$ that L^1 propagation holds for solutions to the homogeneous non-cutoff Boltzmann equation, and in [4] it was further established that L^p-norms of such solutions are both generated and propagated for $p \in (1,\infty]$ when $\gamma \geq 0$. In any case, however, the theory relating to soft potentials ($\gamma < 0$) remains underdeveloped and it is this gap in the literature that we, in some capacity, strive to fill here.

Now, it is the latter paper mentioned above, [4], by Alonso which this thesis is primarily based on and therefore we include here, together with the statements of our main results, the parallel results presented in [4] and provide a rough structure of their proofs so that we may draw comparison and highlight significant differences.

As is implied by the chapter titles, we treat $p < \infty$ and $p = \infty$ separately as the approach varies between these distinct cases. As such, let us consider first the case when $p < \infty$. The a priori estimates given by Alonso in [4, Theorem 1] may then be stated as follows: *for $p \in (1, \infty)$, $s \in (0, 1)$ as in (1.14), and $\gamma \in [0, 1]$, if $f(t, v)$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0, \infty) \times \mathbb{R}^3$ with initial condition $f_0 \in \mathcal{U}(D_0, E_0)$, then there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\|f(t)\|_{L^p} \leq C \left(t^{-\frac{3(p-1)}{2sp}} + 1 \right). \tag{1.24}$$

Moreover, if additionally $f_0 \in L^p$, then

$$\sup_{t \geq 0} \|f(t)\|_{L^p} \leq C \max \left\{ \|f_0\|_{L^p}, E_0 \right\}. \tag{1.25}$$

Here, for $D_0, E_0 > 0$:

$$\mathcal{U}(D_0, E_0) := \left\{ g \in L^1 : \|g\|_{L^1} \geq D_0, \|g\|_{L^1_2} + \|g\|_{L \log L} \leq E_0 \right\} \tag{1.26}$$

where

$$\|g\|_{L \log L} := \int_{\mathbb{R}^3} |g| \log(1 + |g|) dv,$$

and for $p \geq 1$, $r \in \mathbb{R}$, and $\langle v \rangle := \sqrt{1 + |v|^2}$,

$$\|g\|_{L^p_r} := \|\langle v \rangle^r g\|_{L^p}, \tag{1.27}$$

which we commonly refer to as the r^{th} L^p moment of g. We note that with this new notation, the condition $f_0 \in \mathcal{U}(D_0, E_0)$ corresponds to the minimum assumption one may take for the initial data for a system assumed to have zero total momentum and is justified by the conservation laws (1.21) and Boltzmann's H Theorem (1.23).

Alonso proves this result via the weak formulation of (1.12) with the particular choice of test function being pf^{p-1}. Specifically speaking, by multiplying (1.12) by pf^{p-1} and integrating over $v \in \mathbb{R}^3$ Alonso obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f(t)\|^p_{L^p} = p \int_{\mathbb{R}^3} Q(f, f) f^{p-1} dv.$$

Then, using the weak formulation of $Q(f,f)$ given by (1.18), the right hand side is bounded by a linear combination of $\left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}}$ and the Sobolev norm $\left\| f^{\frac{p}{2}} \right\|^2_{H^s_{\gamma/2}}$ given in general by

$$\|g\|_{H^s_r} \coloneqq \|\langle\cdot\rangle^r g\|_{H^s}$$

where, if we denote the Fourier transform of g by $\mathcal{F}[g]$,

$$\|g\|_{H^s} \coloneqq \left(\int_{\mathbb{R}^3} |\langle\xi\rangle^s \mathcal{F}[g](\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Notably, this bounding yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^p_{L^p} + C\left\| f^{\frac{p}{2}} \right\|^2_{H^s_{\gamma/2}} \leq c\left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}}, \quad \text{for } s, r \in \mathbb{R}. \tag{1.28}$$

Obtaining such an estimate, however, relies on the fact that $\gamma \geq 0$. Moreover, via Sobolev embedding and Lebesgue interpolation, Alonso relates this inequality to an ODE of which $\|f(t)\|^p_{L^p}$ is a sub-solution; a procedure which, through inequalities such as

$$\|f\|^p_{L^p} \leq \|f\|^p_{L^p_\gamma}$$

for example, relies further on $\gamma \geq 0$. The final step is then demonstrating that the right hand sides of (1.24) or (1.25), depending on whether or not $f_0 \in L^p$, are super-solutions to that same ODE.

As we will see, in order to obtain a similar estimate to (1.28) and furthermore, a similar type of ODE for $\gamma < 0$, we must restrict ourselves to a fixed finite time interval, impose stronger L^1 moment assumptions on the initial data f_0, and assume the L^2 moments (up to a specified degree) of the solutions $f(t,v)$ are bounded over our given interval of time. More specifically, we assume

$$f_0 \in \mathcal{V}_p(D_0, E_0) \coloneqq \left\{ g \in L^1 : \|g\|_{L^1} \geq D_0, \|g\|_{L^1_{\nu_p}} + \|g\|_{L\log L} \leq E_0 \right\} \tag{1.29}$$

for $\nu_p \coloneqq \max\left\{2, |\gamma|, \frac{3|\gamma|(p-1)}{2sp}\right\}$, and that the solutions $f(t,v)$ satisfy

$$f(t,v) \in \mathcal{T} \coloneqq \left\{ g \in L^2 : \|g(t)\|_{L^2_{|\gamma|}}, \leq C_t, \forall t \in [0,T] \right\} \tag{1.30}$$

where C_t is some monotonically increasing function of t such that

$$C_T \coloneqq \sup_{t\in[0,T]} C_t < \infty \tag{1.31}$$

and depends only on γ, s, D_0, and E_0. Consequently, it no longer becomes appropriate for us to make any claims regarding L^p generation or propagation for $p < 2$. We lastly note that some justification for the latter additional assumption is provided by [9, Theorem 1] and is included in chapter 2. With these added assumptions, by following the structure of Alonso's proof outlined above we are able to obtain a priori estimates for L^p-norms of solutions to the homogeneous Boltzmann equation (1.12) for soft potentials ($\gamma < 0$). These estimates are stated in the first main result of this thesis:

Theorem 1.1. *Let $p \in (2,\infty)$, $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}_p(D_0, E_0)$ be as in (1.29) with $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}_p(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\|f(t)\|_{L^p} \leq C \left(\max\left\{1+T, C_T\right\}\right)^{\frac{3(p-1)}{2sp}+1} \left(t^{-\frac{3(p-1)}{2sp}} + 1\right), \quad \text{for all } t \in (0,T]. \tag{1.32}$$

Moreover, if we additionally assume that $f_0 \in L^p$, then

$$\sup_{t\in[0,T]} \|f(t)\|_{L^p} \leq \|f_0\|_{L^p} e^{C\max\{1+T, C_T\}T}. \tag{1.33}$$

We include also our version of Alonso's regularization result [4, Corollary 1], although our proof has been substantially simplified due to our much stronger assumptions.

Theorem 1.2. *Let $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}_2(D_0, E_0)$ be as in (1.29) with $p = 2$ and $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}_2(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\int_t^T \|f(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq C \left(\max\{1+T, C_T\}\right)^4, \quad t \in [0,T].$$

When $p = \infty$, Alonso is able to obtain very similar a priori estimates for L^∞-norms as was previously achieved for finite p. Indeed, the statement provided in [4, Theorem 2] can be written: *for $s \in (0,1)$ and $\gamma \in [0,1]$, if $f(t,v)$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,\infty) \times \mathbb{R}^3$ with initial condition $f_0 \in \mathcal{U}(D_0, E_0)$, then there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\|f(t)\|_{L^\infty} \leq C \left(t^{-\frac{3}{2s}} + 1\right), \quad \text{for any } t > 0. \tag{1.34}$$

Moreover, if additionally $f_0 \in L^\infty$, then

$$\sup_{t\geq 0} \|f(t)\|_{L^\infty} \leq C \max\left\{2\|f_0\|_{L^\infty}, E_0\right\}, \tag{1.35}$$

where C depends also on $\|f_0\|_{L^2}$.

While the statement of the result is similar to the case when $p < \infty$, the idea of the proof is quite different. This is because when $p < \infty$ we could choose a convenient test function to produce an L^p-norm on the left hand side of the weak homogeneous Boltzmann equation that one could then work to bound, but when $p = \infty$ there is no such convenient test function. What Alonso does instead is introduce the positive level set function

$$f_k \coloneqq \left(f - K(1-2^{-k})\right)\chi_{\left\{f \geq K(1-2^{-k})\right\}}$$

for $k \in \mathbb{Z}_{\geq 1}$ and $K > 0$, so that the weak formulation of (1.12) with test function f_k gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_k(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} Q(f,f) f_k dv.$$

A similar bounding procedure as in the finite p case is employed again to bound the right hand side of the expression above. As before, this procedure relies on the fact that $\gamma \geq 0$ and therefore must be adapted for the $\gamma < 0$ setting. The estimate obtained, once integrated over a time interval $[\xi, t]$, can be written

$$\frac{1}{2}\|f_k(t)\|_{L^2}^2 + C\int_\xi^t \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau \leq \frac{1}{2}\|f_k(\xi)\|_{L^2}^2 + c\int_\xi^t \|f_k(\tau)\|_{L^2_{\gamma/2}}^2 d\tau + cK\int_\xi^t \|f_k(\tau)\|_{L^1_\gamma} d\tau. \tag{1.36}$$

In view of the left hand side above, Alonso defines an energy functional

$$W_k \coloneqq \frac{1}{2}\sup_{t\in[t_k,T]} \|f_k(t)\|_{L^2}^2 + C\int_{t_k}^T \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau$$

where $t_k \coloneqq t_*(1-2^{-k+1})$ for some fixed $t_* > 0$ (or in the propagation case we may simply take instead $t_k \equiv 0$ in the definition above). The strategy is then to show that for a particular choice of K, $W_k \to 0$ as $k \to \infty$ since if that were the case and due to the fact that taking $k \to \infty$ implies $f_k \to (f-K)\chi_{\{f\geq K\}}$, this would then show that

$$\left\|(f-K)\chi_{\{f\geq K\}}\right\|_{L^2} = 0$$

and hence

$$\|f\|_{L^\infty} \leq K.$$

Alonso achieves this by using (1.36) to relate W_k to a recurrence relation of which, similarly to the ODE in the finite p setting, W_k is a sub-solution. Again, the assumption that $\gamma \geq 0$

is used in this process. For a suitable choice of K, Alonso is able to obtain a super-solution to that recurrence relation that can clearly be seen to tend to zero as $k \to \infty$. This choice of K is then precisely what appears on the right hand sides of (1.34) and (1.35).

We follow the same framework to obtain a similar recurrence relation when $\gamma < 0$, however there are multiple arguments that must be modified in the soft potential setting. These new arguments are what make up the list of lemmas appearing in Preliminary Results section of chapter 3. Too, in this case we must still consider only those solutions $f(t,v) \in \mathcal{T}$ over a finite time interval and with initial data satisfying additional L^1 moment requirements. In fact, the assumptions placed on f_0 end up being stronger than before. Specifically, we enforce that

$$f_0 \in \mathcal{V}^*_\zeta(D_0, E_0) \coloneqq \left\{ g \in L^1 : \|g\|_{L^1} \geq D_0, \|g\|_{L^1_{\nu^*_\zeta}} + \|g\|_{L \log L} \leq E_0 \right\}, \tag{1.37}$$

where $\nu^*_\zeta = \max\left\{2, \frac{3|\gamma|}{2s(2-\zeta)}\right\}$ for $\zeta \in (1,2)$. With that in mind, our final main result can be stated as follows:

Theorem 1.3. *Let $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}^*_\zeta(D_0, E_0)$ be as in (1.4) for any $\zeta \in (1,2)$ and with $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}^*_\zeta(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that for any $0 < t_* < T$ sufficiently small we have*

$$\sup_{t \in [t_*, T]} \|f(t)\|_{L^\infty} \leq C \left(\max\left\{1+T, C_T\right\}\right)^{\frac{6+8s(\zeta-1)}{s\zeta}} \left(\frac{1}{t_*}\right)^{\frac{3}{s\zeta}\left(\frac{3}{2s}+1\right)}. \tag{1.38}$$

If we additionally assume that $f_0 \in L^\infty$, then

$$\sup_{t \in [0,T]} \|f(t)\|_{L^\infty} \leq \max\left\{ 2\|f_0\|_{L^\infty}, C\left(\max\{1+T, C_T\}\right)^{\frac{6+8s(\zeta-1)}{s\zeta}} e^{\frac{3}{s\zeta} C \max\{1+T, C_T\} T} \right\}. \tag{1.39}$$

1.5 Essential theorems and inequalities

For the sake of completeness, we include here (without proof) the various theorems, some of which are standard to the field of kinetic theory and others being classical inequalities, which we will make use of in the proceeding chapters.

The first, and perhaps most important, is the *cancellation lemma* due to Alexandre, Desvillettes, Villani, and Wennberg [1]:

Theorem 1.4 (Cancellation Lemma [1])**.** *Let B be as in (1.13) and (1.14). Then, for a.e. $v_* \in \mathbb{R}^3$,*

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} (f' - f)B(v - v_*, \sigma)d\sigma\, dv = (f * S)(v_*),$$

where

$$S(z) := \left|\mathbb{S}^1\right| \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{1}{\cos^3(\theta/2)} B\left(\frac{|z|}{\cos(\theta/2)}, \cos\theta\right) - B(|z|, \cos\theta)\right] d\theta.$$

Another standard result in kinetic theory we will make use of is the following theorem due to Alexandre, Morimoto, Ukai, Xu, and Yang [3].

Theorem 1.5. [3] *Let $s \in (0, 1)$ as in (1.14), $\gamma \in (-3, 1)$, and $g \in \mathcal{U}(D_0, E_0)$ for $\mathcal{U}$ as in (1.26) with $D_0, E_0 > 0$. Then for f sufficiently smooth, there are constants c and C depending only on D_0, E_0 such that*

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \int_{\mathbb{S}^2} g_*(f' - f)^2 |v - v_*|^\gamma b(\cos\theta) d\sigma\, dv_*\, dv \geq c\|f\|^2_{H^s_{\gamma/2}} - C\|f\|^2_{L^2_{\gamma/2}}.$$

We end this section now by listing three classical inequalities so that we may more easily refer to their parameters when used in the following two chapters.

Theorem 1.6 (Sobolev inequality)**.** *Let $f \in H^k(\mathbb{R}^n)$ and ℓ be such that*

$$\frac{1}{\ell} = \frac{1}{2} - \frac{k}{n}.$$

Then, if

$$k < \frac{n}{2},$$

there exists a constant C depending only on k and n such that

$$\|f\|_{L^\ell} \leq C\|f\|_{H^k}.$$

Theorem 1.7 (Hardy-Littlewood-Sobolev inequality)**.** *Let $f \in L^k(\mathbb{R}^n)$ for $k > 1$. Then, there is a constant $C > 0$ such that*

$$\left\| f * |\cdot|^{-\alpha} \right\|_{L^\ell} \leq C\|f\|_{L^k},$$

where $\alpha := n\left(1 + \frac{1}{\ell} - \frac{1}{k}\right)$.

Theorem 1.8 (Young's convolution inequality)**.** *Let $p, q, r \in [1, \infty]$ be such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

and let $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$*. Then,*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Chapter 2

L^p Theory

2.1 Preliminary results

Let us suppose for now that $p \in [1,\infty)$, $\gamma \in (-3,0)$, $s \in (0,1)$ as in (1.14), and $B(v-v_*,\sigma)$ is as given in (1.13), but note that the ranges for p and γ will become more restricted in Theorem 2.1 (the main result of this chapter) below. As it will quickly become relevant, let us also define the class of functions as seen in section 1.4:

$$\mathcal{V}_p(D_0,E_0) \coloneqq \left\{ g \in L^1 : \|g\|_{L^1} \geq D_0, \|g\|_{L^1_{\nu_p}} + \|g\|_{L\log L} \leq E_0 \right\},$$

where $\nu_p \coloneqq \max\left\{2, |\gamma|, \frac{3|\gamma|(p-1)}{2sp}\right\}$.

One of the pivotal ideas we make use of in this thesis is that for a particular choice of test function, we may bound the weak collision operator by a linear combination of the following integrals:

$$I_p(g,f) \coloneqq \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} g_* \left[(f')^p - f^p\right] B(v-v_*,\sigma) d\sigma\, dv_*\, dv, \tag{2.1}$$

$$J_p(g,f) \coloneqq \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} g_* \left[(f')^{\frac{p}{2}} - f^{\frac{p}{2}}\right]^2 B(v-v_*,\sigma) d\sigma\, dv_*\, dv, \tag{2.2}$$

where B satisfies (1.13) and (1.14). This idea is made precise in the following lemma.

Lemma 2.1. *Let Q be as in (1.10) with B satisfying (1.13) and (1.14). Then, for f sufficiently smooth,*

$$\int_{\mathbb{R}^3} Q(f,f) f^{p-1} dv \leq \frac{1}{p'} I_p(f,f) - \frac{1}{\max\{p,p'\}} J_p(f,f),$$

where p and p' are Hölder conjugates.

This lemma is a consequence of [4, Lemma 1] which states that for any $h \geq 0$ and $p \in [1,\infty]$, one has

$$h^{\frac{2}{p'}} - 1 \leq \frac{1}{p'}(h^2-1) - \frac{1}{\max\{p,p'\}}(h-1)^2, \tag{2.3}$$

where equality is obtained when $p = 2$. With this in mind we present the proof of Lemma 2.1 as given in [4].

Proof. By (2.3) with

$$h = \left(\frac{f'}{f}\right)^{\frac{p}{2}},$$

we see that

$$\left(\frac{f'}{f}\right)^{\frac{p}{p'}} - 1 \leq \frac{1}{p'}\left[\left(\frac{f'}{f}\right)^{p} - 1\right] - \frac{1}{\max\{p,p'\}}\left[\left(\frac{f'}{f}\right)^{\frac{p}{2}} - 1\right]^2$$
$$= \frac{1}{f^p}\left(\frac{1}{p'}\left[(f')^p - f^p\right] - \frac{1}{\max\{p,p'\}}\left[(f')^{\frac{p}{2}} - f^{\frac{p}{2}}\right]^2\right).$$

Then since

$$f\left[(f')^{p-1} - f^{p-1}\right] = f^p\left(\frac{f'}{f}\right)^{p-1} - f^p = f^p\left[\left(\frac{f'}{f}\right)^{\frac{p}{p'}} - 1\right],$$

by the weak formulation (1.18) with test function $\varphi = f^{p-1}$, we find that

$$\int_{\mathbb{R}^3} Q(f,f)f^{p-1}dv = \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{\mathbb{S}^2} f_* f\left[(f')^{p-1} - f^{p-1}\right]B(v-v_*,\sigma)d\sigma\, dv_*\, dv$$
$$\leq \frac{1}{p'}I_p(f,f) - \frac{1}{\max\{p,p'\}}J_p(f,f).$$

□

Now, if f solves the homogeneous Boltzmann equation (1.12) with $f_0 \in \mathcal{V}_p(D_0,E_0)$, then for $\mathcal{U}$ as in (1.26), since $\mathcal{V}_p(D_0,E_0) \subseteq \mathcal{U}(D_0,E_0)$ the conservation laws (1.21) imply $f \in \mathcal{U}(D_0,E_0)$ and hence in this case Theorem 1.5 gives

$$J_p(f,f) \geq C\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}} - c\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}}, \tag{2.4}$$

for constants c and C depending only on D_0 and E_0. As for I_p, we may apply the Cancellation Lemma 1.4 with f^p in the place of f to obtain

$$\begin{aligned} I_p(f,f) &= \int_{\mathbb{R}^3} f_*(f^p * S)(v_*)dv_* \\ &= \int_{\mathbb{R}^3\times\mathbb{R}^3} f_*\left(f^p S(v_*-v)\right)dv\,dv_* \\ &= \int_{\mathbb{R}^6} f_* f^p\left|\mathbb{S}^1\right|\int_0^{\frac{\pi}{2}}\sin\theta\left[\frac{1}{\cos^3\frac{\theta}{2}}B\left(\frac{|v-v_*|}{\cos\frac{\theta}{2}},\cos\theta\right) - B(|v-v_*|,\cos\theta)\right]d\theta\,dv\,dv_* \\ &= \left|\mathbb{S}^1\right|\int_{\mathbb{R}^6} f_* f^p|v-v_*|^\gamma\int_0^{\frac{\pi}{2}}\sin\theta\left(\frac{1}{\cos^{3+\gamma}\frac{\theta}{2}} - 1\right)b(\cos\theta)d\theta\,dv\,dv_*. \end{aligned}$$

Then, by writing the MacLaurin series for $\left(\cos^{-(3+\gamma)}\frac{\theta}{2}-1\right)$:

$$\left(\cos^{-(3+\gamma)}\frac{\theta}{2}-1\right)=\left(\frac{1}{\cos^{3+\gamma}(0)}-1\right)+\left(\frac{3+\gamma}{2}\frac{\sin(0)}{\cos^{4+\gamma}(0)}\right)\theta+ \\ +\left[\frac{3+\gamma}{4\cos^{5+\gamma}(0)}\left((4+\gamma)\sin^2(0)+\cos^2(0)\right)\right]\theta^2+O(\theta^3)\quad(\text{as }\theta\to 0),$$

we see that there is a constant, c, depending only on γ such that $\left(\cos^{-(3+\gamma)}\frac{\theta}{2}-1\right)\approx c\theta^2$ for $\theta\approx 0$. Therefore, if we let $\tilde{b}(\cos\theta):=\left(\cos^{-(3+\gamma)}\frac{\theta}{2}-1\right)b(\cos\theta)$, we find that for $\epsilon\ll 1$

$$\begin{aligned}\left\|\tilde{b}\right\|_{L^1(\mathbb{S}^2)}&=2\left|\mathbb{S}^1\right|\int_0^{\frac{\pi}{2}}\sin\theta\left|\tilde{b}(\cos\theta)\right|d\theta\\&=2\left|\mathbb{S}^1\right|\left(c\int_0^{\epsilon}\theta^{1-2s}d\theta+\int_{\epsilon}^{\frac{\pi}{2}}\sin\theta\left|\tilde{b}(\cos\theta)\right|d\theta\right)\\&=2\left|\mathbb{S}^1\right|\left[\frac{c}{2-2s}\left(\theta^{2-2s}\right|_0^{\epsilon}+\int_{\epsilon}^{\frac{\pi}{2}}\sin\theta\left|\tilde{b}(\cos\theta)\right|d\theta\right]<\infty\end{aligned}$$

since $2-2s>0$. Thus, there exists some constant C depending only on γ and s such that

$$\begin{aligned}I_p(f,f)&=\frac{1}{2}\int_{\mathbb{R}^6}f_*f^p|v-v_*|^{\gamma}\int_{\mathbb{S}^2}\tilde{b}(\cos\theta)d\sigma\,dv\,dv_*\\&\leq C\int_{\mathbb{R}^3\times\mathbb{R}^3}f_*f^p|v-v_*|^{\gamma}dv_*\,dv.\end{aligned}\tag{2.5}$$

Lemma 2.2. *Let $p\in[1,\infty)$, $s\in(0,1)$, $\gamma\in(\max\{-3,-2s-\frac{3}{2}\},0)$, f be sufficiently smooth, and I_p be as in (2.1). Then, there exist constants c,C depending only on γ and s such that*

$$I_p(f,f)\leq\max\left\{\|f\|_{L^1_{|\gamma|}},\|f\|_{L^2_{|\gamma|}}\right\}\cdot\begin{cases}c\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}}, & -\frac{3}{2}<\gamma \quad (2.6a)\\ \frac{c}{\epsilon}\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}}+C\epsilon\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}, & \gamma\leq-\frac{3}{2} \quad (2.6b)\end{cases}$$

for any $\epsilon>0$.

Proof. Due to the Cancellation Lemma, it suffices to show that the right hand sides of (2.6a) and (2.6b) provides an upper bound for the right hand side of (2.5). To this end, we first make the observation that if $|v-v_*|\leq 1$ we can write $|v|\leq 1+|v_*|$, so

$$\begin{aligned}1&\geq|v-v_*|^2\\&\geq|v|^2+|v_*|^2-2|v||v_*|\\&\geq|v|^2+|v_*|^2-2|v_*|\left(1+|v_*|\right)\\&=|v|^2-|v_*|^2-2|v_*|,\end{aligned}$$

and hence

$$\begin{aligned}|v|^2+1 &\le |v_*|^2+2|v_*|+2\\ &\le 2\left(|v_*|+1\right)^2\\ &= 8\left(\frac{1}{2}|v_*|+\frac{1}{2}\right)^2\\ &\le 4\left(|v_*|^2+1\right).\end{aligned}$$

In particular, when $|v-v_*|\le 1$ we have $\langle v\rangle \le 2\langle v_*\rangle$.

Let us now consider the case when $\gamma > -\frac{3}{2}$. In this case, with the observation above we see that

$$\begin{aligned}\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^\gamma dv_*\, dv &= \int_{|v-v_*|\le 1} f_* \langle v\rangle^{|\gamma|}\langle v\rangle^{\gamma} f^p |v-v_*|^\gamma dv_*\, dv\\ &\le 2^{|\gamma|}\int_{|v-v_*|\le 1} f_* \langle v_*\rangle^{|\gamma|}\langle v\rangle^{\gamma} f^p |v-v_*|^\gamma dv_*\, dv.\end{aligned}$$

Furthermore, applying Cauchy-Schwarz and using the fact that, for any fixed v, $\{v_*\in\mathbb{R}^3 : |v-v_*|\le 1\}\subseteq \mathbb{R}^3$ we obtain

$$\begin{aligned}\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^\gamma dv_*\, dv \le 2^{|\gamma|}\int_{\mathbb{R}^3}\left(\langle v\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right)^2 \left(\int_{|v-v_*|\le 1} |v-v_*|^{2\gamma} dv_*\right)^{\frac{1}{2}}\cdot\\ \left(\int_{\mathbb{R}^3}\left(\langle v_*\rangle^{|\gamma|} f_*\right)^2 dv_*\right)^{\frac{1}{2}} dv\end{aligned}$$

and since $|\cdot|^\gamma \chi_{\{|\cdot|\le 1\}}\in L^2$ for $-\frac{3}{2}<\gamma$, this shows that

$$\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^\gamma dv_*\, dv \le c\|f\|_{L^2_{|\gamma|}} \left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}}. \tag{2.7}$$

On the other hand, if we let $A_v := \{v_*\in\mathbb{R}^3 : |v_*|\le \frac{1}{2}|v|\}$, then if $|v-v_*|>1$ for $v_*\in A_v$, we must have $|v|>1$ and hence

$$|v-v_*|\ge |v|-|v_*|\ge \frac{1}{2}|v| = \frac{1}{2\sqrt{2}}\sqrt{|v|^2+|v|^2} > \frac{1}{2\sqrt{2}}\langle v\rangle.$$

In this case we have

$$\begin{aligned}\int_{|v-v_*|>1} f_* f^p |v-v_*|^\gamma \chi_{A_v} dv_*\, dv &\le (2\sqrt{2})^{|\gamma|} \int_{|v-v_*|>1} f_* f^p \langle v\rangle^\gamma \chi_{A_v} dv_*\, dv \\ &\le (2\sqrt{2})^{|\gamma|} \int_{|v-v_*|>1} f_* f^p \langle v_*\rangle^{|\gamma|} \langle v\rangle^\gamma \chi_{A_v} dv_*\, dv \\ &\le (2\sqrt{2})^{|\gamma|} \int_{\mathbb{R}^3} (f^{\frac{p}{2}} \langle v\rangle^{\frac{\gamma}{2}})^2 \int_{\mathbb{R}^3} f_* \langle v_*\rangle^{|\gamma|} dv_*\, dv \\ &= (2\sqrt{2})^{|\gamma|} \|f\|_{L^1_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}}.\end{aligned}$$

If instead $v_* \in A_v^c$, then $\langle v\rangle \le 2\langle v_*\rangle$ and in which case

$$\begin{aligned}\int_{|v-v_*|>1} f_* f^p |v-v_*|^\gamma \chi_{A_v^c} dv_*\, dv &= \int_{|v-v_*|>1} f_* f^p \frac{\langle v\rangle^{|\gamma|} \langle v\rangle^\gamma}{|v-v_*|^{|\gamma|}} \chi_{A_v^c} dv_*\, dv \\ &\le 2^{|\gamma|} \int_{|v-v_*|>1} f_* f^p \langle v_*\rangle^{|\gamma|} \langle v\rangle^\gamma \chi_{A_v^c} dv_*\, dv \\ &\le (2\sqrt{2})^{|\gamma|} \|f\|_{L^1_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}}.\end{aligned}$$

Therefore, we see that

$$\int_{|v-v_*|>1} f_* f^p |v-v_*|^\gamma dv_*\, dv \le c \|f\|_{L^1_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}} \tag{2.8}$$

for a constant c depending only on γ. Then, combining inequalities (2.7) and (2.8), we arrive at

$$\int_{\mathbb{R}^3} f_* f^p |v-v_*|^\gamma dv_*\, dv \le c \max\left\{ \|f\|_{L^1_{|\gamma|}}, \|f\|_{L^2_{|\gamma|}} \right\} \left\| f^{\frac{p}{2}} \right\|^2_{L^2_{\gamma/2}}, \quad \text{for } -\frac{3}{2} < \gamma.$$

Hence, we can see that there is a constant c depending only on γ and s such that (2.6a) holds.

Consider now the case when $\gamma \le -\frac{3}{2}$. Then, similarly to above

$$\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^\gamma dv_*\, dv \le 2^{|\gamma|} \int_{|v-v_*|\le 1} f_* \langle v_*\rangle^{|\gamma|} \langle v\rangle^\gamma f^p |v-v_*|^\gamma dv_*\, dv.$$

Now, by Cauchy-Schwarz

$$\begin{aligned}\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^\gamma dv_*\, dv \le 2^{|\gamma|} &\left(\int_{\mathbb{R}^3} \left| f_* \langle v\rangle^{|\gamma|} \right|^2 dv_* \right)^{\frac{1}{2}} \cdot \\ &\left(\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} f^p \langle v\rangle^\gamma |v-v_*|^\gamma dv \right|^2 dv_* \right)^{\frac{1}{2}},\end{aligned}$$

which is the same as

$$\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^{\gamma} dv_*\, dv \le 2^{|\gamma|} \|f\|_{L^2_{|\gamma|}} \left\| (f^p \langle \cdot \rangle^{\gamma}) * |\cdot|^{-|\gamma|} \right\|_{L^2}. \tag{2.9}$$

Then, we may apply Theorem 1.7 (Hardy-Little-Sobolev inequality) with $\alpha \equiv |\gamma|$, $\ell \equiv 2$, and $n \equiv 3$ to obtain

$$\int_{|v-v_*|\le 1} f_* f^p |v-v_*|^{\gamma} dv_*\, dv \le c \|f\|_{L^2_{|\gamma|}} \|f^p\|_{L^{6/(9-2|\gamma|)}_{\gamma}},$$

for a constant c depending only on γ. Furthermore, by rewriting the right hand side of the above inequality and applying Theorem 1.6 (Sobolev inequality) with $n \equiv 3$, $\ell \equiv \frac{12}{9-2|\gamma|}$, and $k \equiv s' := \frac{6|\gamma|-9}{12}$ we get

$$\begin{aligned} \int_{|v-v_*|\le 1} f_* f^p |v-v_*|^{\gamma} dv_*\, dv &\le c \|f\|_{L^2_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{L^{12/(9-2|\gamma|)}_{\gamma/2}} \\ &\le C \|f\|_{L^2_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{H^{s'}_{\gamma/2}}, \end{aligned} \tag{2.10}$$

for a constant C depending only on γ.

Now, since (2.8) remains true for $\gamma \le -\frac{3}{2}$, we also have

$$\int_{|v-v_*|> 1} f_* f^p |v-v_*|^{\gamma} dv_*\, dv \le c \|f\|_{L^1_{|\gamma|}} \left\| f^{\frac{p}{2}} \right\|^2_{H^{s'}_{\gamma/2}},$$

which together iwht (2.10) implies that there is a constant C depending only on γ and s such that

$$I_p(f,f) \le C \max\left\{ \|f\|_{L^1_{|\gamma|}}, \|f\|_{L^2_{|\gamma|}} \right\} \left\| f^{\frac{p}{2}} \right\|^2_{H^{s'}_{\gamma/2}}, \quad \text{for } \gamma \le -\frac{3}{2}. \tag{2.11}$$

Now, because $0 < s' < s$ for $\gamma \in (-2s - \frac{3}{2}, -\frac{3}{2})$, there exists some $\theta \in (0,1)$ which depends only on γ and s such that $s' = \theta(0) + (1-\theta)s$. Therefore, an application of Young's inequality shows that for any $\epsilon > 0$ and $z \in \mathbb{R}^3$,

$$\begin{aligned} \langle z \rangle^{2s'} &= \left(\frac{\epsilon}{\epsilon} \langle z \rangle^2 \right)^{(1-\theta)s} \\ &= \frac{1}{\epsilon^{(1-\theta)s}} \left[\left(\epsilon \langle z \rangle^2 \right)^{\theta(0)} \left(\epsilon \langle z \rangle^2 \right)^{(1-\theta)s} \right] \\ &\le \frac{1}{\epsilon^{(1-\theta)s}} \left[\theta \left(\epsilon \langle z \rangle^2 \right)^{\frac{\theta(0)}{\theta}} + (1-\theta) \left(\epsilon \langle z \rangle^2 \right)^{\frac{(1-\theta)s}{(1-\theta)}} \right] \\ &= \frac{\theta}{\epsilon^{(1-\theta)s}} + (1-\theta)\epsilon^{\theta s} \langle z \rangle^{2s}. \end{aligned}$$

With this in mind, by Plancherel's Theorem we have that for any $\epsilon > 0$

$$\begin{aligned}\left\|f^{\frac{p}{2}}\right\|^2_{H^{s'}_{\gamma/2}} &= \int_{\mathbb{R}^3} \langle z\rangle^{2s'} \left|\left(\langle\cdot\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right)^{\widehat{}}(z)\right|^2 dz \\ &\le \int_{\mathbb{R}^3} \left(\frac{\theta}{\epsilon^{(1-\theta)s}} + (1-\theta)\epsilon^{\theta s}\langle z\rangle^{2s}\right) \left|\left(\langle\cdot\rangle^{\frac{\gamma}{2}} f^{\frac{p}{2}}\right)^{\widehat{}}(z)\right|^2 dz \\ &= \frac{\theta}{\epsilon^{(1-\theta)s}} \left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}} + (1-\theta)\epsilon^{\theta s}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}.\end{aligned}$$

Thus, we may finally conclude from (2.11), that (2.6b) also holds. □

Now, ideally we would like to treat the coefficient $\left(\max\left\{\|f\|_{L^1_{|\gamma|}}, \|f\|_{L^2_{|\gamma|}}\right\}\right)$ as constant in time, however the problem of L^2 moment estimation for solutions to (1.12) with soft potentials is currently open. The way in which we deal with this is by further restricting the a priori estimates provided in this thesis to densities, f, which solve (1.12) with $f_0 \in \mathcal{V}_p(D_0, E_0)$ and additionally satisfy

$$\|f\|_{L^2_{|\gamma|}} \le C_t,$$

where C_t depends on γ, s, D_0, and E_0, and is a monotonically increasing function of t. Moreover, as in Section 1.4, we denote for any given $T \ge t \ge 0$

$$\mathcal{T} := \{g \in L^2_{|\gamma|} : \|g(t)\|_{L^2_{|\gamma|}} \le C_t, \forall t \in [0,T]\},$$

and

$$C_T := \sup_{t\in[0,T]} C_t < \infty.$$

Our assumptions regarding C_t are motivated by the result due to Carlen, Carvalho, and Lu in [9, Theorem 1], the statement of which we present as the following lemma.

Lemma 2.3. [9, Theorem 1] *Let $f(t,v)$ solve the homogeneous Boltzmann equation (1.12) on $(0,\infty)\times\mathbb{R}^3$ with $f_0 \in \mathcal{U}(D_0,E_0)\cap L^1_r$ and Q as in (1.10) with B satisfying (1.13) and (1.14). Then, there exists a constant C depending only on γ, s, D_0, E_0 and $\|f_0\|_{L^1_r}$ such that*

$$\|f(t)\|_{L^1_r} \le C(1+t).$$

Before we may move on to the main result of this chapter, we include one final lemma which serves as the culmination of Lemmas 2.1-2.3.

Lemma 2.4. *Let $p \in (1,\infty)$, $s \in (0,1)$, $\gamma \in (\max\{-3, -2s-\frac{3}{2}\}, 0)$, $\mathcal{V}_p(D_0,E_0)$ be as in (1.29) with $D_0, E_0 > 0$, and $\mathcal{T}$ and C_t be as in (1.30) for a fixed $T > 0$. Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T]\times\mathbb{R}^3$ with $f_0 \in \mathcal{V}_p(D_0,E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14),* *there are constants c and C depending only on γ, s, D_0, and E_0 such that*

$$\int_{\mathbb{R}^3} Q(f,f)f^{p-1}dv \leq \frac{c}{p'}\max\{1+t, C_t\}\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}} - \frac{C}{\max\{p,p'\}}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}.$$

Proof. We first note that by Lemma 2.3, since $f_0 \in \mathcal{V}_p(D_0, E_0)$ and $f \in \mathcal{T}$ we have

$$\max\left\{\|f\|_{L^1_{|\gamma|}}, \|f\|_{L^2_{|\gamma|}}\right\} \leq c\overline{C}_t$$

where $\overline{C}_t \coloneqq \max\{1+t, C_t\}$ and c is a constant depending only on γ, s, D_0, and E_0. Then, by Lemmas 2.1 and 2.2 as well as (2.4), we have for $\gamma \leq -\frac{3}{2}$

$$\begin{aligned}\int_{\mathbb{R}^3} & Q(f,f)f^{p-1}dv \\ &\leq \frac{1}{p'}I_p(f,f) - \frac{1}{\max\{p,p'\}}J_p(f,f) \\ &\leq \frac{\overline{C}_t}{p'}\left(\frac{c}{\epsilon}\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}} + C\epsilon\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}\right) + \frac{1}{\max\{p,p'\}}\left(\tilde{c}\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}} - \tilde{C}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}\right)\end{aligned}$$

for any $\epsilon > 0$. Hence, for ϵ sufficiently small and using the fact that $\frac{1}{\max\{p,p'\}} \leq \frac{1}{p'}$, we get

$$\int_{\mathbb{R}^3} Q(f,f)f^{p-1}dv \leq \frac{c\overline{C}_t}{p'}\left\|f^{\frac{p}{2}}\right\|^2_{L^2_{\gamma/2}} - \frac{C}{\max\{p,p'\}}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}},$$

where c and C depend only on γ, s, D_0, and E_0.

The result follows similarly for the case when $-\frac{3}{2} < \gamma$. □

2.2 Generation and propagation of L^p-norms

With the collection of lemmas in the previous section, we are now in a position to state and prove the a priori estimates comprising our first main result.

Theorem 2.1. *Let $p \in (2,\infty)$, $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}_p(D_0, E_0)$ be as in (1.29) with $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}_p(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\|f(t)\|_{L^p} \leq C\left(\max\{1+T, C_T\}\right)^{\frac{3(p-1)}{2sp}+1}\left(t^{-\frac{3(p-1)}{2sp}} + 1\right), \quad \textit{for all } t \in (0,T]. \tag{2.12}$$

Moreover, if we additionally assume that $f_0 \in L^p$, then

$$\sup_{t\in[0,T]}\|f(t)\|_{L^p} \leq \|f_0\|_{L^p}e^{C\max\{1+T,C_T\}T}. \tag{2.13}$$

Proof. By multiplying (1.12) by pf^{p-1}, integrating over $v \in \mathbb{R}^3$, and applying lemma 2.4, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p + \frac{p\widetilde{C}}{\max\{p,p'\}}\left\|f^{\frac{p}{2}}\right\|_{H^s_{\gamma/2}}^2 \leq \frac{pc\overline{C}_t}{p'}\left\|f^{\frac{p}{2}}\right\|_{L^2_{\gamma/2}}^2 = \frac{pc\overline{C}_t}{p'}\|f\|_{L^p_{\gamma/p}}^p,$$

for $\overline{C}_t \coloneqq \max\{1+t, C_t\}$ as in lemma 2.4 and constants $c, \widetilde{C}$ depending only on γ, s, D_0, and E_0. In particular, since $\langle v\rangle^\gamma \leq 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p + \frac{p\widetilde{C}}{\max\{p,p'\}}\left\|f^{\frac{p}{2}}\right\|_{H^s_{\gamma/2}}^2 \leq \frac{pc\overline{C}_t}{p'}\|f\|_{L^p}^p \tag{2.14}$$

and furthermore, since $1 \leq \frac{p}{\max\{p,p'\}}$ and $\frac{p}{p'} \leq p$, we may more simply write

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p + \widetilde{C}\left\|f^{\frac{p}{2}}\right\|_{H^s_{\gamma/2}}^2 \leq pc\overline{C}_t\|f\|_{L^p}^p. \tag{2.15}$$

Since (2.15) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p \leq pc\overline{C}_t\|f(t)\|_{L^p}^p \leq pc\overline{C}_T\|f(t)\|_{L^p}^p,$$

if $f_0 \in L^p$ then we may apply Grönwall's inequality to directly conclude that

$$\|f(t)\|_{L^p}^p \leq \|f_0\|_{L^p}^p e^{pc\overline{C}_T t}$$

which proves (2.13).

Let us now proceed with the assumption that $f_0 \notin L^p$. If we take $\theta = \frac{3(p-1)}{3(p-1)+2s} \in (0,1)$ so that $1 = p(1-\theta) + p\theta\frac{3-2s}{3p}$, by Lebesgue interpolation we get

$$\begin{aligned}
\|f\|_{L^p}^p &= \int_{\mathbb{R}^3} \langle v\rangle^{|\gamma|}\langle v\rangle^{\gamma} f^p dv \\
&= \int_{\mathbb{R}^3} \langle v\rangle^{|\gamma|}\left(\langle v\rangle^{\frac{\gamma}{p}} f\right)^p dv \\
&= \int_{\mathbb{R}^3} \langle v\rangle^{|\gamma|}\left(\langle v\rangle^{\frac{\gamma}{p}} f\right)^{p(1-\theta)}\left(\langle v\rangle^{\frac{\gamma}{p}} f\right)^{p\theta\frac{3-2s}{3p}\frac{3p}{3-2s}} dv \\
&= \int_{\mathbb{R}^3} \left(\langle v\rangle^{\frac{|\gamma|}{p(1-\theta)}+\frac{\gamma}{p}} f\right)^{p(1-\theta)}\left(\langle v\rangle^{\frac{\gamma}{p}} f\right)^{p\theta\frac{3-2s}{3p}\frac{3p}{3-2s}} dv \\
&\leq \|f\|_{L^1_{\nu_p}}^{p(1-\theta)}\|f\|_{L^{\frac{3p}{3-2s}}_{\gamma/p}}^{p\theta},
\end{aligned}$$

where the final inequality follows from Hölder's inequality and the fact that $\frac{|\gamma|}{p(1-\theta)} + \frac{\gamma}{p} = \frac{3|\gamma|(p-1)}{2ps} \leq \nu_p$, for ν_p as in $\mathcal{V}_p(D_0, E_0)$. Thus, from lemma 2.3 we may further write

$$\|f\|_{L^p}^p \leq [C(1+t)]^{p(1-\theta)} \|f\|^{p\theta}_{L^{\frac{3p}{3-2s}}_{\gamma/p}} \leq C\overline{C}_t^{p(1-\theta)} \|f\|^{p\theta}_{L^{\frac{3p}{3-2s}}_{\gamma/p}}$$

where C depends at most on γ, s, D_0, E_0. Furthermore, by applying Theorem 1.6 (Sobolev inequality) with $n = 3$, $k = s$, and $\ell = \frac{3-2s}{6}$ gives

$$\left\|f^{\frac{p}{2}}\right\|_{H^s_{\gamma/2}} \geq c\left\|f^{\frac{p}{2}}\right\|_{L^{\frac{3-2s}{6}}_{\gamma/2}} = c\|f\|^{\frac{p}{2}}_{L^{\frac{3p}{3-2s}}_{\gamma/p}},$$

with c depending only on s and hence

$$\|f\|_{L^p} \leq \frac{C}{c^{2\theta}} \overline{C}_t^{p(1-\theta)} \left\|f^{\frac{p}{2}}\right\|^{2\theta}_{H^s_{\gamma/2}}, \tag{2.16}$$

or equivalently

$$\frac{c^2}{C} \overline{C}_t^{-\frac{p(1-\theta)}{\theta}} \|f\|^{\frac{p}{\theta}}_{L^p} \leq \left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}. \tag{2.17}$$

Applying Young's inequality to (2.16) yields

$$\begin{aligned}
pc\overline{C}_t\|f\|_{L^p}^p &\leq pc\overline{C}_T\|f\|_{L^p}^p \\
&\leq pC\overline{C}_T\overline{C}_t^{p(1-\theta)} \left\|f^{\frac{p}{2}}\right\|^{2\theta}_{H^s_{\gamma/2}} \\
&= \left[\left(\frac{pC\overline{C}_T}{\widetilde{C}^\theta}\right)^{\frac{1}{1-\theta}} \overline{C}_t^p\right]^{1-\theta} \left(\widetilde{C}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}}\right)^\theta \\
&\leq (1-\theta)\left(\frac{pC\overline{C}_T}{\widetilde{C}^\theta}\right)^{\frac{1}{1-\theta}} \overline{C}_t^p + \theta\widetilde{C}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}} \\
&= (1-\theta)c\overline{C}_T^{\frac{1}{1-\theta}}\overline{C}_t^p + \theta\widetilde{C}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}},
\end{aligned}$$

where $\overline{C}_T \coloneqq \overline{C}_t\Big|_{t=T}$, and the constant c depends only on γ, s, D_0, and E_0. This estimate, together with (2.15) then gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p + (1-\theta)\widetilde{C}\left\|f^{\frac{p}{2}}\right\|^2_{H^s_{\gamma/2}} \leq (1-\theta)c\overline{C}_T^{\frac{1}{1-\theta}}\overline{C}_t^p,$$

and thus (2.17) further implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_{L^p}^p + C\overline{C}_t^{-\frac{p(1-\theta)}{\theta}}\|f\|^{\frac{p}{\theta}}_{L^p} \leq c\overline{C}_T^{\frac{1}{1-\theta}}\overline{C}_t^p.$$

Now, let $X_p(t) := \|f(t)\|_{L^p}^p$ and $Y_p(t) := \overline{C}_t^p$. With this notation, the inequality above can be written

$$\frac{\mathrm{d}X_p(t)}{\mathrm{d}t} + \frac{C(X_p(t))^{\frac{1}{\theta}}}{(Y_p(t))^{\frac{1-\theta}{\theta}}} \leq c\overline{C}_T^{\frac{1}{1-\theta}} Y_p(t),$$

and hence X_p is a sub-solution to the ODE

$$\frac{\mathrm{d}X_p(t)}{\mathrm{d}t} + \frac{C(X_p(t))^{\frac{1}{\theta}}}{(Y_p(t))^{\frac{1-\theta}{\theta}}} = c\overline{C}_T^{\frac{1}{1-\theta}} Y_p(t). \tag{2.18}$$

If we then consider $Y_p^* := \overline{C}_T^p$ and $X_p^*(t) := C_T^* Y_p^* \left(t^{-\frac{\theta}{1-\theta}} - 1\right)$ for

$$C_T^* := \max\left\{ \left(\frac{c\overline{C}_T^{\frac{1}{1-\theta}}}{C} \right)^{\theta}, \left(\frac{\theta}{C(1-\theta)} \right)^{\frac{\theta}{1-\theta}} \right\},$$

where c, C are as in (2.18), then

$$\begin{aligned}
\frac{\mathrm{d}X_p^*(t)}{\mathrm{d}t} + \frac{C(X_p^*(t))^{\frac{1}{\theta}}}{(Y_p(t))^{\frac{1-\theta}{\theta}}} &\geq \frac{\mathrm{d}X_p^*(t)}{\mathrm{d}t} + \frac{C(X_p^*(t))^{\frac{1}{\theta}}}{(Y_p^*)^{\frac{1-\theta}{\theta}}} \\
&= -\left(\frac{\theta}{1-\theta}\right) C^* Y_p^* t^{-\frac{\theta}{1-\theta}} + C(C_T^*)^{\frac{1}{\theta}} Y_p^* \left(t^{-\frac{\theta}{1-\theta}} + 1\right)^{\frac{1}{\theta}} \\
&\geq -\left(\frac{\theta}{1-\theta}\right) C^* Y_p^* t^{-\frac{\theta}{1-\theta}} + C(C_T^*)^{\frac{1}{\theta}} Y_p^* \left(t^{-\frac{1}{1-\theta}} + 1^{\frac{1}{\theta}}\right) \\
&= C(C_T^*)^{\frac{1}{\theta}} Y_p^* + \left[C(C_T^*)^{\frac{1}{\theta}} - \left(\frac{\theta}{1-\theta}\right) C_T^*\right] Y_p^* t^{-\frac{1}{1-\theta}} \\
&\geq C(C_T^*)^{\frac{1}{\theta}} Y_p(t) + \left[C(C_T^*)^{\frac{1}{\theta}} - \left(\frac{\theta}{1-\theta}\right) C_T^*\right] Y_p^* t^{-\frac{1}{1-\theta}}.
\end{aligned}$$

Now, by construction

$$C(C_T^*)^{\frac{1}{\theta}} \geq \left(\frac{\theta}{1-\theta}\right) C_T^*$$

and

$$C(C_T^*)^{\frac{1}{\theta}} \geq c\overline{C}_T^{\frac{1}{1-\theta}},$$

and hence we conclude that

$$\frac{\mathrm{d}X_p^*(t)}{\mathrm{d}t} + \frac{C(X_p^*(t))^{\frac{1}{\theta}}}{(Y_p(t))^{\frac{1-\theta}{\theta}}} \geq c\overline{C}_T^{\frac{1}{1-\theta}} Y_p(t).$$

In particular, this implies that $X_p^*(t)$ is a super-solution to (2.18) and therefore $X_p(t) \leq X_p^*(t)$ for all $t \in (0, T]$. By recalling the definition of θ above, we have thus shown that

$$\|f(t)\|_{L^p} \leq (C_T^*)^{\frac{1}{p}} \overline{C}_T \left(t^{-\frac{3p}{2sp'}} + 1\right)^{\frac{1}{p}} \leq (C_T^*)^{\frac{1}{p}} \overline{C}_T \left(t^{-\frac{3}{2sp'}} + 1\right), \quad \text{for all } t \in (0, T].$$

Finally, since $\frac{\theta}{1-\theta} = \frac{3(p-1)}{2s} \geq 1$ and $\overline{C}_T \geq 1$, we may write

$$C_T^* \leq c\left[\overline{C}_T\left(\frac{3(p-1)}{2s}\right)\right]^{\frac{3(p-1)}{2sp}} \leq C\overline{C}_T^{\frac{3(p-1)}{2sp}}$$

for some constant C depending only on γ, s, D_0, and E_0. Therefore, combining this with the inequality above proves (2.12). □

We conclude this chapter with a regularization result similar to that of [4, Corollary 1]. This result fits nicely into the L^p theory, however we note that it will be of particular importance in the next chapter.

Theorem 2.2. *Let $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}_2(D_0, E_0)$ be as in (1.29) with $p = 2$ and $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}_2(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that*

$$\int_t^T \|f(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq C\left(\max\{1+T, C_T\}\right)^4, \quad t \in [0,T].$$

Proof. Similarly to how we obtained (2.14), by multiplying (1.12) by $2f$, integrating over $v \in \mathbb{R}^3$, and applying lemma 2.4 with $p = 2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{L^2} + C\|f\|^2_{H^s_{\gamma/2}} \leq c\overline{C}_t\|f\|^2_{L^2} \leq c\overline{C}_T\|f\|^2_{L^2},$$

where $\overline{C}_T := \max\{1+t, C_t\}|_{t=T}$ as in the preceding proof. Then, since $f \in \mathcal{T}$ we may further write

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|^2_{L^2} \leq c\overline{C}_T^3.$$

Integrating along $[t,T]$ then gives

$$\|f(T)\|^2_{L^2} + C\int_t^T \|f(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq \|f(t)\|^2_{L^2} + c\overline{C}_T^3(T-t),$$

and in particular

$$\int_t^T \|f(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq \frac{1}{C}\left(\|f(t)\|^2_{L^2} + c\overline{C}_T^3(T-t)\right).$$

Therefore, again since $f \in \mathcal{T}$ we get

$$\int_t^T \|f(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq C\overline{C}_T^4,$$

where C depends only on γ, s, D_0, and E_0. □

Chapter 3

L^∞ Theory

3.1 Preliminary results

We begin by letting $K \geq 0$ and defining $f_K^+ \coloneqq (f-K)\chi_{\{f\geq K\}}$. We will also denote $K_k = K\left(1-2^{-k}\right)$ for $k \in \mathbb{Z}_{\geq 1}$ and write

$$f_k \coloneqq f_{K_k}^+. \tag{3.1}$$

As mentioned in section 1.4, our strategy will be to investigate the weak form of the homogeneous Boltzmann equation (1.12) with the test function being the level set function f_k. As such, we would like to obtain a result similar to that of Lemma 2.4 with f_k in the place of f^{p-1}. The following lemma provides the first step towards that goal.

Lemma 3.1. *Let Q be as in (1.10) with B satisfying (1.13) and (1.14). Then, for f sufficiently smooth and f_k as in (3.1),*

$$\int_{\mathbb{R}^3} Q(f,f)f_k dv \leq KI_1(f,f_k) + \frac{1}{2}I_2(f,f_k) - \frac{1}{2}J_2(f,f_k), \tag{3.2}$$

where I_p and J_p are as in (2.1) and (2.2) with $p = 1, 2$.

Proof. We first observe that

$$\begin{aligned} f(f_k' - f_k) &= (f-K_k)(f_k' - f_k) + K_k(f_k' - f_k) \\ &= (f-K_k)(\chi_{\{f\geq K_k\}} + \chi_{\{f<K_k\}})(f_k' - f_k) + K_k(f_k' - f_k). \end{aligned}$$

Then, since

$$(f-K_k)\chi_{\{f<K_k\}}(f_k' - f_k) = (f-K_k) = (f-K_k)\chi_{\{f<K_k\}}f_k' \leq 0,$$

we further see that

$$\begin{aligned} f(f_k' - f_k) &\leq f_k(f_k' - f_k) + K_k(f_k' - f_k) \\ &= f_k^2 \left(\frac{f_k'}{f_k} - 1 \right) + K_k(f_k' - f_k) \\ &= \frac{1}{2}\left((f_k')^2 - f_k^2 \right) - \frac{1}{2}(f_k' - f_k)^2 + K_k(f_k' - f_k), \end{aligned}$$

where the final equality follows from (2.3) with $p = 2$. In particular, since $K_k \leq K$, one has

$$f(f_k' - f_k) \leq \frac{1}{2}\left((f_k')^2 - f_k^2 \right) - \frac{1}{2}(f_k' - f_k)^2 + K(f_k' - f_k).$$

Now, by the weak formulation of the collision operator (1.18) with test function $\varphi = f_k$, we get

$$\begin{aligned} \int_{\mathbb{R}^3} Q(f,f) f_k dv &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{S}^2} f_* f(f_k' - f_k) B(v - v_*, \sigma) d\sigma \, dv_* \, dv \\ &\leq K I_1(f, f_k) + \frac{1}{2} I_2(f, f_k) - \frac{1}{2} J_2(f, f_k). \end{aligned}$$

□

As we can see, Lemma 3.1 bears a strong resemblance to Lemma 2.1 in the previous chapter with $p = 2$, with the primary distinction being the appearance of the additional I_1 term. Now, as we will see, we are able to use the same procedure as in the proof of Lemma 2.4 for the difference of integrals $I_2 - J_2$ but this, however, requires us to separately bound I_1. We therefore proceed by dealing with this particular obstacle.

Lemma 3.2. *Let $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{U}(D_0, E_0)$ be as in (1.26) with $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{U}(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant c depending only on γ, s, D_0, and E_0 such that for any $0 < t_* < T$ sufficiently small*

$$I_1(f, f_k)(t) \leq c \max\{1 + T, C_T\} t_*^{-\frac{3}{2s}} \|f_k(t)\|_{L^1}, \quad \textit{for all } t \in [t_*, T]. \tag{3.3}$$

Moreover, if we additionally assume that $f_0 \in L^\infty$, then

$$I_1(f, f_k)(t) \leq c \max\{1 + T, C_T\} e^{c\overline{C}_T T} \|f_k(t)\|_{L^1}, \quad \textit{for all } t \in [0, T]. \tag{3.4}$$

Proof. Let us start with the observation that due to Lemma 2.2 with $p = 1$, if $-\frac{3}{2} < \gamma$, then regardless of whether or not $f_0 \in L^\infty$ we have

$$I_1(f, f_k) \leq c\overline{C}_T \|f_k\|_{L^1}, \tag{3.5}$$

where $\overline{C}_T := \max\{1+t, C_t\}|_{t=T}$ and c is some constant depending only on γ, s, D_0, and E_0. Notably, we may also write

$$I_1(f, f_k) \le c\overline{C}_T t_*^{-\frac{3}{2s}} \|f_k\|_{L^1}. \tag{3.6}$$

Let us now consider the case when $\gamma \le -\frac{3}{2}$ and $f_0 \in L^\infty$. Then, since $f_0 \in \mathcal{V}(D_0, E_0)$, we have $f_0 \in L^1 \cap L^\infty$ and hence $f_0 \in L^p$ for all $p \in [1, \infty]$. So, by Theorem 2.1 we deduce that for any $p > 2$,

$$\sup_{t\in[0,T]} \|f(t)\|_{L^p} \le \|f_0\|_{L^p} e^{c\overline{C}_T T}. \tag{3.7}$$

Now by (2.5), Hölder's inequality with q and q', and Theorem 1.8 (Young's convolution inequality) we get

$$\begin{aligned}
I_1(f, f_k) &\le C \int_{\mathbb{R}^3\times\mathbb{R}^3} f_* f_k |v-v_*|^\gamma dv_*\, dv \\
&= C \left(\int_{|v-v_*|\le 1} f_* f_k |v-v_*|^\gamma dv_*\, dv + \int_{|v-v_*|>1} f_* f_k |v-v_*|^\gamma dv_*\, dv \right) \\
&\le C \left(\|f\|_{L^{q'}} \left\| f_k * |\cdot|^\gamma \chi_{\{|\cdot|\le 1\}} \right\|_{L^q} + c\|f\|_{L^1} \|f_k\|_{L^1} \right) \\
&\le c \left(\|f\|_{L^{q'}} \left\| |\cdot|^\gamma \chi_{\{|\cdot|\le 1\}} \right\|_{L^q} + 1 \right) \|f_k\|_{L^1}.
\end{aligned}$$

Now, by choosing $1 < q < \frac{3}{|\gamma|}$, then $|\cdot|^\gamma \chi_{\{|\cdot|\le 1\}} \in L^q$ and $q' > 2$. Hence, by (3.7) we obtain

$$\begin{aligned}
I_1(f, f_k) &\le C \left(\|f\|_{L^{q'}} + 1 \right) \|f_k\|_{L^1} && (3.8) \\
&\le C \left(\|f_0\|_{L^{q'}} e^{c\overline{C}_T T} + 1 \right) \|f_k\|_{L^1} \\
&\le c e^{c\overline{C}_T T} \|f_k\|_{L^1}, && (3.9)
\end{aligned}$$

where c depends only on γ, s, D_0, and E_0. Therefore, (3.8) together with (3.5) imply (3.4) holds for any $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$.

Let us now continue with the case where $\gamma \le -\frac{3}{2}$ but forgo the assumption that $f_0 \in L^\infty$. Then, by Theorem 2.1 we have for t_* small enough

$$\|f(t_*)\|_{L^{q'}} \le C \left(t_*^{-\frac{3}{2sq}} + 1 \right) \le c t_*^{-\frac{3}{2sq}} \le c t_*^{-\frac{3}{2s}}$$

where c depends only on γ, s, D_0, E_0 and q' is as above. By treating t_* as an initial time, we may then invoke (2.13) of Theorem 2.1 to obtain

$$\sup_{t\in[t_*,T]} \|f(t)\|_{L^{q'}} \le \|f(t_*)\|_{L^{q'}} e^{c\overline{C}_T T},$$

and in particular

$$\sup_{t\in[t_*,T]} \|f(t)\|_{L^{q'}} \leq \|f(t_*)\|_{L^{q'}} \leq ct_*^{-\frac{3}{3s}}.$$

Now, since (3.8) remains true even when $f_0 \notin L^\infty$, we then see that

$$\begin{aligned} I_1(f, f_k) &\leq C\left(\|f\|_{L^{q'}} + 1\right)\|f_k\|_{L^1} \\ &\leq C\left(ct_*^{-\frac{3}{2s}} + 1\right)\|f_k\|_{L^1} \\ &\leq Ct_*^{-\frac{3}{2s}}\|f_k\|_{L^1} \\ &\leq C\overline{C}_T t_*^{-\frac{3}{2s}}\|f_k\|_{L^1} \end{aligned}$$

for all $t \in [t_*, T]$. Therefore, together with (3.6) we deduce that (3.3) holds for all $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$. □

Since the upper bound obtained above for I_1 differs when it is additionally assumed that $f_0 \in L^\infty$, we save our discussion regarding the further estimation of the right hand side of (3.2) for the proof of Theorem 3.1 in the proceeding section where the arguments presented are separated into the L^∞-norm generation and propagation settings. Now, we shall see that these arguments follow an identical procedure as in the proof of Lemma 2.4 to bound $I_2 - J_2$, which will consequently yield, up to some constants, the difference of $L^2_{\gamma/2}$- and $H^2_{\gamma/2}$-norms. In this case, it is not immediately clear how the L^1-norm appearing on the right hand side of both (3.3) and (3.4) will be compatible with our $I_2 - J_2$ bound. The next lemma provides an important tool that we may use to make sense of this.

Lemma 3.3. *If $\alpha \geq 0$ and $\beta \in (0, 1]$, then for f_k as in (3.1) we have*

$$\chi_{\{f \geq K_k\}} \leq \left(\frac{1}{2^\beta - 1} \cdot \frac{2^k}{K} f_{k-\beta}\right)^\alpha.$$

Proof. We first notice that since $\{f \geq K_k\} \subseteq \{f \geq K_{k-\beta}\}$, we have $\chi_{\{f\geq K_k\}} \leq \chi_{\left\{f\geq K_{k-\beta}\right\}}$. With this in mind, we compute

$$
\begin{aligned}
f_{k-\beta} &= (f - K_{k-\beta})\chi_{\left\{f\geq K_{k-\beta}\right\}} \\
&= \left[f - K\left(1 - \frac{2^\beta}{2^k}\right)\right]\chi_{\left\{f\geq K_{k-\beta}\right\}} \\
&= \left[f - K_k + K_k - K\left(1 - \frac{2^\beta}{2^k}\right)\right]\chi_{\left\{f\geq K_{k-\beta}\right\}} \\
&= \left(f - K_k + K - \frac{K}{2^k} - K + \frac{K2^\beta}{2^k}\right)\chi_{\left\{f\geq K_{k-\beta}\right\}} \\
&= \left(f - K_k + K\frac{2^\beta - 1}{2^k}\right)\chi_{\left\{f\geq K_{k-\beta}\right\}} \\
&\geq \left(f - K_k + K\frac{2^\beta - 1}{2^k}\right)\chi_{\{f\geq K_k\}} \\
&\geq K\frac{2^\beta - 1}{2^k}\chi_{\{f\geq K_k\}},
\end{aligned}
$$

and therefore

$$
\chi_{\{f\geq K_k\}} \leq \left(\frac{1}{2^\beta - 1}\cdot\frac{2^k}{K}f_{k-\beta}\right)^\alpha
$$

for any $\alpha \geq 0$. □

This lemma essentially means that, up to a constant, we may bound the level set function f_k by $f_{k-\beta}^{1+\alpha}$ for any $\beta \in (0, 1]$ and $\alpha \geq 0$. This increase in power allows us to interpolate f_k between L^p-norms that may not have otherwise been allowed only by moving down a fraction of a level. This idea is made precise in the proof of the following lemma.

Lemma 3.4. *Let $s \in (0, 1)$, $\gamma \in (-3, 0)$, f be sufficiently smooth, f_k be as in (3.1), and denote $q_s := \frac{6}{3-2s}$ and q_s' its Hölder conjugate. Then, for $p \in (1, q_s)$ and $r_s := \frac{1}{p} - \frac{1}{q_s}$, there is a constant C depending only on s such that*

$$
\|f_k\|_{L^p} \leq C\left(\frac{2^k}{K}\right)^{r_s\zeta}\|f\|_{L^1_{|\gamma|/2r_s(2-\zeta)}}^{r_s(2-\zeta)}\|f_{k-1}\|_{L^2}^{2r_s(\zeta-1)}\|f_{k-1}\|_{H^s_{\gamma/2}}, \tag{3.10}
$$

for any $\zeta \in \left(1, \min\left\{2, \frac{1}{r_s}\right\}\right)$ and

$$
\|f_k\|_{L^1} \leq C\left(\frac{2^k}{K}\right)^{\frac{s\zeta}{3}+1}\|f\|_{L^1_{3|\gamma|/2s(2-\zeta)}}^{\frac{2s(2-\zeta)}{3}}\|f_{k-1}\|_{L^2}^{\frac{2s(\zeta-1)}{3}}\|f_{k-1}\|_{H^s_{\gamma/2}}^2, \tag{3.11}
$$

for any $\zeta \in (1, 2)$.

Proof. Let $\zeta \in \left(1, \min\left\{2, \frac{1}{r_s}\right\}\right)$, $q = \frac{q_s}{p}$, and $\ell = \frac{\zeta}{q'}$. Then, by Hölder's inequality

$$\begin{aligned} \|f_k\|_{L^p} &= \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{|\gamma|q_s}{2q}} f_k^{\ell} \langle v\rangle^{\frac{\gamma q_s}{2q}} f^{p-\ell} dv\right)^{\frac{1}{p}} \\ &\leq \left\|f_k^{\ell}\right\|^{\frac{1}{p}}_{L^{q'}_{|\gamma|q_s/2q}} \left\|f_k^{p-\ell}\right\|^{\frac{1}{p}}_{L^{q}_{\gamma q_s/2q}}. \end{aligned} \tag{3.12}$$

We now consider separately first and second terms on the right hand side of (3.12).

Indeed, let us first focus on the first term. First of all, since $\ell q' = \zeta \in \left(1, \min\left\{2, \frac{1}{r_s}\right\}\right)$, we can write

$$\frac{1}{\ell q'} = \frac{1}{\zeta} = (1-\theta) + \frac{\theta}{2}$$

for $\theta = \left(1 - \frac{1}{\zeta}\right) \in (0,1)$. Thus,

$$\begin{aligned} \left\|f_k^{\ell}\right\|^{\frac{1}{p}}_{L^{q'}_{|\gamma|q_s/2q}} &= \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{|\gamma|q_s q'}{2q}} f_k^{\ell q'} dv\right)^{\frac{1}{pq'}} \\ &= \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{|\gamma|}{2r_s}} f_k^{\zeta(1-\theta)} f_k^{\frac{2\zeta\theta}{2}} dv\right)^{r_s} \\ &\leq \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{|\gamma|}{2r_s\zeta(1-\theta)}} f_k dv\right)^{r_s\zeta(1-\theta)} \left(\int_{\mathbb{R}^3} f_k^2 dv\right)^{\frac{r_2\zeta\theta}{2}} \\ &= \|f_k\|^{r_s(2-\zeta)}_{L^1_{|\gamma|/2r_s(2-\zeta)}} \|f_k\|^{2r_s(\zeta-1)}_{L^2} \\ &\leq \|f\|^{r_s(2-\zeta)}_{L^1_{|\gamma|/2r_s(2-\zeta)}} \|f_{k-1}\|^{2r_s(\zeta-1)}_{L^2}. \end{aligned} \tag{3.13}$$

As for the second term, it follows from Lemma 3.3 with $\alpha \equiv \ell q$ and $\beta \equiv 1$, and the Sobolev inequality with $n \equiv 3$, $k \equiv s$, and $\ell \equiv \frac{6}{3-2s}$ (as in Theorem 1.6) that

$$\begin{aligned} \left\|f_k^{p-\ell}\right\|^{\frac{1}{p}}_{L^{q}_{\gamma q_s/2q}} &= \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{\gamma q_s}{2}} f_k^{(p-\ell)q} dv\right)^{\frac{1}{pq}} \\ &\leq \left(\frac{2^k}{K}\right)^{\frac{\ell q}{pq}} \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{\gamma q_s}{2}} f_{k-1}^{(p-\ell)q+\ell q} dv\right)^{\frac{1}{pq}} \\ &= \left(\frac{2^k}{K}\right)^{\frac{\zeta}{pq'}} \left(\int_{\mathbb{R}^3} \langle v\rangle^{\frac{\gamma q_s}{2}} f_{k-1}^{q_s} dv\right)^{\frac{1}{q_s}} \\ &= \left(\frac{2^k}{K}\right)^{r_s\zeta} \|f_{k-1}\|_{L^{q_s}_{\gamma/2}} \\ &\leq C\left(\frac{2^k}{K}\right)^{r_s\zeta} \|f_{k-1}\|_{H^s_{\gamma/2}}, \end{aligned} \tag{3.14}$$

where the constant C depends only on s. Therefore, (3.12) together with (3.13) and (3.14) yields (3.10).

To see (3.11), we first note that by Lemma 3.3 with $\alpha \equiv 1$ and $\beta \equiv \frac{1}{2}$, we have

$$\|f_k\|_{L^1} \leq \frac{1}{\sqrt{2}-1}\frac{2^k}{K}\int_{\mathbb{R}^3} f_{k-1/2}^{1+1}dv = \frac{1}{\sqrt{2}-1}\frac{2^k}{K}\left\|f_{k-1/2}\right\|_{L^2}^2.$$

Now, using (3.10) with $p = 2$ and observing that in this case $r_s = \frac{1}{2} - \frac{1}{q_s} = \frac{1}{2} - \frac{3-2s}{6} = \frac{s}{3}$, we get

$$\begin{aligned}\|f_k\|_{L^1} &\leq C\frac{2^k}{K}\left[\left(\frac{2^k}{K}\right)^{\frac{s\zeta}{3}} \|f\|_{L^1_{3|\gamma|/2s(2-\zeta)}}^{\frac{s(2-\zeta)}{3}} \|f_{k-1}\|_{L^2}^{\frac{2s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}\right]^2 \\ &= C\left(\frac{2^k}{K}\right)^{\frac{s\zeta}{3}+1} \|f\|_{L^1_{3|\gamma|/2s(2-\zeta)}}^{\frac{2s(2-\zeta)}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}^2.\end{aligned}$$

Moreover, since $\frac{1}{r_s} = \frac{3}{s} > 3$ when $p = 2$, we find that $\left(\min\left\{2, \frac{1}{r_s}\right\}\right) = 2$ and hence we see that (3.11) holds for any $\zeta \in (1, 2)$. □

It is due to Lemma 3.4, or more specifically (3.11), that we require the introduction of the class mentioned in section 1.4:

$$\mathcal{V}_\zeta^*(D_0, E_0) := \left\{g \in L^1 : \|g\|_{L^1} \geq D_0, \|g\|_{L^1_{\nu_\zeta^*}} + \|g\|_{L\log L} \leq E_0\right\},$$

where $\nu_\zeta^* = \max\left\{2, \frac{3|\gamma|}{2s(2-\zeta)}\right\}$ for $\zeta \in (1, 2)$, since we control the the weighted L^1-norm appearing on the right hand side of (3.11) when f solves (1.12) with $f_0 \in \mathcal{V}_\zeta^*$. In particular, by Lemma 2.3, if f solves (1.12) with $f_0 \in \mathcal{V}_\zeta^*$ then we may write

$$\|f_k\|_{L^1} \leq C\overline{C}_t^{\frac{2s(2-\zeta)}{3}}\left(\frac{2^k}{K}\right)^{\frac{\zeta}{q_s'}+1} \|f_{k-1}\|_{L^2}^{\frac{2s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}^2 \tag{3.15}$$

in the place of (3.11), where C depends only on γ, s, D_0, and E_0, and $\overline{C}_t = \max\{1+t, C_t\}$ for C_t as in (1.30). It will also be relevant to note for later that $\mathcal{V}_\zeta^*(D_0, E_0) \subseteq \mathcal{V}_2(D_0, E_0)$ since $\nu_2 = \max\left\{2, |\gamma|, \frac{3|\gamma|}{4s}\right\}$ and for any $\zeta \in (1, 2)$

$$\frac{3|\gamma|}{2s(\nu-1)} \geq |\gamma|, \quad \text{and} \quad \frac{3|\gamma|}{2s(\nu-1)} \geq \frac{3|\gamma|}{4s}.$$

To conclude this section, we prove one final and fairly general lemma relating to the energy functional, mentioned in section 1.4, that will be introduced in the proof of our next main result.

Lemma 3.5. *Let $a, b, C > 0$, $c > 1$, $k \in \mathbb{Z}_{\geq 1}$, and W_0 be constant with respect to k. Then,*

$$W_k := W_0\left(2^{-\frac{a}{c-1}}\right)^k$$

is a super-solution to the equation

$$W_k = C2^{ak}K^{-b}W_{k-1}^c \tag{3.16}$$

when

$$K \geq \left(C2^{\frac{ac}{c-1}}W_0^{c-1}\right)^{\frac{1}{b}}. \tag{3.17}$$

Moreover, W_k is a solution to (3.16) when (3.17) holds with equality.

Proof. The proof follows from computation:

$$\begin{aligned} C2^{ak}K^{-b}W_{k-1}^c &= C2^{ak}K^{-b}W_0^c\left(2^{-\frac{a}{c-1}}\right)^{c(k-1)} \\ &= \left(C2^{\frac{ac}{c-1}}W_0^{(c-1)}\right)K^{-b}W_0\left(2^{-\frac{a}{c-1}}\right)^k. \end{aligned}$$

Then,

$$W_k \geq C2^{ak}K^{-b}W_{k-1}^c$$

when (3.17) holds, and

$$W_k = C2^{ak}K^{-b}W_{k-1}^c$$

when (3.17) holds with equality. □

3.2 Generation and propagation of L^∞-norms

Let K, k, K_k, and f_k be as in the previous section.

Theorem 3.1. *Let $s \in (0,1)$, $\gamma \in (\max\{-3, -2s - \frac{3}{2}\}, 0)$, $\mathcal{V}_\zeta^*(D_0, E_0)$ be as in (1.4) for any $\zeta \in (1,2)$ and with $D_0, E_0 > 0$, $\mathcal{T}$ be as in (1.30) for a fixed $T > 0$, and C_T be as in (1.31). Then, if $f(t,v) \in \mathcal{T}$ is a sufficiently smooth solution to the homogeneous Boltzmann equation (1.12) on $[0,T] \times \mathbb{R}^3$ with $f_0 \in \mathcal{V}_\zeta^*(D_0, E_0)$ and Q as in (1.10) with B satisfying (1.13) and (1.14), there is a constant C depending only on γ, s, D_0, and E_0 such that for any $0 < t_* < T$ sufficiently small we have*

$$\sup_{t\in[t_*,T]} \|f(t)\|_{L^\infty} \leq C\left(\max\{1+T, C_T\}\right)^{\frac{6+8s(\zeta-1)}{s\zeta}}\left(\frac{1}{t_*}\right)^{\frac{3}{s\zeta}\left(\frac{3}{2s}+1\right)}. \tag{3.18}$$

If we additionally assume that $f_0 \in L^\infty$, then

$$\sup_{t\in[0,T]} \|f(t)\|_{L^\infty} \leq \max\left\{2\|f_0\|_{L^\infty}, C\left(\max\{1+T, C_T\}\right)^{\frac{6+8s(\zeta-1)}{s\zeta}} e^{\frac{3}{s\zeta}C\max\{1+T,C_T\}T}\right\}. \tag{3.19}$$

Proof. Let us forgo the additional assumption that $f_0 \in L^\infty$ for now and begin by proving (3.18). To this end, by multiplying (1.12) by f_k and integrating over $v \in \mathbb{R}^3$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} Q(f,f) f_k dv &= \int_{\mathbb{R}^3} f_k f_t dv \\ &= \int_{f \geq K_k} (f - K_k) f_t dv \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} (f-K_k)^2 \chi^2_{\{f \geq K_k\}} dv \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_k(t)\|^2_{L^2}.\end{aligned}$$

Now, by Lemma 3.1 we have

$$\int_{\mathbb{R}^3} Q(f,f) f_k dv \leq K I_1(f, f_k) + \frac{1}{2} I_2(f, f_k) - \frac{1}{2} J_2(f, f_k).$$

Then, we may bound $\frac{1}{2}I_2 - \frac{1}{2}J_2$ via the same procedure used in the proof of Lemma 2.4 for $\frac{1}{p'}I_p - \frac{1}{\max\{p,p'\}}J_p$. Indeed, by taking $p = 2$ in the proof of Lemma 2.4 and replacing f with f_k, we obtain

$$\frac{1}{2}I_2(f, f_k) - \frac{1}{2}J_2(f, f_k) \leq c\overline{C}_T \|f_k\|^2_{L^2_{\gamma/2}} - C\|f_k\|^2_{H^s_{\gamma/2}},$$

where $\overline{C}_T := \max\{1+t, C_t\}|_{t=T}$ for C_t as in (1.30) and c and C are constants depending only on γ, s, D_0, and E_0. Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_k(t)\|^2_{L^2} + C\|f_k\|^2_{H^s_{\gamma/2}} \leq K I_1(f, f_k) + c\overline{C}_T\|f_k\|^2_{L^2_{\gamma/2}} \leq K I_1(f, f_k) + c\overline{C}_T\|f_k\|^2_{L^2} \quad (3.20)$$

Moreover, by integrating (3.20) over $\tau \in (\xi, t)$ for any $\xi \leq t$, we get

$$\begin{aligned}\frac{1}{2}\|f_k(t)\|^2_{L^2} + C\int_\xi^t \|f_k(\tau)\|^2_{H^s_{\gamma/2}} d\tau \leq \frac{1}{2}\|f_k(\xi)\|^2_{L^2} + K\int_\xi^t I_1(f, f_k)(\tau) d\tau + \\ c\overline{C}_T \int_\xi^t \|f_k(\tau)\|^2_{L^2} d\tau. \quad (3.21)\end{aligned}$$

Let us now define the times $t_k := t_*\left(1 - 2^{-(k+1)}\right)$. Then, since $0 < t_1 = \frac{3}{4}t_*$, by (3.3) in Lemma 3.2 with $t_* \equiv t_1$ we see that there is a constant c depending only on γ, s, D_0, and E_0 such that when $t_* > 0$ is sufficiently small we have

$$I_1(f, f_k)(t) \leq c\overline{C}_T \left(\frac{3}{4}t_*\right)^{-\frac{3}{2s}} \|f_k(t)\|_{L^1}, \quad \text{for all } t \in [t_1, T],$$

which we may instead simply write as

$$I_1(f, f_k)(t) \leq c\overline{C}_T t_*^{-\frac{3}{2s}} \|f_k(t)\|_{L^1}, \quad \text{for any } t \in [t_1, T]. \quad (3.22)$$

Combining this result with (3.21) then gives

$$\frac{1}{2}\|f_k(t)\|_{L^2}^2 + C\int_\xi^t \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau \le \frac{1}{2}\|f_k(\xi)\|_{L^2}^2 + c\overline{C}_T t_*^{-\frac{3}{2s}} \left(K\int_\xi^t \|f_k(\tau)\|_{L^1} d\tau + \int_\xi^t \|f_k(\tau)\|_{L^2}^2 d\tau\right)$$

for all $t \in [\xi, T]$.

Now, if we define the energy functional

$$W_k := \frac{1}{2}\sup_{t\in[t_k,T]} \|f_k(t)\|_{L^2}^2 + C\int_{t_k}^T \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau,$$

the inequality above implies that for $t_{k-1} \le \xi \le t_k \le t \le T$ we have

$$W_k \le \frac{1}{2}\|f_k(\xi)\|_{L^2}^2 + c\overline{C}_T t_*^{-\frac{3}{2s}} \left(K\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^1} d\tau + \int_{t_{k-1}}^T \|f_k(\tau)\|_{L^2}^2 d\tau\right).$$

Then, since $t_k - t_{k-1} = t_* 2^{-(k+1)}$, taking the mean over $\xi \in [t_{k-1}, t_k]$ gives

$$\begin{aligned} \frac{1}{2}\|f_k(\xi)\|_{L^2}^2 &\le \frac{1}{2|t_k - t_{k-1}|}\int_{t_{k-1}}^{t_k} \|f_k(\tau)\|_{L^2}^2 d\tau \\ &= \frac{2^k}{t_*}\int_{t_{k-1}}^{t_k} \|f_k(\tau)\|_{L^2}^2 d\tau, \end{aligned}$$

and thus

$$\begin{aligned} W_k &\le \frac{2^k}{t_*}\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^2}^2 d\tau + c\overline{C}_T t_*^{-\frac{3}{2s}} \left(K\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^1} d\tau + \int_{t_{k-1}}^T \|f_k(\tau)\|_{L^2}^2 d\tau\right) \\ &\le c\overline{C}_T t_*^{-\frac{3}{2s}} \left[K\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^1} d\tau + \left(\frac{2^k}{t_*} + 1\right)\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^2}^2 d\tau\right] \\ &\le C\overline{C}_T t_*^{-\frac{3}{2s}-1} \left(K\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^1} d\tau + 2^k\int_{t_{k-1}}^T \|f_k(\tau)\|_{L^2}^2 d\tau\right) \end{aligned}$$

for some constant C depending only on γ, s, D_0, and E_0. Furthermore, by Lemma 3.4 with $p = 2$ we see that for any $\zeta \in (1, 2)$ there is a constant C depending only on s such that

$$\begin{aligned} 2^k\|f_k\|_{L^2}^2 &\le 2^k C\left[\left(\frac{2^k}{K}\right)^{\frac{s\zeta}{3}} \|f\|_{L^1_{3|\gamma|/2s(2-\zeta)}}^{\frac{s(2-\zeta)}{3}} \|f_{k-1}\|_{L^2}^{\frac{2s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}\right]^2 \\ &= C\left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{2s\zeta}{3}} \|f\|_{L^1_{3|\gamma|/2s(2-\zeta)}}^{\frac{2s(2-\zeta)}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}^2 \end{aligned}$$

and thus since $f_0 \in \mathcal{V}^*_\zeta(D_0, E_0)$ there is a constant C, by Lemma 2.3, depending only on γ, s, D_0, and E_0 such that

$$2^k \|f_k\|^2_{L^2} \le C\overline{C}_T \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{2s\zeta}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|^2_{H^s_{\gamma/2}}.$$

Lemmas 3.4 and 2.3 similarly show that

$$K\|f_k\|_{L^1} \le C\overline{C}_T \left(2^{\frac{s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|^2_{H^s_{\gamma/2}},$$

and in particular, this implies

$$K\|f_k\|_{L^1} + 2^k \|f_k\|^2_{L^2} \le C\overline{C}_T \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|^2_{H^s_{\gamma/2}}. \tag{3.23}$$

Therefore, we find that

$$W_k \le C\overline{C}^2_T t_*^{-\frac{3}{2s}-1} \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} \int_{t_{k-1}}^T \|f_{k-1}(\tau)\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}(\tau)\|^2_{H^s_{\gamma/2}} d\tau$$

and hence

$$W_k \le C\overline{C}^2_T t_*^{-\frac{3}{2s}-1} \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} 2^{\frac{2s(\zeta-1)}{3}} \left(\frac{1}{2} \sup_{t\in[t_{k-1},T]} \|f_{k-1}(t)\|^2_{L^2}\right)^{\frac{2s(\zeta-1)}{3}} \cdot \int_{t_{k-1}}^T \|f_{k-1}(\tau)\|^2_{H^s_{\gamma/2}} d\tau,$$

from which we see that

$$W_k \le C\overline{C}^2_T t_*^{-\frac{3}{2s}-1} \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} 2^{\frac{2s(\zeta-1)}{3}} W_{k-1}^{\frac{2s(\zeta-1)}{3}+1}. \tag{3.24}$$

Now, we see from Lemma 3.5 by setting $a \equiv \frac{2s(\zeta-1)}{3}$, $b \equiv \frac{s\zeta}{3}$, $c \equiv \frac{2s(\zeta-1)}{3} + 1$ and $C \equiv C\overline{C}^2_T t_*^{-\frac{3}{2s}-1}$, that

$$W^*_k := W_0 \left(\frac{1}{2}\right)^k$$

satisfies (3.24) with equality when

$$K = \left(C\overline{C}^2_T t_*^{-\frac{3}{2s}-1} 2^{\frac{2s(\zeta-1)}{3}+1} W_0^{\frac{2s(\zeta-1)}{3}}\right)^{\frac{3}{s\zeta}} \tag{3.25}$$

and hence

$$W_k \le W^*_k \underset{k\to\infty}{\longrightarrow} 0$$

provided $W_0 < \infty$. Indeed, since

$$W_0 = \frac{1}{2} \sup_{t\in[t_0,T]} \|f(t)\|_{L^2}^2 + C\int_{t_0}^T \|f(\tau)\|_{H^s_{\gamma/2}}^2 d\tau,$$

we see from Theorem 2.2, together with the fact that $f \in \mathcal{T}$, that there is a constant c depending only on γ, s, D_0, and E_0 such that

$$W_0 \le \frac{1}{2}\overline{C}_T^2 + C\overline{C}_T^4 \le c\overline{C}_T^4 < \infty. \tag{3.26}$$

Hence, if we take $k \to \infty$, then $K_k \to K$ and $t_k \to t_*$

$$\sup_{t\in[t_*,T]} \|f_K(t)\|_{L^2} = 0,$$

from which we deduce that

$$f(t,v) \le K$$

for any $t \in [t_*, T]$ and for almost all $v \in \mathbb{R}^3$, where K is as above in (3.25). Therefore, from (3.25) and (3.26) we finally conclude that for t_* sufficiently small and any $t \in [t_*, T]$, we have

$$\|f(t)\|_{L^\infty} \le C\left(\frac{1}{t_*}\right)^{\frac{3}{s\zeta}\left(\frac{3}{2s}+1\right)} \overline{C}_T^{\frac{3}{s\zeta}\left(2+\frac{8s(\zeta-1)}{3}\right)} = C\overline{C}_T^{\frac{6+8s(\zeta-1)}{s\zeta}} \left(\frac{1}{t_*}\right)^{\frac{3}{s\zeta}\left(\frac{3}{2s}+1\right)},$$

where C depends only on γ, s, D_0, and E_0, thus giving (3.18).

Let us now further suppose that $f_0 \in L^\infty$. Now, taking $K \ge 2\|f_0\|_{L^\infty}$ implies $K_k = K(1-2^{-k}) \ge \|f_0\|_{L^\infty}$ since $(1-2^{-k}) \ge \frac{1}{2}$ for every $k \ge 1$. Therefore, integrating (3.20) over $\tau \in [0,t]$ gives

$$\frac{1}{2}\|f_k(t)\|_{L^2}^2 + C\int_0^t \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau \le K\int_0^t I_1(f,f_k)(\tau)d\tau + c\overline{C}_T\int_0^t \|f_k(\tau)\|_{L^2}^2 d\tau.$$

Additionally, by Lemma 3.2 we have the following bound for $I_1(f,f_k)$:

$$I_1(f,f_k)(t) \le c\overline{C}_T e^{c\overline{C}_T T}\|f_k(t)\|_{L^1}, \quad \text{for all } t \in [0,T],$$

where c depends only on γ, s, D_0, and E_0. Thus, we may further write

$$\begin{aligned}\frac{1}{2}\|f_k(t)\|_{L^2}^2 + C\int_0^t \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau &\le c\overline{C}_T e^{c\overline{C}_T T} K\int_0^t \|f_k(\tau)\|_{L^1} d\tau + c\overline{C}_T\int_0^t \|f_k(\tau)\|_{L^2}^2 d\tau \\ &\le c\overline{C}_T e^{c\overline{C}_T T}\int_0^t K\|f_k(\tau)\|_{L^1} + \|f_k(\tau)\|_{L^2}^2 d\tau\end{aligned}$$

for any $t \in [0,T]$. Then similarly to the previous case, by defining the modified energy functional

$$W_{k,0} := \frac{1}{2} \sup_{t\in[0,T]} \|f_k(t)\|_{L^2}^2 + C\int_0^T \|f_k(\tau)\|_{H^s_{\gamma/2}}^2 d\tau$$

we see from the above inequality that

$$W_{k,0} \le c\overline{C}_T e^{c\overline{C}_T T} \int_0^T K\|f_k(\tau)\|_{L^1} + \|f_k(\tau)\|_{L^2}^2 d\tau.$$

Now, since $1 < 2^k$ we see from (3.23) that

$$K\|f_k(\tau)\|_{L^1} + \|f_k(\tau)\|_{L^2}^2 \le C\overline{C}_T \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} \|f_{k-1}\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}\|_{H^s_{\gamma/2}}^2$$

for a constant C depending only on γ, s, D_0, and E_0. With this inequality in mind, we may then write

$$W_{k,0} \le C\overline{C}_T^2 e^{C\overline{C}_T T} \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} \int_0^T \|f_{k-1}(\tau)\|_{L^2}^{\frac{4s(\zeta-1)}{3}} \|f_{k-1}(\tau)\|_{H^s_{\gamma/2}}^2 d\tau,$$

and thus similarly to (3.24) we have

$$W_{k,0} \le C\overline{C}_T^2 e^{C\overline{C}_T T} \left(2^{\frac{2s\zeta}{3}+1}\right)^k \left(\frac{1}{K}\right)^{\frac{s\zeta}{3}} 2^{\frac{2s(\zeta-1)}{3}} W_{k-1,0}^{\frac{2s(\zeta-1)}{3}+1}. \tag{3.27}$$

Therefore, we again see that by Lemma 3.5 with a, b, c as above and $C \equiv C\overline{C}_T^2 e^{C\overline{C}_T T}$, that

$$W_{k,0}^* := W_{0,0} \left(\frac{1}{2}\right)^k$$

satisfies (3.27) with equality when

$$K = \left(C\overline{C}_T^2 e^{C\overline{C}_T T} 2^{\frac{2s(\zeta-1)}{3}+1} W_{0,0}^{\frac{2s(\zeta-1)}{3}}\right)^{\frac{3}{s\zeta}},$$

where

$$W_{0,0} = \frac{1}{2} \sup_{t\in[0,T]} \|f(t)\|_{L^2}^2 + C\int_0^T \|f(\tau)\|_{H^s_{\gamma/2}}.$$

In particular, Lemma 3.5 implies that $W_{k,0}^*$ is a super-solution to (3.27) when

$$K = \max\left\{2\|f_0\|_{L^\infty}, \left(C\overline{C}_T^2 e^{C\overline{C}_T T} 2^{\frac{2s(\zeta-1)}{3}+1} W_{0,0}^{\frac{2s(\zeta-1)}{3}}\right)^{\frac{3}{s\zeta}}\right\}, \tag{3.28}$$

and hence $W_{k,0} \leq W_{k,0}^*$ with this choice of K. Then, since the bounding in (3.26) also applies to $W_{0,0}$, we may conclude that

$$W_{k,0} \leq W_{k,0}^* \underset{k\to\infty}{\longrightarrow} 0$$

and thus, similarly to above, taking $k \to \infty$ yields

$$\|f(t)\|_{L^\infty} \leq K, \quad \text{for all } t \in [0, T]$$

for K as in (3.28). Finally, from (3.26) we conclude that there is a constant C depending only on γ, s, D_0, and E_0 such that

$$\sup_{t\in[0,T]} \|f(t)\|_{L^\infty} \leq \max\left\{2\|f_0\|_{L^\infty}, C\overline{C}_T^{\frac{6+8s(\zeta-1)}{s\zeta}} e^{\frac{3}{s\zeta}C\overline{C}_T T}\right\},$$

which is precisely (3.19). □

Chapter 4

Further research

We dedicate this final chapter to the discussion of various potential future research directions which have emerged as a result of the work presented in Chapters 2 and 3. Throughout this thesis, we have imposed a number of assumptions that may, to some degree, restrict the utility of our main results. In particular, it was mentioned in Section 1.4 that the space $\mathcal{U}(D_0, E_0)$, as defined in (1.26), reflects the minimum assumptions we may take for the initial data of a zero total momentum system and therefore corresponds to the most general set of solutions to the Boltzmann equation (spatially homogeneous or not). However, due our need to control higher moments of solutions to the homogeneous Boltzmann equation (1.12) (a consequence of Lemmas 2.2 and 3.4) we required that the initial data be limited the spaces $\mathcal{V}_p(D_0, E_0)$, as defined in (1.29), in the L^p-norm setting with $p < \infty$, or $\mathcal{V}^*_\zeta(D_0, E_0)$, as defined in (1.4), when studying L^∞-norms. An interesting question then becomes how one might modify Lemmas 2.2 and 3.4 in such a way that these more strict conditions for the initial data, f_0, may be relaxed so that one may, for example, only require $f_0 \in \mathcal{U}(D_0, E_0)$.

Due again primarily to Lemma 2.2, we also consider only those solutions to (1.12) that are also elements of $\mathcal{T}$ as defined in (1.30). This condition essentially corresponds to the propagation of L^2 moments for solutions to (1.12) with soft potentials and, as mentioned in Section 2.2, it is currently unknown whether or not this assumption is reasonable. Consequently, the problem of L^2 moment generation and propagation is of particular interest for future work.

Lemma 2.2 additionally restricts γ from the full range of soft potentials, $(-3, 0)$, to $\Big(\max\{-3, -2s - \frac{3}{2}\}, 0\Big)$. This assumption is enforced so that we may apply Sobolev embedding (see (2.10)) to prove estimate (2.6b). Thus, in order to obtain results similar to Theorems 2.1, 2.2, and 3.1 for the full range of soft potentials, $\gamma \in (-3, 0)$, a different approach must be taken which therefore results in another research direction.

Finally, each of the main results presented in this thesis represent a priori estimates for solutions to the homogeneous Boltzmann equation (1.12) and hence the important question which remains is whether or not these results can be extended to weak solutions of (1.12).

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