A Tri-ple Analysis of Thinking Multiplicatively around/with *TouchTimes*

by

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Abstract

Multiplication is important for future mathematical competency. However, many students have difficulty thinking multiplicatively. Researchers attribute this difficulty to the widespread use of the repeated addition model when introducing learners to multiplication. In this dissertation, I explore how learners' multiplicative thinking emerges around/with a gesture-based, multi-touch, iPad application called *TouchTimes* (TT), which enables learners to create and manipulate a multiplication model that is different from the repeated addition model.

Learning mathematics by using digital tools is a complex phenomenon. Drawing on my diffractive reading of the theory of semiotic mediation through enactivism, my thesis addresses multiple dimensions of this phenomenon, presented in three separate qualitative studies which followed the methods of videography.

The first study explores the semiotic potentials of TT and pencil-and-paper to engage students with multiplicative thinking. The analysis was conducted with respect to the same multiplication task which was initially designed for TT and modified for pencil-and-paper. The data were created through the recordings of the critical gestures that were required to solve the task. The findings show that two artefacts share some semiotic potentials and also that each of them has some singular contributions to students' understanding of multiplication.

The second study examines how young children make sense of TT when they use a duo of artefacts (pencil-and-paper and TT) back-and-forth. The data were created through a video-recording of a five-year-old child using the duo. The findings were strongly related to those of previous research and showed that back-and-forth use of the duo helped the child bring forth different aspects of multiplicative relationships.

The third study attends to how children collaboratively structure quantities in TT. The data were created by a video-recording of two third-graders' interaction around/with TT. The findings showed that the structures of the students' bodies and TT co-evolved through reciprocal interactions. While at the beginning, these structures were aligned with additive relationships, they were multiplicative towards the end. The peer's body contributed to this shift in various ways other than allowing for the verbal exchange of ideas.

Keywords: Multiplicative thinking; Multi-touch technology; Enactivism; The theory of semiotic mediation; A duo of artefacts; Collaborative learning

Dedication

To my best guide and teacher: my lovely mother Cemile

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This dissertation emerged from my interaction with many people each of whom participated in my learning process in a unique way. I want to thank all these people and the many more I could not include in here.

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Chapter 1.

Introduction

This dissertation contributes to the discussion about the relationship between the various bodies (tools, tasks, people) that participate in the mathematical thinking process. It consists of three published articles, each of which focuses on the same iPad application that was designed to engage students in multiplicative thinking. In this chapter, I provide background information about, yet not limited to, the application, the mathematical concept, and the theories I studied. While I do this, I introduce some personal stories that reveal my own relationship with each of them.

Mathematics education researchers as a community have been interested in the use of tools for teaching and learning since its foundation. Upon analyzing the official journal of The International Commission on Mathematical Instruction, ‘‘*L'enseignement mathematique*”, from its creation in 1908, Maschietto and Trouche (2010) identified an association between the use of various tools and an experimental approach in mathematics teaching, including the idea of the mathematics laboratory. This approach continued with respect to the introduction of computers in mathematics lessons, and other digital technologies (including handheld calculators, software and the Internet), which allowed new opportunities for experimentation and for revitalizing the idea of a mathematics laboratory.

The main components of a mathematics laboratory are the presence of tools that can be used for mathematical experiments or constructions (these were original concrete tools such as mechanical or digital calculators), the presence of an expert guide, and a good set of open activities proposed to students to carry out collaboratively (Maschietto & Trouche, 2010). Participating in such a learning environment, pupils construct meaning through open exploration of a given subject without time pressure.

Following Seymour Papert's proposition of microworlds, which were designed by drawing on the situated and embodied nature of cognition (Healy & Kynigos, 2010), researchers developed an interest in building and studying learning environments that might be described as a mathematical laboratory with digital tools. In these environments, digital tools were used to enable students to explore and make sense of “difficult” mathematical concepts through their bodily interactions with the tools. The following list includes just a few examples of such contemporary works:

- Nathalie Sinclair and her team study geometric objects developed through using dynamic geometry environments, that are “computer programs which allow one to create and then manipulate geometric constructions, primarily in plane geometry” (Rakov, Gorokh, & Osenkov, 2009, p. 278);
- Maria G. Bartolini Bussi and Maria Alessandra Mariotti study number sense developed through using an e-pascal (a virtual counterpart of Pascal’s calculator);
- Dor Abrahamson and his team study ratio developed through using a Wiimote (a remote-controller that allows the user to manipulate items on screen via gesture recognition).

This dissertation focuses on one such kind of digital technology, which has emerged relatively recently—touchscreen technology. In particular, it concerns the iPad application *TouchTimes* (Jackiw & Sinclair, 2018), which was designed to help children develop a robust understanding of multiplication. This dissertation will investigate: (1) the affordances of this application to build meaning for multiplication and develop multiplicative thinking; (2) how a student creates meaning when the specific tasks created for this tool are introduced with corresponding pencil-and-paper activities; (3) how students’ multiplicative thinking emerges as they interact with/around *TouchTimes* (TT) collaboratively when they were given a specific multiplication task. Studying this technology from various, yet related angles, this dissertation follows a manuscript-type dissertation, or a dissertation comprised of “a compilation of research articles” (Paltridge & Starfield, 2007, p. 70) to thoroughly examine both TT’s potential affordances and its actual use in developing multiplicative thinking.

With this introductory chapter, I aim to provide readers with sufficient background information for the following chapters that consists of three research articles. To do that, I introduce the rationales, conceptual frameworks, research questions, and overall methodological direction I employed, respectively. I finish this chapter by describing my positionality as a researcher and outlining the organization of my dissertation.

1.1. Rationales

1.1.1. An Introspection of My Motivation

My interest in educational tools dates back to my undergraduate studies in teaching mathematics. My professors emphasized the importance of active learning by using materials, so I was motivated to apply these ideas in my own classroom. However, this was challenging in Turkey because many schools privileged practicing test items to prepare students for national exams. Fortunately, I found a boutique private middle school where activity-based learning through material use was highly valued. As the only mathematics teacher in the school, I felt alone and lonely at the beginning of my teaching career.

Designing effective mathematics tasks and introducing them in the classroom successfully was challenging, in part because I was not familiar with this type of learning myself. Upon watching how primary school teachers used materials in their mathematics lessons, I noticed that activity-based instructions with materials was open to different interpretations. Rather than using materials to build meaning for mathematical concepts, some teachers were using them only after they gave step-by-step instructions. Students were given various materials with various tasks to apply “freshly learned concepts”. In contrast, I was trying to incorporate the materials and tasks into students’ meaning-making process. Even though I was having difficulties to design such a learning environment, the moments when I experienced the power of thinking with tools (Levy, 1993, as cited in Villarreal & Borba, 2010) to learn mathematics motivated me to pursue a master’s degree in primary education.

I still remember one such joyous moment when I prompted a student, who was known as “mathematically weak”, to figure out rounding numbers to the nearest tens. In order to explain this concept, I used a drawing as in Figure 1.1 and asked the student to imagine placing a ball on a number and seeing where it rounded. After she tried this method with different numbers, I witnessed the student’s excitement about figuring out rounding numbers. Then I decided to build a concrete model made of plastic water bottles to build the broken line and an actual ball which was rolled in the bottles. This way students could experience this method physically, rather than only imagining it mentally.

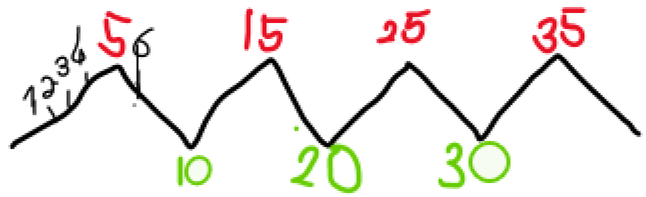


Figure 1.1. **A re-enaction of my drawing for rounding numbers.**

My interest in educational material design continued during my master studies, this time intertwined with technology. I was introduced to the ideas of Seymour Papert, especially the concept of turtle geometry that grew out of the Logo Group at MIT. This concept inspired me to design, though only on paper as a course project, a digital tool (see Figure 1.2) that would help students develop an understanding of integers, including arithmetic operations on them, with which students have difficulty.

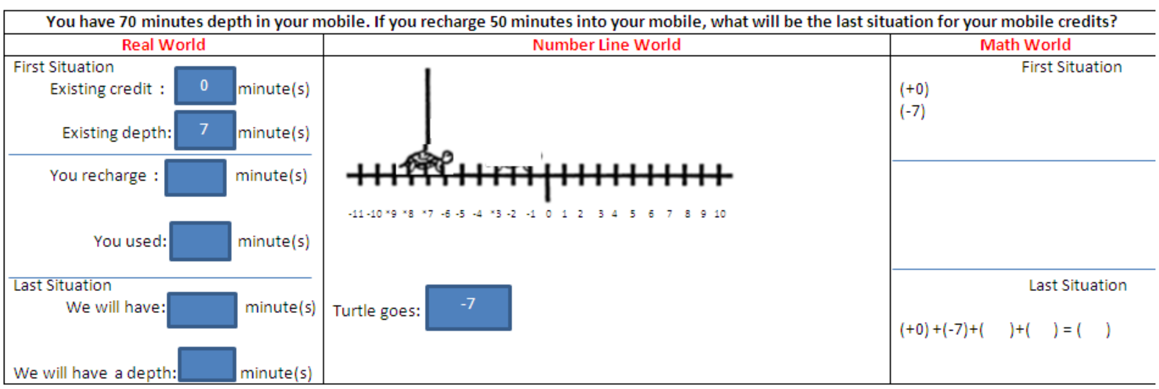


Figure 1.2. **The imagined interface of the digital tool I designed.**

Continuing my education as one of the Ph.D. students of Dr. Nathalie Sinclair, I was involved in a research programme called “Tangible Mathematics Learning”. This programme proposes to provide young learners with an intuitive and embodied interface via the touch-screen iPads enabling them to explore mathematical ideas and express mathematical understandings by using their fingers. It is closely aligned with the pedagogy I wish to enhance. Thus, working as a research assistant in this project, I carried my personal interest into another level by shifting from drafting plans for possible digital tools to studying a real educational tool called *TouchTimes*.

1.1.2. An Inspection of the Motivations behind *TouchTimes*

TouchTimes is designed to mobilise the embodied nature of cognition, particularly in relation to research arising from these three aspects of embodiment:

- a high correlation between spatial reasoning and mathematical achievement;
- the positive impact of gesturing on students' mathematical thinking;
- the research areas that show the kinetic and temporal nature of mathematics.

Drawing on these findings, TT aims to endow children's understanding of multiplication with the representational power of their fingers. The following sub-sub-section elaborates on these areas of research in embodied cognition.

Spatial reasoning and mathematical achievement

The title proposes a concept called spatial reasoning, yet the literature shows that there is not a standardized term for this phenomenon. Although terms such as 'spatial skill', 'spatial ability', and 'spatial thinking' are related, the concept of 'spatial reasoning' draws attention to the inferencing nature of the activities such as imaging, transforming and locating. Thus, reasoning is preferred to the term "ability", which implies an innate trait that cannot be changed through learning (N. Sinclair, personal communication, November 15, 2018). Indeed, research on spatial reasoning shows that it is malleable (Verdine, Irwin, Golinkoff, & Hirsh-Pasek, 2014; Zhang, 2016) and a hypothesis of the TT design is that it can draw on and support spatial reasoning.

A related concept is that of visual-spatial skills, which Zhang (2016) defined as, "the ability to understand the visual-spatial relations among objects" (p.179). Rather than defining what it is, studies mostly explain what it does. For example, Verdine et al. (2014) stated that, "spatial skills support the process of representing, analyzing, and drawing inferences from relations between objects" (p. 39). Similarly, Al-Balushi and Al-Battashi (2013) referred to spatial skills as involving "the retrieval, retention and transformation of visual information in a spatial context" (p. 14). Those descriptions associate spatial skills with manipulation of existing visual information. In addition, Lean and Clements (1981) emphasized "the ability to formulate mental images" (p. 267) as another aspect of spatial skills.

As the plural form of the term “spatial skills” implies, the phenomenon is separated into different categories. According to Linn and Petersen (1985, as cited in Zhang, 2016), there are three types of spatial skills: spatial perception, mental rotation, and spatial visualization. Spatial perception refers to “identifying spatial relations among task components in spite of distracting information” (p. 180). Mental rotation means “mentally rotating a 2-D or 3-D object” (p. 180). Spatial visualization denotes “processing complicated, multi-step manipulations, often analytical, of spatial information” (p. 180).

Studies indicate a positive correlation between the spatial skills and mathematics performance for different age groups from university down to pre-school students. Battista, Wheatley and Talsma (1982) examined the relative importance of spatial visualization (yet another term!) for pre-service teachers’ achievement in a geometry course. They found that nearly a third of the variance in achievement in geometry exams is accounted to spatial visualization (operationalised as mental rotation). Guay and Daniel (1977) distinguished low spatial ability from high spatial ability and found that both are positively correlated with mathematics achievement of elementary school children (grade 5 to 7). Gilligan, Flouri and Farran (2017) investigated the predictive role of spatial skills on 7-year-old children’s performance on comprehensive mathematical topics. They found that spatial skills (operationalised as recreating 3D models) in age 5 explained 8.8% variance in mathematics performance in age 7. Zhang (2016) found that the spatial perception (processing spatial relations between visual forms) was positively correlated with early number competence.

Dehaene, Bossini, Giraux, and Pascal (1993) brought some explanation about the relationship between spatial reasoning and mathematical thinking. They identified an association between space and numbers. When participants were asked to classify numbers by placing them right or left based on a given rule (odd numbers to the right and even numbers to the left), the response rate was shorter for the large numbers that were placed on the right and the small numbers that were placed on the left. Dehaene et al. (1993) attributed this association to a mental number line, which might help learners in their calculations various ways (Marghetis, Núňez, & Bergen, 2014) However, this association between number and space was not universal. The direction of the aforementioned association between numbers and space reversed for the participants who are familiar with right-to-left writing. This shows how bodily acts may shape the conceptualization of mathematical objects by shaping the spatial reasoning (Newcombe

& Frick, 2010). The following section elaborates on this relationship by focusing on the gestures as a specific form of bodily acts.

Gesturing and mathematical thinking

Gestures are generally seen as a tool to represent ideas. For example, McNeill's (1992) categorized gestures into four groups: beat gestures reflect the tempo of the speech; deictic gestures indicate directions or refer to something mentioned previously; iconic gestures represent an object; metaphoric gestures represent an abstract idea. Drawing on this typology, Alibali and Nathan (2012) proposed these gestures as strong evidence for the embodied nature of cognition. Deictic gestures manifest "a mapping between an abstraction and a more concrete, familiar referent" (p. 250). Pointing to the geometrical shapes drawn on a surface while explaining the relationships between their parts is an example for deictic gestures. Representational gestures, which have both iconic and metaphoric character, reflect "mental simulations of action and perception" (p. 252). For example, keeping one's arm upward at an angle and moving it up and down while explaining the change in the slope of a line is described as a representational gesture. Metaphoric gestures manifest the conceptual metaphors that map abstractions to specific bodily experiences. For instance, Alibali and Nathan (2012) identified a hand making hops on a table as a gesture that manifested the conceptual metaphor of time as a movement in space.

According to Radford (2009), the sole emphasis on the expressive role of gestures draws on the theoretical conceptions of thinking as a mental phenomenon. Instead, he conceptualized thinking as a material phenomenon that occurs through intricate interplay among language, body and tools. Therefore, gestures function as one of the "genuine constituents" of thinking, not only as its manifestation (Radford, 2009, p. 113). In alignment with Radford's conceptualization of gestures, Streeck (2009) defined them as:

> […] a constantly evolving set of largely improvised, heterogeneous, partly conventional, partly idiosyncratic, and partly culture-specific, partly universal practices of using the hands to produce situated understandings. (p. 5)

According to this definition, gestures are not only "in the air" forms of communication, but also "on the ground" forms of meaning-making. Sinclair and de

Freitas (2014) were particularly interested in indexical gestures and, following Peirce, their material nature. They drew on Charles Sanders Peirce, whose classification of signs was the basis of McNeill's work, to recover the importance of the trace left by indices, such as the smoke that indexes the fire, or the footprint that indexes the passage of a human. Thus, in TT, a screen gesture, like tapping one's finger, can index the increasing of the unit, but it also leaves a visible trace on the screen.

When the gesture is used "in the air", perhaps to explain to a classmate, then the gesture may no longer leave a visible trace. However, recent technology shows that even these in-the-air gestures may leave coloured traces, as in the case of the "1 Year to Go!" ceremony in 2019, where traces of an athlete's movement were created (see Figure 1.3). These traces index the movement of the hand. I point this out as further evidence of the changing nature of signs as a function of technology, and the materiality of both communicative and meaning-making gestures.

Figure 1.3. Capturing traces of gestures in the air.
Credit: Tokyo 2021 Olympics one year ago ceremony "1 Year to Go!" Opening Performance (Tokyo Olympics 2020 Reloaded, Aug 2, 2019)

There are several examples that illustrate how gestures participate in arithmetic thinking. Radford (2003) documented how one eighth-grade student identified a pattern between the consecutive figures in a sequence as he pointed the figures with his pencil and verbalized the arithmetic actions that reveals the pattern. This "crude pointing" gesture allowed the students to go beyond the given figures and to find the number of items in any specific figure (p. 47). In this particular case, the interaction between gesturing and speech was explained as a means of "making apparent the new relations and objects that needed to be put forward in the generalizing activity" (p. 65).

Marghetis, Núñez, and Bergen (2014) showed how hands participate in arithmetic operations. 44 undergraduate students were shown single digit addition and subtraction number problems (e.g., 4 + 3) and two responses in the top corners of a computer monitor. They were asked to select the exact solution among these responses by using a computer mouse. Their hand movements were recorded based on the streaming *x*, *y* co-ordinates of the computer mouse cursor. These recordings revealed a systematic deflection in hand movements; to the right during addition and to the left during subtraction.

Berteletti and Booth (2015) provided neuroscientific evidence for the link between finger movement and arithmetic thinking, especially in subtraction and multiplication. 39 children between eight and thirteen years of age were shown a subtraction or multiplication word problem and asked if the outcome was correct. During this task, they were in the fMRI scanner and responded by pressing a button. The results showed that finger motor areas in the brain were activated only during subtraction problems. Berletti and Booth (2015) attributed this to how these two operations were processed. While subtraction relies on quantity manipulation, multiplication relies on verbal retrieval, which indicates rote memorization of multiplication facts.

Björklund, Kullberg and Kempe (2018) investigated how young children use their fingers to manipulate quantity during subtraction. In this study, 4–5-year-olds were given a word problem verbally and observed while they solve the problem. Researchers identified three roles of the fingers in solving the subtraction problem: (1) counting the fingers as single units; (2) structuring the parts and the whole; (3) combining these two functions together. They associated the success in subtraction with the second role of fingers, which allowed a simultaneous experience of parts and the whole. Children who used their fingers this way either showed the given part in the problem with their fingers by folding them and raising the remaining fingers at once, or vice-versa. This study supports Bender and Beller's (2012) hypothesis about the role of fingers in learning mathematics. These authors documented the differences between the structures of several finger counting systems and proposed that a comparative study of finger counting systems might reveal important implications for their influence on understanding of various mathematical topics such as infinity, base system, and mental number line.

In light of the previous studies, the design of TT considers the generative role of fingers as constituent of cognition and aims to prompt learners to use their fingers in specific spatio-temporal configurations. This would allow students to experience multiplicative relationships and develop multiplicative thinking through their hands.

1.2. Multiplication and Multiplicative thinking

Multiplication is a mathematical concept first introduced in the early grades (2nd or 3rd grades) of the elementary school in British Columbia. At these grades, it is defined as an arithmetic operation on natural numbers. As the grade level increases, this same operation is also defined on other number sets such as rational numbers, irrational numbers, integers, and on various other mathematical objects such as functions and matrices. Moreover, this concept is important to gain understanding in other mathematical concepts such as proportion (Hino & Kato, 2019; Singh, 2000) and fractions (Hackenberg & Tillema, 2009). Therefore, it is important to understand this concept very well for future achievements in mathematics. However, studies show that students have certain difficulties with multiplication, not only to recall multiplication facts and compute multiplicative expressions, but also to solve word problems.

Students in the lower elementary school tend to add the given numbers in multiplication problems, while in the upper elementary level, they tend to use multiplication inappropriately for the additive situations (van Dooren, de Bock & Verschaffel, 2010). The numbers given in the problems (van Dooren et al., 2010) and key words such as twice (Carrier, 2014) play a role on which operation students choose to solve the word problems. This shows that, rather than using the mathematical structures underlying the word problems, students respond to the mathematical tasks based on the superficial task characteristics.

Some intervention studies demonstrated improvement in students' achievements when they were faced with multiplication problems (e.g., Agaliotis & Teli, 2016; Bakker, van den Heuvel-Panhuizen, & Robitzsch, 2015; Zhang, Xin, & Si, 2013). However, even though a student can use multiplication algorithms properly in multiplicative situations, this does not guarantee that they are engaged in multiplicative thinking (Carrier, 2014), which is necessary to master the conceptual field of multiplicative structures (Vergnaud, 1988). Vergnaud claims that, like other conceptual fields, the field of multiplicative structures consists of, “a set of situations that make the concept meaningful, […] a set of invariants (objects, properties and relationships) that can be recognized and used by subjects to analyze and master these situations, and a set of symbolic representations that can be used to point to and represent these invariants and therefore represent the situations and the procedures to deal with them” (p. 141).

He also proposed that a single concept usually develops not in isolation, but in relationship with other concepts. This explains why students who understand various elements of multiplication, such as factors and multiples, the commutative property of multiplication and the inverse relationship between multiplication and division, might have difficulty to make connections between these concepts (Hurst, 2017), due to a lack of familiarity with the multiplicative structures that emerge from the intricate relationships among these individual elements.

Students are more familiar with additive thinking, in which they measure an amount of quantity by using a single unit-count and compare the amount of two quantities in terms of the difference, not the ratio, between them, which is also measured by the same single unit-count that measures both quantities.

When students are given problems that can be solved both by additive and multiplicative approaches, they tend to approach the quantities additively. Upon facing quantities in the problems, they are more prone to focus on the difference and the sum between the quantities, rather than the multiplicative relationships among them (Degrande, Hoof, Verschaffel, & Dooren, 2018). In the former case, one structures the quantity based on a single unit count. In the latter, one needs to structure a quantity considering the intricate relationships between multiple unit-counts (more in Chapter 2).

Figure 1.4 illustrates the differences between these two approaches to the quantity.

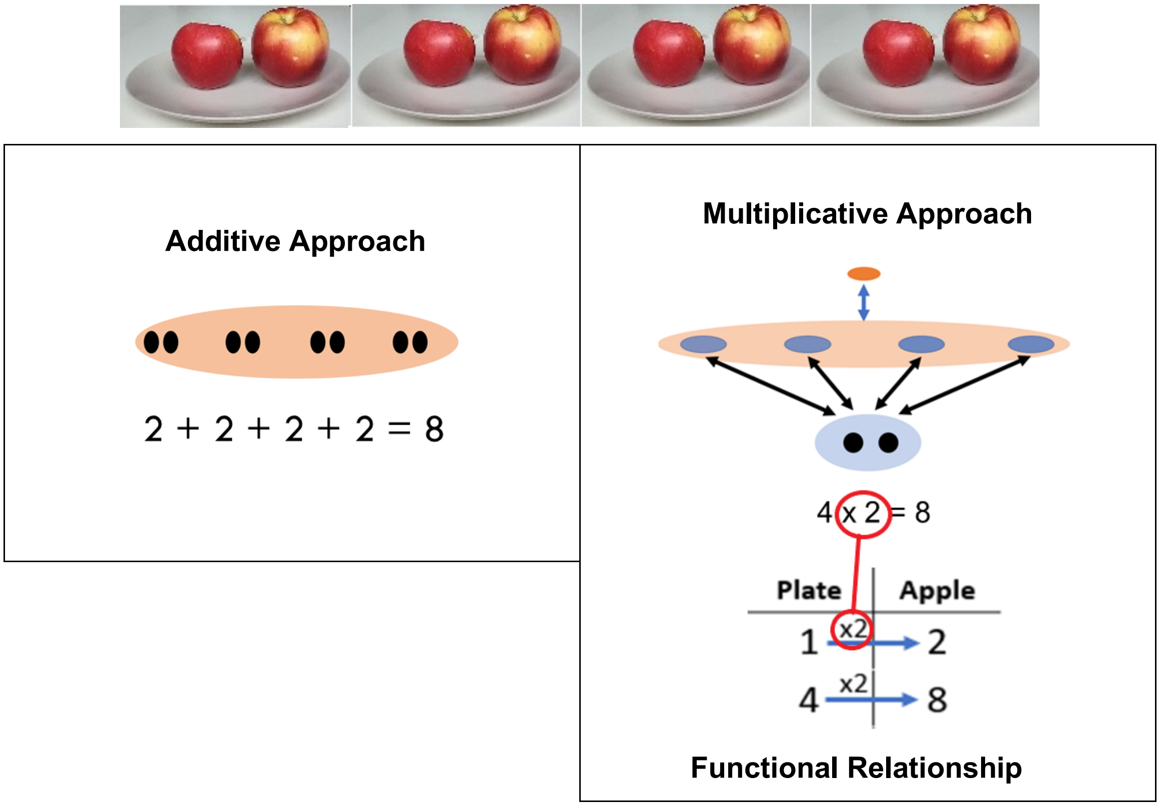


Figure 1.4. Additive and multiplicative approaches to structure quantity.

Looking at the image of the apples in Figure 1.4, we can quantify them in two different ways. First, we can add 2 repeatedly four times. In this case the only quantity we play with is the number of apples which is represented by 2. We add them four times because there are four plates, but the number of plates is not visible in this equation, unlike the number of apples. So, the emphasis is on the apples. The unit we are counting with is 1 apple is represented in the model with the black dots.

If we want to find the total number of apples by multiplying, we do something different. We multiply 4 by 2. In this equation, two different quantities are emphasized at once. Unlike in the additive case, 2 does not represent the number of apples, but the number of apples per plate, a relationship between the plates and the apples. Vergnaud (1988) called this a functional relationship. According to this relationship, one plate corresponds to two apples and this correspondence holds for each plate. Thanks to this relationship, we can calculate the total number of apples not by counting the apples one by one but by counting the plates. Davydov (1992) describes this situation as the indirect measurement of quantity through the transfer of unit count. In brief, when we think multiplicatively, we simultaneously play with two distinct unit counts that have a functional relationship.

Many school curricula (e.g., in British Columbia and Turkey) are aligned with Vergnaud's notion of conceptual fields of multiplicative structures as they introduce multiplication to students in relation to various concepts such as skip counting, arrays, division and repeated addition. Among these concepts, the last one seems to be the fundamental concept on which multiplication is based. However, an over-emphasis on the repeated addition model is thought to have a negative effect on students multiplicative thinking (e.g., Confrey, 1994; Greer, 1994; Maffia & Mariotti, 2018; Schwartz, 1988; Vergnaud, 1988).

The association between multiplication and repeated addition was also very strong for me. When Dr. Sinclair asked me at our very first research meeting "what is multiplication?", I immediately answered as "repeated addition". Moreover, my early writings on multiplication included sentences like "multiplication is repeated addition". Despite this association, which was suggested as a possible limitation in mathematical achievement, I consider myself mathematically literate (I successfully completed my mathematics classes and obtained high marks on the national exam). So, it seems that thinking of multiplication as repeated addition may not be so problematic. Either I managed to pass the exams based on my success in following and applying algorithms for multiplication or, as the above examples of apples show, my spatial skills, especially the spatial perception, helped me to bring forth two-unit counts (as the model on the right in Figure 1.4) even though I engaged with images (as apples on plates in Figure 1.4) that was most likely associated with the repeated addition model (as the model on the left in Figure 1.4). Based on the findings of this study, I am leaning to the second possibility and argue that it is not the model itself, but how learners engage with it that plays a role in building understanding.

When the focus was on finding an answer to multiplication sentences, introducing multiplication through the repeated addition model may work well, because the solution to multiplication sentences can be obtained through repeated addition. This equivalence between the repeated addition and multiplication sentences might lead to the conclusions such that multiplication is a short-cut for repeated addition. Such conclusions only reduce multiplication to an algorithm, obscuring the underlying multiplicative structures that are quite different from the relationships among the addends and the sum of an addition.

When learning multiplication through the repeated addition model, some students might not have the opportunity to build a new approach to structure quantity multiplicatively, which is very important for understanding more advanced mathematical concepts in the upper grades such as linear functions, vector spaces and dimensional analysis (Vergnaud, 1988). The repeated addition model may assist in calculating the answer for the multiplication sentence, but it may not help them develop multiplicative thinking. Therefore, alternative models to repeated addition may be helpful.

There are different representations suggested in the research literature as alternatives to the repeated addition model, such as arrays, double number lines or graphs (more in Chapter 4). In addition to using these static models, Kaput (1985) proposed presenting them on a computer screen simultaneously. This would help students make connections between the models by manipulating one of them and monitoring the change in all representations at once. Moreover, variations of all models would emphasize the invariant relationship between the two unit counts of multiplication. In alignment with, yet not limited to, Kaput's proposition, this study examines how *TouchTimes* provides students with a means to develop multiplicative thinking by using their fingers to manipulate multiple representations of multiplication in a world that embeds multiplicative structures.

1.3. Studying *TouchTimes*

TouchTimes consists of two worlds called Zaplify and Grasplify (more detailed descriptions of these worlds are in Chapters 2 and 4). In the Zaplify world, learners can create horizontal and vertical lines that look like lightning rods which create orange sparks at the points where they intersect (see Figure 1.5a). The Grasplify world allows users to create a collection of individual circular discs with one hand and the multiple copies of this collection with the other hand simultaneously (see Figure 1.5b).

Figure 1.5. Making 3 x 4 a) in Zaplify, b) in Grasplify.

I studied both worlds, but I am more attracted to Zaplify as a researcher for some cultural reasons. In particular, I made exciting connections between the terminology used for multiplication in Turkish and the interface of Zaplify, which illustrates multiplication with crashing lightning rods. The Turkish term for multiply is “çarpmak”. The word “çarpmak” means ‘to hit’, or ‘to crash’ in daily usage, which I associate with the crash between the lightning rods of Zaplify. Also, this Turkish word is used for electric shock, which I associated with the lightning rods of Zaplify. For example, the following sentence and its Turkish translation illustrate how this word is used in these contexts.

He got an electric shock from one of the wires.

Kablolardan birinden dolayı adamı elektrik çarptı.

In the Turkish translation, “elektrik” is the subject of the sentence which hits (çarpmak) the person.

Mustafa Kemal Atatürk, the founder of the Turkish Republic, translated this word into Turkish from its Arabic counterpart. The Arabic word for multiplication is “darp”, which also means ‘to hit’, or ‘to crash’. This association between multiplication and hitting may not be random. Indeed, the book *Fleeting footsteps: Tracing the conception of arithmetic and algebra in ancient China* (Lam & Ang, 2004) documents the multiplication algorithm step-by-step both in Chinese (sun zi suanjig, around the 3rd century AD) and in Islamic resources. The methods are very similar in both cultures. Interestingly, this method is devised by manipulating physical rods in China. Maybe this is the reason why it is called “carpma” (to hit). In order to multiply the numbers, one must hit the relevant rods to each other to track the numerals used in the computation.

This double meaning of the term “multiply” in these languages also reminded me of a multiplication technique that might indicate a relationship between this term and crash. In this method, which is sometimes referred as Japanese Multiplication Method, the numbers are represented with parallel line segments according to base ten. The digit places are identified by the space between the separate chunks of lines (see Figure 1.6). When a number is multiplied by another one, the line segments of one number are drawn in a way that they intersect with the line segments of the other number (see Figure 1.6). The digits of the product are obtained through the number of intersection points accumulated in different sections of the diagram (see Figure 1.6). Looking at this diagram, I imagine the intersection points as the places where the rods crash each other.

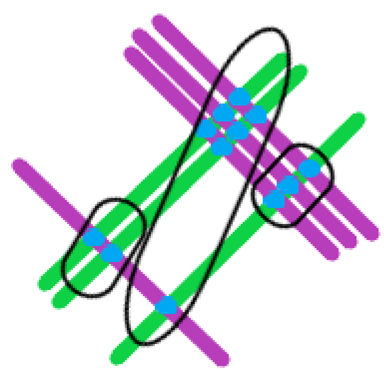


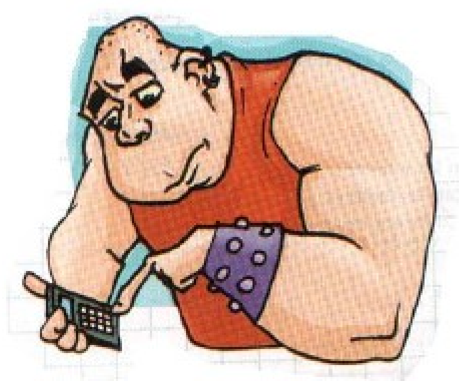
Figure 1.6. 13 x 21 in Japanese multiplication method.

In addition to this possible association between crashing rods and multiplication, I identified another resemblance between Zaplify and the Turkish terminology for multiplication. Unlike in English, the Turkish language has no separate terms for multiplicand and multiplier. They are both called “çarpan”, which phonetically means the object that hits. I associated this with the symmetric nature and functioning of lightning rods in Zaplify. Unlike in the Grasplify World, each hand creates the same type of objects, but in different orientations.

All of these associations prompted me to conduct some research in Turkey to study the relationships between language, multiplication models and multiplicative thinking. I collaborated with an elementary school teacher for six weeks to teach second graders multiplication. In the mornings, she introduced the topic the way she used to do in previous years. In the afternoons. I gave students relevant Zaplify tasks to work in small groups. In between, we either debriefed about our lessons or planned the next lessons. When we talked about the affordances of Zaplify in one of these meetings, one of her first responses was about how Zaplify can create models faster than drawing them on the board.

TT definitely allows users to create numerous multiplication sentences in a short time, yet it consists of other important affordances. However, its interface might obscure them from the user. The two worlds of TT might first recall two different static representations of multiplication: equal groupings and arrays, respectively. Therefore, a new technology, which looks like the virtual counterpart of the existing concrete models, including the drawings on paper, might raise the following questions: Why do we need this? Cannot we do the same with the concrete models? I encountered these questions multiple times in different contexts and they led me to study the similarities and differences of engaging with the same model in two different settings, computerized and non-computerized. In the first article I included in this dissertation, I use the Theory of Semiotic Mediation (TSM) to study some important differences in how students use their bodies to create the same model in distinct settings and how these differences might potentially result in different meanings.

After the computerized artefacts are introduced into the classrooms, their implications are discussed in contrast to non-computerized artefacts, prompting teachers to choose one over the other. Even the mathematics textbooks and some educational magazines participated in this debate through the images they used (see Figure 1.7). The image on the left depicts “a rather unintelligent-looking man holding a calculator” (Villarreal & Borba, 2010, p. 57) and it possibly reveals a negative attitude towards calculator for learning arithmetic operations. On the other hand, the image on the right (see Figure 1.7) illustrates a classroom where a classmate is accused of using pencil-and-paper which is to be considered inappropriate (Villarreal & Borba, 2010).



"Teacher, Gonzalez is using pencil and paper!"

Figure 1.7. Images represent the debate between pen and paper and computer use.

Maschietto and Soury-Lavergne (2013) shifted this debate. They proposed that introducing a digital artefact with its physical counterpart may enrich students' experiences and prompt them to transfer schemes from the physical artefact. Thus, it may extend students' learning experiences by raising an awareness of the epistemic value of their actions with the physical artefact. This assumption about how the duo of artefacts extend students' learning enforces a specific order to introduce the duo to the students: first the physical artefact and then its digital counterpart. This order might withhold the affordances of the physical artefacts, which the digital counterpart cannot provide.

In the second article, drawing on TSM, I studied how one learner built meanings by interacting with a duo of artefacts that was introduced in a cyclical manner (digital-physical-digital-physical…). The findings show that the reciprocal transfer of signs from one medium to the other allowed the student to enrich the meaning she or he created for multiplicatively structured quantities in *TouchTimes.*

TSM hypothesises that students learn mathematics by creating specific signs while they manipulate mathematical artefacts in a social context (more on this in the next section). So, in my first two articles, this theory helped me identify which signs were created. However, it did not assist me to understand the processes that led to these signs that can foster a specific way of mathematical thinking in these interactions. The third study examines this process by drawing on enactivism (more on this in the next section) and sheds some light on how signs emerge and how various bodies participate in one's learning process.

1.4. Theoretical Underpinnings

Since computerized technology entered the classrooms, many theoretical frameworks have been used to study learning mathematics with technology (Drijvers et al., 2010). Some theoretical constructs were adapted from existing higher order theories used in mathematics education (e.g., cognitive, sociocultural and enactivist theories) and tailored for investigating mathematical learning and teaching within technological environments (e.g., webbing and situated abstraction, milieu, perceptuo-motor integration, semiotic mediation). The multiplicity and isolated use of these theoretical frames started to raise some concerns among researchers for being an obstacle to explain all phenomena in the complex setting of learning mathematics in a technology-rich environment (Artigue & Mariotti, 2014; Donevska-Todorova & Trgalova, 2018; Drijvers et al., 2010).

These concerns indicate a movement towards networking theories in the mathematics education research. So, I draw on various theoretical constructs to study different aspects of mathematics learning with/around TT. I chose these constructs both from a grand frame that explains learning in general, and from an intermediate frame that explains learning mathematics by using a tool (Kieran, Doorman, & Ohtani, 2015). These theories are enactivism (Maturana & Varela, 1987) and the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), respectively.

1.4.1. Enactivism

I first encountered enactivism during Dr. Alf Coles' seminar at Simon Fraser University in 2018. The seminar was about re-imagining mathematics pedagogy from an enactivist standpoint. An image in his slides immediately attracted my attention back then (see

Figure 1.8) and I still remember it after three years. The image was depicting a bacterium's movement with and without sugar molecules in the environment. It was unusual to see that learning, which is a phenomenon that is mostly attributed to humans, was explained by referring to a one-cell organism's behaviour.

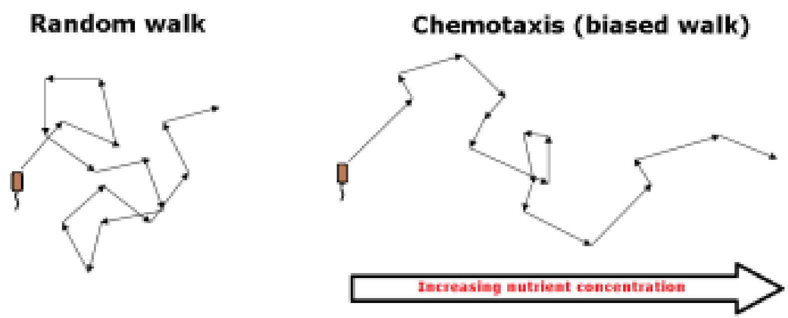


Figure 1.8. Chemotaxis of a bacterium.
Credit: https://thisscienceiscrazy.wordpress.com/tag/random-walk/

This reference to a bacterium's movement to explain learning makes me very excited because this perspective supports my personal beliefs about animal intelligence. I always found it unfair that complex animal behaviour like bees constructing hexagonal structures are mostly associated with genetic coding but not cognition. In a way this association creates a hierarchical relationship between animals and humans, favouring the latter. Seeing the image that acknowledges a bacterium's behaviour as an example of cognition, I become interested in enactivism as explained by Maturana and Varela (1987).

According to enactivism, living beings have specific organizations and structures. Organization is "the relations that must exist among the components of a system for it to be a member of a specific class. Structure denotes the components and the relations that actually constitutes a particular unity and make its organization real" (Maturana & Varela, 1987, p. 43). Organization is specific to a class of unities, while structure is specific to a particular unity within the class. For example, when we talk about a *person*, we distinguish the person as a member of the class of human beings, as we recognize the parts of her/his body and the interaction within that body that keeps the person alive and makes her/him human. However, the specific features of the structure of the person, such as her hair colour, height, and weight, do not change her/his humanness.

Living beings are autopoietic unities which are organized in a way that they maintain their existence by changing their own structure in relation to their environment. In this relationship, the changes in the environment might trigger some changes in the structure of the organism. Even though this change in the organism's structure is triggered by the environment, the organism's structure itself determines the range of its changes. Similarly, these changes in the structure of the organism can trigger some changes in the environment. If these reciprocal changes maintain living being's existence, organism's actions are described as effective and they indicate learning. This process of structural congruence between the organism and its environment is called structural coupling.

Varela, Thompson, and Rosch (1991) based enactive approach on two underlying premises:

- “perception is consisted in perceptually guided action;
- cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (p.173).

Even though these premises were not explicitly stated in Maturana and Varela (1987), I can relate them to how they explained the bacterium's movement towards sugar based on the entanglement between action and perception and the emergent nature of cognitive structures. The flagellum of the bacterium is a tail-like structure that extends out from the base of the cell. It rotates in two directions. While bacterium moves forward as the flagellum rotates in one direction, the organism tumbles at the same place as it rotates on the opposite direction. The membrane of the bacterium has special molecules that are susceptible to sugar molecules. When there is a change in the sugar concentration around the bacterium, the cell structure changes because of the interaction between the bacterium's membrane and the sugar molecules, rotating the flagellum in a different direction.

This correlation between the sensory and motor surface of the bacterium is recurrently established at each moment and allows the emergence of a discriminatory behaviour as the bacterium heads towards the sugar. Thus, as the bacteria cognizes the sugar, it acts effectively to approach the sugar[1]. Even though I write the account of this correlation between the sense and the motor surfaces in a chronological order, it occurs simultaneously. The sense and motor surfaces are not discrete structures that function sequentially. They are opposite ends of a continuous structure. While this continuity is straightforward in one-cell organisms, in more complex organisms, it is maintained through the complex neural system.

The enactivist premise on the emergent nature of cognitive structures shifts the acquisitionist approach of learning to a more participationist one. Rather than emphasizing the constructed mental structures of learners, it associates knowing with doing and being. As the -ing form in “knowing” indicates, learning is conceptualized as an activity that happens in the present, as opposed to the noun “knowledge” that implicates a fixed mental entity acquired by learners. Therefore, enactivism acknowledges the body as one of the constituents of learning. This approach has recently been adopted by many researchers in the field (for examples, see Chapter 4).

Adopting an enactivist approach in my research is an experience that emotionally influences me. Coming from a constructivist tradition, I was always uncomfortable to study schemes which are operationalized as invisible mental structures that cannot be observed but can be inferred from learners’ verbal accounts or from their techniques, “a manner of solving a task” (Artigue, 2002, p. 248). Theorizing cognition as an effective action helps me feel a bit at ease because it allows me to study learners’ actions not as the representation of their mental schemes but as the constituent of their learning. Even though both participationist and acquisitionist approaches study the observable actions to examine learning, differences in the epistemological approaches might result in distinct educational goals and curricular requirements (Drijvers et al., 2010). For example, considering techniques as generative, not representative, elements of learning may encourage curriculum designers to hold bodily activities as the main requirements of mathematics lessons.

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[1]While accounting for cognition, Maturana and Varela are very careful to distinguish the operation of an organism from an observer’s semantic description of the behaviour. While moving its flagellum according to the sugar concentration is the operation of a bacterium, labelling this behaviour as ‘approaching sugar’ is an observer’s semantic description of the bacterium’s movement.

Whether seen as the counterparts of schemes or not, techniques are found to be essential in learning because “technical and conceptual aspects are closely related” (Drijvers et al., 2010, p. 110). For example, one can draw a circle either by circumscribing a bottle cap or by turning the unfixed end of a compass around its fixed end. However, the user experiences the constant distance between the centre and the circumference of the circle only with the second technique which may allow the user to perceive circles as the collection of points equidistant from a fixed point (Bartolini Bussi & Mariotti, 2008). However, this connection between the technical and the conceptual is not trivial.

Researchers have proposed various theoretical metaphors to explain this connection. According to sociocultural perspectives, a more knowledgeable person acts like a scaffolder and enables learners to achieve a task that they would not be able to accomplish alone (Wood, Bruner, & Ross, 1976). In addition to an adult’s scaffolding, tools play an important role to mediate higher forms of mental activity including mathematical ideas (Vygotsky, 1978, as cited in Bartolini Bussi & Mariotti, 2008,). Noss and Hoyles (1996) criticized these two metaphors for attributing agency to the entities external to the learner and proposed another metaphor called webbing. It is a support mechanism that learners can draw on to understand mathematical concepts. It is made of connections embedded in the structure of an environment. However, rather than a physical phenomenon, it denotes a cognitive construct that is a product of learners’ and others’ (for example the designers of the setting) current understanding of it. Thus ‘webbing’ acknowledges the agency of learners and that of the environment.

According to Mariotti (2002), these metaphors effectively highlight the potential of an artefact in learning mathematics, yet fail to explain the complete story of how and why these tools enable learners to access mathematical ideas. Based on this concern, Bartolini Bussi and Mariotti (2008) proposed the theory of semiotic mediation (TSM) which integrated Rabardel’s instrumental genesis with Vygotsky’s approach to artefacts to elaborate on the link between artefacts and mathematical knowledge (more explanation in Chapter 2).

1.4.2. The Theory of Semiotic Mediation

According to Rabardel (1995, as cited in Bartolini Bussi & Mariotti, 2008), an artefact is a concrete or symbolic object. When it is used in a specific way to achieve a specific task, an instrument emerges. The instrument is a psychological construct that consists of the artefact and individual schemes for a given task. The transformation of artefacts into instruments is called instrumental genesis and it consists of two processes: instrumentation and instrumentalization. The former refers to the emergence of utilization schemes; the latter refers to recognizing different components of the artefact. These processes are subject to the characteristics of the context within which the artefact originates.

Individuals may transform several artefacts into mathematical instruments if they get into a situation where they need to use mathematical relationships. For example, during the Covid-19 pandemic, protecting social distance in the community parks has become the responsibility of the local governments. I saw some municipal workers using a string which was tied to a stake and a paint brush at the opposite ends to enclose a circular area with 2 metre diameter. The fixed length of the string and the fixed location of the stake seems to have become salient for these people and spur them to combine these artefacts to enclose an area whose boundary is exactly 1 metre away from the location of the stake at each point. Thus, the bundle of string, the stake, and the paint brush became a tool (like a compass) to draw circles in the parks during the pandemic.

An artefact may be associated with mathematical concepts that are already familiar to its user as in the previous example. Artefacts can also be used to help users access mathematical concepts that are novel to them. According to the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) this happens when individuals collaboratively manipulate an artefact that embeds mathematical relationships (more explanation in Chapters 2 and 3). As students use tools to solve a task, they create some signs both to manipulate the artefact in a specific way and to communicate with others. These signs are firstly related to the specific aspects of the artefact, thus enable students to create a personal meaning for the artefacts. Then these signs *evolve* into signs that are related to mathematical knowledge shared by the mathematical society (see Figure 1.9).

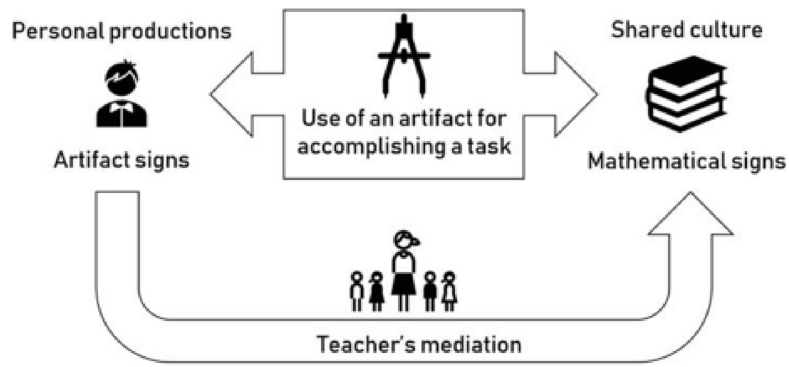

Figure 1.9. Accesing mathematical concepts through artefacts.
Credit: Maffia & Mariotti (2020, p. 27).

Evolution is a term that I feel comfortable to use in a biological context. It is defined as "change in the heritable characteristics of biological populations over successive generations" ("Evolution", 2020). However, seeing this term in relation to the TSM challenged my familiarity with the concept. Does 'the evolution of signs' mean that individuals first create only artefact signs and then create only mathematical signs? Or is it the change in an individual's phenomenological experience with respect to the created signs such that the same sign first refers to the use of an artefact and then to the mathematics culture? Or is it the creation of mathematical signs in addition to artefact signs and using them in relation to each other? Since, according to TSM, the evolution of signs is key to learning mathematics, I have found it important to ask the right question to understand this theory correctly.

An important question posed in Mariotti (2009) directed my attention to learners' phenomenological experiences to explain the process of evolution: "how may personal meanings arising from the use of a certain artefact for the accomplishment of a task become mathematical meanings for students?" (p. 429). For a meaning to change, the interplay between the signs should also change because TSM does not make a strict separation between signified and signifier, and rejects the assumption that meanings exist independent from their signifiers (Mariotti, 2009). Therefore, it is important to understand how signs are created. At this point, TSM draws on Vygotsky's theoretical constructs of internalization and mediation (Mariotti, 2007).

According to Vygotsky, signs are the products of the process of internalization, which is described as the individual elaboration of previous socially lived experiences.

Social cognitive functions that are active in collaboration become internal cognitive functions through internalization.

Even though this description emphasizes social activities as the prerequisite for meaning making and cognitive development, Vygotsky refers to a specific type of social experience in which the interlocutors have different statuses. There must be a collaboration between “one individual, whose cognitive attitude presents a potential in relation to change, and another individual (or a collectivity) who intentionally cooperate to accomplish a task or to pursue a common aim” (Bartolini Bussi & Mariotti, 2008, p. 749). In other words, individuals develop higher cognitive functions as they collaborate with more knowledgeable interlocutors. During this collaboration, signs are used both to accomplish a task and to communicate. It is the latter aim that contributes to the cognitive development because individuals must interpret the signs created by others and respond to them appropriately to communicate. Therefore, the system of signs and semiotic processes constitute the basis for internalization.

For internalization to happen, certain semiotic processes must be stimulated. Considering mathematics learning, TSM suggests that this stimulation is achieved by the semiotic mediation of the artefact and the cultural mediation of teachers. Both mediation processes involve a mediator whether semiotic or cultural, a learner that is subjected to mediation and circumstances for mediation such as the means of mediation and the location in which mediation occur.

A teacher uses an artefact as a semiotic mediator as long as the tool can evoke both personal and mathematical meanings. Such tools are said to have a semiotic potential. Even though an artefact has such a potential, as a cultural mediator, the teacher plays a central role in the evolution of mathematical meanings from personal meanings by unfolding the potential of the artefact through certain activities. TSM categorizes such didactic activities into three groups that should be conducted respectively in a cyclical manner.

The didactic cycle starts with the introduction of an artefact and a task to students. The students use the artefact to solve the mathematical task collaboratively in small groups. This type of social activity promotes the emergence of artefact signs during the achievement of the task. The second phase of the didactic cycle requires that the students individually engage in specific semiotic processes by producing and recording signs related to the use of the artefact, such as writing a reflection on the activities conducted at the previous phase. Even though the signs created in this phase are related to the use of the artefact, as in the previous phase of the didactic cycle, this semiotic process is described as “a first detachment from the contingency of the situated action” (Bartolini Bussi & Mariotti, 2008, p. 755).

According to TSM, keeping a notebook plays a key role in the evolution of signs because the written record of the signs makes them permanent objects, which can be offered as an artefact in the following semiotic activities to create a semantic link to mathematical signs. It is the last phase in which the teacher facilitates the evolution of signs by conducting mathematical discussions that stimulate the collective production of signs. These discussions promote a cognitive dialectics between artefact and mathematical signs. Therefore, it is described as the core of the semiotic processes that constitute the basis of the teaching and learning.

The teacher initiates certain didactic moves to facilitate the evolution of mathematical signs during mathematical discussions. These didactic moves consist of two pairs of actions: the “back to the task” / “focalization” pair, and the “ask for a synthesis” / “provide a synthesis” pair. The first pair of actions has two aims: promoting the students’ production and sharing of artefact signs and directing the students’ attention only to the ones that have potential to evoke mathematical signs. One typical example of a “back to the task” move is asking the students how they accomplished the task by using the artefact. Responding to this question, the students might create various artefact signs some of which do not demonstrate a potential to be linked with mathematical signs. At this point, the teacher “focalizes” the discussion by selecting the pertinent aspects of using the artefact and by directing the students’ attention to them.

While “back to the task”/ “focalization” moves can help students to collaboratively construct shared and stable artefact signs, the “ask for a synthesis” / “provide a synthesis” pair promotes the detachment of these signs from the artefact context. Prompting the students to generalize their personal meanings that cling on the use of the artefact is one way of facilitating this detachment. Generalizations are made with pivot signs, which are related to artefact use but at the same time have potential to be linked to the target mathematical signs. Therefore, this second pair of actions stimulates the production of pivot signs. The “ask for a synthesis” action prompts the students to make a synthesis of the discussion up to a certain point and the students respond to this request by using pivot signs as they make generalizations. Then this generalization is reformed and fixed by the teacher’s integration of the standard mathematical terminology into students’ generalization. The teacher thus “provides a synthesis” for the mathematical discussion.

Even though the notions of internalization and mediation are introduced to explain the process of sign production, they seem to be more successful in disintegrating an event into discrete chunks that is responsible for the creation of signs instead of unearthing a continuous process. For example, mediation is explained through the separation between the mediator, learner, and the conditions of mediations and the mediated. As evident in the main activities of didactic cycle, teacher mediates certain student actions. However, it does not explain how these actions emerge.

Similarly, internalization is explained through a movement from external to internal. However, it does not explain how that movement happens. Instead of studying each chunk and their role separately, examining the process as a whole by focusing on the bi-directional relationships among them might shed more light on the process which starts with the learners’ encounter with an artefact and ends with the emergence of signs that can be associated with the targeted mathematical concepts.

1.4.3. An Alternative Approach to Networking Enactivism and TSM

There have been several attempts in the literature to explain internalization by disrupting the external-internal separation. They focus on the individual experiences that are guided by cultural artefacts and social settings (Zittoun & Gillespie, 2015). Even though this attempt to reinterpret the process of internalization might challenge its dualistic nature, explaining it by clinging on to a theory that is built on this very separation may not break the dualistic habits entrenched within its main constructs. The term “internalization” itself promotes a separation between “in” and “out” of a person. At this point, I took a diffractive approach (Jackson, & Mazzei, 2012) and read TSM through enactivism by interpreting the constructs of TSM based on the enactivist premises which explain cognition by focusing on the inseparable link between organisms and their environments (see Figure 1.10 for a pictorial depiction of this link).

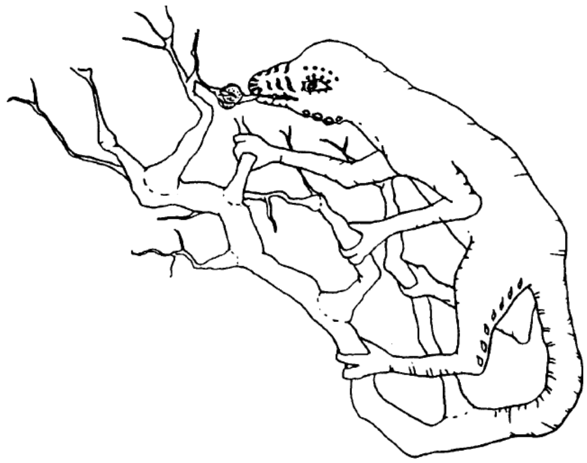

Figure 1.10. Union of organism and environment.
Note: The Image is from Maturana and Varela (1987, p. 240).

In Chapters 2 and 3 there is no explicit reference to enactivism but I used some notions of TSM with an enactivist interpretation. For example, instead of analyzing the utilization schemes as mental constructs stored in the mind, I focused on the emergent bodily actions as the constituent of artefact signs. In Chapter 4, instead of operationalizing multiplication knowledge as an immaterial object to be accessed through internalization, I consider the sensorimotor-integration (as per Nemirovsky, 2013) as one aspect of multiplicative thinking. This notion focuses on the individual and does not explain the social aspects of learning as emphasized by the Vygotskian notion of mediation. However, instead of operationalizing the teacher's contributions to students' tool use as mediation, I identified these instances as "social coupling[2]" (Maturana & Varela, 1987, p. 188) among the people.

The theory of didactic situations (TDS) and activity theory also acknowledge this relationship between the learners and the environment, yet each of them situates the environment differently (Drijvers et al., 2010). While the environment acts like an antagonist to the subjects according to TDS, activity theory describes it as cooperative with the learner. Unlike TDS and activity theory, enactivism does not attribute any agenda either to the environment or to the organism. One does not adapt to the other. They co-evolve as long as they coupled structurally.

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[2] This phrase is used interchangeably with the phrase "third-order coupling" (see Maturana & Varela, 1987, p. 268). I think "social coupling" transmits the social aspect of the phenomenon more explicitly to the reader. Therefore, I chose to adopt it in my writing.

Drijvers (2019) also proposed networking the theories of embodied cognition and instrumentation, which is an important part of TSM, in a complementary manner. This way, one can address the aspects of learning that have not been focused on by the other. According to him, the former theory has not particularly concerned about the convergence of tool techniques and conventional mathematical notions, whereas the latter one tended to neglect the embodied nature of cognition. Thus networking them "reconcile[s] the embodied nature of instrumentation schemes and the instrumental nature of sensorimotor schemes" (Drijvers, 2019, p. 16).

This complementary approach is inspiring for me to examine learning more comprehensively, yet at the same time it is challenging because it prompts me to accept distinct epistemological assumptions simultaneously. The operationalization of schemes as mental phenomena seems to be a barrier for this reconciliation from the beginning as embodied approaches oppose to the dualism between the mind and the body. So rather than bringing these theories together with their assumptions, reading TSM through an enactivist lens allowed me to re-operationalize some theoretical constructs of TSM based on the enactivist premises. Thus, this process made the reconciliation between the two theories epistemologically compatible for me and helped me answer the following main question in this dissertation:

How does multiplicative thinking emerge around/with TouchTimes?

Learning mathematics in a technology-rich environment can be studied from many different points of view. Therefore, I focused on a different aspect of the setting in each study comprising this dissertation. As a whole, they enabled me to answer the above question through a three-dimensional examination as explained below.

1.5. An Overview of the Studies

In the first study, I examined how TT, particularly Zaplify, can contribute to students' multiplicative thinking in a way that differs from a pencil-and-paper environment. The theory of semiotic mediation proposes that if an artefact mediates both personal meanings related to the achievement of a task and mathematical meanings related to the target mathematical concepts, it has a semiotic potential. Therefore, I analyzed the semiotic potential of Zaplify by focusing on its potential. Since the interface of Zaplify has some features that are similar to arrays, which can be drawn with pencil-and-paper, I also analyzed the semiotic potential of pencil-and-paper. This dual analysis helped me understand the unique participation of Zaplify in students' opportunities for meaning making.

In order to conduct the analysis of semiotic potential, I designed a specific task and recorded my actions as I solved the task in these two different settings. My interactions in these two settings resulted in two images that modelled the same multiplicative situation in the task. Even though these models looked similar visually, I discussed how the differences in the actions that created these models can contribute to students' meaning-making process in a unique way.

In the second study, I examined how a 5-year-old child identified multiplicative relationships while reciprocally using a duo of artefacts introduced in the first study, pencil-and-paper and Zaplify. Drawing on the theory of semiotic mediation and using Arzarello, Paola, Robutti, and Sabena's (2009) synchronic and diachronic analysis of semiotic bundles[3], I studied the relationship between the signs the child created in two different settings and argued that the reciprocal use of the artefacts enriched the child's experiences of Zaplify, allowing him to extend his understanding of multiplicative relationships embedded in TT.

In this second study, the child engaged with Zaplify in a social setting. He was interacting both with his father and with the researcher as he manipulated TT. These individuals did not manipulate TT, but as "more knowledgeable" ones they directed the child's attention to specific features of Zaplify by verbal exchanges.

In the third study I examined how a third grader learns to structure a given number in a multiplicative way while collaborating with a "not more knowledgeable" peer with/around TT under the supervision of the researcher. In this study, students used the other world of TT, Grasplify. Although its interface is very different from Zaplify, the underlying design principles of each world are mostly shared. For example, the iconic and symbolic signs are simultaneously presented in each world. It is an important

[3] "A system of signs—with Peirce's comprehensive notion of sign—that is produced by one or more interacting subjects and that evolves in time" (Arzarello, Paola, Robutti, & Sabena, 2009, p. 100).

feature for learning multiplication because the simultaneous transfer of changes across the representations helps learners understand the invariant nature of functional relationships between the multiplicative factors (Kaput, 1985).

The analysis of the data was framed by the enactivist notions of distinctions, structural and social coupling. This approach focuses on the interactions between the individuals and their environment allowing me to understand how learning mathematics by using technology emerges as the students collaborate with both “more” and “not more” knowledgeable individuals. I discussed how others’ bodies participate in one’s learning and presented an alternative argument to Kaput’s (1985) proposal.

1.6. Methodological Considerations

I was interested in the emergence of multiplicative thinking around/with TT across all the articles of this dissertation. Therefore, in addition to the users’ target actions that I identified as multiplicative, I focused on the learning processes “[that] contribute[s] to such a development before the target action has been established” (Shvarts, Alberto, Bakker, Doorman, & Drijvers, 2021, p. 456). Learning is an ongoing process, so it is impossible to pinpoint a specific point in time as the beginning of multiplicative thinking. Similarly, it is not meaningful to state that multiplicative thinking is completely developed at a specific point in time of an individual’s life. Considering this, I studied the process which started with an individual’s encounter with TT and ended with the emergence of the target actions. The ontological and epistemological assumptions I presented in the previous section directed my focus onto the learning process and informed the methodological approaches I adopted.

All three studies in this dissertation are aligned with Hatch's (2002) characterization of a qualitative study. Hatch provided a list of characteristics by synthesizing several widely cited sources on qualitative work, yet he stated that this list does not constitute a norm that each qualitative study should follow. Rather, he proposed this list as a way to understand qualitative research in relation to more traditional forms of scholarship. Below, I present this list and explain how the three studies of this dissertation are aligned with it. According to this list, qualitative methods have the following characteristics:

1. “The intent is to explore human behaviours within the contexts of their natural occurrence” (Hatch, 2002, p. 7).

In this dissertation, my aim was to study how children engaged with TT to solve specific TT tasks. One study examined a child’s interaction with TT at his home. The other study examined two children’s interaction at school, separate from the rest of their classmates. In both cases, the children were with a researcher who interacted with them while they manipulated TT. Even though these instances may be considered as artificial settings arranged by the researcher to create data, the interactions between children and the researcher is authentic to classroom activities, which happen when teacher initiates a mutual inquiry with students about their class work (diSessa, 2007).

2. “It is axiomatic in this view that individuals act on the world based not on some supposed objective reality but on their perceptions of the realities that surround them. […] Qualitative research is about understanding the meanings individuals construct in order to participate in their social lives” (Hatch, 2002, p. 7&9).

I found these characteristics as being aligned with enactivism’s assumption that there is not “the reality”—rather, individuals bring forth their own realities. Following this assumption, I aimed to understand what relationships children bring forth out of their interaction with/around TT, instead of asking to what extent children can represent multiplication.

3. “Data take on no significance until they are processed using the human intelligence of the researcher” (Hatch, 2002, p. 7).

In all three studies, I created the data as a result of my intellectual engagement with the video-recordings of events. For example, a video segment as a collection of pictures became the data as soon as I made sense of the graphics: I described the accumulation of lights with different frequencies on the screen as pips (one of the Grasplify objects).

4. “Researchers […] must spend enough time with those participants in those contexts to feel confident that they are capturing what they claim” (Hatch, 2002, p. 8).

Enough time is a vague expression, so it is open to interpretation. Rather than taking time as a criterion, I considered the amount of interactions observed by the researcher as a measure of credibility of the researcher's claim. Thanks to the nature of TT, which allows users to create countless interactions with different quantities in a short period of time, I observed enough interactions to make sound interpretations, but I don't claim to "capture" the reality.

5. "Qualitative methods provide means whereby social contexts can be systematically examined as a whole, without breaking them down into isolated, incomplete, and disconnected variables" (Hatch, 2002, p. 9).

I find this characteristic relevant to enactivism's emphasis on the inseparable link between the organisms and their environment. In my analysis, I interpreted the events considering wholeness and the complexity of the context (both social and material).

6. "Qualitative research is as interested in inner states as outer expressions of human activity […] Researchers concentrate on reflexively applying their own subjectivities in ways that make it possible to understand the tacit motives and assumptions of their participants" (Hatch, 2002, p. 9).

The studies of this dissertation do not fully reflect this characteristic. I did not distinguish inner states from outer expressions. Rather, I considered the human actions as the very essence of the individual's motives.

7. "The overall pattern of data analysis in qualitative work is decidedly inductive, moving from specifics to analytic generalizations" (Hatch, 2002, p. 10).

My analysis reflects an opposite nature: I started with the theoretical constructs and interpreted the events deductively based on them. Thus, I intend to unpack the process of children's engagement with TT based on specific assumptions of the theories.

8. "Reflexivity [to keep track of one's influence on a setting, to bracket one's biases, and to monitor one's emotional responses] […] is essential to the integrity of qualitative research" (Hatch, 2002, pp. 10-11).

There are different types of qualitative research methods. The three studies included in this dissertation use videography (Knoblauch, 2012) as a method to analyze people acting in social settings by video. This method is similar to ethnography as it addresses the conduct of people in their natural environments; and audiovisual observation is the core activity of this method. However, it differs from ethnography in three aspects. While ethnographical studies require longer periods of fieldwork, data collection is shorter for a videography.

The lack of intensity in data collection of videography might be criticized by others for being superficial. However, videographies create huge amount of data which requires intensive and detailed data analysis, unlike the conventional ethnographies based on written records. Lastly, ethnographies focus on larger, locally distributed social structures. In contrast, a videography encompasses the particulars of situated actions in social interactions. For example, instead of studying mathematics clubs in schools, a videography may try to unpack how a student performs a specific mathematics activity in a mathematics club.

Videographies assume that “actions […] are produced methodologically in certain ways, and it is only being performed in certain ways that certain things are brought about” (Knoblauch, 2012, p. 74). This was aligned with how I approached the children´s actions while they manipulated TT. Instead of regarding these manipulations as background activities to learning multiplicative structures, I considered the series of the children´s actions involved in using TT as the very essence of learning multiplicative structures.

I analyzed three video-recordings, each of which was used for each individual study. The events recorded in the videos shared some characteristics with clinical interviews, yet differed from them in a few aspects. diSessa (2007) described a clinical interview “as one-on-one encounter between an interviewer, who has a particular research agenda, and a subject” (p. 525). The interviewer´s role is to propose interviewees a problematic situation to think about and to encourage them to explore these situations by using materials, if possible. Another role of the interviewer is to prompt the interviewee to explain their thinking by talking aloud or by using available materials. Thus, the aim of the clinical interview is “to allow the interviewee to expose his/her “natural” ways of thinking about the situation at hand”, not the interviewer’s “intended response” (pp. 525–526). Therefore, it has a reflective nature, and it takes time. The interviewer achieves this by avoiding making personal, authoritative challenges and judgments on subjects´ responses; and giving instructions for the solution of the problem.

Like in a clinical interview, the interviewers met the children (one 5-year-old and two 9-year-olds) with a particular research agenda: to understand how children think while solving TT tasks. However, the events recorded in the videos differed from a clinical interview due to the interviewers’ epistemological assumptions. Clinical interviews come from the cognitive tradition. Therefore, they assume that interviewee´s thinking should be surfaced in a mutually intelligible way, which is mostly through verbal exchanges. As a result, a clinical interviewer prompts the interviewee to talk aloud. Whereas the interviewers in the video-recordings were more interested in prompting the interviewees´ bodily action, yet they did not completely neglect the verbal exchanges.

Clinical interviews also aim to unearth the students’ “current knowledge” (Steffe, Thompson, & Glasersfeld, 2000, p. 274). Thus, they involve two assumptions: (1) that knowledge has a static nature, and it exists prior to the interviewing; and (2) that interviewing is a transparent instrument that can “measure” students’ “current knowledge” without interfering with it. However, drawing on new materialism’s approach to the instrument of measurement and the measured phenomena, I do not separate the latter from the former. In other words, I assume that the interviewing process unearths students’ thinking which emerges as a result of the interaction with the interviewer, the interviewee/s and TT. Like in clinical interviews, I did not identify any incident in which the interviewers communicated judgements on subject´s responses. However, I argue that interviewing guided the children to achieve the TT tasks without providing them with direct instructions. Therefore, it can be said that the interviewer and TT acted like teaching agents, unlike in a clinical interview.

The teaching aspect of the interviews might recall teaching experiments. However, the events recorded in the videos do not have all the characteristics of a teaching experiment. According to Steffe, Thompson and von Glasersfeld (2000) a teaching experiment consists of a sequence of teaching episodes each of which involves a teaching agent, one or more students and a witness of the teaching episodes. The sequence of teaching episodes allows the teacher-researcher to generate hypotheses about students’ learning and test them. The witness of the teaching episodes catches important elements of a student's actions and propose further action to contribute to students’ learning in case the teacher-researcher misses them. These two characteristics of the teaching experiment were not evident in the recorded events in my studies.

There are different ways to analyze data in a videography (Knoblauch, 2012). I followed two approaches which are aligned (1) with Siegler and Crowley's (1991) microgenetic analysis and (2) with vom Lehn and Heath’s (2012) interpretive audiovisual analysis. While Sigler and Crowley (1991) used microgenetic analysis to study conceptual development in children, vom Lehn and Heath’s (2012) used interpretive audiovisual analysis to study people´s interactions in museums and science centres. It seems that the former method tends to direct researchers’ focus more on the individual, unlike the latter one. Combining two methods helped me consider the relationships between the individuals’ actions while I attended the changes in the actions of one child to study the process of children’s learning around/with TT.

Microgenetic method was first suggested in the field of developmental psychology to unearth the process of cognitive development (Siegler & Crowley, 1991). This method associates development with change and focuses on the change while it happens. In other words, instead of comparing behaviors before and after a change happens as in the pre and posttest studies, this method is interested in what happens in between.

Siegler and Crowley (1991) described the change as a continual phenomenon, not as occasional episodes that ‘punctuate’ the static developmental states. However, they do not explicitly state what it is that changes. Some of their examples indicate changes in cognitive schemes, others are about changes in children’s strategy to solve a problem. They did state that this method is applicable in any study that focuses on the process of change irrespective of their theoretical orientation. Since I took an enactivist approach to understand how target actions around/with TT emerged once the children were encountered with the tablet, I conducted the analysis by focusing on the change in the children’s actions.

This method has three characteristics:

- observations span the entire period from the beginning of the change to the time at which it reaches a relatively stable state;
- the density of observations is high relative to the rate of change of the phenomenon;
- observed behavior is subjected to intensive trial-by-trial analysis, with the goal of inferring the processes that give rise to both quantitative and qualitative aspects of change. (Siegler & Crowley, 1991, p. 606)

Following this method, I divided the video-recordings into numerous discrete sections (more explanation in Chapters 2, 3 and 4). This helped me focus on a very short section of the recording at a time to make fine distinctions in meaning (Parnafes & DiSessa, 2013). Thus, microgenetic method yielded "more differentiated descriptions of particular changes" (Siegler & Crowley, 1991, p. 608) in children's actions around/with TT. However, this method does not prompt the researcher to understand the change with respect to the actions of others. At this point, vom Lehn and Heath's (2012) interpretive audiovisual analysis allowed me to attend these relationships to understand the change process in a more holistic way.

vom Lehn and Heath (2012) drew on studies that ¨have directed analytic attention towards the action and interaction with and around the material environment and in particular the ways in which tools, technologies, objects and artefacts feature in, and gain their occasioned sense and significance through, practical collaborative activities¨ (p.101). This approach is very appropriate to study how children make meaning while they collaboratively manipulate TT. Moreover, its assumptions are aligned with enactivist and sociocultural theories I draw on. Three main assumptions of Lehn and Heath´s (2012) approach are:

- The intelligibility of action, its sense and significance, is inseparable from the occasion, moment and circumstances in which it is produced.
- Social actions and activities are emergent and contingently accomplished with regard to each other […] The action is both context sensitive and context renewing.
- The analysis is concerned with explicating the organization through which participants produce particular actions and make sense of the actions of the others. (p. 104)

The emergent nature of the actions recalls enactivism and suggests that "whatever is observable and understandable should not be considered as being due to external factors beyond the video recorded scene itself, such as "derives", "subconscious desires", attitudes or interests, but as motivated by the local sequence of actions recorded" (Knoblauch, 2012, p. 74).The specific location and character of an action (vocal, visual or material) and how it is related to preceding and proceeding actions are critical to the sequential analysis. This emphasis on the context is aligned both with sociocultural and enactivist approaches to learning.

As in vom Lehn and Heath (2012), I conducted the analysis on the audiovisual recording, yet used the transcription of the audiovisual data to get familiar with the complexities of a particular fragment in the data corpus. I used a technical instrument called score (Raab & Tanzler, 2012) to translate visual and vocal data into written language. A score consists of a table in which actions are recorded based on their successive and simultaneous occurrences. The events are noted vertically in separate rows to illustrate their temporal succession. The audiovisual dimensions of the events are distinguished according to the research interest and noted in different columns of the same row to illustrate their simultaneity. I froze the video and created motionless stills to record the events into the score.

The analysis I present in this dissertation is not a verbal account of students' mathematical realities, but it is my interpretation of students' mathematics (as per Steffe et al.'s (2000) differentiation of students' mathematics and the mathematics of students). Therefore, it is not objective; indeed, as Maturana and Varela said: "Everything said is said by the observer" (1987/1992, p. 65). It is also not the only possible interpretation. Even though the variation in what researchers emphasize in their analysis might be seen as a weakness of this method, I choose to see it as a strength because "different possibilities for students' mathematics education emerge" from these variations (Steffe et al., 2000, p. 269).

1.7. Dissertation Organization

As a manuscript-based thesis, the organization of this dissertation is different from traditional ones which have IMRaD (Introduction–Method–Result–and–Discussion) structure. Except for the last chapter, each of the following chapters constitutes an independent research study which is already published in a journal. Chapter 2 explores the semiotic potentials of TT and pencil-and-paper. Chapter 3 examines young learners' meaning making process of TT when they use it with pencil-and-paper. Chapter 3 studies primary school students' responses to a TT task while they collaborate with a peer. The chapters are ordered this way to maintain the coherence across the manuscripts. This order also coincides with the chronological order of the articles with respect to their publishing date.

Each of these chapters follows a prelude which consists of my accounts of the creation stories of each manuscript. These preludes also outline the purpose and nature of the manuscripts and how they relate to the aims of the overall program of my research and to the other manuscripts included in the thesis.

As an inevitable consequence of this type of dissertation, the manuscripts must be identical to the published version. This created a tension for me because I wanted to make some changes in the original versions after reengaging with these manuscripts to compile them in my dissertation. I found that some ideas may lead to misunderstandings and that some others are not emphasized enough. Therefore, I provided postludes at the end of each chapter, which aim to clarify or re-emphasize these ideas.

The last chapter of this dissertation is the conclusion chapter, which integrates the main results across the manuscripts. It summarizes key findings; lists the theoretical, methodological, and practical implications of my study; addresses to the limitations and challenges I identified as a researcher and provides directions for future research.

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Prelude to Chapter 2

The following manuscript is a research paper which is an expanded version of a paper I submitted to the Tenth European Society for Research in Mathematics Education Topic Conference: Mathematics Education in the Digital Age (MEDA). In that conference paper, I analyzed the semiotic potential of Zaplify with respect to two different tasks which were designed to engage students with two important key ideas of multiplication: unitizing and spreading.

I expanded the conference paper into the following article which was published in the journal *'Mathematics'* on 9 March 2021. In the article, I analyzed the semiotic potentials of Zaplify and pencil-and-paper with respect to another spreading task, which could also prompt students to experience unitizing. My aim in making a dual analysis was to examine the particular contribution that Zaplify offered in comparison to pencil-and-paper. I decided to make such a comparison after one of the audience members at my MEDA asked me "what is the added value of Zaplify". This was a common reaction I encountered in different settings.

I first heard the following question, which was posed by a colleague when Grasplify (the other world in TT of which interface looks like colourful tokens) was introduced in a research meeting: "Why do we need Grasplify if we have access to physical manipulatives?" Then I encountered a teacher's comment on the affordances of Zaplify: "It does what I do with drawings, but faster". This type of reaction speaks to the tendency people have to consider the affordances of TT mostly based on its static characteristics (such as shape, colour, quantity) of objects they identified on the interface at a specific time point.

As these characteristics are also shared by manipulatives and static models, it is not surprising that TT does not, on the surface, seem to provide students with more experiences than manipulatives and static models would. However, my own perspective was that the way these objects are created also matters when learners build mathematical meanings. Since the analysis of the semiotic potential of an artefact reveals an artefacts' various modes of use, comparing the semiotic potentials of two artefacts would illustrate the "added value" of Zaplify with respect to pencil-and-paper. With this analysis, I aimed to answer the following research questions:

- Which signs might emerge when students use pencil-and-paper to solve a multiplication task?
- Which signs might emerge when students use Zaplify to solve a multiplication task?
- Which meanings of multiplication do these signs relate to?

As the aforementioned question about the added value of Zaplify implies, I was also thinking that Zaplify would extend the affordances of pencil-and-paper. In other words, in addition to all affordances of pencil-and-paper, Zaplify would have some other affordances to prompt multiplicative thinking. However, through the dual analysis, I learned that each artefact allows unique experiences for the learners, rather than one being the continuation of the other. My equal focus on the product and the process allowed me to identify these unique experiences.

This manuscript presents the semiotic potential of Zaplify based on how I used the artefact. This contrasts with the other manuscripts, in which I studied how students actually learned around/with TT. I present this manuscript first for two reasons. Firstly, it familiarizes the reader with the affordances of Zaplify to engage learners with multiplicative thinking. Secondly, while reading the rest of the dissertation, it directs the reader's attention both to the objects students manipulated and to their actions, which is likely to be neglected if it is not explicitly emphasized.

Chapter 2.

The Analysis of a Model–Task Dyad in Two Settings: Zaplify and Pencil-and-Paper

Abstract

This paper examines the added value of a digital tool that constitutes a new model to introduce students to multiplication. Drawing on the theory of semiotic mediation, the semiotic potential of this new model is analysed with respect to the same task that can be solved in two different settings (the digital tool and pencil-and-paper). The analysis shows that the task solutions undergo significant changes depending on to the technological settings. Even though the end product of the model–task dyads might look the same in both settings, the product emerges from the different processes that would mediate quite different meanings for multiplication. This suggests that while designing tasks that involve mathematical models, rather than focusing only on the end product, considering the whole process would reveal the extensive potential meanings the model–task dyad can mediate.

Keywords: digital tools; model–task dyad; multiplication model; the semiotic potential

2.1. Introduction

The repeated addition model is pervasively used in schools around the world to teach multiplication although it cannot fully capture the essence of multiplicative situations (see below for more detail). Students might be able to come up with the right answer to multiplication equations by using this model, yet this does not guarantee an understanding of multiplicative relationships (Hurst & Hurrell, 2016). Additive thinking still overweighs the multiplicative thinking, when students in the upper grades interpret the situations that are open to both ways of thinking (Degrande et al., 2018).

This has in part driven the growth of research around the use of different models (Askew, 2018; Kosko, 2020; Venkat & Mathews, 2019).These models represent multiplicative situations by illustrating a number of pre-given objects that are spatially organized different from that of the repeated addition model. However, this static approach to depict multiplication excludes the processes that are involved in multiplying. This paper describes a new model—henceforth called Zaplify—that draws on but is not identical to existing models. Like the others, Zaplify is spatially organized in order to highlight multiplicative relationships. Unlike the other models, it has been developed by using multi-touch technology affordances so that the model also requires physical multiplicative actions. Zaplify does not draw on game design elements such as time constraint, embedded tasks and level. Instead, it offers an open exploratory environment for students to bodily experience multiplicative structures.

Given existing models, it is reasonable to ask why a new one is necessary and, furthermore, to consider what exactly it can contribute to the teaching and learning of mathematics. This is the basic question that I pursue in this paper (This is an expanded version of the conference paper written by Günes). However, instead of focusing just on the model, I examine how a given task changes according to the models and the technological environments (dynamic or static) in which it is situated. Therefore, I follow the theory of semiotic mediation in analysing the model–task dyad by looking at how the same task might be solved with different models that are enacted in different environments, as a way of understanding how the model might give rise to different solutions and different ways of reasoning about those solutions. Thus, this paper attempts to answer the question Watson and Ohtani (2015) posed to further the research in task design: “How different design principles reflect or generate different perceptions of mathematical concepts” (p. 14).

2.2. The Theory of Semiotic Mediation

Drawing on Vygotsky’s theoretical construct of semiotic mediation, Maria Giuseppina Bartolini Bussi and Maria Alessandra Mariotti developed the Theory of Semiotic Mediation (hereafter, TSM) to explain mathematics learning through collective use of mathematical artefacts. There are four assumptions underlying this theory (Maracci & Mariotti, 2012). First, forming scientific concepts is among the main objectives of the teaching-learning process that involves both acting on artefacts to accomplish some tasks and building mathematical meanings out of these actions. Creating signs within an intricate interplay constitutes the origin of this semiotic process. Second, these scientific concepts cannot be formed without conscious awareness. Individuals create personal meanings by interacting with the artefact. Semiotic processes bring these personal meanings into consciousness and enable individuals to formulate scientific concepts out of these personal meanings. Third, “mathematical meanings are culturally and historically established” and they “can be crystallized and embed into certain artefacts and signs” (Maracci & Mariotti, 2012, p. 22). Unlike most learners, the teacher, as an expert in the field, can identify such embedded mathematical meanings. Lastly, TSM assumes that intensive human mediation is necessary for learners to form mathematical meanings by using such artefacts.

Based on these assumptions, TSM hypothesizes that “meanings are rooted in the phenomenological experience, but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher” (Mariotti & Cerulli, 2001, p. 225) and accounts for this evolution with a theoretical construct called didactic cycle (Bartolini Bussi & Mariotti, 2008). As the aim of this paper is to analyse the potential link between artefacts and mathematical meanings, rather than a teacher’s exploitation of this link, I will only explain TSM’s constructs of “artefact”, “signs” and “the semiotic potential of an artefact”.

2.2.1. Artefact and Signs

Bartolini Bussi and Mariotti (2008) defined an artefact as any object that was made by human beings: “Sounds and gestures; utensils and implements; oral and written forms of natural language; texts and books; musical instruments; scientific instruments; tools of the information and communication technologies” are examples of various types of artefacts (p. 746). The TSM’s use of the term “sign” is aligned with Pierce’s description: “something which stands to somebody for something in some respect or capacity” (Drijvers et al., 2010, p. 118). The sign functions both to represent something, and to create meaning through their intricate interplay (Drijvers et al., 2010). As the teaching/learning process is based on meaning making, signs play an important role in this process. In their seminal work, Bartolini Bussi and Mariotti (2008) referred to Vygotsky’s list of signs that includes “language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs; etc.” (p. 751).

Even though Bartolini Bussi and Mariotti (2008) defined artefacts and signs separately and listed several examples for each construct, it might be difficult to distinguish them in some cases because some of the above-mentioned examples are given both as an artefact and as a sign (e.g., gesture, language). This prompted me to consider artefacts and signs not as disjoint materials but as phenomenological entities. At this point, Bartolini Bussi and Mariotti's (2008) reference to Vygotsky's analogy between tools and signs together with Wartowsky's categorization of artefacts is helpful in explaining how TSM distinguishes the artefact from the sign.

In the Vygotskian account, the analogy between the "tool" and the "sign" might reflect the distinction between TSM's constructs of artefact and sign:

> The invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose, and so on) is analogous to the invention and use of tools in one psychological respect. The sign acts as an instrument of psychological activity in a manner analogous to the role of a tool in labour. (Vygotsky, 1978, as cited in Bartolini Bussi & Mariotti, 2008, p. 753)

This analogy reflects two distinctions between the sign and the tool. First, the nature of the activity differs. Artefacts are used to conduct physical activity, whereas signs play a role in psychological activities. Second, their mediating functions are different. While the tool functions as externally oriented, it exerts human influence on the object of physical activity; the sign functions as internally oriented, it does not change anything in the object of a psychological activity. It is an internal activity aimed at mastering oneself (Falcade, Laborde, & Mariotti, 2007, p. 55). In other words, we change the world via tools, we master ourselves via signs.

Wartowsky (1979, as cited in Bartolini Bussi & Mariotti, 2008) separated the artefacts into three types, according to the user's phenomenological experience. A primary artefact is used to navigate one's environment. A secondary artefact is used to preserve and to transmit skills, which is necessary for the use of primary artefacts. Finally, tertiary artefacts do not have a practical goal in the sense of primary artefacts, yet they "constitute an autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical" (1979, as cited in Bartolini Bussi & Mariotti, 2008, p. 779).

For example, if I move the handle of a door, I can open the door physically. Here, the handle, a human made object, is the primary artefact which helps me to open the door. If I then move my hand in the air, as if grabbing the handle and rotating my fist downwards, this gesture becomes a secondary artefact because it helps me remember how to open the door or to communicate to others how to open the door. In other words, the gesture stands (in part, at least) for the act of opening the door, the same way Pierce characterized signs: “Something which stands to somebody for something in some respect or capacity” (Pierce, 1932, as cited in Drijvers et al., 2010, p. 118). The rotation of the handle might then be modelled as an angle, which is a geometrical construct that conforms to certain geometrical rules. In this case, the notation of angle constitutes a tertiary artefact. Within this categorization, the primary artefacts might correspond to the TSM’s notion of artefact. The secondary artefacts can be associated with what TSM refers to as signs.

2.2.2. The Semiotic Potential of an Artefact

Wartowsky’s categorization of artefacts might present them as mutually exclusive entities. However, the same object might be used both as an artefact and as a sign according to the phenomenological experience of the user (Maschietto & Bartolini Bussi, 2009). For example, for a novice learner, an abacus might be interpreted as a tool to record the counting activity, whereas for an expert, such as a teacher, the abacus might represent place-value. Thus, the same artefact has the potential to mediate two different but related meanings. Bartolini Bussi and Mariotti (2008) described this as the semiotic potential of an artefact.

The artefacts that have semiotic potential can be used to help its users enact mathematical relationships that are novel to them. According to TSM, whenever individuals use an artefact to achieve a mathematical task in a social context, they will be using the artefact in a certain way and will be creating certain signs both to achieve the given task and to create shared meanings. TSM categorizes these signs according to their relationship to the artefact and to the mathematical culture. The artefact sign plays a role in expressing the relationship between the task and the artefact. It is associated with the operations conducted to achieve the task. The mathematical sign expresses the relationship between the artefact use and mathematical knowledge, and it is aligned with the existing mathematical culture. The pivot sign “may refer both to the activity with the artefact; in particular [it] may refer to specific instrumented actions, but also to natural language, and to the mathematical domain” (Bartolini Bussi & Mariotti, 2008, p. 757). It plays an important role in the evolution of artefact signs into mathematical signs.

The evolution of artefact signs into mathematical signs is the aim of mathematics education and this is achieved by the semiotic mediation of the artefacts and the cultural mediation of the teacher (Bartolini Bussi & Mariotti, 2008). At this point, Mariotti (2012) considered the analysis of an artefact's semiotic potential as an a priori phase in designing a successful teaching sequence, because the specific utilisation schemes (Verillon and Rabardel (1995) defined the utilization schemes in the Piagetian tradition as "the structured set of the generalizable characteristics of artefact utilization activities" (p. 86)) can be predicted from examining the tasks in relation to the artefact. In this paper, by comparing the semiotic potentials of Zaplify and pencil-and-paper with respect to the same task, I aim to identify the added value of Zaplify in evoking various meanings of multiplication. The comparison is not meant to argue that one is better than the other; it simply brings to the foreground the particular value of Zaplify that may be difficult to appreciate when it is assumed that pencil-and-paper is the presumed technology of learning multiplication.

2.3. Repeated Addition versus Multiplication

The repeated addition model (RAM) of multiplication, equates a multiplication sentence to an addition sentence. For example, 4 x 3 is arithmetically represented as the repeated sum of 3 taken 4 times. The visual representation of this repeated addition involves 4 groups of 3 objects as in Figure 2.1:

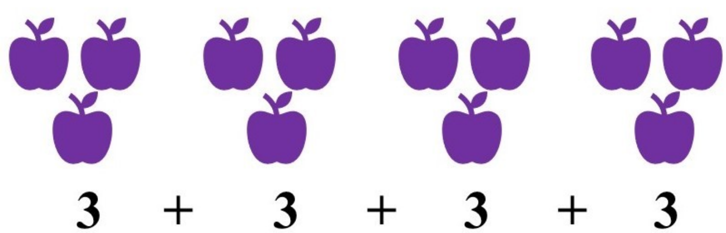


Figure 2.1. RAM of multiplication 4 × 3.

RAM depicts multiplication as the collection of equal groups. Therefore, it may suggest that multiplication involves a spatial organization of pre-existing objects. Even though this model represents objects in groups, the quantity of the groups is not as visible in the mathematical expression as the size of the groups (which is represented by the total number of purple apples in a group, as in Figure 2.1). Therefore, RAM emphasizes only a one-unit count which is represented by purple apples. The groups are spatially separated, and this separation might portray each addend as independent of the others. The RAM might therefore suggest that changing the size of one group would not require a change in the sizes of the other groups.

RAM is widely evoked in mathematics curricula as a bridge between addition and multiplication (e.g., Turkish mathematics curriculum, British Columbia mathematics curriculum). However, multiplication is explicitly distinguished from repeated addition by several researchers. Their definitions of multiplication emphasize both its static and dynamic aspects by pointing to the relationships between the quantities involved in multiplication and to the underlying actions taken in multiplicative situations.

Schwartz (1988) situated the meaning of multiplication within mathematical modelling activity. He claimed that the identification of quantities and referents (the attributes to be measured) is the basis for such an activity. He distinguished two types of quantities: extensive and intensive. Extensive quantities are the ones that are directly counted or measured like weight, height or amount. Intensive quantities cannot be quantified directly but must be calculated through other quantities such as speed (the travelled distance per unit time).

Schwartz (1988) separated mathematical operations into two groups: referent preserving and referent transforming operations. The former produces a third like quantity out of two like quantities. Addition and subtraction are such operations. Referent transforming operations produce a quantity with a different referent. Multiplication enters this second category, and Schwartz (1988) described it as a mapping from “a quantity in one space to another quantity in another space” (p. 50). Thus, all multiplicative situations require the identification of three referents and three relationships between them. For example, the multiplication that reveals the number of points in a 3 x 4 array (see Figure 2.2) can be represented as:

{3, the number of rows} × {4, the number of points per row} = {12, the number of points}

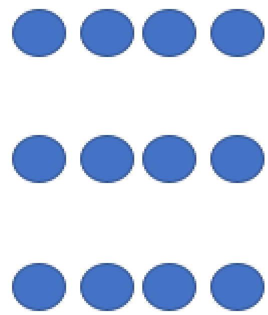


Figure 2.2. **The points in a 3 × 4 array.**

In this situation there are three relationships between the number of rows (R), of points (P), and of points per row (PpR): R = P/PpR, PpR = P/R, P = R PpR.

Vergnaud (1988), like Schwartz, proposed that identifying the relationships between the variables was essential for multiplication. However, he identified two variables and four values attached to them.

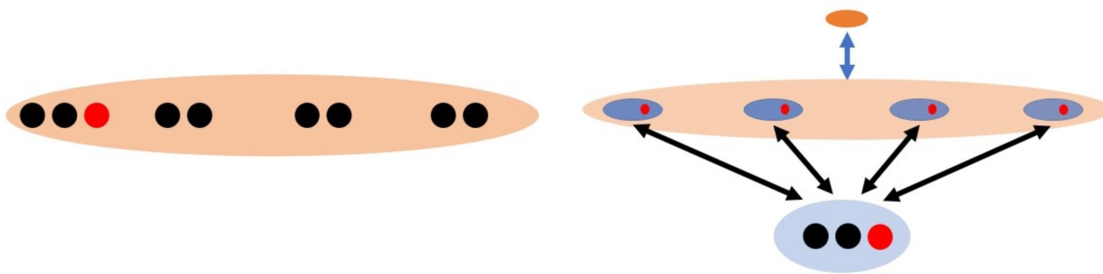
In Figure 2.3, Vergnaud (1988) distinguished two relationships: scalar and functional. The scalar relationship is the ratio between two values of a variable: for example, the ratio between the number of cars is five to one and this is the same for the number of tires. On the other hand, the functional relationship is the ratio between two values of two distinct variables. For example, the functional relationship between the number of cars and the number of tires is four. This is similar to Schwartz's (1988) intensive quantity that builds a many-to-one correspondence between the number of cars and tires. One car corresponds to four tires and this correspondence holds for each car.

The number of car	The number of tires
1	4
5	x

Figure 2.3. **The representation of a multiplicative situation in Vergnaud's T-Table.**

Drawing on Piaget, Clark and Kamii (1996) pointed to two important differences between multiplication and repeated addition. First, multiplication involves two units each of which quantifies two distinct entities. There is a many-to-one correspondence between these units. This correspondence may be associated with Vergnaud's (1988) functional relationship, for example between the number of cars and the number of tires: one car corresponds to four tires. The car represents a composite unit that is made up of four units. Whereas addition emphasizes only one unit that quantifies the size of the groups (as in the above-mentioned case of purple apples). Equal groupings might recall composite units. However, they are not quantified in the addition: the amount of groups is not assigned a numerical value, unlike the size of the groups.

Based on this difference, inclusion relationships between the units of addition and multiplication also have different natures. Since there is only one unit in repeated addition, they are included in one level among themselves as in Figure 2.4a. When we add "2"s, we act on the units that quantify the size of the groups. Thus, these units (they are represented by the individual black discs in Figure 2.4a) construct a collection of multiple individual units. Moreover, each addend acts independently from the others. Therefore, let us say in 2 + 2 + 2 + 2, increasing the size of the first group from two to three (this is represented by the red dot added next to the first two black dots in Figure 2.4a), does not have to change the size of the rest of the groups. Therefore, the groups that are depicted in repeated addition do not necessitate a many-to-one correspondence that holds for each group. Whereas in multiplication, the single units (represented by black discs in Figure 2.4b) that take part in the body of the composite unit create a single multitude (represented by the blue ellipses in Figure 2.4b). As these single units are included among themselves, they are also included in each multitude simultaneously because of the many-to-one correspondence between the composite and the single units (represented by the arrows in Figure 2.4b). Thus, any change in the size of one multitude, let us say from two to three (as represented by the red dot added next to two black discs in Figure 2.4b), is mirrored on the size of the other multitudes (as represented by the red discs added into four blue ellipsis in Figure 2.4b). Therefore, increasing the first factor of the multiplication 2 x 4 = 8 by one unit increases the product by four units.



(a) (b)

Figure 2.4. The representation of inclusion relations:(a) in repeated addition (b) in multiplication.

Note: The diagram is reimagined based on Clark & Kamii (1996) and Confrey (1994).

Confrey (1994) took a dynamic approach and described multiplication with respect to creating a numerical quantity. She described multiplication as, “an action of creating simultaneously multiple versions of an original” and named this action as “splitting” (p. 292). The notion of an original in this definition might recall the single unit of repeated addition. Fortunately, Confrey (1994) explicitly stated the main difference between splitting and repeated addition: they differ based on how the change occurs. In repeated addition, identifying a unit and counting instances of that unit consecutively brings the change. Whereas in splitting, change occurs through a simultaneous one-to-many splitting action. Moreover, Confrey (1994) defined the unit as “the invariant relationship between a successor and its predecessor; it is the repeated action” (p. 312) and distinguishes between additive unit in a repeated addition and a multiplicative unit in splitting. Thus, she considered the unit of the operations in relation to the action. Counting the results of splitting action might also be associated with the model of repeated addition. However, Confrey (1994) warned that, “the cognitive act of recognizing a situation as multiplicative and displaying it appropriately occurs prior to this counting action” (p. 311).

Davydov (1992) explained multiplication with respect to the operations involved in the quantifying activities. He defined such activities as “assigning a numerical value of some magnitude in relation to a given unit” (p. 11). Such activities become cumbersome when the given unit-count is too small. Davydov (1992) described multiplication as a practical tool to quantify magnitudes by transferring the unit count from a smaller to a larger one “for which a relationship to another, smaller unit, is already established” (p.

12). Davydov's conceptualization of multiplication emphasized both a multiplicative action (transfer of the unit count) and multiplicative relationships between the quantities.

For example, let us say a grocery shop employee needs to make an inventory for the amount of honey in stock. Honey is sold in individual jars and they come in boxes of 8. The employee can assign a numerical value to the amount of honey either in relation to a jar or in relation to a box. In this case, the jar and the box are the two distinct unit-counts for quantifying activity, and they have an established relationship because of packaging: each box corresponds to 8 jars. If the employee wants to assign a numerical value to the amount of honey with respect to jars, they can directly count the jars one by one. Alternatively, the employee can indirectly quantify it with respect to jars by counting the boxes and transferring the unit-count from the jars (the smaller unit-count) to the boxes (larger unit-count) thanks to the established relationship between them. If there are 14 boxes of honey jars, the operations that are involved in indirect measurement are represented as "8 x 14 = 112". In this equation, 8 represents the established relationship between the smaller (the jar) and the larger (the box) unit-counts. This is similar to Schwartz's (1988) intensive quantity and Vergnaud's (1988) functional relationship. 14 represents the numerical value of the magnitude (the amount of honey) in relation to the larger unit-count (the box); 112 represents the numerical value of the magnitude (the amount of honey) in relation to the smaller unit-count (the jar).

As shown in the above models of multiplication, the researchers conceptualized multiplication by focusing either on the relationship between the quantities, or on the action underlying the multiplicative situations, or both. Table 2.1 summarizes these approaches.

Table 2.1. The various meanings of multiplication

The Researcher	Key Concepts of Multiplication
Schwartz (1988)	Multiplication is a mapping from “a quantity in one space to another quantity in another space” (p. 50). All multiplicative situations require the identification of three referents and three relationships between them.
Vergnaud (1988)	Multiplicative situations include two variables and a functional relationship (many-to-one correspondence) between them.
Clark and Kamii (1996)	Multiplication has two units which have inclusion relationships in two levels.
Confrey (1994)	Multiplication is “an action of creating simultaneously multiple versions of an original” and named this action as “splitting” (p. 292) .
Davydov (1992)	Multiplication is a practical tool to quantify magnitudes by transferring the unit count from a smaller to a larger one “for which a relationship to another, smaller unit, is already established” (p. 12).

2.4. TouchTimes

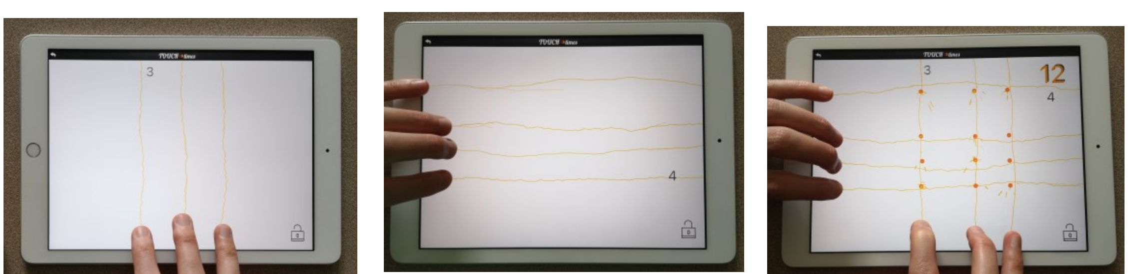
TT (Jackiw & Sinclair, 2019) is an iPad application designed to enhance multiplicative thinking as different from repeated addition. The design of TT draws both on Davydov’s (1992) notion of change in unit-counts and on Vergnaud’s (1988) notion of functional relationship. TT consists of two worlds called Grasplify and Zaplify. Both worlds are designed to convey the multiplicative ideas brought by Davydov and Vergnaud, yet they embedded these ideas in distinct models that prompt learners to embody them in different ways. This paper will focus only on the Zaplify world, which can be described as a dynamic array model.

Zaplify starts with an empty screen. When the tablet is placed horizontally on a surface, four fingerprints appear just above the lower edge of the tablet, and three fingerprints appear on the left edge of the tablet (see Figure 2.5a). Then, a diagonal appears on the screen as the fingerprints gradually fade away (see Figure 2.5b). These fingerprints and the diagonal line are introduced automatically by the app to guide users to place their fingers both horizontally and vertically in the designated areas (upper and lower triangular areas) which are formed by the diagonal. While the fingerprints completely disappear in a few seconds, the diagonal line stays on the screen until the user touches on the screen (see Figure 2.5c). Users can create two types of Zaplify objects: lightening rods and lightening balls (these will be explained in detail below).

(a) (b) (c)

Figure 2.5. (a) Fingerprints, (b) fingerprints and the diagonal line, (c) the diagonal line.

Zaplify can be used in two modes: unlocked and locked. The user can change the screen mode by touching the lock icon at the lower right corner of the screen. This icon becomes visible once the user touches on the screen and creates a Zaplify object (see Figure 2.6). In unlocked mode, the Zaplify object(s) stay(s) on the screen as long as the user maintains the finger contact. In locked mode, the Zaplify objects stay on the screen even after the user lifts their finger-created the object/s. It is possible to shift from unlocked to locked mode any time. When the screen mode changes, the visual characteristics of the Zaplify objects slightly alter.



(a) (b) (c)

Figure 2.6. (a) Vertical lines, (b) horizontal lines (c) both horizontal and vertical lines.

When a user places a finger on the screen in unlocked mode, a yellow object that looks like a “lightening rod” (referred to as “lines” henceforth) appears. The lines extend from one edge of the screen to the opposite edge. These lines are not static objects, rather they continuously tremble like lightning strikes. The lines seem to be passing through the point where the user's fingertip contacts the screen. If the user touches the lower triangular area, Zaplify produces vertical lines (see Figure 2.6a), while touches in the upper triangular area produce horizontal lines (see Figure 2.6b). Screen contact can be made with one finger at a time or with multiple fingers simultaneously. Multiple fingers can create either only vertical lines (see Figure 2.6a), only horizontal lines (see Figure 2.6b) or both (see Figure 2.6c), according to the position of the fingers.

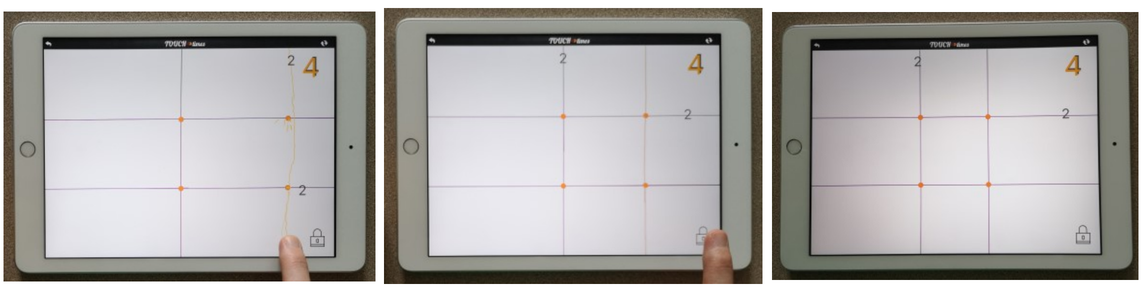
The number of vertical and horizontal lines are indexed separately by two black numerals. The numeral at the top of the screen represents the number of vertical line(s) (see Figure 2.6a) and it continuously moves from left to right on a horizontal path, repeatedly passing over all vertical lines. The numeral at the right edge of the screen represents the number of horizontal lines (see Figure 2.6b) and it follows a vertical path up and down spanning all horizontal lines.

Whenever a horizontal line intersects with a vertical line, a lightening ball (referred to as "points" henceforth) gradually appears at the close vicinity of the intersection point. Like the lines, the points are not static objects. They vibrate like sizzling sparks. The total number of points is represented by an orange numeral that appears at the upper right corner of the screen (see Figure 2.6c). No orange numeral appears at the upper right corner unless there is an intersection (see Figure 2.6a,b).

When a user lifts a finger, the line that is created by that finger and the points that vibrate on it disappear all together. This change is immediately mirrored in the numerals that represent the numbers of lines and the product. The orange numeral that represents the number of points (the ones before the finger is lifted) moves down until the bottom edge of the screen and then disappears. Meanwhile, a new orange numeral that represents the number of points (the ones that are left after the finger is lifted) appears on the upper right corner of the screen.

The Zaplify objects are created according to the same principles in locked mode: (1) Users create line objects when they touch the screen; (2) The location of touch designates the orientation (vertical/horizontal) of the lines; (3) The circular object(s) appear at the intersection(s) of perpendicular lines. In locked mode, the Zaplify objects stay on the screen once they are created. Therefore, they do not disappear, even though the line-making fingers are lifted. The Zaplify objects are also dynamic in this mode, yet they do not vibrate as dramatically as they do in unlocked mode.

Let us say a user first creates two horizontal and one vertical lines in locked mode. When the user touches the screen, a new yellow vertical line appears, and it vibrates as in the unlocked mode (see Figure 2.7a). Once the user lifts that finger, the amplitude of the vibration reduces and the colour of the line changes to a darker hue (see Figure 2.7b); all the lines that are parallel to it (vertical lines in this example) gradually slide (to the left in this example) until the grid becomes symmetrical (see Figure 2.7c).



(a) (b) (c)

Figure 2.7. (a–c): Adding a vertical line in locked mode.

According to TSM, an artefact that is used to solve a task has a semiotic potential if it mediates two meanings: (1) with respect to the achievement of the task; (2) with respect to the mathematical culture (Bartolini Bussi & Mariotti, 2008). Therefore, the semiotic potential of the artefact cannot be identified without considering the specific task. I have analysed the semiotic potential of pencil-and-paper and Zaplify based on the task called “Making 198 with various M-ples”. M-ple is a specific Zaplify term that refers to a line that has exactly M (any positive integer) points on it, as in Figure 2.8. For example, three-ple is a line that has only three points on it. In this analysis, I assume that students are already comfortable with this term.

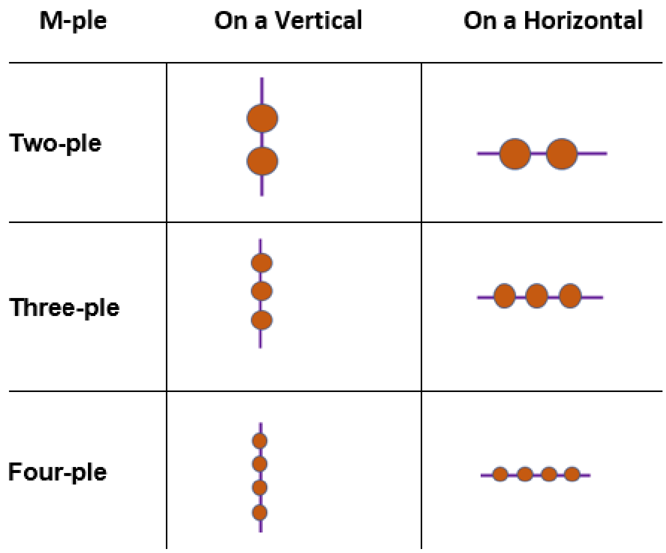


Figure 2.8. Examples of M-ples.

The task consists of the following questions:

(1) Make the product 198 by counting up with two-ples?

(2) Make the product 198 by counting up with three-ples?

(3) Make the product 198 by counting up with four-ples?

(4) Make the product 198 by counting up with five-ples?

(5) Make the product 198 by counting up with nine-ples?

2.4.1. The Underlying Task Design Principles

I chose this task because it followed many task design principles proposed in (Leung & Baccaglini-Frank, 2017). I draw particularly on the two following principles: (1) providing students with tasks that would involve a variety of examples (e.g., Mackrell & Bokhove, 2017; Robotti, 2017) and (2) providing students with tasks that lead to conflicting situations (e.g., Bokhove, 2017; Naftaliev & Yerushalmy,2017; Buchbinder, Zodik, Ron, & Cook, 2017). Keeping certain aspects of the examples intact and changing others prompt discernment and awareness by enabling students to separate "certain aspects of something from other aspects", to contrast their experiences, to make generalizations (Bokhove, 2017, p. 248). Conflicting situations "creat[e] learning opportunities for students to re-evaluate and refine their mathematical and proof-related knowledge" (Buchbinder, Zodik, Ron, & Cook, 2017, p. 217).

This Zaplify task achieves the first principle by prompting students to count up by various M-ples. Both the act of counting up and the target product stay invariant, unlike the M-ples. Moreover, the product is chosen to be a big number so that students will be counting up for an extended time period and creating several products until they reach 198. This is designed to help students focus on the structure rather than on computation (Zazkis, 2001). Repeating the process for a while would create new examples in which the structures they discern hold true. Therefore, this repetition could enable students to make a generalization out of their experiences, thus crystalizing the multiplicative meanings that emerge. This task would also lead to a conflict: the third and the fourth subtasks do not have a solution in Zaplify. Although students first have to create an M-ple and then tap one finger repetitively until they create the target product in each case, this strategy would not work for making 198 with four-ples and five-ples. This task also follows the sequence suggested by Bokhove (2017): “first appropriate pre-crisis items, then the item that intends to intentionally cause a crisis (for some students), and then some post-crisis items” (p. 242). The task starts with counting up by two-ple and three-ple. These are pre-crisis items that would build students’ confidence because it is very easy to reach 198 by repeating taps. Then, the crisis items (counting up with four-ples and five-ples to make 198) follow. They would possibly pose frustration in students’ bodies yet prompt them to ask “why”, which is an important question for learning. Finally, the last subtask includes a post-crisis item that students can solve without having difficulty.

Another design principle highlighted in Leung and Baccaglini-Frank (2017) was that of feedback. The feedback in Zaplify has the “trivial sense in which the appearance and behavior of objects that are constructed is feedback” (Finzer, as cited in Mackrell & Bokhove, 2017, p. 66). For example, students are not given any message saying whether they achieve the task or not. The numeral that represents the product in Zaplify is the feedback for students to judge if they succeed because this type of feedback “offers immediate information, which is directly related to one’s actions, and which can guide further actions, especially if a goal has not been reached” (Sinclair & Zazkis, 2017, p. 190). Moreover, the feedback is “constrained by the mathematics underlying the environment. In this way unexpected implications of the mathematics may be revealed” (Laborde, as cited in Mackrell & Bokhove,2017, p. 62). For example, in Zaplify the M-ples are all connected to each other in the array form. It is impossible to create various types of M-ples at the same time. This multiplicative relationship that underlies Zaplify would reveal that some numbers cannot be reached by counting up with any M-ples as happens in the above-mentioned crisis items.

This task is designed specifically for Zaplify. Therefore, it is not a task that teachers would ask their students to solve with pencil-and-paper. I analyzed the semiotic potential of pencil-and-paper with respect to this task to understand the unique contribution of Zaplify to students' meanings for multiplication.

2.5. Data Analysis

Mariotti (2012) suggested that Rabardel's theoretical notion of utilization schemes that are associated with specific tasks were important tools to analyze the semiotic potential of an artefact. Verillon and Rabardel (1995) defined the utilization schemes in the Piagetian tradition as "the structured set of the generalizable characteristics of artefact utilization activities" (p. 86). Borrowing a theoretical construct from cognitivist approaches might seem to be contradictory for TSM, which draws on a social constructivist approach (primarily through Vygotsky). However, rather than a scheme that was defined as an entity existing in the mind of the person independent from the social context, Mariotti (2012) seemed to be interested in the person's real actions that can be observed by another individual. In the same article in which she promoted the utilization schemes as a tool for analysis of the semiotic potential of an artefact, she wrote:

> Thus, the artefact and the modes of its use may appear as key elements in the emergence of mathematical knowledge in the school context. They become a unit of analysis that can guide the design of teaching and learning activities. (p. 26)

Parallel to Mariotti's (2012) account, I have focused on the real actions (modes of use) conducted around/with artefact in my analysis rather than the schemes that are believed to exist in the mind. However, I decided that the unit of analysis should be the potential signs emerging from the user's potential modes of use. Since the meanings are created through the interplay of signs (Bartolini Bussi & Mariotti, 2008), identifying the signs with respect to this intricate interplay would reveal the potential meanings. These signs may include the potential gestures, potential words to describe both artefact and the interactions with/around the artefact. In order to capture these potential signs, I video recorded an iPad screen as I solved the given Zaplify task and verbally described the steps I took. An M-ple can be created in many ways that might be associated with:

(1) $1 \times (\underbrace{1 + 1+1 + \ldots\ldots + 1}_{\text{The sum is equal to M}})$

(2) $1 \times M$

(3) $M \times 1$

I analyzed the third case that was aligned with Davydov's (1992) argument about the direction of the transfer of unit-counts (from the smaller unit-count to the larger unit-count, not vice versa). Students might create the M-ples in any of these ways. However, a teacher's role is to select the most appropriate signs that emerge from the artefact use and highlight them in the classroom activities to exploit the semiotic potential of the artefact (Mariotti, 2009). Therefore, I chose to illustrate the most appropriate mode of Zaplify use that would be associated with the targeted meanings of multiplication.

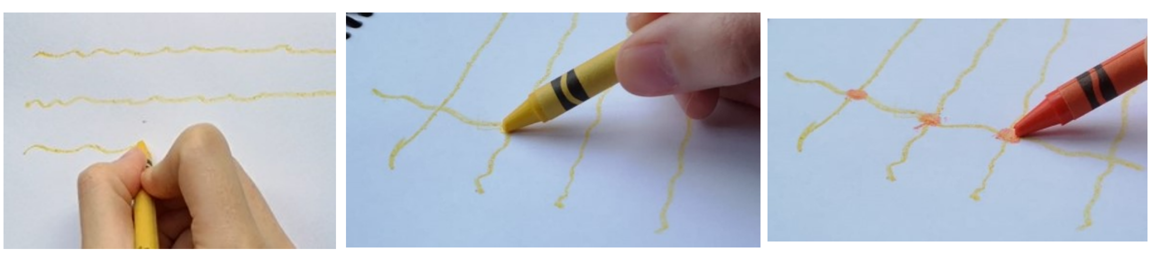
I organized the transcription of the video-recording in a table format that included three parts: the written record of my verbal accounts, written descriptions of my bodily actions and Zaplify objects (see Figure 2.9). I highlighted the transcription with specific colours to match the signs that were simultaneously created in different modalities. For example, the parts highlighted in yellow happened at the same time: as I was saying "then", I was pressing my four fingers on the iPad screen and Zaplify created four horizontal lines. I have compartmentalized the video-recording into ten-second-intervals to systemize the transcription. Surprisingly, this artificial separation did not create a semantic break between the cells except for one incident in which my verbal account "I am going to press my four fingers, at once, on the upper triangular" was in two separate cells. In this case, I moved the first part of the sentence down to the following cell together with the corresponding records of the bodily actions and the screen configurations. In my analysis, I identified the signs that would be associated with various meanings of multiplication and examined the relationship between them.

Time	Verbal account	Bodily account	Zaplify account
00:20-00:30	Then, one by one, I am going to press my finger, but I am going to do it in this locked mode,	As my four left hand fingers touches the screen, I stretch my one right hand finger above the screen, then move it towards the right edge and press it down next to the lock icon. I hold it on the screen as I tap my right middle finger on the lock icon. Then I drag my right index towards the left edge.	The four horizontal lines vibrate and the black numeral "4" moves up and down continuously. One vertical line that passes through below the index finger, the black numeral "1", four points that vibrates around the intersections of the lines and the orange numeral "4" appears on the screen simultaneously. The vertical line together with the dots on it moves towards the left edge of the screen. The black numeral "1" moves left and right on the top of the screen

Figure 2.9. An excerpt from the transcription organized in a table.

2.6. Semiotic Potential of Pencil-and-Paper

In order to create an M-ple that is aligned with the multiplication expression M x 1, students must first draw as many parallel lines as the number of points that should be on the M-ple. Then, they must draw one perpendicular line that intersects with these parallel lines. After students finish drawing the perpendicular line, they must draw individual points at the intersections. Pencil-and-paper allows students to draw each line and point one at a time (see Figure 2.10a,c). Similarly, as they draw the perpendicular line, the intersections of the lines would appear one by one, sequentially (see Figure 2.10b). Students can also start by drawing as many points as there would be on a given M-ple and then draw the lines that would pass through these points.



(a) (b) (c)

Figure 2.10. (a–c) The three steps of drawing an M-ple.

Drawing perpendicular lines would create visual and haptic distinction in students' experiences due to the orientations of lines. Students most probably start drawing the horizontal lines from the left to the right and vertical lines from the top to the bottom. This spatial and orientational separations might be described by two distinct signs (e.g., “vertical” and “horizontal”; tracing gestures from left to right for horizontal lines and top to bottom for vertical lines) that might be associated with two distinct unit-counts of multiplication.

While students draw the vertical line, the trace of the pencil passes over the existing traces of the parallel lines (see Figure 2.10b). This physical interaction between the traces of perpendicular lines might evoke for students the relationship between the multiplicative factors: one factor spreads over the other factor. Moreover, physical contact with the points that are drawn at the intersections might help students identify inclusion relationships in two levels (as per Clark and Kamii, 1996): a point at the intersection is included both in the horizontal and in the vertical line simultaneously.

Drawing points at the intersections of lines might evoke the idea of double unitizing (as per Davydov, 1992). While each horizontal line corresponds to one point, the vertical line corresponds to M points and thus constitutes an M-ple. Therefore, the former might be associated with the smaller unit-count, while the latter might evoke the larger unit-count. The number of points on an M-ple might be associated with the functional relationship between the sizes of unit-counts. As the referent of the intensive quantity is different from the ones that create it (Schwartz, 1988), the points are the objects that are different from the lines that create them. They refer neither to the number of horizontal lines, nor to the number of vertical lines but to the horizontal lines per vertical lines.

Creating points one by one at the intersections of lines might emphasize a sequential nature for unitizing action. Therefore, drawing an M-ple might evoke for students the idea of combining multiple single units instead of creating a single multitude as Confrey (1994) suggested.

In order to count up with the given M-ple, students must repeatedly draw M-ples and add up the points on them until they reach the given product. This might associate multiplication with a counting activity as opposed to Confrey’s (1994) warning that, “the cognitive act of recognizing a situation as multiplicative and displaying it appropriately occurs prior to this counting action” (p. 311).

Every time students draw a new M-ple, the number of intersections on the new M-ple must be the same as the previous M-ples. However, the free nature of the pencil-and-paper might prompt students to draw as many points as they want on a line. Therefore, while they have M points on the previous M-ples, they might draw fewer points on the following vertical line to reach the given product. This allows students to reach 198 irrespective of the M-ples they draw. For example, they can create four-ples to reach 198. In this case, after they make the product 196 by drawing the 49th four-ple, they may not completely draw the 50th four-ple. After they create two points on the 50th vertical line and reach 198, they might simply stop drawing (see Figure 2.11). This gradual increase in the number of points might also be associated with the conception of product as the combination of single units rather than the combination of multitudes. Moreover, this potential inconsistency of the number of points on each M-ple might evoke for students a functional relationship that is not spread equally across each larger unit-count.

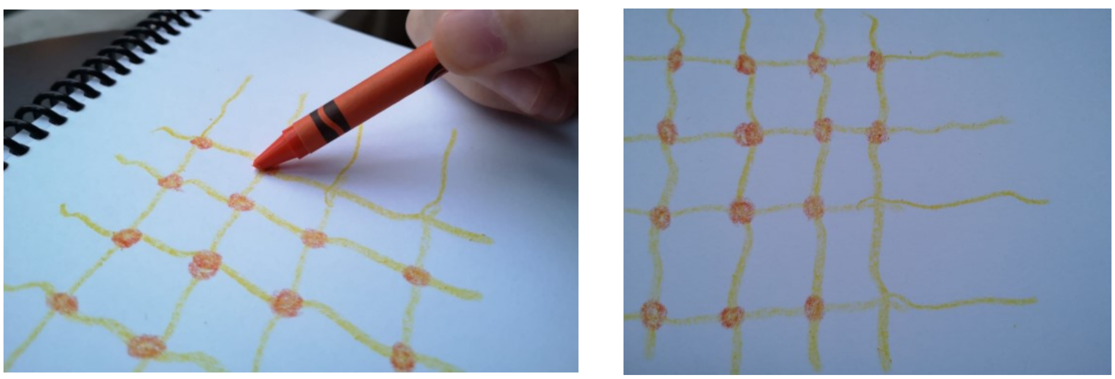


Figure 2.11. An example of gradual increase in the number of points.

2.7. Semiotic Potential of Zaplify

This task must be solved in locked mode, because the target number is very large and creating it by keeping the fingers on the screen is not feasible. When Zaplify is in locked mode, the lines stay on the screen even though the user lifts the finger(s) that create(s) them. When the user holds a line in locked mode, it is yellow and very vibrant. It looks as if it passes below the user's fingertip. As soon as the contact between the line making-finger and the screen diminishes, the line becomes purple and less vibrant. All lines slightly glide across the screen until they become equidistant.

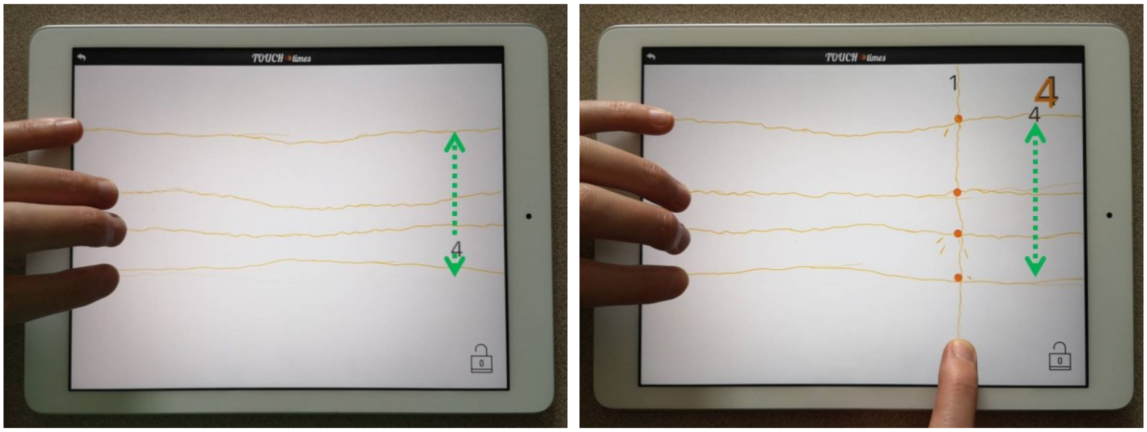
The task starts with an empty screen that is divided by a diagonal (see Figure 2.5c). Before touching the screen, I say, "In order to make, in order to count up, by four-ples until one hundred-ninety-eight”. Starting to “Make one hundred-ninety-eight” on an empty screen might be associated with a meaning for multiplication that refers to creating a quantity from scratch, rather than structuring a given quantity as represented by the static repeated addition model. This dynamism is aligned with Confrey’s (1994) description of multiplication as “creating multiple versions of an original” (p. 292).

The first step of counting by four-ples in Zaplify might be associated with the concept of composite unit.

> 00:10: C: I am going to press, my four fingers (stretching four left hand fingers on the edge of the screen), at once (pressing four fingers down simultaneously), on the upper triangular, this is going to make, four horizontal lightning rods.

During this episode, I first see an empty screen divided by a diagonal (see Figure 2.5c); then four horizontal lines and the black numeral “4” appear on the screen (see Figure 2.12a), and then the lines vibrate, and the numeral continuously moves up and down.

In this episode, the quantity is represented in three modalities: through (1) the utterance of “four fingers”, (2) stretching and pressing four fingers; (3) the appearance of four lines on the screen. While the verbal and the bodily account quantify the fingers, the Zaplify account quantifies the lines. The location of the lines creates a relationship between the lines and the fingers: The lines pass through my fingertips creating a correspondence between the lines and the fingers. This correspondence transfers the quantity of the fingers to the quantity of the lines. This transfer and the appearance of the black numeral “4” on the screen might evoke the mathematical meaning of “fourness” as a composite unit. Moreover, pressing four fingers simultaneously while uttering the word “four” might strengthen this idea of a composite unit.



(a) (b)

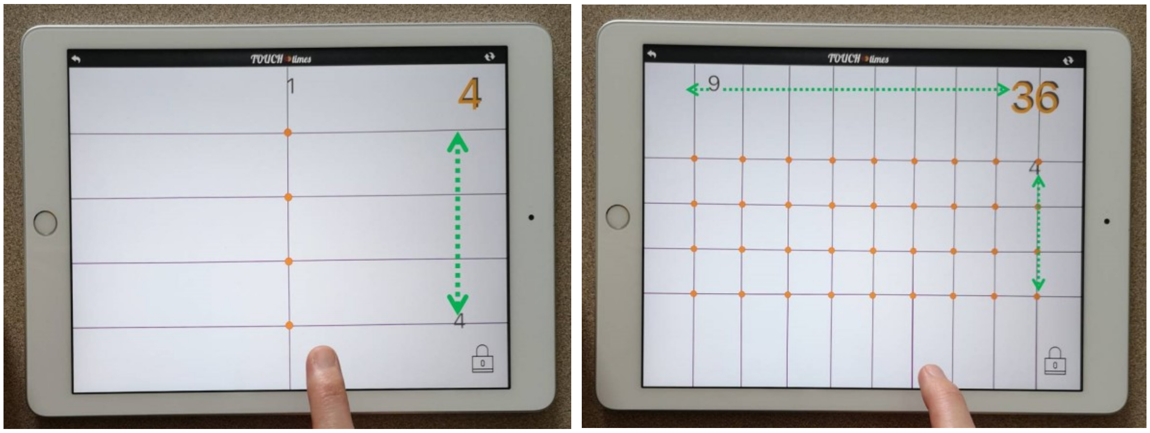
Figure 2.12. (a) Four horizontal lines and the numeral "4", (b) one four-ple on the vertical line.

Note: The green arrows are added on the figures to illustrate the path the numerals follow.

The third step of the task creates a four-ple. Creating an M-ple in Zaplify can be associated with the concepts of factors, product, transfer of unit counts and the functional relationship.

> 00:20: C: Then (holding four left fingers on the screen), one by one, I am going to press my (stretching right index finger above the screen) finger (moving the index finger to the right edge and pressing it down).

During this episode, the four horizontal lines vibrate and the black numeral "4" moves up and down continuously. One vertical line appears as if it passes below the index finger. The black numeral "1", four points that vibrate around the intersections of the perpendicular lines and the orange numeral "4" appear on the screen simultaneously (see Figure 2.12b). The vertical line together with the points on it moves towards the left edge of the screen. The black numeral "1" follows the vertical until it is stabilized (see Figure 2.13a).



(a) (b)

Figure 2.13. (a) 1 four-ple; (b) 9 four-ples.

In this episode, certain signs that are created in each modality reveal contrasting relationships. For example, using right and left hand creates a bodily contrast; the orientation of horizontal and vertical lightning rods, the vertical and horizontal routes of the black numerals create a visual contrast. In addition to these contrasts, placing fingers on two different sides of the diagonal creates a spatial separation between the line-making fingers and the lines they create. These contrasting signs might be related to two distinct quantities that have distinct referents (as per Vergnaud (1988) and Schwartz (1988)). The two numerals that represent these quantities might be associated with two distinct unit-counts of multiplication (as per Davydov, 1992). While one four-ple (the vertical line that has four points on it) might be associated with the larger unit-count, four one-ples (the horizontal lines with one point) might be associated with the smaller unit-count.

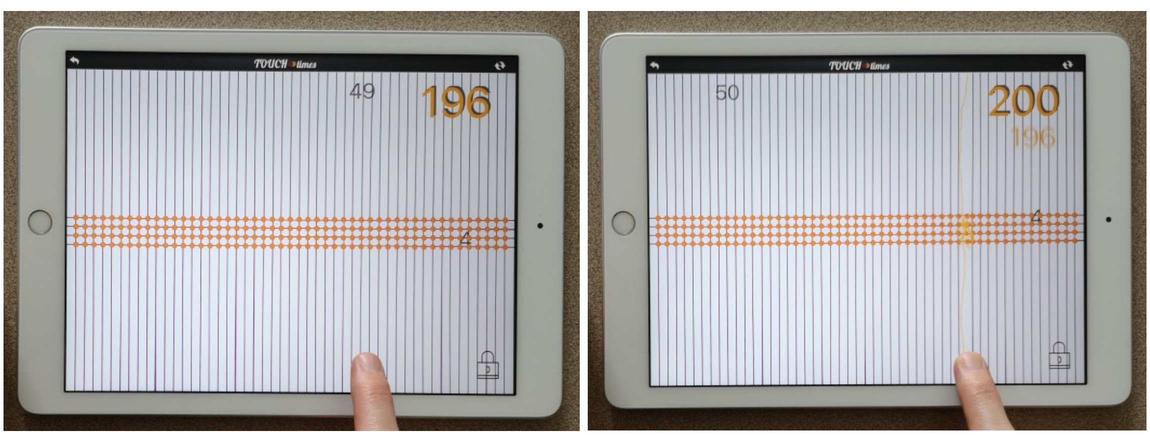
As soon as the vertical line intersects the parallel lines, this creates as many points as the number of parallel lines. Creating multiple points with one finger might be associated with the transfer of unit-counts in multiplication (as per Davydov, 1992). Moreover, as happens in the solution of the task by using pencil-and-paper, the number of points on a four-ple might be associated with the functional relationship between the sizes of the unit counts (as per Vergnaud, 1988). In addition to the functional relationship, four points on the vertical line can also be associated with the product in this episode.

The idiosyncratic character of the Zaplify objects might mediate the difference between the factors and the product. While the lines can be related to the factors of multiplication, points can be related to the product. The difference between the colours and the movement of the numerals contributes to this separation as well: while the moving numerals that represent the number of lines are black, the static numeral that represents the number of points is orange. Thus, it might evoke for students a meaning for multiplication different from repeated addition in which both addends and the sum are represented with the same type of objects. However, the factors and the product do not act as independent entities. As the vertical line moves to the left, the points on it move to the left with the same pace. This might be associated with the co-varying relationship between the product and the factors. In a later episode, I start to increase the product by repeatedly tapping my right index finger along the bottom of the screen as I say, “I press my one finger”. This mode of use might be related to various meanings of multiplication. During this episode, the four-ple and the black numeral “1” stay at the centre of the screen and the black numeral “4” moves up and down continuously on the right edge of the screen (see Figure 2.13a). A new vertical appears on the screen with each tap. Each vertical line passes below the right index finger and slides (either to the right or to the left) when the finger is lifted. Every time a new vertical is created, four new points appear where the vertical line intersects four horizontals. At the end of this episode, there are nine verticals on the screen (see Figure 2.13b). During this episode, the black numeral increases by one starting from “1” until “9”, it moves left and right at the top of the screen (see Figure 2.13a,b). The orange numeral increases by four starting from “4” until “36” (see Figure 2.13a,b).

The repetition of unitizing action creates the same signs again and again that might result in a generalization about multiplication. Every time I place a new finger below the diagonal, a new four-ple (a line with four points) appears. The number of points on each four-ple is the same as the number of horizontal lines. Thus, this repeating four-ples might evoke for the students the functional relationship between the units as a constant entity that is spread across each larger unit count (as per Vergnaud, 1988).

The simultaneous actions may create connections between the Zaplify objects. Both the black numeral that represents the factor and the orange numeral that represents the product simultaneously change with each tap. Thus, the synchronized haptic and visual experiences might create a connection between the factors and the product. However, the amount of change in the numerals is not the same. While the black numeral that represents the number of vertical lines increases by one, the orange numeral increases by four, which corresponds to the number of horizontal lines. Therefore, the horizontal lines might be perceived as playing a role in this connection between the verticals and the orange points. This imbricative connection might be associated with the covariation of product with respect to both factors.

In the above analysis, I illustrate the solution of the third subtask. The rest of the subtasks can be solved in a similar way by creating different M-ples and repeatedly tapping one finger only on one triangular area until reaching the given product. However, among these five M-ples only the two-ples, three-ples, and nine-ples would allow students to create exactly the given target product 198. Working with different M-ples, students would not reach the given product in each case. For example, when they create four-ples to reach 198. In this case, after they make the product 196 by pressing the 49th four-ple, pressing one finger on the screen would create the 50th four-ple. Since the orange numeral would abruptly change from 196 to 200 with the 50th four-ple, the students cannot see 198 on the screen contrary to their expectation (see Figure 2.14a,b). This unexpected change in the product might prompt students to explore the covarying relationship between the factors and the product.



(a) (b)

Figure 2.14. (a) Making the 49th; (b) the 50th four-ples in Zaplify.

2.8. Discussion

As a researcher who works on educational technologies, one of the questions I have been repeatedly asked is “what can this technology do that pencil-and-paper cannot do?”. In this paper, I answer this question by analysing the semiotic potential of both pencil-and-paper and Zaplify. However, the unbreakable link between the task and the semiotic potential of an artefact problematizes this question. As the tasks are created based on their artefacts (e.g., dragging task in a dynamic geometry environment (DGE) is a task specific to DGE), different artefacts prompt distinct modes of use to solve the task (e.g., drawing lines with pencil-and-paper versus pressing fingers on the iPad screen). The different modes of use have potential to create different personal meanings, which can in turn evoke different mathematical meanings. Therefore, rather than asking “what does Zaplify adds to the pencil-and-paper?”, the question should be “what is the unique contribution of each artefact to students’ meanings?”. However, this does not imply that those meanings are always mutually exclusive.

The analysis of the semiotic potentials of Zaplify and pencil-and-paper show that each artefact has the potential to mediate meanings that could help students distinguish multiplication from repeated addition. Both artefacts can mediate multiplication as a binary operation in which each quantity has a distinct referent. While the horizontals quantify one factor as the smaller unit count, the verticals quantify the other factor as the larger unit count (Davydov, 1992). The points on the M-ples represent the functional relationship (as per Vergnaud, 1988) between the smaller and the larger units. In each artefact, the referent of the product is different from the factors.

The binary nature of multiplication can be mediated based on the contrasting signs. Each artefact has the potential to mediate such signs. However, Zaplify can provide students with more means to create contrasting signs. For example, the diagonal in Zaplify that visually divides the screen into two separate areas might also be related to the difference between the referents of two multiplicative factors. Moreover, the numerals in Zaplify quantify two different types of lines. Therefore, they might also be related to the distinction between the referents of the factors. Students may also add numerals on their drawings. However, these signs would be static on paper. Whereas in Zaplify, numerals move in two different directions that might give students another opportunity to discuss the difference between the referents of the factors.

Solving a task in Zaplify results in simultaneous actions. This simultaneity presents another unique contribution of Zaplify in mediating various meanings for multiplication. When using Zaplify, a student can use both of her hands, each spatially separated, at the same time. This has the potential to provide a haptic experience that can differentiate the factors. While drawing, students can also experience a spatial haptic difference because of the orientation of tracing gesture. However, these separations would also be temporally apart, lessening the contrast between the haptic experiences.

The simultaneous actions conducted in Zaplify also support the idea of the transfer of unit counts (as per Davydoy, 1992). In Zaplify, one finger brings M point(s) as soon as the M-ple-making finger touches the screen. This transfer is not simultaneous when using pencil-and-paper. Every point must be drawn sequentially, so the correspondence between the unit counts is revealed after students complete their drawing and count the total number of points on the line. Therefore, this temporal separation between the points might hinder the emphasis on the simultaneous unitizing action of multiplication.

Pencil-and-paper can also contribute to students' meaning making process in a unique way. The intersections between the perpendicular lines and the points in Zaplify appear without users' direct manipulation: students interact with these signs only through visual experience. Whereas when students draw the intersections and the points with pencil-and-paper, they create these signs both with visual and haptic experiences. Therefore, drawing an array with pencil-and-paper might provide more means to mediate the relationship between the factors and the product.

Each artefact has a unique contribution to the mediation of multiplication as different from repeated addition. Therefore, instead of choosing one artefact over the other, combining these similar but unique artefacts in the learning process would enrich students' meaning-making process as Maschietto and Soury-Lavergne (2013) suggested. They conceptualized the combined use of a physical pedagogical artefact and its digital counterpart as duo of artefact and showed that the use of technology added value to the use of physical objects as classroom teaching equipment because different artefacts triggered different signs, and different signs lead to different cognitions.

The analysis of the semiotic potential of the artefacts revealed two types of interplay between the signs: the contrasting play and the parallel play. In a contrasting play, signs are situated as opposed to each other or the signs that signify differences become salient. Whereas in the parallel play, signs that signify a commonality between the entities are related to each other with linguistic, visual, or haptic connections. Even though in each episode the interplay between visual and haptic signs carries the contrasting and the parallel meanings simultaneously, prompting certain linguistic signs might highlight the targeted interplay between the signs. For example, in each episode, while the vertical and horizontal lines present a contrast between the Zaplify objects, the repetition and the simultaneity of actions present a similarity between the objects. Therefore, the users would simultaneously experience these interplays. When a question prompts students to create additional linguistic signs within a parallel interplay, these signs would highlight the experiences related to similarity over contrast for that session.

The semiotic analysis of artefacts based on TSM is conducted with respect to a pre-given task. However, this analysis can also give educators insights about how to design new tasks. Starting the analysis with the targeted mathematical meanings and relating them to the specific modes of artefact use provide educators with a framework to progress in their designs. The task designer first decides which mathematical meanings to evoke and then they consider which mode of artefact use might be related to these meanings, as illustrated in Fahlgren (2016). The compartmentalization of the video-recording into ten-seconds intervals shows that the main task (making 198 in Zaplify) consists of many sub tasks that require different modes of artefact use. For example, creating an M-ple, increasing the number of M-ples, making a specific product are three subtasks embedded in this main task. Each of these subtasks requires a certain mode of artefact use that would mediate specific personal and mathematical meanings. Therefore, the identification of appropriate subtasks and the integration of them in a specific way would evoke the desired mathematical meanings.

Since the meanings emerge from the interplay of the signs (Maracci & Mariotti, 2012), examining the relationships between the signs that participate in the targeted mathematical meanings might help the task designer to prompt an appropriate mode of use for the artefact. Let us take Confrey's (1994) description of multiplication: "creating versions of an original" (p. 292). This expression consists of the interplay between three main signs: “creating”, “version”, and “original”. Here, original refers to the first example of its species. Therefore, the question of “what counts as the original in Zaplify” would be the catalyzer for the designer to choose a specific subtask. However, the separate analysis of these signs would reduce the overall meaning of multiplication. For example, analyzing the sign “original” without considering the sign “version” would be incomplete. It is important to ask, “how are the original and the version related to each other?”. This relationship can be associated with a common notion of multiplication that is emphasized by many researchers, yet with different names: functional relationship (as per Vergnaud, 1988), intensive quantity (as per Schwartz, 1988), many-to-one correspondence (as per Clark and Kamii, 1996). Therefore, the original and its version should be prompted according to the specific relationship between them. For example, if one point is considered as the original, then making an M-ple would be considered as the version of an original. If an M-ple is taken as an original, making multiple M-ples would be a version of it.

2.9. Conclusions

Task design, most of the time, does not explicitly designate a specific technology. However, the pencil-and-paper is generally the implied technology especially when the task prompts students to use static mathematical models. However, with digital technology-specific tasks, we also do not often think about what the task might offer in other technological settings. In this paper, I have analysed the model–task dyad in two distinct settings through the theory of semiotic mediation to examine the affordances of the Zaplify model in mediating multiplication as different from repeated addition. This is similar to identifying the potential of a duo of artefacts in mediating mathematical meanings in the sense that the analysis requires close examination of two similar artefacts. However, unlike a duo of artefact, the analysis of the model–task dyad allows researchers to imagine the possibility of two artefact by considering how a single task might be interesting and relevant to each.

The analysis of the semiotic potential of two artefacts with respect to the same task illustrates how the tasks can undergo some changes when attempted in two different environments. These changes seem to be minor if the focus is directed at the end product, which is the array. However, the processes that result in the array in both environments include many actions that can mediate quite different meanings for multiplication. This analysis suggests that while designing a task, the focus should not be only on the end product; instead, designers should also consider the bodily actions involved in the task solution.

As the term suggests, this analysis identified the potential of these artefacts to mediate various meanings of multiplication. The next step will be to study how these potentials do/not unfold in a real classroom when students attempt to solve the task with each of these artefacts.

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Postlude

Re-engaging with this article a few months after it was published, I found some points that could be re-addressed. These are related to my methods to analyze the data, the illustration of the Zaplify objects and the Davydovian notions of smaller and larger unit-count.

I have not found an explicit account of a method for the analysis of the semiotic potential of an artefact. Based on the reports of such analyses, undertaken by other researchers, I assumed that they simply imagining how a learner might use an artefact. I devised two different methods in which I used the artefact myself to solve the multiplication task. I used one method for the analysis of pencil-and-paper and the other one for the analysis of Zaplify. In the former method, I took the photos of each action that was necessary to complete the task. In the latter method, I video recorded myself while I used Zaplify and created signs in different modalities. These methods allowed me to base my analysis on concrete snapshots of the events.

I realized that I missed some important information in my analysis when I used the images as the basis of my data creation. For example, when I reported on my analysis of the semiotic potential of pencil-and-paper, I wrote that "In order to create an M-ple that is aligned with the multiplication expression M × 1, students must first draw as many parallel lines as the number of points that should be on the M-ple". When I engaged with this finding after my paper was published, I found it problematic because it is impossible to create Mx1 by using pencil-and-paper. Each line had to be created sequentially unless M = 1. Therefore, whoever uses pencil-and-paper has to create (1 + 1 + 1 … + 1) x 1, not Mx1. This is an important difference as it may influence how a learner experiences a quantity (as a single multitude or multiple singles).

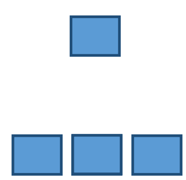
The illustration of M-ples as shown in figure 2.8 on page 65 is also problematic. I would modify this image by adding horizontal lines to the vertical M-ples and vertical lines to the horizontal M-ples. Because these added lines would create intersection points and index the genesis of the dots. Otherwise, the image cannot completely illustrate my following statement: "While each horizontal line corresponds to one point, the vertical line corresponds to M points and thus constitutes an M-ple. Therefore, the former might be associated with the smaller unit-count, while the latter might evoke the larger unit-count”.

After engaging with this statement, I felt that I should clarify the Davydovian notions of smaller and larger unit-count, as well. This statement is true only for the individual M-ples. It might be confusing when the reader engages with this statement with respect to an image as in Figure 2.13b. After the fourth M-ple, the horizontals include more than 4 points on it. So, one might relate horizontals with the larger unit-count and the verticals with the smaller unit-counts. However, the smaller and larger unit-counts do not refer to the units with the larger/smaller quantity on them. The larger unit-count is the composite unit created through unitization of the smaller unit-counts. Therefore, the larger unit count emerges in relation to the smaller unit-count. In other words, it can be identified in Zaplify after the smaller unit-count is designated. The temporality of the actions in Zaplify is one way to distinguish these two unit-counts. If the temporality is disregarded, it is possible to experience either the horizontal or the vertical lines as the larger unit-count. In this respect, arrays prove to be versatile to model multiple multiplicative situations on the same diagram.

It is also important not to confuse the smaller and larger unit-counts with the magnitude of multiplicative factors. Think about the multiplication expression 3 x 1/4 = 3/4. According to Davydov (1992), this equation models a situation in which an amount or quantity is measured by transforming the unit-count from smaller to larger. In this situation, the smaller and larger unit-counts can be represented as in the following:

Smaller unit-count:

Larger unit-count:



The first factor is the measure of larger unit-count with respect to the smaller unit-count. The second factor refers to the measure of quantity with respect to the larger unit count. The product is the measure of quantity with respect to the smaller unit count. For example, in this situation the larger unit-count corresponds to three (as per the first factor) smaller unit-counts. The amount of the quantity to be measured corresponds to one fourth (as per the second factor) of larger unit-count. The amount of the quantity to be measured corresponds to three fourth (as per the product) of smaller unit count. As the example illustrates, the types of the unit-counts are not determined by the magnitudes of the factors. They depend on the experience of the person who conducts the measuring activity.

Prelude to Chapter 3

The following article is a research paper which evolved from one of my comprehensive exam papers. In that paper, I explored the theory of semiotic mediation by analyzing a video-recording of my neighbour's child using TT. In my analysis, I focused on the artefact and pivot signs the child created when he used pencil-and-paper and Zaplify back-and-forth to solve the same tasks. The main tasks were making one, two, three and six dots in Zaplify and they corresponded to unitizing and spreading, two of the key ideas that underlie multiplication.

The combined use of a physical pedagogical artefact and its digital counterpart is described as a duo of artefacts in the literature. In most of the research studies, the duo of artefacts is provided to the students with a specific order in which the digital artefact follows the non-digital one. This order might be related to the assumption that the digital artefact expands the affordances of the non-digital one. However, as the first article in this dissertation shows, each artefact has some unique affordances.

Even before writing the first article, I thought that this non-digital followed by digital artefact order might hinder the potential of physical artefact to enrich the affordances of the digital counterpart. I wondered how using a duo of artefacts back-and-forth might influence students' meaning making. I explored this new way of using a duo by reviewing my aforementioned analysis of the task "making one dot" and shared my findings in a departmental conference. Later on, I extended my conference paper into the following article which was published in the Simon Fraser University Educational Review Journal on August 5, 2021.

Even though this article dates back to my initial exploration of TSM, it relates to the first article by extending its findings. In the first article, I explored the semiotic potentials of Zaplify and pencil-and-paper separately. In the following article, I explored how a child made sense of Zaplify when he used pencil-and-paper and Zaplify back-and-forth; and how his personal meanings were related to multiplicative thinking. The following article answered two research questions:

- How do signs evolve when a child reciprocally uses a duo of artefact?
- How does a child experience the relationships between the Zaplify objects?

Chapter 3.

Reciprocal Influences in a Duo of Artefacts: Identification of Relationships that Serves to Multiplicative Thinking

Abstract

The combined use of a physical pedagogical artefact and its digital counterpart is described as a duo of artefact. In the literature, duos of artefacts are mostly presented with a certain order: non-digital artefact is followed by the digital counterpart. This study examines the influence of reciprocal use of artefacts in a duo on a 5-year-old child's identification of multiplicative relationships between the objects. The data is created through the video record of two clinical interviews with the child. The results showed that the reciprocal use of the artefacts enriched the child's experiences of each artefact and mediated the relationships which were important for multiplicative thinking.

Keywords: duo of artefacts, multiplicative thinking, drawing, educational technology

3.1. Introduction

Studies show that mathematical tasks which require students to manipulate physical artefacts enhance mathematical teaching and learning (Carbonneau et al., 2013). However, the rigid structure of artefacts might prevent teacher from modifying them in a way to increase their mathematical potentials. At this point, their digital counterparts add value to the use of physical objects as classroom teaching equipment because different artefacts trigger different signs (e.g., natural language, gestures, and mathematical semiotic systems), and different signs lead to different meanings. Digital counterpart can achieve this through "offering students a new opportunity to identify the mathematical properties embedded in the artefact behavior and more abstract and conventional representation of mathematical objects" (Soury-Lavergne, 2017, p.1). This combined use of a physical pedagogical artefact and its digital counterpart is described as *duo of artefacts* (Maschietto & Soury-Lavergne, 2013).

Integrating duo of artefacts in mathematics classes is a recent practice, but it has already demonstrated some positive outcomes (see below for more detail). In most of these studies, the duo of artefacts is presented with a certain order: first, students are introduced the physical artefact and then they are given the digital counterpart. This restrictive order suggests that the duo of artefacts enhances mathematical ideas through the added value of digital counterpart only. However, this one-directional approach might hinder the potential of physical artefact to enrich the affordances of the digital counterpart. In the literature, various duos of artefacts have been used to introduce students to various mathematical topics. I will study how reciprocal use of a duo of artefacts enhances the mathematical ideas related to multiplicative thinking which, to the author's knowledge, has not been studied with respect to a duo of artefacts yet. In this study, the digital artefact is a free tablet application called *TouchTimes* (Jackiw & Sinclair, 2019) which is designed to develop multiplicative thinking through creating quantities in specific ways. The physical artefact is the pencil-and-paper, through which students draw the target numbers they created with Zaplify – one of the *TouchTimes* "worlds".

3.2. Duo of Artefacts

Drawing on instrumental approach, Soury-Lavergne (2021) proposes a difference between "two artefacts" and "duo of artefacts". According to the instrumental approach, when individuals encounter an artefact (material entity), they construct utilization schemes (psychological entity) as they interact with the artefact. The combination of the material and the psychological entities generates a specific instrument for the individual. This is called instrumental genesis. For example, upon seeing a plastic circular object (material entity), someone might think of placing it on the paper and circumscribing (psychological entity) to create a geometrical diagram.

The difference between "two artefacts" and "duo of artefacts" depends on the nature of instrumental genesis they prompt. The former suggests two separate instrumental geneses of two separate artefacts. Whereas the duo of artefacts constitutes a system that emerges through the joint instrumental genesis of two artefacts. Soury-Lavergne (2021) acknowledges that the new instruments integrate the previously developed instruments into its form creating a system rather than an isolated independent instrument. As it is not practical to identify all the previous instruments in a system, she proposes to reduce the complex system of instruments into duo of a tangible entity and a digital one to study their influence on learning.

Drawing on Bourmaud, Soury-Lavergne (2021) indicates three conditions for the joint instrumental genesis triggered by a duo of artefact: complementarity, continuity and antagonism. When two artefacts are used together (either simultaneously or successively) they complement each other. However, the complementary use of artefacts may not result in a joint instrumental genesis without a continuity between them. When the artefacts are used in relation to each other, shared characteristics or elements of the artefacts build a continuity. On the other hand, the divergent features/functionalities of the artefacts result in antagonism between them. These divergences create constraints for the users' existing schemes and prompt them to adapt their schemes when passing from one artefact to the other.

These three conditions explain why providing two artefacts may not be effective in creating a system of artefacts that results in joint instrumental genesis. This is illustrated in Lei et al. (2018) that examined an ineffective combination of a material and a digital tool. The material artefacts were a tape measure and theodolite. Whereas the digital artefacts were two apps installed in tablets called EasyMeasure and Angle Meter. The teacher provided the students with this duo of artefacts to introduce the concept of percentage error. One of the main reasons Lei et al. (2018) attributed to the failure of the duo was the difference between the artefacts. Apart from their functions, which was to measure, they did not share any feature. When we consider Lei et al.'s (2018) finding with respect to the conditions cited by Soury-Lavergne (2021), it could be said that there is little opportunity not only for continuity but also for antagonism. So, unlike a duo of artefact, these two tools did not lead to a system of artefacts that triggered a joint instrumental genesis. Therefore, it is not appropriate to call them as duo of artefacts from Soury-Lavergne's perspective. Unlike this counterexample, the literature presents various successful use of duo of artefacts in teaching and learning mathematics. The following section summarizes a few of them. The exemplar studies are chosen to represent the diverse use of duos in mathematics lessons.

3.3. Teaching and Learning through Various Duos of Artefacts

Maschietto (2018) studied how the Pythagorean theorem was introduced to 7-grade students in a composite environment which consisted of a material and a digital tool. One of the material tools was a mathematical machine which consisted of four congruent wooden right triangles that fitted into a wooden square. The square was covered with a red paper and surrounded by a frame. The digital tool was an Interactive Whiteboard (IWB) on which the teacher created the digital version of the mathematical machine. The tasks were (1) to obtain red square areas by placing the triangle prisms into the square frame and (2) to change the configuration to obtain a larger red square which is surrounded by the triangle prisms. While students directly manipulated the mathematical machine, the digital tool was manipulated only by the teacher and a few students to switch between the configurations of the triangles on the board. Even though many students did not manipulate the digital tool directly, Maschietto (2018) proposed that the conservation of the square areas was emphasized through linking the manipulations of the triangles in the digital tool with the manipulations of the triangles in the mathematical machine. This conversation helped students deduce the Pythagorean theorem.

Van Bommel and Palmér (2018) compared six-year-old students' responses to a combinatorial task when they used only physical artefacts and when they used a duo of artefacts. The task was to find how many different ways three toy bears can be arranged in a row on a sofa. The physical artefacts were the toy bears, paper and a number of coloured pencils to record the arrangements. The analysis of the children's drawings revealed many duplicates in students' solutions and thus indicated that students did not systematize their solutions. The digital artefact was designed based on these findings to provide the children with feedback about the duplicates. When the students used the duo of artefacts, they were first introduced the digital artefact and then asked to find the number of seating arrangement by using paper and pencil. The results show that the children who solved the task via the duo of artefacts were found to keep more systematic records of the situations and to enhance their understanding of what a duplicate means in a combinatorial problem.

Soury-Lavergne and Maschietto (2015) studied how a duo of artefacts was used by six years-old students to learn about numbers. The students first worked with pascaline, a mechanical machine made of gears which allowed students to create and to add numbers symbolically by rotating them. The digital counterpart of pascaline was embedded in an e-book. The students were given two tasks. One of them asked students to add two numbers. The other one asked them to write a number with minimum rotations. The findings showed that the duo of artefacts prompted the students to connect the separate conceptualizations of quantity and digit.

All the duos used in these studies conform to the three principles that would result in a joint instrumental genesis. While they differ from each other in terms of mathematical topics they develop, the type of artefacts involved in the duo and the nature of the tasks they posed; the order of the artefacts was the same across all of these studies: either the digital artefact was followed by the non-digital counterpart or vice versa except for one case. In Soury-Lavergne and Maschietto (2015), one teacher made the physical artefact available again after the students had difficulty to solve the tasks in e-pascaline. This bi-directional use of duo is unique among these studies and it suggests a new way to exploit the potential of the duo. Compared to using each element of a duo individually in successive occasions, manipulating them reciprocally during a mathematical activity might enhance the integration of instrumental geneses more strongly.

In this study, I will examine reciprocal use of a duo which involves pencil-and-paper as its non-digital element. Compared to the artefacts like the mechanical machines used in Soury-Lavergne and Maschietto (2015) and Maschietto (2018), pencil-and-paper provides students with a special medium to create meanings with less restrictions that stem from the physical structure of the artefact. This use of pencil-and-paper is different from using drawings only to express and record thoughts after manipulating the mechanical artefact, which was the case in all three studies. However, the unrestricted diagramming might deviate learners from the target mathematical idea unless it is repeatedly restructured based on the manipulation of the digital artefact which embeds the intended mathematical relationships within its design, in this study the multiplicative relationships.

3.4. Multiplicative Thinking

Multiplicative thinking is conceptualized by many researchers in a unique way. Even though they slightly differ from each other and focus on the different aspects of the concept, one thing is shared by all: it is different from additive thinking. Schwartz (1988) focuses on the referents of quantities in these operations. While the quantities refer to the same entity in addition (e.g., 5 apples + 4 apples = 9 apples), different type of quantities are operated on in a multiplication (e.g., 5 kg of apples per bag x 4 bags =20 kg apples). Similarly, Clark and Kamii (1996) points to the abstraction of the number of units involved in both operations. While addition is conducted with only one unit-count (that quantifies only the individual apples in the previous example), in multiplication one operates on two unit-counts (one that quantifies the bags of apples, the other that quantifies the weight of the apples per bag).

Vergnaud (1988) emphasizes the relationship between the unit counts a child establishes in an operation and distinguishes scalar relationships from functional ones. For example, when asked the problem “Amy wants to buy 4 bags of apples. Each bag has 5 kg of apples. How many kilos of apples does she buy in total?”, a student might show the solution either with 4x5=20 or with 5x4=20. Even though they are both multiplications, Vergnaud says that “the relationships that leading to these choices are very different” (p. 145) and illustrates the difference using the following T tables in Figure 3.1.

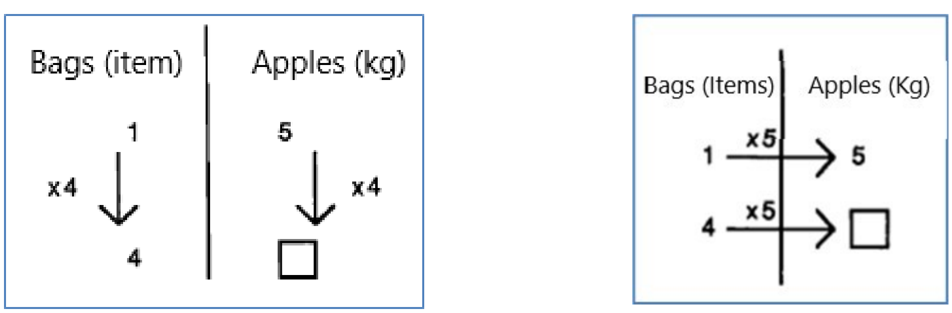


Figure 3.1. Illustration of 4x5=20 and 5x4=20.
Note: Adapted from “Multiplicative structures” by G. Vergnaud, G., in J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161), 1988, Lawrence Erlbaum Associates. Copyright 1988 by The National Council of Teachers of Mathematics.

In the first case students attends to the ratio between the same quantities which is a scalar. Therefore, 4 x 5 is a “concatenation” of 5+5+5+5: the amount of apples = the amount of 1 bag, plus the amount of 1 bag, plus the amount of 1 bag, plus the amount of 1 bag (Vergnaud, 1988, p. 146). Whereas in the second case (5 x 4) the student attends to the ratio between the different quantities. In this case 5 is not a scalar, it is associated with a many-to-one correspondence between the unit counts: 5 kilos per 1 bag.

In addition to this static relationship between the two unit-counts, Davydov (1992) points to a dynamic feature of multiplication when he defines it as the transfer of unit counts. He explains the meaning of multiplication with respect to measuring activities and distinguishes a small and a large unit-count which both quantify a given magnitude of an object. Measuring a magnitude (e.g., apples) with the small unit (kg) would be impractical. Therefore, one indirectly quantifies the magnitude in relation to the smaller unit by transferring the unit count from the smaller to the larger (bags) thanks to the established relationship between the two (5 kg/bag). This transfer implies a simultaneous multiplicative action.

Drawing on Davydov's notion of transfer of unit-count and Vergnaud's notion of functional relationship, Jackiw and Sinclair (2019) designed *TouchTimes* (TT) to enhance multiplicative thinking. TT consists of two models or "worlds" – Zaplify and Grasplify. Davydov's and Vergnaud's multiplicative notions are conveyed in both worlds, yet through distinct models. Thus, Zaplify and Grasplify prompt learners to experience these multiplicative ideas in two different ways. This paper will focus only on the former world (see Bakos & Pimm, (2020) for more details on how Grasplify world prompts these multiplicative notions).

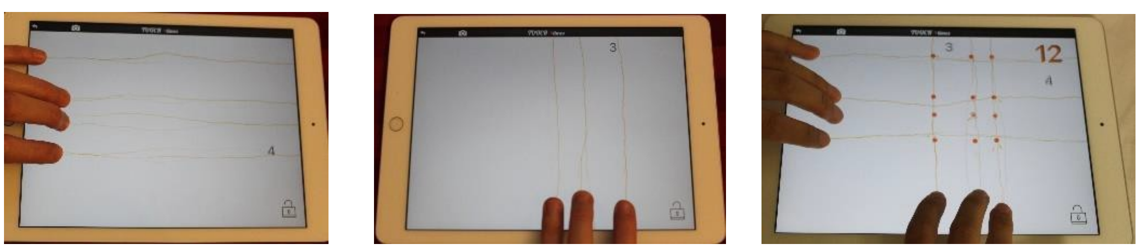
3.5. Zaplify

When entered, this world shows an empty screen. When the tablet is placed horizontally on a surface, seven fingerprints and a diagonal line appear respectively in order to guide users to place their fingers both horizontally and vertically in the designated areas separated by the diagonal (see Figure 3.2a & 3.2b).

(a) (b)

Figure 3.2. (a) Fingerprints and (b) fingerprints and the diagonal.

When a user places and holds any finger on the screen, a “lightening rod” (I will call them “lines” from now on), which passes through the point of touch and crackles dynamically, appears on the screen either horizontally or vertically according to the position of the touch with respect to the diagonal. The upper-left triangular area formed by the diagonal allows horizontal lines (HL), while the lower-right triangular area allows vertical lines (VL). Screen contact can be made with one finger at a time or with multiple fingers simultaneously. Multiple fingers that maintain continuous contact can create either only HL, only VL or both VLs and HLs (see Figure 3.3 a-c). Whenever two perpendicular lines intersect, an orange disc gradually appears on the intersection points. The numerical value of the total number of intersections, which is the product of the two factors, appears in the upper right corner of the screen (see Figure 3.3c). If there is no intersection, only the number of factors appear (see Figure 3.3 a,b).



(a) (b) (c)

Figure 3.3. (a)HLs, (b) VLs (c) VLs and HLs.

There are two modes of manipulation of the app: locked and unlocked. In the unlocked mode, the lines disappear as the fingers separate from the screen, whereas in the locked mode, lines remain on the screen even when the user’s finger is lifted, but no longer crackle dynamically. This allows a user to create products that involve more than ten fingers.

The Zaplify objects, the gestures that create these objects and the relationship between these objects are all associated with various aspects of multiplicative thinking. The vertical and the horizontal lines represent the two unit-counts of multiplication. The orientations of lines may help students distinguish these units-counts. In addition to this visual difference, the separation of the units may be associated with the difference in the haptic experiences. While the horizontal lines can be created only by touching the upper triangular area, one must touch the lower triangular area to create vertical lines. Pressing fingers to create parallel lines on one triangular area and then pressing down a finger on the opposite side to create a perpendicular line can be associated with Davydov's notion of transfer of unit counts. In this case, the unit count is transferred from the parallel lines to the perpendicular line. This transfer results in Vergnaud's notion of many-to-one correspondence between the units: many units represented by the parallel lines correspond to the new unit which is represented by the perpendicular line (see Güneş, 2021, for a more detailed explanation of how Zaplify can prompt multiplicative thinking).

3.6. Theoretical Framework

This study draws on Bartolini Bussi and Mariotti's Theory of Semiotic Mediation (TSM). This theory focuses on the relationship between the representation systems and the human cognition. Human beings create representations through using artefacts and this has two consequences: the modification of the environment and the cognitive development. TSM is based on this double nature of artefacts.

An artefact does not guarantee a specific use for the subject. Indeed, Rabardel (as cited in Bartolini Bussi & Mariotti, 2008) distinguishes artefacts from instruments. An artefact is a concrete or a symbolic object itself. It becomes an instrument by the subject through its particular use. For example, a glass is an object which is designed to carry liquid. If a cook uses it to crash some walnuts into smaller pieces by pressing the walnuts between the bottom of the glass and a cutting plate, the glass becomes an instrument.

The instrumental approach to artefacts can be informative in analyzing the cognitive processes related to the use of a specific artefact and its semiotic potential. However, it is not adequate to analyze the more complex process of teaching and learning mathematics through artefact use. At this point, Bartolini Bussi and Mariotti (2008) resort to Vygotsky's approach to artefacts.

Vygotsky talks about the difference between an individual's developmental levels in two different situations: (1) when an individual is able to accomplish a task him/herself, and (2) when an individual can accomplish a task with the guidance of a more knowledgeable individual (as cited in Bartolini Bussi & Mariotti, 2008). This difference is called the *zone of proximal development*.

Within this zone, the communication between the individual and the more knowledgeable one leads to the cognitive development of the learner. The theory of semiotic mediation elaborates more on the relationship between tasks, signs and mathematical meaning making within this process and distinguishes semiotic mediation of artefacts from teachers' cultural mediation.

Using an artefact in a social context, learners produce certain signs which are essential for semiotic mediation. These signs have a dual role: expressing the relationship between the task and the artefact on the one hand, and the relationship between the artefact and mathematical meaning on the other hand. The former is called an *artefact sign* and their meaning is associated with the operations conducted to achieve the task. The latter is called a *mathematical sign* and it is aligned with the existing mathematical culture. On the way to the evolution of artefact signs into mathematical signs, pivot signs are important. The pivot signs "may refer both to the activity with the artefact...and to the mathematical domain" and they are distinguished from the other signs based on the extent of generalization they carry (Bartolini Bussi & Mariotti, 2008, p.757). In this study I asked how the signs evolved during reciprocal use of a duo of artefact.

3.7. Methods

The data is created through the video-recording of two clinical interviews with a 5-year-old child, whom I name Zach. Both interviews lasted for approximately half an hour. Zach used Zaplify and pencil-and-paper during the interviews. The interviews consisted of number-making tasks, drawing tasks, and what-happens task in which I (denoted as R in the below transcripts) asked Zach (denoted as Z in the below transcripts) to anticipate how the number would change if I added more fingers.

Clinical interviews conducted in this study could be described as the derivative of joint inquiry activities which naturally occur in every individual's life (diSessa, 2007). I conducted the interviews at Zach's home. Zach's father (denoted as F in the below transcripts) was present during the first interview, and he participated in the interview by asking questions to Zach when he seemed hesitant to respond. My goal was to help Zach to make sense of Zaplify and to discover how he makes sense of it. Even though the interviews did not carry an instructional orientation (I avoided evaluative comments based on a normative response to the tasks), it would be problematic to deny that manipulating the artefacts while communicating with the interviewer did not contribute to Zach's learning.

The participant is recruited through convenience sampling. Multiplication is generally introduced in the second and the third grade of elementary schools. However, studies show that before formal schooling, young children can demonstrate some aspects of multiplicative thinking (Bakker et al., 2014), for example by extracting the invariant proportional relationship between two numerical magnitudes (McCrink & Spelke, 2010). Therefore, choosing a young participant, this study also contributes to the discussion of whether multiplicative thinking can be developed with instruction in younger ages (as per Askew, 2018) and whether the ordering of the mathematical topics in the curriculum documents that positions learning of multiplication after addition based on a hypothesized developmental learning progressions can be challenged (as per Bicknell, et al., 2016).

In this analysis, I focused on the signs Zach created via the duo of artefacts, drawing from Arzarello et al.'s (2009) concept of *semiotic bundle*. There are two ways to analyze a semiotic bundle: synchronic and diachronic analysis. The former focuses on a specific moment where the subject produces different signs spontaneously. The latter focuses on the evolution of the signs produced by the subject in successive moments. I also analyzed different signs created by different artefacts at different time points in a synchronic manner in order to examine the relationship between the artefact signs.

3.8. Findings

In the following, I highlight how Zach identified relationship between mathematical objects via duo of artefacts. I characterize the instances with excerpts from the interviews.

At the beginning of the first interview session, Zach randomly made one orange disc on Zaplify. Zach described the orange disc as a dot. When I asked him to make one more, he could not make it. During the following 18 minutes, while Zach was holding HLs, I was adding VLs one by one, making 2, 4, 6, 8, 10, and 3, 6, 9, 12, 15 respectively. Then I asked Zach to make "one" again, assuming that creating numbers repeatedly on Zaplify might have helped Zach to identify the relationship between the lines and the discs. As I pointed to the upper right corner of the screen, I said: "I want to see the [numeral] one here and one orange ball". After a few attempts, he could not make any disc. Then I asked him to draw one disc:

1. R: In order to get one dot, what we should see? How does one dot appear? Can you draw one dot? How was it on the screen when we see one dot?
2. Z: It was small and red [*drawing a circle*]
3. R: Were there anything else other than the dot?
4. Z: A yellow line
5. R: Where was it?
6. Z: … [drawing a curvy line which looks like a wave just below the circle]

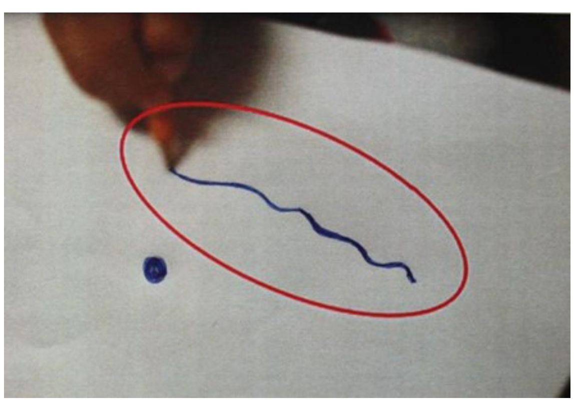


Figure 3.4. Horizontal curly line.
Note: The author retraced the pencil marks in the pictures to improve visibility.

Zach used the words “small” and “red” in order to describe the dot. These artefact signs refer to physical features of the ball unlike its position, which might suggest a relationship between the other artefact signs such as lines and the intersection point. When I drew Zach’s attention to the other artefact signs (line 3), Zach uttered the word “yellow line”. This artefact sign includes a mathematical sign, which is a “line”, yet it also refers to the color of the line in order to describe it. Again Zach created signs related to the physical features of the objects rather than their orientation (e.g. horizontal/vertical), which is important in terms of multiplicative relationships. When I hinted the orientation by asking where it was (line 5), Zach created a sign in another modality. Rather than describing it with verbal signs, he created a visual sign with his drawing (see Figure 3.4). This sign illustrates the line in horizontal orientation as in the Zaplify, yet separate from the disc. So it seems that Zach did not relate the disc with the HL except for their quantities. For one disc, he created one line.

The relationship between the signs appeared in our second trial. After Zach and I together made a disc the second time on Zaplify, I asked him to draw a disc on the paper.

7. R: How did we do one dot? Can you draw it?
8. Z: … [drawing a circle]
9. F: Draw what you saw on the screen. Where were the yellow lines?
10. Z: Where were the yellow lines? One is here and one is here.
11. R: Why don’t you draw it here [*pointing to the paper*]
12. Z: … [drawing one vertical curly line from top to the bottom of the paper, then another one from left to right of the paper crossing over the VL]
13. F: [*pointing to the dot on the paper*] Is this dot on the same spot compared to the screen?
14. Z: No.
15. F: Draw the dot. Where should it be?
16. Z: It should be in the middle of here [pointing the intersection of the lines]

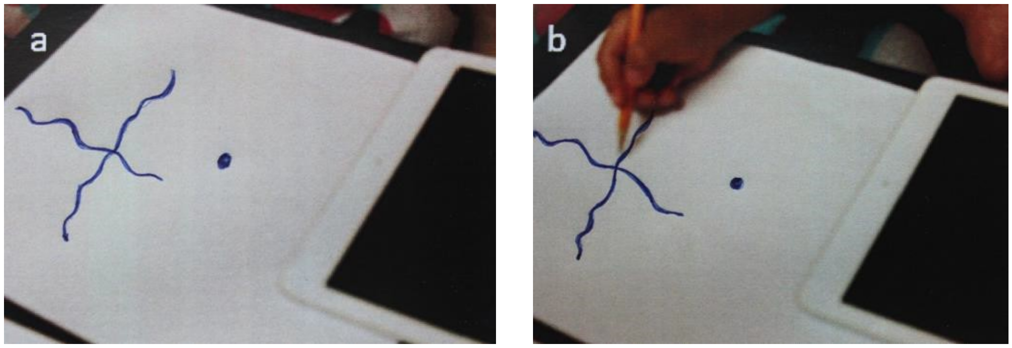


Figure 3.5. (a) Dots and the intersecting lines, (b) pointing to the intersection of the lines.
Note: The author retraced the pencil marks in the pictures to improve visibility.

Compared to the first drawing, Zach produced more signs in this episode. First, he drew one disc and then two lines next to the disc, which intersected each other. So, this physical separation between the lines and the disc in Zach's drawing indicates partial relationship between the artefact signs in that the lines are related to each other, but they are not necessarily related to the disc.

Zach transferred the orientation of the lines from Zaplify to the paper directly. He drew two perpendicular lines as in Zaplify (see Figure 3.5a). When we made one disc together, Zach first held his finger and made a VL, and then I put my finger and made a HL. Similarly, first he drew the VL in this episode. While the order of the lines created in Zaplify was mirrored in his drawing, it was not the case for the order of the disc. In Zaplify, the disc appeared following the lines, but on the paper, he first drew the disc and then the lines. Thus, he did not transfer the location of the disc in relation to the lines in his drawing. Zach connected the disc with the lines (see Figure 3.5b) only after he was asked to compare his drawing of the disc with the diagram in the Zaplify (no. 13-16).

Zach started to create the intersecting lines on the screen after he used his second drawing as a reference to make one disc in Zaplify. However, the relationship between the intersecting points and the discs became solid after we discussed the relationship between the lines at the second interview. Until this episode, Zach answered few "what happens" tasks correctly. After our discussion, he started to demonstrate a consistent strategy to answer these tasks correctly. The following episode presents one such discussion:

After Zach made one disc on the screen, I asked him: “What happens here?” as I pointed to the intersection of the lines.

17. Z: One dot.

18. R: What is happening to the lines here where the dot stays [*pointing to the intersection*]?

19. Z: The dot stays in the middle [*pointing to the dot*] of these [*tracing the VLs and the HLs*] lines

20. R: How did you make this [*pointing to the dot*] in the middle?

21. Z: I put my finger here [pointing to the bottom of the VL] and make the line, and then I put my finger here [pointing the HL] and make the line, and then I make the dot with this line [tracing the HL back-and-forth]

22. R: You made this line [pointing the VL] first, and this one [pointing the HL] second, right?

23. Z: Yes.

24. R: What did the second line do to the first line? What happened here [pointing the intersection]?

25. Z: Second line crossed [*tracing the HL*] the first line [*tracing the VL*]. The dot is with the second line.

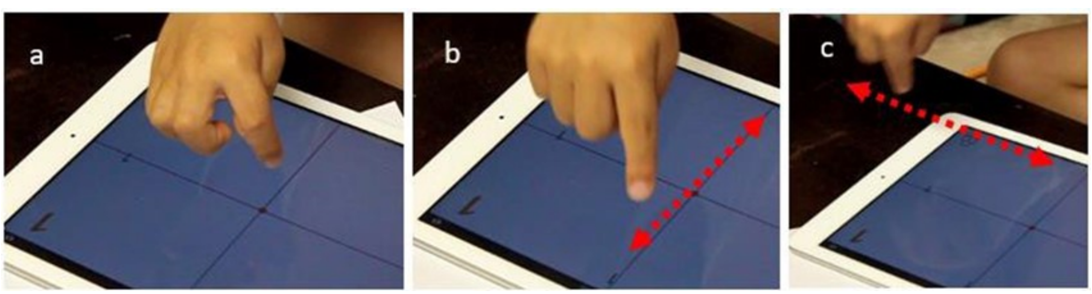


Figure 3.6. (a) Pointing to the dot, (b) tracing the VL, (c) tracing the HL.

Zach referred to the intersection point via a sign “the middle”, which he created during a drawing task in the previous interview (line 16). The verbal sign “the middle” and “these lines” are used together with gestures (line 19). They all together suggest that the orientation and the intersection point of the lines are both related to the location of the disc. The pointing gesture (see Figure 3.6a) and the word “middle” refer to the intersection point, and the tracing of the lines (see Figure 3.6b & 3.6c) refers to the perpendicular lines. According to Zach's verbal accounts, the intersection seems to be necessary for the disc to appear. He stated that he made the disc with the second line, which crossed the first line (line 25). Thus, the sign "cross" points to the relationship between the lines and it is an important sign to create the disc.

3.9. Discussion and Conclusion

In this study I examined the evolution of signs during the reciprocal use of a duo of artefact. The digital artefact was Zaplify which was an iPad application designed to develop multiplicative thinking. The non-digital artefact was pencil-and-paper. The tasks were designed for the duo to help a five-year-old child to identify relationships which can be associated with the two unit counts of multiplication (as per Clark & Kamii, 1996, Davydov, 1992, Schwartz, 1988, and Vergnaud, 1988) and the functional relationship between them (as per Davydov, 1992, and Vergnaud, 1988). So rather than to multiply two numbers correctly, the child was prompted to sense multiplicative notions by distinguishing HL's and VL's of Zaplify which represent two factors of multiplication and by making one object (the dot) out of two objects (the lines), which is contradictory to additive thinking.

The findings show that after manipulating the digital artefact, the child first created the signs which were related to the individual characteristics of the objects such as their shape (e.g., curly lines), their size (e.g., small dot), and their colors (e.g., yellow line), instead of the spatial relationship between the objects. Moreover, the former signs illustrated more additive thinking. The child created one line next to the dot when asked to make the numeral 1. This might indicate that for the child the numeral which symbolizes the dot must be created with one object which is the single line. By interacting reciprocally with each element of the duo, the child started to create signs which expressed the spatial relationships among the Zaplify objects and to create quantities in a way which would challenge the additive relationships between the objects.

The result of this study shows that a child as young as five years old can fluently identify the difference between the referents of the quantities and coordinate them to create a multiplicative product after interacting with a duo of artefact which is designed to prompt multiplicative thinking. Thus, it supports Askew's (2018) finding that under the appropriate instruction younger children can also learn multiplicative concepts which are assumed to be difficult for them. Even though the child might have been introduced some notions related to addition in the kindergarden or by his family, he has not been formally trained on addition which happens in the grade 1. Therefore, like Bicknell et al., (2016), this study also challenges the hypothetical learning trajectory which situates learning of multiplication after the formal introduction of addition.

The findings show that creating dots in Zaplify was not enough for the child to right away identify the multiplicative relationships between the objects. At the beginning of the interview, while exploring the app, Zach created a dot right away probably by chance as he could not achieve it when the interviewer asked him to make a dot again. Then he made many dots with the interviewer for a relatively long time (18 minutes). He started to express the relationships between the Zaplify objects after drawing. However, moving from manipulating the digital artefact to drawing the screen configuration in one cycle was not effective to make the relationships between the objects salient, either. Zach created several pivot signs in different modalities via reciprocal use of this duo of artefacts in several cycles before he fluently answered the “what happens” questions which required identification of the relationships between the lines and the dots.

This study does not propose that the digital artefact must be provided with the non-digital counterpart to develop multiplicative thinking. The child might have identified these multiplicative relationships after interacting only with the digital artefact for a longer time with additional tasks which prompt him to compare various configurations of his fingers with the resulting products. However, I propose that shifting between manipulating the digital artefact and drawing has a potential to speed up the process of identifying the multiplicative relationships.

In addition to accelerating learning process, the reciprocal use of duo helped the child build various meanings for the lines. As soon as a finger is pressed on the screen, a line always appears as a complete discrete object. Whereas the child created a line on paper as the trace of a continuous hand movement. However, these varying meanings attributed to the Zaplify objects were not confined to the specific medium they were created. Zach’s verbal accounts that described the relationship between the Zaplify objects were accompanied with dynamic gestures that mirrored his drawings. These dynamic gestures were accompanied some verbal signs (e.g., “the second line crosses the first line”) which emphasized the relationship between the lines.

Mariotti and Montone (2020) describe this interaction as the synergy between the artefacts of the duo. So, the reciprocal use of the duo enriched the child's experience of multiplicative relationships embedded in the digital artefact through this synergy. In this study pencil-and-paper provided the child with a medium to build and extend meanings in addition to record his interpretation of the digital artefact (de Freitas & Sinclair, 2012; Thouless & Gifford, 2019).

While interacting with the duo of artefacts, Zach was communicating with the adults most of the time. Therefore, discussing with adults (both the researcher and the father) through specific signs seemed to play a role in mediating the relationship between one disc and the intersection point of two lines. While these discussions guided the child to attend to specific relationships, the child's responses did not always indicate an alignment with the intended direction of the adults' questions. For example, the interviewer asked "how" questions to direct the child's attention to the process of making a dot. While the child responded to these questions by creating independent static signs (e.g., drawing of a single dot, saying "a yellow line") at the beginning of the interview, his responses included multiple signs in relation to each other (e.g., tracing gesture on both lines in Zaplify) as his interaction with the duo of artefact progressed.

This study presents the preliminary results of reciprocal use of a duo that is designed to develop multiplicative thinking. These tentative findings show that moving repeatedly back-and-forth between each element of the duo while communicating with others can accelerate students' meaning making process and expand their meanings by prompting a synergy between the two media. The next step will be to analyse the relationship between the signs created through each element of the duo based on extensive data.

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Postlude

In the prelude, I stated that this manuscript extends the findings of the first manuscript of this dissertation. Since they were published independently, I want to use this postlude to elaborate on how this article extends the findings of the first one. Moreover, I want to make some clarifications about a specific finding that might have been misunderstood.

In the first paper, I analyzed the semiotic potential of each artefact separately and proposed that each artefact had unique contributions to students' meaning-making process: while Zaplify could help students experience the spontaneous actions involved in multiplication, the pencil-and-paper would allow learners to directly experience the relationship between the product and the factors. I want to remind the reader of that these contributions are hypothetical because the analysis is based on the potential, not actual, use (by learners) of an artefact. I created the data by recording my own interactions with the artefacts.

In this study, I analyzed how a child actually used Zaplify and pencil-and-paper back-and-forth, rather than simply move from the non-digital to the digital. Thus, it illustrated how the semiotic potentials of two artefacts unfolded in an authentic, real-life, setting. The findings of this study showed that drawing Zaplify with pencil-and-paper allowed the child to attribute a dynamic, traversing movement to the lines, which, on the contrary, appeared in Zaplify as static objects extending from one side to the other. The child said that the second line crossed the first line and made a dot. This description portrays the dot with respect to an interaction between two lines. So, it can be related to the relationships between the multiplicative factors and the product. This is aligned with one of the semiotic potentials of pencil-and-paper which I identified in the previous manuscript.

When I re-engaged with this paper, I felt that some of my statements sounded like I valued some signs over others, which was not my intention. Therefore, I want to present some clarifications. For example, I stated that “The findings show that after manipulating the digital artefact, the child first created the signs which were related to the individual characteristics of the objects such as their shape (e.g., curly lines), their size (e.g., small dot), and their colours (e.g., yellow line) instead of the spatial relationship between the objects.” This statement reports the initial signs in comparison to the latter ones, which addressed the spatial relationships. This might portray the initial signs as being deficient spatial characteristics, so they are less important than the latter signs. However, I find initial signs as important as the latter ones because they refer to the differences between the Zaplify objects. Creating signs referring to different characteristics of the Zaplify objects indicates that these differences constitute a significant part in the child's experience of Zaplify. Since these differences of the Zaplify objects might be associated with the difference between the referents of the multiplicative factors and the product, these initial signs are important for understanding multiplication as a binary operation (as opposed to the unary operation of repeated addition).

Making the product 1 by using Zaplify is equivalent of multiplication by one and it might be argued that such a situation does not involve multiplicative thinking. Even though in this case the product is not large enough to illustrate the power of multiplication as an indirect measurement as Davydov (1992) proposed, the way the product is created various features of multiplicative thinking. First of all, it is counter intuitive to create a quantity of one by using two entities (line making fingers). When a child thinks additively, two fingers correspond to a quantity of two unlike in Zaplify. Secondly, when making the product 1 in Zaplify, each finger acts in a different way, unlike in addition. While one finger sets the stage by determining the functional relationship (one finger corresponds to one dot), the other finger actualizes this relationship by spreading it across the perpendicular line.

Prelude to Chapter 4

The following manuscript is a research paper which examines two students' interactions around/with Grasplify. More specifically, it studies how the students' way of structuring a quantity changes and how this change unfolds during their interactions around/with TT. Compared to the first two manuscripts, this piece focuses on a different dimension of learning mathematics by using digital technology. In the first manuscript, I focused on the semiotic potentials of Zaplify and pencil-and-paper. In the second manuscript, I studied how these potentials unfolded when a young child used pencil-and-paper and Zaplify back-and-forth. In the following manuscript, I explore how a third grader structures a quantity in Grasplify while collaborating with a peer and a researcher. Rather than a student who uses a duo of artefacts, this study focuses on a duo of students who use a single artefact.

As in the second study, this manuscript evolved from one of my comprehensive exam papers in which I explored enactivism by analyzing a video-recording. However, this time I did not recruit my participants myself. Instead, I used an existing video-recording in which two third graders were using Grasplify to structure a specific quantity in a specific way. My fine-grained analysis illustrated the inseparable link between the learners and their environment. In other words, it shows how the students' actions and Grasplify co-evolved through the reciprocal interactions between the students, TT and the researcher. I extended this analysis into an article which was published in the journal 'Digital Experiences in Mathematics Education' on 24 September 2021 and reproduced with permission from Springer Nature. This article pursues to the following research questions;

- How do children collaboratively structure quantities in order to solve a unitizing task in *TouchTimes*?
- How do children couple with their environment, as well as with other individuals also engaged in this same environment, in order to solve a unitizing task in *TouchTimes*?

This co-evolution of students and Grasplify results in the enactivist construct called natural drift. Maturana and Varela (1987) describe this theoretical construct as the process of change the different generations of the same species underwent with respect to the changes in the structure of the nature. These changes do not happen abruptly, they take time. I argue that this theoretical construct can also be used to explain the changes in a learner's behaviour in relation to the characteristics of his/her environment. I could not emphasize that idea in the article as much as I wanted to. Therefore, while engaging with the findings, I invite the reader to pay attention to that the structure of the students' actions drifts towards a more multiplicative nature with respect to the changes in Grasplify.

Chapter 4.

A Quantitative Shift towards Multiplicative Thinking

Abstract

When two third-graders collaboratively manipulated a multi-modal, digital learning device called *TouchTimes* (hereafter, TT), that introduces multiplication through visual, tangible and symbolic means, their thinking about quantity shifted from being additive to being multiplicative. In this study, I examine the children's interactions around/with TT. My goal is two-fold: (1) to demonstrate the shift between the students' additive and multiplicative thinking; (2) to explain how their multiplicative thinking emerged around/with TT. The emergence of multiplicative thinking does not refer to the students' correct computations of multiplicative expressions as a response to verbal or number problems. Instead, drawing on an enactivist perspective, I identify the children's thinking as their effective bodily reactions to a given unitizing task using TT—and I distinguish their multiplicative and additive thinking based on various researchers' conceptions of multiplicative thinking. The data was created by video-recording the children's interaction around/with TT. A retrospective analysis of the data reveals that the children's effective action to solve the unitizing task developed through a history of recurrent interactions in this environment.

Keywords: multiplicative thinking, touchscreen technology, enactivism, collaborative learning, *TouchTimes*

> *One incident with one child, seen in all its richness, frequently has more to convey than a thousand replications of an experiment conducted with hundreds of children. Our preoccupation with replicability and generalisability frequently dulls our senses to what we may see in the unique unanticipated event that has never occurred before and may never happen again. That event can, however, act as a peephole through which we can get a better glimpse at a world that surrounds us but that we may never have seen in quite that way before. (Brown, 1981, p. 11)*

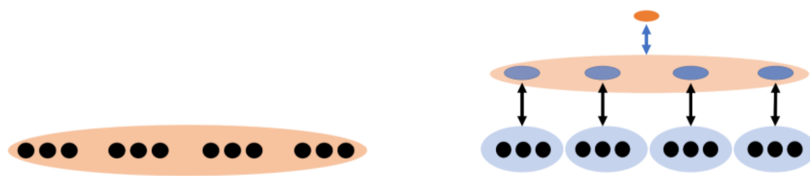
Multiplication is introduced to students at different grade levels within different number domains, such as natural numbers, rational numbers and integers. Studies show that students from varying grade levels have difficulties when they are faced with multiplicative situations (e.g., Brown et al., 2010; Clark & Kamii, 1996; Hackenberg, 2010; Hurst, 2017). Multiplicative thinking is important for students to navigate successfully at each grade level and, relatively recently, there has been wide interest in finding ways to improve it.

Several researchers have attributed children's subsequent difficulty to the introduction of multiplication as repeated addition and illustrated the difference between multiplicative and additive thinking through various models (e.g., Confrey, 1994; Greer, 1992; Maffia & Mariotti, 2018; Schwartz, 1988; Vergnaud, 1988). Another attempt to improve students' achievement in multiplicative situations was to design educational software to provide students with dynamic representations of multiplication (e.g., Kaput, 1985). Thanks to the internet, one can easily access countless online games and apps about multiplication nowadays. However, most of them are based on practice and drill exercises, which strengthen the recall of multiplication facts, while only poorly encouraging students to think multiplicatively.

This article reports on a study about a multi-touch digital device (*TouchTimes*, hereafter TT) that was designed to enable students to think multiplicatively. Its design draws on multiplication models that are different from repeated addition. At the beginning of a long-term research project, two third-graders and a researcher interacted around/with this technology and the children's interaction demonstrated a shift in their approach to quantity. In this article, based on a smaller study which is somewhat tentative in nature, I examine the process of this shift by focusing on how children structured a target quantity in TT, how they interacted around/with TT and how these interactions played a role in the children's approaches to structure quantity multiplicatively.

4.1. Multiplicative Thinking

Multiplication is introduced as repeated addition in the elementary school curricula in many countries, which can often lead students to remain in the mode of additive thinking. However, multiplicative thinking is different from additive thinking in terms of the number of levels of abstraction[4] and the number of inclusion relationships a child makes simultaneously (Clark & Kamii, 1996). Figure 4.1 demonstrates this difference explicitly.



Additive thinking:3+3+3+3 Multiplicative thinking: 3x4

Figure 4.1. Additive and multiplicative thinking.
Note: The figure is modified version of Fig. 1 in Clark & Kamii (1996, p. 42)

When adding three repeatedly four times, the child operates on ones as the only unit count (see Figure 4.1a). Even though the repeating terms of addition are equal, the reason behind this equality is not explicit to the child. In other words, it is a pre-given condition (Davydov, 1992). According to Davydov, in multiplying three and four, the child considers two distinct unit counts, as illustrated in Figure 4.1b by the black dots (units of one) and the blue ellipses (units of three). Unlike the repeated term of addition, three refers to the established relationship between the two unit counts in multiplication, not a pre-given condition. This relationship constitutes a "many-to-one correspondence between the three units of one and one unit of three" (Clark & Kamii, 1996, p. 43). Askew (2018) called this correspondence "the functional relation" between the variables of the multiplication, drawing on Vergnaud's multiplicative conceptual field. According to Davydov, this relationship is established when one unit of count is transferred to another unit of count in a measuring activity. According to Steffe (1994), this transfer constitutes the core of multiplication.

The other difference comes from inclusion relations which play a role in the composition of the product (Clark & Kamii, 1996). When learners add numbers, they combine the units along a single level: one is added to one to make two, one is added to two to make three, and so on up to twelve (see Figure 4.1a). Whereas in multiplication, simultaneous inclusion relationships are identified in two levels: (1) within the units and (2) between the units. Within the units of three, one three is included in two threes, and so on up to four threes (see Figure 4.1b). Similarly, within the units of ones, the child

[4] The term "level of abstraction" was used by Clark and Kamii (1996), but I interpret it here as the number of units to be operated on.

“includes one in two and two in three” (p. 42). As the child includes the smaller units within themselves, these units are also included in the larger unit simultaneously, such that “3 ones are included in each unit of three, and 4 units of three are included in the product” (p. 43), as represented by the bidirectional arrows in Figure 4.1b.

Additive thinking in multiplicative situations may conceal the inclusion relationship between the units such that a change in one group may not be conveyed to all groups of threes at once. For example, if a child is given “6 × 17 is 102” and then asked how the product changes when 17 increases by 1, s/he might suggest that it also increases by one due to the repeated addition model of multiplication. However, when a child thinks multiplicatively, s/he can simultaneously consider the inclusion relationship between the units to evaluate the change in the product (Davydov, 1992).

4.2. Teaching through Multiplication Models

Multiplicative thinking plays an important role in a large number of mathematical situations found in upper grades (Brown et al., 2010). For example, ratio, rate, rational number, linear and non-linear functions, vector space and dimensional analysis are concepts that are all involved in a learner’s organization of actions relevant to multiplicative situations (Vergnaud, 1988). The repeated addition model is not sufficient to explain these concepts, because it lacks an emphasis on the relationship between the distinct units. Therefore, although learning multiplication as repeated addition may not pose a problem in the lower elementary grades, children may well have difficulty in the upper grades if their sole conception of multiplication is restricted to repeated addition (e.g., Siemon et al., 2005).

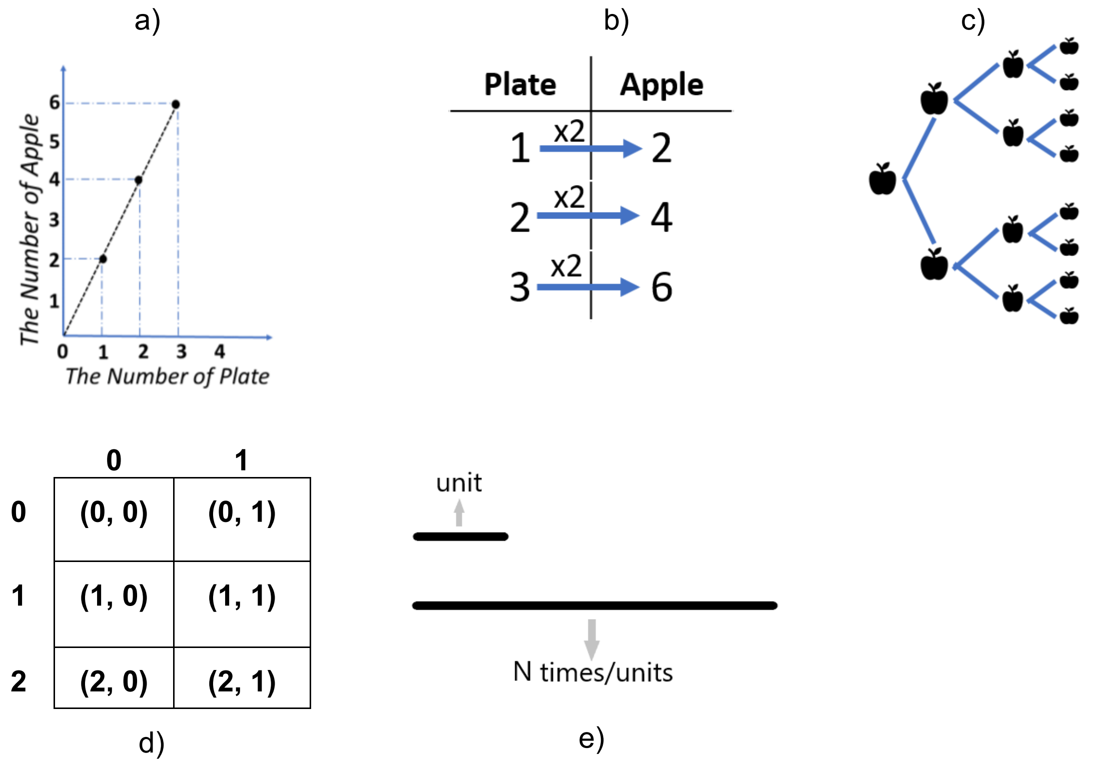


Figure 4.2. Models of multiplication: (a) a graph; (b) a T-table; (c) a tree diagram; (d) an array model; (e) a ratio comparison of lengths.
Note: These images are based, respectively, on Schwartz (1988, p. 45); Vergnaud (1988, p. 146); Confrey (1994, p. 302); Maffia and Mariotti (2018, p. 31); Izsák and Beckmann (2019, p. 92) and Polotskaia and Savard (2021).

Several representations are suggested in the literature to model multiplication as different from repeated addition. Schwartz (1988) defined multiplication as a mapping from "a quantity in one space to another quantity in another space" (p. 51) and suggested that multiplication could be introduced through a graph model, in which axes represent the distinct referents of multiplication (see Figure 4.2a). Vergnaud (1988) framed multiplication within a multiplicative conceptual field and suggested that T-tables (see Figure 4.2b) can explicitly illustrate the ratio between the multiplicative factors (if one unit has two parts, then three units have six parts). Confrey (1994) defined multiplication as, "an action of creating simultaneously multiple versions of an original" (p. 292) and proposed a tree diagram to illustrate this splitting action (see Figure 4.2c). Maffia and Mariotti (2018) offered the array model of multiplication as an intuitive counterpart of the more formal Cartesian product: for example, the array model describes the multiplication 2 × 3 as a rectangular array of three rows with two elements in each row (see Figure 4.2d). The ratio comparison of lengths (see Figure 4.2e) was another representation suggested by various researchers (e.g., Izsák & Beckmann, 2019; Lay, 1963; Polotskaia & Savard, 2021).

Using alternative multiplication models indicated promising outcomes for developing children's multiplicative thinking. Following Vergnaud's functional aspect of multiplicative reasoning (MR)[5], Askew (2018) suggested a cyclical and iterative instruction method to develop MR in young students. In this setting, the teacher first posed a multiplicative word problem and then observed which kinds of informal models students elicited for the problem. Next, the teacher identified which of these informal models could be connected to a mathematical model, such as the double number line or the ratio table, which was hypothesized to stimulate multiplicative reasoning. Lastly, the teacher taught these mathematical models explicitly to the students and encouraged them to solve a variation of the word problem they were given at the beginning by one of these means. Askew applied this instructional method to second- and fourth-graders and showed that the functional approach to MR was effective when the students were given appropriate conditions.

In Venkat and Mathews' (2019) study, seventh-graders were instructed to solve multiplication word problems using an abstract array model, a double number line and T-tables. Their solution strategies were compared via pre- and post-tests. After the intervention, the students answered questions with more abstract models instead of their previous strategies, such as counting models. Even though the authors identified a shift in the students' choice of models, and an increase in their achievement, the use of models did not guarantee success in multiplication word problems. Indeed, some learners could not solve the problems despite using appropriate abstract models for multiplication. Moreover, the modelling may not be transferred to other multiplicative situations. When bare multiplication sentences were given, the students did not make use of the models.

[5] Multiplicative thinking does not necessarily indicate any reasoning. Based on Clark and Kamii's (1996) definition, multiplicative thinking involves identification of two units of multiplication and the two levels of inclusive relationships among them, not an expression of reasoning behind these structures, even though it might implicitly involve some reasoning. Therefore, I do not use the expressions "multiplicative reasoning" and "multiplicative thinking" interchangeably. Rather, I employ the specific terminology used by the authors cited.

Unlike static models created by pencil-and-paper, Kaput (1985) proposed dynamic models to develop MR. He attributed its absence to poor and inflexible cognitive representations of multiplication, division and intensive quantities, which refer to the ratio between two quantities, and suggested that students should be given the opportunity to manipulate multiple representations of these concepts in a co-ordinated window using a computer. Figure 4.3 shows the four different representations starting from the upper left-hand box in clockwise order: concrete iconic representations of the base set of objects; vertical data table in which entries are co-ordinated according to the number of base sets; a co-ordinate graph in which axes are connected to the number of base sets; a semantic calculator which makes the calculations via formal expressions of the number of base sets.

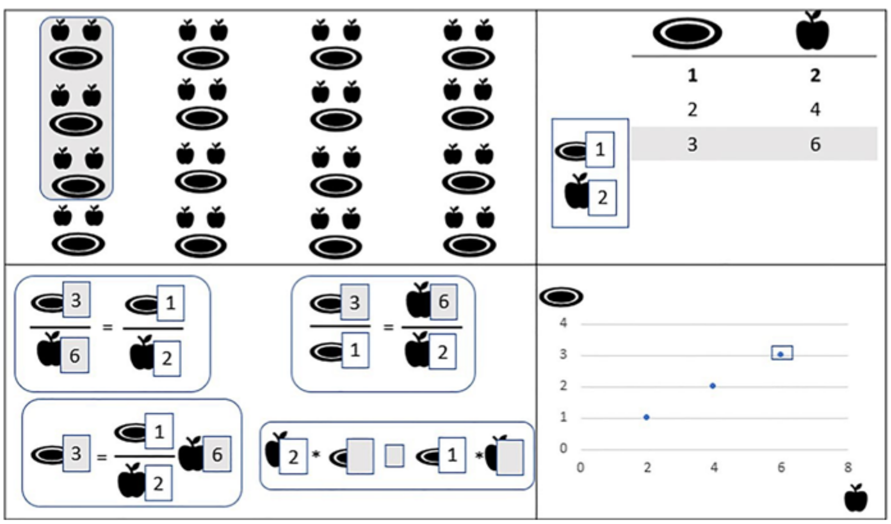


Figure 4.3. Multiple representations of multiplication in a co-ordinated window.
Note: This is based on Fig. 3 in Kaput & Pattison-Gordon (1987, p.13).

Unlike static media, which commonly demonstrate the co-ordination of the representations simultaneously by presenting a table and a graph on the same page next to each other, digital software is open to manipulation and can demonstrate the change in the variables of a multiplication in four representations at the same time. Even though this approach can simultaneously translate the changes from one representation to others, presuming that students build a meaning for a mathematical concept which is invariant across the representations might be problematic (Thompson, 2013). So, in Kaput's co-ordinated window, it is not clear how it draws children's attention to the proportional relationship between the two operands of the multiplication.

For example, one can manipulate an iconic representation of a multiplicative situation by replicating or deleting copies of the base-set objects separately. However, it may not be obvious for the learners to identify the many-to-one correspondence between the variables from the spatial vicinity between the icons in each cell alone. For instance, in Figure 4.3a, changing the number of apple icons can be translated into the other representations. However, representing the ratio between the number of apples and the number of plates only by closely aligning the like icons may not allow children to experience many-to-one correspondence between them.

Since Kaput (1985), few studies have examined dynamic digital technologies that combine different representations of multiplication. Paek et al. (2016) found that, when young children with no prior knowledge of multiplication manipulated the graphical and symbolic representations of multiplication facts in a touchscreen tablet application, they were able to answer multiplication facts they had not studied before. Participants only practiced the two- and three-times tables and they were subsequently asked for multiplication elements from the two-, three-, four- and five- times tables. The graphical representation involved blocks which could be stacked and combined in various ways, while the symbolic representation was in the form of a multiplication sentence. Even though this study suggested promising results in terms of learning specific multiplication elements, combining stacks with fingers via a tablet application does not seem to offer an alternative model to the repeated addition approach. On the other hand, Bolden et al. (2015) introduced young children to array and number-line models in addition to repeated addition. However, the children involved were not able to manipulate the models directly. They were shown multiplication facts with both graphical and symbolic representations using static slides on computer screens.

In the present study, I examine a touchscreen iPad application called *TouchTimes* (Jackiw & Sinclair, 2019). TT provides dynamic and linked representations (as per Kaput, 1985) for students bodily to experience (as per Paek et al., 2016) the functional relations between the multiplicative factors (as per Askew, 2018). This functional relationship is important in learning multiplication, but it is not enough by itself. A multiplicative conceptual field consists of multiplicative situations, multiplicative schemes and symbols (Vergnaud, 1988). Drawing on Seymour Papert's argument that programming through computers may not teach mathematics, but that it may bring a way of thinking that can make mathematics learnable (as cited in Noss & Hoyles, 1996, p.

66), this study aims at extending it to the potential of gesture-based TT in opening windows to make multiplication learnable by shifting students' approach to quantity from being additive to being multiplicative.

4.3. A Brief Description of *TouchTimes*

TT is an iPad application designed to enhance multiplicative thinking. It comprises two verb elements, termed Grasplify and Zaplify. This paper will focus only on the unit-of-units model in the Grasplify world, which draws both on Davydov's (1992) notion of change in unit and on Vergnaud's (1988) notion of functional relations between the variables.

When opened, the layout of this world consists of a permanent vertical line which divides the screen into two equal parts (see Figure 4.4a). These parts contain two different kinds of objects (called "pips" and "pods"), based both on the initial temporality of the touches and then on the spatiality of the related sides of the screen. When one or more fingers first touch one side of the screen, this creates one or more coloured discs (see Figure 4.4b) that are called pips. Pips can stay on the screen only provided finger contact is maintained. If a pip-making finger is lifted, the related pip disappears. If a finger is pressed on the other side of the screen from the pips, this creates bounded collections of colour-matched pips. These collections are called "pods" and they preserve the number, colour and spatial configurations of the original pips (see Figure 4.4c). Unlike pips, pods stay on the screen even if the pod-making fingers are lifted (see Figure 4.4d).

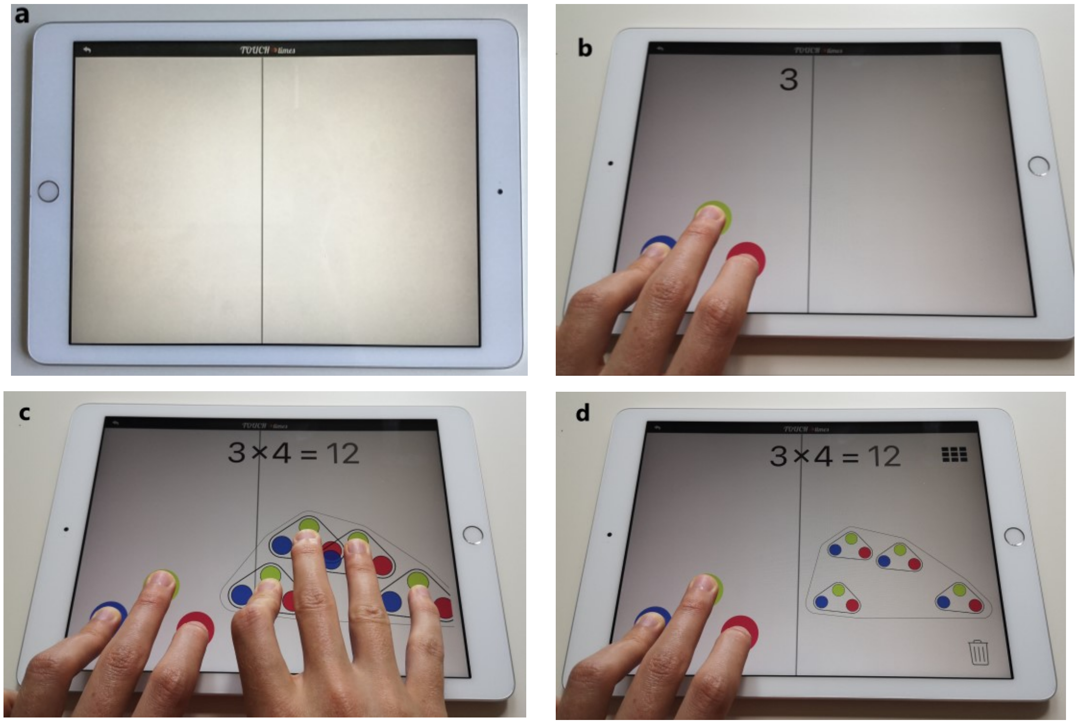


Figure 4.4. (a) Initial TT screen; (b) creating three pips; (c) creating four three-pods; (d) pods after lifting the fingers.

Note: A “three-pod” refers to a pod that includes three pips. In Figure 4.4c, the fingers were first placed on the left side of the screen, but if they had been placed on the right side of the screen, then the pods would appear on the left side of the screen.

One can create multiple pods either by pressing multiple fingers at once or by tapping a single finger repeatedly on the opposite side from that of the pips. If one of the pip-making fingers is lifted, then each pod changes to contain one less pip (of the same colour). If one more pip-making finger is pressed down on the pip side, then each pod changes to contain one more pip (of the same colour). Once all the fingers that maintain the pips are lifted, all TT objects disappear and both sides of the screen become blank again. One can also drag any object around the screen by moving any pip-/pod-making finger without losing contact between the finger and the screen. The total numbers of both TT objects (pips and pods) are presented on the top of the screen as numerals (see Figure 4.4b) and any change in TT objects is mirrored in the numerals simultaneously. As soon as there are both pips and pods on the screen, a multiplication expression (that is generated aligned with the order of the TT objects) appears at the top (see Figure 4.4c).

TT objects, the hand gestures[6] that create these objects and the relationship between these objects are all associated with various aspects of multiplicative thinking. The pips and the pods represent the two unit counts of multiplication. Once pips are created on one side of the screen, a pod-making finger on the opposite side creates a new unit of count. Pressing down a pod-making finger after placing and holding a pip-making finger corresponds to Davydov's notion of transfer of unit counts. The transfer of configurations from pips to pods relates to Vergnaud's notion of many-to-one correspondence between the units. A simultaneous adjustment in the size of each pod according to the number of pips reflects Clark and Kamii's notion of the inclusion relationship between the units.

As a multimodal digital learning environment, TT provides the users with an interactive platform that supplies direct and fast feedback to their gestures. It creates digital objects at the fingertips and adjusts these objects simultaneously according to the many-to-one correspondence between the factors. Thus, following body studies in mathematics education (see the next sub-section), the design of TT aims at enabling children to think multiplicatively as they literally embody the functional relationship between these factors.

4.4. The Body in Mathematics Education Research

Studies which emphasize the role of the body in learning mathematics are not rare in the literature. Having a participationist approach to cognition, these studies bring a new perspective to the meaning of mathematics learning and challenge the educational implications that result from presuming the body to be peripheral to mathematical thinking. The following survey presents studies that examine computerized technology use in the mathematics learning environment.

Nemirovsky et al.'s (2013) theory of perceptuomotor learning shows the importance of integration between motor actions and perceptions in learning the mathematical concept of slope. In their study, two children were introduced to a mathematical instrument which consisted of a computer screen and a mechanical

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[6] In this paper, the term "gesture" refers to the notion of the "tangible gesture" as discussed in Sinclair and de Freitas (2014).

apparatus including two handles, one controlling the x-axis motion and the other the y-axis motion. The researchers showed that the children obtained fluency in using the mathematical instrument as they integrated their motor actions, which consisted of the movement of the body parts such as hand, arm and torso, in conjunction with their visual perception of the graphics on the screen. Attending both to the motor actions and to the perceptions in their microanalysis, the authors suggested a new meaning for mathematics learning: "a process in which the perceptual and motoric aspects of using a mathematical instrument become intertwined" (p. 406).

Abrahamson and Trninic (2015) took one step further, in examining the evolution of perceptuomotor doing into disciplinary knowing from an anti-representationalist perspective. They proposed a view of conceptual learning which goes beyond the cognitive semantic theory of conceptual metaphor. Drawing on a non-dualist approach, they attended to and explained the dynamical, interactionist, emergent, sociocultural, distributed and developmental aspects of teaching and learning in a more comprehensive manner, compared with the construct of conceptual metaphor. According to this account, mathematics learning is based on two actions: (1) engagement in motoric problem solving through goal-oriented situated interactions within a particular ecology and (2) reflection on this engagement. They supported this account empirically by modelling two students' goal-oriented sensorimotor actions in an instrumented field of promoted action, in which the students manipulated technological devices to develop proportional thinking. In order to model students' actions, they conducted a microgenetic analysis which captured students' actions in detail. The results showed that, as the students manipulated the artefacts, both their actions and their articulations co-evolved. As their actions shifted from local adjustments to global action plans, they started to include elements of the mathematics register in their discourse with the guidance of the instructor.

Francis et al. (2016) examined children's spatial reasoning as they used a laptop to program a LEGO Mindstorms EV3 robot. In contrast to other studies on the use of robots in the mathematics classroom, their enactivist approach shifted the focus from symbolic representations to the children's moment-by-moment manipulations. The authors created their data through slow-motion video-recordings to capture the fine-grained interactions between the computer screen and the children. This enabled them to show that enactive-iconic-symbolic representations, which are important aspects of spatial reasoning, develop simultaneously, which, they claimed, conflicted with Bruner's hypothesis of sequential development. As the children manipulated the symbolic representations on the screen to move the robot, their subtle bodily actions accompanied their symbolic manipulations. In other words, spatial reasoning appeared as, "the constrained co-occurrence of sensory flux (sensation), recognition/ discrimination (perception) and the situated movement of a body (or bodies) in the context of a goal-oriented situation" (p. 4).

Unlike the above studies, Lozano (2017) examined learning over a longer time-span, attending to the relationships among tasks, classroom cultures and mathematical learning. Drawing from the enactivist idea that cognition is effective action (more on this in the next section), her method was based on identifying students' effective mathematical actions on computerized mathematics tasks. One of the criteria to identify effective action was the frequency of the action, while the other criterion was the expansion of the actions across several mathematics classes. Mathematical learning was defined as the change in the learners' effective actions when noticing mathematical task features. Lozano found that task characteristics triggered certain effective actions, which were not independent of teacher guidance and students' individual histories.

Overall, these studies challenged both the meaning of mathematics learning and the methods to study it. The majority in this survey focused on fine-grain body interactions through microgenetic analysis, yet the researchers did not discard analyzing bodily changes over relatively longer periods, as in the last example. Irrespective of their specific methodologies, all of these studies challenged the over-emphasis on discourse analysis as the only way to study mathematics learning (Nemirovsky et al., 2013). They showed that an emphasis on the body shifted the meaning of mathematical learning from being a mental action to being a bodily one, which is aligned with the main assumptions of enactivism, the theory this study also draws on.

4.5. Enactivism

Enactivism brings a participationist perspective to cognition, which depends on different epistemological assumptions from representationalist approaches. The latter assumes an objective world, whereas enactivism emphasizes several worlds that are brought forth by living beings—observers. While the representationalist approach associates cognition with the representation of the environment, enactivism describes cognition as the effective action of organisms to maintain their unities in an environment. Such actions can be observed in organisms' behaviour that is defined as, "the changes of a living being's position or attitude which an observer describes as movements or actions in relation to a certain environment" (Maturana & Varela, 1987/1992, p. 136). If behaviour is related to a history of interactions between the organism and its environment, it is categorized as learned; otherwise, it is defined as innate behaviour. In this article, I focus on learned behaviour.

Living beings as autopoietic unities are organized in a way that their actions produce themselves. In other words, as they do certain things, they become those things. One of the things that we, as humans, do is make distinctions. We necessarily and permanently indicate any being, object or unity as separate from its background based on a criterion of distinction. For example, looking closely at a tree, we might distinguish its leaves from the rest of its body, even though there is not a rigid separation between the leaf and the branch it emerges from. Also, looking at a landscape, we might distinguish the tree as separate from the soil it "occupies", despite the blurred material separation between the soil and the roots of the tree.

Similarly, looking at a quadratic equation $9 - b^2 = a^2 + 2ab$, a student might distinguish the equivalence between $a^2 + 2ab$ and $9 - b^2$ if attention were directed towards which terms the equation sign separates. On the other hand, when interacting with the same equation, it is possible to distinguish the equivalence between the square of $a + b$ and 9, if attention is directed towards the relationship between the terms, irrespective of their position with respect to the equal sign.

To explain the dynamics of any system, including those that lead a living being to make certain distinctions and act in certain ways, Maturana and Varela suggested that organisms as autopoietic unities should be considered as operating both in their internal dynamics and in their circumstances—in other words, within the context. Unities are born with an initial structure[7] in a particular environment that also has a particular structure. This initial structure not only conditions how the unity responds to the changes

[7] Maturana and Varela define structure as, "the components and the relations that actually constitute a particular unity" (1987/1992, p. 43).

in its environment, but also determines what kind of changes the environment can trigger within it.

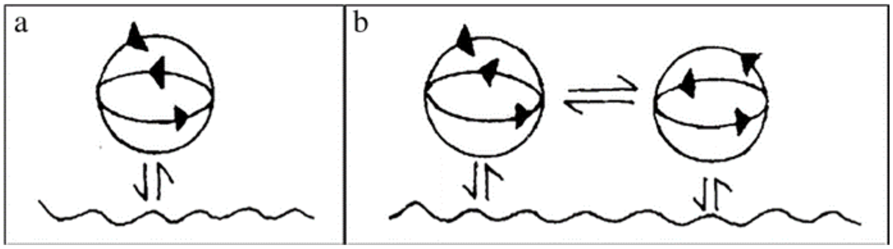


Figure 4.5. Representation of (a) structural coupling and (b) social coupling.
Note: The figure is taken from Maturana & Varela (1987/1992, p. 180).

For example, a pottery master is a pottery master as long as s/he creates ceramics out of clay. The creation of a piece of ceramics depends both on the bodily configuration of the master and on the structure of the clay. Pottery masters with bigger hands would sense the clay differently from those with smaller hands, and also use their hands differently to create the same form of ceramics. Similarly, the hardness of the clay would influence how much pressure should be applied on the clay by the pottery artist.

In a mathematical context, observing a student's problem-solving process might demonstrate this relationship as well. For example, presenting an equation in two different forms $a^2 + 2\mathrm{ab} + b^2 = 9$ and $9 - b^2 = a^2 + 2\mathrm{ab}$ and asking for the value of (a + b) might trigger two different approaches for its solution, unless the student brings forth the binomial $(\mathrm{a} + \mathrm{b})^2$ in her/his world upon interacting with the second equation. Both the context of the problem and the structure of the learner at that moment, which emerged from a history of recurrent interactions, play important roles in explaining this interaction.

For a unity to operate effectively in its environment, there must be a structural congruence between it and its environment. As the environment and the unity act as mutual sources of perturbation, which triggers changes within their structures, this history of recurrent interaction leads to structural congruence between them. This process is called structural coupling. Maturana and Varela (1987/1992, p.180) illustrated this process with a diagram (see Figure 4.5a) in which: (1) the closed curves represent the autopoietic unity and the nervous system; (2) the arrows attached to these closed curves represent their autopoietic organization; (3) the curly line represents the environment; (4) the vertical arrows represent the interaction between the environment and the organism.

When each organism constitutes a source of perturbation for the other organism's structure (as illustrated in Figure 4.5b), this is called social coupling. If organisms demonstrate mutually triggered co-ordinated behaviour in a social unity, then they are said to be communicating. Organisms can communicate their knowing through chemical, visual or auditory interactions.

In brief, enactivism theorizes cognition as living beings' actions that are effective to maintain their unities in their environments. If these effective actions emerge from a history of recurrent interaction between the environment and the living beings, they are said to be learned. Drawing on this theoretical approach to learning, this study examines the interactions between the bodies, in addition to the individual bodies' interaction with their environment, and explores the following research questions:

- How do children structure quantities in order to solve a unitizing task in *Touch Times*?
- How do children couple with their environment, as well as with other individuals also engaged in this same environment, in order to solve a unitizing task in *TouchTimes*?

4.6. Methods

I analyzed a 27-min video-recording[8], which captured interactions between two third-grade students (whom I call Jacy and Kyra) and a researcher[9] around TT on a single iPad. The researcher described this recorded event as an exploratory conversation with the children as part of an iterative design experiment. According to the researcher, the aim of this conversation was potentially to help refine the TT prototype and to develop appropriate tasks for use with grade two and three children. Jacy and Kyra were purposely recruited in this study by their classroom teacher. Following the researcher's request, the teacher chose them firstly because they had never used TT before. Moreover, the teacher found it beneficial to provide Jacy, who often ignored instructions,

[8] This particular video-recording was also analyzed, albeit differently, by Bakos and Pimm (2020).

[9] The author of this article was not present at this recorded event.

with a different kind of mathematical experience. The children had not formally learned multiplication in school yet.

This interaction occurred in an elementary school in a culturally diverse and affluent neighbourhood in British Columbia, Canada. The conversation was conducted outside of their class time and separate from their teacher. The children were encouraged to play freely at the beginning of the session, which they did for some seven minutes. This pedagogical decision was aligned with local teaching practices which aim to allow children to explore and to elicit noticing. In addition, the girls interacted with TT during its design phase, so this free-play period would have helped to study how young children might interact with TT without any guidance. Following the free-play period, the researcher posed a unitizing task to direct the two children's attention to specific features of TT.

I analyzed the children's responses to a mathematical task: "making seven at once". This task was presented orally to the children by the researcher who asked, "I want [Jacy] somehow with one finger to make seven over here [pointing the right side (RS) of the screen which was closer to Jacy]. How can you do that?".

There is only one way to create seven with one finger in TT: first holding seven fingers down on one side of the screen to make seven pips, and then tapping a single finger on the opposite side of the pips to make a pod comprising seven pips. Creating just seven pips and creating one pod in addition to seven pips are two different actions which both produce seven. So, these two distinct actions might generate two different meanings for making seven. From a mathematical perspective, making seven by co-ordinating seven pip-making fingers with one pod-making finger recalls several features of multiplication: operation with two units (Clark & Kamii, 1996), the transfer of unit counts (Davydov, 1992) and the many-to-one correspondences between the variables (Vergnaud, 1988). Thus, this unitizing action is important to challenge learners' additive approach to quantity and to prompt them to embody multiplicative thinking.

4.6.1. Data Analysis

I conducted my analysis drawing on Abrahamson and Trninic's (2015) method. They studied how students learned proportional relations via computer-supported inquiry activity that required embodied interaction with the device, and were inspired by enactivism in their analysis, focusing on transitions among the students' bodily actions which seemed to be key to identifying their learning. Therefore, I also focused on the moments of transition among the different kinds of gestures[10] on the screen. In the analysis of these transition moments, I considered not only interactions between each child and TT, but also the interactions among the two children, the researcher, the iPad, TT and the task.

Enactivism conceptualises cognition in terms of the organisms' effective actions in relation to a specific task. In my analysis, the task was to make seven with one finger in TT. This task was given to the children at the seventh minute of the video. Analyzing their effective actions without studying their history of interaction would be incomplete in explaining their learning process. Indeed, Maturana and Varela distinguished learned from innate behaviour based on the history of interactions with the environment. Therefore, I followed a retrospective approach, also analyzing what happened before the task was given to the children.

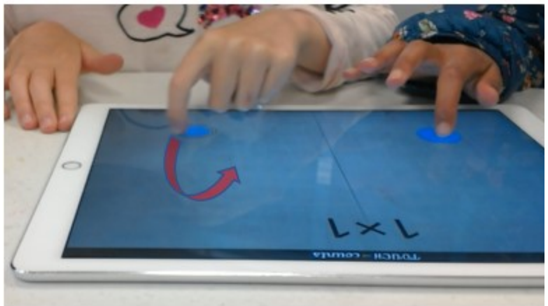
As I mentioned earlier, the children were prompted to explore TT before the researcher gave them the task. During the first four minutes of their free play, the girls demonstrated all types of gestures that they enacted in the rest of the interview. Therefore, I transcribed the video and audio recording of this episode to analyze the characteristics of the first transitions. I organized the data in a tabular form (as in Table 4.1) and recorded each new object (pip/pod) and the pertinent action by freezing the screen. This fine-grained record of the data illustrates the density of the children's interactions and the pattern of shift in their gestures. Therefore, I integrated relevant tables into the findings.

[10] For more discussion on certain aspects of gestures more generally, see Bakos and Pimm (2020).

Table 4.1. Records of actions

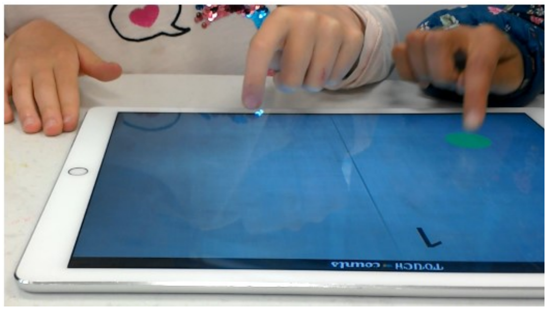
Time Period	Jacy's gesture			Kyra's gesture			Screen view number
	Action type	Object	Side	Action type	Object	Side	
0:05-0:06	Drag "mmmm"	1 pod	RS	Tap	1 pip	LS	1
0:05-0:06	Drag	-	RS	-	-	-	2
0:07-0:08	Lift	-	-	Tap	1 pip	LS	3
0:07-0:08	Tap	1 pip	RS	-	-	-	4
0:07-0:08	Hold	1 pod	RS	Hold	1 pip	LS	5

In Table 4.1, "action type" refers to the way of touching on the screen, "object" shows what an action created on the screen and "side" denotes the specific part of the screen on which the children demonstrated their actions. RS is an abbreviation for "right side" (and LS for "left side") according to each child's point of view. The children's verbal accounts are shown in the table within quotation marks. As it is impossible to record the verbal accounts by freezing the screen every other second, they are indexed at the moment when the verbal accounts start. Screen view numbers are used to index the screen shots of the relevant rows which are presented in Figure 4.6.

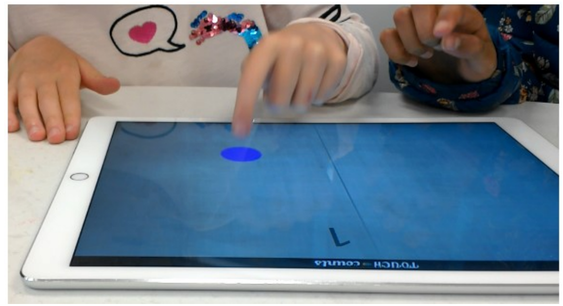


Screen View 1

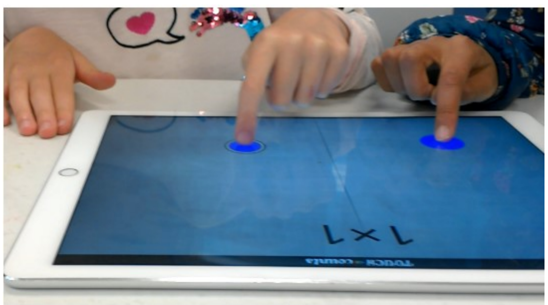
Screen View 2



Screen View 3



Screen View 4



Screen View 5

Figure 4.6. Records of screen views.

Each row of the tabular record of the video refers to a distinct event, and each event is distinguished by freezing the video at each discrete screen touch of either child. Among them, eight different gestural action types were found, which are described as follows.

1. Drag: Moving a finger (or fingers) on the screen without lifting.

2. Lift: Removing a finger (or fingers) from the screen.

3. Tap: Touching and immediately removing a finger (or fingers).

4. Hold: Touching a finger (or fingers) on the screen and keeping it (them) on the screen more than one second.

5. Stand Still: Continuing to hold a finger (or fingers) down.

6. Hold-and-tap: While holding a finger (or fingers), tapping with another finger (or fingers).
7. Hold-and-add: While holding a finger (or fingers), touching on the screen with another finger (or fingers) and holding it (or them).
8. Hold-and-lift: While holding a finger (or fingers), lifting it (or some of them) from the screen.

4.7. Findings

I have divided the findings into two sub-sections: the history and the task, respectively. The history sub-section explains the evolution of the children's touches that were observed in the first four minutes of the video-recording, while the subsequent task sub-section explains how the children responded to the given unitizing task. Before starting to present my analysis, I acknowledge Maturana and Varela's assertion that, as an observer, my analysis is the description of the interactions between the children and the environment from my perspective: "Everything said is said by the observer" (1987/1992, p. 65).

4.7.1. The History of the Recurrent Interactions

The children made various types of gestures repeatedly during their free play. My analysis of the evolution of these gestures is in this sub-section. At the beginning of this episode, the researcher did not give the children any mathematical task. Therefore, there was not an explicit, extrinsic goal for the children that required certain effective actions to achieve it. However, Jacy repeatedly questioned how to make multiple objects and imitated Kyra, who was successful in creating them. Therefore, I distinguished this question as a self-generated task and focused on the shifts in Jacy's gestures, which eventually created multiple objects.

Episode 1: A Shift from Single Pip to Multiple Pips

Both children started to touch the screen with a single finger and thereby created single objects. When Kyra created multiple pods on the screen and an additional pip on Jacy's side at 0:12 for the first time (see Figure 4.7a), Jacy asked Kyra, "How did you do that?", while she repeatedly tapped her finger. After this question, she either held her one finger on the screen while Kyra created multiple pods or she tapped on the screen with a single finger only. Jacy started to hold her finger down at 0:18 again. Between 0:19 and 0:20, Kyra tapped her finger on the RS of the screen for a second time where Jacy was holding a pip. This created one more pip on Jacy's side (see Figure 4.7b).

Figure 4.7. Kyra's taps on "Jacy's side".

After this moment, Jacy made a new gesture, hold-and-tap, and repeated it (see Table 4.2). This new gesture was effective in creating more than one object on Jacy's side. Then Jacy continued this gesture by adding one more finger as she held down the previous pips. As soon as she created an additional pip, she continued her hold-and-add gesture as she counted up: "two, three".

Table 4.2. Records of the children's gestures from the 18th to the 23rd second

	Jacy's Gesture			Kyra's Gesture		
Time Period	Action type	Object	Side	Action Type	Object	Side
0:18-0:19	-			Tap	1 pip	LS
0:18-0:19	Hold	1 pip	RS	(inaudible)	-	
0:19-0:20	Hold	1 pip	RS	Tap	1 pip	RS
0:19-0:20	Hold "one"	1 pip	RS	Tap (inaudible)	1 pip	RS
0:19-0:20	Hold and tap	2 pips	RS	-	-	
0:20-0:21	Hold	1 pip	RS	Tap	1 pod	LS
0:21-0:22	Hold and add 1 finger "two"	2 pips	RS	Tap	1 pod	LS
0:21-0:22	Hold and add 1 finger	3 pips	RS	-	-	
0:22-0:23	Hold "three"	3 pips	RS	Tap	1 pod	LS
0:22-0:23	Hold and add 2 fingers	5 pips	RS	-	-	

The design of TT allowed Kyra to tap on the opposite side of the screen and to change the structure of the screen in a novel way. Jacy's hold-and-add gesture (and the number words she uttered) immediately followed Kyra's tap which changed the number of the pips. Both Jacy's verbal account and the change in her gesture corresponded to a change in Jacy's structure. These changes reflect a synchronization: each number name matched the number of fingers she placed on the screen. Since Jacy repeated this gesture, and uttered consecutive number names as the number of pips increased one by one, although not for a long time, it constituted a history of recurrent interaction between Jacy and the screen. This history suggests that making multiple objects by using multiple fingers was a learned behaviour and the synchronization of Jacy's gesture with her verbal accounts constituted a one-to-one correspondence.

Episode 2: A Shift from Adding One Finger at a Time to Tapping Multiple Fingers at a Time

Jacy touched the screen with multiple fingers at the same time for the first time during her interaction at 0:22. While she was holding three fingers on the screen, she added two more fingers at the same time. Table 4.A.1 (see Supplementary Material A) shows the evolution of this gesture into tapping multiple fingers at once. As Jacy held five fingers at 0:23, she started to add and lift one finger repeatedly fourteen times until 0:27. Then she lifted three fingers at once and tapped them all at once. Next, she first lifted four fingers at once and then the rest of her fingers all-together. Then, the next moment, she did a completely new gesture: tapping five right-hand fingers at once and tapping five left-hand fingers at once. She repeated this gesture at least three times and then tapped all her ten fingers at once creating multiple objects.

Jacy's actions gradually shifted from holding and adding to tapping multiple fingers at the same time. When Jacy started to hold-and-add her fingers one by one, the pips stayed on the screen and their number increased gradually. The changes in Jacy's actions and on the screen followed each other in a cyclic manner. The change in her action triggered a change in the structure of the screen and this in turn created multiple objects. The change in Jacy's gesture occurred right after these multiple objects appeared on the screen, and these reciprocal perturbations ended with a gesture that reflected a slight difference from Jacy's holding-and-adding gesture. Thus, it suggested a distinct gesture, namely a discovery: holding and adding multiple fingers at once. After this discovery, she repeated this gesture for a few times with different variations, which resulted in multiple pips that simultaneously appeared on the screen. Then Jacy discovered another gesture: tapping multiple fingers at once. After this discovery, she repeated this gesture for a few times with either hand, too. At the end, Jacy tapped all her fingers at once and this created ten pips. This new gesture suggested a learned behaviour which was also effective in creating multiple objects because it followed the structural coupling between Jacy and TT.

Episode 3: Another Way to Make Many

At 0:31, Jacy held five fingers at once and this created five pips. Then she tapped her left index finger and held it there while Kyra interchangeably tapped her multiple fingers between 0:32 and 0:33. At 0:33, Jacy asked, “How do you make those many ones?” After asking the question, her gestures changed from holding to repetitive tapping (see Figure 4.8 and Table 4.3). This repetitive tapping created a single pod of two pips. Then Jacy said “Oh” twice, as the pods appeared on the screen. Jacy held her finger on the screen just after a pod appeared, but it disappeared again when Kyra lifted her fingers at 0:35.

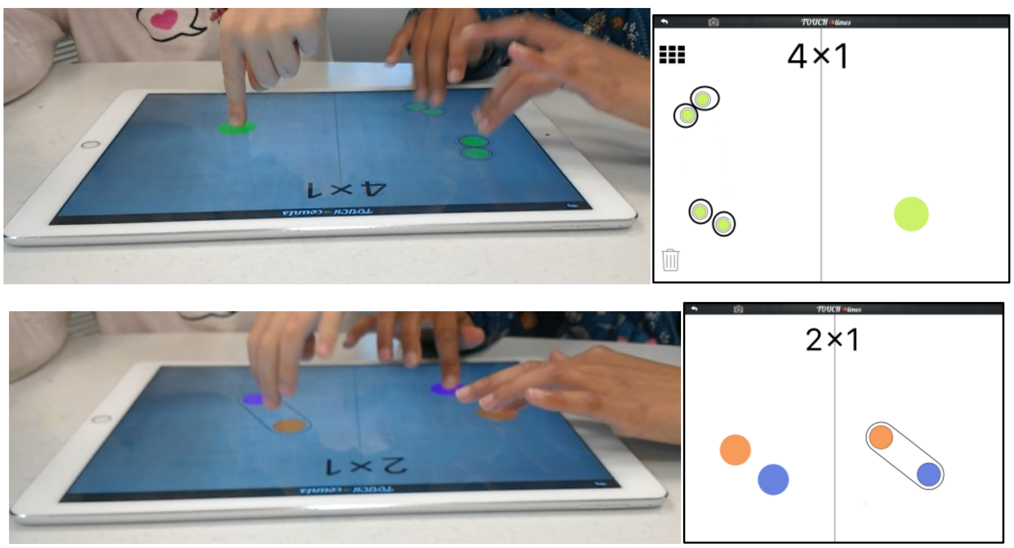


Figure 4.8. Jacy’s shift from holding a pip to tapping a pod.
Note: Black circles added by the author to make the pods more visible

Table 4.3. The records of children's gestures from the 32nd to the 36th second

	Jacy's gesture			Kyra's gesture		
Time period	Action type	Object	Side	Action type	Object	Side
0:32–0:33	Hold five fingers	5 pips	RS	Tap right index	5 pods	LS
0:32–0:33	Hold left index	1 pip	RS	Tap two right two left	4 pods	LS
0:33–0:34	Stand still	1 pip	RS	Tap two right two left	7 pods	LS
0:33–0:34	Stand still	1 pip	RS	Tap four right one left	9 pods	LS
0:33–0:34	Stand still	1 pip	RS	Tap three right two left	9 pods	LS
0:33–0:34	Standstill "how do you get those many ones?"	1 pip	RS	Tap one right one left	9 pods	LS
0:33–0:34	Tap left middle	1 pod	RS	Hold left LI and RI	2 pips	LS
0:34–0:35	Tap left middle "oh"	1 pod	RS	Stand still	2 pips	LS
0:34–0:35	Tap left middle "oh"	1 pod	RS	Stand still	2 pips	LS
0:35–0:36	Hold left middle	/	RS	Lift	/	/
0:35–0:36	Stand still	/	RS	Hold LI and RI	2 pips	LS
0:35–0:36	Lift	/	/	Stand still	2 pips	LS

Note: *LI*, left index finger; *RI*, right index finger.

Jacy's question suggested visual and auditory interactions between the girls. Her reference to Kyra's pods constituted a visual interaction among Jacy, Kyra and the screen. Even though, in the previous episode, Jacy could create multiple objects by tapping multiple fingers, holding multiple fingers down at once was not effective to create "those many ones" that referred to Kyra's pods. This suggested that Jacy distinguished the pips visually from the pods. The transition in Jacy's gestures from holding to tapping followed Kyra's repetitive taps. After Jacy asked the question, she changed her gesture from holding to tapping like Kyra did. This suggested that Jacy imitated Kyra to create pods. As Jacy tapped repetitively, Kyra was holding her fingers on the screen. Thus, Kyra and Jacy socially coupled as they exchanged their previous gestures. Kyra's pip-holding gesture allowed Jacy to create the pods. As soon as the pods appeared on the screen, this triggered a change in Jacy's structure. She said "Oh" and kept her finger on the screen. In this episode, tapping repetitively was effective in creating "many ones". This gesture suggests a learned behaviour, because it followed the history of Jacy's visual interaction with Kyra: Jacy's tapping followed Kyra's repetitive tapping.

After Jacy shifted her gesture from holding to tapping, this created single pods. She kept tapping her single finger repetitively until she created another pod at 0:40. As soon as this pod appeared on the screen, she held the pod-making finger still. Then she said, "Oooooh" once again, as the pitch of her voice increased gradually in an excited manner. During this time, Kyra was changing the number of pips. Then she held her finger on the screen and the pod stayed there until Kyra cleared the entire screen by lifting her finger. At 0:42, Jacy tapped, and this gesture created a pod again. Then she held the pod for five seconds (see Table 4.A.2 in Supplementary Material A) while Kyra was changing the number of the pips (see Figure 4.9). This change in the number of pips was mirrored in the pod and at 0:47 Jacy said, “Wait, get as much as you can” as she held the pod. However, Kyra did not respond to this request.

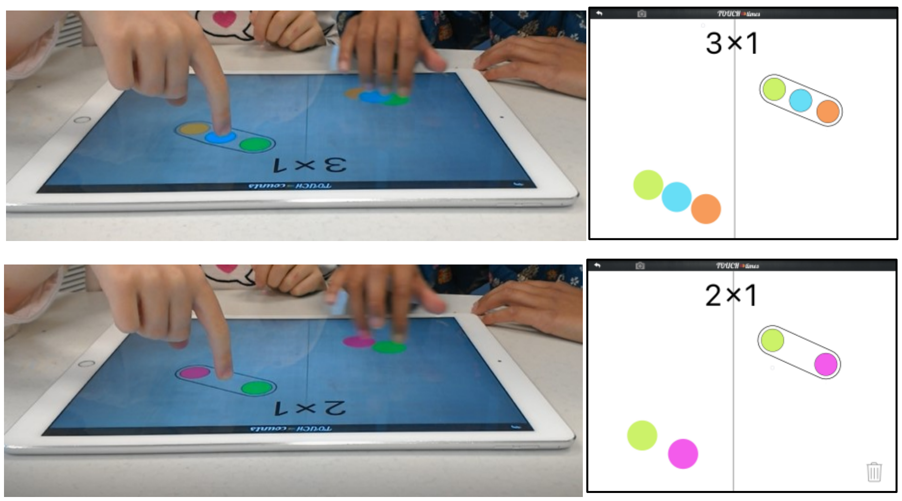


Figure 4.9. Kyra changes the pips while Jacy holds the pod.

Like in the previous episode, Jacy uttered “Oh” as soon as a pod appeared on the screen. This time, her utterance lasted longer, and it was enacted with a higher-pitched tone. During her utterance, Jacy’s pod underwent several changes, even though Jacy did not make anything else other than keeping the pod-making finger still. Rather than the pod itself, the spontaneous changes within the pod might have triggered this exclamation, one that was different from the previous ones. Holding a pod allowed Jacy to observe recurrent changes in her pod during the following five seconds. The request from Jacy to Kyra to “get as much [sic] as you can” suggested a visual interaction among Jacy, Kyra and TT, and constituted a call for a co-ordinated action to achieve a goal that might be growing her pod via “as much [sic] as” pips. She did this request as she was holding her pod. This suggests another distinction to make many.

Episode 4: Changing Pods with Pips

At 3:32, there were thirteen five-pods on the screen (see Figure 4.10a). Kyra was holding one pod, and Jacy was dragging the pods one by one as she held five pips (see Figure 4.10a).

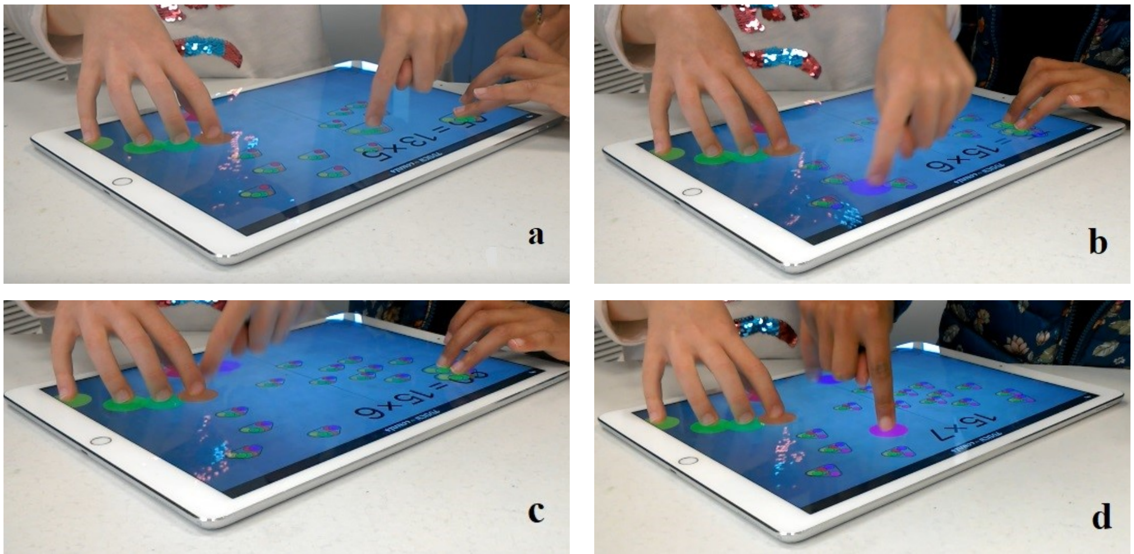


Figure 4.10. Dragging pods and pips.

At 3:40, she made a pip and dragged it across the screen for four seconds, creating circular paths (see Figure 4.10b and c). During this time, Jacy said, “Whaaat” and Kyra said, “Woooooow”. At 3:44, Kyra lifted her pod-holding finger, held a pip on the RS (see Figure 4.10d) and dragged it in a circular motion like Jacy. Then the following conversation emerged as they dragged the pips across the screen.

> [3:46] Jacy: Oh wait, I can change the shape.
>
> [3:54] Jacy: Look [points to one of the pods.], look here [touches the pod]. I am doing the exact same thing [drags her two pip-making fingers towards each other and further away repeatedly].
>
> [3:57] Kyra: Yeaah, and I am doing it with the pink [drags the pink pip with a circular motion]… green and yellow [drags green and yellow pips].

Dragging individual pods across the screen allowed Jacy to discern a change within the pods that resulted from Jacy’s making of another pip. Dragging pods required close eye contact with the pod: to drag the pod, she must have grabbed it, and to grab the pod she must have seen it. So, this eye contact allowed Jacy to discern a change within the pods. This distinction was evident both from the changes in Jacy's gestures and in her verbal accounts. This new pip Jacy created at 3:40 (see Figure 4.10b) halted her pod-dragging gesture and triggered a pip-dragging one instead. The change in Jacy's gesture, the change in the configuration of the pods and the girls' exclamations co-emerged, suggesting a social coupling. In the following conversation, the girls brought forth a connection between the pips and the pods as Jacy "changed the shape" while she dragged the pips and Kyra "did it with the pink, green and yellow" pips.

Following this episode, at 4:25, Jacy said to Kyra, "Wait, do just one". Kyra made one pip and Jacy made seven one-pods. Then they created several more one-pods by taking turns for thirty seconds. At 4:55, Jacy said to Kyra, "Wait, press a lot…press your whole hand". Kyra made five pips and Jacy tapped the pods one by one until Kyra lifted her fingers.

In this episode, Jacy achieved the task that she posed at 0:47. While Kyra did not respond to Jacy's request back then, in this episode their social coupling allowed Jacy to create multiple objects with single touches. This behaviour emerged after each girl created several one-pods. These multiple attempts to create pods constituted a history of recurrent interaction among the girls, the pips and the pods. Therefore, this pod-making gesture suggested a learned behaviour.

4.7.2. The Task Session

After seven minutes of free play, the researcher asked the children to make seven with one finger. This was the first time that they were given a unitizing task to solve. In this episode, I identified three different approaches to structure a quantity based on various researchers' distinctions of additive and multiplicative thinking. These approaches follow a chronological order. Therefore, I share the findings within three sub-sections respectively.

An Additive Approach

Jacy's first response to this question was to tap her left index finger and to hold it there for four seconds. This gesture made one pip on the RS. Then Jacy started to add her thumb, middle and ring finger respectively as she held her index finger. This created four pips on the RS.

Pressing the index finger was ineffective to create seven because it only created one pip on the screen. Even though in the previous episodes I observed that Jacy had created multiple objects either sequentially or simultaneously, as a response to create seven she touched the screen with only one finger. The emphasis on the “one finger” in the researcher’s question might have triggered this ineffective action. After one pip appeared on the screen, Jacy added more pip-making fingers as she had done in “Episode 1: A Shift from Single Pip to Multiple Pips”. The single pip on the screen might have triggered Jacy to use her other pip-making fingers to increase the quantity. Since Jacy had already conducted this gesture before this episode, her structure allowed enactment of this behaviour in this episode as well.

Jacy’s consecutive touches only created pips on the RS. This gesture created a temporal and spatial separation between the pips, indicating them as a single unit of counts (Davydov, 1992), which suggested an additive approach to make a given quantity because the pips were the only referents (Schwartz, 1988) that were combined only on one level (Clark & Kamii, 1996): the second pip was added to the first pip, the third pip was added to the previous two pips and the fourth pip was added to the previous three pips. There was no pod that would include them all.

Spreading Pips Across the Pods

When Jacy lifted her pip-making fingers, Kyra said “one finger” and pointed to the RS of the screen. After Jacy tapped her right index on the RS, and held it there, Kyra started to tap seven times consecutively on the LS. This created one pip and seven one-pods. The researcher said, “Well, you made seven over there, Kyra, but I want Jacy to make seven coming out of her one finger”, as she pointed to Jacy’s index finger, which was holding the sole pip. As soon as the researcher said, “Well, you made seven over there”, Jacy started to tap her thumb twice as she held her index finger (see Figure 4.11a and b). This created another pip in each existing pod. Then, one of the children said, “Hmmm” with a pitch which increased and then decreased, indicating distinctions.

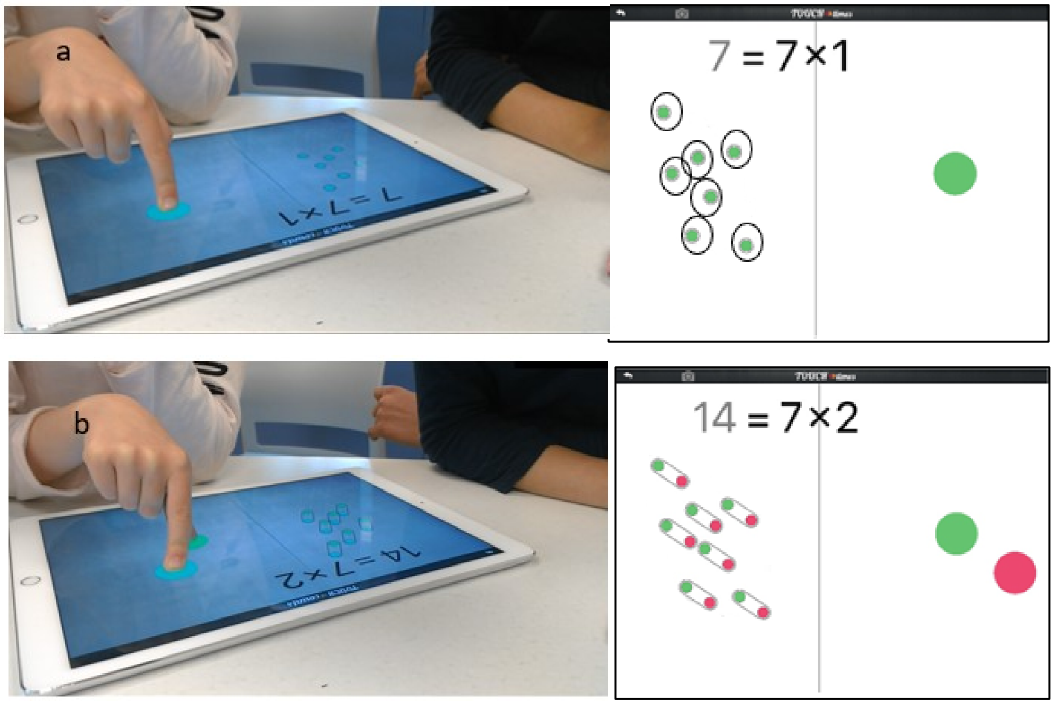


Figure 4.11. (a) Holding a pip; (b) adding another pip.
Note: Black circles added by the author to make the pods more visible.

Both Jacy and Kyra used their single fingers to create seven at the beginning of this episode. The researcher’s and Kyra’s repeated reference to “one finger” might have triggered the children’s co-ordinated, single-fingered gestures to create seven. Even though the children socially coupled with the researcher as they co-ordinated their gestures in relation to her verbal account, they were not effective in making seven coming out of one finger, as the researcher stated.

After making Jacy create a pip with her one finger, Kyra created seven pods sequentially. Increasing the number of pods one by one might recall an additive manner for an observer, but this gesture embodies a multiplicative relationship in many aspects. First, the children operated on two levels (Clark & Kamii, 1996) as they manipulated both pips and pods. Second, making seven from pod-making fingers creates the target quantity indirectly with the new unit of count (Davydov, 1992). Even though the pip-making finger was fixed on the screen, seemingly playing a passive role in quantifying, it was involved in the creation of pods. Every single pod is created through the joint action of pip-making and pod-making fingers. Thanks to this joint action, the single pip was transferred to the pods, thereby expanding the inclusion relationships to a second level (Clark & Kamii, 1996). This expansion became more obvious when Jacy added another pip: while the second pip was being added to the first one, it was simultaneously included in every pod. Rather than being a random act, this gesture embodied the connection among the pip, the pod and the product, as was the case in “Episode 4: Changing Pods with Pips”.

Here Comes the Multiplicative Action

Just before this episode, Jacy was holding five pips on the RS and Kyra tapped her right index finger on the LS. This created one five-pod (see Figure 4.12a). When the children created a five-pod, the researcher said, “Ooooh, Kyra made five with one finger”. Then Kyra tapped her index again and this created two five-pods. Then the researcher said, “Then she made another five”, as Jacy was dragging her right hand in a circular path (see Figure 4.12b).

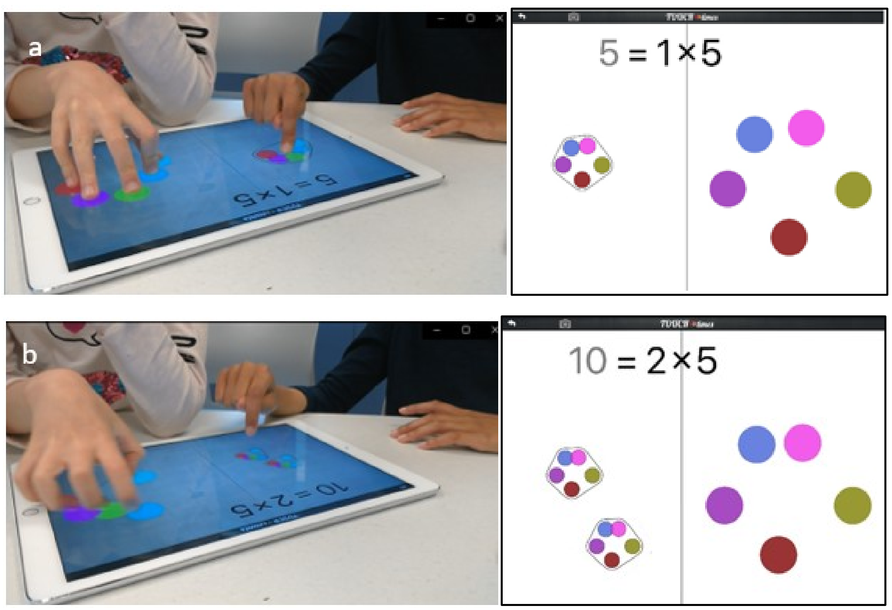


Figure 4.12. (a) One five-pod; (b) two five-pods.

As soon as the researcher finished her sentence, Jacy lifted her right hand immediately and said, “Wait” with a high pitch. Jacy grabbed Kyra’s right hand, pushed Kyra’s right index on the LS of the screen and said, “You start one”. This created one pip on the screen (see Figure 4.13a). Then she lifted Kyra’s finger (see Figure 4.13b) and said, “No, you tap seven fingers”, as she pushed Kyra’s right hand on the screen (see Figure 4.13c). Then Kyra added two left-hand fingers while she also held her five right-hand fingers on the LS and said, “I did it and then you…”. Finally, Jacy tapped her left index finger and created a seven-pod (see Figure 4.13d).

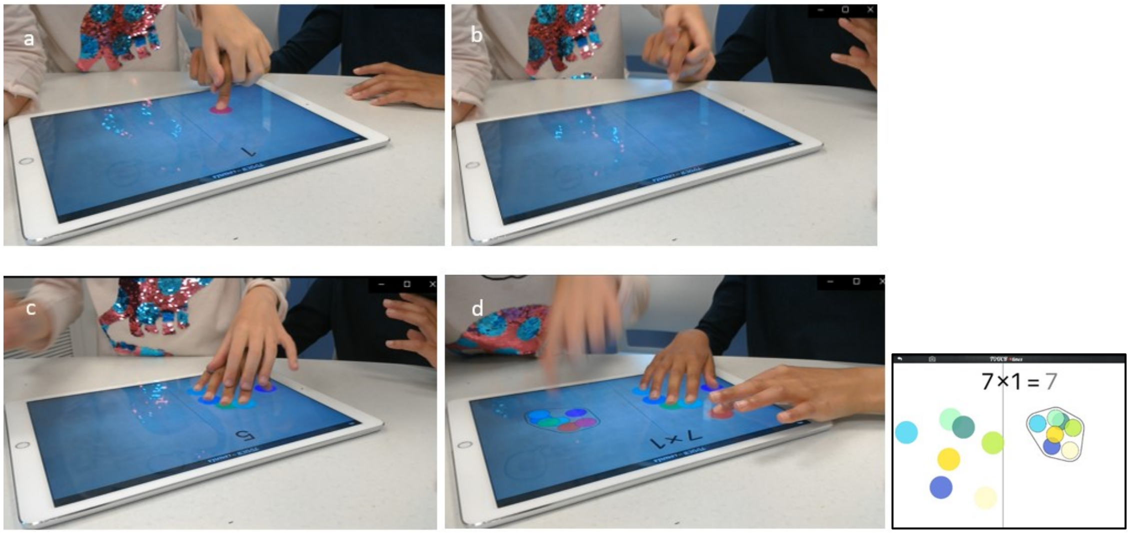


Figure 4.13. (a) Making one pip together; (b) starting from the beginning; (c) holding five pips at once; (d) making one seven-pod.

Even though making a five-pod was not an effective action to create seven with one touch, it was an important action in terms of unitizing the multiple pips. Kyra’s gestures suggested a distinction between pod-making and pip-making, as in “Episode 4: Changing Pods with Pips”. Instead of increasing the number of pips to get closer to the target number, Kyra tapped on the LS and created a five-pod, whose organization was different from the five pips. Kyra’s pod-making fingers created another unit of count at a second level (Clark & Kamii, 1996). Creating a pod, Kyra made a multitude not directly with multiple pip-making fingers, but indirectly with a single pod-making finger, as suggested by the researcher.

Kyra had already created one five-pod before this episode. However, the researcher named this gesture as “five with one touch” only in this episode. The children’s gesture, the image on the screen and the researcher’s verbal and emotional exchange supplied visual, verbal and haptic interactions simultaneously, and this created a communication domain among the researcher and the two girls. The researcher also created a distinction among the TT objects by naming the five-pod, which triggered the girls’ structures to make the same distinction.

After Jacy said, “You start one” and pushed Kyra’s index finger on the screen, this created one pip. Again, Jacy’s verbal account and her gesture reflected the researcher’s emphasis on using “only one finger”. However, seeing one pip on the screen, Jacy immediately lifted Kyra’s finger and said, “No, you tap seven fingers”. Even though Jacy started with one finger, after she saw the single pip on the screen, she acted differently. Consequently, seeing a single pip on the screen might have triggered Jacy to push Kyra’s seven fingers at once. Jacy learned this gesture in “Episode 2: A Shift from Adding One Finger at a Time to Tapping Multiple Fingers at a Time”.

The creation of seven pips was neither sequential nor simultaneous. After pushing Kyra’s index finger, Jacy started to create pips from scratch (see Figure 4.13b) instead of adding six more fingers next to Kyra’s index finger. Jacy lifted Kyra’s index finger immediately and pushed all her five fingers at once onto the screen. By resetting the screen, and creating five pips at once, Jacy created the five not as the combination of separate five objects, but as a single object which was congruent with the notion of a composite unit. This demonstrated a transition from sequential (adding multiple pips one by one) to simultaneous (adding multiple pips at once) action.

After Kyra added two more pips, Jacy placed a pod-making finger onto the screen, thereby creating seven with one touch. Both pressing pips at once and making a pod via these pips were not random movements. Kyra emphasized the temporal order of the touches by saying, “I did this, now you do…”. This suggested that the children communicated a particular order for their gestures, as in “Episode 4: Changing Pods with Pips”. Moreover, after the meaning of making five with one finger was communicated between the girls and the researcher, they were able to create seven in the same way. Therefore, in making seven with one touch, the children embodied the act of unitising and they changed the unit count from one to seven. This indirectly created a quantity that was congruent with Davydov’s (1992) brief description of multiplication: “arriving at the result of a count indirectly” (p. 16).

4.8. Discussion

This study examined how children structure quantities when they are given a unitising task in TT. The “how” question was directed to two things: (1) the specific way the children embodied quantities and (2) the process that led to this specific embodiment.

Rather than suggesting a new theoretical construct, this discussion draws on the main theoretical constructs of enactivism to respond to these questions.

The unitizing task requested students make seven with one finger. This task must be solved by making a single pod after making seven pips, which can be described as a unitizing action. The children's response to this task indicated a shift in how they embodied the quantities. First, they responded to this task by creating a collection of single pips using one hand, which was additive. Then they made a single multitude out of pips and a pod by collaboratively using their hands, which was more aligned with the multiplicative structures, as explained in the task sub-section of the findings section. This shift emerged from on-going interactions among the bodies (TT, the children, the researcher), each of which participated in the multiplicative embodiment of the quantity differently.

The children's unitising act did not emerge as soon as they encountered TT. Their subjective sense of unitising developed through many repeating interactions, thanks to the nature of TT, which allowed children easily to create numerous quantities with varying structures through their fingertips over a brief period. During their free play, the children's actions revealed a pattern: once they shifted a gesture, they repeated it for a while until they shifted it again. (The tabular representation of the data explicitly captures this pattern.) This pattern in the children's body movements constituted what Maturana and Varela suggested was a necessary condition of learned behaviour: the history of recurrent interaction. As this history unfolded, the children's verbal accounts that accompanied their gestures revealed certain distinctions, both of objects and of gestures.

The distinct organization of pips and pods triggered children to distinguish between the TT objects that represent the dissimilar roles of multiplicative factors. This distinction was evident in Jacy's self-set goals during their free play to structure the quantity in a specific way that was more aligned with multiplicative structures. Although they were not given a particular task, interacting freely with TT in collaboration with Kyra triggered Jacy to pose a task of "making many ones", which was closely aligned with the underlying aim of TT. Moreover, she was interested not only in the number of objects, but also in the way she created them: pods proved more interesting than pips in terms of "making many ones".

The design of TT is consistent with Kaput's (1985) argument: manipulating multiple representations of multiplication in a co-ordinated window might help learners understand the invariant nature of functional relationships between the multiplicative factors through the simultaneous transfer of changes across the representations. The iconic representation of TT objects and the symbolic representation of the multiplication equation that mirrored the number of TT objects were present simultaneously. However, this co-ordination feature may not be enough in itself to help learners make meaning of multiplicative relationships out of graphics. As Thompson (2013) wrote, "the core concept of 'function' is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance" (p. 79). In enactivist terms, this might be explained through the phrase "bringing forth a world". The functional relationship is not a pre-given concept that can be represented in the students' mind; rather, students bring forth this relationship through structural coupling with the visual and symbolic representations on a co-ordinated window.

The findings of this study show that, rather than the simultaneous transfer of changes across the multiple representations (iconic TT objects and the equation), the simultaneous transfer of changes within a specific type of representation (iconic TT objects) might also prompt students to experience a correspondence between the number of pips and the size of individual pods. TT is designed to help students bring forth this correspondence by transferring any change in the number, colour and spatial configurations of the original pips to each pod. In addition to engaging with this transfer visually, the major muscle movements of individual bodies played an important role when the children distinguished the correspondence between pips and pods.

Unlike Kaput's design, TT's haptic and temporal affordances allowed these types of major bodily engagement, which were manipulated through the computer mouse. When Jacy and Kyra said they "changed the shape [of the pods] by doing the exact same things [with the pips]", they were dragging the pips. While any change in the pips is visually transferred to each pod since the beginning of the children's engagement with TT, they distinguished pods as objects controlled by pips by moving pips with their hands and tracking pods with their eyes, which can be described using Nemirovsky et al. (2013) term of "perceptuomotor integration".

In addition to structural coupling between the individual bodies and the environment, the social coupling between the individual bodies contributed to the emergence of a unitizing act. The social coupling with a more knowledgeable person also triggered new distinctions that led to effective unitizing action. Jacy intentionally created multiple objects with single touches once she and Kyra had related the pips to the pods during their free play. However, once they were given the unitizing task, they did not "make seven with one finger" by immediately re-enacting this gesture. They did so right after the researcher, as a more knowledgeable person, triggered the girls to distinguish "five with one finger" on TT through her verbal and visual interaction with the girls.

The findings of this study also verify that children can learn through social coupling with a person who is not more knowledgeable, as shown in the previous studies (e.g., Abrahamson et al., 2011; Kelton & Ma, 2018; Nemirovsky et al., 2012, 2013). Among them, I identified only in Nemirovsky et al. (2013) a brief explanation about how the peer's body contributes to one's learning. Below, I extend this explanation.

Social coupling with the peer's body allowed the children to amplify the multiplicative relationships through other's body. When the unitizing task was given, they re-enacted the gestures they had learned during the free play. However, I do not suggest that, in engaging with TT, the children simply reproduced the same gestures that they had used during their free play. The instruction of the task and the girls' structure at that moment might have allowed what Nemirovsky et al. (2013) called, "a non-trivial transformation of actual motor actions" (p. 405).

When Jacy was asked to create seven with her single finger, she held Kyra's fingers and pressed them onto the screen to make pips instead of touching the screen herself. Until this task, the children had not interacted with each other physically. This indirect interaction with the screen suggested that Jacy avoided using more than "one finger to make seven", following the researcher's request. Instead of using her own body to make pips, Jacy used her hand to manipulate Kyra's hand to act on the screen effectively within the restrictions of the task. Thus, by distancing her unitizing finger from the other unit counts, Jacy amplified the distinctions between the multiplicative factors and the functional relationship between them through the other's body.

This is similar to Nemirovsky et al. (2013) account in which the other's body reveals the differences: when two children collaboratively used handles to manipulate the cursor on the screen, "the unidirectional nature of Kayla's left-to-right movement provides a backdrop against which the bidirectional nature of Ivan's upward and downward movements may be made salient" (p. 393).

In addition to amplifying the multiplicative relationships, during social coupling, the other's body changed the individual's environment and triggered discovery of novel gestures that embodied the quantity in a novel way. In Abrahamson et al. (2011) piece, the educator as a more knowledgeable person had a pivotal role in such changes. This study shows that students as "not more knowledgeable" ones also played such a role by making critical changes in the environment. When Kyra broke the boundary between "her side" and "Jacy's side", by pressing her finger on the RS of the screen where only Jacy was making single pips, this innovative gesture increased the number of pips. Only after this change did Jacy start to use her multiple fingers. As Maturana and Varela suggest, each girl's structure determined different courses of interaction with their environments. Kyra's structure enabled her to trigger a change on the other side of the screen. Without Kyra's interaction with the screen, Jacy may not have been able to use her multiple fingers on it. This incident reflected the importance of innovative actions in learning, yet it was unclear what had caused Kyra to break the routine and pass the border between the sides.

When individuals socially coupled with others, TT objects were always involved in these interactions. When the researcher and the children socially coupled to communicate distinctions, the researcher was referring to the TT objects that the children were holding. Similarly, when Kyra crossed "the border" and touched Jacy's side, she created an additional pip. After Kyra's gesture, Jacy's body did something totally different from Kyra's, which served to her self-set goal of "making many ones". Instead of pressing her one finger on the opposite side, as Kyra had, she increased the number of fingers she pressed on the screen. This gesture increased the number of the TT objects. Seeing Kyra's finger had created an additional pip, Jacy interacted with TT in a novel way.

Based on these incidents, and Maturana's (2020) recent clarifications of the notion of "organism", I propose a new interpretation for "social coupling" and argue that children were coupling not with one another's bodies, but rather with the other individual's interaction with TT. Maturana described an organism as, "a living system as it operates as a totality integrated in its ecological niche" (p. 16), and named this operational unity as the ecological organism-niche unity. Therefore, when Maturana and Varela earlier had defined social coupling as structural coupling with another organism, they might have been referring to coupling with the ecological organism-niche unity, not with the other's body as an autopoietic unity separated from the environment through its molecular border. For this reason, it may be more appropriate to distinguish these incidents as socio-structural couplings, in order to emphasize the integration between the ecological organism-niche unities, and re-present this theoretical construct as in Figure 4.14.

Figure 4.14. Re-presentation of socio(-structural) coupling.
Note: Modified from Maturana & Varela (1987/1992, p. 180).

4.9. Towards a Conclusion

This study aimed to examine two children's interactions around/with TT when they were asked to create a quantity in a specific way. In particular, the focus was on the process that preceded the emergence of a specific gesture, which was the first unitising in multiplication. The findings show that engaging with touchscreen technology allowed students literally to be in contact with a world which was designed based on multiplicative relationships. The children's approach to creating a target quantity shifted from being additive towards being multiplicative through such an engagement. Therefore, in addition to providing multiple representations of multiplication, allowing students to engage haptically with the transfer of changes within a specific type of representation is important for them to distinguish the multiplicative factors and the many-to-one correspondence between them.

In addition to TT's structure that embeds various aspects of multiplicative relationships and the children's individual bodies, others' bodies also participated in the children's enactment of a quantity in a multiplicative way by triggering mutual distinctions, by changing one's environment in a way that otherwise would not be possible and by amplifying the multiplicative relationships. All these actors were interwoven into the structural and socio-structural couplings among the children, the researcher and TT. Therefore, it would be unfair to hold one of them alone responsible for the shift in the children's approach to quantity.

Making seven with one touch was not a traditional multiplication task that required calculations or simply a recall of multiplication facts. It offered these students an activity of exploration that triggered a shift from additive thinking and encouraged them to think multiplicatively. The girls still have a long journey to make until they will feel comfortable navigating within the multiplicative conceptual field that consists of various multiplicative situations and symbols in addition to multiplicative thinking. A next step could be to include these situations, as well as the symbols, into the children's interaction around/with TT.

Supplementary Material A

The Records of Children's Gestures

Table 4.A.1. The records of children's gestures from the 21st to the 30th second

Jacy's Gesture				Kyra's Gesture		
Time Period	Action Type	Object	Side	Action Type	Object	Side
0:21–0:22	Hold and add 1 finger	3 pips	RS	-	-	
0:22–0:23	Hold	3 pips	RS	Tap	1 pod	LS
0:22–0:23	Hold and add 2 fingers	5 pips	RS	-	-	
0:22–0:23	Hold and lift "four"	4 pips	RS	-	-	
0:23–0:24	Stand still "five"	5 pips	RS	Tap	1 pod	LS
0:24–0:25	Hold and add "six"	6 pips	RS	Tap	1 pod	LS
0:24–0:25	Hold and lift "two aah"	5 pips	RS	Tap	1 pod	LS
0:25–0:26	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:25–0:26	Hold and lift	5 pips	RS	Tap right index	1 pod	LS
0:25–0:26	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and lift	5 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and lift	5 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and lift	5 pips	RS	Tap right index	1 pod	LS
0:26–0:27	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:27–0:28	Hold and lift	3 pips	RS	Tap right index	1 pod	LS
0:27–0:28	Hold and add	6 pips	RS	Tap right index	1 pod	LS
0:27–0:28	Hold and lift	2 pips	RS	Tap right index	1 pod	LS
0:27–0:28	Lift both hands	-		Tap	-	
0:27–0:28	Tap right five fingers	5 pips	RS	Tap right index	/	/
0:27–0:28	Tap left five fingers	5 pips	RS	Tap right index	/	/
0:28–0:29	Tap right five fingers	5 pips	RS	Tap right index	/	/
0:28–0:29	Tap left five fingers	5 pips	RS	Tap left index	/	/
0:28–0:29	Tap right five fingers	5 pips	RS	Tap right index	/	/
0:28–0:29	Tap left five fingers	5 pips	RS	Tap left index	/	/
0:29–0:30	Tap ten fingers	3 pods	RS	Tap	1 pip	LS

Table 4A.2 The records of children's gestures from the 41st to the 47th second

Jacy's Gesture				Kyra's Gesture		
Time Period	Action Type	Object	Side	Action Type	Object	Side
0:40–0:41	tap RI "oooooohhh"	1 pod	RS	Tap RM & RR	2 pips	LS
0:40–0:41	hold RI	1 pod	RS	Tap RI	1 pip	LS
0:40–0:41	Hold RI	1 pod	RS	Tap RI	1 pip	LS
0:40–0:41	Stand still	1 pod	RS	Hold RI and add RM & RR "look what I did"	3 pip	LS
0:41–0:42	Stand still	/	/	Lift	/	/
0:41–0:42	Stand still	/	/	Hold RI, RM, RR	3 pips	LS
0:41–0:42	Stand still	/	/	Hold RI & RR, lift RM	2 pips	LS
0:42–0:43	Tap and hold RI	1 pod	RS	Hold RI & RR, add RM	3 pips	LS
0:42–0:43	Hold RI	1 pod	RS	Drag RI, RR & RM	3 pips	LS
0:42–0:43	Stand still	1 pod	RS	Drag, RI, RR, lift RM	2 pips	LS
0:43–0:44	Stand still	1 pod	RS	Drag RI, RR	2 pips	LS
0:43–0:44	Stand still	1 pod	RS	Drag RI, RR	2 pips	LS
0:43–0:44	Stand still	1 pod	RS	Drag RI, RR	2 pips	LS
0:44–0:45	Stand still	1 pod	RS	Drag RI, RR	2 pips	LS
0:44–0:45	Drag RI	1 pod	RS	Drag RI, RR	2 pips	LS to RS
0:44–0:45	Drag RI	1 pod	RS	Drag RI, RR	2 pips	RS
0:45–0:46	Drag RI	1 pod	RS	Drag RI, RR	2 pips	RS
0:45–0:46	Drag RI	/	/	Lift	/	/
0:45–0:46	Drag RI	/	/	Hold RI & RR	2 pips	LS
0:46–0:47	Hold RI " wait get as	1 pod	RS	Stand still	2 pips	LS

Note*:* *RI*, right index finger; *RM*, right middle finger; RR, right ring finger

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Declarations

Conflict of Interest The author declares no competing interests.

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Postlude

After re-engaging with this article, I want to emphasize a few points about the linguistic contrast between the phrases “mental action” and “bodily action”, Kaput’s (1987) argument about how children can understand intensive quantities to develop multiplicative thinking, and TT’s affordances with respect to simultaneous gestures.

When I compared the participationist and representationalist approaches in the field of mathematics education, on page 126 I stated that the former one “shifted the meaning of mathematical learning from being a mental action to being a bodily one”. This might sound as if I support the mind/body split by contrasting them. On the contrary, since enactivism associates effective action with cognition, for me the phrase “bodily action” already conveys cognition, not disassociates it. Therefore, I chose to drop the term “mental” because the existence of both terms in my languaging makes me feel like there is a separation between the two.

In the discussion section, I argue that TT’s affordance of simultaneously translating the changes in the pips to the pods constitutes an alternative way to enable students to understand the intensive quantity, the constant ratio between the two quantities which have different referents. I want to clarify that, rather than being an alternative tool to prompt students to experience intensive quantities, it is a tool to prompt students to experience the co-variance between the two unit-counts of multiplication.

In Kaput’s (1987) design of multiple representations of multiplicative structures, the two unit-counts of multiplication are represented as independent icons. Even though these icons are placed close to each other, and the ratio between their quantities is constant, it is not clear why such a proportional relationship exists. Whereas in TT, the two unit-counts have an organic link between them. Pods are created though pips. This organic link is made visible by the fact that the pods take on the same configuration as the pips. Therefore, unlike deciphering a pre-given relationship between the unit counts (as per Davydov, 1992), if we apply Kaput’s (1987) model to TT, users experience co-variance between the two quantities by identifying the organic link between them. I propose that, before students are expected to understand the relationship between the two unit-counts as a fixed ratio, students should be prompted to experience a co-variant relationship between the unit-counts. When students create multiple pods and experience the size of each pod as the same, they can identify the relationship between the unit counts as a constant.

The findings of this study illustrate the importance of the dual analysis I presented in the first article. The analysis of the semiotic potentials of Zaplify and pencil-and-paper revealed how different ways of creating similar images were related to different mathematical meanings. Drawing multiple lines on paper sequentially and creating them in Zaplify simultaneously would relate to the ideas of (1 + 1 + 1 + 1 … + 1) and M, respectively. This study shows that students might create quantities in Grasplify sequentially, too. When Jacy created multiple pips for the first time, her fingers touched the screen sequentially while she recited the number words as if she was counting up. This action is aligned with (1 + 1 + 1 + 1 … + 1). However, the structure of Grasplify allowed this gesture to gradually evolve into simultaneous tapping of multiple fingers. Before the target action (making seven with one finger) emerged, transition from 'tapping single pips' to 'hold and add pips' to 'tap multiple pips' happened. This transition was important because it allowed the first step of unitizing action. Jacy re-enacted simultaneous taps while she was making seven with one finger. Unlike counting, this gesture and the utterance of "seven" without reciting the previous number names suggested that Jacy experienced the pips with respect to cardinality which is aligned with M. The transition from counting up to creating a specific quantity was related to the idea of a composite unit which was finally crystallized with her single unitizing finger.

While Jacy was re-enacting simultaneous tapping during the unitising task, she asked Kyra to tap seven fingers as she pushed Kyra's right hand on the screen. This episode was interesting in terms of the relationship between the children and their fingers. Pressing Kyra's whole hand, no matter how many fingers it has, seemed to be effective for Jacy to create seven in TT. It is likely that Jacy felt Kyra's hand like the 'container' of fingers (singletons), suggesting a composite unit. Moreover, Jacy's asking Kyra to press her multiple fingers first suggested that she avoided pressing more than one finger on the screen even though there was not such an explicit instruction in the task. It seemed like Jacy felt her index finger as a unitizing operator. These feelings seem to participate in students' multiplicative thinking. So, I wonder what kinds of feelings emerge around children's bodies while they use TT. I will elaborate on the question of affect in the following chapter.

Chapter 5. Conclusion

This dissertation sought to study young learners' multiplicative thinking around/with a touchscreen educational technology called *TouchTimes* (TT). My motivation to conduct this research emerged as the result of the following:

1. The existing literature on students' difficulties in multiplicative thinking.
2. The variety of attempts—sometimes quite different one from the other—by researchers to propose alternatives to the model of repeated addition, in order to help students think multiplicatively.
3. A worldwide interest in the benefits of integrating educational technologies in mathematics teaching and learning.
4. An increased awareness of the important role of body in cognition.

I was particularly interested in the semiotic potential of TT to engage young learners with multiplicative thinking in a way that is different from the repeated addition model. Moreover, I was interested in how this semiotic potential unfolded during the young learners' interactions around/with the TT. The aim of this multi-layered qualitative study was to examine what type of bodily engagement TT triggered in students and how these bodily engagements enabled them to experience multiplicative structures. This final chapter first summarizes the key findings in relation to the research questions; provides theoretical, methodological, and educational implications; and ends with limitations, challenges, and future directions for research.

5.1. Summary of the Key Findings

The first study "The Analysis of a Model–Task Dyad in Two Settings: Zaplify and Pencil-and-Paper" explored the added value of a digital tool that constitutes a new model to introduce students to multiplication. Zaplify is the name of one of the two TT models. Some features of its interface are similar to arrays, which are suggested in curriculum documents, textbooks and the research literature as an alternative model to introduce multiplication different from repeated addition. The motivation behind this study came from the question I was asked repeatedly when I introduced this model in various contexts: What can Zaplify do that we can't do with pencil-and-paper?

I drew on the TSM's (Bartolini Bussi & Mariotti, 2008) notion of semiotic potential to explore the added value of Zaplify with respect to pencil-and-paper. The focus of the study was on the potential signs created as the result of the artefacts' different modes of use to solve a task. This task required creating array-like diagrams to model a multiplicative situation. The research questions driving this study were:

- Which signs might emerge when students use pencil-and-paper to solve a multiplication task?
- Which signs might emerge when students use Zaplify to solve a multiplication task?
- Which meanings of multiplication do these signs relate to?

I reported my fine-grained micro genetic analysis of how I used Zaplify and pencil-and-paper to make 198 by making M-ples (a Zaplify term that was created to help students experience composite units). In order to achieve this task in both settings, one must create circular objects at the intersections of perpendicular linear objects. I found that the images created in each setting looked similar. So, both artefacts had potential to prompt the same signs to describe the images. However, bodily actions that created these images differed significantly. While my single hand interacted with pencil-and-paper continuously as I drew with a pencil, I used both my hands in Zaplify as I tapped my fingers on the screen. My analysis showed that these differences in the qualitative characteristics of interactions might also prompt different signs that are related to the different meanings of multiplication.

Zaplify and pencil-and-paper have the potential to prompt common signs that can be related to the same aspects of multiplication. Perpendicular linear objects and intersections, whether created by Zaplify or with pencil-and-paper, might prompt common signs such as "vertical", "horizontal", "dots". These signs could be related to two distinct unit-counts and the product of multiplication, respectively. In addition, the number of circular objects on a linear object might also be associated with common signs such as "M-ple" which might be related to the functional relationships between the two unit counts of multiplication.

Each setting also might prompt different meanings related to multiplication. The circular objects emerged one by one on an M-ple when pencil-and-paper was used to solve the task. Therefore, the temporal aspects of the signs might emphasize a sequential nature for unitizing action and might prompt students to experience multiplication as combining multiple single units. Whereas in Zaplify, intersections were created all-at-once with a single finger touch. This was a simultaneous action which was aligned with Davydov's (1992) notion of transfer of unit count. Thus, an M-ple might be associated with the idea of a single multitude. While the former idea prompts multiplication as repeated addition, the latter one relates multiplication to splitting (Confrey, 1994). Moreover, it was possible to draw different number of intersections on each line with pencil-and-paper and this allowed M-ples with different sizes to emerge side by side. Therefore, the functional relationship between the unit counts may not mean a constant entity, unlike happens in Zaplify.

The second study "Reciprocal influences in a duo of artefacts: Identification of relationships that serves to multiplicative thinking" explored how a child made sense of the relationships between the Zaplify objects when he reciprocally used the pencil-and-paper and Zaplify. When a physical pedagogical artefact used with its digital counterpart, it is called a duo of artefact. Duos are mostly presented in a certain order: non-digital artefact is followed by the digital counterpart. In this case, the digital artefacts are assumed to add on to the affordances of its non-digital counterpart. My study explored an alternative use of duo to exploit the unique contribution of each artefact to the child's meaning-making process. The child used the artefacts in a back-and-forth way, rather than in an ordered manner.

I drew on the TSM's (Bartoloni Bussi & Mariotti, 2008) notion of pivot signs to explore the child's meaning making process. The focus of the study was on the signs the child created to explain the genesis of the Zaplify objects and the child's understanding of them. The research questions driving this study were:

- How do signs evolve when a child reciprocally uses a duo of artefact?
- How does a child experience the relationships between the Zaplify objects?

The analysis of the data showed that the child first produced signs related to physical features of the Zaplify objects. These signs were potentially the pivots to connect the difference between the Zaplify objects to the difference between the referents of a multiplicative situation. The spatial relationship between the Zaplify objects were not signified at the beginning of the episode, even though the adults around the child asked questions that addressed these relationships. The signs related to spatial relationships are important because they might constitute critical pivot signs: they can be related both to the relationship between the Zaplify objects and to the functional relationship between the factors of multiplication.

Findings illustrated two types of transition of signs as the child moved back-and-forth between the two artefacts. The signs were first connected to visual characteristics of the objects and then to the spatial relationships between the objects. This is resonant with Duval's (2006) 'treatments' (see below). Some signs which were created in one setting were re-created in another setting later. This is like Duval's (2006) 'conversions' (see below). It seemed that these transitions together allowed the learner to enrich his meaning-making experience and to identify various aspects of multiplicative structures.

The third study "A Quantitative Shift Towards Multiplicative Thinking" explored how two third graders learned to structure the quantities multiplicatively in TT. My aim was to demonstrate the shift between the students' additive and multiplicative thinking and to explain how this shift emerged around/with TT. Unlike in many studies, I did not operationalize multiplicative thinking based on the students' correct computations of multiplicative expressions as a response to verbal or number problems. After all, using multiplication algorithms properly does not necessitate multiplicative thinking (Carrier, 2014). Instead, drawing on enactivism, I operationalized thinking as the students' bodily reactions to a given TT task. The task is called unitizing and it requires students to create a quantity indirectly by coordinating their two hands. This way of creating a quantity is aligned with the multiplicative structures conceptualized by various researchers (e.g., Confrey, 1994; Davydov, 1992; Vergnaud, 1988). The research questions of this study were:

- How do children collaboratively structure quantities in order to solve a unitizing task in *TouchTimes*?
- How do children couple with their environment, as well as with other individuals also engaged in this same environment, in order to solve a unitizing task in *TouchTimes*?

The children first explored Zaplify without being given any specific task. I examined the children's thinking process by focusing on the transitions among the students' bodily actions both before and after the unitizing task was given. This followed the enactivist assertion that learning can be described based on the history of interactions with the environment (Maturana & Varela, 1987). I distinguished additive thinking from multiplicative thinking by examining the structural relationships among the hands and TT objects with respect to various researchers' conceptualization of additive and multiplicative structures (e.g., Confrey, 1994; Davydov, 1992; Vergnaud, 1988).

The result of the analysis showed that the child's engagement with TT shifted dramatically from the beginning until the end of the free exploration session. At first, Jacy (one of the two children) pressed her single finger on the screen without coordinating her gestures with her peer Kyra. This created "independent" objects on the screen. Single-finger-pressing gesture gradually evolved into multiple-finger-pressing gesture through the interactions between Jacy and Grasplify. It seemed that the peer's finger, which created another object on "her" side, triggered these iterations. The relationship between the pips and the pods emerged after the pods Jacy was holding changed based on how she or Kyra used their pip-making fingers. The analysis of the task session also suggested a similar transition in Jacy's gestures. However, the shift from single finger use to coordinated use of multiple fingers happened faster compared to the free exploration session.

This shift in Jacy's gestures indicated a shift in the structure of the quantity. While Jacy first created quantities by making multiple single objects (pips) herself, later she created quantities as single multitudes (pods) with Kyra who set the size of the multitudes by making multiple single objects. While the former gestures and the objects they created were aligned with additive relationships, the latter ones were aligned with multiplicative structures. In the former case, only one type of object was used to create quantities. The latter case involved two types of objects, one of whose size depended on the amount of the other. This difference was aligned with the contrast between the single independent unit-count of addition and two unit-counts of multiplication which were related to each other through a many-to-one correspondence.

5.2. Implications

This section presents the contributions of this dissertation to the mathematics education literature. I first introduce the methodological and theoretical implications and then conclude the section with the educational ones.

5.2.1. Methodological and Theoretical Implications

I drew on two theoretical frameworks in this dissertation. The way I used them is a method recently emerged in a field different from mathematics education. However, applying it in mathematics education is not common. For this reason, as a whole this dissertation illustrates an example of a method to use multiple theories to examine learning mathematics by using tools. Moreover, each manuscript I included in this dissertation has certain methodological or theoretical contributions. The third manuscript proposes a new interpretation for a theoretical construct of enactivism. The second manuscript extends Arzarello et al.'s (2009) analysis methods of semiotic bundles. The first manuscript presents an explicit documentation of how to create data for the analysis of the semiotic potential of an artefact.

Several researchers who study learning mathematics by using technology have proposed networking theories in the field to examine this complex phenomenon in a more wholistic way. This way, one theory would help to explain what the other theory cannot. Following this proposition, I used certain theoretical constructs of enactivism and TSM. While enactivism as a broad, philosophical frame explains cognition in general by taking a participationist approach, TSM as an educational frame draws on social constructivist perspective to explain learning mathematics by using a tool. Enactivism focuses on the body, which has been mostly considered as periphery to learning by other theories. Therefore, it enabled me to attend to various bodily actions to study the process of learning which has multiple dimensions. However, the theoretical constructs of enactivism are inadequate to match the particularities of learning mathematics by using an artefact. TSM allowed me to attend such particularities. Instead of networking these two theories, which have very different epistemological assumptions, by using their theoretical constructs together like in a bricolage, I adopted a different approach to alleviate epistemological conflicts.

As I explained in the introductory chapter, I was heavily influenced by enactivism's emphasis on the individual actions as the constituent of cognition and started not only to interpret the data I created, but also the world I brought forth for myself based on its assumptions. As a result, I also read TSM through enactivism and interpreted its constructs based on enactivism's assumptions. Jackson and Mazzei (2013) described this method of reading one text through the other as diffractive reading which was first introduced to the literature by new materialist Karen Barad. In the same book they explicitly stated that this approach works against “the stance of the bricoleur and push[es] the concepts with the data to exhaustion” (p. 139).

To my knowledge, this diffractive reading method has not been widely practiced in the field of mathematics education. The only example I came across was Schvarts et al.’s (2021) study in which they proposed the theoretical construct of embodied instrumentation. As an alternative to networking theories as they are, this method might allow mathematics education researchers to use a duo of frames that are theoretically inclusive and epistemologically coherent.

Using theoretical constructs from two theories also helped me make a theoretical contribution in the field by reinterpreting the notion of social coupling, which one of the constructs of enactivism. According to enactivism, social coupling happens when one organism constitutes a resource for perturbations in another organism’s structure. In Maturana and Varela (1987) this phenomenon is depicted such that the resource of perturbation is the organisms’ body, which is bounded by the skin that surrounds it. However, in Chapter 4, I identified some incidents in which it was not only the others’ body but the interactions between the other’s body and the environment that constituted a source for perturbations in the learner’s structure. My simultaneous focus on body (as per enactivism) and the artefact (as per TSM), allowed me to identify these incidents as different from social couplings. Since both the other’s body and the environment take part in these couplings, I name these incidents as socio-structural couplings which have slightly different nature than social couplings.

In Chapter 3, I analyzed the semiotic bundles that emerged while a 5-year-old manipulated Zaplify and pencil-and-paper. I used Arzarello et al.’s (2009) method of analysis by extending it. Semiotic bundles refer to “a system of signs—with Peirce’s comprehensive notion of sign—that is produced by one or more interacting subjects and that evolves in time” and they can be analyzed either synchronically or diachronically (p. 100). In the former method, the focus is on the relationships among the signs that are created synchronously (such as gesture-utterance). Synchronic analysis helps to understand the role of signs in learners’ meaning-making activities. Diachronic analysis considers the relationship between the signs that are produced at different times (such as Duval's conversions[11]) and allows the researcher to gain insight on the evolution of signs.

I attended to the signs that were produced both synchronously and successively as Arzarello et al. (2009) suggested. Moreover, I applied a hybrid method in my analysis by combining aspects of synchronic and diachronic analysis: I treated the conversions, which were produced successively, as if they were produced synchronously. I did this because my aim was not only to gain insight of the evolution of signs, but also to understand how conversions participate in the child's experience of Zaplify. My re-thinking of Duval's conceptualization of conversions prompted this method.

Duval (2006) argued that semiotic representations are essential for mathematical thinking. They stand for mathematical objects, however, he warned that "mathematical objects must never be confused with the semiotic representations that are used" (p.107). The significance of the signs does not rely on their capacity to stand for mathematical objects, but on their capacity to be substituted for other signs. Mathematical processing always involves this activity of transforming signs. It is necessary to use different semiotic systems (registers) as each one provides specific possibilities for different mathematical processing.

There are two types of transformations of semiotic representations: treatment and conversion. While treatments happen within the same register, conversions happen between registers. Duval assumed that when a sign in one register is transformed into another sign in another register, the mathematical object that the signs denote stays the same. This assumption indicates a representational approach to mathematical thinking. It depicts mathematical objects as abstract, fixed entities waiting to be accessed by learners through signs. At this point, I propose re-thinking conversions from an enactivist perspective.

I argue that learners bring forth mathematical objects as they interact with their environments. I agree with Duval that each register provides learners with specific possibilities for action. Thus, the way a sign is created in one register allows a different

[11] Duval (2006) described a conversion as one type of transformations of semiotic representations "that consists of changing a register without changing the objects being denoted" (p. 112).

interaction with the object than the one in another register. Each type of interaction allows one to bring forth a new meaning for the object, not necessarily demolishing the previous ones. Therefore, when a sign is converted from one register to the other, the mathematical object that is brought forth changes. In order to examine the child's understanding of Zaplify objects more holistically, I analyzed the conversions which were produced in different settings at different time points synchronically. This hybrid method can be called a di-chronic analysis.

In Chapter 2, I used two different methods to create data for the analysis of semiotic potential of artefacts. To my knowledge, I have not encountered any explicit documentation of how data should be created to conduct this analysis. I assumed that the researchers who presented such analyses (e.g., Bartolini Bussi & Baccaglini-Frank, 2015; Falcade, Laborde, & Mariotti, 2007; Mariotti, 2010) created a description of using artefacts to solve a task based on their imagining. Instead, I used two other methods. In one of them, I video-recorded my hands while I solved the mathematical task in Zaplify. In this event, I described my actions verbally as I conducted them and then transcribed the video-recording including the bodily, verbal and Zaplify actions.

In the other method, I took photos of each type of action I conducted to solve the mathematical task (such as drawing horizontal lines, drawing vertical lines, drawing dots at the intersection points) and described the events in written language based on these photos. Thus, this dissertation proposes an explicit account of a data creation process for the analysis of semiotic potential of artefacts. Moreover, the comparison of these two methods indicates that creating by using video-recording provides the researcher with a tool to consider the dynamic dimensions of incidents that might not be salient when described based on a photo.

This may also increase the accuracy of the analysis. For example, when I re-engaged with my analysis, I realized that it was impossible to create Mx1 with pencil-and-paper because each line had to be created sequentially. However, the static image which illustrated all lines simultaneously prompted me to see that incident as the depiction of M, while the sequential emergence of them was more aligned with (1 + 1 + 1 + … + 1). The way of experiencing these lines is critical as it may influence how a learner experiences a quantity (as a single multitude or a multiple of singles).

5.2.2. Educational Implications

After working on my dissertation, I realized how strong the association between repeated addition and multiplication is. This definitive link had been strong for me as well, up until my engagement with this research. However, it did not prevent me from acting effectively in multiplicative situations I had encountered so far. This may not be the case for most learners as many students tend to structure quantity additively in situations which are multiplicative (van Dooren, de Bock, & Verschaffel, 2010). Many researchers attribute the difficulty in thinking multiplicatively to the introduction of multiplication through the repeated addition model (e.g., Confrey, 1994; Greer, 1992; Maffia & Mariotti, 2018; Schwartz, 1988; Vergnaud, 1988). In alignment with their proposition, the findings of this dissertation suggest that TT constitutes an alternative model to help learners engage with multiplicative thinking in a way different than the repeated addition model does.

In addition to the repeated addition model, multiplication is strongly associated with abstract equations, which do not express spatio-temporal meanings of the operation. Both research and my personal experience indicate that for most people multiplication is all about abstract equations. The research shows that many people resort to verbal recall of multiplication facts when they conduct multiplication operations, while they use their fingers to subtract numbers (Berteletti & Booth, 2015). When I share my research topic within my community, the first reaction generally happens to be "Oh, four times two is eight what else is there to study". Even though this comment carries humour, my repeated encounter with similar comments suggests that this joke is not random (or funny!). When we asked teachers about teaching multiplication, one of their first responses was that students had difficulty with multiplication facts. In this dissertation, I have focused less on the learning of multiplication facts and more on the development of spatio-temporal meanings of multiplication, which involved using their bodies in new ways to produce multiplicative relationships.

The way the students use their body to structure quantity is quite different from the most common use of hands in our daily mathematical practices. In TT, the product must be created with the coordination of two hands, and it emerges as a result of indirect influence of singletons. The use of fingers for counting and even adding is not uncommon. We use an index finger to point at the objects when counting them. We count on our fingers as well, and sometimes use them to add numbers. But the hand actions required to create a product in TT involves a new bodily practice, one that corresponds to a multiplicative structure. Therefore, TT still allows students to make use of their hands, which are powerful tools for arithmetic, but in ways that are more multiplicative than additive.

The necessity of coordinating representations in different registers to develop mathematical thinking is emphasized by many researchers. Duval (2006) proposed this idea for any mathematical concept and Kaput and Pattison-Gordon (1987) illustrated how it can be achieved with respect to multiplicative thinking. According to Kaput, learners must identify the intensive quantity in multiplicative situations to think multiplicatively. He proposed to provide students with multiple representations of multiplication through the use of a computer program. When they manipulate one type of representation, the computer program would simultaneously transfer this change in all other representations system.

Thus, this co-ordinated transformation would help learners understand the intensive quantity which is the invariant ratio between the two unit-counts. However, before identifying this ratio as a constant entity, it is necessary to understand the co-variance between the unit counts: a change in one unit count depends on the change in the other unit count. At this point, the findings of this dissertation suggest that TT provides students with another way to identify this co-variance: through monitoring the transformations within the same register. This verifies Sinclair's (2018) proposition about affordances of dynamic geometry environments, which allow users to manipulate images in a continuous manner as if they play with a single object that changes in time. These temporal dimensions might engage students with the concepts of invariance and co-variance.

The findings of this dissertation also suggest that while using touchscreen technologies like TT, teachers should encourage pair work not only to promote peer learning through verbal interaction, but also to amplify the children's experience of the in/co-variant relationships embedded in the multiplicative structures through the other's body. For example, in Chapter 4, one child was continuously holding the pods while the other child changed the number of pips with discrete tapping gesture. This was the initial event that triggered them to identify the co-variance between the pods and the pips.

Unlike the on-going debate in the literature about whether phenomena are fundamentally discrete or continuous, this pedagogical approach uses both continuity and discreteness with respect to individuals' phenomenological experience of the screen contact.

Last but not least, after working on this dissertation I realized the importance of using relational language while using dynamic models to teach multiplication. When describing specific incidents, three different approaches are possible to take on. The first is to attend only to our body movements, such as: "I pressed my fingers down". Second, it is possible to describe the same incident as "multiple pips appeared in the pods after I press my multiple fingers on the screen".

Compared with the first one, this account emphasises relation between the individual action and the environment. Third, it is also possible to separate the self as a controlling unity by saying, "multiple pips appeared in pods as soon as fingers contacted the screen". Each account might influence learners' experience by bringing forth a different environment aligned with the description. Since it is the structural coupling between the bodies that allowed children to act effectively in multiplicative situations, a relational approach should be prompted to help learners bring forth multiplicative structures.

5.3. Limitations, Challenges and Future Directions for Research

Writing a manuscript-based dissertation is advantageous for many reasons, yet it also involves some challenges, not just for the dissertation as a whole, but for the individual studies.

In terms of the dissertation as a whole, the studies do not follow one from the other. They all started as a practice of getting familiar with different theories and evolved into their final versions after sharing them with my colleagues and doing more thinking on them as I continued to read. As a whole, they help me understand different aspects of students' engagement with an alternative model of multiplication that is different from the repeated addition model. However, they do not provide various perspectives on one single event or phenomenon. Across the studies, both the multiplication tasks, the interface of the multiplication model, and the participants varied. Since the context is important in learning, I would be interested in answering the same research questions I asked in three studies by analyzing a single case. I wonder how the new findings would compare to the current ones. This would also allow me to re-conduct the studies by considering the limitations and challenges of each study, which I will articulate below.

In the first study, when I reviewed the literature to learn a method to conduct the analysis of semiotic potential of artefacts, I did not find an explicit description of a specific method. Therefore, I devised two methods to create data for the analysis of semiotic potential of artefacts: (1) video-recording my actions and (2) taking the photos of each type of actions. As I explained earlier, I found that the video-recording method might present the data in a way that makes it less likely that the analyst will miss the dynamic aspects of the events. Those aspects might be critical for the analysis of the semiotic potential of the artefacts. Therefore, I recommend using video-recording for future studies.

I also felt another challenge in the first study. Like the lack of description of a method for data creation, the literature lacks the documentation of an explicit method to analyse the semiotic potential of an artefact. I have focused on the mode of use of the artefact to identify potential pivot signs, which function both as artefact and mathematic signs, and help users make mathematical meanings out of the specific use of the artefact. However, it was challenging to decide how to analyze these signs. First, I coded each sign separately by using Nvivo. Even though this method allowed me to identify the potential artefact signs, I found it more useful to analyze the signs by focusing on the interplay between them to identify which mathematical meanings they might be associated with. I believe that I identified the semiotic potentials of each artefact to a great extent by using this method of analysis. However, I think that the field might benefit from the documentation of an explicit methodology to help other researchers engage with this type of research.

The second study was challenging in terms of participant recruitment and the monitoring of the participant's interaction with TT. I used convenience sampling for this study. The participant was the child of my landlord who was also present during the interviews and participated in the child's interactions with TT by asking questions. These questions played a similar role as the ones I was asking: to direct the child's attention to the specific features of TT. In addition, I felt that his questioning might have influenced the child emotionally because even I experienced some tension when I heard the intonation of the father's voice and the child's hesitant actions (e.g., answering with a low voice). Therefore, how this emotional atmosphere played a role in students' engagement with TT remains an unanswered question with respect to the findings.

This study also poses another unanswered question that might explain the child's engagement with TT. After the child encountered TT for the first time and engaged with it while I and the father asked specific questions, I lent the iPad to the child so that he could explore it himself without the social pressure he might feel under the surveillance of adults. However, this prevented me from observing his interactions during this period. It is possible that the father continued to train the child so that he could successfully respond to my future questions. Therefore, a question arises as to what role the parent played in the child's use of the duo of artefacts?

In the third study, we have seen that children discovered new gestures as they structurally coupled with their environment which included both TT and the other individuals who also engaged with TT. TSM was useful to identify these gestures as pivot signs, which were important in developing multiplicative thinking. Enactivism was useful to understand how those gestures emerged. However, these two theories are limited in terms of explaining how *the new* emerges. For example, how come Kyra stopped continuously pressing her single finger only on "her side" which created pods and crossed the boundary between "her side" and "Jacy's side" which created pips? This question can't be answered completely by either of these theories. At this point, theories on affect would perhaps provide an appropriate frame for future research on learning mathematics (with educational technologies) to explore the emergence of the new.

What I meant by affect is different from "a disposition or tendency or an emotion or feeling attached to an idea or object" (Philipp, 2007, p. 259). This type of framing associates affect with "hypothetical psychological constructs internal to an individual that can become active under specific social conditions… and govern a student's in-the-moment engagement for minutes (or even seconds) at a time during a class period" (Goldin, Epstein, Schorr, & Warner, 2011, p. 548). Therefore, unlike participationist approaches, this conceptualization of affect assumes that individuals act based on static psychological states, which would not explain how the new emerges.

McLeod’s (1992) conceptualization of the affective domain is also ineffective in theorizing the new. According to this frame, the affective domain consists of psychological constructs such as beliefs, attitudes and emotions. While beliefs and attitudes are described as more stable constructs that are largely cognitive in nature, emotions are associated with processes that involve physiological responses to the aversive situations. Emotions are depicted as temporary and less cognitive in nature, and they serve as a survival mechanism. Among these constructs, McLeod’s conceptualization of emotions seems to be compatible with participationist approaches to cognition because they are explained as a bodily, temporal and relational phenomenon: they involve physiological responses that emerge with respect to situations and they transit from one to another. However, this framing does not explain how the new emerges, either.

I refer to affect as framed in Sinclair and Ferrara (2021), which drew on Alfred North Whitehead’s work to explore the affective dimensions of a child’s encounter with arithmetic tasks in *TouchCounts* (TC)—a digital tool. The affect was framed in the sense of “the ways he was affecting and being affected by TC, but not necessarily through intentional, sensory perception” (p.22). Focusing on a non-sensuous activity of encountering a world opens space for potentialities, unlike studying the restrictive power of conscious act which presumes a set of a priori rules controlling one’s actions. Therefore, Sinclair and Ferrara’s perspective might be helpful for future research to explain how novice gestures emerge when students manipulate digital mathematical artefacts.

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