APPLICATIONS OF REALIZED VOLATILITY, LOCAL VOLATILITY AND IMPLIED VOLATILITY SURFACE IN ACCURACY ENHANCEMENT OF DERIVATIVE PRICING MODEL

by

Keyang Yan
B. E. in Financial Engineering, South-Central University for Nationalities, 2015

and

Feifan Zhang
B. Comm. in Information Management, Guangdong University of Finance, 2016
B. E. in Finance, Guangdong University of Finance, 2016

PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN FINANCE

In the Master of Science in Finance Program
of the
Faculty
of
Business Administration

SIMON FRASER UNIVERSITY

Term Fall 2017

Approval

Name: **Keyang Yan, Feifan Zhang**

Degree: **Master of Science in Finance**

Title of Project: **Applications of Realized Volatility, Local Volatility and Implied Volatility Surface in Accuracy Enhancement of Derivative Pricing Model**

Supervisory Committee:

Associate Professor Christina Atanasova

Professor Andrey Pavlov

Date Approved: ______________________

Abstract

In this research paper, a pricing method on derivatives, here taking European options on Dow Jones index as an example, is put forth with higher level of precision. This method is able to price options with a narrower deviation scope from intrinsic value of options. The finding of this pricing method starts with testing the features of implied volatility surface. Two of three axles in constructed three-dimensional surface are respectively dynamic strike price at a given time point and the decreasing time to maturity within the life duration of one strike-specified option. General features of implied volatility surface are justified by the real trading data. With the dynamic strike price and variable volatility, option price is assumed to reflect the market expectation towards the performance of underlying asset and accompanied uncertainty when approaching the maturity. Therefore, the applications of implied volatility, local volatility and realized volatility are involved in the pricing of derivatives, because of their respective compatibilities of the forward-looking expectation, the stochastic parameter and the tight fit to the real return distribution. In the researching and analysing process, it is found that the realized volatility and the real return distribution are the derivative pricing combination with highest accuracy in the three categories of volatility. The implied volatility fails to fit the derivative price for its emphasis on market expectation and lack of independence from existing model, at the meantime, the local volatility loses its ground in practical application in pricing derivatives, with insufficient small-interval data of transactions.

Keywords: Implied volatility surface; Local volatility; Realized volatility; Volatility smile; Volatility Skew; Return probability distribution; Derivative pricing; Pricing accuracy

Executive Summary

In our research, we mainly focus on the methodology to enhance accuracy in the pricing of derivatives. A more precise pricing method involves trial applications of the implied volatility surface, local volatility, realized volatility and real return probability distribution. Procedures to improve the Black-Scholes Model include

- Black-Scholes Model with the assumptions that price option volatility strike price and time to maturity are all fixed when pricing options, in which pricing convexity deviates real price from theoretical price;
- Based on the above limitation in the Black-Scholes Model, considering different measurements of volatility used in the model and other parameters may bring effect on pricing;
- Specifying and comparing the pricing effects of three different volatilities respectively applied to the model in the trial and error phase;
- Combination of realized volatility with real return probability distribution shows an advantage in this comparison, for its compatibility of varying volatility and a tight fit to performance of derivatives.

With in-depth analysis, technical modelling and rigid deduction, we conclude that options on index can be more accurately priced under our improved Black-Scholes Model, in which the fixed volatility and standard normal distribution are respectively supplemented by realized volatility and real return distribution. The application of local volatility in pricing is limited to some practical degree.

We would like to express our sincere appreciations to Professor Christina Atanasova, our final project supervisor, who has given us numerous instructions and practical suggestions, modifying our research destination and encouraging us when testing went off the rail.

Table of Contents

List of Figures

List of Tables

1: Introduction

1.1 Research Topic

Asset pricing methods and corresponding risk management have together been an intractable issue for decades, since the profitability of derivatives emerged and prospered, with folded and unfolded risk exposures. In this context, the desperate need to develop appropriate method to price assets and to innovate proper measurements of uncertainty is of great value to asset managers, authorized traders as well as risk analysts at the buy-side.

To better research on the asset pricing theory and risk managing techniques in transactions on options, we studied the relationship between the implied volatility ($\boldsymbol{\sigma_{imv}}$) of options on indexes, the local volatility ($\boldsymbol{\sigma_{loc}}$) and the realized volatility ($\boldsymbol{\sigma_{rea}}$). With deeper understanding about the volatility relationship, the research continues to elaborate on pricing method using a new transformed probability distribution of return, with mapping techniques. At the meanwhile, the implied volatility surface (IVS) is well employed in the research to justify the presence of volatility smile and general feature existing in the implied volatility surface, subsequently, facilitating the measurement of relative risk exposure. The implied volatility surfaces take three series of parameters into 3-dimentional figures, time to maturity ($\boldsymbol{T}$), strike prices ($\boldsymbol{K}$) and implied volatilities ($\boldsymbol{\sigma_{imv}}$).

1.2 Research Motivation

The finding in volatility relationship to price options on indexes is of practical value and forward-looking meaning for asset managers when make decisions to invest in options and monitor the index volatilities (VIX), as an important measurement of market risk and an efficient tool to adjust the transaction price. The pricing method put forth in this research enables asset managers and fund risk managers to monitor the movements of market and calculate the appropriate adjustment on trading prices (or closing prices). After adjusting, the scope between the transaction price and the expected price is narrowed down, in this case, asset managers price options with higher precision.

In the Black-Scholes Model (Black & Scholes, 1973), assumptions are held to calculate option price. In this context, the interest rate and volatility of underlying asset price are neither assumed dynamic and the underlying asset return is assumed to be normal distribution. It is obvious that assumptions are unrealistic in real market, and further adjustments have been widely proposed to improve the pricing accuracy. One among many others is the addition of Stochastic Volatility (SV) (Hull & White, 1987), (Heston, 1993) into the pricing model. Thereafter, the complex reasoning and rigorous assumptions limited the efficiency of option and asset pricing. The historical probability distribution of underlying asset returns is more closed to real asset performance. This probability distribution is mapped to fit the historical movements of individual asset, thus more specialized, or customized, to the researching assets. With the combination of realized volatility, implied volatility and real probability distribution, we are capable to price options on indexes at a higher precision.

For asset managers and financial derivative traders, a more accurate option pricing method that is specialized for trading security is more useful than generalized method. On the other aspect, the risk managers for investment in market gain direct access to the relationship between investors' expectation in the market to the future volatility and the historical realized volatility. This multi-lateral relationship bridges the history and the future of market response and expectation to underlying asset and therefore take investor expectation and behavior into account.

1.3 Summary of Results

Results of our research are displayed in the way of 2-dimentional and 3-dimentional figures, data tables, programming works and a series of numerical outputs.

In the first part of our research, we find several facts that the implied volatility ($\boldsymbol{\sigma_{imv}}$) of a put option is higher than that of a call option on the same underlying asset; simultaneously, the existence of volatility skew[1] in an option that has the feature of carrying not only a relatively lower strike price ($\boldsymbol{K}$) but also a relatively higher implied volatility ($\boldsymbol{\sigma_{imv}}$). This justifies the finding of volatility skew phenomena in the book 'options, futures and other derivatives' (Hull J.

[1] *Volatility skew*

Volatility skew and volatility smile both are a varying volatility process. Volatility smiles are implied volatility patterns that arise in pricing financial options. In given maturity, options which are either deep in-the-money or out-of-the-money command higher prices than that the standard option pricing model estimates, and carry higher implied volatility. Options on foreign currency are typically in volatility smile, while options on equity usually show volatility skew, with the volatility tail skewed to left.

C., 2003) and that in the technical paper of ‘De-arbitraging With a Weak Smile: Application to Skew Risk’ (Mahdavi Damghani & Kos, 2013). The justification of volatility skew effect in equity asset helps our further research on accuracy enhancement in pricing options with volatilities.

With the theoretical support from our first part of research, we proceed to the pricing model improvement in the second part of our research. In our improved option pricing model, we are capable to rise the accuracy level of the estimation on option market value. In both the laboratorial data modelling and the back-testing with trading data, we succeed in modifying the option closing prices to be convergent to its fair value, of which the calibration stems from several modifications to the Black-Scholes Model – the employment of return mapping and realized volatility ($\boldsymbol{\sigma_{rea}}$). Having the outputs and functioning principles of our innovative option pricing model back-tested, it is found that there is still space for improvement for our model and more research can be worked on, besides the innovative aspects.

1.4 Novel Aspect

The novelty and innovation of our pricing method is exhibited in two forms. One is justification on the features of implied volatility ($\boldsymbol{\sigma_{imv}}$) and its surface, which represents the future expectation of underlying asset performance, as well the finding of limitation in applying local volatility ($\boldsymbol{\sigma_{loc}}$) to use. The other one is the merge of historical volatility ($\boldsymbol{\sigma_{rea}}$) and customized return distribution, which stands for the historical investor response and market sensitivity to the underlying asset. The volatilities are obtained from the market movements and trading activities.

Compared to the Black-Scholes model, which is static in terms of time and volatility, the time to maturity and dynamic process of volatility are added into pricing. Thus, the results are more closed to the option market price.

1.5 Thesis Structure

Based on above preparation, research and introduction, the remained thesis is organized in the order of Literature Review, Data and Summary Statistics, Theory and Methodology, Results and Interpretations, and Conclusions. In the first following section, Literature Review, the referred articles and works are partially listed and summarized into the most directly effective conclusions or methods that are regarded as theoretical foundations in our research and thesis.

The next section of Data and Summary Statistics section is examples and samples of researched objective – option prices, implied volatility ($\boldsymbol{\sigma_{imv}}$, inversely deducted from the Black-Scholes model with option market prices), realized volatility ($\boldsymbol{\sigma_{rea}}$, obtained from historical option market return), etc.; figures of implied volatility surface (3-dimensional) of different options; data tables of partial data and charts real return probability distribution; etc. Background Theory and Necessary Definitions is related to useful concepts, relationships and effects as prerequisite and researching environment for our pricing method. With above parts well demonstrated, the core methodology and overall outline are illustrated in the remaining Methodology section, of which the results and corresponding interpretations on the method are subsequently thrown light upon in the Results and Interpretations. Conclusions are drawn thereafter, followed by additional statements on some potential improvements that can be further worked on with this pricing method.

2: Literature Review

Numerous papers have studied the implied volatility surface and its relationship with related fields.

‘Relative Implied Volatility Arbitrage with Index Options – Another Look at Market Efficiency’ (Ammann & Herriger, 2001) discovered whenever the relative implied volatilities were found to violate a specific boundary, a relative implied volatility mispricing was identified. However, after taking bid-ask spread into consideration, the probability of arbitrage is small and not all of the arbitrage opportunities detected could actually have been executed because of various arbitrage barriers in special market situations.

‘Predictable Dynamics in the S&P 500 Index Options – Implied Volatility Surface’ (Goncalves & Guidolin, 2004) found the truth that it is impossible to forecast the S&P 500 Index options implied volatility surface through linear regression of time to maturity and moneyness. He also established an approach to predict implied volatility by modeling the cross-sectional variation of implied volatilities as a function of polynomials in moneyness and time to maturity and by estimating parametric VAR-type models.

In 2011, ‘Implied Volatility Surface: Construction Methodologies and Characteristics’ (Homescu, 2011) offered a great number of methodologies to construct IMV surface. Moreover, they discussed several factors which would impact the accuracy of the construction of implied volatility surface like the put-call parity, bid-ask spread, strike price and time to maturity.

‘Arbitrage-free SVI Volatility Surfaces’ (Gatheral & Jacquier, Arbitrage-free SVI Volatility Surfaces, 2013) illustrated the way to calibrate SVI (Stochastic Volatility Inspired) parameter with the implied volatility surface when there are no arbitrage opportunities in the market. They successfully derived a formula to calibrate them and the quality has been proved through the 2013 SPTSX options data.

‘Analyzing Volatility Risk and Risk Premium in Option Contracts: A New Theory’ (Carr & Wu, 2010) discovered a specific option pricing algorithm for institutional investors to monitor and manage the options position in their portfolio based on short term fluctuation of the IMV, and to find arbitrage opportunities in market. They also constructed and made a comparison between the implied, expected and realized volatilities.

‘The Predictive Power of Implied Volatility of Options Traded OTC and on Exchanges’ (Yu, Lui, & Wang, 2008) proved that option price derived from historical volatility and expected volatility calculated through GARCH model are inferior than market price (implied volatility) which showed the efficiency of the OTC option market.

‘The Relation between Implied and Realized Volatility’ (Christensena & Prabhalab, 1998) investigated the accuracy of implied volatility by comparing it with the realized volatility and found it is inefficient to price options by integrating implied volatility with historical volatility.

‘Dynamic Estimation of Volatility Risk Premia and Investor Risk Aversion from Option-Implied and Realized Volatilities’ (Bollerslev, Gibson, & Zhou, 2005) proposed a new methodology to construct an investor risk aversion index measuring the expected volatility premium. The index was established through calculating high-frequency five-minute-based realized volatility and actual S&P 500 option implied volatility. Extracted volatility risk premium was also proved to help estimate future stock market returns.

‘Volatility Dynamics for the S&P500: Evidence from Realized Volatility, Daily Returns and Option Prices’ (Christoffersen, Jacobs, & Mimouni, 2008) improved the SQR model (Affine Square Root Stochastic Volatility Model) to increase the accuracy of option valuation. They replaced the square root diffusion with linear diffusion for variance and the new model has the lowest option implied volatility mean squared error in and out-of-sample of index returns. It has also been proved to fit the at-the-money options and offers a better volatility term structure and implied volatility skew.

‘A Simple Long Memory Model of Realized Volatility’ (Corsi, 2004) proposed a new realized volatility model to forecast the volatility based on time-series theory. It uses the Heterogeneous Autoregressive model of the Realized Volatility (HAR-RV) to simulate the volatility which includes many characteristics of financial data: fat tail, autocorrelation and long memory.

3: Background Theory and Necessary Concepts

3.1 Background Theory

The Black-Sholes-Merton model (BS model) is a classical option pricing method, firstly proposed by Fischer Black[2], Myron S. Scholes[3] and Merton[4] in 1970s. The BS model has several assumptions on the market for consisting of at least one risky asset (called the stock in most cases), and one risk-free asset (called the money market, cash or bond for most time). To further elaborate on these assumptions in detail, they are listed as follows:

1) R_f, risk-free rate, the rate of return on the riskless asset is constant and thus called the risk-free interest rate. (The constancy and variability of risk-free interest rate would be discussed and explained in remaining parts.)
2) Random walk, the instantaneous log return of sock price is an infinitesimal random walk with drift; more precisely, it is a geometric Brownian motion, and assumes its drift and volatility is constant.
3) The stock does not pay a dividend.
4) There is no arbitrage opportunity (i.e., there is no theoretical way to gain a risk-free profit).
5) It is possible to borrow and lend any amount, even fractional, of cash at the risk-free rate.
6) It is possible to buy and sell any amount, even fractional, of the stock (shorting selling is included).
7) The above transactions do not incur any fees or cost (i.e., frictionless market).

Additionally, the notations that used in the BS model to price options are given below:

S_t, the stock price at time t, which sometimes is a random variable or constant in some cases.

$C(S, t)$, the price of a European call option.

[2] Fischer Black
[3] Myron S. Scholes
[4] Merton

P(S, t), the price of a European put option.

K, the strike price of the option on underlying asset.

r, the annualized risk-free interest rate, continuously compounded (the force of interest).

μ, the annualized drift rate of ***S***.

σ, the standard deviation of the stock's returns, the square root of the quadratic variation of the stock's log price process, the measurement of volatility of stock return.

N(x), the notation of standard normal cumulative distribution function**,**

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

The BS formula calculates the price of European put and call options. The value of a call option for a non-dividend-paying underlying stock in terms of the BS parameters is in the form of:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

For a put option, the pricing formula derived from the BS model takes the form of:

$$P(S_t, t) = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Based on the BS model, with given model parameters including S_t, ***K***, ***r***, ***t***, ***T***, ***μ***, $C(S_t, t)$, $P(S_t, t)$ and the standard normal distribution to generate the $N(d_1)$ and $N(d_2)$, the volatility ***σ*** of the underlying asset price in the time horizon of ***T*** can be backward inducted, which is called the deduction of implied volatility, σ_{imv}.

3.2 Necessary Concepts

3.2.1 Implied Volatility Surface (Surface of Implied Volatility, IVS)

Analyzing Volatility Risk and Risk Premium in Option Contracts – A New Theory (Carr & Wu, 2010) set a series of assumptions on the model to generalize the implied volatility surface model with general features. The assumptions of implied volatility surface model are elaborated on in the research paper of Carr and Wu as follows:

1) The investment market must have at least one risk-free bond, one risky asset and one continuum of vanilla European options with the risky asset as the underlying asset.
2) In the market, interest rate and carrying cost or yield are constant and equal to zero for the risky asset.
3) In practical implementation, investor can readily accommodate a deterministic term structure of financing rates by modeling the forward value of the underlying security and defining moneyness of the option against the forward.
4) Transactions in the market are frictionless and those between the stock and bond have no arbitrage opportunity
5) There exists a risk-neutral probability measure $\mathbb{Q}$, equivalent to the statistical probability measure $\mathbb{P}$, such that the stock price $\boldsymbol{S}$ is a martingale.

The stock price, implied volatility of vanilla option and their correlated relationship in between are in the forms of:

$$dS_t/S_t = \sqrt{v_t}dW_t$$

$$dI_t(K,T) = \mu_t dt + \omega_t dZ_t$$

$$\mathbb{E}_t[dW_t dZ_t] = \rho_t dt$$

Let ***B(S, σ, t; K,T)***: $\mathbb{R}^+ \times \mathbb{R}^+ \times [\mathbf{0}, \boldsymbol{T}) \mapsto \mathbb{R}^+$be the BS model formula for a European put option:

$$B(S,\sigma,t;K,T) \equiv KN\left(\frac{ln(K/S)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}\right) - SN\left(\frac{ln(K/S)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}\right)$$

This can be described as,

$$I_t^2(k) = a_t + \frac{2}{\tau}\sqrt{(k-b_t)^2 + c_t}$$

with

$$a_t = \frac{-2(1 - 2e^{-\eta_t\tau}m_t\tau - e^{-\eta_t\tau}\omega_t\rho_t\sqrt{\nu_t}\tau)}{\omega_t^2\tau^2}$$

$$b_t = -\frac{\rho\sqrt{\nu_t}}{e^{-\eta_t\tau}\omega_t}$$

$$c_t = \frac{(1-\rho_t^2)\nu_t}{e^{-2\eta_t\tau}\omega_t^2} + \frac{(1 - 2e^{-\eta_t\tau}m_t\tau - e^{-\eta_t\tau}\omega_t\rho_t\sqrt{\nu_t}\tau)^2}{e^{-4\eta_t\tau}\omega_t^4\tau^2}$$

where m_t, ω_t, and η_t are stochastic processes that do not depend on ***K***, ***T***, or ***I(K, T)***. ω_t is constricted to be a strictly positive process with no loss of generality.

In the paper of ‘A closed-form solution for options with stochastic volatility with applications to bond and currency options’ (Heston, 1993), the diffusion of the variance is specified as proportional to the square root of the variance, namely the volatility, employing the affine variance rate dynamics. However, it relies heavily on affine setting mainly for pricing tractability.

By the analysis in the book ‘The volatility surface: a practitioner’s guide’ (Gatheral, The Volatility Surface: A practitioner's Guide, 2006), the more general form for the implied variance smile that involves five free covariates in stochastic volatility inspired (SVI) model, but it is not demonstrated that how these five coefficients should vary across different time to maturities.

$$I_t^2(k) = a + b\left[\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right]$$

The theoretical literatures so far are about the induction of dynamic implied volatility. Implied volatility is the estimated or expected volatility of asset price, a looking-forward measurement. Consequently, when the market is bearish, σ_{imv} declines with the market negative expectations and fear to the future market performance; while the market investors expect positively on the future performance of this asset, σ_{imv} rises. As the feature of σ_{imv} that it is a forward measurement of price fluctuating magnificence (deviation from the average).

The implied volatility surface (IVS) is the three-dimensional surface of the multi-lateral relationship among the implied volatility (σ_{imv}, derived reversely from the BS model), the strike price (***K***, of each option), and time to maturity (***T-t***). Typically, for options on an underlying asset of foreign currency forwards, implied volatility (σ_{imv}) displays itself in the shape of “smile” with accordance to strike price, $\boldsymbol{K}$, reaching the bottom point when the option is at-the-money, that is the strike price equals to the stock price at time, $\boldsymbol{K} = \boldsymbol{T}$, and this situation is called volatility smile. For options on the underlying asset of equities, implied volatility ($\boldsymbol{\sigma_{imv}}$) is in the shape of "skewed smile", called volatility skew.

3.2.2 Local Volatility ($\boldsymbol{\sigma_{loc}}$)

Local volatility is a concept and innovative volatility measurement proposed by Bruno Dupire in 1993. In BS model and the backward reasoning of implied volatility, the deduction relies heavily on the strike price of options and respective maturity in the life time of options. To build a spot process with less dependence on option strike price, in the research paper of 'pricing and hedging with smiles' (Dupire, Pricing and Hedging with Smiles, 1993) and its subsequent research 'pricing with smiles' (Dupire, Pricing with Smiles, 2004), the local volatility concept is firstly come up with and improved, which both is compatible with the observed volatility smiles at all maturities and keeps the BS model complete.

Assume that the interest rate is zero, for a given maturity $\boldsymbol{T}$ for a European option, the collection of option price with different strike prices $\boldsymbol{f(K, T)}$ yields the risk-neutral density function $\boldsymbol{\varphi_T}$ of the spot at time $\boldsymbol{T}$, differentiated twice with respect to $\boldsymbol{K}$ to obtain the risk-neutral probability density of the spot being equal to $\boldsymbol{K}$ at time $\boldsymbol{T}$ (take an European call for instance),

$$\boldsymbol{C(K,T) = \int_0^\infty \max(S-K,0)\,\varphi_T(S)\,dS}$$

$$\boldsymbol{\varphi_T(K) = \frac{\partial^2 C}{\partial K^2}(K,T)}$$

Solving this equation of probability density function to get a solution dictated by the spot stock price $\boldsymbol{S}$ and binomial tree models consistent with the volatility smile effect in the article of 'riding on a smile' (Derman & Kani, 1994),

$$\boldsymbol{\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T;S_0)}{2}K^2\frac{\partial^2 C}{\partial K^2} - (r-q)K\frac{\partial C}{\partial K} - qC}$$

The deduction for local volatility given by above Dupire's formula is,

$$\boldsymbol{\sigma(T,K) = \frac{1}{K}\sqrt{2\frac{\frac{\partial C}{\partial T} + (r-q)K\frac{\partial C}{\partial K} + qC}{\frac{\partial^2 C}{\partial K^2}}}}$$

In practically calculating the local volatility of options, it is not accessible to capture the theta of option ($\frac{\partial C}{\partial T}$) or to demonstrate the second derivative of option price on strike price ($\frac{\partial^2 C}{\partial K^2}$). Thus, a supplemented calculating method with more accessible compositions is used in measuring the local volatility.

3.2.3 Realized Volatility (Historical Volatility, σ_{rea})

Realized volatility, also called historical volatility and denoted as $\boldsymbol{\sigma_{rea}}$, is the measurement of deviation magnificence in option price. Realized volatility takes a historical scope and derived from previous trading day data, therefore reflects the fluctuation of historical risk at the most closed degree. It is assumed to be widen with the risk level rising and to be narrowed with the decrease of risk.

4: Data and Statistics Summary

In order to obtain accurate data to calculate the realized volatility, local volatility, implied volatility and to build its surface, we need several components. Since the options we researched are on market index, components include prices, within a range of time to maturity (***T***), of options matured at the same time and on the same underlying index with different strike prices (***K***) and risk-free rate (R_f).

4.1 Underlying asset

In the United States, exchange traded options are available for hundreds of stocks and for several indices. At first, we began our research on options on stocks which belong to the Standard & Poor's 500 index. We downloaded option prices of 500 stocks from Bloomberg[5], all of which matured on July 21st, 2017. In order to collect option price of market meaning, we chose options with high liquidity and with more than 30 days to expire because when the option is closed to the maturity the implied volatility will be extremely high and meaningless due to the price of options would never reach or below zero. Moreover, implied volatility of options with low liquidity cannot precisely reflect market's expectation of stock price. Based on that rule, we picked up both call and put options on 33 stocks, each with seven different strike prices. The implied volatilities are calculated through the closing ask price of options and the underlying stock. The risk-free rate used to calculate the implied volatilities is USD LIBOR overnight rate.

After we found the problems of options on stocks, we decided to change our target to options on index which are more liquid. We restarted data collection process on options on Dow Jones Index which matured during the period between 2014 and 2015 from WRDS[6]. The reason why we choose Dow Jones as our objective is that there have been numerous papers on Standard

---

[5] Bloomberg

Bloomberg L.P. provides financial software tools such as an analytics and equity trading platform, data services, and news to financial companies and organizations through the Bloomberg Terminal (via its Bloomberg Professional Service).

Most of data used in the research and study during the time of this program is from the database of Bloomberg.

[6] WRDS

Wharton Research Data Service (WRDS) provides the broad collection of financial, economic, healthcare, marketing data, analytics, and the most robust computing infrastructure available, all backed by the credibility and leadership of the Wharton School.

Data sources from third-party data vendors include S&P Global Market Intelligence, NYSE, CRSP, Thomson Reuters, etc.

& Poor's 500 index. We used data from WRDS instead of Bloomberg because we have limited access to the data of options which matured after 3 months before the day we downloaded the data while we were able to download the data of options that matured before mid of 2016. We chose the options based on the same rule we used on stock options, after which, we started the analyse on the call and put options matured on Dec.19 2015

4.2 Strike Price

Due to the fact that these options were all issued on Jan 2nd, 2015 and during this period the price of Dow Jones fluctuated between $156.66 to $183.12, so we chose the options with K ranging from $155 to $185 whose interval is $5.

4.3 Risk-free Rate

We chose USD LIBOR overnight rate as the risk-free rate. LIBOR is the mean of interest rates provided by several important lending banks in market and the reason why we chose overnight LIBOR rate as risk free rate is that we suppose the LIBOR rates which have a longer horizon take the credit risk, market risk and liquidity risk into consideration.

5: Methodology

Basically, our analysis can be divided into two major steps. Firstly, we generated the implied volatility surface through Black-Scholes Model and found the possible reasons for this situation. Secondly, we tried to modify the Black-Scholes Model to calculate a more reasonable option price than the market price.

5.1 Building Implied Volatility Surface

Normally, Black-Scholes Model is used to calculate the option price based on the K, underlying price, volatility and time to maturity, however, in the first step, we used strike price, market price of Dow Jones, risk-free rate and market price of option as datum to calculate the implied volatilities through Matlab. After we generate the implied volatility surface, we need to analyze the features of the surface and discuss the possible reasons for this phenomenon that may direct our improvement.

5.2 Improvement of Black-Scholes

Secondly, after our analysis on implied volatility surface and previous papers, two assumptions may result in the biases in Black-Scholes Model: constant volatility and returns which accord with normal distribution. Therefore, we tried to modify these two assumptions to improve the accuracy of Black-Scholes Model.

In terms of volatility, we decided to utilize realized volatility and local volatility which are dynamic. After calculating the realized volatility and local volatility, we need to confirm the rationality of them and choose the better one through the correlation between them and the implied volatility.

Before we find methods to enhance the assumption of returns of normal distribution, it is important to demonstrate the differences between normal distribution and real return distribution. Figure 5-1 shows the cumulative function of normal distribution and real return., while Figure 5-2 exhibits the probability density of real return against the normal distribution. The real return distribution carries the features and moments displayed in Table 1.

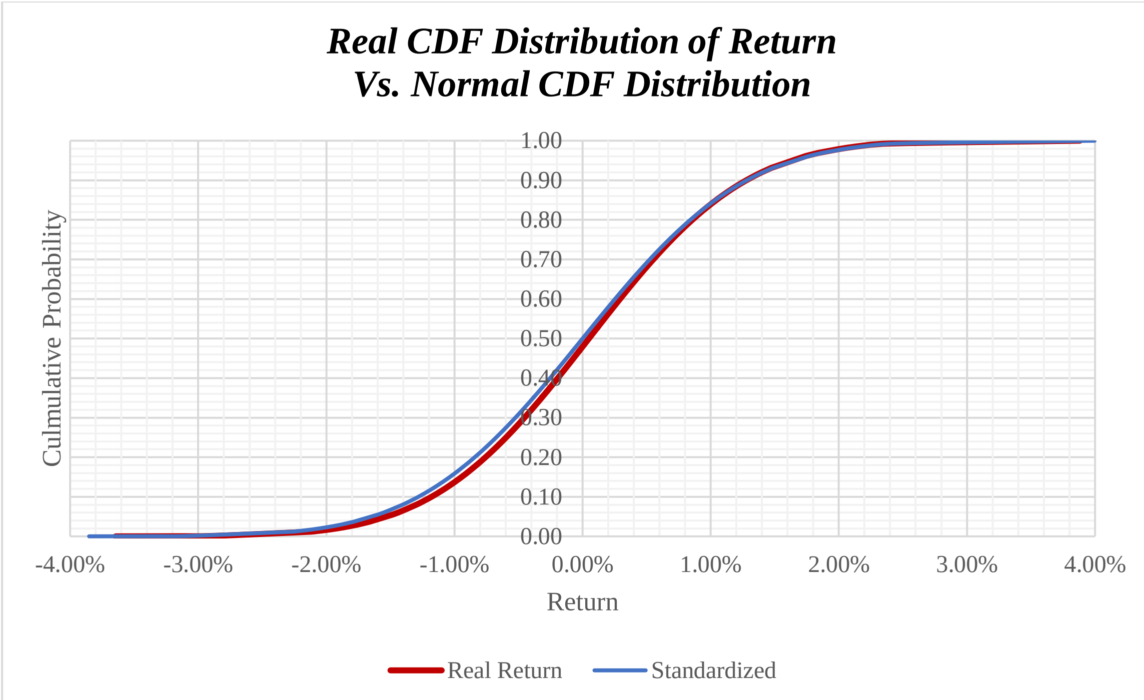

Figure 5-1 Real CDF Distribution of Return Vs. Normal CDF Distribution

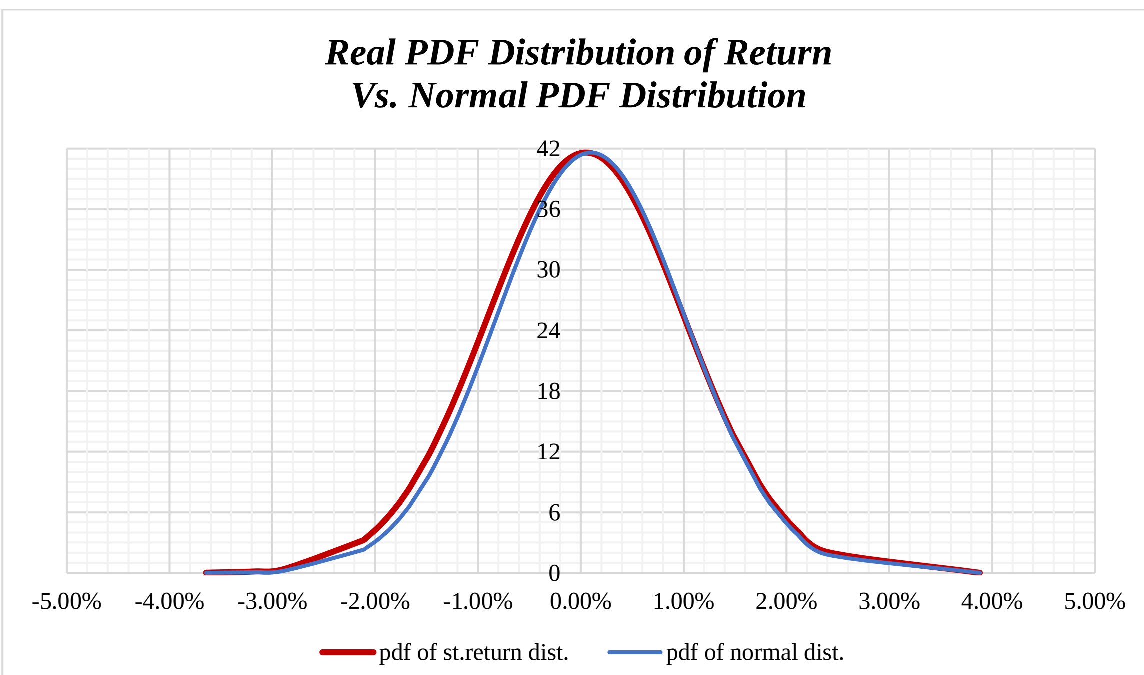

Figure 5-2 Real PDF Distribution of Return Vs. Normal PDF Distribution

Table 1 Moments and Features of Real Return Distribution

Moments & Features	*Values*
Mean	0.0494%
Standard deviation	0.9589%
Skewness	-0.2514
Kurtosis	1.8899
Sample size	232
Maximum in sample	0.0388
Minimum in sample	-0.0364

It is easy to find that the real return tends to have a fatter tail than normal distribution. Methods to improve the Black-Scholes model relies on improvement on several components related to the assumption of normal distribution: $\boldsymbol{N(d_1)}$, $\boldsymbol{N(d_2)}$ and $\boldsymbol{ln(S/K)}$ in call option and $\boldsymbol{N(-d_1)}$, $\boldsymbol{N(-d_2)}$ and $\boldsymbol{ln(S/K)}$ in put option.

In Black-Scholes model, $\boldsymbol{ln(S/K)}$ could be treated as a start point in the random walk, measuring the difference between current underlying asset price and the strike price. Moreover, the distance is measured under the assumption of normal distribution. Therefore, we tried to mend $\boldsymbol{ln(S/K)}$ through the following formula mapping the $\boldsymbol{ln(S/K)}$ from normal distribution to the real return distribution.

$$\boldsymbol{ln(\frac{S}{K})_{new} = InverseRealCDF\left\{NormCDF[\frac{ln(\frac{S}{K})}{t}]\right\} \times t}$$ [7]

Except the $\boldsymbol{ln(S/K)}$, $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ are also based on the assumption of returns of normal distribution as $\boldsymbol{(r + \frac{\sigma^2}{2})(T - t)}$ is the formula for the random walk.

To be specific, $\boldsymbol{\left(r + \frac{\sigma^2}{2}\right)}$ shows the possible movement per day which multiplied by time to maturity to get the possible movement before maturity. $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ measure the probability of the movement to actually happen, in other words, $\boldsymbol{N(d_1)}$ measures the probability of the option is in-the-money (Asset or nothing) at expiry that use stock as a numeraire and $\boldsymbol{N(d_2)}$ measures the probability of the option is in-the-money (Cash or nothing). The

[7] InverseRealCDF
The function to find the corresponding return to the value of cumulative real distribution function.
NormCDF
The function to find the corresponding value of cumulative normal distribution function to the return.

probabilities of them are both calculated under the assumption of returns of normal distribution, therefore, we decided to improve the $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ by following formula.

$$\boldsymbol{N(d_1)_{new} = RealCDF(d_1)}$$ [8]

$$\boldsymbol{N(d_2)_{new} = RealCDF(d_2)}$$

After we determine several possible methods to modify the Black-Scholes model, we need to figure out the best way to improve it through trying on the different combination of the methods. (Excluding local volatility here because it has been proved to be meaningless in our research due to our data.) Following are the combination of the modified input we tried during our experiment.

1) Realized Volatility

2) Realized Volatility & mapped $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$

3) Mapped $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$

4) Realized Volatility & mapped $\boldsymbol{N(d_1)}$**,** $\boldsymbol{N(d_2)}$ and$\boldsymbol{ln(S/K)}$

We use the following formula to compare the deviation scope from intrinsic value of options of market price and new price.

$$\boldsymbol{Error1 = \sum_{i=0}^{n} \frac{(Real\ Payoff_i\ -\ Market\ Price_i)^2}{n}}$$

$$\boldsymbol{Error2 = \sum_{i=0}^{n} \frac{(Real\ Payoff_i\ -\ New\ Price_i)^2}{n}}$$

When *Error2* is smaller than *Error1*, we can state that the New Price is more reasonable than the Market Price. In the end, we need to back test our results to confirm that the improvement makes sense in normal by estimating the market value of another option through our model.

[8] RealCDF
is the function to find the corresponding value of cumulative real return distribution function to the return

6: Results & Interpretations

6.1 Implied Volatility Surface and Volatility Smile

Here's the graph of the implied volatility surface of equity. Figure 6-1 is the implied volatility of call and Figure 6-2 shows the implied volatility surface of put on the same stock.

Due to the fact that the trading volume was very small, therefore, the implied volatility surface of put option looks very strange and we hope that in the future we will further study on the possible arbitrage of under this circumstance and the reason behind this phenomenon. (Strange implied volatility surface and differences between implied volatilities of call and put on the same stock).

Table 2 Implied Volatility of Call Option on Equity (20 trading days)

T-t	***75***	***77.5***	***80***	***82.5***	***85***	***87.5***	***90***
72	20.70%	20.72%	21.90%	19.54%	20.64%	20.19%	18.97%
71	23.59%	23.47%	23.69%	23.22%	21.59%	22.05%	21.02%
70	23.51%	23.58%	24.70%	23.28%	23.24%	22.48%	21.82%
67	21.54%	23.50%	23.48%	23.79%	23.48%	23.14%	22.72%
66	23.08%	23.94%	24.64%	24.58%	24.66%	25.24%	24.39%
65	25.74%	25.48%	25.28%	25.20%	25.13%	25.01%	24.78%
64	25.02%	25.08%	25.22%	25.02%	25.01%	24.61%	24.72%
63	27.31%	26.81%	26.70%	27.50%	27.49%	27.19%	27.85%
60	24.88%	25.70%	25.71%	26.55%	25.60%	26.15%	26.16%
59	21.35%	24.01%	23.37%	24.99%	24.63%	24.92%	25.05%
58	22.39%	25.92%	25.99%	26.11%	25.94%	25.62%	26.00%
57	21.28%	23.61%	24.27%	23.93%	24.68%	25.28%	24.66%
56	22.85%	24.23%	24.44%	24.55%	25.36%	25.14%	24.92%
52	18.28%	20.68%	22.30%	23.23%	23.73%	23.21%	23.53%
51	16.44%	18.97%	16.45%	19.45%	18.27%	17.92%	19.40%
50	14.60%	17.27%	18.74%	18.76%	19.45%	17.95%	19.07%
49	12.77%	19.02%	19.55%	19.78%	20.10%	19.74%	19.70%
46	19.67%	23.17%	24.31%	24.54%	24.30%	23.44%	24.36%
45	20.04%	20.82%	22.69%	23.17%	22.74%	21.65%	23.29%
44	24.45%	24.02%	24.29%	23.50%	23.07%	22.50%	23.17%

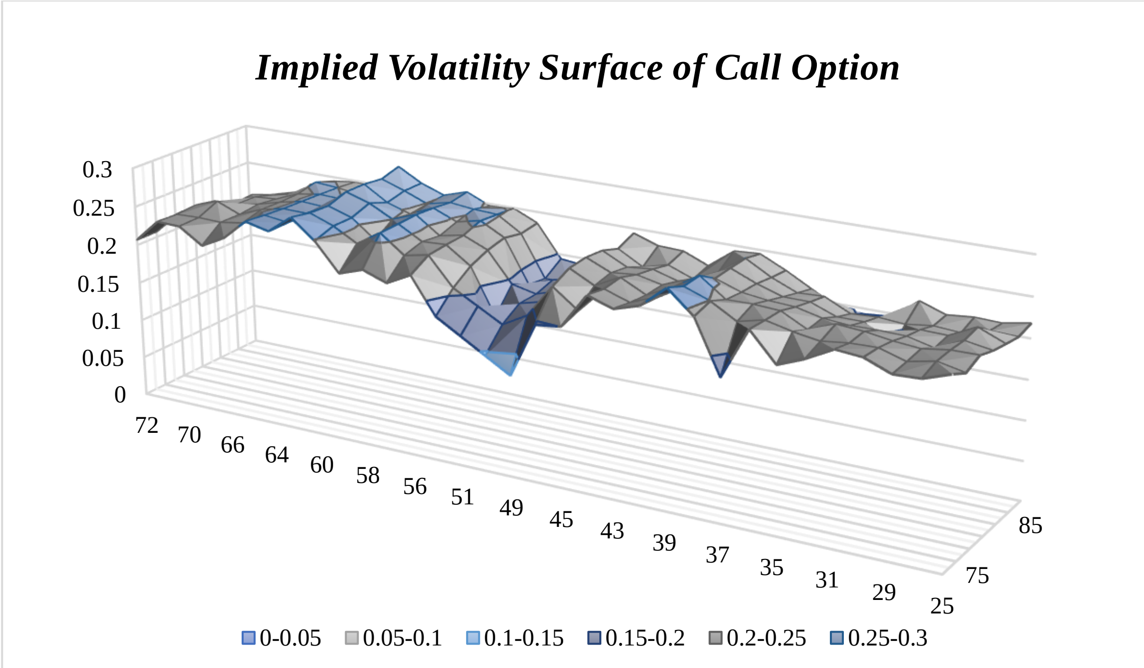

Figure 6-1 Implied Volatility Surface of Call Option

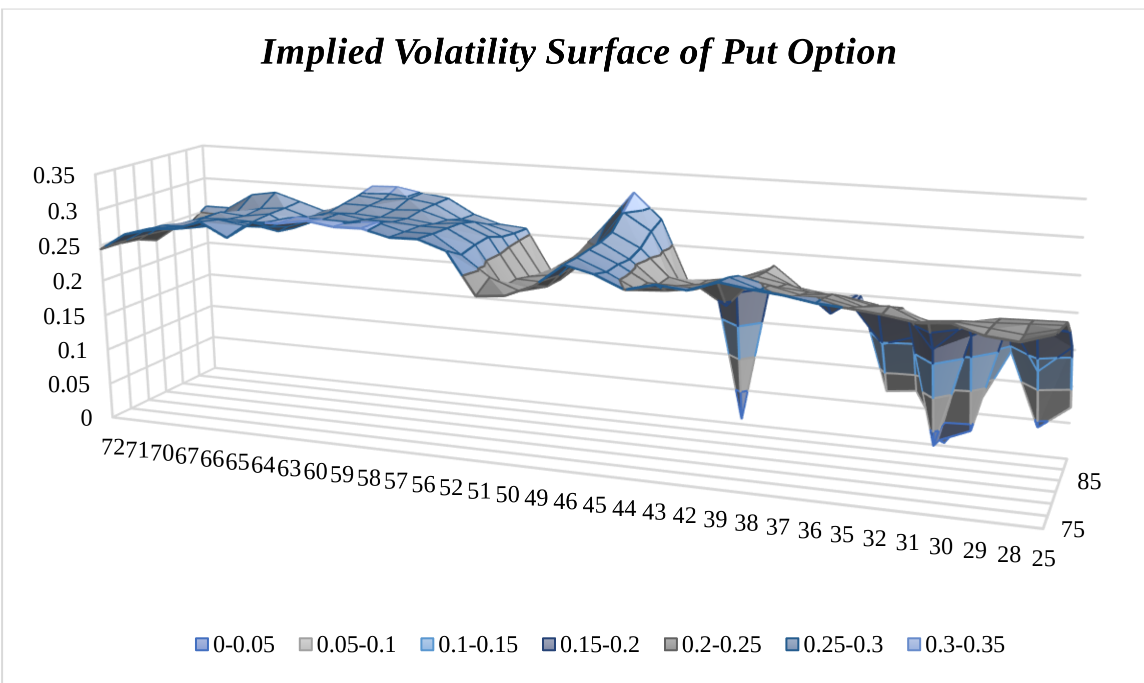

Figure 6-2 Implied Volatility Surface of Put Option

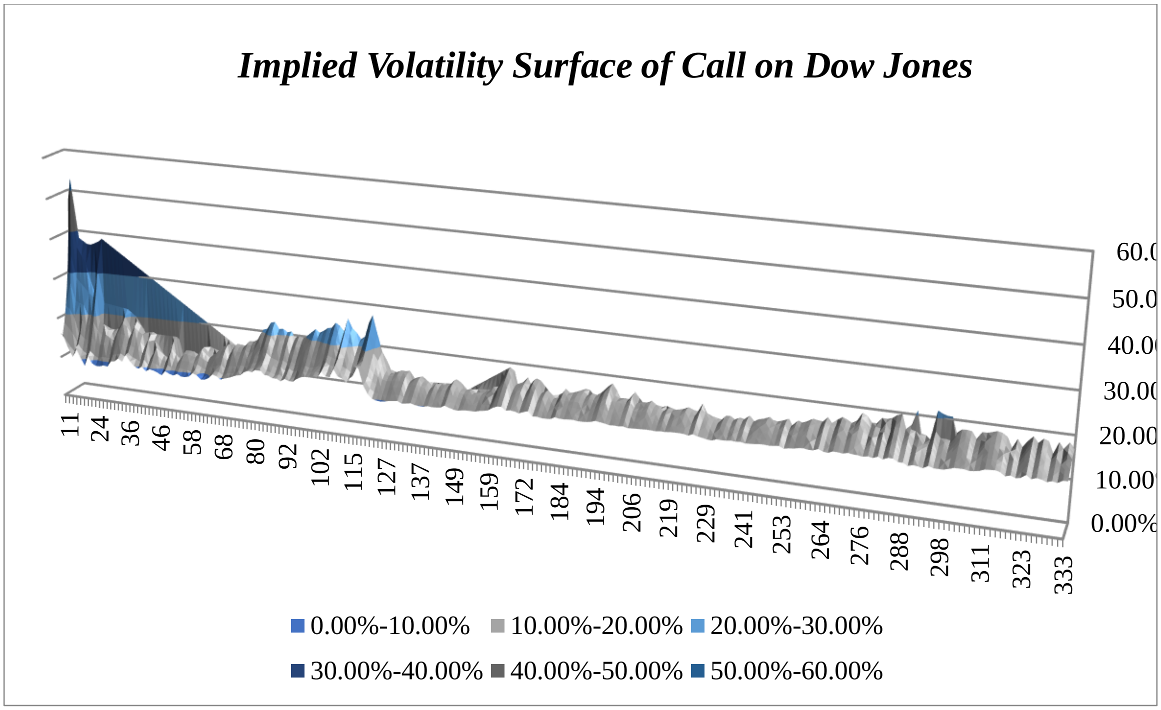

Figure 6-3 Implied Volatility Surface of Call on Dow Jones

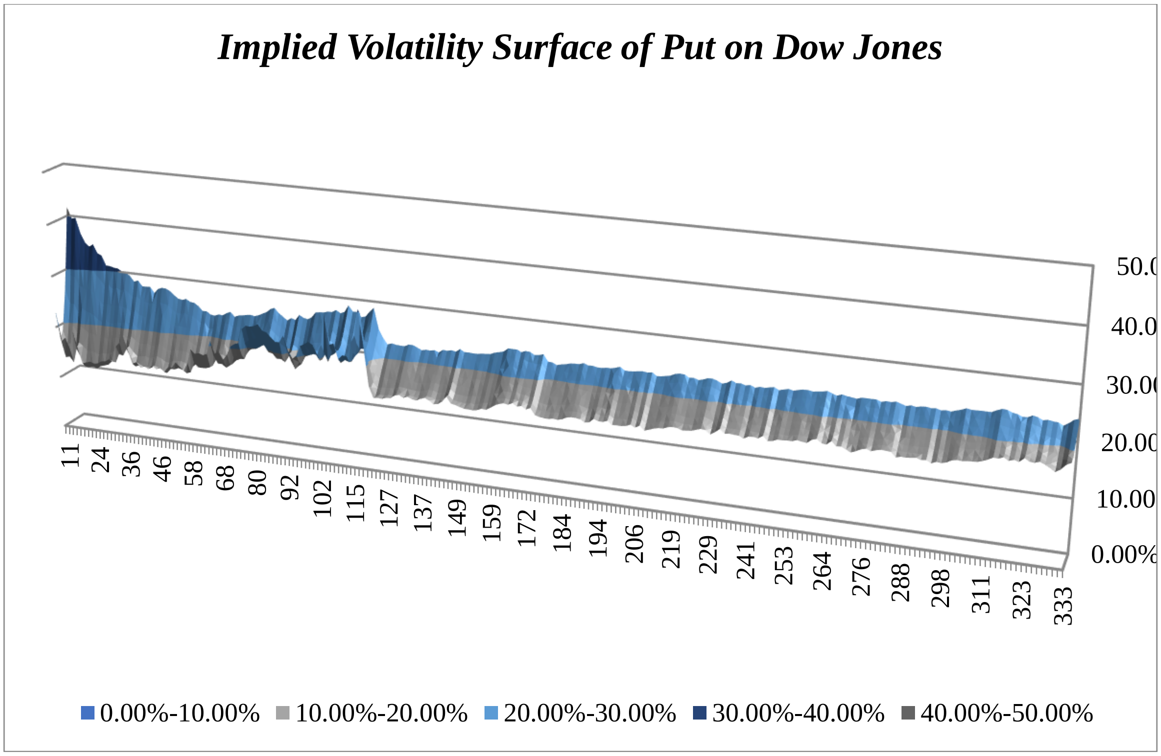

Figure 6-4 Implied Volatility Surface of Put on Dow Jones

The implied volatility surfaces of options which have high trading volume on Dow Jones Index are more reasonable than that on equity. Figure 6-3 and Figure 6-4 demonstrate the implied volatility surface of DJ index.

The implied volatility of call and put show similar trend in this period and the implied volatility of put are slightly higher than that of call. More importantly, it is clear to confirm the existent of Volatility Smile and Volatility skew as in Figure 6 -5, the implied volatility of options with lower K are higher than those with relative higher.

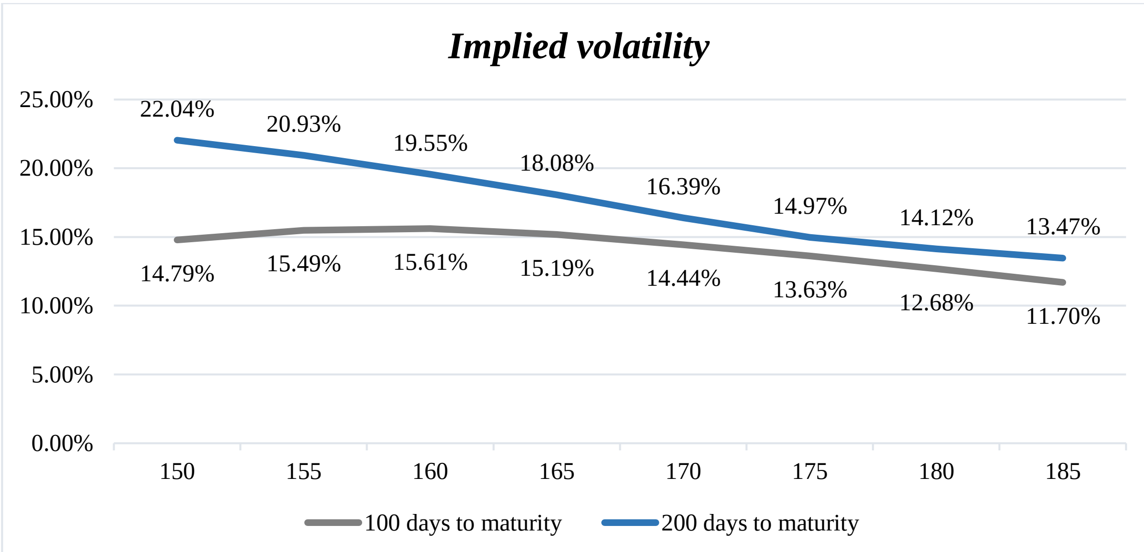

Figure 6-5 Implied Volatility Skew

These two phenomena are both resulted from the same reasons. First of all, implied volatility to some extent reflects investors' expectation to the price of the underlying asset at expiry. Normally, even in bullish market, when the market did decline, those declines were (on average) more sudden and more severe than advances. In other words, the market tends to increase relatively slower than the speed of declining and that's why investors would be concerned about the probability of the sudden decrease in the price of underlying asset. Secondly, the volatility of bear market is much higher than that of bull market according to the historical data. More importantly, the Black-Scholes model assume that the return accord to normal distribution which is different from the expectation of the investors and is different from the real return distribution of the market.

Therefore, the implied volatility of put will be higher than that of call on the same underlying asset and the implied volatility of options with lower ***K*** tends to be higher than that of options with higher ***K***.

6.2 Local Volatility and Realized Volatility

After we confirmed the volatility skew, we tried to figure out if it is reasonable to calculate the option price by local volatility and realized volatility.

Table 3 shows the part of the results of local volatility.

Table 3 Local Volatility of Dow Jones (20 trading days)

Strike price	***155***	***160***	***165***	***170***	***175***	***180***
Time to maturity						
348	11.54%	14.43%	9.21%	8.62%	8.09%	6.94%
347	2.94%	3.72%	2.28%	2.11%	1.22%	1.14%
346	11.90%	12.80%	10.03%	9.22%	7.13%	7.12%
345	7.57%	8.53%	7.21%	4.13%	4.72%	10.70%
344	8.66%	9.54%	8.29%	7.50%	5.93%	5.38%
341	7.21%	5.10%	4.11%	4.78%	3.23%	3.23%
340	8.83%	17.07%	5.10%	6.92%	5.10%	4.21%
339	12.77%	9.46%	7.42%	7.79%	5.89%	5.82%
338	1.86%	7.21%	4.93%	3.95%	4.04%	1.44%
337	2.24%	4.02%	5.27%	4.72%	11.63%	2.59%
333	5.00%	1.86%	4.56%	2.73%	2.99%	1.32%
332	11.63%	18.81%	5.15%	8.16%	5.27%	5.03%
331	2.58%	2.74%	3.58%	4.78%	3.23%	2.40%
330	11.33%	5.85%	7.68%	6.10%	4.93%	4.70%
327	22.12%	6.77%	21.15%	7.42%	6.45%	5.69%
326	14.31%	10.70%	8.61%	9.31%	7.39%	5.75%
325	1.61%	6.16%	6.65%	1.14%	1.22%	3.64%
324	1.61%	1.61%	7.76%	1.77%	1.08%	1.52%
323	6.45%	3.82%	2.73%	2.40%	2.44%	1.77%
320	12.18%	10.40%	9.99%	7.39%	6.27%	6.84%

It is easy to find that the local volatilities are not reasonable as the local volatilities fluctuate significantly. We suppose that the reason for this phenomenon is the large interval of K. Afterwards, we calculate the realized volatilities and here is the line graph (Figure 6-6 & Figure 6-7) of the realized volatilities compared to the implied volatilities of options with different K.

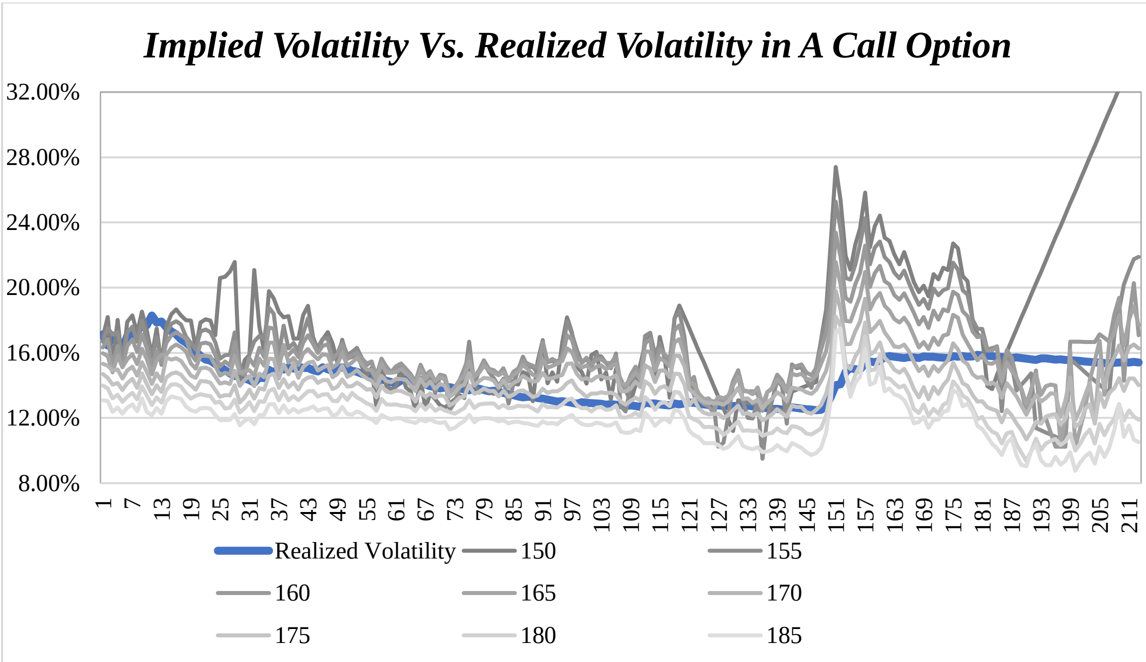

Figure 6-6 Implied Volatility Vs. Realized Volatility in A Call Option

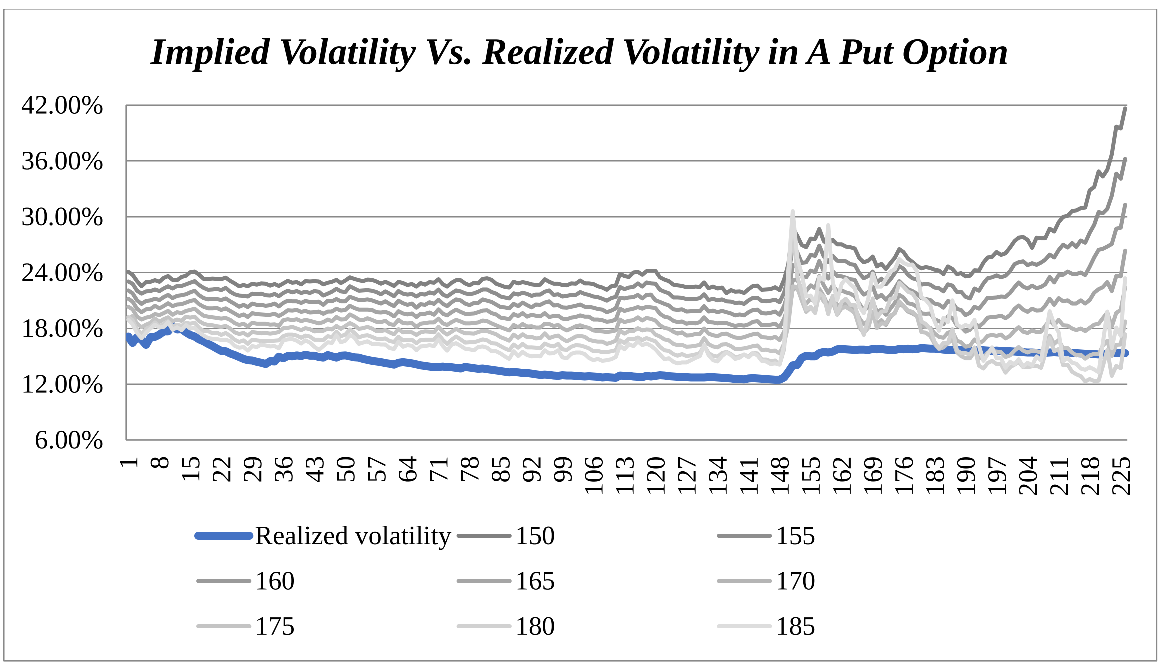

Figure 6-7 Implied Volatility Vs. Realized Volatility in A Put Option

Table 4 Market Performance Exhibit for Spike in Realized Volatility of Call Option on Dow Jones Index

Date	***Underlying Asset Price***	***Realized Volatility***	***150***	***155***	***160***	***165***	***170***	***175***	***180***	***185***
17/08/2015	175.45	12.51%	14.09%	13.86%	14.61%	14.35%	13.49%	12.27%	10.99%	9.73%
18/08/2015	175.11	12.48%	14.21%	15.16%	15.08%	14.60%	13.74%	12.51%	11.16%	9.84%
19/08/2015	173.49	12.49%	16.78%	16.76%	16.17%	15.32%	14.26%	12.80%	11.35%	10.16%
20/08/2015	169.91	12.73%	18.64%	18.04%	17.27%	16.12%	14.89%	13.48%	12.15%	11.11%
21/08/2015	164.6	13.30%	22.92%	21.34%	18.95%	17.71%	16.10%	14.62%	13.72%	13.17%
24/08/2015	158.71	14.02%	27.40%	25.27%	23.37%	21.57%	19.79%	18.22%	17.40%	17.24%
25/08/2015	156.66	14.07%	25.37%	23.61%	21.88%	20.22%	18.67%	17.66%	17.28%	17.50%
26/08/2015	162.86	14.83%	21.95%	20.60%	19.37%	17.96%	16.52%	15.29%	14.66%	14.77%
27/08/2015	166.55	15.05%	21.10%	20.50%	19.13%	17.92%	16.54%	15.01%	13.78%	13.31%
28/08/2015	166.43	15.00%	22.63%	21.40%	20.19%	18.73%	17.30%	15.79%	14.64%	14.17%
31/08/2015	165.28	14.98%	23.64%	22.68%	20.90%	19.47%	17.92%	16.23%	15.22%	14.53%
01/09/2015	160.58	15.35%	25.83%	24.27%	22.58%	20.96%	19.32%	17.87%	17.00%	16.22%
02/09/2015	163.51	15.46%	22.47%	21.43%	20.04%	18.66%	17.28%	15.88%	14.91%	14.05%
03/09/2015	163.75	15.42%	23.77%	22.45%	20.89%	19.27%	17.56%	16.09%	14.97%	14.17%
04/09/2015	161.02	15.51%	24.42%	22.82%	21.33%	19.67%	17.97%	16.63%	15.85%	15.16%

Table 5 Market Performance Exhibit for Spike in Realized Volatility of Put Option on Dow Jones Index

Date	***Underlying Asset Price***	***Realized Volatility***	***150***	***155***	***160***	***165***	***170***	***175***	***180***	***185***
17/08/2015	175.45	12.51%	22.23%	20.99%	19.67%	18.40%	16.94%	15.58%	14.48%	14.16%
18/08/2015	175.11	12.48%	22.41%	21.08%	19.80%	18.50%	17.08%	15.62%	14.53%	14.23%
19/08/2015	173.49	12.49%	22.10%	20.82%	19.48%	18.12%	16.74%	15.31%	14.15%	14.11%
20/08/2015	169.91	12.73%	23.24%	21.84%	20.44%	19.05%	17.66%	16.38%	15.73%	16.19%
21/08/2015	164.6	13.30%	24.85%	23.49%	21.88%	20.38%	19.14%	18.33%	22.14%	24.21%
24/08/2015	158.71	14.02%	28.80%	26.88%	24.80%	23.19%	22.23%	22.51%	29.44%	30.60%
25/08/2015	156.66	14.07%	27.98%	26.22%	24.62%	23.29%	22.36%	22.47%	23.60%	25.58%
26/08/2015	162.86	14.83%	26.98%	25.05%	23.61%	22.13%	21.10%	21.02%	21.88%	23.84%
27/08/2015	166.55	15.05%	26.72%	25.10%	23.50%	22.00%	20.63%	19.76%	20.02%	21.43%
28/08/2015	166.43	15.00%	27.65%	25.90%	24.21%	22.64%	21.22%	20.21%	20.31%	21.54%
31/08/2015	165.28	14.98%	27.60%	25.81%	24.02%	22.31%	20.82%	19.82%	19.64%	20.67%
01/09/2015	160.58	15.35%	28.69%	26.89%	25.23%	23.64%	22.24%	21.69%	21.79%	23.57%
02/09/2015	163.51	15.46%	27.38%	25.73%	24.12%	22.63%	21.46%	20.88%	21.44%	23.22%
03/09/2015	163.75	15.42%	27.01%	25.14%	23.35%	21.79%	20.36%	20.37%	19.53%	29.09%
04/09/2015	161.02	15.51%	27.53%	25.78%	24.13%	22.64%	21.41%	20.83%	21.48%	23.02%

In the Figure 6-6 and Figure 6-7, the spikes present are stemmed from the market sudden surge in the late August 2015, during which the price of call option on Dow Jones index dipped from around USD$175 to less than USD$160 in four trade days. Thereafter, the option price fluctuated around the level of USD$163, instead of the previous "reversed mean" – USD$175. Hence, the realized volatility calculated from the existing market data is accumulative on the base on the volatility of one hundred trading days before the inception of the option. Consequently, this remarkable price change brings material influence on the overall realized volatility, that is sudden spikes in both figures. The related market data around those dates is offered above as the Table 4 and Table 5.

It is obvious that realized volatility shares similar trend with implied volatility and at the same time we could find that the implied volatilities have a higher volatility than the realized volatilities have. Actually, implied volatilities are able to reflect the expectations of investors, while realized volatility actually reflects the historical volatility. According to the paper "Volatility Dynamics for the S&P500: Evidence from Realized Volatility, Daily Returns and Option Prices" written by Christoffersen (Christoffersen, Jacobs, & Mimouni, 2008), *"realized volatility acts perfect in fitting numerous samples of index returns, and it has the lowest option implied volatility mean squared error in-and out-of- sample, and it provides a more realistic volatility term structure."* Therefore, we decided to use the realized volatility to calculate the new option price.

6.3 Results of Different Improvement Methods on Model

In the first place, results of improvement on call option can be seen in Table 6 Trial and Error below.

It is easy to find that if we only map $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ the *Error2* is larger than *Error1* which shows the failure of only mapping the random walk part. Although realized volatility has the lowest option implied volatility mean squared error, using realized volatility only to estimate the option value is not a good choice. The results of the combination of these two changes is also not more precious than the market price.

Only when we map the start point and the random walk part as well as use the realized volatility, we can get a more accurate option value than the market price.

Table 6 Trial and Error

	K=150	*K=155*	*K=160*	*K=165*	*K=170*	*K=175*	*K=180*	*K=185*
Error1	*59.2597*	*63.5848*	*71.2214*	*83.4970*	*102.8287*	*134.5293*	*188.9858*	*278.7836*
Error2$_1$[9]	*64.6016*	*68.5382*	*75.8779*	*99.9243*	*149.0622*	*231.4155*	*331.8590*	*336.1852*
Error2$_2$[10]	*60.0939*	*63.0231*	*68.8651*	*80.6716*	*103.4314*	*144.2960*	*212.1916*	*316.1664*
Error2$_3$[11]	*61.7133*	*65.6911*	*72.8229*	*87.3704*	*139.0549*	*232.6047*	*372.3005*	*404.8855*
Error2$_4$[12]	*49.4533*	*57.5891*	*68.7873*	*83.1860*	*100.7876*	*121.7290*	*147.7702*	*178.7074*

6.4 Back-testing

We back-tested the improvement through the put option on Dow Jones index which matured at the same day and we find that our improvement in total behaved better than the market price except when the $\boldsymbol{K}$ equals 185.

Table 7 Back-testing Error

	K=150	*K=155*	*K=160*	*K=165*	*K=170*	*K=175*	*K=180*	*K=185*
Error1	*556.7564*	*370.0751*	*227.7318*	*126.8845*	*63.1989*	*30.6516*	*20.5360*	*21.4646*
Error2	*434.3270*	*290.50*	*176.3071*	*91.2666*	*35.2838*	*8.1801*	*9.6128*	*38.1632*

[9] *Error2$_1$*
It is the *Error2* when we only map the $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$.

[10] *Error2$_2$*
It is the *Error2* when we only use realized volatility to calculate the new price.

[11] *Error2$_3$*
It is the *Error2* when we map $\boldsymbol{N(d_1)}$ and $\boldsymbol{N(d_2)}$ together with the realized volatility to calculate the new price.

[12] *Error2$_4$*
It is the *Error2* when we map $\boldsymbol{N(d_1)}$, $\boldsymbol{N(d_2)}$ and $\boldsymbol{ln(S/K)}$ together with the realized volatility to calculate the new price.

In order to figure out the reason for the mistake, Table 8 shows part of the option value when ***K*** is 185 after the improvement.

Table 8 Partial Testing Result 1 (K=185)

Index Price	*Market Price*	*New Price*	*Error1*	*Error2*
$179.66	*$10.3*	*$6.39*	*11.63*	*53.62*
$178.9	*$10.8*	*$6.52*	*8.47*	*51.75*
$179.47	*$10.55*	*$6.42*	*2.13*	*53.21*
$175.96	*$13.25*	*$6.28*	*0.66*	*49.21*
$176.2	*$12.9*	*$6.42*	*2.13*	*55.14*

We could find that when the market price is high, the accuracy of the improvement is not satisfying. Following is the table for part of the period when the improvement is more accuracy than market price.

Table 9 Partial Testing Result 2 (K=185)

Index Price	*Market Price*	*New Price*	*Error1*	*Error2*
$163.3	*$23.35*	*$12.06*	*92.93*	*2.71*
$164.33	*$22.35*	*$12.15*	*74.65*	*2.42*
$166	*$22.7*	*$13.05*	*80.82*	*0.43*
$167.4	*$20.35*	*$12.47*	*44.09*	*1.52*
$166.75	*$19*	*$12.46*	*27.98*	*1.56*

It is amazing to find that when the market price is at a relative lower point, our improvement shows a wonderful function to estimate the option value. We are not sure that if it is only an accident and in the future, we need to do more back-testing on our improvement on Black-Scholes model and confirm the accuracy of it.

7: Conclusion

7.1 Justification to Features of Volatility Smile, Volatility Skew and Implied Volatility Surface

During our research, we have several features of implied volatility surface justified and supported by the real market trading data.

7.1.1 Implied Volatility of Put Vs. Implied Volatility of Call

The implied volatility of a put option, in the existing theory, is assumed to be higher than that of a call option on the same underlying asset. This phenomenon results from the fact that the market tends to crash at a higher velocity than the market to boom. Therefore, short position of put option needs to assume a more significant downside risk than short call position. Higher risk forces put sellers to ask for a higher premium and results in a higher implied volatility.

Moreover, options are always used as a method to hedge the risk of downside movement of market. Thus, the demand for put options is larger than that for call options driving higher price and higher implied volatility of put option.

7.1.2 Volatility smile

Implied volatility of options on equity with lower K is always higher than that with higher K. The reason behind is that implied volatility has the characteristic to reflect investors' expectation of market movement and the real return of market always has a fatter negative tail. Sometimes, during the bullish market, options with higher K will have a higher implied volatility than options with lower K. After the announcement of M&A, the implied volatility will show the dual peak feature as investors gamble on the failure or success of M&A.

7.2 Application and Possible Limitations of Local Volatility

7.2.1 Application

Local volatility models are always used to compare the assumptions of derivative valuation models. Local volatility is also utilized to predict volatility while most of pricing models treat volatility as a random number or as a constant number. Its compatibility to stochastic volatility in valuation makes its results and theory more acceptable among the contemporary market.

7.2.2 Possible Limitation

While trying to calculate the local volatility, we find that local volatility requires option data to be high-frequency and high-intensity. In other words, we need a great number of option prices to calculate the local volatility as the interval of ***K*** should be extremely small and the time interval should also be very tiny, to maintain the continuousness of model, which requires substantial data and high quality of data collection process.

However, those requirements do not go along with the real market well. Trading market, or central clearing houses, are always accessible to options with strike prices differ each other by a multiple of USD$2.5 among equities, and this strike price interval goes even wider with options on indexes, usually at USD$5. This *status quo* drives the local volatility model affectless in valuation.

7.3 Higher-Accuracy Pricing Model with Realized Volatility and Return Distribution

The model has been proved to be able to increase the accuracy of Black-Scholes Model using our samples through following two methods.

7.3.1 Replace the Constant Volatility with Realized Volatility

The Black-Scholes Model assumes that the volatility is constant which is not realistic in market. Therefore, we use the realized volatility to calculate the value of options instead of constant volatility. There are several advantages of realized volatility 1) It is dynamic, 2) Short term realized volatility is stronger than that of absolute return volatility, 3) Volatility of realized volatility is lower than that of implied volatility.

7.3.2 Mapping

The Black-Scholes Model is based on the assumption that return accord with normal distribution which is also not realistic in market. Therefore, mapping the components in Black-Scholes model related to normal distribution to real return distribution is an effective way to improve the accuracy of Black-Scholes Model.

7.4 Future Research

Based on our research on the implied volatility surface on options of equity, we could try to figure out the possible risk-free arbitrage opportunity under that circumstance. Moreover, we need to test our improvement through other samples to find the advantages of our model and the limitation of our model.

Bibliography

Ammann, M., & Herriger, S. (2001). Related Implied Volatility Arbitrage with Index Options. (J. Baumberger, Ed.)

Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy, 81*, 637-654. doi:10.1086/260062

Bollerslev, T., Gibson, M. S., & Zhou, H. (2005). Dynamic Estimation of Volatility Risk Premia and Investor Risk Aversion from Option-Implied and Realized Volatilities. *Journal of Econometrics*.

Carr, P., & Wu, L. (2010, October 2). Analyzing Volatility Risk and Risk Premium in Option Contracts: A New Theory. *NYU Tandon Research Paper No. 1701685*.

Christensena, B. J., & Prabhalab, N. R. (1998, November 1). The Relation between Implied and Realized Volatility. *Journal of Financial Economics, 50*(2), 125-150. doi:10.1016

Christoffersen, P., Jacobs, K., & Mimouni, K. (2008). Volatility Dynamics for the S&P500: Evidence from Realized Volatility, Daily Returns and Option Prices. *AFA 2008 NEW ORLEANS MEETINGS*.

Corsi, F. (2004). *A Simple Long Memory Model of Realized Volatility.*

Derman, E., & Kani, I. (1994). Riding on a Smile. *Risk*(7), 32-39.

Dupire, B. (1993). Pricing and Hedging with Smiles. *Proceedings of AFFI Conference.*

Dupire, B. (2004, 10 08). *Pricing with Smiles.* Bloomberg.

Gatheral, J. (2006). *The Volatility Surface: A practitioner's Guide.* Hoboken, New Jersey: John Wiley & Sons.

Gatheral, J., & Jacquier, A. (2013, March 17). Arbitrage-free SVI Volatility Surfaces.

Goncalves, S., & Guidolin, M. (2004). *Predictable Dynamics in the S&P 500 Index OptionsImplied Volatility Surface.*

Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Stuides, 6,* 327-343.

Homescu, C. (2011, July 9). Implied Volatility Surface: Construction Methodologies and Characteristics.

Hull, J. C. (2003). *Options, Futures and Other Derivatives* (5th ed.). Prentice-Hall.

Hull, J., & White, A. (1987, June). The Pricing of Options on Assets with Stochastic Volatilities. *The Journalof Finance, 42*(2), 281-300.

Mahdavi Damghani, B., & Kos, A. (2013, March). De-arbitraging With a Weak Smile: Application to Skew Risk. *Wilmott, 1*, 40-49. doi:10.1002

Merton, R. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science, 4*, 141-183. doi:10.2307/3003143

Yu, W., Lui, E., & Wang, J. (2008, December 5). The Predictive Power of Implied Volatility of Options Traded OTC and on Exchanges. *Journal of Banking and Finance, 34*.

Appendix

Table 10 Realized Volatility and Implied Volatility of Call Option (Excerpts)

T	***Realized Volatility***	***155***	***160***	***165***	***170***	***175***	***180***	***185***
19/06/2015	12.82%	15.95%	15.79%	15.32%	14.63%	13.42%	12.77%	11.73%
22/06/2015	12.79%	12.71%	14.12%	14.16%	13.66%	13.01%	12.05%	11.16%
23/06/2015	12.74%	12.41%	13.83%	14.03%	13.62%	12.81%	12.02%	11.09%
24/06/2015	12.76%	13.92%	14.63%	14.40%	13.82%	13.05%	12.12%	11.10%
25/06/2015	12.73%	14.99%	15.15%	14.79%	14.23%	13.25%	12.32%	11.29%
26/06/2015	12.68%	14.25%	14.61%	14.37%	13.88%	13.07%	12.09%	11.21%
29/06/2015	12.94%	17.04%	16.62%	16.10%	15.23%	14.25%	13.28%	12.31%
30/06/2015	12.89%	17.23%	16.82%	16.06%	15.03%	14.01%	12.99%	11.96%
01/07/2015	12.89%	15.86%	15.21%	14.91%	14.21%	13.47%	12.50%	11.53%
02/07/2015	12.84%	16.59%	16.24%	15.68%	14.92%	13.85%	12.89%	11.78%
06/07/2015	12.79%	15.71%	15.90%	15.48%	14.91%	13.94%	12.96%	11.96%
07/07/2015	12.76%	15.50%	15.61%	15.18%	14.50%	13.69%	12.74%	11.73%
08/07/2015	12.88%	18.11%	17.39%	16.65%	15.81%	14.72%	13.60%	12.50%
09/07/2015	12.84%	18.57%	17.69%	16.84%	15.82%	14.70%	13.56%	12.42%
10/07/2015	12.90%	16.98%	16.50%	15.87%	15.19%	14.05%	13.04%	11.95%
13/07/2015	12.96%	14.42%	14.17%	14.16%	13.78%	12.77%	12.08%	11.22%
14/07/2015	12.92%	13.35%	14.53%	14.07%	13.49%	13.16%	11.92%	10.95%
15/07/2015	12.87%	13.27%	13.14%	13.51%	13.20%	12.57%	11.79%	10.80%
16/07/2015	12.84%	13.18%	12.85%	13.15%	12.98%	12.33%	11.44%	10.46%
17/07/2015	12.79%	13.09%	12.72%	13.20%	12.94%	12.22%	11.46%	10.46%
20/07/2015	12.74%	13.00%	12.30%	12.85%	12.91%	12.24%	11.42%	10.45%
21/07/2015	12.77%	10.24%	13.19%	13.30%	12.92%	12.25%	11.32%	10.34%
22/07/2015	12.73%	10.47%	13.01%	13.20%	12.82%	11.99%	10.96%	10.10%
23/07/2015	12.72%	12.11%	13.20%	13.35%	13.03%	12.18%	11.19%	10.22%
24/07/2015	12.74%	14.09%	14.28%	14.01%	13.38%	12.48%	11.50%	10.53%
27/07/2015	12.73%	14.68%	14.94%	14.51%	13.85%	12.76%	11.79%	10.89%
28/07/2015	12.76%	13.05%	13.71%	13.66%	13.16%	12.24%	11.21%	10.29%
29/07/2015	12.75%	12.00%	13.64%	13.62%	13.21%	12.30%	11.24%	10.16%
30/07/2015	12.71%	11.96%	13.63%	13.45%	12.94%	12.17%	11.21%	10.09%
31/07/2015	12.67%	13.02%	13.47%	13.87%	13.13%	12.28%	11.23%	10.20%
03/08/2015	12.64%	9.50%	12.44%	12.81%	12.58%	11.86%	10.91%	9.91%
04/08/2015	12.61%	12.02%	13.58%	13.60%	12.94%	12.04%	11.02%	9.94%
05/08/2015	12.56%	13.04%	13.84%	13.77%	13.18%	12.22%	11.09%	10.05%
06/08/2015	12.55%	14.33%	14.66%	14.35%	13.57%	12.54%	11.36%	10.33%
07/08/2015	12.52%	13.96%	14.30%	14.11%	13.36%	12.30%	11.15%	10.11%
10/08/2015	12.60%	11.65%	13.63%	13.59%	13.03%	12.10%	11.06%	9.97%
11/08/2015	12.66%	15.28%	15.13%	14.72%	13.86%	12.75%	11.50%	10.46%
12/08/2015	12.62%	15.05%	15.19%	14.62%	13.79%	12.68%	11.46%	10.31%
13/08/2015	12.58%	15.13%	15.28%	14.72%	13.76%	12.67%	11.33%	10.16%
14/08/2015	12.54%	14.32%	14.80%	14.40%	13.58%	12.37%	11.05%	9.91%
17/08/2015	12.51%	13.86%	14.61%	14.35%	13.49%	12.27%	10.99%	9.73%
18/08/2015	12.48%	15.16%	15.08%	14.60%	13.74%	12.51%	11.16%	9.84%

T	Realized Volatility	155	160	165	170	175	180	185
19/08/2015	12.49%	16.76%	16.17%	15.32%	14.26%	12.80%	11.35%	10.16%
20/08/2015	12.73%	18.04%	17.27%	16.12%	14.89%	13.48%	12.15%	11.11%
21/08/2015	13.30%	21.34%	18.95%	17.71%	16.10%	14.62%	13.72%	13.17%
24/08/2015	14.02%	25.27%	23.37%	21.57%	19.79%	18.22%	17.40%	17.24%
25/08/2015	14.07%	23.61%	21.88%	20.22%	18.67%	17.66%	17.28%	17.50%
26/08/2015	14.83%	20.60%	19.37%	17.96%	16.52%	15.29%	14.66%	14.77%
27/08/2015	15.05%	20.50%	19.13%	17.92%	16.54%	15.01%	13.78%	13.31%
28/08/2015	15.00%	21.40%	20.19%	18.73%	17.30%	15.79%	14.64%	14.17%
31/08/2015	14.98%	22.68%	20.90%	19.47%	17.92%	16.23%	15.22%	14.53%
01/09/2015	15.35%	24.27%	22.58%	20.96%	19.32%	17.87%	17.00%	16.22%
02/09/2015	15.46%	21.43%	20.04%	18.66%	17.28%	15.88%	14.91%	14.05%
03/09/2015	15.42%	22.45%	20.89%	19.27%	17.56%	16.09%	14.97%	14.17%
04/09/2015	15.51%	22.82%	21.33%	19.67%	17.97%	16.63%	15.85%	15.16%
08/09/2015	15.74%	21.86%	20.40%	18.88%	17.24%	15.68%	14.44%	13.64%
09/09/2015	15.79%	21.57%	20.19%	18.59%	16.86%	15.40%	14.42%	13.81%
10/09/2015	15.75%	20.93%	19.55%	18.08%	16.39%	14.97%	14.12%	13.47%
11/09/2015	15.73%	20.57%	19.25%	17.87%	16.35%	14.78%	13.88%	13.34%
14/09/2015	15.69%	21.04%	19.65%	18.16%	16.50%	15.01%	14.00%	13.10%
15/09/2015	15.73%	20.27%	19.08%	17.71%	16.10%	14.50%	13.46%	12.54%
16/09/2015	15.72%	19.58%	18.46%	17.06%	15.45%	13.78%	12.56%	11.68%
17/09/2015	15.68%	18.93%	17.73%	16.33%	14.87%	13.27%	12.29%	11.75%
18/09/2015	15.77%	19.29%	18.19%	16.66%	15.16%	13.88%	12.84%	12.07%
21/09/2015	15.76%	18.70%	17.52%	16.24%	14.59%	13.14%	12.15%	11.40%
22/09/2015	15.77%	19.94%	18.58%	16.92%	15.18%	13.89%	12.55%	11.89%
23/09/2015	15.73%	19.53%	18.04%	16.48%	14.83%	13.48%	12.27%	11.91%
24/09/2015	15.69%	19.85%	18.67%	17.05%	15.22%	14.02%	12.82%	12.37%
25/09/2015	15.67%	19.95%	18.55%	17.13%	15.55%	14.37%	12.98%	12.46%
28/09/2015	15.79%	21.53%	19.75%	18.33%	16.58%	15.38%	14.24%	13.70%

Table 11 Local Volatility of Dow Jones (Excerpts)

Date	155	160	165	170	175	180
19/06/2015	5.59%	3.95%	3.82%	4.93%	2.04%	2.79%
22/06/2015	6.03%	6.72%	5.59%	3.86%	3.86%	2.00%
23/06/2015	9.50%	8.06%	7.90%	6.77%	4.56%	3.89%
24/06/2015	10.33%	11.63%	8.78%	7.00%	5.79%	4.56%
25/06/2015	2.79%	3.23%	2.63%	2.50%	1.68%	1.61%
26/06/2015	12.49%	10.45%	8.01%	7.57%	5.48%	3.76%
29/06/2015	15.80%	8.74%	9.12%	6.94%	5.42%	4.32%
30/06/2015	8.93%	6.84%	5.59%	3.72%	3.08%	2.19%
01/07/2015	12.90%	4.27%	5.20%	3.88%	3.40%	2.49%
02/07/2015	5.59%	4.56%	3.95%	3.23%	2.28%	1.89%

Date	155	160	165	170	175	180
06/07/2015	2.11%	2.79%	1.72%	1.32%	1.38%	0.89%
07/07/2015	9.94%	7.03%	5.43%	4.42%	3.53%	2.73%
08/07/2015	11.85%	6.72%	5.85%	5.45%	4.01%	3.23%
09/07/2015	13.87%	7.68%	7.21%	5.82%	4.35%	3.47%
10/07/2015	11.17%	10.50%	8.36%	9.24%	5.18%	4.49%
13/07/2015	21.15%	7.81%	7.79%	8.53%	4.48%	3.84%
14/07/2015	6.03%	9.68%	3.72%	2.38%	4.93%	1.61%
15/07/2015	8.53%	3.95%	3.53%	2.99%	1.02%	1.32%
16/07/2015	4.16%	5.10%	2.63%	2.28%	1.08%	1.59%
17/07/2015	5.10%	4.56%	3.53%	2.94%	2.17%	1.86%
20/07/2015	12.70%	10.20%	6.57%	7.50%	4.89%	4.00%
21/07/2015	12.63%	12.35%	8.43%	7.12%	6.03%	4.59%
22/07/2015	10.86%	9.40%	7.33%	6.34%	4.76%	3.37%
23/07/2015	15.64%	8.83%	7.81%	7.32%	5.01%	3.58%
24/07/2015	12.90%	8.64%	7.61%	6.27%	4.73%	3.52%
27/07/2015	4.08%	3.82%	2.79%	2.55%	1.44%	0.81%
28/07/2015	16.29%	9.03%	7.90%	6.84%	4.82%	3.39%
29/07/2015	7.21%	7.21%	5.03%	4.56%	3.23%	2.40%
30/07/2015	5.89%	6.45%	3.66%	3.40%	2.63%	2.03%
31/07/2015	18.25%	6.08%	10.03%	5.10%	4.27%	3.15%
03/08/2015	12.28%	6.58%	5.70%	5.04%	3.90%	2.94%
04/08/2015	3.95%	3.23%	2.63%	2.04%	1.72%	1.49%
05/08/2015	8.12%	5.59%	4.88%	3.88%	2.79%	2.04%
06/08/2015	8.69%	6.84%	5.97%	4.67%	3.58%	2.59%
07/08/2015	10.20%	6.45%	5.97%	4.56%	3.25%	2.43%
10/08/2015	4.84%	4.56%	3.23%	2.63%	1.93%	1.39%
11/08/2015	10.54%	6.68%	6.34%	4.89%	3.66%	2.66%
12/08/2015	7.94%	1.44%	3.94%	1.02%	0.85%	0.97%
13/08/2015	5.35%	4.56%	3.66%	2.57%	1.76%	0.81%
14/08/2015	7.39%	6.29%	5.03%	4.02%	2.53%	1.70%
17/08/2015	4.56%	3.53%	2.94%	2.15%	1.45%	1.00%
18/08/2015	11.85%	6.72%	5.97%	4.81%	3.40%	2.44%
19/08/2015	14.43%	11.17%	9.12%	7.83%	5.04%	3.48%
20/08/2015	14.37%	13.94%	9.89%	8.06%	5.64%	3.88%
21/08/2015	34.59%	7.75%	9.44%	6.22%	4.47%	3.42%
24/08/2015	10.09%	8.35%	7.01%	5.24%	3.59%	2.76%
25/08/2015	4.81%	3.77%	2.77%	1.74%	1.01%	1.04%
26/08/2015	11.17%	10.20%	7.85%	5.92%	4.37%	3.35%
27/08/2015	13.81%	7.14%	6.53%	5.18%	3.62%	2.65%

Date	155	160	165	170	175	180
28/08/2015	3.45%	3.23%	2.17%	1.60%	0.95%	0.94%
31/08/2015	18.39%	7.28%	6.77%	5.27%	3.39%	2.78%
01/09/2015	7.11%	5.49%	4.86%	3.87%	3.03%	3.06%
02/09/2015	7.09%	4.89%	3.72%	2.66%	1.63%	2.38%
03/09/2015	6.72%	4.93%	4.01%	3.03%	2.28%	1.70%
04/09/2015	3.40%	3.04%	2.30%	1.65%	1.04%	0.52%
08/09/2015	3.95%	2.63%	1.68%	0.38%	1.09%	1.43%
09/09/2015	6.25%	5.59%	4.47%	3.46%	2.88%	2.47%
10/09/2015	5.17%	3.82%	3.14%	2.25%	1.55%	1.36%
11/09/2015	2.63%	1.77%	1.32%	1.00%	0.49%	0.64%
14/09/2015	5.59%	4.33%	3.48%	2.42%	1.78%	1.32%
15/09/2015	8.36%	6.80%	5.42%	3.77%	2.44%	1.75%
16/09/2015	2.63%	1.14%	1.38%	1.42%	1.39%	1.39%
17/09/2015	9.94%	7.60%	5.80%	4.46%	2.94%	2.42%
18/09/2015	6.14%	5.48%	3.84%	3.15%	2.55%	2.21%
21/09/2015	3.23%	2.57%	2.24%	1.74%	1.38%	1.44%
22/09/2015	6.79%	5.30%	3.99%	2.81%	2.28%	1.80%
23/09/2015	5.45%	3.72%	2.89%	2.15%	1.71%	1.22%
24/09/2015	2.38%	2.50%	2.10%	1.64%	1.77%	1.37%
25/09/2015	5.10%	3.95%	2.86%	1.72%	1.20%	0.46%
28/09/2015	7.00%	4.33%	3.90%	2.82%	2.60%	1.71%

Technical notes and software scripts used in the research:

MatLab scripts for accuracy enhanced pricing model:

```
clc
clear
close all
%% Preparing the raw data
% Importing the raw data of options on Dow Jones index
DJ20151219 = xlsread('DJ112');
S = DJ20151219(:,2);
% Calculating the daily return the index
r = diff(log(S));
num = numel(r);
% Setting the return distribution
cdfvalue = ones(num,1);
for i = 1:num
    cdfvalue(i,1) = i/num;
end
distribution = [sort(r),cdfvalue];
%% Setting the parameters of the distribution
DJ20151219 = xlsread('DJ112','DJ20151219');
[row,c] = size(DJ20151219);
```

```
S = DJ20151219(1:237,2);
t = DJ20151219(:,3)/365;
r = DJ20151219(1:237,4)/100;
K = DJ20151219(1:243:end,10)'/1000;
CallPrice = ones(237,8);
PutPrice = ones(237,8);
sig = DJ20151219(12:243,5);
% Putting in the improved Black-Scholes Model with real return
distribution
for i = 1:8
    for j = 12:237
        lnsk = log(S(j,1)/K(1,i));
        judge = lnsk/t(j,1)/365;
        norm = normcdf(judge);
        [v,newrow] = min(abs(distribution(:,2)-norm));
        lnsk = distribution(newrow,1) * t(j,1) * 365;
        d1 = (lnsk + t(j,1) * (r(j,1) + 0.5 * sig(j-
11)^2))/(sig(j-11) * sqrt(t(j,1)));
        d2 = d1 - sig(j-11) * sqrt(t(j));
        [v,newrow] = min(abs(distribution(:,1)-d1/100));
        Nd1 = distribution(newrow,2);
        [v newrow] = min(abs(distribution(:,1)-d2/100));
        Nd2 = distribution(newrow,2);
        CallPrice(j,i) = Nd1 * S(j,1) - Nd2 * K(1,i) * exp(- r(j)
* t(j));
        PutPrice(j,i) = (1 - Nd2) * K(1,i) * exp(- r(j) * t(j)) -
(1 - Nd1) * S(j,1);
    end
end
```