**PORTFOLIO CONSTRUCTION IN TURBULENT MARKETS**
by

Chanyu (Karren) Ma
BSc in Financial Computing, University of Liverpool


and

Lei (Amelia) Zhang
Master of Economics, Capital University of Economics and Business




PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN FINANCE


In the Master of Science in Finance Program
of the
Faculty
of
Business Administration


© Chanyu Ma and Lei Zhang 2017

SIMON FRASER UNIVERSITY

Fall 2017

# Approval

**Name:**                    **Chanyu Ma and Lei Zhang**

**Degree:**                **Master of Science in Finance**

**Title of Project:**        **Portfolio Construction in Turbulent Markets**

**Supervisory Committee:**

_____

**Dr. Peter Klein**
Senior Supervisor
Professor

_____

**Carlos da Costa**
Second Reader
Visiting Lecturer

Date Approved:      _____

**Note (Don't forget to delete this note before printing)**: **The Approval page must be only one-page long.** This is because it MUST be page "ii", while the Abstract page MUST begin on page "iii". Finally, **don't delete this page from your electronic document,** if your department's grad assistant is producing the "real" approval page for signature. You need to keep the page in your document, so that the "Approval" heading continues to appear in the Table of Contents.

# Abstract

We conducted the portfolio optimization on the selected benchmarks for nine asset classes with a time range starting from January 2007 to December 2016 in Canadian Currency, to prove whether the mean-variance approach by Markowitz (1952) combined with a covariance matrix blended from a quiet time and a turbulent time as introduced by Chow, G., Jacquier, E., Kritzman, M., and Lowry, K. (1999) is still valid with recent years' data.

As a result, the optimal portfolios with different covariance matrices blended from turbulent and quiet periods have shown sensitivity of optimal weights to both possibilities of occurrence for the turbulent and quiet periods, and different risk aversion to turbulent and quiet periods. The outlier-sample optimal portfolio is the most conservative one and provides a lowest expected return. Besides, the optimal weights of Cash are much higher due to the higher volatilities of ours benchmarks for US equity, emerging market equity, US bonds, high-yield bonds, and commodities.

**Keywords:** Multivariate Outlier; Mean-Variance Optimization; Blended Covariance Matrix.

# Acknowledgements

First, we would like to thank our thesis instructor Professor Peter Klein of the Beedie School of Business at Simon Fraser University. We are enlightened by his extensive knowledge in finance and professional attitude. We appreciate that he steered us in the right direction when we had questions and ran into some problems.

Second, we would also like to acknowledge Visiting Lecturer Carlos da Costa of Beedie School of Business at Simon Fraser University as the second reader of this thesis project, and we are gratefully indebted to his very valuable comments on this thesis.

Finally, we would also like to thank our families for providing us with their continuous support and encouragement through the Master of Science in Finance program. Thank you!

# Table of Contents

# List of Figures

# List of Tables

# 1: Introduction

Financial markets were shaken by a series of shocks from mid-2007 through the first quarter of 2009. Major indices, such as the S&P 500 Index and the MSCI Index, had suffered tremendous losses during the crisis. Investors had also lost money, and many asset management firms were in survival mode while others had gone bankrupt (Fabozzi, Focardi & Jonas, 2010). The losses during the financial crisis highlighted the fact that risk parameters are unstable and unpredicted. To enhance the performance of investment portfolios, a more informative alternative should be developed to better estimate risk parameters from those event-measured observations, instead of time-measured observations, and construct optimal portfolios which can better represent a portfolio's likely performance during turbulent markets than the time-measured approach.

This report is aimed to replicate the two innovations of procedure coding with MATLAB to the portfolio optimization introduced by those researchers (Chow et al., 1999). The innovations are based on the landmark Markowitz mean-variance approach. Additionally, analyses of empirical results based on the ten-year return data including a wide range of asset classes, starting from the February 2007 to the end of 2016, will be provided to demonstrate both procedures.

# 2: Literature Review

## 2.1 Markowitz Optimization Approach

Harry Markowitz introduced one of the most important and influential theories for portfolio selection (Markowitz, 1952). He states that investors can construct optimal portfolios by using this theory so that the resulting portfolio would achieve an acceptable expected return with minimal volatility. He also introduced the efficient frontier which can be drawn as a curve on a graph of risk against expected return of a portfolio. The frontier is the set of optimal portfolios that can give the maximized expected return for a given level of risk or the lowest level of risk for a given level of expected return.

However, the success of the mean-variance model has inevitably drawn many criticisms. One of its considerable limitations is that the estimation of the requisite risk parameters is inaccurate and unreliable since those parameters are estimated from small samples. Additionally, during the risk estimation procedure, a sample's observation is equally weighted. Therefore, during turbulent periods, the estimates may misrepresent a portfolio's risk attributes and there is arbitrariness to measuring returns simply as a function of units of time (Chow et al., 1999). Furthermore, one of the prerequisite to the use of Markowitz is that the utility is only a function of the first two moments and he did not work out the optimization process considering skewness and higher moments. Thus, the two-moment approach cannot offer guidance for making effective trade-offs between mean, variance, and skewness demonstrated from the empirical summary demonstrated by (Harvey et al., 2010).

## 2.2    Outliers

### 2.2.1    Outliers for a Single Asset

An outlier is an observation that appears to deviate markedly from other and lies an abnormal distance from other values observations in the return series observed (Barnet and Lewis, 1994). It is straightforward to visually identify an outlier in a return series for a single asset. However, the boundary between normal values and outliers is difficult to determine by screening in some cases. It is critical to decide what values will be considered abnormal. For example, a return that falls within the 2.5% of the distribution on either tail can be defined as an outlier, given that the expected continuous return is μ and the standard deviation of the return series is σ. Therefore, a return that is greater than $\mu+1.025\sigma$ or less than $\mu-1.025\sigma$ is identified as outlier (Chow et al., 1999).

### 2.2.2    Multivariate Outliers

A multivariate outlier (MVO) is a combination of contemporaneous unusual returns on at least two types of asset classes. It is more difficult to identify MVO because simple visual detection of the outliers is virtually inapplicable (Majewska, 2015). When there are only two return series in the portfolio, the following procedure can be used to identify an outlier.

For two independent return series Asset A and Asset B with equal variances, a scatterplot is presented below in Figure 1. Firstly, the inside circle was drawn around the mean of the data and its radius was chosen as tolerance for the outliers (Chow et al., 1999). Secondly, to identify the outliers, the equation of circle for each observation was calculated with its centre located at the mean of the data and its perimeter passing through the observations. Thus, if the radius of the circle is greater than the radius of tolerance circle,

the observation is identified as an outlier. The limitation of this method is that it can only be applied to the uncorrelated return series with equal variance.
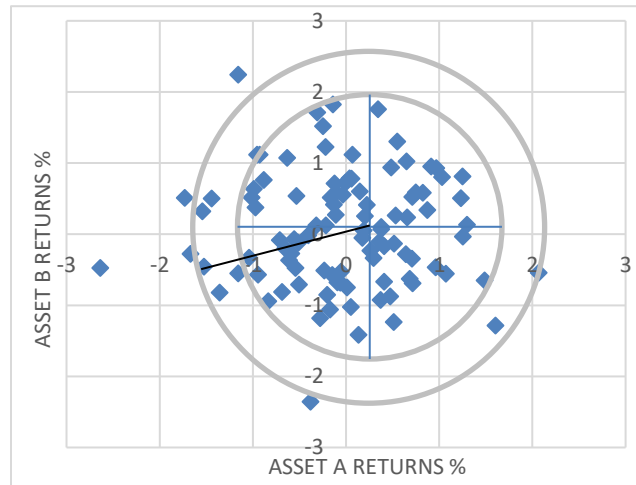


*Figure 2-1: Scatter Plot for Two Independent Return Series with Equal Variances.*

The second type of return series is that two series are uncorrelated, but with unequal variance. In this scenario, an ellipse is a more appropriate shape used to define the outlier boundary. The method to identify outlier is similar to the circle case above. The outliers are found by comparing the boundaries of observation and tolerance ellipses with the same perimeter.
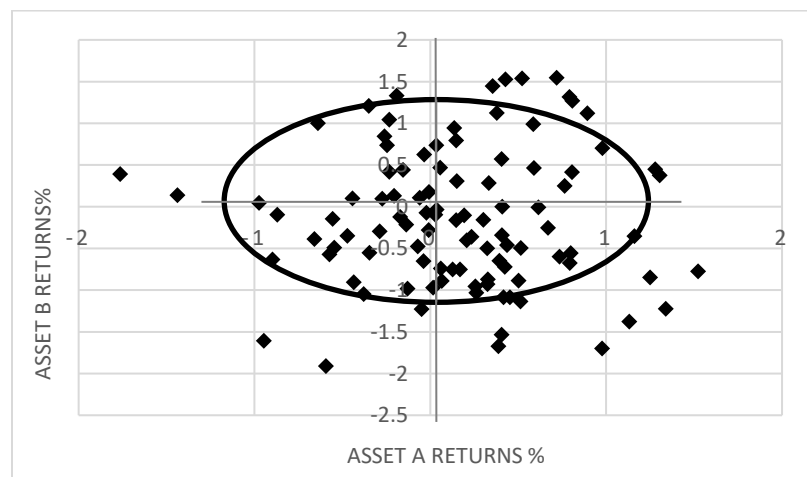


Figure 2-2: Scatter Plot for Two Uncorrelated Return Series with Different Variances.

The scatter plot below indicates that when the return series is positively correlated with unequal variances, the tolerance ellipse generated is positively sloped. The basic method for identifying an outlier is unchanged, however, if the asset returns are correlated or the number of assets in the portfolio exceeds three, the Mahalanobis distance is a more appropriate criterion used to calculate the exact outliers.
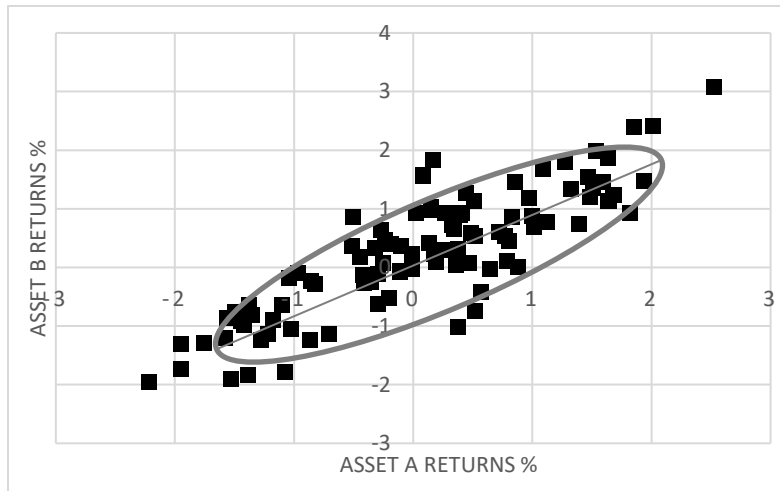


*Figure 2-3: Scatter Plot for Two Correlated Return Series with Unequal Variances.*

### 2.2.3 Mahalanobis Distance

The Mahalanobis distance is a well-known criterion for identification of multivariate outliers and it depends on robust estimated parameters of the multivariate distribution (Majewska, 2015). It is assumed that the return series $y_t$, from a p-dimensional dataset, is multivariate normally distributed. The sample mean vector is denoted by μ and the sample covariance matrix is denoted by $V$. The Mahalanobis distance (MD) for each multivariate data point is denoted by $MD_i$ and given by:

$$MD_i = \sqrt{\sum_{i=1}^{n}(y_i - \mu)V^{-1}(y_i - \mu)'} \qquad\qquad Eq.1$$

Accordingly, the observations with a large MD value can be identified as outliers. For normally distributed data, the MD is approximately Chi-squared distributed with p degrees of freedom. Potential multivariate outliers will typically have large values $MD_i$, and in this situation a comparison with the $\chi_p^2$ distribution can be made. For example, if a tolerance distance was identified as the 97.5%-quantile Q of the Chi-squared distribution with d degrees of freedom, all samples of $MD_i$ which are larger than Q are declared as outliers.

It is also assumed that the square of the Mahalanobis Distance $MD_i^2$ given by the following equation equals the square of the distance from the mean point to each data point, and the use of $MD_i^2$ in replace of $MD_i$ can improve the performance of the detection procedures in presence of outliers (Chow et al., 1999).

$$MD_i^2 = \sum_{i=1}^{n}(y_i - \mu)V^{-1}(y_i - \mu)' = (y_t - \mu) \qquad\qquad Eq.2$$

### 2.2.4 Chi-squared Distribution

In statistics, if m independent random variables $Z_1, Z_2. Z_3, \dots, Z_k$ are normally distributed, then the sum of their squares is shown below:

$$Q = \sum_{i=1}^{k} Z_i^2 \qquad\qquad Eq.3$$

Q follows a Chi-squared distribution with k degrees of freedom denoted as follows:

$$Q \sim \chi^2(k) \hspace{4cm} Eq.4$$

The following two figures are the Cumulative Distribution Function and Probability Distribution Function graphs of the Chi-squared distribution with 1 to 9 degrees of freedom ('Chi-squared distribution', n.d.).
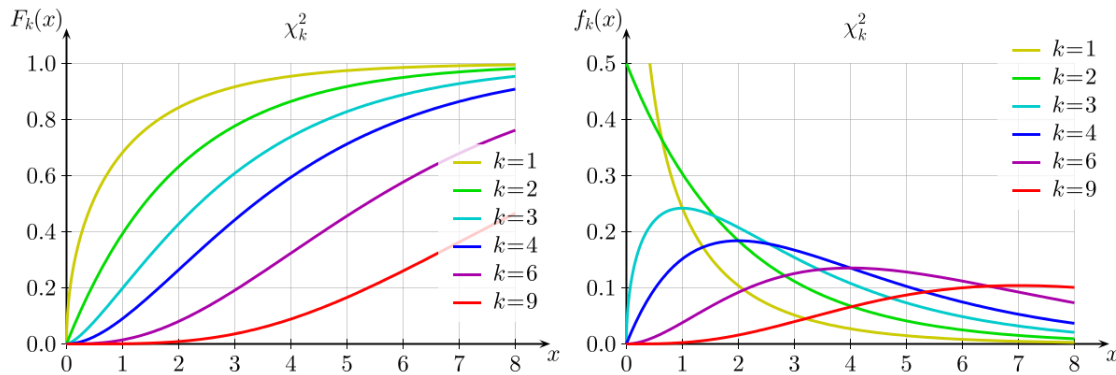


*Figure 2-4: Chi-squared Distribution (left: CDF; right: PDF)*

Chi-squared distribution is a critical part of detecting multivariate outliers. After the calculation of Mahalanobis Distance (MD), which is a set of Chi-squared score, a tolerance boundary can be identified by finding the Chi-squared score, which is defined as the critical value shown below in Table 2-1. If an outlier is defined as falling beyond the outer 10% of the distribution and the number of return series is four, the critical value can be found as 7.78, which is the tolerance boundary of the data set. Finally, all the outliers can be detected by finding all the points with a larger MD value than 7.78 (Chow et al., 1999).

**Table 2-1: Percentage Points of Chi-squared Distribution**

| Percentage Points (Critical Value) | | | | |
|---|---|---|---|---|
| **Degrees of Freedom** | **Probability** | | | |
| | **0.25** | **0.1** | **0.05** | **0.01** |
| **1** | 1.32 | 2.71 | 3.84 | 6.63 |
| **2** | 2.77 | 4.61 | 5.99 | 9.21 |
| **3** | 4.11 | 6.25 | 7.81 | 11.34 |
| **4** | 5.39 | 7.78 | 9.49 | 13.28 |

## 2.3  Blended Covariance Matrices

Chow G., Jacquier E., Kritzman M., and Lowry K. (1999) recommended that investors should take consideration of risks during quiet periods and turbulent periods so that they can achieve their long-term objective while the portfolios can withstand exceptional periods of market turbulence. An innovation in their framework built on the original Markowitz model introduced an approach for selecting portfolios based on a blended covariance matrix shown below including the inside covariance matrix $\sum_i$ and the outlying covariance matrix $\sum_o$, where $p$ is the probability of falling within the inside sample and $1 - p$ is the probability of falling within the outlier sample. The inside covariance matrix represents a quiet risk regime, and the other from outlier observations represents a turbulent risk regime.

$$p\sum_i + (1 - p)\sum_o \qquad\qquad Eq.5$$

Then, the expected utility $EU$ equation can be obtained by substituting these two covariance matrices, with a weight vector $w$ and a probability $p$ of occurrence to the quiet risk regime:

$$EU = w'\mu - \lambda[\lambda_i^* p w'\sum_i w + \lambda_o^*(1 - p)w'\sum_o w \qquad\qquad Eq.6$$

The equation can be recast to the original Markowitz objective function as below:

$$EU = w'\mu - \lambda(w'\sum{}^* w) \qquad\qquad Eq.7$$

The $\lambda$ is defined as the aversion to full-sample risk, and the $\lambda_i^*$ for inside samples and $\lambda_o^*$ for outlier samples are used to differentiate investor's views about the respective probabilities of two risk regimes.

$$\lambda_i^* = \frac{2\lambda_i}{\lambda_i + \lambda_o} \qquad\qquad Eq.8$$

$$\lambda_o^* = \frac{2\lambda_o}{\lambda_i + \lambda_o} \qquad\qquad Eq.9$$

# 3: Data Processing

## 3.1 Data Source

10-year monthly prices beginning in January 2007 and continuing through December 2016 were selected since the global financial crisis period during 2007 and 2008 can be included for a better detection of outliers. These prices are for 4 different kinds of asset classes selected to create a diversified portfolio, which include equities, bonds, commodities, and cash, with a detailed illustration of indices chosen for each class. All the prices were obtained from the Bloomberg platform and have been converted to Canadian Dollars. MATLAB was used to process and analyse the price data as matrix operations will be mainly used to calculate the Mahalanobis distance and find the optimal portfolios based on the new blended covariance matrices.

**Table 3-1: Data Sources**

| | |
|---|---|
| Domestic Equity | SPTSX Index |
| US Equity | S&P 500 Index |
| STOXX Euro Equity | STOXX Europe Total Market Index |
| Emerging Market Equities | MXEF index |
| Domestic Bonds | XBB (replication of FTSE TMX) |
| US Bonds | Bloomberg Barclays US Aggregate Total Return Value Unhedged |
| Global High Yield Bond | Bloomberg Barclays Global High Yield Total Return index |
| Dow Jones Commodity | Dow Jones Commodity Index |
| Cash | Canadian Three-Month T-bill |

## 3.2 MATLAB Modelling

MATLAB was applied to the calculation and modelling process. A MATLAB model was created to process the raw data, identify multivariate outliers, and create blended optimal portfolios.

### 3.2.1 Mahalanobis Distance (MD)

The essential parameters to obtain the MD are mean and covariance matrix of the return series. These two were calculated by MATLAB functions mean() and cov() respectively. Then, the MD was determined by equation 1 and its scatter plot with the date on the x-axis is shown below in Figure 3-1. It indicates that some of the multipliers may be detected visually from the figure, for example, there are 4 points circled between 2008 and 2009 located considerably far away from the mean value. However, it is hard to precisely identify all the outliers in the return series, and the application of Chi-squared distribution will be illustrated in the following paragraphs.
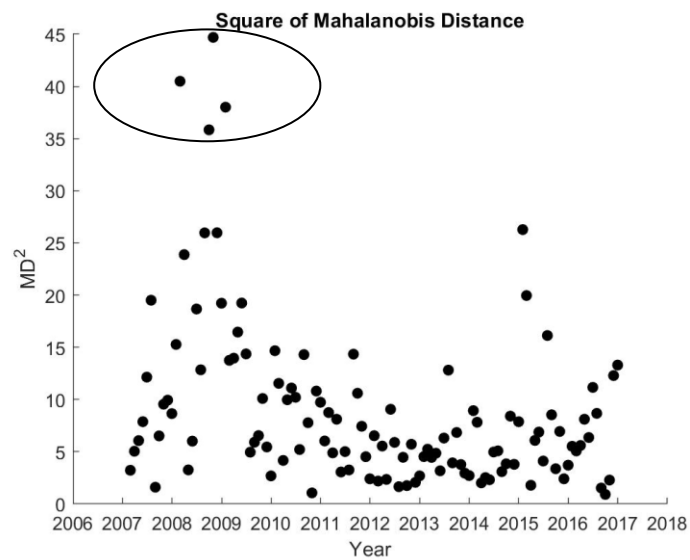


*Figure 3-1: Scatter Plot for the Square of Mahalanobis Distance*

### 3.2.2 Outlier Identification

The most critical step to find all outliers is to calculate the critical value of the return sample. The value is vital for defining the tolerance boundary and all the points which fall beyond the boundary can be defined as outliers.

For general n-return series, the square of Mahalanobis distance is also distributed as a Chi-squared distribution with n degrees of freedom (George et al., 2009). In this case, it is assumed that outliers are defined as falling beyond the outer 25 percent of the distribution with 9 degrees of freedom. It can be found from the distribution table in Appendix 1 that the critical value is 11.39. Therefore, as shown in the cumulative distribution function plot below, all the points with larger $MD^2$ than boundary value of 11.39 are the outliers detected, and the corresponding months and return series are selected to create a new outlier portfolio for further calculation.
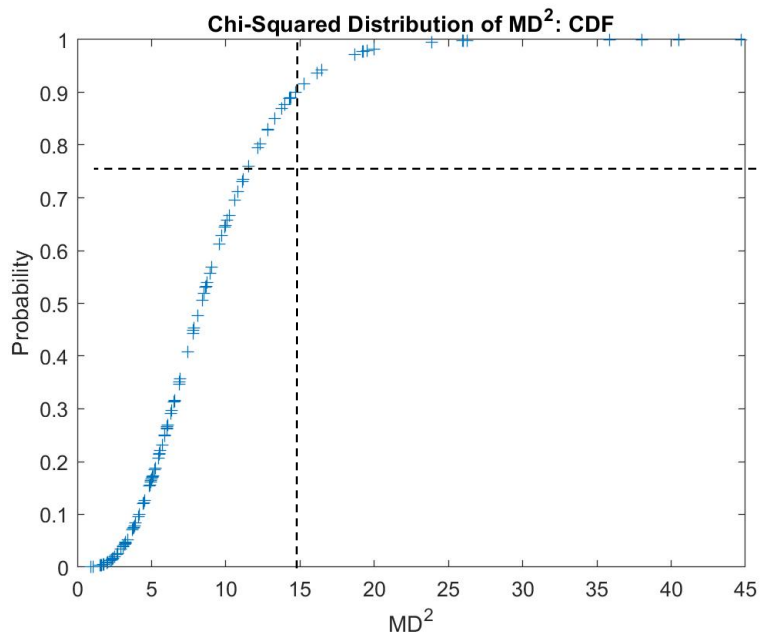


*Figure 3-2: PDF Plot for Chi-squared Distribution of $MD^2$*

### 3.2.3 Blended Optimal Portfolios

In the previous part, multivariate outliers were identified, which are representative of turbulent markets during the global financial crisis with higher-than-normal volatility and correlations. There are four steps to finding the optimal portfolios with event-varying

12

covariance matrices. Firstly, the probabilities of occurrence of each risk regime $p$ and $1 -$ $p$ are set, which are the forecast parameters. Secondly, different degrees of risk aversion toward the two regimes $\lambda_i^*$ and $\lambda_o^*$ are specified, which can be interpreted as investors' attitude toward risks. Then, a single covariance matrix can be calculated by Equation 5, which can reflect one's view about the likelihood of each regime and one's attitude toward each regime at the same time. Finally, the optimal allocations, risk parameters, and returns for the portfolio can be generated by a MATLAB function portfolio(), which implements the Markowitz mean-variance portfolio optimization. There are four types of tests conducted as illustrated below in Table 3-2 based on different expected likelihood of each regime and attitude towards each regime.

**Table 3-2  Type of Tests**

| *Types* | *p* | $\lambda_i$ | $\lambda_o$ |
|---|---|---|---|
| *Full Sample* | N/A | N/A | N/A |
| *Equal Probability, Higher Outlier Aversion* | 50% | 2 | 3 |
| *Equal Probability, Equal Risk Aversion* | 50% | 2 | 2 |
| *Empirical Probability, Higher Outlier Aversion* | 80% | 2 | 3 |

# 4: Result Analysis

## 4.1  Identify Outliers

As shown in Table 4-1, in the year 2008, 9 months are identified as outliers except for May, April and June, based on a 25 percent outlying area of the Chi-squared distribution. During 2007-2008, the global equity and bond market performance were quite volatile due to the global financial crisis starting from the collapse of the US housing market. Commodities also generated huge losses during the second half year of 2018. 14 of the total 24 outliers of the 119 months in the full sample appeared during the period of December 2007 to December 2008.

**Table 4-1: Returns for Nine Asset Classes, January 2008 - December 2008**

*(25 Percent Boundary; Annualized monthly return)*

| Month of 2008 | Domestic Equity | US Equity | Euro Equity | Emerging Market Equities | Domestic Bonds | US Bonds | High-yield Bonds | Commodities | Cash |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -2.18% | -2.21% | -4.07% | -5.32% | -0.09% | 1.25% | 0.06% | 2.44% | 0.47% |
| 2 | 1.39% | -2.53% | -0.42% | 2.05% | 0.20% | -0.93% | -1.40% | 4.05% | 1.81% |
| 3 | -0.75% | 1.76% | 2.00% | -0.39% | 0.52% | 2.17% | 1.97% | -0.20% | -1.23% |
| 4 | 1.87% | 1.21% | 0.86% | 2.49% | -0.10% | -0.89% | 0.70% | 0.31% | 0.07% |
| 5 | 2.36% | -0.11% | -0.73% | 0.10% | -0.25% | -0.89% | -0.32% | 0.53% | 0.18% |
| 6 | -0.74% | -2.92% | -3.13% | -3.67% | -0.02% | 0.95% | -0.24% | 4.51% | 0.12% |
| 7 | -2.71% | -0.14% | -1.04% | -1.55% | -0.39% | 0.26% | -0.21% | 2.04% | 0.00% |
| 8 | 0.57% | 2.14% | -0.39% | -2.11% | 0.32% | 2.03% | 1.44% | -0.84% | 1.03% |
| 9 | -6.88% | -4.30% | -7.17% | -8.63% | 0.24% | -0.75% | -3.91% | -5.66% | -0.20% |
| 10 | -8.06% | -2.66% | -5.20% | -8.56% | -1.34% | 4.37% | -3.55% | -9.08% | 0.29% |
| 11 | -2.25% | -1.95% | -1.84% | -2.02% | -0.48% | 2.82% | -1.60% | -2.38% | 1.19% |
| 12 | -1.35% | -0.48% | 1.54% | 2.37% | 1.05% | 0.77% | 2.40% | -1.93% | 0.03% |

Table 4-2 and Table 4-3 show the risk parameters estimated respectively from the full sample of 119 months and outlier sample of 24 months. The annualized average standard deviation of the full sample is 16.26%, compared to the outlier sample's standard deviation of 25.44%, representing a 56% increase over that of the full sample.

However, the average correlation of full sample is 0.33, which is lower than the 0.36 average correlation of outlier sample. The correlation between different asset classes in the global markets rose during turbulent time mainly because that investors were dumping risky assets indiscriminately during the financial crisis.

From an asset class diversification perspective, Table 4-4 indicates a higher correlation in both the full sample and the outlier sample when commodities are excluded, and the average correlation increases 6 percent and 3 percent respectively for both samples. In addition, the average correlation of commodities with all other assets classes increasing from 0.23 for the full sample to 0.28 for the outlier sample indicates that the diversification function of commodities is weakened during financial crisis periods.

**Table 4-2: Risk Parameters of Full Sample**

| | Domestic Equity | US Equity | Euro Equity | Emerging Market Equities | Domestic Bonds | US Bonds | High-yield Bonds | Commodities | Cash |
|---|---|---|---|---|---|---|---|---|---|
| A. Standard deviation | 19.25% | 17.75% | 22.68% | 25.08% | 5.68% | 17.47% | 13.89% | 20.71% | 3.79% |
| B. Correlation | | | | | | | | | |
| | 1.00 | 0.45 | 0.55 | 0.70 | 0.20 | -0.59 | 0.13 | 0.41 | -0.03 |
| | | 1.00 | 0.76 | 0.49 | 0.10 | 0.04 | 0.54 | 0.11 | -0.24 |
| | | | 1.00 | 0.68 | 0.15 | -0.14 | 0.54 | 0.25 | -0.20 |
| | | | | 1.00 | 0.19 | -0.32 | 0.38 | 0.33 | -0.05 |
| | | | | | 1.00 | -0.11 | 0.24 | 0.07 | -0.03 |
| | | | | | | 1.00 | 0.39 | -0.21 | 0.05 |
| | | | | | | | 1.00 | 0.19 | -0.21 |
| | | | | | | | | 1.00 | 0.15 |
| | | | | | | | | | 1.00 |

**Table 4-3: Risk Parameters of Outlier Sample**

| | Domestic Equity | US Equity | Euro Equity | Emerging Market Equities | Domestic Bonds | US Bonds | High-yield Bonds | Commodities | Cash |
|---|---|---|---|---|---|---|---|---|---|
| A. Standard deviation | 29.54% | 26.54% | 32.32% | 39.40% | 9.35% | 26.93% | 24.98% | 32.32% | 7.59% |
| B. Correlation | 1.00 | 0.45 | 0.57 | 0.71 | 0.24 | -0.54 | 0.27 | 0.60 | 0.00 |
| | | 1.00 | 0.81 | 0.54 | 0.10 | 0.16 | 0.62 | 0.13 | -0.28 |
| | | | 1.00 | 0.75 | 0.08 | 0.03 | 0.65 | 0.23 | -0.24 |
| | | | | 1.00 | 0.22 | -0.15 | 0.54 | 0.48 | 0.03 |
| | | | | | 1.00 | -0.21 | 0.27 | 0.21 | -0.09 |
| | | | | | | 1.00 | 0.26 | -0.26 | -0.03 |
| | | | | | | | 1.00 | 0.20 | -0.32 |
| | | | | | | | | 1.00 | 0.17 |
| | | | | | | | | | 1.00 |

**Table 4-4: Comparison Risk Parameters (Average)**

| Sample | Standard Deviation | Correlation | Correlation of All Assets Excluding Commodities | Correlation of Commodities with All Other Assets Classes |
|---|---|---|---|---|
| Full | 16.26% | 0.33 | 0.35 | 0.23 |
| Outlier | 25.44% | 0.36 | 0.37 | 0.28 |

Additionally, by defining outliers as those falling beyond the outer 25 percent of the distribution, 20.2 percent months (24 months out of 119 months) are detected as outliers, which is less than the 25 percent assumed. This slightly platykurtic distribution indicates that there are more observed months concentrating near the mean return than the number of predicted observations theoretically. The 24 outliers are 2.7 times the number of asset classes, provided 24 observations for and nine-by-nine covariance matrix,

consistent with the validity criteria set forth by Chow G., Jacquier E., Kritzman M., and Lowry K. (1999).

## 4.2 Optimal Portfolios

In our case, the process of portfolio optimizations is conducted under the assumption that investor does not pay taxes, and there's no transactions during the investment horizon.

We use the annualized monthly mean return of the nine asset classes and different blended covariance when determining the optimal portfolio weights in different scenarios as shown in Table 4-5. Because the mean returns are set in a different scenario, the changes in the optimal weights results reflect only differences of risk parameters, which is the covariance matrix in this case. Besides, it is also assumed that the full-sample covariance matrix reflects the risk during the 10-year investment horizon.

**Table 4-5: Annualized Mean return and Standard Deviation for the nine asset classes**

|  | Domestic Equity | US Equity | Euro Equity | Emerging Market Equities | Domestic Bonds | US Bonds | High-yield Bonds | Commodities | Cash |
|---|---|---|---|---|---|---|---|---|---|
| Mean Return | 2.56% | 3.99% | 2.04% | 3.36% | 0.48% | 3.72% | 4.55% | 3.78% | 0.60% |
| Standard Deviation | 19.25% | 17.75% | 22.68% | 25.08% | 5.68% | 17.47% | 13.89% | 20.71% | 3.79% |

Table 4-6 shows the results comparison of optimal portfolio weights for both the full-sample portfolio and the outlier-sample portfolio. In terms of the degree of risk aversion, it is assumed that investors are willing to sacrifice 2.5 units of expected return of portfolio to lower risk, which is the variance, by one unit.

**Table 4-6: Comparison of Optimal Asset Allocation (Full Sample and Outlier Sample)**

| | Domestic Equity | US Equity | Euro Equity | Emerging Market Equities | Domestic Bonds | US Bonds | High-yield Bonds | Commodities | Cash |
|---|---|---|---|---|---|---|---|---|---|
| Full Sample | 20.59% | 3.07% | 0 | 0 | 0 | 34.32% | 28.37% | 13.64% | 0 |
| Outlier Sample | 8.78% | 2.80% | 0 | 0 | 16.73% | 15.74% | 8.48% | 1.70% | 45.78% |

As shown above in Table 4-7, the optimal portfolio weight for the outlier-sample portfolio is more conservative than that of the full-sample portfolio. The weights for different asset classes in the full-sample optimal portfolio would be primarily allocated to Domestic Equity, US Bonds, High-yield Bonds, and Commodities. However, the optimal portfolio based on the outlier covariance matrix would be concentrated in bonds and 3 months Canadian T-bills, and the weight for equities is almost lowered by half to 11.58% of the portfolio, and weights for the commodities and High-yield Bonds are reduced to 1.70% and 8.48% respectively.

As shown in the table below, the expected return of the optimal portfolio constructed only using the outlier sample, mainly representing the global financial crisis period, offers an extremely low expected return of 1.73% and a comparatively low standard deviation of 3.92%. However, while the full-sample optimal portfolio provides a higher expected return, the standard deviation increased by 66.8% in the turbulent environment.

**Table 4-8: Comparison of Optimal Asset Allocation (Full Sample and Outlier Sample)**

| | | Standard Deviation | Expected Return |
|---|---|---|---|
| Full Sample | Normal Environment | 9.05% | 3.73% |

| | | | |
|---|---|---|---|
| Optimal Port. | Turbulent Environment | 15.1% | 3.73% |
| Outlier Sample | Normal Environment | 3.92% | 1.73% |
| Optimal Port. | Turbulent Environment | 6.36% | 1.73% |

In this case, we assumed that the normal environment is associated with the risk parameters reflecting the whole investment horizon, whereas the turbulent environment is associated with the volatility of those 24 months which are identified as outliers. Table 4-7 shows that optimal portfolio based on the outlier sample offers a lower volatility in a turbulent economic environment, reducing 58% of the standard deviation for the full-sample optimal portfolio in the same period (15.1%) to 6.36%. However, the expected return of the optimal portfolio constructed based on the outlier-sample risk parameters offers an extremely low expected return of 1.73%, due to the change in asset allocation with a lower weighting of domestic and US equities in favour of bonds and cash.

So, to avoid the extremely low return of the optimal portfolio that is significantly under investor expectations, we use a blended covariance matrix using inside-sample risk parameters with a lower risk aversion and outlier-sample risk parameters with a higher risk aversion, as introduced by Chow G., Jacquier E., Kritzman M., and Lowry K. (1999).

Table 4-8 displays the results from the portfolio optimization process in different scenarios, as well as the expected performance in a normal environment representing the full-sample months and in a turbulent environment which represents only the outlier months.

**Table 4-9: Comparison of Optimal Asset Allocation (Full Sample and Outlier Sample)**

| Asset Class | Full Sample Optimal (Empirical Probability and Equal Outlier Aversion) | Empirical Probability and Higher Outlier Aversion | Equal Probability Equal Aversion | Equal Probability Higher Outlier Aversion |
|---|---|---|---|---|
| Domestic Equity | 20.59% | 19.54% | 12.00% | 11.71% |
| US Equity | 3.07% | 0.26% | 1.96% | 2.17% |
| Euro Equity | 0 | 0 | 0 | 0 |
| Emerging Market Equities | 0 | 0 | 0 | 0 |
| Domestic Bonds | 0 | 0 | 5.92% | 6.59% |
| US Bonds | 34.32% | 30.28% | 19.65% | 19.77% |
| High-yield Bonds | 28.37% | 18.54% | 12.64% | 12.48% |
| Commodities | 13.64% | 8.46% | 4.10% | 4.13% |
| Cash | 0 | 22.91% | 43.74% | 43.14% |
| *Normal Environment* | | | | |
| Expected Return | 3.73% | 2.938% | 2.137% | 2.136% |
| Standard Deviation | 9.05% | 6.78% | 4.76% | 4.76% |
| *Turbulent Environment* | | | | |
| Expected Return | 3.73% | 2.938% | 2.137% | 2.136% |
| Standard Deviation | 15.1% | 11.15% | 7.78% | 7.78% |

The first column presents the optimal portfolio determined by the full-sample risk parameters. The volatility and covariance matrix estimated from the full sample reflect the empirical probability and equal risk aversion to both inside and outlier months. The second column indicates that, with a higher risk aversion to outlier months, which assumes that investors are 1.5 times as averse to risk in turbulent times as they are during quiet months, the optimal portfolio shifts 22.91% of the portfolio to 3-month Canadian T-bills.

The next column provides the portfolio optimization under the risk parameters of equal frequency for outlier months and inside months, which means a higher volatility compared to the empirical frequency portfolio. It shows an even higher weighting of cash in the portfolio. Moreover, the last optimal portfolio is the most conservative one, under the assumption that turbulent months will occur 50% of the time rather than the actual frequency, which is 20% of the time, and the risk aversion to outlier months is the same as the second one. Compare to the full sample portfolio, this final portfolio reduces the domestic equity weighting from 20.59% to 11.71%, but higher than the pure outlier portfolio with an 8.78% domestic equity. Under the condition of emphasizing both the higher frequency of turbulent period and greater risk aversion, the last optimal portfolio most closely resembles the pure outlier-sample optimal portfolio.

# 5: Conclusion

## 5.1 Conclusion

In this paper, we adapted the Markowitz's mean-variance portfolio optimization theory and blended covariance matrix portfolio optimization procedure introduced by Chow, G., Jacquier, E., Kritzman, M., and Lowry, K. (1999), by using the selected actual indices for 9 asset classes as portfolio benchmark with a time range from January 2007 to December 2016 in Canadian Currency.

We identified total 24 months as outliers falling outside the tolerance boundary in the Chi-squared distribution of the Mahalanobis distance. The outliers are mainly falling during the global financial crisis period during 2007-2008. The risk parameters estimated from the outlier sample more precisely reflect the riskiness during the period of global financial crisis than the risk parameters estimated from the period in full sample. By using the blended covariance matrix representing various investors' risk aversion during quiet and turbulent times, and weighted by various possibilities for the occurrence of quiet and turbulent times, our optimal portfolio results support the key findings concluded by Chow, Jacquier, Kritzman, and Lowry (1999).

In our case, the volatility of full-sample optimal portfolio increased from 9.05% to 15.1% when the portfolio is subjected to the riskiness of outlier sample. As expected, the optimal portfolios with different covariance matrices blended from turbulent and quiet periods have shown sensitivity of portfolio optimal weights to both various possibilities of occurrence for the turbulent and quiet period, as well as investors' different degrees of willingness for taking on risk during turbulent and quiet periods. The outlier-sample

optimal portfolio is the most conservative one among all 5 optimal portfolios under different scenarios and provides the lowest expected return.

The results concluded by Chow, G., Jacquier, E., Kritzman, M., and Lowry, K. (1999) are analysed from a US investor's perspective. In our case, our innovation is that we conducted the research from a Canadian investor's perspective, and the optimal portfolios are invested in the assets classes in Canadian dollars. Our investment horizon covers last ten years including the recent global financial crisis during 2007-2008. Our results have shown positive correlations between commodities and all other equity asset classes, while the data used by Chow G., Jacquier E., Kritzman M., and Lowry K. (1999) shows a negative correlation between commodities and other asset classes.

The results also show a less volatile Canadian Treasury bill, compared to the US cash equivalents with a higher volatility used by Chow G., Jacquier E., Kritzman M., and Lowry K. (1999).

Moreover, for the results of optimal portfolios generated from the different blended covariance matrices, the optimal weights of cash are much higher for our results, as the volatilities of US equity, emerging market equity, US bonds, high-yield bonds, and commodities are all higher than those in Chow G., Jacquier E., Kritzman M., and Lowry K. (1999). Besides, our results show a higher average correlation between asset classes than that in the precedent paper. It might indicate that the acceleration in the pace of globalization and financial product innovations in the last ten years might have deepened the global risky assets' connections and fluctuations, or it's only a matter of different benchmarks the two researches are using. Due to the lack of quantitative academic research on overall trend of global risky assets performance's volatility and correlation over time,

we examined the Chicago Board Options Exchange (CBOE) Volatility Index (VIX), which is constructed using the implied volatilities of a wide range of S&P 500 index options, introduced in 1993. The VIX combines the price of multiple options and derives an aggregate value of volatility. We found out that the average weekly volatility level increased from 18.28 in the period from January 1993 to December 1999 to 20.59 during the period from December 2006 to December 2016. However, this index slumped to 9.14 at the beginning of November 2017, its lowest since December 1993.

For research limitations, we did not incorporate the effects of taxes such as capital gains and transactions costs in the process of portfolio optimization. There are two reasons. First, the focus of this research is on the sensitivity of portfolio optimal weights to both possibilities of occurrence for the turbulent and quiet periods, and risk aversions during turbulent and quiet periods, thus taxation would not be different during quiet and turbulent times. Second, liquidity during the financial crisis is highly restricted and transactions become near impossible during extremely turbulent times. Therefore, we made the assumption that there are no transactions during the investment horizon. These limitations do not have great impact on the key conclusions in our research and the ability to effectively achieve the research goal. However, it would be great explorations for further research on portfolio optimizations.

# Reference List

Chow, G., Jacquier, E., Kritzman, M., & Lowry, K. (1999). Optimal Portfolios in Good Times and Bad. *Financial Analysts Journal*, 55, 65-73.

Chi-squared distribution. (n.d.). In *Wikipedia.* Retrieved December 1st, 2017, from *https://en.wikipedia.org/wiki/Chi-squared_distribution*

Fabozzi, F., Focardi, S., & Jonas, C. (2010*). Investment management after the global financial crisis.* [Charlottesville, Va.]: Research Foundation of CFA Institute.

Harvey, C., Liechty, J., Liechty, M., & Müller, P. (2010). Portfolio selection with higher moments. *Quantitative Finance, 10*(5), 469-485. http://dx.doi.org/10.1080/14697681003756877

Kannan, K., & Manoj, K. (2015). Outlier detection in multivariate data. *Applied Mathematical Sciences, 9*(47), 2317-2324. http://dx.doi.org/10.12988/ams.2015.53213

Majewska, J. (2015). Identification of Multivariate Outliers – Problems and Challenges of Visualization Methods. *Economic Studies. Science Notebooks, Information Technology and Econometrics 4*(247).

Markowitz, H. (1952). Portfolio Selection. *The Journal of Finance*, 7(1), 77-91.

# Appendix

```matlab
%% 1. Input data: full dataset
ret = xlsread('monthly_data.xlsx','10-year ret');

%% 2. Define variables
p = 0.5; % probablility of falling within the inside sample
lambda_in = 2; % aversion to inside risk;
lambda_out = 3; % aversion to outlier risk;

%% 3. Get the average of return
mu = mean(ret);
figure;

% Define x-axis: Date
dateinput= xlsread('Date.xlsx');
year = dateinput(:,3);
mon = dateinput(:,1);
day = dateinput(:,2);
dates = datenum(year,mon,day);

% Plot the return
plot(dates,ret);
datetick;
title('Returns of All Asset Classes');
xlabel('Year');
ylabel('Returns');
legend('Domestic Equity','US Equity','STOXX','Emerging Market Equity',...
    'Domenstic Bonds','US Bonds','High-yield Bond','Commodity',...
    'T-bill','Location','southeast');

%% 4. Get the full-sample covariance matrix
covariance = cov(ret);

%% 5. Calculate the multivariate outliers and find the Outlier Portfolio
dt = (ret-mu) * covariance^(-1)*(ret-mu)';

% Find out the dialogue of the dt matrix: the distance we find
real_dt = diag(dt);

% Plot the distance in a scatter plot
scatter(dates,real_dt,'filled','k');
datetick;
title('Square of Mahalanobis Distance');
xlabel('Year');
ylabel('MD^2');

% Chi-squared Distribution pdf
figure;
```

```matlab
x=real_dt;
y = chi2cdf(real_dt,9);
plot(x,y,'*','k')
title('Chi-squared Distribution of MD^2: CDF');
xlabel('MD^2');
ylabel('Probability');

% Suppose:
% The outlier is defined as falling beyond the outer 25 percent
% of the distribution
% Tolerance score is a Chi-squared score of 11.39 with the degree of
% freedom of 9

% Find the positions of outliers:
pos_outlier = find(x>11.39);

% Creat outlier portfolio
port_outlier =zeros(28,9);
for i = 1:28
  no_rows = pos_outlier(i);
  for j = 1:9
  port_outlier(i,j)= ret(no_rows,j);
  end
end

% Create inside portfolio
pos_inside = find(x<=11.39);
port_inside = zeros(91,9);
for k = 1:91
   no_Rows = pos_inside(k);
   for g = 1:9
   port_inside(k,g) = ret(no_Rows,g);
   end
end

%% 6. Blend
cov_inside=cov(port_inside);
cov_outlier = cov(port_outlier);
% Create an equal-aversion Blended Covariance Matrix
cov_blended_e=p*cov_inside+(1-p)*cov_outlier;

% Define inside risk aversion and outlier risk aversion
lambda_i=2*lambda_in/(lambda_in+lambda_out);
lambda_o=2*lambda_out/(lambda_in+lambda_out);

% Calculate blended covariance for different risk aversion
cov_blended_d=lambda_i*p*cov_inside+lambda_o*(1-p)*cov_outlier;

% Find the optimal portfolio weights for full sample portfolio
port_full = Portfolio('assetmean', mu, 'assetcovar', covariance, ...
'lowerbudget', 1, 'upperbudget', 1, 'lowerbound', 0);
```

```
plotFrontier(port_full);
pwgt_port_full = estimateFrontier(port_full, 10);
```