Conditional statements in mathematics and beyond: Syntax, semantics, and context

by

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Abstract

Logic is an inseparable part of mathematics and is ubiquitous everywhere, such as in definitions, in theorem statements, in proofs, etc. This thesis starts with a discussion about the significant role of logic, both in formal mathematics and in informal mathematics, that includes all mathematical discussions in everyday language, with the main focus on conditional statements. A conditional statement does not have a fixed definition and has different interpretations in colloquial language, philosophy, logic, and mathematics. Mathematics uses the same definition as the one defined in logic, known as a *material conditional*. However, this definition is different from the interpretations in other disciplines, which makes it a topic that causes difficulties for students, especially when conditionals have irrelevant antecedents and consequents. This thesis discusses word problems in mathematics and explains why mathematics needs to include conditionals with irrelevant clauses.

Like any other language, logic has its own semantics and syntax, but, besides the logical form of a statement, its context can influence the way people understand them. The current work adds a third component, namely context, to syntax and semantics of logic as a language, and details two studies to show that the context of a conditional statement is a determinative factor in understanding them.

The first study examines how a mathematician with a good background in elementary logic understands conditional statements in different contexts of logic, mathematics, and colloquial language. The data are created through a clinical interview designed in three formats: structured, semi-structured, and non-structured. I also extend Stephen Toulmin's argumentation scheme to capture more features of the events and prepare data for analysis. The second study examines the language that thirty-four prospective elementary school teachers use and understand when dealing with situations involving conditional sentences; data is collected through a questionnaire. The theory of mental models is used to analyze data for both studies.

The results from the research studies of this thesis show that the context can highly influence people's understanding of a conditional statement, and they mostly choose the type of conditionals present in everyday language.

Keywords: Logic; Conditional statement; Material conditional; Toulmin's argumentation scheme

Dedication

To my incredibly supportive husband, Hooman, who have always been enthusiastic of my passions in life.

Thank you for your continual, unwavering love!

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Chapter 1. Introduction

This chapter starts with the reason I chose this topic to study in this Ph.D. program. It then gives the layout of the thesis and briefly explains what the reader will expect from each chapter.

1.1. Motivation

First-order logic and tables of values, even for compound statements like $(p \wedge q) \Rightarrow \sim q$ and $(p \vee q) \wedge (\sim r)$ had been a part of my high-school lessons in Iran. Even though we practiced well in first-order logic, it was not easy to understand the necessary and sufficient conditions and relations between the premise and conclusion in a conditional statement. While learning this concept, I created my own set of examples to learn which one is necessary and which one is sufficient, and for years I was using my model. The example is:

If it is a rabbit, then it has long ears.

Any rabbit has long ears, so “having long ears” should be a necessary condition. On the other hand, “being a rabbit” is a sufficient condition to have long ears because we do not need any other information and being a rabbit suffices to say that it has long ears. One may argue that a rabbit with small ears could exist (like the fact that black swans exist in Australia), but the purpose of this example for a high-school student was only to learn about necessary and sufficient conditions, and it ignored exceptions.

The relations between “being a rabbit” and “having long ears” also helped me to understand that $q \Rightarrow p$ and $p \Rightarrow q$ are essentially different statements:

If it is a rabbit, then it has long ears.

If it has long ears, then I cannot say what animal it is.

The example I created to learn the meaning or the application of conditional statements in everyday language was not a difficult mathematical example, and I could have learned about it many years earlier. It could simply explain to me when I was a child, and I did not have to wait to understand that interesting relation between some assertions.

There is a common belief that if people study mathematics, then they can improve their logical thinking. Such belief backs up at least to Plato, which is now known as the "theory of formal discipline" (see §§7.2.1).

Maybe learning mathematics and learning logic are somehow related, and this idea is nicely matched with mental logic theory (see §§7.2.2) for those who assume people reason using an in-built logical system. However, this is not what I think about human reasoning, at least not using classical logic to make inferences. Many examples of conditional statements are defined and have meaning in classical logic but are unsound in everyday language. I probably have an in-built logic that I think about based on it, but it may not be classical logic.

I am more inclined to think about what improves mathematical thinking itself and not what is/are improved by mathematical thinking. Through the years of studying and teaching mathematics and creating many theorems and some definitions in advanced analysis, I realized that understanding only the basic rules of first-order logic can facilitate mathematical learning besides improving general reasoning. By understanding these basic rules, I do not mean just knowing the symbolic part and constructing compound and complicated truth-value tables; I mean understanding the applications in everyday language, knowing the actual relation between the premise and the conclusion of a conditional statement.

As part of the curriculum in high school in Iran, I taught first-order logic for years and found the conditional statement to be a topic that many students had difficulty with. Some odd conditional statements like "if $X = 2$, then I am a white rose" might be unsound for most people, but in mathematics, we expect students to know conditionals and the relations between premise and consequent; otherwise, how could they learn mathematical constructions?

Learning logic and the actual meaning of conditional in childhood can facilitate learning mathematics. I am not alone with this story; many of my students have the same difficulty learning the relation between two parts of a conditional. After many years, I figured out that logic is not a different topic separated from mathematics; it is ubiquitous everywhere in mathematics, such as definitions, theorems, proofs. Basically, logic is a non-separable part of mathematics, but previously, when I learned logic in high school, I had learned that “for $a < 1$, $a^2 < 1$”, and “from $x + 3 = 4$, we get $x = 1$” or learned many statements like these examples.

Both statements “for $a < 1$, $a^2 < 1$”, and “from $x + 3 = 4$, we get $x = 1$” are conditional statements. The following is the equivalent form of these statements using if-then:

if $a < 1$, then $a^2 < 1$ and if $x + 3 = 4$, then $x = 1$.

Both are conditionals, but many students look at them as biconditionals. A biconditional statement has the general form of “p if and only if q” and is equivalent to the combined statement of “if p then q” and “if q then p”.

My experience dealing with conditionals shows that many students do not focus on the connective and understand the relation between the premise and conclusion by the meaning. Let me clarify what I mean by using an example. Consider the following statement:

$x - 4 = 1$ in the case that $x = 5$.

Looking at the order of the assertions “$x - 4 = 1$” and “$x = 5$”, or their relation, one may result that the meaning of the above statement is:

If we have $x - 4 = 1$, then $x = 5$.

However, this is not the actual meaning of the given statement “$x - 4 = 1$ in case that $x = 5$”. The actual meaning is “if we have $x = 5$, then $x - 4 = 1$”. Those types of statements might result in assuming conditionals as biconditionals or to confuse a statement and its converse.

A conditional statement in mathematics is the material conditional statement with a certain interpretation that has nothing to do with cause-and-effect or relatedness of the premise and conclusion. However, in everyday language, we have different conditionals, like causals and counterfactual conditional statements. In my opinion, informal mathematics discussing mathematics and explaining what we mean by a certain mathematical construction can be regarded as argumentations in everyday language, such as when a teacher explains a topic in class or a student discusses a problem while looking for a method to solve it. Therefore, mathematics is not a quite different discourse from colloquial language. There is an overlapping between mathematics and everyday language. In my opinion, the central part of mathematics is formal mathematics, including definitions, theorems, proofs, and other mathematical constructions. However, to create such formal constructions, mathematicians use arguments in everyday language.

Knowledge, pragmatics, and semantics may affect an individual’s perception of a conditional’s meaning. This knowledge can also add information about temporal and other relations between premise and consequent. So, it may be natural to ask how people determine conditionals in different areas. These all directed me to assess the role of context in triangular relations in logic, specifically for conditional statements.

The above reasons are why I found conditional statements as a challenging topic for students and an interesting one to research in this program. This thesis investigates the potential issue resulting from different interpretations of a conditional statement and sees how people recognize a conditional statement. In doing so, two studies are reported in this thesis.

1.2. Plan of the thesis

This thesis is organized into several different parts:

- Background: Chapters 2, 3 and 4
- Literature: Chapter 5
- Methods and Theories: Chapters 6 and 7
- Empirical research: Chapters 8 and 9
- Discussion: Chapter 10

Chapter 2 introduces different interpretations of a conditional statement in colloquial language, philosophy, logic, and mathematics. It also discusses several models of conditionals that are mainly applied in philosophy, namely indicative, subjunctive, counterfactual, and causal conditionals. The last type of conditional statement discussed in Chapter 2 is the material conditional, the conditional used both in logic and mathematics. This chapter also discusses why mathematics needs material conditional on interpreting conditional statements and distinguishes predicates from conditionals in terms of “examples”, “non-examples”, and “counterexamples”.

Since the definition and semantics of the material conditional are aligned with logic, **Chapter 3** indicates the role of logic in the foundations of mathematics. This chapter explains the significant role of the material conditional in both informal and formal mathematics. It then classifies formal logics into classical and non-classical logics. The logic applied in mathematics is predicate logic, which is a classical logic.

Chapter 4 specifies the role of logic in mathematics. Mathematics is not only the formal construction such as theorems, definitions, and axioms; it also includes all mathematical discussions known as informal mathematics. This chapter discusses the role of conditional statements in mathematical constructions. Then it introduces different types of reasoning models: inductive, deductive, and abductive reasoning, and discusses the role of logic in each of them.

Chapter 5 examines the previous research studies on conditional statements. The topic of conditional statements has been the focus of many studies, including psychology, philosophy, physics, and computer science. This chapter begins by situating this work in mathematics education. It briefly reviews the role of logic in mathematics education and in other sciences and addresses the difficulties people may encounter when dealing with conditional statements. It then discusses how people do not reason based on general rules of inference. Chapter 5 also distinguishes two different types of logical meaning: formal meaning of symbols in a logical context that could be the truth values of a statement (semantics) and interpretation of logical statements in its content (context). It explains that any statement has three main components, namely syntax, semantics, and context.

Chapter 6 reviews some methods to create data focusing on different types of clinical interviews: structured, semi-structured and non-structured clinical interviews. The chapter then introduces some types of questions used in clinical interviews, including performance questions, unexpected “Why” questions, twist questions, give-an-example tasks, and reflection questions. It also explains Toulmin's argumentation scheme as a method to prepare data for analysis by highlighting features of the data using a visualization technique. I was not sure in which chapter to include this model, but since, in the current work, this model is solely applied as a mediator between collected data and data analysis to organize the gathered information, I decided to include this model in this chapter. At the end of the chapter, Toulmin's scheme is extended by adding some features to adjust the model for the current work.

Chapter 7 briefly discusses some popular theories of reasoning working specifically with conditional statements and their deficiencies. These theories include the dual process theory of reasoning, mental logic, mental model theory, theory of formal discipline, and information gain model. This chapter provides details for the choices that guided my investigation: mental model theory.

Chapter 8 reports the first study. It investigates how a mathematician determines a conditional statement and investigates the reasoning behavior of a successful mathematician aware of inference rules. In this study, the participants have a good background in mathematics and at least basic knowledge in first-order logic. They are provided with a set of statements and are asked to determine the conditional statements and, in particular, to address the source of possible confusion, conditionals are provided in three different contents, namely mathematics, logic, and colloquial language. Toulmin's scheme is extended and used to prepare a few parts of data for analysis and to organize the argumentation. This chapter reports a case study, and the model to capture the aspects of this certain argumentation is mental model theory.

Chapter 9 reports the second study. The purpose of this chapter is to investigate how people identify conditional statements. Are there any keywords guiding them to recognize conditionals? Are they comfortable determining conditionals in the context of mathematics? I investigate data from elementary perspective teachers' responses for conditional statements. They were given a questionnaire including different tasks about conditional statements that were mainly in two contexts: mathematics and everyday language. The theory applied to explain the created data is the theory of mental models.

The last chapter, **Chapter 10,** begins with a brief review of the two studies. Then it draws conclusions based upon the results and analysis from Chapters 8 and 9; for example, participants could not recognize conditionals with unrelated statements. The chapter explains why mathematics uses material conditionals and why unrelatedness and unsoundness cannot be excluded from the features of a conditional in mathematics. Then, it lists the contributions of this thesis to methods, theory, and mathematics education.

Chapter 2. What Is a Conditional?

This chapter briefly explains different interpretations of a conditional statement in colloquial language, philosophy, logic, and mathematics. Then I introduce the relationships between "predicates" and "examples, non-examples, and counterexamples" in mathematics.

2.1. Conditional statements

Suppose we google for the different types of conditional statements. In that case, there are lots of pages categorizing conditionals in different contents of computer programming (C programming, Python, JavaScript, …), the grammar of the English language, logic, mathematics and philosophy, though the main part is almost the same in different branches. This section summarizes different models of implications that are mainly applied in mathematics, psychology, and philosophy. Note that the word 'conditional' and the term 'conditional statement' are used interchangeably in this work.

A statement is an assertion that can either be true or false; therefore, a statement is something that has a truth value. We typically use letters to denote such statements, like p and q. A compound statement is a proposition formed by joining other statements using logical connectives. In what follows, some of such connectives are defined.

Let p and q be two statements. The *conjunction* of the two statements, denoted by $p \wedge q$, read as "p and q", is true when p and q are both true. The *disjunction* of two statements is denoted by $p \vee q$, read as "p or q", which is true when either p is true, or q is true, or both are true; this is the inclusive sense of the word "or".

Another type of compound statement is the *implication*, denoted $p \rightarrow q$. It can be read as "p implies q", "if p, then q" or simply "q, if p", and it asserts that if p is true, then q must also be true; otherwise, the implication is false (when p is true, but $q$is false). It is a conventional agreement in mathematics and philosophy that the implication $p \rightarrow q$ is true when p is false. The statement p is called the *antecedent*, *hypothesis*, or the *premise* of the implication, and the statement q is called the *consequent* or the *conclusion* of the implication.

2.1.1. If – then form of a conditional

The standard form of a conditional statement used in different branches, as well as in colloquial language, is the if-then form, “if p, then q”, when p (what follows the “if” part) is the antecedent or assumption, and q (what follows the “then” part) is the consequent or the conclusion.

Since the sentence “If p, then q” means something very different from the sentence “If q, then p”, it is essential to identify the antecedent and consequent of a conditional correctly. However, recognizing the antecedent and the conclusion might not be very straightforward because there are several *non-standard* ways to express conditional statements. To be more detailed, Table 2.1 represents some of such forms.

Table 2.1. Some of the ways to express a conditional statement

If p, then q	If not p, then q
p implies q. p only if q. q, if p. q, assuming that p. q, given that p. Provided that p, q.	q unless p. Unless p, q. Until p, q Without p, q. q, else p.

2.1.2. Implications versus conditionals

There is a framework for grammar that is appropriate in understanding logical statements. Quine (1953) makes much of the grammatical distinction between “if-then” (a connective) and “implies” or its synonym “entails”. To distinguish a conditional from an implication, consider the following statements:

(1) If you cheat in an exam, then you will fail the course,

(2) “cheating in an exam” implies “failing the course”,

(3) That you cheat on the exam implies that you fail the course.

The tradition (Quine, 1953) states that (1) is called a conditional and that (2) and (3) are called implications, and Quine clarified that not every conditional statement is an implication. To clarify the distinction, consider Austin’s (1961) example below:

There are biscuits on the sideboard if you want some.

This statement sounds very odd indeed when phrased as an implication (… implied that …). Anderson and Belnap (1975) state that:

> every use of “implies” or “entails” as a connective can be replaced by a suitable “if-then” form of the statement; however, the converse may not be true. (p. 491)

In the current study, there is no significant difference between the two terms: conditional and implication, and I will specify cases in which it is needed to distinguish a conditional with an if-then form from an implication.

There is no standard system to classify conditional statements. Linguists/grammarians have proposed different typologies based on verb tense, level of reality, and conditional construction. However, this thesis does not focus on different classifications for conditionals, so only a few classifications from the English language, and philosophy are discussed in this section.

In the next three sections, I briefly review the different types of conditional statements that are mainly used in the English language, philosophy, psychology, and mathematics. Even though there is a subtle distinction between an implication and a conditional statement, it is not important for this study, and the words are used interchangeably.

2.2. Various kinds of conditionals in the English language

Depending on the language, the classifications could be different: in the English language, the conditional statement is distinguished by the grammar of the sentence, while in Greek, one typology is based on the mood of the verb in the premise.

Like other languages, a conditional statement in English, also known as “Conditional Clauses” or “If Clauses”, consists of two clauses: a condition clause specifying a condition/s or hypothesis; and a consequence clause specifying what follows from the condition/s.

In the English language, we often have five types of conditional sentences based on the grammar and verb tense of the statement, namely *zero conditional*, *first conditional*, *second conditional*, *third conditional*, and *mixed* conditional. It is important to review this classification because this classification may affect people's understanding of a conditional in an informal discussion about a mathematics problem. The definitions provided below are the ones that most people learn first. This classification of conditionals is based on the grammar of the sentence.

Zero conditional

This type of conditional sentence is usually used to describe scientific facts or currently accepted truths. The tense for both parts of a zero conditional is the simple present. For example, *if ice is heated, then it turns into water*.

First conditional

The first conditional refers to predictive conditional sentences, that is, those that concern consequences of a probable future. In these sentences, the *if* clause is in the simple present, and the consequence clause is in the simple future. For example, *if Mary gets the job, she will move to Toronto*.

Second conditional

The second conditional, also known as a hypothetical conditional statement, is used to describe hypothetical or counterfactual situations with a present or future time frame. In such conditional sentences, the if clause is in the past tense, and the consequence clause uses the <u>conditional</u> construction with an auxiliary verb. For example, *if I were an animal, I would be a lioness*.

Third conditional

This type of conditional refers to a time **in the past** or a situation **opposite to reality.** It describes hypothetical situations, typically counterfactual. It is an unreal past condition, and its probable result is in the past. In this type, the tense for the premises is in the past perfect, and the consequence is expressed using the conditional perfect. For example, if Kaela had studied harder for the exam, then she might have passed.

Mixed conditional

As the name suggests, a mixed conditional is a mixture of the second and third conditionals. In this type, either the premise or the consequence has a time-past reference. The premise can be an unreal condition, and the consequence can be its possible result in the present. For example, *if I had not missed my plane, I would not be in this situation*.

Looking at the structure of each type and the truth-value definition of a conditional statement, it is obvious that the second, third, and mixed conditionals are all technically true because, in all three cases, the assumption is false, meaning such conditionals are true statements.

2.3. Various kinds of conditionals in philosophy

This section discusses several models of conditionals that are mainly applied in philosophy, namely indicative, subjunctive, counterfactual, and causal conditionals. In some cases, comparisons are made between them.

2.3.1. Indicative conditional

English conditionals discussed in the previous section can be distinguished into two basic classes. The first class includes zero and first conditionals, while the other one includes second, third, and mixed conditionals. So, the second class has past tense in the antecedent and a modal auxiliary (see Glossary) in the consequent. Such difference between the two classes of conditionals results in different uses of such statements. Indicative conditionals are typically used for stating facts, while subjunctive conditionals state possibilities, such as hypothetical and counterfactual things.

In philosophical studies, we frequently see these two types of conditionals, namely indicative and subjective conditionals. A conditional is called indicative if it can happen, and often its antecedent is an indicative mood. For example, if Sara is home, then she can open the door.

2.3.2. Subjunctive conditional

A subjunctive conditional is one having an antecedent in the subjunctive mood, such as, *if I had studied, then I would have passed the exam*. To distinguish the two types, consider Ernest Adams' (1970) statements:

(1) If Oswald did not shoot Kennedy, someone else did. *Indicative conditional*

(2) If Oswald had not shot Kennedy, someone else would have. *Subjunctive conditional*

The first conditional suggests that it is possible that Oswald did not kill Kennedy. This assertion could be judged differently as true or false by the readers, depending on their information about the premise. On the other hand, the second assertion assumes that Oswald did kill Kennedy and makes the dubious claim that Kennedy's assassination was inevitable; for more details, see von Fintel (2012). Indicative conditionals are truth-functional; however, subjunctive conditionals are not.

At the beginning of "On the Plurality of Worlds", Lewis (1986) defines our world as "I and all my surroundings", and possible worlds are the ways things could be. For example, the statement "it is possible for me to be a mentalist" is a true statement in one or more possible worlds. He is also a pioneer of modal realism and believes that modal operators can be regarded as operators over possible worlds.

Another way to explain a subjunctive conditional is within possible worlds. In this typology, there is some dependency between the content of the premise and the content of the conclusion. This type of conditional originally did not discuss statements like "if you ask me, she never comes back", whether she comes back or not, cannot be a part of the situation. However, now by the possible world, it means all possible situations, even those imagined worlds that are different from the real world.

It is still controversial among philosophers whether the two main kinds of conditionals, namely indicatives and subjunctives, are semantically unified. For example, Stalnaker (1978) and Edgington (1995) claim that there is no significant difference between these two types; they argue that the differences between the two kinds of conditionals should be explained within one theory that captures the behaviour of both types of conditionals. For instance, Stalnaker explains both types in terms of possible worlds. However, those who assume a strict distinction between the two classes, including Lewis (1973) and Bennett (2003), argue that indicative and subjunctive conditionals should have fundamentally different analyses.

2.3.3. Counterfactual conditional

A counterfactual conditional is one whose antecedent is false and whose consequent describes how the world would have been had the antecedent occurred, and this term comes from the idea that the premise can be false, that it is somehow counter to fact. For example, *if I were an animal, I would be a lioness*.

The fact that counterfactual conditionals follow the grammar of subjunctive conditionals may suggest that ‘counterfactual conditional’ is another label for the “would” type of conditionals. True that counterfactual conditionals are sometimes just subjunctive conditionals, but subjunctive is determined based on the grammar of the statement, and counterfactual is determined in terms of the degree of reality of the antecedent.

Most of the time, the subjunctive mood is used to discuss something that is uncertain or is hypothetical, and the “would” is used to express counterfactual possibility. However, not all subjunctives are counterfactual, because they do not always presuppose that the antecedent is false. For example, Anderson (1951) states that counterfactual conditional is a subjunctive conditional with a denial of the premise. He distinguishes counterfactual from subjunctive in four following ways:

> (i) subjunctive conditionals do not entail the denial of their antecedents. (ii) To every subjunctive conditional sentence, there corresponds a unique counter-factual sentence, and conversely; and further (iii) the class of counterfactual conditionals and the class of subjunctive conditionals are mutually exclusive. Finally, (iv) an adequate logical analysis of subjunctive conditionals (or of some subclass of the subjunctive conditionals) will yield as a trivial consequence an adequate analysis of counterfactual conditionals (or some subclass of the counterfactual conditionals). (p. 38)

2.3.4. Causal/temporal conditional

In a causal conditional, the connection between antecedent and consequent is discovered empirically, perhaps based on some experience in the real world, or a possible or hypothetical world that is the world that all hypothetical situations can occur for a non-existing object like unicorns. For example, if I mix yellow and blue-colored water together, then I get green. So, there is a causal relationship if the occurrence of the one statement, namely the premise, causes the other. In this sense, the first event is called the cause, and the second one is called the effect.

An implication is said to be more general than causality, because any causality can be regarded as conditional, but not vice versa. For example, being a sheep implies being a mammal, but it does not cause it. So, this is an implication that is not causal. In other words, the implication is a relation to statements, and causality is a relation among facts in a world.

There is also a close connection between counterfactuals and causations; however, counterfactual conditionals speak about possible but not true cases. For example, when we say, “if it were to rain, then I would open my umbrella”, this is regarded as a casual conditional. Also, it may mean that it does not rain, and in such a case, the antecedent would be false; that is a counterfactual conditional.

Frosch and Byrne (2012) discuss the possibilities people encounter when they understand a causal counterfactual, for example, “if I had been at home yesterday, I would have opened the door for Carl”. Their study suggests that the causal counterfactual appears to convey something different from its conditional counterpart: that is, in this case, if I were at home, then I would have opened the door (but I was not home, so I could not open the door).

2.4. Material conditional

A **material conditional** is a conditional statement that describes a relationship between the antecedent/premise and the consequent. This relationship is denoted by the symbol “ $\Longrightarrow$”. In this type of implication, there could be no real connection between antecedent and consequent. For example, if I learn philosophy, then more white roses will grow on earth.

Among all different types of implications, the material conditional is well-suited for mathematics. This type of implication is used in the classical, two-valued, truth-functional logic. In logic, a conditional statement is always true except when a true hypothesis leads to a false conclusion. Table 2.2 defines a conditional.

Table 2.2. Definition of the Material Conditional

p	q	$p \Rightarrow q$
True	**False**	**False**
True	**True**	**True**
False	**True**	**True**
False	**False**	**True**

Consider the statement "If 7 is prime, then 1007 is prime". Direct computation can be used to show that 1007 is a prime number. Therefore, this statement is true using a material conception of implication. However, if this statement is assumed to be a material conditional, then the value of the antecedent does not affect the truth value of the statement because, in the truth value table, a conditional is false if and only if the antecedent is true and the conclusion is false. So, the statement is true when the conclusion is true. For example, consider the below statement

If 6 is prime, then 1007 is prime.

Even though the number 6 is not a prime, the statement is true because the conclusion is true. This explanation may not seem very clear, but this is the definition of the material conditional in logic and the one we use in mathematics.

I emphasize that all kinds of implications except materials do not need any truth value operation/function. This is because, based on the nature of such implications, if the antecedent happens then without any argument, it entails the consequent, therefore they are always "true". For example, consider the following statement:

if it is raining, then the streets are wet.

If we regard this statement as a casual implication, then there is nothing to discuss on the value of the statement, as it is always true. If it rains, then we expect that the street is wet, so the condition is satisfied. If it is not raining, then again, the conditional itself is true because the antecedent is false, and this makes sense because the antecedent did not happen. Now, consider this statement as a material conditional, where the truth-value table determines the value of conditionals. In this instance, the truth-value table appears in Table 2.3:

Table 2.3. Truth value table for "if it is raining then the streets are wet"

It is raining	Streets are wet	If it is raining, then the streets are wet
True	False	False
True	True	True
False	True	True
False	False	True

When we look at the statement as a conditional in logic, we also consider the case in which it rains, but the street is not wet, notice that this could happen in a possible world, for example, if all streets have a roof, or all streets are under the ground. So, the material conditional considers all possibilities, even those that have not happened yet, or those that slipped from our eyes.

Often, when there is a relation between the two parts of a conditional statement, the material conditional can more precisely analyze the situation. However, there are many examples of statements with no relation between antecedent and consequent. For example, consider "if it is raining in Vancouver, then Pixi is a dog". Since there is no obvious causal connection between the weather in Vancouver and the name of my elderly dog, then the conclusion may not result from the antecedent. However, this statement can be regarded as a material conditional. Then, using the truth-value table for a material conditional, it is true irrespective of whether it is raining in Vancouver.

Through this work, I mean material conditional that is more appropriate for this study by a conditional. In the next sub-section, I give more details regarding this conditional type as a logical statement.

2.4.1. Why mathematics uses material conditionals?

Among all logical constants, conditionals are the most researched topic. Chapter 5 presents only a part of research studies addressing serious problems students experience when dealing with conditionals, like the issues with irrelevant antecedents and consequents and true conditionals with false antecedents.

The only type of conditional statement used in mathematics is material conditional. It has a context-free rigid definition and is only based on a logical relationship between two statements known as antecedent and consequent, and as long as it follows the correct syntax, it would be accepted as a material conditional. For example,

If the sun is blue, then my dog is a bird.

At first glance, the above statement may seem odd, but let me rewrite this statement in one of its equivalent forms:

If my dog is not a bird, then the sun is not blue.

This statement that is the contrapositive of “If the sun is blue, then my dog is a bird” may now make a little more sense. The reason can be that the statement “if the sun is blue” is a false statement, however the statement “my dog is not a bird” is a true statement, and people are usually more sensitive to invalid statements than to valid ones.

Now let me look at a material conditional in a different way. The following table reiterates the semantics or meaning of a material conditional defined in Table 2.2:

Table 2.4. Truth values for a conditional statement

p	q	$p \Longrightarrow q$
True	False	False
True	True	True
False	True	True
False	False	True

As the table suggests, the only possibility that $p \Longrightarrow q$ is false is when p is true, and q is false. In all other three cases, $p \Longrightarrow q$ is true. In other words,

$p \Longrightarrow q$ is true when q is at least as true as p.

These relationships between the truth values lead us to some ordering among the antecedent and the consequent: $p \Longrightarrow q$ is true means that “q is not less true than p”. This property of ordering between antecedent and consequent is the one we need specifically in sound and valid reasoning. This is because, in a valid argument, we do not want to reach a false conclusion based on a true assumption. Even though the statement if the sun is blue then my dog is a bird seems unsound, the assertion my dog is a bird is no less true than the sun is blue, because both are false.

The table indicates that if p is false, then the conditional $p \Rightarrow q$ is true, no matter what the value of q is. Also, an explosion in logic is when a contradiction results in any conclusion, that is:

$$p \wedge \sim p \Rightarrow q$$

This statement is true because the antecedent is false, and this conditional would be true using the truth value table of conditionals. There are many research studies addressing people's difficulties in accepting this rule, e.g. Sharpe & Lacroix (1999). However, that principle only specifies that if we have a contradiction in a system – meaning that for some p, both p and $\sim p$ are valid – then you can conclude anything. In a consistent system, $p \wedge \sim p$ is not true. The explosion principle states that, if there is any inconsistency in a system, like both p and the $\sim p$ are valid, that system would be trivial, meaning that any conclusion would be true within this system.

2.4.2. Statements involving a variable

In the introductory part of §2.1, I introduced a statement as an assertion with a truth value. In mathematics, understanding statements, also known as propositions, may not be very straightforward. Let me begin with the equation $x + 1 = 3$. Like any other equation, this equation is not a statement and cannot have any truth values because x is a variable and only acts as a placeholder for numbers from a certain set of numbers. The equation $x + 1 = 3$ is known as a predicate in logic used in mathematics, i.e., predicate logic (see §§3.2.4).

A predicate does not have any truth values *per se,* because the truth values depend on the value that the variable takes, and not only in equations. Assuming that the domain of the variable x in the equation $x + 1 = 3$ is all real numbers, the truth set (see §§3.2.4) is the same as the solution set for the equation, that is, $\{2\}$. For any real value $x \neq 2$, this conditional statement is false. Table 2.5 shows the truth values for the predicate $x + 1 = 3$.

Table 2.5. Truth values for the predicate $x + 1 = 3$

	$x + 1 = 3$
$x = 2$	**T**
$x \neq 2$	**F**

$x = 2$ is an example for which the equation $x + 1 = 3$ is satisfied, and for all $x \neq 2$ the equation, or the rule, is violated. To focus on the role of a variable's domain, I examine the predicate:

for two whole numbers, a and b, if $a \leq b$ then $ax^2 \leq bx^2$. (1)

by comparing the truth values of the following sentences:

- For any real number x, if $a \leq b$, then $ax^2 \leq bx^2$. **True**
- For any complex number x, if $a \leq b$, then $ax^2 \leq bx^2$. **False**
- For some complex number x, if $a \leq b$, then $ax^2 \leq bx^2$. **True**
- if $a \leq b$, then $ax^2 \leq bx^2$. **No truth value**

The expression "if $a \leq b$ then $ax^2 \leq bx^2$ " is not a statement, so it cannot have any truth values because in the above statement, the domain for x values is not known. If we assume that x is a real number, then $x^2 \geq 0$ and the statement "if $a \leq b$ then $ax^2 \leq bx^2$" will be true. The domain is important to be specified, because if the domain is extended to complex numbers, then for some values for x, the statement "if $a \leq b$, then $ax^2 \leq bx^2$" would be false. So, sentence (1), without knowing the domain, cannot have any truth values, but once the domain is specified it turns to a predicate and will have truth values for each x in the domain.

Once we define the domain of the variable, we are putting up a condition. Besides the domain that is usually presupposed as the set of real numbers, there are quantifiers that could be ignored. Both the following statements are true:

For any real number x, if $a \leq b$, then $ax^2 \leq bx^2$.

For some complex number x, if $a \leq b$, then $ax^2 \leq bx^2$.

2.4.3. A statement or a predicate?

Not all conditional relations including variables have truth values. Let me open the discussion with an example. Consider the sentence:

(1) if *n* is an even number, then *n* is divisible by 4,

when the domain of n is all integers. Even though it is not stated, it may be presupposed that the conditional (1) is equivalent to the following statement:

(2) For each *n*: if *n* is an even number, then *n* is divisible by 4.

It is worth mentioning that sentence (1) is a predicate when the domain is all integers, but sentence (2) that begins with the universal quantifier is a false conditional statement. Table 2.6 shows the truth values of the statement (1).

Table 2.6. Truth values for "if n is an even number, then n is divisible by 4"

n	p	q	$p \rightarrow q$
Multiple of 4	T	T	T
Even & not divisible by 4	T	F	F
Odd	F	F	T

Like many other predicates (discussed in more detail in sub-section §§2.4.3), this sentence is sometimes true. In such cases, we use an existential quantifier to write sentence (1) as a true statement. For example, the following statement is a true conditional statement:

(3) For some *n*, if *n* is an even number, then *n* is divisible by 4.

Since "if *n* is an even number, then *n* is divisible by 4" is true for all multiples of 4, then statement (3) is a true conditional statement. But, statement (2), "for each *n*: if *n* is an even number, then *n* is divisible by 4", is false, because there is a counterexample, an even number that is not divisible by 4, for example, 6 or 10.

Therefore, the truth value of this statement is false if there is at least one counterexample that makes the premise true and the conclusion false. Without any quantifier, the statement (1) could be regarded as the predicate "if P(*n*) then Q(*n*)". The truth set (see §§3.2.4) of this predicate is all values in the domain of *n* that make the statement true. This set is the set of all examples that make both the premise and the conclusion true.

Therefore, the two following sentences are logically different – the first sentence is a conditional statement, while the second one is a conditional predicate:

For each *n*: if *n* is an even number, then *n* is divisible by 4.

If *n* is an even number, then *n* is divisible by 4.

The first statement is false because there are counterexamples to make it false. The second sentence is sometimes true, but not always true. This suggests that, depending on different values within the domain of the variable, there are three possibilities for truth values of a predicate:

(1) The sentence is “always true”;

(2) The sentence is “never true”;

(3) The sentence is “sometimes true”.

What I defined here is different from the truth values of a statement. The definitions of “always true”, “never true”, and “sometime true” are only used for predicates not statements. It may be worth to remind that, unlike statements, a predicate does not have a truth value. Therefore, when a predicate is sometimes true, it means that it is true for some values within the domain of the predicate’s variable, and is false for other values in the domain.

For example, the statement “if n is an odd number, then $n^2 + 1$ is an even number” is always true, no matter what the value of n is. The statement “if n is an odd number, then $n^2 + 1$ is an odd number” is never true. One may assume that an always true predicate, could be regarded as a true statement, and a never true predicate is a false statement, but it is important to notice that predicates do not have any truth values and the terms “true” and “false” are only used in their meanings in daily language not in logic.

For case (III), consider the statement “if $n^2 = 1$, then $n = 1$”. This statement is true for any real number except when $n = -1$. Table 2.7 Shows the truth values for this statement based on different values for n:

Table 2.7. Truth values for "if $n^2 = 1$, then $n = 1$"

	$n^2 = 1$	$n = 1$	$n^2 = 1 \Rightarrow n = 1$
$n = 1$	T	T	T
$n = -1$	T	F	F
$n \neq 1, -1$	F	F	T

Table 2.7 indicates that the conditional is not always true. Using the universal quantifier (or existential quantifier), this sentence can be converted to a false (or true) conditional statement.

Besides this, it is useful to discuss the number of possible values that make the statement true. Because if we know there is only a finite number of values making the statement false (counterexamples), we can exclude them from the domain of the variable and convert such statement to either "always true" or "never true" without adding any quantifier. For example, in the statement "if $n^2 = 1$, then $n = 1$", if I restate it as any of the following cases, then it will always be true:

for almost all integer numbers for n, if $n^2 = 1$, then $n = 1$.

for $n \epsilon N$, if $n^2 = 1$, then $n = 1$.

for $n \epsilon Z - \{-1\}$, if $n^2 = 1$, then $n = 1$.

Therefore, it is important to know "to what extent" a predicate is true or false. It is important because based upon it we can limit the domain and make the statement either "always true" or "never true", then give a proof to the restated statement and make it a theorem. This is a usual way to derive a theorem from a conjecture.

If the set of values making the statement false is nowhere dense in the domain of the variable, i.e. its closure has an empty interior, then we say that the claim is true for "almost all" values in the domain. For example, consider the statement:

(4) if $f(x) = \sqrt{\sin x}$, then, for all x in the domain of f, $f'(x)$ exists.

This statement is false for infinitely many values for x. To be more accurate, for any *x*-value from the set $\{0, \pm\pi, \pm 2\pi, \ldots\}$, the statement is false. Therefore, the conditional statement (4) is false. The following is a revised version of the above statement:

if $f(x) = \sqrt{\sin x}$, then, for almost all x in the domain of f, $f'(x)$ exists.

This modified version of the original statement (4) statement is now "a true conditional statement".

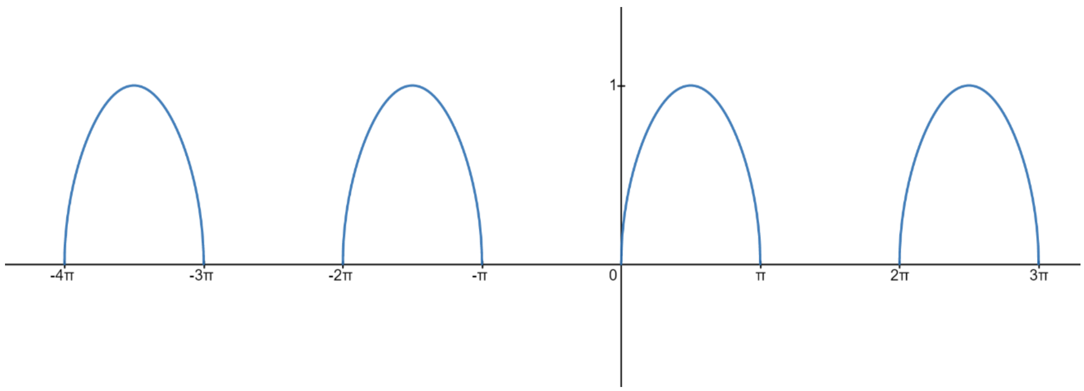

Figure 2.1. The graph of $f(x) = \sqrt{sin\ x}$

When a conditional statement is "always true", it needs mathematical proof, otherwise there is no guarantee that a counterexample does not exist. Similarly, a proof may support a conditional statement that is "never true". Case (3) is the only one that needs at least one example and one counterexample.

2.4.4. Truth tables, examples, non-examples, and counterexamples

In Table 2.6, the truth value when n is an odd number is true, but since the premise is false, this case is not relevant to the rule. Whatever is in this category, when the premise is false and the conclusion true, is defined as non-examples. In the following table, I categorize example, non-example, and counterexamples concerning the truth table 2.6:

Table 2.8. Truth table, Examples, Non-examples, and Counterexamples

$n \epsilon Z$	p	q	$p \rightarrow q$	
Multiple of 4	T	T	T	Example
Even & not divisible by 4	T	F	F	Counterexample
Odd	F	F	T	Non-example

For a conditional predicate "if P(*x*), then Q(*x*)", I suggest the following table to categorize examples, non-examples and counterexamples. It is defined based on the following facts:

- a counterexample is an example that makes the premise true but the conclusion false,
- a non-example is irrelevant to the rule/condition, meaning that with it the premise is false,
- an example satisfies the rule/condition and makes the conclusion true.

Table 2.9. Truth values for a conditional statement involving a variable

p	q	$p \rightarrow q$	
T	T	T	Example
T	F	F	Counterexample
F	T	T	Non-example
F	F	T	Non-example

The following examples show that not all three categories of example, non-example, and counterexample exist for any mathematical predicate or statement.

Table 2.10 specifies different possible categories of typical cases for the predicate $x + 1 = 3$. It does not have any non-examples, as there is no condition to be violated.

Table 2.10. Categories of examples

$x \in R$	p	q	$p \rightarrow q$	
$x = 2$	T	T	T	Example
$x \neq 2$	T	F	F	Counterexample

The conditional "if n is an odd number, then n^2 is an odd number" has the following table assuming the domain of n is all integers:

Table 2.11. Truth values for a conditional statement involving a variable

$n \in Z$	p	q	$p \rightarrow q$	
n is odd	T	T	T	Example
n is even	F	F	T	Non-example

The truth table for this statement only includes examples and non-examples. Since there is no counterexample for the statement "if n is an odd number, then n^2 is an odd number”, then it is always true. Such tables introduce theorems. If we extend the domain of n to all real numbers, then this extension of the domain only adds to the category of non-examples, while the category of examples will not change.

2.5. Summary of Chapter 2

In this chapter, I specified the meaning of a conditional statement in this work and that there is no significant difference between the two terms: conditional and implication in the current study. Also, I introduced several different types of conditional statements, mainly applied in the English language, philosophy, and logic/mathematics.

There are five types of conditionals in English language grammar that are mainly classified in terms of the verb tense of the statement. In philosophy, the main classes of conditionals are called indicative and subjunctive. This classification is based on the tense and mood of the verb in both antecedent and consequent. Another type of conditional is counterfactual implications that are essentially a type of subjunctive implications. I also went through the causal implications and discussed that this type could be regarded as a cause-and-effect relation. Finally, I introduced a conditional statement in the logic used in mathematics, namely material conditional. This type of conditional represents a certain relationship between two propositions, namely the antecedent and the consequence. Regardless of the meanings of the propositions, the truth values are used to specify the relationship between the two propositions, in formal mathematics a true hypothesis does not entail a false conclusion. In other words, the consequence is no less true than the antecedent.

In §§2.4.2, I discussed a conditional statement involving a variable. When there is a variable in a conditional statement, the truth values will depend on the variable's value. Therefore, any statement involving a variable is either “always true”, “never true”, or “sometimes true”. Then, I involved examples, non-examples, and counterexamples in the truth values of a predicate in mathematics.

The type of conditional statement that is regarded for the aim of this research is the material implication that is most fitted to mathematics. Since the definition and semantics of material conditional (see the beginning of §2.4) are along with logic, the next chapter looks at different types of logic and picks the one used in the current study.

Chapter 3. Logic and its Main Branches

In this study, the terms “formal logic”, “informal logic”, and “mathematical logic” are used frequently, and since there might be different interpretations, these notions are briefly explained in the first section. Before we can evaluate a given piece of reasoning, we need a way to represent that piece of reasoning or argument. In general, logic can be divided into Formal Logic, Informal Logic, Symbolic Logic, and/or mathematical Logic. In the following, I give a brief review of these main branches.

3.1. What does logic mean?

> *To discover truths is the task of all sciences; it falls to logic to discern the laws of truth, and logic, I assign to logic the task of discovering the laws of truth, not of assertion or thought. The meaning of the word “true “ is explained by the laws of truth.*
>
> *Gottlob Frege (1848-1925)- from the paper 1956, pp. 289–290.*

The term “logic” originally came from the Greek word *logos*, which is translated as word, sentence, reason, rule, and ratio. However, these translations do not provide any definition of logic as it is used today. The exact definition of logic is a controversial matter. However, roughly speaking, we can define logic as “*the study of the principles of correct reasoning”, and* the logician’s concern is with the validity of the relation of consequence between premises and conclusions. Understanding this helps to avoid making mistakes in reasoning, and it allows us to evaluate the reasoning of others. Hence, it can make us better thinkers. Therefore, logic is neither a theory of what conclusions people can draw from a set of assertions, nor a theory of how they do that. Rather, using formal rules of inference states which conclusions can be drawn from the premises in a formalized language.

The history of logic has roots back to Aristotle, and the first explicit work in formal logic could be where Aristotle regarded logic as the theory of syllogism and the study of argument from a view with the correctness of argumentation. At that time, his theory had had much influence, and for many years the study of logic was part of the classical trivium, which also included grammar and rhetoric. The initial work on logic was to produce correct laws of mathematical reasoning. To be more precise, the basic principles of logic are derived from the law of contradiction and the law of the excluded middle. The law of contradiction states that any statement cannot be both true and false, and the law of the excluded middle asserts that a statement must be either true or false. So, logic was originally derived from mathematics, which was then separated as a study independent of mathematics (Kline, 1972).

However, the logic we use and study these days – sometimes called “Modern Logic” or “Contemporary Logic” – was essentially developed in the 19^{th} and early 20^{th} centuries and is discussed in sub-section §§3.2.2.

3.1.1. Systems of formal logic

Today, logic is considered a special kind of a formal system. To be more precise, in any formal system, we start with a language, some set of axioms, and some inference rules. The language allows us to express propositions, which may or may not be true. The axioms comprise the propositions that are accepted as true without any proof, and the deduction rules provide us with a mechanism to combine true propositions to produce other true propositions. Hence, a logical system, or simply logic, is a formal system together with a form of semantics, which usually assigns truth values to sentences of the formal language.

3.1.2. Formal logic

Formal logic, also called deductive logic, is a branch of logic concerned with deductive reasoning and the forms of logical inference. These inference rules are defined in a way to formalize the content of propositions set of rules for making deductions, like Aristotelian syllogisms, and, therefore, formal logic is a logic where all statements are expressed using a special formal language, and reasoning in such logic is performed by using formal rules.

A syllogism is an argument with a specific form when two premises result in a conclusion that is usually represented in a three-line form. It constitutes three assertions and is based on the meaning of the assertions, and each is either false or true. For example,

All musicians are smart

Sara is a musician

Therefore, Sara is smart.

Any syllogism consists of a major premise, a minor premise, and a conclusion. A major premise usually makes a general rule, like "all musicians are smart", a minor premise gives an assertion about a particular thing/s or person/s like "Sara is a musician", and in conclusion, both major and minor premises are connected. Therefore, knowing that all musicians are smart (major premise) and that Sara is a musician (minor premise), we may validly conclude that Sara is smart.

Syllogism dates back to Aristotle; he defined a syllogism as a conditional statement with two premises and one conclusion that follow a set of rules, for example:

(1) If some As are B and every B is C, then some As are C, or

(2) If every A is B and some Bs are C, then some As are C.

Aristotle classified these syllogisms into two categories: those that lead from truth to truth, whatever be the interpretation of the terms A, B, and C as in (1); and those which may have true premises and a false conclusion in some interpretation as in (2), for instance, assume that A is the set of all birds, B is the set of all animals, and C is the set of all reptiles, then every bird is an animal, and some of the animals are reptiles, but since birds and reptiles are separate categories of animals, no bird is a reptile. Therefore, in this example, the premises of (2) are true, but the conclusion is false.

Rules of Inference

Logic is about validity, and systems of inference rules are one way of studying validity. In logic, a rule of inference is used to draw a conclusion from premises using their syntaxes. For example, the rule of inference called *modus ponens* takes two premises, one in the form "if p, then q" and another in the form "p", and returns the conclusion "q". There are four popular rules of inference related to using a material conditional, two legitimate and two fallacious.

The two consistent logical argument constructions are *modus ponens* (the way that affirms by affirming) and *modus tollens* (the way that denies by denying):

Modus ponens: "If A is true, then B is true. A is true. Therefore, B is true."

Modus tollens: "If A is true, then B is true. B is not true. Therefore, A is not true."

Modus tollens is an application of the general truth that if a statement is true, then so is its contrapositive. So, the form of *modus tollens* is: "If p, then q. Not q, therefore not p."

The two incorrect constructions are affirming the consequent and denying the Antecedent.

Affirming the Consequent: "If A is true, then B is true. B is true; therefore, A is true."

Denying the Antecedent: "If A is true, then B is true. A is not true. Therefore, B is not true."

3.1.3. Informal logic

Informal logic, also known as critical thinking, shares a premise and conclusion conception of an argument. In contrast to formal logic, it usually operates by an everyday spoken language and is studying natural language[1] arguments. Attempts to develop a logic to analyze and improve everyday reasoning mainly emerged in the last half-century when many philosophers and logicians paid more attention to analyzing, evaluating, and improving daily-life arguments.

Informal logic is usually known as inductive reasoning, a reasoning based on informal, inductive logic that moves from premises to a conclusion that generalizes the premises. An important branch of informal logic is the study of fallacies. The dialogues of Plato are some examples of informal logic. For example, "All humans are mortal. Socrates is a human. Therefore, Socrates is mortal". Here, the reasoning relies on our understanding of such words like "all, people, human, therefore". Accordingly, "formal logic has to do with the forms of argument (the syntax of the language) and truth values (semantics of the language), while informal logic has to do with the use of argumentation in a context of dialogue.

1 Natural language here means a language that is spoken, written or signed by humans for general-purpose communication, as distinguished from formal languages (such as computer-programming languages) or constructed languages (such as Esperanto).

3.2. Classical logic

With formal logic, there could be several ways to validate an inference. Classical logics use truth functionals as the guiding rules. A traditional introduction to logic only covers classical logic, and it has been classified at several levels, including propositional, predicate, modal.

Classical logic, developed by Frege, Russell, and others, is originally the reconciliation of Aristotelian logic, and in most of the last two millennia, it has been the dominant paradigm of logic. In classical logic, the guiding principle for semantics is that the connectives in the inference are truth-functional tables of values. This means that, for example, whether or not $p \wedge q$ is true depends only on whether or not p and q are true. Specifically, $p \wedge q$ is true means that both p and q are true. In this sense, the study of classical logic uses a truth table that summarizes the possibilities of the statements to determine whether they are true or false.

Therefore, classical logics are those with truth-functional values, and that the "excluded middle principle" is held in any classical logic, it states that $p \vee \sim p$ holds, i.e., is always true, no matter if p holds or not. Using the truth-table method, we can replace statements or predictions with letters and symbols, which is called symbolic logic.

It is worth to note any rule of inference can be explained through classical logic. For example, *modus ponens* is also valid concerning the semantics of classical logic and the semantics of many other non-classical logics, in the sense that if the premises are true, then the conclusion will be true.

Any classical logic employs only two truth values, namely true denoted by T, and false denoted by F; however, there could be the third truth value "undefined", denoted by U, and maybe the fourth one "contradictory", denoted by C. So, for such logics with more than two truth values, the classical type of logic does not work.

Although classical logic is adequate for mathematics needs, it cannot capture most of the "implications" in everyday non-mathematical language; to be more specific, it disregards causality, relevance, and other ingredients of everyday reasoning. It is also worth mentioning that ordinary human thought is a mixture of many logics, not just classical logic; it could be a mixture of different formal logic systems, both classical and non-classical.

3.2.1. Main branches of classical logic

Logic aims to develop a theory of valid reasoning. It is the study of correct reasoning, not a study of how this reasoning originates or its effects in persuading people, but it is a study of what makes some reasoning "correct", as opposed to "incorrect". Classical logic aims to capture such validity by a truth-functional table of values. There are different deductive systems within this category, such as propositional and predicate logic; the latter is also known as first-order logic and is an extension of propositional logic.

Some other branches of classical logic are not explained in this work. For example, in 1934, Hilbert and Bernays extended predicate logic by adding a function, namely epsilon ε. This function takes a formula $\varphi(x)$ and produces an expression $\varepsilon x. \varphi(x)$, that is an arbitrary element satisfying the formula $\varphi(x)$. This extension of predicate logic is called "epsilon calculus". Some classical logics are extensions of predicate logic by adding some types of quantifiers.

In what follows, I briefly introduce propositional and predicate logic, but I first need to explain what symbolic logic is.

3.2.2. Symbolic logic

Symbolic logic is the study of symbolic abstractions that consider the formal features of logical inference and deals with the relations of symbols to each other. Following Aristotle and philosophers from other ancient civilizations, logic was further developed and systematized by the medieval scholastic philosophers. However, in the late 19th and 20th centuries, it experienced substantial growth, which has continued until the present.

Symbolic logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic, and it relies on the works of George Boole and Friedrich Ludwig Gottlob Frege. However, Frege's work in logic had little international attention until 1903, when Bertrand Russell began to promote it and wrote an appendix to *The Principles of Mathematics*.

Before this emergence, logic was studied with rhetoric and with calculations through syllogism. Modern logic begins with Boole's (first published in (1847) and reprinted in (1951)) The Mathematical Analysis of Logic, when the algebra of logic was developed so that classical logic syllogisms were proven as algebraic equations, and the turn from the logic of classes to propositional logic was suggested. Hence, mathematical logic formalizes deductions like a syllogism.

Although the term "Mathematical logic" is a different branch from Symbolic logic, it is an extension of symbolic logic into other areas, but in many texts, they are considered one thing, and they will be considered the same in this current study.

Whatever we call it, by using symbolical logic, many concepts of mathematics such as theorems, deductions, axioms, sets, functions, graphs, and representations can be defined rigorously.

3.2.3. Propositional logic

Symbolic logic is often divided into two main sub-branches, propositional logic and predicate logic (also known as first-order logic). Predicate logic is used in the current work to define and evaluate conditional statements. Even though first-order logic is not the most suitable type of logic with human reasoning, this type of logic is the most popular one. This sub-section focuses on propositional and predicate logic, and the next sub-section introduces modal logic and relevant logic, logics that are seemingly more fitted with the human way of reasoning.

Also known as zeroth-order logic or sentential logic, as it is clear from the name, propositional logic is the study of propositions, where a proposition is a statement that is either true or false, such as "Tehran is the capital of Iran", "3 + 6 = 9", or "It rains".

Elementary propositions, also called atomic, are usually denoted by letters, such as lower-case p and q. These propositions are connected and manipulated using simple logical connectives, such as:

$\sim$ for not,

$\wedge$ for and,

∨ for or,

⇒ for if, then,

⇔ for if and only if.

Using these symbols, we can combine atomic propositions by using logical connections between them. For example, we can generate a more complex proposition, "If Tehran is the capital of Iran, then it rains", which can be denoted by $p \Rightarrow q$, where p is the proposition "Tehran is the capital of Iran," and q is "it rains". We then use truth tables to evaluate the truth of this proposition.

Therefore, propositional logic is a system in which propositions can be formed by combining atomic propositions using logical connectives, and a system of formal proof rules allows certain forms to be considered as theorems.

3.2.4. Predicate logic

Propositional logic cannot capture the statement like "if $-1 < x < 1$, then $x^2 < x$" to formalize such statements involving variables and expressed using "all", "some" or "every": it is needed to add a concept and quantifiers to propositional logic.

Predicate logic, also known as first-order logic, is an extension of propositional logic by adding the concept of predicate and quantifiers. A "predicate" is a statement including one or more variables with/without quantifiers whose truth value depends on the variables specified in the statement, where the quantifiers explain to what extent a predicate is true over a range of elements. Two main types of quantifiers are universal and existential quantifiers and are denoted by the two symbols:

Universal quantifier ∀, meaning all, every

Existential quantifier ∃, meaning some, or there exists.

To compare propositional and predicate logic, consider the following example: $3 + 6 = 9$ is a statement in propositional logic, and $x + 6 = 9$ is a **predicate** whose value depends on the value of x. A predicate including a variable x is denoted by $\mathrm{P}(x)$ that is either true or false depending on different values for x, and the domain of a predicate variable is the set of all values that may be substituted in place of the variable. For a predicate $P(x)$, the **truth set** is the set of all elements in the domain of x that make

$P(x)$ true. So, the propositional logic allows us to discuss such statements, including quantifiable variables.

Even though, unlike propositional logic, predicate logic can deal with sets of entities and make them useful in mathematics, we cannot employ truth tables for a variable, because the truth values depend on the value that the variable takes.

Material conditional statements in predicate logic

The term “material implication” has been introduced in §2.4; this expression is named by Russell to refer to a special class of formulas, i.e., those of the form:

$$\sim p \vee q$$

Russel further introduced the connective “⟹” so that

$p \Longrightarrow q$ is defined as $\sim p \vee q$

So, the symbolic form of “material implication” is a way of expressing the disjunction without using the negation sign, allowing the move from the truth of p to the truth of q using the symbol $p \Longrightarrow q$.

Material implication is a relation between the truth values of two propositions, namely antecedent and conclusion. Table 3.1 summarizes the truth value of symbol $p \Longrightarrow q$; this is the same as Table 2.2 in Chapter 2.

Table 3.1 Truth values for conditionals in predicate logic

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \Longrightarrow q}$
True	False	False
True	True	True
False	True	True
False	False	True

The truth value of $p \Longrightarrow q$ is content-independent and is not based on the logical relation between the meanings of the propositions, so they could be unrelated in terms of their contents. However, the process of an inference can be based on the content. For example, consider the disjunctive syllogism in its abstract form,

premise 1: either A is true, or B is true

premise 2: A is not true

conclusion: B is true.

This disjunction is valid for both interpretations for “or”; namely inclusive and exclusive. Even though the above argument is equivalent to $(p \vee q) \wedge \sim p \implies q$, this may be expected to be dependent on the content. In an inference, the main attention is on the resultant part, that is B, while in a material implication, the truth value of the conditional depends on truth values of the premise and conclusion.

3.3. Non-classical logic

Many of the developments in logic in the last fifty years have mainly occurred in areas like conditional logic, relevant logic, free logic, quantum logic, fuzzy logic. Such logics are intended either to augment classical logic or to replace it where it goes wrong. These logics are known as non-classical logics, and most of them are based on propositional logic, as the main expectation from a non-classical logic is typically at the propositional level.

Logical quantifiers may resemble probability in a sense. They both assume a variable and its domain and put restrictions on some predicate of that variable over that domain. The relation between logic and probability is not a new idea. In 1973, Lotfi Zadeh proposed his theory of fuzzy logic closely related to probability theory, and the key difference is their meaning. Also, the term “probabilistic logic”, coined by Nilsson (1986), proposes that the truth values of sentences are probabilities.

The semantic technique in most non-classical logics is possible-world semantics. However, this does not mean that there is no non-classical logic based on first-order logic. For example, free logic is a non-classical logic with the same interpretations for quantifiers and is based on predicate logic, but the objects could be out of the domain of argument; in other words, free logic removes the existential import on the quantifier, meaning that in free logic, when it is said “something exists”, it could be out of the domain. In classical logic, when we say $\exists x,$ it means that there exists something, but free logic is called "free logic" because it is free from the requirement of a referent. Regarding this logic, Honderich (1995) states that:

> free logics do justice to the intuitive view that from the fact that the concept of something exists it does not follow that thing exists: from "A unicorn has a single horn" it does not follow that there are any unicorns. The intuitive view is not beyond question. (p. 316)

Even though classical logic seems appropriate for classical physics, it cannot describe some atomic events. For example, it is possible for a disjunction in atomic scale to be true even if neither of its components is true[2]. Quantum Logic was proposed by Birkhoff and Von Neumann in 1936 in a paper by defining some logical rules that could explain the events in quantum mechanics.

Quantum logic is not a classical logic as it violates some principal rules for all classical logics; the excluded middle principle and the distributive law, i.e., in this logic $p \wedge (q \vee r)$ is not always equal to $(p \wedge q) \vee (p \wedge r)$. This logic defines its own rules in a way that properly explains atomic and sub-atomic events.

Like Quantum logic that can capture quantum mechanics events, there might be a logic explaining arguments in everyday language, and that most of the everyday reasoning can be analyzed in this logic. To fulfill this aim, philosophical logicians have developed non-standard logics and different extensions of classical logic. In what follows, I briefly explain the two non-classical logics that are related to this study, namely modal logic and relevant logic.

3.3.1. Modal logic

Propositional logic and predicate logic work very well with mathematics, but ordinary language is ambiguous. The analysis of colloquial language and its arguments lacks some operators, as there are many sentences in ordinary language that cannot be expressed in

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[2] For example, in Quantum level, electrons, protons, and neutrons are all spin ½ particles which is in a linear combination of two states of *up* and *down*. While dealing such a state, both propositions, "the state is up" and "the state is down" have no certain truth value, this violates the excluded middle principle in classical logic, but the disjunction "the state is up or the state is down" is the same as of classical logic. Therefore, it is possible for a quantum disjunction to be true even if neither of its members is true.

classical logic: for example, "if it rains, you may get wet". In propositional logic, you can express the conditional "if it rains, then you will get wet" but cannot express the possibility of getting wet. By modal logic, we can eliminate some of these ambiguities.

Modal logic is an extension of classic propositional and predicate logic: the study of modal propositions and the logical relationships between them. So, with modal logic, we can express conditionals with modal operations such as: possibly, can, could, may, might, must, etc.; for example, "if you fall, you may get hurt".

Aristotle originally discussed this type of logic. He developed a modal syllogistic in Book I of his *Prior Analytics*[3] and noted that the following modal propositions are both true:

If it is necessary that if-p-then-q, then if p is possible, so is q.

If it is necessary that if-p-then-q, then if p is necessary, so is q.

The most well-known modal propositions are propositions about what is necessarily the case and what is possibly the case. For instance, the following are all examples of modal propositions:

It may rain tomorrow.

It is possible for humans to travel to Mars.

It is necessary that either I am at home now or I am not at home now.

Modal logic originated back to Aristotle; he developed the modal syllogisms containing "necessary" and "possible". In terms of both syntactically and semantically, the first development of modal logic was made by C.I. Lewis (1912). Since the 1970s, modal logic has developed through mathematics, philosophy, computer science, linguistics, political science, and economics and its scope is still expanding.

3.3.2. Relevance logic

Relevance logic, also known as relevant logic, was proposed by Ivan E. Orlov in 1928. It is a kind of non-classical logic requiring the premise and conclusion of conditional statements to be related.

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[3] Bobzien, Susanne. "Ancient Logic". In Zalta, Edward N. Stanford Encyclopaedia of Philosophy.

In classical logic, we have some paradoxes like those of material implication. There are conditional statements that are conditional by logic rules, but are not sound. For example, in classical logic, a false premise always makes an argument valid, equivalent to “any false premise results any conclusion“. Even though a statement with a false premise is a logically valid argument, it is not sound. For example, the statement “if the earth is flat, then today is Friday” is always true, whether today is Friday or not. Some logicians and philosophers, e.g., MacColl (1908), reason that such statements are unsound because the premise is irrelevant to the conclusion. Relevant logic fixes such paradoxes for material conditionals using the notion of relevance between premises and conclusions.

3.4. Summary of Chapter 3

Chapter 3 provided a brief explanation of formal and informal logic as sub-divisions of logic. Formal logic is known as deductive logic. The type of reasoning used in formal logic is deductive reasoning and is based on logical inference. In contrast to formal logic, informal logic is known as critical thinking and is concerned with inductive reasoning. Informal logic depends on the context and focuses on analyzing arguments in everyday language and the study of fallacies.

Different ways to validate an inference result in different types of formal logic, namely classical and non-classical logics. Classical logics follow some rules, for example, any proposition is either true or false, and $p \vee \sim p$ is always true, no matter if p is true or not. The two main classical logics are propositional logic and predicate logic. Propositional logic, known as zeroth-order logic, is the study of propositions. Predicate logic, known as first-order logic, is an extension of propositional logic by adding variables and quantifiers.

Any logic that violates at least one principal rule for classical logics is named a non-classical logic. There are many non-classical logics, such as quantum logic, that explain the events in quantum mechanics. This chapter briefly introduced two non-classical logics that can capture arguments in daily language, namely modal logic and relevance logic. The research conducted in this dissertation is based on a predicate or first-order logic that is a type of classical logic.

The main purpose of chapter 3 is to draw the readers' attention to non-classical logics that could explain some events more adequately than classical logics. Chapter 4 introduces "informal mathematics" and discusses the role of logic in formal and informal mathematics.

Chapter 4. The Role of Logic in Mathematics

Even though conditional statements have a large variety of linguistic constructions to express the logical consequence relation between statements, in mathematics discourses, we have almost clearly stated conditionals. The purpose of this chapter is to signal such logical forms in both formal and informal mathematics.

In this chapter, before moving forward and discussing mathematical constructions of conditionals, I first address why conditionals are essential to mathematics. Then, I briefly go over the precise meaning of some of the most common words that appear in mathematics discourse. In particular, the following terms are defined: axiom, definition, conjecture, theorem, proof, lemma, and corollary. Then examples of each are presented, and the basic structure of conditionals in each construction is demonstrated:

4.1. The role of conditionals in formal mathematics

In formal mathematics, constructions of the form “If-then” are complemented by many other forms; however, some are expressed without cue words, which could be the reason for underestimating the role of conditionals. In the following, I discuss the role of logic in formal mathematics, and, in doing so, I first need to introduce what I mean by a logical form of a statement and the logical structure of main mathematical constructions. In §4.2, I turn to informal mathematics, describe what is intended by the phrase, and discuss the role of logic in different types of reasoning.

4.1.1. Logical form

The logical form of a sentence (also known as proposition or statement), or logical form of a set of sentences, is the form obtained by abstracting from the subject matter of its content: for example, the logical form of “if it rains, the steer will get wet” has the logical form of “if p, then q”. Each sentence in an argument has a particular logical form, and the argument is a pattern of such forms. The logical form of an argument is called the argument form. This translation from a natural language to the language of logic enables us to discuss the truth of the conclusion given the truth of the premises. More precisely, the logical form of a sentence of natural language determines both its logical properties and its logical relations to the other sentences.

To determine the form of an argument, we can substitute letters for similar items throughout the sentences in the original argument. For example, consider the below argument

All humans are mortal.

Socrates is a human.

Therefore, Socrates is mortal.

Then, the argument form is:

All *H* are *M*.

S is *H*.

Therefore, *S* is *M*.

As discussed in §§3.1.2, some examples of valid argument forms are *modus ponens* and *modus tollens*, and two instances of invalid argument forms are affirming the consequent and denying the antecedent.

4.1.2. Basic terminology

This sub-section introduces some essential mathematical terminology, namely mathematical definition, axiom, conjecture, lemma, theorem, corollary, and proof. The reason to select these terms is in the following quotation from Rossi (2006) on the first page of his textbook "Theorems, Corollaries, Lemmas, and Methods of Proof":

> Whereas the roots of mathematics are based on counting and the study of geometric shapes, modern mathematics is much more than just the study of numbers and shapes. In particular, modern mathematics is the science of operations on collections of arbitrary objects. Modern mathematics, or axiomatic mathematics, is developed according to the following structure:
>
> Axioms => definitions => conjectures => proofs => theorems => generalizations and extensions =>.

Even though Gödel's incompleteness theorems4 show the limitations of a formal axiomatic system, in each theory of modern mathematics, we follow the axiomatic

[4] Gödel showed that in a mathematical axiomatic system, there are some statements which can neither be proved nor disproved using the set of axioms. The original paper Gödel (1931) was in German and translated by Meltzer (1962) into English.

structure begins with a set of axioms and definitions. Then, from these axioms and definitions, new mathematics is created through deductive reasoning. For instance, in Euclidean geometry, the words "line" and "point" are primitive concepts. However, despite Euclid's careful work, he made assumptions that were neither stated as axioms nor previously proven to be true. In 1899, the mathematician David Hilbert revised the Euclidean axiomatic system by proposing three primitive concepts: point, straight line, plane. I begin this sub-section with an axiom, then a definition, because these are the building blocks of a mathematical theory. The mathematical constructions discussed in this section are in the order Rossi (2006) suggested, and the role of implication in each construction will be discussed.

Axiom

Euclid proposed a valuable model to construct an axiomatic system, and he was the first person known to apply the axiomatic method to study the field of geometry, which then this structure is applied in different areas of modern mathematics. The term 'axiom' that is used throughout the whole of mathematics can be defined as follows: "an axiom is a mathematical statement that is taken to be self-evidently true without proof." (Rossi, 2006, p. 4). Therefore, an axiom is a statement that is accepted as true for that particular branch, though we cannot freely set the axioms, and they must satisfy some conditions, i.e. the number of undefined terms and axioms has to be as few as possible. For instance, the following are three of the Zermelo-Fraenkel axioms that are the basis for Zermelo-Fraenkel set theory:

- The Axiom of Infinity: there exists an infinite set.
- Unordered Pair Axiom: for any *a* and *b*, there exists a set {*a*, *b*} that contains exactly *a* and *b*.
- Empty Set Axiom: there is a set with no members, written as { } or ∅.

Different parts of mathematics may have different sets of axioms. Nevertheless, this does not mean that we can never go beyond the axioms, parallel mathematical theories built on alternate sets of axioms. For example, non-Euclidian geometries are introduced by re-describing the properties of primitive concepts like points, lines, and other shapes.

Definition

Along with the axioms, the other fundamental component in an axiomatic system is the definition. Cupillari (2005), in his introductory to his mathematical proof chapter, describes a definition as follows:

> A definition is an unequivocal statement of the precise meaning of a word or phrase, a mathematical symbol or concept, ending all possible confusion. (p. 2)

With this description, a mathematical definition is designed to prevent mathematicians from using the same word to represent different mathematical ideas. Rossi (2006, p. 4) defines a mathematical definition as “a statement that gives precise meaning to a mathematical concept or word” and is used to give explicit conditions for the mathematical term being defined

Cupillari (2005) explains that mathematicians do not create new definitions without giving the process much thought and that, usually, a definition arises in a theory to capture the properties of some concept. Therefore, it is not unusual to expect a definition to follow the form “if-then”. For example,

- Definition of “subset” in Set Theory: If A and B are two sets, we define $A \subseteq B$ to mean that every element of A is an element of B. That is, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.
- Definition of a prime number: A positive integer p other than 1 is said to be prime if its only positive divisors are 1 and p.
- Definition of relatively prime numbers: We say that positive integers *a* and *b* are relatively prime if $GCF(a, b) = 1$.

In each of these definitions, a name is given to a property. To find the logical form of each of the above statements, I first restate them as follows,

- If A and B are two sets, and if every element of A is an element of B, then A is called the subset of B.
- If *p* is a positive integer other than 1, and if its only positive divisors are 1 and *p*, then *p* is said to be a prime.
- If *a* and *b* are positive integers and if $GCF(a, b) = 1$, then we say that *a* and *b* are relatively prime.

The logical form of each of the definitions follow this structure:

if … satisfies some conditions, then it is called …

and vice versa. Therefore, not all but most of the definitions can be regarded as biconditional statements. Although there are alternative definitions to the above definitions, they are all equivalent and describing the same concept. To define a symbol or a concept, we have an almost similar structure to define a word. For instance, the following is the first definition in a Number Theory textbook written by F. Jarvis (2014):

Definition: *Let a and b be integers. Then b divides a, or b is a factor or divisor of a, if $a = bc$ for some integer c. Write $b \mid a$ to mean that b divides a and $b \nmid a$ to mean that b does not divide a. When b divides a, we also say that a is a multiple of b.*

In the structure of this definition, we have six conditional statements in the following if …, then … form:

If a and b are integers. **Then**:

If $a = bc$ for some integer c, **then** b divides a or b is a factor or divisor of a,

If b divides a, **then** we write $b|a$, and

If b does not divide a, **then** we write $b \nmid a$.

If b divides a, **then** we say that a is a multiple of b.

A definition classifies the domain of the variables into two subsets of the universal set; one includes all elements satisfying the definition, called the domain of definition, while the other includes the rest of the elements of the universal set. If the domain is empty or not large enough, it may not be worth creating theorems based on such definitions. To clarify, consider the definition of “zero divisors” for rings in abstract algebra,

An element a in a ring is called a right zero divisor if there is a non-zero element x in the ring, for which $xa = 0$.

The set of integer numbers with natural addition and multiplication is a ring, but why do we not have this definition in integer numbers? It might well be because the set of numbers with the property given in the definition of zero divisors is empty! So, to create a definition, it is crucial to consider some features, like that the domain of the variable is not empty. However, in this example for the integers, by manipulating the definition of addition and multiplication, the domain of the variable will be affected, and there could be infinite many integers satisfying the definition. An easy example is quotient rings, like Z_3, where any multiple of 3 equals 0, so any multiple of 3 would be a zero divisor.

Conjecture

An axiomatic mathematical system is based on the explicitly stated axioms and definitions, and new mathematical results can be obtained from these foundations using deductive reasoning. To develop a mathematical conjecture, one may use inductive reasoning, and make up several specific examples and look for a pattern. However, inductive reasoning does not rely on acceptable proof. Until the truth of a hypothesized result is not known, it is called a conjecture. Rossi (2006) asserted that:

> A conjecture is any mathematical statement that has not yet been proved or disproved. (p. 6)

Therefore, we need to find a way to determine whether a conjecture is always true for all possible cases rather than enumerate many confirmatory cases. An example is “Goldbach's Conjecture”. This is one of the most famous (so far) unproven mathematical conjectures. It states that:

> Every even integer greater than 2 can be expressed as the sum of two prime numbers.

Even though this statement is not clearly stated as a conditional, it constitutes a necessary assumption required to understand its meaning, the definition of a prime number. So, the assumption is hidden in the statement and could be restated as follows,

> **If** a number is an even integer greater than 2, **then** it can be expressed as the sum of two prime numbers.

In his textbook about prime numbers, Ribenboim (2004) gives an interesting example about a hypothesis made in the mid-19th century and published as a theorem, but in a few months had been disproved. In 1869, Li Shanlan proposed a conjecture known as the “Chinese hypothesis”. He claimed that an integer number *n* is prime if and only if $2^n - 2$ is divisible by n. If n is prime, then the conjecture is true, but the converse is not always true. For example, $2^{341} - 2$ is divisible by 341, $341 = 11 \times 31$ is not a prime. So, the whole conjecture is false.

The logical form of the Chinese Conjecture could be regarded as:

If n is an integer, **then** n is prime **if and only if** $2^n - 2$ is divisible by n.

Therefore, equivalently, we have two following conditionals:

If n is a prime, then $2^n - 2$ is divisible by n and

If n is an integer and divides $2^n - 2$, then n is a prime.

The logical form of the above conjecture is a bi-conditional statement. A biconditional statement has the general form of “p if and only if q”, and denoted by $p \Leftrightarrow q$, that is equivalent to the combination of “if p, then q” and “if q, then p”. A bi-conditional statement is true when the truth value of both sides is the same and is false otherwise.

Theorem

When a new result, also known as a conjecture, is proved rigorously, it is accepted as a new contribution to mathematics and is called a theorem. The term ‘theorem’ may be used throughout the whole of mathematics much more than the term ‘conjecture’. This might be because, often, when a conjecture is refuted, it would not be published. However, any proof or disproof requires plenty of experience and hard work. Mostly, the proved conjectures are available, while a disproof may not be published.

While many theorems and conjectures are directly stated in the form “if-then”, this is not always the case; for example, Goldbach's Conjecture is not explicitly in the “if-then” form. For example, the following theorem is selected from Jarvis’ (2014) *Number Theory* textbook.

Theorem*: Diagonals of a square are equal in length.* (p. 2*)*

This statement is not in “if-then” form; however, it has a necessary and a sufficient part and could be restated as follows,

If a polygon is a square, **then** its diagonals are equal in length.

Proof

Even though the development of mathematics is a creative process, and it might be based on trial and error, a new result in mathematics must be proved rigorously to get accepted; otherwise, there is always a danger for the result to be refuted by a counterexample. Regarding a mathematical proof, Hardy (1929) states that:

> A mathematical theorem is a proposition, and a mathematical proof is clearly in some sense a collection or pattern of propositions. (p. 3)

As an example, let us see a proof structure the following theorem:

Theorem: *If a, b, and c are integers such that $a|b$, and $b|c$, then $a|c$.*

Proof:

Since $a|b$ and $b|c$, there exist integers k_1 and k_2 for which $b = k_1 a$ and $c = k_2 b$. By combining these equations, we get $c = k_2 b = k_2(k_1 a) = (k_2 k_1)a$, and hence $a|c$.

The above proof is only one example of a mathematical argument that proves a theorem. Since the conditional statements are often hidden in the meaning of the sentence, in the following, I restate the above argument in the form of "if-then" statements, so that it is easier to recognize conditional statements in the above proof:

> **If** $a|b$ and $b|c$, **then** there exist integers $k_1$and k_2 for which $b = k_1 a$ and $c = k_2 b$.
>
> **If** $b = k_1 a$ and $c = k_2 b$, **then** $c = k_2 b = k_2(k_1 a) = (k_2 k_1)a$.
>
> **If** $c = (k_2 k_1)a$, **then** $a|c$.

While there are different proof structures, discussing them is beyond the scope of this thesis.

Lemma and corollary

A *lemma* is an auxiliary theorem proved beforehand to be used to prove another theorem, and a *corollary* is a theorem that follows logically from a proved theorem. There is no formal distinction among the structures of a lemma, a corollary, and a theorem. Therefore, the if-then structure for both of them is similar to the statement of theorems.

4.2. The role of conditionals in informal mathematics

So far, this chapter has only focused on the written and formal parts of mathematics. However, mathematics does not only consist of definitions, theorems, proofs, examples, counterexamples, etc.; there is always a substantial hidden part on the back of each formal construction. To be more specific, for instance, consider a definition. A mathematician does not always come up with a perfect definition without delving into definitions and theorems already existing in the same content.

The current section discusses the role of logic in informal mathematics. In this work, by "informal mathematics" I mean the use or integration of mathematics in everyday language; it is not as rigorous as formal mathematics[5]. Informal mathematics addresses mathematical arguments in everyday language. It does not necessarily include formal mathematics: for example, it includes all discussions to search for a counterexample for a conjecture or all discussions in order to find a method to prove a conjecture.

The idea of this phrase is from a paper about "the effects of informal mathematics activities on the beliefs of pre-service elementary school teachers" by Roscoe and Sriraman (2011). What they call 'informal mathematics' is close to what I am introducing but not the same. The following is what they mean by informal mathematics activity:

> informal mathematics activity is defined as a mathematics problem-solving activity, which emphasizes that the creative and investigate processes in mathematics and requiring students to communicate mathematically. (p. 604)

Any definition needs a thorough consideration to check some essential properties as discussed in section §§4.1.2, however this part is mainly a discussion among mathematicians (or with themselves) and is never shown in the created definition.

The same is true for theorems: a theorem is not just simply a conditional statement in mathematics discourse. A theorem adds a new result to the body of mathematics that works appropriately with the other related parts. There could also be a long process before stating a theorem. The arguments mathematicians use to create mathematical constructions and objects are mainly in the territory of informal mathematics.

In contrast to formal mathematics, which uses rigid rules to verify arguments and make it impossible for a conclusion to be false when the premise is true, daily language accepts the conclusions with some level of certainty that can range from low to high. Do people apply conditional statements while discussing mathematical concepts and proofs?

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[5] The term "informal mathematics" is also used in a project for formalizing mathematics in computer science cited below, they also compare formal and informal mathematics in a table (https://www.mat.univie.ac.at/~neum/FMathL/inForm).

To see the role of conditionals in informal mathematics, I briefly go over the structure of common types of reasoning in daily language.

4.2.1. Reasoning

Reasoning is part of a set of skills required to help us develop mathematically and allow individuals to think critically. Johnson-Laird (2000) defines reasoning as the mental process of deriving consequences from given information, where the premises can be perceptions, descriptions, or memories, and the conclusion can be a statement, a thought, or a decision. The three main methods of reasoning are the deductive, inductive, and abductive approaches, and these are three stages in scientific research.

Deductive reasoning

A **deduction** or *deductive argument* is an argument whose premises provide strong support for the conclusion. In other words, if the premises are true, then it would be impossible for the conclusion to be false; in such cases, we say that the argument is valid or deductively valid. For example, consider the following:

if $\mathrm{x} = 3$ and, if $\mathrm{y} = 2$,

then $2\mathrm{x} = 3\mathrm{y}$.

In this example, it is a *logical necessity* that $2x = 3y$.

A deductive argument is an invalid argument if the truth of the premises does not result in the truth of the conclusion. In the definition of a valid argument, a false premise can result in a false conclusion, For example:

Milo is a cat.

All cats are insects.

Therefore, Milo is an insect.

This argument is valid because it is regarded invalid only when the false premise results in a true conclusion, and in this example, the conclusion is false, so the argument is valid. Although this is an example of a valid argument, it is not sound. Soundness is different from validity. A sound argument is a valid argument with true premises. However, any valid argument is not necessarily sound.

Most everyday reasoning is based on uncertain premises; this influences the degree of certainty in the conclusion of inference (Stevenson & Over, 1995). Therefore, to investigate everyday reasoning, we may need to consider the uncertainty of premises and how it affects the conclusion. Everyday reasoning shows itself up in the process of generating mathematical objects. To clarify, in Lakatos' (1976) *Proofs and Refutations*, the role of everyday reasoning in mathematics is specified. In this book, Lakatos creates a conversation, which resembles Socratic dialogues, among a group of fictional students to discuss Euler's characteristic of polyhedra ($V - E + F = 2$), which occurs for other mathematical objects, a transition from uncertainty to certainty. By this I mean that mathematics is a process of conjecture, discovery, proofs and refutations.

Inductive reasoning

An **inductive argument** is one for which, if the premises are true, then it will be *unlikely* that the conclusion is false. So, roughly speaking, an inductive argument claims its conclusion follows with some degree of probability that can range from very low to very high, but always less than 100%. For example, I have seen many persons with broken nails who have anemia (low number of red blood cells), so I conclude that (all) persons who have broken nails are prone to have anemia. Therefore, unlike with deductive arguments, there is a matter of degree of certainty for the success of an inductive argument. Therefore, induction is an argument in which the truth of the premises results in a conclusion that is probably true.

Based on the degree of such probability, there are different inductive arguments, namely strong and weak inductive arguments. **A strong inductive argument is an** argument in which the truth of the premises makes it improbable for the conclusion to be false, **and in a weak Inductive Argument**, the truth of the premises does not result in the conclusion that is probably true. **A cogent argument is a convincing argument, and** it is a strong inductive argument in which all the premises are true. An **un-cogent argument** is a weak argument or an argument with at least one false premise. Mathematically speaking, inductive reasoning is a type of reasoning that arrives at a conclusion using patterns and observations. Therefore, inductive reasoning is not an accurate method for arriving at accurate conclusions, like those we name *theorems*. Even though inductive reasoning cannot be used as the sole basis to prove an idea, it can help to generate ideas and conjectures which could be proved or refuted through deductive methods or using counterexamples, respectively.

Converting an inductive reasoning to a deductive reasoning

Sometimes we can turn an inductive argument into a deductive argument by making some changes in the premises. For example:

Dan is a man.

Dan is 95 years old.

Therefore, Dan cannot run a mile in 6 minutes.

We can always add a premise like Dan has a broken leg; this makes the result more probable. However, if we add the premise that Dan has paraplegia, then the argument becomes deductive because the result is 100% true. Inductive logic, which is the study of probable reasoning, is not very well understood, though some special cases are well developed, such as applying the probability calculus to gambling.

Deductive versus inductive reasoning

Consider the following argument,

If the stapler is empty, then it will not work.

The stapler is empty.

So, the stapler will not work.

The above inference is valid because the conclusion holds in any case in which the premises hold. Now consider:

If the stapler is empty, then the stapler will not work.

The stapler will not work.

So, the stapler is empty.

In this case, the inference is not valid because there may be another reason that the stapler will not work. For example, it could be jammed, yet the conclusion may still be true. In this example, the conclusion does not follow inevitably from the given assumptions. In a conditional statement, the validity of the conclusion cannot guaranty the validity of the assumption. Even though the above inference is not valid, I would not be surprised if one regards it as a valid argument. Because most human reasoning is based on experience, either individual experience or collective experience, the kind of reasoning based on experience is called *inductive reasoning* (Taylor & Garnier, 2014). For many theorists, like Aristotle, all inferences fall into one of these two categories: deduction and induction. Aristotle defined induction as an inference from a particular assertion to a universal one, such as the inductive reasoning about the stapler in the second example, that only uses the experience to conclude.

An inference to a conclusion, that refers to all the same possibilities as the premises do, or at least includes them all in what it refers to, is a deduction. In the first above example, the premises refer just to one possibility: the stapler is empty, and the stapler will not work. Hence, this inference is valid because its conclusion holds in the one possibility to which the premises refer. This contrasts with an inference to a conclusion that refers only to some of the premises' possibilities.

Abductive reasoning

Another form of reasoning that does not fit with inductive or deductive reasoning is abductive. This concept is developed by Peirce (Collected Papers (1931-1958)) as a logic of discovery within scientific inquiry to explain how new concepts are created and where ideas are coming from. It starts with one or more observations, then discovers the most likely reasons, or the best guess, to explain those observations; however, it does not verify the conclusion. Despite the weakness of abduction reasoning, it is very important, because it is a type of inference that creates new ideas. To have a better understanding of abductive reasoning, DeMichele (2018) compared these three types of reasoning with the following example:

Table 4.1. Three types of reasoning in terms of a syllogism

	Deductive	Inductive	Abductive
Major Premise	All humans are Mortal	Most Greeks Have Beards	Observation: That Man Has a Beard
Minor Premise	Socrates is a human	Socrates is a Greek	Known Fact: Most Greeks Have Beards
Result	It is certain that: Socrates is Mortal	It is “likely” that: Socrates has a beard	Perhaps: This Man is Greek

The following figure summarizes these three types of arguments:

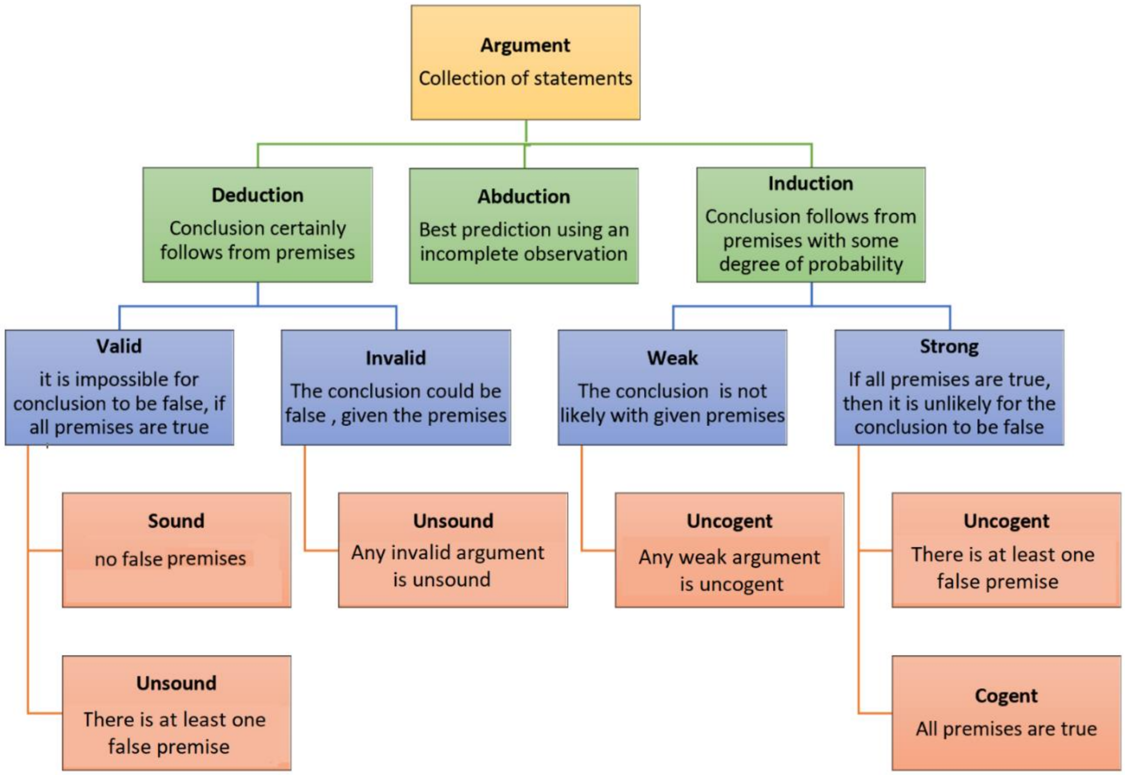

Figure 4.1. Three types of reasoning

4.2.2. Conditional reasoning

Conditional reasoning is based on the construction "if p, then q" when the premise is true, the conclusion will be true. However, this leaves open the question of what happens when p is false, which means that, in this case, q can logically be either true or false. Studies are abundant about four main conditional inferences: *modus ponens*, modus tollens, affirmation of the consequent and denial of the antecedent. Johnson-Laird and Byrne (2002) discuss that, among all four conditional inferences mentioned in §§3.1.2, only *modus ponens* and *modus tollens* are valid for the conditional interpretation. The following is an example of modus ponens:

If it rains, then you get wet.

It rains.

Therefore, you get wet.

While conditional reasoning using *modus ponens* may seem easy, *modus tollens* (if p, then q: not q, therefore, not p) instead could be difficult. In the following argument, I formulate the above *modus ponens* in terms of *modus tollens*:

If it rains, then you get wet.

You are not wet.

Therefore, it does not rain.

The reason for the difficulty in making inferences in such arguments could be dealing with such inferences, and individuals may only focus on the antecedent possibilities, then a certain inference will be difficult.

4.2.3. Reasoning in daily life

Logic captures the implications among sentences that are usually expressed in a formalized language. In contrast, reasoning is the mental process of drawing conclusions from sets of propositions in everyday language. Many theories of reasoning based on formal rules cannot explain how reasoners determine the logical form of propositions and why they should do so. Unlike logic, the grammar of a sentence in natural language cannot alone yield the logical forms of propositions that the sentence can express. They can be determined only from the meaning of the sentence in context, which depends on knowledge and even on the particulars of an inferential task.

Logical form, and therefore formal rules of inference, are unlikely to play any significant role in the mental processes underlying reasoning in daily life. Johnson-Laird (2014) proposes that reasoners build models of premises and base their inferences on them. He criticizes the idea that logic provides the basis for human reasoning and states:

> [this article] does not argue that everyday propositions lack a logical form, but merely that the task of recovering it is very difficult and probably unnecessary. If this criticism is correct, does it follow that psychologist should dispense with logic? Not at all. There is much that students of reasoning can learn from logic, including which implications among sentences are valid. The mistake is to import logic directly into psychological theory, and to assume that the mental processes of everyday reasoning extract the logical form of premises and use it to reason. (p. 195)

It may be worth mentioning that there is nothing against logic in the above quotation, and he is not criticizing the use of logic or the existence of logical forms.

However, it might not be easy to determine the logical form of sentences and appropriate inference rules to assess them in everyday language. Also, it might not be necessary to reason in everyday language as accurately as in mathematics. If this is true, and people are not using logical rules, how do they reason in daily life?

The following question is examined in the current work:

> Do people recover the logical form of a sentence and apply inference rules, or use a different strategy if any?

4.3. Summary of Chapter 4

Chapter 4 first defined the logical form of a statement as its logical structure that is obtained by abstracting from the subject matter of the content.

In mathematics, constructions of the form "If – then" are complemented by many other forms, but sometimes without the use of cue words. This chapter discussed the significant role of conditional statements in both formal and informal mathematics.

To discuss the role of logic in formal mathematics, the logical form of some mathematical terminologies (namely, definition, axiom, conjecture, lemma, theorem, corollary and proof) has been discovered. The investigation shows that conditional statements are used to describe most of the mathematical constructions.

In §4.2, the chapter turned the focus from formal parts of mathematics to informal mathematics, including arguments about mathematics in colloquial language. I then discussed different types of reasoning used in informal and formal mathematics, namely deductive, inductive and abductive approaches.

Abductive reasoning gives the best guess fit with one or more observations. Inductive reasoning claims the conclusion follows with some degree of confidence. Neither abductive nor inductive reasoning verify the conclusion. In deductive reasoning, it is impossible for the conclusion to be false when the premises are true.

Deductive reasoning is the only reasoning used in formal mathematics, but informal mathematics uses a mixture of the three types of reasoning: deductive, inductive, and abductive. This conflict between the reasoning types used in formal and informal mathematics may be why some students give many examples to prove a mathematical claim. The next chapter aims to examine previous studies on logic in mathematics education and some other fields.

Chapter 5. Literature Review

The purpose of this chapter is to examine prior research on logic, especially conditional statements. It begins by focusing on the role of logic in mathematics and different fields to answer the question, “Why is this topic considered in mathematics education?”. Since the focus of this work is on conditional statements, I aim for the literature review in this direction.

5.1. The role of logic in mathematics education and other disciplines

Alfred Tarski (1940) defines logic in his “Introduction to logic and the methodology of deductive sciences” as follows:

> [Logic is] the name of a discipline that analyses the meaning of the concepts common to all the sciences and establishes the general laws governing the concepts. (p. xi)

Logic applies to different branches. However, there may be different interpretations for its applications in different contexts that make it challenging to capture the logical form of a statement. This chapter outlines some potential obstacles encountered when applying logic in mathematics and other disciplines.

5.1.1. Logic in mathematics education

There are different views among researchers and mathematics educators regarding the role of formal logic and the necessity of learning rules of inference in reasoning and proving in mathematics.

Some mathematics educators, for example, Hanna and de Villiers (2008) and Selden and Selden (2009), support the idea that introducing formal logic to students does not necessarily build up their ability at proof construction because of the distinction between “proving” and “reasoning logically” as different mental activities. For example, Hanna and de Villiers (2008) state that:

> It remains unclear what good comes from teaching students the elements of formal logic, because many mathematicians are willing to admit they rarely use formal logic in their research. (p. 311)

Many other researchers, such as Epp (2003), and Hoyles and Küchemann (2002), have stressed the role of logic in students' understanding of mathematical proof and stated that we need to focus on introducing logical relations implications: for example, presenting logic in a way that links it to language and both mathematics and real-world subject matter can help understand logical relations (Epp, 2003).

Also, Milbou, Deprez, and Laenens (2013) emphasized that teaching logic supports students in developing mathematical skills. While a significant portion of students have trouble in understanding aspects of logic (ibid.), there are studies, such as Goetting (1995), Harel and Sowder (1998), and Jungwirth (1990), that have shown frequent uses of logical misconceptions by mathematics and science teachers might be the reason for students' logic misconceptions. Even though there are different points of view concerning the importance of formal logic in learning mathematics, they seem unanimous in the central role of informal logic; this type of logic was discussed in §§3.1.3.

Among all logical constants in mathematical logic, including connectives and quantifiers, conditionals have been most extensively researched, and it is repeatedly reported that "implication" is a topic that causes students serious difficulties (e.g., Durrand-Guerrier, 2003; Hoyles & Küchemann, 2002).

Regarding the importance of conditional statements – specifically, modus tollens (discussed in §§3.1.2) – for mathematical success, there are two different views. The first one is from Durand-Guerrier (2003), who asserts that *modus tollens* inference is essential for mathematical success and that this is, even more, the case at more advanced levels of mathematics. However, on the other hand, Hoyles and Küchemann (2002) argue the opposite: they state that making the *modus tollens* inference is mainly irrelevant to mathematical success.

In order to investigate this problem, Inglis and Attridge (2016) conducted a comprehensive study to examine the role of modus tollens deduction in advanced mathematics, and the results were consistent with Hoyles & Küchemann's point of view (2002). However, the results of their study have been inconsistent with Durand-Guerrier's view (2003) that the material conditional is necessary for a successful engagement with advanced mathematics. According to Inglis and Attridge, *modus tollens* might be necessary to understand proof by contrapositive.

5.1.2. Logic in other disciplines

Formal systems of logic are used in many sciences; for example, linguists use it to study natural languages, computer scientists employ formal systems of logic in research relating to Artificial Intelligence and programming, and philosophers apply symbolic logic when dealing with complicated philosophical problems, in order to make their reasoning more explicit.

Many authors and researchers in psychology, such as Wason and Johnson-Laird (1977), address the issues in applying mathematical logic and inference rules in everyday language. Notably, Wason (1966) introduced the "selection task" to study reasoning about conditionals that the researchers in cognitive psychology have consistently used for the last forty years.

In this task, four cards bearing the symbols D, K, 3, and 7 are shown to the participants; each card has a letter on one side and a number on the other side. The rule is that "if a card has a D on one side, then it has a 3 on the other side". The question is, "which card(s) should be turned over in order to determine if the rule is true or false?". This task focused on how people reason about conditional statements. Wason's selection task is a famous one that has contributed to the development of many competing theories of reasoning and led to the rejection of many others. The correct solution of the Wason's task that is based on logical rules is given in the introductory part of section §5.3.

Researchers in other areas, such as physics and computer science – for example, Greene, Devlin, Cannata, and Gomez (1990) and Pane and Myers (2000) – have complained that students cannot properly apply principles of logic. For example, *conditional* is an essential aspect of any programming language. It makes programmers able to determine if a condition is true or false and then perform certain actions based on the outcome of the condition test. Almstrum (1993, 1996) investigated students' performance for the Computer Science Advanced Placement Exam and stated that students generally perform worse on the logic-based questions than on any other type of question on the exam, while the concepts of logic are key building blocks in the knowledge that computer science students must acquire.

Also, some other studies confirmed that a student's understanding of logic is correlated with success in science (Piburn, 1990), and computer science classes (Kim, 1995). To study students' difficulties with logical statements, researchers (e.g. Piburn, 1989) at Rutgers University developed the Propositional Logic Test (PLT). This test is designed to examine students' understanding of four logic operations namely: AND, OR, if-then, and if-an-only if. Similar to Wason's (1966) task, PLT gives students a propositional statement, and they must select which of the four conditions could violate the propositional statement. Therefore, there are 16 possibilities to answer each question, because each question has four conditions and students could either select or deselect each condition. However, some of these studies, for example, Owens and Seiler (1996), have revealed that students' progress does not increase significantly after instruction on formal logic.

The PLT is a 16-item instrument that tests students' understanding of four logic operations (AND, OR, if-then, and if-and-only-if) through four 4-item subtests.

Research studies conducted on logical reasoning in psychology suggest that our emotions remarkably influence our reasoning. Several research studies confirm that emotions play an important role in deciding and even in problem solving. For example, in a study by Oaksford, Morris, Grainger, and Williams (1996), participants with some pre-existing emotional states reasoned less normatively while manipulating Wason's selection tasks based on their mood.

In a study by Channon and Baker (1994), participants diagnosed with depression were asked to reason with syllogisms in different topics, such as those with sad contents. The results show that depression resulted in a deterioration of reasoning performance compared with non-depressed participants. Addressing the results showing that mood or affective state influences reasoning behaviour, Baddeley (2003) explains that emotions affect working memory by putting an additional load on working memory.

Even though many studies suggest that emotions affect the way we think and decide – such as Jung, Wranke, Hamburger and Knauff (2014) – there is some evidence suggesting that this outcome might be depending on content. For example, in an experiment, they asked participants with spider phobia and people with exam anxiety to solve logical problems with content about their affective state. The results showed that participants with spider phobia perform less normatively on spider-content problems, while those with exam anxiety were not affected by the content.

5.1.3. Why conditional statements?

In philosophical logic literature, logical and linguistic uses of conditional statements have been two of the main subjects for research (e.g. Edgington, 1995). Also, conditional reasoning (§§4.2.2) has received particular attention in the psychological literature, and a significant number of research studies on conditionals in psychology are related to the Wason's selection task, the truth-table tasks, and the logical inference of conditionals (e.g. Evans, Newstead & Byrne, 1993; Johnson-Laird, 2010).

Conditionals and the mathematical concept of implication in argumentation and proof are necessary for students to understand various kinds of reasoning. Sometimes, even to apply a simple definition, they need to distinguish between necessary and sufficient conditions, and most of the mathematical propositions can be expressed as follows: "If p, then q" (§4.1), and this statement corresponds to a formal understanding of implications as $p \Longrightarrow q$. Rodd (2000) argues that a logical implication in the form of reasoning ($p \Longrightarrow q$, p so q) is one of the most basic structures for establishing a mathematical truth.

5.2. Situating conditionals in the literature

Research about an understanding of logical reasoning has started with the seminal work of Inhelder and Piaget (1958) in the field of developmental psychology, who used propositional calculus as a basis for their analysis of participants' responses. For example, they argued that understanding an implication $p \Longrightarrow q$ required an appreciation of its equivalence to the appropriate four combinations of truth values of the two propositions p and q.

Hoyles and Küchemann (2002) observed that students tend to treat a conditional statement and its converse as equivalent, a phenomenon which is described as "child logic" by O'Brien, Shapiro and Reali (1971). This has also been reported by Tall (1989) that students have issues in distinguishing between the statements "if p, then q" and "if q, then p". However, Tall declared there is some evidence that the use of child logic decreases with age.

In mathematics education, it has been argued that understanding logical implications is one of the most important prerequisites for understanding mathematics and constructing proofs. For example, Hoyles and Küchemann (2002) emphasized the importance of logical implications for success in mathematics. However, by comparing the inferences drawn from abstract conditional statements by advanced mathematics students and well-educated arts students, Inglis and Simpson (2008) showed no simple relationship between the study of advanced mathematics and conditional inference behaviour. However, the results depend mainly on the particularly designed tasks.

It has also been recognized that logical implication is a topic that causes difficulties for students (e.g., Deloustal-Jorrand, 2002; O'Brien, 1973). For instance, Durand-Guerrier (2003) investigated first-year university students' difficulties in understanding logical implications. She emphasized that students' difficulties were because of introducing conditional statements along with quantifiers in high school mathematics classrooms, instead of learning conditional statements in the general and abstract form of $p(x) \Longrightarrow q(x)$. It is also evidenced by Hoyles and Küchemann (2002) that only the minority of advanced students can use deductive methods or even determine the negation of a given implication.

Cheng and Holyoak (1985) claimed that a typical college student does not reason well with formal logic and fewer than 10% of the participants reason correctly about the statement "if A, then B". Also, in another study of college students about understanding and applying the logical implication and its contrapositive in a real-world situation, Romano and Strachota (2016) showed that students had insufficient knowledge of the logical concepts and argued that they struggled to distinguish between the definitions of mathematical concepts and processes involving those concepts, such as statements or theorems that are instances of formal implications.

Conditional is a critical concept, not only in logic and mathematics, but in other sciences too. For example, to understand the reason for students' difficulties in programming in computer science, Almstrum (1999) discusses students generally understand the AND concept while they struggle to learn OR concepts and conditional statements. Most students mistranslated if-then as AND (Cheng & Holyoak,1985).

These instances suggest that logical implications play an essential role in learning mathematics, specifically in grasping different types of reasoning and understanding and constructing proofs. Nevertheless, the application of conditionals is not restricted to mathematics. As follows from the studies mentioned above, despite the critical role of logic in learning mathematics and other disciplines, student achievement related to logical structures is relatively low. However, identifying misconceptions alone is not enough, and we need to understand the sources of students' reasoning.

5.3. Formal logic, informal logic, and everyday reasoning

Although logical implication plays a fundamental role in real-world situations, identifying conditional statements in colloquial language may depend on people's understanding of the elements of implication. Even in mathematics communication, to answer a question, students must deduce conclusions from hypotheses by interpreting ideas through written text or verbal communication.

Piaget (1958) argued that it is possible to talk about general reasoning that is domain independent. However, Inglis and Attridge (2016) discuss Wason's selection task (§§5.1.2), showing that people do not reason based on general rules of inference. Less than 10% of well-educated participants could make a correct solution to Wason's task, and the most common response was that it is necessary to turn over D and 3, while this answer is not consistent with logical rules. Based on formal logic, a conditional statement is logically equivalent to its contrapositive (The contrapositive of the conditional statement of the form "if $p,$ then q" is "if $\sim q,$ then $\sim p$ "; these two statements are logically equivalent, meaning that they have the same truth table.) Since the negation of 7 is 3 and the negation of D is K, therefore "if D, then 3" is equivalent to "if 7, then K", so the correct selections have to be D and 7.

The above discussion suggests that formal logic might not be applicable in everyday reasoning, but we mainly use examples from the real world to learn logic. Using such examples may suggest a logical form for any statement; however, rather than using logic as a base in everyday language, we call sentences from daily language to give meaning to logical forms.

Pimm (1987) states that multiple meanings present a problem in learning many mathematical terms, and non-mathematical meanings can influence mathematical understanding and can be a source of confusion. He also states that, to give meaning to an expression or usage, students apply the world knowledge they acquired and manipulated and knowledge of the language.

There can be a similar situation for logic; it is not always true that any sentence in a daily language has an interpretation consistent with its logical form. In everyday language, statements are often ambiguous, as there are different acceptable interpretations for the statements in different contexts. To exemplify, the statement "I must go up or down" implies an exclusive-OR in an everyday context, even though the verbal construction is an inclusive-OR. In contrast, mathematical language is unambiguous as the logical form determines one interpretation for every proposition. The disjunction $\vee$ is an inclusive or, and for exclusive or logic uses another symbol, XOR or $\oplus$, which has a different truth table than $\vee$.

In further detail, OR is a term in logic, but it is also a word used in daily speech, and whatever meaning it has in logic may not correspond to its meaning in everyday language. In other words, the different meanings for terms in logic and everyday language may result in misunderstanding and confusion. On the other hand, since logic is symbolized for mathematics, the logic register may be included in the mathematics register[6]. Therefore, a connective like "if-then" (the same meanings in both mathematics and logic) may not correspond with its meaning in everyday language.

More specifically, with respect to propositional statements, Epp (2003) noted that a possible reason for misconceptions about conditionals could be different possible interpretations in everyday language and mathematics. Therefore, students need to know that some words have different meanings in mathematics and everyday language.

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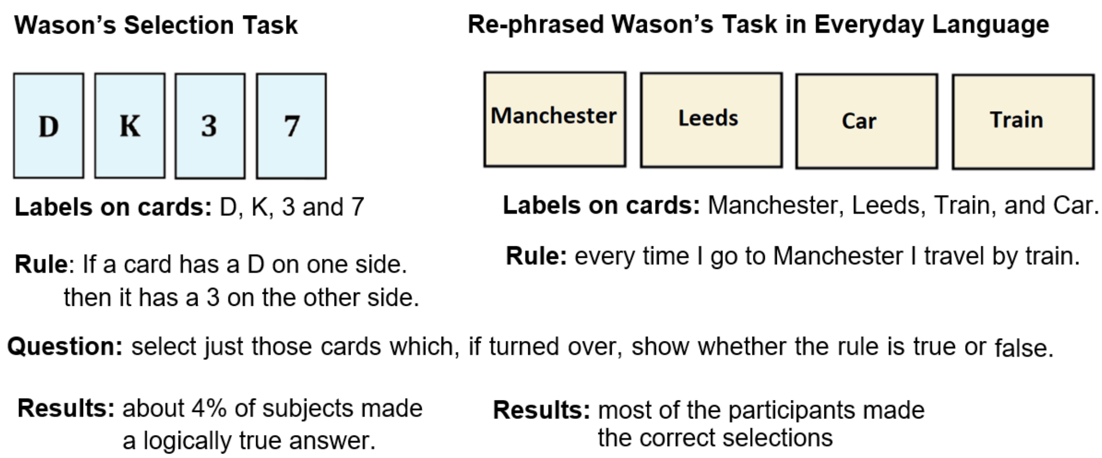
[6] Pimm (1987) proposes that the "mathematics register" includes the meanings that belong to the language of mathematics; it is the mathematical use of natural language. He then discusses differences in meaning between "borrowed" elements in the mathematics register and the use of the same terms in everyday language.

In 1971, Wason and his doctoral student, Diana Shapiro, changed the wording in the Selection task (§§5.1.2) and designed a version of the "Four cards problem" in in a familiar context in everyday language. The rule in the new version is "every time I go to Manchester I travel by train" and the cards are labeled with Manchester, Leeds, Train, and Car. Wason and Shapiro's (1971) explanation for the improved performance was that while the British participants did not have experience reasoning about relations between numbers and letters, they did have experience with traveling by car or train to cities in the UK. This may suggest that the way individual reason is context-dependent.

Contrary to the original version, most participants made the correct selections this time: see the following table. This suggests that the way that a statement is worded has an important role in determining the sentence's logical form.



By comparing students' interpretations of everyday and mathematical claims of the same logical form, Dubinsky and Yiparaki (2000) suggested that, instead of trying to make everyday-life analogies between ordinary English statements and mathematical statements, we need to remain in the mathematical context and focus on helping students to understand mathematical statements in the mathematical context.

Regarding quantified logic, which is a part of logic dealing with the existential and universal quantifiers, Durand-Guerrier (2008) argues that, despite the difficulty to master quantifiers, the benefit it can bring to mathematics education makes this approach worthwhile to study. Also, Lin, Lee, and Wu (2003) showed that many students struggle with applying quantified logic and proof by contradiction that is an application of the fact that a conditional statement is logically equivalent to its contrapositive. Despite the results, they stated that, under proper support, the students could understand these concepts (proof by contradiction and quantified logic) and apply them in real-world situations. This claim is not far-fetched, and their idea gains strength by looking at the results of a study in probability and statistics by Fong, Krantz, and Nisbett (1986). They showed that having a good background in statistics can result in its applications in everyday language, and people with a higher level of education could give a more statistical answer closer to their actual answers in statistics.

5.3.1. The conflict between mathematics and everyday language about conditionals

In classical logic, the truth value of "$p \Longrightarrow q$, when p is false" is true, regardless of the consequent truth. Lewis' (1917) main aim was to find a satisfactory theory of implication. He argued that material implication is unsatisfactory to capture implication adequately because it involves claiming that a false proposition entails any conclusion.

For example:

- If today is Friday, then 1 + 1 = 2.
- If 1 + 1 = 3 then today is Friday.
- If the earth is flat, then today is Friday.

In the above examples, all the statements are always true whether or not today is Friday. The premise and consequent are irrelevant, which might be one of the sources of misunderstanding of implications. Consider today is Wednesday, so by logic rules, the first conditional is true, and the second one is false. The third statement is always true since the premise is false, no matter whether today is Friday or not.

These examples may seem weird because the premises have nothing to do with the consequences. They talk about entirely different things. So how can these arguments be valid (§§4.2.1)? Some philosophers and logicians have indeed argued that we need a better definition of validity to ensure that the conclusion is relevant to the premises.

As a result, many philosophers, beginning with MacColl (1908), have claimed that statements in the above example are counter-intuitive. They assert that these statements fail to be valid if we interpret $\Longrightarrow$ as representing the concept of an implication that we have in the English language before we learn classical logic. *Relevance logic* (§§3.3.3) is a non-classical logic (§3.3) and is developed to avoid such paradoxes of material conditionals.

A conditional statement in daily language connects two parts in a certain way. Similarly, we may expect a relation between the premise and the consequent for a conditional statement in mathematics.

Lewis (1917) stated that:

> A relation that does not indicate the relevance of content is merely a connection of 'truth values', not what we mean by a 'logical' relation or 'inference'. (p. 355)

The following conditional statement exemplifies Lewis' idea:

if the elephants could fly, then you win the lottery.

Under some conditions, say in a lab, there might be an elephant that can fly. In this example, I take a general belief that an elephant cannot fly. Therefore, the above statement is a true proposition, regardless of whether you win the lottery or not. To eliminate this problem and exclude such statements as implications, Mitchell (1962) defined the "hypothetical proposition" only to assert the consequent q is true when the premise p is realized; in this definition, cases where p is false but irrelevant are ignored. The hypothetical implication defined by Mitchell is similar to the definition of an implication in relevant logic.

Regarding material and hypothetical implication, it has been argued that while the material conception of implication is the one most commonly used in formal mathematics, the latter conception – namely hypothetical proposition – plays a more important role in school mathematics (Hoyles & Küchemann, 2002). However, most new research discoveries are very specialized and advanced to be appropriate for beginners, and the textbooks only change slowly. Therefore, most of today's introductory logic textbooks are focused on classical logic. Now we are at the beginning of the way, and the non-classical view of logic can be reformulated for beginners, though there have to be many objections to change.

5.4. Syntax and semantics of a language

Languages, in general, can be considered to consist of two main components: semantics (or meanings) and syntax (or structural rules). To apply them, first, we examine how to form sentences in the language using the rules of syntax, or grammar in the case of natural languages, and then we interpret sentences by introducing a semantic. It is not surprising to have a sentence with different syntaxes but the same semantics. For example, there are many different computer programming languages, and to perform a certain task, there could be many programs, each written in a different language, but all perform the same task. Even though the programs written in different languages are different in syntax, they have the same semantic.

The main idea of mathematical logic is the distinction between syntax and semantics. The syntax here is a formal language to represent mathematics, and semantics is the mathematical structures such that sentences of this language are true or false.

Particularly, there are many sentences with different syntax and the same meaning in the context of logic, such as a conditional and its contrapositive. Also, the conditional $p \Longrightarrow q$ may have different syntaxes in another language; for instance, the conditional sentence $p \Longrightarrow q$ have some equivalent syntaxes in the English language like "the conditional of q by p", "p implies q", "q is resulted from p", "p only if q", and "if p, then q".

The multiple semantics for one syntax may result in some difficulties determining the logical form of a sentence in a language or its interpretation/translation in another language. Inglis and Simpson (2008) reported that students experience considerable difficulty accepting that "p only if q" is logically equivalent to "if p, then q", though they have the same semantics. They proposed that one reason may be that the statements are not interchangeable in certain real-world contexts, and many cannot distinguish between a material conditional and an implication.

Regarding semantics for logic as a language, there are different approaches. In this study, by "semantics", I mean "Model theoretic semantics of truth" initiated by Alfred Tarski (1940). This theory applies to formal language. For some reason, Tarski did not extend his theory to natural language. For instance, he asserted there is no systematic way of deciding whether a given sentence of a natural language is well-formed and that a natural language can describe the semantic characteristics of its elements. This explanation might explain one of the sources of confusion while situating a conditional in the English language.

In Tarski's model, semantics for a sentence is its truth value. For instance, to determine the meaning of ¬W (not-W) in logic, we need to determine the truth value of ¬W. In doing so, we must determine the truth value of W, that is I(W), where "I" is a meta-symbol defining the value of W. Then, I(¬W) is true if and only if I(W) is false, and I(¬W) is false if I(W) is true. This gives a good specification of the semantics of ¬W, but the following tabular representation is more often used:

Table 5.1. Truth values for negation

W	¬W
True	False
False	True

Kahle and Keller (2015) used colours to distinguish syntax and semantics as an educational tool in logic classes and discussed some of the issues related to that distinction. They stated that "semantics comes first", specifically in the first-order logic and mathematics: "we presuppose the meanings of the logical and mathematical expressions before we start to manipulate them" (p. 80), where semantics means the interpretation and not the truth values of the statements. The conditional statements show that, even though syntax and semantics are important components of a conditional, they may not be enough to understand them, and even educated people make inferences based on the context. This suggests adding the third component, namely context, to specify the subject area in the logical statement.

5.4.1. A third component: context

About 2,500 years ago, in *Republic* Book 7, Plato (427 – 347 BC) stated that:

> those who have a natural talent for calculation are generally quick at every other kind of knowledge, and even the dull if they have had an arithmetical training, although they may derive no other advantage from it, always become much quicker than they would otherwise have been. (p. 354)

The idea that studying mathematics can develop other logical skills was dominant for centuries until the turn of the twentieth century, when this view was attacked by the new discipline of psychology and learning theories of the 1920s and 1930s. Then this idea drew the attention of cognitive scientists of the 1960s and 1970s, and they rejected the view that general rules mainly influence everyday reasoning. For example, Piaget (1958) believed that people reason inconsistently with formal mathematical logic rules. Even though Lehman and Nisbett (1990) discussed that taking a course in formal logic improves conditional reasoning and reasoning about events in everyday life, it can be effective for some types of syllogistic reasoning.

Many studies in cognitive psychology support that reasoning skills are highly dependent on context, and it is not easy to apply a rule learned on a topic in another topic. Barwise (1989) argues that the inference “All humans are mortal. Socrates is a human. So, Socrates is mortal” (§§3.1.3) is judged valid by any user of the English language, which means if both hypotheses are true, so is the conclusion. He states that this inference is judged valid because of what the sentences mean, not because it is valid in a formal system of logical inference.

Many researchers in mathematics education, such as Selden and Selden (1995), Dubinsky and Yiparaki (2000), Hanna (2000), and Inglis and Simpson (2006), have emphasized the difference between common sense and mathematical logic. Regarding the nature of an individual’s conceptual knowledge, Carey (1999) argues that an individual’s misconceptions are coherent and consistent. On the other hand, diSessa, Gillespie, and Esterly (2004) suggest that an individual’s knowledge is fragmented. The second view states that knowledge is dispersed in smaller distinctive parts that could be regarded as an individual’s mental models gained by experience.

Although deductive reasoning is considered abstract, even educated people make deductive inferences based on the premise’s content. Regarding the content of an inference, Johnson-Laird and Byrne (2002) declare that the meaning of a conditional could be modulated by the meanings of the premise and conclusion, and their referential relations. They also describe core meanings of conditionals and argue that the meaning can be modulated by semantics and pragmatics:

> Philosophers and logicians distinguish between the meaning of an expression and its reference. The classic illustration contrasts the Morning star and the Evening star, which differ in meaning but have the same reference, namely, the planet Venus. We assume that the meaning of a

> sentence when it is used in a particular context functions to refer to a context or to a set of contexts. (p. 648)

They discuss that general knowledge and knowledge of context could be additional factors that modulate the individual's interpretation of a conditional. This may suggest that knowledge about the premises can shape the processes of deductive reasoning. For example:

- Alan lives in Vancouver, or he lives in Canada;
- He does not live in Canada. Therefore, he lives in Vancouver,

is a valid inference: however, it is impossible for Alan to be in Vancouver and not in Canada. This statement is only sensible in the context of logic when only the logical form of the statement decides about its validity.

Oaksford, Chater and Larkin (2000) propose that, when people make conditional inferences, they do not attempt to make logical inferences; instead, they use a kind of probabilistic reasoning that is applicable in many everyday contexts with direct access to knowledge about premises.

Baldwin (2009) added a third component to syntax and semantics, namely situation to discuss the learning of algebra. He distinguished between "semantics" and "situation" in algebra and proposed that these are two sorts of mathematical meaning: meaning of symbols in a mathematical context and interpretation of mathematical statements in the physical world. He also considered "syntax", "semantic," and "situation" as three needed components for learning variables in algebra.

5.5. From literature to the current study

Cummins, Lubart, Alksnis and Rist (1991) argue that a characteristic of human reasoning performance is that it is influenced by the content or subject matter of the reasoning task. They state that problems with identical logical forms, but different subjective contents often result in different performance levels.

Many conditional statements are conditional by logic rules, but do not make sense in either the realm of mathematics or colloquial language. It seems there must be some relation between the meanings of premise and conclusion. As discussed in §§5.3.1, Lewis

(1917) stressed that to justify something, there must be some relevance of content or meaning and that the material implication cannot properly capture this feature, because it involves the claim that a false premise can result in any proposition. Any conclusion entails a true proposition. For example, the statement “if the earth is flat, then today is Friday” is always true, whether today is Friday or not. Here seems to be a failure of relevance. The conclusion seems to have nothing to do with the premise. Some logicians and philosophers (e.g., MacColl, 1908) claim that what is unsettling about these so-called paradoxes is that the premise seems irrelevant to the consequent.

Especially in the context of mathematics, we may expect some sort of relevance between the premise and the conclusion in a conditional statement. For example, if we consider a conditional as a connector between different parts of a proof, then the premises and consequents are expected to be relevant. This may suggest that the hypothetical implication is the one that is applied in mathematics, rather than the material implication (see §2.2 and §2.3).

In this study, I use the word “context” as the third important component in first-order logic. In other words, along with syntax and semantics in first-order logic, we can also consider “context”. Here, by the context, I mean the discourse of the logical statement. For example, the statement “If $(x-2)(x+1)=0$, then $x = 2$ or $x = -1$” is presented in the context of mathematics.

The problem arising from the previous section is that a symbolic expression in logic may be ambiguated by situating it in either mathematics or everyday language. If so, how do people determine (or evaluate) conditional statements in different contexts? Do individuals with a good background in either formal logic or mathematics apply inference rules to draw a conclusion in daily life reasoning?

To reiterate, there are at least three main components in the first-order logic:

- First, the “syntax” determines which collections of symbols are legal expressions; for example, in logic, $\exists p \implies \sim\forall$ is incorrect syntax.
- Second, the “semantics”, which determine the meanings behind these expressions. By semantics, I mean Tarski’s model, which is the truth-value function.

- Third, the “context”, that is, the context of the statement, could be in the context of mathematics, in everyday language, or the context of logic.

To clarify the above classification, I give three examples in three mathematics, colloquial language, and formal logic contexts.

First, consider the following statement:

If you see a purple crew, then Bertrand Russel will have dinner with you.

The **syntax** of the above sentence is “p ⟹ q” which is correct in grammar. Its logical **semantics** is “p implies q” which is a conditional statement with truth value as its semantics (see Table 4.1). Also, its **context** is “everyday language”.

As another example, consider the sentence $p \Longrightarrow q$; this statement is conditional and has the above truth value of semantics in the table, and its context is in logic. Consider the statement “=7+8”. It has incorrect syntax, so it has no meaning, but its context is nonetheless mathematics.

In my research, I aim to investigate “how does context affect the ways in which people discuss conditional statements?” and “how do people recognize conditionals in different contexts?”.

In the previous section, I noted that being unaware of different interpretations of conditional statements in different contexts might be a reason for misunderstanding conditionals. For example, in the context of mathematics, individuals with a background in both formal logic and mathematics may have at least two different interpretations of a conditional statement at the same time, i.e., semantics and context. This could be a source of confusion. To clarify, consider the following statement, assuming that today is Monday:

If today is Saturday, then $\sqrt{-9} = 2$

Then, using Tarski’s model as the semantics for a conditional statement, the above statement is true. But attending to basic mathematics, one may only look at the consequent and evaluate the statement as false because $\sqrt{-9}$ is not equal to 2. Do people read the statement in the content of mathematics? Or do they consider that as a logical statement, namely a conditional? Does context come first, compared with semantic and syntax?

In this study, I examine whether situating a conditional statement in the realms of mathematics and colloquial language can influence the way people determine them. I am inquisitive to see how a mathematician with a good background in elementary logic classifies such statements.

5.6. Summary of Chapter 5

Conditionals are recognized as the most highly researched topic among all logical operations, and the literature shows that conditional statements are a topic with which students have serious difficulties, especially when conditionals have irrelevant parts. The literature suggests that some other factors can modulate people's performance on tasks involving syllogism and conditionals, like emotions and content.

An understanding of logical implication is one of the essential prerequisites for understanding mathematics and constructing proofs. However, formal logic is not entirely applicable in everyday reasoning, and it cannot capture many sentences in daily language. This may be the reason that everyday language cannot be easily used to teach mathematical logic.

The context of a conditional statement can be a determinative factor to recognize and evaluate conditionals. For example, when Wason changed the wording in the Selection task from a logical puzzle into everyday language, most participants could make the correct selections. It is also frequently addressed that even educated people make inferences based on the context. This suggests adding the third component to syntax and semantics of logic, namely context.

The next chapter discusses some methods used to create data, and a model to prepare and visualize the created data for analysis.

Chapter 6. Methods

The first section of this chapter aims to review some of the data collection methods used in the current study, including different types of clinical interviews. The second section reviews Toulmin's argumentation scheme. This model can either be applied as a framework to analyze data or also as a method to organize the created data. Toulmin's argumentation scheme can prepare data for analysis by highlighting and relating important features of the data using a visualization technique. In this work, this model is solely applied as a mediator between collected data and data analysis to organize the gathered information.

6.1. Methods of data collection

Data collection methods can be divided into two main categories: primary and secondary data collection methods. Primary data are the original observations collected by the researcher for the first time for any investigation. The secondary data collection is to collect data from second-hand information. This data is already collected from another researcher for some purpose and is now available for other purposes. The data applied for this study is original and created by me.

6.1.1. Interviews

There are three main types of research interviews: structured, semi-structured, and unstructured. Structured interviews, also known as standardized interviews, are verbally and/or written administered questionnaires. In such methods to collect data, the interviewees are asked to answer a list of questions, with little or no variation and no follow-up questions to responses. Therefore, such interviews are relatively quick and easy to administer. However, the gathered data within this method only includes limited participant responses, and it does not make enough space for more profound ones. This approach aims to ensure that each interview is presented with the same questions in the same order.

Conversely, in an unstructured interview, there are no predetermined series of questions. It usually starts with an opening question, for example, “what do you know about irrational numbers?” Then the interviewer asks the following questions based on the interviewee's response. Such interviews are performed with little or no organization. Since there are no fixed interview questions in unstructured interviews, they can be time-consuming and difficult to moderate. However, this type of interview might be helpful when a significant 'depth' is required, or when nothing is known about the subject area, or a different perspective on a known subject area is required.

Semi-structured interviews can be regarded as a combination of structured and unstructured interviews. In this method, some main questions help answer the research question(s), but, like an unstructured interview, it allows to go further by asking for more explanation or creating spontaneous questions by the interviewer based on the interviewee's response.

6.1.2. Clinical interviews

Many things are not measurable such as mathematical thinking and mathematical understanding. These are intangible things, and we have access to what people write or say and not to what they think. Therefore, in mathematics education, we often need to use some substitutions to observe thoughts, such as speech and writing.

The word clinical here refers to a deep observation and usually, and the style of this method dates back to the time of Piaget. This method was adapted and developed by Piaget himself as an instrument for psychological research in 1920. He discussed clinical interviews as a research method in his introduction to The Child’s Conception of the World (1929). This method was the basis of his work for 50 years and was used to understand children's cognitive development better. The use of the flexible style of questioning allowed him to observe children's problem-solving behaviour while working on tasks and then asking questions fitted with the child's observed behavior. The ability to explore the interviewee's thinking in such a manner distinguishes the clinical interview from other methods. Hunting and Doig (1997) suggest that a clinical interview can be defined as follows:

> A dialogue or conversation between an adult interviewer and a subject. The dialogue centers around a problem or task chosen to give the subject every opportunity to display behavior from which to infer which mental processes are used in thinking about that task or solving that problem. (p. 3)

This method allows for some flexible questions, and the interviewer should be prepared for different answers provided by the interviewee and orient the interview efficiently to determine what he/she thinks.

Because of their flexibility, clinical interviews have been mainly used by researchers in qualitative research in mathematics education (e.g., Confrey, 2006; and diSessa, 2007), as they allow researchers to give the interviewer the option to make spontaneous decisions about when and how to explore an individual's thinking.

Ginsburg (1981) suggested that research into mathematical thinking has three basic aims: the discovery of cognitive processes, the identification of cognitive processes, and the evaluation of competence. He also claimed that the clinical interview is the most appropriate method for accomplishing these aims. The major property of this method is the contingency of questions, and the interview cannot be thoroughly planned in advance due to responding subsequently to the participant's responses.

In the context of mathematics education, among the three different sets of goals for a clinical interview – discovery, identification, and competence – clinical interview may especially work very well for the stage of identification, as a clinical interview can provide a close observation to individuals' ways of thinking which can reveal and describe the underlying cognitive process. It is typically not easy to recognize the cognitive process in an individual's activity, as sometimes the participants are not aware of such process. Goldin (2000) suggests that follow-up questions are designed to discover the interviewee's thought-process.

6.1.3. Different types of questions in a clinical interview

By reviewing some research studies in mathematics education that used clinical interviews as a data collection tool, Zazkis and Hazzan (1998) classified some types of possible questions and discussed their role in learning about students' thinking. They address clinical semi-structured interviews with cognitive orientation on the matter with students from kindergarten to university.

In what follows, I will briefly review some types of potential questions posed to the participants in clinical interviews proposed by Zazkis and Hazzan, including performance questions, unexpected “Why” questions, twist questions, give an example tasks, and reflection questions.

Performance questions

This type of question is designed to reveal the interviewee's familiarity and understanding of a specific topic learned before. This helps to have an understanding of the background knowledge of the interviewee about the subject matter. Such questions are often followed by the interviewer's request to explain how the answer was found, why an action or procedure was chosen, and how a decision was reached. Researchers who apply performance questions are usually interested in students' strategies, approaches, and conceptions rather than their performance.

Unexpected “why” questions

This type of question is one the most frequent question in interviewing about mathematics. “Why questions” are often asked to elucidate interviewees' responses. Such questions are typically asked in unexpected situations. For example, questions such as “why infinity is not a number?” or “why 2 cannot be divided by zero?” may look obvious. However, to answer such questions, the interviewee needs to provide an acceptable mathematical reason, and this communication may reveal the participant's thinking and understanding beyond the successfully applied algorithms or memorized rules.

Twist questions

Twist questions present a variation on a familiar situation, for example, “If $3x - 6 = y + 6,$ what is the value of $\frac{8^x}{2^y}$?”. This problem might be a twist question for a student who just learned the rules of exponents, but it could be a performance question for a calculus student. Twist questions can reveal some connections or ideas and promote a clinical interview (diSessa, 2007). Since “twist questions” are those within the boundary of interviewees' knowledge, to make sure that they are chosen/designed with an appropriate level of difficulty, we need to have prior knowledge about the interviewee's background. The difference between “performance questions” and “twist questions” is that the latter is on the boundary of the interviewee's knowledge, so they are often difficult questions to answer or not very straightforward to respond to.

“Give an example” tasks

A mathematical example can be an instance of a mathematical class with specified properties (a definition), a solution to a problem, an instance of a theorem, etc. In an interview, a researcher can ask the participant to create an example of a mathematical object or a mathematical property. For instance, “Give an example of a six-digit number divisible by 9”. Such questions are called “Give an example task”, and they present an alternative wording for a construction task or represent some examples and non-examples of a definition. More specifically, to see if one knows what a prime number is, the question could be “give an example of a prime number” and “give an example of a number that is not a prime”.

Construction tasks

In this type of question, participants are asked to build mathematical objects under some conditions. For example, define a polynomial $f(x)$ of the 4^{th} degree with x-intercepts of $x = 0,\ x = 1,$ and $x = 2,$ such that it passes through the point (4, $\sqrt{3}$). Such questions are helpful to check the interviewee's knowledge on a specific topic.

Reflection questions

In reflection questions, participants are required to reflect on a solution provided by someone else rather than to solve a problem. In such questions, the interviewee is asked to reflect on the solution provided by someone else; or even by themselves.

6.2. Toulmin's model of argumentation

In §§4.1.1, I introduced an argument in the context of logic, which is a collection of statements, and usually one of the statements is the conclusion, and the rest are the premises. It is expected that the premises of the argument justify the conclusion.

One of the first modern accounts of argumentation was developed by British philosopher Stephen Toulmin (1958) in his work on logic and argument, “The Uses of Argument”. It is usually referred to as “Toulmin's model” or “Toulmin’s scheme”. He offered a general account of the layout of an argument, and this method can be used as a tool for developing, analyzing, and categorizing arguments. Toulmin, Riek and Janik (1979) proposed that this model can be applied to any form of argument, including mathematics.

Toulmin's argumentation model has been used in mathematics education in order to analyze arguments: for example, Stephan and Rasmussen (2002), Inglis, Mejia-Ramos, and Simpson (2007), and Evans and Houssart (2004). According to Toulmin, argumentation consists of at least three essential parts, called the core of the argument: data, conclusion, and warrant, along with three additional, optional parts: backing, modality/qualifier, and rebuttal.

When one presents an argument, one is trying to convince audiences of a specific assertion. In Toulmin's framework, this assertion is referred to as the conclusion (C). To support the conclusion, the presenter makes use of evidence or data (D). The presenter's explanation for why the data prove the conclusion is referred to as the warrant (W). At this stage, the audience can accept the data but reject the explanation that the data establishes the conclusion; in other words, the authority of the warrant can be challenged. If this occurs, and for some reason, the warrant is not immediately obvious, then some justification or backing (B) is required. The modality/qualifier (Q) indicates the level of certainty contained in the argument. The final part, the rebuttal (R), occurs when the conviction in the argument is non-absolute. Figure 6.1 illustrates Toulmin's model.

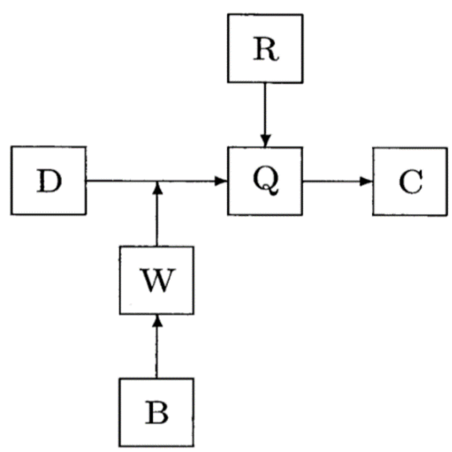


Figure 6.1. Toulmin's model

Each method analyzes and represents some aspects of an event, and Schoenfeld (2002) warned us to be careful in adopting/using different lenses, because with more than one camera, one can get more "coverage", but issues of focus remain. For the current research, in order to organize data and find the pattern of the interviewee's thinking model while engaging with the tasks, I use Toulmin's scheme model as a tool to prepare data for analysis. So, it is not used as a framework, and it only provides mediation between transcription and analyzing data.

There are two compelling reasons to choose this method. One is to organize and somehow visualize the collected data. The other reason is to depict conditional reasoning starting with the premise in the diagram and ending with one or several conclusions.

Toulmin's model adequately describes argumentation in the context of mathematics. Moreover, in this work, I found it useful in organizing some critical parts of the interview by visualizing the transcription to find patterns. However, we cannot simply use Toulmin's model for disagreement arguments or those with uncertainty and no conclusion. Because arguing for something does not have the same structure as arguing against it. In a refutation scheme, the arguer can claim a rebuttal at any point of the argument. Specifically, in a mathematical argument the rebuttal could be a counterexample which puts an end to the argument, meaning that with a counterexample, it is impossible to prove the conclusion.

6.2.1. Applying Toulmin's model to a formal mathematical argument

Many mathematical education researchers have used this model to analyze the process of formulating proofs, constructing definitions, and solving mathematical problems. For example, Simpson (2015) analyzed a model solution to a proof using Toulmin's scheme to predict what examiners may expect of students.

Also, Laamena and Nusantara (2019) investigated the use of backing and its relation to rebuttal and qualifier in prospective mathematics teachers' argumentation when constructing a mathematical proof about algebraic functions and applied Toulmin's model for the data analysis. The results show that the participants used specific types of backings when constructing a proof.

I noted in Chapter 4 that a formal mathematics argument is a *deductive argument* – an argument that the arguer intends to be deductively valid, that is, to provide a guarantee of the truth of the conclusion provided that the argument's premises are true and that such a guarantee is called a proof. Regarding formal mathematics, that includes proofs and is based on definitions, theorems, axioms, etc., there may not be any rebuttal or modal qualifiers. In what follows, I provide an example of a proof argumentation depicted by Toulmin's model.

Example 1. Prove the statement: If $a, b,$ and c are integers and $a|b$, then $a|bc$.

Proof: To prove this proposition, we begin with the assumption that $a|b$. Using the definition of "I": for some integer *k*, and $b = ka$. By multiplying both sides by c, we get $bc = (ka)c$. Since multiplication has the commutative and associative properties in the set of integer numbers, we conclude that $bc = (kc)a$. We, again, use the definition of "|", then $a|bc$.

Regarding Toulmin's model of the proof of this proposition, we begin with the premise and should end at the conclusion.

Figure 6.2a. Toulmin's model for Example 1

The following diagram shows the above proof; the boxes in gray represent backings:

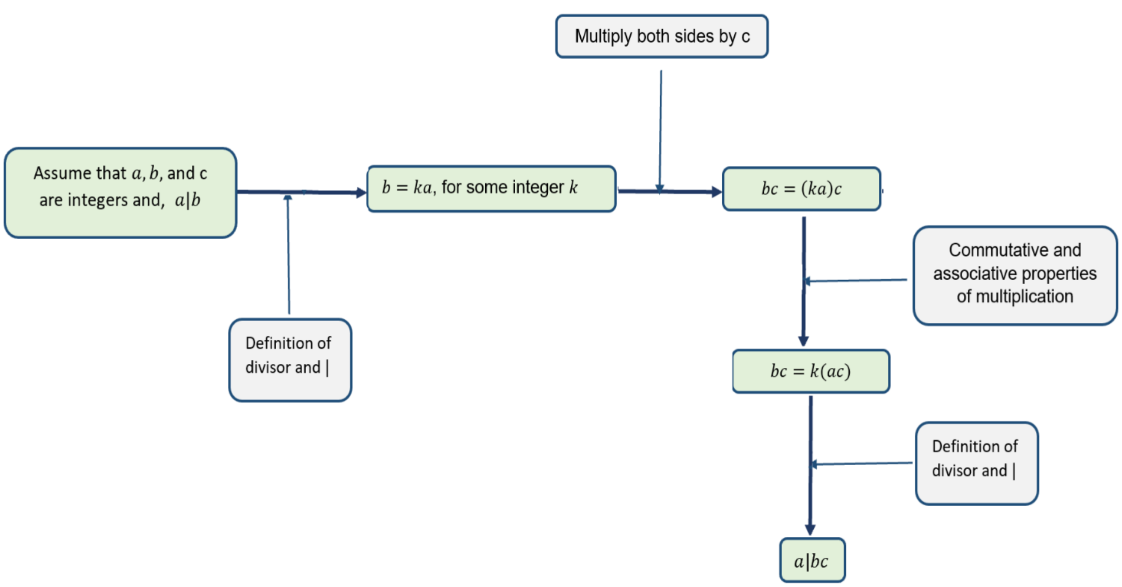


Figure 6.2b. Toulmin's model for Example 1 – proof

To fit the aim of research and data, some researchers suggested some changes to Toulmin's model (e.g. Langsdorf, 2011; and Prusak, Hershkowitz, & Schwarz, 2012). More related to our discussion, Inglis, Mejia-Ramos, and Simpson (2007) reason that, if the backing component is assumed to be a theorem or a definition, there would not be any rebuttals in the argument.

Regarding rebuttals, specifically for a theorem, if there are any, it must be considered in the argument's assumptions. So, under the given assumptions of the argument, there would not be any rebuttals. Therefore, we can delete boxes for qualifiers and rebuttals in mathematical proofs.

There is a subtle difference between rebuttal and counterexamples. A wrong mathematical statement is expected to have counterexamples, however in such cases, the mathematical argument can also move to adjust the premises/conditions, or the domain of the variables involved in the statement for which the conclusion holds, these will serve as a rebuttal in Toulmin’s scheme. Therefore, rebuttals suggest alternatives or new conditions under which the conclusion holds, but counterexample is the one that refutes the conclusion. The existence of counterexample can affect the argument; either counterexample disproves the conclusion and stops the argument, or results in revising the conditions and proving the conclusion under the new conditions.

6.2.2. Applying Toulmin's model to an informal mathematical argument

Chapter 4 discussed that mathematics is not only definitions, theorems, proofs, examples, counterexamples; there is a huge hidden part on the back of each of these formal constructions, which I refer to here as “informal mathematics”. There is a long process while discovering such mathematical objects. This discussion may not entirely be in the realm and language of formal mathematics. There are many hedges in such parts, and there is no guarantee about the truth of the claim until it is formally proved. In such cases, it could be said “p probably results in q”, “there might be a contradiction for this conjecture”, “this must be true”, “we can prove this”, and there are many other instances of hedging, however even when they say “this conjecture is true”, it is still hedged, because only those with mathematical proof are acceptable as true conjectures, known as theorems.

Therefore, for the purpose of informal mathematics, we need boxes for **modal qualifiers** to represent the level of certainty. Also, for the mathematics context, we may need a box for **backing the rebuttals**. For instance, for analyzing the structure of an argument in informal mathematics, Laamena and Nusantara (2019), the following problem was given to the participants:

> Suppose the function f: $\mathbb{R} \longrightarrow \mathbb{R}$ by the formula $f(x) = x^2$ and g: $\mathbb{R} \longrightarrow \mathbb{R}$ by the formula $g(x) = x$. Investigate whether $f(x) \geq g(x)$, for all x real numbers?

The participants provided different arguments, some including uncertain modal qualifiers and some with rebuttals. Since it is not always true that $f(x) \geq g(x)$, then there must be a rebuttal.

I selected the following two diagrams from Laamena and Nusantara (2019). In the first one, the participant could figure out that for negative numbers $f(x) \geq g(x)$; however, the modal qualifier he used, “maybe”, implies that he had doubts about the statement's truth. In the second model, the participant provided a counterexample to disprove the conjecture.

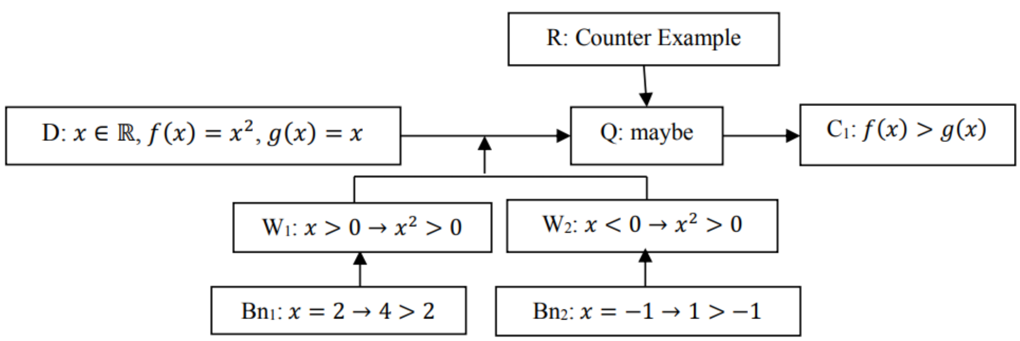


Figure 6.3a. Toulmin's model a

One of the numerical backings in the below diagram is a rebuttal, $x = \frac{1}{2}$. It is a counterexample that assures the participant to claim that for any real number x, it does not always happen that $x^2 \geq x$. It is the reason they do not try other real numbers.

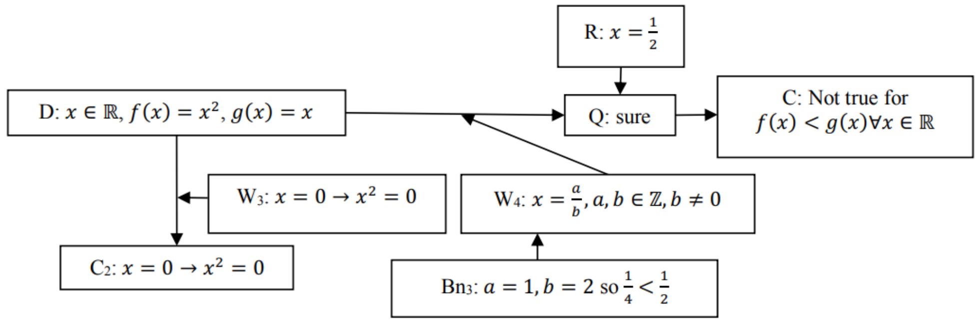


Figure 6.3b. Toulmin's model b

6.2.3. Adjusting Toulmin's argumentation scheme

Toulmin's scheme can provide a method to separate and organize parts of arguments, namely, data, qualifiers, conclusions, rebuttals, backing, and warrants. However, we are not limited to its original structure introduced in the introductory part of the current section. For example, there have to be some extra parts in a refutation scheme because there is always a reason or backing to refute something. So, we may need a box for the "backing of rebuttals". In the remaining part of this sub-section, I suggest adjusting Toulmin's scheme applicable to my research.

Campione and Véronis (2002) observed that a typical speech consists of pauses, and they classified them into three types: short (0.15 seconds), medium (0.50 seconds), and long (1.50 seconds) pauses. Further, they note that spontaneous speech (speaking without reading) shows more frequent use of medium and long pauses. Also, Maclay and Osgood (1959) classified hesitation pauses into "filled pauses" and "unfilled pauses". Filled pauses are all occurrences of the English hesitation sounds such as "errr", and "umm". Unfilled pauses include silence.

Osgood and Sebeok (1954) suggested that the hesitation pauses anticipate sudden increases in information or uncertainty in the message being produced, and such pauses will tend to occur at points of highest uncertainty in spontaneously produced utterances. Furthermore, Grosjean and Deschamps's (1975) research shows that we expect more pauses if the communicative task is more complex. Considering the nature of the clinical interview, I decided to consider long pauses (at least 1.50 seconds), both filled and unfilled pauses and added them to Toulmin's diagram. My reason for such changes is that when there are such pauses, this may mean uncertainty in the answers, and the level of certainty is important to make sure the participant is confident about the answer.

The participant's arguments are in informal mathematics and in such circumstances, there could be a significant number of hedges. Therefore, for the purpose of informal mathematics, we need boxes for modal qualifiers to represent the level of certainty. Also, for the mathematics context, we may need a box for backing the rebuttals. We do not typically accept a counterexample without reason. So, I will use all six components of Toulmin's model with no restrictions, namely data, qualifiers, conclusions, rebuttals, backing, and warrants.

Furthermore, it may be useful to add the other person's idea to Toulmin's diagram or sometimes depict two different arguments in one diagram. In such a case, when a sentence or any part of an argument made by the interviewer is to be added to the argumentation scheme of the interviewee, I make it in a double lined box.

In a Toulmin diagram, the boxes through the main line represent “conclusions”, above/below the main line represent “warrants” or “rebuttals”. Also, each rebuttal and warrant can be justified by a box for “backing”. The number of conclusions, the duration of pauses, and voice pitches show the level of uncertainty in the response. To represent the voice pitches, I use ↓ or ↑. The following diagram shows the adjusted Toulmin’s scheme:

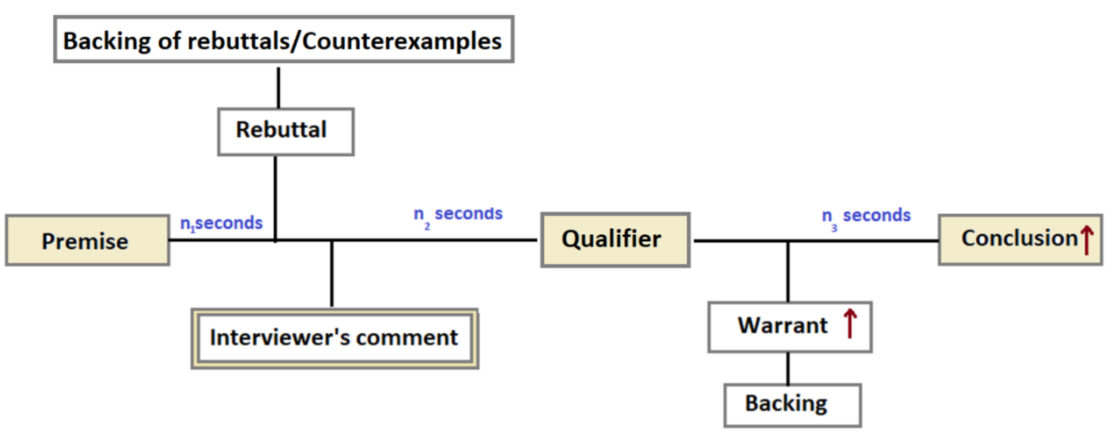


Figure 6.4. Adjusted Toulmin's model

6.3. Summary of Chapter 6

One of the most popular methods among data collection methods in mathematics education is the clinical interview. In this method, the interviewer prepares some problems in advance, but not just limited to those questions. In order to have a more extensive observation of the individual's thinking, this method allows for a flexible style of questioning.

Based on research in mathematics education, different types of questions in a clinical interview with the type of questions fitted with the purpose of the study have been discussed in this chapter.

In §§4.1.1, I introduced an argument in the context of logic, which is a collection of statements, and usually one of the statements is the conclusion, and the rest are the premises. It is expected that the premises of the argument justify the conclusion.

§6.1 introduces Toulmin's argumentation model that is a general account of the layout of an argument, and this method can be used for any form of argument, including mathematics and has been used by many mathematics educators. According to this model, an argumentation consists of data, qualifiers, conclusions, rebuttals, backing, and warrants.

Toulmin's scheme is an effective method to separate and organize different parts of an argument and is not limited to the original structure introduced by Toulmin. In order to fit the Toulmin's model with the research reported in this thesis, I add some extensions to the original structure such as:

- Including some extra boxes in a refutation scheme namely "backing of rebuttals".
- To depict a conversation using Toulmin's model, or add some other person's comment, I use a double lined box.
- Forasmuch as the duration of pauses and voice pitches show the level of uncertainty, I add them on the models if applicable.

The next chapter will review some popular theories of reasoning in mathematics education.

Chapter 7. Theories of Reasoning

Reasoning is the process of using existing knowledge to draw conclusions, make predictions, or construct explanations. Johnson-Laird and Byrne (2002) argue that people reason about conditional relations because much of our knowledge is conditional. For example, if one cuts his finger, then it bleeds; if you study hard, you pass the exam; or you can buy a coffee if you have a $10 bill.

Section §3.1 briefly discussed four conditional inferences: *modus ponens*, *modus tollens*, affirmation of the consequent and denial of the antecedent – and that only *modus ponens* and *modus tollens* are valid for the conditional interpretation.

Conditional reasoning is a central part of our thinking; however, people do not always reason correctly based on logic rules. Many studies have investigated the four conditional inferences, and some psychologists (e.g., Macnamara, 1986) argued that the validity of *modus ponens* and *modus tollens* depends on their logical form. On the other hand, Lycan (1993) refutes that *modus ponens* inferences are valid only in virtue of their form. He declares that in certain contexts and with certain interpretations, such inferences are valid. This suggests that formal rules are blind to content, but individuals are sensitive to content.

Johnson-Laird and Byrne (2002) suggest that reasoners usually reject unbelievable conclusions, even when these are logically correct. For instance, in a study carried out by Santamaría, García-Madruga and Johnson-Laird (1998), participants were given the following premises:

If Ann is hungry, then she has a snack.

If she has a snack, then she eats a light supper.

Using the transitive property of conclusion, it is inferred that:

If Ann is hungry, then she eats a light supper.

However, the participants declared that nothing followed from the premises. The reason could be that having a light supper seems unbelievable to people for the assumption that "Ann is hungry". This may violate what we naturally learned and the presupposed causal relation. Johnson-Laird and Byrne (2002) argue that people's beliefs modify the models they construct, and they construct the null model when there is some conflict. Null models represent contradictions when no further conclusions can be drawn.

In this chapter, I describe some theories proposed to explain conditional reasoning that successfully attract supporters; however, they explain general reasoning, and their application to conditional statements is not straightforward.

Most of theories of human reasoning have attempted to explain Wason's selection task results. One reason might be related to the observation that people make systematic and non-random errors while doing this task. There are two hierarchically related distinctions between the different theories. Some philosophers and psychologists see Wason's task as a problem in formal logic. From this view, non-probabilistic theories of reasoning have been developed to explain the systematic errors people make. These include the mental model theory of reasoning (Johnson-Laird & Byrne, 1991) and mental logic (Rips, 1994).

Alternatively, some researchers see the problem as a probability question. For example, regarding Wason's selection task, Oaksford, Chater, and Grainger (1999) state that:

> We do not deny that the selection task has a logical interpretation: If the rule is interpreted as only applying to the four cards presented in the task, then the task can be construed purely deductively. However, we do deny that this is the most natural interpretation of the task. Interpreting an if … then statement as restricted to a domain of only four objects is pragmatically odd. (p. 236)

Such views resulted in probabilistic theories, such as the information-gain model, Bayesian rationality, and dual-process theory of reasoning. Among all potential theories of reasoning, I briefly review a few that appear related to my research area, namely dual-process theory of reasoning, information-gain model schemas, Bayesian rationality, theory of formal discipline, mental logic theory, and mental model theory.

7.1. Probabilistic theories of human reasoning

In probabilistic theories, everyday reasoning is best explained probabilistically rather than logically, and even though each statement has its own logical form, logic may not account for conditionals in everyday reasoning. These theories of reasoning explain the errors people make while reasoning in terms of different normative standards, that is not based on the logic rules. This section explains three probabilistic theories, namely “dual-process theory of reasoning”, “information gain model”, and “Bayesian rationality”.

7.1.1. Dual-process theory of reasoning

Recent research in the psychology of reasoning suggests two processing systems involved in human reasoning, namely system-1 and system-2. These two systems are more like the two poles of a continuum. This allows having many dual-processing accounts of human reasoning, because there could be multiple interpretations of system-1 processes resulting in developing different kinds of dual-process theories (e.g., Evans,1989; Wason & Evans, 1975). However, all such theories make a distinction between the two systems, one with fast and unconscious processes mainly based on implicitly acquired knowledge, and the other with the slower and deliberative cognitive process with more analytic uses of formal rules (Evans, 2007; Stanovich 2004). Even though system-1 is a faster process, the experience can influence the system-2 process, and it can be accessed even faster than system-1. My particular interest seems to be close to dual-process accounts of human reasoning and related higher cognitive processes, such as decision-making.

Regarding how reasoners determine whether statements are true or false, Evans, Handley, and Over (2003) note that reasoners do not use truth values; rather, possibilities highly depend on their probability. This probability is mostly derived from the background knowledge of phenomena and their relations. §5.1 and §5.3 included a discussion of the systematic errors made by participants in Wason's task. Regarding such errors, Evans and Stanovich (2013) explain that:

> we are aware of too much specific evidence of qualitative differences between reasoned and intuitive responding to find this argument plausible . Also, when given the opportunity, most participants can explain the reasoning that led to a correct answer, but we are not aware of a single instance of a participant reporting an established bias like belief bias or matching bias as the basis for a wrong one. On the contrary, participants giving a matching response on the Wason's selection task are known to rationalize their answer with reference to the logic of the task .And, as already mentioned, the two kinds of answers are associated with different neural regions and differentially correlated with cognitive ability. (p. 235)

The dual-process theory is a general theoretical framework, and there is no explanation as to where system-1 heuristics come from, whether they are innate or learned, and how one can measure this. Also, it does not explain how the two sorts of reasoning in system-1 and -2 work together. Other theories focus on two thinking processes, namely system-1 and -2; however, not all of them have the probabilistic approach. For example, the mental model theory also distinguishes between the two systems regarding mental models and working memory.

7.1.2. Information-gain model of reasoning

Evans, Newstead, and Byrne (1993) reviewed studies about reasoning in Wason's selection task, the decision making, and the rational nature of reasoning in such realistic contexts. They indicate that most people consistently provide incorrect answers according to formal logic and, surprisingly, they make errors that are not random. The point here is that such non-normative answers are called “errors” because the performance is compared with formal inferences that are “correct” according to logic rules.

Rather than viewing the systematic errors in Wason's task as error-prone, Oaksford and Chater (2001) argue that performance has been compared with the wrong normative standard. The information-gain model is one of the probabilistic theories arguing that this behaviour could be explained in terms of the rationality of people's reasoning, which is uncertain reasoning, rather than certain reasoning that is based on formal logic rules.

To explain such errors, several models relying on probability theory rather than on logic have been proposed. The optimal data-selection model, also known as the “information-gain model” proposed by Oaksford and Chater (1994), explains the errors made in Wason’s selection task by interpreting such a task as a decision-making problem, not a problem related to logical reasoning. They also indicate that:

> We suggest that people are rational but must define rationality in terms of optimal performance in real-world, uncertain, inductive tasks rather than purely in terms of deductive logic. (p. 628)

This model proposes the rational strategy assuming that people independently select cards based on the expected information gain of turning a particular card. Therefore, this theory suggests that people will only turn over cards with a high probability of resulting in information gain by reducing uncertainty. In this theory, probability refers to a reasoner's degree of belief.

7.1.3. Bayesian rationality: the probabilistic approach to human reasoning

Stanovich and West (1998) maintain that, while many theories are about System-2 processes, results from Wason's task suggest that only about 10% of participants that are university students apply System-2 processes when reasoning. Oaksford and Chater (2001) argue that a probabilistic approach to these processes can explain people's performance as a reasonable attempt to make sense of the task, and they adapt and learn such reasoning to cope with the uncertainty of the everyday world.

Oaksford and Chater (2007) developed Bayesian probabilistic models of data selection (Wason's selection task), conditional inference, and quantified syllogistic reasoning tasks. Bayesian Rationality mainly focuses on the success of human reasoning under uncertainty. From this view, everyday thought is a rich source of probabilistic reasoning. The probabilistic approach to human reasoning is exemplified by the information-gain model for Wason's selection task. In recent years the probabilistic approach of Wason's task has been extended to syllogisms and conditional inference.

Bayesian rationality extends the information gain model to conditionals; however, Johnson-Laird and Byrne, who developed the theory of mental models, suspect that people base their choices on conditionals on the extended gain in information. They declare that if the probabilistic approach of the selection task is correct, then manipulating the probabilities of the premise, and the consequent of the conditionals, would affect participants' performance. In other words, the logically correct selections should be more frequent when the probabilities of both parts of a conditional are high than when they are low. However, some research studies refute this idea (e.g., Oberauer, Wilhelm & Diaz, 1999).

7.2. Non-probabilistic theories of human reasoning

The theory of Bayesian Rationality provides a model that can explain people's errors in Wason's task better than its logic rivals. However, it still has no answer to the question:

Where do the conclusions come from? That is, given a set of premises, how do reasoners produce their conclusion? The main theoretical competitors to the probabilistic approach are mental logic theory and mental model theory. Whatever their deficiencies, these theories can provide answers to such questions. Besides these theories, another one is the theory of formal discipline, empowered from a common belief that learning logical reasoning can develop general reasoning skills.

7.2.1. Theory of formal discipline

In the late 1980s, some psychologists suggested that some reasoning skills could be developed in teaching. Larrick, Nisbett, and Morgan (1993) argue that people do have inferential rules corresponding to some inferential rule systems, including probability and statistics, and the rules of cost-benefit decision theory.

Theory of formal discipline maintains that studying mathematics skills, specifically logical reasoning and problem solving, can develop students' general reasoning skills[7] useful in many areas of life. Although this theory is endorsed by many influential figures in philosophy, mathematics, and educational policymakers, many psychologists disagree.

Inglis and Attridge (2016) compared the performance of mathematicians with that of literature students and reported that mathematicians could provide more normative answers between groups, and the difference was larger on the conditional inference than on the Aristotelian syllogism task. However, there might be an alternative reason for the observation that mathematicians are better at reasoning than other people. The authors then came up with a new question, namely "filtering hypothesis": Is it true that those who choose to study mathematics already respond differently, compared with those who do not? Given this question, they conducted another piece of research. In this one, they focused on abstract conditional inference and found that the two groups, as mentioned earlier, did not differ. Therefore, it was concluded that studying mathematics could help students provide more normative answers.

However, in the study carried out by Inglis and Attridge (2016), there is a common mathematical format for each of the designed tasks. More particularly, all thematic

[7] In Chapter 2 of their book, Inglis and Attridge (2016) discuss that what exactly these skills are and how they can be measured.

conditionals to assess how participants apply logical rules follow a certain style. Therefore, there is no guarantee that mathematics students can do better than literature students in more general tasks with different patterns in everyday language. In Appendix B of their book, all conditionals are designed in a mathematics context. Also, even though all conditionals in Appendix A are in everyday language, their specific style ensures participants follow a certain pattern and may set them in a mathematics context or check with the rules of inference. This may explain why the difference between the two groups' responses was smaller on the Aristotelian syllogism than on the conditional inference task. If mathematics can improve students' reasoning in the context of mathematics or even with rules of inference, this does not mean that it improves general reasoning. In the following, I recite two of these syllogism tasks:

(1) The premises: If oil prices continue to rise then UK petrol prices will rise.

Oil prices continue to rise.

The conclusion: UK petrol prices rise.

(2) The premises: If oil prices continue to rise then UK petrol prices will rise.

Oil prices do not continue to rise.

The conclusion: Then, UK petrol prices do not rise.

7.2.2. Mental logic theory[8]

Many psychologists have taken the Aristotelian view related to human reasoning. In this view, people first recognize the logical form of the premises and then apply rules of inference that match this form to prove that the conclusion can be derived from the premises. For instance, everyone agrees that the following syllogism is valid:

If I have $10, then I will buy a pen.

I have $10.

I will buy a pen.

[8] The mental logic theory is also sometimes referred to as "natural deduction theory", "inference rule theory" or "mental rules theory".

In this model, the question is not about the possibilities in which I change my mind and, rather than buying a pen, I go and buy a car. The question is, how does the mind carry out such reasoning? How does the mind draw inferences in real-time?

Logical inference can be seen as applying rules to sentences. So, a possible answer to this question would be that the mind reasons by applying logical rules to sentences, which is what a mental logic approach involves. The main idea of this model of reasoning goes back to Piaget, who claimed that individuals could construct an unconscious logical model that enables them to reason, which can then be developed and allow them to learn and use logic and mathematics (Piaget & Inhelder, 1958).

Regarding mental logic theory, Braine (1978) and Rips (1994) claim an inbuilt logical system that guides human reasoning. This theory argues that a person constructs a sort of mental proof and then verifies it. The mental logic theory distinguishes between direct rules of inference, which are applied effortlessly, and indirect rules, which require conscious effort and are much more error prone. Braine and O'Brien (1991) explain errors in reasoning by reference to the acquired experience and other contextual effects. However, many people have never learned logic, but they can nevertheless reason well (Stanovich, 1999). Also, Chao and Cheng (2000) argued against this theory by showing that pragmatic rules develop before generalized logical rules. This finding may contradict Rips' (1994) claim that the natural deduction system is innate.

Johnson-Laird (2010) criticizes the theory of mental logic and argues that formal rules of inference do not play any important role in the mental processes applied in everyday reasoning. Even though learning logic is necessary and very useful and it can enable reasoners to evaluate the sentences, we cannot deduce that people use logical forms in their everyday reasoning.

Every method emphasizes and de-emphasizes things. Nevertheless, we may ignore whatever that is de-emphasized because we cannot catch it properly. Even though logical form plays role in theory of mental logic, it cannot explain how people identify the logical form of a sentence in daily language.

7.2.3. Mental model theory of reasoning

The mental model theory of reasoning proposed by Johnson-Laird (1983) is one of the famous theories of cognition and reasoning. It is an alternative theory to mental logic, and it almost denies that reasoning involves formal operations over logical forms. In this view, instead of following logical rules during reasoning, people reason over models in which such forms are true. Such models are constructed in individuals' minds which they modify and reason; these models are concrete representations of situations rather than abstract things assumed in mental logic.

Within the mental model theory, individuals may not be aware of the logical form of the sentences, and they consider possibilities fitted with the information using the meanings, context, and their own knowledge. Barwise (1989) stated that everyday reasoning is not a formal process, and psychological theories based on formal rules cannot give any method to recognize a logical form of a sentence in everyday language. Unlike logic, the grammar of a sentence in everyday language is not sufficient to determine its logical form, whereas it can be determined from the meanings of the sentence in context and the individual's knowledge about the topic. In mental model theory, a conclusion is necessary if it holds in all the models of the premises, a conclusion is probable if it holds in most of the models of the premises, and a conclusion is possible if it holds in at least some model of the premises (Johnson-Laird, 1997).

Although the theory of mental models gives a dual-process account of reasoning, it is mainly explained in terms of mental models. It shows that system-2 has access to working memory and searches for alternative models, while system-1 does not have access to working memory and only works with one model at a time, one that is most easily accessible. (Johnson-Laird & Khemlani, 2014).

Sections §7.1 and §7.2 briefly introduced some popular theories of reasoning working specifically with conditional statements and their deficiencies. Among all competing theories of reasoning discussed in this section, I choose the theory of mental models as the framework to explain the research studies in the next two chapters. The next sections provide details for the choices that guided my investigation and more details about the theory of mental models.

7.3. Mental model theory as the framework for the current study

The theory of mental models (Johnson-Laird, 1983) mostly accounts for the informal arguments in science and daily life, counterfactual thinking (Byrne, 2005), and probabilistic inference (Johnson-Laird, 2006). Counterfactual thinking is a concept in psychology involving the human tendency to create possible alternatives to life events that have occurred and focus on how the past might have been. The most common example of such thinking is evaluating past decisions that were once possible but are now impossible as their time has passed. Considering how past decisions might have worked out is a common human thought process that may improve decision-making abilities. So far, the theory of mental models has been used to explain performance in some areas such as language comprehension, analogical reasoning, and deductive reasoning.

Johnson-Laird (1983) notes that, when a new context is encountered, the reasoner goes through the following three stages:

- They look at the premises and create a mental model of the possible context they find themselves in.
- They form a non-trivial conclusion that is based upon the premises of their model.
- They look for counterexamples to their model and conclusion. If they cannot find any, then they accept the conclusion.

Johnson-Laird and Khemlani (2014) discuss that reasoners build models of premises and base their inferences on them. In this view, mental models can be constructed from perception, imagination, or the comprehension of the context.

Mental model is a theory that allows individuals to withdraw conclusions that conflict with the facts. However, theories based on formal rules of inference, such as the theory of mental logic, do not explain how individuals discover that the assertions are inconsistent and how they can rectify this inconsistency.

7.3.1. How does mental model theory work?

Johnson-Laird (1983) proposes that, with this theory, individuals assess a sentence in everyday language, and they consider the meanings of the premises, take context and knowledge into account, and then outline the possibilities fitted with this information. Therefore, if a conclusion holds in each of such possibilities, they then consider that the inference is valid.

According to mental model theory, reasoners usually work with just a single mental model, which suffices for intuitions. Intuitions are judgments that are made by the mind that is perceived by the unconscious. Such judgments may exhibit intelligence, but the processes by which they are generated are not well understood. Although intuition is sometimes taken lightly, it has played a significant role in scientific discovery.

In mental model theory, individuals use their own experience to decide. As an example, Johnson-Laird (2010) invites us to consider the following logical form:

If she played a musical instrument, then she did not play the flute.

She played the flute.

So, she didn't play a musical instrument.

He argues that people are less likely to make this inference because they know that a flute is a musical instrument. Therefore, if she plays the flute, she already played a musical instrument, and as a result, the above conclusion would be false. The meaning of the clauses in a conditional, the relations between antecedent and consequent, and the individual's knowledge about the context can all affect the core meaning of the conditional. To be more precise, knowledge about the context can add information to models of possibilities, and the meaning of clauses and the relationship between them can block the construction of models, which can reduce the number of possibilities. For instance, in the above example, the knowledge that a flute is a musical instrument can block the construction of the possibility that she did not play a musical instrument. As a result, individuals only consider the possibilities with a true conclusion; in other words., “either she played a musical instrument and didn't play the flute, or she didn't play a musical instrument and didn't play the flute”.

Regarding how the theory of mental models works, Johnson-Laird (1997) explains that this theory integrates deduction, probabilistic reasoning, and modal reasoning. A conclusion is *necessary* if it holds in all the models of the premises; a conclusion is *probable* if it holds in most of the models of the premises; a conclusion is *possible* if it holds in at least some models of the premises.

Although mental logic and mental model theories both give logic a central role in human reasoning, they explain irrationalities differently. For example, mental logic theory may explain errors in terms of the accessibility of different rules, whereas mental models explain errors in terms of limitations in how mental models are constructed and checked and how many models must be considered (Oaksford & Chater, 2009a).

7.3.2. Relevance between the parts of a conditional Statement

Some theorists have defended a defective truth table in which a conditional has no truth value when its antecedent is false. The following table represents a defective truth table:

Table 7.1. Defective truth values

p	q	$p \Rightarrow q$
True	False	False
True	True	True
False	True	Irrelevant
False	False	Irrelevant

For example, Braine and O'Brien (1991) considered some sort of relevance between antecedent and consequent, and they claim that conditionals are only asserted if the consequent follows with certainty from the antecedent. So, conditionals such as "if 2 is odd, then the sky is purple" are no longer true because of the false antecedent. However, using their definition, conditionals such as "if 2 is even, then the sky is purple" can still be asserted. Also, any conditional whose conclusion is known to be true, such as "if the sky is blue, then 2 is even", will be asserted too; nevertheless, this type of conditional still violates intuitions of relevance. Oaksford and Chater (2009b) criticized such a system and said that "despite Braine and O'Brien's intentions (ibid.), it does not seem to enforce relevance between antecedent and consequent" (p. 107).

Regarding this issue, Johnson-Laird claims that people usually judge that the conditionals with false antecedents, like "if there is a circle, then there is a triangle" when there is no circle there, are neither true nor false. Instead, they assume it as *irrelevant* (e.g., Johnson-Laird & Tagart, 1969; Evans, 1972). Therefore, in mental model theory, no explicit mental model represents the possibilities in which the antecedent is false, so that people may assume the conditional to be neither true nor false but irrelevant (Johnson-Laird & Byrne, 2002).

7.3.3. What is a mental model and what does make it different from other mental representations?

Imagine a triangle that is on the right of a circle. This may have a single mental model for each person. The model represents some possibilities that have in common only that a triangle is on the right of a circle. Of course, some factors like the relative sizes of the figures in the model and their distance apart play no role in reasoning from the model, but everyone prefers to think about just one possibility at a time. Mental models are psychological representations of real, hypothetical, or imaginary contexts. They can be used to reason deductively by ensuring that a conclusion holds in all the models of the premises (Johnson-Laird & Byrne, 1991).

Three main assumptions distinguish mental models from mental representations. First, each mental model is a possibility that is common to a whole set of possibilities. For example, when we toss a die, there are six different mental models to represent all possibilities, e.g., one model represents that the die came up one, the next model can represent that it came up two, and so forth. Also, mental models are based on the truth principle; in other words, they represent only two possible contexts – true propositions and false propositions – and each model of a possibility represents only true propositions. There are two possibilities to come up with 1 in tossing a die: it comes up 1, and it does not come up 1. The possibility considered as a mental model is that the die comes up 1. Third, mental models are iconic, and it means that each representation corresponds to each part of what it represents.

7.3.4. Where do the mental models come from?

Some mental models are acquired through experience particular to the individual. Also, many mental models come from experiences that are particular to an environment and so are widely shared within one society but not necessarily in others (Berger & Luckman, 1966). When people's mental models are well adapted to the task at hand, familiar possibilities can put them in a better position. Ross and Nisbett (1991) maintain that time and energy are saved in such a context, hesitation and doubt are reduced, and nothing important is lost.

Although mental models enable thought and action, they can also constrain them. Since we have limited power of observation, we may ignore many details that can be observed in an event and violate our assumptions based on what our mental models suggest. They might not be in accordance with the real world and may limit the amount of information that decision-makers use and cause them to fill in uncertain details of a context with incorrect assumptions. So, it is still likely to miss some event features and just rely on mental models. Therefore, mental models are influenced by reasoners' memory and experiences and their own reasoning strategies (Vandierendonck, Kemps, Fastame, & Szmalec, 2004).[9]

7.3.5. Illusory inferences

Johnson-Laird and Savary (1999) introduced “illusory inferences” that arise from mental model theory. They define an illusion as one for which the mental models of the premises yield a conclusion that is different from the correct conclusion. Since there are a few known examples of illusions, I recall the one discussed by Johnson-Laird and Savary (1999) and Johnson-Laird and Byrne (2002). In the following task, the participants were given the following assertions for a certain hand of cards and were asked to choose the correct answer:

Suppose that you are playing cards with Billy, and you get two clues about the cards in his hand. You know that one of the clues is true and that one of them is false, but you do not know which one is true and which one is false:

- If there is a king in his hand, then there is an ace in his hand.
- If there is not a king in his hand, then there is an ace in his hand.

Now, select the correct answer:

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[9] More explanation about different sources of such models can be found in World Development Report (2015). But, since the report is provided by the World Bank’s behavioural sciences team, who works with project teams and governments to diagnose, design, and evaluate behaviourally informed interventions, I found some of the discussed sources unrelated to the purpose of this work.

a) There is an ace in Billy's hand.

b) There is not an ace in Billy's hand.

c) There may, or may not, be an ace in Billy's hand.

Most of the participants selected (a), i.e. there is an ace in Billy's hand.

In the given problem, there is a condition, in the beginning, that is, "only one of the given conditionals is true". Therefore, only one of them could be false. Let the first conditional be false, that is, "If there is a king in his hand, then there is an ace in his hand". Therefore, the second conditional must be true (recall that a conditional is false only if the premise is true and the conclusion is false). In such a case, if we assume that there is a king in Billy's hand since the first conditional is false, it results that there is not an ace in Billy's hand. In the other possibility, if we assume that there is not a king in Billy's hand since the second conditional is true, then there is an ace in his hand. Therefore, there is no guarantee that there is an ace in the hand even when there is a king in the hand. If the second conditional is false, with an analogous argument, we cannot say for sure that there is or is not an ace in Billy's hand.

Such inferences have conditionals that may appear compelling to individuals but have invalid conclusions. None of the theories based on formal rules can explain the errors in such inferences; however, mental model theory can explain this phenomenon.

7.3.6. Mental model theory's deficiencies

The theory of mental models is not without its critics. It is incomplete and likely to have its own problems and deficiencies. One issue with this view is that it offers no clear explanation of how individuals were able to devise logic and mathematics if they were incapable of deductive reasoning beforehand because individuals untrained in logic are still able to assess whether or not a set of assertions is consistent (Johnson-Laird, Girotto, & Legrenzi, 2004).

Also, since models are created via meaning, a second issue is how meaning and context influence representation and how reasoners translate world knowledge into mental models, and how the models are constructed in reasoners' minds (Morris, 2010).

7.4. Why mental model theory?

This section attempts to provide some reasons why all the theories explained in this chapter are rejected, and finally chooses mental model theory as the theory applied for the current work.

7.4.1. Reason not to apply dual process theory

I believe that system-1 plays an important role in explaining the way individuals think in many situations; however, this may still not work for many conditionals. So, reasoners may not have any starting point, and therefore, there is nothing to direct them. However, the question is that does this (having less or no information about context) make the problem more confusing? I think it mostly depends on the task; for example, consider the following conditional:

If A is a Banach algebra, then for every Banach $A-$bimodule $X, H\ (A, X') = 0$.

I can ask one either to “evaluate it” or to “determine whether it is a conditional statement or not”. Having no relevant information may make it impossible to evaluate such a statement, but to decide if it is conditional or not, more information may result in more confusion. It might be easier to decide about the above statement than the statement “if the elephants could fly, then you win the lottery”.

I also believe that the nature of a question (the area and the way we design conditional statements) can affect the way people think, which then they may pose another situation in thinking about different conditionals that result in different types of argumentations. Therefore, this theory does not fit very well with the current study.

Although the dual process is on the stronger theoretical ground than many other theories, choosing problems from a different area or with a particular design may result in different types of argumentations. Therefore, the dual-process theory might not be the most fitted lens with the designed tasks in the empirical part of this study. My preliminary investigation indicated that the interviewee does not rely on his intuition; rather, he was more dependent on his working memory, even more than logical inference rules. Because of the focus on meaning rather than underlying syntactic structure, mental model theory can better account for the interviewee's performance than the mental logic.

7.4.2. Reason not to apply information gain model, and Bayesian rationality

In Chapter 5, I noted that in many reasoning tasks, people fail to select the response that standard logic dictates. When comparing probabilistic theories rather than logic, such as Bayesian rationality and more specificaly the information-gain model, participants' reasoning is seen in a more positive light because they can explain people's performance as a reasonable attempt to make sense of Wason's selection task and have been extended to syllogisms and conditional inference. Nevertheless, this theory does not explain the resources of the conclusions and does not explain why they answer in such a manner.

7.4.3. Reason not to apply theory of formal discipline

In a mathematics or business situation, people may show different reactions to conditional statements compared with everyday situations. Also, when mathematicians or students with some background in logic encounter a conditional statement, and have no doubt that it is conditional, they may evoke logical inference rules; however, we cannot generalize this result to the other contexts.

If advanced mathematics can improve logical reasoning, then one may expect mathematicians to apply their logical reasoning skills in everyday language; however, this may not occur in reality!

7.4.4. Reason not to apply mental logic theory

A common problem with the theory of mental logic and other theories of reasoning based on formal rules is how people determine the logical form of the premises encountering a sentence in daily life. However, regarding mental model theory, Johnson-Laird (2014) proposes that reasoners build models of premises and base their inferences on them. According to mental model theory, individuals do not only rely on the logical form of the sentences. Rather, they consider the meaning of all words, the grammatical structure of the sentence, and their knowledge to construct models of all the possibilities to which propositions refer – and a conclusion is valid if it holds in all of these possibilities.

One main issue for most of the theories of reasoning based on the formal logic rules of inference is to determine the dominant logical form. For the statements in the context of formal logic, the grammar of the sentence represents the logical form, but the issue can come up in everyday language, where the underlying grammatical form of sentences may not match the logical form that applies formal rules of inference (Chomsky, 1995). Since the logical form depends on the meaning and context, so it transcends the grammatical form. To clarify, consider the following premise:

If Hanna pays me $10, then she will have my pen.

Hanna paid $10.

Among many, three possible conclusions among many are “Hanna would have my pen”, “Hanna would have my pen or Hanna did not pay $10” and “Hanna would have my pen and Hanna would have my pen, or Hanna would not have my pen”. Even though this statement may not be the one we use in everyday language, it is true in logic. The second and third conclusions do not make sense in everyday language, and the most possible answer is “Hanna would have my pen”. However, using rules of inference, all three possible conclusions are logically true, though, among them all, the only one fitting with common sense is “Hanna would have my pen”. This may offer that people draw the conclusion from the grammatical form of a sentence and consider the meaning and the relationship between the premises. The following argument is valid in logic rules but not sound:

If Hanna pays me $10, then she will have my pen.

Hanna paid $10.

Hanna would have my pen, or Hanna would not have my pen.

In both mental logic and mental model theories, logic has a main role in human reasoning. These are similar theories; however, mental logic explains errors in terms of the accessibility of different rules, whereas mental models explain errors in terms of limitations in how mental models are constructed and checked and how many models must be considered (Oaksford & Chater, 2009a). Instead of following logical rules during reasoning, people reason over models in which such forms are true in the theory of mental models.

Considering the above reasons, the theory of mental logic is not the theory I apply to account for the studies reported in this thesis, and mental model theory seems more fitted than mental logic.

7.4.5. The successful theory: mental model theory

Which psychological theory provides a better account of human reasoning is a controversial topic, but the studies have confirmed signs of the use of mental models (Evans, Newstead & Byrne, 1993). As discussed in the previous section, mental model theory allows individuals to withdraw conclusions that conflict with the facts. In contrast, theories based on formal rules of inference do not explain how individuals discover that assertions are inconsistent and how they can rectify this inconsistency.

Chase, Hertwig, and Gigerenzer (1998) note that people do not have the time to employ the process proposed by mental logic or probability theories in real life. Instead, people judge and make decisions by the use of heuristics and biases.

7.5. Summary of Chapter 7

This chapter briefly described some theories proposed to explain conditional reasoning and are classified into two main categories: probabilistic and non-probabilistic theories of reasoning. In probabilistic views, a problem is regarded as a probability question; this is the case in the Information gain model, Bayesian rationality, and dual-process theory of reasoning. However, in non-probabilistic approaches, a problem is mainly assumed as a logic problem; this is the case in the theory of pragmatic reasoning schemas (Cheng & Holyoak, 1985); mental model theory of reasoning (Johnson-Laird & Byrne, 1991); mental logic (Rips, 1994).

Mental models are concrete representations of situations constructed in an individual's mind, like the concept image for mathematical objects. These models are acquired through experience particular to the individual or from experiences particular to an environment that could be shared in one society but not necessarily in others (Berger & Luckmann, 1966). When the mental models that people use are well adapted to a task, then familiar possibilities can put them in a better position to decide.

Section §7.4 provided some reasons not to apply the theories introduced in sections §7.1 and §7.2. Even though all theories of reasoning explored in this chapter can explain reasoning about conditionals and are extensively used by the researcher in topics similar to the current work, mental model theory is the theory that reveals more about the underlying cognitive process to determine the conditionals.

Chapter 8. Study 1: how a mathematician determines a conditional in different contexts

This chapter reports on a qualitative study and has four main sections.

Laying the groundwork: the first section of the current chapter gives an outline of the research and sets out the following research questions:

- How does a mathematician determine conditional statements in different contexts, namely classical logic, mathematics, and colloquial language?
- What mental models can explain the participant's decision-making?

Tasks: §8.2 gives a brief review of the applied method(s) of data collection through an interview broken into the three following parts. Each phase of the interview has its own tasks. It also invokes methods appropriate to conduct each phase of the interview, namely structured, semi-structured and un-structured.

- The first phase of the interview assesses the participant's knowledge about conditional statements and follows a structured interview format.
- Phase 2 includes 14 conditional statements in different contexts: mathematics, logic, and everyday language. Phase 2 will be taken through a semi-structured clinical interview.
- Phase 3 is mainly based on the interviewee's responses in phase 2 and asks for more explanations for some tasks in phase 2. The tasks for this phase are not structured and will be designed based on the interviewee's responses to phase 2 tasks in a short break between phase 1 and phase 2.

Methods: §8.3 uses a method, namely Toulmin's argumentation scheme, to prepare the data for analysis. Then, it poses mental model theory as the framework for data analysis.

Data analysis: §8.4 analyses data from one interview within the adopted framework, namely the theory of mental models. To address the research questions, one interviewee's responses for phases 2 and 3 will be analyzed.

8.1. Laying the groundwork

A conditional statement has the general form of "if p, then q" and, at first glance, may seem straightforward. However, it is evidenced in the literature in Chapter 5 that a conditional statement is a topic that causes students serious difficulties. The following are some requirements needed to understand a conditional statement:

- Determine that the given sentence is a "conditional statement.
- Differentiate between "premise" and "consequent".
- Determine the relation between "premise" and "consequent".

Chapter 4 explored the role of logic in formal and informal mathematics, while Chapter 2 specified that there are different types of conditional statements in everyday language, but only one in mathematics, namely material conditional. Any mathematical question is also worded in a language, though it is in the discourse of mathematics. The type of conditional in mathematical questions is material, and in this type of conditional, the two clauses may or may not be related.

This chapter looks at situations where people with a background in logic and mathematics look at conditional statements worded in different contexts of mathematics, logic, and everyday language. It aims to see if they use material conditional for conditional statements in the context of mathematics and, if there are any criteria, they apply to recognize conditional statements.

§5.3 included the main components that constitute a language, namely syntax and semantics. Then, in §5.4, I added a third component, namely "context", and provided some research evidence where the interest of the current study is set out. The following are the research questions addressed in this chapter:

- How does a mathematician determine conditional statements in different contexts, namely classical logic, mathematics, and colloquial language?
- What mental models can explain the participant's decision-making?

The reason I considered different contexts for conditional statements is to target both formal and informal mathematics. To address these questions, I conducted interviews with some mathematicians familiar with material conditionals. Based on the broad engagement of participants with mathematics, I refer to them as "mathematicians".

I chose one participant, a mathematician, Hugo (not the actual name of the interviewee), who has a good background in first-order logic and a solid knowledge of mathematical reasoning. The reasons for interviewing mathematicians and choosing Hugo's arguments to be analysed are explained in sections §§8.3.1 and §§8.4.1.

8.2. Designed tasks

As summarized in §6.1, the basic aims of mathematical thinking include discovering cognitive processes, identifying cognitive processes, and evaluating competence (Ginsburg, 1981). In the current study, I include a combination of two activities, "discovery" and "identification", in three phases, which I explore in a clinical interview.

In the **first phase**, for discovery, I examine the participant's background information about conditional statements in both colloquial and mathematical contexts.

In the **second phase**, I focus on how the participants determine conditionals in different contexts and whether there is any difficulty experienced in conditional statements with irrelevant premises and conclusions.

In the third phase, which begins after about a 10–20 minute break, I pick some of the interviewees' responses to elucidate the underlying process and ask some general questions.

8.2.1. Phase 1, discovery

In the interview's discovery stage, I aimed to determine the participant's prior knowledge regarding conditional statements. For this phase, I designed the following questions to be presented in a flexible order. Regarding the type of questions used in this part, I am using some "performance questions" and "give an example tasks" (see Chapter 6):

1) Give an example of a conditional statement in general.

2) Give an example of a conditional statement in mathematics.

3) Give an example of a conditional in everyday language.

4) Give an example of a statement in mathematics that is not conditional.

5) Does a conditional statement have a certain form (follow a certain rule) in mathematics/ colloquial language?

Depending upon the interviewee's responses, several multiple-choice questions were prepared as a follow-up that were used for a few of the interviewees:

1) Which of the following is a conditional statement?

   a) Sina plays guitar or Shan plays piano.

   b) Sina plays guitar when Shan plays piano.

   c) If Sina plays guitar, then Shan plays piano.

   d) None of the above.

2) What is the if-then form of the following Conditional statement?

"It is time for dinner when it is 6 pm."

   a) If it is 6 pm, then it is time for dinner.

   b) If it is time for dinner, then it is 6 pm.

   c) If you want to eat dinner, then you must eat at 6 pm.

   d) None of the above.

8.2.2. Phase 2, identification

This stage is the major part of the interview, as it is aimed to answer the main questions that this work is set to answer. To design the task, three following components have been considered.

- The appropriate type/s of questions (discussed in §§6.1.3) for this stage.
- Questions must be designed in three different contexts, mathematics, logic, and everyday language.

- There should be some conditionals with both relevant and irrelevant premises and conclusions.

I first determined which type of questions would be applied in designing tasks. Then, considering the type of questions, at least two conditional statements in each of the three different contexts (mathematics, colloquial language, and logic) were designed. For each case, I prepared one conditional statement with a relevant premise and conclusion and the other one with irrelevant parts.

In the previous phase, the method to generate data was a structured interview, and the questions were designed to satisfy the aim of phase 1, that is, discovery, and to ensure that the interviewees have basic knowledge about conditionals. However, for identification, phase 2, a semi-structured clinical interview was the most suitable method to collect accurate data. The follow-up questions in a semi-structured interview provide a closer observation of individuals' ways of thinking to reveal and describe their underlying cognitive process. Again, since the participants are sometimes unaware of such a process, I needed to be prepared for different responses. So, the following were only opening questions that directed the discussion based on the responses.

Like in phase 1, participants' background knowledge played a significant role in designing the tasks in this stage. The reason is that the level of difficulty of questions is vitally important in such interviews, as very easy questions may not provide the opportunity to focus on their thinking, and on the other hand, very difficult questions may only be confusing or result in no response.

Therefore, I needed to know the participants' knowledge of the subject matter and design this stage's tasks fitted to their knowledge. The questions were to vary from easy (not too much) to reasonably difficult (difficult but manageable). More specifically, by a reasonable difficult question, here I mean the questions in the boundary of the interviewee's knowledge.

Regarding the type of questions in this part, I designed some performance questions to see how the participants dealt with conditional statements, and there are some follow-up questions to find out their idea or see if there is any pattern to determine the conditionals. The other types appropriate for this phase are "why questions" and "twist questions".

Twist questions may seem difficult to design, as I needed to make them within the boundary of the interviewee's background knowledge of conditional statements, and without prior knowledge about their experiences, it could be confused with "performance questions". For instance, the following question could be a twisted problem for many people:

> Is the statement "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" conditional?

Also, because of the nature of this part, there will be many "why questions" being asked in unexpected contexts. More specifically, when the interviewees answer the questions without considering the reasons for them. For instance,

> why the statement "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" is/isn't conditional?

Different contexts

In §5.4, I discussed the syntax and semantics of a language. In §5.5, I added a third component to logic as a language, namely "context". These three components: "syntax", "semantics" and "context" are needed to understand conditional statements, and this chapter investigates how a mathematician dealt with conditional statements in different contexts.

Compared with everyday context, in mathematics, people may show different reactions to conditional statements. §5.5 explained what I mean by the different contexts by setting conditionals in the contexts: mathematics, logic, and everyday language.

For a statement in the context of logic, I have not used the general form of $p \Longrightarrow q$. Rather, I used a statement with unknown content. This is because, with no information about the subject of a conditional statement, it can be regarded as mere "p $\Longrightarrow$ q", with no influence of potential interpretation derived from the interviewee's knowledge about the content. In such questions with unknown content, the relatedness is not applicable.

Since this chapter focuses on reasoning while dealing with conditionals in certain contexts, the conditional statements in the context of logic refer to some unknown topics from mathematics and everyday language. With prior knowledge of the interviewee's background, it may not be difficult to find unknown (to the interviewee) concepts in mathematics and everyday language. For instance:

- If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the earth.
- Let A be a Banach algebra, then $P(A)$ is a $\wedge$-semi-lattice.
- If A is a commutative Banach algebra and A″ is (-1)–weakly amenable, then for each $T \in BL(A, B)$ with a dense range, $Z^1(A'', B_T') = \{0\}$.

Therefore, the questions are designed in mathematics and everyday language; however, there are no relationships between the premise and conclusion for some of them.

Design and organize questions for Phase 2

In preparing questions for phase 2 of the interview, I excluded false conditional statements. Also, there could be some different ways to represent a conditional statement. For example, each below statement is conditional (more specifically, an implication), but with different wording:

- A number is divisible by 5 when its last digit is 5.
- We always have a fraction when we divide an integer number by a non-zero integer.
- For all $x,\ y$ in R, we have $sin\ (x + y) = sin\ x\ cos\ y + cos\ x\ sin\ y$.
- $x = 25$ implies that $\sqrt{x} \neq 5$.
- He spoke to reporters on condition that he was not identified.
- You can go out to play football provided that you have finished all your homework.
- To have my pen, you need to pay $10.

To focus on context, all the statements are presented in an if-then form and are designed under the following conditions.

- Premise and conclusion are unknown but seem relevant (representing a general form of a logical conditional),

- Both Premise and conclusion are in the same realm, either mathematics or colloquial language.

- Premise and conclusion are known, but there is no relevance between premise and conclusion.

The following is the list of the questions for phase 2:

- If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*.
- If $4x^2 - 5x - 6 = 0$, then $sin\,\theta = 0.546$.
- If $cos\,\theta = 0.81$, then $\theta = 2\pi \pm \frac{13\pi}{173}$.
- If A is a Banach algebra, then for every Banach A-bimodule $X, H\ (A, X') = 0$.
- If you pay $\$10$, you will have my pen.
- In triangle ABC, if we have $A + B + C = 180^o$, then the area of a circle is pi times the radius squared ($\pi\, r^2$).
- If you are late for the meeting, then you will be fired.
- If $x + 1 = 0$, then $z = 5$.
- If $(x - 2)(x + 1) = 0$, then $x = 2$ or $x = -1$.
- If the sun erupts from an active region called $AR\ 2673$, then there will be loops of plasma tens of times the size of the earth.
- If a population consists of 40% men, then 60% of the population must be women.
- If elephants could fly, then you win the lottery.
- If ABC is a triangle, then $A + B + C = 180^o$. (Assuming A, B, C are angles and that the triangle is on a plane.)
- If the discovery of galaxies without dark matter holds up, then astronomers will have to seriously consider what this growing population of galaxies without dark matter means.

Table 1 shows the statements designed for phase 2. It assures that I considered all possibilities in almost the same weight.

Table 8.1. Designed Tasks for Phase 2

Contexts	Colloquial	Mathematics	Logic
Related	If you pay \$10, you will have my pen. If you are late for the meeting, then you will be fired.	If a population consists of 40% men, then 60% of the population must be women. If $(x-2)(x+1)=0$, then $x = 2$ or $x = -1.$ If ABC is a triangle, then $A+B+C=180^{\circ}.$	If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the earth. If the discovery of galaxies without dark matter holds up, then astronomers will have to seriously consider what this growing population of galaxies without dark matter means.
Unrelated	If elephants could fly, then you win the lottery. If you buy a fresh fish tonight, then Bill Gates meets Aamir Khan.	If $x+1=0$, then $z=5.$ In triangle ABC, if we have $A + B + C = 180^{o}$, then the area of a circle is pi times the radius squared. If $4x^2 - 5x - 6 = 0$, then we have $\sin\theta = 0.546$.	If $\cos\theta = 0.81$, then $\theta = 2\pi \pm \frac{13\pi}{173}$. If A is a Banach algebra, then for every Banach A-bimodule $X, H(A, X') = 0.$

In the above table, the statements in their column are examples of conditional statements in the context of logic. However, these statements are subject to change to be fitted with the interviewee's knowledge. Regarding the statement "If $\cos\theta = 0.81$, then $\theta = 2\pi \pm \frac{13\pi}{173}$", the actual value is close to the suggested solution for θ. Therefore, even though all mathematicians know about trigonometric functions, and even if they can approximate the answer in a limited time, they may not ensure that the solution is correct; what they may be sure is that the premise and conclusion are relevant and the provided solution could be a possible answer.

8.2.3. Phase 3, re-evaluation

After conducting an interview with a master's student in mathematics as the pilot study and listening to the audios, I decided to add the third phase to the interview process to focus on unclear or challenging statements in phase 2. This phase is mainly unstructured, and the questions are designed during a 10- to 20-minute break between phase 2 and phase 3 of the interview. In fact, after the second phase of the interview, I used the break time to pick some of the interviewee's responses for more explanation and a possibility to revisit few statements.

The interviewees were also asked some general questions about the second phase. For example, they were asked to compare two contexts of mathematics and colloquial language regarding the determination of conditional statements, though nothing was presupposed. In the following, I list some potential questions to open the discussion.

- All statements were in the realms of mathematics and everyday language. In which realm could you recognize conditionals easier than the other one?
- Do you think if there is any difference between conditionals in mathematics language and everyday language?
- Do you think there are any conditional statements that do not make sense in reality, but they are still conditional?

Such questions that ask for a comparison between the two provided solutions need more reasoning. This phase places the interviewees at a distance from a few of their responses to produce new or more explanations by viewing their responses to those questions. Therefore, phase 3 cannot be structured; it depends on the interviewee's responses in phase 2 and the type of questions appropriated in this part; there will be many "reflection questions", especially to compare two answers.

Therefore, to discover more about their thinking process and possible patterns, I gave them a few questions they had done in the previous phase and asked them to either re-evaluate them or re-explain why? I also gave them similar questions that were evaluated differently and asked them to reflect on their answers and explain why they seemed different to them? By presenting their responses, these questions could shift the focus of the interview to the reasons for the answers rather than to the answer itself.

This stage is designed to examine if the interviewee has any pattern in determining conditional statements in different contexts. In the case where there is a pattern, is there only one pattern working with different contexts, or does the interviewee have one pattern for each context?

8.3. Methods

This section gives information about the participants, how the data were collected, and the selected theory for data analysis in the next section, §8.4. It then explains the plan in which the three phases of interviews were carried out.

8.3.1. Participants and data collection

What I am interested in needs closer observation. Thus, for the current study, I chose clinical interviews as my method to collect data. For the interview, I designed tasks suited to the purpose of this study.

A clinical interview is the most appropriate method for addressing my research questions among all different methods to collect data. This is because, in such a method, some fixed questions are designed in advance, yet, depending on the participant's responses, the questions may change. This ability in the interview allows me a much more detailed exploration of what thinking processes may be happening.

I conducted 14 interviews, the participants were interviewed individually, and the interviews were audio-recorded. Later on, some parts were transcribed for analysis.

I selected the interviewees among graduate and graduated students in mathematics. Some of them were teaching mathematics in post-secondary institutions in Canada. For this thesis, I focus on one participant, Hugo, as explained in §8.1 and §8.4.

I decided to interview mathematicians since a mathematician is surely aware of the truth values of material conditional, and what the antecedent and conclusion are, and their difference. Such people are actively dealing with mathematics for years and they are either research students or those with advanced education in mathematics.

8.3.2. Theoretical framework

Even though predicting to adopt a theory is important, since human behaviour is not predictable, the literature can only suggest the theory for similar events and under some constraints. In Chapter 7, I discussed some potential theories tailored to the current study's aim and, reflecting on the literature; I decided to use the theory of mental models to analyze data.

However, it may not be sufficient merely to find what theory literature suggests, because each research question may be approached with a different theory. For that reason, in the pilot study, I assessed theories using the collected data to see which one could explain the non-normative answers provided by the participants. After going through the audio recordings, it was clear that Hugo had a mental process that he was using to assess each given statement. Therefore, this comparison suggested that, among all potential theories, the mental model theory of reasoning provides the best framework within which to analyze how a mathematician recognizes a conditional statement.

To analyze the data from a clinical interview, I organized data according to Toulmin's scheme (see Chapter 6).

8.3.3. Plan for interviews

During phases 1 and 2 of the interview, I gave participants cards with one question on each. However, in the second phase, the interviewees were only asked to determine if any of the statements were conditional statements or not. Some participants made notes during the course of the interview that were collected and used in data analysis whenever needed. Like any other semi-structured clinical interview, when necessary, I asked the participant for clarification within each question in phase 2. However, the main discussion was only in the last phase, when needed to provide more convincing reasoning for a few questions. During the interview, there were no hints to push the interviewees to the correct answers.

The tasks were designed in three following phases explained in section §8.2, and the following gives the plan for the interview in three parts:

Phase 1. In the first phase focused on the discovery, I examined participants' background information about conditionals in both realms of everyday language and mathematics. A structured clinical interview was adopted for this part; it lasted for about five minutes.

Phase 2. The focus of the second phase was mainly on the identification, and fourteen questions had been designed to explore the interviewee's understanding of a conditional statement in different contexts: mathematics, logic, and everyday language. For this part, I used a semi-structured clinical interview, which took between 30 and 40 minutes.

Phase 3. The last phase was the re-evaluation stage. For this part, I used an unstructured clinical interview format, focusing on unclear or challenging statements that had come up in phase 2, and the questions were designed during a 10 to 20 minute break between phases 2 and 3.

8.4. Data analysis

In this section, the reasoning behaviour of a successful mathematician aware of rules of inference is being investigated. Mental model theory is used.

For the current case study, I picked one interviewee's responses to be analyzed. Though he was not the only participant who responded in a specific manner addressing the issues, he was the one with the most elaborate explanations, which allowed for a detailed analysis of his reasoning. In the following section, I explain the reasons I chose Hugo's responses for the data analysis.

8.4.1. How was Hugo selected?

This chapter is aimed to answer, "How the context modulates interviewees' understanding of conditionals, how they determine conditionals in different contexts and what mental models they use to do so if there are any". Specifically, which type of conditional do they choose to interpret the statements where the language and logic coincide? The interviewees' responses suggest that the following statement is one example of a challenging question in the designed tasks:

Is the statement "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" a conditional?

A challenging problem is a subjective thing and is highly dependent on the interviewee's knowledge. In §§5.4.1, I discussed that general and contextual knowledge could be regarded as an additional factor influencing the individual's understanding of a conditional statement. A person with good background knowledge in language or in logic may look at the statements in a more abstract form. By this, I mean they may be able to recognize automatically the logical form of the statement, and if the definition of material conditional is still active in their mind, they can recognize them no matter how it is worded and what the context is. Therefore, based upon their experience and background knowledge, interviewees react to challenging questions differently.

I classified the interviewees' responses into three categories based on how they react/respond when encountering a challenging problem:

(1) With no or small hesitation, they determine the correct answer and know how they did the problem.

(2) Either they refuse to answer the challenging questions, or their answer is not correct, and in the follow up question, they do not provide any explanation for them.

(3) They are uncertain about the answer, and/or in the follow up questions in phase can discuss their idea.

It is my opinion that, when there is an issue to understand a topic or a challenge to solve a problem, and some people can manage it easier or faster than others, it might be because at some point they had cleared that challenge up. So, I explain the three different types of responses as follows:

In the first category of responses, they had figured the issue out and resolved it before they were given the problem. As a result, the problem was not an actual problem for them and such responses may not be very useful to be discussed in the current section.

In the second category, they were not aware of the issue and there was still a long way to reach that point to figure the issue out. In such cases, they usually insisted on the incorrect answer and the reasonings they made for the follow-up questions in phase 3 of the interview was not very useful, though they still had a pattern to recognize the conditionals, but mostly they preferred not to answer that question

In the last category, they were in a small neighborhood of figuring the issue out. These were the ones who still had the problem distinguishing material conditionals and had uncertainty between logic or mathematical interpretation of a conditional statement and its meaning in daily language. In such cases, they clearly showed the confusion during the interview while looking at the statement with a non-trivial structure, and this hesitation usually was along with the long pauses in speech and voice pitches.

As a researcher, among all responses, I was specifically interested in those within the third category, where the issue had not been resolved yet and where they were experienced enough to see the problem. If there was no hesitation that a statement is a conditional or not, no matter whether the answer is correct or wrong, there would be nothing to be discussed. These criteria narrowed my choices to a few interviewees who could discuss their ideas, and among them, I chose the one who could discuss more. I selected Hugo's responses to be discussed in this chapter not because they had been extreme or just like the other responses; I chose this interviewee because I could extract more to address the research question from his responses. Let me briefly explain my criteria to choose Hugo's responses to be analyzed in this chapter.

The selected interviewee, Hugo, has a master's degree in applied mathematics and is the author of several books on different mathematical topics, including calculus and differential equation. He has been teaching mathematics for years. Given his mathematical background, he was familiar with the truth values of a material conditional.

8.4.2. A determinative factor: context

In this part, I compare and discuss Hugo's responses to the conditionals in three different contexts. That is, in each case, a request was to determine whether the given statement was conditional. In all cases in logic and mathematics with related parts, the interviewee could provide the answer with certainty. He determined the conditionals with unknown context normatively and promptly. For the conditionals in mathematics, he first checked if the statement was mathematically correct or that there was no reason to refute it. In cases where Hugo could not refute a mathematical statement, he recognized it as conditional. However, mathematically wrong statements were not considered as conditionals.

Regarding the statements in everyday language, Hugo directly checked for two requirements: "if-then" form and "relatedness". In this sub-section, I only discuss the conditionals in the colloquial language where the antecedent and consequent are related, and where he could determine all conditionals.

Statements in logic

The following statements in the designed task, with unknown antecedent and consequent, or an unknown relationship, simulate assertions in logic:

- If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the earth.
- If A is a Banach algebra, then for every Banach A-bimodule $X, H\ (A, X') = 0$.
- If the discovery of galaxies without dark matter holds up, then astronomers will have to seriously consider what this growing population of galaxies without dark matter means.
- If $cos\ \theta\ =\ 0.81$, then $\theta\ =\ 2\pi\ \pm\ \frac{13\pi}{173}$.

The above statements have the form of "if p, then q" and Hugo presumably did not have any further information about antecedent and consequent; however, both parts appear as relevant in each statement. This interviewee responded to these questions normatively and with certainty! This may suggest that when there is no information about the subject area, the sentence format shows itself up, and the interviewee decides based on the logical form of the sentence. The transcriptions of the argument can be found in Appendix.

Statements in mathematics

The interviewee's responses on almost all the following mathematical statements were normatively correct:

- If $4x^2\ -\ 5x\ -\ 6\ =\ 0$, then $sin\ \theta\ =\ 0.546$.
- If $cos\ \theta\ =\ 0.81$, then $\theta\ =\ 2\pi\ \pm\ \frac{13\pi}{173}$.
- If ABC is a triangle, then $A + B + C\ =\ 180^o$. (Assuming A, B, C are angles and that the triangle is on a plane)
- In triangle ABC if we have $A + B + C\ =\ 180^o$, then the area of a circle is pi times the radius squared ($\pi\ r^2$).

- If $x + 1 = 0$, then $z = 5$.
- If $(x-2)(x+1) = 0$, then $x = 2$ or $x = -1$.
- If A is a Banach algebra, then for every Banach A –bimodule $X, H(A, X') = 0$.

In what follows, I discuss some excerpts of Hugo's responses in the context of mathematics by sharing excerpts for the interview transcripts and designing a related Toulmin's diagram

Excerpt 1.

Interviewer**:** [handed a card with the statement "If $4x^2 - 5x - 6 = 0$, then $sin\,\theta = 0.546$ "]

Hugo: [15 secs] Mathematically [4 secs] it's wrong,

Interviewer**:** But my question is not to determine if it is wrong or correct,

Hugo: No, this sentence [7 secs], is it a conditional statement? [5 secs] No, it's not, no, it's not a conditional statement, not at all.

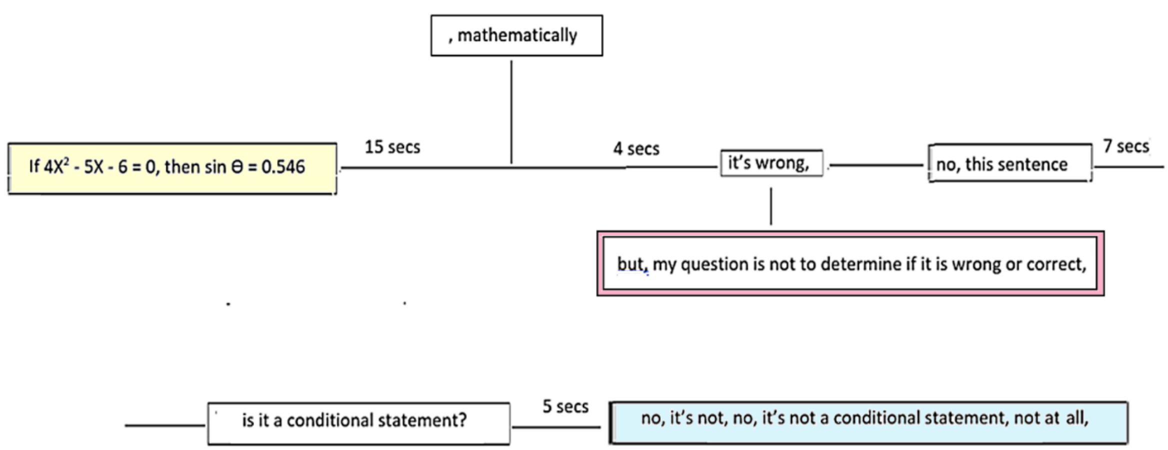


Figure 8.1. The Toulmin's Model 1

In this case, the answer is not normative, but the interviewee could answer with certainty in the end. Although he managed to decide this case, the number of pauses and their durations shows his hesitation. He first specified that this statement is mathematically wrong. However, both parts are unrelated, not wrong. Even when he determined this statement as wrong, long pauses show he still needed time to make his decision as to whether it was conditional or not. This hesitation might be because of the "if …, then …" form of the sentence suggests it could be conditional. A mathematically wrong statement is different from a conditional with totally unrelated parts. This statement with unrelated parts was recognized as non-conditional.

Excerpt 2.

> Interviewer: [handed a card with the statement "If cos θ = 0.81, then θ = 2π ± $\frac{13\pi}{173}$"]
>
> Hugo: I don't know if it is correct or not, but for sure, this is conditional.

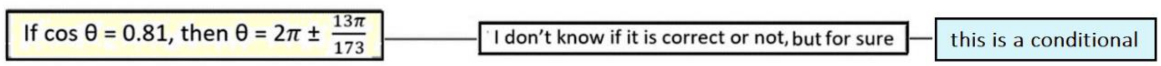


Figure 8.2. The Toulmin's Model 2

The given statement is a conditional that is mathematically wrong because $\cos(2\pi \pm \frac{13\pi}{173}) \approx 0.97$. However, the provided solution set is close to the correct solution, and, with a quick look at the equation, it is not difficult to guess that the answer is likely correct. In this case, the interviewee promptly responded with no hesitation because he was unsure whether the statement was mathematically wrong.

In Excerpts 1 and 2, Hugo determined the statement "if $4x^2 - 5x - 6 = 0$, then $sin\ \theta = 0.546$" as non-conditional and the statement "If cos θ = 0.81, then θ = 2π ± $\frac{13\pi}{173}$ " as a conditional. Both statements are conditionals in the context of mathematics, and the antecedents and consequents are in symbols. What makes a difference is that the first statement has unrelated parts.

To recognize the following conditionals:

- In triangle ABC, if we have $A + B + C = 180^o$, then the area of a circle is $\pi\, r^2$.
- If A is a Banach algebra, then for every Banach A-bimodule $X, H\ (A, X') = 0$.

Hugo used similar reasoning to excerpt 1 and excerpt 2, respectively. He determined the first as a non-conditional and the latter as a conditional. Both statements are in mathematics and include fewer symbols, but the second statement has an unknown context.

Excerpt 3:

Interviewer**:** [handed a card with the statement "If ABC is a triangle, then $A + B + C = 180^o$"]

Hugo: It is very obvious that this is a conditional,

Interviewer**:** why do you think so?

Hugo: It has "if … then", this is related to this, it satisfies each condition, so it is a conditional,

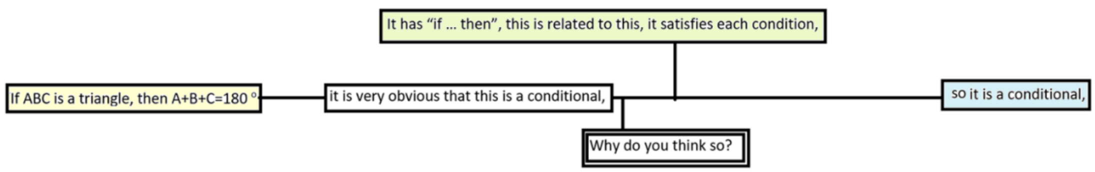


Figure 8.3. The Toulmin's Model 3

This is one of the most obvious cases for him to recognize; that could be because of the "if-then" form and the correctness of the statement.

Statements in everyday language

In everyday language, Hugo was asked to consider the following statements, and he did recognize them as conditional statements, except the last two items; these items are discussed in §§8.4.2.

- If you are late for the meeting, then you will be fired.
- If you pay 10$, you will have my pen.
- If a population consists of 40% men, then 60% of the population must be women.
- If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*.
- If elephants could fly, then you win the lottery.

Notice that in any of the below diagrams, there is at least one box for the warrant. This may suggest that the interviewee needed a reason for the conclusion. Also, in the modal boxes, he used models with a high level of certainty.

Excerpt 4:

Hugo: "If you are late for the meeting, then you will be fired" well, this is conditional because it has "if… then"; it is very clear that because of "if" and "then", it is conditional.

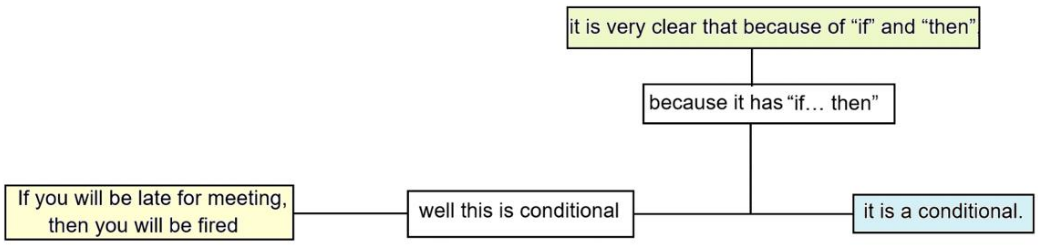


Figure 8.4. The Toulmin's Model 4

Excerpt 5:

Hugo: "If you pay 10$, you will have my pen." [4 secs] Yeah it is conditional, it has "if", and this is related to this, so, yes it is conditional, first "if", and then relatedness, these are definitely related,

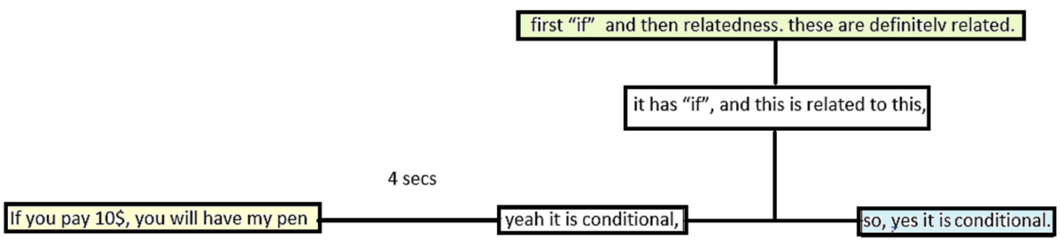


Figure 8.5. The Toulmin's Model 5

Excerpt 6:

Hugo: "If a population consists of 40% men, then 60% of the population must be women." Yeah, this is conditional. If 40% men, then 60% women.

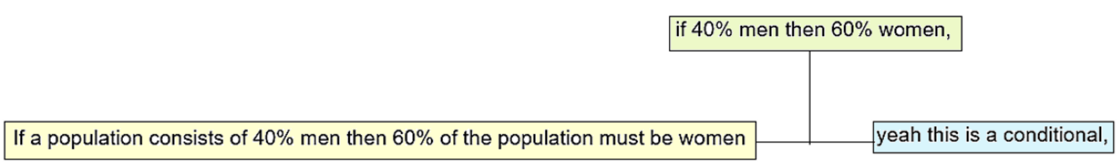


Figure 8.6. The Toulmin's Model 6

The above excerpts show that the interviewee provided a backup or justification for the response: however, he did not do this in the same manner in the context of mathematics. In fact, in the context of mathematics, when the two parts were related, he was confident about the answer. This suggests that, in any context, he would check the statements' validity if possible. Since he is a mathematician, he may be able to check the correctness of the assertion just by looking at the statement, and in an everyday context, he may check the validity by comparing the case with his experiences and knowledge in such situations, and in case there is nothing to violate the conclusion, it would be valid. Therefore, it seems that he determines the conditionals not just based on the logical form of the sentence, but also on its soundness.

As it is clear from the excerpts, compared with everyday context, Hugo decided much faster and with a higher level of certainty for the statements in the context of mathematics. Even when he made a non-normative/incorrect response, it was with certainty. There is no box for warrants in the associated Toulmin's diagram of these arguments.

All the above comparisons suggest that, while working on the statements with the unknown subject areas, the interviewee's reasons were based on the logical form of the sentence, and his responses are normative within predicate logic (mathematical logic). However, when he reasoned either in colloquial language or in mathematics, the context took priority over the logical form, and his rationale was based on the content or sometimes a mixture both of content and of logical form. Furthermore, he reasoned with higher certainty in mathematical than in everyday statements. The results reported in this section are consistent with those many studies in the literature showing that the content of the conditional influences conditional reasoning (§5.1 and §5.4).

8.4.3. A Conflict between two different contexts

The excerpts in the previous section suggest that the interviewee gave more credit to situated meanings of a conditional than semantics (the meaning of a material conditional in first-order logic). To be more specific, in some cases the interviewee demonstrated inconsistency between his mathematics view and daily language: for example, see Figure 8.1 representing the first Toulmin diagram. In that excerpt, the interviewee used the word "mathematically" to specify the subject area, and it seems that, because that statement is in mathematics, it is not conditional. The if-then form of the conditional is not as strong as its situated meaning: that is, "$sin\ \theta = 0.546$" is resulted from "$4x^2 - 5x - 6 = 0$". If he did not know mathematics, he might have reasoned differently.

This dominant situated view is not only for the statements in the context of mathematics. In the following, two statements in everyday language are exemplified. It is also worth noticing that, for this interviewee, the whole interview for the second phase lasted about 22 minutes, and we spent about 7 minutes on these statements.

- If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan.*
- If elephants could fly, then you win the lottery.

Although these two assertions are clearly in colloquial language, the interviewee explicitly looked at them from two different perspectives: mathematics and daily language. He stressed that they are not conditional in mathematics, but in colloquial language, they might be. It is likely that by referring to mathematics, he addressed the certain validity of the sentence. In the following, I provide the excerpt from our discussion on the first statement coded using Toulmin's model.

Excerpt 7:

Hugo: "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" [6 sec]. Yes. It doesn't make sense, but it is conditional ↓.

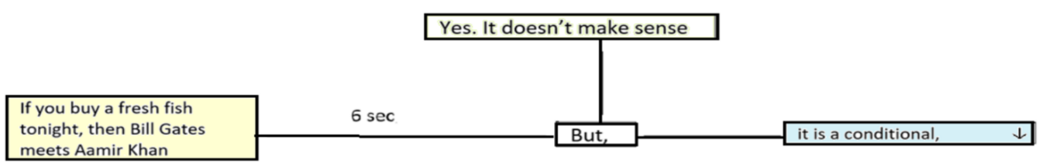


Figure 8.7a. The Toulmin's Model 7a

Because the interviewee used the word "but", and there was a pitch fall in his voice, I asked him to elucidate his response, and then he argued as follows:

Interviewer: Can you say why?

Hugo: If-then. If you buy something, then [stressed], something happens, though they are not very related, errr, it is conditional, though meaningless.

Interviewer: So, what is this?

Hugo: Only an idiot would say that [laugh]. (6sec) In fact, mathematically, it's not conditional, but in daily language, we take it as a conditional.

Interviewer: So, is it conditional or not?

Hugo: With regards to math or daily language? Actually, in mathematics, it's not conditional, but for daily language, it is ↓.

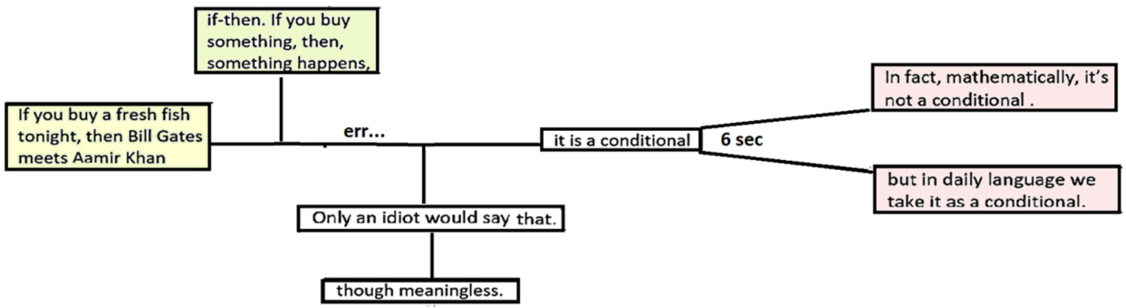


Figure 8.7b. The Toulmin's Model 7b

As indicated in his argumentation model, different mathematics and colloquial language situations explicitly influenced his decision. He separated the results when situating the sentence in mathematics from when situating it in everyday language. In phase 3 of the interview, I reviewed some statements and asked Hugo for more explanations. In that part, I reviewed this statement as below:

Interviewer: Let me go back to this question [If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*], you said before, mathematically it's not conditional, but as a sentence it is, but why did you say this statement is conditional?

Hugo: I again say that mathematically neither is conditional, but both are ambiguous [pointing to statements: "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan" and* "if elephants could fly then you win the lottery[10]"],

Interviewer: What about non-mathematically?

Hugo: They're very unclear, I don't know, I cannot say anything, but there is no reason to be a conditional,

[10] The interviewee's response for this statement will be discussed in the next section.

This conflict might be because people expect more accuracy in mathematics than in everyday language. There are many instances of statements that are conditionals by the definition of a material conditional but may seem unsound. In §§8.3.4 I review such statements, they are conditionals, but the conclusion cannot be derived logically from the premises.

8.4.4. Conditionals with irrelevant parts

The statements listed below have the form of "if p, then q" and the interviewee seemingly did not have any further information about antecedent and consequent. However, in each statement, the connection between the antecedent and the consequent appears relevant.

- If A is a Banach algebra, then for every Banach A-bimodule X, $H\ (A, X') = 0$.
- If $cos\ \theta\ =\ 0.81$, then $\theta\ =\ 2\pi\ \pm\ \frac{13\pi}{173}$.
- If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the earth.
- If the discovery of galaxies without dark matter holds up, then astronomers will have to seriously consider what this growing population of galaxies without dark matter means

In all the above cases, the interviewee determined the conditional statement normatively. However, he explicitly stated that "I do not have information, but it is a conditional". The corresponding transcript and Toulmin's diagram are discussed in §§8.4.1. In what follows, I analyze Hugo's response to the third statement:

Excerpt 8:

Hugo: "If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the Earth", [36 secs], since I guess these two parts are related, though I'm not sure, but I would say this is conditional, if I read this in a newspaper, I am sure that it is conditional, but if these are unrelated, then it is not, but this is conditional because the sun and earth are two related things, and the sun can influence the earth.

Interviewer: So, is this a conditional?

Hugo: Yes, it is.

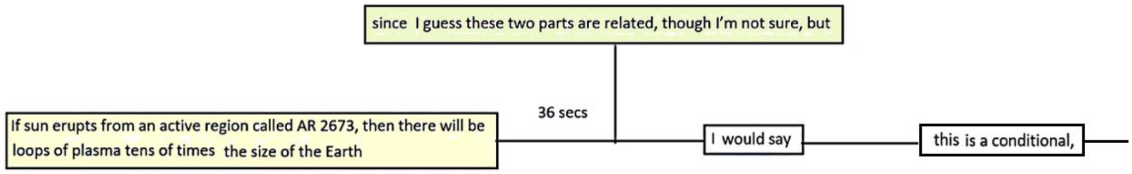


Figure 8.8a. The Toulmin's Model 8a

In the above excerpt, the interviewee first specified that the given statement was a conditional because he guessed that the premise and conclusion were related things. Then he added some extra explanation, that if they were not related things, then it would not be a conditional statement; see the below flowchart.

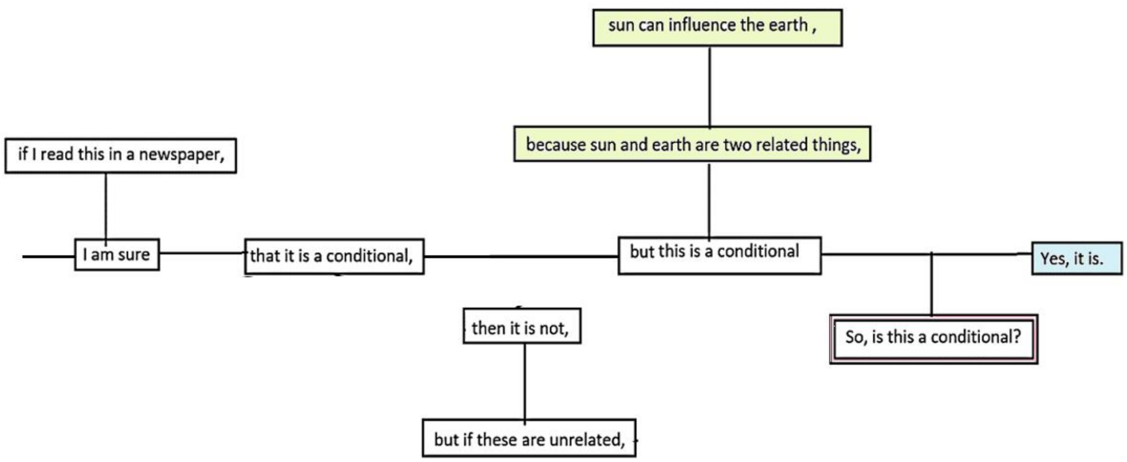


Figure 8.8b. The Toulmin's Model 8b

It seems from this argument – and similar cases in the data – that, for Hugo, the relatedness of the premise and conclusion is a very important feature, and each statement violating such a condition cannot be a conditional. The interviewee could not recognize conditional statements only for those statements with unrelated premise and conclusion, i.e., all statements with unrelated parts determined as non-conditionals.

In both mathematics and daily language contexts, the interviewee responded normatively correct with certainty when antecedent and consequent were relevant. Nevertheless, for all statements with irrelevant connections between antecedent and consequent, his responses were either non-normative or he did not give any response. Specifically, he was certain about his non-normative answer in a mathematics context and preferred to not give any answers in a colloquial context.

Also, Hugo went further and explicitly made a distinction between an impossible assertion and unrelatedness. In the following, I focus on the statement, “If elephants could fly, then you win the lottery”.

Excerpt 9:

Hugo: “if elephants could fly, then you win the lottery”, [4 secs], this means that the chance to win the lottery is very narrow, you cannot win, [11 secs] yeah this is a conditional since this results this [pointing to the premise and conclusion], [4sec] yes, [12sec] no it’s not a conditional, [2sec] I don’t know, [3sec] I’m not sure …

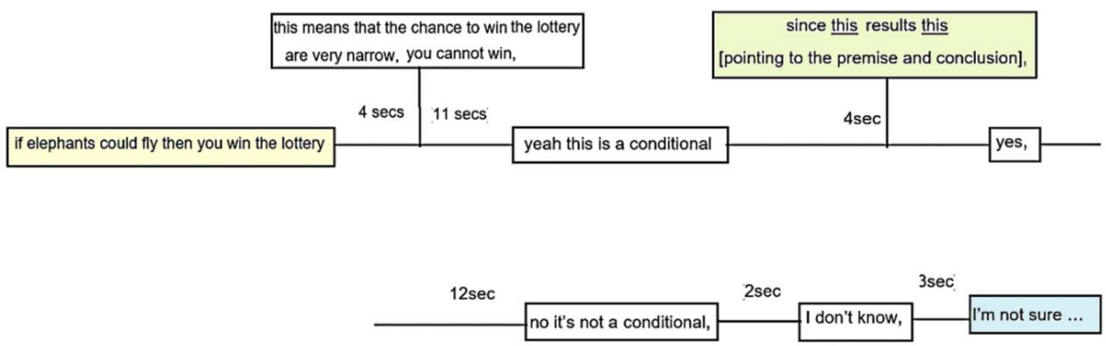


Figure 8.9. The Toulmin’s Model 9

In the above argument, there are long pauses and a high level of uncertainty. To figure out the reason, in phase 3 of the interview, I asked Hugo to revisit the cards with the two following statements and to compare his answers:

- If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan.*
- If elephants could fly, then you win the lottery.

Both statements are in everyday language and have unrelated antecedent and consequent, but initially, Hugo reasoned differently. For the first statement, he stated that “in mathematics, it is not a conditional but a conditional in daily language”. However, for the second statement, he was not sure. In what follows, I investigate the reason for these different answers.

Excerpt 10:

Interviewer: Why are you not sure?

Hugo: To win a lottery is only by chance; they are unrelated because wining a lottery is just luck [16sec] errr [8sec] no it's not a conditional [he drew ※ in Figure 1].

Interviewer: Does this symbol [pointing to ※] mean that you are sure that this is not a conditional?

Hugo: Yes, almost ↓.

Interviewer: Do you have a reason why it is not a conditional?

Hugo: First, these two [pointing to premise and conclusion] are very unrelated, then winning the lottery is all by chance and cannot be conditioned.

Interviewer: So, they are unrelated?

Hugo: Yeah↓, in general, they are terribly unrelated. One other thing is that I don't know what we call them in grammar, do we take each if-then a conditional or not? I don't know (3sec), maybe it is, but in my view, it is not.

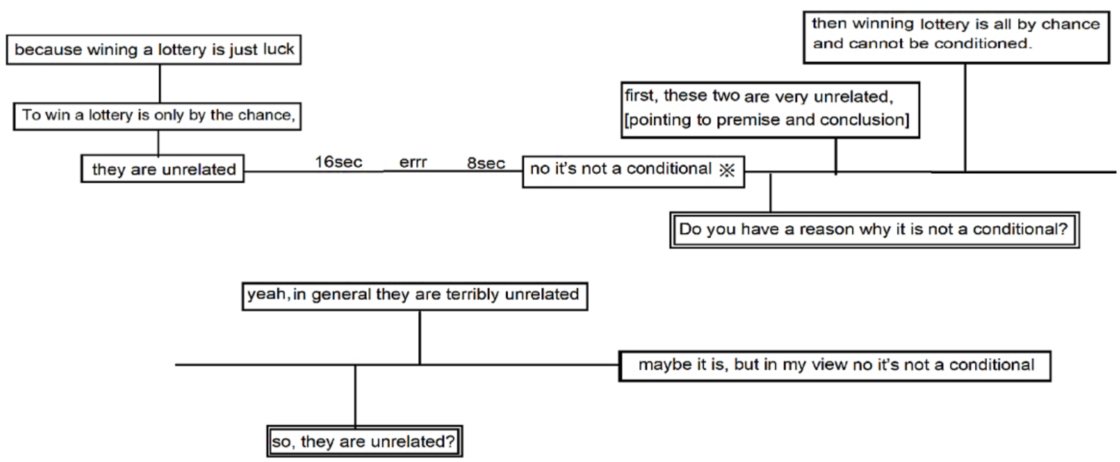


Figure 8.10. The Toulmin's Model 10

For the statement "if elephants could fly, then you win the lottery", Hugo distinguished between impossibility and unrelatedness of statements. Some parts of this assertion seemed to him "impossible", but he did not use this word first; rather, he used "very unrelated" or "terribly unrelated". These extreme words for unrelatedness using a special sign ※ directed me to ask for more explanation, which then eventually he used the word "impossible":

Hugo: maybe because in this statement [pointing to the second one], the sentences are impossible and unrelated, but these are things that can happen [pointing to the first statement].

The use of sign ※ by this interviewee with a solid background in mathematics is not unusual. In logic, a contradiction consists of a logical incompatibility between two or more propositions, and sometimes this symbol is used when there is a contradiction in mathematics problems.

In line with the other studies in philosophy and psychology – that people do not assume a conditional statement as to material conditionals – this section has produced evidence that a mathematician with a background in logic does not regard a conditional as a material conditional. Not only may the fallacies result in some confusion, but, in this section, I signaled statements with irrelevant parts were not taken as conditionals and that, in different contexts, he reasoned differently. The above excerpts are not the only ones for which the interviewee reasoned strongly on the relatedness of the two parts of the statement, and there are more instances of similar arguments (Appendix).

8.4.5. Evidence of existing mental models

Within the mental model theory, individuals may not be aware of the logical form of the sentences, and they consider possibilities fitted with the information using the meanings, context, and their own knowledge. Barwise (1989) stated that everyday reasoning is not a formal process, and the theories based on formal rules cannot give any method in order to recognize the logical form of a sentence in everyday language. Unlike logic, the grammar of a sentence in everyday language is not enough to determine its logical form, and it can be determined from the meanings of the sentence in context and the individual's knowledge about the topic. The results from the created event in this chapter show no convincing evidence that logical forms play any role in the interviewee's reasoning; rather, it suggests that the interviewee's reasoning could be explained within the theory of mental models.

When there is a mental model for something, it can influence the way of arguing. Different examples are verifying the existence of the mental models for this interviewee. Toulmin's models provided in Figures 8.4, 8.5, 8.6, 8.8, and 8.10 show Hugo checked some criteria to decide. In all corresponding Toulmin's schemes, the mental model posed by Hugo is highlighted in green box above the main route of the diagram (the given question that is the first box is highlighted in yellow and the last box that is interviewee's final conclusion is in blue). In addition to those excerpts, the excerpts below suggest that the interviewee decided about a conditional by checking the two following conditions:

- an existing “if-then” form in the sentence;
- the relatedness of the premise and conclusion.

The mental model posed by the interviewee is nicely indicated in the following two schemes, the Toulmin models 11 and 12:

Excerpt 11:

Interviewer**:** [handed a card with the statement “If ABC is a triangle, then $A + B + C = 180^o$”]

Hugo: It is very obvious that this is a conditional,

Interviewer: Why do you think so?

Hugo: It has “if … then”, this is related to this, it satisfies each condition, so it is a conditional,

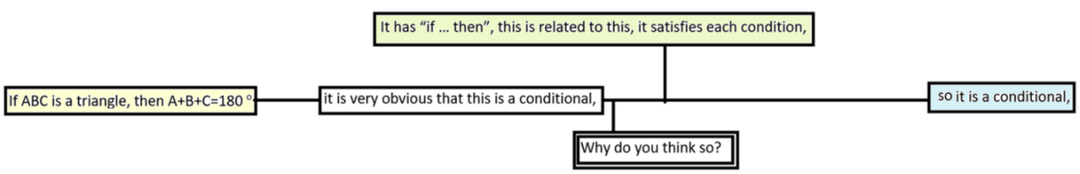


Figure 8.11. The Toulmin’s Model 11

Excerpt 12:

Hugo: “If $(x-2)(x+1) = 0$, then $x = 2$ or $x = -1$”, of course, this is a very simple math statement, obviously a conditional, they are related, and it in “if …, then …” form.

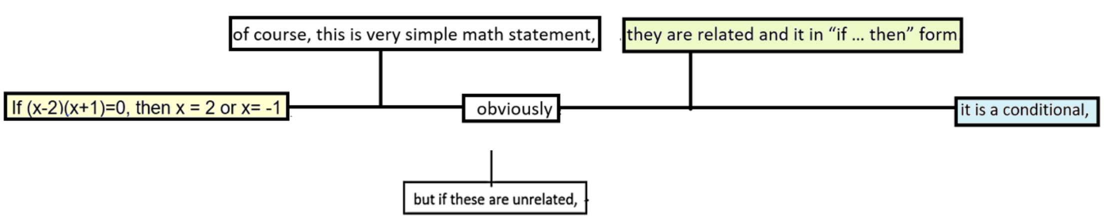


Figure 8.12. The Toulmin’s Model 12

In order to make sure that the interviewee had such a model for a conditional statement, right after his response for the above statement, I asked him for a false but very similar statement, which again he determined the conditional normatively:

Interviewer: What if it was "If $(x-2)(x+1)=0$, then $x = 2$ or $x = 3$"?

Hugo: That's okay. It is still a conditional,

Interviewer: Why?

Hugo: It is only solved incorrectly, but still a conditional,

Interviewer: So, being unrelated is different from being false?

Hugo: Yes, of course,

In the Toulmin diagrams, for most of the straightforward statements there are only four boxes: the first box includes the premise, the second warranting with "if-then", the third for checking relatedness, and the last one is the conclusion.

To determine if a sentence is a conditional or not, Hugo added information to the statement, in order to decide if it were a conditional or not. In other words, to see each statement is a conditional, Hugo used his mental model: he first looked for an *if-then pattern* and then for *relatedness* in order to reach his decision. Hugo argued that those with unrelated parts are not conditional, even if they are in an if-then form.

Figure 8.2 shows the Toulmin's model for the statement "If cos θ = 0.81, then θ = $2\pi \pm \frac{13\pi}{173}$". In this statement, the antecedent and consequet are related. Even though the statement is mathematically wrong, it is difficult to guess that $\cos(2\pi \pm \frac{13\pi}{173}) \neq 0.81$. Hugo recognized this conditional with no doubt because there was nothing to violate his mental model of the solution of such a trigonometric equation.

I exemplified cases in which the interviewee used a certain model for a conditional. This could also be explained in the light of the theory of mental logics, but mental logic theory explains the non-normative responses in terms of the accessibility of different models; which is not the case in this case study. The interviewee has his model: "if-then" form and relatedness of the premise and the conclusion", and seemingly he used this model with no hesitation. The mental model theory addresses such reasoning errors.

Perhaps the interviewee's knowledge and pragmatics shaped his perception of a conditional, and while reasoning, he added his information to the statement, then decided if it were a conditional or not. To reiterate, to see each assertion is an implication, the interviewee had his mental model; he first looked for an *if-then pattern* and then determined *relatedness* in order to make his decision.

Interviewing with one, two, or even 10,000 people does not guarantee that the results can be generalized for all people, and I can never claim that the results discussed in the current work suggest a way that reasoners apply to determine a conditional statement. However, I claim that my research presents one possible way of thinking about conditional statements. To be more precise, this is only one possibility at some certain time that highly depends on the designed tasks and the interviewer's follow-up questions. Hence, the interviewee in this small-scale study may have responded to the questions differently, either at a different time or with a different interviewer. That being said, this sub-section (§§8.4.4) argues that a particular mental model of thinking exists.

A brief look through some other theories

Although the conditional statement **"**If $4x^2 - 5x - 6 = 0$, then $sin\,\theta = 0.546$**"** is in the context of mathematics, the interviewee ignored the logical form of the sentences and only considered possibilities fitted with his information using: the meanings, context, and his knowledge. This may suggest that if there is any innate logic, it cannot be the same as what we know as formal logic, or at least confirming the formal definition of a conditional statement. Therefore, there is no convincing evidence that logical forms play any role in the participant's reasoning, as the selected interviewee noticeably ignored the logical form of this conditional statement and reasoned based solely on his own knowledge. This confirms that **mental logic theory** cannot explain the current data.

The current data also confirms that the **theory of formal discipline** cannot account for the reasoning process of the interviewee. This theory specifies that, if studying advanced mathematics can improve logical reasoning skills, people with good mathematics backgrounds are expected to work with conditionals more normatively. However, the data in this study is not in accord with the purpose of this theory. Furthermore, in a very clear way, most participants distinguish between conditionals in the contexts of mathematics and daily life, even though they might be able to work more normatively on conditionals in both contexts of mathematics and logic, but not in everyday language.

According to the **Information-gain model** and **Bayesian rationality,** the key aim of participants is to increase the amount of information they have about the context by reducing uncertainty, so extra information is defined to be less uncertain. However, the results from this interview do not confirm this idea, and extra information does not always result in more frequent, logically correct solutions. For instance, regarding the statement "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*", the interviewee's knowledge about material conditional conflicted with his knowledge about causal conditionals (see the Toulmin models in Figures 8.7a and 8.7b).

8.5. Summary of Chapter 8

Any language has its own syntax and semantics. In the current work, I added the "context" as the third component of logic as a language. Chapter 8 reported a case study addressing the following questions:

- How does a mathematician determine conditional statements in different contexts, namely classical logic, mathematics and colloquial language?
- What mental models can explain the participant's decision-making?

Sections §8.2 and §8.3 explain the tasks and the data collection methods. The tasks are designed in three phases considering the research questions and satisfying three aims for a clinical interview: discovery, identification, and competence. Phase 1 is fully structured, and interviewees were given certain questions to assess their knowledge about conditional statements. Phase 2 needs more depth in responses and uses a semi-structured clinical interview to determine the interviewee's thinking model pattern while doing tasks. Phase 2 includes 14 questions. In phase 3, the methodology to create data is mainly an unstructured interview, and the interviewee was asked to review and/or compare some of the answered questions in phase 2. Phase 3 is designed to reveal more about the underlying cognitive process to determine the conditionals. Such questions that ask for a comparison between two provided solutions need more reasoning and help discover a thinking process that the interviewees may not know themselves.

Toulmin’s argumentation scheme was applied to prepare data for analysis, and some adjustments were appropriated on Toulmin’s model, in order to fit the diagrams with the interviewees’ reasoning and to capture more features important for this study. Then the data were analyzed using the mental model theory.

I demonstrated that the interviewee follows a certain mental model to recognize conditional statements. He first checks for the “if-then” form, then the relatedness of the two clauses. Regarding the role of context to recognize conditionals, Hugo recognized all conditionals in the context of logic and those with related parts in the other contexts of mathematics and colloquial language. He also reasoned differently for the conditionals in the context of mathematics and colloquial language, but still expected relatedness and correctness for the conditional statements in mathematics.

Chapter 9. Study 2: Different Words to State a Conditional Statement

The first study shows how context can affect reasoning about the conditionals and suggests a mental model for the interviewee. The second study reported in this chapter, takes a sample from prospective elementary school teachers, and confirms the role of context as well as the mental model suggested by the first study. More specifically, this chapter examines the language that prospective elementary school teachers use and understand when dealing with situations involving conditional sentences. It also investigates if they are able to differentiate between "premise" and "consequent" in conditionals that are stated using certain words, like "if-then", "when", "implies", and so on.

This chapter has three main sections.

- Laying the groundwork: this section explains what guided me to conduct the current study and its relation to other studies within mathematics education.
- Methods: this section explains how the tasks were designed and the way data was collected.
- Results: this section analyses data within the adopted framework. As previously, I use mental model theory for data analysis.

The tasks and the populations for the two studies are different, but the results look similar.

9.1. Laying the groundwork

Chapter 2 discussed different types of conditional statements that are mainly applied in English, philosophy, psychology, and mathematics. I also noted that there is no standard system to classify conditional statements. Linguistics have proposed different typologies based on verb tense, level of reality, and construction of conditionals. There is a distinction among indicative, subjunctive, counterfactual, causal, and material conditionals in psychology and philosophy. However, the material conditional is the only one that is used in mathematics.

Chapter 4 discussed the role of conditional statements in formal mathematics (§4.1) and various types of reasoning that involve creating mathematical statements (§4.2). For example, to apply a definition, we need to distinguish between necessary and sufficient conditions. Most of the mathematical propositions are expressed as "If p (hypothesis), then q (conclusion)". In mathematics education, it has been argued that an understanding of logical implication is one of the most important prerequisites for understanding mathematics and constructing proofs or, for example, Hoyles and Küchemann (2002) emphasized logical implication's importance for success in mathematics.

Different kinds of conditionals, other than material conditionals, do not need any truth-value function. This is because, based on the nature of such implications, if the premise happens, then, without any argument, the consequent will happen; therefore, they are always "true". For example, in causal implication ("if it is raining, then the streets are wet"), there is nothing to discuss concerning the statement's truth value. But the implication "if it is raining in Vancouver, then Dakota is a dog" needs to have a truth value because there is no obvious causal connection between the weather in Vancouver and the name of my cousin's dog. So, it is important to distinguish material conditionals from other implications that are always true.

9.1.1. Different ways to state a conditional statement

A conditional statement can be represented using different cue words. For instance, "if p, then q" can be stated by any of the following equivalent forms.

Table 9.1. Alternative cue words to represent a conditional statement

If $\boldsymbol{p}$, then $\boldsymbol{q}$
$\boldsymbol{p}$ implies $\boldsymbol{q}$.
$\boldsymbol{p}$ only if $\boldsymbol{q}$.
$\boldsymbol{q}$, if $\boldsymbol{p}$.
$\boldsymbol{q}$, when $\boldsymbol{p}$.
$\boldsymbol{q}$, assuming that $\boldsymbol{p}$.
$\boldsymbol{q}$, given that $\boldsymbol{p}$.
Provided that $\boldsymbol{p}$, $\boldsymbol{q}$.

Some research studies address the issues of understanding these equivalent forms (e.g. Tall, 1989). The results show that, even when people are familiar with conditional statements, identifying conditionals with equivalent forms can still be challenging. However, not just the equivalent forms mentioned above, but also how a conditional is worded, may affect an individual's decisions. For example, the word "implies" and "when/where" are frequently used in mathematics to represent a conditional statement. That is, "p implies q" is another wording for "if p, then q", and "p when/where q" is another wording for "if q, then p".

In a mathematical context,

$\sin\theta = 1,$ when $\theta = \frac{\pi}{2} + 2k\pi$ is equivalent to "if $\theta = \frac{\pi}{2} + 2k\pi$, then $\sin\theta = 1$"

and

$x2^x = 0$ implies $x = 0$ is equivalent to "if $x2^x = 0$, then $x = 0$".

For as much as most of the conditional statements in mathematics are also statements in everyday language, it may be possible to confuse material conditionals with some other types of conditionals in everyday language, like causals.

According to Pimm (1987), multiple meanings could be a problem for many mathematical terms, and non-mathematical meanings can influence mathematical understanding and be a source of confusion. More specifically, concerning conditional statements, Epp (2003) reported that students develop misconceptions because, in many cases, the interpretation of a statement in everyday language is different from that in mathematics, which confuses them.

Therefore, students need to know how mathematical language is linguistically different from everyday language. Some studies address the importance of semantics in decisions about a conditional, for example, Buchbinder and McCrone (2020). Not only is the context important to determine and assess a conditional, but also the way it is stated can modulate the participant's interpretation of a conditional.

9.1.2. Research questions

In a pilot investigation, I noted that participants were confused with the following conditional that had been worded with 'implying':

"$x^2 + y^2 = 0$ implies that $x = 0$ and $y = 0$".

This led me to the current study that is aimed to investigate the following.

- How do participants recognize conditional statements?
- How do the particular key words influence the participants' responses?

9.2. Design of tasks

To assess participants' understanding of conditionals, I designed a questionnaire that included conditionals in mathematics and everyday language contexts. In order to fit the tasks with the purpose of the current work, I conducted a pilot study with 6 participants, all from the class for mathematics for prospective elementary teachers. The initial questionnaire was modified based on the responses; in particular, I added conditionals with the cue words of "when" and "imply".

For some reasons I decided to design different tasks for the two studies. One reason is that study 1 and 2 have different populations. The first study's population is mathematicians and the population in the second study is prospective elementary school teachers. What is simple in one population maybe twisted and challenging in the other population, and what is known context for one population maybe unknown for the other one.

Another reason is: even though the tasks for both studies are designed to address the main research question the thesis is set out to answer, the second study- besides the overarching question- examines the way different words to indicate conditionals influence people's understanding of them. So, I decided to exclude conditionals with unknown context and add conditional worded by different indicators, like when and implies.

The questionnaire included 6 tasks.

The first task was designed to ensure the participants know what a conditional is. The following multiple-choice problem was given as the first question under the title of warm-up. It asks to determine the conditional and justify the answer.

Task 1: Which of the following is a conditional statement?

a) Sina plays guitar, or Shan plays piano.

b) Sina plays guitar when Shan plays piano.

c) If Sina plays guitar, then Shan plays piano.

d) None of the above.

The second task includes nine conditional statements with different cue words to indicate conditionals. This task was aimed to see if they knew conditionals could be stated in different ways and were familiar with some of these keywords.

Task 2a: Read the following statements and determine conditional statements. If it is conditional, determine the conclusion.

1) If I am late for dinner, then my friend will be sad.
2) A triangle has a right angle; then it is a right-angle triangle.
3) To be a multiple of 18, the number must be divisible by 2 and 9.
4) To have my pen, you need to pay $10.
5) As the name implies, oilstones require oil as a lubricant.
6) If a number is divisible by 45, then it is divisible by both 9 and 5.
7) 9 divides *n*, implies that 45 divides *n*.
8) Two lines *l* and m are skew lines if they are in two different planes.
9) A number is divisible by 5 when its last digit is 5.

Task 2b: Which of the following sentences is the if-then form of the conditional statement “It is time for dinner when it is 6 pm.”

a) If it is 6 pm, then it is time for dinner.

b) If it is time for dinner, then it is 6 pm.

c) If you want to eat dinner, then you must eat at 6 pm.

d) None of the above.

Towards Tasks 3 and 4

The converse of the conditional “if p, then q” is “if q, then p”. A conditional and its converse do not necessarily have the same truth values. The following truth values table represents the truth values for the conditional “if p then q” and its converse “if q, then p”.

Table 9.2. Truth values for a conditional statement and its converse

p	q	$p \implies q$	$q \implies p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Depending on the truth values of p and q, not always do $p \to q$ and $q \to p$ have the same truth values.

- In the case where both p and q have the same truth value (both are true or both are false), $p \to q$ and $q \to p$ will have the same truth value of true,
- In the case where p and q have different truth values, $p \to q$, and $q \to p$ will have different truth values.

The confusion occurs when p and q have different truth values (for example, “if $x^2 = 4$ then $x = 2$”), then the statement is not always true, i.e. for some values of x it is true and for some values is not true (§§2.4.2). In this instance, it is true when $x = 2$ and false when x = –2. The following table shows the truth values for the conditional “if $x^2 = 4$, then $x = 2$”:

Table 9.3. Truth values for a conditional statement involving a variable

x	$x^2 = 4$	$x = 2$	$x^2 = 4 \implies x = 2$	$x = 2 \implies x^2 = 4$
$x = 2$	T	T	T	T
$x = -2$	T	F	F	T
$x \neq 2, -2$	F	F	T	T

The table shows that the two statements (“if $x = 2$, then $x^2 = 4$” and “if $x^2 = 4$, then $x = 2$”) do not have the same truth values. Even though “if $x = 2$, then $x^2 = 4$” is always true, its converse "if $x^2 = 4$, then $x = 2$" is not always true, and indeed is false when $x = -2$.

In fact, a conditional $p \rightarrow q$ and its converse $q \rightarrow p$ are equivalent (have the same truth tables) if p and q are equivalent. However, even if p and q are equivalent, the conditional $p \rightarrow q$ has still different meaning from the conditional $q \rightarrow p$. For example, the two propositions $x = 3$ and $x + 2 = 5$ are equivalent, but the conditional "if $x + 2 = 5$, then $x = 3$" is different from the conditional "if $x = 3$, then $x + 2 = 5$". In logic, these two conditionals have the same semantics, but different syntaxes. In mathematics, the conditional "if $x + 2 = 5$, then $x = 3$" resembles one of the models to describe "subtraction", known as missing addend, but the conditional "if $x = 3$, then $x + 2 = 5$" represent an addition.

To design the third question, I combined this idea about the difference between a conditional $p \rightarrow q$ and its converse $q \rightarrow p$, even with equivalent clauses, by applying "implies" instead of "if-then" to state a conditional statement.

For task 3, the participants were asked to rephrase the conditional statement "$x = 3$ implies $x + 2 = 5$" in the "if-then" structure. Those who are not familiar with the role of the word "imply" as "then" may be more inclined to choose the statement "$x + 2 = 5$ implies $x = 3$" as the equivalent in rephrasing the statement "$x = 3$ implies $x + 2 = 5$".

Task 3: Rephrase the statement "$x = 3$ implies $x + 2 = 5$" to a conditional.

In task 4, the participants were asked to find a mismatch to a conditional. In this task, the conditionals are worded with different cue words to represent a conditional.

Task 4a. Find a mismatched rephrasing of the statement "$ax = ac$ (*a* non-zero) implies that $x = c$" among the following statements:

a) If we consider $ax = ac$, then $x = c$.

b) If $x = c$, then solve $ax = ac$.

c) For $ax = ac$, we have $x = c$.

d) $x = c$ is a necessary condition for $ax = ac$.

Task 4b. Find a mismatched rephrasing of the statement "If a polygon has two right angles, then it is separable" among the following statements:

a) A polygon is separable in the case that it has two right angles.

b) Granted that A is a separable polygon, we have A has two right angles.

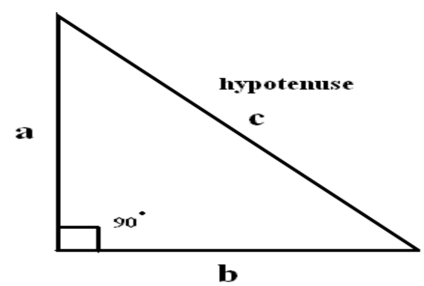
c) A is a separable polygon when it has two right angles.

d) A is a separable polygon that has two right angles.

Task 4c. Find a mismatched rephrasing of the statement, "In a right-angled triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides".

a) If *a*, *b* and *c* are the lengths of the sides of a right-angle triangle, then $a^2 + b^2 = c^2$, where *c* represents the length of the hypotenuse.

b) In the triangle below, we have $a^2 + b^2 = c^2$.



c) Show that the given triangle has a right angle, then prove that the square of the hypotenuse is equal to the sum of the squares of the other two sides.

d) $a^2 + b^2 = c^2$ is a necessary condition for a right-angle triangle when *a*, *b* and *c* are the lengths of its sides.

In the next task, task 5, I asked the participants to write examples of conditional statements without using if–then. This task aims to see what the most active alternative word is to indicate a conditional.

Task 5. State one conditional statement using if–then, and two conditional statements without using if–then, if there are any.

Task 6 explicitly asked the participants if they had any method to determine conditional statements.

9.3. Methods

This study investigates and analyzes the prospective teachers' responses to a written questionnaire on conditional statements. This section explains the method used to collect data and details a theory to analyze the data.

9.3.1. Participants and data collection

The participants in this study were 34 prospective elementary teachers taking a 'mathematics for teachers' course from a college in BC. The pre-requisites to take this course are "Principles of Math 11", "Precalculus 11 or 12" (C or better), and some of them had not been taken any mathematics course for years. The questionnaire was administered in the last week of the academic term, and the participants were asked to complete the tasks when I was invigilating them, and there was no extra explanation or any hints when they were writing the tasks.

They had been introduced and practiced problems including simple equations, cancellation laws and the divisibility tests during the course. These are topics the tasks are designed based upon them. It took them about an hour to finish the tasks. Participants' written responses comprise the data analyzed in this study.

9.3.2. Theoretical framework

Chapter 7 discussed some popular theories of reasoning that can explain people's errors in conditional reasoning. These included the information gain model, Bayesian rationality, and mental models' theory of reasoning. These theories argue that people reason not in line with formal logic rules; instead, they reason using uncertainty and/or considering the possibility of alternative assumptions.

I chose the theory of mental models because the data in the current work do not strongly support uncertainty while reasoning: rather, it suggests a presupposed idea/definition of a conditional statement that may vary depending on the individual's experience and pragmatic knowledge. Within mental models' theory, people may not be aware of the logical form of a sentence, and they just consider possibilities fitted with the given information using the meanings, context, and their own knowledge. This theory also accounts for the informal arguments in both science and daily life. Considering these affordances, the theory of mental models was used in data analysis to address the research question/s.

9.4. Data analysis

To analyze the data, the participant's responses were summarized in different tables, and some of the tasks are selected to discuss in this section. For task 1, all participants except one (who is excluded from the data reported here) identified the normatively correct answer, i.e., part c, and 70.7% (23 out of 33) students added some additional explanation to reason why it is a conditional. The following are three excerpts of their justifications,

> **Case 1:** Something happens as a result of something else,
>
> **Case 2:** This (the third item) is a conditional because the result is caused by a prior action. The result is Shan playing the piano if Sina plays guitar. A condition that needs to be met for something else to be true.
>
> **Case 3:** This is conditional because Shan playing the piano is dependent on Sina playing the guitar.

The results from the first task suggest that almost all of them knew what a conditional is when it is in the "if-then" form. However, a few also considered the statement in part (b) as conditional. The source of conflict in the second item is the word "when", which could refer to either the time, or a condition. For example, a participant explained that: "this statement is a conditional statement because Sina plays the guitar under the condition that Shan is playing the piano". With such an interpretation, it could also be regarded as a conditional. In mathematics, the word "when" is frequently used to represent a conditional statement, for example:

$\sqrt{x^2} = x$ when $x \geq 0$.

In what follows, conditionals worded with "when" are discussed in more detail.

In the second task, the participants were provided with statements that had different cue words to indicate conditionals. The statements are classified in the following table to ensure that I covered each area almost equally and I considered all possibilities:

Table 9.4. The Table Summarizing Task 2a

	Colloquial	Mathematics
If – then	If I am late for dinner, then my friend will be sad. (1) If Sina plays guitar, then Shan plays piano. (Task 1)	If a number is divisible by 45, then it is divisible by both 9 and 5. (2) If a triangle has a right angle; then it is a right-angle triangle. (14)
Implies	As the name implies, oilstones require oil as a lubricant. (29)	9 divides n, implies that 45 divides n. (25)
When	Sina plays guitar when Shan plays piano. (Task 1)	A number is divisible by 5 when its last digit is 5. (8).
None	To have my pen, you need to pay 10 $. (11)	To be a multiple of 18, the number must be divisible by 2 and 9. (10) Two lines l and m are skew lines if they are in two different planes. (12)

The number in brackets at the end of each statement shows the number of non-normative answers. As it is clear from the table, the majority of the participants could not correctly determine the conditionals stated with the word "imply". The results are summarized with a bar graph in Figure 9.1 The vertical axis represents the number of participants who provided a normatively correct answer, and the horizontal axis shows the cue words used to indicate the conditional:

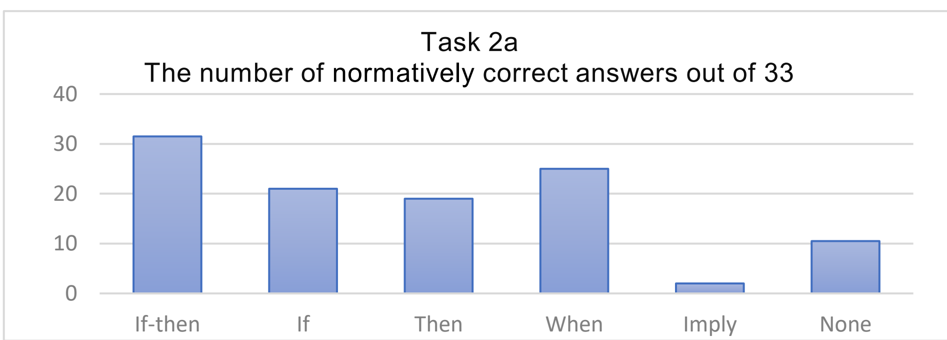


Figure 9.1. The bar graph summarizing task 2a

The lowest number of the normatively correct answers was for the conditionals stated with the word "implies". In the following sub-section, I discuss this word in more detail.

9.4.1. The Word “imply” as an indicator of a conditional statement

The bar graph for task 2a (Figure 9.1) suggests that the word “imply” may not be an active indicator of a conditional for the participants. For instance, 25 participants stated that the statement “9 divides *n* implies that 45 divides *n*” is not a conditional. However, all except two students could recognize the following statement (3) as a conditional:

“If a number is divisible by 45, then it is divisible by both 9 and 5”.

The high failure rate of 76% (25 out of 33) to recognize the statement “9 divides *n* implies that 45 divides *n*” as a conditional might be because of the use of the word “implies” rather than “if-then”. However, this could also be because of the participants’ partial knowledge about the content, as the statement “9 divides *n* implies that 45 divides *n*” is not true. There are many numbers divisible by 9 but not divisible by 45 – for example, 9, 18 and 27. In order to see whether the source of such a result is the “content” or the word “imply” itself, the participants were provided with two more items, both including the word “imply”.

Task 3: Rephrase the statement “$x = 3$ implies $x + 2 = 5$” to a conditional.

This task specifies that the given statement is a conditional, and the participants needed to determine its “if-then” form. As the following pie chart suggests, about half of them were able to rephrase the statement correctly, while the rest wrote either its converse “if $x + 2 = 5$, then $x = 3$” or a bi-conditional. The results are summarized in the following pie chart.

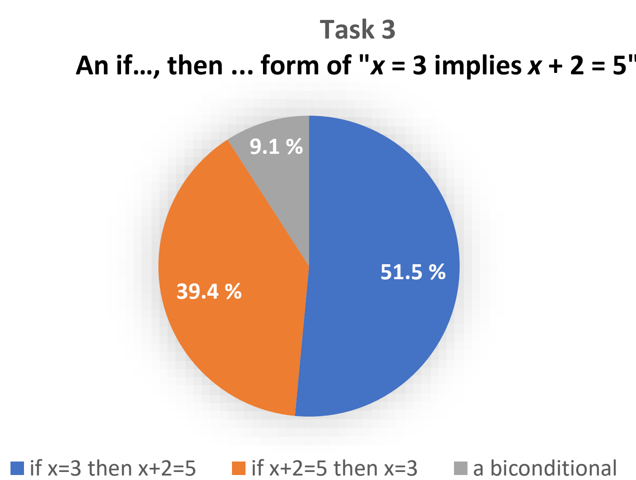


Figure 9.2. The pie chart summarizing Task 3

The result suggests that, even knowing that the statement is a conditional, many participants could not successfully determine which part is the premise.

Task 4a. In this task, the participants were asked to find a mismatched rephrasing of the statement "$ax = ac$ (a non-zero) implies that $x = c$" among the following statements:

a) If we consider $ax = ac$, then $x = c$.

b) if $x = c$, then solve $ax = ac$.

c) For $ax = ac$, we have $x = c$.

d) $x = c$ is a necessary condition for $ax = ac$.

The results summarized in the chart show that only 15.2% (5 out of 33) participants could determine the correct answer.

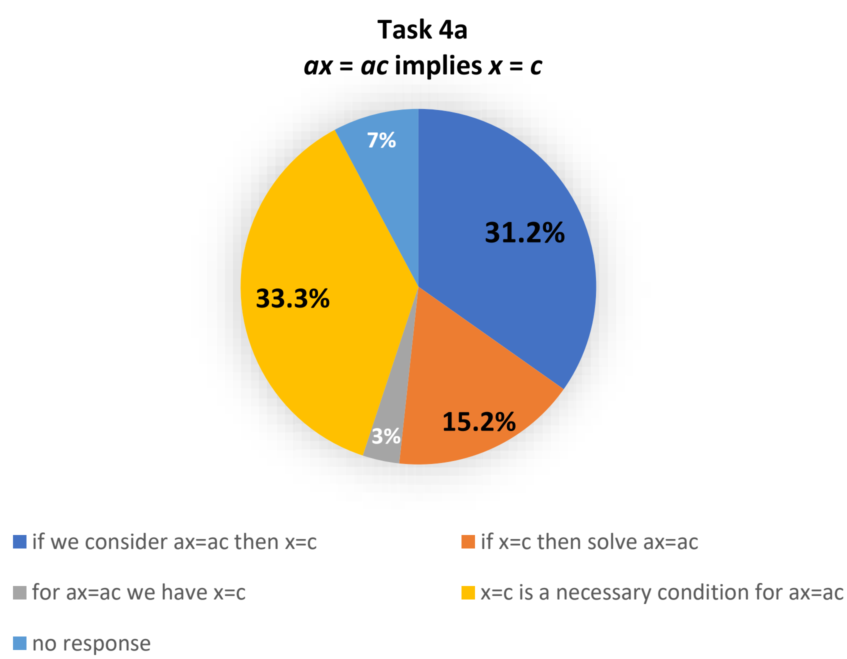


Figure 9.3. The pie chart summarizing Task 4a

In the case of conditionals including "implies" in Tasks 3 and 4a, the antecedent and consequent are equivalent mathematical statements, meaning that it is not possible to determine the exact relation between the antecedent and the consequent by just looking at the mathematics. Therefore, to figure the correct relations out, one needs to know the meaning of "imply".

The results from Tasks 1 and 2 suggest that almost all the participants knew what a conditional is. They had also been introduced and practiced such problems, including simple equations and cancellation laws and the divisibility tests during the course. In other words, the situation includes familiar and clear content. Therefore, the confusion is attributed to the word "implies". However, this word has been frequently used in mathematical propositions and proofs, and if one does not know about its use, it is not always straightforward to guess which is the premise and which is the conclusion. That is very important in mathematics, as most of the theorems in mathematics and many mathematical rules are true conditionals, but their converses are not true.

9.4.2. The word "when" as an indicator of a conditional statement

In Task 1, 15.2% (5 out of 33) of the participants selected both parts (b) and (c) as a conditional. They reasoned that these statements are conditionals because one part of the statement is contingent on the other part. For the statement in part (b) (Sina plays guitar when Shan plays piano), some participants suggested that Sina's act of playing guitar is **contingent upon** when Shan plays the piano. For example, one of the participants explained that, "this statement is a conditional statement because Sina plays the guitar under the condition that Shan is playing the piano". With such an interpretation, it can be regarded as a conditional, and its restated form would be "If Shan plays piano, then Sina plays guitar".

Also, a participant gave more details on using the word "when" in a conditional. She explained that part (b) could be a conditional or could be referring to time. She added that if we had "only when", rather than "when", it would be a conditional. Based on her reasoning, the second statement means "Sina plays guitar at the same time as Shan plays the piano", and only the following statement will be a conditional:

Sina plays guitar only when Shan plays piano.

Though some participants did not consider part (b) in task 1 as a conditional, the bar graph for task 2a (Figure 9.1) shows that the word "when" could be an active word to represent a conditional statement. Also, for the statement (9): "the number is divisible by 5 when the last digit is 5", there are some examples for which the one's digit is not 5, but the number is divisible by 5 like 20. This shows that the converse of this statement is not true. However, this should not be a problem, because even if the word "when" is misinterpreted as “then”, the statement should still be regarded as a conditional, unless they check for causation rather than conditional. 26 students (out of 33) identified this statement as a conditional. Even the student who specified the use of “only when” rather than “when” to represent a conditional, determined the statement (“the number is divisible by 5 when the last digit is 5”) as a conditional without any explanation. The participants may expect conditionals in mathematics to be different from conditionals in everyday language, but they may not know that there is only one conditional in mathematics, namely material conditional.

9.4.3. Causation or a material conditional

A pattern to determine a conditional can be regarded as a mental model. Although there might be a personal mental model for each individual, it is worth investigating what is common among the participants’ mental models for a conditional and how they differ. In order to see if participants’ responses were guided by some method, in **Task 5**, I explicitly asked them if they had any method to determine conditional statements.

Their responses show that most of them considered some criteria for both the syntax and the semantics of the statement. All of them were following a model for a conditional indicated by some certain words, such as “if-then” and “when”, and some sort of relevance between the premise and the conclusion; the conclusion/outcome is dependent on whether or not some other condition (the premise) is satisfied. This suggests that many of them search for a cause-and-effect relationship between the parts of a conditional so that they may be more inclined to causal conditionals introduced in §§2.3.4. Also, a few participants explained that if there is a counterexample that violates the statement’s rule, it cannot be a conditional.

Whatever model they chose to determine a conditional statement is the one to identify conditionals in everyday language. Even when the statement is entirely in the context of mathematics, they still searched for a contingent. As the mental model theory suggests, people prefer one model, and this may not be people’s error to determine all the conditionals with one model, if they are not aware of the definition of material conditionals and that this is the only type of conditional used in mathematics which is the only needing truth values.

Searching for a connection and checking to see if there is a counterexample to refute the conclusion are the criteria for causal conditionals, and these have nothing to do with material conditionals. The following conditional statements are in the context of number theory:

- 9 divides *n* implies that 45 divides *n*.
- A number is divisible by 5 when its last digit is 5.

The participants reacted differently to these statements. Even though the concept of divisibility had been introduced in class and they had practiced the topic, 76% (25 out of 33) of the participants stated that the first statement is not conditional, which is a high rate compared with 24% (8 out of 33) of the non-normative answers for the second statement.

In mathematics designed for the prospective elementary teachers' course, the students learned and practiced the definition of divisibility and divisibility tests. This may suggest that they could figure out that, in the first statement, the conclusion cannot always be derived from the premise. For example, 9 divides 18, but 45 does not divide 18, which could be a reason for the high rate of incorrect responses. This confirms that they expected a cause-and-effect relationship between the two parts of a conditional.

9.5. Summary of Chapter 9

There are different keywords to indicate a conditional statement, and "if p, then q" has some equivalent forms like "p implies q" and "q when p". It is important to recognize the conditional and its premise and conclusion.

This chapter investigated how prospective elementary school teachers made sense of various forms of conditional statements in the contexts of mathematics and colloquial language and which keywords they recognized to represent a conditional. The data was comprised of responses of 34 prospective elementary teachers from a college in BC to a written questionnaire.

The theory of mental models has been applied to analyze the data. The main focus of the analysis was on the understanding of different wording to represent a conditional, e.g. "if-then", "when", and "imply". Even though almost all the participants knew what a conditional is, and the word “implies” is frequently used in mathematics to indicate if – then, many participants were not aware of that even when the context of the conditional was a familiar topic.

The chapter also examined the patterns that individuals use to recognize a conditional and suggested some mental model presence. The results indicate that participants’ mental models used to determine the conditionals are more fitted with causal implications than with material conditionals, even when a conditional is presented in a mathematics context. To understand the conditionals, they searched for a connection and contingent, and some also checked to see if there is a counterexample to refute the conclusion; and these are the criteria to determine causal conditionals.

Chapter 10. Discussion

Theories of human reasoning distinguish between deductive and pragmatic inference, namely between reasoning based solely on the logical form of the sentence (deductive inference) and reasoning based on content alone or some mixture of the two (pragmatic inference).

This thesis focuses on the way people understand conditional statements while reasoning about conditionals, namely which type of reasoning do they use? Do they use pragmatic or deductive reasoning or a mixture of both? With this focus in mind, two different research studies have been carried out and were reported in Chapters 9 and 10. The participants for studies 1 and 2 are drawn from different populations. The tasks for each study is designed in a way to make it appropriate for each group, and used different methods to collect data. Even though I had been open to other theories of reasoning but for both studies I came with the same theory, namely theory of mental models.

This final chapter first gives a brief overview of each study in sections §10.1 and §10.2, then draws conclusions from them in section §10.3 to address the research questions. Section §10.4 discusses my ideas and further comments after doing this work. Section §10.5 lists the contributions of the current work, and the final section, §10.6, gives the final quotation.

10.1. Study 1: how a mathematician determines a conditional statement in different contexts

The main goal of Study 1 was to explore how a successful mathematician reacted to a conditional statement in different situations, namely classical logic, mathematics, and colloquial language. To this end, I conducted a qualitative study that applied the mental model theory of reasoning to analyze a clinical interview with a selected interviewee named Hugo.

The results suggest that his knowledge and pragmatics might shape the interviewee's perception of a conditional. It is indicated in his model of argumentation (Figure 8.7b) that different contexts of mathematics and colloquial language explicitly influenced his decision, and there were no signs of him applying the criteria for material conditionals.

The interviewee ignored the logical form of the sentences and only considered possibilities fitted with his information using the meanings, context, and his own knowledge. This may suggest that, if there is any innate logic, it cannot be the same as what we know as formal logic. Therefore, there is no convincing evidence that logical forms play any role in the participant's reasoning.

The mental model posed by the interviewee is indicated in the schemes coded using Toulmin's model. As Figures 8.5, 8.11 and 8.12 represent, he first was looking for an *if-then pattern* and then for the *relatedness* between the parts of a conditional to reach a decision. He argued that statements with unrelated parts are not conditional statements, even if they are in an if-then form.

10.2. Study 2: different words to state a conditional statement

Even though the tasks for both studies are designed to address the main research question the thesis is set out to answer, the second study besides the overarching question (that is discussed in Chapter 9) investigated individuals' awareness related to key terms that indicate a conditional statement, namely "if-then", "when", and "imply". While the main focus was on understanding different wording to represent a conditional, it also examined the patterns individuals use to recognize a conditional.

The data consisted of responses of 34 prospective elementary teachers to a written questionnaire that included tasks involving conditional statements in the contexts of mathematics and everyday language. The mental model theory was used in data analysis.

The results showed that, even when people were familiar with conditional statements, identifying conditionals with equivalent forms can be challenging. The context is important to determine and assess a conditional, but the way it is stated can modulate the participant's interpretation of a conditional. The results indicate that the word "imply" is not as easy as its counterpart format "if-then" to address a conditional statement. Even though the word "implies" is ubiquitous in mathematics to address a conditional statement, the majority of the participants were not aware of that. Among all alternative words (discussed in §§9.1.1) to represent a conditional, the word "when" was the most popular one.

10.3. Conclusions

This section attempts to draw conclusions from the results of both Study I and II in order to address the research questions this thesis aimed to investigate, namely "how do people recognize conditional statements?". To respond to this question, I conducted two studies, the first one aimed to engage with these questions:

- How does a mathematician determine conditional statements in different contexts, namely classical logic, mathematics, and colloquial language?
- What mental models can explain the participant's decision-making?

Moreover, the second one was to address these questions:

- How do participants recognize conditional statements?
- How do the particular key words influence the participants' responses?

10.3.1. The role of context besides semantics and syntax

In the introductory part of §5.4, I discussed two components of a formal language: syntax and semantics. In mathematics discourse, Baldwin (2009) suggests a difference between the meaning of symbols in algebra and their interpretation in the physical world, to which he refers as "situation". Then he adds it to syntax and semantics of mathematical expressions in algebra and discusses that these are necessary components for learning variables in algebra (§5.5).

The research studies reviewed in §5.3 suggest that identifying the connections between the parts of a conditional statement in everyday language depends on people's understanding and knowledge about the context. Baldwin's idea about the three main components of learning algebraic variables (§5.5), and the results of the two different versions of Wason's Selection task (§§5.1.2 and §5.3), directed me to consider the "context" as an important component in understanding conditional statements.

Chapter 5 pointed out that general knowledge and knowledge of context can affect the individual's interpretation of a conditional statement. The studies reported in Chapters 8 and 9 are original, in that participants were asked to recognize conditionals in different contexts, and they show how their background knowledge can mislead them to specify conditionals.

For all statements in the second phase of Study 1 (§§8.2.2), in both logic and mathematics, the interviewee, Hugo, could answer with certainty, but he experienced uncertainty in the daily language with unrelated clauses. The following table (Table 10.1) shows the statements classified into three discourses and distinguished by the relatedness of the premise and conclusion. The interviewee recognized all as conditionals, except the statements in two boxes shaded in blue.

Table 10.1. The Table analyzing Task 2a in Study 1

Contexts	Colloquial	Mathematics	Logic
Related	If you pay 10$, you will have my pen. If you are late for the meeting, then you will be fired.	If a population consists of 40% men, then 60% of the population must be women. If $(x-2)(x+1)=0$, then $x = 2$ or $x = -1$. If ABC is a triangle, then $A+B+C=180^{\circ}$.	If the sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the earth. If the discovery of galaxies without dark matter holds up, then astronomers will have to seriously consider what this growing population of galaxies without dark matter means.
Unrelated	If elephants could fly, then you win the lottery. If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*.	If $x+1=0$, then $z=5$. In triangle ABC, if we have $A + B + C = 180^{o}$, then the area of a *circle* is pi times the radius squared. If $4x^2 - 5x - 6 = 0$, then we have $sin\,\theta = 0.546$.	If $cos\,\theta = 0.81$, then $\theta = 2\pi \pm \frac{13\pi}{173}$. If A is a Banach algebra, then for every Banach A-bimodule $X, H\,(A, X') = 0$.

Conditionals in Logic: The interviewee responded with no hesitation and promptly for the conditionals in logic, with unknown context, and only checked the "if-then" criteria for each statement.

Conditionals in Mathematics: For a conditional in mathematics, the interviewee first checked to see if the statement were mathematically correct and whether there was no reason to refute it. In cases he could not refute a mathematical statement, it was labeled as conditional, but those that were mathematically wrong or with unrelated clauses were labeled as non-conditional.

Conditionals in Colloquial Language: Regarding the statements in everyday language, Hugo directly checked for two requirements: "if-then" form and "relatedness" of the antecedent and the consequent.

The model of the interviewee's argumentation in Figure 8.7b shows that the different situations of mathematics and everyday language explicitly influenced the interviewee's reasoning behaviour. He reasoned differently when situating the sentence in mathematics from when situating it in everyday language. For example, the interviewee recognized "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" as conditional in everyday language, but not as conditional in mathematics. Maybe by mathematics, he addressed the certain validity of the sentence or expected more accuracy.

10.3.2. Does knowledge of mathematics mislead the participants?

Besides syntax and semantics as significant components of a conditional statement, another important factor, namely context, needs to be carefully considered. When dealing with conditional statements, even educated people make inferences based on the context. This raises the question that "does more information result in more certainty?".

In probabilistic theories of reasoning discussed in §7.1, knowledge about the context and extra information result in less uncertainty. However, the results from the studies discussed in this thesis do not confirm this idea, and extra information does not always result in more frequent logically correct answers.

Although system-1 plays an important role in explaining the way individuals think in many situations, this may still not work for many conditionals. So, depending on the content of a conditional statement, reasoners may not have any starting point to evoke their intuitions, and therefore there would be almost nothing to direct them. However, the question is, does this lack of knowledge or intuition (having less or no information about content or subject matter) make the problem more confusing? I think it mostly depends on the task. For example, consider the following statement:

If A is a Banach algebra, then for every Banach A-bimodule X, $H(A, X') = 0$.

Regarding this statement, one can ask different questions. Consider two possible tasks:

1) Evaluate the above statement.
2) Determine whether it is a conditional statement or not.

Having no relevant information may make it impossible to evaluate such a statement, but to decide if it is a conditional or not, more information may result in more confusion. In other words, it might be easier to decide about the above statement compared with the statement "if the elephants could fly, then you win the lottery". Because with no information about the mathematical subject matter, Hugo only focused on the syntax of the conditional.

Study 1 (in Chapter 8) shows the selected interviewee could successfully determine the conditionals that he did not have any knowledge about the context or partial knowledge that was not enough to decide its validity. In such cases, he only checked the if-then relationship between the antecedent and the consequent, which are the criteria to determine a material conditional. I believe that the nature of the question, the content, and how a conditional statement is stated can affect how people understand it. The interviewee's knowledge about mathematics misled him to recognize the mathematical conditionals with unrelated parts (Table 10.1). This suggests that the interviewee has a problem to recognize conditionals that is related to the context of the statement.

Also, the results from Study 2 suggest that the participants' knowledge about the divisibility rules may have influenced the way they recognize a conditional statement. The bar graph in Figure 9.1 shows that 25 participants (out of 33) stated that the statement "9 divides *n* implies that 45 divides *n*" is not a conditional statement, while all except 2 students could recognize the following statement as a conditional:

"If a number is divisible by 45, then it is divisible by both 9 and 5".

The high rate of 76% failure to recognize the statement "9 divides *n* implies that 45 divides *n*" as conditional might be because of the participants' knowledge about the divisibility rules since the statement "9 divides *n* implies that 45 divides *n*" is not always true. In such cases, when they have some expertise about the subject matter of the content, they check for a causal relationship between the premise and the consequent.

10.3.3. Task 3: causal conditionals or material conditionals?

Chapter 2 reviewed different types of conditional statements applied in colloquial language, philosophy, and logic. In all types of conditionals except material ones, there is some relevance between the antecedent and the consequent. The relation between the two parts of a conditional is somehow presumed in all conditionals except material conditionals. Not many people are aware that the material conditional is the one that is used in mathematics, and it is the only conditional with truth values as semantics. In fact, material conditionals include all the other types of conditionals and are the most general definition of a conditional statement.

Research studies on conditional reasoning have focused predominantly on causal conditionals. In such statements, the premise of a conditional provides a cause of the effect in the consequent. People mostly think about conditionals as expressing causal dependencies, which allows them to consider possible alternatives for the cause or possible hinges for the conclusion, known as disablers. For example, Markovits (2000) suggests that both adults and children are more inclined to accept conditionals in the form of causals, specifically causals with a few disablers. The presence of disablers can increase doubt on the sufficiency of the premise to result in the conclusion (Oaksford & Chater, 2010). Disablers and the presence of alternative reasons for the conclusion may increase doubt to accept the conclusion as an acceptable result for the given assumption.

Both studies (discussed in Chapters 8 and 9) show a high tendency of participants to take conditional as implication with related antecedent and consequet. In the second task of Study 1, all fourteen statements were conditional, and the interviewee determined normatively only nine of them. After decoding the statements, the results demonstrated that all five non-normative responses pertained to the second row of Table 1 in Chapter 9, where there were no evident relationships between the premise and the conclusion.

The interviewee evaluated three of these five statements in the realm of mathematics as non-conditionals, and for the two other statements (both in colloquial language), he preferred not to give any answer.

Also, among the fourteen statements in task 2 in Study 1, there was only one with the impossibility for its premise or conclusion, that is, “If elephants could fly, you win the lottery”. The interviewee reacted differently to this, not just in his verbal response, but he also drew the contradiction symbol ※ and stated that “this is terribly unrelated”.

The results from Study 2 show that most of the participants were following a model for a conditional indicated by some certain words, such as “if-then” and “when”, and some sort of relevance or dependence between the premise and the conclusion. The model they used to determine a conditional statement, even in the context of mathematics, is the one to recognize conditionals in everyday language.

A conditional in mathematics does not completely fit with conditionals in everyday language because, for a conditional in everyday language, we expect some sort of relevance between the antecedent and the conclusion, but in mathematics a conditional statement is a material conditional that is a specific relationship between two propositions. However, we use the same term “conditional statement” in both, and many people may not be aware of material conditionals. When we have a statement in mathematics, we may not automatically switch to material conditionals. For example, evaluating the statement “If $X = 2$, then I am a white rose”, one may think of material conditional if this conditional has been introduced to them.

For many of the participants in Study 2, the word “implies” was not an active indicator of a conditional, perhaps because they had not seen many conditionals worded with “imply”. It may be worth mentioning that most of the students I taught had not been aware of the meaning of the symbol “$\Longrightarrow$”, which can also be read as “implies” or “concludes”.

As the mental model theory suggests, people prefer a single model, and this may not be people’s error to evaluate all the conditionals with one model if they are not aware of the definition of the material conditional; that is the only type used in mathematics.

Mathematics problems can also be stated in words, so there should be a similar form for conditionals in everyday language and mathematics. Which type of conditional should we consider? If it is material conditional, then it is not applied to everyday language, and the intuitive conditionals for everyday language do not meet the criteria for a mathematical conditional. So normative mathematical evaluation of some conditionals makes no sense in everyday language, and what appears as reasonable in everyday language is non-normative in mathematics.

10.4. Does classical logic adequately explain mathematics?

Among all logical constants, conditional statements are the most researched topic. Chapter 5 represents only a very small part of research studies addressing serious problems students experience dealing with conditionals, like the issues with irrelevant antecedents and consequents, and true conditionals with false antecedents. The results from the two studies of this thesis show that recognizing conditional statements with unrelated clauses or stated with alternative keywords other than "if–then" were challenging practices for the participants.

In the introductory part of §3.3 about non-classical logics, I mentioned that the logic that can describe the events in quantum mechanics is termed "quantum logic", though the Boolean structure of classical logic seemed the appropriate one in that case. However, some events in atomic scales violate some principal rules of classical logic. Even though classical logic adequately describes classical physics' events, it cannot properly explain events in atomic and sub-atomic scales, as some events violate the two main rules of classical logic, namely, the law of the excluded middle and the distributive law.

In the next two subsections I discuss my comments on possible non-classical logics to explain the relations in mathematics, and explain unsoundness is an inseparable part of mathematics.

10.4.1. Other non-classical logics

Even though an argument in informal mathematics can be considered in both contexts of colloquial language and mathematics, it also frequently happens to make a conditional with a true premise and a false conclusion in a mathematical argument and it must be evaluated as a false statement. Material conditionals can explain the conditional statements in formal mathematics and in the arguments about mathematics. But there is still the question: is material conditional the most adequate type of conditionals to describe conditional statements in mathematics?

Quantum logic (§3.3) is an interesting instance of a non-classical logic that could successfully explain the events in atomic scales. Its rules are defined to describe events in quantum mechanics. Also, even at an advanced level, mathematics is used to describe events in this field. The influence of quantum physics on technology like nanotechnology and quantum computers is undeniable, and more will be expected in the near future.

Like quantum logic that can explain quantum mechanics events, some non-classical logics and some extensions of classical logic have been developed to explain arguments in everyday language, for example, modal logic and relevant logic (see §§3.3.1 and §§3.3.2).

Classical logic is adequate for mathematics needs, but it does not work properly for the "conditionals" of the everyday, non-mathematical, English language. It disregards causality, relevance, and other ingredients of everyday reasoning. In classical logic, we have some paradoxes like those of material conditionals (§§2.5.1). If a non-classical logic like relevant logic can fix those problems, why do we continue to use classical logic? Quantum mechanics uses quantum logic to explain events, and it nicely fits and describes the relations on an atomic scale.

In classical logic, $p \rightarrow q$ is true whenever at least one of q or $\sim p$ (the negation of p) is true, regardless of how p and q are related. So, the statement "if the earth is square, then today is Tuesday" is true to a classical logician (since the earth is not square); but it is nonsense to anyone else. Relevant logic is designed to avoid implications with superfluous hypotheses or other irrelevant clauses, e.g. as in the formula $p \rightarrow (q \rightarrow p)$ or the statement "If today is Tuesday, then $a^2 + b^2 = c^2$ is satisfied by right triangles". Such clauses are permitted by classical logic, but most people in everyday conversation avoid them. They are also avoided by most mathematicians, who believe themselves to be practicing classical logic plus "good taste," and relevant logic codifies some of that good taste. Here is an example:

A new Pythagorean theorem: If $\lim_{\theta \to 0} \theta \csc \theta = 1$, then the sides of a right triangle with hypotenuse b satisfy $a^2 - b^2 = c^2$.

A theorem of this type would not be accepted for publication in any research journal. The editor or referee might respond that, "one of the hypotheses of the theorem has not been used" or may say the premise and the conclusion are irrelevant.

Some non-classical logics enables us to avoid paradoxes and fallacies with classical logics like relevance logic, but it might be at a high price! To be more specific, in typical systems of relevance logic, we lose disjunctive syllogism, i.e. the rule from A∨B and ~A, you can infer B; intuitively, that is a valid fundamental rule. So, it seems that still, classical logic offers great proof-systems, semantics, etc. This may not suggest that classical logic is the best, as we still have some issues with classical logic, and this is still an open problem (e.g. Burgess,1981, 1983).

10.4.2. Unsoundness could be possible in mathematics

Based on mental model theory, when dealing with conditionals, people construct mental models based upon the premises, search for similar situations they may have encountered, and check for the validity of the consequents based on the given premises. If they find situations with similar premises, but refute the consequent (a counterexample), they do not accept it as a conditional. Within the lens of mental model theory, people construct mental models based on true possibilities; this may be the reason that the following conditional seems unsound:

If it is a compactor, then its range is the subset of the interval [0, 1].

This statement may seem odd in everyday language, because we do not expect a compactor to have a range, especially when stated as a mathematical object. However, it is still a conditional statement, not just in classical logic but also in mathematics. In classical logic, we accept any relation with the syntax $p \to q$ as a conditional, no matter whether it sounds reasonable or not. In mathematics, we define conditionals the same way as in logic, but mathematics is where we can define new objects. For example, I can define a "compactor" as a function with a domain of counting numbers, $\{1, 2, 3, \dots\}$ whose range is a subset of the interval [0,1]. There are many functions with that property, such as $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{2x}$, etc. So, there are many compactors whose range is a subset of [0,1]. The above statement is a true conditional statement with the provided definition, and there is nothing strange about it.

By this example, I am conveying that a conditional statement that seems unsound and may make no sense based on our experience in the real world could nevertheless be possible in mathematics. So, in mathematics, we have different expectations from a conditional statement than conditionals in everyday language. Some statements that seem odd in everyday language may be valid and sound (§§4.2.1) in mathematics.

It is also important to note that, if in a mathematics course, students encounter the statement "If it is a compactor, then its range is the subset of the interval [0, 1]", they know what it is, and it would not be an unsound statement. To clarify, consider the following two statements:

If you have a ring, then you are engaged.

If you have a ring, then associativity holds.

For a student in an algebra lecture on ring theory, the second statement seems reasonable, and the first one seems odd. However, for people not aware of ring theory, the second statement looks unsound. Therefore, what is 'reasonable' depends on knowledge and a particular interpretation of words.

Let me give another example. The following statement may not seem sound for many people.

If $X = 2$, then I am a white rose.

However, we can make many examples in which this statement is sound and valid. For instance, suppose there are six players in a game, each labeled with a number 1, 2, 3, 4, 5 and 6. My number is 2, and the name "white rose" is given to the person whose number is rolled on the die. Rolling a die is a random variable denoted by X, and in case $X = 2$, I will get the label and play the "white rose" role in the game.

I can give many such examples of conditional statements with seemingly unrelated antecedents and consequents, but which have meanings in mathematics or in a particular context. Even when we say $2 = 15$, one may ask, "Which group are you talking about, maybe Z_{13}? Because in Z_{13} we have $2 = 15$". Using such mathematical objects, for any natural numbers n and m, the equation $n = m$ would be valid in some mathematical groups. Even though those mathematical objects seem abstract, they have applications in business, computer science, chemistry, physics, etc.

10.4.3. Which type of conditional is used in word problems?

Some logical connectives are well-fitted with everyday language, like quantifiers and conjunctions, but some connectives are not, like disjunctions and conditionals. Differences in everyday language and mathematics may make understanding conditionals and disjunctions more difficult than quantifiers.

Another situation where everyday language and mathematics meet is in word problems. Any word problem is either a conditional statement or its solution includes conditionals. Most of the word problems ask to find or show something based on some given assumptions. For instance, consider the word problem "A bacterial colony grows by 6.5% every hour. Show that after about 11 hours the colony is doubled in size". This problem can be restated as follows:

IF growth rate is 10%, **THEN** the colony is doubled in size after 11 hours

However, not all word problems are conditional statements per se. For example, the problem "Anna is 17 and she is 10 years younger than three times her sister's age. How old is Anna's sister?" is not a conditional statement.

This problem asks to find n that makes the following (predicate) conditional true:

If Anna is 17 & Anna is 10 years younger than three times her sister's age **THEN** Anna's sister is n years old.

One possible solution to this question is:

IF n is Anna's sister age, **THEN** 3n-10=17, **THEN** n=9, **THEN** Anna's sister is 9.

The solution has some interconnected conditional statements. I also draw your attention to the last conditional statement "If n=9, then Anna's sister is 9". It may look unsound in everyday language but is sound and valid in mathematics.

The point is that word problems are a part of mathematical problems which are worded in everyday language. In such cases, we need to translate the problem into the language of mathematics. At the end of the solution, when we get the answer, we need to translate the final answer back to the language of the given word problem, as the final answer is likely just a number or a mathematical symbol, but the answer of the word problem is the interpretation of that number or formula.

While translating a word problem into mathematics, and then translating the final answer back to the language of the word problem, we need to use the same type of conditional, and we cannot switch from mathematical conditionals to other types, because the solution is situated in mathematics.

Example: Suppose we have 93 marbles shared consistently with people in a room, with the rule that if we order people in a sequence, then each person has twice as many as the previous one, and the first person has three marbles. How many people are in the room?

Assume that: x is the number of people in the room and n_i is the number of marbles for the i^{th} person.

We know: $n_1 + n_2 + \cdots + n_k = 93$, and we know $n_1 = 3$.

We want: $k = ?$

There are different methods to find the following answer:

$$3 + 6 + 12 + 24 + 48 = 93$$

Then $k = 5$.

So, from $n = 93$ and $n_1 = 3$, we got $k = 5$.

Therefore, there are five people in the room.

Let me restate the above statements:

If $n = 93$ and $n_1 = 3$, then $k = 5$.

If $n = 93$, then five people are in the room.

If $n_1 = 3$, then five people are in the room.

If $n = 93$ and $n_1 = 3$, then five people are in the room.

If we just look at these statements without knowing the situation described, then they may look unsound with unrelated parts, but when we look at them as a part of the solution of the given word problem, then we see the relations between the antecedents and consequents. For example, the statement “If $n = 93$, then five people are in the room” is a conditional and is regarded as a connector from mathematics to the language of the word problem. It now makes sense, but when the statement is considered without its context, it seems unsound.

Conditional statements are used in word problems or in their solutions, this raises the question which type of conditional is assumed for mathematical word problems. The logic applied in mathematics is different from the logic in everyday language; which logic wins the battle? In fact, since a mathematical word problem is basically a mathematics problem, we may expect a material conditional for such statements. However, the results from the current work and the others in the literature suggest that what people mostly choose is the type of conditionals in everyday language.

10.5. Contributions

This section outlines the contributions of the current work to theory, methods, and mathematics education.

10.5.1. Theoretical contributions

In sub-section §§2.4.1, I explained the reasons mathematics needs material conditional among different conditionals. Then, in sub-section §§2.4.3, I investigated the relationships between “predicates” and “examples, non-examples, and counterexamples” in mathematics, and finally in sub-section §§2.4.4 I integrated them all in the truth values of a predicate.

I opened the discussion in sub-section §§2.4.3 with this fact the “a statement in mathematics including variables may not have a fixed truth value”. For example, the equation $x^2 = 9$ is not a statement and is called a predicate. Similarly, the conditional “If n is a natural number divisible by 3, then n is odd” is a conditional predicate, because it does not have any truth values *per se,* and the truth values depend on the value that

n takes. Using the universal quantifier (or existential quantifier), this sentence can be converted to a false (or true) conditional statement.

I also discussed cases when the proof is needed (sub-section §§2.4.3). Depending on the values in the truth set of a conditional predicate, it can be either always true, never true, or sometimes true. The first two cases, namely "always true" and "never true," can be supported by proof. Depending on the number of counterexamples (or examples), by narrowing the domain for the variable, the third case, "sometimes true", may be converted to "always true" (or "never true"), which then could be proved. The following are examples for three possibilities for truth values of a predicate:

- The sentence is "always true"; Ex: if n is an odd number, then $n + 1$ is an even number
- The sentence is "never true"; Ex: if n is an odd number, then $n + 1$ is an odd number
- The sentence is "sometimes true". Ex: if x is an integer, then $\frac{x}{2}$ is an integer.

The following table represents when the last sentence is true and when it is false:

	p: x is an integer	q: $\frac{x}{2}$ is an integer	$p \longrightarrow$ q
x is even	T	T	T
x is odd	T	F	F

In sub-section §§2.4.4, I discussed the number of possible values that made the statement true (truth set) and used this idea to distinguish between examples, non-examples, and counterexamples.

For a conditional predicate "if P(*x*), then Q(*x*)", I categorized examples, non-examples, and counterexamples as follows:

- a counterexample is an example that makes the premise true but the conclusion false;
- a non-example is irrelevant to the rule/condition, meaning that with it, the premise is false;

- an example satisfies the rule/condition and makes the conclusion true.

This assignment is summarized in Table 10.2 (repeated from sub-section §§2.4.4):

Table 10.2. Truth values for a conditional statement involving a variable

$p(x)$	$q(x)$	$p(x) \rightarrow q(x)$	
T	**T**	T	Example
T	**F**	F	Counterexample
F	**T**	T	Non-example
F	**F**	T	Non-example

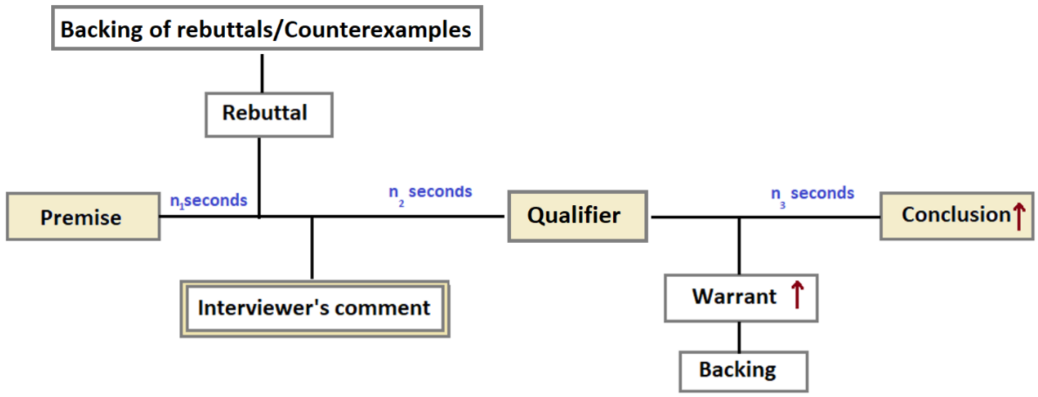
This work's contributions to the theory, is to discuss when a predicate is true and based upon that it classified predicates into three categories, namely "always true", "never true", and "sometimes true". It also integrated different types of examples with a truth table for a conditional involving a variable.

10.5.2. Methodological contributions

Toulmin's argumentation scheme is a visualization technique to depict important parts of an argument. In §§6.2.3, in order to capture more from an argument and add more details to its diagram, I extended Toulmin's scheme so that it is easily applicable in any research about arguments. The following adjustment adds more details from the argument to Toulmin's scheme:

- including some extra box in a refutation scheme, namely "backing of rebuttals", this box can include counterexamples;
- use of a doubled box to display a conversation, or to add some other person's comment;
- adding the duration of pauses and voice pitches to the diagram.

The adjusted Toulmin's model is shown in the following figure, Figure 6.4 "Adjusted Toulmin's model" from section 6.2,



10.5.3. To mathematics education research

Section §5.3 discussed that any language has two main components: semantics (or meanings) and syntax (or grammar). Regarding the language of logic, although syntax and semantics are important components of a conditional, they may not be enough to understand them and for even educated people to make inferences based on the context. This suggests adding the third component, namely context, to specify the subject area in the logical statement.

The examples explained in §§5.4.1 guided me to the idea that besides syntax and semantics of logic, the context is also an important factor in recognizing and evaluating a conditional statement. The results of this thesis summarized in section §10.3 confirmed this idea that the context could highly influence people's understanding of a conditional statement.

The current work is one among many works on conditional statements, but is unique and different in terms its focus, that is adding the third component to logic and study the way people recognize conditionals in different contexts. This thesis considered "syntax", "semantic," and "context" as three vertices of a triangle that are essential elements to understand a conditional statements. The results show the context plays a determinative role and for most cases the context came first.

10.6. Final note

In chapter 4, I discussed the role of logic in both formal and informal mathematics. Classical logic, or at least material conditionals, is at the core of mathematics and is the skeleton of all proofs, definitions, and other mathematical constructions. Also, rather than discussing different types of conditionals, we only have one type of conditionals in mathematics, namely material conditional.

Many logics can fix the fallacies, explosion principle, the law of the excluded middle, etc. In the current chapter, I explained how we need material conditionals in mathematics: more accurately, how all other sciences need mathematics as a very creative and flexible subject to explain different events in the real world and relate them to formulas.

I am not claiming that classical logic is the adequate logic to describe different mathematical constructions; neither am I saying that we already have a non-classical logic satisfying all we need in mathematics. The relevant logic I exemplified in this chapter is only one among many, and my knowledge about logic is ever growing. In the future, I may learn about some non-classical logics satisfying certain properties like those that fix the explosion principle and do more research on people's understanding of conditionals, like true conditionals with false antecedents.

Word problems have been the most difficult type of question in any mathematics course I taught. A word problem is worded in everyday language; the formula, equations, and assumptions are hidden in the problem, and students must recognize them by translating the word problem from everyday language to mathematics. The first attempt is to recognize assumptions and conclusions, namely "what we know" and "what we want". This important part needs to determine the logical form of the statements, what is/are the antecedent/s and what is/are the conclusion/s? If students do not know the assumption and conclusion of a problem, they cannot solve it.

As a teacher, I learned from this study to focus on the logical form of questions and help students learn to determine the premise and conclusion, and explicitly recognize "what is known in a problem?" and" what is asked?". This might be half the battle!

As a researcher, I still suspect that material conditional is the most reasonable interpretation of a conditional in mathematics. However, I could come up with some reasons to support some features of material conditionals as a necessary feature for a conditional in mathematics.

Study 2 specifies that applying alternative cue words rather than “if-then” makes it difficult to recognize the logical form of the sentences. It is possible to state conditionals in mathematics in “if-then” form, but there is no limitation for word problems, and conditionals could be stated by any alternative cue words rather than “if-then”. The following are the next potential research questions:

- How students recognize the assumption/s and conclusion/s in a word problem?
- How do students do a word problem differently when they are provided with a problem with hints to translate it to mathematics?

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Appendix: Hugo's transcripts

This appendix includes the transcription of the entire interview with Hugo for Study 1.

Legend (written devices)

[ ] Indicates Transcriber's description

( ) A number inside brackets denotes a timed pause

↓ When a downward arrow appears, it means there is a drop in intonation

Underline When a word or part of a word is underlined, it denotes a raise in volume or emphasis

Part 1

Interviewer: Please write an implication in the content of mathematics, a conditional statement.

HUGO: If $a^2 + b^2 = c^2$ then the triangle has a right angle.

Interviewer: Okay, thank you. Now please give a conditional statement in everyday language, not necessarily in mathematics.

HUGO: [12 secs] If I can finish chapter 4 of the textbook by tonight, then as a prize I will have a lot of food.

Interviewer: [h]

Part 2

Interviewer: I am giving you some cards with one statement on each of them; the statement is in either mathematics or in colloquial language, and ask you to determine if it is a conditional statement or not.

HUGO: okay,

Interviewer: and that if they are conditional, explain why, and how would you get that?

HUGO: "If you will be late for the meeting, then you will be fired", well this is conditional because it has "if… then", it is very clear that because of "if" and "then", it is a conditional.

HUGO: "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*" [6 sec], Yes [3 secs], it doesn't make sense, but it is a conditional ↓.

Interviewer: Can you say why?

HUGO: if-then. If you buy something, then [stressed], something happens, though they are not very related, errr, it is a conditional, though meaningless.

Interviewer: so, you say these are unrelated?

HUGO: of course, they are unrelated,

Interviewer: so, what is this? This is a sentence in everyday language.

HUGO: yes it is, but it doesn't make sense, only an idiot would say so [laugh].

Interviewer: so, is it conditional?

HUGO: [6sec] In fact, mathematically, it's not a conditional, but in daily language we take it as a conditional; yeah, it is a conditional ↓.

Interviewer: you look unsure. Is it a conditional?

HUGO: with regards to math or daily language?

Interviewer: as a statement,

HUGO: with regards to math it's not, but as a sentence, yes it is, it is a conditional,

Interviewer: so, you sure?

HUGO: Yeah, almost↓.

Interviewer: When you say almost, then there is a level of uncertainty; if you want to be more precise what is its likelihood in percentage?

HUGO: say, 60% or 70%.

Interviewer: Was the likelihood 100% in the last statement?

HUGO: Yes, it was, because they were related, but these are unrelated, this is approximately conditional,

Interviewer: But why approximately?

HUGO: Because it has "if … then", a conditional doesn't have to be true or false,

Interviewer: [hand in the third card]

HUGO: " If $4X^2 - 5X - 6 = 0$, then $\sin \theta = 0.546$ ", [15 secs], mathematically [4 secs] it's wrong,

Interviewer: So, is it false?

HUGO: yes, but since I know mathematics, I recognized this statement is wrong,

Interviewer: but, my question is not to determine if it is wrong or correct,

HUGO: no, this sentence [7 secs], is it a conditional statement? [5 secs] no, it's not, no, it's not a conditional statement, not at all,

Interviewer: Okay, [hand in the fourth card]

HUGO: "If $\cos \theta = 0.81$, then $\theta = 2\pi \pm \frac{13\pi}{173}$ ", I don't know if it is correct or not, but for sure, this is a conditional,

Interviewer: [hand in the fifth card]

HUGO: "If A is a Banach algebra, then for every Banach A-bimodule X, H (A, X') = 0", I don't but it is, since both have "a", they are related, so it is a conditional,

Interviewer: what do you mean by having "a"?

HUGO: if **A** is a Banach algebra then for every Banach **A**-bimodule, so these are related, so this is a conditional,

Interviewer: so, you checked that these are related?

HUGO: yes, the second part is related to the first part; both have "A", both have "Banach",

Interviewer: is this the only reason to be a conditional for this statement, or there is also another side reason,

HUGO: if … then

Interviewer: So, which one do you check first, relatedness or if-then?

HUGO: of course, first if-then.

Interviewer: [hand in the sixth card]

HUGO: "If you pay 10$, you will have my pen", [4 secs], yeah it is conditional, it has "if", and this is related to this, so, yes it is conditional, first "if", and then relatedness, these are definitely related,

Interviewer: [hand in the seventh card]

HUGO: "In triangle ABC we have A + B +C = 180 °, then the area of a *circle* is pi times the radius squared (πr^2)", [19 secs], no, this is not a conditional, because these are very unrelated, absolutely unrelated, this is neither sufficient nor necessary condition for this,

Interviewer: So, your reason is because these are unrelated,

HUGO: Yes, relatedness is more important than "if … then"

Interviewer: you sure that this is not a conditional

HUGO: Yes, I'm sure, it's not a conditional.

Interviewer: [hand in the eighth card]

HUGO: "If ABC is a triangle, then A+B+C=180 °", it is very obvious that this is a conditional,

Interviewer: Why do you think so?

HUGO: It has "if … then", this is related to this, it satisfies each condition, so it is a conditional,

Interviewer: [hand in the ninth card]

HUGO: "if elephants could fly then you win the lottery", [4 secs], this means that the chance to win the lottery are very narrow, you cannot win, [11 secs] yeah this is a conditional since <u>this</u> results <u>this</u> [pointing to the premise and conclusion], [4sec] yes, [12sec] no it's not a conditional, [2sec] I don't know, [3sec] I'm not sure …

HUGO: To win a lottery is only by the chance, they are unrelated because wining a lottery is just luck [16sec] errr [8sec] no it's not a conditional [he drew ※ in Figure 1].

Interviewer: [hand in the tenth card]

HUGO: "If (x-2)(x+1)=0, then x = 2 or x= -1", of course, this is very simple math statement, obviously a conditional, they are related and it in "if … then" form.

Interviewer: What if it was "If (x-2)(x+1)=0, then x = 2 or x= 3"?

HUGO: That's okay, it is still a conditional,

Interviewer: Why?

HUGO: It is only solved incorrectly, but still a conditional,

Interviewer: So, being unrelated is different from being false?

HUGO: Yes, of course,

Interviewer: [hand in the eleventh card]

HUGO: "If sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the Earth", [36 secs], since I guess these two parts are related, though I'm not sure, but I would say this is a conditional, if I read this in a newspaper, I am sure that it is a conditional, but if these are unrelated, then it is not, but this is a conditional because sun and earth are two related things, and sun can influence the earth ,

Interviewer: So, is this a conditional?

HUGO: Yes, it is.

Interviewer: [hand in the twelfth card]

HUGO: "If a population consists of 40% men, then 60% of the population must be women", yeah this is a conditional, if 40% men, then 60% women,

Interviewer: [hand in the thirteenth card]

HUGO: "If a population consists of 40% men, then 60% of the population must be women", yeah this is a conditional, if 40% men, then 60% women,

Part 3

Interviewer: in the previous part you were not sure that "if elephants could fly then you win the lottery "is a conditional or not? Why you are not sure?

HUGO: To win a lottery is only by the chance, they are unrelated because wining a lottery is just luck [16sec] errr [8sec] no it's not a conditional [he drew ※ in Figure 1].

Interviewer: Does this symbol [pointing to ※] mean that you are sure that this is not a conditional?

HUGO: Yes, almost ↓.

Interviewer: Do you have a reason why it is not a conditional?

HUGO: first, these two [pointing to premise and conclusion] are very unrelated, then winning lottery is all by chance and cannot be conditioned.

Interviewer: so, they are unrelated?

HUGO: yeah↓, in general they are terribly unrelated. One other thing is that I don't know what we call them in grammar, do we take each if-then a conditional or not? I don't know (3sec), maybe it is, but in my view it is not.

No because in daily conversation, mostly we talk about very simple and clear things,

Interviewer: Okay, does this make it difficult to recognize conditionals,

HUGO: No, but the problem is we don't really pay that much attention to what we say, we only have some rough sentences. We don't use serious things in daily language.

Interviewer: Let me go back to this question [If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*], you said before, mathematically it's not conditional, but as sentence it is, but why did you say this statement is a conditional?

HUGO: I again say that mathematically neither is conditional, but both are ambiguous [pointing to statements: "If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan" and "*if elephants could fly then you win the lottery*"*],

Interviewer: what about non-mathematically?

HUGO: They're very unclear, I don't know, I cannot tell anything, but there is no reason to be a conditional,

Interviewer: You before said that as a sentence it is a conditional,

HUGO: I changed my mind, this is not a conditional, doesn't make sense as a conditional, not at all, this is unclear, I cannot give you any answer.