Mathematics as the Science of Material Assemblage: Enactivist, Quantum Theoretical, and Educational Perspectives

by

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Abstract

This dissertation explores implications of reconsidering the nature of mathematics from a conjoined enactivist/quantum theoretical perspective. The research is motivated by the view that reinterpreting the nature of mathematics through more inclusive materialisms has the potential to reinforce the fundamental relationship between mathematics and the material world, and to deepen our collective understanding of the ways in which our models of mathematics ultimately take on the meanings that they do. Four core themes underpin the overall research trajectory (i.e., new materialisms and issues of dualism, epistemological uncertainty, matters of agency, and complexity associated with emergent systems); however, only the initial two will be directly addressed within this particular document. Both yield insights into how specific facets of quantum theory and enactivism might supplement the more traditional discourse surrounding the nature of mathematics, and, in so doing, set conceptual groundwork for a broader mathematical (or rather *material–mathematical*) worldview.

In light of these diverse themes, the program of research is necessarily interdisciplinary in scope, synthesizing literature from the interconnected domains of physics, mathematics, educational psychology, and philosophy more generally. Considering this literature alongside works from the established discourse of mathematics education, and reading it through a conjoined enactivist/quantum theoretical perspective, the dissertation elaborates points of disciplinary confluence, whilst expressing how such confluence might inform or reshape the sense of what mathematics is. By drawing upon the *assemblage theory* of Gilles Deleuze and Félix Guattari, Elizabeth de Freitas and Nathalie Sinclair's characterization of *the body in/of mathematics*, and the concept of *quantum entanglement*, the research also articulates a perspective regarding the mathematical structure of reality, and levies a view in which mathematics itself may be perceived as *the science of material assemblage*. A driving tenet of the research is the notion that a changed view of the material also changes one's view of the mathematical.

Keywords: Material Assemblage; Nature of Mathematics; New Materialism, Enactivism; Quantum Theory; Entanglement

Dedication

This dissertation is dedicated to my dear parents and sister, whose love, support, and understanding have sustained me throughout this endeavour.

Thank you for instilling in me a great love of learning and discovery, and for enabling this deeply theoretical exploration.

You have my eternal gratitude and utmost respect.

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I greatly appreciate the encouragement that has been offered throughout this exploration, the wealth of information that has been shared, and the permission/freedom that was granted to look outside of our field in an effort to enrich the views within it.

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Table of Contents

List of Figures

Preface

Despite what the reader might naturally presume (given the program to which this dissertation is submitted in partial fulfillment, and the likelihood of a certain kind of relationship between the author and the discipline of interest), I would not actually consider myself a *mathematician* in either a traditional or professional sense of the word. Nor would I count myself among the many educators and practitioners of mathematics who can profess a profound and genuine *love* of mathematics itself. That said, I qualify these remarks immediately. This is certainly not to suggest that I am ambivalent toward mathematics, that I am unable to derive pleasure from mathematical exploration and discovery, that I do not recognize the beauty of mathematics in its various forms, or that I have somehow distanced myself from the mathematical field at large. Quite the contrary; it is, instead, simply an open acknowledgment that my reasons for *engaging* with mathematical subject matter may not necessarily correspond to those *expected* by the reader. I grant that this is an unusual, perhaps even dispassionate and abstruse, qualification to make in the introductory passages of a research document pertinent to mathematics education; but, I believe it to be a necessary one, as it is intended to clarify that it is my *fascination* with mathematics (not a love of it) that has prompted and shaped the exploration manifested within this doctoral dissertation.

Though my experiences as a mathematics teacher prior to embarking on graduate studies in mathematics education were relatively limited, my postsecondary educational pursuits have been closely intertwined with mathematics for a significantly longer period extending back to an undergraduate degree in physics. Some years ago, I might have been inclined to say that it was my interest in physics that subsequently motivated my interest in mathematics; however, that would not properly capture my *present* frame of mind, from which I tend to think of mathematics as the more intriguing (and more foundational) of the two disciplines. If anything, I would now be more disposed to foreground the *ontological* and *epistemological* motivations behind my pursuit of both physics and mathematics, and it is within the current document that I formalize at least some of these.

Perhaps more than anything else, it is the reciprocal interplay between mathematics/physics and the material world that strikes me as simultaneously compelling and curious, and I very often find myself wondering why it is that mathematics and the material world should share any connection at all, let alone one that proves to be so revealing, generative, and fruitful for us (human beings). Moreover, why human beings should have any sort of capacity for *thinking* mathematically remains something of an ongoing conundrum. In an age when particle physics has probed further into the structure of matter than ever before, and mathematics itself has, in many regards, become more and more deeply embedded in the metaphorical DNA of our digital technology and tools, the question of how the mathematical and the material are intertwined seems especially salient, as does the even more fundamental question of exactly what mathematics *is*.

Largely because such questions are often considered to fall within the purview of philosophy and, occasionally, pure mathematics, and tend to go unasked within the space of mathematics teaching, I believe there is particular value in emphasizing their importance here, within the discourse of research in mathematics education. In fact, I suggest that it is equally, if not more, important for mathematics educators/researchers to contemplate the nature of mathematics, if only because doing so can reveal and clarify foundational beliefs that inevitably manifest within teaching practice and other mathematical endeavours. Furthermore, since the *teaching of mathematics* is influenced by underlying philosophical commitments regarding the *nature of mathematics* (i.e., what mathematics is, how it is instantiated in the world, what it means to engage with mathematical principles, et cetera), it behooves all professionals within our field to give thoughtful consideration to the mathematical worldviews that inform, support/constrain, and otherwise shape the mathematical practices in which we engage, including the endeavour of mathematics teaching itself.

Whether voiced or unvoiced, conscious or subconscious, implicit or explicit, commitments regarding the nature of mathematics pervade all mathematical activity and are likely to play a significant role in shaping the mathematical discourses that evolve within the classroom setting and elsewhere. These discourses have subsequent implications at not only the level of the personal, but the institutional and societal as well.[1] Characterizing what it means to think mathematically, evaluating problem-solving strategies, establishing the conditions that constitute valid mathematical proofs, determining curricular objectives and learning outcomes, deciding which mathematical skill sets should be emphasized or underplayed, attending to or neglecting the aesthetic, and even judging which mathematical processes are worth engaging with, all stem from more foundational perspectives concerning the nature of mathematics and mathematical activity. As a result, I stress the need to engage more deeply with these foundational perspectives, and to examine the implications that arise from them.

With the ultimate aim of elaborating a fresh perspective on the nature of mathematics and its relationship with matter, I first contextualize the major themes underlying this program of research, by presenting a brief historical touchstone intended both to communicate the impetus for my discussion, and to act as a preface to the early chapters of this document. For the reader who has anticipated the flow of my reasoning, it may come as no surprise that the notion of *the unreasonable effectiveness of mathematics* will be utilized as a point of entry.

A contemporary and colleague of physicists Albert Einstein and Werner Heisenberg, applied mathematician John von Neumann, and various other renowned science and mathematics researchers of the early to mid-20th century, Hungarian-born mathematician and theoretical physicist Eugene Wigner is widely known for his contributions to atomic theory, solid-state physics, and mathematical formulations of the wave theory that would become integral to subsequent developments in quantum mechanics (Seitz, 1995). It was on May 11th, 1959 that Wigner delivered the inaugural Courant Lecture at New York University, the lecture series itself having been established in honour of German-American mathematician Richard Courant. During his talk, Wigner would not only address the strong interplay between mathematics and the physical sciences, and discuss the notion of mathematical structures underlying physical theories, but also highlight the incredible efficacy of mathematics in capturing and characterizing various facets of material reality.

[1] Similar sentiments are expressed byErnest (1991).

Those familiar with the inaugural Courant Lecture are sure to recognize the title of Wigner's talk; however, it is likely Wigner's identically titled paper published the *following* year that is better known for elaborating *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*.

Considered by many as a seminal piece of literature in both the philosophy of physics and the philosophy of mathematics, Wigner's paper has also become more widely influential in the discourse of mathematics education. The sometimes-earnest and sometimes-playful manner in which Wigner presents his insights into the mathematical underpinnings of physical theories communicates a less conventional view of what mathematics is, of what physics is, and of the mutually informative roles they play in shaping one another. Much as the title of the paper connotes, Wigner (1960) speaks to the seemingly "uncanny usefulness of mathematical concepts" in the physical sciences (p. 2), and speculates about how it is that those very same concepts can be so powerful and consistently efficacious in modeling the diverse and multifaceted aspects of our collective worldly experience. In so doing, he articulates the need for a form of reflection in which even his mathematical contemporaries were perceived as somewhat lacking. How is it indeed, Wigner ponders, that mathematics is so unreasonably effective in the realm of scientific investigation? What epistemological significance might be surmised/gleaned from such consideration, and what ontological implications might accompany it in turn? The current dissertation proceeds with an eye toward precisely these queries, and the lens of mathematics education providing a focus.

As noted by Wigner (1960):

> it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language. (p. 8)

Not only do the above assertions embed the language of mathematics squarely within an ontological and epistemological discourse, they also more pointedly draw out the profound connections that exist between mathematical and scientific activities as they are realized or enacted by human beings. A core implication put forth by Wigner would seem to be that the robustness of mathematics as a means of modeling physical phenomena and the applicability of mathematical language to descriptions of those phenomena are not simply *coincidental*, but possibly *fundamental*. Reading more deeply into Wigner's remarks, I push this perspective further by claiming that this is the case entirely because mathematics *itself* is an ontological fundament, and not merely a tool for mediating the human experience of reality (although it undoubtedly serves that purpose as well).

There are, of course, some like Richard Feynman who are known for having expressed largely *pragmatic* sensibilities about mathematics, according to which the analogy of the useful mathematical toolkit might be entirely appropriate.[2] Nevertheless, viewing mathematics simply as a convenient toolkit that facilitates other scientific activity and is useful for solving certain kinds of problems does not strike me as an interpretation that adequately accounts for the unreasonable effectiveness characterized by Wigner, for it essentially relegates or demotes mathematics to a status from which it is, at best, a means to an end. As an historical anecdote that may help to contextualize the perceived inadequacy of this interpretation, the following humorous exchange between Feynman and mathematician Mark Kac comes to mind (see Nahin, 2011):

> The great probabilist Mark Kac (1914–1984) once gave a lecture at Caltech, with Feynman in the audience. When Kac finished, Feynman stood up and loudly proclaimed, "If all mathematics disappeared, it would set physics back precisely one week". To that outrageous comment, Kac shot back with that yes, he knew of that week; it was "Precisely the week in which God created the world". (p. xxiv)

Feynman's prioritization of the practical yields *some* insight into his subordination of mathematics to physics, and the accompanying suggestion that the latter is not as reliant on the former as might typically be thought; yet, it also gives an indication of his broader mathematical worldview (i.e., presumably, one in which mathematics exists primarily as a tool whose application *enables* scientific development but is not *essential* for that development). Even though I do not intend to invoke any deist/theological overtones

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[2] See the edited volume *"Surely You're Joking, Mr. Feynman!": Adventures of a Curious Character*, from W. W. Norton & Company (1985/1997).

myself, I do appreciate the manner in which Kac's rejoinder swings Feynman's pragmatic view toward a deeper and more expansive sensibility about the nature of mathematics. Kac seems to hit directly at the kind of sentiment implicit within Wigner's remarks (quoted earlier), and the sense of mathematics having much greater universal import. While Feynman's "outrageous comment" suggests that mathematical understanding does little to inform physics, Kac, as in my reading of Wigner, would appear to advocate a very different view (i.e., one in which the mathematical is not only essential for understanding the physical, but integral to its very make-up as well). This is an idea that I shall revisit throughout the course of this dissertation.[3]

As I have already noted, the reciprocal interplay between mathematics/physics and the material world is of particular interest within the context of this dissertation, and while I share Wigner's sensibilities regarding the mathematical underpinnings of physical theories, I also believe that the relationship between these disciplines is such that developments within the realm of the physical sciences can do much to retroactively inform mathematical theory (and even the philosophy of mathematics) as well. Certainly, explorations in both areas can be *mutually* informative, and the directionality of their relationship need not be construed as one-way. That is to say, just as the mathematical informs our physical worldviews, so too may the physical inform our mathematical worldviews.

At the time of writing this introductory frontmatter, news of the first ever black hole image, captured by the Event Horizon Telescope array (EHT) on April 10th, 2019, is still relatively fresh. Only one month shy of the 60th anniversary of Wigner's inaugural Courant lecture, this significant event comes as a timely and fortuitous example of the deep interconnection between mathematics/physics and the material world. It wonderfully illustrates a case in which a physicist's *theory* concerning the nature of gravity (i.e., Einstein's General Relativity) would lead to the development of *mathematical models* that would predict the effects of extreme gravitational forces on the geometry of spacetime (i.e., as with the phenomenon of gravitational collapse, for which John Archibald Wheeler would coin the

[3] Cf. Meillassoux (2008), which upholds the thesis that "all those aspects of the object that can be formulated in mathematical terms can be meaningfully conceived as properties of the object in itself", and that "the mathematizable properties of the object are [...] effectively in the object in the way in which [we] conceive them, whether [we are] in relation with this object or not" (p. 3).

term 'black hole'), and in which the accuracy of said models would, in turn, be largely verified over a century later by ground-breaking *physical observations* that had previously been unattainable (see Ghosh, 2019, Landau, 2019, and Mortillaro, 2019).

In the months following the acquisition of the radio telescope images, the collective scholarly community continued to revise its models of black hole phenomena in light of the newly acquired observational evidence. Thus, in addition to confirming that black holes are not simply theoretical abstractions, and that they are very *real* phenomena that exist in our universe, this revolutionary radio telescope data has also exemplified an instance in which mathematical theory has informed physical observation, and physical observation has subsequently fed *back* into mathematical theory. One can imagine how Wigner might delight at these developments were he still alive today.[4]

With the physical science that is so deeply intertwined with our mathematical endeavours proving to be such an informative and revealing pursuit, and with its continued evolution under various contexts, the insights arising from it can retroactively enlighten the mathematical thinking upon which that same science is founded and with which it also co-evolves. In particular, extending Wigner's discussions of *subatomic* physics allows for certain connections to be made to larger philosophical themes. Also speaking to Wigner's influential 1960 paper, and quantum theory specifically, Dutch mathematical physicist and director of Princeton's Institute for Advanced Study, Robbert Dijkgraaf (2017) presents the following perspective in a brief article written for *Quanta Magazine*:

> The mathematical physicist and Nobel laureate Eugene Wigner has written eloquently about the amazing ability of mathematics to describe reality, characterizing it as "the unreasonable effectiveness of mathematics in the natural sciences." The same mathematical concepts turn up in a wide range of contexts. But these days we seem to be witnessing the reverse: the unreasonable effectiveness of quantum theory in modern mathematics. Ideas that originate in particle physics have an uncanny tendency to appear in the most diverse mathematical fields. (para. 4)

[4] One might also take into account the importance of the complex mathematical algorithm devised/written by MIT doctoral candidate and programmer Katherine Bouman, which was integral to the process of *resolving* the EHT data into a single coherent image (see Bouman, 2017).

The increasing awareness of this sense of reversal alluded to by Dijkgraaf, this *unreasonable effectiveness of quantum theory in modern mathematics*, is particularly intriguing; for it is suggestive of an unprecedented kind of disciplinary convergence. It is true that the quantum phenomena common at the level of the subatomic do not *seem* to encroach upon our everyday experiences at the level of the macroscopic; yet, as Dijkgraaf (2017) notes, "The bizarre world of quantum theory […] not only represents a more fundamental description of nature than what preceded it, it also provides a rich context for modern mathematics" (para. 2). His reference to quantum theory as *a more fundamental description of nature* accords very nicely with Wigner's sense of mathematical language being *the correct language* with which to describe reality. In fact, Dijkgraaf even puts forward the idea that the logical structures of quantum theory might eventually inspire a hybrid disciplinary form, which he tentatively refers to as *quantum mathematics*.

Although my own explorations will not go to the point of attempting to unpack Dijkgraaf's quantum mathematics, I do suggest that there is sufficient reason to believe that adapting aspects of quantum theory to the aims of research in mathematics education might have the capacity to bring about (or at least foster) a singular set of insights into *the nature of mathematics* and possibly mathematical sense-making as well. This view is partially predicated upon the thought that *a changed view of the material also changes one's view of the mathematical*.

It could be said that ontological questions along the lines of "What is 'X'?" and epistemological questions regarding *what we know* about 'X' and *how we know* about 'X' had largely fallen out of fashion in the mid- to late-20th century, due to postmodernist trends favouring skepticism, a general movement away from absolutism toward relativism, and more contingent perspectives concerning the nature of knowledge. Nevertheless, such questions have seen something of a resurgence within the last decade or so, and, through the recent "ontological turn", the scholarly community is, in a sense, newly able (or newly allowing itself) to re-engage with them (N. Sinclair, personal communication, May 13th, 2019).

Granted, from a certain point of view, seeking to address what something *is* can be misconstrued as connoting, even presupposing, a rigid construct or a static object/phenomenon of interest, in much the same way that some *definitions* might connote a kind of rigidity or inflexibility. There may well be instances in which such connotations are appropriate, even desired; however, I do not wish to impose them upon my phenomenon of interest (i.e., the nature of mathematics) at the present time. Rather, I aim to put forth a perspective that is descriptive and thought-provoking, while also being suitably inclusive and responsive to the different contexts in which human beings engage with and think about mathematical subject matter.

As already noted in the preceding abstract, a core theme underpinning this program of research is that of epistemological uncertainty. Although the scope of this document and the prioritization of other themes precludes a focused treatment of it, uncertainty inevitably dovetails with other facets of my discussion. Most notable are those surrounding the notion of material indeterminacy, which arises from the quantum theoretical discourse and has subsequently entered into the new materialist discourse. Due partly to the emphasis placed on statistical interpretations of quantum theory, and the usefulness of characterizing observable system states in terms of probability distributions, indeterminacy is often represented as a purely *quantitative* consideration. It can also be discussed in more *qualitative* terms, though, which are rooted in wave–particle duality. While I will briefly make mention of certain statistical themes when engaging with quantum theory in **Chapter 4**, it is actually this latter qualitative view that is of greater interest within the space of this dissertation, and which I expect to foreground.

Infusing an interdisciplinary exploration with elements of the quantum theoretical discourse affords one the opportunity to address material indeterminacy and epistemological uncertainty in both literal and figurative terms. Markedly more literal than figurative is the view that uncertainty might already be written into the very structure of matter. Indeed, as Ryerson professor and science/literary scholar Jennifer Burwell points out in her 2018 publication *Quantum Language and the Cultural Migration of Scientific Concepts*:

> The indeterminism of subatomic phenomena […] may be explained as an existing uncertainty that is "encoded" within matter (ontic), or it may be explained as a limitation of human observation (epistemic). If some aspects of the quantum world can never be known, and this unknowability is not a function of any sort of limiting factor such as insufficient intelligence or lack of technological sophistication, then it would seem that this unknowability is encoded within its nature – that it is ontological. (pp. 257–258)

Such assertions are in keeping with standard quantum theory, for quantum indeterminacy *is* currently believed to be a fundamental property of matter, in that (subatomic) matter will not exhibit fixed characteristics (be they particulate or wave-like, localized or distributed) until it has been subjected to some manner of measurement or observation. The precision of such measurement/observation will then be constrained by fundamental uncertainties (as in the case of Werner Heisenberg's uncertainty principle).[5]

Despite the possibility of encoded uncertainty, I nevertheless point out that indeterminacy and uncertainty at the quantum scale do not necessarily preclude determinacy and certainty in an absolute sense. As a result, I avoid committing to a *purely* indeterministic worldview and instead consider a stance from which the material world is capable of supporting both pairings (indeterminism and determinism, uncertainty and certainty), depending upon the contexts and circumstances being dealt with at any given time. Essentially, I remain *agnostic* as to whether the material world is exclusively in/deterministic and un/certain, and I remain open to the possibility that nature may well accommodate all of these. While it is not yet clear to me if my agnosticism might be inherently flawed or inconsistent, I currently believe it to be the more prudent option when counted against alternatives that favour exclusivity over inclusivity. This more contingent perspective will ultimately be in keeping with the overall worldview that I develop in the forthcoming chapters.

If even uncertainties at the quantum level do not directly correspond to uncertainties at the macroscopic level, it is clear that human beings routinely and unavoidably grapple with uncertainty when reasoning. Dealing with, or perhaps *coping* with, uncertainty would seem to be innate to our processes of cognition. That said, our reasoning with and about

[5] Here, measurement/observation is understood to include not only instances in which conscious human agents participate, but also scenarios involving interactions between classical and quantum objects.

uncertainty does not always adhere to the principles that one might logically assume. Notably, as some research has borne out, human beings often fail to observe the rules of classical probability, as in cases that illustrate widespread conjunction biases and disregard for commutativity principles (see, for instance, de Freitas and Sinclair, 2018 and Wendt, 2015 for involved discussions of Tversky and Kahneman, 1983).

Within their discussion of reasoning with and about uncertainty, de Freitas and Sinclair (2018) examine some of the failings of *classical probability* when it comes to capturing more complex facets of human reasoning and judgment, and engage with the concept of *quantum probability* "as a possible alternative formalization" (p. 272). In so doing, they also point out that researchers across a range of fields, such as cognitive science, information theory, machine learning, economics, and many others, are currently exploring broader implications of quantum probability as a possible means through which to rethink human cognition. As with Dijkgraaf's *quantum mathematics*, this is territory into which I will not delve too deeply in the current document, but I draw attention to it as a burgeoning facet of the discourse in mathematics education research that also leverages quantum theoretical perspectives in an effort to move away from more classically, or traditionally, constrained sensibilities.[6]

I am entirely open to the possibility that indeterminacy at the quantum level could ultimately give way to determinacy at the macroscopic level, in much the same manner that most quantum effects are less apparent above a particular threshold (of approximately 10^{-7}m) than below it. However, even though I remain agnostic as to whether or not uncertainty *supersedes* certainty in a universal sense, my broader mathematical worldview embraces the fundamental indeterminacy of matter, and conceives of mathematical processes as ones that involve the *organization and reorganization* of matter (to greater or lesser degree). What this fundamental material indeterminacy might mean with regard to

[6] To name only a few, see de Freitas and Sinclair (2018), Wendt (2015), Aerts et al. (2013), Pothos and Busemeyer (2013), Wang et al. (2013), Trueblood and Busemeyer (2011), and Busemeyer et al. (2011) as works that exemplify how elements of the quantum theoretical discourse are becoming more widely adopted into the spaces of mathematics education, cognitive science, and psychology.

the nature of mathematics, and how it might be implicated in mathematical cognition will be a recurring theme in the coming chapters.

Even though I shall not focus specifically on the *teaching* of mathematics in this dissertation, and primarily remain attendant to more theoretical considerations, I do not believe that this lessens/decreases the relevance or value of the current exploration. In keeping with my introductory preamble, I re-emphasize the importance of actively questioning the nature of mathematics, not only at the level of the personal, but the institutional and societal as well; for responses to the question of what mathematics *is* invariably underlie the (re)presentation of mathematical subject matter, the communication of values associated with mathematical thinking, and related commitments concerning the teaching of mathematics itself. All of that said, with Wigner's inaugural Courant lecture (1959) and subsequent publication (1960) acting as invitations to rethink the (seemingly) *unreasonable effectiveness of mathematics in the natural sciences*, and with Dijkgraaf (2017) identifying a compelling reversal of circumstances through which quantum theory has more recently begun to feed back into a range of mathematical contexts, a basic philosophical and historical preface has now been established for my overall discussion.

Chapter 1.
Philosophical and Theoretical Orientation

"One weird phenomenon of modern philosophy is that philosophy of science and philosophy of mathematics are almost disjoint. Authors in philosophy of science rarely refer to philosophy of mathematics, and vice versa. An author who writes on both subjects, in any one article sticks to one or the other."

–Reuben Hersh (1995)
Fresh Breezes in the Philosophy of Mathematics, p. 590

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In the excerpt above, mathematician Reuben Hersh articulates an unfortunate disjunction that often seems to arise within scholarly treatments of the philosophy of science and the philosophy of mathematics. As I have already alluded to in the introductory pages of this dissertation, I do not believe that this disjoint relationship need be the case (particularly when discussing mathematics and physics), and a guiding principle behind my research is that these respective philosophies may (and indeed should) be rejoined. It is, for instance, worthwhile to recall that the Pythagoreans essentially viewed mathematics as the first philosophy, in that *number* bridged the qualitative distinction between the material realm of the senses (the realm of seeming) and the ideational realm of the intellect (the realm of being). This quintessential Pythagorean sensibility is implicit/encapsulated within the oft-cited maxim that *all things accord in number*.[7] Despite the tendency toward disjunction identified by Hersh, this is a key perspective from which it could/should be more reasonable for modern authors in the philosophy of science and the philosophy of mathematics to reunite these areas of interest. Moreover, across these and other disciplines, it may even encourage broader recognition and prioritization of mathematics as the *primary* science.

Within the latter pages of his unfinished treatise *Die Krisis der Europäischen Wissenschaften und die Transzendentale Phänomenologie: Eine Einleitung in die Phänomenologische Philosophie* (*The Crisis of European Sciences and Transcendental*

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[7] As expressed by Iamblichus (3rd century BCE/1919, p. 76)

Phenomenology: An Introduction to Phenomenological Philosophy), philosopher and mathematician Edmund Husserl (1970) is known for having expressed the view that, "we must engross ourselves in historical considerations if we are to be able to understand ourselves as philosophers and understand what philosophy is to become through us" (p. 391). This comes as a keen reminder to not only acknowledge and respect the long history of thought and deliberation that precedes contemporary philosophical works, but also to reconsider how modern worldviews might actually be informed (and perhaps even bettered) by revisiting the great thinkers of the past. Notable as well is its suggestion that philosophy is an ever-evolving practice, and an ongoing process of exploration and inquiry. By virtue of my own adherence to, and advocacy for, this view, I aim to express its great significance within the complementary spaces of educational theory and praxis (both of which are undergirded by philosophical foundations and historical legacies).

Husserl's viewpoint resonates with me in that it parallels a sentiment that has been extremely influential throughout my studies in physics and mathematics, as well as in my early years of teaching and tutoring those same subjects. In my more recent postgraduate endeavours, an Husserlian outlook has also informed my overall approach to research, my engagement with literature, and the continuing development of my relationship with mathematics. At various points in my academic and professional growth, I have been reminded of a personal view parallel to Husserl's, which addresses the disciplines of mathematics and science in much the same way that he addresses the subject matter of philosophy. Though I have not previously articulated this view in the form of a mantra, I purposefully mirror Husserl's language here:

> *We must engross ourselves in philosophical considerations if we are to be able to understand ourselves as mathematicians and scientists and understand what mathematics and science are to become through us.*

Of course, in light of Husserl's suggestion that historical considerations must infuse the philosophical, I am open to the idea that my thoughts concerning mathematics and physics (which some might consider a branch of applied mathematics) will inevitably be informed by the historical as well. Nevertheless, by voicing this sensibility, I reiterate the notion that the disciplines of mathematics and physics are deeply intertwined at the level of their philosophical underpinnings, and I foreground the view that *what we know about mathematics is mutually implicated in what we know about ourselves as mathematicians*. Whereas the former ontological sensibility is grounded in the aforementioned Pythagorean worldview, the latter epistemological stance stems more from an enactivist approach to embodied cognition, which will be elaborated elsewhere in this document.

I recognize that some mathematicians may conceive of mathematics, its idealized objects, processes, and assertions, as being inherently "disconnected" from reality, in that mathematics can be seen simply as a robust and generalized system of abstractions with purely representational significance. This is, however, a perspective that I move away from for much the same reasons cited in my allusions to Feynman and the useful mathematical toolkit. Indeed, in so far as mathematics and physics are concerned with describing various facets of reality (modeling might be the appropriate term), I suggest that both disciplines invariably draw us toward, and offer opportunities to engage with, basic ontological and epistemological truths (although these are rarely, if ever, addressed in the teaching of these subjects). The sense of mathematics being the *primary* science through which these truths can be addressed is integral to the discussions within this document; however, it is also important to note that no discipline evolves in isolation. While I uphold the status of mathematics as primary to physics and other natural sciences, I still seek to avoid the disjunction spoken of by Hersh, by acknowledging that these disciplines have co-evolved (and continue to co-evolve) in such a manner that they are, and should be recognized as, *mutually informative* practices/pursuits.

The potential for this sort of mutual/reciprocal contribution is borne out in works by Campbell (1999, 2002), who recapitulates the strong association between Western mathematics, physics, logic, and the emergence of ancient Greek philosophical thought. In particular, he points out:

> It is possible that without the emergence of science in ancient Greece, there would be no such thing as Western Philosophy as we have come to know it. Nor, I would like to suggest, could mathematics have developed as we have come to know it either. The origins of Western philosophy and mathematics are not unrelated. (2002, p. 6)

Adding that the science of ancient Greece was itself dependent upon a confluence of ideas and perspectives drawn from other cultures, I echo this sense of rich interconnectivity and mutually informative emergence, and also emphasize that modern perspectives in mathematics education might be enriched or amended by re-embedding such insights into the current discourse of our field. More to the point of this particular program of research, I ask what we might stand to learn about the nature of mathematics, the process of mathematization, and possibly mathematical thinking as well, by reinvesting in them our scientific and philosophical perspectives. It is with this thought in mind, and the commentaries of Wigner and Dijkgraaf providing additional impetus, that I draw upon the quantum theoretical and enactivist discourses already mentioned.

Research Aim and Motivation

As I have prefaced in the preceding pages, a driving tenet of the research represented by this dissertation is that a changed view of the material also changes one's view of the mathematical. In keeping with this idea, I shall make a case for moving away from traditional materialisms and toward more *inclusive materialisms* that speak to modern sensibilities about human and non-human bodies, classical and non-classical interactions, problematic dualisms, and agency. Though I draw from de Freitas and Sinclair's (2014) characterization of *inclusive materialism*, the manner in which I borrow (or perhaps co-opt) the term is quite limited, for I shy away from much of the socio-political subject matter which they address directly. Whereas these authors propose "an inclusive materialism that emphasizes the biopolitics of all phenomenological studies, with the aim of mapping the links between the micro-visceral activity of the body and the political forces that flow across and through these bodies" (p. 41), I remain focused on a more compartmentalized or singular interrogation of the nature of matter itself, leveraging the terminology of *inclusive materialism* to examine the paradigm shift in the physical sciences (i.e., from classical theory to quantum theory), the subsequent renegotiations of what *constitutes* matter, and how these renegotiations influence sensibilities about the *nature of mathematics*. The reader may contrast this with the more expansive work of de Freitas and Sinclair, who characterize their inclusive materialism as "part of a paradigm shift in the social sciences more generally" (p. 230). Essentially, I adopt the adjective 'inclusive' to acknowledge the manner in which quantum theory supports a broader notion of matter than do its more classical counterparts.

de Freitas and Sinclair (2014) point out that two central notions from the philosophy of Gilles Châtelet inform their conceptualization of inclusive materialism: "The first is the very notion of the human body […] The second is the nature of materiality and its relation to the human body, the social and the conceptual" (p. 2). Rather than Châtelet's notion of the *human body*, I would suggest that the origins of my own work are more closely rooted in Bohr's model of the *atom*. Thus, the starting points of our explorations are somewhat different. This research is motivated by the view that reinterpreting the nature of mathematics through more inclusive materialisms has the potential to reinforce the fundamental relationship between mathematics and the material world, and to deepen our collective understanding of the ways in which our models of mathematics ultimately take on the meanings that they do. In the pages to come, I hope to communicate how appealing to elements of the quantum theoretical and enactivist discourses (both of which draw into question how human beings engage with the material world) grants access to alternative ways of approaching and conceptualizing the fundamental nature of mathematics, and the role that mathematics plays in establishing a structural basis for reality.

As a relatively new field arising out of the social science and humanities conversations of the late 1990s and early 2000s, *new materialism* exemplifies a theoretical turn that refocuses on matter and embodiment while simultaneously pivoting away from classical dualisms and traditions that prioritize the human over the non-human. Along with the human/non-human binary, it also troubles many other established oppositions such as the organic/inorganic, active/passive, male/female, continuous/discontinuous, and so on. Indeed, feminist scholar Judith Butler's explorations of sex and gendering played a prominent role in the early development of the new materialist space,[8] and much of the subsequent discourse has remained closely connected to matters of inclusivity/exclusivity,

[8] See, for instance, Butler (1990) and Butler (1993).

social justice, equity, and the politics of power im/balance.[9] Beyond perturbing common binary distinctions, new materialism also inquires into more general boundary-making practices, and the wide range of ways in which it does this is partly responsible for its suffusion into a multitude of fields/disciplines.

As I have come to know it, the new materialist discourse does not currently have well-defined disciplinary boundaries. In itself, this is largely to be expected, as the majority of new materialist scholars actively engage in the deconstruction and reconsideration of commonly accepted boundaries within and between their respective fields of interest, as well as the judgments/distinctions/discernments from which those boundaries arise. Consequently, the themes explored (and the works produced) by new materialist scholars are highly varied. Consider, for instance, Barad (2007), de Freitas and Sinclair (2013, 2014), and Dolphijn and van der Tuin (2012), who delve into the physical sciences, mathematics education, general philosophy, and socio-politics as well. Interestingly, the contributions of scholars such as Wendt, Merleau-Ponty, and Deleuze, some of whom *predate* new materialist trends, have been retroactively incorporated into new materialism by virtue of the themes with which they engage. The reader may wish to keep this in mind, as certain discussions have been *adopted into* the new materialist discourse rather than *born of it*.

My primary interest in new materialism relates to entanglements of matter, and the principles according to which those entanglements organize and reorganize themselves. In that regard, I leverage the portion of the new materialist discourse that skews more toward relational ontology than object-oriented ontology. Furthermore, since the notion of entanglement immediately implicates quantum theory, I also wish to more thoroughly unpack the significance of their connection. If only because quantum theory necessitates a rethinking of matter, it might be considered as fundamentally new materialist; however, it should be recognized that quantum theory is an established field with its own discourse, and that it is not *synonymous* with new materialism. That said, there are numerous scholars

[9] By questioning the nature of mathematics, it will not be possible for me to avoid all associated political themes, particularly in terms of possible implications for institutionalized education; however, I do intend to maneuver around them as best I can, in favour of maintaining a more focused discussion.

across a variety of fields (for example, physicist and feminist theorist Karen Barad, mathematics education scholars Elizabeth de Freitas and Nathalie Sinclair, professor in theory of cultural inquiry Iris van der Tuin, and philosopher Quentin Meillassoux) who readily explore ramifications and affordances of quantum theory in relation to (or adjacent to) the new materialist discourse. Thus, quantum theory has been *partially* absorbed into new materialism, and exists within its discursive space. To be clear, for the purposes of my own research, I have not found it particularly necessary to emphasize this distinction; but I acknowledge it here in the interest of clarity. Much as will be the case with enactivist theories of embodied cognition, I offer only a brief introduction to quantum theory in this early chapter, providing a more extensive discussion of its significance elsewhere in the dissertation.

Extending well beyond the classical worldviews that preceded it, quantum theory has ushered in a new era of conceptual understanding and revolutionized human comprehension of the material world in many respects. Subatomic theory bolstered by quantum perspectives has prompted advances in microprocessing, computing science, materials design and engineering, and fostered developments in microscopy, thermometry, and precision time measurement. Reconceptualized cryptographic procedures and even new insights into certain biological processes have also accompanied the evolution of quantum science (Mansfield, 2013).[10] In even more profound ways, quantum theory has troubled long-standing classical views concerning the determinacy of matter and the nature of spacetime. In much the same way that Gödel's incompleteness theorems raise issues about the nature of knowledge and the logical systems from which we derive meaning, so too do quantum theoretical principles like Heisenberg uncertainty and quantum entanglement bring into question the nature of reality and our perceptions of it. Both mathematicians and physicists are confronted with analogous (and equally substantial) philosophical problematics as a result. Certainly, quantum theory has fundamentally changed the ways in which human beings think about, express ideas about, interact with, and coexist within the world of today. While the various implications of quantum theory

[10] To date, quantum electrodynamics (QED) remains the most accurate physical theory devised by human beings.

are undeniably counterintuitive, and perhaps even disquieting or unnerving for some, they are nevertheless intrinsic to (and inseparable from) the human experience of both space and time.

In his own discussions of quantum theory, Alexander Wendt (2015) refers to space and time as universal facets of human experience; however, he also qualifies this by stating that he "[does] not mean that we all experience time and space in the same way, but that by virtue of sharing the same physics of the body our experience of time and space has a universal aspect" (p. 190).[11] Wendt's reference to *the body*, might suggest a fairly traditional object-oriented ontology. At the same time, in light of his broader constructivist views concerning the relationality of identity, and considering that the body is, itself, part of what constitutes identity, I offer that it would not be inconsistent to view Wendt's perspective as aligning with an underlying *relational* sensibility about matter. Moreover, it could easily be argued that physics, as a systematized science, is rarely (if ever) concerned with objects in isolation, and is more concerned with the relations that exist *between* its objects and phenomena of interest. As a result, I choose to read Wendt's perspective (from which we share *the same physics of the body*) as both materially grounded *and* relational, and I shall revisit aspects of it within this document.

Similar to quantum theory, enactivist theory is *not* innate to the new materialist discourse, and constitutes its own variegated/multifaceted philosophical orientation. At a certain level, though, it does engage with comparable themes. Of some interest within the current dissertation are Husserl's *natural* and *phenomenological* attitudes (which influence enactivism). Despite any suggestion of dichotomization that might be provoked/prompted by their names, these complementary modes of being are *not* actually framed in opposition to one another, and are articulated in such a way as to be ontologically unified within the embodied self (i.e., the thinking, knowing, sensing self). Indeed, this focus on an ontologically unified, rather than divided, self has provided me with further motivation to consider enactivism alongside quantum theory. Perhaps of even greater interest than the

[11] Though Kant espoused the phenomenological and mathematical relevance of space and time centuries ago (see Sutherland, 2004), he spoke of space and time themselves as *a priori* foundations, as pure intuitions obtainable independently of experience. In contrast, Wendt grounds space and time in the very substance of our lived experience.

Husserlian attitudes, though, is the classical Cartesian dualism between mind and body. Commitments to Cartesian dualism remain both pronounced and persistent in the modern day, and much of Western language is still rife with dualist connotations that accompany the bifurcation of mind and body. I have found that attempting to navigate past this dualism can be useful at the level of identifying and deconstructing certain inconsistencies common to specific mathematical worldviews.

It is in their seminal volume *The Tree of Knowledge: The Biological Roots of Human Understanding* (originally published in English in 1987 and re-released as a revised edition in 1992), that Humberto Maturana and Francisco Varela elaborate much of their enactivist perspective. In conjunction with Evan Thompson and Eleanor Rosch, Varela would also further develop this perspective in the 1991 volume *The Embodied Mind: Cognitive Science and Human Experience*. As a theory of cognition, enactivism engages strongly with the notion of *embodied action*, and a characteristic term associated with this theory is 'enaction', itself a conceptual (if not-quite-linguistic) portmanteau of these two words.[12]

Varela et al. (1991) establish that their use of the term 'embodied' is intended to highlight two points: "that cognition depends upon the kinds of experience that come from having a body with various sensorimotor capacities, and […] that these individual sensorimotor capacities are themselves embedded in a more encompassing biological, psychological, and cultural context" (p. 173). In addition to communicating that cognition is grounded in the material, this characterization of embodiment also speaks to the relationality between bodily matter and the larger differentiated systems within which it is embedded. In a way, I find this reminiscent of (or rather precursory to) Wendt's qualifying remarks about universal facets of human existence, and *of sharing the same physics of the body*; for Varela et al. also seem to eschew a traditional object-oriented ontology by emphasising an underlying *relational* sensibility about (bodily) matter. Thus, as I did with Wendt's

[12] Were an historical amendment possible, I would prefer an etymology that characterizes 'enaction' in terms of 'entangled action', allowing for a linguistic *and* conceptual portmanteau much more in keeping with my own reading of the theory. To be fair, Maturana and Varela's enactivism is very much grounded in a *biologically oriented* perspective. I am opting, however, to engage with enactivism from a *physics-based* perspective.

perspective above, I choose to read their approach to embodiment as being simultaneously materially grounded *and* relational.

Varela et al. (1991) also contextualize their use of the term 'action' by emphasizing that "sensory and motor processes, perception and action, are fundamentally inseparable in lived cognition" (p. 173), such that they "are not merely contingently linked in individuals; they have also evolved together" (p. 173). This characterization has provided me with early motivation for considering the possibility of a conjoined enactivist/quantum theoretical perspective; as the language used by Varela, Thompson, and Rosch (alluding to fundamental inseparability and co-evolution) already seems somewhat evocative of an *entangled* worldview. The potential for connectivity between enactivism and quantum theory is further supported by Maturana and Varela's (1992) discussion of *structural coupling*, a phenomenon they describe in terms of recurrent interactions between inseparable unities and their environments, and "a history of mutual congruent structural changes" (p. 75). While I grant that it may be due to my own scientific predilections, I am inclined to envision structural coupling as something akin to quantum entanglement. This is not a stance that I will speak to in great detail, but I do mention it as yet another conceptual similarity that has prompted me to consider parallels between enactivism and quantum theory.[13]

Much as quantum theory engages with wave–particle duality regarding the nature of matter, enactivist theory engages with mind–body dualism regarding the nature of the thinking, knowing, sensing self, and I suggest that it is through a conjoining of monist sensibilities afforded by these respective frameworks that a basis for an alternative mathematical worldview that expresses a more accurate and inclusive material–mathematical relationship might be established. With both the enactivist and quantum theoretical discourses contributing to this discussion, the reader should note that I do not subsume enactivism into new materialism, as I do quantum theory, in large part because

[13] It is worth noting that Maturana and Varela's structural coupling is framed according to *classical interactions between unified systems and their environments*. In light of both the new materialist and quantum theoretical perspectives that contribute to my own worldview, I would lean toward reframing structural coupling so as to include *non-classical intra-actions within fundamentally inseparable systems* (invoking terminology more akin to Karen Barad's); but this distinction will be addressed in a later chapter.

there are areas where enactivism and new materialism rub up against one another. Where possible, I shall attempt to indicate how some of these frictions might arise.

To reiterate, the idea of revising perspectives on the nature of mathematics by infusing them with modern scientific and philosophical orientations about matter, embodiment, interaction, and agency, holds particular appeal, primarily because of the significant shifts in worldview required by the enactivist and quantum theories when compared to more classical worldviews steeped in Cartesian and Newtonian biases. Additionally, whereas Cartesian and Newtonian worldviews tend toward characterizations of an objective observer who is distanced from the objects and phenomena of interest, the modern relational worldviews I draw upon frame human beings as active (and, in some instances, *enactive*) contributors/participants within a larger unity of emerging phenomena, such that we exist *in relation to the world* and not separate from it. The importance of this conceptual shift will be come clearer throughout the course of the current dissertation.

Early engagement with the new materialist, enactivist, and quantum theoretical literature has suggested four core themes upon which to build my proposed exploration. These are, in (descending) order of priority: 1) new materialisms and issues of dualism, 2) epistemological uncertainty, 3) matters of agency, and 4) complexity associated with emergent systems. In light of the need to maintain a focused discussion, and the desire to avoid overburdening the dissertation as a whole, only the first *two* of these themes will be directly addressed in the coming chapters, with the first (new materialisms and issues of dualism) grounding the bulk of the discussion, and the second (epistemological uncertainty) being incorporated intermittently. Both shall be explored within the broader context of inquiring into the nature of mathematics from a conjoined enactivist/quantum theoretical perspective.

It is entirely possible that this conjoined perspective will also have implications for mathematics education theory and pedagogical approaches to mathematics teaching/learning. However, because this dissertation focuses foremost on the nature of mathematics, and treats this as a precursor to other discussions, the possibility of reworking mathematics education theory more generally will be postponed for future efforts. It is expected that the two remaining themes not directly addressed in this document (matters of agency, and complexity associated with emergent systems) may be central to those explorations. That said, brief comments/interjections regarding these secondary themes may be woven into the primary discussions when appropriate.

Alongside careful reconsideration of the *nature of mathematics*, this program of research also calls for thoughtful re-engagement with *the nature of matter*. Throughout the upcoming chapters, I endeavour to convey a view that re-embeds the material in the mathematical (by way of quantum theory), and that draws the mathematical back into us as well (by way of enactivism). As a result, this dissertation is part *deconstruction of norms*, part *tentative exploration*, and part *attempt to formalize a line of thought that has emerged during the course of the investigation*. Offering up a conceptual treatment of the nature of mathematics from a conjoined enactivist/quantum theoretical perspective, I hope to substantiate a new materialist view from which mathematics itself may be reconceived as *the science of material assemblage*. Though I can not delve into a proper explication of this phrase without first laying out other conceptual background, I emphasize it here as an indication of its importance to this entire document. Precisely what it means will be elaborated throughout the dissertation, with the following overview more clearly delineating the structure of the surrounding discussion.

Structural Overview

As previously stated, this dissertation will be driving toward the articulation of a fresh perspective regarding the nature of mathematics. At a certain level, this will be reflected in the unconventional structure of the document. There will, for instance, not be a self-contained literature review, as each chapter of the work is informed by its own source material. I also draw attention to the brief metacommentaries interspersed throughout the document as a means of expressing other considerations that are related to, yet somewhat removed from, the core discussions. I remain indecisive regarding the terminology I wish to attach to these supplementary contributions, and have toyed with the idea of referring to them as *quantum interludes*; however, the notion of something existing in spaces *between* the discrete, integer-numbered chapters is rather antithetical to the entirety of the quantum theoretical discourse. Referring to them, instead, as *intraludes* is more in keeping with both the enactivist stance I adopt and the notion of quantum entanglement that I inject into my discussion. As *intraludes*, these contributions can be seen as existing *within* the body of the larger unified document, whilst also occupying a metacommentative space from which to reflect upon selected facets of (or ideas surrounding) the primary exploration.

Essentially, these intraludes are intended to supplement the main chapter offerings with somewhat less formal (potentially more open-ended) treatments of related thematic content. In addition to discussing the process of engaging with the core themes that underlie this research, they also yield insight into the emergence of the dissertation document itself. Thus, while the intraludes may not be as extensive or as fully formed as the primary chapters, they nevertheless encapsulate important aspects of my thinking with respect to material entanglements and the mathematical (or rather *material–mathematical*) worldview I wish to communicate. By exploring concepts such as discretization and continuity, the emergence of new discursive spaces, and order/disorder, these intraludes reveal additional layers of conceptual strata to be found within the larger structures of the discussion. In keeping with the concept of quantum discontinuity, and assemblage theory more generally, it is possible to jump between the intraludes and to read them in an order other than they appear. I would, however, encourage the reader not to do this upon the first pass through the document. Having said that, engaging with the intraludes *out of sequence* could prove interesting upon subsequent readings.

As a unified whole, the dissertation spans six primary chapters (including the present one), with three accompanying intraludes and a brief postface. The five primary chapters to come are organized according to major motifs that surfaced throughout the course of my research. Although the dissertation has not been written as a chronological narrative, it is intended to reflect the conceptual journey on which I have embarked, while further substantiating my broader mathematical worldview. In addition to elaborating upon the themes of interest, it also acknowledges and justifies the exclusion of other themes from the same discussion.

Intralude A constitutes a metacommentary concerning the process of writing this dissertation document, and discusses the metaphors of "borderless puzzles" and "defragmentation", both of which have been particularly useful in terms of viewing the dissertation's evolution as a process of material assemblage. By linking these metaphors to an overarching thematic of granularity, I contextualize specific mathematical considerations with which I have grappled throughout the writing process. Notably, the complementary pairings of *the discrete* and *the continuous*, *the part* and *the whole*, provide connections between the quantum theoretical and mathematical discourses.

Chapter 2 establishes my overall research approach, while addressing associated methodological considerations. The chapter largely articulates the manner in which I have opted to navigate the literary terrain in which the exploration is based, and offers insight into factors that have influenced what I have attended to. At a certain level, the chapter also clarifies my choice *not* to engage deeply with specific philosophical content, namely the posthumanist discourse. Similar clarification is provided concerning Karen Barad and Donna Haraway's method of diffractive analysis, to which my research approach bears some resemblance.

Chapter 3 examines the (implicit) mathematical worldview of Richard Skemp in an effort to reveal how certain classical modes of thought still pervade modern perspectives regarding the nature of mathematics. By identifying what I believe to be internal inconsistencies inherent to Skemp's implicit worldview, I aim to highlight foundational characteristics of mathematics that are often overlooked, and to which greater attention could be paid. In conjunction with remarks from David Wheeler, Skemp's articulations are used as illustrative examples against which to contrast my own sensibilities about the nature of mathematics, and to help paint the broad strokes of the perspective from which I see mathematics as *the science of material assemblage*.

Albeit briefly, **Intralude B** touches on a matter of significance to all three of the enactivist, quantum theoretical, and new materialist discourses. As a prelude to **Chapter 4's** heavier treatment of quantum theory and paradigm change within the physical sciences, it offers metacommentary on linguistic commitments and the emergence of new discursive spaces.

Extending the metacommentary further, tentative connections are also made to a specific form of self-organizing behaviour, which recent physics research reveals to be an indicator of persistent quantum entanglement under certain conditions.

Chapter 4 presents key features of the quantum theoretical view of matter, and recounts particular historical and conceptual problematics that led to the inception of Niels Bohr's innovative (albeit controversial) model of atomic structure. Principles of quantization and their links to foundational mathematical notions of continuity/discontinuity and discretization are explored, as are certain number theoretical constructs as well as ideas salient to other socio-cultural contexts. By attending to the principle of wave–particle duality, the phenomenon of the quantum leap, notions of coherence and decoherence, and a specific formulation of entanglement, this chapter also harkens back to the motivating commentaries of Wigner and Dijkgraaf, indicating *how* and *why* the quantum theoretical discourse provides a powerful means of reconceptualizing the role that mathematics plays in establishing a structural basis for reality. It also articulates a sense of how mathematics can be seen as embodying the very principles according to which matter organizes and reorganizes itself.

With the preceding chapters establishing much of the interdisciplinary backdrop for the dissertation as a whole, **Chapter 5** forges direct connections to the discourse of mathematics education. By engaging with Paul Ernest's survey of prominent perspectives in the *philosophy of mathematics education*, it situates my overall worldview with respect to the absolutist and fallibilist orientations, with *Logicism*, *Formalism*, *Constructivism*, and Ernest's *Social Constructivism* being featured/highlighted as key points of reference. Appealing to Reuben Hersh's notion of *pluralism* in the philosophy of mathematics, it also suggests a means of addressing (and potentially bypassing/overcoming) certain frictions between the realist and idealist perceptions of mathematics, mathematical knowledge, and mathematical modeling. These same considerations ultimately help to substantiate the idea that *how we come to know mathematics is intimately entwined with how we come to know ourselves*, which builds upon enactivist sensibilities expressed throughout the dissertation. Lastly, a revisitation of the quantum leap, reconsidered through the contexts of elementary arithmetic, provides a concrete example of how mathematical structures and processes are seen as permeating matter at a foundational level.

Intralude C communicates how the thermodynamic principle of *entropy* (as a fundamental measure of disorder) might be reconceived as an innately mathematical characteristic of material assemblages. Though decidedly open-ended, the tentative thoughts expressed here are also intended to tie together strands of discussions formed elsewhere in the dissertation.

As the final primary chapter offering, **Chapter 6** more tightly weaves together elements of the enactivist, quantum theoretical, and new materialist discourses already presented, by synthesizing the main discussion points of the preceding chapters and intraludes. It also provides an addendum to them, reiterating key features of the conjoined enactivist/quantum theoretical perspective that has been proposed, and articulating further reflections on the overall characterization of mathematics as *the science of material assemblage*. Lastly, the chapter addresses specific limitations of the present study, giving voice to my thoughts about the potential for further research in this area. A summative recap of the primary chapters and secondary intraludes acts as the companion to this preliminary structural overview.

Something of a counterpart to the opening preface, a complementary postface gives brief closing remarks on the dissertation as a whole, and comments on the process of engaging in this program of research.

Intralude A | Granularity

A Commentary on the Fundamental Tension Between the Discrete and the Continuous

While I can not pretend to know how it is that the process of research writing unfolds for others, it has more recently occurred to me that the earlier stages of my dissertation work unfolded in a manner largely analogous to the construction of a jigsaw puzzle that had no border pieces (or rather, one whose borders were not apparent at the onset). The puzzle pieces also seemed to vary not only in shape but in size as well. The initial absence of an outer bound, compounded by this variation in granularity essentially meant that neither the scale of the puzzle image nor its resolution were entirely obvious, and the required scope and depth of the research task did not become evident until much later on. It was only when smaller clusters of related ideations began to accrete and connective tissues emerged from the interstices *between* neighbouring clusters that the general features of the emerging puzzle image (and their orientations with respect to one another) started to reveal themselves. In addition to indicating how the complexity of a given research topic might impact the features of the puzzle being constructed, these factors also helped to make clear the necessity of oscillating back and forth between single puzzle pieces (i.e., individual words and ideas) and the larger puzzle image (i.e., the overarching vision for the dissertation as a unified entity). The need for this oscillation was not always immediately apparent at the time of writing; however, in retrospect, it does seem to have been an essential activity, as both *the part* and *the whole* demanded similar (if not equal) consideration.

With this in mind, my perception of the early dissertation as a *borderless puzzle* can be seen as thematically derived from (perhaps even fundamentally rooted in) mereological[14] underpinnings, and as marking the importance of attending to part–whole relationships. Of course, while the contexts of scholarly writing involve markedly different moment-to-moment concerns when compared to explorations in number theory, this metaphor

[14] For the reader who may be unfamiliar with the term, mereology is the study of *parthood*, or of the relation between parts and the wholes they constitute. Beyond mathematics, it enters into the considerations of numerous other disciplines as well, such as linguistics and metaphysics, to name only a couple.

nonetheless draws attention to the mathematically significant notions of *the discrete* and *the continuous*, and indicates how they may actually be implicated in activities that are not typically associated with mathematics.

In fairness, while the notion of writing as akin to puzzle construction is likely very common amongst writers, I have a sense that a direct association with mereological themes, specifically, might be less so. That said, just as fragmentary elements at a very small scale support more cohesive, unified structures at a larger scale, I would suggest that the reverse relationship is equally important. Often, the role of the solitary parts in shaping the intended whole is given the greater priority; yet, the reciprocal contribution of the intended whole in shaping the solitary parts should not be overlooked, as both facets of the relationship are mutually informative. Mechanically, structural concerns at the level of individual characters, words, phrases, and so on, inevitably precede more expansive concerns at the level of sentences, paragraphs, chapters, et cetera, and all of these ultimately contribute to a single unified document whose subsections have the potential to cohere structurally and thematically. At the same time, ideationally or conceptually, the emerging composite features of the document at large invariably shape the singular choices that are made as writing progresses. In this way, larger structures that are *envisioned* (before and as the work takes shape) impact smaller structures that are *realized* from moment to moment. Thus, rather than viewing these as two opposed perspectives on (or approaches to) writing, I instead prefer to think of them as complementary facets of one unified process of *becoming*.

Of crucial importance to my own work is the assemblage theory of French philosophers Gilles Deleuze and Félix Guattari, which emphasizes the importance of structural interplay and reconfiguration within complex systems. As with entangled quantum systems, assemblages involve/necessitate scenarios wherein contributing components co-evolve with and within the larger systems in which they are embedded, and it is by so doing that they "form emergent unities that nonetheless respect the heterogeneity of their components" (Smith, 2013, "Deleuze's Readings of Other Philosophers", para. 7). The features of these unified systems are neither static nor predetermined, such that contributing objects or processes can play different roles and exhibit greater or lesser prominence at different times. As John Macgregor Wise writes in editor Charles Stivale's *Gilles Deleuze: Key Concepts* (2011):

> Assemblage, as it is used in Deleuze and Guattari's work, is a concept dealing with the play of contingency and structure, organization and change […] The term in French is *agencement* […] It is important that *agencement* is not a static term; it is not *the arrangement* or *organization* but the *process* of arranging, organizing, fitting together. The term […] is commonly translated as *assemblage*: that which is being assembled. An assemblage is not a set of predetermined parts […] Nor is an assemblage a random collection of things […] An assemblage is a becoming that brings elements together. (p. 91)

At its heart, my broader mathematical worldview is also deeply concerned with ongoing processes of reorganization/structural change and the *bringing together of elements*. One of the key ways I shall articulate this is by working to substantiate the claim that *mathematics embodies the very principles according to which matter organizes and reorganizes itself*. In keeping with this motif, my worldview leverages a portion of the new materialist discourse that skews less toward object-oriented ontology and more toward relational ontology (with a specific interest in entanglements stemming from the quantum theoretical discourse).

Embracing the implications of assemblage theory and the associated notions of structural change and reconfiguration, I have inevitably begun to think about the *dissertation writing process* in similar terms. More specifically, I have come to envision this dissertation as an evolving assemblage of entangled ideas, and I am increasingly concerned with the structuring principles at play within it, and through which the embedded discussions are being realized. It is for this reason that I have more recently reconceived of *metaphors* in a manner analogous to my overall mathematical worldview. Carrying forward the notion that mathematics might be seen as embodying the principles according to which *matter* might be reorganized, I here entertain an extended notion in which metaphors embody the principles according to which *meaning* might be reorganized. Granted, this may strike the reader as being rather removed from a mathematically grounded discussion; but I shall shortly revisit this analogy in terms that should allow the underlying mathematical significance to become more evident.

The excerpt from Wise already given is rather emblematic of my thoughts about the activity of scholarly writing. Indeed, I believe that the writing process is wonderfully illustrative of the sensibilities Deleuze and Guattari express about the nature of assemblage (or *agencement*); for the composition of coherent, engaging, and accessible text with sensible structure, agreeable flow, and rigorous discussion clearly has much to do with the processes of *arranging*, *organizing*, and *fitting together* ideas (and symbols). There also exists a corresponding and ever-present need to be cognizant of structural interplay and possibilities for reconfiguration at multiple levels. Whether *literally* (at the level of word choice and grammar), or *figuratively* (at the level of connotation and intentionality), the conveyance of meaning through written text is very much an act of *bringing together* and entangling ideas in such a way that a larger, unified whole emerges from more singular elements. Syntactic/technical and aesthetic/stylist considerations are all deeply implicated in the negotiation of meaning common to written communication, and it is by *facilitating* the negotiation of meaning that *metaphors* are of immense value to author and reader alike. As an example, the borderless puzzle metaphor already described has been particularly useful to me in the endeavour to craft a cohesive and (hopefully) well-structured research document.

Though the turn may be abrupt, at this juncture I ask the reader to move with me into the informational realm of binary data and digital file management. More specifically, it is the processes of data storage and retrieval that I wish to highlight. When dealing with digital storage space under specific file management systems, the effects of *data fragmentation* can be an occasional concern. Data fragmentation occurs when a file system is unable to reserve enough contiguous file space for data to be written as a unified "block", and must instead reallocate data fragments elsewhere on the storage medium. While not all file management systems operate in this manner, those that do experience substantial file fragmentation can suffer a number of adverse effects. The most apparent of these will manifest as general performance issues resulting from the scattered distribution of data and less efficient use of the available storage space. Essentially, the more fragmented a file system, the greater the time and effort necessary for a processor to access stored data and to reconstitute the bits (binary digits) pertaining to any given file. As a preventative measure, *defragmentation* can be performed in order to combat these effects.

Simply put, defragmentation is the redistribution of binary data, a reallocation of digital information into more contiguous "chunks". Inasmuch, it is also an extremely powerful and highly illustrative example of what I might refer to as *material assemblage in action*,[15] and another instance that nicely embodies the play of contingency and structure, organization and change spoken to by Deleuze and Guattari. It is a literal *bringing together of elements*.

This notion of redistributing/reallocating and "tightening up" information has been integral to the latter stages of my dissertation writing. In a way, the defragmentation metaphor is merely a variation on the puzzle metaphor offered earlier; however, the contexts that motivate it are quite different and I have found it useful in characterizing *another* aspect of the writing process. Whereas the "dissertation as borderless puzzle" metaphor could be seen as relating to the resolution of a coherent puzzle image with clear boundaries (i.e., a unified document with well-defined scope), this "defragmentation" metaphor might be more comparable to sharpening the image (i.e., of restructuring/refining the document in order to further consolidate the content within). It could even be said that the former is concerned with the overall *construction* of the dissertation, while the latter is concerned with the *efficiency* of that construction. I grant that these analogies might be a little loose, but it has been no less helpful to think of the evolving dissertation work in these terms.

To be clear, by drawing upon the notion of defragmentation, I am in no way meaning to invoke the long-standing and potentially problematic "brain as a computer" metaphor that continues to be debated. I am not suggesting that human brains *actually* reorganize written information in the same way that the operating systems of digital computers defragment file space. Rather, I am simply articulating a figurative connection that has informed my own writing efforts. In fact, I would also suggest that the defragmentation metaphor I have put forth does *not* carry any of the problematic baggage associated with the aforementioned debate; for it makes no real claims about brain *function* and merely speaks to an underlying mathematical process. Indeed, if defragmentation is reconsidered through the assemblage

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[15] My use of this phrase here is slightly anachronistic/premature, in that it precedes the characterization of material assemblage that is developed throughout the rest of this document. The reader may wish to revisit this passage after reading the entire dissertation.

theory of Deleuze and Guattari, and the broader worldview to be elaborated within this dissertation (i.e., where mathematics embodies the principles according to which matter might be organized/reorganized), then defragmentation is an *inherently* mathematical process, albeit not necessarily a *digital* one. Alternately put, one might think of the defragmentation of a written document as solving a specific kind of *optimization problem*, where the optimization constraints are not only associated with grammatical syntax and the economy of word use, but also stylistic, semantic (and various other) social factors as well. In this way, the notion of defragmentation (applied to the scholarly writing process) retains a mathematical basis. I will even go so far as to preface that this is in keeping with the manner in which I see mathematics as being encoded in matter, a perspective to be discussed within **Chapters 3**, **4**, and **5**.

Thus far, I have framed this intralude in terms of my overall dissertation writing process, largely because it is the materially grounded activity that has occupied the greatest part of my attention in recent years. However, since mulling over the two metaphors addressed in the previous pages, I have also begun to wonder if such metaphors, or others like them, might be more generally applicable in terms of characterizing how human beings organize and reorganize, configure and reconfigure, our *mathematical knowledge* as well. As indicated earlier in this document, my dissertation research draws upon the quantum theoretical discourse. Accordingly, I believe that both of the metaphors discussed herein might also fall under the more encompassing thematic of *granularity*. By employing this term, I mean to emphasize the fundamental tension between the discrete and the continuous that resurfaces in both metaphors, and which seems to speak to a key aspect of human sense-making activities.

In his 2018 publication *The Order of Time*, theoretical physicist Carlo Rovelli (director of the quantum gravity research group at Aix-Marseille University's *Centre de Physique Théorique*), puts forth the view that continuity "is only a mathematical technique for approximating very finely grained things. The world is subtly discrete, not continuous" (p. 75). Also noting that granularity "is ubiquitous in nature" (p. 75), he remarks that, "The good Lord has not drawn the world with continuous lines: with a light hand, he has sketched it in dots, like Seurat" (p. 75). Beyond his artful allusion to Seurat's pointillist style as a proxy for the quantum paradigm, it is Rovelli's reference to continuity as a *mathematical technique* that I find especially compelling. While I can not commit to the stance that continuity is completely illusory, I am intrigued by the notion that the *perception* of continuity might arise from (or be grounded in) some sort of mathematical process. In a manner of speaking, this would suggest that the perception of continuity is not only a means of making sense of the lived experience of time (i.e., the *passage* of time), but also a powerful form of *mathematization* that is integral to the ways in which human beings make sense of the material world as a whole.

It is conceivable that a metaphor which characterizes continuity as a form of mathematization might actually be closely aligned with the two metaphors already discussed in this intralude; however, I shall not delve deeper into that possibility here. Instead, I encourage the reader to consider the sense in which written articulations depend upon some degree of continuity emerging from the combination of discrete elements. Individual units of text (i.e., characters, punctuation, special symbols, et cetera), which are largely devoid of meaning when removed from the systems they comprise, can be assembled according to various technical and stylistic constraints so as to produce words and passages designed to communicate intentionality and meaning, evoke emotional experiences, and so on. Via a surface reading of Rovelli's remarks, any semblance of continuity that accompanies a given text *could* be interpreted as a simple "trick of perception"; but I find it far more compelling to interpret it as *mathematically driven activity*. The case of speaking could also be treated in a similar manner, as longer and more significant utterances are built up from individual phonemes. Thus, I am left wondering if it might be appropriate/fitting/reasonable to discuss both writing and speaking as activities that are also realized or enacted, in some sense, through the capacity to *mathematize*.

I have chosen to describe the use of linguistic metaphors as a process consistent with the assemblage/*agencement* described by Deleuze and Guattari, and as one that facilitates the negotiation of meaning common to written (and possibly spoken) communication. By extension, I have also attempted to make a case for treating two specific metaphors as being innately mathematical. In the case of the "dissertation as borderless puzzle" metaphor, the construction of coherent and unified written work from singular characters/words/ideas has been reframed as a process that is profoundly concerned with mereology and the complementary relationship between *the part* and *the whole*. Via the "defragmentation" metaphor, the redistribution of scattered words/ideas into more contiguous passages has been likened to an optimization problem whose constraints involve a range of social as well as technical considerations. Both instances entail the *rearranging*, *reorganizing*, and *fitting together* of text and/or ideas, and both appear to be fundamentally rooted in considerations falling under the greater thematic of *granularity*. In itself, this evokes the central tension between *the discrete* and *the continuous* (a key component of the quantum theoretical discourse); yet, some scholars like Rovelli question classical ontological commitments to continuity. Through my reading of Rovelli, I tentatively proposed that the perception of continuity might actually involve a powerful form of mathematization.

As a closing remark for this intralude, I openly acknowledge that the perspectives I have shared within are themselves largely built up from metaphor, and the *rearranging*, *reorganizing*, and *fitting together* of ideas that I have been mulling over for some time. As the reader continues to engage with this research document, as more pieces of the larger dissertation puzzle fall into place, and as the emerging ideational content is gradually defragmented, I am confident that the deeper significance of these perspectives will become clearer.

Chapter 2.
Engaging with Theory and Discourse

"When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong."

–Arthur C. Clarke (1962)
Profiles of the Future: An Inquiry into the Limits of the Possible, p. 14

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Commonly referred to as Clarke's Law, the above assertion can be found in Chapter 2 of author and futurist Arthur Clarke's *Profiles of the Future: An Inquiry into the Limits of the Possible* (1962). A collection of essays concerning what Clarke believed might be the upper limits of human technological innovation, the volume extrapolates forward from the trends and influences of Clarke's present day, in an effort to indicate the pitfalls, the possibilities, the perils, and the promise of our ongoing development as a technologically inclined species. Alongside various speculations regarding transportation, local and deep space travel, planetary colonization, revolutions in design and manufacture, increased mastery over the environment, the advent of machine intelligence, the eventual obsolescence of human beings, and a host of other themes, Clarke also articulates a number of cautionary examples with much broader applicability/relevance. Specifically, the excerpt quoted above appears in the chapter entitled *Hazards of Prophecy: The Failure of Imagination*, and alludes to the lingering effects that might manifest through dogmatism and adherence to outmoded sensibilities.

As with the Husserlian outlook expressed in **Chapter 1**, Clarke's assertion has also done much to inform my overall approach to research, my engagement with literature, and my ongoing relationship with mathematics. Setting aside its playful intonation and brevity, which might lead some to disregard it as little more than a clever witticism, I consider this a highly incisive statement about institutionalized knowledge, and a compelling plea to be mindful of the ways in which established worldviews might colour our thinking. In terms more germane to the current dissertation, it has also encouraged me to more deeply consider the lasting influence of *theory*, and I purposefully extend Clarke's Law toward theories concerning *the nature of mathematics*.

The Oxford Dictionary of English Etymology (1966) points out that 'theory' might be defined as a "mental conception" or "scheme of thought", and the word itself can be traced back to both the Latin '*theōria*' and the Greek '*theōriā*', meaning "contemplation, speculation, sight" (p. 916). By extension, the latter vision-related connotations might include the notion of *looking upon* or *seeing*. Indeed, visual and optical metaphors seem quite pervasive in language associated with the processes through which we come to know and understand our world and ourselves. Understanding tends to imply increased degrees of *clarity* rather than obscurity/opacity. We regularly characterize metacognition as an inherently *reflective* activity. It is not uncommon, in research practice, to speak of *looking at data through a particular lens*. We might talk of *surveying* a body of literature, of *putting 'something' under a microscope* for observation, or of *narrowing the focus* of our research. We even conceive of *concept images* and refer to *the mind's eye*. Such visual/optical metaphors are abundant.

As a result, I tend to construe theories concerning *the nature of mathematics* as *ways of "seeing" mathematics*. At the same time, I question if and how mathematics might constitute a theory of its own, such that it too may be understood as a particular *way of seeing*.[16] As broad a characterization as this might be, I nevertheless find it a useful/generative starting point; for how we *see* mathematics is intimately connected with the greater sense of what we think mathematics *is*. At the level of theory, the perspective I work to develop in this document is strongly motivated by the idea that a changed view of the material will also change one's view of the mathematical. By this, and in keeping with the spirit of Clarke's Law, I mean to say that re-evaluating and revising long-standing notions of what constitutes matter might also reshape the perspectives that inform our sensibilities about mathematics. This is a key part of my rationale for drawing on the

[16] Skemp (1987) directly asks the question, "What kind of theory is mathematics?" (p. 142). I present a response to his subsequent discussion in **Chapter 3**.

enactivist and quantum theoretical discourses when elaborating my overall mathematical worldview.

Research Approach and Methodological Considerations

This dissertation has emerged chiefly through the examination and consideration of literature from the interconnected domains of physics, mathematics, educational psychology, and philosophy. As a result, I am inclined to treat the various perspectives, suppositions, assertions, and commitments encapsulated by that literature as my “data”. That said, since I do not employ statistical tools or analytical methods associated with empirical investigation, my entire program of research, and the interpretations I put forward, might be classified as strictly theoretical. In accordance with the nature of the questions and curiosities that motivate my work, it also seems appropriate to acknowledge that this dissertation constitutes a *conceptually* motivated exploration more than it does a *methodologically* motivated one.

Driven by the aims/objectives outlined in **Chapter 1**, my engagement with literature has been geared toward the elaboration of a fresh perspective on the nature of mathematics and its relationship with matter. Out of necessity, this has meant engaging with *existing* perspectives on the nature of mathematics and its relationship with matter, and subsequently using these as a basis for comparison. Pursuant to the focal themes already identified (i.e., new materialisms and issues of dualism, and epistemological uncertainty), I attempt to represent key literary findings by foregrounding prominent perspectives whose influences are widespread and long-standing. For instance, the Cartesian dualism between mind and body exemplifies an omnipresent and persistent philosophical bifurcation whose effects are manifested across multiple domains. As one alternative, enactivist theory offers a less restrictive means of characterizing the embodied self. Similarly, Newtonian physics is still an extremely pervasive paradigm, despite its limited applicability, an inability to account for a range of known phenomena, and the articulation of a worldview that distances the observer from the objects and phenomena of interest. In contrast, its modern successors (i.e., relativity theory and quantum theory) not only express more accurate and inclusive material–mathematical relationships, but also establish paradigms that frame human beings as active (possibly even *enactive*) participants within emerging phenomena, such that we exist *in relation to the world* and not separate from it.

Other perspectives (discussed in **Chapter 3**) have remained highly influential within the realms of mathematics education and educational psychology, even in light of what I believe to be problematic philosophical commitments, and some notable inconsistencies regarding the nature of mathematics. By speaking to these perspectives and contrasting them with my own, I hope to provide some different insights in response to Wigner's deliberations about the *unreasonable effectiveness of mathematics*, and to better communicate/substantiate my overall sense of the fundamental relationship that exists between mathematics and the material world. In the interest of maintaining a manageable scope for this program of research, there will be limits to the rigour of my discussion; however, I aim to provide a well-rounded treatment that adequately addresses the identified themes.

When engaging with the literature for the purposes of comparison and critique, a number of conceptual tools have proven useful. Perhaps the most immediate of these has been attending closely to wording/commentary that *explicitly* addresses the nature of mathematics, the nature of matter, and/or the material–mathematical relationship. More demanding and more generative, though, has been the complementary task of attempting to reveal the *implicit*, or what is *not* said about these same topics. In general, this has meant "reading between the lines" of existing works in order to uncover deeper philosophical commitments that have not been overtly voiced by the authors, but nevertheless shape the sensibilities that are ultimately articulated. In the sense that what is *not* said can, at times, be just as informative as what *is* said, the implicit or unspoken offers a way of tempering the explicit, and of indicating other deeply rooted predispositions from which particular biases may have germinated. Thus, I attend to both the explicit and the implicit when possible.

A more general conceptual tool has been to continually "bounce" the literary perspectives I encounter off of the enactivist, quantum theoretical, and new materialist discourses, in order to see how they might compare to (or be reframed by) the conjoined perspective that

I propose. In addition to indicating points of similarity, this "bouncing" of ideas off of one another also illuminates areas of difference or contention, helping to identify ideational structures around which to build the necessary discussion. In cases when my own assertions differ from more established scholarly perspectives, my assessments of those perspectives are certainly not intended to be unduly critical or unfairly dismissive. Rather, I prefer to indicate how their underlying commitments have the *potential* to perpetuate problematic biases or outmoded sensibilities regarding the nature of mathematics, the nature of matter, and/or the material–mathematical relationship. Acknowledging such difference/dissensus is integral to this process of "bouncing" established literary perspectives off of the discursive spaces with which I am concerned.

As indicated in **Intralude A**, appealing to the power of metaphor as a tool for reorganizing meaning has also been valuable within the contexts of this exploration. It is by discussing the metaphorical alongside the literal that I hope to maintain a high degree of accessibility throughout the dissertation, particularly in light of the various domains of interest from which perspectives are drawn, and with which the reader may not be as familiar. Indeed, the overall structure of the document reflects this consideration, with the primary chapters communicating more literal and formalized content, and the secondary intraludes representing looser, more open-ended metaphorical treatments. Thus, the explicit, the implicit, the literal, and the metaphorical shall all factor into the coming chapters.

Drawing Upon Multiple Theories

Because I have opted to engage with enactivism and quantum theory simultaneously, and with that pairing being rather unorthodox within the discourse of mathematics education, I would like to be as transparent as possible about how these theories have informed my mathematical worldview. Although I can describe them both as being *integral* to that worldview, it is somewhat difficult to communicate precisely how these theories have ultimately been managed. In order to shed more light on that facet of my thought process, I again rely upon some useful metaphors.

In considering a conjoined enactivist/quantum theoretical perspective, it becomes clear that there are a number of different ways in which such a relationship might be manifested. It is, for instance, entirely possible to read into the nature of mathematics by "layering" these theories atop one another, as one might do with the compound lenses of a microscope when magnifying an object of interest. The reader may note that this would be in keeping with some of the optical/visual metaphors mentioned earlier, specifically the notion of *looking at data through a particular lens*. It occurred to me, however, that this layered-lens metaphor might be somewhat misrepresentative, for it evokes an optical principle that has the potential to be misleading in a research context. At least *mathematically speaking*, determining the overall magnification of a compound lens assembly is a purely *commutative* operation, in that the *order* in which the multiplication of magnifying powers occurs (i.e., m_1 x m_2 or m_2 x m_1) does not influence the resultant magnification value. That said, it would be a sizable oversight to extend an analogous assumption of commutativity to the ordering of one's *theoretical* lenses when conducting research as well. By expressing this, I draw attention to the likelihood that an enactivist perspective read through the lens of quantum theory could very well yield markedly different results when compared to a quantum theoretical perspective read through the lens of enactivist theory. At the very least, the reversal of priority would speak to vastly different research goals and a corresponding set of alternative research questions.

Recognizing the possibility of prioritizing these frameworks differently, or of filtering one through the other in an inverted hierarchical arrangement, I began to think of the conjoined frameworks in metaphorical terms more akin to a nesting of functions, or, rather, a *composition of functions*, where results can depend greatly upon the ordering of the composition, and where "*g(f(x))*" will not necessarily equate to "*f(g(x))*" for all, or indeed any, *x*. Pondering over how this composition metaphor might apply to the two theoretical frameworks with which I am interested has helped me to conceptualize how those frameworks might actually interact with one another. It seems to me that an enactivist perspective metaphorically composed with quantum theory, or "*e(q(x))*", could result in a rather new and interesting theory of *embodied cognition*. Conversely, it does not seem unreasonable to think that a quantum theoretical perspective metaphorically composed with enactivism, or "*q(e(x))*", could have the potential to yield a different theory of *matter*.

Indeed, some of the new materialist work already cited seems to engage with very similar ideas; however, neither of these metaphorical compositions aligns with my intended research aims.

Despite the importance of acknowledging the ramifications of ordering combined theoretical frameworks (or perhaps *because* of these considerations), I have come to realize that my intent is *not* to prioritize one theoretical framework over the other at all, but to read both frameworks *alongside* one another. More accurately, by looking to build out a theory concerning the nature of mathematics in the manner that I envision, I apply both the enactivist theory and the quantum theory to *it*, and not to each other. Believing that both of these monist sensibilities actually suggest very interesting things about mathematics itself, I would instead prefer to read mathematics through *them*. As a result, it could be said that *neither* theory is viewed as being subordinate (or in service) to the other. It is by attending to similar features and seeking out points of confluence/convergence/intersection that these theoretical frameworks are tentatively drawn together.

The Posthumanist Discourse and Diffractive Reading

As a final acknowledgment within this chapter, I briefly touch upon the choice *not* to engage with a specific subset of philosophical content, namely the posthumanist discourse, which some readers might be readily inclined to attach to aspects of this research. While the new materialist work of physicist and feminist theorist Karen Barad (2003, 2007) articulates an openly posthumanist stance with regard to *material agency*, I have already noted that I will not be engaging with agency in the current document. I also do not believe that the other facets of this discussion need necessarily invoke posthumanist language or assertions, particularly given that the new materialist discourse speaks to an ideological program that shares a number of thematic similarities with posthumanism (at the level of de-centering the human, for example, revising notions of the extended self, and of reconceptualizing human relationships and interactions within the material world).

In no way should this be taken as a slight against the posthumanist orientation, or as an attempt to disregard the relevance of posthumanist perspectives to the discussion at hand.

Rather, with the bulk of my exploration already stemming from the quantum theoretical and enactivist discourses, I fully expect that the introduction/inclusion of yet *another* discourse would severely overburden this dissertation, and very likely lead to the unwanted conflation of numerous issues. In essence, the intent is to lighten the conceptual load of this work by restricting my theoretical baggage to that associated with quantum theory and enactivism, without also taking on the additional encumbrance required for an earnest foray into posthumanism. In the event that the reader is not willing to accommodate this exclusion, I respectfully request the leeway to carry forward, after first acknowledging the likelihood that my future research will turn a more attentive eye toward a posthumanist treatment of the same subject matter.

The choice not to engage with posthumanism in the current document accompanies another methodological consideration that also bears relation to the work of Karen Barad, whose interdisciplinary extension of Bohr's quantum theory provided much early inspiration for the research presented in this dissertation. Strongly associated with Barad's work and her underlying theoretical stance is the methodological technique of *diffractive analysis*, an approach first proposed by Barad and fellow feminist theorist Donna Haraway, which makes a deliberate move away from standard optical metaphors (specifically reflection) as a means of characterizing the processes through which human beings come to know. In particular, Barad and Haraway's diffractive analysis renounces the oversimplifications of *geometric optics* and *ray-based reflection*, in favour of emphasizing the more accurate principles derived from *physical optics* and *wave-based interference phenomena*. As Barad (2007) and Haraway (2004) note:

> So unlike the phenomenon of reflection, which can be explained without taking account of the wavelike behavior of light […] diffraction makes light's wavelike behavior explicit. (Barad, 2007, p. 81)
>
> Diffraction does not produce "the same" displaced, as reflection and refraction do. Diffraction is a mapping of interference, not of replication, reflection, or reproduction. A diffraction pattern does not map where differences appear, but rather maps where the *effects* of difference appear. (Haraway, 2004, p. 70)

In a research context such as mine, where two theoretical frameworks are being read *alongside* one another so as to inform a single, broader worldview, it is likely that Barad and Haraway would advocate a diffractive analytical approach, emphasizing the significance of the interference that might arise from the interplay between the two frameworks. As Barad (2007) makes clear, her diffractive approach to reading social and scientific theories through one another differs significantly from more traditional methods.

> I am not interested in reading, say, physics and poststructuralist theory against one another, positioning one in a static geometrical relation to the other, or setting one up as the other's unmovable and unyielding foil. Nor am I interested in bidirectional approaches that add the results of what happens when each takes a turn at playing the foil, as it were […] my approach is to place the understandings that are generated from different (inter)disciplinary practices in conversation with one another. (pp. 92–93)

In light of my opening comments about Clarke's Law and the lasting influence of theory, I greatly value Barad's explicit move away from the more traditional analytical approaches that she describes above. However, while I believe that my own sensibilities are very much aligned with Barad's efforts to bring interdisciplinary practices into conversation with one another, I do not believe that I can fairly/rightly characterize my approach to theory as being *diffractive*. I say this largely because the possibility of reading literary perspectives diffractively came to my attention rather late in the process of my research, and I was not consciously working to apply such an approach from the onset.

By reading quantum theory and enactivist theory *alongside* one another, it would be very easy to *assume* that I have also been reading diffractively. As much as I would like to make the retroactive claim of this intention, it would be misrepresentative to do so, and such a claim would simply do a disservice to the very intentional/deliberate techniques of Barad and Haraway. At least within the current document, I focus more on exploring points of similarity between my two theoretical frameworks, whilst acknowledging relevant differences where applicable. I have not consciously been looking for interference effects; however, Haraway's notion of mapping *where the effects of difference appear* is a strategy that I anticipate employing in future research. My own method of "bouncing" ideas/perspectives off of one another seems to share similar characteristics, albeit in a much less sophisticated form.

In the event that Karen Barad or Donna Haraway engage with this document, I leave it to them to decide if the research approach described within this chapter adequately captures the essence of their diffractive analytical method, or if it is, as I suspect, simply demonstrating similar sensibilities without the same degree of depth. If some future examination of this work reveals that I have, indeed, read my two theoretical frameworks diffractively, then I look forward to amending these pages. For the time being, I certainly acknowledge the strong surface resemblance between my more tentative conceptual tools and the meticulous program entailed by Barad and Haraway's diffractive analysis. A *truly* diffractive analysis, in keeping with their approach, may enter into later explorations.

Through the offerings expressed in the preceding pages, the current chapter has aimed to provide the reader with more developed insight into various considerations that have informed this program of research, and the choices that have guided my engagement with literature. In conjunction with the introductory preamble and condensed structural overview of **Chapter 1**, it is hoped that this has adequately set the stage for the remaining chapters of this work. With this conceptual and theoretical backdrop established, I now enter into a more direct discussion of key perspectives regarding the nature of mathematics and the material–mathematical relationship. The viewpoints examined within the following chapter are intended to act as illustrative examples, indicating how a small sampling of prominent mathematical perspectives compare/contrast with my own. By the end of **Chapter 3**, the reader will have been exposed to a number of mathematical implications that arise from these influential viewpoints, with the notion of *mathematization* being used to operationalize the discussion.

Chapter 3.
Regarding Skemp and Wheeler

"Because mathematics is recognizable but not easily defined, we replaced it by a process or processes which can be made more tangible and that we named "mathematization". Hence the general title of this volume, THE AWARENESS OF MATHEMATIZATION."

–Caleb Gattegno (1988)
The Science of Education Part 2B: The Awareness of Mathematization, p. vii

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The above excerpt from Caleb Gattegno's posthumously published work draws attention to an interesting quality of mathematics, namely its elusiveness with respect to definition. As Gattegno points out, the difficulty in pinpointing exactly what mathematics *is*, in a definitional sense, leads us to look for processes that might help to reveal its nature (I paraphrase, of course). At a certain level, Gattegno's sense of needing to look elsewhere mirrors my interest in the *implicit*, and, like Gattegno, I also appeal to the process(es) of *mathematization* in order to tease out the elusive character of mathematics.

Skemp's Implicit Mathematical Worldview

Numerous perspectives concerning the nature of mathematics have been proposed within the discourse of our field. Some of these I have spoken to already; however, here, I highlight an especially salient one set out within the work of Richard Skemp. As shall be seen, Skemp's articulations are of particular interest because of the manner in which they present a tentative and somewhat unconventional perspective on the nature of mathematics whilst simultaneously being bound to problematic (classical) dualisms. Despite, or perhaps because of, certain internal inconsistencies inherent to Skemp's implicit mathematical worldview, his discussion ultimately highlights some foundational characteristics of mathematics that are worth attending to more closely. To be clear, Skemp's mathematical worldview is a singular example that I leverage in order to illustrate how classical modes of thought still pervade modern thinking within our field. Considering the prominence of Skempian thought in the more recent history of mathematics education, it also presents an opportunity to demonstrate the value of considering an alternative perspective. Thus, it is in the following section that I not only identify inconsistencies embedded within Skemp's mathematical worldview, but also aim to rectify those inconsistencies through an interpretation aligned with the sense in which I see mathematics as *the science of material assemblage*. This latter sensibility is only prefaced in this early chapter and shall be developed throughout the body of the dissertation, the intent being to provide a sort of anticipatory set for the reader, and a means of foreshadowing the discussions that appear in subsequent chapters.

There are various instances throughout the expanded American edition of his seminal work, *The Psychology of Learning Mathematics*, in which Skemp (1987) orbits around the nature of mathematics without speaking to it explicitly. Mathematical concept formation, for instance, is addressed within Skemp's Chapter 2. Discussions of mathematical symbolism and structure appear in Chapter 5, with the communication of surface structure and deep structure of mathematics treated much later in Chapter 14. With all of these subthemes helping to illuminate Skemp's mathematical worldview, his notion of what mathematics actually *is* takes shape as the book progresses, with less incisive remarks appearing in formative chapters and more telling comments in ensuing ones. As an example of the former, Skemp alludes to the nature of mathematics through *non-example* when he makes the following simple distinction: "The automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols, which is not mathematics" (p. 62). The subsequent discussion revolves around the meaning making that accompanies mathematical work, and while the point Skemp makes here is not especially revelatory when taken at face value (i.e., it is one that the vast majority of mathematics educators are likely to agree upon), it is his quiet assertion of what is *not mathematics* that poses something of a problem.

In reference to the language of quantum mechanics, which I would consider to be deeply and inherently mathematical, Wigner (1967) claimed that "quantum mechanics teaches us to store and communicate information, to describe the regularities found in nature" (p. 170). With a very similar sentiment driving his own perspectives on the nature of mathematical modeling, Campbell (2001, 2003) explores Wigner's statements further, and proceeds to draw the associated discourse out of the conventional realist and idealist philosophies that are often intermingled with matters of scientific and mathematical investigation. Through a radical enactivist approach to mathematical thinking and modeling (2001), he puts forward a view of mathematics as “the science of organization” or “the science of processes and structures” (p. 4), a view somewhat aligned with that proposed by Wigner. Both perspectives entail a representation of mathematics, and quantum mechanics by association, as means of managing or systematizing information in order to describe/characterize the structures that constitute the natural world.

When reinterpreted through a view of mathematics as the science of organization, or as *the science of material assemblage* (as I hope to substantiate), I would suggest that even Skemp’s (1987) “mechanical manipulation of meaningless symbols” (p. 62) *ought* to be considered mathematics (or at least mathematical). That is, in terms of the organization and reorganization of material processes and structures, even basic symbol manipulation unaccompanied by understanding is still recognizable as inherently mathematical activity (albeit activity that may not be very *meaningful*). To be fair, Skemp frames his own discussion around the psychology of *learning* mathematics, and his primary interest at the point from which the given assertion was excerpted is in the role that work with mathematical symbols plays in developing *understanding* of mathematical concepts. Thus, the distinction he makes between the automatic and the mechanical (the former being attributed to thinking, knowing mathematicians, and the latter to unknowing machines that are unaware of the processes they enact) is a convenient foreshadowing of the way in which he eventually adopts and applies Stieg Mellin-Olsen’s notions of relational and instrumental understanding; but it ultimately conflates the *understanding* of mathematics with the *nature* of mathematics.[17]

It seems to me that such conflation might stem from the very same quality singled out by Gattegno and reiterated within the opening quotation of this chapter: that *mathematics is*

[17] Skemp’s articulations about machines and mechanical manipulation are not only found in the 1987 expanded edition of *The Psychology of Learning Mathematics*, but in the original 1971 text as well (see Chapter 5 of both publications). This is notable given that computer-assisted proofs began to emerge in the intervening years, providing another impetus to question what might be considered valid mathematical activity.

recognizable but not easily defined. This almost seems to speak to a sort of *indeterminacy* with respect to the nature of mathematics itself, which then filters into/throughout the remainder of the mathematical discourse. Indeed, in their own discussion of aesthetics and the mathematical mind, Pimm and Sinclair (2006) point out that "neither of the terms 'mathematics' nor 'mathematician' have had stable or firm meanings – nor even constant resonance – across the ages" (p. 243), and it seems likely that this lack of stable meaning or constant resonance has also impacted the ways in which human beings have come to categorize/identify mathematical understanding.

In establishing *his* view of instrumental understanding, Skemp (1987) offers that: "I would until recently not have regarded [it] as understanding at all. It is what I have in the past described as 'rules without reasons' […]" (p. 153). The chapter in which this excerpt appears is a reprint of Skemp's article *Relational and Instrumental Understanding*, published in the December, 1976 issue of *Mathematics Teaching*. Its inclusion in the expanded 1987 edition of *The Psychology of Learning Mathematics* is a clear indication of shifts in Skemp's thought since 1971. Through his use of the qualifying phrase "until recently", he acknowledges the change to his own preconception, essentially conceding recognition that instrumental understanding is still a *form* of understanding, if only one that is less meaningful than the relational (i.e., less *full* of meaning). In light of this, I would suggest that Skemp's earlier statement about what is *not* mathematics requires a similar amendment. Granted, the "mechanical manipulation of meaningless symbols" may not be particularly *meaningful* mathematics, but it is mathematics nonetheless. Via my mathematical worldview, its grounding in processes of material organization and reorganization allows this, and making such an amendment would bring Skemp's earlier remark about the nature of mathematics more in line with his later assertions about mathematical understanding. Indeed, reading between the lines of Skemp's writing bears out the interesting entanglement between the material and the meaningful, both of which are closely linked to (and possibly emergent from) the mathematical.

Skemp's Type 1 and Type 2 Theories

> A *type 1 theory* is an abstract, general, and well-tested mental model of regularities in the physical world. It embodies what are sometimes called *laws of nature*, and to qualify for this description it must have explanatory and predictive power. (Skemp, 1987, pp. 129–130)

> A *type 2 theory* is a model of regularities in the ways in which type 1 theories are constructed […] It is a mental model of the mental-model-building process. (Skemp, 1987, p. 130)

Following his discussions of type 1 and type 2 theories in Chapter 10 of the expanded edition of *The Psychology of Learning Mathematics*, it is in Chapter 11 that Skemp (1987) probes more deeply into the heart of his mathematical worldview by posing to readers, and possibly himself, a crucial underlying question: "What kind of theory is mathematics?" (p. 142). It is notable that Skemp ultimately concedes that mathematics does not fit into *either* of the categories proposed above, and that the same query is reiterated elsewhere. Though the question of theory is not answered in any conclusive sense, Skemp's motivating educational concern *is* made clear: "[H]ow can we usefully think about teaching it [mathematics] if we do not know what kind of subject it is that we are trying to teach?" (p. 149). Even within the readership comprised of mathematics educators and teacher-researchers, I imagine that much of Skemp's intended audience might initially be taken aback by the posing of this question, if only momentarily; for a nondescript yet somehow-taken-as-shared sense of what mathematics is seems fairly commonplace/ubiquitous, and the question itself is rarely raised outside of documents like the current doctoral dissertation. Skemp (1987) further notes that theories about the *teaching and learning* of mathematics, as "theories about how we construct mathematical theories" (p. 149), should be distinguished from theories about mathematics itself. This is consistent with his working definitions of type 1 and type 2 theories given above, but it more importantly makes clear Skemp's conviction that a taken-as-shared sensibility about the nature of mathematics is not sufficient. This is a conviction with which I empathize, and the broader question from Skemp does align with some of my own reasons for perturbing common notions of what mathematics is; however, there are also ways in which Skemp's view diverges significantly from my own, or rather mine from his.[18]

Just as I have invoked Wigner's reference to the unreasonable effectiveness of mathematics in the natural sciences, Skemp (1987) draws upon a similar query attributed to Einstein: "How can it be that mathematics, as a product of human thought independent of experience, is so admirably adapted to the objects of reality?" (p. 149).[19] Though the heart of this question may be likened to that of Wigner's, I find both the *disembodiment* and *depersonalization* of mathematics within to be concerning, and I would submit that neither of these is tenable. Moreover, Skemp himself also refers to mathematics as "an activity of our intelligence" (p. 142). Though there is some value in this second claim, I amend it and the Einstein quotation quite strongly by noting that mathematics should not be seen as an activity of our intelligence *alone*. This is to say that mathematics is as much of the body as it is the mind; for both are integral to the coherent sense of being/becoming/knowing that underlies our embodied and unified experiences of mathematical structure.

de Freitas and Sinclair (2013) provide a powerful counterpoint to the Einsteinian and Skempian perspectives expressed above when they elaborate their notion of "the body in and of mathematics" (p. 454), and clarify that this new materialist approach to embodied cognition is one that "helps us rethink the body in/of mathematics so that embodiment entails mathematical concepts and artifacts, as well as human learners" (p. 468). The view they forward is one in which mathematics is neither divorced from experience, nor entirely driven by the intellect. Rather, mathematics is not only immanent in, but closely entangled with, the unified material self. As they note: "Instead of seeing [mathematical] concepts as entirely discursive or abstracted and dislocated from an inert matter, we have argued that activity should be studied for the way that new learner-concept assemblages emerge" (p. 468). Thus, by entangling the mathematical with the material, de Freitas and Sinclair shift

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[18] Much like the comments from Pimm and Sinclair offered earlier, Davis (1994) notes that "there is surprisingly little consensus on such "straight forward" topics as the nature of mathematics" (p. 4).

[19] As I understand it, Einstein first expressed his query in German, when delivering a lecture for the Prussian Academy of Science on January 27th, 1921. An expanded form of this address, translated into English and worded slightly differently than Skemp has noted, can also be found in Einstein's *Sidelights on Relativity*, first published in 1922 and reprinted numerous times since.

the emphasis away from the *purely* or *exclusively* intellectual, negating the sense of disembodiment and depersonalization that characterizes the Einsteinian and Skempian perspectives illustrated above.

Indeed, much of the present dissertation has been devoted to justifying a perspective in which mind and body are *not* taken to be ontologically distinct. Thus, to speak of mathematics only as a function of intellect, or as independent of experience, reverts us back to the classical dualisms that have proven so restrictive. To be fair, Skemp acknowledges that he does not fully answer Einstein's question; so I allow that there is a possibility he may not have been *entirely* committed to these dualisms, and that the overarching issue may be more closely related to, or grounded in, the dualism-preserving Cartesian phrasings that permeate the English language.

In close proximity to the excerpts discussed above, Skemp also claims that mathematics is not, itself, one of the natural sciences. While this could be considered true from a traditional perspective that maintains strict boundaries between the disciplines of physics, chemistry, biology, astronomy, geology, et cetera, it is inadequate within the material–mathematical worldview that I envision. It not only deprives mathematics of its primacy, but also elides any sense of mathematics being *natural*, of being *innate* to the structures and processes of matter. In contrast, from the perspective I am working to develop, it would actually be much more appropriate to conceive of mathematics (the discipline) as the *most foundational* natural science. If only to reiterate the decidedly Pythagorean roots underlying this sensibility, I will even suggest that *reinstating it* as such would not be at all unreasonable. In fact, this alternative view of mathematics as the most foundational natural science (i.e., what I refer to as *the science of material assemblage*) restores mathematics to a position of prominence that supersedes even the philosophy-physics of the atomists Democritus and Leucippus, who many consider to have laid the foundations for modern Western science.

Interestingly, Skemp (1987) does offer that mathematics:

> can be regarded as a conceptual kit of great generality and versatility, so valuable to anyone who wants to construct a scientific theory as to be almost

> indispensable. Did I say "almost"? Francis Bacon wrote: "For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity nor accommodated into use with sufficient dexterity without the aid and intervention of mathematics." Likewise, Jeans: "All the pictures which science now draws of nature and which alone seem capable of according with observational fact are mathematical pictures." (pp. 149–150)

Unlike Skemp's allusion to the almost-indispensable *conceptual kit of great generality*, which is reminiscent of Feynman's outrageous comment to Mark Kac at Caltech, the more evocative reflections from Bacon and Jeans suggest that mathematics is indeed much more fundamental than the natural sciences of which we normally speak, and the above excerpt consequently sets out a slightly inconsistent pairing. Bacon and Jeans would seem to be pointing toward the primary status of mathematics, or its significance/immanence as the underlying structure of material reality, whereas Skemp's characterization of it as a *conceptual kit* is more suggestive of a convenient toolset that simply *happens* to be incredibly robust/efficacious. This is to say that the latter speaks simply to the *coincidental utility* of mathematics much more than it does its *unreasonable effectiveness*. Again, it is notable that Skemp admits that mathematics does not fit the criteria for either of the type 1 or type 2 theories outlined earlier in his work. As he says of type 1 theories: "Collectively, they form the natural sciences" (p. 130). However, via his conceptual kit analogy and the quotes from Bacon and Jeans, Skemp also appears to be saying that the natural sciences are built upon mathematical foundations; yet, mathematics is not itself a type 1 theory. Nor is it characterized by Skemp as a *collection* of type 1 theories. Ultimately, I choose to favour the voices of Bacon and Jeans, which Skemp also gives more prominence than his own in the cited passage. Nevertheless, Skemp's overall message about mathematics comes across as somewhat mixed.

Revisiting the Question of Theory

So, as does Skemp (1987), let us also return to the inciting question: What kind of theory is mathematics? In spite of the fact that the previous inconsistencies are not addressed by Skemp himself, his exploration of this query continues. He ventures into a space to which his choice of language might not be entirely suited, yet it is worth quoting him at length.

> I suggest that we regard mathematics as a theory of a unique kind, having all the characteristics of a type 1 theory except mode 1 testing. It is the mental stuff of which type 1 theories are made; or to put this differently, it is pure form […] I have said on many occasions that I regard mathematics as a particularly pure and concentrated example of the functioning of human intelligence. This suggests that it is a kind of essence […].
>
> It is still hard to say why this should be. I still have not answered Einstein's question, so I offer the following as a beginning. Here is another quotation, this time from Galileo. "Where our senses fail us, reason must step in." With the help of our senses, we perceive regularities of our physical environment. These regularities are embodied in what I have called *primary concepts*. Next, by the use of our intelligence, we find regularities among these regularities – we form secondary concepts. In mathematics, we repeat this process to form more abstract concepts, representing regularities of great generality, and relations between these. All this time we are getting further away from what is accessible to our senses; yet, paradoxically, we seem to be getting closer to the essential nature of the universe. (p. 150)

As with his earlier description of mathematics as an activity of our intelligence, Skemp still adheres to an exclusively mental characterization, belying the classical underpinnings of his perspective, and a bias toward (if not a compete commitment to) Cartesian dualism. That said, his interpretation of Galileo in the final paragraph reworks the perspective by speaking to a more circulatory exchange between sense and intellect, and a more complementary interplay between inner and outer experience. The repetition of the process that he describes here is in keeping with the entangled sense of being/becoming/knowing that I have come to align with the enactive monist approach to cognition. In my reading of Skemp, it is the curious combination of his classically dualistic statements about the exclusively mental character of mathematics, and the subsequent assertions regarding abstraction and generality that suggest he might have one foot planted on fairly classical Cartesian ground, with the other inching toward terrain where the ontological distinctions already addressed hold less sway.

At least with respect to his articulations about the nature of mathematics and its relation to the material world, I sense a sort of indecision on Skemp's part. In his discussions of abstraction, he clearly indicates that, "'more abstract' means 'more removed from experience of the outside world', which fits in with the everyday meaning of the word 'abstract'" (1987, p. 14). He also notes that mathematical concepts are "far more abstract than those of everyday life", such that the "communication of mathematical concepts is therefore much more difficult" (p. 14). At a certain level, his concession that mathematics is neither a type 1 theory, nor a type 2 theory, but a *theory of a unique kind*, *the stuff of which type 1 theories are made*, or *pure form* raises even more questions than his primary inquiry into what mathematics is. While it does seem reasonable that mathematical abstraction initially draws us further from the everyday experience of the outside world (i.e., into the idealized spaces of the inner world), I would also suggest that the subsequent capacity to generalize from abstractions and to use generalizations as the bases for *further* abstractions (possibly under different contexts) ultimately brings us right back to the outer world again, and I hesitate to break apart these two processes, preferring instead to envision something cyclic or circulatory and potentially even self-perpetuating.

For Skemp, it is paradoxical that the move away from sensory experience in the outer world into the abstracted inner space of the intellect should draw us closer to the essence of the universe (I have paraphrased heavily, here). However, from a perspective aligned with the worldview I wish to advocate, there is really no need to consider this circumstance as paradoxical at all; for just as mind and body are ontologically unified in this worldview, so too are inner and outer experience. From a *classically grounded* perspective, there may be a certain irony accompanying the sense that higher order abstraction distances us from the outer world whilst simultaneously bringing us closer to our experience of the inner world; but, from the conjoined enactivist/quantum theoretical perspective that underlies my broader mathematical worldview, that irony dissipates completely. Even reconceptualising Skemp's notion of schema such that it is not driven solely by intellect should not be overly problematic, particularly in light of the manner in which scholars like de Freitas and Sinclair re-imbue the cognitive *with* the material. Essentially, Skemp's paradox is only paradoxical when one's worldview is constrained by classical modes of thought.

Other scholars whose works I reference in this dissertation also speak to matters of paradox and contradiction, expressing potential "remedies" to such issues which are entirely apropos to the current discussion despite the differing contexts in which they appear. Lakatos (1976), for instance, furnishes us with a lighthearted tête-à-tête between his imaginary pupils SIGMA and PI, who debate the meaning of the term ‘polyhedron’ in *Proofs and Refutations: The Logic of Mathematical Discovery*.

> SIGMA: I think your views are paradoxical!
>
> PI: If you mean by paradoxical ‘an opinion not yet generally received’, and possibly inconsistent with some of your ingrained naïve ideas, never mind: you only have to replace your naïve ideas with the paradoxical ones. This may be a way to ‘solve’ paradoxes. (p. 91)

Maturana and Varela (1992) articulate a very similar idea when discussing the difficult philosophical problem of navigating the sharp edge between representationism and solipsism, and they share the following perspective in *The Tree of Knowledge: The Biological Roots of Human Understanding*.

> The solution, like all solutions to apparent contradictions, lies in moving away from the opposition and changing the nature of the question, to embrace a broader context. (p. 135)

Both of these excerpts are striking, not only because of the dry wit and subtle sarcasm of the former or the unembellished succinctness of the latter, but because they openly acknowledge the constraints that are imposed (at times unknowingly) by entrenched worldviews. Read together, this trio of scholars (Lakatos, Maturana, and Varela) seems to indicate that navigating beyond paradox can be achieved by a shift in paradigm. Lakatos’ *opinion not yet generally received*, and Maturana and Varela’s move to *embrace a broader context* by *changing the nature of the question* convey the value inherent to such shifts in mindset and, as I have tried to illustrate above, a similar shift in mindset might be useful in reframing the paradox characterized by Skemp.

Even Einstein’s question of how mathematics can be *so admirably adapted to the objects of reality* can be revised in a similar way. In fact, I completely co-opt Einstein’s phrasing by asserting that mathematics need *not* actually be “adapted to the objects of reality”; for it is *already* innate to those very same objects (i.e., it is immanent in matter). Rather, it is the viewpoint that *divorces* the mathematical from the material in which the ontological problems rest. As *the science of material assemblage*, mathematics can instead be described as the essential framework upon which material reality is built, from which its objects emerge, and according to which they (co-)evolve. In the view I put forth, mathematics is the *embodiment* of the underlying structures of material reality, and of the dynamic principles according to which those structures organize and reorganize themselves. This is to say that mathematics embodies the very principles according to which *matter* organizes and reorganizes itself, and in that sense, the unreasonable effectiveness of mathematics spoken to by Wigner is not unreasonable at all. In this view, mathematics not only *underlies* the material structure of reality, but also *encapsulates* the conditions and constraints through which the dynamic processes of material assemblage are manifested.

Wheeler's Notion of Mathematization

As a comparative case, I now consider a perspective elaborated by David Wheeler, whose work on *mathematization* extended across much of his career in mathematics education. The specific commentary from Wheeler that interests me also invokes notions of structure and organization, but moves along a trajectory that may only be parallel to my own, rather than intersecting with it. Acknowledging that his view of mathematization continued to evolve throughout his career, Wheeler drafted a condensed lineage of his thoughts on the matter, highlighting certain notable attempts to characterize the phenomenon of mathematization along the way. Of particular significance is the following stance, which Wheeler (1982) identifies as having been worked on during the Canadian Mathematics Education Study Group (CMESG) in 1977:

> We may think of a young child playing with blocks and using them to express awareness of symmetry, of an older child experimenting with a geoboard and becoming interested in the relationships between the areas of the triangles he can make, an adult noticing a building under construction and asking himself questions about the design, etc. We notice that mathematization has taken place by the signs of organization, of form, of additional structure, given to a situation. I use these tenuous clues to suggest that mathematization is the act of *putting structure onto a structure*. (p. 47)

There are two facets of this excerpt that I wish to deconstruct, although I do so in the reverse order of their appearance. One of these is the significance of the term *awareness* in the opening sentence above. This I shall return to later. The other more prominent facet, which

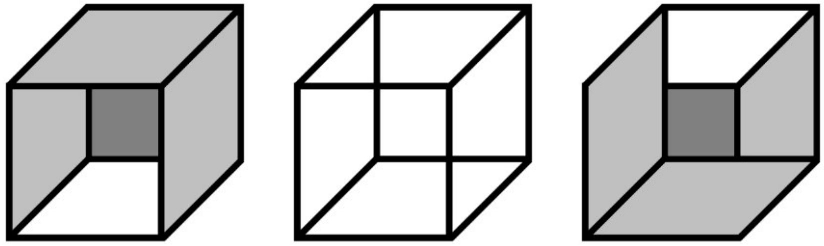
I shall speak to first, is Wheeler's "definition", or metaphor, of mathematization as the act of *putting structure onto a structure*. This description is both intriguing and troubling. At one level, it directly *implicates* the mathematical in the perception of the material, yet it also *relegates* the mathematical to an external position, or one in which it is somehow distanced or removed from the material. This is concerning in much the same way as Skemp's claim that mathematics is not, itself, one of the natural sciences. As in that case, this also suppresses the sense of mathematics being innate to the structures of matter, and serves to *detach* mathematics from the "objects of reality" (to call back Einstein's phrasing as well). So, while Wheeler's allusions to structure may initially suggest a close connection to my own attempt to characterize mathematics as *the science of material assemblage*, I would argue *against* making such a connection.

To be clear, I reiterate that my own work seeks to shed light upon the *nature of mathematics*, while the quoted passage from Wheeler characterizes the *nature of mathematization*. By pointing this out, I do not suggest that our discussions are entirely unrelated; rather, I wish to signal the recognition on my part that our objects/phenomena of interest differ slightly. Still, though the focus of Wheeler's interest is not identical to my own, his discussions do reveal tacit assumptions about the nature of mathematics (just as Skemp's discussion of the psychology of *learning* mathematics does). By this, I mean to say that delving into the nature of mathematization *also* exposes implicit (even explicit) commitments regarding the nature of mathematics.

I share Wheeler's sense that mathematization might be noticed, or recognized, via *signs of organization*, *of form*; however, from the point of view that he expresses, mathematization also involves applying *additional structure* (i.e., additional *mathematical* structure) to a given situation, suggesting that the structure *onto* which mathematics is added or layered is not itself mathematical. Thus, a question arises as to the fundamental nature of the underlying base structure. To phrase this differently, if mathematical structure is to be conceived of as an *additional* structural layer that is placed overtop that of reality during the process of mathematization, then what is the base structure of reality itself, and why should a mathematical overlay reveal anything of consequence about it? From what I have encountered thus far, Wheeler does not cast any light upon these questions. In essence, this is the way in which Wheeler's perspective also elides any sense of mathematics being "natural", and it illustrates a key manner in which his notion of mathematization as the act of *putting structure onto a structure* is incompatible with my own sensibilities about mathematics as *the science of material assemblage*. Upon *first* encountering Wheeler's approach to mathematization, I admit to having found it quite agreeable; yet, this particular quandary has since caused me to distance myself from it.

Were I to adopt Wheeler's notion of layering structure upon structure, I would lean toward characterizing mathematics itself as the base structure atop which *other* structures might be added or layered (for example, linguistic structures, notational structures, social structures, et cetera), but I am not yet convinced that this would be sufficient to bring Wheeler's perspective and my own together. Wheeler's notion of mathematization emphasizes the *imposition of external structure*, whereas my view of mathematization would emphasize the *recognition of innate structure* (or at least an increased cognizance of it).

Consider, for instance, the well-known optical illusion instantiated by the ambiguous figure of the Necker Cube, originally devised by crystallographer and geographer Louis Albert Necker. In its most basic/traditional representation, the Necker Cube is simply a wireframe line drawing where no (internal or external) visual cues for depth or orientation are apparent. This is to say that no shading, texture gradients, occlusions, foregrounding or backgrounding of features, motion or parallax information, positioning with respect to other figures, et cetera, are rendered in the drawing (Anderson, 2000). As an ambiguous, or gestalt figure, the Necker Cube lends itself to multiple visual interpretations, much like those below.



(a) (b) (c)

Figure 1: An illustration of the standard Necker Cube (centre) with two alternate perspectives exaggerated for effect (left & right)

Shown as a basic line drawing, the Necker Cube is fundamentally a flat, two-dimensional figure (possibly seen as a hexagon with crisscrossing lines in its interior); however, the perception of three-dimensional depth within the figure is common. It is the innate potential for the figure to assume any of a number of stable, yet incompatible, states when perceived that demonstrates its ambiguous character. As an example, perceiving the figure as an "open-faced cube angled downward and to the left" (as in (a) of Figure 1 above) is incompatible with the perception of it as an "open-faced cube angled upward and to the right" (as in (c) of Figure 1 above); for each of these relies upon a different, and largely exclusive, set of orientation cues. Though depth cues remain invariant across these two resolved states, specific faces of the cube are taken to be more prominent in each instance and not the other. As noted in Anderson (2000):

> the two-dimensional image is consistent with an infinite number of three-dimensional realities. For instance, the lines that look like a cube could actually represent a flat pattern on a page. To overcome this ambiguity, assumptions must be made about the likely cause of a given image, and the different cues to depth must be combined into a single, three-dimensional representation. (p. 479)

While Wheeler might agree that the drawing is inherently two-dimensional, I imagine that he would also frame the possible three-dimensional interpretations of it in terms of putting specific mathematical structures *onto* the flat drawing (namely mathematical structures associated with *depth* and *perspective*). Thus, *mathematizing* the drawing, for Wheeler, would include the imposition of external mathematical structure that is somehow different from the existing structure of the drawing itself. As already noted, I consider this problematic, and would be more inclined to treat mathematics as the *foundational structure* underlying the entire scenario, and as being immanent within the so-called ambiguous figure.

Of particular importance when considering the example of the Necker Cube is that the drawing itself does not "favour" any perceptual state over another. The core sensory information that can be gathered by looking at the drawing does not change; rather, it is the *perception* of that sensory information that has the potential to vary. In the aforementioned Figure 1, two common three-dimensional interpretations of the Necker Cube have been exaggerated for effect. In the interest of clarity, certain distinguishing lines and occlusions have been added to the basic line drawing; but these three-dimensional interpretations are known to emerge directly from the image (for some observers) even without the artificial foregrounding and backgrounding of features. As a *drawing*, the Necker Cube is inherently two-dimensional, but the mathematics that underpins the figure supports the potential for both two- and three-dimensional interpretations (perhaps even four-dimensional, if one includes the possibility of shifting *between* representations over time), and those respective forms emerge when certain perceptual predispositions manifest themselves.

Indeed, it could be argued that the ambiguous figure of the Necker Cube only presents an *optical* illusion when it is *being perceived* in some manner or another, but the figure does not specify which interpretations emerge at any given time for any given observer perceiving its mathematical structures. Rather, it is the observer/perceiver that *organizes* these structures by favouring a given perspective over another, and *reorganizes* the same structures when shifting between perspectives. I readily acknowledge that the *visual language* of depth perception and orientation could be taken as additional structure layered overtop the mathematical structure of the Necker Cube drawing; but this would not constitute mathematization in a Wheelerian sense; for it reconceives the mathematical structure as being immanent within the figure, and not external to it. I proffer that is this already-extant structure which supports the multitude of potential interpretations. So, while Wheeler has framed his notion of mathematization through the metaphor of *imposing structure*, I would instead suggest reframing mathematization via the metaphor of *exposing structure*. It is the capacity to recognize or attend to this exposed structure, and to develop an *awareness* of the potential to organize/reorganize it that is, in my way of thinking, a more appropriate, way of characterizing the phenomenon of mathematization.

Referring to the Necker Cube as an *ambiguous* figure (as in Anderson, 2000), or even as a *gestalt* figure, captures part of its relevance to the current discussion, but not the entirety of it; for both of these terms characterize the Necker Cube from a primarily *psychological* standpoint. Drawing from the conjoined enactivist/quantum theoretical perspective that informs my mathematical worldview, I would suggest that the ambiguity of the Necker Cube has a much more fundamental root, in particular one that is grounded in its underlying *material indeterminacy*. One certainly might ask if any particular perception of the Necker Cube is mathematically *primary* to the others. In my reading of Wheeler's notion of mathematization, it seems likely that he might characterize the two-dimensional hexagonal interpretation as mathematically primary, with the three-dimensional cubes as secondary interpretations (in that perceiving them would require the imposition of additional layers of structure). There may be a certain utility to this perspective, but it still does not account for any sort of base structure in a way that I would find satisfactory. To quote Anderson (2000) again:

> One of our most remarkable perceptual capacities is our ability to recover the three-dimensional structure of our environments. All of our actions rely on the ability to recover information about the positions, shapes, and material properties of objects and surfaces as they exist in three-dimensional space. (p. 476)

A drawing of the Necker Cube may be resolved (through perception) into a number of different and incompatible states, including those already described; yet, it may also, arguably, occupy/inhabit an indeterminate middle-ground, where no single state is clearly resolved and no stable determination is made of its dimensional characteristics. This is to say that a viewer of the image (which I will here take to be a thinking, knowing, sensing human being), may not actually *know* or even *understand* what it is they are looking at. They may be unclear which visual information to *recover* from the figure, in the sense that Anderson uses the term above, or they may even find the visual information too ambiguous/confusing to resolve into a single stable image state. It is this unresolved scenario, or the indeterminate middle-ground, that I would consider mathematically primary to the others. Subsequently, I would also suggest that it is the Necker Cube's fundamental indeterminacy at the *material* level that gives rise to its interpretational ambiguity at the *perceptual* level. In a manner of speaking, this helps to illustrates the sense in which dynamic processes of material assemblage (i.e., the configuration and reconfiguration of matter) could be seen to motivate acts of mathematization.[20]

Wheeler's Appeal to Awareness

Though Wheeler (1982) discloses his "lessening of confidence" (p. 47) in the notion of mathematization as the act of *putting structure onto a structure*, he does revisit this metaphor in later works as well. In a piece published posthumously in the pages of *For the Learning of Mathematics* in the year following his passing, Wheeler (2001) informs the reader that he still has "no solid conclusions to offer" regarding mathematization (p. 50), yet it is here that his evolving discussion of structure also enters into a more thorough treatment of a related theme that was not featured as prominently in the 1982 piece previously cited (namely the theme of *awareness*). Thus, it is at this juncture that I finally return to that other facet of the extended Wheeler quotation given earlier and partially repeated below; but I now draw further attention to its relevance by also offering up the statement that immediately precedes the excerpt I have already discussed. To reiterate Wheeler (1982):

> Although mathematization must be presumed present in all cases of "doing" mathematics or "thinking" mathematically, it can be detected most easily in situations where something not obviously mathematical is being converted into something which obviously is. We may think of a young child playing with blocks and using them to express awareness of symmetry, of an older child experimenting with a geoboard and becoming interested in the relationships between the areas of the triangles he can make, an adult noticing a building under construction and asking himself questions about the design, etc. (p. 47)

As already expressed, there are ways in which my sensibilities about what constitutes mathematization differ from Wheeler's, but it is interesting to note that I am in full

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[20] It is in the upcoming **Chapter 4**, which engages directly with the history and development of quantum theory, that I more fully explicate the concept of material indeterminacy.

agreement with his assertion that mathematization *must be presumed present in all cases of "doing" mathematics or "thinking" mathematically* (with the addendum that it is conceived of as *exposing innate mathematical structure*). That said, I have also tried to be clear that my sense of what mathematics *is* differs from Wheeler's as well. At first glance, his opening remark above might be taken as implying that there is a clear distinction between situations that are *obviously mathematical* and ones that, by their very nature, are not. This would be to say that only certain things *are* mathematical by nature. However, I do not believe that this is the intended thrust of Wheeler's remark; for the qualifying term "obviously" plays a doubly important role in the assertion and could be seen as shifting the discussion in a slightly different direction. By referring to situations that are or are not *obviously* mathematical, I believe that Wheeler is articulating a distinction based upon the *awareness* of mathematical structure as opposed to one based upon an innate ontological difference between the situations themselves (i.e., the mathematical ones and the non-mathematical ones). Certainly, what is *obviously mathematical* to a given individual may not be to another, and Wheeler's distinction takes on the role of a highly subjective qualifier. It connotes an awareness that is deeply grounded in perception of the material, and not simply grounded in abstractions of the intellect alone. The subsequent references to the child playing with blocks and expressing an *awareness of symmetry*, and the older child *experimenting with a geoboard and becoming interested* in certain relationships also speak to this interpretation.

In fact, while Wheeler's early characterization of mathematization as the act of *putting structure onto a structure* is troubling for the reasons already put forth, this introductory preamble concerning the imagined mathematical play and explorations of children and adults carries similar overtones to the senses in which I see mathematics as *the science of material assemblage*, and mathematization as the capacity to recognize/expose and attend to innate mathematical structure (with an awareness of the potential to organize/reorganize that structure). By this, I mean to suggest that it is not the "situation" that is any more or less mathematical; rather it is the awareness of the mathematics *embedded* in a given situation that manifests to greater or lesser degree. As I have read it, a portion of Wheeler's 2001 paper would seem to support this stance, for he attaches a very similar sensibility to his notion of imposed structure.

> In analyzing acts of mathematisation, it is not particularly difficult to describe the structures that the mathematiser imposes, but it seems much more difficult to get close to the 'reasons why', to the source of the particular decisions that the mathematiser takes. It seems clear that the energy source that powers the structuring activity is awareness – awareness of some feature of the situation that suggests the structure to be imposed. (p. 51)

As I have previously stated, and as the above excerpt reiterates, Wheeler's approach to mathematization necessitates *imposing external mathematical structure*. In contrast, my own approach is more in keeping with the idea of *exposing innate mathematical structure*. Nevertheless, the senses in which we both speak to the role of *awareness* in processes of mathematization are surprisingly similar/simpatico. In fact, I am tempted to adopt the structure of Wheeler's passage, and modify its phrasing ever so slightly, if only to consider the implications that doing so might have for the broader mathematical worldview I am working to establish. The reader is asked to consider the following adjusted passage in light of the discussion thus far:

> In analyzing acts of mathematisation, it is not particularly difficult to describe the structures that the mathematiser **exposes**, but it seems much more difficult to get close to the 'reasons why', to the source of the particular decisions that the mathematiser takes. It seems clear that the energy source that powers the **structuring and restructuring** activity is awareness – awareness of some feature of the situation that suggests the structure to be **exposed**. (Modified from Wheeler, 2001, p. 51)

These minor modifications are certainly not sufficient to bring our two perspectives together completely, but the resulting similarities are notable, and at least suggestive of reconciliatory possibilities. In a manner of speaking, favouring *innate* mathematical structure over *imposed* mathematical structure not only reverses the flow of Wheeler's mathematization, but also emphasizes that this innate structure is the very same structure in which the situational awareness originates and around which it might expand. As with my reading between the lines of Skemp's work, this also seems evocative of an entangled relationship between the material and the meaningful (i.e., where meaning has its origin in an awareness of the innate mathematical structures of matter). Is it possible that reworking Wheeler's perspective in this way could bring something more conclusive into his view of mathematization? Might reinstating the primacy of mathematics as innate to the structure of matter allow Wheeler to more adequately address his lessening of confidence in the metaphor of mathematization as *placing structure onto a structure*? Were I, in some way, to have the luxury of conversing with Wheeler in person, I like to think that he might entertain this possibility.

Notably, some additional remarks from Wheeler do suggest that this could be a viable alternative. Within the same exploration of mathematization, Wheeler (2001) incorporates a discussion of *generalisation*, which he first describes as a "well known 'process' word frequently applied to mathematical activity" (p. 51). Soon thereafter, though, he also specifies that:

> Generalisation is an instance of putting a structure onto a structure (even though the general is less highly structured than the particular). The generaliser structures the situation by becoming aware of those relationships that define the 'essence' of the situation and emphasising them to the exclusion of all others. (p. 51)

It is this slightly different appeal to awareness, accompanied by Wheeler's mention of *the 'essence' of the situation*, that leads me to wonder if his lessening of confidence might have resulted from the manner in which his earlier approach to mathematization *displaces* mathematics, *distancing it* from the material rather than allowing it to *emerge* from the material. Were he to instead reassert the primacy of mathematics in the manner that I propose, perhaps some greater confidence might be possible. As a case in point, Wheeler uses somewhat vague terms when he refers to *those relationships that define the 'essence' of the situation*, and I suspect that this might also be due to the unavoidable disconnect his metaphor enforces between mathematical structures and the situations onto which he imposes them.

Under my own mathematical worldview, the *relationships that define the 'essence' of the situation* might actually be reframed as the very same principles that motivate the dynamic processes through which matter organizes and reorganizes itself. Exactly how it is that these principles operate is something about which I may only speculate as of yet, but a major portion of the current dissertation has formed around the assertion that such principles are inherently mathematical in nature. Recall that Skemp (1987) also spoke to the nature of mathematics by regarding it as "a kind of essence" or "pure form" (p. 150). By reading this insight into Wheeler's reference to the *'essence' of the situation*, I attempt to erect a bridge between Wheeler's perspective and my own, and in so doing, I wish to assert that mathematics itself *is* the essence of the situation.

There is certainly a sense in which the natural laws of physics could be seen as expressions of the motivating principles described above, or perhaps as reifications of them. Indeed, a pertinent example from classical physics might be the mathematical formulation of Newton's second law, F = ma, which expresses the direct proportionality between the net force 'F' (in Newtons) applied to a given mass 'm' (in kilograms) and the acceleration 'a' (in metres per second squared) that the mass experiences. Casper and Noer (1972) make the interesting observation that, while it is generally referred to as a *law*, the equation "is in fact a definition, and hence has no physical content by itself. Essentially it is a prescription for how to go about studying the interactions of matter" (p. 247). They also note that its fundamental importance to physical theory is "as an organizing principle" (p. 247), for "it singles out the relevant questions to ask and the kinds of concepts that are appropriate to use in answering them" (p. 247). Though this interpretation speaks more to the role that physical theory plays in organizing and facilitating the process of *inquiry*, it does share some similarities with the broader worldview I seek to explicate. However, with classical Newtonian theory already known to be quite limited in its descriptive capacity (compared to more modern scientific worldviews), it seems likely that *other* organizing principles may live much closer to the '*essence' of the situation* and the *pure form* to which Wheeler and Skemp respectively allude.

This would align with my suggestion that Wheeler's mathematization metaphor might be less troubled if mathematics *itself* could be seen as the base structure atop which *other* structures might be added or layered. In fact, Wheeler (2001) does put forward a tentative hypothesis wherein additional structures associated with "language, notation, graphical representation and imagery" could be conceived of as "facilitating awareness" (p. 51). As he says, they seem like "carriers of awarenesses, or at least like media which facilitate the achievement of awarenesses because they mediate between the mathematiser and the situation he is mathematising" (p. 51).[21] His perspective is still driven by the imposition of *external* mathematical structure, yet the above quotation is suggestive of the ways in which language, symbolism, and a host of other representational systems might be of great use in helping the mathematiser to *become more aware* of the structures that are of interest. I believe it is also quite telling that Wheeler invokes the plurality of 'awarenesses' as opposed to the singular 'awareness', as this could, once again, be seen as alluding to the great power of metaphor as a tool for reorganizing meaning. I encourage the reader to consider that it may be a kind of metaphorical bridging, or transition, between structural representations that facilitates the multiple awarenesses of which Wheeler speaks.

From the literature I have explored, Wheeler's approach to awareness appears to have been strongly influenced by Gattegno. However, as Hewitt (2001, 2009) points out, it would seem that Gattegno never actually put forth a clear definition of the word 'awareness'. To a certain degree, Hewitt (2001) leaves it to the reader to decide if his own discussion yields further clarity. Acknowledging that I have not provided a conclusive definition for 'awareness' myself, I make a concession similar to Hewitt's, although I also note that I am currently content to evoke the *everyday* connotations associated with 'awareness', alongside the brief characterization I have communicated in the previous pages. For the time being at least, I prompt the reader to draw upon their own sensibilities about awareness to see if doing so might better help them to understand my rationale for moving away from Wheeler's view of mathematization as the act of *putting structure onto a structure*.

To reiterate, whereas Wheeler (1982) claims that we "notice that mathematization has taken place by the signs of organization, of form, of additional structure, given to a situation" (p. 47), I instead propose that, "we notice that mathematization has taken place by *recognizing* the signs of organization, of form, of *internal* structure, *innate to* a situation". Without preserving Wheeler's sentence structure, I might articulate this view slightly differently (i.e., as I have done throughout this section); however I mirror his language here in order to emphasize the contrast. There is a chance that Freudenthal's brief commentary *Why to Teach Mathematics So as to Be Useful* (1968) might offer insights that

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[21] This is not so different from the manner in which Casper and Noer (1972) characterize the fundamental importance of Newton's second law as an *organizing principle*.

could help to reconcile the lingering differences between Wheeler's perspective and my own; for it is there that Freudenthal raises the compelling question of whether it is possible to *mathematize mathematics*. In response to this question, Freudenthal (1968) articulates aspects of his own material–mathematical worldview via the following claims:

> Mathematics is a peculiar subject. Arithmetic and geometry have sprung from mathematizing part of reality. But soon, at least from the Greek antiquity onwards, mathematics itself has become the object of mathematizing. Arranging and rearranging the subject matter, turning definitions into theorems and theorems into definitions, looking for more general approaches from which all can be derived by specialization, unifying several theories into one – this has been a most fruitful activity of the mathematician […] (p. 6)

The reader might notice that Freudenthal's use of the phrase *arranging and rearranging the subject matter* is somewhat reminiscent of Deleuze and Guattari's characterization of assemblage as "the process of arranging, organizing, fitting together" (Wise, 2011, p. 91). Similarly, his notion of *unifying several theories into one* would not be out of step with their concept of a dynamic process of becoming "that brings elements together" (Wise, 2011, p. 91). Indeed, the entire passage above seems to speak to the dynamism inherent to mathematical being/becoming/knowing, and the means through which different layers of mathematical meaning might emerge through the reorganization of existing structures. Slightly later in the same work, Freudenthal (1968) also communicates the need for human beings to learn mathematics not as a closed system, "but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (p. 7). Admittedly, I am inclined to suggest that reality is *already* mathematical, such that it does not actually need to be *mathematized*, at least in the sense represented by Wheeler's perspective. However, if by *mathematizing reality*, Freudenthal actual means *exposing innate mathematical structures*, and *developing an awareness of the potential to organize/reorganize those structures*, then I would agree with his assessment. Of course, the notion of drawing mathematics out of an innately mathematical world speaks to a fundamentally different mode of engagement than the notion of imposing/forcing mathematics onto an innately *non-mathematical* world, and it is not clear to me at present how Freudenthal might align himself with respect to these conflicting stances.

Still, even without over-extending the connections identified above, or drawing *too* heavily upon the power of metaphor, it seems to me that Freudenthal's comments about *mathematizing mathematics* accord quite nicely with the assemblage theory of Deleuze and Guattari. Although I find it necessary to revise Wheeler's approach to mathematization by reversing the directionality of his process, Freudenthal's insights seem to present a means of tentatively spanning the remaining gap between Wheeler's perspective and my own. Moreover, the manner in which Freudenthal's notion of *mathematizing mathematics* harkens back to the assemblage theory of Deleuze and Guattari captures a key aspect of the sense in which I re-envision mathematics as *the science of material assemblage*.

Throughout this chapter, an exploration of the implicit mathematical worldview of Richard Skemp (1987) has revealed a number of inconsistencies stemming from tacit assumptions about the nature of mathematics. By identifying and addressing these inconsistencies, I have attempted to provide an alternative interpretation aligned with my own worldview. This alternative interpretation has been discussed alongside David Wheeler's (1982, 2001) view of mathematization as the act of *putting structure onto a structure*, which also speaks to a number of underlying assumptions regarding the nature of mathematics. Despite broader assertions made by both Skemp and Wheeler that distance the mathematical from the material, aspects of their respective commentaries do appear to be seeking out new ways of accounting for the *essence* of the objects and situations about which we aim to develop mathematical understanding.

In large part, the discussion surrounding Skemp and Wheeler has aimed to clarify the sense of mathematical primacy I wish to advocate, as well as illustrate the discursive spaces into which I extend my mathematical worldview. In addition to serving as examples against which to compare/contrast my sensibilities about material assemblage, the cited works have also been instrumental in indicating that a view which reconceives of mathematics as the *most foundational* natural science (i.e., *the science of material assemblage*) has the potential to rectify some of the inconsistencies stemming from classical philosophical commitments. As in the case of the Necker Cube, I have even expressed that aspects of the mathematization process might be rethought by appealing to an underlying material indeterminacy.

With the notion of material indeterminacy rooted in the domain of quantum theory, this is an appropriate juncture to provide a more focused treatment of key quantum theoretical principles that have bearing on this dissertation. To that end, **Chapter 4** presents a condensed historical overview that should help to contextualize the significance of the worldview change that accompanied the early development of quantum theory, as well as brief introductions to key quantum theoretical principles salient to the overall themes of this document. It is hoped that the current chapter's engagement with selected insights from two well-known mathematics education theorists whose works are *already* widely influential in the field of mathematics education will provide a counterbalance to the more technical elements of the upcoming discussion. Before shifting into that adjacent material, though, I offer the reader a small palate cleanser in the form of another intralude.

Intralude B | Entanglement

A Commentary on Linguistic Commitments and the Emergence of New Discursive Spaces

A particular concern that has weighed on my mind throughout the writing of this dissertation relates directly to the use of language, more specifically the need to use language that moves away from classical commitments and biases whilst simultaneously respecting the historical and philosophical lineage of those commitments and biases. What has become somewhat clearer during this exploration is the difficulty inherent to articulating concepts that push back against traditional modes of thought while trying to avoid much of the conceptual baggage already bound to the language that accompanies those modes of thought. During the drafting of this document, I have found it necessary to mull over the existing meanings attached to specific words and phrases, as a related goal is to modify, adapt, and occasionally even co-opt those meanings so as to forward an alternative set of ideas. This drive to modify/adapt/co-opt or otherwise reconfigure meanings may well be why the notion of *metaphor* has been so valuable throughout the preceding chapters.

At a certain level, this entire dissertation has emerged from efforts to articulate a number of non-classical ideas using fairly classical language. In particular, as a result of engaging with the physics discourse of the upcoming chapter, I have become increasingly curious about the struggles faced by the early quantum theorists attempting to make clear a range of new and unusual ideas regarding the nature of matter without having the security or affordances of an established discourse through which to express those ideas. Though elements of quantum theory have circulated widely enough to enter into the more general discourses of the modern day, this would clearly not have been the case at the time of the theory's inception, or for a significant interval afterward. Consequently, beyond problematizing our general understanding of the nature of matter, there are also ways in which quantum theory problematizes *language*. Speaking to this same idea, Jennifer Burwell (2018) offers up the following perspective, which coincides with my concern.

> Heisenberg, Schrödinger, and Bohr were highly sensitive to the fact that even assigning a name to a quantum phenomenon was already to impose

> habitual notions onto utterly novel circumstances, and Bohr spoke for both Heisenberg and Schrödinger when he observed that conventional terminology had to be retained "because you haven't got anything else". The necessity of using conventional terms such as "wave", "particle", "complementary", "observer", "uncertainty", and "entanglement" introduced into quantum concepts what might be called an "originary drift" – originary, because the drift emerges simultaneously with the naming of the concept. (p. 12)

While I had not encountered a formalized expression of "originary drift" prior to engaging with Burwell's work, an *informal* sense of it might have been influencing aspects of my earlier thinking. This notion of being bound to an existing discourse despite the need/intent to break away from it certainly encapsulates part of the challenge faced by many theorists in the physical sciences, as well as new materialists and enactivist scholars alike. Given how prevalent/pervasive the classical Newtonian mindset continues to be, it is no surprise that many implications of the quantum theoretical worldview remain difficult to parse even now.[22] Beyond offering a possible reason why this is the case, Burwell's originary drift also seems to illustrate another way in which meanings might be reorganized, albeit slightly differently from the application of metaphor. I would suggest that it is the use of conventional terminology in unconventional ways, or applied to unconventional circumstances, that distances originary drift from the simple use of metaphor as a figurative comparison, and which emphasizes the *disruption* of prior meanings to greater or lesser degree. As Burwell notes by sharing Bohr's remark, terms such as 'wave', 'particle', 'complementary', 'observer', et cetera, already had accepted scientific meanings prior to the inception of the quantum theoretical discourse; yet, new meanings emerged concurrently with/within that new discourse. Thus, much like the borderless puzzle metaphor presented in **Intralude A**, originary drift can also be read as evoking mereological themes, and of illustrating the co-emergence of the part and the whole. Additionally, it hints at the manner in which the space of the quantum theoretical discourse

[22] Speaking to the role played by language in the mediation and construction of reality, Burwell (2018) also goes on to express that much of quantum theory is "especially subject to misinterpretation" (p. 12) once linguistic expression becomes involved, citing what she describes as a "structural bias" in Indo-European language that precludes the accurate description/representation of quantum phenomena (p. 12).

might be self-organizing to some degree, its implications emerging at the very same time as the meanings associated with the discrete terminology upon which it is built.[23]

With much of quantum theory being elaborated outside the contexts of everyday language, and with the theory addressing themes/concepts to which it may be difficult to relate, I suspect that the originary drift that coincided with the *inception* of the theory is still very much in effect today, as the discourse continues to emerge both at the level of its terminology and at the level of its collective assertions/implications. This ongoing emergence (or perhaps continual *re-emergence*) of part, of whole, and of meaning, as a function of originary drift, is somewhat reminiscent of the assemblage theory already discussed; for it also incorporates concepts of structural interplay and reconfiguration within complex (unified) systems.

Interestingly, Burwell's notion of drift could be seen to parallel a different form of drift described within the enactivist literature, namely *the natural drift of living beings*, about which Maturana and Varela (1992) speak during their discussion of evolutionary change. In fairness, with Maturana and Varela's *natural drift* springing from the biological roots of their enactivist stance, the impetus for the discussion is completely different from that behind Burwell's language-driven account of *originary drift*, and I certainly can not claim full parity between the two. However, there is one particular way in which I see resonance between these concepts. According to Maturana and Varela, evolution is essentially a process through which a unity (be it organism or machine) and its environment remain in a continuous *structural coupling*, with structural coupling referring to "a history of mutual congruent structural changes" (p. 75), or a mode of constructive interaction through which "environment and unity act as mutual sources of perturbation" (p. 99). By this, the authors indicate the manner in which organism and environment emerge simultaneously, with perturbations in one influencing the development of the other, and vice versa. Moreover, they later assert that "evolution is a *natural drift*, a product of the conservation of

[23] One might consider that young mathematics learners are regularly required to deal with forms of originary drift as the meanings of mathematical terms and objects are adjusted/changed along with their contexts of use. As a perhaps overly simplified example, the reader might reflect upon how notions of number change when moving from the contexts of natural numbers to integers, integers to rationals, rationals to irrationals, irrationals to reals, and so on.

autopoiesis and adaptation" (p. 117). Though somewhat easy to overlook in light of the more accessible and recognizable notion of adaptation, their allusion to autopoiesis is significant.

In a highly abbreviated sense, Maturana and Varela's autopoiesis refers to a unified system's capacity to produce, regulate, maintain, and generally organize *itself*. Via the more elaborate formalization in their 1980 work *Autopoiesis and Cognition: The Realization of the Living*, an autopoietic unity/machine is:

> a network of processes of production (transformation and destruction) of components that produces the components which: (i) through their interactions and transformations continuously regenerate and realize the network of processes (relations) that produce them; and (ii) constitute it (the machine) as a concrete unity in the space in which they (the components) exist by specifying the topological domain of its realization as such a network. It follows that an autopoietic machine continuously generates and specifies its own organization through its operation as a system of production of its own components [...]. (p. 79)

As was pointed out slightly earlier, Burwell's originary drift hints at the self-organizing capacity of the evolving quantum theoretical discourse. While it might be counterintuitive to describe the discourse as a "machine" in the same way that Maturana and Varela do, the idea of it being a *network of processes of production that continuously generates and specifies its own organization* does not seem unreasonable, and I believe that it can be seen as adhering to the two criteria provided by these authors in the excerpt above (particularly if their concept of a *space* can be extended to discursive spaces). In keeping with this interpretation, I suggest that Burwell's *originary drift* also involves a form of self-organizing behaviour. Whereas Maturana and Varela's *natural drift* captures the evolution of complex systems responding to perturbations from the environment with which they are coupled, Burwell's *originary drift* conveys the evolution of a discourse, in this case the physics discourse, where perturbations within the accepted scientific milieu to which it is coupled incite its terminology and associated meanings to change in a manner that is ultimately dictated by or within the discourse itself. Were one to approach Burwell's *originary drift* as a discourse-specific form of *natural drift*, and to reframe the evolving quantum theoretical discourse as a complex unity in its respective domain, I propose that there could be ground for addressing its emergence in terms of self-organizing capacities (or autopoiesis).

I confess that this may be a somewhat tenuous link to draw between these disparate forms of drift, and one with largely open-ended ramifications; but, in each case, it seems to me that the self-organizing capacities of the unities in question point toward a more primary thematic principle. Just as the two metaphors discussed in **Intralude A** were brought together under an overarching thematic of *granularity*, I now wish to draw these two forms of drift together in a similar fashion, under the overarching thematic of *entanglement*. By doing this, I mean to highlight the sense in which the self-organizing processes of both forms of drift might actually belie deeper connections to the quantum theoretical discourse.

It may be of interest to the reader to note that there is already a precedent for discussing self-organizing behaviours in terms of quantum entanglements. A landmark study from Witthaut, Wimberger, Burioni, and Timme published in 2017 indicates that coupled quantum systems (where 'coupled' is used in the mechanical sense and not the sense of Maturana and Varela's 'structural coupling') can synchronize with one another without the interference of external forces. Such synchronizing behaviour was previously known to be common across a wide range of *classical* phenomena, such as the swinging of jointly suspended pendula and the sympathetic resonance of connected crystalline structures; however, the findings of Witthaut et al. establish that this behaviour also occurs in the *non-classical* quantum domain. As indicated by the authors, self-synchronization at the quantum level proves to be a hallmark of *quantum entanglement*, and the unique contribution of their study is that it has "unearthed the manifestation of classical synchronization in a class of quantum many-body systems, providing a direct link between collective classical and quantum dynamics" (p. 5). Although the technical details of their paper extend well beyond the intended scope of this intralude, I briefly touch upon the main themes/findings below.

Unlike synchronized classical systems, such as swinging pendula, synchronized non-classical (quantum) systems become entangled, which is to say that their components can no longer be described independently of one another. Within the specific class of many-body quantum systems explored in the study from Witthaut et al. (2017), synchronization is found to emerge as an *intrinsic* feature of the systems, and, as the authors clarify, "quantum coherence and entanglement arise persistently through the same transition as synchronization" (p. 1). The importance of this finding can not be understated, as it has previously been notoriously difficult to find correlations between behaviours occurring at the non-classical quantum level and those at the classical non-quantum level. This new evidence that synchronization may point to, or signify, the presence of entanglement, despite the former initially being predicated on a broader understanding of *classical* phenomena, is a significant result, and sheds light on how a more unified view of the physical world has begun to take shape within the quantum theoretical discourse. Although "synchronization and entanglement have been mostly studied as two separate phenomena in the classical and quantum worlds, respectively" (Witthaut et al., 2017, p. 5), these recent findings suggest that the classical and non-classical realms may not be quite as distinct as was once presumed.[24] They also illustrate an interesting circumstance in which an existing piece of scientific terminology has taken on additional meaning in tandem with the evolution of the larger theoretical discourse in which it is embedded. Though the connotations of the term 'synchronization' are largely the same, the contexts to which the term applies have been greatly expanded.

By virtue of the conversations that have emerged within their respective discourses, it could be said that both forms of drift spoken to in this intralude also share a specific implication concerning the foundations of *thought*. Indeed, both biological evolution and the evolution of new discursive spaces suggest the emergence of new modes of thought, or at least the potential to be *thinking differently* in and about the world. I briefly expand upon this idea by bringing up the notion of *discernment* as a thoughtful activity.

Be it in terms of structures or processes, I do not believe it would be objectionable to assert that we can not give names to things without first having *discerned* them, which is to say,

[24] Note that the revolutionary theories of physicist David Bohm, to be discussed in **Chapter 4**, also point to more fundamental levels of connectivity manifesting through quantum effects.

having *distinguished* or *foregrounded* them in some meaningful way.[25] The ability to discern, to distinguish foreground from background, signal from noise, particular from general, can be taken as a foundation of thought in this regard, and as a key way in which the simple *reception* of sensory input differs from subsequent acts of *perception*. New language and new discourses are unavoidably accompanied by new discernments, new modes of categorization, new forms of meaning-making, et cetera, which point toward an expanded capacity to *think* in and about the world of our experiences. As a result, Burwell's *originary drift* has notable implications regarding not only the evolution of language and discourse, but also the foundations of thought itself. Similarly, accounting for the biological roots of Maturana and Varela's enactivist perspective, the notion of *natural drift* implies changes to ways in which living beings (or unities) might engage with and within the world, and differing modes of engagement also suggest new discernments, new modes of categorization, new behaviours, new forms of meaning-making, et cetera. Thus, in both instances of drift, the notion of *discernment* as a thoughtful activity that is responsive to the processes of evolutionary change, would seem to apply.

Though I have ultimately expressed views about quantum entanglement, processes of self-organization, and foundations of thought, this brief intralude began with ruminations about the nature of language and the *originary drift* that has accompanied the evolution of the quantum theoretical discourse. Contemplating Jennifer Burwell's concept of *originary drift* alongside Maturana and Varela's *natural drift of living beings* has also led me to consider possible points of commonality, despite the vastly different contexts in which they are presented and discussed. Emphasizing the notion of self-organization in relation to the evolving space of the quantum theoretical discourse, and indicating similarities to the autopoiesis of Maturana and Varela as it relates to *structural coupling*, reveals that a broader notion of *entanglement* might actually come into play as a common, overarching thematic.[26] As conveyed in **Chapter 1**, one of my keys interests in the new materialist discourse relates directly to the notion of entanglement, and the principles according to

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[25] Though I here use the pronoun 'we' to refer to thinking, knowing, sensing human beings, I am comfortable with the notion of discernment including a much broader array of thinking, knowing, sensing entities.

[26] The reader will recall that I tend to view Maturana and Varela's structural coupling as a phenomenon akin to quantum entanglement.

which entanglements organize and reorganize themselves. Albeit in an unexpected way, this commentary on linguistic commitments and the emergence of new discursive spaces has also touched upon that same motivating interest.

In closing, I offer another quotation from theoretical physicist Carlo Rovelli, which not only encapsulates the language-oriented impetus of this intralude, but also speaks to the core themes of the dissertation as a whole. As Rovelli (2018) observes, “Our thinking is prey to its own weakness, but even more to its own grammar. It takes only a few centuries for the world to change: from devils, angels and witches to atoms and electromagnetic waves” (p. 180).

Chapter 4.
Quantum Theory and Historical Perspectives

"Despite scholarly consensus on the empirical facts, ever since its inception in the 1920s there has been deep disagreement about how to interpret quantum theory […] All scientific theories require interpretation, since strictly speaking what they describe is our experience of the world rather than the world itself. Thus, whether we are trying to explain or merely describe the world, we are always engaged in inference […] New theories often require adjustments to nearby theories, but in doing so we can rely on paradigms to make them cohere. More rarely paradigmatic assumptions are themselves challenged, but in that case scientists can fall back on their worldview to make sense of the needed changes. By implication, the most difficult interpretive problems arise when our worldview is called into question, which leaves us without any frame of reference on which to fall back […] Quantum theory poses a worldview problem."

–Alexander Wendt (2015)
Quantum Mind and Social Science: Unifying Physical and Social Ontology, p. 58

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As with Jennifer Burwell's notion of *originary drift* explored in the preceding intralude, the excerpt from Wendt above also attends to a particular challenge associated with the inception and emergence of the quantum theoretical discourse, specifically the difficulty that quantum theory poses with respect to prior scientific worldviews. While Burwell's drift primarily addresses epistemological issues arising from linguistic considerations, Wendt gives the issue expressed here a much weightier grounding, noting that quantum theory troubles our understanding of the world at a far more fundamental level. It is by virtue of the *worldview problem* he identifies that a deeper look into the background of quantum theory and a brief foray into surrounding historical and philosophical contexts are warranted.

In keeping with aspects of Wendt's remarks, professor in logic and philosophy of science at the Federal University of Santa Catarina, Décio Krause (2000) establishes the following perspective in the frontmatter of his article *Remarks on Quantum Ontology*.

> It is generally agreed that we cannot describe the world in the absence of any theory or of any prior way of understanding it. Even in the particular case of the microscopic world, despite the well known discrepancies between the kind of 'objects' that inhabit that world and the standard objects

> described by classical physics, we still keep our prejudices based on classical logic and mathematics when considering the stuff on which to base the theory we are considering. (p. 155)

These quotes from Wendt and Krause point toward counterintuitive aspects of quantum theory, and conflicts that can emerge when the quantum domain is scrutinized with a classically trained eye. In particular, Krause indicates how ingrained modes of thought continue to be deeply influential, despite the differences that exist between the "objects" of the classical and quantum domains. To be clear, although Krause uses the adjective 'microscopic' in multiple instances throughout the aforementioned article, the reader may already be aware that not all microscopic objects are necessarily quantum-scale objects, and that quantum effects are generally not *assumed* to manifest above a particular threshold (of approximately 10^{-7}m). While not exactly *erroneous*, this latter assumption is somewhat misleading, as quantum effects *can* manifest at macroscopic scales as well. Though the circumstances under which this occurs tend to be highly restricted, as in the case of extremely cold Bose-Einstein condensates or superfluids/superconductors more generally, the work of Witthaut et al. (2017) discussed in **Intralude B** also indicates that specific quantum phenomena may have classical, macroscopic analogues. Further deconstructing assumptions about quantum effects, Karen Barad (2007) issues the following cautionary note.

> There is a common misconception (shared by some physicists as well as the general public) that quantum considerations apply only to the micro world […] But this is to confuse practical considerations with more fundamental issues of principle. No one would suggest that because atoms are too small to see with the naked eye, we are therefore entitled to deny their existence and their relevance to our everyday lives (although we do at times successfully ignore their existence). The entity in question may be small, but its consequences may be quite profound. (pp. 109–110)

Via this brief passage, Barad acknowledges the sense in which the quantum domain remains largely invisible and inaccessible to normal human perception, while emphasizing that *implications* arising from quantum phenomena may nevertheless play out within the immediately accessible realm of everyday, macroscopic existence. Of course, with Wendt's and Barad's remarks in mind, I do not assume that efforts to draw from quantum theory so as to enrich perspectives within mathematics education will necessarily appeal to common intuition; and I endeavour to illustrate how quantum theory has already begun to reshape worldviews within related disciplines. Though the current dissertation has remained focused on the articulation of a purely theoretical exploration, there are modern scholars, like physicist and futurist Michio Kaku (2011), who note that a myriad of human *activities* and *practices* in the next century may well "depend on the bizarre and counterintuitive principles of the quantum theory" (p. 201), and I would suggest that it is not unreasonable to consider the *possibility* that mathematics education (as well as the research that develops within and alongside it) could be included amongst the range of activities/practices to which he refers. This might even be read alongside Dijkgraaf's tentative allusion to the hybrid discipline of *quantum mathematics* (mentioned in the opening preface of this dissertation).

The fact that quantum ontologies are not currently well represented in the broader discourse of our field suggests that common thought has not yet turned in that direction; however, the literature that does draw from quantum theory whilst also being grounded in (or circulating around) the contexts of mathematics education would seem to point toward a growing interest in shifts from classical to more relational worldviews.[27] Out of the necessity to "ease into" these discussions, the present chapter engages with quantum theory not in terms of its complex technical language and strict mathematical formalisms, but in terms of how it has prompted changes to the ways in which human beings think about, express ideas about, interact with, and coexist within the material world. Beyond this, it also articulates how the unparalleled descriptive efficacy of quantum theory could be taken as support for the view that mathematics provides a structural basis for material reality. The principle of *quantization*, whose foundations lie in the mathematical notions of continuity/discontinuity and discretization, is discussed alongside certain number theoretical constructs as well as ideas salient to other socio-cultural contexts. The phenomenon of the quantum leap is also used as a more specific framing device, following an introduction to the revolutionary Bohr model of the atom.

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[27] See Aerts et al. (2013), Busemeyer et al. (2011), de Freitas and Sinclair (2013, 2014, 2018), Pothos and Busemeyer (2013), Trueblood and Busemeyer (2011), and Wendt (2015) as examples.

Basic Quantum Theoretical Principles

At times described as concerning the physics of *the very*, *very small* and *the very*, *very fast*, quantum theory exposes facets of material reality that are typically not considered in the day-to-day dealings of most individuals. In its broadest strokes, the theory asserts that the natural world operates according to an underlying physics of *the discrete* as opposed to *the continuous*, such that specific physical properties are permitted to manifest only in integer-valued multiples of indivisible, unitized amounts (known as quanta). Aside from "running against the grain" of a significant portion of human perceptual experience, this constraint is also fundamentally at odds with classical Newtonian theory, which is much more limited in its descriptive scope yet demands no such restriction. Though different programs of research in quantum physics might focus on a variety of quantized properties, including the spin of subatomic particles (i.e., angular momentum), extension in space (i.e., length), and even duration of or between events (i.e., time), quantum theoretical perspectives ultimately stem from a single foundational quantization constraint, namely the quantization of *energy*, and this constraint is itself rooted in the need to address a profound problem faced by the classical Newtonian worldview.

What is commonly referred to as the *ultraviolet catastrophe* of the late 19th century encapsulates a concern associated with classical predictions of energy emission from ideal black-body radiators, and is, perhaps, the most significant historical impetus behind the inception of quantum theory (Ackroyd et al., 2009; Faye, 2008; Gribbin, 1984; Schroeder, 1999). In physics, a perfect black body is essentially an idealized object that has the capacity to absorb as well as emit electromagnetic radiation (i.e., light) across *all* frequencies, for *all* angles of incidence, without any loss due to reflection or internal scattering. This may be contrasted with the inverse case of the perfect white body, which would *reflect* electromagnetic radiation across all frequencies, for all angles of incidence, in the same lossless fashion. The emission spectra of ideal black-body radiators vary only with temperature, and are in no way dependent upon the general composition or other physical properties of the black bodies themselves. Thus, for any given temperature, a perfect black body is expected to radiate a continuous stream of light at a specific frequency. Indeed, many physical objects, including stars like Earth's sun, the active heating elements of regular household ovens, the electrified filaments of incandescent lightbulbs, and even warm-blooded animals such as human beings, behave as *approximate* black-body radiators over limited frequency ranges, emitting different forms of light in accordance with their respective temperatures.

Classical predictive models of the emission spectra for black-body radiators (most notably the Rayleigh–Jeans Law)[28] tend to accord with empirical observations of physical objects radiating at relatively high wavelengths (or low frequencies); however, marked discrepancies arise when these models are applied to comparably low-wavelength (or high-frequency) emissions at the other end of the electromagnetic spectrum. While observations indicate that the intensity of radiation in the ultraviolet range decreases as wavelength decreases, classical models like the Rayleigh–Jeans law instead predict that the intensity of ultraviolet emissions will exponentially *increase* as wavelength decreases, approaching infinity as wavelength approaches zero, and violating fundamental energy conservation laws in the process (hence the moniker of 'the ultraviolet catastrophe'). It was the German physicist Max Planck who (in 1900) first proposed the concept of energy quantization as an *ad hoc* solution to this problem.

A previous underlying assumption about black-body radiators was that the energetic vibrations of very small particles (atoms), caused objects to glow. To avoid the troublesome predictions of exponentially increased emissions at ultraviolet wavelengths, Planck proposed that the vibrational energies of these atoms, and the frequencies of their corresponding emissions, might actually be constrained to very *specific*, *discrete* values. The subsequent formalization of this discretization condition is represented by Planck's Law for black-body radiation, which expresses the spectral radiance of a body as a function of its temperature. Though the law has appeared in a number of mathematically equivalent forms, Einstein's later work on the photoelectric effect eventually saw it subsumed into the simplified energy–frequency relation $E = nhf$, for energy 'E' in joules, the integer-valued number of quanta 'n', Planck's constant 'h' in joule-seconds, and frequency 'f' in hertz). Attending to the incredibly small value of Planck's constant 'h' (approximately

[28] Attributed to John William Strutt (3rd Baron Rayleigh) and Sir James Jeans

$6.62607004 \times 10^{-34}$ joule-seconds) helps to convey the idea that energy is quantized at such a minute scale that the gradations between allowable energy levels manifest many, many orders of magnitude below what is perceptible in our everyday macroscopic existence (leading, in part, to the assumption/appearance of continuity).

Although initially put forward for purely practical purposes, in response to the problem of the ultraviolet catastrophe, Planck's quantization constraint ultimately facilitated the *accurate* prediction of black-body energy emissions, and the remarkable precision of the law, which hinges upon the constant 'h', has led some to consider that the mathematical formulation of the constraint actually provides *direct* insight into one of the most basic features of our material reality. As noted by Argentine-Canadian philosopher and physicist Mario Bunge (2003), a peculiarity of 'h' is its apparent *universality*, which is to say that "its value does not depend on the kind of matter" (p. 448).[29]

By pointing toward the very fine granularity of our material existence, the concept of energy quantization and its accompanying formalisms, not only moved the physical sciences into a new era, but also reshaped broader sensibilities about how human beings perceive and understand reality. Historically, the disparity between theoretical predictions and empirical observations for low-wavelength ultraviolet emissions came as a clear sign that the preceding theories of black-body radiation were deeply flawed and in need of revision. However, it also posed a second conceptual issue for the physicists of the day, for it brought into question not only the *mathematical* models of electromagnetic phenomena, but the prevailing *physical* model of matter as well. Indeed, this historical impetus to rethink the nature of matter is likely a key reason for quantum theory's more recent absorption into the new materialist discourse.

In order to account for Planck's *ad hoc* assertion of energy quantization, a new model of atomic structure was required, as the leading models of Planck's time could not adequately support the unconventional constraints that had been proposed. Prior to the 1900s, the

[29] Beyond its role in Planck's solution to the ultraviolet catastrophe and Einstein's later work on the photoelectric effect, Planck's constant 'h' also appears in Bohr's equation for the quantization of atomic orbits (to be discussed momentarily), and even in the uncertainty principle of Werner Heisenberg.

physical structure of matter was understood only in a very rudimentary sense, with John Dalton's model of the atom forwarding the ancient Greek philosophers Democritus and Leucippus' notion of *indivisibility*, and assuming that these very small particles had zero substructure. It would not be until J. J. Thomson's discovery of the electron in 1897 that the indivisible atoms of these earlier worldviews were found to be more nuanced in their composition.

In the interest of brevity, I avoid a lengthy description of Thomson's 1904 "plum pudding" model of the atom (which characterized electrons as negatively charged particles embedded in a larger, positively charged sphere), but I do emphasize the importance of Ernest Rutherford's subsequent 1911 nuclear model (which stratified the internal structure of the atom significantly). As the immediate precursor to Bohr's atomic model, the latter emerged as a consequence of Rutherford's famed gold foil experiments, during which a beam of heavy alpha articles (i.e., positively charged helium nuclei) emitted by a radioactive source was fired at a very thin sheet of gold foil, with a movable zinc sulfide screen acting as a detection apparatus.[30] Though I omit the details of the experiment itself, the primary findings resulting from it were essentially threefold: 1) Atoms are composed of largely empty space; 2) The bulk of an atom's mass, and the entirety of its positive charge, are located within a dense central core (or nucleus); and 3) Negatively charged electrons occupy stable orbits around the positively charged nucleus.[31]

While Thomson's earlier "plum pudding" model conceived of the atom simply as a positively charged primary mass with negatively charged secondary masses embedded in it (but no further substructure), Rutherford's revised atomic model redistributed Thomson's electrons into elliptical orbits around a dense, positively charged central nucleus, similar to a miniature solar system with planets circulating around a central star. Indeed, it is by virtue of this convenient analogy that Rutherford's nuclear model has also

[30] The experiments were actually conducted by undergraduate researcher Ernest Marsden, under the supervision of Hans Geiger and Rutherford (Ackroyd et al., 2009).

[31] The reader may wish to note that an anecdotal account of the more remarkable/surprising experimental results can be found in Edward Andrade's 1964 volume *Rutherford and the Nature of the Atom*, prefaced by Andrade and retold by Rutherford himself.

come to be known as the *planetary model* of the atom.[32] Though this model quickly supplanted the less sophisticated (and less accurate) predecessor from Thomson, this is not to say that it was without issues of its own. In particular, the newly surmised distribution of protons and electrons *within* the atom proved problematic as a result of known electrodynamic principles, and it was by addressing this problem that Niels Bohr would establish a modified form of quantization from which interesting new implications emerged. As indicated in **Chapter 1** of the current dissertation, the Bohr model of the atom has acted as something of a starting point for the exploration contained herein.

Superficially similar to Rutherford's planetary model in its diagrammatic/pictorial representations, Bohr's model is nevertheless characterized by hitherto unprecedented technical assertions that place highly specific constraints upon the allowable dynamics of the electrons in orbit about atomic nuclei. These constraints bear some resemblance to Planck's original discretization condition, yet Bohr would necessarily invoke a revised form of quantization, given that the atomic model had since evolved. Considering that the particles of interest (i.e., protons and electrons) have masses and charges, and are subject to forces like other material bodies, Rutherford's nuclear/planetary model offered more than simply a convenient metaphor for re-imagining atomic structure. However, the proposed elliptical orbits of his model required that electrons moving about the nucleus would exist in a state of constantly accelerated motion. Since charged particles are known to emit electromagnetic radiation when accelerated, Rutherford's electrons would face the continual *depletion* of their electromagnetic energy, causing them to spiral into the center of the atom during a process of orbital/structural decay. Moreover, with protons and electrons carrying opposite charges, this same decay process would be hastened by the electrostatic force of attraction that exists between opposite charges. Essentially, the physical structure of Rutherford's atomic model foreshadowed the inevitable collapse of most material bodies, posing a sizable issue quite different from the ultraviolet catastrophe faced by earlier physicists.

[32] Considered in light of the *worldview problem* identified by Wendt (2015), this reframing of circumstances harkens back to my earlier remarks concerning the power of metaphor as a tool for reorganizing meaning.

Bohr would overcome this obstacle by quantizing the electron orbits themselves, requiring that all electrons traverse discrete paths (or shells) of *fixed* size and energy. With higher energy states corresponding to shells at greater radial distances from the nucleus and lower energy states corresponding to those in closer proximity to it, the electron orbits were redefined as *non-radiating* steady states, granting the electrons "safe" paths to traverse, and effectively reworking Rutherford's preceding planetary model so as to avoid the problematic emissions that would lead to the systemic collapse predicted by classical theory.[33] A particularly unusual consequence of Bohr's quantization constraint is that electrons are not permitted to occupy spaces anywhere *between* the stable, non-radiating shells; yet, despite this stipulation, there exists no condition preventing them from *transitioning* from one shell to another. It is precisely this theorized process of transition that constitutes the distinctly *non-classical* phenomenon of the quantum leap (addressed later in this chapter).

By requiring that electrons exist in stable, non-radiating paths, Bohr's steady states violate the principles of classical electromagnetism and contradict the preceding Newtonian framework, yet they prove to be in complete accordance with certain physical observations that remain irreconcilable within classical frameworks. In this latter regard, the deep historical and mathematical salience of Bohr's atomic model may not be entirely obvious, until one considers its success in *retroactively* providing a materially grounded explanation for the wavelength calculations that had emerged from Johannes Rydberg's work with the hydrogen atom over a decade earlier.

Rydberg's formula for determining the wavelengths of light emitted or absorbed during the energy level transitions of electrons in the basic hydrogen atom appears as follows,

$$\frac{1}{\lambda} = \mathrm{R}\left(\frac{1}{\mathrm{n}_1^2} - \frac{1}{\mathrm{n}_2^2}\right)$$

with wavelength 'λ' in metres, Rydberg constant 'R' per metre (approximately 1.09677583 $\times 10^7$), and integer-valued principal quantum numbers 'n_1' and 'n_2', which designate the

[33] This hybridization of the planetary model and Bohr's revised quantization constraints is known, synonymously, as both the Rutherford-Bohr model and simply the Bohr model.

discrete energy levels between which a given electron may transition.[34] While accurate in terms of predicting experimental results, Rydberg's formula originally had no accompanying physical explanation, and could not be substantiated via the atomic models that existed at the time. It was the energy level quantization proposed by Bohr that yielded the much-desired reconciliation between theory and observation, revealing how the mathematical underpinnings of Rydberg's formula could actually be encoded within the very *structure* of the atom, and allowing for a deeper understanding of a particular facet of the material–mathematical relationship at play within the universe. Indeed, as evidenced by the field of radio astronomy, which uses spectroscopic analyses of electromagnetic emissions to identify the chemical compositions of deep space objects/phenomena, the same material–mathematical relationship seems to manifest throughout the totality of the observable universe, not simply on Earth or in the vicinity of its local solar system.

Beyond dramatically altering concepts of atomic structure/substructure and informing broader perceptions of principles that appear to underlie our material reality, the introduction of quantization constraints by Planck and Bohr compelled the scientific and mathematical communities to re-evaluate long-standing assumptions of continuity that had carried forward from earlier worldviews. Models of the atom have continued to evolve since the time of Bohr's theorizations, yet the notion of discretized energy states has persisted as a foundational element inherent to subsequent quantum frameworks. Though the inception of quantum theory brought with it a new degree of mathematical clarity concerning the inner substructure of matter, it also introduced innate *uncertainties* and *indeterminacies* that remain somewhat intractable. These shall be spoken to in the remaining sections of this chapter; however, the interested reader is encouraged to see Bohm and Hiley (1993), Bunge (2003), Camilleri (2009), Gribbin (1984), and von Neumann (1955) for more involved technical discussions of the topics addressed in the previous pages.

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[34] In the basic Rydberg equation, both 'n_1' and 'n_2' are greater than or equal to '1', with 'n_1' being less than 'n_2', and 'n_1' being equal to '1' in the specific case of an electron occupying the lowest possible ground state energy.

With its centennial milestone having been widely recognized in the year 2000, honouring Planck's solution to the classical problem of black-body radiation, quantum theory has existed in the public consciousness long enough that much of its terminology, its iconography, some of its basic premises, and even associated *Gedankenexperiments*[35] are bandied about within popular culture and the more general media of the 21st century. Nevertheless, the quantum domain presents a number of unusual features and phenomena that run contrary to everyday intuition and seem to defy the logic upon which a good deal of "common-sense reasoning" is based. Instantaneous action-at-a-distance, the inherent discontinuity of spacetime, the possibility of backward causation, and even notions of "particles" that aren't really particles all enter into the quantum theoretical discourse at some level, collectively exemplifying the unusual and varied landscape the theory spans.[36] Moreover, the so-called "objects" of quantum theory can not be approached by traditional investigative means. As Krause (2000) observes, we can not engage with them directly, touch them, "look at them, analyse their properties in order to form our theories. On the contrary, we approach them indirectly, by making theories, and not by infering [*sic*] properties from sensible impressions" (p. 156). Thus, in the absence of technological intermediates and related theoretical constructs that facilitate the investigative process, the quantum domain can be rather reluctant to reveal its inner workings to researchers confined/bound to the macroscopic world.

Though this is an interesting problem to face, it is certainly not one unique to the domain of quantum physics. The world of education presents its own classes of "objects" and phenomena with similarly elusive traits, perhaps most notably the *thoughts* and *thought processes* of individual learners. Arguably, even educational assessment tools do not yield that much insight into the nature of thought, or the ways in which sense-making activity *actually* emerges within the embodied self. In the realm of mathematics education specifically, matters are further complicated by the fact that the entities of interest can be

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[35] Thought experiments

[36] Indeed, Krause (2000) attempts to address "some possibilities for accommodating the logico-mathematical frameworks of the theories which deal with such strange ontology where the inhabitants are things devoid of identity and both having and not having certain properties" (p. 155).

just as “slippery” in their own ways, perhaps even more so.[37] Thus, Krause’s observation about engaging with objects that inhabit the quantum domain seems somewhat applicable to the space of mathematical cognition and sense-making as well; for educators can not gain unmediated or unfettered access to learners’ mathematical thoughts/thinking, let alone directly gauge their understandings or clearly apprehend their intuitions. Furthermore, even in the case of an individual’s *metacognitive* practice or other forms of mindfulness, the question might be raised as to whether such inward-looking reflection or contemplation could be subject to observer effects analogous to those of quantum theory, such that the act of *self-observation* inevitably alters or influences the characteristics of the very thing being observed.[38]

Wave–Particle Duality

Bohr’s model of atomic structure was extremely useful in explaining the absorption and emission spectra for the hydrogen atom; however, its earliest instantiation was not robust enough to properly account for observations derived from more complex atoms. At least initially, the model could not be extended across the entire set of known elements, and faltered when applied to the spectra of atoms containing *multiple* electrons. While the recurrent properties and family resemblances found within the periodic table of elements would later be explained by a refined version of Bohr’s model, the original nevertheless constituted a significant step forward in terms of establishing a more accurate material worldview. Interestingly, this more accurate worldview also blurred the boundary between matter and light, which had traditionally been seen as distinctly different ontological entities, subject to separate physical theories and laws. Planck (in 1900), Einstein (in 1905), and eventually Bohr (in 1913) were all faced with an array of experimental results and accompanying theoretical implications that compelled them to re-evaluate the validity of this distinction, and the revelations of the evolving quantum theory gradually eroded the

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[37] The reader may consider, for instance, how the meanings of even basic mathematical concepts such as ‘number’ and ‘shape’ can be quite malleable, not to mention the variability arising from associated definitions and terminology.

[38] I elaborate upon the relevance of observation a little later in the chapter, by way of David Bohm’s explorations with renowned Indian philosopher Jiddu Krishnamurti.

ontological bifurcation between matter and light, particle and wave, with Einstein's work on the photoelectric effect, Thomas Young's double-slit experiment, and the electron scattering experiments of Clinton Davisson and Lester Germer ultimately indicating that the classical distinction was no longer tenable.[39]

The pioneering experimental work of Thomas Young and Augustin-Jean Fresnel in the early 1800s, and the later mathematical/theoretical contributions of Planck, Einstein, and de Broglie in the early 1900s confirmed the need for a *unified* wave–particle interpretation (Ackroyd et al., 2009; Gamow, 1940/2013). In particular, Young's widely referenced double-slit experiment of 1801 (now considered something of a seminal experiment in quantum physics, despite having been performed almost a century before Planck's proposal of quantization) was instrumental in establishing that both light and matter have the capacity to exhibit particulate as well as wave-like characteristics, depending upon the circumstances under which they are observed. Furthermore, it reiterated how limited the classical concepts of particles and waves were, for neither could *fully* account for the outcomes demonstrated by the experiment.

Aside from proving that light does, in fact, diffract around sufficiently small objects (a point that had long been in contention in classical physics), the double-slit experiment also yields the critical result that light emitted from adjacent point-like slits generates an interference pattern (or diffraction pattern) in precisely the manner of propagating waves (as in Figure 2 below).

[39] This same blurring of distinction or erosion of bifurcation within the physical sciences also troubles the traditional Cartesian dualism between mind and body.

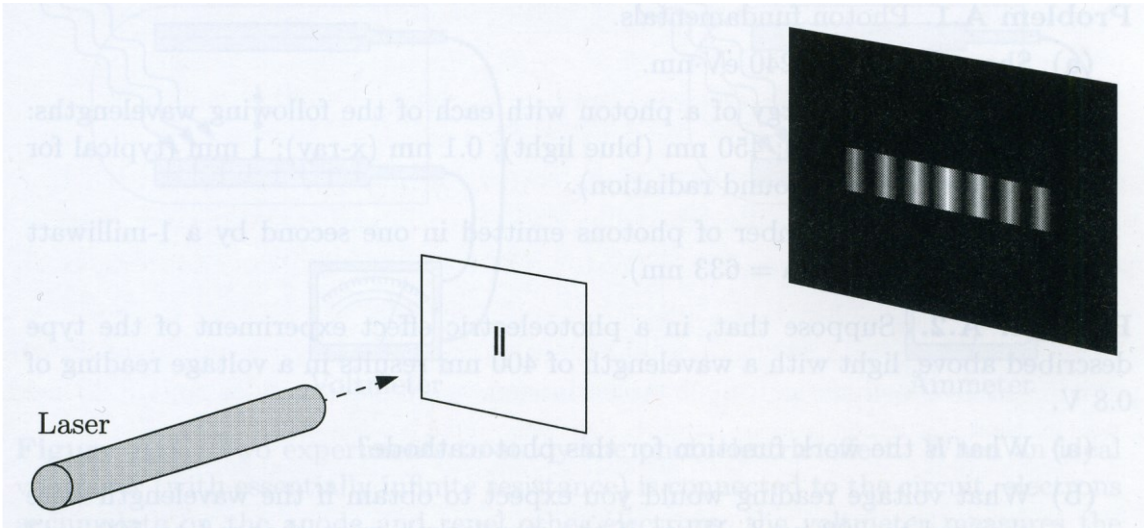

Figure 2: Monochromatic light passing through a double-slit apparatus generates an interference pattern on a viewing screen, demonstrating the wave-like properties of electromagnetic radiation.[40]

Perhaps the more surprising outcome, though, is that particles such as electrons *also* generate the same kind of interference pattern when passing through a double-slit apparatus, refuting classical assumptions about such particles, demonstrating the shared wave-like behaviours of matter and light, and further reinforcing the importance of a unified wave–particle interpretation.

Between 1925 and 1927, the wave-like nature of electron beams, and the phenomenon of electron diffraction, were observed and verified experimentally by American physicists Clinton Davisson and Lester Germer, with English physicist George Thomson independently yielding findings that corroborated their results.[41] Figure 3 below, while not from the original Davisson-Germer or Thomson experiments, illustrates the formation of an electron interference pattern, with the right-most image presenting a visual not unlike that on the viewing screen of Figure 2.

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[40] From Figure A.2 of Schroeder (1999, p. 360)

Image Credit: Schroeder, Daniel V., *An Introduction to Thermal Physics*, 1st, ©2000., p. 360. Reprinted by permission of Pearson Education, Inc., New York, NY.

[41] As noted by Ackroyd et al. (2009), perhaps one of the great ironies in the history of physics is that George Thomson played a key role in confirming the wave-like behaviour of electrons, while his father J. J. Thomson had advocated for a particle view of the electron some three decades earlier.

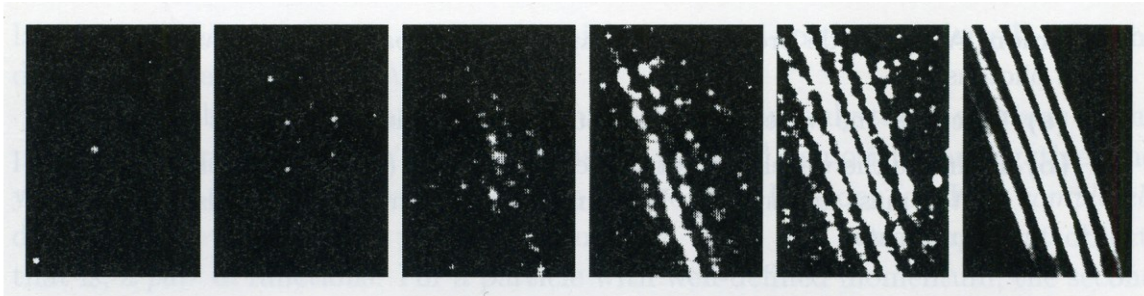

Figure 3: From left to right, photographic images illustrate the gradual build-up of an electron interference pattern similar to that produced by light, but generated by passing the beam of an electron microscope through a diffracting apparatus.[42]

Inexplicable under classical Newtonian theory, the similitude between the diffractive behaviours of light and electrons speaks to a defining feature of quantum theory, namely *wave–particle duality*, which acknowledges that quantum entities can be described as *either* particles or waves, with neither description being capable (on its own) of rendering a *full* account of the entities in question. Wave and particle properties are both implicated in the nature of matter, yet they do not manifest simultaneously. This particular caveat is quite profound, as it communicates that the wave and particle natures of matter are not to be taken as *diametric opposites*, but as *complementary facets* of a unified whole. As Wendt (2015) recapitulates, "they are mutually exclusive but jointly necessary for a total picture" (p. 48). Oddly, it is in this sense that the principle of wave–particle duality actually allows quantum theory to express a kind of ontological *monism*, although this may become more obvious when wave–particle duality is read alongside the relativistic principle of mass–energy equivalence (embodied by Einstein's ubiquitous equation $E = mc^2$).[43]

The underlying principle of wave–particle duality also points toward a special class of fundamental interactions that can occur *between* matter and light, and for this reason, I would be remiss in discussing Young's double-slit experiment without also speaking to

[42] From Figure A.3 of Schroeder (1999, p. 361)

Image Credit: Schroeder, Daniel V., *An Introduction to Thermal Physics,* 1st, ©2000., p. 361. Reprinted by permission of Pearson Education, Inc., New York, NY.

[43] Considering that the 'dualism' of Cartesian dualism implies a sort of *rift* or *separation* between mind and body, while the 'duality' of wave–particle duality suggests a kind of *multifaceted unity* between wave and particle interpretations of matter, it may be worthwhile (in future research) to more closely examine differences in connotation associated with the nouns 'dualism' and 'duality'.

Einstein's ground-breaking work on the photoelectric effect (for which he was awarded the Nobel Prize in 1921). Through the mechanism of the photoelectric effect, certain metallic surfaces may be induced to libertate/eject electrons, and even establish an electric current, when light of or beyond a particular threshold frequency is incident upon them. Essentially, these liberated/ejected electrons (referred to as *photoelectrons*) are freed from their parent atoms after absorbing specific amounts of energy from the incoming light rays with which they interact. For light below the threshold frequency, which varies according to the metal in question, electron liberation/ejection will *not* occur. Classical theories suggest that increasing the amplitude (or brightness) of the incident light should result in photoelectrons becoming more energetic, or in the induction of a larger current; yet, this proves not to be the case. In fact, while a brighter incident light source will induce a greater number of electrons to be liberated from a metallic surface in a given amount of time, the intensity of the light in no way influences the maximum (kinetic) energy gained by any single electron (Ackroyd et al., 2009; Schroeder, 1999).

It was this known (yet initially perplexing) phenomenon that prompted Einstein to consider whether the discrete packets of energy (quanta) proposed by Planck might also be implicated in the transfer of energy between incoming light rays and the electrons within the metallic surfaces upon which they were incident. As communicated by Bunge (2003), Einstein postulated that a form of quantization also holds for electromagnetic radiation, and that "the total energy of a light beam of frequency υ is nhυ, where n is a positive integer. In other words, radiation is composed of […] electromagnetic field quanta" (p. 448). The reader may recognize this as the simplified energy–frequency relation into which Planck's Law had been subsumed, with the algebraic symbol for frequency having changed from 'f' to 'υ', and the notion of discrete quanta having expanded to include discretized "packets" of light. Ackroyd et al. (2009) summarize as follows:

> In 1905 Einstein extended this quantum theory by proposing that light is emitted in quantized, tiny, massless particles, which are now called photons. Planck's original theory thus evolved into the currently accepted quantum model of light, which is a combination of both the particle and wave models. (p. 640)

Under Einstein's interpretation, then, light is reconceived not as a continuous wave propagating through space, but as a collection of discretized energy packets, such that every photon of light is envisioned as an individual "particle" carrying precisely one indivisible, unitized quantum of energy. Despite being massless, these "particles" also have momentum, which can, like their energy, be transferred to other material bodies. Thus, just as Young's double-slit experiment reveals that entities traditionally referred to as particles may exhibit *wave-like* properties, Einstein's novel approach to electromagnetic radiation establishes a theoretical basis through which phenomena traditionally referred to as waves may exhibit *particle-like* properties, effectively breaking down the bifurcated ontology of classical physics. British science writer and University of Sussex astrophysics research fellow John Gribbin (1984) expresses a similar sentiment by remarking that:

> The complete break with classical physics comes with the realization that not just photons and electrons but all "particles" and all "waves" are in fact a mixture of wave and particle. It just happens that in our everyday world the particle component overwhelmingly dominates the mixture […]. (pp. 91–92)

By virtue of the fact that quantum entities can not be said to manifest their wave-like or particle-like properties *prior* to some form of observation or measurement (i.e., an interaction with classical, non-quantum objects), the material world as it is known at the level of the macroscopic might be described as *emerging from* an inherently indeterminate quantum background where neither facet of the wave–particle duality necessarily exhibits primacy over the other. Elsewhere in this document (as with the discussion of the Necker Cube in **Chapter 3**), I have extended this notion of an underlying material indeterminacy to the processes of mathematization and mathematical sense-making more generally. While a more technical treatment of material indeterminacy is certainly possible, it is hoped that these introductory explorations will suffice for the current document. In an effort to carry forward with this brief overview of basic quantum theoretical principles, I now turn more direct attention to the unusual phenomenon of the quantum leap, which was prefaced earlier in this chapter.

The Quantum Leap

Through the advent and development of its underlying quantization constraints, the atom (as a quantum object) was gradually recognized as belonging to a very different class of material entity than had previously been considered. Perhaps literally as well as figuratively, the quantization of atomic structure brought back under the microscope some of the most fundamental assertions upon which modern theories concerning material reality had been grounded, such that principles governing the organization and reorganization of matter (and the mathematics that embody them) were reopened to scrutiny. As a phenomenon that emerges as a consequence of atomic energy level discretization, the quantum leap is a notable example of the dynamic processes through which matter organizes and reorganizes itself, and an instance of what I have previously termed *material assemblage in action* (i.e., a process of structural change and reconfiguration). It also happens to exemplify a scenario that reveals variations *within* the larger quantum theoretical worldview.

It was mentioned earlier that the electrons of Bohr's atomic model are prohibited from existing *between* the sharply delineated energy levels that characterize their non-radiating steady states. Nevertheless, they are permitted to *transition* from one steady state to another by either absorbing or releasing appropriate (i.e., quantized) amounts of energy. Rightly or wrongly, the everyday contexts of the term 'transition' tend to carry connotations of *smoothness* and *continuous flow*; however, under the standard Copenhagen interpretation of quantum theory (principally attributed to Niels Bohr, Werner Heisenberg, and Max Born)[44] the quantum leap is most commonly described as a kind of *sudden* or *abrupt* change of state. More specifically, this is to say that the energy level transitions are theorized as being both *random* and *discontinuous*. With regard to the former condition (randomness), standard theory holds that there is no way of ascertaining precisely *when* a quantum leap will actually occur, making the event itself entirely unpredictable. With regard to the latter

[44] Faye (2008) refers to the Copenhagen interpretation as "the first general attempt to understand the world of atoms as [it] is represented by quantum mechanics" (para. 2). I would suggest that this allows it to maintain a certain weight/significance even in light of the various other interpretations of quantum theory that have since arisen. This is also why I use it as the primary touchstone in the current subsection.

condition (discontinuity), transitioning electrons are essentially envisioned as ceasing to exist in one particular energy level (or shell) whilst *instantaneously* (re)appearing in another, without traversing the intermediate space (i.e., not unlike the graphical representation of a jump discontinuity in a step function). In these conjoined senses, the quantum leap of the Copenhagen interpretation implies both *spatial* and *functional* discontinuity, disrupting the connotations of smoothness and continuous flow already mentioned.

Instancing one of various ways in which quantum theory has extended beyond the technical register into the informal, everyday parlance, the discontinuities of the quantum leap are somewhat evocative of a phenomenon commonly reported as accompanying mathematical sense-making activity. In so far as they both imply an abrupt, and seemingly unexpected/unpredictable change of state, the quantum leap might actually be likened to the sudden flash of insight often referred to as the 'AHA!' moment or the 'Eureka!' moment (the latter famously being connected to the popularized narrative of Archimedes' discovery of fluid displacement principles). In keeping with this tentative connection, during a discussion of the French mathematician Edmond Maillet, methods through which mathematicians conduct their work, problem-solving activity, and mathematical dreaming, Jacques Hadamard (1945/1954) offers a personal anecdote concerning what he refers to as "the sudden and immediate appearance of a solution at the very moment of sudden awakening" (p. 8). Elaborating on his experience, he remarks that:

> On being very abruptly awakened by an external noise, a solution long searched for appeared to me at once without the slightest instant of reflection on my part – the fact was remarkable enough to have struck me unforgettably – and in a quite different direction from any of those which I had previously tried to follow. (p. 8)

The unexpected quality of Hadamard's epiphany, and its apparent disconnect from his prior solution approaches, are reminiscent of the discontinuities described above, with Hadamard's new state of clarity perhaps being loosely analogous to a new steady state achieved after an electron transitions to a higher energy level. That said, though Hadamard's account does not address it outright, it is interesting to speculate whether the abruptly realized solution *persisted* in his thoughts, or if it eventually faded/dissipated as waking insights are sometimes wont to do; for, just as an electron can leap *back* to a previous energy level, an individual might *revert* from a state of clarity if the realization that first facilitated it proves to be fleeting, or if the newly gained insight remains disconnected from broader contexts to which it might otherwise be anchored. If the 'Eureka!' moment implies a sudden *realization* or *gaining of insight*, it may also be reasonable to consider a sudden *misapprehension* or *loss of insight.* I draw attention to this thematic similarity, in part because Hadamard's anecdotal account could easily be read alongside Wheeler's appeal to *awareness* (as I have outlined in **Chapter 3**), and possibly interpreted as an historical example in line with my assertion that it is not the "situation" that is any more or less mathematical; rather it is the awareness of the mathematics embedded in a given situation that manifests to greater or lesser degree (also in **Chapter 3**).

Though I will not go to the point of suggesting that epiphanies like Hadamard's may be drawn into *complete* analogy with the quantum leaps of electrons to higher energy states, I am, in effect, questioning what the *inverse* of Hadamard's revelatory experience might look like, and whether an analogy with quantum leaps to lower energy states might be appropriate, in some sense or another, if even only phenomenologically/experientially.[45] In any event, I do not overextend this tentative connection, as the theoretical foundations concerning the quantum leap have very recently met with profound change, and it seems likely that associated connotations in both the technical and informal registers may soon need to be revised as a result. While it was *once* firmly believed that the energy level transitions of electrons occurred instantaneously, as of mid-2019 this no longer appears to be the case.

Ground-breaking research published by Minev et al. (2019) has radically altered the modern perception of the quantum leap, by providing evidence that the energy level

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[45] Moving beyond this kind of analogical treatment of the quantum leap into a much more literal space, the rapidly emerging field of *quantum cognition* draws on principles from quantum theory to examine cognitive processes such as information processing and decision-making. Not entirely synonymous with, and often distanced from, more controversial theories of *quantum mind*, which posit that consciousness itself is a quantum phenomenon, theories of quantum cognition emerge from the observation that aspects of cognition are better described by the mathematical formalisms of quantum probability than they are the formalisms of classical probability. Wendt (2015) engages with this topic extensively.

transitions of electrons do *not* occur instantaneously, and are instead manifested over *finite*, *measurable* intervals of time. Moreover, quoting Minev et al. (2019), the experimental results demonstrate that "the evolution of the jump is coherent and continuous" (p. 203). Within the context of the current discussion, these results are of interest primarily because of how sharply they contrast with the aforementioned Copenhagen interpretation, and how they ultimately point toward a decidedly different kind of material–mathematical relationship.

As Ball (2019) notes in a *Quanta Magazine* article covering the breakthrough of Minev's research group, the "abruptness of quantum jumps was a central pillar of the way quantum theory was formulated by Niels Bohr, Werner Heisenberg and their colleagues in the mid-1920s" ("All Too Random", para. 1); however, this theorized abruptness had been deeply troubling to the Austrian physicist Erwin Schrödinger, who is known to have expressed his distaste/disdain for the idea in the two-part paper *Are There Quantum Jumps?* published in 1952.[46] Schrödinger's discomfort with both the proposed randomness and discontinuity of the quantum leap is thought to be partly responsible for his formulation of an alternative mathematical approach to the phenomenon of electron energy level transition, an approach which favours fluid behaviour over the instantaneity of the leap proposed by Bohr and Heisenberg. Again, as noted by Ball (2019), "Schrödinger's theory represented quantum particles in terms of wavelike entities called wave functions, which changed only smoothly and continuously over time, like gentle undulations on the open sea" ("All Too Random", para. 3).

While incompatible with the underlying assumptions of the older Copenhagen interpretation of quantum theory, the experimental findings of Minev's research team appear to be consistent with specific predictions derived from a wave function interpretation. There is, however, much that remains to be unpacked with regard to these relatively recent experimental findings. Quoting once again from Ball (2019):

> The gradualness of the "jump" is just what is predicted by a form of quantum theory called quantum trajectories theory, which can describe

[46] While I have not referenced this paper explicitly, I do note that its two parts may be found in volume 3 of *The British Journal for the Philosophy of Science*, issues 10 and 11, respectively.

> individual events like this. "It is reassuring that the theory matches perfectly with what is seen" said David DiVincenzo, an expert in quantum information at Aachen University in Germany, "but it's a subtle theory, and we are far from having gotten our heads completely around it." ("Flash of Insight", para. 2)[47]

Minev et al. (2019) have even noted that it may eventually be possible to predict (within a certain window of opportunity) when a quantum leap will occur, pointing out that, under quantum trajectory theory, "there is always a latency period prior to the jump, during which it is possible to acquire a signal that warns of the imminent occurrence of the jump" (p. 200).

The scope of the present chapter will not accommodate a detailed discussion of the research methodology employed by Minev's team; nevertheless, the implications of their experimental results are compelling, particularly considering how significant a shift in worldview they exemplify, when compared to the assertions of the prominent Copenhagen interpretation. Essentially, as of mid-2019, the phenomenon of the quantum leap no longer carries the same technical implications that it once did (i.e., implications of abruptness and unpredictability, discontinuity and instantaneity), and it is conceivable that the phrase 'quantum leap' could soon experience a disruption in its broader scientific and socio-cultural contexts as a result. Curiously, unlike the originary drift described by Burwell (2018), where "drift emerges simultaneously with the naming of the concept" (p. 12), this seems to constitute a case in which the drift emerges because the concept itself has *changed*, with a new material–mathematical relationship being realized in the process.

Much as occurred with Planck's initial implementation of quantization constraints and Bohr's modified approach to discretized orbits, the mathematics of Schrödinger's wave functions also necessitated a different physical characterization of atomic structure, with the overall concept of the atom undergoing yet another revision as a result. Like its immediate predecessor, Schrödinger's model of the atom retained the concept of discretized orbits; however the orbits themselves were approached in a novel way. No

[47] As a relatively new branch of quantum theory (developed in the 1990s), quantum trajectory theory posits, among other things, that the quantum leap *should* be both a continuous and coherent process. The experimental results of Minev et al. (2019) have provided the first empirical support for this theory.

longer conceived of as elliptical/cyclic, paths along which electrons move in fixed trajectories, they were instead re-envisioned as three-dimensional *regions* of space within which electrons could freely manifest as waves. Though each of these regions surrounding the atomic nucleus still corresponds to a quantized energy level and a non-radiating steady state, the idea of electrons moving along *set paths* within these quantized regions no longer applies.

With general concepts of space and time being integral to the ways in which human beings perceive, cognize, and characterize material reality (see Sutherland, 2004), the new organizing principles expressed by Schrödinger's theory of wave functions upset common views of these concepts in particular ways, in large part because it is difficult to rely upon intuitions about *position*/*location* when the wave-like properties of quantum entities (such as electrons) are embraced. With waves manifesting more as *distributed phenomena* than as *localized objects*, appealing to the wave-like nature of matter confounds the notion of electrons having precise positions in space at any given time (even when they are moving within the steady states of their quantized shells), and this inherent *uncertainty* motivates the modern conceptualization of atomic orbitals as a sort of blurry "electron cloud", an indistinct region surrounding the atomic nucleus, wherein electrons are characterized more by *probability densities* (i.e., mathematical descriptions of *potential* behaviours) than the kinds of traits that would classically be associated with physical objects. In this way, electrons have been reconceived as entities far more mathematically abstract than ever before, whilst still being material.[48]

Commonly represented by the Greek letter psi (Ψ), wave functions are complex-valued functions of position (i.e., x, y, and z coordinates) that describe the evolution of quantum entities/systems. In the case of wave functions with well-defined oscillating behaviour, momentum information may also be extracted from their wavelengths via the de Broglie

[48] At a certain level, these circumstances bring back to mind the question raised by Skemp (1987) about what kind of theory mathematics is (or might be), and his paradoxical sense of "getting further away from what is accessible to our senses" (p. 150) whilst simultaneously "getting closer to the essential nature of the universe" (p. 150). In light of the current chapter's examination of the history and development of atomic theory, the reader might now recognize that Skemp's concern actually speaks more to the underlying *worldview problem* identified by Wendt (2015) than it does anything inherently paradoxical.

relation $p = \frac{h}{\lambda}$ (for momentum 'p' in kilogram metres per second, wavelength 'λ', and Planck's constant 'h'). Together, position and momentum are the principal properties of interest when considering the overall states of electrons, and, as noted by Schroeder (1999), "the wavefunction serves the same purpose in quantum mechanics that the position and momentum vectors serve in classical mechanics: It tells us everything there is to know about what the particle is doing at some particular instant" (p. 362). In fairness, though it may be convenient to discuss the *purpose* of the wave function in terms analogous to those of classical mechanics, this quotation from Schroeder does come with important caveats resulting from the uncertainty described in the preceding paragraph. In particular, the idea of knowing *everything there is to know about what the particle is doing at some particular instant* must be heavily qualified.

Under Schrödinger's wave function interpretation, measurable variables (such as position and momentum) take the form of linear operators. Depending upon the specific pairing of variables in question, their corresponding operators may or may not commute, and when the latter proves to be the case (as with the operators for position and momentum), it is not possible to simultaneously measure both variables to an arbitrary degree of precision. Pairs of such non-commutating variables are said to be *mutually complementary*, and the non-commutativity (or mutual complementarity) of the position-momentum pair establishes the basis of Heisenberg's widely known uncertainty principle (which articulates the fundamental limit to the precision of simultaneous measurement for these two variables).[49]

A wide range of wave functions is possible, many of which are not particularly convenient to analyze. Perhaps the most convenient (and useful) wave functions are those for which either position or momentum is well-defined. The overarching limits of Heisenberg uncertainty prevent there from being cases in which both characteristics are well-defined;

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[49] Position and momentum are also referred to as *conjugate variables*, as they are related to one another via Fourier transforms. Pairings of any two conjugate variables imply corresponding degrees of computational uncertainty, and, in the case of variables with *physical* significance, observational uncertainty as well. Heisenberg's uncertainty principle for the position-momentum pair is neatly encapsulated by the inequality $\sigma_x\sigma_p \geq \frac{h}{4\pi}$ for the standard deviation in position 'σ_x', the standard deviation in momentum 'σ_p', and the eponymous Planck's constant 'h' (resurfacing again).

however, there are an abundance of instances in which *neither* is well defined.[50] In a manner of speaking, the complementarity that underlies Heisenberg's uncertainty principle is not unlike the complementarity inherent to wave–particle duality. In the case of the former, both position and momentum information are implicated in the full characterization of an electron's state at a given time, yet they can not be *measured* simultaneously with arbitrary precision. In the case of the latter, both wave and particle properties are implicated in the full characterization of the nature of matter, yet they do not *manifest* simultaneously (and hence can not be observed or measured simultaneously). Although the background contexts driving these respective situations differ, both inevitably speak to complementary facets of unified wholes.

With the order of their application mattering, the non-commutativity of the position and momentum operators has immediate ramifications for the order in which desired information about quantum systems might be collected (or if it might be collected at all); for making measurements on quantum systems interrupts the continuous evolution of a quantum wave function, leading to an *irreversible* process known as wave function collapse, and preventing additional measurements on the same system state.[51] Different measurements may be made as a *newly emerging* wave function evolves, yet current theory holds that there is no way of recovering a *prior* system state once an act of measurement collapses a particular wave function. This contributes to a larger quantum theoretical issue referred to as *the measurement problem*, which arises from the fact that measurements are physically significant, deterministic events whereas wave functions are strictly probabilistic functions that have no physical significance in and of themselves. This raises the questions of *precisely* what is happening when physical measurements are made on

[50] Cf. Figures A.5 through A.7 of Schroeder (1999, pp. 362–363), where these scenarios are illustrated in graphical representations of wave functions for particles moving along a single dimension.

[51] Again, slightly anachronistically, I introduce a term well in advance of its deeper treatment. The broader significance of *reversible* and *irreversible* processes is discussed in **Intralude C**, toward the end of this document.

quantum systems, what it means for probabilistic wave functions to collapse, and how these two phenomena are actually related.[52]

A second important caveat to the earlier quotation from Schroeder (1999) also speaks to the probabilistic nature of wave functions. As opposed to describing where electrons *are* at any given time, Schrödinger's wave functions instead deal with the *probability* of electrons being in particular regions of space at a given time. Thus, while the wave function model is incapable of revealing *precise* location information (as one might expect in the case of classical particles), it does reveal where an electron is *likely* to be, thereby treating the electron as a fundamentally *statistical* entity and mathematically expressing the conceptual uncertainties described in earlier paragraphs. It is also worth clarifying that, although the complex-valued wave function (Ψ) has no physical significance of its own, the square of the wave function does, with Ψ^2 being directly proportional to the probability of finding a particle described by Ψ in a given region at a given time. Thus, in addition to highlighting significant differences between the standard Copenhagen interpretation of quantum theory and Schrödinger's wave function interpretation, this interesting comparison between Ψ and Ψ^2 also helps to offer another sense of how the material might be seen as *emerging from* the mathematical (or at least how material properties might be seen as emerging from mathematical underpinnings).[53]

Coherence and Decoherence

The reader may recall from the discussion of **Intralude B**, that Witthaut et al. (2017) were quoted as stating that "quantum coherence and entanglement arise persistently through the same transition as synchronization" (p. 1). Within the more recent subsection concerning the changed contexts of the quantum leap, key findings from Minev et al. (2019) also revealed that the evolution of the quantum leap is both "coherent and continuous" (p. 203). Ball (2019) speaks further to the relevance of quantum coherence in the following abridged

[52] Burwell (2018) points out: "As inappropriate and inaccurate as it is in the quantum context, the logic of "thingness" that is embedded into our language remains inescapable" (p. 37).

[53] As Wendt (2015) describes it, "wave function collapse is a process that brings forth actuality from potentiality" (p. 263).

account of a significant practical challenge faced by Minev's research group during their efforts to "watch" an electron transition between discrete energy states:

> they couldn't monitor that transition directly, because making a measurement on a quantum system destroys the coherence of the wave function – its smooth wavelike behavior – on which quantum behavior depends. To watch the quantum jump, the researchers had to retain this coherence. Otherwise they'd "collapse" the wave function […] This is the problem famously exemplified by Schrödinger's cat, which is allegedly placed in a coherent quantum "superposition" of live and dead states but becomes only one or the other when observed. ("Seeing Without Looking", para. 4)

Not yet having addressed this important concept of coherence, but with a reasonable amount of theoretical background now established, I use the present opportunity to do so, and to elaborate on how the accompanying concept of *decoherence* is closely related to wave function collapse, yet also subtly different from it.

In fairly broad terms, coherence is an indication of the phase difference between two or more waves (typically measured in degrees). When waves of the same type are totally coherent, or completely *in phase*, they exhibit no phase difference whatsoever and interfere with one another only constructively. Conversely, waves that are totally decoherent, or completely *out of phase*, exhibit maximum phase difference and interfere with one another only destructively. As one might expect, innumerable cases involving *partial* coherence/decoherence are also possible, with the associated waves manifesting various combinations of constructive and destructive interference. These same general scenarios apply in the quantum domain as well, with the added stipulation that the coherence of quantum wave functions accompanies the phenomenon of quantum entanglement.

As noted above, *making a measurement on a quantum system destroys the coherence of the wave function*, reducing the wave function from a superposition of various quantum states to a single classical state. It could even be said that, by interrupting the wave-like evolution of a quantum system, the process of measurement *forces* the system into a singular state. In keeping with this distinction, but in a somewhat simplified sense, coherence can be thought of as expressing the degree to which bodies contributing to quantum systems remain entangled, or the degree to which entangled systems are free to continue evolving in accordance with their wave functions. Mathematically speaking, a quantum system's characteristic wave function results from the superposition of the wave functions of the various entangled bodies that constitute the system. When entangled, these bodies can no longer be described in isolation, as their ongoing evolution is tied to that of the system at large; however, it is possible for such bodies to *disentangle* from the system, with their wave functions decohering in the process. Interestingly, with varying degrees of decoherence being admissible, it is also possible for *portions* of quantum systems to disentangle without the wave functions of the larger systems collapsing entirely. Put slightly differently, the manifestation of decoherence does not necessarily *cause*, or even *guarantee*, wave function collapse; however, it could be described as a necessary precursor to it.

As the *loss* of quantum coherence, decoherence arises as an effect of quantum systems interacting with the classical environment, which is to say that the classical domain tends to *disentangle* quantum systems. Karen Barad (2007) offers the following insight:

> quantum behavior is difficult to observe because of the difficulty of shielding an object, especially a relatively large object, from interactions with its "environment", which continually fluctuates in an erratic fashion in such a way that a superposition is "randomized" into a mixture […]. This randomization process is called "decoherence". (p. 279)

If coherence is taken to be indicative of a movement toward *highly ordered* states (where multiple wave functions are unified through superposition), then decoherence might be comparable to a contrasting tendency toward *disordered states* (or the randomized mixture to which Barad alludes). Such a reading would suggest that decoherence is itself a particular manifestation of the all-encompassing thermodynamic principle of *entropy*, which has powerful implications for the ways in which the matter in our known universe organizes and reorganizes itself.[54]

As with my readings of certain other processes (discussed in **Intralude A**, the current chapter, and the upcoming **Intralude C**), I have been inclined to think of both coherence and decoherence as additional instances of what I refer to as *material assemblage in action*.

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[54] The third and final intralude of this dissertation (**Intralude C**) is devoted entirely to a discussion of entropy.

While this remains true of the former (coherence), the description from Barad given above reveals that it *may* not be appropriate to include decoherence under the same classification. At different points in this document it has been noted that the principles according to which entanglements organize and reorganize themselves are of significant interest; however, by virtue of its role in prefacing (but not directly causing) *disentanglement*, decoherence would seem to come as something of an antithetical principle. To reiterate a passage from Wise's entry in Charles Stivale's *Gilles Deleuze: Key Concepts* (2011), "An assemblage is not a set of predetermined parts […] Nor is an assemblage a random collection of things […] An assemblage is a becoming that brings elements together" (p. 91). Via Barad's interpretation, decoherence is inherently a process of *randomization*, which fundamentally places it at odds with Wise's characterization of assemblage. More crucially, as a process that signals and accompanies the *breakdown* of entanglements, decoherence can not exactly be seen as a kind of *becoming that brings elements together*. Thus, despite exemplifying/expressing the idea of structural change and reconfiguration, decoherence nevertheless remains difficult to reconcile with the manner in which I have already been discussing material assemblage, and it is conceivable that it might be worthwhile to articulate a corresponding process of material *disassemblage*.[55]

Conceptually, coherence and decoherence (as a connected pair) are concerned with the degrees of interaction that might or might not manifest between quantum systems and the classical environment. Inasmuch, they are also an attempt to understand how and why disentanglement leads to the emergence of classical behaviours. With acts of measurement, or observation, serving as catalysts for the decoherence of quantum wave functions and the breakdown of entanglements more generally, the wave function interpretation of quantum theory does away with the classical assumption of a detached and purely objective observer whose presence and watchful gaze have no impact on the phenomena of interest. Indeed, the manner in which the observer of quantum phenomena *participates in* (and reshapes) the processes through which matter organizes and reorganizes itself is very much aligned with the enactivist sensibilities that I express elsewhere in this document: namely that

[55] An alternative interpretation that *might* carry some potential for resolution is presented in the aforementioned **Intralude C** and its discussion of entropy. In the event that the reader opted to engage with the intraludes out of order, they may have already encountered it.

human beings exist *in relation to the world* and not separate from it. Within ***footnote 12*** of **Chapter 1**, I mentioned the desire to (somehow) retroactively amend the etymology of Maturana and Varela's 'enaction' such that it might enhance the idea of *embodied action* by incorporating the idea of *entangled action*, which could be more profound. This cursory overview of coherence and decoherence, measurement and observation, offers additional insight into why that is the case.

To be clear, I acknowledge that shifting the focus to *entangled action* would necessitate a deeper adherence to quantum theoretical formalisms, which is to say that a *non-classical*, entangled relationality would differ fundamentally from a classical relationality. As an example, it was previously noted that Maturana and Varela's structural coupling, which emerges from the biological roots of their enactivism, is framed according to *classical interactions* between unified systems and their environments. Though I view structural coupling as a construct *comparable* to quantum entanglement, I must also concede that its classical basis prevents it from properly accounting for the strangeness of actual quantum phenomena (such as instantaneous action at a distance and superpositions of states). It is difficult to describe what a non-classical, entangled relationality might "look like"; but, Karen Barad's *agential realist* ontology has emerged as a preeminent modern example. An extremely important foundational concept within this ontology is that of *intra-action*, which Barad (2007) describes as follows:

> "intra-action" *signifies the mutual constitution of entangled agencies*. That is, in contrast to the usual "interaction", which assumes that there are separate individual agencies that precede their interaction, the notion of intra-action recognizes that distinct agencies do not precede, but rather emerge through, their intra-action. It is important to note that the "distinct" agencies are only distinct in a relational, not an absolute, sense, that is, *agencies are only distinct in relation to their mutual entanglement*; *they don't exist as individual elements*. (p. 33)

Barad's agential realism invokes a relationality that accounts for the ongoing materiality of not only the human, but the *non-human* and the *conceptual* as well, shifting from "a metaphysics of things" (p. 33) to a metaphysics of phenomena. Since this metaphysics of phenomena *includes* quantum phenomena, it automatically moves Barad's framework beyond the scope of the enactivist orientation as it was originally conceived by Maturana and Varela. Again, tying back to ***footnote 12***, this raises the interesting question of whether it might be possible to reformulate the classical relationality and interactions of enactivism *through* Barad's non-classical relationality and the intra-actions of agential realism. Regardless, in keeping with assertions made in **Chapter 1**, a proper discussion of Barad's agential realist account and the overarching topic of *material agency* is deferred for future explorations and shall not be presented in this document.

As enactments that force quantum systems out of their entanglements, measurement and observation take on radically different meanings compared to those they may have had in the scientific worldviews that preceded quantum theory; yet, even *within* quantum theory, the notion of observation (in particular) carries a range of connotations, and not all quantum theoretical perspectives consider the role of the observer to be one that is necessarily *detrimental* to quantum entanglements. In the final section of this chapter, I will communicate one such existing perspective, which converges with certain ideas that I advocate within this document, whilst simultaneously diverging from others.

Undivided Wholeness

In previous chapters and intraludes, I have spoken to the concept of entanglement through other constructs that I believe to be closely related, most notably the assemblage theory of Deleuze and Guattari (in **Chapter 1**, **Chapter 3**, **Intralude A**, and **Intralude B**), and Maturana and Varela's structural coupling (in **Chapter 1** and **Intralude B**). Even Jennifer Burwell's originary drift (which resurfaces on multiple occasions) is thematically similar in some respects. In turn, the current chapter has lightly touched upon entanglement as it pertains to the wave function interpretation of quantum theory and the accompanying principles of quantum coherence/decoherence and measurement/observation. Though each of these is invariably beholden to its own lineage of ideas and discourse-specific commitments, there is nevertheless a sense in which such constructs might be taken as variations on a common theme, for all of them evoke ideas of deep interconnectedness and mutual emergence.

While the material–mathematical worldview that underpins this program of research primarily leverages principles rooted in enactivism and quantum theory; it has also been influenced by perspectives that transcend the disciplinary borders regularly associated with these theories, and which reflect broader sensibilities about entanglement and material being/becoming/knowing. One such alternative perspective is exemplified by the work of the American theoretical physicist and philosopher David Bohm, which provides a compelling counterpoint to some of the more established interpretations of quantum theory. I here offer an abbreviated sketch of Bohm, whose contributions to physics are (now) widely regarded as highly insightful and even revolutionary, despite having been dismissed by prominent members of the scientific community in the 1950s and 1960s.

David Bohm is regularly characterized as something of a visionary whose explorations exceeded the scope of the physical sciences alone. In an effort to seek out a deeper truth regarding the foundations of reality, he advocated for the convergence of theoretical and spiritual perspectives at a time when doing so was largely shunned/dismissed by an established scientific community that deemed such endeavours to be overly esoteric and misguided. During the era of Cold War politics marked by McCarthyism, anti-communist rhetoric, and significant social unrest in the United States, Bohm was subjected to discrimination and institutional blackballing as a result of his political stance and specific communist group affiliations. Although much of Bohm's early academic work was either suppressed or ignored by the Western academic orthodoxy of the day, it has since seen a major resurgence, with a wider array of scholars and practitioners from a variety of fields/disciplines re-engaging with Bohm's perspectives in recent decades (Freire, 2019). At the moment, I can not definitively say that all of Bohm's theoretical, philosophical, and spiritual commitments align with my own, however certain elements of his work do make assertions in keeping with those I wish to forward. Bohm's approach to the concept of entanglement is of primary interest here, as it speaks directly to the manner in which human beings coexist within the material world, and how a significant (yet subtle) aspect of the material–mathematical relationship might underlie that coexistence.

As is evident from Freire's biography of Bohm, Bohm's career was one marked by extensive interdisciplinary exploration, a rare kind of open-mindedness, and a profound drive to unite disparate perspectives that he considered to be equally relevant to the realms of theoretical physics and everyday life. His sense of the deeply interconnected nature of all things, or what he would refer to as the *undivided wholeness of the universe*, would be a conceptual hallmark that informed Bohm throughout his career.[56] He was motivated by fundamental questions about the nature of existence and reality, and searched for wholeness at the intersection of science and spirituality. His overlapping explorations in quantum physics and the nature of consciousness bear this out, as do his recurrent dialogues with Indian philosopher Jiddu Krishnamurti, who is known to have played a considerable role in the development of Bohm's thinking regarding the interconnectedness of mind and matter (Freire, 2019).

Presented in Chapter 5 of *Wholeness and The Implicate Order* (1980/2002) is Bohm's characterization of quantum theory "AS AN INDICATION OF A NEW ORDER IN PHYSICS" (p. 141; capital letters in original), through which he communicates the idea that "[r]evolutionary changes in physics have always involved the perception of new order and attention to the development of new ways of using language that are appropriate to the communication of such order" (p. 141). Despite its succinctness, the latter passage resonates with a number of themes already addressed in this dissertation. Bohm's reference to *new ways of using language* can be read through Burwell's originary drift with respect to evolving discursive spaces, and his mentions of *perception* and *attention* bring back to mind the notion of awareness forwarded by Wheeler. More importantly, though, Bohm's overall thought regarding the emergence of *a new order* (in physics) might be considered through the lens of Deleuze and Guattari's assemblage theory, as well as my own sensibilities about the principles underlying the organization and reorganization of matter.

Shortly after the offering above, Bohm (1980/2002) raises the question of what 'order' actually is, at the same time acknowledging that the word spans a wide range of contexts, in many of which "its meaning can be seen fairly clearly from its usage" (p. 146). However,

[56] See Bohm's seminal 1980 publication *Wholeness and The Implicate Order* (republished in 2002), as well as the subsequent 1993 volume *The Undivided Universe: An Ontological Interpretation of Quantum Theory*, cowritten by Bohm and long-time colleague/collaborator Basil Hiley.

he also notes that these more transparent usages do not capture the broader contexts in which he intends to apply the term. Bohm (1980/2002) elaborates:

> we do not restrict order to some regular arrangement of objects or forms in lines or in rows […] Rather, we can consider much more general orders, such as the order of growth of a living being, the order of evolution of living species, the order of society, the order of a musical composition, the order of painting, the order which constitutes the meaning of communication, etc. If we wish to inquire into such broader contexts, the notions of order to which we have referred earlier in this chapter will evidently no longer be adequate. We are therefore led to the general question: 'What is order?' (p. 146)

Having raised this question, Bohm (1980/2002) proceeds to point out the difficulty inherent to supplying a proper response, suggesting that:

> The notion of order is so vast and immense in its implications […] that it cannot be defined in words. Indeed, the best we can do with order is to try to 'point to it' tacitly and by implication, in as wide as possible a range of contexts in which this notion is relevant. We all know order implicitly, and such 'pointing' can perhaps communicate a general and overall meaning of order without the need for a precise verbal definition." (p. 146)

In light of the diverse scenarios of interest to Bohm, and considering that the notion of order might also incorporate/implicate processes such as *arranging*, *organizing*, *sequencing*, *structuring*, et cetera, Bohm's paired ideas of *knowing order implicitly* and *tacitly pointing to order* are similar to Gattegno's (1988) observation that "mathematics is recognizable but not easily defined" (p. vii), leading us to access it through different constructs. Though Bohm does not go to the point of identifying an alternate construct through which to access order, as Gattegno does with mathematics and *mathematization*, I would suggest that the passage above might nevertheless be read as an example in support of Gattegno's view. By this, I mean to imply that, by grappling with the notion of order, Bohm is *also*, if indirectly (or unintentionally), grappling with the elusive character of *mathematics* (or at least a facet of it).

While discussing aesthetic dimensions of 'pattern' across mathematics and the arts, Martin Schiralli (2006) speaks to the nature of order as well, with the following excerpt providing a provisional account that emphasises the non-arbitrary relationality inherent to ordered arrangements.

> to discern a pattern is to see or consider something as part of an ordered arrangement such that it is possible to identify at least one of the principles constituting that order. To say an arrangement is ordered is to claim that the relationships among the arranged phenomena are not arbitrary, that the arrangement may be at least partially described in terms of one or more relational principles or themes. (pp. 118–119)

Schiralli also directly contrasts 'pattern' with 'order', asserting that, "Unlike 'order', which may often be considered in almost exclusively abstract and formal terms, there is something palpable about 'pattern' that reaches directly into the world of the senses and experience" (p. 119). Read in concert with Bohm's commentary, this suggests that 'pattern' may be one of the constructs that *points to order tacitly*, and through which we come to *know order implicitly*, although it is clear that Schiralli is seeking more precise language to characterize these concepts.

Perhaps as a result of my own efforts to articulate precisely what 'mathematics' is, and what I mean when referring to 'the principles according to which matter organizes and reorganizes itself', I am sympathetic to Bohm's concession that the notion of 'order' can not be defined in words. At the same time, and in light of efforts like Schiralli's to express mathematical/aesthetic sensibilities about 'order', I am not entirely convinced that this is the case, preferring instead to absorb 'order' into the notion of mathematization discussed in **Chapter 3**. This informs my reading of Bohm's (1980/2002) view of quantum theory "AS AN INDICATION OF A NEW ORDER IN PHYSICS" (p. 141), as it captures the sense in which quantum theory as a whole might be treated as a new kind of *awareness* about *innate mathematical structures*.

Despite its incredible descriptive efficacy, it is important to note that quantum theory (the physics of *the very*, *very small* and *the very*, *very fast*) remains irreconciled with general relativity (which describes how gravitational effects influence the geometry of spacetime). To date, these theories appear to be fundamentally incompatible with one another, as the former invokes a discrete (or discontinuous) field theory, while the latter entails a continuous field theory.[57] In part, Bohm's notion of *undivided wholeness* was proposed as a foundational principle that both theories could potentially have in common (i.e., a unifying ordering principle), conveying Bohm's thought that a deeper level of order might actually underlie *all* material structures and processes. This notion of a deeper level of order would eventually lead to the formalization of his paired ontological concepts of the *implicate* (or enfolded) order and the *explicate* (or unfolded) order, which together offer an account of how/why certain phenomena (such as quantum phenomena), manifest differently under different contexts. Though the terms 'implicate', 'explicate', 'enfolded', and 'unfolded' might be suggestive of a binary opposition, Bohm (1980/2002) clarifies that the relationship between the implicate order and the explicate order is more appropriately conceived of as a kind of nesting or embedding, such that the explicate may be "regarded as a particular or distinguished case of a more general set of implicate orders" (p. 226). Of the explicate order, he makes the following assertions:

> explicate order arises primarily as a certain aspect of sense perception and of experience with the content of such sense perception. It may be added that, in physics, explicate order generally reveals itself in the sensibly observable results of functioning of an instrument.
>
> What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. (p. 200)

By emphasizing the relevance of *sense perception*, *sensibly observable results*, *the functioning of instruments*, and *a Euclidean system of order and measure*, Bohm forges a strong link between the explicate order and the material world as it is known/understood through the everyday experience of space and time. Considering that the human body can itself be treated as one of the "instruments" whose functioning yields sensibly perceptible content, this might even be read in concert with Wendt's (2015) remarks about space and time as universal facets of human experience (see **Chapter 1**). That is, with knowledge of Bohm's explicate order emerging primarily through sensory perception and observation, it

[57] Modern research in the field of *quantum gravity* seeks to reconcile these disparate frameworks.

seems closely aligned with Wendt's (2015) reference to human beings sharing *the same physics of the body*.

With regard to the implicate, or enfolded, order, Bohm (1980/2002) states that:

> space and time are no longer the dominant factors determining the relationships of dependence or independence of different elements. Rather, an entirely different sort of basic connection of elements is possible, from which our ordinary notions of space and time, along with those of separately existent material particles, are abstracted as forms derived from the deeper order. These ordinary notions in fact appear in what is called the explicate or unfolded order, which is a special and distinguished form contained within the general totality of all the implicate orders. (p. xviii)

Under Bohm's formulation, then, the everyday notions of space and time emerge from a deeper fundamental order, with the primary principles of the implicate order underlying the secondary (but perhaps more accessible) principles of the explicate. This would seem to be the distinction that motivates Bohm's (1980/2002) reference to quantum theory "AS AN INDICATION OF A NEW ORDER IN PHYSICS" (p. 141), and he later clarifies that the laws of physics have typically "referred mainly to the explicate" (p. 189), whereas he proposes a formulation in which the laws of physics attend primarily to the implicate.[58]

The manner in which Bohm embeds the explicate in the implicate is very much how I envision the material being embedded in the mathematical (or of mathematical structures and processes *underlying* material reality). In keeping with that theme, it is also notable that the extended quotation from Bohm above speaks to the plurality of the implicate (i.e., *the general totality of all the implicate orders*). Read through Wheeler's (2001) plurality of 'awarenesses' and my own approach to mathematization, this immediately suggests that different types of awareness may be developed with respect to the organizing principles of Bohm's implicate order (and that those awarenesses do not necessarily need to converge to a single coherent outlook at the level of the explicate).

[58] To quote Bohm (1980/2002) directly, "we are proposing that in the formulation of the laws of physics, primary relevance is to be given to the implicate order, while the explicate order is to have a secondary kind of significance" (p. 190).

As was previously expressed, the paired concepts of measurement and observation carry great importance in quantum physics, particularly under Schrödinger's wave function interpretation. Interestingly, Bohm and Hiley point out that Bohm's account of quantum theory does *not* assume the participation of an *outside observer*.[59] Though this may initially seem odd/contradictory, given Bohm's characterization of the explicate order and its ties with sense perception and observation, it actually is not, as Bohm's approach to quantum entanglement under the *implicate* order ultimately reframes the relationship between the observer and the observed.

Unlike the wave function interpretation outlined in the previous subsections of this chapter, Bohm does not draw a causal link between measurement/observation and wave function collapse. Rather, he describes the dynamics of the situation differently, such that measurement/observation does not involve the *breakdown* of a given entanglement, but instead *expands the scope* of the entanglement, with the observer entering into the entangled system itself. Essentially, Bohm avoids the issues associated with wave function collapse and the associated measurement problem by attributing a different meaning to the phenomenon of measurement/observation. This is why Bohm does not (and need not) assume the participation of an outside observer, for the notion of the observer *being* outside of an observed quantum system is no longer tenable. As expressed in the introduction of Bohm and Hiley's 1993 collaborative work:

> The probability of a particular result of the interaction between the instrument and the observed object is shown to be exactly the same as that assumed in the conventional interpretation. But the key new feature here is that of the undivided wholeness of the measuring instrument and the observed object, which is a special case of the wholeness to which we have alluded in connection with quantum processes in general. Because of this, it is no longer appropriate, in measurements to a quantum level of accuracy, to say that we are simply 'measuring' an intrinsic property of the observed system. Rather what actually happens is that the process of interaction reveals a property involving the whole context in an inseparable way. Indeed it may be said that the measuring apparatus and that which is

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[59] See p. i in the introductory frontmatter of Bohm and Hiley's *The Undivided Universe: An Ontological Interpretation of Quantum Theory* (1993).

> observed participate irreducibly in each other, so that the ordinary classical and common sense idea of measurement is no longer relevant. (p. 5)

In the 1999 volume *The Limits of Thought: Discussions Between J. Krishnamurti and David Bohm*, Bohm reiterates this idea more simply by stating that, "In some sense there is no division of the observer and the observed" (p. 34), with the even more condensed articulation that *the observer is the observed* often being attributed (informally) to Krishnamurti.

This view, in which there is ultimately no separation between the observing entity or measuring device and the objects/phenomena with which it interacts, encapsulates Bohm's broader notion of entanglement, as well as the sense in which his implicate order operates at a deeper level than is generally accounted for within the natural sciences. It also helps to communicate a key principle with which I connect the concept of quantum entanglement (specifically, Bohm's *undivided wholeness*) to the enactivist sensibilities forwarded in this document.[60]

[60] Bohm points out that the Chapter 6 appendix of his *Wholeness and the Implicate Order* (1980/2002) specifically discusses the implicate order from a *mathematical* standpoint. While I have not engaged with this content directly, I anticipate that the ideas Bohm articulates within will offer more insight regarding the extent to which his perspective might be compatible with my own material–mathematical worldview.

Chapter 5.
Educational Perspectives

"We are more complex than our mental faculties are capable of grasping. The hypertrophy of our frontal lobes is considerable, and has taken us to the moon, allowed us to discover black holes and to recognize that we are cousins of ladybirds. But it is still not enough to allow us to explain ourselves clearly to ourselves."

–Carlo Rovelli (2018)
The Order of Time, pp. 179–180

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In the opening chapter of this dissertation, I foregrounded the epistemological stance that *what we know about mathematics is mutually implicated in what we know about ourselves as mathematicians*. Though the brief passage from Rovelli above appears in the contexts of a somewhat different discussion, I nevertheless read the same sentiment into it as well, particularly when considering that (despite the great depth and breadth of humanity's collective mathematical know-how), we continue to grapple with much uncertainty regarding how we, as individuals, come to know mathematics at all. This is to say, even when individuals are believed to demonstrate forms of mathematical understanding, it is never entirely obvious, from an educational standpoint, *how* or *why* said understanding manifests.[61] It is in the current chapter that I revisit the aforementioned epistemological stance by speaking more directly to the manner in which the material and the mathematical are at a confluence within the embodied self, and how the mathematical sensibilities of the embodied self might develop *alongside* sensibilities about the material world with which we engage. This chapter also communicates a tentative view of how mathematical sense-making practices are rooted in the dynamics of the material–mathematical relationship.

The reader will recall that **Chapter 3** teased apart some well-known perspectives about the *nature of mathematization* and the *psychology of learning mathematics*. In so doing, it revealed a number of implications arising from deeper commitments, and also indicated

[61] Depending, perhaps, upon the specific parties and organizations concerned, this might (re)present an educational problematic of greater or lesser significance.

how such commitments, be they implicit or explicit, inevitably seep into the broader educational practices through which mathematical ideas are presented/communicated (thus shaping the educational discourse surrounding mathematics itself). Extending forward from that discussion, this new chapter draws upon insights from the philosophy of mathematics education, in order to establish scholarly grounding for the notion that *how we come to know mathematics is intimately entwined with how we come to know ourselves*. This slightly reframes the epistemological stance expressed previously, by shifting the focus from the *objects* of knowledge (i.e., *what* we know) to the *processes* of knowing (i.e., *how we come* to know).

Within his influential 1991 volume *The Philosophy of Mathematics Education*, social constructivist and mathematics educator Paul Ernest provides a comprehensive survey of relevant perspectives, a sampling of which is shared here in an effort to situate the alternative perspective emerging from the current exploration. Beyond that, through a somewhat more practical offering, a revisitation of the quantum leap introduced in **Chapter 4** will illustrate how this alternative perspective might be used to express key principles from elementary arithmetic by emphasizing specific aspects of the material–mathematical relationship. The reader may find that this latter offering allows the more prominent perspectives described by Ernest to be contextualized slightly differently.

Standard Theories in the Philosophy of Mathematics Education

In the introductory pages of *The Philosophy of Mathematics Education*, Ernest (1991) remarks that "[a]t least four sets of problems and issues for the philosophy of mathematics education can be distinguished" (p. xii). These include *the philosophy of mathematics*, *the nature of learning*, *the aims of education*, and *the nature of teaching*.[62] When speaking to the philosophy of mathematics specifically, Ernest raises the following primary questions: "What is mathematics, and how can we account for its nature? What philosophies of mathematics have been developed? Whose?" (p. xii). The reader will note that these questions closely accord with the themes of the current dissertation, and reflect a number

[62] Clearly, it can be argued whether these are mutually exclusive or overlapping sets.

of the topics already addressed. In particular, the philosophical and theoretical backdrop provided in **Chapter 1** establishes a basis for inquiries akin to Ernest's. Although literature from Ernest did not inform the earlier chapters of this document, it should quickly become apparent why his work is being incorporated into the discussion at this point.

Concerning the second issue for the philosophy of mathematics education*, the nature of learning*, Ernest (1991) goes on to ask: "What philosophical assumptions, possibly implicit, underpin the learning of mathematics? Are these assumptions valid? Which epistemologies and learning theories are assumed?" (p. xiii). **Chapter 3** of this dissertation touched upon similar questions by engaging with the implicit mathematical worldview of Richard Skemp and more explicit assertions from David Wheeler and Hans Freudenthal. Portions of that discussion will be reiterated and supplemented in the closing chapter of this document, but are not expressly re-entered here.

Ernest also breaks down the remaining sets of problems/issues for the philosophy of mathematics education according to their own foundational queries; however, in keeping with the acknowledgments offered in my opening preface (i.e., that I shall not focus on the teaching of mathematics) and in **Chapter 1** (i.e., that I wish to maneuver around some of the more politically oriented themes), *the nature of teaching* and *the aims of education* are issues that I will not speak to directly. As a general concession, since it will not be possible to delve into the philosophical milieu as deeply or as broadly as Ernest does, I offer only a highly condensed review of his insights related to *the philosophy of mathematics*, as those are the most closely aligned with the themes/aims of this document.

To be clear, Ernest (1991) initially orients readers to his viewpoint by explicitly characterizing the philosophy of mathematics as "the branch of philosophy whose task is to reflect on, and account for the nature of mathematics […] a special case of the task of epistemology which is to account for human knowledge in general" (p. 3). Alongside the primary questions raised earlier, he also acknowledges that the philosophy of mathematics asks: "What is the basis for mathematical knowledge? What is the nature of mathematical truth? What characterises the truths of mathematics? What is the justification for their assertion? Why are the truths of mathematics necessary truths?" (p. 3). Though this brief excerpt emphasizes epistemological considerations, Ernest attends closely to ontological considerations as well, by asserting that *any* proposed philosophy of mathematics must also account for the nature/origins of mathematical objects, as well as the broader applicability/effectiveness of mathematics "in science, technology, and other realms" (p. 27). In point of fact, when discussing his own social constructivist view of mathematical knowledge, Ernest himself calls back to Wigner's seminal 1960 publication *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* (which has resurfaced a number of times throughout this dissertation).[63]

The opening chapters of Ernest's *The Philosophy of Mathematics Education* (1991) and his later volume *Social Constructivism as a Philosophy of Mathematics* (1998) are devoted to identifying and unpacking various "isms" that encapsulate the major perspectives within the philosophy of mathematics. Chief among these are *absolutism* and *fallibilism*, which are typically played against one another in binary opposition. From the former perspective, mathematics tends to be portrayed as a discipline whose truths are entirely beyond reproach and whose assertions are undeniable, thus allowing for a kind of certainty not common in other disciplines. Moreover, knowledge that stems from accessing or engaging with absolute mathematical truths is taken to be both *objective* and *immune to criticism*. In Ernest's words (1991):

> The absolutist view of mathematical knowledge is that it consists of certain and unchallengeable truths. According to this view, mathematical knowledge is made up of absolute truths, and represents the unique realm of certain knowledge, apart from logic and statements true by virtue of the meanings of terms [i.e., analytic statements]. (p. 7)

In direct contrast to the (supposed/alleged) objective certainty of absolutism, fallibilism admits that mathematical truths are not only subject to the possibility of *error*, but also *corrigible* (a term used frequently by Ernest), such that associated assertions are perpetually open to correction, revision, negotiation, and even contradiction. Thus, for the fallibilist, mathematical knowledge can *not* be equated to absolute truth, nor can it have or exhibit unquestionable validity. Essentially, by virtue of the uncertainty it engenders,

[63] Not actually having *set out* to propose a philosophy of mathematics, I will not claim to be doing so here, despite the similarities between Ernest's considerations and my own.

fallibilism imbues mathematical knowledge with qualities of fluidity and open-endedness that are unheard of within the absolutist dogma.[64]

Ernest (1991) describes the rejection of absolutism as the *negative form* of the fallibilist thesis, whereas embracing the corrigibility and negotiability of mathematical knowledge is expressed as the corresponding *positive form*. These are taken by Ernest to be functionally equivalent; however, a more nuanced explication emerges from his 1998 work, where he further distinguishes between different interpretations of fallibilism.

> The first is the doctrine that mathematical knowledge is or may be false. This implies that absolute true/false judgements about mathematical knowledge claims can be made, that is, there is absolute truth, but mathematics may fail to attain it. The second version rejects the assumption that absolute judgements regarding truth/falsity and correctness/incorrectness can be made, on the grounds that the relevant criteria and definitions, including the rules of truth and proof, change and will never attain a final state. This version of fallibilism […] leads to the view that mathematical knowledge is a relative, contingent, historical construct. (p. 36)

Ernest aligns himself with the second of these interpretations, and the final sentences of the passage above point toward the basis of his social constructivist account (in some ways an amalgam/synthesis of features from other fallibilist philosophies of mathematics).

Without doubt, absolutist representations of mathematics and mathematical knowledge have met with significant criticism in the twentieth and twenty-first centuries, much of which has been raised by Ernest himself. Ernest (1991) posits that, "the choice of which of these two philosophical perspectives is adopted is perhaps the most important epistemological factor underlying the teaching of mathematics" (p. 3), and he is unambiguous about his opposition to the absolutist doctrine. At the same time, Ernest is careful to acknowledge that the rejection of absolute mathematical truth "should not be seen as a banishment of mathematics from the Garden of Eden, the realm of certainty and truth" (p. 20), and that the *loss of certainty* "does not represent a loss of knowledge" (p.

[64] Cf. Lakatos (1976) and Szpiro (2008), both of whom skillfully illustrate this quality of *corrigibility*.

20).[65] His subsequent commentary is of particular interest here, in that it draws analogy between the epistemological uncertainty that characterizes fallibilism and a parallel form of uncertainty within the quantum theoretical discourse, namely Heisenberg's uncertainty principle, which I have touched upon in the preceding chapter. While two similar commentaries can be found in Ernest's 1991 and 1998 publications, the slightly more refined elaboration below appears in the latter:

> In quantum theory, Heisenberg's uncertainty principle means that the notions of precisely determined measurements of position and momentum […] had to be given up. But this does not represent the loss of knowledge of absolute frames and certainty. Instead it represents the growth of knowledge, bringing with it a realization of the limits of what can be known, given present theories. Relativity and uncertainty in physics represent major advances in knowledge which take humanity to the limits of what can be known (while the theories are retained). Analogously in mathematics, with increased knowledge of the foundations of mathematics it appears to many that the absolutist view is an idealization, more a myth than a reality. To fallibilists, this represents an advance in knowledge, not a retreat from past certainty. (pp. 34–35)

Though perhaps not obvious, this move to reframe the presumption of loss in terms that instead speak to the growth of knowledge through the recognition of limitations indicates how the fallibilist perspective opens up the space of mathematical meaning-making possibilities, while at the same time qualifying the range across which those possibilities might extend at any time. Ernest's use of such measured phrases as "given present theories" and "while the theories are retained" conveys the idea that the scope of mathematical knowledge will always be subject to particular constraints, if even those constraints are continually evolving and changing in response to (or in tandem with) larger contexts.

Owing to the more flexible and contextually dependent notion of mathematical truth instanced by the fallibilist perspective, Ernest (1991) suggests that it is worthwhile to revisit underlying assumptions associated with the philosophy of mathematics itself. Particularly when maintaining an absolutist mindset, it is common to accept the hypothesis that "mathematical knowledge is a set of truths, in the form of a set of propositions with

[65] Ernest attributes the phrase "loss of certainty" to mathematician Morris Kline, author of the 1980 volume *Mathematics: The Loss of Certainty*.

proofs, and that the function of the philosophy of mathematics is to establish the certainty of this knowledge" (p. 23). However, as Ernest urges, "Having found that this hypothesis is untenable we are forced to reconsider the nature of the philosophy of mathematics" (p. 23), both in terms of its function and purview. This deeper degree of cognizance regarding the changed philosophical backdrop prompts a new range of questions, which Ernest (1991) describes as representing "a broadening of the scope of the philosophy of mathematics from the internal concerns of absolutism" (p. 25). Elaborating further:

> Mathematics is multi-faceted, and as well as a body of prepositional knowledge, it can be described in terms of its concepts, characteristics, history and practices. The philosophy of mathematics must account for this complexity, and we also need to ask the following questions. What is the purpose of mathematics? What is the role of human beings in mathematics? How does the subjective knowledge of individuals become the objective knowledge of mathematics? How has mathematical knowledge evolved? How does its history illuminate the philosophy of mathematics? What is the relationship between mathematics and the other areas of human knowledge and experience? Why have the theories of pure mathematics proved to be so powerful and useful in their applications to science and to practical problems? (p. 25)

This revised series of questions suggests a much more relational approach to the philosophy of mathematics that is intimately associated with the attitudes, aims, and activities of human beings, and which immediately counters the absolutist assumption of objectivity by openly acknowledging the *subjectivity* of the many thinking, knowing, sensing human beings who are individually and collectively coming to know mathematics and participating in the evolution of mathematical activity at large. From Ernest's perspective, the amended queries above represent "the proper task of the philosophy of mathematics, which was obscured by the mistaken identification [on the part of the absolutists] of the philosophy of mathematics with the study of the logical foundations of mathematics" (p. 27). He earlier cites a passage from British philosopher Stephan Körner, which also feeds into this same line of reasoning:

> As the philosophy of law does not legislate, or the philosophy of science devise or test scientific hypotheses—the philosophy of mathematics does not add to the number of mathematical theorems and theories. It is not mathematics. It is reflection upon mathematics, giving rise to its own particular questions and answers. (p. 23)

Within the philosophy of mathematics, absolutism manifests in a number of ways, and by countering its core assumptions, fallibilism essentially speaks to three constituent "isms". Together, *Logicism*, *Formalism*, and *Constructivism* (of which *Intuitionism* is one particular subclass) account for the major schools of thought within the absolutist tradition, all of which seek to uphold the certainty of mathematical knowledge.

Logicism subordinates mathematical truth to axiomatic rules of inference, such that logical principles supersede the assertions of pure mathematics. In this way, *Logicism* subsumes the mathematical *into* the logical, with the certitude of mathematical knowledge being claimed as a result of its rooting in logical foundations. Historically, major proponents of this view have included Alfred North Whitehead, Gottfried Liebniz, Gottlob Frege, Rudolph Carnap, and Bertrand Russell (among others), and according to Ernest (1991), it was Carnap and Russell who were largely responsible for formalizing the primary logicist knowledge claims: 1) All mathematical concepts can be reduced to logical foundations, provided these include set theoretic, or similar, principles; and 2) All mathematical truths can be proven by logical inference alone (paraphrased from Ernest, 1991, p. 9). As scholars such as Whitehead and Russell would discover, however, it is when the second of these knowledge claims comes under scrutiny that the logicist thesis falters and ultimately fails, for there *are* cases in which logical inference alone is not sufficient to establish foundational mathematical truths.[66] In the succinct phrasing of Ernest (1991), "This means that the axioms of mathematics are not eliminable in favour of those of logic" (p. 10), and "Logic does not provide a certain foundation for mathematical knowledge" (p. 10).

With the program of *Logicism* asserting that mathematics can be known *a priori* (i.e., by virtue of its deductive basis), it also becomes clear that this paradigm disregards experiential factors and any role that the senses might play in the processes of coming to know mathematics. Consequently, *Logicism* ignores the possibility of engaging with mathematical structures and processes at the material level, and pays no attention to the

[66] Two commonly cited examples involve the set theoretic *Axiom of Infinity* and the *Axiom of Choice*, both of which are discussed at length in Ernest (1991) and Ernest (1998).

broader experience of the embodied self. As a philosophical orientation, *Logicism* does not, indeed can not, entertain notions of embodied (mathematical) cognition.

There are notable respects in which the paradigm of *Formalism* bears a resemblance to the logicist tradition, for it is also characterized by a kind of reductionist view of mathematics. As Ernest (1998) offers, "formalism is the view that mathematics is a meaningless formal game played with marks on paper" (p. 16), which is to say that mathematics is devoid of any inherent meaning beyond what is extracted from its symbolic systems and their associated rules of use. [67] It could be said that this notion of a formal game with fixed rules of play mirrors the logicist affinity for *rules of inference*, at least in so far as rules imply a set of consistent, overarching principles that limit or constrain action in a coherent manner.

Alongside David Hilbert, scholars such as John von Neumann and the American mathematician Haskell Curry are recognized as prominent figures within the formalist tradition (see Ernest, 1991 and Ernest, 1998), with Hilbert acting as something of a proxy/representative for the formalist program as a whole. Similar to *Logicism*, the formalist thesis consists of two primary knowledge claims. Quoting Ernest (1998): 1) "Pure mathematics can be expressed as uninterpreted formal systems in which the truths of mathematics are represented by formal theorems" (p. 18); and 2) "The safety of these formal systems can be demonstrated in terms of their freedom from inconsistency, by means of metamathematics" (p. 19). As it is here understood, the *uninterpreted formal systems* to which Ernest refers are the collections of mathematical symbols, or the same "marks on paper" alluded to previously. Thus, put slightly differently, at its core, *Formalism* seeks to express mathematical truths in purely symbolic terms that encode formal theorems within strings of symbols whose overall consistency should guarantee the certitude of mathematics as a whole. The subsequent appeal to *metamathematics* (or what Hilbert might call *proof theory*) in the second formalist knowledge claim indicates the further need to adhere to specific syntactic rules when representing mathematical truths in symbolic form. Beyond this, the concept of metamathematics also shows that mathematics

[67] It is worth noting that, compared to the logicist tradition, *Formalism* does entertain a superficial form of materiality, in that it recognizes the status of mathematical symbols as objects unto themselves. This is a limited recognition, though, and does not seem to extend to broader principles of embodiment or material engagement.

itself can be subjected to study using mathematical methods. This is revealing in terms of earlier discussions, as it harkens back, at a certain level, to Freudenthal's question of whether it is possible to *mathematize mathematics* (see **Chapter 3**).

There is some small irony in the fact that metamathematics makes mathematics the subject of its own study, for it draws upon mathematical tools as a means of commenting on the coherence of mathematics' own formal structures. That said, it also casts light upon the possibility of looking at mathematics from *outside* of itself. Historically, this second activity has revealed issues with the formalist doctrine as a whole. Without going into significant detail, suffice it to say that the formalist paradigm was largely countered/overturned by the incompleteness theorems of logician/mathematician and philosopher Kurt Gödel, which essentially characterize proof limits within formalized axiomatic systems, and refute the formalist knowledge claims by demonstrating that not all mathematical truths *can* be reduced to formal theorems in the manner required by Hilbert and his compatriots.[68]

Other scholars have criticized the formalist doctrine by commenting on its substantial omissions. In Lakatos' (1976) appraisal, for example, *Formalism* "disconnects the history of mathematics from the philosophy of mathematics" (p. 1), and "denies the status of mathematics to most of what has been commonly understood to be mathematics" (p. 2). Thus, it says nothing about the evolution and growth of mathematics as a field, or of the complex processes associated with mathematical discovery and negotiation of meaning.

It may be that the program of *Constructivism* better maintains the historical/developmental connections ignored by *Formalism*, and this shall be the last of the absolutist "isms" that I include in this condensed overview. It should be clarified straight away that the constructivist tradition carries somewhat different connotations in the philosophy of mathematics than it does within the philosophy of education more widely. Whereas the latter is known for Piagetian roots that emphasize how meaning-making emerges (or is constructed) from ongoing interactions between personal experience and prior ideations, the former is generally seen as having Kantian roots that seek to preserve the certainty of

[68] See Ernest (1991, pp. 10–11) and Ernest (1998, pp. 18–20).

existing mathematical meaning by continually *reconstructing* mathematical truths and *reconstituting* mathematical practice (paraphrased from Ernest, 1991, p. 11).[69] This is perhaps best exemplified by the intuitionist branch of *Constructivism*, which is known for demanding direct, demonstrable proofs, with early intuitionists expressing a concern that mathematics would simply regress to a meaningless rhetorical exercise in the absence of such demonstration. This coincides with the broader constructivist concern that the assertions of classical mathematics might, in some way, turn out to be *invalid* or *unsafe*.

Unlike classical approaches to mathematics that invoke deductive proof arguments or proofs by contradiction, *Constructivism* necessitates the construction of mathematical objects in order to prove their existence. More specifically, as noted in Ernest (1991), "Constructivists claim that both mathematical truths and the existence of mathematical objects must be established by constructive methods" (p. 11). Thus, *Constructivism* has a sort of reiterative, or "reconstitutive" dimension (if the reader will allow the term), in that it regularly re-imbues current mathematical thought and practice with an historical subset of mathematical thought and practice. In that sense, (mathematical) *Constructivism* is something of a backward-looking paradigm as opposed to a forward-moving one.[70]

The constructivist paradigm in the philosophy of mathematics is closely associated with scholars such as mathematician/philosopher Luitzen Brouwer (with whom the intuitionist strand is strongly identified), analytic philosopher Sir Michael Dummett (known, among other things, for his comprehensive explorations of Frege's work), and mathematician Leopold Kronecker (a known critic of Cantor's set theoretic paradigm). In Ernest's (1998) estimation, it is *Intuitionism* that represents "the most fully formulated constructivist philosophy of mathematics" (p. 21), and he cites Dummett's account of two key, but separable, intuitionist knowledge claims: 1) As a whole, intuitionist theory is intelligible, and the intuitionist construal of mathematical ideas and logical operations is both coherent

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[69] Note that there does seem to be some debate about Kant's status as a constructivist/intuitionist (see Shabel, 2016); however, since I am not specifically looking to deconstruct Kantian perspectives, for the sake of the current exploration I simply offer Ernest's insights, which have already been widely disseminated.

[70] Contrary to the "marks on paper" from which the formalists derive mathematical meaning, the constructive/reconstitutive activities of the constructivists need not necessarily be realized in a written or concretized form.

and legitimate; and 2) As a whole, classical mathematics is unintelligible, and classical construals of mathematical ideas and logical operations are neither coherent nor legitimate (paraphrased from Ernest, 1998, p. 22).

Intuitionist assurances of the certainty of mathematical knowledge are grounded in the belief that pure and primordial intuitions (about reality) reveal the axiomatic/self-evident truths from which valid mathematical knowledge can emerge. However, this reliance on *belief* also foreshadows the primary critique of the intuitionist view. Although it draws on certain foundations of classical logic, a key feature of *Constructivism* is that it rejects the Law of the Excluded Middle over infinite domains. Commonly referred to as the third of the traditional/fundamental laws of thought, this law essentially states that *either* a logical proposition or its negation may be true. While Aristotelian thought accepts the Law of The Excluded Middle as axiomatic and generalizable to the infinite case, the constructivist paradigm does not, calling into question the omission/exclusion of a third, logically viable alternative. By and large, it is this rejection of the Law of the Excluded Middle over infinite domains that prompts the constructivist aversion to traditional deductive proof arguments and proofs by contradiction, as well as the subsequent claim that the existence of mathematical objects must be proven by *construction* (at least in *principle*, if not in *practice*). As a result, *Constructivism* and *Intuitionism* remain grounded in the constraints of finite experience and face additional difficulties when tasked with mathematical problems or proof scenarios that extend into infinite spaces. Indeed, in critiquing constructivist philosophy, non-constructivists might cite the broader implication that mathematical meaning-making arrived at through constructivist means is inherently limited by virtue of the constrained methodologies it employs, and which can not adequately account for the wider range of proofs achievable through classical means.

An Alternative Perspective

Having established a rough backdrop of standard perspectives in the philosophy of mathematics, I now look to more fully explicate the alternative perspective that underpins the mathematical worldview embodied by this document. Though I intend no disrespect to Ernest, I do not devote a great deal of time to his social constructivist account, which is largely framed through its opposition to the absolutist paradigm. With it being the subject of its own volume (see Ernest, 1998) and much subsequent literature, I opt instead to direct the reader toward Ernest's already-cited works so as to avoid diverging from my main thread of discussion. However, to offer *some* sense of how social constructivism is situated with respect to absolutism, I do offer the following introductory excerpt from Ernest (1991), and occasionally bounce my own perspective off of it.

> Social constructivism views mathematics as a social construction. It draws on conventionalism, in accepting that human language, rules and agreement play a key role in establishing and justifying the truths of mathematics. It takes from quasi-empiricism its fallibilist epistemology, including the view that mathematical knowledge and concepts develop and change. It also adopts Lakatos' philosophical thesis that mathematical knowledge grows through conjectures and refutations, utilizing a logic of mathematical discovery. Social constructivism is a *descriptive* as opposed to a *prescriptive* philosophy of mathematics, aiming to account for the nature of mathematics understood broadly […]. (p. 42)

By and large, the thrust of Ernest's perspective is predicated on the belief that mathematical knowledge is rooted in linguistic systems and their associated conventions, and that ongoing social interaction and interpersonal communication allow the subjective knowledge of the individual to enter into and commingle with the shared, objective knowledge of the social collective. Thus, for Ernest, objectivity is itself understood to be *socially negotiated*, clearly exemplifying its fallibilist attributes.[71] That being the case, the concept of a socially negotiated objectivity may not sit well with those who subscribe to absolutist doctrine. Indeed, it can be difficult to parse Ernest's (1991) assertion that interpersonal social processes "are required to turn an individual's subjective mathematical knowledge […] into accepted objective mathematical knowledge" (p. 42), for it insinuates that knowledge can *only* be objective if it is agreed upon by the larger community (i.e., as though the consensus of the group is itself somehow impartial or free of bias). Certainly, to a Platonist, for whom the notion of objectivity connotes the existence of universal truths that are *independent* of the thinking, knowing, sensing human being, negotiated objectivity would be problematic, and some debate might be raised as to whether group acceptance actually constitutes an adequate criterion for determining the objectivity of (mathematical)

[71] Cf. Ernest (1991, p. 42)

knowledge. Giving Ernest the benefit of the doubt, this obvious friction with the Platonist epistemology suggests that his sense of objectivity likely has different connotations attached, and it would not be unreasonable to consider that his usage of the term actually implies *intersubjective agreement*, rather than the more rigid, detached meaning by which 'objectivity' might traditionally be understood.[72]

In a wider context, philosophical orientations such as social constructivism help to illustrate the shift toward recognizing the role of *social practice* in the judgment and acceptance of mathematical knowledge, by shedding light not only upon the gatekeeping of knowledge that manifests at the socio-cultural level, but also the tremendous normative impact of established paradigms within (and upheld by) communities of shared practice.[73] Regarding communities of shared practice within the space of mathematics education, one might initially consider that groups such as *preservice mathematics teachers*, *in-service mathematics teachers*, *mathematics teacher educators*, *mathematics education researchers*, and even *professional mathematicians* all encompass or engage in different forms of social practice in accordance with their respective aims and responsibilities (with certain overlaps being assured). However, it is also clear that mathematics education, as a field, draws upon and incorporates the practices of numerous *other* social groups as well, many of which inform the field by contributing different ideas of what constitutes mathematical activity.

Accounting for these varied disciplinary contributions and the multitude of social practices that influence mathematics education, Reuben Hersh (2017) expounds the *pluralism* of the philosophy of mathematics, a sort of metaphilosophy through which he extends the ideas that, "In philosophy of mathematics, mathematics is the thing being modeled" (p. 21), and that numerous different models of mathematics may *peacefully coexist*. Hersh's pluralism allows not only for the coexistence of *complementary* models of mathematics, but also

[72] von Glasersfeld's (1999) review of Ernest's *Social Constructivism as a Philosophy of Mathematics* would seem to support this reading (see p. 72).

[73] Sociologist David Bloor is known for elaborating the *strong programme in the sociology of knowledge*, which examines the various ways in which knowledge claims and their associated inclusions and exclusions are rooted in social practice (see Bloor's 1976 volume *Knowledge and Social Imagery*, later republished in 1991).

incompatible or *conflicting* ones as well. As an illustration of this "peaceful coexistence" (p. 19), Hersh (2017) remarks that:

> History, logic, neuroscience, psychology, and other sciences offer different models of mathematics, each focusing on aspects accessible to its method of investigation. Different studies of mathematical life overlap, and they have interconnections, but, still, each works to its own special standards and criteria. Historians are historians first of all and likewise educators, neuroscientists, and so on. Each special field studying math has its own model of mathematics. (p. 21)

Thus, whereas the historian may model mathematics "as a segment of the ongoing story of human culture" (Hersh, 2017, p. 25), the logician models mathematics "as a web of formal inscriptions" (p. 25), and the neuroscientist models mathematics as "electrochemical activity in the nervous system" (p. 24). Through Hersh's perspective, none of these models is necessarily *the* correct model of mathematics, nor are any of them necessarily *incorrect*. Under a pluralist philosophy of mathematics, they need not be framed as competing paradigms vying for dominance, or as worldviews to which one must commit exclusively. Instead, they can be conceived of simply as *different* models of mathematics, each of which emphasizes particular facets of mathematical practice, while de-emphasizing others. When clarifying how he perceives the notion of modeling, Hersh (2017) further acknowledges that:

> It is never necessary to argue that any model is incomplete or partial, because it is in the very nature of modeling to be incomplete and partial, just as any map of any region of the Earth is incomplete as a description of that region. It is the very purpose of a model or a map to be incomplete, to select certain features of its subject to represent and theorize. A complete description would be just a duplicate of the original and therefore useless. *Models are naturally pluralistic*. (p. 26)

Though it is debateable whether it is the models themselves or the creators of the models that are responsible for *selecting certain features of the subject to represent and theorize*, it is clear that Hersh means to highlight the affordances of adopting different models of mathematics according to context and/or need. By virtue of its overall flexibility and inclusiveness, his sense of the inherent pluralism of models can even be useful in recontextualizing Ernest's survey of prominent educational perspectives. In the case of the absolutist/fallibilist overview presented earlier, the following remarks *soften* the distinctions between some of the "isms" that were identified (as well as a number of others). Again drawing from Hersh (2017):

> If we set philosophy of mathematics among the whole group of studies of mathematics, both humanistic and scientific, we can see the significance of formalism, logicism, nominalism, structuralism, naturalism, and even social constructivism, in a new light. Each of them is in fact a model of mathematics. In that way, each is legitimate, and none prevents another from carrying on its work. (p. 21)

It is possible, for instance, to "admire logic without succumbing to logicism" (Hersh, 2017, p. 22), and to find it *useful* and *illuminating* without necessarily subscribing to it fully or conclusively. Indeed, Hersh speaks to these paired qualities of usefulness and illumination as being, perhaps, more suitable valuations for models of mathematics than their finality or completeness, and Canadian mathematician and philosopher William Byers (2017) echoes some of this sentiment with the following passage (from the same compiled volume):

> Subscribing to a philosophy of mathematics [...] creates a context that gives meaning to mathematics. For example, formalism tells you that mathematics consists of proving theorems and setting up deductive systems. Doing mathematics is not really possible without a philosophy that tells you what mathematics is, even if this philosophy is not consciously embraced but is only implicit in what you do. (p. 46)

Beyond reinforcing the idea that philosophies of mathematics are, at some level, *implicit in what we do* (i.e., in our practices), Byers subsequently speaks to the multiplicity of philosophies of mathematics in a manner that might be likened to Hersh's pluralism. At the same time, his foregrounding of a *specific* implication of accepting a pluralist view, reintroduces a concept briefly touched upon much earlier in this dissertation (albeit with a different framing device motivating its discussion). As expressed by Byers (2017):

> We should expect that embracing the idea that there can be multiple useful philosophies of mathematics will inevitably force us to accept the existence (and even the usefulness) of ambiguity. We gain from being able to look at mathematics in multiple ways, and so we should always be open to new philosophies of mathematics and try not to get trapped within a rigid and supposedly definitive philosophy. (p. 55)

Thus, pluralism in the *philosophy* of mathematics, and ambiguity regarding the *nature* of mathematics would seem to coincide with one another. As with the material indeterminacy discussed in **Chapter 3**, both point toward fundamental uncertainties that infuse the human experience of coming to know mathematics as a whole. Though human beings must regularly grapple with various forms of uncertainty when engaging in reasoning, Byers suggests that a more open-ended, pluralistic view of mathematics can actually be a great asset in managing that uncertainty.

To reiterate an assertion from the opening preface of this dissertation, I remain agnostic as to whether the material world itself is *exclusively* in/deterministic and un/certain; and am open to the possibility that all of these can be accommodated in different ways, under varying circumstances. I extend essentially the same agnosticism to the absolutist/fallibilist opposition of the current chapter, and as contradictory as it might initially seem, the mathematical perspective I propose incorporates elements that are *both* absolutist and fallibilist in nature. Such a pairing is far less contradictory, however, when considered from a pluralist orientation like Hersh's. It is with his metaphilosophy of mathematics, and Byers' remarks on ambiguity, in mind that I finally discuss mathematics as *the science of material assemblage*, one possible view of mathematics within a larger, pluralistic space of possibility. Though this perspective straddles the traditionally perceived gap between absolutism and fallibilism, it also draws these classically divergent perspectives closer together by emphasizing the sense in which both *pluralism* and *ambiguity* might be related to the potential for organizing and reorganizing mathematical structures and ideas.

As expressed within/across the previous chapters and intraludes, my broader mathematical worldview embraces the fundamental indeterminacy of matter, and conceives of mathematical processes as ones that involve the organization and reorganization of matter to varying degrees (hence the alternative phrasing of 'material–mathematical worldview', which I use synonymously). Viewed more generally as being inherent to (or encoded within the structure of) matter, mathematics is itself described as *embodying* the principles according to which matter might be organized and reorganized. In large part, it is by appealing to the quantum theoretical discourse and *the unreasonable effectiveness of mathematics in the natural sciences* that I have endeavoured to make a case for thinking of mathematics as innate to the structure of *all* matter, and not simply as a layer of formal structure that is imposed *atop* material reality when human beings engage in processes of mathematization. This particular ontological claim is reminiscent of absolutism, as it retains the presence of a persistent and self-consistent world of mathematical truths (for which many of the natural laws of physics *might* be reasonable proxies); however, it does not assume that human knowledge *about* that persistent/consistent world of mathematical truths is absolute, or even accurate. Thus, while the material world itself is seen as being fundamentally mathematical, what is ultimately *known* about its underlying structures and processes at any given time is relative, fallible, corrigible (to use Ernest's terminology), and very much contingent upon the materiality of lived experience. In this way, my perspective emphasizes that mathematical structures and processes are to be distinguished from *knowledge about* mathematical structures and processes, which is to say that mathematics and *what we come to know about/through* mathematics are rather different things. Of course, being mindful of linguistic commitments such as those discussed in **Intralude B**, I am aware that this distinction can easily be occluded by our everyday (and even some technical) language.

In no way does this perspective deny the value or power of abstraction and imagination, which have the capacity to extend mathematical meaning-making into a broader conceptual space, beyond the contexts of immediate, localized experience. It does, however, assert that the *most essential* meaning-making is rooted in the dynamics of our ongoing materiality and our engagement with the larger material world.[74] This offers additional insight into my choice to conceive of mathematics as the *most foundational natural science* (as expressed in **Chapter 3**), while also communicating a perspective that would, for comparison, readily conflict with Ernest's (1991) social constructivist stance below.

> The social constructivist view is that mathematical entities have no more permanent and enduring self-subsistence than any other universal concepts such as truth, beauty, justice, good, evil, or even such obvious constructs as 'money' or 'value'. Thus if all humans and their products ceased to exist, then so too would the concepts of truth, money and the objects of

[74] The reader is asked to recall that the new materialist discourse, into which quantum theory has been partially absorbed, expands the scope of terms such as 'matter', 'material', and 'materiality'.

> mathematics. Social constructivism therefore involves the rejection of platonism. (p. 57)

Since I uphold the notion of matter being inherently mathematical, I do not take issue with Ernest addressing mathematical entities as/alongside *universal concepts*; however, I do believe that he may be using the term 'universal' in a fairly restricted sense that differs significantly from my own. While I am in agreement that there is no reason to believe that the so-called *obvious constructs* of 'money' and 'value' would persist in the event that *all humans and their products ceased to exist* (as Ernest considers above), these are largely *social* constructs and do not exhibit the same fundamentality as mathematical entities (from which they are at least a step removed). Similarly, while the concepts of 'beauty', 'justice', 'good', and 'evil', might have maintained some degree of consistency in their meanings throughout the span of human history and across its various cultures, to describe them as being *universal* in any sense greater than that seems like an over-extension of the adjective. Ernest's expression of universality appears to share the same boundaries as his social constructivist account, in that it is applicable to human social constructs alone. In contrast, the universality I express is in relation to the material structures and processes of the known universe (which encompass much more than human civilization and its various social practices). Since our intended usages may differ, I shall not suggest that Ernest's comparison is misrepresentative or that it belies some form of anthropocentric bias, but I do question if the concepts and constructs that Ernest mentions actually partake in the *same* kind of universality (and hence, generality) that can be ascribed to mathematics.

Despite the incredible scale, complexity, and dynamism of our material reality, the general expectation is that the universe accords with certain assumptions of *stability* and *predictability*. Indeed, in so far as the modern sciences can currently claim, the underlying structures and processes of the known universe appear to be not only *consistent*, but also *persistent* and *ubiquitous*. Under my interpretation, and unlike one such as Ernest's, this entails that mathematical entities *also* exist and perdure in the same manner, affording them a status of universality that is *not* contingent upon human existence and insight. Again, this is not to deny that mathematical *knowledge* is relative, fallible, and corrigible. On the contrary, it is an acknowledgment that even the negotiations and practices that serve as the basis for mathematical knowledge claims will ultimately be constrained by an underlying ontology (i.e., a *material–mathematical* ontology).

Ernest (1991) points out that:

> there are variations in the degree of relativism ascribed to knowledge. In the extreme case, all human knowledge is seen as relative to social groups and their interests, and physical reality itself is regarded as a social construction. More moderate positions regard knowledge (and not reality) as a social construction, and accept an enduring world as a constraint on the possible forms of knowledge. (p. 94)

Considering Ernest's comments, it is somewhat unclear to me if the acknowledgment I have offered would be better aligned with a relativist or realist orientation. Though the preceding paragraphs may seem suggestive of adherence to a fairly standard Platonic worldview, I caution against such a reading and take a moment to distance myself somewhat from Platonic realism in particular. *Unlike* Platonic realism, which asserts that mathematical entities are inaccessible to the senses and may be approached only via the intellect, I argue that our sensory experience is both embedded in and ultimately resultant/emergent from those same mathematical entities and the relationships that exist between them. This is to say that they are crucial for our material experience to be reified as it is, and to evolve as it does; for thinking, knowing, sensing human beings are subject to the very same material–mathematical ontology as the rest of the known universe.

Under Platonic realism, *ideas* (or purely abstract ideal Forms) encapsulate the true nature of reality, while physical experience of that reality is, at best, illusory.[75] Similar distinctions between the (perceptual) realm of the senses and the (cognitive) realm of the intellect form the basis of Plato's famed *Allegory of the Cave*, which details the process by which the "unenlightened" might be liberated from the metaphorical shackles of false perception and ascend to true illumination through reason. In light of the stance I have already expressed with regard to the embodied self (i.e., the thinking, knowing, sensing self), suffice it to say that I do not entertain so sharp a distinction between what I envision as entangled aspects

[75] The reader may recognize the potential for confusion stemming from terminology/phrasing, as the universal, timeless, immutable essences of Plato's *realist* view are also referred to as *ideal* Forms.

of being/becoming/knowing. Instead, I advocate an interpretation in which the information gathered through our perceptual (hence, embodied) experience is not an illusion, but an integral facet of coming to know the mathematical structures and processes that govern our material reality.[76]

Clearly, there are historical precedents for worldviews that conceive of material reality as being fundamentally mathematical. Both Pythagoreanism and Platonism have been mentioned within this document; however, other modern perspectives also maintain similar assertions. Though somewhat provocative and not without its criticisms, the Mathematical Universe Hypothesis (MUH) of Swedish-American physicist and cosmologist Max Tegmark exists among these. Tegmark's (1998) perspective, which he describes as "a form of radical Platonism" (p. 4), articulates a relational ontology that offers what could be seen as a partial response to Wigner's (1960) question of why "the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena" (p. 8). In the conclusion of his 2008 paper *The Mathematical Universe*, Tegmark provides the following summation of his worldview, which essentially postulates that the universe itself *is* a mathematical structure, and not simply a system that *lends itself* to representation by mathematical means.

> This paper has explored the implications of the Mathematical Universe Hypothesis (MUH) that our external physical reality is a mathematical structure (a set of abstract entities with relations between them). I have argued that the MUH follows from the external reality hypothesis (ERH) that there exists an external physical reality completely independently of us humans, and that it constitutes the opposite extreme of the Copenhagen interpretation […] of physics where human-related notions like observation are fundamental." (p. 139)

Elaborating on the main argument above, Tegmark's (2008) MUH treats the surprising utility of mathematics (i.e., its unreasonable effectiveness) as a "natural consequence" (p. 107) of the fact that physical reality is *already* mathematically structured, and that human beings, in his words, "are simply uncovering this bit by bit" (p. 107). Although Tegmark's later works would reiterate, refine, and further substantiate the idea, one of his earliest

[76] Cf. de Freitas and Sinclair (2013) and their discussion of "the body in and of mathematics" (p. 454).

publications includes an assertion that encapsulates an especially powerful implication of his worldview: "Everything that exists mathematically exists physically" (Tegmark, 1998, p. 2). This is put slightly differently in Tegmark's 2008 article, when he states that "mathematical existence and physical existence are equivalent, so that *all* mathematical structures have the same ontological status" (p. 125).[77] To be clear, Tegmark qualifies this assertion by indicating that it can only be the case if the mathematical structures constituting physical reality are consistent with certain Gödelian constraints; however, that discussion exceeds the scope of this dissertation. Regardless, for Tegmark, mathematics does not simply *describe* our physical reality; it *is* our physical reality, and his notion of uncovering this reality *bit by bit* is very much in keeping with my own sensibilities about mathematization as the process of *exposing innate mathematical structure*.

At present, I can not be as bold as Tegmark is when he states that mathematical existence and physical existence are *equivalent*, for I ascribe a primacy to mathematical structures that surpasses the ontological status of material structures. This is to say that I conceive of matter as having *mathematical underpinnings*, as opposed to mathematics having material underpinnings.[78] As a result, I am inclined to offer the mirror image of Tegmark's assertion, by stating that "everything that exists physically exists mathematically". I accept that I have not yet resolved whether Tegmark's original claim would hold under the worldview that I propose. At a certain level, Tegmark's assertion of equivalence between mathematical existence and physical existence connotes that human *experiences* of the mathematical and the physical (or what I am reading as *the material*) could also be identified with one another, and this is not a connotation that I am currently prepared to adopt. At least from an educational standpoint, it seems quite evident that human awarenesses of mathematical structures and processes do not always coincide with human

[77] The reader may note that this bears some resemblance to the viewpoint expressed by Meillassoux (2008), and reiterated in ***footnote 3*** of the current dissertation, namely that "all those aspects of the object that can be formulated in mathematical terms can be meaningfully conceived as properties of the object in itself" (p. 3).

[78] While I do speak of a *material–mathematical ontology*, I should, perhaps, also be clear that this refers to a relationship in which the material is grounded in the mathematical and not the reverse. I also acknowledge that there could be discrepancies between Tegmark's intended meaning of the word 'physical' and my intended meaning of the word 'material'; however, my reading of Tegmark suggests that the former might actually contain the latter.

awarenesses of physical structures and processes.[79] Nor are those awarenesses necessarily equally well developed. With this being the case, I hesitate to draw their ontological statuses into equivalence.

As does the traditional Platonic realism, Tegmark's MUH also forwards the notion of a *completely independent* physical reality, which stands apart from the thinking, knowing, sensing human being. I have already pointed out that I conceive of mathematical entities and relations as *persisting* in a way that does not depend upon the existence of human beings, and while Tegmark's assertion of an independent physical reality intersects with this, his commitment to an *external* reality does not (not entirely, at any rate). Ernest (1991) proposes that one of the main problems associated with Platonism is that "it is not able to offer an adequate account of how mathematicians gain access to knowledge of the platonic realm" (p. 29). I would suggest that a possible reason for this is that Platonism conceives of its external world as being "out there" (in the sense of being *outside* of the thinking, knowing, sensing human being). As a result, it removes us from the world (or it from us) in a way that prevents what is "out there" from sharing a unified ontology with what is "in here", by which I mean the embodied experiences (i.e., sensations, actions, thoughts, reflections, et cetera) that mark our engagement with the material world. Despite its more radical moves beyond traditional Platonism, Tegmark's MUH (which arises from the External Reality Hypothesis) still appears to carry the encumbrance of this ontological separation.

To speak of external reality and internal reality as fundamentally different from one another calls back classical Cartesian dualisms that my worldview seeks to avoid, and, in accordance with the enactivist commitments expressed throughout this document, I do not divorce the external (i.e., the "out there") from the internal (i.e., the "in here"). For the moment, the reader is asked to consider (again) that human beings are subject to the very same material–mathematical ontology as the rest of the known universe, and that its underlying mathematical relations are at play *within* us (i.e., as part of our very make-up and structure) just as they are within the world that is perceived to be *external to* us.

[79] This is intended to evoke the same plurality of 'awarenesses' expressed by Wheeler (2001) and referred to in **Chapter 3** of this dissertation.

Revisiting Ernest's remark about the problems with Platonism, it would follow that mathematicians are able to gain access to knowledge of the idealized Platonic realm of mathematical entities and relations because the foundational structures of which mathematicians are composed are based in the very same realm, such that it is not exactly "external" to them, nor they from it. Though I knowingly co-opt Tegmark's phrasing in order to offer an analogous version of his statement, it could be said that knowledge of the mathematical realm is simply a "natural consequence" of the fact that physical reality and the mathematicians seeking to engage with it occupy the same material–mathematical ontology. It is for this reason that I earlier put forth the idea that *how we come to know mathematics is intimately entwined with how we come to know ourselves*.

The perspective I espouse shares certain similarities with Tegmark's Mathematical Universe Hypothesis; yet, as I have pointed out, there are subtle points of difference that prevent our worldviews from overlapping completely. As a result, I acknowledge that the MUH is attractive at a conceptual level, whilst also hedging my commitment to specific assertions (as discussed in the preceding paragraphs). While some detractors of Tegmark's Mathematical Universe Hypothesis point toward the difficulty of defining just what is meant by 'external physical reality', others such as Gil Jannes (2009) question Tegmark's (2008) assertion that "the MUH follows from the external reality hypothesis" (p. 139). Jannes (2009) also addresses the overall problem of testing (and potentially falsifying) the Mathematical Universe Hypothesis itself. In particular, he cites Wittgenstein, who is known to have "defended the view that it is impossible to give conclusive rational arguments for the objective existence of an external reality" (p. 399), while simultaneously acknowledging "that it is nevertheless an essential presupposition or background for any acquisition of knowledge" (p. 399).

Granted, ambiguity associated with some of Tegmark's terminology prompts additional questions about *exactly how* the mathematical structures of his proposed universe become known to (human) mathematicians, and the general issue associated with proving/falsifying the Mathematical Universe Hypothesis does pose something of a philosophical problem. Though I do acknowledge and respect these critiques, they are not ones that I seek to resolve in the current document. I do, however, respond to them by revoicing Hersh's pluralist refrain of *peaceful coexistence*. In spite of the criticisms levied by scholars such as Jannes, and the unresolved issues inherent to the MUH, Tegmark's overall worldview remains compelling by virtue of the manner in which it addresses Wigner's foundational questions about the nature of mathematics and its relationship with the physical/material reality that human beings perceive and inhabit. As Jannes points out by way of Wittgenstein, there may be no way of conclusively proving the existence of an external reality that accords with Tegmark's hypothesis, but I do not believe this diminishes the *useful* or *illuminating* qualities of his MUH as a singular exploration of the ways in which human beings access and engage with mathematical ideas.

While perspectives such as Tegmark's are not common within the field of mathematics education, the foundational questions raised by the MUH are ones that could be asked more frequently. The scope of the current chapter prevents me from delving deeper into Tegmark's hypothesis, however I do wish to highlight its salience as a modern mathematical worldview that attends closely to the role that mathematical entities (i.e., structures and processes) play in the grander scheme of our material reality. I have suggested that it is by being entangled with that same reality (not removed from it), and by enacting processes of material assemblage, that human beings are assured some form of connection to its underlying mathematical structures and processes.[80]

Revisiting the Quantum Leap

With **Chapter 4** having examined in detail some of the basic principles of modern quantum theory, evolving models of atomic structure, and a number of phenomena that have recontextualized how human beings perceive the material–mathematical relationship, it should now be much clearer why the early chapters of this document forwarded the idea that *a changed view of the material also changes one's view of the mathematical*. This very much speaks to the idea of how mathematics is *seen*, in the sense of addressing what mathematics is and how it is instantiated in the world; yet, it was also suggested in **Chapter 2** that mathematics itself might be understood as a particular *way of seeing*. Though not

[80] In Wise's (2011) words, we should keep in mind "that we are always caught up in and constituted by multiple assemblages." (p. 94)

necessarily framed in terms of mathematics teaching specifically, this related stance speaks more to the consideration of what it means for human beings to *internalize* mathematical principles, and what our (multiple) awarenesses of those principles might in turn reveal about the world.

The phenomenon of the quantum leap discussed in **Chapter 4**, provided a useful case study through which to illustrate the deep connectivity between matter and mathematics, and there may yet be more to draw from this example now that I have better outlined my material–mathematical worldview. Consider, for instance, that there is a sense in which the quantum leap might actually be thought of as a physically manifested analogue of the arithmetic carry and borrow, in that it reorganizes quantities (of electrons) at higher or lower energy levels, with respect to different base representations (S. Campbell, personal communication, May 15th, 2019).

As noted earlier, higher energy electron states correspond to shells at greater radial distances from the atomic nucleus, while lower energy states correspond to those in closer proximity to it. In addition to their individual energy constraints, each shell is capable of accommodating only a specific number of electrons, with the maximum number differing across the shells. For instance, the first (and innermost) shell, corresponding to principal quantum number $n = 1$, may accommodate a maximum of 2 electrons, while the second shell, corresponding to principal quantum number $n = 2$, may accommodate 8. The third shell may hold a maximum of 18 electrons, the fourth 32, and so on, in accordance with the generalized expression $2n^2$. When atoms form in nature, these shells tend to be "filled" from innermost to outermost, in a manner resembling the construction of whole numbers beginning with unitary digits and moving through successively higher place values. As with numerical additions involving sums that exceed the single-digit allowance of a given place value (under base-10 arithmetic), electrons that exceed the allowance of a given shell will occupy the *next* available position with the lowest possible energy. These additional electrons are effectively "carried" over to a new shell, or figurative "place value".

This logic holds for quantum leaps to higher energy levels as well. When an electron is excited via the absorption of electromagnetic radiation and a quantum leap to a more energetic state is induced, the discrete quantum of energy "carries" the electron into a higher order "place value". Conversely, the electron that loses a discrete quantum of energy when leaping to a lower energy level, is essentially "borrowed" by that lower order position in the atomic configuration, not unlike the borrowing that might occur during numerical subtractions.

Interestingly, with each atomic shell capable of holding a different number of electrons, the filling of shells involves something akin to a change of base. For example, the n = 1 shell may hold, at most, 2 electrons (similar to a number system under base-3 addition, where each place value accommodates one of three possible "entries"). A position for a fourth electron would not be available within that shell, and the electron would instead occupy the next available position in the higher-energy n = 2 shell, which accommodates up to 8 electrons. Likewise, attempting to place a ninth electron in a full n = 2 shell would see it "carried" into the higher-energy n = 3 shell, where the maximum allowance of 18 electrons would be more analogous to a system under base-19 addition. It is in this way that the shells act somewhat like place values across which quantities of electrons are reorganized according to different base representations. I contrast this, of course, with the standard base-10 number system, where *every* place value supports the *same* numerical base and any of ten possible digits (i.e., 0 through 9).

To be fair, this structural analogy is limited, as there do appear to be natural bounds governing the total number of electrons to be found within a single atom, as well as the extent to which electron energy level transitions can be realized. Currently, the synthetic atom Oganesson has the highest electron count of any known element, with its 118 electrons distributed across 7 shells (not all of which are filled to capacity). While it is theorized that the synthesis of atoms with a greater electron count could be possible, these are expected to be highly unstable/reactive, with exceptionally short half-lives, and finding them under natural circumstances (i.e., those outside of a laboratory setting) would be extremely unlikely. There are also limits to how "far" electrons might leap during their energy level transitions. This is to say that electrons can not continue leaping to higher and higher (or lower and lower) energy levels *indefinitely*. At the upper limit, infusing an electron with too much energy will ultimately cause it to break free of its parent atom entirely, in which case, the analogy I have offered breaks down. Similarly, every stable atom has a lowest-energy state (known as the ground state), which expresses the minimum allowable energy for electrons bound to the atomic nucleus. As a result of these two considerations, I can not extend my analogy so far as to include the equivalent of infinitely high place values, or of decimal place values either. Nevertheless, I offer the analogy as an attempt to articulate a different kind of awareness about the mathematical structures that permeate our material reality, and as an example of sense-making practice that attends to the dynamics of the material–mathematical relationship at the level of individual quanta.

In the opening commentary of **Chapter 1**, in response to Hersh's identification of an unfortunate disjunction between the philosophy of science and the philosophy of mathematics, the Pythagorean maxim that *all things accord in number* was presented as an historical touchstone through which to revisit and reassert the common essences of these respective philosophies.[81] Variations on this idea have resurfaced in different ways throughout the subsequent chapters, with the discretization conditions of quantum theory offering significant modern support for such an interpretation. Indeed, the material–mathematical worldview that I elaborate is of a very similar spirit; however, it should be clear that I do not restrict this worldview to considerations of number alone. Though not intending to detract from the Pythagorean sensibilities in any way, I instead prefer to forward the maxim that *all things accord in mathematics*, so as to explicitly communicate the intent of encompassing a much broader range of mathematical structures and processes. It is this sentiment, in particular, that suggests some manner of kinship with Tegmark's Mathematical Universe Hypothesis.[82]

Not unlike David Bohm, other modern physicists have worked to articulate how organizing principles apparent at the quantum level might also be active in the macroscopic domain. Bunge (2003), for instance, appeals to the long history of mathematical and scientific

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[81] As physicist James Clarage (2013) eloquently expresses, "Pluck a string, or perturb an atom, and number attends" (p. 533).

[82] Schiralli (2006) reminds us that the ancient Greek/Pythagorean concept of number was actually much more robust and expansive than is often acknowledged in the modern day. He also notes the Pythagorean view that the universe is revealed in "discernible patterns or ordered arrangements" (p. 116), with particular attention paid to the concept of number as *ordered plurality*.

endeavours when championing the idea that “the first to discover quanta was not Planck in 1900, but Pythagoras in the 6th century B.C.” (p. 445). He later expresses that “[t]his must be emphasized to debunk the myth that only exotic microphysical objects have quantal properties. Harps, drums, crystals, beams, bridges and many other large objects have some of them too” (p. 446).

With the quantum leap and its reorganization of quantity across atomic shells illustrating processes of structural change and reconfiguration enacted at an extremely small scale, I pause before moving into the final chapter of this document, so as to offer the reader insights into a similar process enacted at a much grander scale. As a commentary on the paired notions of *order* and *disorder*, **Intralude C** discusses how the thermodynamic concept of entropy might also be recontextualized though the action of material assemblage.

Intralude C | Entropy

A Commentary on the Notions of Order and Disorder

Rooted in thermodynamics and the pioneering work of Rudolf Clausius, the concept of entropy was first formulated in order to explain why a portion of any thermodynamic system's total energy is *unavailable* to do work. Clausius is said to have named 'entropy' after the Greek word for 'transformation', and in the parlance of classical thermodynamics, entropy is strongly bound to ideas of energy transformation/transfer and heat flow within isolated systems (i.e., systems whose boundaries do not permit the exchange of either matter or energy with the surrounding environment). Though I shall not go into the formalisms behind this characterization, I do note that entropy, at its heart, arises from considerations associated with energy *conservation* and the fundamental *directionality* of natural processes. Alongside the notions of energy transformation/transfer and heat flow, conservation principles establish a basis for describing entropy in terms of *irreversible processes*, the characteristics of which are essentially encapsulated by the second law of thermodynamics. In an abridged sense, entropy can be interpreted as a measure of energy consumption through which irreversible processes are distinguished from reversible ones.

Theoretically, reversible processes would require that both a thermodynamic system and its surrounding environment be restorable to their initial conditions *after* a spontaneous process of energy transformation/transfer has occurred. Something like a perfectly elastic collision, in which no energy is lost to heat or friction, would exemplify this nicely.[83] In such cases, the system entropy would remain unchanged and the reverse process would be just as likely to occur spontaneously as the original. This implies that the system undergoing the process would be in *perfect* thermodynamic equilibrium with its surroundings, which makes *truly* reversible spontaneous processes an impossibility. Though there is a non-zero probability that spontaneous processes may be *partially*

[83] It is extremely important to note that not all processes are spontaneous. This terminology applies specifically to those which can proceed without the need for an external impetus or energy source. Moreover, classifying a process as spontaneous does not imply that the process *will* occur; rather, it is simply an indication that it *can* occur.

reversible, such process will (in nature) not run to completion. On the contrary, spontaneous *irreversible* processes are abundant in nature. Including inelastic collisions, combustion reactions, spontaneous chemical reactions, and a wide variety of other cases, these are processes in which a thermodynamic system and its surrounding environment are *not* restorable to their initial conditions once the processes of energy transformation/transfer have occurred. To date, these account for the vast majority of spontaneous processes in the known universe. Unlike their theoretical fully reversible counterparts, such processes are always accompanied by a dissipation of heat energy, an *increase* in entropy, and the tendency for a given system's internal temperature, pressure, and density to equalize over time.

At a certain level, the concept of entropy emerged from efforts to explain the observation that particular types of reactions and/or internal system changes will occur spontaneously, while their time-reversed counterparts will not. Although both sets of processes would theoretically be subject to the same conservation principles expressed by the second law of thermodynamics, all indications are that only those spontaneous processes characterized by *increasing* entropy actually manifest. In effect, entropy helps to explain why explosions do not spontaneously "unexplode", shattered vases do not spontaneously reassemble themselves, and lightning does not spontaneously "unstrike". The strictures of increasing entropy not only prohibit the realization of perpetual motion (since irreversible processes are always accompanied by the expenditure or dissipation of some amount of unrecoverable heat energy), but also indicate that the transformation and exchange of energy has a fundamental and unidirectional *flow*, namely one that accords with the "forward-moving" flow of time.

All of this carries interesting implications for our known universe, which is currently thought to be the prime example of an isolated thermodynamic system. Just as leading cosmological theory posits an extremely hot and dense early universe emerging from a singularity commonly referred to as the *Big Bang*, related theories point toward the continued *expansion* and gradual *cooling* of our universe, resulting in its eventual "heat death". Whereas the highly volatile and compact/condensed primordial universe at (or moments after) the *Big Bang* might be characterized as a state of *minimum entropy*, where the potential to redistribute energy and do work would be at its highest, the aptly named *Big Chill* would correspond to an opposing state of *maximum entropy*, at which point all thermodynamic processes would equalize and all energy would be evenly distributed throughout the universe, allowing no further work to be done (i.e., a state of complete thermodynamic equilibrium).

Perhaps due to something akin to Burwell's *originary drift* (see **Intralude B**), this thermodynamic concept of entropy has, over time, been adopted by numerous other domains, including quantum mechanics, information theory, economics, et cetera. Though Clausius' early work led to the first mathematical formulations of entropy, formal definitions of it would gradually expand beyond those grounded in the dynamics of *energy transformation* and *heat flow* to include other more general forms emerging from statistical mechanics (via Josiah Gibbs' and John von Neumann's work) and the abstract notions of *order* and *disorder* (as with Boltzmann's characterization). Perhaps not surprisingly, it is these latter notions in which I am most interested. To illustrate how entropy might be recontextualized in terms of order and disorder, I return once again to theoretical physicist Carlo Rovelli (2018), who addresses the concept in the following manner:

> The entire coming into being of the cosmos is a gradual process of disordering, like the pack of cards that begins in order and then becomes disordered through shuffling. There are no immense hands that shuffle the universe. It does this mixing by itself, in the interactions between its parts that open and close during the course of the mixing, step by step. Vast regions remain trapped in configurations that remain ordered, until here and there new channels are opened through which disorder spreads.
>
> What causes events to happen in the world, what writes its history, is the irresistible mixing of all things, going from the few ordered configurations to the countless disordered ones. The entire universe is like a mountain that collapses in slow motion. Like a structure that very gradually crumbles. (pp. 143–144)

Were either/both of them to engage with Rovelli's richly articulated passage, I believe that Deleuze and Guattari could easily read their own assemblage theory into it. The *gradual process of disordering*, the *irresistible mixing of all things*, the *shuffling* at a universal scale, the *slow-motion collapse* and *crumbling*: all of these scenarios echo the sense in which Deleuze and Guattari refer to assemblage/*agencement* as "dealing with the play of contingency and structure, organization and change" (Wise, 2011, p. 91). Moreover, the grand timescale over which entropy unfolds, and the all-encompassing extent to which it is enacted both speak to its pre-eminent role as a dynamic process of *becoming*. Much like the notion of defragmentation discussed in **Intralude A**, I treat Rovelli's interpretation of entropy as another instance of *material assemblage in action*. Indeed, given its overall ubiquity and inevitability, it occurs to me that entropy could very well be the *most* apparent and pervasive instance of material assemblage in action, realized at the grandest possible scale and extended across the greatest conceivable span of time (i.e., the lifetime of the material universe).

At different points in this dissertation, I have described mathematics as embodying the very principles according to which matter organizes and reorganizes itself, and the "dance of ever-increasing entropy", to use Rovelli's phrasing (2018, p. 144), is a key example to which I apply this interpretation. Mathematically speaking, entropy *is* the primary physical measure that quantifies nature's tendency toward disorder. Though Wheeler might say that such quantification constitutes the imposition of an external mathematical structure onto a natural, but inherently *non-mathematical*, phenomenon, the discussion presented in **Chapter 3** of this dissertation establishes an alternative perspective from which to view the situation.[84] Particularly in light of that chapter's content, and the parallels between Rovelli's characterization of entropy and the assemblage theory of Deleuze and Guattari, I have subsequently come to wonder if/how the reality of ever-increasing entropy might have any bearing on the human capacity to *mathematize* or to develop awarenesses of mathematical structures more generally. By expressing this curiosity, I also underline the interesting inverse relationship that exists between entropy and order.

Consider that, in the very early universe, when entropy was at a minimum, matter is theorized to have existed in a *highly organized* or *highly ordered* state, by virtue of which the allowable configurations of matter were *highly restricted*. It was only after (or rather, because of) the occurrence of the Big Bang, which could be interpreted as the *prototypical* spontaneous process, that opportunities to reorder/reorganize matter into a wider range of

[84] The reader is encouraged to revisit the subsection entitled *Wheeler's Notion of Mathematization*.

novel configurations became a possibility.[85] In essence, while states of comparably low entropy correspond to higher degrees of order and a *more restricted* range of possible material configurations, states of comparably high entropy correspond to lower degrees of order and a *less restricted* range of possibility. As a result, it might be asked whether or not a wider assortment of material configurations might (gradually) become available for human examination and contemplation as the universe continues to age and the effects of entropy continue to manifest. If matter itself is being opened up to new possibilities for configuration, then so too, perhaps, will be the collective human capacity for thinking about the *mathematics* that underlies our embodied cognitive experiences.

As more disorder manifests in the material world, it seems feasible that we could potentially encounter new opportunities to *find* different forms of order within that disorder, or to *become aware* of different material configurations from which to glean new mathematical insights. As pointed out in the earlier quotation from Rovelli (2018), there are, relatively speaking, "few ordered configurations compared to the countless disordered ones" (p. 144). In the broader probabilistic sense, then, the flow of entropy would suggest that disorder is becoming *increasingly* more probable than order (even if vast regions temporarily remain trapped in orderly configurations). If only due to the unavoidable consequences of this flow, it is conceivable that theoretical frameworks and branches of mathematics/physics that engage closely with disorder and complexity (i.e., chaos theory, complexity theory, fluid dynamics, combinatorics, et cetera) could become even more relevant than they already are, and this gives rise to a related question about what we generally *attend to* in mathematics education.

With cognition and sense-making involving an array of ordering/organizing activities, *patterns* are certainly deeply implicated in the work and thought of most (if not all) mathematicians to some degree or another, and developing *awarenesses* of patterns is widely regarded as an important aspect of mathematical understanding. Indeed, in his oft-cited essay *A Mathematician's Apology*, Hardy (1940/2012) is known for having referred

[85] To be fair, discussing the theoretical circumstances that are thought to have existed at or near the origin of the universe can be problematic. For the sake of the current intralude, I offer only modest speculation, and do not push the discussion too far.

to the mathematician (like the painter and the poet) as "a maker of patterns" (p. 84). Within the same work from Hardy, the phrase is later amended to the more specific form, "a maker of patterns of ideas" (p. 98). The current discussion of entropy and order/disorder brings these sentiments from Hardy back to mind, and prompts some additional reflection on *how* mathematicians make their patterns of ideas, and what they might be thinking *about* when doing so.[86] Seeing that the awareness of patterns is at least somewhat dependent upon the capacity to foreground specific entities (i.e., objects, processes, concepts, relationships, et cetera), while backgrounding others, it could be said that the general universal tendency toward disorder poses something of a practical problem; for it would seem that discerning foreground from background might be an increasingly challenging cognitive task as time progresses. This is to say that, as disorder becomes more widespread/prevalent, the noticeable differences between ordered and disordered configurations could, correspondingly, become more subtle.

When discussing the extension of entropy into information theory and the communication of meaning, Umberto Eco (1989) briefly touches upon the manner in which increasing disorder manifests as communicational "noise" and a potential hindrance to the meaning-making process. As he describes the associated complications, "If the meaning of the message depends on its organization [...], then "dis-order" is a constant threat to the message itself, and entropy is its measure" (pp. 50–51). Eco speaks of the need to "protect the message against consumption [by entropy], so that no matter how much noise interferes with its reception the gist of its meaning (of its order) will not be altered" (p. 51). In the context of the current intralude, I interpret Eco's remarks as a cautionary note to be mindful of the ways in which increasing entropy might impact meaning-making practices, and perhaps even as an earnest reminder not to take for granted the developmental capacity to *discern* meaningful foreground elements from a more expansive and disruptive background.

Although entropy does not enter into his analysis directly, Skemp (1987) raises points very similar to Eco's when discussing the relevance of 'signal' and 'noise' to mathematical

[86] Cf. Schiralli (2006), which also discusses Hardy's remarks about mathematicians as *makers of patterns*.

concept formation (see pp. 16–19). Initially indicating the deleterious effects of excessive noise, he also acknowledges that "some noise is necessary to concept formation" (p. 19). Moreover:

> In the early stages [of concept formation], low noise – clear embodiment of the concept, with little distracting detail – is desirable; but as the concept becomes more strongly established, increasing noise teaches the recipient to abstract the conceptual properties from more difficult examples and so reduces dependence on the teacher. (p. 19)

Read alongside Eco's observations, this passage from Skemp not only points out the changing contexts of the meaning-making process, but also reveals a possible pedagogical weight that might be associated with entropy and the manifestation of noise in the educational setting. Clearly, there are cases in which an overabundance of either order or disorder can stifle the meaning-making activities within a given educational setting; however it is also important to recognize that particular forms of disorder/disorganization may prove to be educationally *useful*. In itself, this is not a new idea, and it is a given that educators regularly strive to optimize the balance between order and disorder in any learning environment; yet, more closely exploring the implications of reframing mathematical meaning-making practices through the concept of *entropy* may shed additional light on this routine facet of the educational endeavour. Indeed, considering the sense in which an educational setting can be viewed as an evolving assemblage of learners, educators, materials, concepts, et cetera, it seems plausible that there might be pedagogical value in deliberately and judiciously *increasing* the entropy of the learning environment, provided educators can promote opportunities for learners to meaningfully reorder and reconfigure the spaces of possibility that subsequently emerge.

Having ventured into markedly more abstract territory compared to the introductory material concerning Clausius' thermodynamics and the principles of energy conservation, the current intralude expresses a perspective on the wider relevance of entropy as one particular facet of the larger material–mathematical relationship. In fact, it has here been proposed that entropy might even be thought of as the most pervasive and largest scale example of *material assemblage* enacted by/within our known universe. Following that same line of thought, before concluding this brief commentary on the notions of order and disorder, I revisit a minor issue identified in **Chapter 4**, which also has bearing on this interpretation of entropy.

The reader may recall that, despite an early inclination to conceive of wave function decoherence as an instance of *material assemblage in action*, I have conceded that it may *not* actually be appropriate to think of it in these terms. This retraction comes in response to Karen Barad's interpretation of decoherence, which describes it as a process of *randomization*, causing my initial assumption to come into conflict with Wise's (2011) assertion that an assemblage is not "a random collection of things" (p. 91). At first glance, this same conflict might seem to apply to entropy as well, as entropy could also be characterized as a process of randomization (i.e., by virtue of increasing disorder). However, there is an unusual sense in which the designation of *material assemblage in action* might still be suitable in each of these two instances, particularly because the increasing disorder associated with entropy also connotes increasing *homogeneity*. Once again enlisting the power of metaphor as a tool for reorganizing meaning, I would suggest that increased homogeneity also implies increased *uniformity*, and increased uniformity is itself suggestive of regularity, consistency, conformity, and (to a degree) *unity*. In this way, both entropy and decoherence can still be interpreted as processes that ultimately involve the *bringing together of elements*, although perhaps not in as literal (or as obvious) a manner as with quantum entanglements and structural couplings.

Chapter 6.
The Science of Material Assemblage

"Then, as the thousands of centuries trickled by, and the gods retired on a more or less adequate pension, and human calculations grew more and more acrobatic, mathematics transcended their initial condition and became as it were a natural part of the world to which they had been merely applied. Instead of having numbers based on certain phenomena that they happened to fit because we ourselves happened to fit into the pattern we apprehended, the whole world gradually turned out to be based on numbers, and nobody seems to have been surprised at the queer fact of the outer network becoming an inner skeleton."

–Vladimir Nabokov (1942/2000)
The Creative Writer, in B. Boyd and R. Pyle's *Nabokov's Butterflies: Unpublished and Uncollected Writings*, pp. 243–244

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Despite reaching the final chapter of this work, I imagine that the reader might still find the idea of reconsidering *the nature of mathematics* through the construct of *material assemblage* to be a somewhat nebulous and open-ended proposition. This is understandable, as I have largely worked to build out this material–mathematical worldview by appealing to two different theoretical frameworks (i.e., enactivism and quantum theory), the latter of which has *not* been widely adopted by the field of mathematics education. Other complexities emerge from the fact that drawing upon this pair of frameworks necessitates an interdisciplinary approach, and the judicious consideration of perspectives/commitments whose origins and discursive contexts differ substantially. I also remind the reader that, of four core themes identified in the early phases of this research, only *two* have been treated within the preceding chapters, namely new materialisms and issues of dualism, and epistemological uncertainty. Thus, with the remaining two themes (matters of agency, and complexity associated with emergent systems) having been postponed for future explorations, this dissertation represents only the *first* steps toward a proper articulation of the proposed worldview.

In light of the conceptual ground that *has* been covered thus far, I use the current chapter to synthesize main discussion points as well as other more tentative speculations, to supplement and clarify where possible, to offer additional reflections on the study as a whole, and to indicate avenues through which this research might be furthered. At times, this will include the posing of new questions that have been raised in response to the study, and the voicing/revoicing of concerns that have not yet been fully reconciled. With this intent in mind, **Chapter 6** looks backward and forward, in order to (partially) solidify a perspective in the present.

The richly articulated passage that opens this chapter has its origin in a larger composition written by Russian novelist, poet, and lepidopterist Vladimir Nabokov in the summer of 1941, for a creative writing course at Stanford University. Despite the markedly different milieux from which this passage emerged, it resonates remarkably well with the overall themes of this dissertation. In favour of allowing the reader the opportunity to internalize and reflect upon Nabokov's eloquent writing, I do not offer an explicit interpretation of the quoted passage, opting instead to give its evocative phrasing room to breathe. That being the case, I encourage the reader to consider the passage as a sort of mnemonic prompt with respect to earlier discussions, and to allow its connotations to linger as they move through the rest of this chapter.

As a companion to the structural overview offered in **Chapter 1**, the next subsection briefly re-enters the preceding chapters and intraludes. In a manner of speaking, it doubles as a reference guide for the reader who may wish to dip back into those discussions, and can be treated (collectively) almost as a conceptual map of the dissertation as a whole.

Regarding the Primary Chapters and Secondary Intraludes

Beyond (re)presenting a theoretically driven program of research, the current dissertation has been further classified as a conceptually motivated piece, in that it seeks to communicate a new perspective on the nature of mathematics and its relationship with matter. Though the document does not adhere to the structural conventions *typically* associated with doctoral dissertations in our field, or express a chronological account of the research itself, it *has* been designed to convey how engaging with a particular set of ideas ultimately facilitates the deconstruction of certain normative beliefs about *what* mathematics is, *where* mathematics can be found, and *how* it is instantiated in the world.

As a result, it formalizes a number of more tentative ideas that have arisen throughout my graduate pursuits.

Published in 1960 as a follow-up to the inaugural Courant Lecture delivered the previous year, Eugene Wigner's seminal paper *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* acts as a major impetus behind this doctoral research, as does Robbert Dijkgraaf's more recent 2017 article *Quantum Questions Inspire New Math*, in which Dijkgraaf identifies an interesting kind of reversal leading to what he refers to as "the unreasonable effectiveness of quantum theory in modern mathematics" (para. 4). While Wigner (1960) addresses how "the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena" (p. 8), Dijkgraaf (2017) notes that we are now seeing many more instances in which ideas born of modern particle physics "have an uncanny tendency to appear in the most diverse mathematical fields" (para. 4). As well as emphasizing that physics and mathematics are deeply intertwined and mutually informative, this encapsulates the first of two significant reversals that contribute to my material–mathematical worldview, and coincides with my more general assertion that *a changed view of the material also changes one's view of the mathematical*.

Before establishing the philosophical and theoretical orientations that impel this study, **Chapter 1** first speaks to an unusual phenomenon identified by Reuben Hersh, in which modern philosophy tends to perpetuate a disjunction between the philosophy of science and the philosophy of mathematics. By appealing to Pythagorean sensibilities regarding the material realm, the ideational realm, and number, I assume a stance through which these philosophies may be considered in a more unified way, and express a view that looks to *reinstate* mathematics as *the most foundational natural science*. It is in the same chapter that I also give voice to the tentative epistemological claim that *what we know about mathematics is mutually implicated in what we know about ourselves as mathematicians*, which offers early insight into the later discussion of my enactivist stance. **Chapter 5** sees this epistemological claim amended slightly, in favour of a more dynamic characterization.

In keeping with my overall interest in *the nature of mathematics*, implications emerging from the (inherently mathematical) quantum theory raise additional questions about the fundamental nature of matter, some of which foreshadow how mathematics might be seen not simply as a representational tool for mediating the human experience of reality, but as providing a *structural basis* for material reality. With the new materialist discourse playing a key role in the ensuing discussions, I draw upon de Freitas and Sinclair's (2014) notion of *inclusive materialism*, although the manner in which I do so is fairly limited, as I primarily leverage the term to examine evolving views of what constitutes matter at the atomic/subatomic level, whereas de Freitas and Sinclair engage deeply with its socio-political dimensions as well.[87] Unlike these authors, my approach is not informed by Châtelet's sensibilities about the human body and its relation to the material; yet, appealing to Alexander Wendt's (2015) notion of human beings "sharing the same physics of the body" (p. 190) does convey an underlying relationality at the level of lived experience. Although Châtelet's work pertaining to *the virtual*, *the actual*, *the possible*, and *the real* is expected to factor into future explorations, for the time being, my own perspective remains tightly bound to Bohr's model of the atom and its associated uncertainties/indeterminacies. At the same time, I have expressed agnosticism regarding whether or not the universe itself is *exclusively* in/deterministic and un/certain.

Congruent with my focus on the underlying *physics* of materiality, the embodied cognitive experience of mathematical structures and processes is also of great importance, as I endeavour to re-embed the material in the mathematical (via quantum theory) and to draw the mathematical *back* into thinking, knowing, sensing human beings as well (via enactivist theory). Brief introductions to quantum theory and enactivism provided in **Chapter 1** indicate the potential for connectivity between these two discourses, with Maturana and Varela's phenomenon of *structural coupling* sharing certain conceptual similarities with the phenomenon of *quantum entang*lement. It is suggested that a revised concept of 'enaction' that extends beyond the sense of *embodied action* to include a sense of *entangled action* might have merit. That said, in an effort to avoid misrepresenting or conflating the

[87] Though efforts have been made within the current dissertation to steer clear of political themes, I readily acknowledge that forthcoming work (particularly that concerning material agency) would do well to more fully embrace the wider implications of de Freitas and Sinclair's *inclusive materialism*.

enactivist and new materialist orientations, the later quantum-theory-focused discussion of **Chapter 4** addresses a significant point of contrast between them, whilst also suggesting that a *reformulation* of the enactivist stance might be possible via Karen Barad's agential realist ontology. This too has been left for future investigation.

Intralude A offers the first of the supplements to the primary chapter material, in the form of a commentary on the fundamental tension between the discrete and the continuous. In so doing, it addresses how two specific metaphors (involving borderless puzzles and defragmentation) have been extremely useful during different stages of the scholarly writing process. Each metaphor is discussed in terms of part–whole relationships; but, more importantly, they are used to introduce the assemblage theory of Deleuze and Guattari, as well as the related ideas of structural change and reconfiguration, which are integral to the main thread of this dissertation and should be read into my assertion that *mathematics embodies the very principles according to which matter organizes and reorganizes itself*. Via the broader metacommentary of **Intralude A**, the entire dissertation is seen as an evolving assemblage of entangled ideas, with the two metaphors mentioned above acting as examples of what I refer to as *material assemblage in action*.[88]

The various perspectives set out within this supplementary exploration are drawn together under the overarching thematic of *granularity*, and the concepts of continuity and discontinuity are ultimately recontextualized via a unique perspective from theoretical physicist Carlo Rovelli (2018), for whom continuity is "only a mathematical technique for approximating very finely grained things" (p. 75). This interesting assertion leads me to consider that the *perception* of continuity might actually be (or involve) a powerful form of *mathematization*. Though I offer only my initial speculation in **Intralude A**, I suggest that pursuing a more involved treatment of this idea may be worthwhile.

It is in **Chapter 2** that my overall research approach and accompanying methodological considerations are fully detailed, with particular emphasis placed on describing how the

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[88] The reader may recall from ***footnote 15***, that I used the phrase 'material assemblage in action' somewhat prematurely within **Intralude A**, and offered an invitation to revisit the intralude after a complete reading of the dissertation. I repeat that invitation here, as the subsequent chapters have provided much additional context through which to parse the earlier discussion.

relevant discourses have been drawn together. An existing (and possibly historically motivated) inclination toward the use of visual/optical metaphors when discussing processes through which human beings come to know and understand reality has led me to think of theories as *ways of seeing*, and theories concerning the nature of mathematics as *ways of seeing mathematics*. Moreover, the question of whether mathematics might itself constitute a particular kind of theory (or a particular way of seeing) arises as well.

Acknowledging that a conjoined enactivist/quantum theoretical perspective on the nature of mathematics could potentially draw from enactivism and quantum theory in a number of different ways, I clarify that these theories are not being layered *atop* one another in an indiscriminate manner; nor are they being filtered *through* one another hierarchically. Rather, they are being read *alongside* one another, such that neither framework is subordinated or prioritized and mathematics may be read through them both *simultaneously*. This has ultimately involved "bouncing" various insights off of established perspectives from the interconnected domains of physics, mathematics, educational psychology, and philosophy, and leveraging the power of metaphor as a tool for reorganizing meaning has been extremely valuable in terms of negotiating different points of view emerging from these domains. A general appeal to Clarke's Law offered at the beginning of the chapter acts as a reminder to remain attentive to the ways in which established and/or traditional perspectives continue to exert an influence on modern thought.

For transparency, I point out similarities between my approach to theory and the method of diffractive analysis formulated by Karen Barad and Donna Haraway. However, as a cautionary note to the reader, I also acknowledge that the possibility of a diffractive reading had *not* occurred to me until rather late in the process of my research, such that I can not appropriately claim to have been reading diffractively from the beginning. Interestingly, at this late stage in the writing of the dissertation, I have begun to reconsider whether *portions* of the document might actually demonstrate a diffractive reading after all. Regardless, as a matter of paying the proper respect to Barad and Haraway, I am not prepared to rescind my original disclaimer. Instead, I look forward to *re*-engaging with aspects of this discussion through an intentional and premeditated diffractive reading, with an eye toward unveiling different insights. This would simultaneously afford an opportunity to enter into the posthumanist discourse that informs Barad's stance on material agency, but which I have not touched upon here.

Chapter 3 opens with a perceptive passage from Caleb Gattegno, who highlights the generally elusive character of mathematics. Indeed, the idea that mathematics is *easily recognizable*, but *difficult to define* resurfaces throughout the dissertation, prompting discussions of the nature of mathematics in terms of both the explicit and the implicit, the literal and the metaphorical. The chapter takes the unusual tack of presenting an initial analytical sample that embraces my material–mathematical worldview *before* I have fully elaborated its terms of reference. This unconventional choice was made with the intent of prefacing certain motifs that recur in subsequent chapters, and of expeditiously demonstrating my approach to engaging with existing scholarly perspectives.

As a singular example, the implicit mathematical worldview of Richard Skemp is explored by examining his perspectives on the psychology of *learning* mathematics, and perceived inconsistencies in his worldview are used to shine light upon foundational characteristics of mathematics that may warrant greater attention. Particularly notable is Skemp's discussion of type 1 and type 2 theories, which subsequently leads him to consider what kind of theory mathematics itself might be. Though it is telling that Skemp's descriptions of these two types of theory belie an exclusively mental bias (i.e., such that they largely neglect/discount *bodily* ways of knowing), his concession that mathematics does not seem to fit into *either* category is perhaps more intriguing.

The previously mentioned passage from Gattegno points to *mathematization* as a more tangible process through which to gain access to the true nature of mathematics. This has prompted an exploration of David Wheeler's (1982, 2001) metaphor of mathematization as the act of *putting structure onto a structure*. Although I am in agreement with Wheeler's (1982) claim that mathematization "must be presumed present in all cases of "doing" mathematics or "thinking" mathematically" (p. 47), it is the *directionality* of his metaphor with which I take issue. As I have read it, Wheeler's view of mathematization is of a process that involves *imposing external mathematical structure*. Contrary to this, I reframe mathematization through a metaphor of *exposing innate mathematical structure*, such that mathematics is already immanent within matter. This expresses the second significant reversal that underlies my mathematical worldview, and (in advance of **Chapter 4's** explicit treatment of quantum theory) establishes the sense in which I conceive of mathematics as providing a structural basis for reality. It also motivates my decision to use the phrases 'mathematical worldview' and 'material–mathematical worldview' interchangeably.

A subsequent discussion of the ambiguous figure known as the Necker Cube contextualizes this reversal, simultaneously drawing attention to the importance of *awareness* with respect to the process of mathematization. It would appear that Wheeler and I share similar sensibilities about the part played by awareness in mathematical sense-making activity; yet, the very same reversal I have described impacts what that awareness pertains to. I uphold that it is not the "situation" that is any more or less mathematical; rather it is the awareness of the mathematics embedded *in* a given situation that manifests to greater or lesser degree. Accordingly, mathematization hinges upon the capacity to *recognize* and *attend* to innate mathematical structure. In addition to pointing out Wheeler's (1982) "lessening of confidence" (p. 47) in his metaphor of mathematization as the act of putting structure onto a structure, this chapter also offers my thoughts as to why that might be the case.

Via its commentary on linguistic commitments and the emergence of new discursive spaces, **Intralude B** allows me to express concerns about the need to utilize language that deliberately moves away from classical commitments whilst respecting the lineage of ideas from which those commitments originally arose. While this second supplement to the primary chapter offerings begins with a reflection on the nature of language and speculation about the inception/development of the early quantum theoretical discourse, Jennifer Burwell's robust concept of *originary drift* broadens its scope significantly. On its own, originary drift offers yet another construct through which to consider part–whole relationships. However, when read in concert with Maturana and Varela's *natural drift* (which contributes to their formulation of enactivism), it also points toward an overarching thematic of *entanglement* and incites additional questions about whether notions of self-organizing behaviour (or autopoiesis) might be extended to quantum systems. Coincidentally, a compelling study from Witthaut et al. (2017) indicates that there is *already* a precedent for discussing self-organizing behaviours in quantum systems, as the phenomenon of *self-synchronization* proves to be a defining feature of quantum entanglement. Scientifically speaking, this is highly significant, as it points toward a correlation between a certain class of behaviours that manifest in both the quantum domain and the classical domain. Within the context of this dissertation, it also acts as an early setup for the discussion of coherence and decoherence within **Chapter 4**, which addresses how entangled systems organize and reorganize themselves, as well as how such entanglements might be disrupted or broken.

It is this connection to ideas of structural change and reconfiguration that is of chief interest within **Intralude B**; however, with both forms of drift suggesting the emergence of new modes of thought (or the potential to be *thinking differently*), I offer a brief aside regarding the role played by *discernment* in biological evolution and the evolution of new discursive spaces. In closing, an insightful passage from Carlo Rovelli encapsulates the historical significance of *discernment* as a foundation of (critical) thought.

Recognizing that quantum ontologies are, generally speaking, not well represented in mathematics education, I do not assume any prior familiarity on the part of the reader with the quantum theoretical discourse. Also realizing that the more technical aspects of quantum theory can be a deterrent to engagement, I devote the entirety of **Chapter 4** to elaborating key features of the quantum view of matter, and to recounting particular historical issues and experiments that instigated/provoked its revolutionary impact on modern models of atomic structure. In light of the *worldview problem* identified by Wendt in the opening passage of the chapter, I also work to demystify salient features of quantum theory, to better communicate how the quantum view of matter expresses deeper connections to foundational mathematical principles, and to clarify why I have incorporated the theory into my conjoined perspective. Thus, in what may be the most conceptually demanding portion of the dissertation, **Chapter 4** explores principles of quantization and their links to mathematical notions of continuity/discontinuity and discretization. Efforts are made to keep the discussion both accessible and intuitive.

With the ultraviolet catastrophe of the late 19th century acting as a pivotal historical point of reference, I explore how a major discrepancy between classical predictions and physical observations of the black-body radiation spectrum would eventually lead to Max Planck's proposal of energy quantization constraints, which describe a microscopic world that operates according to a physics of the discrete, rather than of the continuous. Beyond confirming the flaws of previous radiation theory, this radical shift in worldview would also make clear the inadequacies of atomic structural models, again illustrating the intimate links between the material and the mathematical. Tracing subsequent changes to models of the atom reveals how the mathematics of quantization foster an expanded concept of what constitutes matter, with the photoelectric effect and Young's double slit experiment fueling the need for a unified wave–particle interpretation. In like fashion, the principle of wave–particle duality heavily blurs the distinction between matter and light by establishing that a full characterization of matter depends upon *both* particle-like properties and wave-like properties, despite the fact that these manifest independently of one another.

Perhaps the most widely referenced phenomenon associated with quantum ontologies, the quantum leap is used to illustrate one of the stranger behaviours that result from the discretization of atomic energy levels. It also proves useful in demonstrating variation within the quantum worldview, as not all theorists interpret the quantum leap in the same manner. Specifically, Bohr, Heisenberg, and Born's vision of the quantum leap as an abrupt (i.e., *random* and *discontinuous*) change of state, is contrasted by the smooth (i.e., *continuous*) transition afforded by Schrödinger's competing wave function interpretation, once again foregrounding the ongoing tension between discrete and continuous worldviews. However, very recent findings from Minev et al. (2019) provide evidence that a quantum theory aligned with the wave function interpretation may be more viable than the long-standing Copenhagen interpretation primarily attributed to Bohr, Heisenberg, and Born.

In conjunction with the research of Witthaut et al. (2017) discussed in **Intralude B**, the findings of Minev et al. (2019) also prompt an examination of the paired concepts of *coherence* and *decoherence*, which emerge from Schrödinger's wave function interpretation and seem to signal either the stability or potential instability of entanglements (respectively). Although both coherence and decoherence are fundamentally concerned with structural change and reconfiguration, I find that decoherence is more difficult to reconcile with the notion of material assemblage that I develop elsewhere in the dissertation. The introductory discussion of **Chapter 4** does not delve too deeply into this inconsistency, yet the later metacommentary of **Intralude C** follows up in more detail.

The final subsection of **Chapter 4** speaks directly to David Bohm's sense of the *undivided wholeness of the universe*, which conveys a unique approach to quantum entanglement that I choose to read alongside aspects of the enactivist perspective (despite their different discursive origins). Within the same discussion, I also examine Bohm's (1980/2002) view of quantum theory "AS AN INDICATION OF A NEW ORDER IN PHYSICS" (p. 141), which I consider through Deleuze and Guattari's assemblage theory and the thoughts I have previously expressed about structural change and reconfiguration. Moreover, Bohm's ensuing remarks regarding how 'order' is known implicitly are likened to Gattegno's (1988) observation that "mathematics is recognizable but not easily defined" (p. vii), and I interpret Bohm's efforts to define what 'order' is as an instance that supports Gattegno's stance on *mathematization*. This is to say that, by struggling to define 'order', Bohm is actually struggling to articulate an awareness about innate mathematical structures. Specifically in terms of how the particularity of the *explicate* (or unfolded) order emerges from the generality of the *implicate* (or enfolded) order, Bohm's distinction between the *implicate* and the *explicate* mirrors the sense in which I conceive of the material being embedded in the mathematical, with the latter exemplifying a deeper level of order that underpins our reality. **Chapter 4** concludes with a brief discussion of the status of the observer in Bohm's quantum ontology, namely the manner in which the observer and the observed are entangled within the implicate order.

Prompted by Carlo Rovelli's interesting remarks about the inability of human beings to clearly explain ourselves to ourselves (despite our well-developed mental faculties), **Chapter 5** revisits my earlier epistemological claim that *what we know about mathematics is mutually implicated in what we know about ourselves as mathematicians*. However, so as to shift the focus from *objects of knowledge* toward *processes of knowing* (or *coming to know*), I amend this epistemological stance slightly, instead offering that *how we come to* *know mathematics is intimately entwined with how we come to know ourselves*, which leverages connotations drawn from the entanglement theory discussed in **Chapter 4**. The majority of **Chapter 5** is structured so as to situate this view with respect to established orientations in the philosophy of mathematics education. To that end, it draws from the frequently referenced work of Paul Ernest (1991, 1998). Absolutist and fallibilist orientations are used as points of reference/comparison, with *Logicism*, *Formalism*, and *Constructivism* stratifying the former. Reuben Hersh's view on *pluralism* in the philosophy of mathematics provides a powerful tool with which to negotiate the many (sometimes conflicting) perceptions of mathematics, mathematical knowledge, and mathematical modeling, and I enlist this pluralist philosophy to suggest that my own material–mathematical worldview might *peacefully coexist* alongside the larger multitude of perspectives. In particular, I embrace Hersh's sense that models of mathematics can be judged according to how *useful* and *illuminating* they are, rather than according to the more traditional criteria of *finality* and *completeness*.

In contrast to the well-established, standardized theories in the philosophy of mathematics education, Max Tegmark's Mathematical Universe Hypothesis (MUH) is presented as a compelling alternative perspective with which my own worldview shares certain conceptual links. That said, **Chapter 5** does outline differences between Tegmark's assertions and my own, and I emphasize that I am currently unable to support/advocate his claims that *mathematical existence and physical existence are* equivalent, and that *everything that exists mathematically exists physically*. In a direct follow-up to this discussion of Tegmark's hypothesis, the final subsection of **Chapter 5** revisits the quantum leap of **Chapter 4**, and reframes it according to elementary arithmetical principles. The revised interpretation of the energy level transition scenario illustrates how processes of structural change and reconfiguration are enacted at the subatomic level, as well as how I see mathematical structures and processes infusing the very building blocks of material reality. I envision this discussion yielding much additional fruit when read alongside Châtelet's work on *the virtual*, *the actual*, *the possible*, and *the real*.

As the last of the supplements to the primary chapter material, **Intralude C** presents a commentary on the paired notions of order and disorder, both of which are discussed in terms of the more encompassing thematic of entropy. Though the concept of entropy originates in the Victorian-era thermodynamics of Rudolph Clausius, the contexts to which it applies have since expanded greatly, such that entropy is now relevant within a much broader range of fields/disciplines. Of chief interest within **Intralude C** is the cosmological interpretation that conceives of entropy as a fundamental measure of *disorder*. Current theory holds that the increasing disorder of the known universe, or what Rovelli (2018) calls the "dance of ever-increasing entropy" (p. 144), is a direct consequence of the second law of thermodynamics, which speaks to the directionality of natural processes.

It would appear that the known universe is inevitably evolving from its original, highly ordered configuration (corresponding to a state of *minimum* entropy) toward a highly disordered configuration (corresponding to a state of *maximum* entropy), and it is by manifesting this universal tendency toward disorder, this "irresistible mixing of all things" (Rovelli, 2018, p. 144), that entropy might exemplify the single most pervasive manifestation of material assemblage. To be fair, I acknowledge that entropy, as with the phenomenon of decoherence, implies increasingly *randomized* configurations, which is somewhat problematic if I am to retain it as an example of "material assemblage in action"; for, as Wise (2011) points out, material assemblages are not random collections of things. As a result, I offer that it may be necessary to further build out my material–mathematical worldview by incorporating a complementary process of *material disassemblage*. That said, **Intralude C** does articulate one possible means through which to avoid introducing this new wrinkle.

Within the space of information theory, Umberto Eco characterizes increasing disorder (or noise) as a threat to communicational clarity. In his own work on the psychology of learning mathematics, however, Skemp (1987) also acknowledges that a certain amount of noise "is necessary to concept formation" (p. 19). Though this does not contradict Eco's view, it does suggest the possibility of positively leveraging the concept of entropy in the educational setting.

Regarding the Nature of Mathematics

With this document representing only the initial stages of what is expected to be a more significant exploration, it is difficult to make definitive statements about my research thus far. Instead, it seems more appropriate to offer *closing comments*, whilst reiterating the motivating questions behind the program of research. At its heart, this entire exploration is driven by a *fascination* with mathematics, a related curiosity about the reciprocal interplay between mathematics/physics and the material world, and a question of *why* mathematics shares any sort of connection with the material world at all (let alone one that is so useful and meaningful to human beings). Indeed, why it is that human beings should even demonstrate a capacity for *thinking* mathematically poses a secondary quandary that has influenced this overall discussion. In large part, these queries and curiosities are rooted in a physical sciences perspective, and further inspired by corresponding ontological and epistemological interests, all of which have led me to raise the more fundamental question of exactly what mathematics *is*.

The pursuit of this fundamental question has resulted in the proposal of a conjoined enactivist/quantum theoretical perspective that reconceives of mathematics as *the science of material assemblage*, and this perspective is largely dependent upon three primary commitments. Firstly, I reprioritize mathematics as an ontological fundament, such that mathematical structures and processes are seen as the *basis* of material reality, and as encoded within matter itself. As a result, the discipline of mathematics is reinstated as the *most foundational natural science*, in that it is intrinsically concerned with the same structures and processes noted above. Secondly, I reverse the directionality of Wheeler's widely representative view of mathematization, such that acts of mathematization do not impose *external* mathematical structure (as though it were divorced from the objects/phenomena of interest), but rather expose *innate* mathematical structure (of which varying degrees of awareness are possible). Thirdly, I describe mathematics as *embodying* the very principles according to which matter organizes and reorganizes itself (which encapsulates/expresses the inherent and ongoing dynamism of the material–mathematical relationship).

Different portions of this document have interpreted certain mathematical and material scenarios through these three primary commitments, with the intent of offering a sort of response to Wigner's (1960) ponderings over the "uncanny usefulness of mathematical concepts" in the physical sciences (p. 2), as well as Dijkgraaf's (2017) more recent observations concerning the "unreasonable effectiveness of quantum theory in modern mathematics" (para. 4). By appealing to the entanglements of quantum theory and the structural couplings of enactivism, I have also attempted to indicate how the same tenets of material assemblage (the same organizing principles) apply to human beings as well, by virtue of our deep interconnectedness with other facets of the material world.

This is to say that human beings have emerged (and continue to emerge) from the same mathematical underpinnings as the rest of the known universe, and that we are innately mathematical entities with a capacity to recognize and attend to the organizing principles at play within and around us. It is this sense of mathematically *being* and *becoming* in an inherently mathematical reality that I read into Nabokov's sentiment about some future time when mathematics *transcend their initial condition and become a natural part of the world*. As the discussions contained within this doctoral dissertation will attest, I very much aspire to the day when nobody will be "surprised at the queer fact of the outer network becoming an inner skeleton" (Nabokov, 1942/2000, p. 244).

Postface

Many pages and many, many months ago, I offered the following insight within the metacommentary of **Intralude A**:

> Embracing the implications of assemblage theory and the associated notions of structural change and reconfiguration, I have inevitably begun to think about the dissertation writing process in similar terms. More specifically, I have come to envision this dissertation as an evolving assemblage of entangled ideas, and I am increasingly concerned with the structuring principles at play within it, and through which the embedded discussions are being realized.

This sentiment has remained at the forefront of my thoughts throughout the writing of this document, and I find that it now resonates even more strongly as I compose these final paragraphs and reflect upon the perspective that I have endeavoured to build out across the preceding chapters and intraludes. It is interesting to internalize how this very particular piece of academic writing has emerged over the course of recent years, especially considering that the form it has ultimately taken was not predetermined at any single point along the way.

Even though the completed dissertation exhibits a range of well-defined features (i.e., a finite number of pages, a definitive word count, symmetrical chapter structure, primary themes delineated by chapter titles, secondary themes delineated by subheadings, a table of contents outlining the sequence of the discussions, et cetera), the extended process of *arranging*, *organizing*, and *fitting together* its various "components" is unlikely to be as apparent to the reader as it has been to the author. That said, as the reader engages with this work, they will inevitably contribute to (and participate in) the emergence of a *new* material assemblage, one with which their own ideas will become entangled, reshaping the totality of the work in ways that I can neither predict nor imagine. Despite the *appearance* of closure that might be associated with this finished document, it is my sincere hope that the ideas articulated within will expand beyond the boundaries of its pages to incite additional reflection on the nature of mathematics and the organizing principles governing our material reality.

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