

Among graphs, groups, and latin squares

by

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M.Sc., Simon Fraser University, 2017

B.Sc., University of Puget Sound, 2014

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

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SIMON FRASER UNIVERSITY
Summer 2021

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Abstract

A latin square of order n is an $n \times n$ array in which each row and each column contains each of the numbers $\{1, 2, \dots, n\}$. A k -plex in a latin square is a collection of entries which intersects each row and column k times and contains k copies of each symbol. This thesis studies the existence of k -plexes and approximations of k -plexes in latin squares, paying particular attention to latin squares which correspond to multiplication tables of groups.

The most commonly studied class of k -plex is the 1-plex, better known as a transversal. Although many latin squares do not have transversals, Brualdi conjectured that every latin square has a near transversal—i.e. a collection of entries with distinct symbols which intersects all but one row and all but one column. Our first main result confirms Brualdi's conjecture in the special case of group-based latin squares.

Then, using a well-known equivalence between edge-colorings of complete bipartite graphs and latin squares, we introduce Hamilton 2-plexes. We conjecture that every latin square of order $n \geq 5$ has a Hamilton 2-plex and provide a range of evidence for this conjecture. In particular, we confirm our conjecture computationally for $n \leq 8$ and show that a suitable analogue of Hamilton 2-plexes always occur in $n \times n$ arrays with no symbol appearing more than $n/\sqrt{96}$ times. To study Hamilton 2-plexes in group-based latin squares, we generalize the notion of harmonious groups to what we call H2-harmonious groups. Our second main result classifies all H2-harmonious abelian groups.

The last part of the thesis formalizes an idea which first appeared in a paper of Cameron and Wanless: a (k, ℓ) -plex is a collection of entries which intersects each row and column k times and contains at most ℓ copies of each symbol. We demonstrate the existence of $(k, 4k)$ -plexes in all latin squares and $(k, k+1)$ -plexes in sufficiently large latin squares. We also find analogues of these theorems for Hamilton 2-plexes, including our third main result: every sufficiently large latin square has a Hamilton $(2,3)$ -plex.

Keywords: latin square (05B15); group-labelled graphs (05C25); Hamilton cycles (05C45); k -plex; partial transversal; harmonious group

Dedication

To Mom, Dad, Caitlin and Kenzie.

*Kindness eases Change.
Love quiets fear.
And a sweet and powerful
Positive obsession
Blunts pain,
Diverts rage,
And engages each of us
In the greatest,
The most intense
Of our chosen struggles*

- Octavia E. Butler, from *Parable of the Talents*

Acknowledgements

There are many people responsible for this thesis coming together. Above all, I would not have made it anywhere near this far in my academic career without the support of my parents. I am incredibly privileged to have two parents who have not only walked this path before, but are eager to share advice and support whenever I need it. Thank you both, I love you so much.

Special mention must also go to my partner Kenzie Parry, whose tender care and loving support has given me the strength I needed to get through this process. We got this!

Thank you to Maureen Callanan, Stefan Hannie, Tara Petrie, and Alexandra Wesolek for reading drafts of various parts of this thesis and offering helpful comments. Thank you to Peter Bradshaw, Jesse Champion Loth, Sebastian Gonzalez Hermosillo de la Maza, Stefan Hannie, Avi Kulkarni, Tabriz Popatia, Stefan Trandafir, and Alexandra Wesolek for discussions of the mathematics behind ideas which appeared here.

Thank you to Luis Goddyn for supervising this project, and to Matt Devos, Bojan Mohar, and Ladislav Stacho for guidance on mathematical matters. Thank you to Veselin Jungic for a year of mentorship that far exceeded that usual bounds of a TA-instructor relationship, to Rob Beezer for introducing me to group theory and computer algebra, to Douglas Cannon for helping me discover the beauty of pure mathematics, and to Jonathan Jedwab for giving me fantastic advice when I was not yet mature enough to receive it. And an extra special thank you to Marni Mishna for helping me navigate the politics of academia.

Finally, I'd like to thank all of my dear friends, including extreme gratitude for those of you mentioned mentioned above, as well as Daoud Anthony, Chelsea Cater, Caro Deady, Danielle Rogers, Jarin Schexnider, Isadore Schexnider Hannie, Mark Sutherland, and Sara Yeomans, for the fun hangs, chats, and emotional support I needed to get through when the work involved in this process felt overwhelming. Similar gratitude must also be extended to family members not yet mentioned: Catilin, Rob, and Amos; Grandma Marilyn; Grandpa Billy (I miss you so much already) and Aunt Beth; Mary, Dave, Siobhan, Jacob, Max, and Fiadh; Scott and Sue; John, Lisa, Eli, and David—I love you all so much.

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List of Symbols

$[a, b]$	For integers $a < b$, the set of integers $\{a, a + 1, \dots, b\}$
$(a_i)_{i=k}^{\searrow 1}$	The sequence $a_k, a_{k-1}, a_{k-2}, \dots, a_1$
$(a_i, b_i)_{i=1}^k$	The sequence $a_1, b_1, a_2, b_2, \dots, a_k, b_k$
$\langle b \rangle$	The cyclic subgroup of a group G generated by the element $b \in G$
$C(L)$	The set of columns in the array (latin square) L
χ_L	The edge-coloring of $K_{n,n}$ corresponding to the order- n latin square L
D_n	The dihedral group of order n
$\Delta(G)$	The maximum degree of the graph G
$\delta(G)$	The minimum degree of the graph G
$\text{dist}_G(u, v)$	The number of edges in the shortest path between u and v in G
$(d \pm \varepsilon)n$	For $d, n \in \mathbb{N}$ and $\varepsilon > 0$, the interval $((d - \varepsilon)n, (d + \varepsilon)n)$
$e_S(A, B)$	The number of edges with a color in S , one end in A , and the other end in B
$[g]_C$	The vertex of $K_{n,n}(G)$ corresponding to the column of L_G with index g
$\Gamma(L)$	The latin square graph of the latin square L
$[g]_R$	The vertex of $K_{n,n}(G)$ corresponding to the row of L_G with index g
$G \cup H$	For graphs G and H , the disjoint union of G and H
$K_{n,n}(L)$	The edge-coloring of $K_{n,n}$ induced by the $n \times n$ array (latin square) L
$ L $	The order, or number of rows/columns, in a square array (latin square) L
L_G	The latin square given by the multiplication table of the group G
$L_{i,j}$	The symbol appearing in row i and column j of the array L .
M_n	The Möbius ladder on $2n$ vertices
$[n]$	The set of integers $\{1, 2, \dots, n\}$
$O(f(n))$	The class of functions g such that $\limsup_{n \rightarrow \infty} g(n)/f(n) < \infty$
$o(f(n))$	The class of functions g such that $\lim_{n \rightarrow \infty} f(n)/g(n) = \infty$
P_ℓ	The path graph on ℓ vertices
$R(L)$	The set of rows in the array (latin square) L
$S(L)$	The set of symbols appearing in the array (latin square) L
$\text{Syl}_2(G)$	For a group G , the isomorphism class of its Sylow 2-subgroups

Glossary

2-bounded

An edge-colored graph containing at most 2 edges of any given color.

2-group

A group in which the order of every element is a power of 2.

balanced bipartite

A balanced bipartite graph is a bipartite graph with bipartition (R, C) satisfying $|R| = |C|$.

bipartition

In a bipartite graph G , there are disjoint sets $R, C \subseteq V(G)$ such that every edge has one end in R and the other end in C . We refer to (R, C) as the (vertex) bipartition of G .

Cayley table

Of a group G , a latin square whose rows and columns are indexed by the elements of G with the entry in row g column h containing the symbol gh .

complete mapping

Of a group G , a bijective function $\theta : G \rightarrow G$ such that $g \mapsto g\theta(g)$ is also a bijection. Similarly, a *near complete mapping* of G is a bijection $\phi : G \setminus \{a\} \rightarrow G \setminus \{b\}$ such that $g \mapsto g\phi(g)$ is a bijection between $G \setminus \{a\}$ and $G \setminus \{c\}$ for some $a, b, c \in G$. If θ is a complete mapping of G , then $\{(g, \theta(g), g\theta(g)) \mid g \in G\}$ is a transversal of L_G . Similarly, near complete mappings of G are equivalent to near transversals of L_G .

entry

One may think of a rectangular array L as a collection of triples $\{(r, c, s) \mid L_{r,c} = s\}$. We refer to each such triple $(r, c, s) \in L$ as an *entry*.

equi- n square

An $n \times n$ array in which each symbol appears exactly n times.

equipartition

Of a set S , a collection of disjoint subsets $\{S_i\}_{i=1}^k$ such that every element of S appears in exactly one S_i and $||S_i| - |S_j|| \leq 1$ for all $i, j \in [k]$.

extension

A group G is an extension of N by Q if G has a normal subgroup $N_0 \triangleleft G$ which is isomorphic to N and the quotient group G/N_0 is isomorphic to Q .

Hamilton cycle

A connected 2-regular spanning subgraph.

harmonious sequence

A list g_1, g_2, \dots, g_n of the elements of a group G for which the consecutive products $g_1g_2, g_2g_3, \dots, g_{n-1}g_n, g_n g_1$ are all distinct.

hypergraph

A pair $H = (V, E)$ of vertices V and hyperedges E , which are subsets of V . A hypergraph is *3-uniform* if each edge has size 3, it is *tripartite* if the vertex set can be partitioned into three sets such that each edge contains exactly one vertex from each of these sets, and it is *linear* if any pair of vertices is contained in at most one edge.

isotopy class

If L_1 and L_2 are latin squares of order n , we say that L_1 and L_2 are *isotopic* if there is an automorphism of $K_{n,n}$ that fixes the vertex bipartition and maps each color class of $K_{n,n}(L_1)$ to a color class of $K_{n,n}(L_2)$. An isotopy class is a maximal collection of isotopic latin squares.

 k -bounded n -square

An $n \times n$ array in which no symbol appears more than k times.

 k -plex

For a positive integer k , a collection of entries in a latin square which intersects each row and column k times and contains k copies of each symbol.

 (k, ℓ) -plex

For positive integers k and ℓ , a collection of entries in a latin square, or other square array, which intersects each row and column exactly k times and contains at most ℓ copies of each symbol. In a latin square, a (k, k) -plex is always a k -plex, but this may not be the case in arrays with more symbols than rows.

Möbius ladder

The cubic graph formed from a cycle of length $2n$ —referred to as the *rim* of M_n —by adding n edges, one joining each pair of vertices at distance n in the initial cycle.

main class equivalent

Two latin squares L_1 and L_2 are main class equivalent if their latin square graphs are isomorphic, i.e. $\Gamma(L_1) \cong \Gamma(L_2)$.

main diagonal

Of a square array, the set of entries whose column index is equal to their row index.

multigraph

A loopless graph in which there may be multiple edges corresponding to the same pair of vertices.

near transversal

A partial transversal of size $n - 1$ in an $n \times n$ array.

partial k -plex

For a positive integer k , a collection of entries in a latin square, or other square array, which intersects each row and column at most k times and contains at most k copies of each symbol.

partial transversal

A collection of entries in a latin square, or other rectangular array, that intersects each row and each column at most once and contains at most one copy of each symbol.

rainbow

An edge-colored graph with exactly one edge of each color.

row-latin array

A rectangular array in which no symbol appears more than once in any row.

solvable group

A group for which there exists a subnormal series $G = G_a \triangleright G_{a-1} \triangleright \cdots \triangleright G_1 \triangleright G_0 = \{1\}$ such that the quotient groups G_i/G_{i-1} are all abelian.

square latin array

An $n \times n$ array containing symbols from a set of size $m \geq n$ such that no symbol appears more than once in any row or column.

Sylow 2-subgroup

Of a group G , subgroups whose order is the largest power of 2 dividing $|G|$. They are named after Peter Ludwig Sylow, who showed that such groups exist and that any two Sylow 2-subgroups of G are isomorphic.

transversal

In an $m \times n$ array, a collection of $\min\{m, n\}$ entries that intersects each row and column at most once and contains at most copy of each symbol.

Chapter 1

Introduction

Currently well-known to the general public in the special case of Sudoku puzzles, latin squares are mathematical objects whose rigorous study goes back over 300 years. While the earliest known appearance in the literature was an early 18th century monograph of Choi Seok-Jeong, constructions of 4×4 latin squares appear in the 13th century writings of Ahmad al-Buni [43, p. 12]. Today, latin squares are used by a wide range of scientists and engineers. They are a well-known and oft-utilized tool in the design of both statistical experiments and digital communications technology [84, Sec. 11.4], and are currently being studied in myriad applied mathematical contexts (see e.g. [72, 109, 122]).

This thesis studies balanced substructures of latin squares. A *latin square* L of order n is an $n \times n$ array in which each row and column is a permutation of some set $S(L)$ of size n (usually the set of integers $[n] := \{1, 2, \dots, n\}$). Indexing rows and columns with $[n]$, latin squares can be treated as collections of n^2 ordered triples $L \subseteq [n]^3$. We refer to each triple $(r, c, s) \in L$ as an *entry*. For positive integers $k \leq n$, a *k-plex* of L is a collection of entries which intersects each row and column k times and contains k copies of each symbol. Our main results concern the existence of k -plexes in latin squares and their generalizations. The first half of this thesis considers group-based latin squares, resolving the restriction to this setting of a well-known old conjecture due to Brualdi and establishing fundamental results concerning a new form of combinatorial group sequence. The second half poses and explores a graph-theoretic strengthening of a less famous conjecture due to Rodney and proves several existence results for approximations of plexes.

The story of latin squares as an object of modern combinatorial mathematics begins with work of Leonard Euler from the late 18th century [60] on a notion now known as orthogonality: two latin squares of order n are said to be *orthogonal* if the array one obtains by superimposing them contains each possible pair of symbols exactly once.¹ Then, in

¹According to Andersen [120, Ch. 11], the somewhat peculiar name “latin square” was introduced by Euler, who was using latin squares to construct magic squares—squares with integer entries wherein the sum of each row, column, and main diagonal is the same—and following the algebraic convention of using latin letters for algebraic indeterminates. Although the term graeco-latin square was later used to describe

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

Figure 1.1: A pair of orthogonal latin squares of order 4.

the late 19th and early 20th centuries, work of Arthur Cayley [41] and Ruth Moufang [95], among others, further established latin squares as important objects of pure mathematics by drawing connections between latin squares and several well-known geometric and algebraic structures. As indicated above, we are particularly interested in the connection between groups and latin squares demonstrated by Cayley: every finite group G can be represented by a *group-based* latin square L_G , with rows and columns indexed by the elements of G and the entry in row g and column h containing the symbol gh . Perhaps the central motivating conjecture for the study of latin squares in the early-to-mid twentieth century, however, goes all the way back to Euler; Euler's conjecture that there are no orthogonal latin squares of order $4n + 2$ inspired a significant amount of work (see [84, Section 5.1]) up until it was definitively disproven by Bose, Shrikhande, and Parker [27] in 1960.

Although we will not discuss orthogonal latin squares explicitly in what follows, their study is the source of the problems considered by this thesis. In particular, if L_1 and L_2 are orthogonal latin squares and $s \in S(L_2)$, then the set T of entries in L_1 corresponding to the positions of entries with symbol s in L_2 is a 1-plex (see Figure 1.1). We will usually refer to 1-plexes by their more common, and much older, name: *transversals*. The key observation to be made here is: for a latin square L_1 of order n , there is a latin square L_2 orthogonal to L_1 if and only if we can partition L_1 into n disjoint transversals. While it was this connection to orthogonal latin squares that first inspired the study of transversals in latin squares [116], we will see shortly that the study of transversals and their various generalizations is a very rich topic in its own right.

1.1 Transversals

Already present in the work of Euler [60] was the implicit observation that, for every even integer n , there exists a latin square of order n with no transversal. However, to this day, there are no latin squares of odd order known to not have a transversal. It has now been over

pairs of orthogonal latin squares, and this name was inspired by Euler's concurrent use of greek and latin letters to represent an orthogonal pair, Andersen states clearly [120, p. 261] that Euler, who introduced the term latin square, never used the term graeco-latin square.

five decades since the mathematical community began trying to prove that this observed pattern extends to all latin squares of odd order.

Conjecture 1.1 (Ryser [107]). *Every latin square of odd order has a transversal.*

Ryser may have made the stronger conjecture that the number of transversals in every latin square of order n is congruent to n modulo 2 (see [25] for a discussion of what exactly Ryser may or may not have conjectured). Balasubramanian [14] provided a partial result in the direction of this stronger conjecture by showing that every latin square of even order has an even number of transversals. Unfortunately, it is not true that all latin squares of odd order have an odd number of transversals because, as was noted in e.g. [25, 36, 115], there exist latin squares of order 7 with an even number of transversals.

Though there has been some very interesting recent work on the enumeration of transversals in latin squares (see [49, 52, 53, 66, 110]), because we are here interested in questions of existence, we will leave an exploration of this topic (via the papers just cited) to the reader. One enumerative result, however, is worth discussing in this context. Although Conjecture 1.1 is concerned only with latin squares of odd order, and it is known that for all even n there are at least $n^{3/2(1/2-o(1))}$ latin squares of order n with no transversal [40], a landmark paper of Kwan [88] shows that almost every latin square has a transversal. More precisely, it is shown in [88] that the probability of a uniformly random latin square of order n containing $\left((1 - o(1))\frac{n}{e^2}\right)^n$ distinct transversals approaches 1 as n approaches infinity. This is one of the most compelling pieces of positive evidence in support of Conjecture 1.1 known to date.

It is worth noting that some evidence indicating Conjecture 1.1 may not be true was given by Egan and Wanless in [57]. In that paper, it was shown that for all odd $m \geq 3$, there exists a latin square of order $3m$ which contains an $(m - 1) \times m$ latin subrectangle all of whose entries are not in any transversal. Moreover, it is shown that, for all $n \geq 4$, there exists a latin square of order n with at least 7 entries which do not appear in any transversal. Nonetheless, a preponderance of the currently available evidence seems to be in support of Conjecture 1.1.

1.2 Partial transversals

Much of the work towards Conjecture 1.1 has considered the slightly weaker notion of *partial transversals*, or collections of entries which intersect each row and column at most once and contain at most one copy of each symbol. There is a growing body of literature studying when partial transversals are properly contained in larger partial transversals [23, 37, 63, 64, 73], and there have been several papers considering decompositions of latin squares into disjoint partial transversals [21, 38, 68, 96]. In this thesis, though, we are most interested in partial transversals of maximum size.

Over the years, a series of papers has provided increasing bounds on the size of partial transversals which must exist in every latin square. Picking up the story partway through, it was shown concurrently and independently by Brouwer et al. [30] and Woolbright [121] that every latin square of order n has a partial transversal of size at least $n - \sqrt{n}$. After a couple of erroneous claimed improvements, Hatami and Shor [78] were then able to improve the bound to $n - 11.053 \log^2 n$. The currently best known bound is due to Keevash et al. [85], who showed that every latin square of order n has a partial transversal of size $n - O\left(\frac{\log n}{\log \log n}\right)$. Given that not all latin squares have a transversal, one may wonder how close to n this bound will get. Referring to a partial transversal of size $n - 1$ as a *near transversal*, we now arrive at the first of the three main motivating conjectures for this thesis.

Conjecture 1.2 (Brualdi [31]). *Every latin square has a near transversal.*

An even stronger conjecture concerning the minimum covering radius (with respect to Hamming distance) of a set of permutations, and the progress towards that conjecture [36, 118, 79], is worth noting. Nonetheless, we will here focus on Brualdi’s conjecture as stated. In Chapter 2 we will prove that the conjecture holds for all group-based latin squares. Further evidence for this conjecture was recently given by Best et al. [24], who confirmed that Conjecture 1.2 holds for all latin squares of order at most 11. However, there has been recent work indicating that Brualdi’s conjecture may not be true.

Conjecture 1.2 has often been attributed to Stein—and is often stated alongside Conjecture 1.1 as the Ryser-Brualdi-Stein conjecture (see e.g. [67, 85, 94])—but the oft-cited paper of Stein makes only the stronger conjecture that every *equi- n square*, or every $n \times n$ array in which each symbol in $[n]$ appears n times, has a near transversal. This distinction is worth noting because Stein’s conjecture was recently shown to be false when Pokrovskiy and Sudakov [103] constructed equi- n squares with no partial transversal of size greater than $n - \frac{1}{42} \log n$ for all $n \geq e^{120}$. When combined with Aharoni et al.’s [2] proof that every equi- n square has a partial transversal of size at least $\frac{2}{3}n$, this work raises the interesting question of whether every equi- n square has a partial transversal of size $n - o(n)$. It is shown in [103] that this question has a positive answer under the additional assumption that no symbol appears too many times in any row or column. It is also known [100] that almost every equi- n square has a transversal.

1.3 Generalizations of latin squares

Equi- n squares are just one of many generalizations of latin squares whose partial transversals have been extensively studied. They can be naturally generalized to what we call *k -bounded n -squares*, which are $n \times n$ arrays in which no symbol appears more than k times. A famous paper of Erdős and Spencer [59] shows that all $\frac{n-1}{4e}$ -bounded n -squares have a transversal. This was later improved upon by Biassot et al. [26], who showed that every

$\frac{27n}{256}$ -bounded n -square has a transversal, and Perarnau and Serra [100], who gave an asymptotically precise enumeration of the transversals in $\frac{n}{10.93}$ -bounded n -squares. We will study k -bounded n -squares below in Section 4.2.

The spirit of latin squares is perhaps better retained by considering arrays with various “latin” properties. A rectangular array is referred to as *row-latin* (respectively *column-latin*) if no symbol appears more than once in any row (resp. column). An array that is both row-latin and column-latin is often called a *latin array*. In $m \times n$ arrays with $m \neq n$, the term transversal is used for partial transversals of size $\min\{m, n\}$. Transversals in such latin arrays are well-understood. Drisko [51] showed that, whenever $m \geq 2n - 1$, every $m \times n$ row-latin rectangle has a transversal and provided examples showing that the given lower bound on m is tight. It is worth noting that Drisko’s proof of the lower bound depends upon columns containing many copies of the same symbol. When we restrict to arrays which are also column-latin, Häggkvist and Johansson showed that all $(n + o(n)) \times n$ latin arrays can be decomposed into disjoint transversals, and there have subsequently been several further, graph-theoretic strengthenings of these results [4, 5, 101, 102].

There has also been much recent work on transversals in square latin arrays (which are more general than latin squares in that they may contain more than n symbols). Inspired by a conjecture of Akbari and Alipour [6] that every $n \times n$ latin array with at least $\frac{n^2}{2}$ distinct symbols contains a transversal, Best et al. [22] showed that every square latin array containing at least $(2 - \sqrt{2})n^2$ symbols has a transversal. Akbari and Alipour’s conjecture was subsequently proven for sufficiently large n concurrently and independently by Keevash and Yepremyan [86], who proved the stronger result that $n^{399/200}$ symbols suffice for large n , and by Montgomery et al. [94].

The proof of the sufficiently large version of this conjecture by Montgomery et al. was only a small portion of a wide-ranging paper; the main result of [94] is also of significant interest here: every $n \times n$ latin array in which at most $n - o(n)$ symbols occur more than $n - o(n)$ times has $n - o(n)$ disjoint transversals. An alternative proof of this result was given by Ehard et al. [58], and a proof of a similar result (where all symbols must occur at most $n - o(n)$ times in the array but rows and columns can contain up to $o(n/\log^2 n)$ copies of a given symbol) was given by Kim et al. in [87]. These results concerning many disjoint transversals in latin arrays are especially interesting to us because they imply the existence of k -plexes, a type of generalized transversal defined above which will play an important role throughout this thesis.

1.4 Plexes

Generalizing the definition given in the second paragraph of this thesis, we define a k -plex in an $n \times n$ array to be a collection of entries which intersects each row and each column k times and contains k copies of each symbol in some n -subset of the array’s symbols. The

term “ k -plex” was introduced fairly recently, by Wanless [115] in 2002, although it was noted in [115] that 2, 3, and 4-plexes have been discussed in the statistical literature for many decades. The study of k -plexes as combinatorial objects has been around since at least the mid-90s when, according to Dougherty [50], our second main motivating conjecture was given—though never published—by the late Peter Rodney.

Conjecture 1.3 (Rodney). *Every latin square has a 2-plex.*

Mirroring the current status of Brualdi’s conjecture, Rodney’s conjecture was shown to hold for groups by Vaughan-Lee and Wanless [112], and is known to hold for all latin squares of order at most 9 [56]. For general latin squares, a weaker, weighted version of Conjecture 1.3, as well as Conjecture 1.1, was proven by Pula [106].

While the attribution of Conjecture 1.3 to Rodney is well-established (appearing as such in the fairly authoritative Handbook of Combinatorial Designs [43, p. 143]), Dougherty actually attributes to Rodney the stronger conjecture [50, p. 130] that every latin square can be partitioned into disjoint 2-plexes with an additional transversal left over for latin squares of odd order. Notice that the union of an a -plex and a disjoint b -plex forms an $(a + b)$ -plex. Combining this observation with the stronger conjecture of Rodney, we obtain the third main motivating conjecture in this thesis.

Conjecture 1.4. *For all even $k \leq n$, every latin square of order n has a k -plex.*

Although Conjecture 1.4 is also due to Rodney, we will use the name “Rodney’s conjecture” only to refer to Conjecture 1.3. Wanless showed that Conjecture 1.4 holds for group-based latin squares in [115]. Moreover, the computation in [56], cited above as confirming Rodney’s conjecture, was in fact a confirmation that all latin squares of order at most 9 have $\lfloor n/2 \rfloor$ disjoint 2-plexes. Thus, it is also known that all latin squares of order $n \leq 9$ have a k -plex for every even $k \leq n$.

When k is odd and n is even, there does not seem to be any consistent pattern concerning the existence of k -plexes in latin squares of order n . Indeed, for every positive even integer n , Egan and Wanless [55] constructed a latin square of order n that has a k -plex for every odd value of k between $\lfloor n/4 \rfloor$ and $\lfloor 3n/4 \rfloor$ but no k -plex for any odd number k outside that range. Furthermore, Cavenagh and Wanless [40] have shown that, for all even $n > 4$, there is a latin square of order n which contains a 3-plex but no transversal. An interesting question which was raised by both of these papers, but remains unanswered, is whether there exists a latin square with an a -plex and a c -plex but no b -plex for odd integers $a < b < c$.

There have also been several papers [33, 54, 56] studying *indivisible plexes*, or k -plexes which do not contain an ℓ -plex for any positive integer $\ell < k$, and others [16, 39, 48, 109, 115] treating k -plexes as a primary object which may or may not be embeddable in a latin square. We refer the reader to [12] for an interesting description of plexes in the language of spectral graph theory, and [89] for a weaker form of orthogonality stated in terms of k -plexes.

In this thesis we coin the term (k, ℓ) -plex to describe a collection of entries in a square array which intersects each row and column exactly k times and contains at most ℓ copies of each symbol. Notice that, in an $n \times n$ array containing exactly n symbols—so, in particular, in a latin square—a (k, k) -plex is always a k -plex, but this is not the case in arrays with more than n symbols. Although the term “ (k, ℓ) -plex” term does not seem to have appeared previously in the literature, there have been at least three papers which considered $(1, 2)$ -plexes. Cameron and Wanless [36, Prop. 7] were the first to show that every latin square has a $(1, 2)$ -plex. This result has since been extended by Aharoni et al. [3, Theorem 1.16], who showed that every $n \times n$ row-latin array with exactly n symbols has a $(1, 2)$ -plex, and Best et al. [24, Lemma 13], who showed that every entry in every latin square is contained in a $(1, 2)$ -plex.

Ian Wanless has also pointed out that, though not part of the published literature, the PhD thesis of Kyle Pula [105] discusses a closely related notion. Pula defined a *weak k -plex* in a latin square as a collection of entries which intersects each row and column exactly k times and contains between $k - 1$ and $k + 1$ copies of each symbol. Notice that a weak k -plex is a $(k, k + 1)$ -plex in which each symbol appears at least $k - 1$ times. Pula proved that every latin square has a weak 2-plex and sketched a proof that every sufficiently large latin square has a weak k -plex when $k = O(1)$. Moreover, in discussing collections of disjoint weak 1-plexes, Pula showed that every latin square of order $n \geq 73$ has a $(k, 2k)$ -plex for all $k \leq \left\lceil \frac{1}{11}n - \frac{8}{11}\sqrt{n} - \frac{3}{11} \right\rceil$. Because we were unaware of Pula’s work until the defence of this thesis, we will often work in parallel to, rather than building upon, the results just mentioned.

This thesis’ work on (k, ℓ) -plexes will appear in Chapter 4. We show that every latin square has a $(k, 4k)$ -plex for all $k \leq \frac{n}{4}$ and every sufficiently large latin square has a $(k, k + 1)$ -plex for $k = O(1)$. Notice that the second of these results follows from Pula’s result on the existence of weak k -plexes. Nonetheless, we offer a novel proof of our weaker result. We also demonstrate the existence of several special families of $(2, \ell)$ -plexes to be introduced in the next section. Finally, in Chapter 5, we raise several questions concerning (k, ℓ) -plexes (most of which are weaker versions of Conjecture 1.4).

1.5 Graph-theoretic representations of latin squares

A significant portion of the work in this thesis is given in the language of graph theory. Indeed, there are several means of representing latin squares graph-theoretically. One which has proven particularly useful (see e.g. [79, 88, 3]), but will not be discussed here, is the equivalence between latin squares and 3-uniform tripartite linear hypergraphs (an excellent explanation of this connection is given in [1]). There are also several connections between certain special families of latin squares and decompositions of complete graphs into cycles

(see [84, Section 8.3]). We are interested here in two particular representations, though another will be worth introducing to motivate some portions of our work.

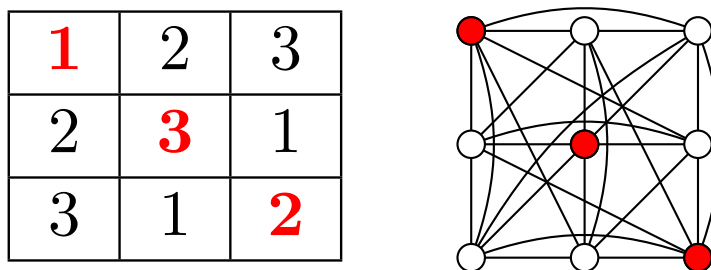


Figure 1.2: A latin square L and its associated latin square graph $\Gamma(L)$. The entries of a transversal are highlighted, as are the vertices of the corresponding independent set.

Given a latin square L , we write $R(L)$ for its set of rows, $C(L)$ for its set of columns, and $L_{r,c}$ for the symbol in row r and column c . The *latin square graph* $\Gamma(L)$ is defined on the vertex set $R(L) \times C(L)$ with $(r,c) \sim (s,d)$ if and only if one of $r = s$, $c = d$, or $L_{r,c} = L_{s,d}$ holds. Figure 1.2 contains an example of this construction. Latin square graphs were introduced by Bose [28] in the 1960s as an example of nontrivial strongly regular graphs. They have attracted attention from researchers (e.g. [12, 47, 104, 98]) interested in strictly graph-theoretic questions as an interesting family of symmetric graphs. They have also been utilized as a tool in the study of latin squares [21, 65, 68]. We will use them in Chapter 2 to prove that Brualdi’s conjecture is true for group-based latin squares, utilizing the crucial observation that partial transversals in L correspond to independent sets, i.e. sets of vertices no two of which are adjacent, in $\Gamma(L)$.

A much more common way of graph-theoretically representing latin squares is as edge-colorings of a balanced complete bipartite graph. Given a latin square L of order n , we define $K_{n,n}(L)$ to be the proper n -edge-colored copy of $K_{n,n}$ such that: the vertex bipartition is labeled by $R(L) = \{r_1, r_2, \dots, r_n\}$ and $C(L) = \{c_1, c_2, \dots, c_n\}$, and the edge $r_i c_j$ is colored by the symbol $L_{i,j}$. We also write $\chi_L : E(K_{n,n}) \rightarrow S(L)$ to denote the edge-coloring of $K_{n,n}(L)$. See Figure 1.3 for an example of this construction. This representation of latin squares has been profitably used to study uniquely completable partial latin squares [34, 83, 91], to study partial transversals in latin squares [3, 45, 94], and to construct latin squares with nice symmetry properties [108, 114, 117]. Notice that we can similarly construct an edge-coloring of $K_{n,n}$ using any $n \times n$ array; we will consider $K_{n,n}(S)$, where S is a k -bounded n -square, in Section 4.2.

Harary [77] was the first to point out the connection between partial transversals in latin squares and rainbow matchings, i.e. matchings containing at most one copy of each color, in proper n -edge-colorings of $K_{n,n}$. In Chapters 3 and 4 we will make use of an extension of this observation. We say that an edge-colored graph is k -bounded if it contains at most k edges of each color (1-bounded graphs are more commonly referred to as *rainbow* graphs).

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

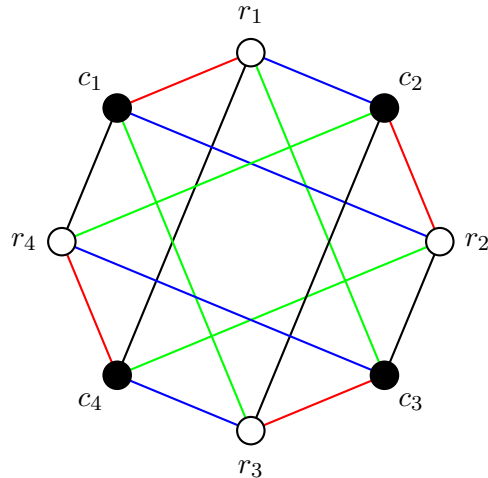


Figure 1.3: A latin square L of order 4 and the associated edge-colored graph $K_{4,4}(L)$.

Following Harary, we observe that k -plexes in (a latin square) L are equivalent to k -regular k -bounded spanning subgraphs of $K_{n,n}(L)$. In a slight abuse of terminology, we will also refer to such subgraphs in edge-colorings of $K_{n,n}$ as k -plexes.

1.6 Main classes and isotopy

We may use the graph-theoretic representations introduced in the previous section to define two important notions of equivalence for latin squares. First, we say that two latin squares L_1 and L_2 are *main class equivalent* if $\Gamma(L_1) \cong \Gamma(L_2)$. Notice that, if L_1 and L_2 are main class equivalent, then there is bijective correspondence between the partial transversals in L_1 and the partial transversals in L_2 —namely, they both correspond to the independent sets of $\Gamma(L_1)$. A *main class* of latin squares is a maximal collection of main class equivalent latin squares. When we discuss partial transversals in what follows, we will not distinguish among latin squares in the same main class.

Second, we say that two latin squares L_1 and L_2 satisfying $|L_1| = |L_2| = n$ are *isotopic* if there is an automorphism of $K_{n,n}$ that fixes the vertex bipartition and maps each color class of $K_{n,n}(L_1)$ to a color class of $K_{n,n}(L_2)$; we refer the map from L_1 to L_2 corresponding to this graph automorphism as an *isotopy*. Notice that two latin squares may be isotopic even when they contain distinct sets of symbols. Moreover, isotopic latin squares are main class equivalent. However, one can find two latin squares in the same main class which are not isotopic (see e.g. [84, Chapter 1]). We refer to a maximal collection of isotopic latin squares as an *isotopy class*.

In what follows, we will consider only properties of latin squares which are invariant under isotopy. Indeed, if L_1 and L_2 are isotopic, then the k -plexes of L_1 and the k -plexes of L_2 induce the same collection of edge-colored graphs (up to graph isomorphism). Therefore,

we will treat isotopic latin squares as equivalent, and we may choose to reorder rows and columns without fundamentally changing the latin square being discussed.

The notion of isotopy allows us to state more clearly what we mean by the group-based latin square L_G . When defining group-based latin squares above, we avoided mentioning that the layout of L_G depends upon an ordering of the elements of G . This omission can be justified by the fact that we will only be considering properties of latin squares which are invariant under isotopy. Nonetheless, it is worth formally defining the *Cayley table* of G , with respect to the ordering $G = \{g_1, g_2, \dots, g_n\}$, as the unique latin square satisfying $L_{i,j} = g_k$ if and only if $g_k = g_i g_j$. Then, following [62], we refer to a latin square as *group-based* if it is isotopic to a Cayley table. It is known that, if L is a group-based latin square, then L is isotopic to the Cayley table of exactly one group (see [84, p. 13]). Moreover, if a group-based latin square L is main class equivalent to some other latin square L' , then L and L' are in fact isotopic (see [84, p. 126]). Therefore, we will write L_G to denote the latin square based on the group G with no possibility of ambiguity.

1.7 This thesis's novel conjecture

We noted above that 2-plexes in the latin square L correspond to 2-regular 2-bounded spanning subgraphs of $K_{n,n}(L)$. Perhaps the most famous type of 2-regular spanning subgraph of a graph G is the *Hamilton cycle*, which is a cycle containing every vertex of G . We define a *Hamilton $(2, \ell)$ -plex* in a square array S as an ℓ -bounded Hamilton cycle in $K_{n,n}(S)$; in the case $\ell = 2$ and L has n symbols, we simply use the term *Hamilton 2-plex*. The first main contribution of this thesis is the following novel strengthening of Rodney's conjecture.

Conjecture 1.5. *Every latin square of order $n \geq 5$ has a Hamilton 2-plex.*

The condition $n \geq 5$ in Conjecture 1.5 is necessary because there is a latin square of order 4 which does not have a Hamilton 2-plex. Indeed, we have shown this latin square in Figure 1.3. It is also straightforward to check that this latin square is isotopic to $L_{\mathbb{Z}_2 \times \mathbb{Z}_2}$. We believe that this is the only latin square which does not have a Hamilton 2-plex, and we have confirmed this for all latin squares of order at most 8.

Proposition 1.6. *Every latin square of order at most 8, besides $L_{\mathbb{Z}_2 \times \mathbb{Z}_2}$, has a Hamilton 2-plex.*

We do not present a formal proof of this proposition here, but we refer the reader to:

<https://sites.google.com/view/kevinhalasz/combinatorial-data>

for the exhaustive data which establish the result. These data come in the form of three documents: for each $n \in [6, 8]$, we give a collection of lists of ordered triples corresponding to the entries of a Hamilton 2-plex in a representative of each isotopy class of order n latin

squares. We use the isotopy class representatives given by McKay in [92], and the ordering of the lists in our data corresponds to McKay’s ordering. As noted in [84, Sec. 4.2], there is exactly one isotopy class of order at most 5 which does not contain a latin square based upon an abelian group. We give a Hamilton 2-plex for that square in Figure 1.4, and we prove that all latin squares based on abelian groups have Hamilton 2-plexes in Section 3.2 below.

0	1	2	3	4
1	3	0	4	2
2	4	3	1	0
3	0	4	2	1
4	2	1	0	3

Figure 1.4: A Hamilton 2-plex in the unique (up to isotopy) latin square of order less than 6 which is not group-based.

The idea of considering Hamilton 2-plexes in latin squares was motivated by the literature on large rainbow paths and cycles in edge-colored complete graphs. There is a vast body of literature concerning rainbow Hamilton cycles in edge-colored graphs (see e.g. [10, 13, 44, 46, 81]) of which we will discuss here only a small, especially relevant portion. An order n latin square L is *symmetric* if $L_{i,j} = L_{j,i}$ for all $i, j \in [n]$. One may associate an edge-coloring of K_n with each symmetric latin square by giving the edge $v_i v_j$ the color $L_{i,j}$. This connection was noticed by Andersen [11], who adapted Brouwer et al.’s [30] proof that every latin square has a partial transversal of size $n - \sqrt{n}$ to show that every proper edge-coloring of K_n has a rainbow path-forest (union of disjoint paths) of size $n - \sqrt{2n}$. In terms of latin arrays, Andersen showed that every symmetric square latin array has a partial transversal of size $n - \sqrt{2n}$.

Andersen then conjectured that every proper edge-coloring of K_n has a rainbow path of length $n - 2$, noting that Maamoun and Meyniel [90] had constructed edge-colorings of K_n with no rainbow path of length $n - 1$ whenever n is a power of 2. As described in the introduction of [8], there was a series of papers working towards this conjecture, with various authors proving the existence of rainbow paths of length αn for increasing constants $\alpha < 1$. A breakthrough paper of Alon et al. [8] showed that every proper edge-coloring of K_n has a rainbow path of length $n - O(n^{3/4})$, and this was quickly improved using similar methods to $n - O(n^{1/2} \log n)$ by Balogh and Molla [15]. It has also been shown by Gould et al. [71] that almost every optimal (where the definition of optimal depends on the parity of n) proper edge-coloring of K_n has a rainbow Hamilton path, and by Albert et al. [7] that every $\frac{n}{64}$ -bounded proper edge-coloring of K_n has a rainbow Hamilton cycle.

An additional colored-graph representation of latin squares is worth mentioning briefly. One can associate with a latin square L an arc-coloring of the complete directed graph \vec{K}_n by giving the edge $v_i v_j$ the color $L_{i,j}$. Writing $\vec{K}_n(L)$ for this coloring, notice that partial transversals in L correspond to rainbow subgraphs of $\vec{K}_n(L)$ with maximum in-degree and maximum out-degree 1. The original motivation for considering Hamilton 2-plexes was a paper of Gyárfás and Sárközy [74] on the structure of the subgraphs of $\vec{K}_n(L)$ corresponding to partial transversals of L . In that paper they conjectured that all latin squares have a partial transversal of size $n - 2$ which corresponds to a directed path in $\vec{K}_n(L)$. This conjecture remains open, although Benzing et al. [19] showed that there exists such a partial transversal of size $n - O(n^{2/3})$ in every latin square. Moreover, a recent paper of Gould and Kelly [70] demonstrates that almost every latin square has a transversal corresponding to a rainbow Hamilton cycle in $\vec{K}_n(L)$.

1.8 Complete mappings and group sequences

It is possible to discuss transversals in group-based latin squares using purely group-theoretic language. A *complete mapping* of a group G is a bijection (but not necessarily a homomorphism) $\theta : G \rightarrow G$ such that $g \mapsto g\theta(g)$ is also a bijection. Observe that, if θ is a complete mapping of G , then $\{(g, \theta(g), g\theta(g)) \mid g \in G\}$ is a transversal of L_G . The study of complete mappings in groups goes back to at least the 1940s [99], though work on similar ideas appeared in the late 19th century (see [62, Chapter 3]). Important results from the mid-twentieth century include Bruck's proof [32] that all groups of odd order have complete mappings, Paige's proof [99] that an abelian group has a complete mapping if and only if it does not have a unique element of order 2, and Bateman's proof [17] that every countably infinite group has a complete mapping.

In 1955, Hall and Paige [76] proved that groups with nontrivial cyclic Sylow 2-subgroups do not have complete mappings and conjectured that the converse to this statement holds. It took over 50 years before the Hall-Paige conjecture was finally resolved in three separate papers by Wilcox, Evans, and Bray.

Theorem 1.7 (Hall and Paige [76]; Bray [29]; Evans [61]; Wilcox [119]). *A group has a complete mapping if and only if its Sylow 2-subgroups are trivial or non-cyclic.*

While Theorem 1.7 fully resolved the existence question for transversals in group-based latin squares, the existence of near transversals was not resolved until last year [67] with work that will be presented in this thesis. Given a group G and three (not necessarily distinct) elements $a, b, c \in G$, we say that $\phi : G \setminus \{a\} \rightarrow G \setminus \{b\}$ is a *near complete mapping* if $g \mapsto g\phi(g)$ is a bijection between $G \setminus \{a\}$ and $G \setminus \{c\}$. Notice that near complete mappings of G are equivalent to near transversals of L_G . Moreover, given any complete mapping θ , one can define a near complete mapping by removing any element from the domain of θ .

This means that every group with a complete mapping also has a near complete mapping. For abelian groups, the existence of near complete mappings in groups with no complete mapping follows from a theorem of Hall [75]. We will extend this result to all groups in Chapter 2.

One may also describe Hamilton 2-plexes in group-based latin squares using purely group-theoretic language. Before we introduce this new terminology, it is worth briefly mentioning the notions after which it is modelled. A *sequencing* of a group G is an ordering of its elements g_1, g_2, \dots, g_n such that the partial products $g_1, g_1g_2, \dots, g_1g_2 \cdots g_n$ are all distinct. If G has a sequencing, we refer to it as a *sequenceable* group. Notice that, given a sequencing of G , the map $g_1g_2 \cdots g_i \mapsto g_{i+1}$ (for each $i \in [n-1]$) is a near complete mapping. Sequencings were introduced by Gordon [69], who fully classified the sequenceable abelian groups. There has since been a substantial body of work on sequencings of groups and various generalizations of this notion. We will focus on one of these generalizations, as it will be particularly important in what follows, while referring the reader to a comprehensive dynamic survey due to Ollis [97] for more information.

A *harmonious sequence* of a group G is an ordering of its elements g_1, g_2, \dots, g_n such that the consecutive products $g_1g_2, g_2g_3, \dots, g_{n-1}g_n, g_ng_1$ are all distinct; if G has a harmonious sequence we call it a *harmonious group*. Notice that, given a harmonious sequence of G , the map $g_i \mapsto g_{i+1}$ is a complete mapping. Moreover, this complete mapping corresponds to a rainbow Hamilton cycle in $\vec{K}_n(L_G)$. Harmonious groups were introduced by Beals et al. [18], who completely classified harmonious abelian groups and showed that all groups of odd order are harmonious. Subsequently, Wang and Leonard [113] have shown that dicyclic groups are harmonious if and only if their order is divisible by, but not equal to, 8. It is worth observing that a harmonious sequence of G corresponds to the vertex sequence of a rainbow Hamilton cycle in $\vec{K}_n(L_G)$. This means that the conjecture of Gyárfás and Sárközy [74] mentioned in the previous section holds for all latin squares based on harmonious groups.

Inspired by the fact that harmonious sequences corresponds to the vertex sequences of a rainbow Hamilton cycles in $\vec{K}_n(L_G)$, we introduce the notion of H2-harmonious sequences to study Hamilton cycles in $K_{n,n}(L_G)$. Notice that we could define $K_{n,n}(G) := K_{n,n}(L_G)$ without reference to latin squares: define the bipartition (R, C) of $K_{n,n}$ by $R = \{[g]_R \mid g \in G\}$ and $C = \{[g]_C \mid g \in G\}$ and label the edge $[g]_C[h]_R$ with the group element gh . Saving a formal definition for Chapter 3, we can here define a *H2-harmonious sequence* of G as the sequence of group elements given by removing the brackets from the vertex sequence of a Hamilton 2-plex in $K_{n,n}(G)$. Notice that, by the discussion in Section 1.7, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not H2-harmonious. Moreover, in light of Conjecture 1.5, we believe that all other groups are H2-harmonious.

It is worth briefly discussing the relationship between harmonious groups and H2-harmonious groups. It is certainly not the case that every H2-harmonious group is harmonious: we will show in Theorem 3.11 that an infinite family of groups which are known

to not be harmonious are H2-harmonious. It may be the case that all harmonious groups are H2-harmonious—in fact, our conjecture concerning H2-harmonious groups implies that this should be the case—but this implication is not direct. Indeed, although we will use harmonious sequences to show that certain families of groups are H2-harmonious, these proofs depend upon various additional properties of the groups in question.

The next two chapters of this thesis study group-based latin squares using the language just introduced. In Chapter 2, we prove that every group has a near complete mapping. In Chapter 3, we establish the existence of H2-harmonious sequences for many families of groups, including all abelian groups besides $\mathbb{Z}_2 \times \mathbb{Z}_2$ and all groups of odd order. We then turn to purely combinatorial questions in Chapter 4, using tools from extremal and probabilistic combinatorics to establish the existence of (k, ℓ) -plexes and Hamilton $(2, \ell)$ -plexes in various families of combinatorial arrays.

Chapter 2

Every group has a near complete mapping

The goal of this chapter is rather transparent: we provide a proof of its titular statement. Our proof is given in the language of graph theory, utilizing relatively elementary structural results for finite groups to show that all group-based latin square graphs have large independent sets. Although our argument depends upon Theorem 1.7, it is independent of—and indeed offers a new proof for—the theorem of Hall [75] that all abelian groups have near complete mappings. The work in this chapter has already been published in [67], where the main result is stated as follows.

Theorem 2.1. *Every group-based latin square has a near transversal.*

2.1 Reduction to a question about induced subgraphs

Recall the definition of latin square graphs from Section 1.5. In particular, in the graph $\Gamma(L)$ we have $(r, c) \sim (s, d)$ if and only if one of $r = s$, $c = d$, or $L_{r,c} = L_{s,d}$ holds. This defines a natural tripartition of $\Gamma(L)$'s edges into, respectively, *row edges*, *column edges*, and *symbol edges*. Moreover, recall that there is a bijective correspondence between near transversals of L and independent sets of size $n - 1$ in $\Gamma(L)$.

Given graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, the *disjoint union* of Γ and Γ' is $\Gamma + \Gamma' := (V \sqcup V', E \sqcup E')$. For a positive integer k , we write $k\Gamma$ for the disjoint union of k copies of Γ . Given a set $W \subseteq V$, the *induced subgraph* of Γ with respect to W is

$$\Gamma[W] := (W, \{e \in E : e \subseteq W\}).$$

The *Möbius ladder* of order $2n$, denoted M_n , is the cubic graph formed from a cycle of length $2n$ —referred to as the *rim* of M_n —by adding n edges, one joining each pair of vertices at distance n in the initial cycle. The *prism graph* of order $2n$, denoted Y_n , is the graph obtained from two copies of the n -cycle by adding edges between corresponding vertices.

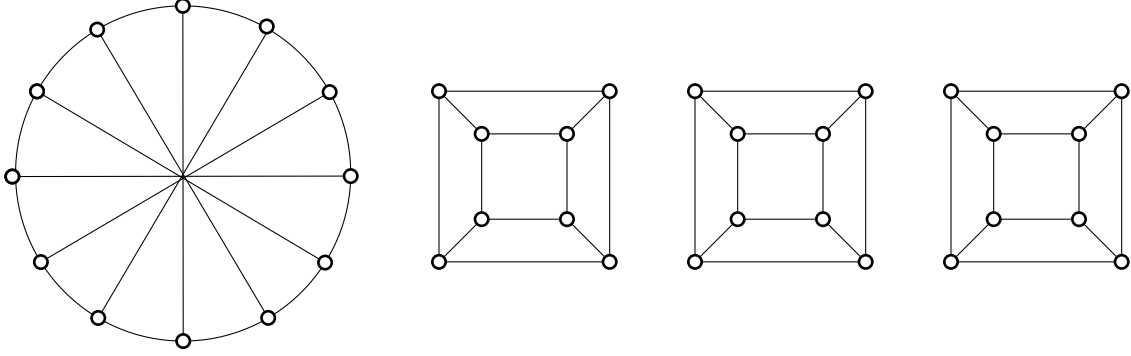


Figure 2.1: The graph $\Lambda_{18,3}$

Lemma 2.2. *Let L be a group-based latin square of even order n , let k be the greatest power of 2 dividing n , and let $l := n/k$. If L does not have a transversal, then there is a positive integer m dividing l such that $\Gamma(L)$ has an induced subgraph isomorphic to*

$$\Lambda_{n,m} := M_{km} + \left(\frac{l-m}{2}\right) Y_{2k}.$$

See Figure 2.1 for an example of the graph $\Lambda_{n,m}$. Though an induced copy of $\Lambda_{n,m}$ in $\Gamma(L_G)$ does not necessarily correspond to a 2-plex of L , the proof of Lemma 2.2 finds a 2-plex inducing $\Lambda_{n,m}$. We now show that the challenge of proving Theorem 2.1 can be passed on to Lemma 2.2.

Proof of Theorem 2.1. Let L be a group-based latin square of order n . We may assume L does not have a transversal. As first shown in [32], this implies n is even. We may therefore apply Lemma 2.2 to find an induced copy of $\Lambda_{n,m}$ in $\Gamma(L)$. Because the $(2k(l-m))$ -vertex graph $\left(\frac{l-m}{2}\right) Y_{2k}$ is bipartite, it contains an independent set of size at least $k(l-m)$. Moreover, one can find an independent set of size $km-1$ in M_{km} by greedily selecting vertices in cyclic order around its rim. Thus $\Lambda_{n,m}$ contains an independent set of size $(l-m)k + km - 1 = n - 1$ which corresponds to a near transversal of L . \square

2.2 Proof of Lemma 2.2

Let G be a group of order n with identity element 1 and let $\text{Syl}_2(G)$ denote the isomorphism class of G 's Sylow 2-subgroups. It is a nice algebraic exercise to check that, if H is a group of odd order, then the identity map is a complete mapping. Indeed, if H has order $2k+1$, $\theta : H \rightarrow H$ is the identity map, and $g\theta(g) = h\theta(h)$ for some $g, h \in H$, then

$$g = g \cdot g^{2k+1} = g^{2k+2} = (g^2)^{k+1} = (h^2)^{k+1} = h^{2k+2} = h.$$

Combining this with Beals, Gallian, Headley, and Jungreis's theorem that all groups of odd order are harmonious, we obtain the following theorem.

Theorem 2.3 (Beals et al. [18]). *For every group H of odd order m , there exists an ordering $H = \{h_1, h_2, \dots, h_m\}$ such that, taking indices modulo m , both $h_i \mapsto h_{i+1}$ and $h_i \mapsto h_i$ are complete mappings.*

Given two subsets $X_1, X_2 \subseteq G$ the *product set* of X_1 by X_2 is

$$X_1 X_2 := \{x_1 x_2 : x_1 \in X_1, x_2 \in X_2\}.$$

We write Xy for the product set $X\{y\}$. If K is a subgroup of G and H is a normal subgroup of G , then we say that G is *the semidirect product* of K and H —written $G = K \ltimes H$ —if $K \cap H = \{1\}$ and $KH = G$. The following was noted in [76] as following from a theorem due to Burnside. We refer the reader to [123, p. 139] for a proof.

Lemma 2.4 (Burnside). *Let G be a finite group and let K be a Sylow 2-subgroup of G . If K is cyclic and nontrivial, then there is a normal subgroup of odd order $H \triangleleft G$ such that*

$$G = K \ltimes H.$$

These are all the group-theoretic preliminaries we need to prove that, for every group G with no complete mapping, $\Gamma(L_G)$ contains an induced subgraph which is the disjoint union of one Möbius ladder and several bipartite prisms.

Proof of Lemma 2.2. Let L be a latin square based on a group G of order $n = kl$, where $k \geq 2$ is a power of 2 and l is odd. We may assume that G does not have a complete mapping. Theorem 1.7 tells us that $\text{Syl}_2(G) = \mathbb{Z}_k$. It then follows from Lemma 2.4 that G contains a normal subgroup H of order l and an element b of order k such that

$$G = \langle b \rangle \ltimes H.$$

Let $a := b^{k/2}$. As $H \triangleleft G$ and a has order 2, H has an automorphism

$$\alpha : h \mapsto aha.$$

Let

$$H^* := \{h \in H : \alpha(h) = h\}$$

and observe that H^* is a subgroup of H . Let $m := |H^*|$. As m divides l and l is odd, m is odd. By Theorem 2.3, there is an ordering $H^* = \{h_1, h_2, \dots, h_m\}$ for which the map $h_i \mapsto h_{i+1}$ is a complete mapping. Here and throughout the rest of this proof, indices are taken modulo m .

<u>1</u>	bc	c^2	b	c	bc^2	d	d^2	cd	cd^2	c^2d	c^2d^2	bd^2	bd	bcd^2	bcd	bc^2d^2	bc^2d
bc	<u>c^2</u>	b	c	bc^2	1	bcd	bcd^2	bc^2d	bc^2d^2	bd	bd^2	cd^2	cd	c^2d^2	c^2d	d^2	d
c^2	b	<u>c</u>	<u>bc^2</u>	1	bc	c^2d	c^2d^2	d	d^2	cd	cd^2	bc^2d^2	bc^2d	bd^2	bd	bcd^2	bcd
b	c	bc^2	<u>1</u>	bc	c^2	bd	bd^2	bcd	bcd^2	bc^2d	bc^2d^2	d^2	d	cd^2	cd	c^2d^2	c^2d
c	bc^2	1	bc	<u>c^2</u>	b	cd	cd^2	c^2d	c^2d^2	d	d^2	bcd^2	bcd	bc^2d^2	bc^2d	bd^2	bd
<u>bc^2</u>	1	bc	c^2	b	<u>c</u>	bc^2d	bc^2d^2	bd	bd^2	bcd	bcd^2	c^2d^2	c^2d	d^2	d	cd^2	cd
d	bcd^2	c^2d	bd^2	cd	bc^2d^2	d²	1	cd^2	c	c^2d^2	c^2	bd	b	bcd	bc	bc^2d	bc^2
d^2	bcd	c^2d^2	bd	cd^2	bc^2d	1	d	c	cd	c^2	c^2d	b	bd²	bc	bcd^2	bc^2	bc^2d^2
cd	bc^2d^2	d	bcd^2	c^2d	bd^2	cd^2	c	c^2d^2	c^2	d^2	1	bcd	bc	bc^2d	bc^2	bd	b
cd^2	bc^2d	d^2	bcd	c^2d^2	bd	c	cd	c^2	c^2d	1	d	bc	bcd^2	bc^2	bc^2d^2	b	bd^2
c^2d	bd^2	cd	bc^2d^2	d	bcd^2	c^2d^2	c^2	d^2	1	cd^2	c	bc^2d	bc^2	bd	b	bcd	bc
c^2d^2	bd	cd^2	bc^2d	d^2	bcd	c^2	c^2d	1	d	c	cd	bc^2	bc^2d^2	b	bd^2	bc	bcd^2
bd	cd^2	bc^2d	d^2	bcd	c^2d^2	bd²	b	bcd^2	bc	bc^2d^2	bc^2	d	1	cd	c	c^2d	c^2
bd^2	cd	bc^2d^2	d	bcd^2	c^2d	b	bd	bc	bcd	bc^2	bc^2d	1	d²	c	cd^2	c^2	c^2d^2
bcd	c^2d^2	bd	cd^2	bc^2d	d^2	bcd^2	bc	bc^2d^2	bc^2	bd^2	b	cd	c	c^2d	c^2	d	1
bcd^2	c^2d	bd^2	cd	bc^2d^2	d	bc	bcd	bc^2	bc^2d	b	bd	c	cd^2	c^2	c^2d^2	1	d^2
bc^2d	d^2	bcd	c^2d^2	bd	cd^2	bc^2d^2	bc^2	bd^2	b	bcd^2	bc	c^2d	c^2	d	1	cd	c
bc^2d^2	d	bcd^2	c^2d	bd^2	cd	bc^2	bc^2d	b	bd	bc	bcd	c^2	c^2d^2	1	d^2	c	cd^2

Figure 2.2: The Cayley table of $S_3 \times \mathbb{Z}_3 = \langle b, c, d \mid b^2 = c^3 = d^3 = 1, bc = cb, bd = d^2b \rangle$ with **T** and **U** highlighted. Here $k = 2$, $H = \langle c, d \rangle$, and $H^* = \langle c \rangle$, with H^* ordered by $h_i = c^i$. The first six rows and columns are indexed by $\langle b \rangle H^*$ and the main diagonal is $T_1 \cup U_1$.

Let $\Gamma := \Gamma(L)$. Toward defining a set $W \subseteq V(\Gamma)$ which induces $\Lambda_{n,m}$, let

$$\begin{aligned}
T_1 &:= \left\{ (b^i h_i, h_i b^i) : i \in [km] \right\}, \\
T_2 &:= \left\{ (b^i h_i, h_{i+1} b^{i+1}) : i \in [km] \right\}, \text{ and} \\
T &:= T_1 \cup T_2.
\end{aligned}$$

Furthermore, let $F := H \setminus H^*$ (i.e. F is the set of elements of H which are not in H^*) and let

$$\begin{aligned}
U_1 &:= \left\{ (b^i f, f b^i) : f \in F, i \in [k] \right\}, \\
U_2 &:= \left\{ (b^i f, f b^{i+1}) : f \in F, i \in [k] \right\}, \text{ and} \\
U &:= U_1 \cup U_2.
\end{aligned}$$

Finally, let

$$W := T \cup U.$$

An example of these sets in the latin square $L_{S_3 \times \mathbb{Z}_3}$ is given in Figure 2.2. We show the induced subgraph $\Gamma[W]$ is isomorphic to $\Lambda_{n,m}$ via the following three claims.

Claim 1. $T \cap U = \emptyset$ and there is no edge between T and U .

As $G = \langle b \rangle \times H$, every element of G has a unique representation of the form $g = b^i h$ for $i \in [k]$ and $h \in H$. Therefore, the definition of F implies

$$\langle b \rangle H^* \cap \langle b \rangle F = \emptyset. \quad (2.1)$$

But for every $(t, s) \in T$ and every $(u, v) \in U$ we have $t \in \langle b \rangle H^*$ and $u \in \langle b \rangle F$. Thus $T \cap U = \emptyset$ and there are no row edges between T and U .

As $H \triangleleft G$ we have $bHb^{-1} = H$. Moreover, as $a = b^{k/2}$, for every $h \in H^*$ we have $\alpha(bhb^{-1}) = bhb^{-1}$, so $bhb^{-1} \in H^*$. Thus

$$bH^*b^{-1} = H^*. \quad (2.2)$$

It then follows from the definition of F , and the fact that both H and H^* are both fixed under conjugation by b , that

$$bFb^{-1} = F \quad (2.3)$$

and, as the identity map is a complete mapping of both H and H^* ,

$$f \mapsto f^2 \text{ is a permutation of } F. \quad (2.4)$$

Thus for every $(u, v) \in U$, both v and uv are in $\langle b \rangle F$. But (2.2) tells us that for every $(t, s) \in T$, both s and ts are in $\langle b \rangle H^*$. It then follows from (2.1) that there are no column edges and no symbol edges between T and U .

Claim 2. $\Gamma[U]$ consists of $\frac{l-m}{2}$ disjoint copies of Y_{2k} .

Observe that, when enumerating the vertices in U , every element of $\langle b \rangle F$ occurs exactly twice as a first coordinate and, by (2.3), exactly twice as a second coordinate. Thus, each vertex in the induced subgraph $\Gamma[U]$ is incident to exactly one row edge and exactly one column edge, so that the row and column edges in $\Gamma[U]$ form a 2-factor (of $\Gamma[U]$). Specifically, they form $l-m$ disjoint $2k$ -cycles $\{C_f : f \in F\}$, with each C_f defined by the vertex-sequence

$$(f, f), (f, fb), (bf, fb), (bf, fb^2), \dots, (b^{k-1}f, fb^{k-1}), (b^{k-1}f, f).$$

It follows from the definitions of H^* and F that $\alpha|_F$ is a fixed-point free involution. Thus, to establish Claim 2 it suffices to show that for every $i, j \in [k]$, every $f, h \in F$, and every $\varepsilon, \delta \in \{0, 1\}$, the vertices $(b^i f, fb^{i+\varepsilon})$ and $(b^j h, hb^{j+\delta})$ are joined by a symbol edge if and only if $j \equiv i + k/2 \pmod{k}$, $h = \alpha(f)$, and $\varepsilon = \delta$.

The “if” direction of this equivalence follows directly from the definition of α . For the converse direction we assume

$$b^i f^2 b^{i+\varepsilon} = b^j h^2 b^{j+\delta}$$

and, as latin square graphs are loopless, $(b^i f, f b^{i+\varepsilon}) \neq (b^j h, h b^{j+\delta})$. It follows from (2.3) and (2.4) that

$$b^i f^2 b^{i+\varepsilon} \in b^{2i+\varepsilon} F \text{ and } b^j h^2 b^{j+\delta} \in b^{2j+\delta} F.$$

Thus $\varepsilon = \delta$ and $|i - j| \in \{0, k/2\}$.

Now if $i = j$, then $b^i f^2 b^{i+\varepsilon} = b^i h^2 b^{i+\varepsilon}$ and (2.4) implies $f = h$, contradicting the fact that $(b^i f, f b^{i+\varepsilon}) \neq (b^j h, h b^{j+\delta})$. It follows that j is the unique element of $[k]$ satisfying $j \equiv i + k/2 \pmod{k}$. Thus

$$b^i h^2 b^{i+\varepsilon} = b^{i+k/2} f^2 b^{i+\varepsilon+k/2} = b^i \alpha(f^2) b^{i+\varepsilon},$$

so $h^2 = \alpha(f^2) = (\alpha(f))^2$ and (2.4) implies $h = \alpha(f)$.

Claim 3. $\Gamma[T]$ is isomorphic to M_{km}

Observe that, when enumerating the vertices in T , every element of $\langle b \rangle H^*$ occurs exactly twice as a first coordinate and, as $\langle b \rangle$ and H^* commute by (2.2), exactly twice as a second coordinate. Thus, as is the case for $\Gamma[U]$, each vertex in $\Gamma[T]$ is incident to exactly one row edge and exactly one column edge. Unlike in $\Gamma[U]$, the row and column edges of $\Gamma[T]$ form a single cycle of length $2mk$. Indeed, as m is odd and the order of b is k , which is a power of 2,

$$(h_m, h_m), (h_m, h_1 b), (b h_1, h_1 b), \dots, (b^{k-1} h_{m-1}, h_{m-1} b^{k-1}), (b^{k-1} h_{m-1}, h_m)$$

is a Hamilton cycle in $\Gamma[T]$ which contains all of $\Gamma[T]$'s row and column edges.

To establish Claim 3 it suffices to show that for every $i, j \in [km]$ and every $\varepsilon, \delta \in \{0, 1\}$, the vertices $(b^i h_i, h_{i+\varepsilon} b^{i+\varepsilon})$ and $(b^j h_j, h_{j+\delta} b^{j+\delta})$ are joined by a symbol edge in $\Gamma[T]$ if and only if $i \equiv j + \frac{k}{2} m \pmod{km}$ and $\varepsilon = \delta$.

Indeed if $i \equiv j + \frac{k}{2} m \pmod{km}$, then $i \equiv j \pmod{m}$ and, as m is odd, $i \equiv j + k/2 \pmod{k}$. Together with $\varepsilon = \delta$ and the definition of H^* , this implies

$$b^i h_i h_{i+\varepsilon} b^{i+\varepsilon} = b^j a h_j h_{j+\delta} a b^{j+\delta} = b^j h_j h_{j+\delta} b^{j+\delta},$$

which establishes the ‘‘if’’ direction of the desired equivalence.

For the converse direction consider $(b^i h_i, h_{i+\varepsilon} b^{i+\varepsilon}), (b^j h_j, h_{j+\delta} b^{j+\delta}) \in T$ and assume that the group elements defining this pair of distinct vertices satisfy

$$b^i h_i h_{i+\varepsilon} b^{i+\varepsilon} = b^j h_j h_{j+\delta} b^{j+\delta}.$$

From (2.2) we see that

$$b^i h_i h_{i+\varepsilon} b^{i+\varepsilon} \in b^{2i+\varepsilon} H^* \text{ and } b^j h_j h_{j+\delta} b^{j+\delta} \in b^{2j+\delta} H^*.$$

Thus $\varepsilon = \delta$ and $i \equiv j \pmod{k/2}$. Now $b^j \in \{b^i, b^{i+k/2}\}$ and as H^* is pointwise fixed by the automorphism $\alpha : h \mapsto b^{k/2}hb^{k/2}$, both possible values of b^j yield $h_i h_{i+\varepsilon} = h_j h_{j+\varepsilon}$. Theorem 2.3 then implies $h_i = h_j$, so $i \equiv j \pmod{m}$.

Suppose $b^j = b^i$, which is equivalent to $i \equiv j \pmod{k}$. As $\gcd(k, m) = 1$ and $i, j \in [km]$, this implies $i = j$, contradicting the fact that $(b^i h_i, h_{i+\varepsilon} b^{i+\varepsilon})$ and $(b^j h_j, h_{j+\delta} b^{j+\delta})$ are distinct vertices. Therefore $j \equiv i + k/2 \pmod{k}$ and, as $k/2$ and m are coprime, we conclude that $i \equiv j + \frac{k}{2}m \pmod{km}$. \square

2.3 Toward a sturdier theory of near complete mappings

The proof of Theorem 1.7 relies upon the classification of finite simple groups, a theorem whose sprawling proof covers hundreds of articles. There is a distinct possibility that this proof contains a mistake. Indeed, Peter Cameron [35] has stated that “the discovery of a serious mistake in the proof [of the classification of finite simple groups] is on the cards.” Because we rely upon Theorem 1.7 to prove Theorem 2.1, there is reason to worry that our theorem may one day be reduced to the statement: every group with nontrivial cyclic Sylow 2-subgroups has a near complete mapping.

One way out of this problem would be a study of near complete mappings in groups which have complete mappings. An interesting question in this direction was raised by Evans in [62]. A partial transversal is *non-extendable* if it is not contained in any larger partial transversal. It is straightforward to check that the near transversals induced by sequencings of groups are non-extendable. The following conjecture was noted by Evans [62, p. 470] as a weak version of Keedwell’s conjecture [82] that every nonabelian group of order at least 10 is sequenceable.

Conjecture 2.5 ([62]). *For every finite non-abelian group G , the latin square L_G has a non-extendable near transversal.*

If true, Conjecture 2.5 cannot be extended to any abelian groups: an old result of Paige [99] implies that, if G is abelian, then L_G has either a transversal or a non-extendable near transversal, but it cannot have both. On the other hand, it is known that, for every integer $k \geq 1$, the Cayley table of the dihedral group D_{4k+2} has both a transversal and a non-extendable near transversal [80].

We have just shown that Conjecture 2.5 holds for those non-abelian groups that do not have complete mappings. Perhaps our techniques can be used to find maximal independent sets of size $n - 1$ in other latin square graphs based upon non-abelian groups, but additional insight would be needed for such a result. As far as we know, Conjecture 2.5 has not been attacked directly. However, many partial results are known due to its connection to sequenceable groups.

There is also hope that another way out of our dependence on the classification of finite simple groups is possible. A recent paper of Eberhard, Manners, and Mrazović [53] uses

Fourier analytic techniques to prove that Theorem 1.7 holds for all groups of order at least 10^5 . This may be enough for Theorem 1.7 to survive any problems with the proof of the classification of finite simple groups.

The main result of [53] is an asymptotically exact enumeration of the complete mappings of any group G with trivial or non-cyclic Sylow 2-subgroups. This raises the question of whether one could prove a similar result concerning near complete mappings. As near complete mappings are obtained from complete mappings by ignoring an element of the domain, the main result of [53] gives a lower bound for the number of near complete mappings in sufficiently large groups with trivial or non-cyclic Sylow 2-subgroups. Perhaps the techniques we used to prove Theorem 2.1 could be extended to find a lower bound for the number of near complete mappings in groups with nontrivial cyclic Sylow 2-subgroups.

Chapter 3

H2-harmonious groups

Continuing our study of group-based latin squares, we turn now to the question of Hamilton 2-plexes. As noted in Section 1.8, this question can be restated in terms of a novel type of group sequencing for which we now give a full, formal definition. Throughout this chapter, we will use indices which “wrap-around,” so that if g_{m+1} is undefined, then $g_{m+1} = g_1$.

Definition 3.1. Let G be a group of order n . An *H2-harmonious sequence* of G is a sequence $S = (g_1, g_2, \dots, g_{2n}) \in G^{2n}$ such that every element of G appears

- (a) once in the subsequence $e(S) := (g_{2i})_{i=1}^n$,
- (b) once in the subsequence $o(S) := (g_{2i-1})_{i=1}^n$, and
- (c) twice in the sequence $\pi(S) := (g_{2i} g_{2i-1}, g_{2i} g_{2i+1})_{i=1}^n$.

A group which has an H2-harmonious sequence is called *H2-harmonious*.

An H2-harmonious sequence S of G corresponds to the vertex-sequence of a Hamilton 2-plex in $K_{n,n}(G)$ by letting $e(S)$ and $o(S)$ denote, respectively, the row vertices and the column vertices. Because $L_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is the only latin square of order less than 5 which does not have a Hamilton 2-plex, we have the following special case of Conjecture 1.5.

Conjecture 3.2. *Every finite group except $\mathbb{Z}_2 \times \mathbb{Z}_2$ is H2-harmonious.*

The goal of this chapter is to offer evidence in support of Conjecture 3.2. Using ideas similar to those presented in the previous chapter, it will be straightforward to show that all groups of odd order and all groups with no complete mapping are H2-harmonious. It will take a bit more work to show that all dihedral groups—besides the (debatable) dihedral group of order 4—are H2-harmonious. A majority of the work in this chapter, though, will be directed towards resolving the special case of Conjecture 3.2 for abelian groups.

Theorem 3.3. *Every finite abelian group except $\mathbb{Z}_2 \times \mathbb{Z}_2$ is H2-harmonious.*

Several of the proofs in this chapter, including the proof of Theorem 3.3, will proceed inductively using group extensions. We say that the group G is an *extension of N by Q* if G has a normal subgroup $N_0 \triangleleft G$ which is isomorphic to N and the quotient G/N_0 is isomorphic to Q . We can also use the language of group extensions to bring all of this chapter's nontrivial theoretical results together into a single theorem.

Theorem 3.4. *Let N be a group of odd order, let Q be a finite group, and let G be an extension of N by Q . If $|G| > 4$ and Q either has a cyclic subgroup of index 2 or is abelian, then G is H2-harmonious.*

We begin this chapter with a few basic results which apply to general (not necessarily abelian) groups. We then proceed with the proof of Theorem 3.3. We end with a list of computer-generated sequencings for all groups of order less than 32 which were not covered by previously proven theorems. Throughout this chapter we will use the term *permutation of G* to refer to a sequence $(g_i)_{i=1}^{|G|}$ in which every element of G appears.

3.1 Basic results for general groups

When first encountering a claim like Conjecture 3.2, a natural first step would be to search for an H2-harmonious sequence for \mathbb{Z}_n . This turns out to be a rather simple exercise, but we will enshrine it as a proposition to use as the base case of future proofs by induction.

Proposition 3.5. *Every finite cyclic group is H2-harmonious.*

Proof. Letting a denote the generator of a cyclic group G of order n . To see that

$$S := 1, 1, a, a, a^2, a^2, \dots, a^{n-1}, a^{n-1},$$

is an H2-harmonious sequence, observe that $\pi(S) = (a^i)_{i=0}^{2n-2}, a^{n-1}$. □

That a similar construction works for all groups of odd order is far from obvious. Indeed, our short proof of this fact uses a nontrivial result of Beals et al. [18]. Having already stated that result in Theorem 2.3, the following is essentially a corollary.

Theorem 3.6. *Every group of odd order is H2-harmonious.*

Proof. Given a group G of odd order, let g_1, g_2, \dots, g_n be the harmonious sequence of G guaranteed to exist by Theorem 2.3. This means that $g_i \mapsto g_i$ and $g_i \mapsto g_{i+1}$ are both complete mappings. That $S := (g_1, g_1, g_2, g_2, \dots, g_n, g_n)$ is an H2-harmonious sequence then follows from the fact that $\pi(S) = (g_i^2, g_i g_{i+1})_{i=1}^n$. □

Another result concerning H2-harmonious sequences which we obtain by leaning heavily upon the extensive literature concerning complete mappings is the following.

Theorem 3.7. *Every finite group with no complete mapping is H2-harmonious.*

The proof of Theorem 3.7 hides an even greater mass of deep group-theoretic work than the proof of Theorem 3.6: in its present form, Theorem 3.7 depends upon the monumental Theorem 1.7, and therefore upon the classification of finite simple groups. Of course, we could avoid this burdensome dependence by stating Theorem 3.7 as: every group with nontrivial cyclic Sylow 2-subgroups is H2-harmonious. However, the strongest possible statement pairs nicely with the following account of the relevant combinatorial machinery. Though fairly elementary, this result will prove to be a powerful tool in the construction of H2-harmonious sequences.

Lemma 3.8. *If a finite group G is an extension of a group of odd order by an H2-harmonious group, then G is H2-harmonious.*

Proof. Let $N \triangleleft G$ denote the given normal subgroup of odd order. By hypothesis, there are ordered sets of coset representatives for N in G , say r_1, r_2, \dots, r_ℓ and c_1, c_2, \dots, c_ℓ , such that $c_1N, r_1N, \dots, c_\ellN, r_\ellN$ is an H2-harmonious sequence for G/N . Moreover, via the assumption that $|N|$ is odd and Theorem 2.3, let a_1, a_2, \dots, a_m be a harmonious sequence of N . We claim that

$$S := \left((a_j c_i, r_i a_j)_{i=1}^\ell \right)_{j=1}^m$$

is an H2-harmonious sequence of G . Indeed, it is straightforward to check that $o(S) = \left((a_j c_i)_{i=1}^\ell \right)_{j=1}^m$ and $e(S) = \left((r_i a_j)_{i=1}^\ell \right)_{j=1}^m$ are both permutations of G .

To see that $\pi(S)$ has two copies of every element of G , we begin by observing that every element of N has a unique representation of the form a_j^2 and a unique representation of the form $a_j a_{j+1}$ for $j \in [m]$. Therefore, for all $i \in [m]$, every element of the coset $r_i c_i N$ has a unique representation of the form $r_i a_j^2 c_i$ and every element of the coset $r_i c_{i+1} N$ has a unique representation of the form $r_i a_j^2 c_{i+1}$. Moreover, every element of the coset $r_\ell c_1 N$ has a unique representation of the form $r_\ell a_j a_{j+1} c_1$. It then follows that the sequence

$$\pi(S) = \left(\left(r_i a_j^2 c_i, r_i a_j^2 c_{i+1} \right)_{i=1}^{\ell-1}, r_\ell a_j^2 c_\ell, r_\ell a_j a_{j+1} c_1 \right)_{j=1}^m$$

contains two copies of each element of G . □

The rest of the proof of Theorem 3.7 is an application of group-theoretic ideas which have already appeared in Section 2.1.

Proof of Theorem 3.7. Letting G be a group with no complete mapping, Theorem 1.7 tells us that $\text{Syl}_2(G) = \mathbb{Z}_{2^t}$ for some integer $t \geq 1$. Lemma 2.4 then tells us that there is a normal subgroup of odd order $N \triangleleft G$ for which $G/N \cong \mathbb{Z}_{2^t}$. Therefore, G/N is H2-harmonious by Proposition 3.5 and the result follows from Lemma 3.8. □

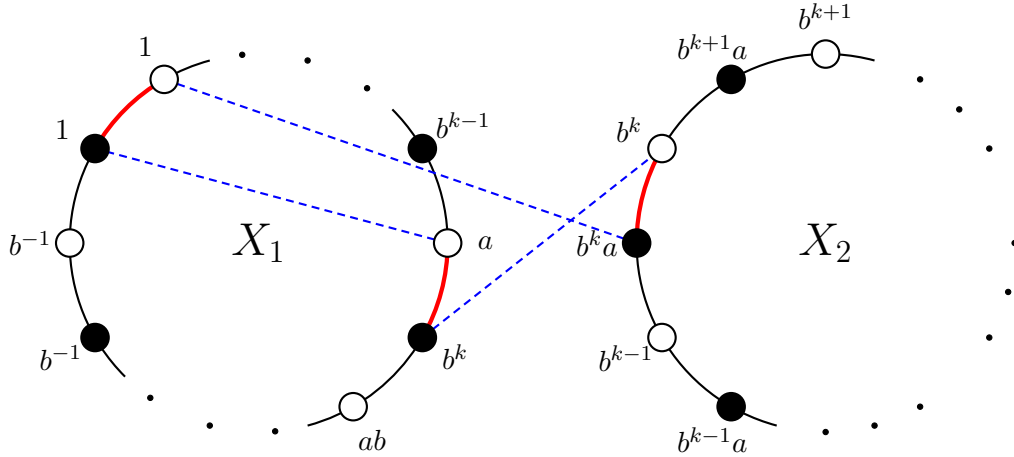


Figure 3.1: An illustration of how X_1 and X_2 are tied together into the Hamilton 2-plex $E(Y)\triangle E(X_1 \cup X_2)$. Here, X_1 and X_2 are the cycles formed by black and thick-red edges and Y is formed by the thick-red and dashed-blue edges. In this figure, and all subsequent figures, we draw vertices of C in black and vertices of R in white.

So far we have shown that many groups are H2-harmonious without expending too much effort. Unfortunately, for most Hall-Paige groups of even order it does not seem to be possible to find H2-harmonious sequences via straightforward extensions of results concerning complete mappings. However, when the group in question is relatively uncomplicated, constructive proofs often arise quite naturally. We will see this when we address H2-harmonious sequences in abelian groups, but first we provide such a result for an infinite family of not necessarily abelian groups.

Theorem 3.9. *If a finite group G of order at least 6 has a cyclic subgroup of index 2, then G is H2-harmonious.*

Proof. Let b generate the cyclic subgroup of index 2 in G , let a denote an element of $G \setminus \langle b \rangle$, and observe that $\langle b \rangle \triangleleft G$. We may assume that the order of b is $m = 2k$ for some $k \geq 2$, as otherwise Lemma 3.8 tells us that G is H2-harmonious. We will show that $K_{n,n}(G)$ has a Hamilton 2-plex. Toward this end, we claim that both of the following $2m$ -cycles are rainbow:

$$X_1 := \left([b^{i-1}]_C, [b^{i-1}]_R \right)_{i=0}^{k-1}, [b^{k-1}]_C, \left([ab^i]_R, [b^{k+i}]_C \right)_{i=0}^{k-2}, [ab^{k-1}]_R, [b^{-1}]_C,$$

and

$$X_2 := \left([b^{k+i-1}a]_C, [b^{k+i-1}]_R \right)_{i=0}^{k-1}, [b^{-1}a]_C, \left([ab^{k+i}]_R, [b^i a]_C \right)_{i=0}^{k-2}, [ab^{-1}]_R, [b^{k-1}a]_C.$$

Notice that the condition $k \geq 2$ is necessary for these two cycles to be well-defined. Moreover, X_1 and X_2 are disjoint.

Recall that G is not necessarily abelian, and the label on the edge $[x]_C[y]_R$ is the group element corresponding to the product yx . The sequence of edge labels in X_1 , starting with $[b^{-1}]_C[b^{-1}]_R$, is

$$\left(b^{i-2}\right)_{i=0}^{2k-1}, \left(ab^{k-1+i}\right)_{i=0}^{2k-1},$$

while the sequence of edge labels in X_2 , starting with $[b^{k-1}a]_C[b^{k-1}]_R$, is

$$\left(b^{i-2}a\right)_{i=0}^{2k-1}, \left(ab^{k-1+i}a\right)_{i=0}^{2k-1}.$$

To see that these two sequences are permutations of G , we use the fact that $\langle b \rangle$ has index 2 to conclude that there is a nonnegative integer t such that $a^2 = b^t$. This implies that every element of G has a unique representation of the form $a^\delta b^i$ and a unique representation of the form $b^i a^\delta$ for $\delta \in \{0, 1\}$ and $i \in [m]$. Moreover, because normal subgroups are closed under inner automorphisms, there is a positive integer ℓ such that $(m, \ell) = 1$ and $a^{-1}ba = b^\ell$. It follows that $ab^i a = a^2 b^{\ell i} = b^{\ell i + t}$ for all $i \in [m]$ and, because ℓ is coprime to m , every element of $\langle b \rangle$ has a unique representation of the form $ab^i a$ for $i \in [m]$.

So $X_1 \cup X_2$ is a 2-plex. To find a Hamilton 2-plex we will tie these two cycles together using the $(X_1 \cup X_2)$ -alternating cycle

$$Y := [1]_C, [1]_R, [b^k a]_C, [b^k]_R, [b^k]_C, [a]_R, [1]_C.$$

See Figure 3.1 for a picture of the subgraph induced by $E(X_1 \cup X_2)$ and $E(Y)$. It follows from $(\ell, 2k) = 1$ that ℓ is odd, so that $ab^k = b^{\ell k} a = b^k a$. Therefore, the set of edge labels in $E(Y) \cap E(X_1 \cup X_2)$ is the same as the set of edge labels in $E(Y) \setminus E(X_1 \cup X_2)$, namely $\{1, a, b^k a\}$. This tells us that the graph induced by $E(Y) \triangle E(X_1 \cup X_2)$ is a 2-plex. The result then follows from the observation that the cycle in the graph induced by $E(Y) \triangle E(X_1 \cup X_2)$ containing the edge $[1]_C[a]_R$ has length $4m$. \square

The original motivation for the argument in the proof of Theorem 3.9 was to show that dihedral groups are H2-harmonious. The more general result we have obtained allows us to conclude that several well-known infinite families of groups are H2-harmonious. Towards showing that Conjecture 3.2 holds for small groups, we are particularly interested in the *dicyclic groups* of order $4n$, which can be defined by $\text{Dic}_{4n} := \langle a, b \mid a^n = b^2, a^{2n} = abab^{-1} = 1 \rangle$, and the *semidihedral groups* of order 2^n , which can be defined by $SD_{2^n} := \langle a, b \mid a^{2^{n-1}} = b^2 = 1, abab = a^{2^{n-2}} \rangle$ (see e.g. the documentation for the named constructions of both of these groups in Sage [111]). We end this section with a corollary which will be useful in Section 3.3 below.

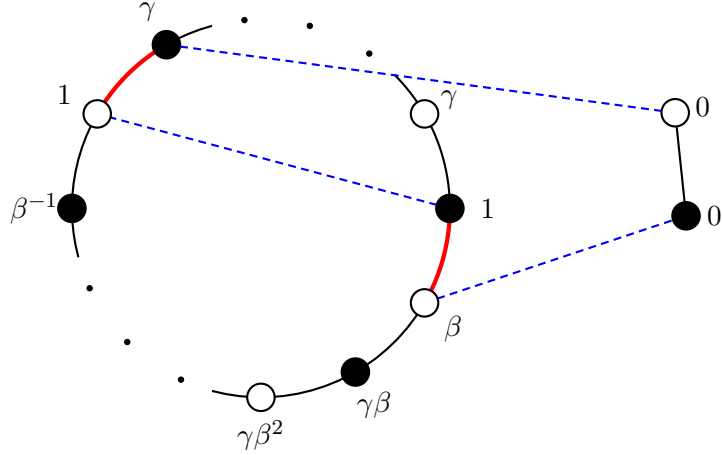


Figure 3.2: An illustration of the Hamilton 2-plex $X \Delta Y$. Here, $X \cup \{[0]_R[0]_C\}$ is the graph formed by black and thick-red edges and $Y \setminus \{[0]_R[0]_C\}$ is the path formed by thick-red and dashed-blue edges.

Corollary 3.10. *All dihedral groups, dicyclic groups, and semidihedral groups of order at least 6 are H2-harmonious.*

3.2 Proof of Theorem 3.3

We now restrict our attention to abelian groups. Roughly speaking, we will prove Theorem 3.3 in two steps: First, we will prove by induction that all abelian 2-groups except $\mathbb{Z}_2 \times \mathbb{Z}_2$ are H2-harmonious. Second, we will break an arbitrary abelian group G into the direct product of a group of odd order and a 2-group, then apply Lemma 3.8 to show that G is H2-harmonious. For the first step of this process, we will need to consider several possible base cases beyond the cyclic groups taken care of by Proposition 3.5. Two of these additional base cases are the groups \mathbb{Z}_2^3 and \mathbb{Z}_2^4 . Although it would be sufficient for the proof of Theorem 3.3 to simply give the H2-harmonious sequences of these groups and move on, we have an interesting algebraic construction which applies to all elementary abelian 2-groups of order at least 8.

It is worth noting that this construction, now in some sense superfluous, was in fact crucial to the formulation of Conjecture 3.2. Because $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not H2-harmonious, it was important to ensure that this example does not generalize to an infinite family of counterexamples. As such, the following result was an important proof of concept which motivated much of the work in this chapter.

Theorem 3.11. *For $k \geq 3$, the group \mathbb{Z}_2^k is H2-harmonious.*

Proof. Consider \mathbb{Z}_2^k as the additive group of the finite field \mathbb{F}_{2^k} . Cohen proved in [42] that \mathbb{F}_{2^k} has a primitive element, say α , whose trace is 0 (i.e. $\alpha + \alpha^2 + \alpha^4 + \dots + \alpha^{2^{k-1}} = 0$)

whenever $k \geq 3$. We claim that the polynomial $p(x) := x^2 + x + \alpha \in \mathbb{F}_{2^k}[x]$ has a root in \mathbb{F}_{2^k} . Indeed, we know that $p(x)$ has a root β in some extension of \mathbb{F}_{2^k} , say $K \supseteq \mathbb{F}_{2^k}$, so we only need to show that, in fact, $\beta \in \mathbb{F}_{2^k}$. Working now over K , which has characteristic 2, it follows from $\alpha = \beta^2 + \beta$ and $\alpha + \alpha^2 + \cdots + \alpha^{2^{k-1}} = 0$ that $\beta^{2^k} + \beta = 0$. This means β is a root of $x^{2^k} + x$, whose splitting field over \mathbb{F}_2 is \mathbb{F}_{2^k} . We may therefore conclude that

$$\beta, \alpha \in \mathbb{F}_{2^k} \text{ and } \alpha = \beta(\beta + 1).$$

We will show that $K_{n,n}(\mathbb{Z}_2^k)$ has a Hamilton 2-plex. For the sake of simplifying notation, let $\gamma := \beta + 1$ and let $m = 2^k - 1$. Consider the $2m$ -cycle

$$X := [1]_C, [\beta]_R, \left([\gamma^i \beta^i]_C, [\gamma^i \beta^{i+1}]_R \right)_{i=1}^{m-1}, [1]_C.$$

The sequence of edge-labels in X can be obtained by interleaving $((\beta\gamma)^i(\beta + 1))_{i=0}^{m-1}$ and $((\beta\gamma)^i(1 + \gamma^{-1}))_{i=1}^m$. But $\beta\gamma = \beta^2 + \beta = \alpha$, and because α is a primitive element of \mathbb{F}_{2^k} , every nonzero element of \mathbb{F}_{2^k} has a unique representation of the form $(\beta\gamma)^i$ for some $i \in [m]$. Moreover, right-multiplication by a fixed nonzero element is a permutation of $\mathbb{F}_{2^k} \setminus \{0\}$. It follows that X has 2 edges of color g for every $g \in \mathbb{F}_{2^k} \setminus \{0\}$.

As we did in the proof of Theorem 3.9, we find a Hamiltonian 2-plex by augmenting X with a cycle of length 6. In particular, consider the cycle

$$Y := [0]_C, [\beta]_R, [1]_C, [1]_R, [\gamma]_C, [0]_R, [0]_C.$$

We claim that the symmetric difference of $E(X)$ and $E(Y)$ corresponds to a Hamilton 2-plex (see Figure 3.2). Indeed, notice that $E(X) \cap E(Y) = \{[\beta]_R[1]_C, [1]_R[\gamma]_C\}$, so that in passing from $E(X)$ to $E(X) \triangle E(Y)$ we lose an edge of color β and an edge of color $\beta + 1$. However, we also pick up the edges $[\beta]_R[0]_C$, $[1]_R[1]_C$, $[0]_R[\gamma]_C$, and $[0]_R[0]_C$, which have edge labels β , 0, $\beta + 1$, and 0, respectively. This shows that $E(X) \triangle E(Y)$ corresponds to a 2-plex. The result then follows from the observation that the cycle in the graph induced by $E(X) \triangle E(Y)$ containing the edge $[0]_R[0]_C$ has length 2^{k+1} . \square

It would have been interesting to extend the field-theoretic construction in the proof of Theorem 3.11 to all elementary abelian groups, but unfortunately this is not possible: the construction relies on the identities $1 + \gamma = \beta$ and $1 + \beta = \gamma$ being concurrently true, which implies $\beta = \beta + 2$. Of course, this does not affect the list of groups we can identify as H2-harmonious given the fact that Theorem 3.6 takes care of all (elementary abelian) groups of odd order.

The final base case we will need for the proof of Theorem 3.3 was obtained using the backtracking algorithm presented in Appendix A.

Proposition 3.12. *The group $\mathbb{Z}_4 \times \mathbb{Z}_4$ is H2-harmonious.*

Proof. Let a and b generate $\mathbb{Z}_4 \times \mathbb{Z}_4$. Checking that

$$S := 1, 1, b^2, a^3b^3, a^3b^3, b^3, ab^2, a^2b, a, a^3b, a^3b, a^2, a^3b^2, a, a^2b^2, ab^2, \\ b, a^3b^2, a^2b, b^2, a^2b^3, ab, ab^3, b, b^3, a^2b^2, a^3, ab^3, ab, a^3, a^2, a^2b^3$$

is an H2-harmonious sequence of $\mathbb{Z}_4 \times \mathbb{Z}_4$ is straightforward given

$$\pi(S) = 1, b^2, a^3b, a^2b^2, a^3b^2, ab, a^3b^3, a^3b, b, a^2b^2, ab, ab^2, b^2, a^3b^2, a^3, ab^3, \\ a^3b^3, ab^3, a^2b^3, a^2b, a^3, a^2, a, 1, a^2b, ab^2, b^3, a^2, b, a, b^3, a^2b^3. \quad \square$$

We now turn to the key lemma in our inductive proof that all sufficiently large abelian 2-groups are H2-harmonious. Although we use group-theoretic language in the proof of this lemma, it is worth giving a brief graph-theoretic description of the underlying ideas. Inspired by the proof of a similar lemma concerning harmonious groups in Section 6 of [18], our H2-harmonious sequences consist of two pairs of long periodic subsequences connected by two short, ad-hoc subsequences. These pairs of long sequences correspond to long 2-bounded paths in $K_{n,n}(G)$, and the challenge of establishing the following result was to find short paths which use the outstanding edge labels while tying these two long paths together into a Hamilton cycle.

Lemma 3.13. *If a finite abelian group G is an extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by a nontrivial H2-harmonious group, then G is H2-harmonious.*

Proof. Letting $m := |G|/4$, we may assume that $m \geq 4$ by Proposition 3.5 and Theorem 3.9. Let a and b generate the given subgroup $N \triangleleft G$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and let c_1, \dots, c_m and r_1, \dots, r_m be ordered sets of coset representatives for N in G such that $c_1N, r_1N, \dots, c_mN, r_mN$ is an H2-harmonious sequence of G/N . In the case $m \equiv 1 \pmod{3}$, we let $k = \frac{m-1}{3}$ and claim that

$$(c_i, r_i)_{i=1}^m \tag{3.1}$$

$$(bc_{3i+1}, ar_{3i}, abc_{3i}, br_{3i-1}, ac_{3i-1}, abr_{3i-2})_{i=k}^2, bc_4, ar_3, abc_3, br_2, ac_2, \tag{3.2}$$

$$br_1, abc_1, ar_m, ac_1, abr_1, \tag{3.3}$$

$$bc_2, abr_2, ac_3, br_3, abc_4, (ar_{3i-2}, bc_{3i-1}, abr_{3i-1}, ac_{3i}, br_{3i}, abc_{3i+1})_{i=2}^k, \tag{3.4}$$

$$(br_{3i+1}, ac_{3i+1}, abr_{3i}, bc_{3i}, ar_{3i-1}, abc_{3i-1})_{i=k}^1 \tag{3.5}$$

$$ar_1, bc_1, abr_m \tag{3.6}$$

is an H2-harmonious sequence of G (when $k = 1$ the periodic portions of (3.2) and (3.4) have length zero). Towards establishing the claim, notice that every element of G has exactly

two representations of the form

$$a^i b^j r_k c_{k+\delta} \text{ for } i, j, \delta \in \{0, 1\} \text{ and } k \in [m]. \quad (3.7)$$

Therefore, it suffices to show that every expression of the form (3.7) appears exactly once as a product of consecutive elements of the given sequence. Notice that we may take consecutive products, rather than following Definition 3.1 precisely, since G is abelian. And indeed, taking consecutive products in (3.1) gives us a copy of every expression of the form $r_k c_{k+\delta}$ with $k \in [m]$ except $r_m c_1$. Similarly, (3.2), (3.4), and (3.5) together give all expressions of the form $ar_k c_{k+\delta}$, $br_k c_{k+\delta}$, and $abr_k c_{k+\delta}$ with $k \in [2, m-1]$ and $\delta \in \{0, 1\}$. Moreover, the product of the first two terms in (3.5) gives us the expression $abr_m c_m$, we obtain $br_m c_m$ in the step from (3.1) to (3.2), and we obtain $ar_m c_m$ in the step from (3.4) to (3.5). Finally, we see that the sequence of consecutive products from the final term of (3.2), through (3.3), to the first term of (3.4) gives us

$$abr_1 c_2, ar_1 c_1, br_m c_1, r_m c_1, br_1 c_1, ar_1 c_2,$$

while taking consecutive products from the final term of (3.5), through (3.6), to the first term of (3.1) gives $br_1 c_2, abr_1 c_1, ar_m c_1, abr_m c_1$. Thus, taking cyclically consecutive products in the sequence given by (3.1)-(3.6), we obtain exactly one copy of each expression of the form (3.7).

In the case $m \equiv 2 \pmod{3}$ let $k := \frac{m-2}{3}$. A similar argument establishes that

$$(c_i, r_i)_{i=1}^m, \quad (3.8)$$

$$bc_m, (ar_{3i+1}, abc_{3i+1}, br_{3i}, ac_{3i}, abr_{3i-1}, bc_{3i-1})_{i=k}^{\searrow 1}, \quad (3.9)$$

$$abr_1, ac_1, abr_m, abc_1, br_1, \quad (3.10)$$

$$ac_2, br_2, (abc_{3i}, ar_{3i}, bc_{3i+1}, abr_{3i+1}, ac_{3i+2}, br_{3i+2})_{i=1}^k, \quad (3.11)$$

$$abc_m, (br_{3i+1}, ac_{3i+1}, abr_{3i}, bc_{3i}, ar_{3i-1}, abc_{3i-1})_{i=k}^{\searrow 1}, \quad (3.12)$$

$$ar_1, bc_1, ar_m \quad (3.13)$$

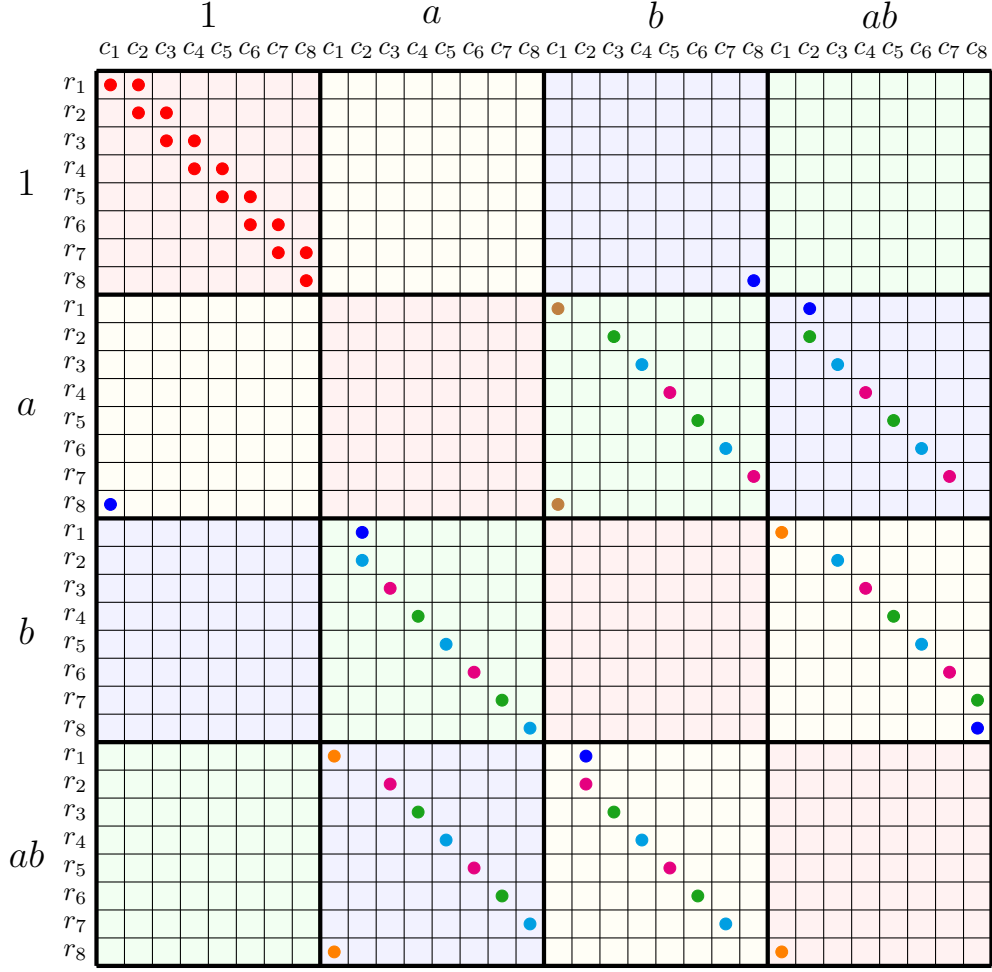


Figure 3.3: The Hamilton 2-plex in L_G corresponding to the H2-harmonious sequence of G given in the $m \equiv 2 \pmod{3}$ case of Lemma 3.13. Consecutive products within the various lines are color-coded as follows: (3.8), (3.9), (3.10), (3.11), (3.12), (3.13). Blue circles, moreover, correspond to products taken between the last term in one line and the first term in the next.

is an H2-harmonious sequence of G (see Figure 3.3 for an illustration of this case). Finally, a similar argument also establishes that, in the case $m \equiv 0 \pmod{3}$, if we let $k := \frac{m}{3}$, then

$$\begin{aligned}
& (c_i, r_i)_{i=1}^m, \\
& (bc_{3i}, ar_{3i-1}, abc_{3i-1}, br_{3i-2}, ac_{3i-2}, abr_{3i-3})_{i=k}^{\searrow 2}, bc_3, ar_2, abc_2, \\
& br_1, ac_1, ar_m, abc_1, ar_1, \\
& bc_2, abr_2, ac_3, br_3, (abc_{3i-2}, ar_{3i-2}, bc_{3i-1}, abr_{3i-1}, ac_{3i}, br_{3i})_{i=2}^k, \\
& abc_m, br_{m-1}, ac_{m-1}, (abr_{3i+1}, bc_{3i+1}, ar_{3i}, abc_{3i}, br_{3i-1}, ac_{3i-1})_{i=k-1}^{\searrow 1}, \\
& abr_1, bc_1, abr_m
\end{aligned}$$

is an H2-harmonious sequence of G . □

We now have all the machinery in place to carry out the two step process outlined at the start of this section. First, a proof by induction that the conclusion of Theorem 3.3 holds for abelian 2-groups.

Theorem 3.14. *All finite abelian 2-groups besides $\mathbb{Z}_2 \times \mathbb{Z}_2$ are H2-harmonious.*

Proof. We proceed by induction on $|G|$, with Proposition 3.5, Theorem 3.11, and Theorem 3.9 establishing the result in the case $|G| \leq 8$. For the induction step, let G be an abelian 2-group of order at least 16. We may assume G is not cyclic by Proposition 3.5, so the fundamental theorem of finite abelian groups tells us that $G \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^\ell} \times A$ for some integers $1 \leq k \leq \ell$ and some abelian 2-group A which is either trivial or isomorphic to $\mathbb{Z}_{2^{t_1}} \times \mathbb{Z}_{2^{t_2}} \times \cdots \times \mathbb{Z}_{2^{t_m}}$ for some sequence of integers $(t_i)_{i=1}^m$ satisfying $\ell \leq t_1 \leq t_2 \leq \cdots \leq t_m$. Letting a denote the unique element of order 2 in the \mathbb{Z}_{2^k} factor of G and b the unique element of order 2 in the \mathbb{Z}_{2^ℓ} factor, the normal subgroup $N = \langle a, b \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the factor group $Q = G/N$ is an abelian 2-group satisfying $|Q| < |G|$. If $Q \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we know by induction that Q is H2-harmonious and Lemma 3.13 tells us that G is also H2-harmonious. If $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we have either $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, in which case we may apply Proposition 3.12 to see that G is H2-harmonious, or $G \cong \mathbb{Z}_2^4$, in which case we may apply Theorem 3.11 to obtain the desired result. □

Finally, the hard work is done and we can enjoy the fruits of our labor with a very short proof of this chapter's main theorem.

Proof of Theorem 3.3. Let G be a finite abelian group and let S be a Sylow 2-subgroup of G . By the fundamental theorem of finite abelian groups, there is a subgroup N of odd order such that $G \cong S \times N$. We may assume both S and N are nontrivial by Theorems 3.6 and 3.14. If $S \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we may apply Theorem 3.14 and Lemma 3.8 to see that G is H2-harmonious. If $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the result follows from Theorem 3.6 and Lemma 3.13. □

It is worth noting that we have also just completed the proof of Theorem 3.4, which follows directly from Lemma 3.13 and Lemma 3.8 via Theorems 3.3 and 3.9.

3.3 Small H2-harmonious groups

We end this chapter with additional evidence for Conjecture 3.2. By combining the above work with several computational searches, we obtained the following exhaustive result.

Proposition 3.15. *Every group of order less than 32 except $\mathbb{Z}_2 \times \mathbb{Z}_2$ is H2-harmonious.*

A complete proof of Proposition 3.15 is given in Appendix B. This proof comes in the form of a list containing every group of order less than 32 not already shown to be H2-harmonious by Theorems 3.3 and 3.6 or Corollary 3.10. In other words, we give either an H2-harmonious sequence or a theoretical explanation for every nonabelian group of even order less than 32 which is not dihedral, semidihedral, or dicyclic. We determined these groups using The Gap Small Groups Library [20], and refer the reader to [20] for confirmation that we have indeed considered all groups of order less than 32. The H2-harmonious sequences were generated by running the algorithm given in Appendix A with the computer algebra system Sage [111].

The choice to stop at groups of order 31 was not quite a matter of computational feasibility—it took just under 3 minutes to find an H2-harmonious sequence for the group with GAP ID [32,43] on a laptop with a 2.3 GHz Intel i7 processor and 16 GB of RAM—but rather a qualitative cost-benefit analysis. Indeed, there are more nonabelian groups of order 32 than there are nonabelian groups of all smaller orders combined. Moreover, similar to the case of abelian groups, it is particularly difficult to show that small groups whose order is a power of 2 are H2-harmonious. This is largely because we cannot apply Lemma 3.8 to groups whose order has no odd divisors. Table 3.1 gives, for each even n between 6 and 30, the number of nonabelian groups of order n that can be shown to be H2-harmonious by various available tools. We see from this figure that 24 of the 31 considered groups whose order is not a power of 2 satisfy the hypotheses of Lemma 3.8, while more than half of the nonabelian groups of order 16 required a computer search to determine the existence of an H2-harmonious sequence.

	6	8	10	12	14	16	18	20	22	24	26	28	30
number of nonabelian groups	1	2	1	3	1	9	3	3	1	12	1	2	3
Lemma 3.8	1	0	1	1	1	0	3	2	1	9	1	1	3
Theorem 3.9	1	2	1	2	1	4	1	2	1	6	1	2	3
Computer Search	0	0	0	1	0	5	0	0	0	3	0	0	0

Table 3.1: For every even order less than 32 (such that there exists at least one nonabelian group), the number of nonabelian groups which are known to be H2-harmonious by various methods. The sum of values in several columns is larger than the total number of nonabelian groups because some groups satisfy the hypotheses of both Lemma 3.8 and Theorem 3.9.

The most significant reason to doubt Conjecture 3.2 is that none of the results above apply to non-solvable groups. It is for this reason that we also ran the algorithm in Appendix A on the alternating group A_5 , well-known to be the only non-solvable group of order less than 100. After several days, the computer found an H2-harmonious sequence for A_5 , and we include this sequence in Appendix B. Nonetheless, resolving the following special case of Conjecture 3.2 would be significant, as it would give an infinite family of H2-harmonious non-solvable groups.

Conjecture 3.16. *All symmetric and alternating groups are H_2 -harmonious.*

Perhaps the most obvious next step towards Conjecture 3.2, though, is to demonstrate that it holds for solvable groups. Inductive techniques similar to those used above are likely capable of establishing such a result.

Chapter 4

Variations on plexes in latin squares

In this chapter we offer further evidence in support of Conjecture 1.5—every latin square of order at least 5 has a Hamilton 2-plex—while mapping out a basic theory of (k, ℓ) -plexes. Moving on from the group-theoretic tools of the two previous chapters, most of the proofs below will use ideas from extremal and probabilistic combinatorics. The general strategy of finding an approximation of the target structure before finishing with local changes, however, will again be heavily utilized. Most of the work in this chapter was initially motivated by the question(s): when can we guarantee that, for a generalized latin square L , there is a Hamilton cycle in $K_{n,n}(L)$ whose color distribution has certain desirable properties? Notice that Conjecture 1.5 can be stated in this manner: for every latin square L , there is Hamilton cycle in $K_{n,n}(L)$ with two edges of every color.

We will begin by showing that, for every latin square L , there is a Hamilton cycle in $K_{n,n}(L)$ containing at least one edge of each color. This is in some sense a warm-up for our first main result, which uses similar techniques to find Hamilton $(2,2)$ -plexes in arrays which do not contain too many copies of any symbol. Recall that a k -bounded n -square is an $n \times n$ array in which no symbol appears more than k times. In Section 4.2, we use the lopsided Lovász Local Lemma to prove the following.

Theorem 4.1. *Every $\frac{n}{\sqrt{96}}$ -bounded n -square has a Hamilton $(2,2)$ -plex.*

It seems likely that this result is not tight, though it is not clear how much it could be improved. One may ask: what is the largest c such that every cn -bounded n -square has a Hamilton $(2,2)$ -plex? The existence of n -bounded n -squares, i.e. equi- n squares, with no near transversals [103] indicates that perhaps some equi- n -squares do not have Hamilton 2-plexes (recall that Hamilton 2-plexes and Hamilton $(2,2)$ -plexes are equivalent in equi- n squares). However, we were unable to modify the construction from [103] to find equi- n -squares with no Hamilton 2-plex despite spending a significant amount of time attempting to do so. This inspires an interesting question about which we cannot yet confidently form a conjecture.

Question 4.2. *Does every equi- n square have a Hamilton 2-plex?*

The second half of this chapter is concerned mostly with (k, ℓ) -plexes in latin squares, or collections of entries which intersect each row and each column k times and contain at most ℓ copies of each symbol. We also consider *partial k -plexes*, or collections of entries which intersect each row and column at most k times and contain at most k copies of each symbol. Conjecture 1.4 asserts that, when k is an even integer, there is a k -plex in every latin square of order at least k . Therefore, when k is even, it should be possible to find both large partial k -plexes and (k, ℓ) -plexes for all $\ell \geq k$. But Egan and Wanless showed that, when k is any odd integer, there exist infinitely many latin squares which do not have a k -plex [55]. This means it is not clear whether $(k, k + 1)$ -plexes or large partial k -plexes should always exist. The second main result of this chapter shows that one can find both of these structures so long as n is sufficiently large with respect to k .

Theorem 4.3. *For all $k \geq 1$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,*

- (a) *every latin square of order n has a partial k -plex of size $kn - o(n)$, and*
- (b) *every latin square of order n has a $(k, k + 1)$ -plex.*

Recall from Section 1.4 that we have recently learned of unpublished work of Pula [105] that contains an alternative proof of Theorem 4.3(b). Our proof of this result will be extended to show that, in the $k = 2$ cases of Theorem 4.3, we can always find plexes which are in some sense Hamiltonian. We then end the chapter with two (k, ℓ) -plex existence results which hold for all latin squares, rather than just latin squares which are sufficiently large. In particular, we show that every latin square has a Hamilton $(2, 5)$ -plex and, for all $k \leq \frac{n}{4}$, every latin square of order n has a $(k, 4k)$ -plex. The second of these results is similar to another result of Pula [105]: for all $k < \frac{n}{11} - o(n)$, every latin square of order n has a $(k, 2k)$ -plex. Throughout this chapter, we will often identify paths and cycles with the set of edges comprising them.

4.1 Local switching and Hamilton covers

Throughout this section we let (R, C) denote the bipartition of $K_{n,n}$, where $R = \{r_1, \dots, r_n\}$ and $C = \{c_1, \dots, c_n\}$. Moreover, unless noted otherwise, we use H to denote a Hamilton cycle of $K_{n,n}$. In order to define our local switching procedure we must first define a notion of congruence on H .

Definition 4.4. We say that two edges $r_i c_j, r_k c_\ell \in H$ are *congruent modulo H* , denoted $r_i c_j \equiv r_k c_\ell \pmod{H}$, if when cyclically orienting H in the direction from r_i to c_j and walking along H from c_j according to this orientation, we reach r_k before c_ℓ .

See Figure 4.1 for an example of two edges which are not congruent. As every Hamilton cycle in $K_{n,n}$ has even length, this notion of congruence is an equivalence relation with two equivalence classes of size n .

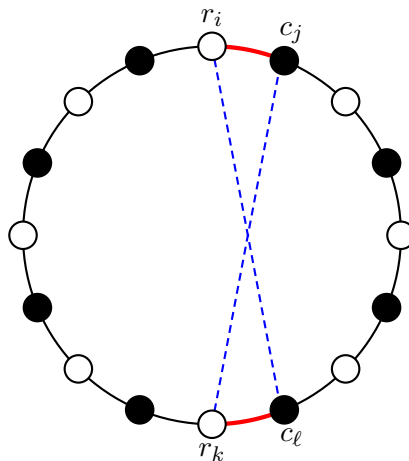


Figure 4.1: A Hamilton cycle H containing the edges $r_i c_j$ and $r_k c_l$. These edges are not congruent modulo H , i.e. $r_i c_j \not\equiv r_k c_l \pmod{H}$. We obtain the switch $\sigma(H; r_i c_j, r_k c_l)$ from H by removing the thick-red edges and replacing them with the dashed-blue edges.

Definition 4.5. Suppose $e, f \in H$ and $e \not\equiv f \pmod{H}$, where $e = r_i c_j$ and $f = r_k c_l$. The σ -switch of H with respect to e and f is the Hamilton cycle

$$\sigma(H; e, f) := (H \setminus \{e, f\}) \cup \{r_i c_l, r_k c_j\}.$$

Notice that the condition $e \not\equiv f \pmod{H}$ is necessary, as if $e \equiv f \pmod{H}$ then the corresponding operation would break H into two disjoint cycles. Moreover, σ -switches are not well-defined for adjacent edges. Figure 4.1 contains illustration of the procedure.

As a first, relatively straightforward application of our switching procedure we consider *Hamilton covers* of latin squares L , i.e. Hamilton cycles of $K_{n,n}(L)$ which contain at least one edge of each color. More generally, a *cover* of a latin square—introduced by Best et al. in [23]—is a collection of entries which intersects every row and column and contains every symbol. It was shown in [23] that every latin square of order n has a cover of size at most $n + 12 \log^2(n)$. So, the existence of a Hamilton cover (which has size $2n$) should not be particularly surprising. Nonetheless, it is a nontrivial generalization of Hamilton 2-plexes which offers a chance to ease into use of our local switching procedure.

Theorem 4.6. *Every latin square has a Hamilton cover.*

Proof. Suppose for the sake of contradiction that there were some latin square L of order n with no Hamilton cover and consider the corresponding proper n -edge coloring χ_L :

$E(K_{n,n}) \rightarrow [n]$. Define the weight of a subgraph $G \subseteq K_{n,n}$ as the number of distinct colors it contains; symbolically $w(G) := |\{\chi_L(e) \mid e \in G\}|$. Let H be a Hamilton cycle with maximum weight, say $w(H) = w_0 \leq n - 1$. Without loss of generality we may assume H misses color n . By relabelling vertices we may assume $H = \{r_1c_1, r_2c_1, r_2c_2, \dots, r_1c_n\}$, so that the two congruence classes modulo H are $F_0 := \{r_i c_i : i \in [n]\}$ and $F_1 := \{r_{i+1} c_i : i \in [n]\}$. We want to find an edge $r_i c_j$ with color n such that $w(H \setminus \{r_i c_i, r_{j+1} c_j\}) = w_0$, as the existence of such an edge implies $\sigma(H; r_i c_i, r_{j+1} c_j)$ has weight $w_0 + 1$, contradicting the maximality of $w(H)$.

For $i \in \{1, 2\}$, let $A_i \subseteq [n - 1]$ denote the set of colors used exactly i times in H . Moreover, let $B \subseteq E(H)$ denote the set of edges whose color is not in A_1 (i.e. edges whose color is used at least twice in H). Notice that, as $|E(H)| = 2n$ and $w(H) \leq n - 1$, there is some $k \in [2, n]$ such that $|A_1| = n - k$ and $|B| = n + k$. Moreover, because $|A_1| + |A_2| \leq w(H)$, we have $|A_2| \leq k - 1$. Given vertex sets $S \subseteq R$ and $T \subseteq C$, we write $E_n(S, T)$ for the set of edges $st \in E(K_{n,n})$ such that $s \in S$, $t \in T$, and $\chi_L(st) = n$. In particular, we are interested in the sets $U_0 := \{r_i : r_i c_i \in F_0 \cap B\}$ and $U_1 := \{c_j : r_{j+1} c_j \in F_1 \cap B\}$. Because χ_L is derived from a latin square, both $E_n(U_0, C)$ and $E_n(R, U_1)$ are matchings which perfectly cover U_i (for $i \in \{0, 1\}$). In particular, we have $|E_n(U_0, C)| = |U_0|$ and $|E_n(R, U_1)| = |U_1|$. We also know $|U_i| = |F_i \cap B|$ and $B = (F_0 \cap B) \cup (F_1 \cap B)$, so $|U_0| + |U_1| = |B| = n + k$. We can therefore use the principle of inclusion-exclusion to conclude that

$$\begin{aligned}
|E_n(U_0, U_1)| &= |E_n(U_0, C) \cap E_n(R, U_1)| \\
&= |E_n(U_0, C)| + |E_n(R, U_1)| - |E_n(U_0, C) \cup E_n(R, U_1)| \\
&\geq |E_n(U_0, C)| + |E_n(R, U_1)| - |E_n(R, C)| \\
&= |U_0| + |U_1| - n \\
&= |B| - n \\
&= k.
\end{aligned}$$

It is left to show that, for some $r_i c_j \in E_n(U_0, U_1)$, we have $w(H \setminus \{r_i c_i, r_{j+1} c_j\}) = w_0$. Because U_0 and U_1 were defined so that $\chi_L(r_i c_i), \chi_L(r_{j+1} c_j) \notin A_1$, this will be the case unless $\chi_L(r_i c_i) = \chi_L(r_{j+1} c_j) \in A_2$. However, as $|A_2| \leq k - 1$, we can run into this issue with at most $k - 1$ of the (at least) k edges in $E_n(U_0, U_1)$. \square

4.2 Hamilton 2-plexes in color-bounded arrays

The idea of combining local switching techniques with the Lopsided Lovász Local Lemma (LLLL) to show the existence of rainbow perfect matchings and Hamilton cycles in bounded edge-colorings of graphs was introduced by Erdős and Spencer [59]. The proof of Theorem 4.1 which we provide in this section gives another application of the LLLL in the study of plexes in generalizations of latin squares.

We will use graph-theoretic language throughout this section. To state the version of the LLLL which we will use, we first need to define the type of negative-dependancy graph we will be using. Our definition, as well as our statement of the lemma, follows [46]. Given a graph G and vertex $v \in V(G)$, we denote by $N_G[v]$ the closed neighborhood of v (i.e. the set containing v and all vertices adjacent to v in G).

Definition 4.7. Given $p \in (0, 1)$ and a set $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ of events in some probability space, a graph $D = (V, E)$ is a p -dependency graph for \mathcal{A} if $V = [k]$ and, for each $i \in [k]$ and each $S \subseteq V \setminus N_D[i]$ satisfying $\Pr(\cap_{j \in S} \overline{A_j}) > 0$, we have

$$\Pr(A_i \mid \cap_{j \in S} \overline{A_j}) \leq p.$$

It is possible to state the LLLL in more generality than we will below. We state only the “symmetric” version because it is sufficient to prove Theorem 4.1. For a proof, as well as the more general statement, see [9, Ch. 5].

Lemma 4.8 (p -Lopsided Lovász Local Lemma). *Given $p \in (0, 1)$, let D be a p -dependency graph for the collection $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$. If D has maximum degree Δ and $4p\Delta \leq 1$, then*

$$\Pr\left(\cap_{j \in [k]} \overline{A_j}\right) > 0.$$

Although the following proof was initially inspired by a paper of Albert, Frieze, and Reed [7] concerning rainbow Hamilton cycles, the current presentation—in particular the use of an auxilliary bipartite graph to establish that our claimed p -dependency graph is indeed a p -dependency graph—follows [46, Section 3]. Although the statement of Theorem 4.1 above concerns k -bounded n -squares, we prove the equivalent statement that there exists a 2-bounded Hamilton cycle in every $\frac{n}{\sqrt{96}}$ -bounded edge-coloring of $K_{n,n}$.

Proof of Theorem 4.1. Let $c = \sqrt{96}$, fix a $\frac{n}{c}$ -bounded proper edge-coloring of $K_{n,n}$, and let \mathcal{T} denote the set of all monochromatic (unordered) edge triples in this coloring. The set \mathcal{T} has a natural tripartition into three sets according to the structure of the corresponding induced subgraphs: let \mathcal{T}_0 denote the set of monochromatic matchings of size 3, \mathcal{T}_1 the set of monochromatic copies of $K_2 \cup P_3$, and \mathcal{T}_2 denote the monochromatic paths of length 3. Now, select a Hamilton cycle H uniformly at random from among the $\frac{n!(n-1)!}{2}$ Hamilton cycles in $K_{n,n}$. For each triple $\{e, f, g\} \in \mathcal{T}$, let A_{efg} denote the event that e, f , and g all belong to H and let $\mathcal{A} = \{A_{efg} : \{e, f, g\} \in \mathcal{T}\}$. Observe that H is 2-bounded if and only if the event $\cap_{A \in \mathcal{A}} \overline{A}$ holds. Thus, $\Pr(\cap_{A \in \mathcal{A}} \overline{A}) > 0$ implies the existence of a 2-bounded Hamilton cycle.

Arbitrarily order $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ and $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ so that the event A_i corresponds to the triple T_i . Let D be the graph with $V(D) = [k]$ and edges defined by $ij \in E(D)$ if and only if some vertex is incident to both an edge in T_i and an edge in T_j .

To see that

$$\Delta = \Delta(D) \leq \frac{3n(2n-1)(n-c)}{2c^2}, \quad (4.1)$$

observe that each edge in $K_{n,n}$ shares at least one vertex with exactly $2n-1$ edges (including itself) and, given $T \in \mathcal{T}$ and any edge $e \in E(K_{n,n})$ which is adjacent to at least one of the edges in T , there are at most $\binom{n/c}{2}$ monochromatic triples containing e .

Let $p = \frac{8}{(n-4)^3 + 6(n-3)^2}$. To establish the desired result it suffices to show that D is a p -dependency graph for \mathcal{A} . Indeed, if D is a p -dependency graph, then $\Pr(\bigcap_{A \in \mathcal{A}} \bar{A}) > 0$ follows from Lemma 4.8 and the fact that

$$4p\Delta \leq \frac{96}{2c^2} \frac{n(2n-1)(n-c)}{(n-4)^3 + 6(n-3)^2} = \frac{2n^3 - (2c+1)n^2 + cn}{2n^3 - 12n^2 + 24n - 20} \leq 1,$$

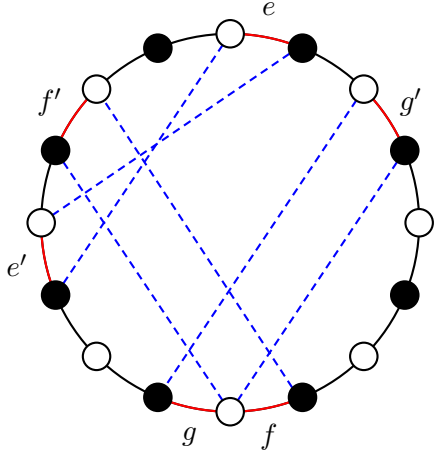
where the last inequality holds for all $n \geq 1$ because $c = \sqrt{96} \approx 9.798$. Towards showing that D is a p -dependency graph, fix an arbitrary $x \in [k]$, let $T_x = \{e, f, g\}$, and fix a set $S \subseteq V(D) \setminus N_D[x]$ satisfying $\Pr(\bigcap_{j \in S} \bar{A}_j) > 0$. We will call a Hamilton cycle S -good if it satisfies $\bigcap_{j \in S} \bar{A}_j$; equivalently, H is an S -good Hamilton cycle if $T_j \not\subseteq H$ for all $j \in S$. Notice that, because S is comprised of non-neighbors of x , any graph obtained from an S -good graph via a series of σ -switches which only introduce edges adjacent to one of e, f , or g must also be S -good.

Let \mathcal{H} denote the collection of S -good Hamilton cycles and, for each $i \in [0, 3]$, let \mathcal{H}_i denote those S -good Hamilton cycles H satisfying $|H \cap T_x| = i$. In particular, \mathcal{H}_3 denotes the set of S -good Hamilton cycles for which A_x holds, so that $\Pr(A_x | \bigcap_{j \in S} \bar{A}_j) = \frac{|\mathcal{H}_3|}{|\mathcal{H}|}$. We would like to show that

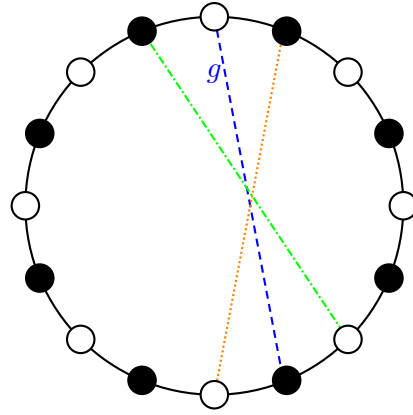
$$\frac{|\mathcal{H}|}{|\mathcal{H}_3|} \geq \frac{|\mathcal{H}_0|}{|\mathcal{H}_3|} + \frac{|\mathcal{H}_1|}{|\mathcal{H}_3|} \geq \frac{(n-4)^3}{8} + \frac{3(n-3)^2}{4} = \frac{1}{p}. \quad (4.2)$$

To establish (4.2) we construct two auxiliary bipartite multigraphs, G_0 and G_1 , with vertices corresponding to Hamilton cycles of $K_{n,n}$. Specifically, for $i \in \{0, 1\}$, the bipartition defining $V(G_i)$ is $(\mathcal{H}_3, \mathcal{H}_i)$. Adjacency in G_0 is defined by adding an edge between $H \in \mathcal{H}_3$ and $H' \in \mathcal{H}_0$ for each ordered triple (e', f', g') such that $e', f', g' \in H$ and $H' = \sigma(\sigma(\sigma(H; e, e'); f, f'); g, g')$. See Figure 4.2(a) for an example of such an iterated switch. Similarly, in G_1 we add an edge between $H \in \mathcal{H}_3$ and $H' \in \mathcal{H}_1$ if one of the following three conditions holds: (1) there exist $e', f' \in H$ such that $H' = \sigma(\sigma(H; e, e'); f, f')$, (2) there exist $e', g' \in H$ such that $H' = \sigma(\sigma(H; e, e'); g, g')$, or (3) there exist $f', g' \in H$ such that $H' = \sigma(\sigma(H; f, f'); g, g')$. We write $\delta_i(\mathcal{H}_3)$ for the minimum degree of a vertex in the part of G_i corresponding to \mathcal{H}_3 and $\Delta_i(\mathcal{H}_i)$ for the maximum degree of a vertex in the part corresponding to \mathcal{H}_i .

By double counting the edges in G_i , for $i \in \{0, 1\}$, we see that $\delta_i(\mathcal{H}_3)|\mathcal{H}_3| \leq \Delta_i(\mathcal{H}_i)|\mathcal{H}_i|$, so $\frac{|\mathcal{H}_i|}{|\mathcal{H}_3|} \geq \frac{\delta_i(\mathcal{H}_3)}{\Delta_i(\mathcal{H}_i)}$. To determine $\delta_0(\mathcal{H}_3)$, consider an arbitrary $H \in \mathcal{H}_3$. In choosing e' we may not select either of the edges adjacent to e , nor may we select f or g . This forbids at



(a) A choice of edges e' , f' , and g' such that, if H is the cycle on black and thick-red edges, then the cycle $\sigma(\sigma(\sigma(H; e, e'); f, f'); g, g')$ on dashed-blue and black edges is S -good.



(b) If $\sigma(\sigma(\sigma(H; e, e'); f, f'); g, g')$ is the cycle on black edges and g is the dashed-blue edge, then g' could have been either the dotted-orange edge or the dash-dotted-green edge.

Figure 4.2: An illustration of the switches used to prove $\delta_0(\mathcal{H}_3) \geq (n-4)^3$ and $\Delta_0(\mathcal{H}_0) \leq 8$.

most 4 possible choices of e' ; any other edge in H which is not congruent to e modulo H is a valid choice. Therefore, we have at least $n-4$ choices for e' .

As we are only looking for a lower bound for $\delta_0(\mathcal{H}_3)$, we may choose to avoid selecting for f' either of the edges in $\sigma(H; e, e')$ adjacent to both e and e' , say e'' and e''' . We do this to ensure that we do not reintroduce e when switching on f and f' . Observe that exactly one of e'' and e''' is not congruent to f modulo $\sigma(H; e, e')$, so that our choice eliminates exactly one option. Furthermore, as above, we may not select as f' either of the edges adjacent to f in $\sigma(H; e, e')$ and, if $g \not\equiv f \pmod{\sigma(H; e, e')}$, we may not select g for f' . These are again the only possible obstructions, giving us at least $n-4$ valid choices for f' .

Finally, in choosing g' , we will not select e'' or e''' , nor may we select the analogously defined f'' and f''' . This eliminates exactly two of the n edges not congruent to g modulo $\sigma(\sigma(H; e, e'); f, f')$. As the only other edges we may not select are those adjacent to g in $\sigma(\sigma(H; e, e'); f, f')$, we have at least $n-4$ choices for g' . This shows that $\delta_0(\mathcal{H}_3) \geq (n-4)^3$. The proof that $\delta_1(\mathcal{H}_3) \geq 3(n-3)^2$ is essentially identical, with the factor of 3 coming from the fact that we have 3 choices for which of e , f , and g will not be involved in a σ -switch.

To determine $\Delta_0(\mathcal{H}_0)$, consider an arbitrary $H' \in \mathcal{H}_0$. Let $g = r_0c_0 \notin H$ and notice that $H' \setminus \{r_0, c_0\}$ has two connected components. Moreover, there are two edges incident to r_0 in H' , say r_0c_1 and r_0c_2 , and two edges incident to c_0 , say r_1c_0 and r_2c_0 , where r_1 and c_1 are in one component of $H' \setminus \{r_0, c_0\}$ and r_2 and c_2 are in the other. If there is some $H \in \mathcal{H}_3$ for which $H' = \sigma(\sigma(\sigma(H; e, e'); f, f'); g, g')$, then we may obtain $\sigma(\sigma(H; e, e'); f, f')$ from H' by switching on either r_1c_0 and r_0c_2 or on r_2c_0 and r_1c_2 (see Figure 4.2(b)). Thus, there are two possible choices for $\sigma(\sigma(H; e, e'); f, f')$. Similarly, for each choice of $\sigma(\sigma(H; e, e'); f, f')$,

there are two possibilities for $\sigma(H; e, e')$, obtained by selecting one of the two edges crossing f in $\sigma(\sigma(H; e, e'); f, f')$ to be f' . Finally, for each choice of $\sigma(H; e, e')$ we get two choices for H in a similar manner. In total, this gives us at most 8 options for $H \in N_{G_0}(H')$, so $\Delta_0(H_0) \leq 8$. A similar argument tells us that $\Delta_1(\mathcal{H}_1) \leq 4$, completing the proof. \square

4.3 Proof of Theorem 4.3

Over the next two sections we will prove Theorem 4.3 and its analogues for Hamilton plexes. These proofs all utilize the rainbow blow-up lemma of Ehard, Glock, and Joos [58]. To state this fairly technical lemma, we first need to introduce some terminology. Throughout the rest of this chapter, given $d, n \in \mathbb{N}$ and $\varepsilon > 0$ we write $(d \pm \varepsilon)n$ for the interval $((d - \varepsilon)n, (d + \varepsilon)n)$. Following [58], we say that a bipartite graph G with bipartition (V_1, V_2) is (γ, d) -super-regular if

- for all $S \subseteq V_1$ and $T \subseteq V_2$ with $|S| \geq \gamma|V_1|$, $|T| \geq \gamma|V_2|$, we have $e(S, T) \in (d \pm \gamma)|S||T|$,
- and for all $i \in [2]$ and $x \in V_i$, we have $\deg(x) \in (d \pm \gamma)|V_{3-i}|$.

Moreover, we refer to $(H, G, (X_i)_{i=1}^\ell, (V_i)_{i=1}^\ell)$ as a (γ, d) -super-regular blow-up instance if

- G and H are graphs, $(X_i)_{i=1}^\ell$ and $(V_i)_{i=1}^\ell$ are partitions of $V(H)$ and $V(G)$, respectively, such that $|X_i| = |V_i|$ and X_i is an independent set of H for all $i \in [\ell]$, and
- for all $ij \in \binom{[\ell]}{2}$, the subgraph of G induced by $E(V_i, V_j)$ is (γ, d) -super-regular.

We call an edge-coloring *locally Λ -bounded* if each vertex is incident to at most Λ edges of each color. Finally, if χ is an edge-coloring of G , we say that $\phi : V(H) \rightarrow V(G)$ is a *rainbow embedding* of H into G if ϕ is injective, $\phi(x)\phi(y) \in E(G)$ for all $xy \in E(H)$, and the multiset $\{\chi(\phi(x)\phi(y)) \mid xy \in E(H)\}$ has no repeated elements. When there is a chance of confusion, we will write $e_G^a(S, T)$ to denote the number of edges in G with color a , one end in S , and the other end in T , and $e_H(S, T)$ to denote the number of edges in H with one end in S and the other end in T . We may now state the rainbow blow-up lemma.

Lemma 4.9 (Ehard et al. [58]). *For all $d, \varepsilon \in (0, 1]$ and $\Delta, \Lambda, \ell \in \mathbb{N}$, there exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Suppose $(H, G, (X_i)_{i=1}^\ell, (V_i)_{i=1}^\ell)$ is a (γ, d) -super-regular blow-up instance. Assume further that*

- (i) *the maximum degree of H satisfies $\Delta(H) \leq \Delta$;*
- (ii) *$|V_i| \in (1 \pm \gamma)n$ for all $i \in [\ell]$;*
- (iii) *$\chi : E(G) \rightarrow [q]$ is a locally Λ -bounded edge-coloring such that the following holds for all $a \in [q]$:*

$$\sum_{ij \in \binom{[\ell]}{2}} e_G^a(V_i, V_j) e_H(X_i, X_j) \leq (1 - \varepsilon)dn^2. \quad (4.3)$$

Then there exists a rainbow embedding ϕ of H into G such that $\phi(x) \in V_i$ for all $i \in [\ell]$ and all $x \in X_i$.

Rather than using the full power of the rainbow blow-up lemma repeatedly below, we will use the following more concise corollary which applies to the proper colorings of dense balanced bipartite graphs we will be considering.

Corollary 4.10. *For all $\varepsilon \in (0, 1]$ and all $\Delta \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. If G and H are balanced bipartite graphs of order $2n$ satisfying $\Delta(H) \leq \Delta$ and $\delta(G) \geq n - n^{3/4}$, then every $(1 - \varepsilon)n^2/e(H)$ -bounded proper edge-coloring of G contains a rainbow copy of H .*

Proof. Let $d = \Lambda = 1$, let $\ell = 2$, and let ε and Δ be as in the hypotheses of this theorem. Plugging these values into Lemma 4.9, we obtain $n_0 \in \mathbb{N}$ and $\gamma > 0$. Then, fix $n \geq \max\{n_0, \gamma^{-4}\}$ and let (R, C) and (X, Y) denote, respectively, the bipartitions of G and H . We claim that $(H, G, (X, Y), (R, C))$ is a $(\gamma, 1)$ -super-regular blow-up instance. By definition, we have $|R| = |C| = |X| = |Y| = n$, so it suffices to show that G is $(\gamma, 1)$ -super-regular. Indeed, because $n \geq \gamma^{-4}$, for all $v \in V(G)$ we have $\deg(v) \in [n - n^{3/4}, n] \subseteq (1 \pm \gamma)n$. Moreover, for any $S \subseteq R$ and $T \subseteq C$ satisfying $|S|, |T| \geq \gamma n$, we have

$$\frac{e(S, T)}{|S||T|} \geq \frac{|S||T| - |S|n^{3/4}}{|S||T|} \geq 1 - n^{-1/4} \geq 1 - \gamma.$$

This proves the claim. To complete the proof, notice that proper colorings are locally 1-bounded and, for our particular blow up instance, (4.3) is equivalent to the statement that G is $(1 - \varepsilon)n^2/e(H)$ -bounded. Therefore, we may apply Lemma 4.9 to find a rainbow copy of H in G . \square

Now, because we are looking for k -bounded subgraphs in $K_{n,n}(L)$, rather than rainbow ones, we need to decompose $K_{n,n}(L)$ into balanced subgraphs before applying Corollary 4.10. We find such a decomposition exists for all sufficiently large latin squares using the following form of the Chernoff bound (for a proof, see [93]). Recall that a random variable X has a binomial distribution $\text{Bin}(n, p)$ for $n \in \mathbb{N}$ and $p \in (0, 1)$ if $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for all $k \in [0, n]$.

Theorem 4.11 (The Chernoff bound). *Let X be a random variable with distribution $\text{Bin}(n, p)$ and write $\mu := \mathbb{E}(X) = np$. For any $\varepsilon > 0$, we have*

$$\Pr(|X - \mu| \geq \varepsilon \mu) \leq 2e^{-\varepsilon^2 \mu / 3}.$$

The following decomposition lemma comes from a standard application of the Chernoff bound, similar to multiple applications appearing in [58].

Lemma 4.12. *For all $\varepsilon > 0$ and all $k \geq 2$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. For every latin square L of order n , there are equipartitions $\{R_i\}_{i=1}^k \vdash R(L)$ and $\{C_i\}_{i=1}^k \vdash C(L)$ such that, for all $(i, j) \in [k]^2$ and all symbols $t \in [n]$, the number of copies of t in the latin subarray $A_{i,j} := L[R_i \times C_j]$ lies in the interval $(1 \pm \varepsilon) \frac{n}{k^2}$.*

Proof. Let $n_1 := \lceil \frac{n}{k} \rceil$, $n_2 := \lfloor \frac{n}{k} \rfloor$, and $\ell := n \pmod{k}$. We want to find $\{R_i\}_{i=1}^k \vdash R(L)$ and $\{C_i\}_{i=1}^k \vdash C(L)$ satisfying $|R_i| = |C_i| = n_1$ for $i \in [\ell]$ and $|R_i| = |C_i| = n_2$ for $i \in [k] \setminus [\ell]$. We do so probabilistically, independently assigning each $r \in R(L)$ and each $c \in C(L)$ a uniformly random value from $[k]$. Then, for each $i \in [k]$, let Q_i be the event $|R_i| = |C_i| = n_j$, where $j = 1$ if $i \in [\ell]$ and otherwise $j = 2$. By iteratively applying the definition of conditional probability, we have $\Pr\left(\bigcap_{i \in [k]} Q_i\right) = \Pr(Q_1) \cdot \prod_{i=2}^k \Pr\left(Q_i \mid \bigcap_{j=1}^{i-1} Q_j\right)$.

Our random process was designed so that the random variables $|R_1|$ and $|C_1|$ both have distribution $\text{Bin}(n, \frac{1}{k})$. Moreover, when $2 \leq i < k$ and we are conditioning on the event $\bigcap_{j=1}^{i-1} Q_j$, the random variables $|R_i|$ and $|C_i|$ both have distribution $\text{Bin}\left(\alpha n, \frac{1}{k-i+1}\right)$ for some $\alpha \in \left(\frac{1}{k}, \frac{k-1}{k}\right)$. Because $\{R_i\}_{i=1}^k$ and $\{C_i\}_{i=1}^k$ are determined independently, we have $\Pr(Q_k \mid \bigcap_{j=1}^{k-1} Q_j) = 1$, and we are assuming n is arbitrarily large while k is fixed, we may apply the DeMoivre-Laplace approximation (see [93, Sec. 9.2]) to obtain

$$\Pr\left(\bigcap_{i \in [k]} Q_i\right) = \Omega(n^{1-k}).$$

Now, for each symbol $t \in [n]$ and each $i, j \in [k]$, let $S_{i,j}^{(t)}$ be a random variable giving the number of occurrences of t in $A_{i,j}$. Fixing arbitrary i, j , and t , we can further decompose $S_{i,j}^{(t)}$ into indicator variables $S_{i,j}^{(t)} = \sum_{r \in R(L)} X_r$, where $X_r = 1$ if the unique entry with symbol t in row r of L appears in $A_{i,j}$. Notice that the X_r are independent with mean $\frac{1}{k^2}$. This means $S_{i,j}^{(t)} \sim \text{Bin}\left(n, \frac{1}{k^2}\right)$, and the Chernoff bound gives

$$\Pr\left(\left|S_{i,j}^{(t)} - \frac{n}{k^2}\right| \geq \frac{\varepsilon n}{k^2}\right) \leq 2e^{-\varepsilon^2 n / 3k^2}.$$

Bringing everything together, we see that our random partitions fail to have the desired properties with probability at most

$$1 - \alpha n^{1-k} + 2nk^2 e^{-\varepsilon^2 n / 3k^2},$$

for some constant α , and this probability is less than 1 for sufficiently large n . \square

Proving Theorem 4.3(a) now amounts to combining Lemma 4.12 with Corollary 4.10 to establish the following, more general result. Recall that, for a square array A , we use $|A|$ to denote the order of A , which is equal to its number of rows.

Theorem 4.13. *For all integers $k \geq 1$ and all $\varepsilon \in (0, 1]$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let L be a latin square of order n , write (R, C) for the vertex bipartition of $K_{n,n}(L)$, and let $\{G_i\}_{i=1}^k$ be a collection of balanced bipartite graphs. If*

(i) $\{V(G_i) \cap R\}_{i=1}^k$ is an equipartition of R ,

(ii) $\Delta(G_i) \leq k$, and

(iii) $e(G_i) \leq (1 - \varepsilon)n$ for all $i \in [k]$,

then L has a partial k -plex isomorphic to $\bigcup_{i=1}^k G_i$.

Proof. Assume n is sufficiently large and let L be a latin square of order n . Denote by $(A_{i,j})_{i,j \in [k]}$ the partition of L into latin subarrays given by Lemma 4.12, so that each symbol appears at most $(1 + \varepsilon) \frac{n}{k^2}$ times in each $A_{i,j}$. Fixing an arbitrary $i \in [k]$, set $\ell = |A_{i,i}|$. Because $\ell \geq \lfloor n/k \rfloor$ and we may assume n is large enough for $\frac{n^2}{(n-k)^2} \leq \frac{1-\varepsilon^2/2}{1-\varepsilon^2}$ to hold, we have

$$\frac{(1 - \varepsilon^2/2) \ell^2}{e(G_i)} \geq \frac{(1 - \varepsilon^2/2) \ell^2}{(1 - \varepsilon)n} \geq \frac{(1 - \varepsilon^2/2)(n - k)^2}{(1 - \varepsilon)nk^2} \geq \frac{(1 + \varepsilon)n}{k^2}.$$

We may therefore apply Corollary 4.10 to find a rainbow copy of G_i in $K_{\ell,\ell}(A_{i,i})$. It follows that $\bigcup_{i=1}^k G_i$ is a k -bounded subgraph of $K_{n,n}(L)$ with maximum degree at most k , and as such its edges correspond to the entries of the desired partial k -plex. \square

A partial Hamilton 2-plex is a 2-bounded union of paths in $K_{n,n}(L)$. By letting $k = 2$ and both G_1 and G_2 be the union of a path of order $(1 - \varepsilon)n$ and however many isolated vertices are needed to satisfy condition (i) in Theorem 4.13, we obtain the following approximation of Conjecture 1.5.

Corollary 4.14. *Every latin square has a partial Hamilton 2-plex of size $2n - o(n)$.*

We can complete the proof of Theorem 4.3 with an argument very similar to the one used to prove Theorem 4.13.

Proof of Theorem 4.3(b). Let L be a latin square of sufficiently large order n , fix $\varepsilon = \frac{1}{2k+2}$, and partition L into the latin subarrays $(A_{i,j})_{i,j \in [k+1]}$ via Lemma 4.12, so that each symbol appears at most $(1 + \varepsilon) \frac{n}{(k+1)^2}$ times in each $A_{i,j}$. Fixing $i \in [k+1]$, let $\ell = |A_{i,i}|$ and let G_i be a k -regular spanning subgraph of $K_{\ell,\ell}$. Because we may assume $n \geq (k+1)(2k+1)$, we have

$$\begin{aligned} \frac{(1 - \varepsilon^2) \ell^2}{e(G_i)} &= \frac{(1 - \varepsilon^2) \ell}{k} \geq \frac{(1 - \varepsilon^2)(n - k - 1)}{k(k+1)} \\ &= \frac{(1 + \varepsilon)(2k+1)(n - k - 1)}{2k(k+1)^2} \geq \frac{(1 + \varepsilon)n}{(k+1)^2}. \end{aligned}$$

It then follows from Corollary 4.10 that there is a rainbow copy of G_i in $K_{\ell,\ell}(A_{i,i})$. The subgraph $\bigcup_{i=1}^{k+1} G_i \subseteq K_{n,n}(L)$ is the desired $(k, k+1)$ -plex. \square

4.4 Further explorations of (k, ℓ) -plexes

For the rest of this chapter we restrict our attention to (k, ℓ) -plexes, starting with the Hamilton analogue of the $k = 2$ case of Theorem 4.3(b). Unfortunately, this proof is nowhere near as direct as the Hamilton extension of the $k = 2$ case of Theorem 4.3(a). We will utilize a theorem of Alon, Pokrovskiy, and Sudakov which shows that the graph formed by the edges from a random selection of color classes in a proper edge-coloring of $K_{n,n}$ has strong expansion properties. In order to give a restatement of Theorem 5.1 from [8] we must provide two additional definitions: an event occurs “with high probability” if the probability of its occurring is $1 - o(1)$ and, for functions $f(n)$ and $g(n)$, the symbol $f \gg g$ means $\lim_{n \rightarrow \infty} f/g = \infty$.

Theorem 4.15 (Alon, Pokrovskiy, and Sudakov [8]). *Given a proper edge-coloring of $K_{n,n}$ with bipartition classes R and C , let G be a subgraph obtained by choosing every color class randomly and independently with probability p . Then, with high probability, all vertices in G have degree $(1 - o(1))np$ and for every two subsets $A \subseteq R, B \subseteq C$ of size $x \gg (\log n/p)^2$, $e_G(A, B) \geq (1 - o(1))px^2$.*

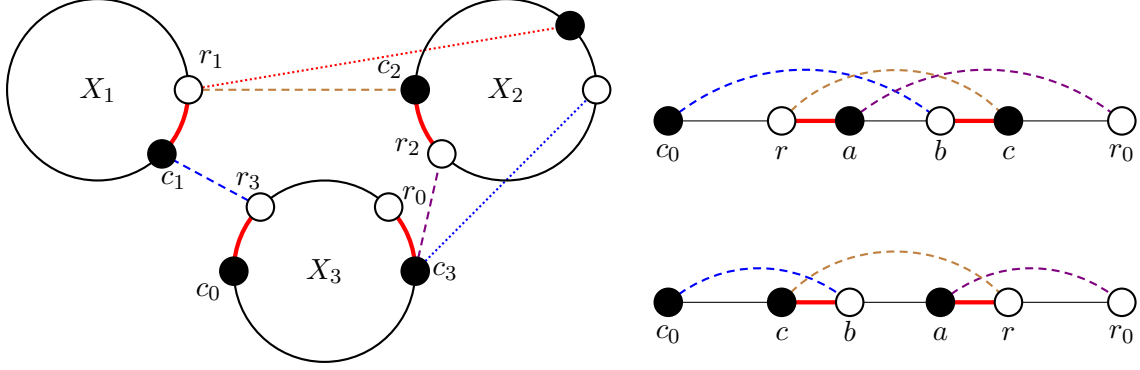
By setting $p = n^{-1/3}$, setting $x = \lceil \frac{n}{10} \rceil$, and using the fact that $\frac{(1-\varepsilon)n^{5/3}}{100} \geq n^{3/2}$ for any $\varepsilon \in (0, 1)$ and all sufficiently large n , we obtain the following corollary tailored to our particular application. Note that, in the case where $|A|$ or $|B|$ is strictly greater than $\lceil \frac{n}{10} \rceil$, we can apply Theorem 4.15 to arbitrary subsets $A' \subseteq A$ and $B' \subseteq B$ satisfying $|A'| = |B'| = \lceil \frac{n}{10} \rceil$ to get the desired result. In what follows, we write $e_T(A, B)$ for the set of edges with a color in the set T , one end in A , and the other end in B .

Corollary 4.16. *For every latin square L of sufficiently large order n , there is a set $T \subseteq S(L)$ of size $\lfloor n^{2/3} \rfloor$ such that, for every pair of sets $A \subseteq R(L)$ and $B \subseteq C(L)$ satisfying $|A|, |B| \geq \frac{n}{10}$, we have $e_T(A, B) \geq n^{3/2}$ in $K_{n,n}(L)$.*

Now, combining Corollary 4.16 with the full power of Corollary 4.10—rather than just the special case where the host graph is $K_{n,n}$ —we prove this chapter’s third main result.

Theorem 4.17. *Every sufficiently large latin square has a Hamilton $(2, 3)$ -plex.*

Proof. Let L be a latin square of sufficiently large order n , let $\varepsilon > 0$ be sufficiently small, and use Lemma 4.12 to find a partition $(A_{i,j})_{i,j \in [3]}$ of L such that each symbol in $S(L)$ appears between $(1 - \varepsilon)\frac{n}{9}$ and $(1 + \varepsilon)\frac{n}{9}$ times in each $A_{i,j}$. Next, pick a set $T \subseteq S(L)$ of size $\lfloor n^{2/3} \rfloor$ using Corollary 4.16, so that the subgraph induced by the edges with colors in T has good expansion properties. Fix $i \in [3]$, let $\ell = |A_{i,i}|$, and consider the subgraph $G_i \subseteq K_{\ell, \ell}(A_{i,i})$ obtained by removing all edges with colors in T . Because $\ell \leq \lceil n/3 \rceil$ and n is sufficiently large, we have $\delta(G_i) \geq \ell - \lfloor n^{2/3} \rfloor \geq \ell - \ell^{3/4}$. Moreover, as we may assume



(a) P_0 is the graph on black and thick-red edges. Because r_1 has (at least) two neighbors in W , drawn here as the two black vertices on X_2 , we can find c_2 so that the Hamilton path P_1 on black and dashed edges is 3-bounded.

(b) Both cases of the transformation from P_1 , the graph on the black and thick-red edges, to H , the graph on the black and dashed edges. On bottom is case (1)— $\text{dist}_{P_1}(c_0, b) < m'$ for all $b \in B$ —and on top is case (2).

Figure 4.3: An illustration of the two steps of switches used to tie the three cycles in P_0 into a single Hamilton (2,3)-plex.

$n \geq 3$ and $1 - \varepsilon \geq \frac{4}{27}$, we have

$$\frac{(1 - \varepsilon^2) n^2}{2\ell} \geq \frac{(1 - \varepsilon^2) 3n^2}{2(n + 3)} \geq \frac{(1 - \varepsilon^2) 3n}{4} \geq \frac{(1 + \varepsilon)n}{9}.$$

We may therefore apply Corollary 4.10 to find a rainbow Hamilton cycle $X_i \subseteq G_i$. Then, taking the union of the three cycles just obtained, say $P_0 := \bigcup_{i=1}^3 X_i$, we have a spanning set of 2-regular rainbow subgraphs, i.e. a (2,3)-plex of $K_{n,n}(L)$. To obtain a Hamilton (2,3)-plex we will tie these three cycles together into a single cycle.

Our first step in this direction is to claim that $K_{n,n}(L)$ has a 3-bounded Hamilton path. Indeed, let F denote the set of colors used fewer than three times in P_0 . Because $e(P_0) = 2n$, we have $|F| \geq \lceil n/3 \rceil$. Having defined $(A_{i,j})_{i,j \in [3]}$ via the equipartitions $\{R_i\}_{i=1}^3 \vdash R(L)$ and $\{C_i\}_{i=1}^3 \vdash C(L)$ given by Lemma 4.12, we know that at least $\frac{(1-\varepsilon)n}{9}$ copies of each symbol in F appear in $A_{3,1}$. In graph-theoretic terms, this means $e_F(R_3, C_1) \geq \frac{(1-\varepsilon)n|F|}{9}$. But $|R_3| = \lfloor n/3 \rfloor$, so by averaging there must be some $r_3 \in R_3$ such that $e_F(\{r_3\}, C_1) \geq \frac{(1-\varepsilon)n}{9}$. Similarly, there exists $c_3 \in C_3$ such that $e_F(R_2, \{c_3\}) \geq \frac{(1-\varepsilon)n}{9}$.

Let U denote the set of vertices $r_1 \in R_1$ such that $K_{n,n}(L)$ has a 2-edge path r_1, c_1, r_3 satisfying $r_1c_1 \in X_1$ and $\chi_L(r_3c_1) \in F$. Similarly, let W denote the set of vertices $c_2 \in C_2$ such that $K_{n,n}(L)$ has a 2-edge path c_2, r_2, c_3 satisfying $r_2c_2 \in X_2$ and $\chi_L(r_2c_3) \in F$. By the definitions of r_3 and c_3 and the fact that n is sufficiently large, we have $|U|, |W| \in [\frac{n}{10}, \frac{n}{3} + 1)$ and can apply Corollary 4.16 to see that $e_T(U, W) \geq n^{3/2}$. It follows that there must be some $r_1 \in U$ such that $e_T(\{r_1\}, W) > 1$. This allows us to find vertices $c_2 \in W$, $c_1 \in C_1$, and $r_2 \in R_2$ such that $r_i c_i \in X_i$ for $i \in [2]$, $\chi_L(r_1c_2) \in T$, $\chi_L(r_3c_1), \chi_L(r_2c_3) \in F$, and $\chi_L(r_3c_1) \neq \chi_L(r_2c_3)$. As illustrated in Figure 4.3(a), this is enough to prove the claim, as

there exist $r_0 \in R_3$ and $c_0 \in C_3$ such that

$$P_1 := (P_0 \setminus \{r_0c_3, r_3c_0, r_1c_1, r_2c_2\}) \cup \{r_3c_1, r_2c_3, r_1c_2\}$$

is a 3-bounded Hamilton path. Notice that the endpoints of P_1 are r_0 and c_0 .

Now, let $F' = F \setminus \{\chi_L(r_3c_1), \chi_L(r_2c_3)\}$ and let m denote the median value in the set $D := \{\text{dist}_{P_1}(r_0, c) \mid c \in N_{F'}(r_0)\}$. Observe that $\text{dist}_{P_1}(c_0, v) = 2n - 1 - \text{dist}_{P_1}(r_0, v)$ for all $v \in V(K_{n,n}(L))$, and let $m' := 2n - 1 - m$. By the pigeonhole principle and the fact that $|F'| \geq \frac{n}{3} - 2$, there is a set $B \subseteq N_{F'}(c_0)$ of size $\lceil \frac{n}{7} \rceil$ such that either (1) $\text{dist}_{P_1}(c_0, b) < m'$ for all $b \in B$ or (2) $\text{dist}_{P_1}(c_0, b) > m'$ for all $b \in B$. In case (1), let $A = \{c \in N_{F'}(r_0) \mid \text{dist}_{P_1}(r_0, c) < m\}$ and observe that $|A| \geq \frac{n}{7}$. Then, let C_A denote the set of vertices in C which are adjacent on the r_0 side to a vertex in A (i.e. $c \in C_A$ if $ca \in P_1$ for some $a \in A$ and c lies on the subpath of P_1 between r_0 and a) and let R_B denote the set of vertices in R which are adjacent on the c_0 side to a vertex in B . Because $|C_A|, |R_B| \geq \frac{n}{7}$, we may again apply Corollary 4.16 to see that $e_T(C_A, R_B) \geq n^{3/2}$, and therefore there is some $c \in C_A$ with $e_T(\{c\}, R_B) > 1$. This means there exists $a \in A$, $b \in B$, and $r \in R_B$ such that $ra, bc \in P_1$, $\chi_L(r_0a), \chi_L(bc_0) \in F'$, $\chi_L(r_0a) \neq \chi_L(bc_0)$, and $\chi_L(rc) \in T$. The Hamilton cycle

$$H := (P_1 \setminus \{ra, bc\}) \cup \{r_0a, bc_0, rc\},$$

shown in Figure 4.3(b), uses at most two edges with colors in T . As the other two edges added in passing to H from (the 3-bounded graph) P_1 were carefully chosen to not introduce more than 3 copies of any color in $S(L) \setminus T$, we may conclude that H is the desired (2,3)-plex.

In case (2), where $\text{dist}_{P_1}(c_0, b) > m'$ for all $b \in B$, we define A to be the set $\{c \in N_{F'}(r_0) \mid \text{dist}_{P_1}(r_0, c) > m\}$, let C_A denote the set of vertices in C which are adjacent on the c_0 side to a vertex in A , and let R_B denote the set of vertices in R which are adjacent on the r_0 side to a vertex in B . Using a similar argument to case (1), we can find $c \in C_A$, $a \in A$, $b \in B$, and $r \in R_B$ such that $ra, bc \in P_1$, $\chi_L(r_0a), \chi_L(bc_0) \in F'$, $\chi_L(r_0a) \neq \chi_L(bc_0)$, and $\chi_L(rc) \in T$, then conclude that the Hamilton cycle $H := (P_1 \setminus \{ra, bc\}) \cup \{r_0a, bc_0, rc\}$ is the desired Hamilton (2,3)-plex. \square

Because the proof of Theorem 4.17 relies upon the very technical proof of Corollary 4.10, it is not clear exactly how large n must be before we can be certain that every latin square of order n has a Hamilton (2,3)-plex. Of course, if Conjecture 1.5 is true, then every latin square would have a Hamilton (2,3)-plex. While we do not yet have a proof of this weaker version of Rodney's conjecture (Conjecture 1.3), we can show that all latin squares have a Hamilton (2,5)-plex.

The essential idea behind this proof, as well as the following general existence result for (k, ℓ) -plexes, is to manipulate (k, n) -plexes (i.e. selections of entries in intersecting each row and each column exactly k times) to remove edges with colors appearing too many times.

Given a (k, n) -plex $P \subseteq L$, let $A_i(P)$ denote the set of colors which appear i times in P and for all $\ell > k$ let the ℓ -deficit of P be the value

$$d_\ell(P) := \sum_{i=\ell+1}^n |A_i(P)|.$$

Both this notion of deficit and our use of it is essentially a generalization of the ideas used by Cameron and Wanless in [36] to show that every latin square has a $(1, 2)$ -plex.

Theorem 4.18. *Every latin square has a Hamilton $(2, 5)$ -plex.*

Proof. Let L be a latin square of order n and, supposing for the sake of contradiction that L has no Hamilton $(2, 5)$ -plex, let H_0 denote the Hamilton cycle which minimizes $d_5(H)$ among all Hamilton cycles $H \subseteq K_{n,n}(L)$. Let $a_i := |A_i(H_0)|$ for all $i \in [0, n]$, let $B = \bigcup_{i=6}^n A_i(H_0)$, and fix an edge $r_1 c_1 \in H_0$ such that $\chi_L(r_1 c_1) \in B$. Then, relabel the vertices of $K_{n,n}(L)$ so that H_0 is given by the vertex sequence $c_1, r_1, c_2, r_2, \dots, c_n, r_n, c_1$. Consider the index sets

$$U := \left\{ j \in [3, n] \mid \chi_L(r_1 c_j) \in \bigcup_{i=0}^3 A_i(H_0) \right\} \text{ and}$$

$$W := \left\{ j \in [3, n] \mid \chi_L(r_{j-1} c_1) \in \bigcup_{i=0}^4 A_i(H_0) \right\}.$$

If $U \cap W \neq \emptyset$, then there exists $r_{j-1} c_j \in H_0$ such that—recalling Definition 4.5 and Figure 4.1—the Hamilton cycle $H_1 := \sigma(P_0; r_1 c_1, r_{j-1} c_j)$ satisfies $d_5(H_1) < d_5(H_0)$, contradicting the minimality of $d_5(H_0)$.

Because we have assumed $\chi_L(r_1 c_1)$ appears at least 6 times in H_0 , we have

$$|U| \geq n - 2 - \left(-1 + \sum_{i=4}^n a_i \right) = n - 1 - \sum_{i=4}^n a_i$$

and, similarly, $|W| \geq n - 1 - \sum_{i=5}^n a_i$. We can therefore apply the inclusion-exclusion principle to see that

$$|U \cap W| \geq n - 2 - a_4 - 2 \sum_{i=5}^n a_i. \quad (4.4)$$

Counting colors in H_0 , we see that (i) $\sum_{i=1}^n i a_i = 2n$ and (ii) $\sum_{i=0}^n a_i = n$. Plugging the first of these sums into (4.4) gives $|U \cap W| \geq \frac{1}{2} a_1 + a_2 + \frac{3}{2} a_3 + a_4 + \sum_{i=5}^n \left(\frac{i-4}{2} \right) a_i - 2$. Then, using the sum (ii), we see that it suffices to show

$$\frac{1}{2}(n - a_0) + \frac{1}{2} a_2 + a_3 + \frac{1}{2} a_4 + \sum_{i=6}^n \left(\frac{i-5}{2} \right) a_i > 2. \quad (4.5)$$

We may assume $a_i > 0$ for some $i \geq 6$, as otherwise H_0 is a $(2, 5)$ -plex and we are done. This means that (4.5) holds so long as $n - a_0 \geq 4$. Moreover, because each color appears

at most n times in $K_{n,n}(L)$, we must have $n - a_0 > 1$. This leaves two cases to consider. In the case $a_0 = n - 3$, we have $\frac{1}{2}(n - a_0) = \frac{3}{2}$ and may assume $a_i = 0$ for all $i \geq 7$, as otherwise (4.5) must hold. But we may assume $n \geq 9$ by Proposition 1.6, so the only way we can cover all $2n$ edges of H_0 using 3 colors fewer than 7 times each is if $n = 9$ and $a_6 = 3$, in which case (4.5) holds. The case $a_0 = n - 2$ is even more straightforward: if we use 2 colors to cover the $2n$ edges in H_0 , then $a_n = 2$ and the assumption that $n \geq 9$ implies $\left(\frac{n-5}{2}\right) a_n \geq 4$. \square

We end this section by showing that every latin square has a $(k, 4k)$ -plex for all $k \leq n$. As was the case with Theorem 4.18, it seems likely that this result is far from best possible, especially in light of Theorem 4.3(b) and Pula's work on disjoint weak transversals in [105]. Although the argument we use is very similar to the one just presented, we now use the language of latin squares, rather than of edge-colored graphs. In particular, we discuss entries in latin squares as ordered triples, and follow [24] in using \bullet for exactly one of the three indices in a triple, a convention justified by the fact that every entry in a latin square is uniquely defined by any pair of its indices.

Theorem 4.19. *Every latin square of order n has a $(k, 4k)$ -plex for all positive integers $k \leq n$.*

Proof. Notice that the result holds trivially if $k \geq \frac{n}{4}$. Let L be a latin square of order n and suppose for the sake of contradiction that there is some $k < \frac{n}{4}$ such that L does not have a $(k, 4k)$ -plex. Let P_0 be the (k, n) -plex which minimizes $d_{4k}(P)$ among all (k, n) -plexes $P \subseteq L$, let $a_i := |A_i(P_0)|$ for all $i \in [0, n]$, and for all $s \in S(L)$ let $\nu(s) = j$ whenever $s \in A_j(P_0)$. By permuting rows, columns, and symbols if necessary, we may assume that the main diagonal of L is in P_0 , that $L_{1,1} = 1$, and that $\nu(1) > 4k$.

Consider the sets

$$U := \left\{ i \in [n] \mid (1, i, \bullet) \notin P_0, \nu(L_{1,i}) < 4k - 1 \right\} \text{ and}$$

$$W := \left\{ i \in [n] \mid (i, 1, \bullet) \notin P_0, \nu(L_{i,1}) < 4k \right\}.$$

If $U \cap W \neq \emptyset$, then for any $i \in U \cap W$ the (k, n) -plex

$$P_1 := (P_0 \setminus \{(1, 1, 1), (i, i, \bullet)\}) \cup \{(1, i, \bullet), (i, 1, \bullet)\}$$

satisfies $d_{4k}(P_1) < d_{4k}(P_0)$, contradicting the minimality assumption in the definition of P_0 and thereby establishing the desired result.

Now, we know $|U| \geq n - k + 1 - \sum_{j=4k-1}^n a_j$ and $|W| \geq n - k + 1 - \sum_{j=4k}^n a_j$ by definition. Therefore, by the principle of inclusion-exclusion, we have

$$|U \cap W| \geq n - 2k + 2 - 2 \sum_{j=4k-1}^n a_j. \tag{4.6}$$

Moreover, because P_0 is comprised of kn entries, we have $kn = \sum_{j=1}^n ja_j$. From this we see that $(4k-1) \sum_{j=4k-1}^n a_j \leq kn$, and therefore

$$\sum_{j=4k-1}^n a_j \leq \frac{kn}{4k-1} = n \left(\frac{k-1}{4k-1} + \frac{1}{4k-1} \right) \leq n \left(\frac{1}{4} + \frac{1}{4k-1} \right). \quad (4.7)$$

Plugging (4.7) into (4.6) and simplifying, we see that $|U \cap W| > 0$ so long as we have $n \left(\frac{1}{2} - \frac{2}{4k-1} \right) > 2k-2$. To confirm that this is indeed the case, recall that the results holds trivially unless $n \geq 4k+1$. Moreover, we may assume $k \geq 3$ by Theorem 4.18 and Cameron and Wanless' proof in [36] that every latin square has a (1,2)-plex. Thus, to complete the proof, we simply note that

$$n \geq 4k+1 > 4k + \frac{4}{7} \geq 4k + \frac{4}{4k-5} = \frac{(2k-2)(8k-2)}{4k-5}. \quad \square$$

Chapter 5

Concluding remarks

With all three of our motivating conjectures still wide open, the general theme of the future work suggested by this thesis is clear. With respect to 2-plexes, we have in some sense raised more questions than answers. Indeed, while Wanless and Vaughan-Lee showed in [112] that, modulo the now completed proof of Theorem 1.7, every group-based latin square has a 2-plex, our study of H2-harmonious groups in Chapter 3 has introduced a new open question: are all nonabelian groups of even order H2-harmonious? It is clear from the conjecture with which we began Chapter 3 that we believe this question has an affirmative answer.

Conjecture 5.1. *Every finite group except $\mathbb{Z}_2 \times \mathbb{Z}_2$ is H2-harmonious.*

We showed in Chapter 3 that many families of groups are H2-harmonious, including all abelian groups, all groups of odd order, all groups with a cyclic subgroup of index 2, and all extensions of odd-order groups by H2-harmonious groups. As was mentioned in Section 3.3, we believe that the study of H2-harmonious groups would be best served by answers to the following pair of questions:

- Are all solvable groups besides $\mathbb{Z}_2 \times \mathbb{Z}_2$ H2-harmonious?
- Are all symmetric and alternating groups H2-harmonious?

With respect to the second of these questions, it is worth noting that we have found H2-harmonious sequences for S_3 , S_4 , A_4 , and A_5 (see Appendix B). However, we have not yet found any pattern among these sequences which points towards a general construction. Perhaps the reader will be able to find one?

More generally, there are many interesting questions one could ask about Hamilton $(2, \ell)$ -plexes. In Section 4.4, we showed that all sufficiently large latin squares have a $(2, 3)$ -plex. While we were able to prove that all latin squares have a Hamilton $(2, 5)$ -plex, a reasonable step towards a proof that all latin squares have a Hamilton 2-plex would be to:

- Show that all latin squares have a Hamilton $(2, \ell)$ -plex for some $\ell < 5$.

While it is in some sense preferable to have theorems that hold for every latin square, there is reason to believe that considering large latin squares could be very fruitful. Indeed, several recent papers [8, 15, 71, 70] have used probabilistic methods to make substantial progress in the study of rainbow Hamilton cycles in sufficiently large edge-colored complete graphs. Recall from Section 4.3 that a partial Hamilton 2-plex is a 2-bounded union of paths in $K_{n,n}(L)$. Could we adapt their methods to prove the following weaker version(s) of Conjecture 1.5?

Conjecture 5.2. *Every latin square has a partial Hamilton 2-plex of size $2n - O(\log n)$. Moreover, almost every latin square has a Hamilton 2-plex.*

It would also be very interesting to determine if all equi- n squares have Hamilton 2-plexes and, if not, what is the largest value of c such that every cn -bounded n -square has a Hamilton $(2,2)$ -plex. Theorem 4.1 shows that this value is greater than 0.102. And what about k -plexes in equi- n squares for $k > 2$? Wanless [115] showed that there exist infinitely many latin squares of even order with no odd plexes, but the strongest version of Rodney's conjecture would imply that every latin square of odd order n has a k -plex for every $k \in [n]$. This raises the question: is there an infinite family of odd order equi- n squares with no 3-plex? We suspect that there is.

Most generally, it would be interesting to see the theory of (k, ℓ) -plexes further developed. An interesting avenue we have not explored is whether all latin squares can be decomposed into (k, ℓ) -plexes for various values of k and ℓ . One could also give a definition of indivisible (k, ℓ) -plexes and ask related existence questions. However, there is much to explore even with respect to the sort of straightforward existence questions considered above. Theorem 4.3(b), which shows that all sufficiently large latin squares have a $(k, k+1)$ -plex for $k = O(1)$, indicates that, for some fixed constant c , it may be possible to find $(k, k+c)$ -plexes in all latin squares, even in the case where k is odd. We end with a question whose answer would indicate the degree to which (k, ℓ) -plexes behave differently than k -plexes.

Question 5.3. *Does every latin square of order $4n$ have a $(2n-1, 2n)$ -plex?*

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Appendix A

Code

This backtracking algorithm, written for the computer algebra system Sage [111], was used to prove Propositions 1.6 and 3.15.

```
#####  
# The technical part  
# Input: S, a list of lists giving a latin square; Path, a  
# list giving a 2-bounded path in  $K_{n,n}(S)$ ; CC, an array  
# documenting how many times each color has been used in Path;  
# aR and aC, sets giving the rows and columns, respectively,  
# not used by Path; n, the order of S; k, the length of Path  
# Output: if Path can be extended by 1, (True,P) where P  
# is the extension. Otherwise, (False,Path)  
#####  
  
def Ham2PlexUtil(S,Path,CC,aR,aC,n,k):  
  
#####  
# Base case: if Path has length  $2n-1$ , check that the edge  
# connecting its end to  $S[0][0]$  has the correct symbol  
#####  
  
    if k==2*n-1:  
        r = Path[k-1][0]  
        s = S[r][0]  
        if CC[s]==1:  
            Path.append((r,0,s))  
            return True,Path  
        else:  
            return False,Path
```

```
#####
# For paths of even length, randomly select a row candidate
# for the next step in the path, then attempt to extend
#####
```

```
if k%2==0:
    c = Path[k-1][1]
    aR2 = Permutations(list(aR)).random_element()
    for r in aR2:
        s = S[r][c]
        if CC[s]<2:
            Path.append((r,c,S[r][c]))
            CC[s]+=1
            aR.remove(r)
            if Ham2PlexUtil(S,Path,CC,aR,aC,n,k+1)[0]:
                return True,Path
            Path.pop()
            CC[s]-=1
            aR.add(r)
    return False,Path
```

```
#####
# For paths of odd length, randomly select a column candidate
# for the next step in the path, then attempt to extend
#####
```

```
else:
    r = Path[k-1][0]
    aC2 = Permutations(list(aC)).random_element()
    for c in aC2:
        s = S[r][c]
        if CC[s]<2:
            Path.append((r,c,S[r][c]))
            CC[s]+=1
            aC.remove(c)
            if Ham2PlexUtil(S,Path,CC,aR,aC,n,k+1)[0]:
                return True,Path
            Path.pop()
            CC[s]-=1
            aC.add(c)
    return False,Path
```

```
#####
# The top part
# Input: S, a list of lists giving a latin square
# Output: A Hamilton 2-plex if one exists, otherwise False
#####
```

```

def Ham2Plex(S):
    n = len(S)
    CC = [0 for i in range(n)]
    CC[S[0][0]]=1
    aR = set(range(1,n))
    aC = set(range(1,n))
    H = Ham2PlexUtil(S,[(0,0,S[0][0])],CC,aR,aC,n,1)
    if H[0]:
        return H[1]
    else:
        return False

```

Appendix B

Proof of Proposition 3.15

All abelian groups and all groups of odd order were shown to be H2-harmonious by Theorems 3.3 and 3.6, respectively. Moreover, we noted in Corollary 3.10 that all dihedral, semidihedral, and dicyclic groups are H2-harmonious. Proving Proposition 3.15 therefore amounts to showing that all groups not considered by these three results are H2-harmonious. It turns out that most of small groups can be taken care of using Lemma 3.8 and Theorem 3.9.

As noted in Section 3.3, the sequencings presented below were found by running the algorithm in Appendix A using Sage [111] and the list of groups we needed to consider comes from The GAP Small Groups Library [20]. We will identify groups using their GAP ID, before giving the more common name (when applicable). For many of the groups we will also give a presentation so as to identify the letters used in our exposition.

Groups of order 12

- [12, 3], the alternating group A_4 :

$$S := (), (), (12)(34), (234), (134), (12)(34), (14)(23), (243), (234), \\ (14)(23), (124), (124), (132), (142), (123), (143), (143), \\ (123), (243), (13)(24), (142), (134), (13)(24), (132)$$

$$\pi(S) = (), (12)(34), (124), (13)(24), (123), (13)(24), (142), (), (124), \\ (234), (142), (243), (14)(23), (143), (14)(23), (134), (12)(34), \\ (143), (123), (134), (132), (243), (234), (132)$$

Groups of order 16

- [16, 3], $\langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, bc = cb, cac = ba^{-1} \rangle$:

$$S := 1, 1, bc, b, a^2b, a^3b, a^2bc, ac, a^2c, a^2, b, a^3bc, c, a^2bc, a^3c, bc, a^3bc, \\ a, a^3b, a^2c, ab, a^3, a, a^3c, abc, ab, ac, a^2b, a^3, c, a^2, abc$$

$$\pi(S) = 1, bc, c, a^2, a, ac, a^3b, a^3, c, a^2b, a^3c, a^3b, a^2b, a^3, a, ab, bc, b, a^3c, ac, \\ b, 1, a^2bc, a^2, a^2c, a^2bc, a^3bc, ab, abc, a^2c, a^3bc, abc$$

- [16, 4], $\langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$:

$$S := 1, 1, ab, a^2b, a^2b^2, b^3, a^2, ab^2, b^2, a^3b^3, a, ab^3, a^2b, b, ab^2, a^3b^2, a^2b^3, \\ ab, b, a^2, a^3b^3, a^2b^3, a^3, a^3, a^3b^2, a^2b^2, ab^3, a^3b, a^3b, a, b^3, b^2$$

$$\pi(S) = 1, ab, ab^2, b^3, a^2b, a^2b^3, a^3b^2, a, a^3b, a^2b^3, b^3, a^3, a^2b^2, a^3b^3, 1, ab, a^3, \\ ab^2, a^2b, ab^3, a^3b^2, a^3b^3, a^2, a^2b^2, a, a^3b, a^2, b^2, b, ab^3, b, b^2$$

- [16, 6], $\langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle$:

H2-harmonious by Theorem 3.9 as $\langle a \rangle$ is a cyclic subgroup of index 2

- [16, 11], $D_8 \times \mathbb{Z}_2$, $\langle a, b, c \mid a^4 = b^2 = c^2 = 1, bab = a^{-1}, ca = ac, bc = cb \rangle$:

$$S := 1, 1, a^2b, ab, a, abc, a^3bc, a^2bc, a^3b, a^3b, bc, ac, abc, a^2b, b, c, a^2, \\ bc, a^3, a^3c, a^2bc, a^3, ac, b, a^2c, a^2, ab, a^2c, a^3c, a^3bc, c, a$$

$$\pi(S) = 1, a^2b, a^3, b, bc, a^2, a^3, a^3c, 1, a^3c, ab, a^2b, ac, a^2, bc, a^2c, a^2bc, \\ abc, a^2c, ab, abc, c, a^3bc, a^2bc, c, a^3b, a^3bc, a, b, a^3b, ac, a$$

- [16, 12], $Q \times \mathbb{Z}_2$, $\langle a, b, c \mid a^4 = c^2 = 1, a^2 = b^2, bab^{-1} = a^{-1}, ac = ca, bc = ca \rangle$:

$$S := 1, 1, ab, bc, a^2, a^2bc, a^2c, a^2c, a, ac, b, a^3bc, ac, a^3b, a^2bc, c, abc, \\ a^3c, a^3c, a^2b, a^3, a^3, bc, abc, a^3b, b, c, a^2, a^3bc, ab, a^2b, a$$

$$\pi(S) = 1, ab, ac, a^2bc, bc, b, 1, a^3c, a^2c, abc, ac, a^2b, a^2bc, a^3c, a^2b, ab, \\ b, a^2, a^3bc, a^3b, a^2, a^3bc, a^3, c, a^3, bc, a^2c, abc, c, a, a^3b, a$$

- [16, 13], $\langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, abc = a^2b \rangle$:

$$S := 1, 1, b, a^3c, a^2c, a, a^2, a^2, a, bc, a^3b, a^3b, bc, b, a^3c, a^3bc, a^3, a^3, \\ a^2b, c, ab, ac, abc, ab, ac, a^2bc, c, abc, a^2bc, a^2b, a^3bc, a^2c$$

$$\pi(S) = 1, b, abc, a, a^3c, a^3, 1, a^3, abc, ac, a^2, a^3c, c, a^3bc, a^2b, a^2bc, a^2, \\ ab, bc, a^3bc, bc, b, a^2c, a^2bc, a^3b, a^2b, ab, a, c, ac, a^3b, a^2c$$

Groups of order 18

- [18, 3], $S_3 \times \mathbb{Z}_3$:

H2-harmonious by Corollary 3.10 and Lemma 3.8

- [18, 4], $\langle a, b, c \mid a^3 = b^3 = c^2 = 1, ab = ba, ac = ca^{-1}, bc = cb^{-1} \rangle$:

H2-harmonious by Proposition 3.5 and Lemma 3.8
as $\langle a, b \rangle \triangleleft G$, $|\langle a, b \rangle| = 9$, and $G/\langle a, b \rangle \cong \mathbb{Z}_2$

Groups of order 20

- [20, 3], $\langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$:

H2-harmonious by Proposition 3.5 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 5$, and $G/\langle a \rangle \cong \mathbb{Z}_4$

Groups of order 24

- [24, 1], $\langle a, b \mid a^8 = b^3 = 1, aba^{-1} = b^{-1} \rangle$:

H2-harmonious by Proposition 3.5 and Lemma 3.8
as $\langle b \rangle \triangleleft G$, $|\langle b \rangle| = 3$, and $G/\langle b \rangle \cong \mathbb{Z}_8$

- [24, 3], the special linear group $SL_2(3)$:

$$S := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

$$\pi(S) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$$

- [24, 5], $S_3 \times \mathbb{Z}_4 = S_3 \times \langle a \mid a^4 = 1 \rangle$:

H2-harmonious by Theorem 3.9 as $\langle\langle(123), a\rangle\rangle$ is a cyclic subgroup of index 2

- [24, 7], $\langle a, b, c \mid a^6 = c^2 = 1, a^3 = b^2, b^{-1}ab = a^{-1}, ac = ca, bc = cb \rangle$:

H2-harmonious by Theorem 3.3 and Lemma 3.8
as $\langle a^2 \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

- [24, 8], $\langle a, b, c \mid a^3 = b^4 = c^2 = 1, ac = ca, a^{-1}ba = ab, cbc = b^{-1} \rangle$:

H2-harmonious by Corollary 3.10 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle \cong D_4$

- [24, 10], $D_4 \times C_3 = D_4 \times \langle a \mid a^3 = 1 \rangle$:

H2-harmonious by Corollary 3.10 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle = D_4$

- [24, 11], $Q \times C_3 = \text{Dic}_2 \times \langle a \mid a^3 = 1 \rangle$:

H2-harmonious by Corollary 3.10 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle \cong \text{Dic}_2$

- [24, 12], the symmetric group S_4 :

$S := ()$, (1) , (1243) , (243) , (13) , (132) , (1342) , (24) , (134) , (1342) , $(14)(23)$, (124) , (24) ,
 $(12)(34)$, (134) , (1423) , (13) , $(13)(24)$, $(14)(23)$, (243) , (23) , (123) , (234) , (234) ,
 (1234) , (143) , (1423) , (12) , (1324) , (1234) , (34) , (1432) , (1432) , $(12)(34)$,
 $(13)(24)$, (132) , (14) , (1324) , (123) , (142) , (1243) , (34) , (143)

$\pi(S) = ()$, (1243) , (1234) , (1324) , (23) , (1423) , (134) , (1342) , (1423) , (1243) , (132) , (14) ,
 (1432) , (143) , $()$, (24) , (124) , (132) , (1234) , (23) , (234) , (24) , $(12)(34)$, (134) , (34) ,
 (12) , $(12)(34)$, (243) , (1324) , (12) , (1342) , $(14)(23)$, $(13)(24)$, (142) , (123) , (142) ,
 $(13)(24)$, (13) , $(14)(23)$, (124) , (1432) , (243) , (14) , (234) , (34) , (123) , (13) , (143)

- [24, 13], $A_4 \times \mathbb{Z}_2$, $\langle a, b, c \mid a^3 = b^2 = (ab)^3 = c^2 = 1, ac = ca, bc = cb \rangle$:

$S := 1$, 1 , ac , abc , b , bc , a^2c , aba , bc , a^2b , aba^2 , aba^2c , a^2bc , ac , a , $abac$, aba^2c , ba^2c ,
 bab , a^2bc , a^2ba , ba^2 , bac , a^2bac , abc , a^2 , $abac$, a^2c , aba , c , c , bab , a^2b ,
 a^2ba , ba^2c , bac , ba , aba^2 , ab , b , bab , ab , a^2bac , bab , a^2 , ba , ba^2 , a

$\pi(S) = 1$, ac , aba , ac , c , ba^2 , abc , ba^2c , a^2c , aba , c , ba^2 , b , a^2c , aba^2c , a^2b , a^2 , a^2bac ,
 bc , ba^2c , a^2b , aba^2c , ab , ba , bc , bac , ba , bac , $abac$, 1 , bab , aba^2 , a^2 , $abac$,
 a^2ba , a^2bc , bab , a , bab , abc , a^2bc , bab , ab , a^2bac , b , a^2ba , aba^2 , a

- [24, 14], $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle \times \langle c, d \mid c^2 = d^2 = 1, cd = dc \rangle$:

H2-harmonious by Theorem 3.3 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle \cong \mathbb{Z}_2^3$

Groups of order 30

- [30, 1], $D_5 \times C_3 = D_5 \times \langle a \mid a^3 = 1 \rangle$:

H2-harmonious by Corollary 3.10 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 3$, and $G/\langle a \rangle = D_5$

- [30, 2], $D_3 \times C_5 = D_3 \times \langle a \mid a^5 = 1 \rangle$:

H2-harmonious by Corollary 3.10 and Lemma 3.8
as $\langle a \rangle \triangleleft G$, $|\langle a \rangle| = 5$, and $G/\langle a \rangle = D_3$

The alternating group A_5

Rather than list out the corresponding permutations, which would occupy a significant amount of space, we provide a more concise description: the list of triples, corresponding to entries in L_{A_5} , obtained from running the commands:

```
G = AlternatingGroup(5)
L = G.cayley_table().table()
Ham2Plex(L)
```

```
[(0, 0, 0), (0, 11, 11), (33, 11, 39), (33, 29, 49), (22, 29, 14), (22, 9, 55), (54, 9, 27), (54, 52, 44),
(18, 52, 14), (18, 34, 49), (24, 34, 17), (24, 57, 11), (26, 57, 9), (26, 10, 25), (19, 10, 52),
(19, 40, 22), (14, 40, 18), (14, 54, 35), (8, 54, 53), (8, 21, 16), (57, 21, 43), (57, 19, 48),
(48, 19, 53), (48, 1, 24), (1, 1, 2), (1, 55, 56), (20, 55, 30), (20, 46, 54), (55, 46, 2), (55, 56, 19),
(32, 56, 36), (32, 2, 56), (53, 2, 40), (53, 7, 20), (34, 7, 28), (34, 42, 0), (42, 42, 50), (42, 47, 4),
(51, 47, 23), (51, 23, 34), (15, 23, 58), (15, 31, 40), (13, 31, 44), (13, 28, 15), (2, 28, 29),
(2, 26, 25), (37, 26, 17), (37, 6, 24), (7, 6, 3), (7, 30, 27), (9, 30, 35), (9, 32, 28), (41, 32, 6),
(41, 13, 26), (52, 13, 50), (52, 3, 43), (31, 3, 22), (31, 48, 15), (17, 48, 1), (17, 12, 4), (21, 12, 6),
(21, 41, 16), (29, 41, 52), (29, 50, 37), (44, 50, 19), (44, 39, 26), (59, 39, 54), (59, 27, 45),
(47, 27, 57), (47, 58, 3), (4, 58, 51), (4, 59, 55), (39, 59, 20), (39, 14, 36), (3, 14, 21), (3, 38, 45),
(36, 38, 30), (36, 22, 41), (38, 22, 39), (38, 43, 8), (50, 43, 57), (50, 18, 29), (16, 18, 38),
(16, 45, 48), (10, 45, 42), (10, 8, 5), (12, 8, 38), (12, 37, 7), (25, 37, 31), (25, 44, 58),
(45, 44, 13), (45, 4, 21), (23, 4, 34), (23, 53, 12), (56, 53, 7), (56, 20, 10), (40, 20, 32), (40, 49, 37),
(43, 49, 42), (43, 16, 59), (30, 16, 46), (30, 51, 8), (49, 51, 33), (49, 36, 31), (58, 36, 33),
(58, 15, 59), (46, 15, 47), (46, 25, 51), (35, 25, 41), (35, 24, 23), (28, 24, 13), (28, 35, 46),
(27, 35, 47), (27, 33, 10), (6, 33, 32), (6, 17, 12), (11, 17, 18), (11, 5, 1), (5, 5, 9), (5, 0, 5)]
```

This data is also available at

<https://sites.google.com/view/kevinhalasz/combinatorial-data>