On the Recognition and Characterization of M-partitionable Proper Interval Graphs

by

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Abstract

For a symmetric $\{0, 1, \star\}$-matrix M of size m, a graph G is said to be M-partitionable, if its vertices can be partitioned into sets $V_1, V_2, \ldots, V_m$, such that two parts V_i, V_j are completely adjacent if $M_{i,j} = 1$, and completely non-adjacent if $M_{i,j} = 0$ (V_i is considered completely adjacent to itself if it induces a clique, and completely non-adjacent if it induces an independent set). The complexity problem (or the recognition problem) for a matrix M asks whether the M-partition problem is polynomial-time solvable or NP-complete. The characterization problem for a matrix M asks if all M-partitionable graphs can be characterized by the absence of a finite set of forbidden induced subgraphs. These forbidden induced subgraphs are called obstructions to M.

In the literature, many results were obtained by restricting the input graphs. In this thesis, we survey these results when the questions are restricted to the class of perfect graphs. We then study the recognition problem and the characterization problem when the inputs are restricted to proper interval graphs. The recognition problem can be solved by an existing algorithm, but we simplify its proof of correctness. As our main result, we prove that all the matrices of size 3 and size 4 with constant diagonal, have finitely many minimal proper interval obstructions. We also obtain partial results about matrices of arbitrary size if they have a zero diagonal.

Keywords: matrix partitions, graph partitions, homomorphism, interval graphs, proper interval graphs, minimal obstructions, perfect graphs

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Chapter 1

Introduction

The vertex coloring problem is one of the most extensively studied problems in graph theory and has wide applications over various fields. In the vertex coloring problem, the vertices of a graph G have to be colored in such a way that no two adjacent vertices have the same color. Alternatively, the vertex coloring problem can also be defined as finding a k-partition $V_1, V_2, \ldots, V_k$ of the vertices of G, such that each V_i is an independent set, so the k-coloring problem can be viewed as seeking such a partition.

When $k = 1$, finding whether a graph G is k-colorable or not is a trivial problem. When $k = 2$, the k-coloring problem is equivalent to checking if the graph is bipartite. A graph G is bipartite if and only if G does not have any odd cycle. Let us look at the algorithm for recognizing a bipartite graph. We apply the breadth-first search algorithm on its vertices, and if there is an edge between two vertices of the same layer in the BFS tree, this implies that G has an odd cycle and hence not a bipartite graph. If there are no edges between the vertices of the same layer, then we can color all the vertices of the even numbered layers with one color and the vertices of the odd numbered layer with another. Therefore, the 2-coloring problem can be solved in linear time and can be characterized by the absence of cycles of odd length. Notice that even though a bipartite graph is characterized by the absence of infinitely many subgraphs, it can be recognized in linear time.

For $k \geq 3$, the k-coloring problem is NP-complete, but sometimes k-coloring can be solved in polynomial time when there are restrictions on the input graph. For example, the 4-coloring problem is NP-complete for a general graph, but if the input graph is planar, then it is 4-colorable. Similarly, if we restrict the input graphs to the class of perfect graphs which will be defined in later sections, the k-coloring problem is polynomial time solvable and can be characterised by the absence of one induced subgraph.

A *biclique or a complete bipartite graph* is a bipartite graph with a condition that every vertex of the first independent set must be adjacent with all the vertices of second independent set. Alternatively, the *biclique recognition problem* asks whether or not the vertices of a graph can be partitioned into two sets V_1 and V_2 such that V_1, V_2 are stable sets and are completely adjacent to each other.

The *split graph recognition problem* asks whether or not the vertices of a graph can be partitioned into two sets X, Y, such that X is a clique, and Y is a stable or an independent set. A graph G is called a split graph if and only if G does not have a C_4, C_5 or $2K_2$ as induced subgraphs [17]. Here C_4, C_5 are cycles of length 4 and 5 respectively, and a $2K_2$ is a graph on 4 vertices with two disjoint edges. Such a characterization allows us to see that a split graph can be recognized in polynomial time, in fact split graphs can be recognised in linear time [17]. An (a, b)-graph is a generalization of split graph which can be partitioned into a-stable sets and b-independent sets. Recognising the (a, b)-graphs, when $(a, b) \leq 2$ can be solved in polynomial time [28], and if $(a, b) \geq 3$, then the recognition problem is NP-complete [2, 4].

All the problems that we have seen can be visualized as finding partitions with some constraints, such as some parts must induce cliques and some independent sets. Then there may also be constraints on the adjacencies between the parts, such as two parts being completely adjacent or non-adjacent. Feder et al. in their paper "List Partitions" [10], formulated the partition problem in which the constraints of a pattern are visualized using a matrix. A partition $V_1, V_2, \ldots, V_k$ can be represented by a symmetric matrix M of size k, such that the adjacencies within a set V_i is defined by the entry $M_{(i,i)}$ and adjacency between any two sets V_i, V_j is defined by the entry $M_{(i,j)}$. If V_i is a stable set then $M_{(i,i)} = 0$, if V_i is a clique then $M_{(i,i)} = 1$, and if there are no restrictions on V_i then $M_{(i,i)} = \star$. Similarly, if the sets V_i, V_j are completely adjacent then $M_{(i,j)} = 1$, and if the sets V_i, V_j are completely non-adjacent then $M_{(i,j)} = 0$, and if there is no restriction on adjacency between V_i, V_j then $M_{(i,j)} = \star$.

Refer to Figure 1.1, for examples of matrices that correspond to various partition problems that are discussed above. For the bipartite recognition problem and the 3-coloring problem, we need a partition of size two and three respectively, such that each part is independent. So the corresponding matrix has 0 on the main diagonal and $\star's$ on the off-diagonal. In the biclique recognition problem, we need a partition of size two, such that each part is independent and these two parts are completely adjacent. So, the corresponding matrix has 0's on the main diagonal and 1's on the off-diagonal. Similarly, in the split graph recognition problem, we require a partition into a clique and an independent set. Hence, the corresponding matrix has one 0 and one 1 on the main diagonal and $\star's$ on the off-diagonal.

Formally, we define the M-partitionability of a graph G as follows.

Definition 1.0.1. *Let M be a $\{\star, 0, 1\}$-symmetric matrix of size m. A graph G is said to be M-partitionable, if its vertices can be partitioned into sets $V_1, V_2, \ldots, V_m$, such that V_i is a stable set if $M_{i,i} = 0$, and a clique if $M_{i,i} = 1$ and V_i, V_j must be completely adjacent if $M_{i,j} = 1$ and completely non-adjacent if $M_{i,j} = 0$.*

For a given $\{0, 1, \star\}$-symmetric matrix M, we are interested in finding

$$\begin{bmatrix} 0 & \star \\ \star & 0 \end{bmatrix}$$
(a) Bipartite graphs

$$\begin{bmatrix} 0 & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{bmatrix}$$
(b) 3-coloring

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
(c) Biclique

$$\begin{bmatrix} 0 & \star \\ \star & 1 \end{bmatrix}$$
(d) Split graphs

Figure 1.1: Matrices corresponding to the partition problems

1. if we can decide whether a graph G is M-partitionable or not in polynomial time.

2. if M-partitionable graphs can be characterized by the absence of a finite set of forbidden induced subgraphs.

Definition 1.0.2. *A graph G is said to be a* minimal obstruction *to M, if G is not M-partitionable but every proper subgraph of G is M-partitionable.*

For a given matrix M, the number of minimal obstructions are finite if and only if any M-partitionable graph G can be characterized by the absence of a finite set of forbidden induced subgraphs. We have seen that any split graph can be characterized by the absence of induced C_4, C_5 and $2K_2$. Hence, the matrix in Figure 1.1(d), has finitely many minimal obstructions. For any graph G, we can check if there is an induced subgraph of size q in $\mathcal{O}(n^q)$ time. So, if a matrix M has finitely many minimal obstructions, then we can check if any graph is M-partitionable or not in polynomial time. However, the converse is not true. For example, bipartite graphs have infinitely many minimal obstructions but can be recognized in polynomial time.

If there is a $\star$ on the main diagonal of a matrix M, then any graph G is M-partitionable, because all the vertices can be placed in that part. So, most of the time, we consider those matrices that have only 0 or 1 on the main diagonal. Lets now look at some more graph theoretic problems that can be viewed as M-partition problems but with some side conditions.

A *cutset* of a connected graph G is subset of vertices C, such that $G - C$ is a disconnected graph. A *clique cutset* of a connected graph G is a complete subgraph C such $G - C$ is disconnected. The *clique cutset problem* can be viewed as partitioning the vertices of a connected graph into three non-empty parts V_1, V_2, V_3, such that V_1 and V_3 have no edges between them and V_2 is a clique. The matrix M that corresponds to the clique cutset problem can be found in Figure 1.2. Note that if a graph G has a clique cutset then it is M-partitionable, but the converse is not necessarily true, because the M-partition can have one or more parts that are empty. So, we include a condition that in the M-partition we must have at least one vertex in each part. Such a partition problem is called the surjective M-partition problem which requires that all parts must be non-empty. Finding a clique cutset of a graph can be solved in polynomial time [38, 41]. Analogously, a *stable cutset* is defined as a stable subgraph C that disconnects the graph. Although the stable cutset problem seems similar to the clique cutset problem, it is NP-complete [39]. This shows that the matrix partition problem may behave very differently for two matrices, even if they differ by just one entry.

A *homogeneous set* X of a graph G is defined as a subset of vertices, such that any vertex from $G - X$ is either adjacent to all the vertices of X or non-adjacent to all the vertices of X. Finding a homogeneous set in a graph can be formulated as finding a partition V_1, V_2, V_3 of its vertices, such that V_1, V_2 are completely adjacent and V_2, V_3 are completely non-adjacent. Note that for V_2 to be homogeneous set, V_2 must have two vertices and V_1, V_3 must have at least one vertex each. Finding a homogeneous set helps in finding solutions for several combinatorial problems on various graph classes, one such example is in the recognition of comparability graphs [32].

$$\begin{bmatrix} \star & 1 & \star \\ 1 & \star & 0 \\ \star & 0 & \star \end{bmatrix} \qquad \begin{bmatrix} \star & \star & 0 \\ \star & 0 & \star \\ 0 & \star & \star \end{bmatrix} \qquad \begin{bmatrix} \star & \star & 0 \\ \star & 1 & \star \\ 0 & \star & \star \end{bmatrix}$$

(a) Homogeneous Set (b) Stable Cutset (c) Clique Cutset

Figure 1.2: Matrices representing the corresponding partitions with some side conditions.

To capture the condition that all the sets of the partition to be non-empty, we can consider the surjective version of M-partition problem. Even more complex situations can be captured by introducing lists. In the list M-partition problem, the input graph G has lists $L(v)$, for every $v \in V(G)$, such that v can only be placed in a part from $L(v)$. If the version of the problem is not specified, then it is considered as the basic version. Notice that in the surjective version or in the list version of the M-partition problems, having a $\star$ on the main diagonal is not a trivial case.

1.1 Scope of the Thesis

In this thesis, we study the recognition problem and the characterization problem when the inputs are restricted to proper interval graphs. The set of proper interval graphs is a subclass of interval graphs. The complexity problem for the list M-partition problem when restricted to interval graphs was studied in [40]. The authors have shown that the list M-partition problem when restricted to various graph classes like interval graphs, circular arc graphs, and permutation graphs is polynomial time solvable. In Section 2.4.4, we discuss few of the techniques from [40], and present a simplified proof for the complexity result when restricted to proper interval graphs.

The M-partition problem restricted to chordal graphs was studied in [33]. The authors have obtained a complete characterization of matrices of size ≤ 3 with finitely many minimal chordal obstructions. We discuss their results in Section 2.4.2. In particular, the authors have shown that there are two matrices of size 3 that have infinitely many minimal chordal obstructions. In Chapter 3, we prove that the above two matrices have only finitely many minimal proper interval obstructions, thus completing the solution of the characterization problem for matrices of size 3. In Sections 4.1 and 4.2, we extend the study to matrices of size 4 and prove our characterization results for matrices with constant diagonal. In Section 4.3, we discuss some results for matrices of arbitrary size.

In Section 2.1, we give the preliminaries of various graph classes and in Section 2.4, we survey the results of the M-partition problems when restricted to these graph classes. In Section 2.2, we survey graph homomorphisms and its known results. We also discuss how the M-partition problem generalizes the graph homomorphism problem. In Section 2.3, we discuss some of the known results of the M-partition problem.

Chapter 2

Background and Literature Survey

2.1 Preliminaries

A *digraph* G consists a set of vertices V and a binary relation $E \subseteq V \times V$. In other words, E is a set of ordered pairs (u, v); each ordered pair is called an *arc*. An *undirected graph* or a *graph* is a digraph whose binary relation is symmetric, i.e., $(u, v) \in E$ if and only if $(v, u) \in E$. For an undirected graph G, if $(u, v) \in E$, we say that uv is an *edge* and consequently represent it as $uv \in E$ or $vu \in E$. The arc (u, u) is called a *loop* and an undirected graph without loops is known as a *simple graph*. Two different vertices u, v of a simple graph G are called *adjacent* if there exists an edge $uv \in E$. The vertices that are adjacent to a vertex u are called its *neighbours*, and the set of neighbours of u is denoted by $N(u)$. In a graph G, the set of vertices and edges are denoted by $V(G)$ and $E(G)$ respectively. The *complement* of a graph G is the graph $\bar{G}$ with $V(G) = V(\bar{G})$ such that any two vertices u, v are adjacent in $\bar{G}$ if and only if they are non-adjacent in G.

A *subgraph* H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is called an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $E(H)$ is the set of all edges that have endpoints in $V(H)$. If S is a subset of vertices of a graph G, then the *subgraph of G induced by S* is the induced subgraph H of G with $V(H) = S$, and is denoted by $G[S]$. A *proper induced subgraph* of G is a subgraph induced by a proper subset of vertices.

A *complete graph* or a *clique* is defined as a graph G such that any two vertices of G are adjacent. A clique on m vertices is denoted by K_m. A *maximal clique* H of a graph G is a subgraph of G, such that H is a clique and $H \cup \{v\}$, for any $v \in G - H$, is not a clique. A *maximum clique* of a graph G is a subgraph that is a clique and has the largest size possible. The size of a maximum clique in a graph is known as the *clique number* and is denoted by $\omega(G)$.

An *independent set* or a *stable set* is defined as a set of vertices in which no two vertices are adjacent to each other. A *maximum independent* set of a graph G is the largest possible subset of vertices that are independent. The size of a maximum independent set of a graph

G is called the *independence number*, and is denoted by $\alpha(G)$. Observe that the complement of an independent set is a clique and vice versa, therefore we have $\omega(G) = \alpha(\bar{G})$.

A *path* is a graph on n vertices $\{v_1, v_2, v_3, \dots, v_n\}$ such that $E = \{v_i v_{i+1}$ for $1 \leq i \leq n-1\}$. Length of a path P is defined as the number of edges in the path. A path of length ℓ (path on $\ell + 1$ vertices) is denoted by P_ℓ. A graph G is called *connected* if there exists a path between any two vertices of G.

A *cycle* is a graph C on n vertices $\{v_1, v_2, v_3, \dots, v_n\}$ such that $E(C) = \{v_i v_{i+1}$ for $1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. The length of a cycle C is defined as the number of vertices in the cycle. A cycle of length n is denoted by C_n. Any connected graph G that does not contain a cycle as a subgraph is called a *tree*.

2.1.1 Perfect Graphs

Recall from Chapter 1 that a k-coloring of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices get the same color. The minimum number of colours required to properly color a graph is called the *chromatic number* of that graph and is denoted by $\chi(G)$.

Since each vertex of a clique must be colored with a different color, it is easy to see that $\omega(G) \leq \chi(G)$. We are interested in those graphs that have $\omega(G) = \chi(G)$.

An undirected graph G is called a *perfect graph*, if every induced subgraph H of G satisfies the following condition

$$\omega(H) = \chi(H)$$

Perfect graphs were introduced by Berge in 1961, and Berge posed two conjectures. The first conjecture states that a graph is perfect if and only if its complement is perfect. This conjecture was proved by Lovász in 1972 and is known by the Perfect Graph Theorem.

Theorem 2.1.1 (Perfect Graph Theorem [31])**.** *A simple undirected graph G is perfect if and only if the complement $\bar{G}$ is also perfect.*

A *hole* is defined as a cycle without any chords. An *antihole* is defined as the complement of a hole. A cycle with odd number of vertices and its complement is called an *odd hole* and *odd antihole* respectively. Chromatic number of an odd hole is 3, but the clique number is only 2. Hence, any graph that contains an odd hole is not a perfect graph. An odd antihole on $2k + 1$ vertices has clique number k and chromatic number $k + 1$, and hence an odd antihole is not perfect. The second conjecture posed by Berge states that a graph is perfect if and only if it does not contain any induced odd holes or odd antiholes. Chudnovsky et al. proved this conjecture in 2003 and is known by the Strong Perfect Graph Theorem.

Theorem 2.1.2 (Strong Perfect Graph Theorem [5])**.** *A graph G is perfect if and only if it does not contain any induced odd hole or induced odd antihole.*

Many NP-complete decision problems restricted to perfect graphs are polynomial time solvable, for instance, $k-$coloring, finding a maximum independent set, and clique number are polynomial time solvable when the input graph is perfect [20]. Lets look at a few graph classes that are perfect and are of interest in this thesis.

2.1.2 Chordal Graphs

An undirected graph G is called a *chordal graph* if G does not contain any induced cycle of length ≥ 4. If a graph G is chordal then every induced subgraph of G is also chordal, because if G does not contain any induced cycle of length ≥ 4 then no induced subgraph of G will contain cycles of length ≥ 4. A vertex v of a graph G is called a *simplicial vertex* if the set of its neighbours $N(v)$ induces a clique in G. A *perfect elimination ordering* is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for any $1 \leq i \leq n$, v_i is simplicial in the graph induced by the vertices $v_i, v_{i+1}, \ldots, v_n$ of the ordering. In a connected graph G, a subset of vertices S of $V(G)$ is called a *vertex separator* for non-adjacent vertices a, b if the removal of S from G disconnects a, b. The set S is called a *minimal vertex separator* if it is a vertex separator for some non-adjacent vertices but no proper subset of S is a separator for any pair of non-adjacent vertices. The following theorem shows a few characterizations of a chordal graph.

Theorem 2.1.3 ([19])**.** *For an undirected graph G, the following statements are equivalent.*

1. *G is a chordal graph.*
2. *G admits a perfect elimination ordering.*
3. *Every minimal vertex separator induces a clique in G.*

We can now prove that chordal graphs are perfect.

Theorem 2.1.4 ([19])**.** *All chordal graphs are perfect.*

Proof. Let G be any connected chordal graph, to prove that G is a perfect graph we show that for every induced subgraph H of G, we have $\omega(H) = \chi(H)$. If G is a clique, then the theorem clearly holds as complete graphs are perfect. So we assume that G is not a clique. Any subgraph of G induced by a single vertex is perfect. Assume that every chordal graph that is smaller than G is also perfect, we now prove that G is perfect. Let S be a minimal vertex separator of G, and $H_1, H_2, H_3, \ldots, H_k$ be the components of $G - S$. Note that the existence of a minimal separator follows from the fact that G is not a clique and it implies that $k > 1$. From Theorem 2.1.3, we know that S is a clique and since each H_i is disconnected from one another, we have the following

$$\omega(G) = \max_{1 \leq i \leq k} \omega(G[S + H_i]) \qquad \text{and,}$$

$$\chi(G) = \max_{1 \leq i \leq k} \chi(G[S + H_i])$$

From the induction hypothesis, each $G[S+H_i]$ is perfect, since $k > 1$ implies that it has fewer vertices than G. Hence, we have $\max_{1\leq i\leq k} \omega(G[S+H_i]) = \max_{1\leq i\leq k} \chi(G[S+H_i])$. Therefore, G is a perfect graph. □

Chordal graphs can be recognised in linear time [35, 21]. Given a graph, we first apply the lexicographic breadth-first search algorithm given by Rose et al. [35], then check if the ordering given by this algorithm is a perfect elimination ordering which can also be done in linear time.

Chordal graph on n vertices has at most n maximal cliques [18]. All the maximal cliques can be found using perfect elimination ordering. For every vertex v, find a clique containing v and the vertices that occur after v in the ordering, then check if the clique is maximal. Once all the maximal cliques are found, the maximum clique can be obtained from that. Hence, finding a maximum clique is polynomial time solvable for chordal graphs. In the later section, we will see that listing all maximal cliques of a chordal graph is used for recognizing interval graphs.

2.1.3 Split Graphs

An undirected graph G is called a *split graph* if $V(G)$ can be partitioned into two sets (C, I), such that the vertices of C induce a clique and the vertices of I induce an independent set in G. It is easy to observe that a graph G is a split graph if and only if $\bar{G}$ is a split graph, because cliques and independent sets are complements of each other.

Theorem 2.1.5. *For an undirected graph G, the following statements are equivalent.*

1. *G is a split graph.*
2. *G does not contain $2K_2, C_4,$ and C_5 as induced subgraphs.*
3. *G and $\bar{G}$ are chordal graphs.*

From this characterization, we conclude that all split graphs are chordal graphs. Because the absence of an induced $2K_2$ implies that there are no induced cycles of length greater than 6, and it does not have C_4, C_5 as induced subgraphs. Therefore, split graphs are chordal and hence perfect.

Another characterization of split graphs is that a graph G is a split graph if and only if both G and $\bar{G}$ are chordal. If G is a split graph then G and $\bar{G}$ are trivially chordal. Now assume that G and $\bar{G}$ are chordal, in order to prove that G is a split graph we only have to show that G does not contain a $2K_2$. If G contains an induced $2K_2$, then $\bar{G}$ will have an induced C_4 which is a contradiction.

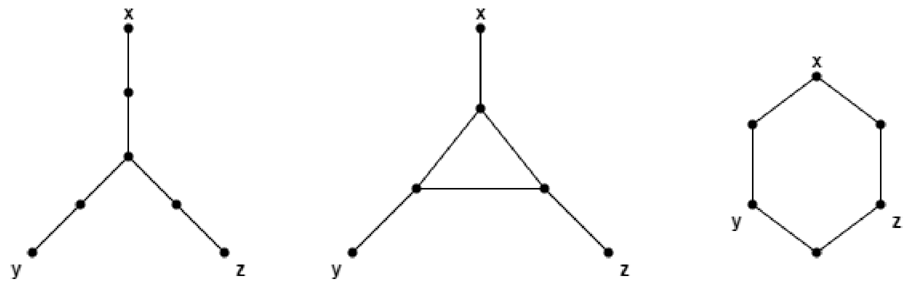


Figure 2.1: Examples of asteroidal triples

2.1.4 Interval Graphs

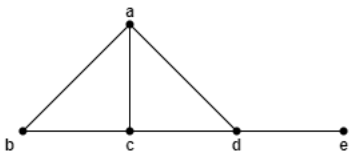
Let $\mathcal{F}$ be a family of sets. The intersection graph of $\mathcal{F}$ is obtained by assigning a vertex for each set of $\mathcal{F}$ and connecting two vertices by an edge if and only if the corresponding sets intersect. An *interval graph* is an intersection graph of a family of intervals on a real line. In other words, a graph G is called an *interval graph*, if its vertices can be associated by a closed intervals on real line, so that two vertices u, v are adjacent if and only if their corresponding intervals intersect. We will usually denote the interval associated with a vertex u by I_u. For an interval I_u, we denote by $l(u)$ and $r(u)$ the left and right endpoints of the interval.

A subgraph of an interval graph is also an interval graph, and it is easy to see that the interval representation of the vertices of a graph can also be used for the interval representation of its subgraphs.

Theorem 2.1.6. *Each interval graph is a chordal graph.*

Proof. To prove that every interval graph is chordal, we show that for $k \geq 4$, C_k is not an interval graph. Consider that a C_k on vertices $v_1, v_2, v_3 \ldots, v_k$ has an interval representation. Let I_{v_1} be the interval with the least right endpoint in the representation. The intervals I_{v_2} and I_{v_k} must not intersect, but both the intervals must intersect I_{v_1}. This cannot happen unless one of them has the least right endpoint, which is a contradiction. Therefore, any graph with an induced C_k is not an interval graph, and hence any interval graph does not contain induced C_k, for $k \geq 4$. Thus all interval graphs are chordal graphs. □

An *asteroidal triple* is a set of three independent vertices of a graph such that there exists a path connecting any two of these vertices which does not contain neighbours of the third one. In Figure 2.1 we give three examples of asteroidal triple x, y, z. A graph without an asteroidal triple is called an AT-free graph. If a graph contains an asteroidal triple, then it cannot have an interval representation. This can be easily proved by considering an asteroidal triple with independent vertices x, y, z and their interval representations I_x, I_y, I_z. Without loss of generality, assume that I_x, I_y, I_z is the ordering of intervals on the real line from left to right. The intervals of the vertices in the path connecting x, z must lie in between I_x, I_z, this implies that there will be an interval intersecting I_y which is a contradiction that



$$\begin{array}{ccccc} a & b & c & d & e \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

Figure 2.2: (a) Example of an interval graph and its corresponding clique matrix that has the property of consecutive 1's for the columns. (b) Clique matrix corresponding to the hexagon graph which does not satisfy the property of consecutive 1's for the columns.

it is an asteroidal triple. Lekkeikerker and Boland have also proved that any chordal graph that is AT-free has an interval representation. Hence, we have the following characterization for interval graphs.

Theorem 2.1.7 ([29])**.** *An undirected graph G is an interval graph if and only if it is chordal and AT-free.*

Another characterization of interval graphs was given by Fulkerson and Gross [18]. All the maximal cliques of an interval graph G can be linearly ordered. For every vertex v of G, the maximal cliques containing v occur consecutively. This characterization lead to an interesting matrix formulation. A $\{0, 1\}$ matrix M is said to have the property of consecutive 1's for the columns if its rows can be permuted such that 1's in each column occur consecutively. Let G be a graph on n vertices with m maximal cliques. The clique matrix of G is a $\{0, 1\}$-matrix A, with m rows and n columns such that the entry $A_{ij} = 1$ if the vertex v_j is in the maximal clique c_i and $A_{ij} = 0$ otherwise. For example, refer to the Figure 2.2 for constructing clique matrices of a graph. Booth and Lueker developed an algorithm to test for the property of consecutive 1's for the column in linear time [1].

Theorem 2.1.8 ([18])**.** *An undirected graph G is an interval graph if and only its clique matrix has the property of consecutive 1's for the column.*

One way to recognize an interval graph is to first check if it is chordal. If it is chordal, then list all the maximal cliques and construct a clique matrix. Finally, check if the clique matrix has the property of consecutive 1's for the columns. There are many known algorithms to recognize interval graphs in linear time [1, 6, 21].

For an interval graph G, a *right end-point ordering* is an ordering $v_1, v_2, \ldots, v_n$ of its vertices, such that $r(v_i) \leq r(v_{i+1})$. Similarly, $v_1, v_2, \ldots, v_n$ is a *left end-point ordering* if $l(v_i) \leq l(v_{i+1})$.

Let G be an interval graph. Let a, b, c be three ordered vertices from a right end-point ordering of $V(G)$, i.e., $r(a) \leq r(b) \leq r(c)$. If $ac \in E(G)$, then $ab \in E(G)$. Because $ac \in E(G)$

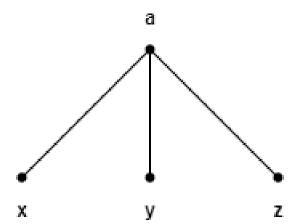


Figure 2.3: A claw

implies that $r(c) \leq l(a)$. Since $r(b) \leq r(c)$, we have $r(a) \leq r(b)$ł(a). Therefore, ab is an edge. This property of interval graphs is used in [40], to show that the list M-partition problem when the input is restricted to interval graphs is polynomial time solvable.

2.1.5 Proper Interval Graphs

An interval graph is called a *proper interval graph* if there exits an interval representation of G such that no interval is properly contained in another interval. Since every proper interval graph is an interval graph, every proper interval graph must be AT-free. The graph $K_{a,b}$ is a complete bipartite graph with a partition V_1, V_2 such that $|V_1| = a$ and $|V_2| = b$. The graph $K_{1,3}$ is called a claw. Refer to Figure 2.3

A claw is not a proper interval graph. Consider a claw with the vertex a being adjacent to three independent vertices x, y, z. Let I_a, I_x, I_y, I_z be an interval representation of the vertices. The intervals I_x, I_y must not intersect with each other but must intersect with the interval I_a. Without loss of generality, assume that $l(x) < l(a) < r(x) < l(y) < r(a) < r(y)$. Since I_z must intersect with I_a but not with I_x and I_z we must have $l(a) < r(x) < l(z) < r(z) < l(y) < r(a)$ but this results in I_z being properly contained in I_a. Therefore, a proper interval graph does not contain a claw. In [34], the authors have also shown that any interval graph that is claw-free is a proper interval graph. Hence, we have the following theorem.

Theorem 2.1.9 ([34])**.** *An interval graph is a proper interval graph if and only if it is a claw-free graph.*

Let u, v be two vertices of a proper interval graph G. Let I_u, I_v denote the intervals corresponding to u, v in an interval representation of G. If $r(u) \leq r(v)$, then $l(u) \leq l(v)$. Because if $l(u) \geq l(v)$, then the interval I_v is properly contained in the interval I_u, contradicting that G is a proper interval graph. Therefore, for any proper interval graph, the right end-point ordering and the left end-point ordering are same.

Corollary 2.1.9.1. *Let G be any proper interval graph. In any proper interval representation of G, the right end-point ordering and the left end-point ordering of $V(G)$ are same.*

Another view of this fact is expressed in the following result from [30].

Theorem 2.1.10 ([30])**.** *A graph G is a proper interval graph if and only if there exists an ordering of its vertices such that for any three ordered vertices a, b, c, if $ac \in E(G)$, then $ab, bc \in E(G)$.*

We will prove the easier part of the above theorem. Let G be a proper interval graph. Let $v_1, v_2, \cdots v_n$ be a right end-point ordering of $V(G)$. Consider three ordered vertices a, b, c. Assume that $ac \in E(G)$, we will now show that $ab, bc \in E(G)$. Because of the ordering, we have $r(a) \leq r(b) \leq r(c)$ and $l(a) \leq l(b) \leq l(c)$. Since $ac \in E(G)$, we have $r(c) \leq l(a)$. Therefore, $r(a) \leq r(b) \leq r(c) \leq l(a) \leq l(b) \leq l(c)$. This implies that ab and bc are also edges in G.

For a vertex $v \in G$, recall that $N(v)$ denotes the set of neighbours of v in G. The closed neighbourhood of v is the set $N(v) \cup \{v\}$, and is denoted by $N[v]$. Two vertices u, v are said to be equivalent if $N[u] = N[v]$. Each equivalence class of G is called a *block* in G. Observe that each block is a homogeneous clique. Two blocks A, B are called neighbours if they are completely adjacent. An ordering ϕ of the blocks in G is called a *straight enumeration* if for any block B, its neighbours and B occur consecutively in ϕ.

Theorem 2.1.11 ([8])**.** *A graph G is a proper interval graph if and only if G has a straight enumeration.*

There are many algorithms to recognize proper interval graphs in linear time [7, 8, 24, 30]. There are also many such algorithms to recognize interval graphs in linear time [1, 6, 21].

2.2 Homomorphisms

In the introduction, we have seen the vertex coloring problem and introduced the matrix partition problem as a generalization of it. In this section, we will be looking at graph homomorphisms and some known results on graph homomorphisms. Graph homomorphisms can be viewed as being in between vertex colorings and matrix partitions because homomorphisms generalize vertex colorings and matrix partitions generalize homomorphisms.

Definition 2.2.1. *Let G and H be two simple graphs. A homomorphism of G to H is defined as a mapping $f : V(G) \longrightarrow V(H)$ such that, if the vertices u, v are adjacent in G, then $f(u), f(v)$ are adjacent in H.*

We say that G is homomorphic to H or G is H-colorable, if there exists a homomorphism of G to H.

Proposition 2.2.1. *A homomorphism of G to K_m is an m-coloring of G.*

Let S_v be the set of vertices in G that are mapped to a vertex $v \in H$. If the vertex v doesn't have a self loop, then S_v must be stable. Therefore, if a graph G is homomorphic to K_m then it has at most m stable sets and hence m-colorable.

Matrix partition of a graph generalizes graph homomorphisms

Let $f : V(G) \longrightarrow V(H)$ be a homomorphism of a graph G to a graph H. Let $v_1, v_2, \dots, v_m$ be the vertices of graph H. For any $v_i \in V(H)$, let S_{v_i} be the set of vertices $f^{-1}(v_i)$ in G. Homomorphism of G to H can be viewed as partitioning the vertices of G into sets $S_{v_1}, S_{v_2}, \dots, S_{v_m}$ such that, S_{v_i} is stable if v_i doesn't have a loop and if there is an edge between the sets S_{v_i}, S_{v_j} in G, then v_i, v_j are adjacent in H.

For a fixed graph H, we denote the matrix M_H as the matrix obtained by replacing 1's with $\star$'s from its adjacency matrix.

Proposition 2.2.2. *An H-coloring of G is an M_H-partition of G.*

Conversely, let M be a $\{0, \star\}$-symmetric matrix with only 0's on diagonal. The M-partition of a graph G is equivalent to homomorphism to a graph H whose adjacency matrix is obtained by replacing $\star$'s with 1's in M. Hence, we can say that all the M-partition problems where M is a $\{0, \star\}$-matrix with only 0's on the main diagonal are homomorphism problems and vice versa.

The Pentagon Homomorphism Problem: The problem is to check if a graph G has a homomorphism to a pentagon graph which is a cycle on 5 vertices v_1, v_2, v_3, v_4, v_5. The adjacency matrix of a pentagon graph and the matrix corresponding to the homomorphism problem can be seen in Figure 2.4.

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a)

$$M' = \begin{bmatrix} 0 & \star & 0 & 0 & \star \\ \star & 0 & \star & 0 & 0 \\ 0 & \star & 0 & \star & 0 \\ 0 & 0 & \star & 0 & \star \\ \star & 0 & 0 & \star & 0 \end{bmatrix}$$

(b)

Figure 2.4: (a) Adjacency matrix corresponding to the pentagon graph. (b) Matrix partition corresponding to the pentagon homomorphism obtained by replacing 1's by $\star$'s

As with the matrix partition, in graph homomorphism, we are also interested, for a graph H, in the following two problems:

1. can we deiced in polynomial time if an input graph G has homomorphism to H.

2. can H-colorable graphs be characterized by the absence of finitely many forbidden induced subgraphs.

The complexity question for H-homomorphism problems was solved by Hell and Nešetřil in [26] and is stated below.

Theorem 2.2.3 ([26])**.** *If H is a bipartite graph, then H-homomorphism is polynomial time solvable, else it is NP-complete.*

The above result is also known as the H-coloring dichotomy, we will not be getting into the details of the proof here. For a shorter proof, refer to [37]. Let H be a simple graph with at least one edge. If the H-homomorphism problem has finitely many minimal obstructions, then it will be polynomial time solvable. However, because of Theorem 2.2.3 this would mean P=NP. Hence, H-homomorphism problem for a non-bipartite graph is unlikely to be characterized by the absence of finitely many minimal obstructions. We now formally prove that it cannot be so characterized.

Proposition 2.2.4 ([27]). *If there exists a homomorphism $f : G \longrightarrow H$, then $\chi(G) \leq \chi(H)$.*

Proof. Let the chromatic number of H be k. This implies that there exists a homomorphism $h : H \longrightarrow K_k$. Then $h \circ f$ is a homomorphism G to K_k. Therefore, G is k-colorable which implies that $\chi(G) \leq \chi(H)$. □

The *girth* of a graph G containing cycles is the minimum length of a cycle in G. Similarly, the odd girth of a non-bipartite graph G is the minimum length of an odd cycle in G. We have a similar result as Proposition 2.2.4 with the odd girth of the graph.

Proposition 2.2.5 ([27]). *If there exists a homomorphism $f : G \longrightarrow H$, then odd girth$(G) \geq$ odd girth(H).*

Let G and H be two graphs such that $\chi(G) \geq \chi(H)$ and *odd girth*$(G) \geq$ *odd girth*(H). From Proposition 2.2.4 we know that $G \not\longrightarrow H$ and from Proposition 2.2.5 we have $H \not\longrightarrow G$. The existence of graphs with high chromatic number and odd girth follows from the result of Erdös.

Theorem 2.2.6 ([9]). *For any positive integers k, ℓ there always exists a graph G with chromatic number k, and with girth at least ℓ.*

Using these propositions and the theorem, we can solve the characterization problem for the graph homomorphism problem.

Theorem 2.2.7 ([22]). *If H is a graph with no edges then H-homomorphism problem has finitely many minimal obstructions. Else, H-homomorphism problem has infinitely many minimal obstructions.*

Proof. If H has no edges then K_2 is the only minimal obstruction for H-homomorphism problem. Let H be a graph that has at least one edge. Assume that H-homomorphism problem has finitely many minimal obstructions G_i, for $1 \leq i \leq n$. Let ℓ be the largest girth among these minimal obstructions and k be the chromatic number of H. From Theorem 2.2.6, there exists a graph G with chromatic number greater than $k+1$ and girth greater than ℓ. From Proposition 2.2.4, we know that $G \not\longrightarrow H$. We now have to show that G does not have any G_i as an induced subgraph. If G_i is an induced subgraph of G, then *girth*$(G) \leq$ *girth*$(G_i) \leq \ell$ which is a contradiction with Proposition 2.2.5. Hence, we have proved that H-homomorphism problem has infinitely many minimal obstructions if H has edges. $\square$

2.3 Matrix Partitions

In this section, we will look at few known results for matrix partition problems. For a more extensive survey on the matrix partition problems, refer to [23]. Recall that we only consider $\{0, 1, \star\}$-symmetric matrix with no $\star$ on diagonal because if there is a $\star$ on the diagonal, then all the vertices can be placed in that part. For the rest of the thesis, we will assume that the matrix M has k rows with 0 on diagonal and ℓ rows with 1 on diagonal where $k+\ell = m$. Also, when a graph G is M-partitionable, the parts are denoted by $V_1, V_2, \ldots, V_m$. Let M be a $\{0, 1, \star\}$-symmetric matrix with k diagonal $0's$ and ℓ diagonal $1's$ such that the first k rows have $0's$ on diagonal and next ℓ rows have $1's$ on diagonal. Then M can be represented by a (A, B, C)-block structure with A as the submatrix containing the rows and the columns $1, 2, 3, \ldots, k$ with $0's$ on diagonal and, B as the submatrix containing the rows and the columns $k+1, k+2, \ldots, k+\ell$ with $1's$ on diagonal and C as the submatrix containing the rows $1, 2, \ldots, k$ and the columns $k+1, k+2, \ldots, k+\ell$.

$$M = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}$$

The matrix obtained from M by replacing all the $0's$ by 1 and all the $1's$ by 0 is called the *complement* of M and is denoted by $\bar{M}$. We have the following result.

Theorem 2.3.1. *Let M be a $\{0, 1, \star\}$-symmetric matrix and $\bar{M}$ be its complement. A graph G is M-partitionable if and only if $\bar{G}$ is $\bar{M}$-partitionable.*

As seen in the introduction, list M-partition problems have a list of parts associated with each vertex. The goal is to find a partition where each vertex is placed in one of the parts mentioned in its list. Observe that if a list M-partition problem can be solved in polynomial time then the general M-partition problem can also be solved in polynomial time, because we can assume that the list contains all the parts. On the other hand, if a list M-partition problem is NP-complete, then we cannot conclude anything about the M-partition problem without lists.

2.3.1 Matrix without $\star's$

In Section 2.2, we have seen that homomorphism problems correspond to matrix partitions without $1's$. From Theorem 2.3.1, we conclude that for any matrix without $0's$ similar results hold. We will now prove that the M-partition problems are all polynomial time solvable when M does not contain $\star's$.

Theorem 2.3.2 ([12]). *If a matrix M does not contain any asterisk entry, then there are only finitely many minimal obstructions.*

Proof. In order to prove that M has only finitely many minimal obstructions it is sufficient to prove that the number of vertices in any minimal obstruction to M is upper bounded. Let M be a $\{0,1\}$-matrix with k-rows of 0 diagonal entries and ℓ rows of 1 diagonal entries; without loss of generality we can assume that $\ell \leq k$ because we can instead focus on the complement of M. We will prove that any minimal obstruction to M has at most $2(k+1)(k+\ell)+1$ vertices. Assume that there is a minimal obstruction G with more than $2(k+1)(k+\ell)+1$ vertices. Let x be any vertex in G then $G-x$ is M-partitionable. From the pigeon hole principle, there exists at least one part S that contains more than $2(k+1)$ vertices in any partition of $G-x$. Since M has no $\star's$, the set S is a homogeneous clique or stable set in $G-x$ i.e., the vertices of S have the same neighbours and non-neighbours in $G-x$.

At least $k+2$ vertices of S are either adjacent to x or non-adjacent to x. Let C be the subset of vertices of at least $k+2$ vertices in S that have the same adjacency with x i.e., C is a homogeneous set in G. We will consider the case of C being a stable set or a clique separately.

First, lets assume that C is a clique. Let y be a vertex from C, we know that $G-y$ is M-partitionable. Note that the set $C-y$ is a homogeneous set and has a cardinality of at least $k+1$. Since any partition has only k independent parts, at least one vertex u of $C-y$ will belong to a clique in the partition. Since u, y belong to a homogeneous set in G, y can also be placed in that part which is a contradiction that G is a minimal obstruction.

Now, lets assume that C is a homogeneous stable set in G. Let y be a vertex in C, then $G-y$ is M-partitionable and has at least $k+1$ vertices that are independent. Since $\ell < k$, it implies that at least one vertex u from $C-y$ will belong to an independent set in the partition. Since C is a homogeneous set, y can be placed in that part, contradicting the fact that G is a minimal obstruction. Hence, we have proved that any minimal obstruction to M has at most $2(k+1)(k+\ell)+1$ vertices. Therefore, M has finitely many minimal obstructions. □

A tighter bound for the maximum number of vertices in a minimal obstruction for matrices without $\star$ was obtained by Feder et al.

Theorem 2.3.3 ([12]). *If M is a matrix without any asterisk entry, then all the minimal obstructions to M have at most $(k+1)(\ell+1)$ vertices. There are at most two minimal obstructions with exactly $(k+1)(\ell+1)$ vertices.*

A special case of matrices called friendly matrices were considered in [15]. A matrix M is called *unfriendly* if $M(i,i) = M(j,j) \neq \star$ and $M(i,j) = M(j,i) = \star$, for some i, j, and it is called *friendly* otherwise. Note that from the block representation of M, M is called *friendly* if the block A and B does not contain a $\star$, and is *unfriendly* otherwise. The following two theorems show the properties of friendly and unfriendly matrices.

Theorem 2.3.4 ([15]). *If M is an unfriendly matrix, then the M-partition problem has infinitely many minimal obstructions.*

Theorem 2.3.5 ([15]). *Assume that block C of a matrix M contains either only $\star's$ or no $\star's$. Then the M-partition problem has finitely many minimal obstructions if and only if M is a friendly matrix.*

In their paper [15], Feder et al. have also shown that if M is a friendly matrix in which neither the block A nor the block B has three identical rows then M-partition problem is polynomial time solvable.

2.3.2 Small Matrices

In this section, we will discuss the matrices of size at most 5. It is easy to see that any matrix of size 1 has exactly one minimal obstruction, either K_2 or $\bar{K}_2$.

Matrices of size 2

If M is a matrix of size 2, then the M-partition problem is polynomial time solvable. To prove this, we construct an equivalent instance of 2-SAT problem with polynomially many clauses. For every $v \in V(G)$, introduce a clause variable x_v whose value corresponds to the part it belongs to, i.e., if $x_v = 0$ then v will be placed in V_1 and if $x_v = 1$ then v will be placed in V_2. Clauses are constructed as follows. For every $uv \in E(G)$, if $M(1,1) = 0$ we add the clause $(x_u \vee x_v)$, if $M(2,2) = 0$ then add the clause $(\bar{x_u} \vee \bar{x_v})$, and $(x_u \vee \bar{x_v}) \wedge (\bar{x_u} \vee x_v)$ when $M(1,2) = 0$. Similarly, for every non-edge $uv \notin E(G)$, we add the clause $(x_u \vee x_v)$ if $M(1,1) = 1$, $(\bar{x_u} \vee \bar{x_v})$ if $M(2,2) = 1$, and $(x_u \vee \bar{x_v}) \wedge (\bar{x_u} \vee x_v)$ if $M(1,2) = 1$. For every edge or non-edge, we add at most 3 clauses. Therefore, the constructed 2-SAT problem has at most polynomially many clauses. A 2-SAT problem is polynomial time solvable, hence we can conclude that the M-partition problem of size 2 is polynomial time solvable.

Matrices of size m with $3 \leq m \leq 5$

Recall that the matrices corresponding to the 3-coloring problem and its complement are as follows

$$M = \begin{bmatrix} 0 & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{bmatrix} \quad \bar{M} = \begin{bmatrix} 1 & \star & \star \\ \star & 1 & \star \\ \star & \star & 1 \end{bmatrix}.$$

Since the 3-coloring problem is NP-complete, we can conclude that the M-partition and the $\bar{M}$-partition problems are NP-complete. We now prove that if a 4×4 matrix contains M

as a principal submatrix, then it is NP-complete [10]. Let M' be a 4×4 matrix containing M as a principal submatrix.

$$M' = \begin{bmatrix} 0 & \star & \star & x_1 \\ \star & 0 & \star & x_2 \\ \star & \star & 0 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$$

Let G be simple graph, and G' be the union of two disjoint copies of G. It is easy to observe that if G is 3-colorable, then G' is M'-partitionable. We will now prove that if G' is M'-partitionable then G is 3-colorable. First, let us assume that $x_4 = 1$. Observe that the vertices from different copies in G' cannot be placed in part 4 of the partition. So, all the vertices of one copy must be partitioned in the first three parts. Therefore, if G' is M'-partitionable then G is 3-colorable. The same argument holds even if one of x_1, x_2, x_3 is equal to 1. Let us now consider the case when $x_4 = 0$. If additionally any of x_1, x_2, x_3 is 0, say $x_1 = 0$, then the vertices of $V_1 \cup V_4$ will form a stable set, and any vertex in V_4 can also be placed in V_1 and hence parts V_1 and V_4 can be combined. In this case, the M'-partition is the same as the 3-coloring problem. If all of x_1, x_2, x_3 are equal to $\star$, then it is equivalent to 4-coloring problem. Therefore, G' is M' partitionable if and only if G is 3-colorable and this implies that the M'-partition problem is NP-complete.

Feder et al. also proved that for a matrix of size at most 4 that does not contain M or $\bar{M}$ as principal submatrix, the partition problem is polynomial time solvable. Thus, we have the following theorem.

Theorem 2.3.6 ([10])**.** *Suppose M is a $\{0, 1, \star\}$-symmetric matrix of size at most four. Then the M-partition problem is NP-complete if there is a principal submatrix corresponding to 3-coloring or its complement, and is polynomial time solvable otherwise.*

Recall that a matrix is called friendly, if the blocks A and B has $\star$ entries, and unfriendly otherwise. It is easy to observe that a matrix is unfriendly if it either has $\begin{bmatrix} 0 & \star \\ \star & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & \star \\ \star & 1 \end{bmatrix}$ as a diagonal submatrix. The authors of [16] have systematically examined all small friendly matrices and verified the following fact.

Theorem 2.3.7 ([16])**.** *Suppose M is a friendly matrix of size at most 5, then the M-partition problem has finitely many minimal obstructions.*

From the Theorem 2.3.4 and 2.3.7, we can deduce the full characterization of matrices of size up to 5 as mentioned in the following corollary.

Corollary 2.3.7.1. *For a matrix M of size at most 5, the M-partition problem has finitely many minimal obstructions if and only if M is friendly.*

$$\begin{bmatrix} 0 & \star & \star \\ \star & 1 & \star \\ \star & \star & 1 \end{bmatrix}$$

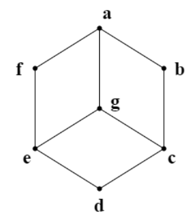


Figure 2.5: Matrix and the corresponding minimal perfect obstruction.

2.4 Matrix Partitions Restricted to Graph Classes

2.4.1 Perfect Graphs

In the previous sections, we have seen a few results for matrix partitions on general graphs. The characterization problem was solved for small matrices, of size up to 5 [15]. In this section, we will look at some results for the matrix partition problem when restricted to perfect graphs. This problem was first considered in [11], where the authors have shown that it is hard to classify the complexity of a matrix partition problem on perfect graphs. Therefore, we will now look at characterizations.

Recall that a perfect graph is a graph with the clique number equal to the chromatic number. Therefore, any perfect graph with clique number k is k-colorable, but not $(k-1)$-colorable. Hence, we have the following theorem.

Theorem 2.4.1. *A perfect graph is k-colorable if and only if it does not have an induced K_{k+1}.*

The analogue of Theorem 2.3.1 holds also when restricted to perfect graphs, because the complement of a perfect graph is also perfect. Let (A, B, C) be the block structure representation of M. Let $E(A)$ denote the set of off-diagonal entries in A, $E(B)$ denote the set of off-diagonal entries in B, and $E(C)$ denote the set of entries in C. A subset of the set $\{0, 1, \star\}$ is called *normal*, if it is either subset of $\{0, 1\}$ or $\{\star\}$. A matrix M is called a *normal matrix*, if the sets $E(A), E(B), E(C)$ are all normal. The following theorems are about finite minimal perfect obstructions to normal matrices.

Any matrix without a $\star$ is a normal matrix. Theorem 2.3.3 states that, any minimal obstruction to a normal matrix with $E(A), E(B), E(C)$ subset of $\{0, 1\}$ has at most $(k+1)(\ell+1)$ vertices. When we consider only perfect graphs, we can extend this result to other cases of normal matrices.

Theorem 2.4.2 ([11])**.** *Let M be a normal matrix with $E(C) = \{0\}$ or $\{1\}$. Then any minimal perfect obstruction to M has at most $(k+1)(\ell+1)$ vertices.*

When $E(C) = \{\star\}$, the size of a minimal perfect obstruction need not have the bound given in the above theorem. For example, consider the matrix in Figure 2.5 from [11], where $(k+1)(\ell+1) = 6$. There exists a minimal perfect obstruction with 7 vertices. Let us prove that the graph G in Figure 2.5 is a minimal obstruction. Since the size of the maximum clique of G is 2, the parts 2, 3 can have at most two vertices each. This implies that part 1 must have at least 3 vertices. There are two maximal independent sets of size greater than 3 in G, namely the set $\{b, d, f, g\}$ and the set $\{a, c, e\}$. So, even if we place any 3 independent vertices in the 1^{st} part, there will be at least three more independent vertices remaining that cannot be partitioned to two cliques. If we remove any vertex from G, it is partitionable. A tighter bound for the size of a minimal obstruction for this class of matrices is not known, but an exponential bound is known. Note that an exponential bound is also sufficient to prove that there are only finitely many minimal perfect obstructions.

Theorem 2.4.3 ([11]). *Let M be a normal matrix with $E(C) = \{\star\}$, and assume that $\ell \leq k$. Any minimal perfect obstruction to M has at most $2(k+1)^{(2k\ell+1)}$ vertices.*

Theorem 2.4.4 ([11]). *Let M be a normal matrix with $E(C) = \{0, 1\}$, and assume that $\ell \leq k$. Any minimal perfect obstruction to M has at most $2(2k+1)^{(2k\ell+1)}$ vertices.*

2.4.2 Chordal Graphs

In this section, we will look at matrix partition problems restricted to the class of chordal graphs. Recall that a graph G is called chordal, if it does not have any induced cycle of length ≥ 4. The complement of a chordal graph is not necessarily a chordal graph, and therefore, the analogue of Theorem 2.3.1 does not hold. Since chordal graphs are perfect graphs, the results mentioned in the section 2.4.1 also holds true for chordal graphs.

For a 0-diagonal matrix with all off-diagonal entries $\star$, there is only one chordal minimal obstruction, namely K_{k+1}. Any 1-diagonal matrix with all off-diagonal entries $\star$, has only one minimal chordal obstruction, namely $\bar{K}_{\ell+1}$. Hell et al [25], showed that there is only one chordal minimal obstruction for partitioning a graph into k cliques and ℓ independent sets. This problem corresponds to a matrix partition problem with k 0-diagonal entries, ℓ 1-diagonal entries and all the off-diagonal entries $\star$.

Theorem 2.4.5 ([25]). *A chordal graph can be partitioned into k independent sets and ℓ cliques, if and only if it does not contain $(\ell+1)K_{k+1}$.*

Partitioning into k independent sets and ℓ cliques is known to be NP-complete for general graphs unless $k \leq 2$ and $\ell \leq 2$ [3]. However, for a chordal graph, such a partition can be found in linear time using the perfect elimination ordering [25]. We call a matrix *0-diagonal* if every entry on the main diagonal is equal to 0. Similarly, we call a matrix *1-diagonal* if every entry on the main diagonal is equal to 1.

Theorem 2.4.6 ([13]). *Assume that M is either a 0-diagonal or a 1-diagonal matrix. Then the list M-partition problem is polynomial time solvable for chordal graphs. Moreover, if the matrix M is 0-diagonal, then the list M-partition problem can be solved in linear time.*

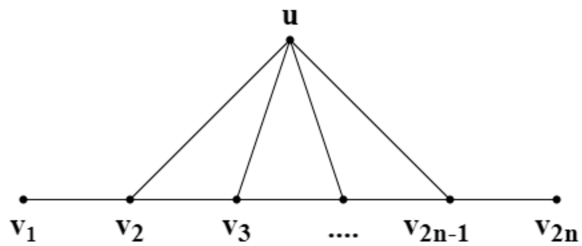


Figure 2.6: Family of minimal chordal obstructions for M_1 and M_2

We will now look at a class of matrices known as *crossed matrices*. A matrix C is called *crossed*, if each non-$\star$ entry of C belongs to row or a column without $\star$. In all the normal matrices, the block C either contains only $\star$ or no $\star$. Therefore, all normal matrices are crossed.

From Theorem 2.3.6, we know that 3-coloring and its complement are NP-complete. Since they are polynomial time solvable in the case of chordal graphs, for any matrix of size at most 4, the M-partition problem is polynomial time solvable for chordal graphs. Classifying the complexity of M-partition problems for larger matrices, even when restricted to the class of chordal graphs, is hard. However, we have the following theorem when we consider crossed matrices.

Theorem 2.4.7 ([13]). *Let M be a matrix in which the block C is crossed. Then the list M-partition problem can be solved in polynomial time for chordal graphs.*

We will now look at the minimal chordal obstructions for the matrix partition problem. Minimal chordal obstructions for small matrices were first handled in [33], where the authors have shown that even for small matrices there are infinitely many minimal chordal obstructions.

Theorem 2.4.8 ([14]). *Let M be a matrix of size $m < 4$. Then M has finitely many minimal chordal obstructions except for the following two matrices.*

$$M_1 = \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 1 \end{bmatrix}$$

Figure 2.6, from [14], depicts an infinite family of minimal chordal obstructions to M_1 and M_2. In order to prove that the remaining 3×3 matrices have finitely many minimal chordal obstructions, the authors used the technique of finding finitely many labeled minimal obstructions. We will briefly show the technique of using labeled minimal obstructions.

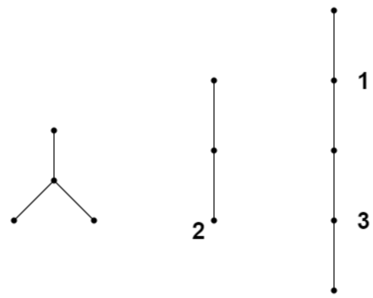


Figure 2.7: The labeled minimal chordal obstructions for M_3

Theorem 2.4.9 ([14]). *The matrix* $M_3 = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ * & 0 & 1 \end{bmatrix}$ *has finitely many minimal chordal obstructions.*

Proof. Let G be a chordal graph with independence number α. If $\alpha = 2$, G can be partitioned using the first and the third parts V_1, V_3. Since $\bar{K_4}$ is a minimal obstruction, any graph with $\alpha \geq 4$, is not partitionable. This leaves us the case when $\alpha = 3$.

Let v_1, v_2, v_3 be a fixed set of independent vertices in G. We place the vertices v_1, v_2, v_3 in the parts V_1, V_2, and V_3 respectively, and show that there are finitely many minimal chordal obstructions. These obstructions are called labeled minimal obstructions. For M_3, the labelled minimal chordal obstructions are shown in Figure 2.7. If each labelled minimal obstruction has at most N vertices, then any minimal obstruction can only have at most $6N$ vertices. The vertices v_1, v_2, v_3 can be placed in three parts in 6 ways bijectively, so for each such labelling, there can only be N vertices preventing them to be partitioned. Therefore, each minimal obstruction can have at most $6N$ vertices.

By assuming that the vertices v_1, v_2, v_3 are placed in V_1, V_2, V_3 respectively, we obtain a bound for N. Let $S(v_1)$ denote the set of vertices that are adjacent only to the vertex v_1, and the sets $S(v_2)$, $S(v_3)$ denotes the set of vertices adjacent only to vertex v_2 and v_3 respectively. The set $S(v_1, v_2)$ denotes the set of vertices adjacent to both v_1, v_2, but not v_3. The sets $S(v_2, v_3)$ and $S(v_1, v_3)$ are also defined in the similar fashion. Finally, the set $S(v_1, v_2, v_3)$, denotes the set of vertices adjacent to all the three vertices.

Assume that the graph G does not have the three labelled minimal obstructions shown in Figure 2.7. It is easy to see that $S(v_1, v_2, v_3) = \emptyset$, because of the first obstruction. Due to the second obstruction, we have $S(v_1, v_2) = \emptyset$ and $S(v_2, v_3) = \emptyset$. Since G is a chordal graph, there will be no induced cycle of length ≥ 4. This implies that the vertices of the set $S(v_1, v_3)$ must induce a clique. Since we also assumed that the independence number of G is 3, the vertices of the sets $S(v_1)$, $S(v_2)$, $S(v_3)$ must induce a clique.

Due to the third minimal obstruction, we can claim that every vertex of $S(v_1, v_3)$ is either adjacent to all the vertices of $S(v_1)$ or adjacent to all the vertices of $S(v_3)$. Let X be the set of vertices of $S(v_1, v_3)$, that are adjacent to every vertex of $S(v_1)$, and Y be set of vertices of $S(v_1, v_3)$, that are adjacent to every vertex of $S(v_3)$. Then the graph G can be partitioned with $V_1 = S(v_1) \cup X$, $V_2 = S(v_2)$ and $V_3 = S(v_3) \cup Y$. Hence, we proved that M_3 has finitely many labeled minimal chordal obstructions, which in turn proves that there are finitely many minimal chordal obstructions. □

2.4.3 Split Graphs

Recall that a graph G is called a split graph, if its vertices can be partitioned into two sets (C, I) such that C induces a clique and I induces an independent set. Matrix partition problem restricted to split graphs is considered in [36]. The authors showed that for any matrix M there are only finitely many minimal split obstructions. It is interesting to note that even for a matrix of size 3, there are infinitely many minimal chordal obstructions, but for any matrix M there are only finitely many minimal split obstructions. Let (A, B, C) be the block structure of M, if C has a $\star$ entry then the matrix M will have $\begin{bmatrix} 0 & \star \\ \star & 1 \end{bmatrix}$ as a principle submatrix. Hence, any split graph will be partitionable. In the remaining of this section, we will assume that C has no asterisk entry. If C has no $\star$, then it is a crossed matrix, therefore from Theorem 2.4.7, the M-partition problem is polynomial time solvable for split graphs. We will now look at some results concerning the finiteness of the set of minimal split obstructions.

Theorem 2.4.10 ([36]). *Let M be a 0-diagonal matrix of size k. Let G be any M-partitionable split graph. Then in any part P in the M-partition of G, there exists a homogeneous set of size at least $\frac{|P|-1}{2^{k-1}}$.*

Proof. Let (C, I) be a split partition of G, where C induces a clique and I an independent set. Let $P_1, P_2, \dots, P_k$ be an M-partition of G. Since each P_i is an independent set, we have $|P_i \cap C| \leq 1$. Lets assume that $|P_i \cap C| = 1$ and $P_i \cap C = \{u_i\}$. Note that, $P_1 \cap I = P_1 - \{u_1\}$. The vertices of $P_1 - \{u_1\}$ are non-adjacent to all but these $k-1$ vertices, $u_2, u_3, \dots, u_k$. For $2 \leq i \leq k$, u_i is either adjacent to at least $\frac{|P_1|-1}{2}$ vertices or non-adjacent to at least $\frac{|P_1|-1}{2}$. So, we have a homogeneous set of size at least $\frac{|P_1|-1}{2^{k-1}}$ in the part P_1. The same argument can be applied to any other part, hence we proved that every part has a homogeneous set of size at least $\frac{|P|-1}{2^{k-1}}$. □

The complement of a split graph is also a split graph, therefore we have the same result for 1-diagonal matrices.

Corollary 2.4.10.1 ([36]). *Let M be a 1-diagonal matrix of size ℓ. Let G be any M-partitionable split graph. Then for any part P in the M-partition of G, there exists a homogeneous set of size at least $\frac{|P|-1}{2^{\ell-1}}$.*

Theorem 2.4.11 ([36]). *Let M be a matrix with $k \geq \ell$, where k is the number of 0's on the diagonal and ℓ is the number of 1's on the diagonal. Any minimal split obstruction to M has at most $2^{k-1}(k+\ell)(2k+3)+1$ vertices.*

Proof. Let (A, B, C) be the block structure of M. Assume that G is a minimal split obstruction to M with at least $2^{k-1}(k+\ell)(2k+3)+2$ vertices. For any vertex $v \in G$, $G - v$ is M-partitionable. Since there are $k+\ell$ parts, there exists a part P with at least $2^{k-1}(2k+3)+1$ vertices in the M-partition of $G - v$. The block C does not have any asterisk entry, so from the above theorem and corollary, we know that there exists a homogeneous set of $G - v$ in P with at least $\frac{|P|-1}{2^{k-1}} = 2k+3$ vertices. The vertex v will have the same adjacency with at least $k+2$ vertices of P i.e., there will be a homogeneous set of size at least $k+2$ in G, say H. The part P can either induce a clique or an independent set, so H will either be a clique or an independent set. Let w be a vertex in H, lets now consider the partition of $G - w$.

Case 1: First, assume that H induces a clique. In any M-partition of $G - w$, at least one vertex, say u, of $H - w$ must be part of a clique, because $k + 1 \geq \ell$. Since u, w belong to a homogeneous set and are adjacent to each other, w can also be placed in the same part as u, contradicting the fact that G is a minimal obstruction.

Case 2: Now assume that H induces an independent set. In any M-partition of $G - w$ at least one vertex, say u, of $H - w$ must be part of an independent set. Since u, w belong to a homogeneous set and are non-adjacent to each other, w can also be placed in the same part as u, contradicting the fact that G is a minimal obstruction. □

2.4.4 The Complexity Problem for Geometric Graph Classes

List M-partition problems restricted to various graph classes like interval graphs, circular arc graphs that represent geometric structures were studied in [40]. Interesting techniques were developed by exploiting the geometric structure of these graph classes. In this section, we will solve the complexity problem for proper interval graphs by adopting the algorithm from [40] for interval graphs, but presenting a simplified correctness proof for proper interval graphs. We denote the index set $\{1, 2, 3 \ldots, m\}$ by $[m]$, and the set of all subsets of $[m]$ by $2^{[m]}$.

Recall that an instance $I = (G, L)$ of a list M-partition problem includes a graph G, and a list function L of $V(G)$. Let m be the size of M. An M-partition f of G is a valid solution for I, if for every $v \in V(G)$, $f(v) \in L(v)$. For a set $D \subseteq V(G)$, $(G[D], L)$ denotes a sub-instance induced by the vertices of D, with the same lists. If a vertex v is placed in the part i, we say that the color i is assigned to v. Let f be a solution of list M-partition problem for the sub-instance $(G[D], L)$(when no matrix is mentioned, we assume that the solution f of $(G[D], L)$, is for the list M-partition problem). If c is the color assigned to a vertex $v \in D$ by f, then any neighbour of v must be assigned a color i such that $M(c, i) \neq 0$, where $M(c, i)$ is the entry in row c and column i of the matrix M. Similarly, any non-neighbour of v must be assigned a color i such that $M(c,i) \neq 1$. A color i assigned to u *conflicts* with $f(v)$ if $M(i, f(v)) = 0$, when uv is an edge or $M(i, f(v)) = 1$, when uv is not an edge. In order to find a solution for (G, L), we first find all possible solutions for certain sub-instances $(G[D], L)$ and extend it to (G, L).

To make sure that we store the necessary information to extend a solution f of $(G[D], L)$ to (G, L), we introduce a pair of functions $P_D^f, \bar{P}_D^f : V(G) - D \longrightarrow 2^{[m]}$.

- $P_D^f(v) = \{f(u) : u \in D \text{ and } uv \in E(G)\}$, i.e., $P_D^f(v)$ stores the set of colors assigned to the neighbours of v in D by f.
- $\bar{P}_D^f(v) = \{f(u) : u \in D \text{ and } uv \notin E(G)\}$, i.e.,$\bar{P}_D^f(v)$ stores the set of colors assigned to the non-neighbours of v in D by f.

The pair $\mathbf{P} = (P_D^f, \bar{P}_D^f)$, is called the *profile* corresponding to the M-partition f of $G[D]$. Note that for every vertex in $V(G) - D$, the profile stores all the colors that would cause a conflict with some $f(u)$, where $u \in D$. We say that a color c assigned to a vertex $v \in V(G) - D$ *conforms* with the profile P, if for every $i \in P_D^f(v)$, $M(c,i) \neq 0$ and for every $i \in \bar{P}_D^f(v)$, $M(c,i) \neq 1$.

Let $v_1, v_2, \ldots, v_n$ be an ordering of $V(G)$, the set D_i denotes the set of first i vertices in the ordering.

Theorem 2.4.12 ([40]). *Let $I = (G, L)$ be any instance of list M-partition problem. Given an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$, let n_i denote the total number of profiles of the solutions of $(G(D_i), L)$ for the list M-partition problem. Then there exists an algorithm to solve the list M-partition problem for I in $\mathcal{O}(n \cdot \sum_{i=0}^{n} n_i^2)$ time.*

Proof. Let $\tilde{\mathcal{P}}_i$ denote the set of all profiles for $(G[D_i], L)$. Note that, when $i = 0$, D is empty, and hence $\tilde{\mathcal{P}}_0$ has only an empty profile. When $i = 1$, all the possible solutions for $(G[D_1], L)$ are all the colors present in the list $L(v_1)$. For every solution f of $(G[D_1], L)$, the corresponding profile can be constructed trivially. We will now show how to construct the set of profiles $\tilde{\mathcal{P}}_{i+1}$ from $\tilde{\mathcal{P}}_i$. First consider a profile $\mathbf{P^f} = (P_{D_i}^f, \bar{P}_{D_i}^f)$ corresponding to a partition f of $(G[D_i], L)$ that is present in $\tilde{\mathcal{P}}_i$.

1. For $c \in L(v_{i+1})$, check if assigning c to v_{i+1} conforms with the profile $\mathbf{P^f}$. Checking if a color conforms or not can be done in $\mathcal{O}(1)$ time and there are $\mathcal{O}(1)$ choices for selecting a color. If there is no color in the list $L(v_{i+1})$ that conforms with $\mathbf{P^f}$, then we drop that profile. For every color c that conforms with $\mathbf{P^f}$, perform the following step.

2. Let f' denote the solution obtained for $(G[D_{i+1}], L)$ by assigning c to v_{i+1}. Construct the profile $\mathbf{P^{f'}}$ from the profile $\mathbf{P^f} = (P_{D_i}^f, \bar{P}_{D_i}^f)$ as follows. Add the color c to $P_{D_i}^f(v)$ if v_{i+1} is adjacent to v, and to $\bar{P}_{D_i}^f(v)$ if it is non-adjacent to v. Once the profile is

constructed, check if it is already present in the set $\tilde{\mathcal{P}}_{i+1}$ and add it. Comparing two profiles can be done in $\mathcal{O}(n)$ time. Since the total number of profiles already present in $\tilde{\mathcal{P}}_{i+1}$ is at most n_{i+1}, the time to check if a profile is already present can be done in $\mathcal{O}(n \cdot n_{i+1})$.

To construct $\tilde{\mathcal{P}}_{i+1}$, iterate the above steps for every profile in $\tilde{\mathcal{P}}_i$. The total number of profiles in $\tilde{\mathcal{P}}_i$ is n_i. Therefore, iterating over all the profiles would take $\mathcal{O}(n \cdot n_{i+1} \cdot n_i)$ time. To find the solution for I, we have to repeat the above process till we construct $\tilde{\mathcal{P}}_n$. For $0 < i < n$, if any of $\tilde{\mathcal{P}}_i = \emptyset$, then it implies that there is no solution for I. If $\tilde{\mathcal{P}}_{n-1}$ is non-empty, it implies that there is a solution f for $(G[D_{n-1}], L)$. From f, it is trivial to check if there a color in the list $L(v_n)$ that conforms with the corresponding profile. If such a color exists, then it will be a solution of I as there are no other vertices remaining to be colored.

To obtain a solution for I, we can keep track of the colors assigned to every vertex while constructing the profiles. Since we are repeating the above process till D_n, it would take $\mathcal{O}(n \cdot \sum_{i=0}^{n} n_i^2)$ time. □

In order to prove that the list M-partition problem is polynomial time solvable, it is sufficient to prove that there are only polynomially many profiles for each D_i.

A sequence of sets $S_1, S_2, \ldots, S_n$ is called an *inclusive sequence*, if $S_1 \subseteq S_2 \ldots, \subseteq S_n$. If each $S_i \subseteq [m]$, then the total number of different inclusive sequences is upper bounded by $(n+1)^m$. This can be easily checked by defining a function $f : \{1, 2 \ldots, m\} \longrightarrow \{1, 2, \ldots, n+1\}$, such that $f(c)$ is the smallest index i for which $c \in S_i$. If c does not appear in any set, then $f(c)$ is set to $n+1$. For each $c \in [m]$, there are $n+1$ choices for $f(c)$. Therefore, the total number of inclusive sequences is at most $(n+1)^m$.

We show that for any proper interval graph G, there exists an ordering of its vertices such that there are only polynomially many profiles for each D_i.

Theorem 2.4.13. *Let $I = (G, L)$ be any instance of a list M-partition problem, where G is a proper interval graph. Then there exists an ordering of $V(G)$ such that the total number of profiles for any D_i is at most n^{2m}.*

Proof. Let $v_1, v_2, \ldots, v_n$ be a proper interval ordering of G. From Theorem 2.1.10, if a vertex v_j is adjacent to v_i, then v_i must also be adjacent to all the vertices that are present between v_i and v_j of the ordering, i.e., if $i < j$ then v_i is adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$. This implies that $N(v_n) \cap D_i \subseteq N(v_{n-1}) \cap D_i \subseteq \cdots N(v_{i+1}) \cap D_i$. Refer to Figure 2.8.

Now, consider an M-partition f of D_i. Recall that $P^f_{D_i}(v)$ is the set of colors assigned to the neighbours of v in D_i. Therefore, we have $P^f_{D_i}(v_n) \subseteq P^f_{D_i}(v_{n-1}) \subseteq \cdots P^f_{D_i}(v_{i+1})$, which is an inclusive sequence. Each set $P^f_{D_i}(w_i)$ is a subset of $[m]$, so the total number of such inclusive sequences is at most $(n-i+1)^m$. This implies that the total number of possible functions for $P^f_{D_i}$ is at most $(n-i+1)^m$. Similarly, we can argue that the total number of possible functions for $\bar{P}^f_{D_i}$ is at most $(n-i+1)^m$. Hence, the total number of profiles $\mathbf{P} = (P^f_{D_i}, \bar{P}^f_{D_i})$ is at most $(n-i+1)^{2m}$. □

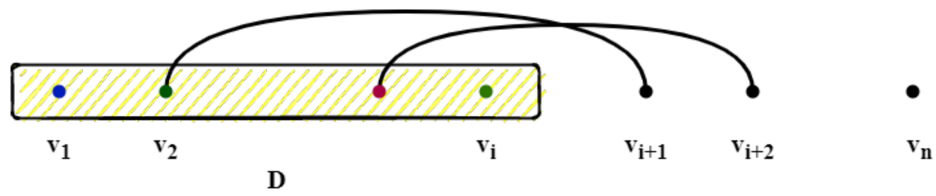


Figure 2.8: Explanation for inclusive sequence for $P^f_{D_i}$ and $\bar{P}^f_{D_i}$

Corollary 2.4.13.1 ([40]). *For any M, the restriction of the M-partition problem to instances (G, L) where G is a proper interval graph, can be solved in polynomial time.*

Proof. From Theorem 2.4.13, we know that there exists an ordering of $V(G)$ such that the number of profiles for each D_i is bounded by n^{2m}. From Theorem 2.4.12, we can conclude that the list M-partition for interval graphs can be solved in $\mathcal{O}(n \cdot \sum_{i=0}^{n}((n-i+1)^{2m})^2) = \mathcal{O}(n^{4m+1})$ time. □

The proof of Theorem 2.4.13 can be generalized to more graph classes by defining the following. For a graph G, an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$ is called a *t-bounding* ordering, if for any $1 \le i \le n$, the set $V(G) - D_i$ can be partitioned into at most t parts, such that for any two vertices u, v from the same part we must have one of $N(u) \cap D_i$, $N(v) \cap D_i$ to be the subset of the other, i.e., for all vertices w from the same part the sets $N(w) \cap D_i$ must form an inclusive sequence. Observe that, proper interval graphs have a 1-bounding ordering as seen in the proof of Theorem 2.4.13. The following theorem states that t-bounding ordering is a sufficient condition to have polynomially many profiles for each D_i.

Theorem 2.4.14 ([40]). *Let (G, L) be any instance of list M-partition problem. If G admits a t-bounding ordering, then the number of profiles for each D_i is at most n^{2mt}.*

In [40], it was shown that various graph classes admit a t-bounding ordering, which can be found in polynomial time. For example, permutation graphs have a 1-bounding ordering, and circular arc graphs have a 2-bounding ordering.

Chapter 3

Minimal PI Obstructions for 3×3 Matrices

Recall that a graph G is called a proper interval graph, if G is an interval graph such that in some interval representation of G, no interval is properly contained in any other interval. There exists an ordering $v_1, v_2, \ldots, v_n$ of vertices of a proper interval graph G, such that if $v_i,\ v_j$ are adjacent, then all the vertices that appear between $v_i,\ v_j$ in the ordering are also adjacent to each other.

In this chapter, we will prove that all matrices of size 3 have only finitely many minimal proper interval obstructions (minimal PI obstructions). In Section 2.4.2, Theorem 2.4.8, states that there are only finitely many minimal chordal obstructions for any matrix of size 3 except for the following two matrices.

$$M_1 = \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 1 \end{bmatrix}$$

Every proper interval graph is a chordal graph. Therefore, to prove that all matrices of size 3 have finitely many minimal PI obstructions, it is sufficient to prove that M_1 and M_2 have finitely many minimal PI obstructions.

3.1 Minimal Obstructions to M_1

Before showing that M_1 has only finitely many minimal PI obstructions, we prove an interesting result which states that any proper interval graph that is M_1-partitionable cannot have more than 9 vertices. We note that bipartite graphs of any size are M_2-partitionable.

Proposition 3.1.1. *Let G be any connected proper interval graph that is not bipartite. If G is M_1-partitionable then G has at most 9 vertices.*

Proof. Let V_1, V_2, V_3 be an M_1-partition of G. Since G is not bipartite, it must contain a C_3, because a proper interval graph cannot have any cycle of length ≥ 4. Thus, the parts

V_1, V_2, V_3 will have at least one vertex each. The graph induced by the vertices of parts V_2, V_3 must be a complete bipartite graph. Hence, at least one of the parts must contain only one vertex, else G will have an induced C_4, contradicting that G is an interval graph. Without loss of generality, assume that V_3 has only one vertex. Observe that V_2 can have at most two vertices, else there will be an induced claw. So $V_2 \cup V_3$ can have at most 3 vertices. Each vertex of V_2, or V_3 can have at most two neighbours in V_1. Therefore, the total number of vertices in G is at most 9. □

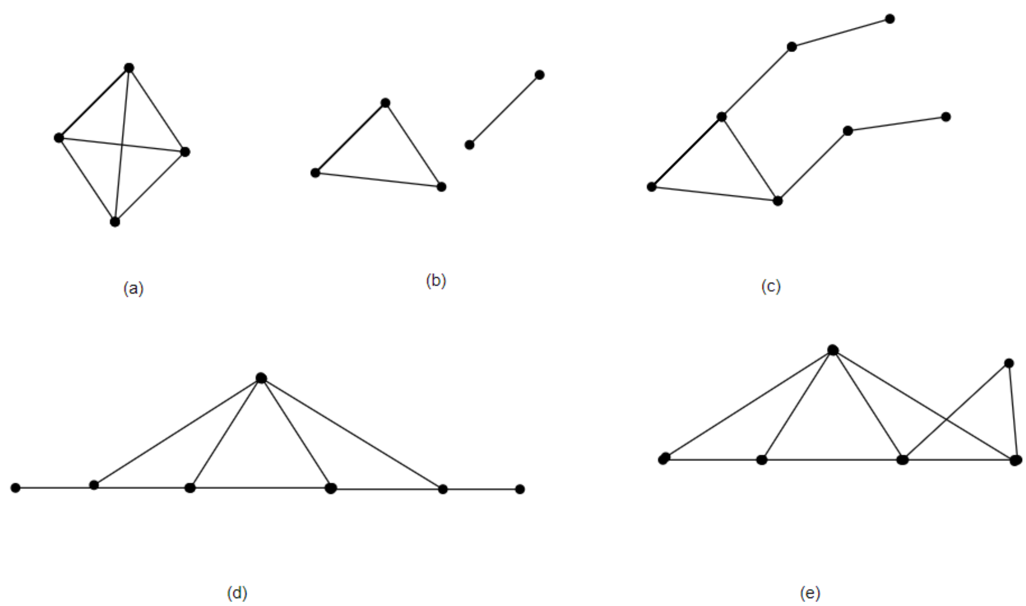


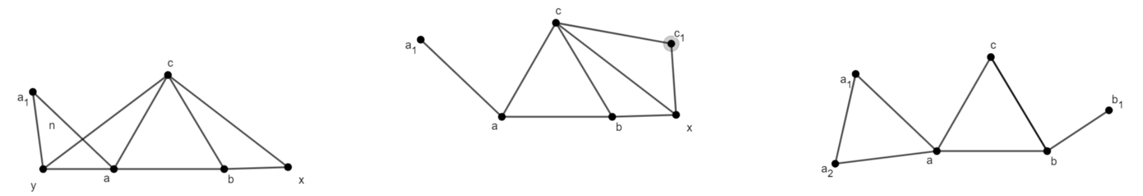
Figure 3.1: Minimal PI obstructions to M_1.

Note that Proposition 3.1.1 implies that all minimal PI obstructions for M_1 have fewer than 10 vertices, and hence there are only finitely many. We now explicitly describe these obstructions: they are depicted in Figure 3.1.

Theorem 3.1.2. *The matrix M_1 has 5 minimal PI obstructions, shown in Figure 3.1.*

Proof. All the minimal PI obstructions to M_1 are shown in Figure 3.1. Consider a proper interval graph G. If G has no C_3, then G is a bipartite graph, and hence can be partitioned into parts V_1 and V_2. Thus, assume that G has at least one C_3 induced by the vertices a, b, c. Considering that G does not have any minimal obstruction shown in Figure 3.1, we will find an M_1-partition of G. Now consider the following sets

- Let W be the set of vertices that are not adjacent with any of a, b, c. The set W must be an independent set, else an edge in W along with a, b, c will induce a minimal obstruction shown in Figure 3.1(b).



(a) The sets X, Y are non empty, corresponds to Case 1

(b) The set X is non-empty, then A can be non-empty one of B or C can be non-empty.

(c) The sets A, B are non-empty.

Figure 3.2

- Let A be the set of vertices that are adjacent to a but not with b and c. The vertices of A must induce a clique, because if there are two vertices u, v in A that are not adjacent, then u, v, b, a will induce a claw.
- Let B be the set of vertices that are adjacent to b but not with a and c. The vertices of B must induce a clique, else there will be an induced claw.
- Let C be the set of vertices that are adjacent to c but not with b and a. The vertices of C must induce a clique, else there will be an induced claw.
- Let X be the set of vertices that are adjacent to b and c but not with a. The vertices of X must induce a clique, else there will be an induced claw. Moreover, if X has two vertices then G will have an induced K_4 which is a minimal obstruction. Thus, X cannot have more than one vertex.
- Let Y be the set of vertices that are adjacent to a and c but not with b. The set Y cannot have more than one vertex.
- Let Z be the set of vertices that are adjacent to b and a but not with c. The set Z cannot have more than one vertex.
- Let S be the set of vertices that are adjacent to a, b, and c. The set S must be empty, else there will be an induced K_4.

We consider various cases based on the cardinality of X, Y, Z. The sets X, Y, Z cannot be all non-empty, because it will form an asteroidal triple contradicting the fact that G is a proper interval graph. In Case 1, we assume that exactly two of X, Y, Z are non-empty, in Case 2, we assume that exactly one of X, Y, Z is non-empty, and in Case 3 we assume that all the sets X, Y, Z are empty.

Case 1: Exactly two of X, Y, Z are non-empty.

Without loss of generality, assume that X and Y are non-empty. Let x be a vertex in X and y be a vertex in Y. Consider the set A which has vertices that are adjacent to a but not b or c. Let a_1 be a vertex in A. Then a_1 must be adjacent to y else the vertices a, y, b, a_1

induces a claw. If $a_1,\ y$ are adjacent then the graph induced by the vertices a, b, c, x, y, a_1 is same as the minimal obstruction in 3.1(e). This implies that the set A must be empty. Due to symmetry, the set B must also be empty. Finally, assume that the set C is non-empty. Let c_1 be a vertex C. If c_1 is not adjacent to x or y, then G will have the claws c, c_1, x, a and c, c_1, y, b respectively, but if c_1 is adjacent to x and$y$, then G will have the cycle c_1, x, b, a, y. Thus, the sets $A,\ B,$ and C are all empty.

Now consider the set W which contains the vertices that are not adjacent to any of a, b, c. A vertex from W cannot be adjacent to both x and y at the same time, because it will induce a cycle with the vertices y, a, b, x. If there are two vertices in W such that one vertex is adjacent to x and the other with y then G will have the minimal obstruction shown in Figure 3.1(d). Therefore, all the vertices of W are adjacent to either x or y but not both. Without loss of generality, assume that the vertices of W are adjacent to x.

In this case, G is M_1-partitionable with $V_1 = W \cup \{b, y\}$, $V_2 = \{x, a\}$, $V_3 = \{c\}$.

Case 2: Exactly one of X, Y, Z is non-empty.

Without loss of generality, assume that X is non-empty. Let x be a vertex in X. Any vertex of B must be adjacent with x, else it will form a claw with the vertices b, a, x. Similarly, any vertex from C must also be adjacent with x, else it will form a claw with the vertices c, a, x. If both B and C are non-empty, say $b_1 \in B$ and $c_1 \in C$, then it will result in an asteroidal triple a, b_1, c_1. Without loss of generality, assume that only C is non-empty. Since $C \cup X$ induces a clique, C has only one vertex. Recall that the set A induces a clique. If a vertex in A is either adjacent to x or c_1 then G will have an induced C_4. Moreover, if A has more than one vertex then the edge in A along with the triangle c, x, c_1 will be a minimal obstruction. This implies that A can only have one vertex.

Observe again that W must be an independent set. No vertex in W can be adjacent to a_1 and c_1 as it will result in a cycle of length greater than 4. No vertex in W can be adjacent only with a_1, because it will result in the minimal obstruction with the triangle c, x, c_1. No vertex in W can be adjacent only with c_1 because it will result in the minimal obstruction shown in Figure 3.1(d). Finally, no vertex in W can be adjacent only with x because it will result in an asteroidal triple. Now we can give a partition of G.

In this case, G is M_1-partitionable with $V_1 = W \cup \{b, a_1, c_1\}$, $V_2 = \{a, x\}$ and $V_3 = \{c\}$. If C is empty the the vertices of W can be adjacent with x in which case G is M_1-partitionable with $V_1 = W \cup \{b, a_1\}$, $V_2 = \{a, x\}$ and $V_3 = \{c\}$

Case 3: All the sets X, Y, Z are empty.

Observe that the sets A, B, C cannot be all non-empty, else the graph G will have an asteroidal triple. Recall that these sets induce cliques, therefore, the cardinality of these sets is at most 2. Now, lets consider the following subcases

Case 3a: Exactly two of A, B, C are non-empty.

Without loss of generality, assume that A and B are non-empty. Recall that the sets A and B induce cliques. Vertices of A cannot be adjacent with any vertex from B as it will induce a C_4. Both A and B cannot have more than one vertex at the same time because if a_1, a_2 are the vertices in A and b_1, b_2 are the vertices in B. The edge b_1, b_2 along with triangle a, a_1, a_2 induce a minimal obstruction, and the edge a_1, a_2 along with the triangle b, b_1, b_2 also induce a minimal obstruction. Thus, only one of A,B can have more than one vertex. Without loss of generality, assume that B has one vertex, b_1. Observe that the vertices of W cannot be adjacent with b_1 and with some vertex in A at the same time, because it will result in a C_4. If there is a vertex in W that is adjacent to b_1 it will induce an minimal obstruction along with the triangle a, a_1, a_2. The vertices of W cannot be adjacent with both a_1 and a_2 at the same time because then G will again have a minimal obstruction with the edge bc. Therefore, all vertices of W can only be adjacent with one vertex in A, let that vertex be a_1.

In this case, G is M_1-partitionable with $V_1 = W \cup \{b_1, c, a_2\}$, $V_2 = \{a_1, b\}$ and $V_3 = \{a\}$.
Case 3b: Exactly one of A, B, C is non-empty.

Without loss of generality, assume that A is non-empty and let a_1, a_2 be the vertices in A. The vertices of W cannot be adjacent with both a_1 and a_2 at the same time, because if there is such a vertex $u \in W$, then the triangle a_1, a_2, u and the edge b, c will result in a minimal obstruction. Therefore,any vertex from W can only be adjacent with one vertex in A, let that vertex be a_1. In this case, G is M_1-partitionable with $V_1 = W \cup \{c, a_2\}$, $V_2 = \{a_1, b\}$ and $V_3 = \{a\}$.
Case 3c: All the sets A, B, C are empty. In this case, G is not a connected graph and it has a independent set W and a triangle(a, b, c). Thus, G is M_1-partitionable with the vertices of W in V_1 and the vertices a, b, c in three different parts. □

3.2 Minimal Obstructions to M_2

For any graph G, the size of a maximum clique is called the clique number, and is denoted by ω. Recall the matrix

$$M_2 = \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 1 \end{bmatrix}.$$

Proposition 3.2.1. *Let G be any connected proper interval graph that is M_2-partitionable. If G has more than $\omega + 5$ vertices, then it is either a split graph or a bipartite graph.*

Proof. Assume that G is a proper interval graph that is not a bipartite or a split graph. Assume that G is M_2-partitionable; we will show that it cannot have more than $\omega + 5$ vertices. Let V_1, V_2, V_3 be an M_2-partition of G. The sets V_1, V_2, V_3 will be non-empty. Note that the parts V_1 and V_2 are independent sets and the part V_3 a clique, and the sets V_2, V_3 are completely adjacent.

The set V_2 cannot have more than two vertices, because if it has three or more vertices, then these vertices will induce a claw with some vertex from V_3. Furthermore, if V_2 has exactly one vertex, then that vertex can be placed in V_3. This implies that G is a split graph, contradicting our assumption. Therefore, in the M_2-partition of G, the set V_2 will have exactly two vertices, say u, v.

We can now prove a bound on the number of vertices in V_1. The set V_1 cannot have vertices that are adjacent to some vertex in V_3 but not adjacent to u or v, as this will result in a claw. The vertices u, v can have at most two neighbours each in V_1, else a claw is present. Therefore, V_1 can have at most 4 vertices.

The vertices in V_3 along with one vertex, say v in V_2, induces a clique. The set V_2 has another vertex u, and V_1 has at most 4 vertices. Therefore, G has at most $\omega+5$ vertices. ☐

Theorem 3.2.2. *The matrix M_2 has three minimal PI obstructions, depicted in Figure 3.1(b), (c), (d).*

Proof. Assume that G is a proper interval graph without any of these minimal obstructions. If G is a split graph, then it can be partitioned into parts 1 and 3. If it is a bipartite graph, then it can be partitioned into parts 1 and 2. So consider that G is neither a split graph nor a bipartite graph.

First, consider that G is connected. Let $v_1, v_2, \ldots, v_k$ be a maximum clique in G. In any proper interval ordering, the vertices of a maximum clique must occur contiguously. Without loss of generality, assume that in any proper interval ordering, $v_1, v_2, \ldots, v_k$ is the order in which they appear, and we denote it by C. Since G is not a bipartite graph it must have a C_3, therefore, $k \geq 3$. Recall that in any proper interval ordering, if a vertex u is adjacent to v, then u is also adjacent to every vertex that is present between u and v in the ordering.

Case 1: Four or more vertices cannot occur before or after C

Let $a, b, c, d, v_1 \ldots, v_k$ be a proper interval ordering of G. The vertex b cannot be adjacent with the vertices v_{k-2}, v_{k-1}, v_k, else the size of the maximum clique will be greater than k. This implies that the edge ab and the triangle v_{k-2}, v_{k-1}, v_k induces a minimal obstruction shown in Figure 3.1(b). We can argue the same for vertices that occur after C. Therefore, it is not possible to have four or more vertices either before or after the vertices of a maximum clique in the ordering.

Case 2: Three vertices before C and two vertices after C cannot occur

Let an ordering of the vertices be $a, b, c, v_1, v_2, \ldots, v_k, d, e$. The vertex b must be adjacent to the vertex v_{k-2}, else the edge ab and the triangle v_{k-2}, v_{k-1}, v_k induces a minimal obstruction. Now, observe that the triangle b, c, v_1 and the edge de will induce a minimal obstruction, because v_1 cannot be adjacent to d. Therefore, this case cannot occur.

Proposition 3.2.1 states that any M_2-partitionable proper interval graph that is neither a split nor a bipartite graph cannot have more than $k+5$ vertices. This can also be proved from the above two cases. We can now assume that G has at most $k+5$ vertices. Upto symmetry, we may assume that either exactly 3 vertices precede C or exactly 2 vertices precede C.

Case 3: Exactly three vertices precede C

Case 3a: No vertex occurs after C

Let an ordering of the vertices be $a, b, c, v_1, v_2, \ldots, v_k$. The vertex b must be adjacent to the vertex v_{k-2} else the edge ab, and the triangle v_{k-2}, v_{k-1}, v_k induces a minimal obstruction. If the vertices a, c are not adjacent to each other then it is easy to see that G can be partitioned with $V_1 = \{a, c, v_k\}$, $V_2 = \{b, v_{k-1}\}$ and $V_3 = \{v_1, v_2, \ldots, v_{k-2}\}$. Now if a, c are adjacent to each other then c must also be adjacent to v_{k-1} else the edge v_{k-1}, v_k and the triangle a, b, c induces a minimal obstruction. In this case, G can be partitioned with $V_1 = \{a, v_k\}$, $V_2 = \{b, v_{k-1}\}$ and $V_3 = \{c, v_1, v_2, \ldots, v_{k-2}\}$.

Case 3b: One vertex occurs after C

Let $a, b, c, v_1, v_2, \ldots, v_k, d$ be an ordering of the vertices. The vertex b must be adjacent to the vertex v_{k-2}, else the edge ab and the triangle v_{k-2}, v_{k-1}, v_k induce a minimal obstruction. Note that the vertex c cannot be adjacent to v_k. So if a, c are adjacent, then the triangle a, b, c and the edge $v_k d$ will be a minimal obstruction. Therefore, a, c cannot be adjacent. Similarly, d cannot be adjacent to v_{k-1} because of the edge ab and triangle v_{k-1}, v_k, d. If the vertex c is adjacent with v_{k-1}, the path on vertices a, b, c, v_{k-1}, v_k, d along with a vertex v_i will induce the minimal obstruction shown in the Figure 3.1(d). The existence of the vertex v_i follows from the fact that $k \geq 3$. Hence, c, v_{k-1} are not adjacent to each other. We can partition G with $V_1 = \{a, c, v_{k-1}, d\}$, $V_2 = \{b, v_k\}$ and $V_3 = \{v_1, v_2, \ldots, v_{k-2}\}$.

Case 4: Two vertices precede C and two vertices succeed C

Let $a, b, v_1, v_2, \ldots, v_k, c, d$ be the ordering of vertices. The vertex b must be adjacent to v_{k-1} and the vertex c must be adjacent to v_2, else there will be a minimal obstruction shown in Figure 3.1(c). The vertices a, v_1 cannot be adjacent to each other, else a, b, v_1 and the edge c, d will be a minimal obstruction. Similarly, the vertices d, v_k cannot be adjacent. Therefore, G can be partitioned with $V_1 = \{a, v_1, v_k, d\}$, $V_2 = \{b, c\}$ and $V_3 = \{v_2, v_3, \ldots, v_{k-1}\}$.

Now, consider that G is not connected. Since we assumed that G is not a bipartite graph, there must exist a C_3. Let X be a connected component in G that contains C_3. No other connected component can have an edge, because it will result in a minimal obstruction shown in Figure 3.1(b). This implies that G can only have one connected component with edges, the remaining vertices must form an independent set. Therefore, G is M_2-partitionable if and only if X is M_2-partitionable. Hence, we proved that M_2 has finitely many minimal PI obstructions.

□

Chapter 4

Minimal PI Obstructions for 4×4 Matrices

In this chapter, we will look at minimal PI obstructions for matrices of size 4. There are too many $\{0, 1, \star\}$-symmetric matrices of size 4 that are possible, so as a first step we considered those matrices that have a constant diagonal. We call a matrix *0-diagonal*, if all the entries on the main diagonal are 0. Similarly, we call a matrix *1-diagonal*, if all the entries on the main diagonal are 1. In Section 4.1, we will look at 0-diagonal matrices of size 4 and in Section 4.2, we will look at 1-diagonal matrices of size 4.

4.1 0-diagonal Matrices

We divide all the 0-diagonal matrices into three subsections based on the off-diagonal entries. If the set of off-diagonal entries is a subset of $\{0, 1\}$ or $\{\star\}$, then the matrix will be a normal matrix. In Section 2.4.1, we have seen that any normal matrix has finitely many minimal perfect obstructions and hence finitely many minimal PI obstructions. From Theorem 2.4.1, we know that any perfect graph with clique number k can be partitioned into k independent sets, and K_{k+1} is the only minimal obstruction. As we are only looking at 0-diagonal matrices, K_5 will be an obstruction to every such matrix.

4.1.1 Off-diagonal Entries Subset of $\{1, \star\}$

In this section, we will look at 0-diagonal matrices of size 4 with no off-diagonal entry equal to 0.

If all the off-diagonal entries are equal to $\star$, then the partition problem is same as the vertex coloring problem. Hence, there is only one minimal PI obstruction, K_5. Before proving that there are only finitely many minimal PI obstructions for other matrices, we will first look at a result that is very useful. We denote by $K_p + K_q$ the disconnected graph consisting of K_p and K_q.

Theorem 4.1.1. *Let G be any proper interval graph with clique number p, and let $q \leq p$. If G does not contain an induced $K_p + K_q$, then there is a set S of at most $3p + 2q$ vertices such that $G - S$ contains no K_q. Furthermore, the vertices of S occur consecutively in any proper interval ordering of G.*

Proof. Assume that $v_1, v_2, \ldots, v_n$ is a proper interval ordering of G. Let $v_i, v_{i+1}, \ldots, v_{i+p-1}$ be a fixed K_p in G. Let S be the set of vertices, v_j with $i - p - q \leq j \leq i + 2p + q - 1$. Note that S has $3p + 2q$ consecutive vertices. Since the clique number is p no vertex in $G - S$ can be adjacent to v_k, where $i \leq k \leq i + p - 1$. Thus, any K_q in $G - S$ would yield a $K_p + K_q$ in G. □

In the above theorem, even if the clique number of G is r instead of p, with $r > p$ and G is a $(K_p + K_q)$-free graph, we have an upper bound of $3r + 2q$ for the number of vertices in S.

Let us look at the case of exactly one off-diagonal entry equal to 1. Up to symmetry there is only one such matrix, M_1

$$M_1 = \begin{bmatrix} 0 & \star & \star & \star \\ & 0 & \star & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

Theorem 4.1.2. *The matrix M_1 has finitely many minimal PI obstructions.*

Proof. We know that K_5 is a minimal obstruction to M_1. Another minimal obstruction to M_1 is $K_4 + K_3$. Each vertex of K_4 or K_3 must go to a different part and the vertices of parts 3 and 4 must be completely adjacent, therefore $K_4 + K_3$ is an obstruction. If one vertex is removed from K_4, then it will be 3-colorable and hence M_1-partitionable. If a vertex is removed from K_3 we have a $K_4 + K_2$ which is M_1-partitionable. Note that any other minimal obstruction to M_1 cannot contain a K_5 or $K_4 + K_3$. If a graph does not have a K_4, then it is 3-colorable, and hence M_1-partitionable. Thus, any obstruction to M_1 must have a K_4. Assume that there is a minimal PI obstruction G of size 19 and let $v_1, v_2, \ldots, v_{19}$ be a proper interval ordering of $V(G)$. From Theorem 4.1.1, any clique larger than K_2 belongs to a set of at most $18 (= 3 \times 4 + 2 \times 3)$ consecutive vertices, so either v_1 or v_{19} will only have one neighbour. Without loss of generality, assume that v_1 has only one neighbour v_2. Since G is a minimal obstruction, $G - v_1$ is M_1-partitionable and let V_1, V_2, V_3, V_4 be its M_1-partition. Note that the vertex v_1 can also be placed in one of V_1 or V_2 based on where its neighbour v_2 is present. This contradicts the fact that G is a minimal obstruction. Hence, we have proved that any minimal PI obstruction to M_1 has at most 18 vertices which in turn implies that they are finitely many minimal PI obstructions to M_1. Note that even if K_5, $K_4 + K_3$ are obstructions that are not necessarily minimal, the bound on the size of a minimal obstruction is still valid. □

We will now look at the matrix M_2 that has two off-diagonal entries equal to 1.

$$M_2 = \begin{bmatrix} 0 & \star & \star & \star \\ & 0 & 1 & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

Theorem 4.1.3. *The matrix M_2 has finitely many minimal PI obstructions.*

Proof. It can be easily proved that K_5, $K_4 + K_1$ and $K_3 + K_2$ are minimal obstructions to M_2, and any minimal obstruction must contain a K_3, else it is 2-colorable and hence M_2-partitionable. We now claim that any minimal PI obstruction to M_2 has at most 14 vertices. Assume that there is a minimal obstruction G with at least 15 vertices. We consider two cases when the clique number of G is 4 and 3 respectively. First, lets consider that G has a K_4. Then from Theorem 4.1.1, G cannot have more than 14 vertices. Now, consider that G has a K_3 but not a K_4. Then from Theorem 4.1.1, every edge of G belongs to a set of at most 13 vertices. This implies that there are two vertices u, v in G that have no neighbours. If $G - u$ is M_2-partitionable, then v can also be placed in the same part as u which is a contradiction to our assumption that G is a minimal obstruction. In both the cases we have proved that the number of vertices in any minimal PI obstruction of M_2 is bounded. □

Using the techniques from Theorem 4.1.2 and 4.1.3, we can prove that every matrix of this type has finitely many minimal PI obstructions by constructing 2 or 3 minimal obstructions. In Table 4.1, we give the minimal obstructions to the remaining matrices that are sufficient to prove that the number of vertices in any other minimal obstruction is also upper bounded.

4.1.2 Off-diagonal Entries Subset of $\{0, \star\}$

We say that a row i dominates a row j, if for each column k with $1 \leq k \leq m$, either $M_{i,k} = \star$ or $M_{i,k} = M_{j,k}$. Note that in a matrix M, if there is a row i that dominates row j, then any vertex that can be placed in the part j can also be placed in the part i. Hence, we can reduce the problem to an equivalent smaller size matrix by deleting the row j and the column j.

Lets look at the matrices where off-diagonal entries are $\{0, \star\}$. From Table 4.1, we can construct all such matrices by replacing all the 1's by 0's. All the cases have at least one

Number off-diagonal entries 1	**Matrix**	**Minimal obstructions**
1	$\begin{bmatrix} 0 & \star & \star & \star \\ & 0 & \star & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_3$
2	$\begin{bmatrix} 0 & \star & \star & \star \\ & 0 & 1 & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_1,\ K_3 + K_2$
2	$\begin{bmatrix} 0 & \star & \star & \star \\ & 0 & \star & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_2$
3	$\begin{bmatrix} 0 & \star & \star & \star \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_3 + K_2$
3	$\begin{bmatrix} 0 & 1 & \star & \star \\ & 0 & 1 & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_1,\ K_3 + K_2$
3	$\begin{bmatrix} 0 & \star & \star & 1 \\ & 0 & \star & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_1$
4	$\begin{bmatrix} 0 & 1 & \star & \star \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_4 + K_1,\ K_3 + K_2$
4	$\begin{bmatrix} 0 & 1 & \star & 1 \\ & 0 & 1 & \star \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_3 + K_1$
5	$\begin{bmatrix} 0 & \star & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ K_3 + K_1$
6	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	$K_5,\ \overline{K_2} = K_1 + K_1$

Table 4.1: Minimal obstructions that are used to prove the upper bound on the number of vertices in every other minimal obstruction.

dominating row except the matrix below

$$M = \begin{bmatrix} 0 & 0 & \star & 0 \\ & 0 & 0 & \star \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

In this case, K_3 is the only minimal PI obstruction, because any graph without K_3 is 2-colorable and hence M-partitionable(can be partitioned either between 1 and 3 parts or between 2 and 4). Hence, we have the following result.

Theorem 4.1.4. *Any 0-diagonal matrix of size 4 with no off-diagonal entry 1 has finitely many minimal PI obstructions.*

4.1.3 Off-diagonal Entries Include all of 0, 1, and $\star$

In this section, we will consider 0-diagonal matrices such that off-diagonal entries include all of 0,1, and $\star$; all the other cases of 0-diagonal matrices have been considered in Sections 4.1.1 and 4.1.2. These matrices correspond to a partition into four independent sets. To construct all such possible partitions, we consider four parts that are independent and explore all the connections between them. A 1-connection between two parts denotes that they are completely adjacent, a 0-connection denotes that they are completely non-adjacent and a $\star$-connection denotes that there are no restrictions on the adjacency between the two parts. Using this notation, we construct all possible partitions in the following proof.

Theorem 4.1.5. *The M-partition problem for any 0-diagonal matrix of size 4 that includes all of 0, 1 and $\star$ as off-diagonal entries has finitely many minimal PI obstructions.*

Proof. Consider four parts A, B, C, D that are independent. Since there exists at least one 0 in the off-diagonal entries, without loss of generality, we can assume that A has a 0-connection to B in all the cases. Note that K_4 is always an obstruction irrespective of the other connections.

Consider the case where there are two 0-connections that are disjoint. Without loss of generality, assume that there is a 0-connection between A, B and between C, D, refer to Figure 4.1(a). Since there must exist at least one $\star$-connection, a graph is M-partitionable if and only if it is bipartite. Thus K_3 will be the only minimal obstruction.

We will now look at the cases where there are no disjoint 0-connections. First, assume that a part is joined to three other parts with 0-connection. Assume that A has 0-connection with B, C, and D. There must exist at least one 1-connection and a $\star$-connection between B, C, and D, so one of the partitions from Figure 4.1(d),(e) will be present. In either case, $K_3 + K_2$ will be an obstruction.

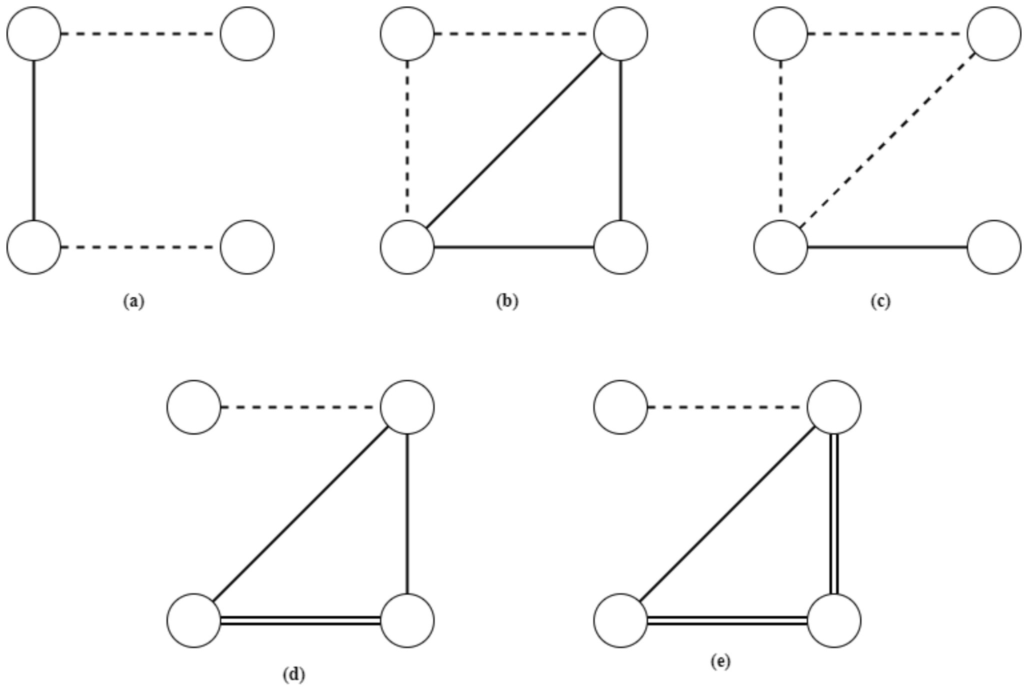


Figure 4.1: Various possible cases for partitions when off-diagonal entries have at least one 0, 1, $\star$. A dotted line denotes 0-connection, single solid line denotes $\star$-connection, and a double solid line denotes 1-connection.

Now, assume that a part is joined to two other parts with 0-connections. Without loss of generality, assume that A has 0-connection to B and D. If either of B, C, D has a 0-connection, we will either have a case with two disjoint 0-connections or the one shown in Figure 4.1(c). Then K_3 will be the only minimal obstruction. If there is no 1-connection among B, C, D, then K_4 will be the only minimal obstruction, refer to Figure 4.1(b). So, this leaves the case where there is at least one 1-connection among B, C, D in which case, $K_3 + K_2$ will be an obstruction.

Finally, if there is a part that is joined to one other part with 0-connection and two parts with $\star$-connections, then that part will be dominating. Part A is also dominating if it has a 0-connection to B, 1-connection to D, and $\star$-connection to C, as long as B and D have a 1-connection as well. So we consider that B has a $\star$-connection to D and 1-connection to C as all the other cases have already been considered. In this case, $K_3 + K_2$ is always an obstruction. If A has one 0-connection to B and two 1-connections, then based on connections of B either we have a case where one of A, B dominates or the case that we have previously considered.

In all the above cases, we either have K_4 or K_3 as the only minimal obstruction or the case where $K_3 + K_2$ as an obstruction. From Theorem 4.1.1, we know that in a $(K_3 + K_2)$-free proper interval graph every edge is contained in a set S of at most 16 vertices. Thus any proper interval graph with more than 17 vertices cannot be a minimal obstruction. Hence, we have proved that there are only finitely many minimal PI obstructions. □

4.2 1-diagonal Matrices

In this section, we will look at 1-diagonal matrices of size 4. If the set of off-diagonal entries is a subset of $\{0, 1\}$ or $\{\star\}$, then the matrix will be a normal matrix, and hence will have finitely many minimal PI obstructions. As in Section 4.1, we divide the discussion into subsections based on the off-diagonal entries.

4.2.1 Off-diagonal Entries Subset of $\{0, \star\}$

From Theorem 2.4.1, we know that any perfect graph with clique number k is k-colorable, i.e., can be partitioned into k independent sets. Recall that the independence number of a graph G is the size of a maximum independent set. Since the complement of a perfect graph is also a perfect graph, from Theorem 2.3.1, we can conclude that any perfect graph with independence number k can be partitioned into k cliques.

Given an ordering of vertices of G, a clique is said to be *contiguous*, if it is induced by a contiguous set of vertices in the ordering.

Lemma 4.2.1. *Any proper interval graph with independence number k and a proper interval ordering, can be partitioned to k contiguous cliques.*

Proof. Let G be a proper interval graph with independence number k, and let $\mathcal{O}$ be a proper interval ordering of its vertices. Let v_1 be the first vertex in the ordering $\mathcal{O}$, and let v_1' be the rightmost neighbour of v_1. All the vertices from v_1 till v_1' in $\mathcal{O}$ induces a clique and we place them in the clique C_1. Note that C_1 is a contiguous clique with respect to $\mathcal{O}$. Let v_2 be the vertex next to v_1' in $\mathcal{O}$, and v_2' be the rightmost neighbour of v_2, then all the vertices between v_2 and v_2' induces a clique and we place them in clique C_2. We repeat the process till there is no vertex remaining in the ordering. Let v_i be the first vertex that is placed in C_i and v_i' be the rightmost neighbour of v_i. Note that the vertices $v_1, v_2, \ldots, v_i, \ldots, v_k$ are independent. If the rightmost neighbour of v_k is v_k', then there will not be any vertex after v_k' in $\mathcal{O}$ as the independence number is k. Hence, we proved that any proper interval graph with independence number k can be partitioned into k contiguous cliques. □

The sequence of contiguous cliques $C_1, C_2, \ldots, C_k$ obtained in the above proof is called the *canonical sequence of contiguous cliques,* and the corresponding sequence of vertices $v_1, \ldots, v_1', v_2, \ldots, v_2', \ldots, v_k, \ldots, v_k'$ is called the *canonical sequence of vertices*. Note that C_i is induced by the vertices $v_i, v_{i+1}, \ldots v_i'$ of the ordering. The above proof also shows that any proper interval graph with an independence number greater than k cannot be partitioned into k cliques. Since we consider 1-diagonal matrices of size 4, $\overline{K_5}$ will be an obstruction to every matrix considered in this section.

Corollary 4.2.1.1. *Any proper interval graph with independence number k, can be partitioned into k cliques $C_1, C_2, \ldots C_i, \ldots C_k$, such that for $1 < i < n$, C_{i-1} and C_{i+1} are completely non-adjacent.*

Proof. Let G be a proper interval graph with independence number k, and a proper interval ordering $\mathcal{O}$. From Lemma 4.2.1, we know that G can be partitioned into k contiguous cliques. Let $C_1, C_2, \ldots, C_k$ be the canonical sequence of contiguous cliques and let the canonical sequence of vertices be $v_1, ..., v'_1, v_2, ..., v'_2, ..., v_k, ..., v'_k$. A vertex $u \in C_{i-1}$ cannot be adjacent to a vertex $v \in C_{i+1}$, this is because the rightmost neighbour of v_i is v'_i. Therefore, the cliques C_{i-1} and C_{i+1} are completely non-adjacent. It is easy to observe that in the canonical sequence of contiguous clique, any clique that is present before C_i is completely non-adjacent to any clique that appears after C_i. □

Lemma 4.2.2. *Any connected proper interval graph with independence number at least 3 has an induced P_5 or induced bull graph.*

Proof. The bull graph is shown in Figure 4.2(c). Consider a proper interval graph G with independence number 3. From Lemma 4.2.1, we know that G can be partitioned into 3 contiguous cliques. Let C_1, C_2, C_3 be the canonical sequence of contiguous cliques, and let $a, \ldots, a', b, \ldots, b', c, \ldots, c'$ be the canonical sequence of vertices. Let x be a vertex in C_1 that is adjacent to b, and let y be a vertex in C_2 that is adjacent to c. If the vertices x and y are non-adjacent, then a, x, b, y, c induces a path of length 5. If the vertices x and y are adjacent, then a, x, b, y, c induces a bull graph. The existence of vertices x and y follows from the fact that G is connected. Hence, we proved that any connected proper interval graph with independence number at least 3 will have a P_5 or a bull graph. □

Lemma 4.2.3. *Let G be any proper interval graph without an induced $P_3 + P_3$. Then G can only have one connected component that is not a clique. Moreover, no connected component can have an independence number greater than 3.*

Proof. Any connected graph that is not a clique must have an induced P_3, because there exists two vertices that are not adjacent but are joined by a path. Hence, there will be an induced P_3. Now, consider a proper interval graph G that does not have an induced $P_3 + P_3$. Assume that there are two connected components X, Y in G that are not cliques. We know that both X and Y will have an induced P_3, which will result in an induced $P_3 + P_3$ in G. Therefore, G cannot have more than one component that is not a clique. Now, let us assume that the connected component X in G has independence number 4. Let C_1, C_2, C_3, C_4 be the canonical sequence of contiguous cliques and $a, \ldots, a', b, \ldots, b', c, \ldots, c', d, \ldots, d'$ be the canonical sequence of contiguous vertices. Let x be a vertex in C_1 that is adjacent to b, and y be a vertex in C_3 that is adjacent to d. The existence of x and y follows from the fact that X is connected. The vertices a, x, b and c, y, d must induce $P_3 + P_3$, because the vertices

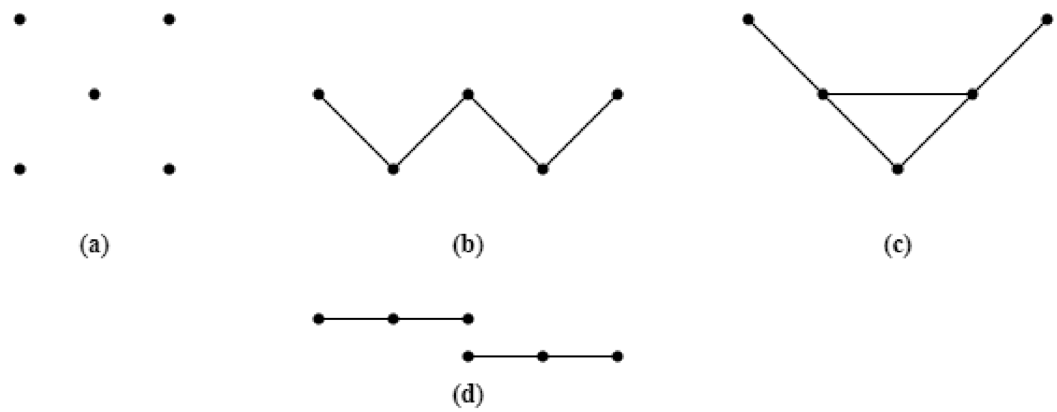


Figure 4.2: (a) $\overline{K_5}$ (b) P_5 (c) Bull graph (d) $P_3 + P_3$

before b cannot be adjacent to any vertex after c. Hence, we proved that any connected component cannot have independence number greater than 3. □

Theorem 4.2.4. *Any 1-diagonal matrix of size 4 with no off-diagonal entry 1 has finitely many minimal PI obstructions.*

We can obtain all the possible matrices by replacing 0's with 1 and 1's with 0 from Table 4.1. We will first consider the following four matrices.

$$M_1 = \begin{bmatrix} 1 & \star & \star & 0 \\ & 1 & \star & \star \\ & & 1 & \star \\ & & & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & \star & 0 & 0 \\ & 1 & \star & \star \\ & & 1 & \star \\ & & & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & \star & 0 & \star \\ & 1 & \star & 0 \\ & & 1 & \star \\ & & & 1 \end{bmatrix} \quad M_4 = \begin{bmatrix} 1 & \star & 0 & 0 \\ & 1 & \star & 0 \\ & & 1 & \star \\ & & & 1 \end{bmatrix}$$

Lemma 4.2.5. *For matrices $M_1, M_2, M_3,$ and M_4, $\overline{K_5}$ is the only minimal PI obstruction.*

Proof. Let G be a $\overline{K_5}$-free proper interval graph. From Corollary 4.2.1.1, we know that G can be partitioned into at most four cliques C_1, C_2, C_3, C_4, such that C_1 is non-adjacent to C_3, C_4 and C_2 is non-adjacent to C_4. Notice that such a partition is a valid partition for all the four matrices. □

$$M_5 = \begin{bmatrix} 1 & \star & \star & 0 \\ & 1 & \star & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \quad M_6 = \begin{bmatrix} 1 & \star & 0 & 0 \\ & 1 & \star & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

Lemma 4.2.6. *The matrices M_5 and M_6 have two minimal PI obstructions, $\overline{K_5}$ and P_3+P_3.*

Proof. Let G be any proper interval graph that does not contain a $\overline{K_5}$ or a $P_3 + P_3$. From Lemma 4.2.3, we know that all the connected components of G must be cliques except for one. Let X be that component which is not a clique. We also know that the independence number of X cannot be greater than 3. Assume that the independence number of X is exactly 3. From Corollary 4.2.1.1, X can be partitioned into 3 cliques C_1, C_2, C_3 such that C_1 and C_3 are non-adjacent. Since G is $\overline{K_5}-$free, there can be at most one more component Y, that is a clique. Note that C_1, C_2, C_3, Y is a valid partition for both the matrices. Even if the independence number of X is less than 3, a partition can be easily found because all the other components are cliques.

□

$$M_7 = \begin{bmatrix} 1 & 0 & 0 & \star \\ & 1 & 0 & \star \\ & & 1 & \star \\ & & & 1 \end{bmatrix} \quad M_8 = \begin{bmatrix} 1 & \star & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & \star \\ & & & 1 \end{bmatrix} \quad M_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & \star \\ & & & 1 \end{bmatrix}$$

Lemma 4.2.7. *The matrices M_7, M_8 and M_9 have finitely many minimal PI obstructions.*

Proof. The matrix M_8 has three minimal PI obstructions, namely $\overline{K_5}, P_5$ and the bull graph. Let G be a proper interval graph that does not contain any of these obstructions. The absence of a P_5 and the bull graph guarantees that any connected component in G cannot have an independence number greater than 2. In this case, it is easy to note that G can be partitioned into four cliques such that one component can be partitioned into parts 1,2, and the other into parts 3 and 4.

The matrices M_7, M_9 have four minimal PI obstructions $\overline{K_5}, P_5$ and a bull graph and $P_3 + P_3$. Assume that G is as above and has no induced $P_3 + P_3$. The minimal obstruction $P_3 + P_3$, guarantees that G has only one connected component that is not a clique. So we can have one connected component with independence number 2 which can be partitioned between the parts 3 and 4, and the remaining components which are cliques can be partitioned into other parts. □

4.2.2 Off-diagonal Entries Subset of $\{1, \star\}$

As in Section 4.1.2, there is only one matrix of this type where there is no dominating row. The matrix is as follows

$$M = \begin{bmatrix} 1 & 1 & \star & 1 \\ & 1 & 1 & \star \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

We claim that M has only three minimal PI obstructions $\overline{K_5}$, P_5 and the bull graph. Absence of P_5 and a bull graph implies that each connected component can only have at most two independent vertices and G cannot have more than 4 independent vertices because there is no $\overline{K_5}$. Therefore, any proper interval graph without these three obstructions is M-partitionable. This leads to the following theorem.

Theorem 4.2.8. *Any 1-diagonal matrix of size 4 with no off-diagonal entry equal to 0, has finitely many minimal PI obstructions.*

4.2.3 Off-diagonal Entries Include all of 0, 1 and $\star$

In this section, we will consider 1-diagonal matrices that include all of 0,1 and $\star$. As in Section 4.1.3, we consider four cliques and explore all the connections between them.

Theorem 4.2.9. *Let P be any partition that corresponds to four cliques such that there is at least one 0,1, and $\star$-connections among them, then it has finitely many minimal PI obstructions.*

Proof. Let A, B, C, D be four cliques. Since there must exist at least one 1-connection, without loss of generality, assume that A has a 1-connection to B. A $\overline{K_4}$ cannot be partitioned into 4 cliques such that at least two of them are completely adjacent. Therefore, $\overline{K_4}$ is an obstruction.

Consider the case where there are two 1-connections that are disjoint. Without loss of generality, assume that there is a 1-connection between A, B and between C, D. Since there must exist a $\star$-connection, $\overline{K_3}$ is the only minimal PI obstruction. A $\overline{K_3}$ is not partitionable, but any proper interval graph without $\overline{K_3}$ can be partitioned into two cliques. We will now construct the cases where there are no disjoint 1-connections. We divide the discussion into different cases based on the connections of A. Case 1 corresponds to A having three 1-connections which is solved in Lemma 4.2.10. Case 2 corresponds to A having exactly two 1-connections which is solved in Lemma 4.2.11. Finally, Case 3 corresponds to A having exactly one 1-connection which is solved in Lemma 4.2.12. □

Lemma 4.2.10. *Assume that there is a part that has three 1-connections, then there are only finitely many minimal PI obstructions*

Proof. Without loss of generality, assume that A has 1-connection to B, C, D. If there is a 1-connection among B, C, D, then it will result in a disjoint 1-connections case. Therefore, assume that there is no 1-connection among B, C, D. Since there must exist at least one 0-connection and one $\star$-connection we have two possible cases corresponding to Figure 4.3.

Case 1a: 0-connection between C, D; $\star$-connection between B, C and B, D.

We claim that $\overline{K_4}$ is the only minimal PI obstruction. From Theorem 4.2.1, any graph with independence number at most 3, can be partitioned into three cliques. Let C_1, C_2, C_3 be the canonical sequence of contiguous cliques. From Corollary 4.2.1.1, the cliques C_1 and C_3 are completely non-adjacent. Note that this corresponds to a valid partition with $C_1 = B$, $C_2 = C$ and $C_3 = D$. Therefore $\overline{K_4}$, is the only minimal PI obstruction.

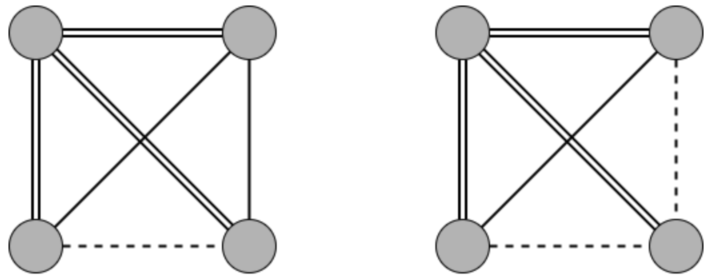


Figure 4.3: Partitions considered in Lemma 4.2.10.

Case 1b: 0-connection between B, C and C, D; $\star$-connection between B, D.
We claim that $\overline{K_4}$, P_5 and the bull graph are the only minimal PI obstructions. If a graph does not contain a $\overline{K_3}$, then it can be partitioned among B, D. Let G be a proper interval graph with independence number 3, and let p, q, r be fixed independent vertices. Note that in any partition of G, no vertex from p, q, r can be placed in A, as it must be completely adjacent to B, C, D. Without loss of generality, assume that p is placed in B, q is placed in C and r is placed in D. If there is a vertex in A, then G will have a claw. Therefore, A must be empty. Since A must be empty it is easy to observe that P_5 and bull graph are minimal obstructions.

From Lemma 4.2.2, any graph that does not contain a P_5 or a bull graph does not have any connected component that has independence number greater than 2. Assume that G has independence number at most 3 and two components X and Y, such that X has independence number 2. Vertices of X can be partitioned into B, D, and the vertices of Y can be placed in C. □

Lemma 4.2.11. *Assume that there is a part that has exactly two 1-connections, then there are only finitely many minimal PI obstructions.*

Proof. Without loss of generality, assume that A has 1-connection to B and D but not to C. If there is a 1-connection among B, C, D, then $\overline{K_3}$ will be the only minimal PI obstruction. If there is only $\star$-connection among B, C, D, then $\overline{K_4}$ will be the only minimal PI obstruction. If there are two $\star$-connections among B, C, D, then either B or D dominates A. Therefore, without loss of generality, assume that there is a $\star$-connection between B, C and 0-connection between B, D and C, D We have the following two cases based on the connection between A, C. Refer to Figure 4.4.
Case 2a: $\star$-connection between A, C.
We claim that $\overline{K_4}$ and P_6 are the only minimal PI obstructions. Let G be a proper interval graph that does not contain $\overline{K_4}$ or P_6. If the independence number of G is 2, then it is partitionable. So assume that the independence number of G is 3. If G is not connected then it is partitionable between B, C, D. Therefore, assume that G is connected. Let $B_1, B_2, \ldots, B_n$ be a straight enumeration of G, we denote it by ϕ.

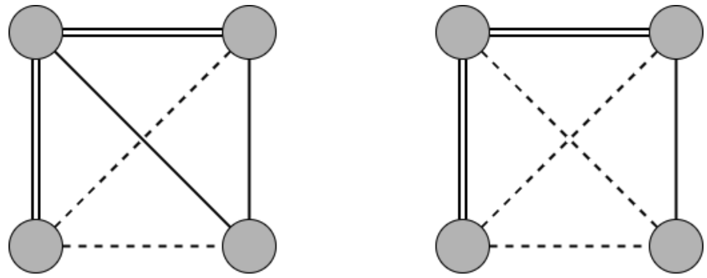


Figure 4.4: Partition considered in Lemma 4.2.11.

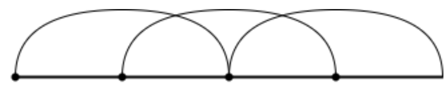


Figure 4.5: Labelled minimal obstruction for Case 2b

Let B_i be the rightmost block in ϕ that is adjacent to B_1, and B_j be the rightmost block in ϕ that is adjacent to B_2. Let B_p be the first block in ϕ that is adjacent to B_{n-1} and let B_q be the first block in ϕ that is adjacent to B_n. Note that $B_i < B_j$ and $B_p < B_q$. Then at least one of the following statements must be true.

1. The block B_{j+1} is adjacent to B_n.

2. The block B_{p-1} is adjacent to B_1.

If $i+1 = q-1$, then $i+1 = j = p = q-1$. This implies that both the above statements are true. Therefore, assume that $i+1 \neq q-1$.

Assume that both the above statements are false. This implies that $B_{j+1} < B_q < B_n$ and $B_1 < B_i < B_{p-1}$ Consider the blocks $B_1 < B_2 < B_{i+1} < B_{q-1} < B_{n-1} < B_n$. The blocks B_{i+1} and B_{q-1} must be adjacent else G will have a $\overline{K_4}$ corresponding to $B_1, B_{i+1}, B_{q-1}, B_n$. Since B_{i+1} is adjacent to B_{q-1}, G will have a P_6 corresponding to $B_1, B_2, B_{i+1}, B_{q-1}, B_{n-1}, B_n$. Therefore, at least one of the above statements must be true.

If statement 1 is true, then we partition G with $D = B_1$, $A = B_2 \cup B_3 \cdots \cup B_i$, $B = B_{i+1} \cup B_{i+2} \cup \cdots \cup B_j$ and $C = B_{j+1} \cup \cdots \cup B_n$. If statement 1 is false, then statement 2 must be true. Therefore, G can be partitioned with $D = B_n$, $A = B_q \cup \cdots \cup B_{n-1}$, $B = B_p \cup \cdots \cup B_{q-1}$, $C = B_1 \cup \cdots \cup B_{p-1}$.

Case 2b: 0-connection between A, C

We denote this partition by P. Observe that P_6 and $\overline{K_4}$ are minimal obstructions even for this partition. The graph shown in Figure 4.5 is a labelled minimal obstruction, we denote this graph by Z. We claim that any minimal PI obstruction for this partition has at most 10 vertices.

Assume that G is a connected proper interval graph. Any connected proper interval graph G has a unique straight enumeration up to reversal [8]. It is easy to observe that in any P-partition of G, either B_1 or B_n must be placed in D. This is because if a vertex $v \in B_i$ is colored D, then there cannot be a vertex colored B both in a block before and a block after B_i. Suppose, without loss of generality, that it is a block after B_i that contains colour B, then all vertices coloured A are also in blocks after B_i, and similarly for colour C. Therefore all blocks before B_i, including B_1, are coloured D. Therefore, either the vertices of B_1 or B_n must be colored D.

Applying the same analysis as in Case 2a produces either a P_6, $\overline{K_4}$ or one of the following partition

$$D = B_1,\ A = B_2 \cup B_3 \cdots \cup B_i,\ B = B_{i+1} \cup B_{i+2} \cup \cdots \cup B_j C = B_{j+1} \cup \cdots \cup B_n.$$

or

$$D = B_n,\ A = B_q \cup \cdots \cup B_{n-1},\ B = B_p \cup \cdots \cup B_{q-1},\ C = B_1 \cup \cdots \cup B_{p-1}.$$

First, assume that it produces the first partition. If A and C are non-adjacent then we are done. If A and C are adjacent, it implies that B_i must be adjacent to B_{j+1}, hence $2 \neq i$. In this case, there is a Z corresponding to the blocks $B_1, B_2, B_i, B_j, B_{j+1}$ with $B_1 \in D$. Denote by U the set of these 5 vertices in Z. If the first partition is not a P-partition, then consider the second partition. Similarly, we either find a valid P-partition or we identify a set W of 5 vertices in a copy of Z if B_n is coloured D. Therefore we proved that if G is not P-partitionable then it either has P_6, $\overline{K_4}$, or Z with $B_1 \in D$ or $B_n \in D$.

Therefore, if G is a minimal PI obstruction, then either G is P_6 or $\overline{K_4}$ or has Z with the sets U and V. Therefore, G has at most $|U \cup V|$ vertices. Possibly there is overlap between U and W, but the number of vertices is at most 10. □

Lemma 4.2.12. *Assume that each part has at most one 1-connection, then there are only finitely many minimal PI obstructions.*

Proof. Without loss of generality, assume that A has 1-connection to B. Since, each part has at most one 1-connection, A and B cannot have 1-connections to C or D. If there is a 1-connection between C and D, it will result in two disjoint 1-connections. Therefore, assume that there is 1-connection only between A and B.

If A has $\star$-connection with C and D, then A dominates B. Similarly, if B has $\star$-connection with C and D, then B dominates A. The part A also dominates B if its has $\star$-connection to C, 0-connection to D, as long as B has a 0-connection to D. Thus we assume that B has a $\star$-connection to D and 0-connection to C. This corresponds to the partition shown in Figure 4.6. We will show that $\overline{K_4}$ and the bull graph are the only minimal PI obstructions for this partition. Consider a proper interval graph G that does not contain a $\overline{K_4}$ or a bull graph.

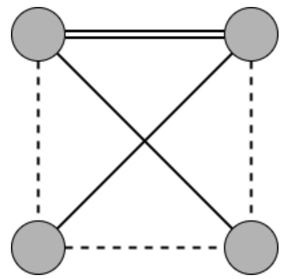


Figure 4.6: Partition corresponding to Lemma 4.2.12.

If the independence number of a G is 2, then it is partitionable. Assume that the independence number of G is 3. If G is not connected, then it is easy to observe that G can be partitioned among B, C, D. Now, assume that G is connected. From Lemma 4.2.1, we know that G can be partitioned into three cliques. Let $v_1, \ldots, v_1', v_2, \ldots, v_2', v_3 \ldots, v_3'$ be the canonical sequence of vertices and C_1, C_2, C_3 be the canonical sequence of contiguous cliques. Let $S(v_1, v_2)$ be the set of vertices that are adjacent to v_1, v_2 but not v_3, and let $S(v_2, v_3)$ be the set of vertices that are adjacent to v_2, v_3 but not v_1. If there is a vertex $a \in S(v_1, v_2)$ that is adjacent to a vertex $b \in S(v_2, v_3)$, then the vertices v_1, a, v_2, b, v_3 induce a bull graph. Therefore, the sets (v_1, v_2) and $S(v_2, v_3)$ are completely non-adjacent. We can partition G as follows. $A = C_2 - S(v_2, v_3)$, $B = C_2 \cap S(v_2, v_3)$, $C = C_1$ and $D = C_3$. Since C_1 and C_3 are completely non-adjacent, C and D are also completely non-adjacent. Note that A does not have any vertex that is adjacent to v_3, therefore, A and D are completely non-adjacent. If there is a vertex in C that is adjacent to a vertex $b \in B$, then v_1' must also be adjacent to b, but this implies that there exits a vertex in $S(v_1, v_2)$ that is adjacent to a vertex in $S(v_2, v_3)$, which is a contradiction. Therefore, B and C are completely non-adjacent.

□

4.3 Large Matrices

In this section, we consider 0-diagonal matrices of any size that have no off-diagonal 0's. Any such matrix can be expressed in the following block structure

$$\begin{bmatrix} P & X & Y \\ X^t & Q & Z \\ Y^t & Z^t & R \end{bmatrix}$$

where P and Q are square matrices that have 0's on the diagonal and $\star's$ on the off-diagonal, R is a square matrix that has 0's on the diagonal, the blocks X and Y (X^t and Y^t) consists only $\star's$, and the block Z (Z^t) consists of at least one 1 per row and column. To see this, first identify all parts which do not have an entry 1 in their row (and column); these will form the blocks P, X, Y. From the remaining parts, find a square submatrix that does not contain an entry 1; this will form the block Q, and also guarantees that the block Z will have at least one 1 in a row/column. All the remaining parts form the block R.

The following matrix is an example of such a block structure where '?' can either be $\star$ or 1.

$$\left[\begin{array}{ccc|ccc|ccc} 0 & \star & \star & \star & \star & \star & \star & \star & \star \\ \star & 0 & \star & \star & \star & \star & \star & \star & \star \\ \star & \star & 0 & \star & \star & \star & \star & \star & \star \\ \hline \star & \star & \star & 0 & \star & \star & 1 & ? & ? \\ \star & \star & \star & \star & 0 & \star & ? & 1 & ? \\ \star & \star & \star & \star & \star & 0 & ? & ? & 1 \\ \hline \star & \star & \star & 1 & ? & ? & 0 & ? & ? \\ \star & \star & \star & ? & 1 & ? & ? & 0 & ? \\ \star & \star & \star & ? & ? & 1 & ? & ? & 0 \end{array}\right]$$

In the theorems below, we assume that M is a symmetric 0-diagonal matrix without off-diagonal 0's, already expressed in the above block form.

Theorem 4.3.1. *If the size of R is* 1*, then M has finitely many minimal PI obstructions.*

Proof. Since R is a matrix of size 1, $R = [0]$. The block Z is a column matrix with all entries equal to 1 because every row of Z must contain 1. Let the size of the block P be p, and the size of the block Q be q. The graph K_{p+q+2} is a minimal obstruction to M, because a K_{p+q+2} cannot be partitioned into $p+q+1$ independent sets. Another minimal obstruction to M_1 is $K_{p+q+1}+K_{p+1}$. Since each vertex of K_{p+q+1} and K_{p+1} must be placed in different parts, there will be a vertex in K_{p+1} that is placed in a part that has at least one 1-connection to other parts. Therefore, $K_{p+q+1}+K_{p+1}$ is an obstruction.

We claim that any minimal PI obstruction to M has at most $t = 7p + 3q + 5$ vertices. Let G be a minimal PI obstruction to M with $t+1$ vertices. The graph G must contain a K_{p+q+1}, else it is $(p+q)$-colorable and hence M-partitionable. Since G does not contain an induced $K_{p+q+1} + K_{p+1}$, from Theorem 4.1.1, there exists a set S of at most $t - 2p (= 3(p+q+1) + 2(p+1))$ vertices such that $G - S$ does not contain a K_{p+1}. Let $v_1, v_2, \ldots, v_n$ be a proper interval ordering of G. The vertices of S occur contiguously in the ordering. Since $G - S$ has $2p+1$ vertices, at least $p+1$ of them must occur either before or after the vertices of S in the ordering. Without loss of generality, assume that the first $p+1$ vertices of the ordering are not in S. Therefore, v_1 has at most $p-1$ neighbours, else it will result in a K_{p+1} outside of S. Since $G - v_1$ is M-partitionable, in any M-partition of $G - v_1$ there exists at least one part I from the block P, such that $N(v_1) \cap I = \emptyset$. This implies that v_1 can also be placed in I contradicting that G is a minimal obstruction. Hence, any minimal obstruction to M has at most $5p+3q+6$ vertices which in turn implies that M has finitely many minimal PI obstructions. □

Theorem 4.3.2. *If all off-diagonal entries of R are 1 and the size of Q is 1, then M has finitely many minimal PI obstructions.*

Proof. Since the size of Q is 1, we have $Q = [0]$, and Z must be a row matrix with all entries equal to 1. Let the size of the block P be p, and the size of the block R be r. The graph K_{p+r+2} is a minimal obstruction to M. Observe that the parts corresponding to the blocks Q and R are pair wise completely adjacent. Therefore, $K_{p+2} + K_{p+1}$ is a minimal obstruction to M.

We claim that any minimal PI obstruction to M has at most $t = 7p + 3r + 5$ vertices. Let G be a minimal PI obstruction with $t+1$ vertices. Let $v_1, v_2, \dots, v_n$ be a proper interval ordering of G. The graph G will not contain a $K_{p+2} + K_{p+1}$, and the clique number of G is at most $p + r + 1$. Therefore, from Theorem 4.1.1, there exits a set S of at most $t - 2p(= 3(p + r + 1) + 2(p + 1))$ vertices, such that $G - S$ does not contain a K_{p+1}. The rest of the proof is same as the proof of Theorem 4.3.1. □

In addition to these results, it can be shown that there are finitely many minimal PI obstructions when the size of R is 2. We omit the technical proof.

Chapter 5

Conclusions

In this thesis we discussed the recognition and characterization problems for matrices M when the input graphs are restricted to be proper interval graphs.

We mostly focused on the characterization problem for small matrices. For matrices of size 3, we proved that all have finitely many minimal PI obstructions. Moreover, we listed all minimal PI obstructions for the special matrices M_1 and M_2, which have infinitely many minimal chordal obstructions.

For matrices of size 4, we considered only those that have constant diagonal. The techniques are markedly different for 0-diagonal matrices and for 1-diagonal matrices. The technique developed for 0-diagonal matrices is also used to prove that some of the larger 0-diagonal matrices have only finitely many minimal PI obstructions. For 1-diagonal matrices, we developed two different techniques one based on the canonical sequence of contiguous cliques and the other using straight enumeration of proper interval graphs.

Our work suggests many questions. Is there one universal technique that can be applied to derive the result for all 1-diagonal matrices of size 4? Can we extend these results to larger constant diagonal matrices? Can we extend these results to mixed diagonal matrices of size 4?

Finally we conjecture that all constant diagonal matrices (of arbitrary size) have only finitely many minimal PI obstructions; possibly this is so for all matrices.

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