Combinatorial Methods for Integer Partitions

by

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Abstract

Integer partitions, while simply defined, are associated with some of the most celebrated results in mathematics. Despite their simple definition, many results on integer partitions can be shockingly difficult to obtain. In this thesis, we use elementary and combinatorial methods to make progress on some fundamental problems related to linear Diophantine equations and integer partitions.

We find an efficient method for finding the number of nonnegative integer solutions (x, y, z) of the equation $ax + by + cz = n$ for given positive integers a, b, c, and n. Our formula involves summations of floor functions of fractions. To quickly evaluate these sums, we find a reciprocity relation that generalizes a well-known reciprocity relation of Gauss related to the law of quadratic reciprocity. Furthermore, we use our result for the number of solutions to a particular equation to prove that the above result of Gauss is equivalent to a well-known result of Sylvester related to the Frobenius coin problem. Moreover, using this equivalence and our generalization of the reciprocity relation of Gauss, we obtain a nice generalization of Sylvester's result.

In a different problem, we prove four conjectures of Berkovich and Uncu regarding some inequalities about relative sizes of two closely related sets consisting of integer partitions whose parts lie in the interval $\{s, \ldots, L + s\}$. Further restrictions are placed on the sets by specifying impermissible parts as well as a minimum part. Our methods consist of constructing injective maps between the relevant sets of partitions.

We obtain a very natural combinatorial proof of Euler's recurrence for integer partitions using the principle of inclusion and exclusion. Using our approach, we are able to generalize Euler's recurrence in the sense that for sufficiently large n, we can express $p(n)$ explicitly as an integer linear combination of $p(n - k), p(n - k - 1), \ldots$ etc. Using such recurrences, we obtain results related to Ramanujan's congruences. For example, if $p_m(n)$ denotes the number of partitions of n that have largest part at most m, we show that for $m > 5$, the numbers $p_m(5n + 4)$ are not divisible by 5 for infinitely many values of n.

Keywords: Integer partitions; linear Diophantine equations; combinatorial proofs; q-series; generating functions; recurrence relations; reciprocity relations; Frobenius coin problem; Ramanujan's congruences; Berkovich and Uncu's conjectures; Jacobi symbol

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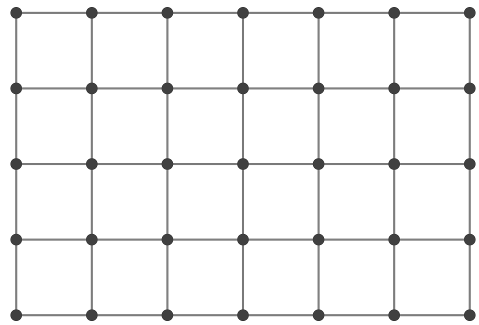
Chapter 1

Introduction

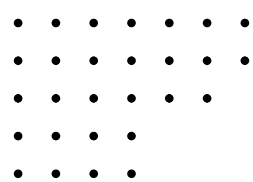
Let n be a nonnegative integer. An integer *partition* $\pi = (\pi_1, \pi_2, \dots)$ *of* n is a weakly decreasing list of positive integers whose sum is n, and we write $|\pi| = n$ to indicate this. Each π_i is known as a *part* of π. We allow the empty partition to be the unique partition of 0. The set of partitions of n is denoted by $Par(n)$ and the number of partitions of n is denoted by $p(n)$. The study of partitions has led to a vast collection of beautiful and intriguing theorems. To the best of our knowledge, the first known non-trivial result is by Euler [Aig07, page 124] in the eighteenth century. He noticed that the number of partitions of a number into odd parts is the same as the number of partitions into distinct parts. Further, he found an expression for the generating function of $p(n)$ as an infinite product. However, most remarkably, he discovered the pentagonal number theorem that expresses this infinite product as an infinite sum. This theorem immediately leads to an amazing recurrence relation for $p(n)$. Though all of these results were initially proved using algebraic manipulations of generating functions, beautiful combinatorial proofs of these were later found.

Researchers always look for combinatorial proofs, even if they have already proved the results using other methods, because of their unparalleled beauty and the ability to easily demonstrate the truth of the results using counting techniques only. Moreover, combinatorial proofs also expose interesting substructures and statistics. For example, suppose that we want to demonstrate the commutativity of multiplication by showing that 5×7 is the same as 7×5. A priori, it is not clear why $5+5+5+5+5+5+5$ happens to be the same number as $7+7+7+7+7$. The easiest way to demonstrate this is to construct a rectangle with sides 6 units and 4 units as follows.

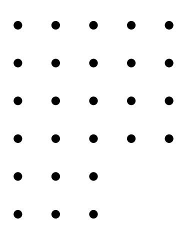


We count the number of lattice points that lie on the boundary or inside this rectangle. If we count the points on each vertical line and then add them, we find that the number of lattice points is equal to $5+5+5+5+5+5+5$. However, if we count the points on each horizontal line and then add them, we find that the number of lattice points is equal to $7+7+7+7+7$. Thus these expressions are equal to each other. In this sense, combinatorial proofs help us see in a natural way why the results are true.

Generally, new discoveries on integer partitions are achieved via algebraic manipulations of generating functions, q-series, and techniques from analytic number theory, especially modular forms. Once the results are established, researchers look for combinatorial proofs of these results. However, there are some results that are obtained more naturally from combinatorics. A typical example is that the number of partitions of n with largest part k is equal to the number of partitions of n with k parts. This follows quite naturally by looking at the *Ferrers diagram* for integer partitions. Let $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_k)$ be a partition. Then the Ferrers diagram for λ is constructed by drawing λ_1 dots in the first line, λ_2 dots in the second line, and so on. For example, consider the partition $(7, 7, 6, 4, 4)$ of 28. Its Ferrers diagram is given as



Note that this partition consists of 5 parts. If we consider the conjugate of this partition – that is, interchange the rows and columns of this diagram – we get the diagram



Thus, we get the partition $(5, 5, 5, 5, 3, 3, 2)$ of 28 whose largest part is 5. The simple conjugation map is therefore the requisite bijection showing that the number of partitions of n with largest part k is equal to the number of partitions of n with k parts. Similarly, there are many other results that follow easily from Ferrers diagrams. In this spirit, the main focus of this thesis is to prove new results on integer partitions that follow naturally using combinatorial methods.

Our methods mostly involve constructing bijections and injections between relevant sets. A novelty about our techniques is that to obtain these maps, we heavily rely on the properties of *linear Diophantine equations*, especially *Frobenius numbers*, defined below. The study of linear Diophantine equations is another focus of this thesis and is intimately connected to the study of integer partitions. For example, for given positive integers a, b, c and n, the number of nonnegative integer solutions (x, y, z) of $ax + by + cz = n$ is equivalent to the number of partitions of n with all parts lying in the set $\{a, b, c\}$. Among linear Diophantine equations, we mainly focus on the renowned *Frobenius coin problem.* Suppose that $a_1, a_2, \dots, a_k$ are given natural numbers such that $\gcd(a_1, a_2 \dots, a_k) = 1$. In the Frobenius coin problem, the following three questions are generally asked. In these questions, we are looking for nonnegative integer solutions $(x_1, x_2 \dots, x_k)$ only.

Question 1.1. *What is the largest integer that cannot be expressed in the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k$?*

This number is known as the *Frobenius number* of the set $\{a_1, a_2 \dots a_k\}$. Sylvester proved that Frobenius number of $\{a_1, a_2\}$ is given as $a_1a_2 - a_1 - a_2$, as shown in the following result, which we refer to as *Sylvester's lemma.*

Lemma 1.2 (Sylvester (1882))**.** *For natural numbers a and b such that $\gcd(a, b) = 1$, the equation $ax + by = n$ has a solution (x, y), with x and y nonnegative integers, whenever $n \geq (a - 1)(b - 1)$.*

The problem of finding the Frobenius number for $k \geq 3$ is wide open and is an active area of research (see [Alf05], [Röd79], [Tsa88] and [Tri17]).

Question 1.3. *For how many natural numbers n is there no solution to the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$?*

Once again, this problem has a complete answer only in the case $k = 2$. Sylvester proved there is no solution for exactly half of the numbers till the Frobenius number. We state this in the following theorem, which we refer to as *Sylvester's theorem.*

Theorem 2.19 (Sylvester (1882))**.** *If a and b are coprime numbers, the number of natural numbers that cannot be expressed in the form $ax + by$ for nonnegative integers x and y is equal to $\frac{(a-1)(b-1)}{2}$.*

This result can be found in [Syl82]. Moreover, Sylvester posed this as a recreational problem, and Curran [S^+84] published a short proof based on generating functions.

Question 1.4. *For a given natural number n, how many solutions are there to the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$?*

Let $N(a_1, \ldots, a_k; n)$ be the number of nonnegative integer solutions to the equation in Question 1.4. Once again, this problem has a full solution only in the case $k = 2$. In this case, for coprime numbers a and b, the number of solutions to $ax + by = n$ has been known for a long time. For history of this problem, refer to [Dic19, pages 64-71]. In 2000, Tripathi [Tri00] used generating functions to obtain a formula for the number of solutions $N(a, b; n)$ of $ax + by = n$ in nonnegative integer pairs (x, y) for given natural numbers a, b and n such that $\gcd(a, b) = 1$. There is no loss of generality in this since $ax + by = n$ is solvable if and only if $d = gcd(a, b)$ divides n, so that the number of solutions in general would be given by $N(\frac{a}{d}, \frac{b}{d}; \frac{n}{d})$. For each n, define the quantities $a'(n)$ and $b'(n)$ that are determined uniquely by the following conditions.

- $a'(n) \equiv -na^{-1} \pmod{b}$, with $1 \leq a'(n) \leq b$.
- $b'(n) \equiv -nb^{-1} \pmod{a}$, with $1 \leq b'(n) \leq a$.

With this notation, Tripathi [Tri00] obtained the following formula.

Theorem 1.5 (Tripathi (2000))**.**

$$N(a, b; n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1.$$

For $k \geq 3$, Beck and Robins [BR04] expressed the number of solutions in terms of Fourier Dedekind sums. They also gave some complicated reciprocity relations for these sums. Komatsu [Kom03] expressed these Fourier Dedekind sums as sums of complicated expressions involving trigonometric functions that can be simplified for small values of the coefficients. However, the sums become intractable as the coefficients become larger. A main focus of this thesis is to generalize Tripathi's methods to obtain a formula for the number of solutions in the case $k = 3$.

In the next two sections, we describe the main results of this thesis in detail. We restate many theorems of later chapters in the next two sections. In Section 1.1, we describe our results on linear Diophantine equations, and in Section 1.2, we describe our results on integer partitions.

1.1 Main results on linear Diophantine equations

Here, we focus on the number of solutions $N(a, b, c; n)$ of the equation $ax + by + cz = n$ in nonnegative integer triples (x, y, z), where a, b, c, and n are given positive integers. The author's work on this problem is published in [Bin20] and [Bin21] and is the main content of Chapter 2. Note that if $\gcd(a, b, c)$ does not divide n, then the equation cannot have any solutions; if it does divide n, then we can divide both sides of the equation by this common factor. Thus, without loss of generality, we can assume that $\gcd(a, b, c) = 1$. We show further that there is also no loss of generality in making the assumption that a, b, and c are pairwise coprime. This allows us to generalize Tripathi's [Tri00] methods, where he finds $N(a, b; n)$, to find an explicit formula for $N(a, b, c; n)$ as given below. We let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. To describe our formula, we need to introduce some more notation.

- Define b_1' such that $b_1' \equiv -nb^{-1} \pmod{a}$, with $1 \le b_1' \le a$. Moreover, define c_1' such that $c_1' \equiv bc^{-1} \pmod{a}$, with $1 \le c_1' \le a$.
- Define c_2' such that $c_2' \equiv -nc^{-1} \pmod{b}$, with $1 \le c_2' \le b$. Moreover, define a_2' such that $a_2' \equiv ca^{-1} \pmod{b}$, with $1 \le a_2' \le b$.
- Define a_3' such that $a_3' \equiv -na^{-1} \pmod{c}$, with $1 \le a_3' \le c$. Moreover, define b_3' such that $b_3' \equiv ab^{-1} \pmod{c}$, with $1 \le b_3' \le c$.
- Define $N_1 = n(n + a + b + c) + cbb_1'(a + 1 - c_1'(b_1' - 1)) + acc_2'(b + 1 - a_2'(c_2' - 1))$ $+baa_3'(c + 1 - b_3'(a_3' - 1))$.

Theorem 2.2 (Binner (2020))**.** *Let a, b, c, and n be given positive integers such that* $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$. *With the notation above, the number of nonnegative integer solutions $N(a, b, c; n)$ of the equation $ax + by + cz = n$ is given by*

$$N(a, b, c; n) = \frac{N_1}{2abc} + \sum_{i=1}^{b_1'-1} \left\lfloor \frac{ic_1'}{a} \right\rfloor + \sum_{i=1}^{c_2'-1} \left\lfloor \frac{ia_2'}{b} \right\rfloor + \sum_{i=1}^{a_3'-1} \left\lfloor \frac{ib_3'}{c} \right\rfloor - 2.$$

Next, we discovered the following reciprocity relation that can be used to quickly evaluate the summations appearing in Theorem 2.2.

Theorem 2.3 (Binner (2020))**.** *Let a, b, c, and K be positive integers such that $b < a$, $c < a$,* $\gcd(a, c) = 1$, *and $K = \left\lfloor \frac{bc}{a} \right\rfloor$. Then*

$$\sum_{i=1}^{b} \left\lfloor \frac{ic}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{c} \right\rfloor = bK.$$

In Chapter 2, we use this reciprocity relation to easily calculate the number of solutions of equations with large coefficients. For example, we show that there are 22 solutions of the equation $4452x + 8030y + 9945z = 3870422$ in nonnegative integer triples (x, y, z). We also obtain, by two applications of this reciprocity relation followed by the division algorithm, another summation of the same form, but with the index of summation as well as the denominator reduced to less than half of their original values. Using the reciprocity relation in Theorem 2.3, we can calculate $N(a, b, c; n)$ in $O(\log t)$ steps, where $t = \max(a, b, c)$, and by a step we mean a basic arithmetic operation on the bits of a, b and c.

The reciprocity relation in Theorem 2.3 is in fact a generalization of a well-known reciprocity relation of Gauss, stated in Theorem 2.7, related to the law of quadratic reciprocity. To describe this, we briefly recall some main results from the theory of quadratic residues. More details about these results can be found in [NZM91, Chapter 3].

Definition. Let a and m be integers such that $\gcd(a, m) = 1$. Then a is called a *quadratic residue modulo m* if the congruence $x^2 \equiv a \pmod{m}$ has a solution. If the congruence has no solution, then a is called a *quadratic nonresidue modulo m*.

Definition. Let p denote an odd prime. The *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue modulo p, -1 if a is a quadratic nonresidue modulo p, and 0 if p divides a.

By Fermat's little theorem, for any a not divisible by p, the congruence $a^{p-1} \equiv 1 \pmod{p}$ holds. Thus, either $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ or $a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$. Euler [NZM91, Theorem 3.1] related this quantity to the Legendre symbol.

Theorem 1.6 (Euler (1748)). *If p is an odd prime, then the Legendre symbol $\left(\frac{a}{p}\right)$ satisfies*

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

A second expression for $\left(\frac{a}{p}\right)$ was given by Gauss in 1808.

Theorem 1.7 (Gauss' lemma for Legendre symbols). *Suppose p is an odd prime and* $\gcd(a, p) = 1$. *Consider the integers $a, 2a, 3a, \cdots, \left(\frac{p-1}{2}\right)a$ and their least positive residues modulo p. If n denotes the number of these residues that exceed $\frac{p}{2}$, then the Legendre symbol $\left(\frac{a}{p}\right)$ satisfies $\left(\frac{a}{p}\right) = (-1)^n$.*

Gauss used this expression to find a third expression for $\left(\frac{a}{p}\right)$. However, this third expression is commonly referred to as Eisenstein's lemma.

Theorem 1.8 (Eisenstein's lemma for Legendre symbols). *If p is an odd prime and a is any odd number not divisible by p, then $\left(\frac{a}{p}\right) = (-1)^t$, where*

$$t = \sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{ia}{p} \right\rfloor.$$

Gauss also found a reciprocity relation for the summations of these floor functions.

Theorem 2.7 (Gauss (1808))**.** *For distinct odd primes p and q,*

$$\sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{iq}{p} \right\rfloor + \sum_{i=1}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{4}.$$

Remark 1. Theorem 2.7 and its proof hold verbatim for any odd positive coprime integers a and b.

Gauss [Gau08] used Theorem 2.7 to complete his third proof of the celebrated *law of quadratic reciprocity.*

Theorem 1.9 (Law of quadratic reciprocity for Legendre symbols)**.** *For distinct odd primes p and q,*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

Eisenstein [Eis44] gave a beautiful geometric proof of Theorem 2.7 in 1844. We refer the reader to Baumgart [Bau15, pp. 15–20] for more information about these classical proofs. Theorem 1.9 is often used to calculate the quadratic residue $\left(\frac{p}{q}\right)$ by using $\left(\frac{q}{p}\right)$. For example, it is somewhat lengthy to calculate $\left(\frac{3}{53}\right)$ directly from the definitions. However using Theorem 1.9, we find that $\left(\frac{3}{53}\right)$ is equal to $\left(\frac{53}{3}\right)$, which by definition is same as $\left(\frac{2}{3}\right)$. One easily finds $\left(\frac{2}{3}\right) = -1$.

Observe that Theorem 2.3 generalizes Theorem 2.7, which is obtained by setting $a = p$, $b = \frac{p-1}{2}$ and $c = q$ in Theorem 2.3. This also suggests that our formula for the number of solutions of the three variable linear Diophantine equation involves summations similar to those in the theory of quadratic residues. We make the connection precise by proving the following result. Let $N_{p,q}$ denote the number of nonnegative integer solutions of the equation $px + qy + z = \frac{q(p-1)}{2}$.

Theorem 2.5 (Binner (2020))**.** *For distinct odd primes p and q, the Legendre symbol $\left(\frac{q}{p}\right)$ is given by*

$$\left(\frac{q}{p}\right) = (-1)^{N_{p,q} - \frac{p+1}{2}}.$$

In [Bin20], by counting the number of nonnegative integer solutions of the equation

$$px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$$

in two different ways, we show that a special case of Sylvester's theorem (the case when a and b are distinct odd primes) is equivalent to the reciprocity relation of Gauss in Theorem 2.7 in the sense that each of these statements can be easily proved using the other. Since we have already generalized Theorem 2.7 in Theorem 2.3 above, it is natural to whether some special case of Theorem 2.3 is equivalent to the general version of Sylvester's theorem. In

[Bin21], we show that the following special case of Theorem 2.3 is precisely the generalization of Theorem 2.7 that is equivalent to Sylvester's theorem.

Theorem 2.18. *Let a and b be coprime positive integers. Then*

$$\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} \left\lfloor \frac{ia}{b} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor.$$

Theorem 2.18 was known before (see [NZM91, Section 3.2, Exercise 23]). However, we provide a proof using Theorem 2.3 in Section 2.8.

As mentioned above, we show that Theorem 2.18 is equivalent to Sylvester's theorem. Note that Theorem 2.3 is a generalization of Theorem 2.18. Thus, it is natural to wonder if the equivalence leads to a generalization of Sylvester's theorem that is equivalent to Theorem 2.3. This is in fact true and leads to an interesting result, described below in Theorem 2.27. We make this sequence of equivalence of results more clear in Figure 1.1 below.

$$\begin{array}{ccc}
\text{Theorem 2.6 (Special Case of Sylvester's theorem)} & \overset{A}{\Longleftrightarrow} & \text{Theorem 2.7 (Gauss (1808))} \\
\bigcap & & \bigcap \\
\text{Theorem 2.19 (Sylvester's theorem (1882))} & \overset{B}{\Longleftrightarrow} & \text{Theorem 2.18} \\
\bigcap & & \bigcap \\
\text{Theorem 2.27 (Binner (2021))} & \overset{C}{\Longleftrightarrow} & \text{Theorem 2.3 (Binner (2020))}
\end{array}$$

Figure 1.1: By Theorem X $\subset$ Theorem Y, we mean that Theorem X is a special case of Theorem Y, and by Theorem X $\Leftrightarrow$ Theorem Y, we mean that Theorems X and Y are equivalent. The present author [Bin20, Section 2.3] proved Equivalence A. We prove Equivalences B and C in Sections 2.8 and 2.10, respectively. These result are published in [Bin21].

Sylvester's theorem shows that exactly one-half of the integers lying in the interval $[0, ab - a - b)$ can be expressed in the form $ax + by$. It is natural to ask a more general question.

Question 1.10. *For given coprime positive integers a and b, and given k such that $0 \leq k < (a-1)(b-1)$, what is the number of nonnegative integers $\leq k$ that can be expressed in the form $ax + by$ for nonnegative integers x and y?*

We denote this number by $N_0(a,b;k)$. For $k<0$, we define $N_0(a,b;k)=0$. By Sylvester's theorem, $N_0(a,b;ab-a-b)=\frac{(a-1)(b-1)}{2}$. In [Bin21], we answer this question for a specific family of values of k. This result is given below and, from Equivalence C in Figure 1.1, is equivalent to Theorem 2.3.

Theorem 2.27 (Binner (2021))**.** *Let a and b be coprime positive integers with $b<a$. Further, let $0\le\alpha<a$ be such that $\alpha\equiv a\pmod 2$, and $\beta=2\left\lfloor\frac{b(\alpha+a)}{2a}\right\rfloor-b$. Then*

$$N_0\left(a,b;\frac{b\alpha+a\beta}{2}\right)=\frac{(\alpha+1)(\beta+1)}{2}.$$

Setting $\alpha=a-2$ in the above theorem gives Sylvester's theorem. Note that

$$\beta\ge 2\left\lfloor\frac{b}{2}\right\rfloor-b\ge -1.$$

Thus, for $\frac{b\alpha+a\beta}{2}<0$, it must be true that $\beta=-1$, and then by Theorem 2.27, $N_0\left(a,b;\frac{b\alpha+a\beta}{2}\right)=0$, which is consistent with our definition of $N_0(a,b;k)$ for negative values of k.

We demonstrate Theorem 2.27 with an example. Suppose $a=29$ and $b=23$. Then Sylvester's theorem shows that $N_0(29,23;615)=308$. However, Theorem 2.27 gives us $N_0(a,b;k)$ for many values of k. See Table 1.1.

α	k	$N_0(29,23;k)$
1	-1	0
3	49	4
5	101	12
7	153	24
9	205	40
11	228	48
13	280	70
15	332	96
17	384	126
19	436	160
21	459	176
23	511	216
25	563	260
27	615	308

Table 1.1: The values of $N_0(29,23;k)$ versus k, as α varies from 1 to 27 such that α is odd.

We prove Theorem 2.27 in Section 2.10. For other values of k not covered by Theorem 2.27, we can calculate $N_0(a,b;k)$ by observing that for $k<ab$, $N_0(a,b;k)$ is equal to the

number of nonnegative integer solutions of $ax + by + z = k$, which can be easily calculated using the algorithm described in Section 2.3. For example, suppose we want to calculate $N_0(29, 23; 257)$. Then, by Theorem 2.2, we find

$$N_0(29, 23; 257) = 15 + \sum_{i=1}^{8} \left\lfloor \frac{23i}{29} \right\rfloor + \sum_{i=1}^{18} \left\lfloor \frac{4i}{23} \right\rfloor .$$

By repeated applications of Theorem 2.3 and the division algorithm, as described in Section 2.3, we easily get that $\sum_{i=1}^{8} \left\lfloor \frac{23i}{29} \right\rfloor = 24$ and $\sum_{i=1}^{18} \left\lfloor \frac{4i}{23} \right\rfloor = 21$, and thus $N_0(29, 23; 257) = 60$.

We conclude our discussion of Question 1.10 by mentioning that for given coprime natural numbers a and b, the study of properties of numbers that cannot be expressed in the form $ax + by$, where x and y are nonnegative integers, continues to be an active area of research. Let $\mathrm{NR}(a, b)$ denotes the set of nonnegative integers nonrepresentable in terms of a and b. That is, it is the set of nonnegative integers n that cannot be expressed in the form $ax + by$. Then, by Sylvester's theorem, we know $|\,\mathrm{NR}(a, b)| = \frac{(a-1)(b-1)}{2}$. Brown and Shiue [BS93] discovered the sum $S(a, b)$ of nonrepresentable numbers. They proved that

$$S(a, b) := \sum_{n \in \mathrm{NR}(a,b)} n = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1).$$

Rødseth [Röd94] considered a generalization of this sum

$$S_m(a, b) = \sum_{n \in \mathrm{NR}(a,b)} n^m.$$

These sums $S_m(a, b)$ are commonly known as the Sylvester sums. Rødseth discovered a formula for these sums in terms of Bernoulli numbers and found that

$$S_2(a, b) = \sum_{n \in \mathrm{NR}(a,b)} n^2 = \frac{1}{12}(a-1)(b-1)ab(ab - a - b).$$

Tuenter [Tue06] related the Sylvester sums to power sums over the natural numbers. Wang and Wang [WW08] studied the alternate Sylvester sums

$$\sum_{n \in \mathrm{NR}(a,b)} (-1)^n n^m.$$

They obtained explicit expressions for the alternate Sylvester sums and related these to the Bernoulli polynomials, the Euler polynomials, and the (alternate) power sums over the natural numbers. Komatsu and Zhang [KZ21] considered the weighted Sylvester sums

$$S_m^{(\lambda)} = \sum_{n \in \mathrm{NR}(a,b)} \lambda^{n-1} n^m.$$

They gave explicit expressions for these sums in terms of the Apostol-Bernoulli numbers. However, in Question 1.10, we consider the number of nonrepresentable numbers below a given number k, instead of considering all the nonrepresentable numbers.

Recall that the theory of quadratic residues can be extended further by generalizing Legendre symbols to Jacobi symbols.

Definition. Let b be an odd number and $b = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be its prime factorization. Then the *Jacobi symbol* $\left(\frac{a}{b}\right)$ is defined as

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)^{r_1} \left(\frac{a}{p_2}\right)^{r_2} \cdots \left(\frac{a}{p_k}\right)^{r_k},$$

where $\left(\frac{a}{p_i}\right)$ is the Legendre symbol.

Some basic properties of Jacobi symbol are given as:

1. For any odd positive integer b,

$$\left(\frac{-1}{b}\right) = (-1)^{\frac{b-1}{2}}.$$

2. For any odd positive integer b,

$$\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}}.$$

3. If a, b and c are integers such that b is positive and odd, then

$$\left(\frac{ac}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{c}{b}\right).$$

We refer the reader to [NZM91, Section 3.3] for proofs of these properties. However, Jacobi symbols do not satisfy Euler's criterion for Legendre symbols (Theorem 1.6). For example, the Jacobi symbol

$$\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) = (-1)(-1) = 1,$$

whereas

$$2^{\frac{15-1}{2}} = 2^7 = 128 \equiv 8 \pmod{15}.$$

Another property satisfied by Legendre symbols not satisfied by Jacobi symbols is demonstrated by the following statement. While the Jacobi symbol $\left(\frac{a}{b}\right) = -1$ implies that a is not a quadratic residue modulo b, the Jacobi symbol $\left(\frac{a}{b}\right) = 1$ does not necessarily imply that a is a quadratic residue modulo b. For example, as seen above, $\left(\frac{2}{15}\right) = 1$, but 2 is not a quadratic residue modulo 15, as it is not a quadratic residue modulo 3 or 5.

Gauss' lemma was generalized to the Jacobi symbol independently by Schering [Sch82] and Jenkins [Jen66]. Recall that $\{x\}$ denotes the fractional part of x, i.e. $\{x\} = x - \lfloor x \rfloor$.

Theorem 2.14 (Gauss' lemma for Jacobi symbols)**.** *For an odd positive integer b and an integer a that is coprime to b, the Jacobi symbol $\left(\frac{a}{b}\right)$ is given by*

$$\left(\frac{a}{b}\right) = (-1)^{|A|}$$

where

$$A := \left\{ i : 1 \leq i \leq \frac{b-1}{2} \text{ and } \left\{\frac{ia}{b}\right\} \geq \frac{1}{2} \right\}.$$

However, direct proofs of Gauss' lemma for Jacobi symbols (see also [Car70, KK09]) are quite technical. Zolotarev [Zol72] observed that Legendre and Jacobi symbols are connected to signatures of naturally associated permutations. Using this approach, other proofs [DH05, BC15] show that Gauss' lemma can be generalized to the Jacobi symbol. Using the standard techniques in the proof of Theorem 1.8 [NZM91, Theorem 3.3], it is straightforward to deduce Eisenstein's lemma for Jacobi symbols from Gauss' lemma for Jacobi symbols, and vice versa.

Theorem 2.15 (Eisenstein's lemma for Jacobi symbols)**.** *For positive odd coprime integers a and b, the Jacobi symbol $\left(\frac{a}{b}\right)$ is given by*

$$\left(\frac{a}{b}\right) = (-1)^{\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor}.$$

Then, through Eisenstein's lemma, these methods give proofs of the law of quadratic reciprocity for Jacobi symbols at the cost of introducing some auxiliary abstract algebraic concepts.

Theorem 1.11 (Law of quadratic reciprocity for Jacobi symbols)**.** *If a and b are positive odd coprime integers, then*

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{(a-1)(b-1)}{4}}.$$

Related information can also be found in [Rou94, Szy11, Tan00, McA].

In contrast, our approach to quadratic reciprocity for Jacobi symbols is directly through Eisenstein's lemma for Jacobi symbols (Theorem 2.15). In Section 2.7, we provide a new and short elementary proof of Theorem 2.15 using floor function sums. This proof also appears in [Bin21]. The law of quadratic reciprocity for Jacobi symbols is then immediately obtained using Theorem 2.15, and Theorem 2.7 and Remark 1 from Page 7. Thus our short proof of Theorem 2.15 provides a natural straightforward generalization of the Gauss-Eisenstein proof of the law of quadratic reciprocity for Legendre symbols to Jacobi symbols.

In Section 2.9, we give another proof of Theorem 2.15. While this second proof is longer, we show that the reciprocity relations satisfied by Jacobi's symbols (Theorem 1.11) and summations of floor functions (Theorem 2.18) force the relationship in Theorem 2.15 to hold true. In particular, we show that a modified version of the Jacobi symbols satisfies four properties (on Page 42) that characterize them, and show that the floor function sums also satisfy these properties modulo 2.

1.2 Main results on integer partitions

Recently, Berkovich and Uncu [BU19] conjectured some intriguing inequalities regarding the relative sizes of certain sets of partitions. Our results related to these conjectures are published in [BR21] and are the main content of Chapter 3. We recall their definitions. For positive integers L and s,

- $C_{L,s}$ denotes the set of partitions where the smallest part is s, all parts are $\leq L+s$, and $L+s-1$ does not appear as a part;
- $D_{L,s}$ denotes the set of nonempty partitions with parts in the set $\{s+1,\ldots,L+s\}$.

Conjecture 1.12 is found in [BU19, Conjecture 3.2].

Conjecture 1.12 (Berkovich and Uncu (2019))**.** *For positive integers $L \geq 3$ and s, there exists an M, which only depends on s, such that*

$$|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for every $N \geq M$.

They proved in [BU19, Theorem 1.1, Theorem 3.1] Conjecture 1.12 for $s = 1$ (with bound $M = 1$) and $s = 2$ (with bound $M = 10$). In both cases, the authors found a suitable injection. Conjecture 1.12 is therefore a natural generalization of those theorems. Their investigations suggested further conjectures, three of which we give below. To state the first, for positive integers L and s,

- if $L \geq s+1$, then $C^*_{L,s}$ denotes the set of partitions where the smallest part is s, all parts are $\leq L+s$, and L does not appear as a part.

The next conjecture is found in [BU19, Conjecture 3.3].

Conjecture 1.13 (Berkovich and Uncu (2019))**.** *For positive integers $L \geq 3$ and s, there exists an M, which only depends on s, such that*

$$|\{\pi \in C^*_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for every $N \geq M$.

In the definition of $C^*_{L,s}$, we must have $L \geq s+1$, so the inclusion of $L \geq 3$ in the conjecture is to exclude the case $L = 2$ and $s = 1$.

Conjectures 1.12 and 1.13 are part of a broader body of recent work concerning sets of partitions whose parts come from some interval. See for example [ABR15, BU16, Cha16]. While we further resolve additional related conjectures from [BU19], there are a number of other research directions suggested in that article that we do not pursue here.

While Conjectures 1.12 and 1.13 motivated our work, we in fact prove a stronger result. For positive integers L, s and k, with $s+1 \leq k \leq L+s$,

- $I_{L,s,k}$ is the set of partitions where the smallest part is s, all parts are $\leq L+s$, and k does not appear as a part.

Whenever a part cannot occur from a range of allowable parts, as with k in the definition of $I_{L,s,k}$, we refer to that as an *impermissible part*. The sets $C_{L,s}$ and $C^*_{L,s}$ above are the special cases of $I_{L,s,k}$ given by $I_{L,s,L+s-1}$ and $I_{L,s,L}$, respectively. Thus, the parameter k allows us to deal with impermissible parts in the set $\{s+1,\ldots,L+s\}$ collectively. Our next theorem generalizes Conjectures 1.12 and 1.13.

Theorem 3.1 (Binner and Rattan (2021))**.** *For positive integers L, s and k, with $L \geq 3$ and $s+1 \leq k \leq L+s$, we have*

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for all $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (3.16).

At this point, the precise value of $\Gamma(s)$ is not important. We have, however, stated Theorem 3.1 with the constant $\Gamma(s)$ inserted to emphasize that it is explicitly known and only depends on s. It also allows us to easily reference this bound when using the partition inequality presented in Theorem 3.1 to prove other results. While our methods are elementary and involve constructing injective maps between the relevant sets, they entail analyzing many cases.

For the remaining conjectures of Berkovich and Uncu considered here, define

- the *q-Pochhammer symbol* by

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

  for an integer $n \geq 1$, with $(a;q)_0 := 1$;

- the series $H_{L,s,k}(q)$ by

$$H_{L,s,k}(q) := \frac{q^s(1-q^k)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right),$$

  for positive integers L, s and k.

A series $\sum_{n\geq 0} a_n q^n$ is said to be *eventually positive* if there exists some $l \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq l$. The next conjecture is found in [BU19, Conjecture 7.1].

Conjecture 3.2 (Berkovich and Uncu (2019))**.** *For positive integers L, s and k, with $L \geq 3$ and $k \geq s+1$, the series $H_{L,s,k}(q)$ is eventually positive.*

As stated, the bound l guaranteeing the coefficient of q^N in $H_{L,s,k}(q)$ is positive for all $N \geq l$ may depend on L, s or k in Conjecture 3.2. When $s+1 \leq k \leq L+s$, elementary partition theory gives the coefficient of q^N in $H_{L,s,k}(q)$ as

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| - |\{\pi \in D_{L,s} : |\pi| = N\}|. \tag{1.1}$$

Hence, Theorem 3.1 proves Conjecture 3.2 when $s+1 \leq k \leq L+s$, and indeed Conjectures 1.12, 1.13 and 3.2 motivated Theorem 3.1. However, Conjecture 3.2 is valid for values of k that do not have the combinatorial interpretation specified in (1.1). In [BR21] and Chapter 3 here, we prove a result stronger than Conjecture 3.2 that also generalizes Theorem 3.1.

Theorem 3.3 (Binner and Rattan (2021))**.** *For positive integers L, s and k, with $L \geq 3$ and $k \geq s+1$, the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in* (3.16)*.*

Again, we emphasize that the bound given in Theorem 3.3 only depends on s, is explicitly known, and is the same as the bound in Theorem 3.1.

Next, we study another related q-series introduced by Berkovich and Uncu [BU19]. Our results related to this series are described in Section 3.4 and Chapter 4. Given a positive integer L, they defined

- $G_{L,1}(q)$ is the series

$$G_{L,1}(q) := \sum_{\substack{\pi\in\mho,\\ s(\pi)=1,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq 2,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

- $G_{L,2}(q)$ is the series

$$G_{L,2}(q) := \sum_{\substack{\pi\in\mho,\\ s(\pi)=2,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq 3,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

  where $s(\pi)$ and $l(\pi)$ denote the smallest and largest parts of π, respectively, and $\mho$ denotes the set of partitions π with $|\pi| > 0$.

Further, Berkovich and Uncu related the series $G_{L,1}(q)$ and $G_{L,2}(q)$ to the series $H_{L,1,L}(q)$ and $H_{L,2,L}(q)$, respectively in [BU19, Theorem 5.1] and [BU19, Theorem 5.2].

Theorem 1.14 (Berkovich and Uncu (2019)). *For $L \geq 1$,*

$$G_{L,1}(q) = \frac{H_{L,1,L}(q)}{1-q^L},$$

$$G_{L,2}(q) = \frac{H_{L,2,L}(q)}{1-q^L}.$$

A series $S = \sum_{n\geq 0} a_n q^n$ is said to be *nonnegative* if $a_n \geq 0$ for all n. The nonnegativity of the series S is denoted by $S \succeq 0$.

Berkovich and Uncu used Theorem 1.14 to prove that the series $G_{L,1}(q)$ is nonnegative and to make the following conjecture about $G_{L,2}(q)$.

Conjecture 3.4 (Berkovich and Uncu (2019)). *For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0;$$

for $L = 4$,

$$G_{L,2}(q) + q^3 + q^9 \succeq 0;$$

and for $L \geq 5$,

$$G_{L,2}(q) + q^3 \succeq 0.$$

Conjecture 3.4 is proved in [BR21] and also here in Section 3.4. Analogous to $G_{L,1}(q)$ and $G_{L,2}(q)$, for any $s \geq 1$,

- $G_{L,s}(q)$ is the generating series

$$G_{L,s}(q) = \sum_{\substack{\pi\in\mho,\\ s(\pi)=s,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq s+1,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|},$$

We begin the study of this series by noticing that it is easy to generalize Theorem 1.14 to get the following result.

Theorem 4.3. *For $L \geq 1$,*

$$G_{L,s}(q) = \frac{H_{L,s,L}(q)}{1-q^L}.$$

We explore the nonnegativity properties of $G_{L,s}(q)$. In particular, we show that for any $L \geq s+1$, the series $G_{L,s}(q)$ is eventually positive and the bound after which the coefficient of q^N becomes positive can be written explicitly in terms of s only.

Theorem 4.1. *If s and $L \geq s+1$ are given positive integers, then the coefficient of q^n in $G_{L,s}(q)$ is positive whenever $n \geq \delta'(s)$, where $\delta'(s)$ is as defined in* (4.2).

Next, we restrict our attention to the case $s = 3$ and obtain an extension of Conjecture 3.4.

Theorem 4.2. *For $L \geq 10$,*

$$G_{L,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + q^{16} \succeq 0.$$

For $5 \leq L \leq 9$, we have the following:

$$\begin{gathered} G_{9,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} \succeq 0, \\ G_{8,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} + q^{20} \succeq 0, \\ G_{7,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + 2q^{14} + q^{16} + q^{20} \succeq 0, \\ G_{6,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{13} + 2q^{14} + 2q^{16} + q^{18} + 2q^{20} + q^{22} \succeq 0, \\ G_{5,3}(q) + q^4 + q^5 + q^8 + q^{10} + 2q^{12} + q^{13} + q^{14} + 2q^{16} + q^{17} + q^{18} + 3q^{20} + q^{22} + q^{24} \succeq 0, \end{gathered}$$

and for $L = 4$,

$$\begin{aligned} G_{4,3}(q) &+ q^4 + q^5 + q^8 + q^{10} + q^{11} + 2q^{12} + 2q^{14} + 3q^{16} + q^{17} \\ &+ 2q^{18} + q^{19} + 4q^{20} + 3q^{22} + q^{23} + 4q^{24} + q^{25} + 4q^{26} + 5q^{28} \\ &+ q^{29} + 3q^{30} + 6q^{32} + 3q^{34} + 4q^{36} + 2q^{38} + 4q^{40} + 2q^{44} \succeq 0. \end{aligned}$$

In Chapter 5, we demonstrate how our technique of constructing injective maps between relevant sets of partitions leads us to some recurrence relations for integer partitions. Recall that Euler (see [And76, Corollary 1.8] or [Aig07, Page 130]) discovered the celebrated recurrence relation

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots \tag{1.2}$$

A close look at the recurrence (1.2) suggests that it is not easy to get a bijective proof. The right hand side of (1.2) contains many negative terms, and it is not clear if the right hand side is counting the cardinality of a set with a natural combinatorial interpretation. One would then be required to produce a bijection from this set to the set of all partitions of n. Bressoud and Zeilberger [BZ85] overcame this difficulty by rewriting the recurrence as follows.

$$p(n) + p(n-5) + p(n-7) + \cdots = p(n-1) + p(n-2) + p(n-12) + p(n-15) + \cdots . \tag{1.3}$$

That is, the equation has been rearranged so that all terms are positive. Let $b_j = \frac{3j^2+j}{2}$ for all integers j. Bressoud and Zeilberger found a bijection between the sets

$$\bigcup_{j \text{ even}} Par(n - b(j)) \to \bigcup_{j \text{ odd}} Par(n - b(j)), \tag{1.4}$$

giving a bijective proof of the recurrence (1.2).

However, it is still natural to ask for a bijective proof of (1.2) without rewriting the terms. That is, we can ask to find a bijection between two sets whose cardinalities are the left hand side and right hand side of (1.2). This is one of our goals in Chapter 5. This method also allows us to generalize Euler's recurrence.

For a given k, we describe a procedure to find a linear recurrence that, for sufficiently large n, expresses $p(n)$ in terms of $p(n-k)$, $p(n-k-1)$, $p(n-k-2)$ etc. To be clear, we note that we are looking for an expression for $p(n)$ that does not involve $p(n-1), p(n-2), \ldots, p(n-(k-1))$, but does involve $p(n-k)$. To do this, we construct an injective map and then measure the size of the image to obtain some partition equalities that give us the desired recurrences.

A feature of our methods is that they also provide similar recurrences for restricted partitions, i.e. for partitions whose parts come from a specified interval. These recurrences also have some arithmetic implications for restricted partitions. For example, using such recurrences, we obtain results related to Ramanujan's congruences. We recall these congruences below.

Ramanujan ([Ram19], [Ram20], [Ram21], [Ram27]) discovered several divisibility properties of integer partitions. Some of these congruences, which hold for all nonnegative integers n, include

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Ramanujan's congruences have remained a topic of central interest in integer partitions throughout the twentieth century, in particular leading to the concepts of rank and crank of partitions. The results of Ono [Ono00] and Ahlgren [Ahl00] proved that in fact such congruences hold modulo any number coprime to 6.

For a given natural number m,

- Let $p_m(n)$ denote the number of partitions of n with largest part at most m.
- Let $p_{=m}(n)$ denote the number of partitions of n with largest part exactly m.
- Let $p(n, \leq m)$ denote the number of partitions of n with at most m parts.
- Let $p(n, m)$ denote the number of partitions of n with exactly m parts.

Using our recurrences for restricted partitions along with Ramanujan's congruences for integer partitions, we obtain the following result.

Theorem 5.2. *Let $m \geq 6$ be a fixed positive integer. Then $p_m(5n+4)$ is not divisible by 5 for infinitely many positive integers n.*

We prove Theorem 5.2 in Section 5.4. One can easily obtain similar divisibility results for $p_m(7n+5)$ and $p_m(11n+6)$. For a general k, we obtain the following result.

Theorem 5.4. *For given positive integers k, l, r and m such that $m > k$ and $r < k$, either $p_m(kn+r)$ is divisible by l for all integers n, or $p_m(kn+r)$ is not divisible by l for infinitely many positive integers n.*

We also prove Theorem 5.4 in Section 5.4. By elementary theory of partitions, the quantities $p(n, \leq m), p_{=m}(n)$ and $p(n, m)$ are connected to $p_m(n)$ and therefore satisfy similar recurrences and congruences, as described in Section 5.4.

Chapter 2

The Number of Solutions to $ax + by + cz = n$

The purpose of this chapter is to calculate the number of solutions $N(a, b, c; n)$ of the equation $ax + by + cz = n$ in nonnegative integer triples (x, y, z), where a, b, c, and n are given positive integers. As described in Chapter 1, there is no loss of generality in assuming that $\gcd(a, b, c) = 1$. We begin by showing that there is also no loss of generality in making the assumption that a, b, and c are pairwise coprime. With the exception of Section 2.9, the work done in this chapter can be found in [Bin20] and [Bin21].

2.1 Reduction to an equation with pairwise coprime coefficients

Let a, b, c, and n be positive integers and, as justified above, we assume that $\gcd(a, b, c) = 1$. We define the following symbols:

- Let g_1, g_2, and g_3 denote $\gcd(b, c)$, $\gcd(c, a)$, and $\gcd(a, b)$, respectively. Note that $\gcd(g_1, g_2) = \gcd(g_2, g_3) = \gcd(g_3, g_1) = 1$.
- Let a_1, b_2, and c_3 denote the modular inverses of a with respect to the modulus g_1, b with respect to the modulus g_2, and c with respect to the modulus g_3, respectively.
- Let n_1, n_2, and n_3 denote the remainders upon dividing na_1 by g_1, nb_2 by g_2, and nc_3 by g_3, respectively.
- Let $A = \frac{a}{g_2 g_3}$, $B = \frac{b}{g_3 g_1}$, and $C = \frac{c}{g_1 g_2}$. Note that A, B and C are pairwise coprime.
- Let $N = \frac{n - an_1 - bn_2 - cn_3}{g_1 g_2 g_3}$. Note that N is an integer.

Lemma 2.1. *With the notation above, the number of solutions of the equation $ax+by+cz = n$ in nonnegative integer triples (x, y, z) is equal to the number of solutions of the equation $Ax + By + Cz = N$ in nonnegative integer triples (x, y, z).*

Proof. Let S and T denote the solution sets of $ax+by+cz=n$ and $Ax+By+Cz=N$, respectively. Then the function $\phi : S \to T$ such that

$$(x,y,z) \mapsto \left(\frac{x-n_1}{g_1}, \frac{y-n_2}{g_2}, \frac{z-n_3}{g_3}\right)$$

provides the required bijection. $\square$

Since A, B, and C are pairwise coprime positive integers, Lemma 2.1 shows that there is no loss of generality in making the assumption that a, b, and c are pairwise coprime.

Remark 2. We briefly describe the motivation behind the bijection in the proof of Lemma 2.1. Reducing the equation $ax+by+cz=n$ modulo g_1, g_2, and g_3 gives the congruences $x \equiv n_1 \pmod{g_1}$, $y \equiv n_2 \pmod{g_2}$, and $z \equiv n_3 \pmod{g_3}$, respectively. Thus, we have the expressions $x = n_1 + g_1 u$, $y = n_2 + g_2 v$, and $z = n_3 + g_3 w$ for some non-negative integers u, v, and w. Substituting these expressions back in the given equation $ax+by+cz=n$ yields the equation $Au+Bv+Cw=N$, which has pairwise coprime coefficients.

2.2 Proof of main theorem of this chapter

As justified above, we may assume that $\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$. Recall the following symbols from Chapter 1.

- Define b_1' such that $b_1' \equiv -nb^{-1} \pmod{a}$ with $1 \le b_1' \le a$. Moreover, define c_1' such that $c_1' \equiv bc^{-1} \pmod{a}$ with $1 \le c_1' \le a$.

- Define c_2' such that $c_2' \equiv -nc^{-1} \pmod{b}$ with $1 \le c_2' \le b$. Moreover, define a_2' such that $a_2' \equiv ca^{-1} \pmod{b}$ with $1 \le a_2' \le b$.

- Define a_3' such that $a_3' \equiv -na^{-1} \pmod{c}$ with $1 \le a_3' \le c$. Moreover, define b_3' such that $b_3' \equiv ab^{-1} \pmod{c}$ with $1 \le b_3' \le c$.

- Define $N_1 = n(n+a+b+c) + cbb_1'(a+1-c_1'(b_1'-1)) + acc_2'(b+1-a_2'(c_2'-1))$ $+baa_3'(c+1-b_3'(a_3'-1))$.

With this notation, we restate from Chapter 1 our main formula.

Theorem 2.2 (Binner (2020)). *Let a, b, c, and n be given positive integers such that* $\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$. *With the notation above, the number of nonnegative integer solutions $N(a,b,c;n)$ of the equation $ax+by+cz=n$ is given by*

$$N(a,b,c;n) = \frac{N_1}{2abc} + \sum_{i=1}^{b_1'-1} \left\lfloor \frac{ic_1'}{a} \right\rfloor + \sum_{i=1}^{c_2'-1} \left\lfloor \frac{ia_2'}{b} \right\rfloor + \sum_{i=1}^{a_3'-1} \left\lfloor \frac{ib_3'}{c} \right\rfloor - 2.$$

Proof. By elementary combinatorics, we know that the number of nonnegative integer solutions of $ax + by + cz = n$ is equal to the coefficient of q^n in

$$\frac{1}{(1-q^a)(1-q^b)(1-q^c)}.$$

Let ζ_m denote $e^{\frac{2\pi i}{m}}$. We know that

$$(1-q^a)(1-q^b)(1-q^c) = (1-q)^3 \prod_{k=1}^{a-1}(1-\zeta_a^{-k}q) \prod_{k=1}^{b-1}(1-\zeta_b^{-k}q) \prod_{k=1}^{c-1}(1-\zeta_c^{-k}q).$$

Since a, b, and c are pairwise coprime, $1-\zeta_a^{-k}q$, $1-\zeta_b^{-k}q$, and $1-\zeta_c^{-k}q$ are distinct for all values of k. Thus, we obtain the partial fraction decomposition

$$\begin{aligned}\frac{1}{(1-q^a)(1-q^b)(1-q^c)} &= \frac{d_1}{1-q} + \frac{d_2}{(1-q)^2} + \frac{d_3}{(1-q)^3} \\ &\quad + \sum_{k=1}^{a-1}\frac{A_k}{1-\zeta_a^{-k}q} + \sum_{k=1}^{b-1}\frac{B_k}{1-\zeta_b^{-k}q} + \sum_{k=1}^{c-1}\frac{C_k}{1-\zeta_c^{-k}q}.\end{aligned} \tag{2.1}$$

On comparing the coefficients of q^n on both sides of (2.1), we find

$$N(a,b,c;n) = d_1 + (n+1)d_2 + \frac{(n+2)(n+1)}{2}d_3 + \sum_{k=1}^{a-1}A_k\zeta_a^{-nk} + \sum_{k=1}^{b-1}B_k\zeta_b^{-nk} + \sum_{k=1}^{c-1}C_k\zeta_c^{-nk}. \tag{2.2}$$

If we substitute $q = 0$ in (2.1), we obtain

$$1 = d_1 + d_2 + d_3 + \sum_{k=1}^{a-1}A_k + \sum_{k=1}^{b-1}B_k + \sum_{k=1}^{c-1}C_k. \tag{2.3}$$

Upon subtracting (2.3) from (2.2), we obtain

$$\begin{aligned}N(a,b,c;n) - 1 &= nd_2 + \frac{n(n+3)}{2}d_3 - \sum_{k=1}^{a-1}A_k(1-\zeta_a^{-nk}) \\ &\quad - \sum_{k=1}^{b-1}B_k(1-\zeta_b^{-nk}) - \sum_{k=1}^{c-1}C_k(1-\zeta_c^{-nk}).\end{aligned} \tag{2.4}$$

The usual procedure for finding coefficients of a partial fraction expansion gives the following equations.

$$\begin{aligned}d_3 &= \frac{1}{abc}, \\ d_2 &= \frac{a+b+c-3}{2abc}, \\ A_k &= \frac{1}{a(1-\zeta_a^{bk})(1-\zeta_a^{ck})},\end{aligned}$$

$$B_k = \frac{1}{b(1-\zeta_b^{ck})(1-\zeta_b^{ak})},$$
$$C_k = \frac{1}{c(1-\zeta_c^{ak})(1-\zeta_c^{bk})}.$$

Substituting these back into (2.4), we have

$$N(a,b,c;n) = \frac{n(n+a+b+c)}{2abc} + 1 - \left(\frac{S_1}{a} + \frac{S_2}{b} + \frac{S_3}{c}\right), \tag{2.5}$$

where

$$S_1 = \sum_{k=1}^{a-1} \frac{1-\zeta_a^{-nk}}{(1-\zeta_a^{bk})(1-\zeta_a^{ck})},$$

$$S_2 = \sum_{k=1}^{b-1} \frac{1-\zeta_b^{-nk}}{(1-\zeta_b^{ck})(1-\zeta_b^{ak})},$$

and

$$S_3 = \sum_{k=1}^{c-1} \frac{1-\zeta_c^{-nk}}{(1-\zeta_c^{ak})(1-\zeta_c^{bk})}.$$

Next, we find S_1, S_2, and S_3. By definition of b_1', we have $bb_1' \equiv -n \pmod{a}$. So $\zeta_a^{-nk} = \zeta_a^{bb_1'k}$, and thus,

$$\begin{aligned}
S_1 &= \sum_{k=1}^{a-1} \frac{1-\zeta_a^{bb_1'k}}{(1-\zeta_a^{bk})(1-\zeta_a^{ck})} \\
&= \sum_{k=1}^{a-1} \sum_{j=0}^{b_1'-1} \frac{\zeta_a^{jbk}}{1-\zeta_a^{ck}} \\
&= \sum_{k=1}^{a-1} \sum_{j=0}^{b_1'-1} \frac{1}{1-\zeta_a^{ck}} - \sum_{k=1}^{a-1} \sum_{j=0}^{b_1'-1} \frac{1-\zeta_a^{jbk}}{1-\zeta_a^{ck}}.
\end{aligned} \tag{2.6}$$

It is well-known that

$$\sum_{k=1}^{a-1} \frac{1}{1-\zeta_a^{ck}} = \frac{a-1}{2},$$

and changing the order of summations yields

$$\sum_{k=1}^{a-1} \sum_{j=0}^{b_1'-1} \frac{1}{1-\zeta_a^{ck}} = b_1'\left(\frac{a-1}{2}\right). \tag{2.7}$$

By definition of c_1', we have $cc_1' \equiv b \pmod{a}$. So $\zeta_a^{jbk} = \zeta_a^{jcc_1'k}$, and thus

$$\begin{aligned}
\sum_{k=1}^{a-1} \sum_{j=0}^{b_1'-1} \frac{1-\zeta_a^{jbk}}{1-\zeta_a^{ck}} &= \sum_{k=1}^{a-1} \sum_{j=1}^{b_1'-1} \frac{1-\zeta_a^{jbk}}{1-\zeta_a^{ck}} \\
&= \sum_{k=1}^{a-1} \sum_{j=1}^{b_1'-1} \frac{1-\zeta_a^{jcc_1'k}}{1-\zeta_a^{ck}} \qquad (2.8) \\
&= \sum_{k=1}^{a-1} \sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \zeta_a^{lck}.
\end{aligned}$$

From (2.6), (2.7), and (2.8), we obtain

$$S_1 = b_1'\left(\frac{a-1}{2}\right) - \sum_{k=1}^{a-1} \sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \zeta_a^{lck}. \qquad (2.9)$$

Now,

$$\begin{aligned}
\sum_{k=1}^{a-1} \sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \zeta_a^{lck} &= \sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \sum_{k=1}^{a-1} \zeta_a^{lck} \\
&= \sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \sum_{k=0}^{a-1} \zeta_a^{lck} - \frac{c_1' b_1'(b_1'-1)}{2}.
\end{aligned} \qquad (2.10)$$

We know that $\sum_{k=0}^{a-1} \zeta_a^{lck} \neq 0$ only if a divides l, and in that case, the sum is a. Note that here we have again used the fact that $\gcd(a,c)=1$. Therefore

$$\sum_{l=0}^{jc_1'-1} \sum_{k=0}^{a-1} \zeta_a^{lck} = a\left(\left\lfloor \frac{jc_1'-1}{a} \right\rfloor + 1\right).$$

Since $\gcd(a, c_1') = 1$ and $j \leq b_1' - 1 \leq a-1$, we have

$$\left\lfloor \frac{jc_1'-1}{a} \right\rfloor = \left\lfloor \frac{jc_1'}{a} \right\rfloor.$$

Hence

$$\sum_{j=1}^{b_1'-1} \sum_{l=0}^{jc_1'-1} \sum_{k=0}^{a-1} \zeta_a^{lck} = a \sum_{j=1}^{b_1'-1} \left(\left\lfloor \frac{jc_1'}{a} \right\rfloor + 1\right). \qquad (2.11)$$

From (2.9), (2.10), and (2.11), we have

$$\frac{S_1}{a} = b_1'\left(\frac{a-1}{2a}\right) + \frac{c_1' b_1'(b_1'-1)}{2a} - \sum_{j=1}^{b_1'-1} \left\lfloor \frac{jc_1'}{a} \right\rfloor - (b_1'-1).$$

We combine the first and last terms to get

$$\frac{S_1}{a} = \frac{c_1' b_1'(b_1' - 1)}{2a} - \sum_{j=1}^{b_1'-1} \left\lfloor \frac{jc_1'}{a} \right\rfloor + 1 - b_1' \left(\frac{a+1}{2a} \right). \tag{2.12}$$

Symmetrically, we also have

$$\frac{S_2}{b} = \frac{a_2' c_2'(c_2' - 1)}{2b} - \sum_{j=1}^{c_2'-1} \left\lfloor \frac{ja_2'}{b} \right\rfloor + 1 - c_2' \left(\frac{b+1}{2b} \right), \tag{2.13}$$

and

$$\frac{S_3}{c} = \frac{b_3' a_3'(a_3' - 1)}{2c} - \sum_{j=1}^{a_3'-1} \left\lfloor \frac{jb_3'}{c} \right\rfloor + 1 - a_3' \left(\frac{c+1}{2c} \right). \tag{2.14}$$

The result now follows from (2.5), (2.12), (2.13), and (2.14). □

2.3 An algorithm to find $N(a, b, c; n)$

In this section, we describe an efficient way to find the sums in Theorem 2.2. Theorem 2.3 will aide us in finding the sums.

Theorem 2.3 (Binner (2020)). *Let a, b, c, and K be positive integers such that $b < a$, $c < a$, $\gcd(a, c) = 1$, and $K = \left\lfloor \frac{bc}{a} \right\rfloor$. Then*

$$\sum_{i=1}^{b} \left\lfloor \frac{ic}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{c} \right\rfloor = bK.$$

Remark 3. As explained in Chapter 1, Theorem 2.7 below (a theorem of Gauss) can be obtained by setting $a = p$, $b = \frac{p-1}{2}$ and $c = q$ in Theorem 2.3.

Proof of Theorem 2.3. We have

$$\sum_{i=1}^{b} \left\lfloor \frac{ic}{a} \right\rfloor = \sum_{t=1}^{K} t n_t,$$

where n_t is the number of i such that $1 \le i \le b$ and $\left\lfloor \frac{ic}{a} \right\rfloor = t$. Clearly, if $t < K$, then

$$n_t = \left\lfloor \frac{(t+1)a}{c} \right\rfloor - \left\lfloor \frac{ta}{c} \right\rfloor;$$

if $t = K$, then

$$n_t = b - \left\lfloor \frac{Ka}{c} \right\rfloor.$$

Therefore,

$$\sum_{i=1}^{b}\left\lfloor\frac{ic}{a}\right\rfloor=\sum_{t=1}^{K-1}\left(\left\lfloor\frac{(t+1)a}{c}\right\rfloor-\left\lfloor\frac{ta}{c}\right\rfloor\right)t+\left(b-\left\lfloor\frac{Ka}{c}\right\rfloor\right)K.$$

We rearrange the terms and solve the summation using telescoping sums to obtain

$$\sum_{i=1}^{b}\left\lfloor\frac{ic}{a}\right\rfloor=\sum_{t=1}^{K-1}\left(\left\lfloor\frac{(t+1)a}{c}\right\rfloor(t+1)-\left\lfloor\frac{ta}{c}\right\rfloor t\right)-\sum_{t=1}^{K-1}\left\lfloor\frac{(t+1)a}{c}\right\rfloor+bK-K\left\lfloor\frac{Ka}{c}\right\rfloor.$$

By cancelling terms and solving, we obtain the required result. □

Theorem 2.3 is helpful to calculate summations of the form $\sum_{i=1}^{b}\left\lfloor\frac{ic}{a}\right\rfloor$ because a summation can be reduced to another of the same form but with a smaller upper limit and a lower denominator.

Remark 4. Similar to Eisenstein's proof of Theorem 2.7 [Bau15, pp. 19–20], we can also give a geometric proof of Theorem 2.3 by counting the number of points under the straight line $y=\frac{c}{a}x$. This proof is omitted.

2.3.1 The algorithm

Our algorithm for finding the number of nonnegative integer solutions $N(a,b,c;n)$ of the equation $ax+by+cz=n$ is as follows:

1. Reduce the given equation to an equation with $\gcd(a,b,c)=1$ as described in Section 1.1. Then reduce it to an equation with pairwise coprime coefficients as described in Section 2.1.

2. Apply the formula in Theorem 2.2 to express the number of solutions in terms of the three summations involving floor functions.

3. Suppose the first summation has the form $\sum_{i=1}^{b_1}\left\lfloor\frac{ic_1}{a_1}\right\rfloor$ for some positive integers a_1, b_1, and c_1 such that $b_1<a_1$, $c_1<a_1$. Then apply Theorem 2.3 to express the summation in terms of the summation $\sum_{i=1}^{K_1}\left\lfloor\frac{ia_1}{c_1}\right\rfloor$, where $K_1=\left\lfloor\frac{b_1c_1}{a_1}\right\rfloor$.

4. To calculate the sum $\sum_{i=1}^{K_1}\left\lfloor\frac{ia_1}{c_1}\right\rfloor$, we cannot apply Theorem 2.3 since $a_1>c_1$. However, by the division algorithm, we have $a_1=c_1q+r$ for some quotient q and remainder r. Then
$$\sum_{i=1}^{K_1}\left\lfloor\frac{ia_1}{c_1}\right\rfloor=\frac{qK_1(K_1+1)}{2}+\sum_{i=1}^{K_1}\left\lfloor\frac{ir}{c_1}\right\rfloor.$$
Since $r<c_1$, we can use Theorem 2.3 again to find this sum.

5. Repeat Steps 3 and 4 until the first summation in Step 2 is fully solved. Then follow the same procedure to find the other two summations and hence the number of solutions.

2.3.2 An example

Let us apply this algorithm to an example. Consider the equation

$$4452x + 8030y + 9945z = 3870422.$$

For brevity, let N denote the number of nonnegative integer solutions of this equation. Observe that $\gcd(4452, 8030, 9945) = 1$, so first part of step 1 of the algorithm is done. Next, we reduce this equation to an equation with pairwise coefficients. Note that $\gcd(4452, 8030) = 2$, $\gcd(4452, 9945) = 3$, and $\gcd(8030, 9945) = 5$. By Lemma 2.1, the number N is equal to the number of nonnegative integer solutions of

$$742x + 803y + 663z = 128598.$$

Next, we apply Theorem 2.2 to get

$$N = \sum_{i=1}^{129} \left\lfloor \frac{281i}{742} \right\rfloor + \sum_{i=1}^{539} \left\lfloor \frac{621i}{803} \right\rfloor + \sum_{i=1}^{335} \left\lfloor \frac{602i}{663} \right\rfloor - 166300. \tag{2.15}$$

In order to solve the first sum, we apply Theorem 2.3 to get

$$\sum_{i=1}^{129} \left\lfloor \frac{281i}{742} \right\rfloor = 6192 - \sum_{i=1}^{48} \left\lfloor \frac{742i}{281} \right\rfloor . \tag{2.16}$$

Then, by the division algorithm,

$$\begin{aligned} \sum_{i=1}^{48} \left\lfloor \frac{742i}{281} \right\rfloor &= \sum_{i=1}^{48} \left(2i + \left\lfloor \frac{180i}{281} \right\rfloor \right) \\ &= 2352 + \sum_{i=1}^{48} \left\lfloor \frac{180i}{281} \right\rfloor . \end{aligned} \tag{2.17}$$

Repeated applications of Theorem 2.3, followed by the division algorithm, give the following equations.

$$\begin{aligned} \sum_{i=1}^{48} \left\lfloor \frac{180i}{281} \right\rfloor &= 1440 - \sum_{i=1}^{30} \left\lfloor \frac{281i}{180} \right\rfloor \\ &= 975 - \sum_{i=1}^{30} \left\lfloor \frac{101i}{180} \right\rfloor , \end{aligned} \tag{2.18}$$

$$\begin{aligned} \sum_{i=1}^{30} \left\lfloor \frac{101i}{180} \right\rfloor &= 480 - \sum_{i=1}^{16} \left\lfloor \frac{180i}{101} \right\rfloor \\ &= 344 - \sum_{i=1}^{16} \left\lfloor \frac{79i}{101} \right\rfloor , \end{aligned} \tag{2.19}$$

$$\begin{aligned}\sum_{i=1}^{16}\left\lfloor\frac{79i}{101}\right\rfloor &= 192-\sum_{i=1}^{12}\left\lfloor\frac{101i}{79}\right\rfloor \\ &= 114-\sum_{i=1}^{12}\left\lfloor\frac{22i}{79}\right\rfloor,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\sum_{i=1}^{12}\left\lfloor\frac{22i}{79}\right\rfloor &= 36-\sum_{i=1}^{3}\left\lfloor\frac{79i}{22}\right\rfloor \\ &= 18-\sum_{i=1}^{3}\left\lfloor\frac{13i}{22}\right\rfloor,\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}\sum_{i=1}^{3}\left\lfloor\frac{13i}{22}\right\rfloor &= 3-\sum_{i=1}^{1}\left\lfloor\frac{22i}{13}\right\rfloor \\ &= 2.\end{aligned} \tag{2.22}$$

From (2.16) to (2.22), we obtain

$$\sum_{i=1}^{129}\left\lfloor\frac{281i}{742}\right\rfloor = 3111.$$

Repeating the same procedure with the other two summations leads to

$$\sum_{i=1}^{539}\left\lfloor\frac{621i}{803}\right\rfloor = 112277,$$

and

$$\sum_{i=1}^{335}\left\lfloor\frac{602i}{663}\right\rfloor = 50934.$$

Substituting these values back in (2.15), we find that $N = 22$, which means there are 22 solutions of $4452x + 8030y + 9945z = 3870422$ in nonnegative integer triples (x, y, z).

2.3.3 Efficiency of the algorithm

We want to find an upper bound for the number of steps required to calculate the number of nonnegative integer solutions of the equation $ax + by + cz = n$. By a step, we mean a basic arithmetic operation on $O(\log r)$ bits where $r = \max(a, b, c)$. Suppose we want to find the sum $\sum_{i=1}^{b}\left\lfloor\frac{ic_1}{a_1}\right\rfloor$ for some positive integers a_1, b, and c_1 such that $b < a_1, c_1 < a_1$, and $\gcd(c_1, a_1) = 1$.

According to Step 3 of the algorithm, we need to apply Theorem 2.3 to get $\sum_{i=1}^{b}\left\lfloor\frac{ic_1}{a_1}\right\rfloor$ in terms of the sum $\sum_{i=1}^{K_1}\left\lfloor\frac{ia_1}{c_1}\right\rfloor$ for some $K_1 < c_1$. Then, as Step 4 in the algorithm describes, we need to apply the division algorithm to obtain $a_1 = c_1q_1 + a_2$, where $a_2 < c_1$. Since $\gcd(c_1, a_1) = 1$, we have $\gcd(a_2, c_1) = 1$. Thus, the sum $\sum_{i=1}^{K_1}\left\lfloor\frac{ia_1}{c_1}\right\rfloor$ can be obtained in terms of the sum $\sum_{i=1}^{K_1} \left\lfloor \frac{ia_2}{c_1} \right\rfloor$. Note that since $c_1 < a_1$, we have $q \geq 1$, and thus,

$$a_1 \geq c_1 + a_2 > 2a_2,$$

or equivalently $a_2 < \frac{a_1}{2}$.

According to Step 3 of the algorithm, we again apply Theorem 2.3 to get the sum $\sum_{i=1}^{K_1} \left\lfloor \frac{ia_2}{c_1} \right\rfloor$ in terms of the sum $\sum_{i=1}^{K_2} \left\lfloor \frac{ic_1}{a_2} \right\rfloor$ for some $K_2 < a_2$. Then we again apply the division algorithm to obtain $c_1 = a_2 q_2 + c_2$ for some $c_2 < a_2$; we can then express the sum $\sum_{i=1}^{K_2} \left\lfloor \frac{ic_1}{a_2} \right\rfloor$ in terms of the sum $\sum_{i=1}^{K_2} \left\lfloor \frac{ic_2}{a_2} \right\rfloor$. Since a_2 is coprime to c_1, it is also coprime to c_2. Finally, since $K_2 < a_2$, $c_2 < a_2$, and $\gcd(c_2, a_2) = 1$, we return to Step 3 of the algorithm to find the sum $\sum_{i=1}^{K_2} \left\lfloor \frac{ic_2}{a_2} \right\rfloor$.

Thus, with two applications of Steps 3 and 4 of the algorithm (i.e., two applications of Theorem 2.3, with each one followed by an application of the division algorithm), we can obtain the sum $\sum_{i=1}^{b} \left\lfloor \frac{ic_1}{a_1} \right\rfloor$ in terms of the sum $\sum_{i=1}^{K_2} \left\lfloor \frac{ic_2}{a_2} \right\rfloor$, where $a_2 < \frac{a_1}{2}$. It is also easy to see that $K_2 < \frac{b}{2}$. This ensures that the Steps 3, 4, and 5 of the algorithm terminate in $O(\log a)$ steps. Hence, the algorithm terminates in $O(\log t)$ steps, where $t = \max(a, b, c)$.

2.4 Relationship with quadratic residues

Let us recall Eisenstein's lemma (Theorem 1.8), which states that for given distinct odd primes p and q, the Legendre symbol $\left(\frac{q}{p}\right)$ is given by

$$\left(\frac{q}{p}\right) = (-1)^t,$$

where

$$t = \sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{iq}{p} \right\rfloor.$$

Remark 5. By a similar analysis to that in Section 2.3.3, it follows that the Legendre symbol $\left(\frac{q}{p}\right)$ can be calculated in $O(\log s)$ steps, where $s = \max(p, q)$.

Thus, Eisenstein's lemma relates Legendre symbols to summations that we have been dealing with while attempting to solve the equation $ax + by + cz = n$. This suggests the existence of an equation whose number of solutions gives the Legendre symbol $\left(\frac{q}{p}\right)$. This connection is made precise by Theorem 2.5, as described in Chapter 1. To prove Theorem 2.5, we need the following helping lemma. Let $N_{p,q}$ denote the number of nonnegative integer solutions of the equation $px + qy + z = \frac{q(p-1)}{2}$.

Lemma 2.4. *For positive odd integers p and q, we have*

$$N_{p,q} = \frac{p+1}{2} + \sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{iq}{p} \right\rfloor.$$

Proof. Clearly, one way of proving this is by applying Theorem 2.2. However, we could also prove it directly by fixing y and then calculating the number of possible values for x. For given x and y, z is automatically determined. $\square$

Theorem 2.5 (Binner (2020)). *For distinct odd primes p and q, the Legendre symbol $\left(\frac{q}{p}\right)$ is given by*

$$\left(\frac{q}{p}\right) = (-1)^{N_{p,q} - \frac{p+1}{2}}.$$

Proof. This follows directly from Eisenstein's lemma (Theorem 1.8) and Lemma 2.4. $\square$

2.5 Equivalence between two well-known results

In this section, the following special case of Sylvester's theorem will be of central importance.

Theorem 2.6 (Sylvester (1882)). *If p and q are distinct odd prime numbers, the number of natural numbers that cannot be expressed in the form $px + qy$ for nonnegative integers x and y is equal to $\frac{(p-1)(q-1)}{2}$.*

Recall Theorem 2.7, stated in Chapter 1.

Theorem 2.7 (Gauss (1808)). *For distinct odd primes p and q,*

$$\sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{iq}{p} \right\rfloor + \sum_{i=1}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{4}.$$

The aim of this section is to establish the equivalence between Theorems 2.6 and 2.7. That is, we prove the Equivalence A in Figure 1.1. Throughout this section, let p and q denote distinct odd primes. We require a few helping lemmas. Let N_0 denote the number of natural numbers that cannot be expressed as $px + qy$ for some nonnegative integers x and y.

Lemma 2.8. *The number of nonnegative integer solutions of the equation*

$$px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$$

is equal to

$$\frac{p(q-1)}{2} + \frac{q(p-1)}{2} + 1 - N_0,$$

Proof. We first fix z and then calculate the number of solutions of the equation

$$px + qy = \frac{p(q-1)}{2} + \frac{q(p-1)}{2} - z.$$

Thus, the number of solutions of the equation $px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$ is equal to

$$\sum_{n=0}^{\frac{p(q-1)}{2}+\frac{q(p-1)}{2}} s_n,$$

where s_n is the number of solutions of the equation $px + qy = n$. Clearly, $s_0 = 1$. We require the following well-known results (see [Tri00]), which we shall also reprove in Section 2.6.2 using the methods developed in Section 2.1.

(i) Whenever $1 \leq n \leq (p-1)(q-1)$, s_n is either 0 or 1.

(ii) Whenever $(p-1)(q-1) < n < pq$, $s_n = 1$.

Thus, by (i) and the definition of N_0,

$$\sum_{n=1}^{(p-1)(q-1)} s_n = (p-1)(q-1) - N_0.$$

Moreover, by (ii),

$$\sum_{n=(p-1)(q-1)+1}^{\frac{p(q-1)}{2}+\frac{q(p-1)}{2}} s_n = \frac{p(q-1)}{2} + \frac{q(p-1)}{2} - (p-1)(q-1).$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\frac{p(q-1)}{2}+\frac{q(p-1)}{2}} s_n &= s_0 + \sum_{n=1}^{(p-1)(q-1)} s_n + \sum_{n=(p-1)(q-1)+1}^{\frac{p(q-1)}{2}+\frac{q(p-1)}{2}} s_n \\ &= \frac{p(q-1)}{2} + \frac{q(p-1)}{2} + 1 - N_0. \end{aligned}$$

□

We calculate the number of solutions of the equation $px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$ in another way by considering four separate cases. Recall that $N_{p,q}$ denotes the number of nonnegative solutions of the equation $px + qy + z = \frac{q(p-1)}{2}$.

Lemma 2.9. *The number of nonnegative integer solutions of the equation*

$$px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$$

is equal to

$$2(N_{p,q} + N_{q,p}) - \left(\frac{p+1}{2} + \frac{q+1}{2} + 1\right).$$

Proof. Let X, Y, and Z denote $\frac{q-1}{2} - x$, $\frac{p-1}{2} - y$, and $\frac{q(p-1)}{2} - z$, respectively. Then the given equation can be rewritten as $pX + qY + Z = \frac{q(p-1)}{2}$. We split our calculation into four different cases according to

1. $X \geq 0, Y \geq 0, Z \geq 0$,
2. $X \geq 0, Y \geq 0, Z < 0$,
3. $X \geq 0, Y < 0$, or
4. $X < 0$.

We define the following sets:

- Let S_1, S_2, S_3, and S_4 denote the set of nonnegative integer solutions of $px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$ in Cases 1, 2, 3, and 4, respectively.
- Let T_1 denote the set of nonnegative integer solutions of $px + qy + z = \frac{q(p-1)}{2}$.
- Let T_2 denote the set of nonnegative integer solutions of $px + qy + z = \frac{p(q-1)}{2}$.
- Let U denote the set of solutions in T_2 that satisfy $z = 0$.
- Let V denote the set of solutions in T_2 that satisfy $y = 0$.
- Let W denote the set of solutions in T_1 that satisfy $x = 0$.

Clearly, $|T_1| = N_{p,q}$ and $|T_2| = N_{q,p}$. Using (i) in the proof of Lemma 2.8 above, we obtain $|U| = 1$. Moreover, it is straightforward to see that $|V| = \frac{q+1}{2}$, and $|W| = \frac{p+1}{2}$. Next, we find the cardinalities of the sets S_1, S_2, S_3, and S_4 by defining the following maps from these sets to T_1 and T_2.

- Define $\phi_1 : S_1 \to T_1$ such that $(x, y, z) \mapsto (X, Y, Z)$.
- Define $\phi_2 : S_2 \to T_2$ such that $(x, y, z) \mapsto (x, y, -Z)$.
- Define $\phi_3 : S_3 \to T_2$ such that $(x, y, z) \mapsto (x, -Y, z)$.
- Define $\phi_4 : S_4 \to T_1$ such that $(x, y, z) \mapsto (-X, y, z)$.

It is easy to verify that ϕ_1, ϕ_2, ϕ_3, and ϕ_4 are well-defined injective maps and their images are given as follows:

- $\phi_1(S_1) = T_1$.
- $\phi_2(S_2) = T_2 \setminus U$.
- $\phi_3(S_3) = T_2 \setminus V$.

- $\phi_4(S_4) = T_1 \setminus W$.

Thus, $|S_1| = N_{p,q}$, $|S_2| = N_{q,p} - 1$, $|S_3| = N_{q,p} - \frac{q+1}{2}$, and $|S_4| = N_{p,q} - \frac{p+1}{2}$. Hence, the total number of nonnegative integer solutions of the equation $px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$ is equal to $|S_1| + |S_2| + |S_3| + |S_4| = 2(N_{p,q} + N_{q,p}) - \left(\frac{p+1}{2} + \frac{q+1}{2} + 1\right)$. $\square$

Upon comparing the number of nonnegative integer solutions of the equation $px + qy + z = \frac{p(q-1)}{2} + \frac{q(p-1)}{2}$ obtained in Lemmas 2.8 and 2.9, and then using Lemma 2.4, we find

$$N_0 + 2\left(\sum_{i=1}^{\frac{p-1}{2}} \left\lfloor \frac{iq}{p} \right\rfloor + \sum_{i=1}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor\right) = (p-1)(q-1).$$

This establishes the required equivalence between Theorems 2.6 and 2.7 (Equivalence A in Figure 1.1).

Remark 6. The proofs in Lemmas 2.4, 2.8 and 2.9 can be easily generalized to any coprime numbers a and b with small modifications, as described in Section 2.8. We presented proofs for odd primes p and q here to keep the logical flow of the presentation.

2.6 Some applications of the above developed techniques

2.6.1 Another proof of Theorem 2.7

In this section, we prove Theorem 2.7 by counting the number of solutions of an equation in two different ways. Without loss of generality, we can assume $q < p$. Recall that in Lemma 2.4, we counted the number of solutions of the equation $px + qy + z = \frac{q(p-1)}{2}$. Now we count these in another way.

Lemma 2.10. *If p and q are distinct odd primes such that $q < p$, then the number of nonnegative integer solutions of the equation $px + qy + z = \frac{q(p-1)}{2}$ is given by*

$$N_{p,q} = \frac{p+1}{2} + \frac{(p-1)(q-1)}{4} - \sum_{i=1}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor.$$

Proof. The maximum possible value for x is

$$\left\lfloor \frac{q(p-1)}{2p} \right\rfloor = \left\lfloor \frac{(q-1)}{2} + \frac{p-q}{2p} \right\rfloor = \frac{q-1}{2}.$$

Now we consider two cases.

Case 1: Let $x = 0$. The number of solutions in Case 1 is equal to $\frac{p+1}{2}$.

Case 2: Let $x \geq 1$. Fix $x = i$. Then the number of possible values for y is equal to

$$1 + \left\lfloor \frac{\frac{q(p-1)}{2} - ip}{q} \right\rfloor = \frac{p-1}{2} - \left\lfloor \frac{ip}{q} \right\rfloor.$$

Hence, the total number of solutions in Case 2 is equal to

$$\sum_{i=1}^{\frac{q-1}{2}} \left(\frac{p-1}{2} - \left\lfloor \frac{ip}{q} \right\rfloor \right) = \frac{(p-1)(q-1)}{4} - \sum_{i=1}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor.$$

Combining the number of solutions in both the cases, we obtain the required result. $\square$

Theorem 2.7 now easily follows from Lemma 2.4 and Lemma 2.10.

2.6.2 An application to the equation $ax + by = n$

As described in Chapter 1, the number of solutions of the equation $ax + by = n$ has been known for a long time. In this section, we show that our technique in Section 2.1 leads to a novel method of solving the equation $ax + by = n$. In addition to the author's paper [Bin20], a similar method has also recently been published independently in [Ali19]. Note that if $\gcd(a, b)$ does not divide n, then there is no solution to $ax + by = n$; otherwise we can divide both sides of the equation by $\gcd(a, b)$. Thus, without loss of generality, we can assume that $\gcd(a, b) = 1$. We define the following symbols:

- Let a^{-1} and b^{-1} denote the modular inverses of a with respect to b and b with respect to a, respectively.
- Let a_1 and b_1 denote the remainders when na^{-1} is divided by b and nb^{-1} is divided by a, respectively.
- Let M denote $\frac{n - aa_1 - bb_1}{ab}$. Note that $aa_1 \equiv n \pmod{b}$ and $bb_1 \equiv n \pmod{a}$. Thus a and b both divide $n - aa_1 - bb_1$. Since $\gcd(a, b) = 1$, it follows that M is an integer.

We obtain a complete list of nonnegative integer solutions of the equation $ax + by = n$.

Theorem 2.11. *Let a, b, and n be given positive integers such that $\gcd(a, b) = 1$. With the notation above, the nonnegative integer solutions of the equation $ax + by = n$ are given as*

$$\{(bi + a_1, (M - i)a + b_1) : 0 \leq i \leq M\}.$$

Proof. Let S and T denote the nonnegative solution sets of $ax + by = n$ and $x + y = M$, respectively. Then the function $\phi : S \to T$ such that

$$(x, y) \mapsto \left(\frac{x - a_1}{b}, \frac{y - b_1}{a} \right)$$

is a bijection with $\phi^{-1} : T \to S$ given by

$$(x, y) \mapsto (bx + a_1, ay + b_1).$$

Clearly, $T = \{(i, M - i) : 0 \leq i \leq M\}$. Then ϕ^{-1} gives the required form for S. $\square$

Remark 7. We briefly describe the motivation behind this bijection. Reducing the given equation $ax + by = n$ modulo b and a gives the equations $x \equiv a_1 \pmod{b}$ and $y \equiv b_1 \pmod{a}$, respectively. Thus, we have the expressions $x = a_1 + bu$ and $y = b_1 + av$ for some nonnegative integers u and v. Substituting these expressions back in the given equation $ax + by = n$ yields the equation $u + v = M$.

Corollary 2.12. *Let a, b, and n be given positive integers such that $\gcd(a, b) = 1$. With the notation above, the number of nonnegative integer solutions $N(a, b; n)$ of the equation $ax + by = n$ is given by*

$$N(a, b; n) = 1 + \frac{n - aa_1 - bb_1}{ab}.$$

This formula is equivalent to the one given in [Tri00].

Corollary 2.13. *The equation $ax + by = n$ has a unique nonnegative integer solution if $(a - 1)(b - 1) \leq n < ab$.*

Proof. If $n < ab$, then clearly $N(a, b; n) < 2$. Moreover, since $a_1 \leq (b - 1)$ and $b_1 \leq (a - 1)$, we have

$$N(a, b; n) \geq \frac{n + a + b - ab}{ab}.$$

Therefore, if $(a-1)(b-1) \leq n$, then $N(a, b; n) > 0$. Thus, whenever $(a-1)(b-1) \leq n < ab$, $N(a, b; n) = 1$. $\square$

Recall that the Frobenius number of a set $\{a_1, a_2, \ldots, a_l\}$ such that $\gcd(a_1, a_2, \ldots, a_l) = 1$ is defined as the largest integer that cannot be expressed in the form

$$k_1 a_1 + k_2 a_2 + \cdots + k_l a_l,$$

where $k_1, k_2, \ldots, k_l$ are nonnegative integers. The proof of Corollary 2.13 shows that if $(a - 1)(b - 1) \leq n$, then $N(a, b; n) > 0$. Moreover, using Corollary 2.12, we can easily show that $N(a, b; ab - a - b) = 0$. Combining these results gives another proof of the fact that for a and b such that $\gcd(a, b) = 1$, the Frobenius number of the set $\{a, b\}$ is equal to $ab - a - b$.

2.7 A short elementary proof of Eisenstein's lemma for Jacobi symbols

In this section, we provide a new elementary proof of Eisenstein's lemma for Jacobi Symbols. This proof also appears in [Bin21]. As described in Chapter 1, we can deduce Gauss' lemma for Jacobi symbols and Eisenstein's lemma for Jacobi symbols from one another. These lemmas were described in Chapter 1. We restate them here.

Theorem 2.14 (Gauss' lemma for Jacobi symbols)**.** *For an odd positive integer b and an integer a that is coprime to b, the Jacobi symbol $\left(\frac{a}{b}\right)$ is given by*

$$\left(\frac{a}{b}\right) = (-1)^{|A|}$$

where

$$A := \left\{ i : 1 \leq i \leq \frac{b-1}{2} \text{ and } \left\{\frac{ia}{b}\right\} \geq \frac{1}{2} \right\}.$$

Theorem 2.15 (Eisenstein's lemma for Jacobi symbols)**.** *For positive odd coprime integers a and b, the Jacobi symbol $\left(\frac{a}{b}\right)$ is given by*

$$\left(\frac{a}{b}\right) = (-1)^{\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor}.$$

To prove Theorem 2.15 using Eisenstein's lemma for Legendre symbols (Theorem 1.8), it suffices to prove the following result.

Lemma 2.16. *For odd positive integers a, b and c such that b and c are coprime with a, we have*

$$\sum_{i=1}^{\frac{bc-1}{2}} \left\lfloor \frac{ia}{bc} \right\rfloor \equiv \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor + \sum_{i=1}^{\frac{c-1}{2}} \left\lfloor \frac{ia}{c} \right\rfloor \pmod{2}.$$

Using Theorem 2.7 and Remark 1 from Page 7, we can easily express all of the sums above in terms of summations of fractions having denominator a and summation index $\frac{a-1}{2}$. From there, Lemma 2.16 reduces to proving the following result.

Lemma 2.17. *For odd positive integers a, b and c such that b and c are coprime with a,*

$$\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ibc}{a} \right\rfloor \equiv \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ic}{a} \right\rfloor \pmod{2}.$$

Note that here we have used that

$$\frac{(a-1)(bc-1)}{4} \equiv \frac{(a-1)(b-1)}{4} + \frac{(a-1)(c-1)}{4} \pmod{2}.$$

To see this, observe that the difference between the LHS and the RHS of the above equation can be rewritten as

$$\frac{(a-1)(b-1)(c-1)}{4},$$

which is clearly an even number for odd numbers a, b and c.

Proof of Lemma 2.17. For brevity of notation, let $r(m)$ denote the remainder when m is divided by a. Since a, b and c are odd, we obtain

$$\begin{aligned}
\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ibc}{a} \right\rfloor &\equiv \sum_{i=1}^{\frac{a-1}{2}} a \left\lfloor \frac{ibc}{a} \right\rfloor \pmod 2 \\
&\equiv \sum_{i=1}^{\frac{a-1}{2}} ibc - r(ibc) \pmod 2 \\
&\equiv \sum_{i=1}^{\frac{a-1}{2}} i - \sum_{i=1}^{\frac{a-1}{2}} r(ibc) \pmod 2. \qquad (2.23)
\end{aligned}$$

Next, we study the latter sum. Let c_i denote $r(ic)$. Note that

$$\begin{aligned}
c_i &= ic - a \left\lfloor \frac{ic}{a} \right\rfloor \\
&\equiv i - \left\lfloor \frac{ic}{a} \right\rfloor \pmod 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
r(ibc) &= r(bc_i) \\
&= bc_i - a \left\lfloor \frac{bc_i}{a} \right\rfloor \\
&\equiv c_i - \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod 2 \\
&\equiv i - \left\lfloor \frac{ic}{a} \right\rfloor - \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod 2. \qquad (2.24)
\end{aligned}$$

Thus, from (2.23) and (2.24), we obtain

$$\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ibc}{a} \right\rfloor \equiv \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ic}{a} \right\rfloor + \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod 2.$$

Therefore, to complete the proof of the lemma, it suffices to show that

$$\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{bc_i}{a} \right\rfloor \equiv \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ib}{a} \right\rfloor \pmod 2. \qquad (2.25)$$

Let $d_i = \min(c_i, a - c_i)$. Since a does not divide bc_i, it follows that

$$\left\lfloor \frac{b(a-c_i)}{a} \right\rfloor = b + \left\lfloor \frac{-bc_i}{a} \right\rfloor = b - 1 - \left\lfloor \frac{bc_i}{a} \right\rfloor \equiv \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod 2.$$

Therefore, for each i,

$$\left\lfloor \frac{bd_i}{a} \right\rfloor \equiv \left\lfloor \frac{bc_i}{a} \right\rfloor \pmod 2. \tag{2.26}$$

Moreover, note that for $1 \le i \le \frac{a-1}{2}$, we also have $1 \le d_i \le \frac{a-1}{2}$. Further, it is easy to verify that the numbers c_i are all distinct and that $c_i \neq a - c_j$ for $1 \le i, j \le \frac{a-1}{2}$. To see this, note that $c_i = a - c_j$ implies that a divides $i + j$, which is not possible. This shows that as i varies from 1 to $\frac{a-1}{2}$, the numbers d_i are a permutation of the numbers $1, 2, \cdots \frac{a-1}{2}$. Therefore,

$$\sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{bd_i}{a} \right\rfloor = \sum_{i=1}^{\frac{a-1}{2}} \left\lfloor \frac{ib}{a} \right\rfloor. \tag{2.27}$$

(2.26) and (2.27) complete the proof of (2.25), and thus of Lemma 2.17. □

2.8 Equivalence between Sylvester's theorem and Theorem 2.18

We have shown that Theorems 2.6 and 2.7 are equivalent (Equivalence A in Figure 1.1). We also know that Theorem 2.6 is a special case of Sylvester's theorem (Theorem 2.19). Moreover, we generalized Theorem 2.7 to get Theorem 2.3. Thus, it is natural to wonder if Sylvester's theorem is equivalent to Theorem 2.3. However, a close look suggests that Theorem 2.3 is possibly too general for producing a result that is equivalent to Sylvester's theorem. Therefore, we need to choose an appropriate special case of Theorem 2.3. As mentioned in Chapter 1, the following special case of Theorem 2.3 is precisely what we need.

Theorem 2.18. *Let a and b be coprime positive integers. Then*

$$\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} \left\lfloor \frac{ia}{b} \right\rfloor = \left\lfloor \frac{a}{2} \right\rfloor \left\lfloor \frac{b}{2} \right\rfloor.$$

Recall also from Chapter 1 that Theorem 2.18 was known before. In Section 2.9.3, we show that Theorem 2.18 naturally leads us to another proof of Gauss' lemma and Eisenstein's lemma for Jacobi symbols (Theorems 2.14 and 2.15).

Throughout Section 2.9, we assume that the reader knows the basic properties of Jacobi symbol from Section 1.1.

We begin by proving Theorem 2.18.

Proof of Theorem 2.18. Without loss of generality, suppose $b < a$. By setting the index of summation equal to $\lfloor \frac{a}{2} \rfloor$ in Theorem 2.3, we obtain

$$\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor = K \left\lfloor \frac{a}{2} \right\rfloor, \tag{2.28}$$

where

$$K = \left\lfloor \frac{\left\lfloor \frac{a}{2} \right\rfloor b}{a} \right\rfloor.$$

We split the calculation into three cases based on the parity of a and b.

Case 1: If a is even, then $K = \left\lfloor \frac{b}{2} \right\rfloor$, and we are done.

Case 2: If a and b are both odd, then

$$K = \left\lfloor \frac{(a-1)b}{2a} \right\rfloor = \left\lfloor \frac{b-1}{2} + \frac{a-b}{2a} \right\rfloor = \frac{b-1}{2} = \left\lfloor \frac{b}{2} \right\rfloor.$$

Case 3: If a is odd and b is even, then

$$K = \left\lfloor \frac{(a-1)b}{2a} \right\rfloor = \left\lfloor \frac{b}{2} - \frac{b}{2a} \right\rfloor = \frac{b}{2} - 1 = \left\lfloor \frac{b}{2} \right\rfloor - 1,$$

and thus (2.28) becomes

$$\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} \left\lfloor \frac{ia}{b} \right\rfloor = \left(\left\lfloor \frac{b}{2} \right\rfloor - 1 \right) \left\lfloor \frac{a}{2} \right\rfloor. \tag{2.29}$$

From (2.29), the theorem easily follows in this case. □

We establish the equivalence between Sylvester's theorem and Theorem 2.18. That is, we prove Equivalence B in Figure 1.1. Sylvester's theorem was introduced in Chapter 1. We restate it here for easy reference.

Theorem 2.19 (Sylvester (1882))**.** *If a and b are coprime numbers, the number of natural numbers that cannot be expressed in the form $ax + by$ for nonnegative integers x and y is equal to $\frac{(a-1)(b-1)}{2}$.*

As described in Remark 6 on Page 33, the proofs of Lemmas 2.4, 2.8 and 2.9 can be easily generalized to give Lemmas 2.20, 2.21 and 2.22 respectively. We skip the details here. For the remainder of this section, suppose a and b are coprime positive integers.

Lemma 2.20. *The number of nonnegative integer solutions $N_{a,b}$ of the equation $ax + by + z = b\left\lfloor \frac{a}{2} \right\rfloor$ is given by*

$$N_{a,b} = \left\lfloor \frac{a}{2} \right\rfloor + 1 + \sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor.$$

Lemma 2.21. *The number of nonnegative integer solutions of the equation*

$$ax + by + z = a\left\lfloor \frac{b}{2} \right\rfloor + b\left\lfloor \frac{a}{2} \right\rfloor$$

is equal to

$$a\left\lfloor \frac{b}{2} \right\rfloor + b\left\lfloor \frac{a}{2} \right\rfloor + 1 - N_0,$$

where N_0 is the number of natural numbers that cannot be expressed as $ax + by$ for any nonnegative integers x and y.

Lemma 2.22. *The number of nonnegative integer solutions of the equation*

$$ax + by + z = a\left\lfloor \frac{b}{2} \right\rfloor + b\left\lfloor \frac{a}{2} \right\rfloor$$

is equal to

$$2(N_{a,b} + N_{b,a}) - \left(\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor + 3\right).$$

We are now ready to show the equivalence between Sylvester's theorem and Theorem 2.18 (Equivalence B in Figure 1.1). Upon comparing the number of nonnegative integer solutions of the equation $ax + by + z = a\left\lfloor \frac{b}{2} \right\rfloor + b\left\lfloor \frac{a}{2} \right\rfloor$ obtained in Lemmas 2.8 and 2.9, and then using Lemma 2.4, we find

$$N_0 + 2\left(\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} \left\lfloor \frac{ia}{b} \right\rfloor\right) = (a-1)\left\lfloor \frac{b}{2} \right\rfloor + (b-1)\left\lfloor \frac{a}{2} \right\rfloor. \tag{2.30}$$

By taking three cases based on the parity of a and b, it can be easily verified that

$$(a-1)\left\lfloor \frac{b}{2} \right\rfloor + (b-1)\left\lfloor \frac{a}{2} \right\rfloor = \frac{(a-1)(b-1)}{2} + 2\left\lfloor \frac{a}{2} \right\rfloor\left\lfloor \frac{b}{2} \right\rfloor. \tag{2.31}$$

From (2.30) and (2.31), we obtain

$$N_0 + 2\left(\sum_{i=1}^{\lfloor \frac{a}{2} \rfloor} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{\lfloor \frac{b}{2} \rfloor} \left\lfloor \frac{ia}{b} \right\rfloor\right) = \frac{(a-1)(b-1)}{2} + 2\left\lfloor \frac{a}{2} \right\rfloor\left\lfloor \frac{b}{2} \right\rfloor.$$

The equivalence between Sylvester's theorem and Theorem 2.18 (Equivalence B in Figure 1.1) now readily follows.

2.9 Another proof of Eisenstein's lemma for Jacobi symbols

Next, we show how Theorem 2.18 naturally leads us to another proof of Gauss' lemma and Eisenstein's lemma for Jacobi symbols (Theorems 2.14 and 2.15). Though we have given a short proof of Theorem 2.15 in Section 2.7, we give an alternate proof in this section to emphasize that the reciprocity relations satisfied by Jacobi's symbols (Theorem 1.11) and summations of floor functions (Theorem 2.18) force the relationship in Theorem 2.15 to hold true.

As described above, our intuition is to use the fact that Jacobi symbol satisfies the same reciprocity relations as the summation of floor functions that appears in Eisenstein's lemma. Thus, we want to characterize Jacobi symbol based on its properties, especially the law of quadratic reciprocity.

A difficulty that we face is that the Jacobi symbol $\left(\frac{a}{b}\right)$ is defined only when b is odd, whereas in the summation of floor functions involved in Eisenstein's lemma, we should not have such a restriction, and the formula in Theorem 2.15 should be valid for all coprime integers a and b. This becomes a problem especially when we need to apply the law of quadratic reciprocity repeatedly to characterize the Jacobi symbol.

We define a quantity that is closely related to the Jacobi symbol and avoids this issue. Recall that for any natural number m, $v_2(m)$ denotes the exponent of 2 in the prime factorization of m. For positive coprime integers a and b, define $T(a,b)$ to be the Jacobi symbol $\left(\frac{a}{b/2^{v_2(b)}}\right)$. Note that if b is odd, $T(a,b)$ is just the Jacobi symbol $\left(\frac{a}{b}\right)$.

Remark 8. The Kronecker symbol also generalizes the Jacobi symbol to include the case when b is even. However, it is different from our generalization $T(a,b)$. The main difference is that $T(a,2)=1$ for all odd numbers a whereas the corresponding Kronecker symbol is defined as

$$\left(\frac{a}{2}\right) = (-1)^{\frac{a^2-1}{8}}.$$

For more information about Kronecker symbols, see [AG18].

2.9.1 Properties of $T(a,b)$

We derive a reciprocity relationship for $T(a,b)$. If a and b are both odd, we already know that they satisfy the law of quadratic reciprocity for Jacobi symbols. Thus, we can assume that at least one of a and b is even. Since a and b are coprime, they are not both even. So exactly one of these is even and the other is odd. First, suppose that a is even and b is odd. Then we have

$$T(a,b) = \left(\frac{a}{b}\right) = \left(\frac{2}{b}\right)^{v_2(a)} \left(\frac{a/2^{v_2(a)}}{b}\right) = (-1)^{\frac{b^2-1}{8}v_2(a)} \left(\frac{a/2^{v_2(a)}}{b}\right),$$

and

$$T(b,a) = \left(\frac{b}{a/2^{v_2(a)}}\right).$$

From the law of quadratic reciprocity for Jacobi symbols, we obtain

$$T(a,b)T(b,a) = (-1)^{\frac{b^2-1}{8}v_2(a) + \frac{b-1}{2}\frac{\left(a/2^{v_2(a)}\right)-1}{2}}.$$

Similarly, we obtain a reciprocity relation in the case where a is odd and b is even. We can combine these relations into the following general reciprocity relation. Suppose a and b are positive coprime integers, then

$$T(a,b)T(b,a) = (-1)^{\frac{b^2-1}{8}v_2(a) + \frac{a^2-1}{8}v_2(b) + \frac{\left(a/2^{v_2(a)}\right)-1}{2}\frac{\left(b/2^{v_2(b)}\right)-1}{2}}.$$

Note that if b is even, then $\frac{b^2-1}{8}$ is not an integer, but in this case a has to be odd (since a and b are not both even), and thus $v_2(a) = 0$. Consequently, the term $\frac{b^2-1}{8}v_2(a)$ becomes 0. Similarly, if a is even, then the term $\frac{a^2-1}{8}v_2(b)$ becomes 0. Thus, in all the cases, the right hand side of the above equation is well-defined. Also, note that we have imposed the condition that a and b should be coprime in the definition of $T(a,b)$ in order to ensure that the above reciprocity relation for $T(a,b)$ holds.

Next, we describe how this reciprocity relation can be used to quickly calculate $T(a,b)$. Suppose b is even. Then divide b by the largest power of 2 dividing b to make it odd. Therefore, we may assume b is odd. If $a > b$, we know from the properties of the Jacobi symbol that we can reduce a modulo b. Thus, we can assume $a < b$. Finally, if $a < b$, we can use the above reciprocity relation and then repeat the procedure.

For example, suppose we want to calculate $T(65, 34)$. Firstly, we divide 34 by the largest power of 2 dividing 34, that is we use $T(65, 34) = T(65, 17)$. Then we reduce 65 modulo 17 to get 14, that is we have $T(65, 17) = T(14, 17)$. Using the reciprocity relation, we express this in terms of $T(17, 14)$. That is,

$$T(14, 17)T(17, 14) = 1,$$

or equivalently $T(14, 17) = T(17, 14)$. Then we repeat the procedure. We have

$$T(17, 14) = T(17, 7) = T(3, 7) = -T(7, 3) = -T(1, 3) = -1.$$

Therefore, $T(65, 34) = -1$. This shows that for positive coprime integers a and b, $T(a,b)$ is uniquely characterized by the following properties.

1. If $b = 2^{v_2(b)}b'$, then $T(a,b) = T(a,b')$.

2. $T(a,b)$ satisfies the reciprocity relation

$$T(a,b)T(b,a) = (-1)^{\frac{b^2-1}{8}v_2(a)+\frac{a^2-1}{8}v_2(b)+\frac{\left(a/2^{v_2(a)}\right)-1}{2}\frac{\left(b/2^{v_2(b)}\right)-1}{2}}.$$

3. If b is an odd number and $a \equiv c \pmod{b}$, then $T(a,b) = T(c,b)$.

4. $T(a, 1) = 1$ for any a and $T(1, b) = 1$ for any b.

Next, we construct another quantity $S'(a,b)$ that satisfies all of these properties. To that end, for positive coprime integers a and b, define

$$S'(a,b) := (-1)^{\frac{b^2-1}{8}v_2(a)+S(a,b)},$$

where

$$S(a,b) := \sum_{i=1}^{\left\lfloor \frac{b}{2^{v_2(b)+1}} \right\rfloor} \left\lfloor \frac{ia/2^{v_2(a)}}{b/2^{v_2(b)}} \right\rfloor.$$

2.9.2 Properties of $S'(a,b)$

From Theorem 2.18, one immediately gets the following reciprocity relation for $S(a,b)$.

Theorem 2.23. *For coprime positive integers a and b,*

$$S(a,b) + S(b,a) = \frac{\left(a/2^{v_2(a)}\right) - 1}{2} \frac{\left(b/2^{v_2(b)}\right) - 1}{2}.$$

Corollary 2.24. *For coprime positive integers a and b,*

$$S'(a,b)S'(b,a) = (-1)^{\frac{b^2-1}{8}v_2(a) + \frac{a^2-1}{8}v_2(b) + \frac{\left(a/2^{v_2(a)}\right)-1}{2}\frac{\left(b/2^{v_2(b)}\right)-1}{2}}.$$

Thus, $S'(a,b)$ satisfies the same reciprocity relation as $T(a,b)$. It is also easy to see that $S'(a,1) = 1$ for any a and $S'(1,b) = 1$ for any b. Hence, $S'(a,b)$ satisfies both 2 and 4 above.

Next, we show that $S'(a,b)$ also satisfies the first property satisfied by $T(a,b)$. Let $b = 2^{v_2(b)}b'$. If $v_2(b) = 0$, we are done. Otherwise, suppose $v_2(b) \geq 1$, that is b is even. Then a is odd, and thus $S'(a,b) = (-1)^{S(a,b)}$. Note that $b' = \frac{b}{2^{v_2(b)}}$ is odd, and thus

$$\frac{b}{2^{v_2(b)}} = \frac{b'}{2^{v_2(b')}},$$

and

$$\frac{b}{2^{v_2(b)+1}} = \frac{b'}{2^{v_2(b')+1}}.$$

This shows that $S(a,b) = S(a,b')$, and thus $S'(a,b) = S'(a,b')$ as required.

Finally, we show that $S'(a,b)$ also satisfies the third property satisfied by $T(a,b)$. Suppose b is an odd number and $a \equiv c \pmod{b}$. We need to prove that $S'(a,b) = S'(c,b)$. Note that it is sufficient to prove that $S'(a,b) = S'(a-b,b)$ and $S'(a,b) = S'(a+b,b)$ for all odd numbers a and b. Firstly, we prove the former relation $S'(a,b) = S'(a-b,b)$ for all odd numbers a and b. Since a and b are both odd, we have

$$S(a,b) = \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor.$$

Moreover, $a-b$ is even, so let $a-b = 2^r s$ for some $r \geq 1$ and some odd number s. Then

$$S(a-b,b) = \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{is}{b} \right\rfloor. \tag{2.32}$$

Using $a = b + 2^r s$, we can rewrite $S(a,b)$ as

$$
\begin{aligned}
S(a,b) &= \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{i(b+2^r s)}{b} \right\rfloor \\
&= \sum_{i=1}^{\frac{b-1}{2}} i + \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{i(2^r s)}{b} \right\rfloor \\
&= \frac{b^2-1}{8} + \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{i(2^r s)}{b} \right\rfloor .
\end{aligned} \tag{2.33}
$$

Using (2.32) and (2.33), along with the definition of $S'(a,b)$, we find that in order to complete the proof of $S'(a,b) = S'(a-b,b)$, we need to show that for all odd numbers s and all positive integers r,

$$
\frac{b^2-1}{8} + \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{i(2^r s)}{b} \right\rfloor \equiv \frac{b^2-1}{8} r + \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{is}{b} \right\rfloor \pmod 2,
$$

that is,

$$
\sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2^r s)}{b} \right\rfloor - \left\lfloor \frac{is}{b} \right\rfloor \right) \equiv \frac{b^2-1}{8}(r-1) \pmod 2. \tag{2.34}
$$

This will follow from the next lemma.

Lemma 2.25. *For an odd positive integer b and an integer t coprime to b,*

$$
\sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2t)}{b} \right\rfloor - \left\lfloor \frac{it}{b} \right\rfloor \right) \equiv \frac{b^2-1}{8}(t-1) \pmod 2.
$$

Firstly, we will complete the proof of $S'(a,b) = S'(a-b,b)$ using the lemma and then we will prove the lemma. In order to apply Lemma 2.25 to prove (2.34), we rewrite the left hand side of (2.34) as follows.

$$
\begin{aligned}
\sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2^r s)}{b} \right\rfloor - \left\lfloor \frac{is}{b} \right\rfloor \right) &= \sum_{i=1}^{\frac{b-1}{2}} \sum_{j=1}^{r} \left(\left\lfloor \frac{i(2^j s)}{b} \right\rfloor - \left\lfloor \frac{i(2^{j-1} s)}{b} \right\rfloor \right) \\
&= \sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2s)}{b} \right\rfloor - \left\lfloor \frac{is}{b} \right\rfloor \right) + \sum_{i=1}^{\frac{b-1}{2}} \sum_{j=2}^{r} \left(\left\lfloor \frac{i(2^j s)}{b} \right\rfloor - \left\lfloor \frac{i(2^{j-1} s)}{b} \right\rfloor \right) \\
&= \sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2s)}{b} \right\rfloor - \left\lfloor \frac{is}{b} \right\rfloor \right) + \sum_{j=2}^{r} \sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2^j s)}{b} \right\rfloor - \left\lfloor \frac{i(2^{j-1} s)}{b} \right\rfloor \right) .
\end{aligned} \tag{2.35}
$$

Since s is odd and coprime to b, by Lemma 2.25 the first term in the above sum is 0 (mod 2). Moreover, the inner sum in the second term can also be simplified modulo 2 by Lemma 2.25. Since $j \geq 2$, $2^{j-1}s$ is even and coprime to b. Then applying Lemma 2.25, we obtain that for all j,

$$\left(\left\lfloor \frac{i(2^j s)}{b} \right\rfloor - \left\lfloor \frac{i(2^{j-1} s)}{b} \right\rfloor\right) \equiv \frac{b^2-1}{8} \pmod 2. \tag{2.36}$$

Upon substituting (2.36) in (2.35), we obtain (2.34). This completes the proof of $S'(a,b) = S'(a-b,b)$ assuming Lemma 2.25. A very similar proof, using Lemma 2.25 and making some obvious modifications, shows that $S'(a+b,b) = S'(a,b)$. Hence, we have shown that $S'(a,b)$ satisfies all the four properties that uniquely determine the function $T(a,b)$. We are only left to prove Lemma 2.25.

Proof of Lemma 2.25. Recall $\{x\}$ denotes the fractional part of x. For any positive real number x, we have

$$\lfloor 2x \rfloor = \begin{cases} 2\lfloor x \rfloor + 1, & \text{if } \{x\} \geq \frac{1}{2}, \\ 2\lfloor x \rfloor, & \text{if } \{x\} < \frac{1}{2}. \end{cases}$$

Thus,

$$\lfloor 2x \rfloor - \lfloor x \rfloor = \begin{cases} \lfloor x \rfloor + 1, & \text{if } \{x\} \geq \frac{1}{2}, \\ \lfloor x \rfloor, & \text{if } \{x\} < \frac{1}{2}. \end{cases} \tag{2.37}$$

Let

$$A := \left\{ i : 1 \leq i \leq \frac{b-1}{2} \text{ and } \left\{\frac{it}{b}\right\} \geq \frac{1}{2} \right\},$$

and

$$B := \left\{ i : 1 \leq i \leq \frac{b-1}{2} \text{ and } \left\{\frac{it}{b}\right\} < \frac{1}{2} \right\}.$$

Substituting $x = \frac{it}{b}$ in (2.37) gives us

$$\left\lfloor \frac{i(2t)}{b} \right\rfloor - \left\lfloor \frac{it}{b} \right\rfloor = \begin{cases} \left\lfloor \frac{it}{b} \right\rfloor + 1, & \text{if } i \in A; \\ \left\lfloor \frac{it}{b} \right\rfloor, & \text{if } i \notin A. \end{cases}$$

Thus,

$$\sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{i(2t)}{b} \right\rfloor - \left\lfloor \frac{it}{b} \right\rfloor \right) = \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{it}{b} \right\rfloor + |A|. \tag{2.38}$$

The arguments after this point are an easy generalization of the proof of [NZM91, Theorem 3.3, Page 135], but we provide all the details here for the sake of completeness. Since we build on our notation so far, our notation here is quite different from that in [NZM91], but the arguments are similar. Define the following sets.

- $A' = \{b\{\frac{it}{b}\} : i \in A\}$.

- $B' = \{b\{\frac{it}{b}\} : i \in B\}$.
- $C' = \{b - x : x \in A'\}$.

Note that for each $1 \le i \le \frac{b-1}{2}$,

$$\frac{it}{b} = \left\lfloor \frac{it}{b} \right\rfloor + \left\{ \frac{it}{b} \right\},$$

and thus

$$it = b \left\lfloor \frac{it}{b} \right\rfloor + b \left\{ \frac{it}{b} \right\}.$$

Summing this over $1 \le i \le \frac{b-1}{2}$, we obtain

$$\sum_{i=1}^{\frac{b-1}{2}} it = b \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{it}{b} \right\rfloor + \sum_{x \in A'} x + \sum_{y \in B'} y.$$

Therefore,

$$t \left(\frac{b^2 - 1}{8} \right) = b \sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{it}{b} \right\rfloor + \sum_{x \in A'} x + \sum_{y \in B'} y. \tag{2.39}$$

Thus, we need to know more about the parity of the sum $\sum_{x \in A'} x + \sum_{y \in B'} y$. For this, we make the following observations which are easy to verify.

- B' and C' are disjoint sets.
- All elements of B' and C' lie between 1 and $\frac{b-1}{2}$.
- $|B'| + |C'| = \frac{b-1}{2}$.

Thus, we find that $B' \sqcup C' = \{1, 2, \ldots, \frac{b-1}{2}\}$. Therefore,

$$\sum_{x \in C'} x + \sum_{y \in B'} y = \sum_{i=1}^{\frac{b-1}{2}} i = \frac{b^2 - 1}{8}. \tag{2.40}$$

We also have

$$\sum_{x \in C'} x = \sum_{x \in A'} (b - x) = b|A'| - \sum_{x \in A'} x. \tag{2.41}$$

Since $|A'| = |A|$, substituting (2.41) in (2.40), we obtain

$$b|A| - \sum_{x \in A'} x + \sum_{y \in B'} y = \frac{b^2 - 1}{8}. \tag{2.42}$$

Since b is odd, from (2.39) and (2.42), we obtain

$$\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{it}{b} \right\rfloor + |A| \equiv \frac{b^2-1}{8}(t-1) \pmod 2. \tag{2.43}$$

The lemma now follows from (2.38) and (2.43).

□

2.9.3 Proofs of Theorems 2.14 and 2.15

We have shown that $S'(a,b)$ satisfies all the four properties that uniquely characterize the function $T(a,b)$. Therefore, these are same as expressions. That is, for all coprime positive integers a and b,

$$T(a,b) = (-1)^{\frac{b^2-1}{8}v_2(a)+S(a,b)},$$

where

$$S(a,b) = \sum_{i=1}^{\left\lfloor \frac{b}{2^{v_2(b)+1}} \right\rfloor} \left\lfloor \frac{ia/2^{v_2(a)}}{b/2^{v_2(b)}} \right\rfloor.$$

In particular, if b is odd, then $T(a,b)$ is just the Jacobi symbol $\left(\frac{a}{b}\right)$ given by

$$\left(\frac{a}{b}\right) = (-1)^{\left(\frac{b^2-1}{8}v_2(a)+\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia/2^{v_2(a)}}{b} \right\rfloor\right)}. \tag{2.44}$$

The equation (2.44) can be simplified using (2.34). Firstly, note that if a is odd, then from (2.44), we obtain

$$\left(\frac{a}{b}\right) = (-1)^{\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor}. \tag{2.45}$$

Next, suppose a is even. Then, from (2.34), we obtain

$$\sum_{i=1}^{\frac{b-1}{2}} \left(\left\lfloor \frac{ia}{b} \right\rfloor - \left\lfloor \frac{ia/2^{v_2(a)}}{b} \right\rfloor \right) \equiv \frac{b^2-1}{8}(v_2(a)-1) \pmod 2. \tag{2.46}$$

Then, from (2.44) and (2.46), we obtain

$$\left(\frac{a}{b}\right) = (-1)^{\left(\frac{b^2-1}{8}+\sum_{i=1}^{\frac{b-1}{2}} \left\lfloor \frac{ia}{b} \right\rfloor\right)}. \tag{2.47}$$

We can combine the cases when a is odd and even together. From (2.45) and (2.47), we obtain

$$\left(\frac{a}{b}\right) = (-1)^{\left(\frac{b^2-1}{8}(a-1)+\sum_{i=1}^{\frac{b-1}{2}}\left\lfloor\frac{ia}{b}\right\rfloor\right)}.$$

Next, we observe that this formula holds even when a is negative, because then

$$\begin{aligned}
\left(\frac{a}{b}\right) &= (-1)^{\frac{b-1}{2}}\left(\frac{-a}{b}\right)\\
&= (-1)^{\frac{b-1}{2}}(-1)^{\left(\frac{b^2-1}{8}(-a-1)+\sum_{i=1}^{\frac{b-1}{2}}\left\lfloor\frac{-ia}{b}\right\rfloor\right)}\\
&= (-1)^{\left(\frac{b-1}{2}+\frac{b^2-1}{8}(a-1)+\sum_{i=1}^{\frac{b-1}{2}}\left(-\left\lfloor\frac{ia}{b}\right\rfloor-1\right)\right)}\\
&= (-1)^{\left(\frac{b^2-1}{8}(a-1)+\sum_{i=1}^{\frac{b-1}{2}}\left\lfloor\frac{ia}{b}\right\rfloor\right)}.
\end{aligned}$$

Suppose a is an odd number. Then, the above result completes the proof of Theorem 2.15.

From Lemma 2.20 and Theorem 2.15, we immediately get the following connection between Jacobi symbols and linear Diophantine equations. Recall that $N_{b,a}$ denotes the number of nonnegative integer solutions of $ax + by + z = a\left\lfloor\frac{b}{2}\right\rfloor$.

Corollary 2.26. *The Jacobi symbol $\left(\frac{a}{b}\right)$ is given by*

$$\left(\frac{a}{b}\right) = (-1)^{N_{b,a}-\left\lfloor\frac{b}{2}\right\rfloor-1}.$$

Note that Corollary 2.26 generalizes Theorem 2.5. Using (2.43) and Theorem 2.15, one immediately gets the proof of Theorem 2.14.

2.10 Proof of a generalization of Sylvester's theorem

Recall from Chapter 1 that for given coprime positive integers a and b, and given k such that $0 \le k < (a-1)(b-1)$, the symbol $N_0(a, b; k)$ denotes the number of natural numbers $\le k$ that can be expressed in the form $ax + by$ for nonnegative integers x and y. In this section, we prove Theorem 2.27 and the Equivalence C in Figure 1.1. We restate Theorem 2.27 here.

Theorem 2.27 (Binner (2021))**.** *Let a and b be coprime positive integers with $b < a$. Further, let $0 \le \alpha < a$ be such that $\alpha \equiv a \pmod 2$, and $\beta = 2\left\lfloor\frac{b(\alpha+a)}{2a}\right\rfloor - b$. Then*

$$N_0\left(a, b; \frac{b\alpha + a\beta}{2}\right) = \frac{(\alpha+1)(\beta+1)}{2}.$$

Recall from Chapter 1 that Sylvester's theorem can be obtained by setting $\alpha = a - 2$ in Theorem 2.27.

For coprime positive integers a and b and any natural number n, recall that $N(a, b; n)$ denotes the number of nonnegative integer solutions of $ax + by = n$. Then it is well known that $N(a, b; n+ab) = N(a, b; n)+1$ (see [Tri00, Lemma 1]). Using this fact while generalizing the proof of Lemma 2.21, we easily get the following result.

Lemma 2.28. *Let a, b, d, and K be positive integers such that $b < a$, $\frac{a}{2} < d < a$, $\gcd(a, b) = 1$, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. The number of nonnegative integer solutions of the equation*

$$ax + by + z = bd + aK$$

is equal to

$$bd + aK + 1 - \frac{(a-1)(b-1)}{2} + N_0(a, b; bd + aK - ab).$$

Generalizing the proof of Lemma 2.22 and then using Theorem 2.3, we obtain the following result.

Lemma 2.29. *Let a, b, d, and K be positive integers such that $b < a$, $d < a$, $\gcd(a, b) = 1$, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. The number of nonnegative integer solutions of the equation*

$$ax + by + z = bd + aK$$

is equal to

$$2\left(\sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor\right) + d + K + 1.$$

The following lemma immediately follows from Lemmas 2.28 and 2.29.

Lemma 2.30. *Let a, b, d, and K be positive integers such that $b < a$, $\frac{a}{2} < d < a$, $\gcd(a, b) = 1$, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. Then*

$$N_0(a, b; bd + aK - ab) = 2\left(\sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor - dK\right) + \frac{(2d - a + 1)(2K - b + 1)}{2}.$$

Using Lemma 2.30, it is clear that Theorem 2.3 is equivalent to the following theorem.

Theorem 2.31. *Let a, b, d, and K be positive integers such that $b < a$, $\frac{a}{2} < d < a$, $\gcd(a, b) = 1$, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. Then*

$$N_0(a, b; bd + aK - ab) = \frac{(2d - a + 1)(2K - b + 1)}{2}.$$

Observe that Theorem 2.27 is just another version of Theorem 2.31 obtained by setting $\alpha = 2d - a$ and $\beta = 2K - b$ in Theorem 2.31. This completes the proof of Theorem 2.27 and its equivalence with Theorem 2.3 (the Equivalence C in Figure 1.1).

2.11 Future directions

A combinatorial proof of Theorem 2.2 will be very interesting as it may also provide a complete list of nonnegative integer solutions (x, y, z) of $ax + by + cz = n$. Moreover, it may also help in providing answers to several other questions related to the equation $ax + by + cz = n$, such as the Frobenius Coin Problem and its generalizations. For the combinatorial proof of Theorem 2.2, a starting point might be to express the summations of floor functions appearing in Theorem 2.2 as number of solutions of given linear Diophantine equations, as demonstrated in the proof of Lemmas 2.4 and 2.20.

Chapter 3

Berkovich and Uncu's Conjectures Regarding Partition Inequalities

3.1 Introduction

Let n be a nonnegative integer. Recall from Chapter 1 that a *partition* $\pi = (\pi_1, \pi_2, \dots)$ *of* n is a weakly decreasing list of positive integers whose sum is n, and we write $|\pi| = n$ to indicate this. We allow the empty partition as the unique partition of 0. Each π_i is known as a *part* of π. For the remaining chapters, it is more convenient to use the notation that expresses the number of parts of each size in a partition. In this notation, we write $\pi = (1^{f_1}, 2^{f_2}, \dots)$, where f_i is the *frequency* of i or the number of times a part i occurs in π. Thus, each frequency f_i is a nonnegative integer, and when $f_i = 0$ this expresses that π has no part of size i. When the frequency of a number is 0, it may or may not be omitted in the expression. In the latter notation, it is clear that $|\pi| = \sum_i i \cdot f_i$. Thus $(4, 4, 2, 2, 1)$, $(1^1, 2^2, 3^0, 4^2, 6^0)$ and $(1^1, 2^2, 4^2, 5^0)$ all represent the same partition of 13.

In this chapter, our main goal is to prove four conjectures of Berkovich and Uncu regarding partition inequalities. In fact, we prove stronger results. The work done in this chapter is published in [BR21]. We restate our main results, described in Chapter 1 and then prove them. We begin by recalling some definitions from Chapter 1. For positive integers L, s and k,

- $I_{L,s,k}$ is the set of partitions where the smallest part is s, all parts are $\leq L + s$, and k does not appear as a part.
- $D_{L,s}$ denotes the set of nonempty partitions with parts in the set $\{s + 1, \dots, L + s\}$.

Theorem 3.1 (Binner and Rattan (2021))**.** *For positive integers L, s and k, with $L \geq 3$ and $s + 1 \leq k \leq L + s$, we have*

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for all $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (3.16)*.*

At this point, the precise value of $\Gamma(s)$ is not important. We have, however, stated Theorem 3.1 with the constant $\Gamma(s)$ inserted to emphasize that it is explicitly known and only depends on s.

As remarked in Chapter 1, Theorem 3.1 generalizes Conjectures 1.12 and 1.13. We prove Theorem 3.1 in Section 3.3.2. For the next conjecture, recall

- the series $H_{L,s,k}(q)$ given by

$$H_{L,s,k}(q) := \frac{q^s(1-q^k)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right),$$

  for positive integers L, s and k.

Elementary partition theory gives that for $s+1 \leq k \leq L+s$, the coefficient of q^N in $H_{L,s,k}(q)$ is

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| - |\{\pi \in D_{L,s} : |\pi| = N\}|.$$

Conjecture 3.2 below is resolved by Theorem 3.3.

Conjecture 3.2 (Berkovich and Uncu (2019))**.** *For positive integers L, s and k, with $L \geq 3$ and $k \geq s+1$, the series $H_{L,s,k}(q)$ is eventually positive.*

As remarked in Chapter 1, Theorem 3.3 below is stronger than all of these conjectures, and Theorem 3.1, and will be a main focus of this chapter.

Theorem 3.3 (Binner and Rattan (2021))**.** *For positive integers L, s and k, with $L \geq 3$ and $k \geq s+1$, the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in* (3.16)*.*

Again, we emphasize that the bound given in Theorem 3.3 only depends on s, is explicitly known, and is the same as the bound in Theorem 3.1.

We use Theorem 3.1 along with other results to prove Theorem 3.3 in Section 3.3.3. Another main focus of this chapter is to prove Conjecture 3.4 in Section 3.4.

Conjecture 3.4 (Berkovich and Uncu (2019))**.** *For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0;$$

for $L = 4$,

$$G_{L,2}(q) + q^3 + q^9 \succeq 0;$$

and for $L \geq 5$,

$$G_{L,2}(q) + q^3 \succeq 0.$$

Remark 9. Our statement of Conjecture 3.4 differs slightly from the one given in [BU19], as their statement is not strictly correct. In the case $L = 3$, the conjecture in [BU19] is stated as $G_{3,2}(q) + q^3 + q^9 \succeq 0$. However, it can be checked, either through machine computation or by hand that the coefficient of q^{15} in $G_{3,2}(q)$ is -1, and hence the discrepancy between our statement and theirs. Subject to this minor modification, their conjecture is as stated above.

Our proofs rely heavily on two results, the first of which is Sylvester's lemma. As described in Chapter 1, Sylvester's lemma states that for natural numbers a and b such that $\gcd(a,b) = 1$, the equation $ax + by = n$ has a solution (x, y), with x and y nonnegative integers, whenever $n \geq (a-1)(b-1)$.

In addition to Sylvester's lemma, we also require the following simple lemma.

Lemma 3.5. *Let s and n be positive integers such that $n \geq s+1$. Then the equation*

$$(s+1)X_{s+1} + (s+2)X_{s+2} + \cdots + (2s+1)X_{2s+1} = n$$

has a solution $(X_{s+1}, X_{s+2}, \ldots, X_{2s+1})$, where X_i is a nonnegative integer for all i.

Proof. We use the division algorithm to write $n = (s+1)q + r$ for some $q \geq 1$ and $0 \leq r \leq s$. If $r = 0$, setting $X_{s+1} = q$ with all other $X_i = 0$ gives a suitable solution. Otherwise $1 \leq r \leq s$, and then

$$n = (s+1)(q-1) + (s+1+r).$$

Note that $s+2 \leq s+1+r \leq 2s+1$, so setting $X_{s+1} = q-1$ and $X_{s+1+r} = 1$ with all other $X_i = 0$ gives a solution, completing the proof. $\square$

3.1.1 Recent proofs of the above conjectures

About eight weeks before we put a preprint of [BR21] on the arxiv, Zang and Zeng [ZZ20] gave proofs of Conjectures 1.12, 1.13 and 3.2 in a preprint. We highlight the specific similarities and differences between their proofs and ours, as well as the strengths of both approaches. In our case, our approach almost always involves constructing an injective map between the relevant sets of partitions, and those maps heavily rely on the properties of Frobenius numbers. Furthermore, our approach allows us to give, in all cases, explicit bounds for when the partition inequalities hold. In addition, our methods also allow us to prove Conjecture 3.4.

In their work and ours, a crucial step is to prove Conjecture 3.2 and then use it to prove the other conjectures. Another similarity is that the proofs of Conjecture 3.2 are separated into two cases: the case with large L and k (compared to s) and the case with small L or k. The comparisons between the two approaches to Conjecture 3.2 are as follows.

- An important achievement in both papers is to show that there exists an M, which depends only on s, such that the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geq M$. Thus, our work and theirs both achieve this strengthening of Conjecture 3.2.

They prove this strengthening only for $\max(s+1, L) \leq k \leq L+s$ ([ZZ20, Theorem 1.1]), while we show this for any $k \geq s+1$ (Theorem 3.3).

- For large L and k, their proofs and ours are different, as are the lower bounds on L and k for when these results hold. The bounds on N guaranteeing the coefficient of q^N in $H_{L,s,k}(q)$ is positive are similar. The lower bounds on L and k are lower in our case ($L \geq s+2$ and $k \geq 2s+2$) versus their case ($k \geq L \geq 2s^3+5s^2+1$). For their results and ours, the coefficient of q^N in the series $H_{L,s,k}(q)$ is positive when N exceeds a lower bound of order $O(s^5)$. However, their restriction on k, that $k \geq L$, means that for a given s, if L becomes arbitrarily large, their result does not have this lower bound on N for an arbitrarily large set of values of k, whereas in our case the bound is valid whenever $k \geq 2s+2$. This is especially important in the limiting case $L \to \infty$. For example, for a given s, for any fixed $k \geq 2s+2$, and any $N > (25s)^5$, our approach shows that the number of partitions of N with smallest part s and no part equal to k is more than the number of partitions of N with smallest part greater than s. On the other hand, their approach does not yield any such result since they require $k \geq L$ and here $L \to \infty$.

- For small L or k, their approach and ours differ greatly. In [ZZ20], the authors use a celebrated result of Frobenius and Schur (related to the *Frobenius coin problem*), which states that for a set $A = \{a_1, \ldots, a_m\}$ of positive integers whose greatest common divisor is 1, the number of partitions of a positive integer n whose parts are restricted to A is approximately
$$\frac{n^{m-1}}{(m-1)!a_1 \cdots a_m}.$$
This result is asymptotic in n, and it is unknown when this approximation is accurate. In fact, even finding for which n onwards there is at least one such partition is a well-known open problem; see [Alf05]. Therefore, the result of Zang and Zeng is also asymptotic. In contrast, our methods are combinatorial, and we produce explicit bounds on when $H_{L,s,k}(q)$ is eventually positive. For large L and small k ($L \geq 3s+3$ and $s+1 \leq k \leq 2s+1$), our bounds are $O(s^{10})$, while for small L ($L \leq 3s+2$), our bounds are of the order $O\Big((6s)^{(6s)^{18s}}\Big)$.

  Some advantages of the proof of Zang and Zeng in this case is that it is short, elegant, and easily understood. Also, their methods show an intriguing connection between the present problems and the above mentioned theorem of Frobenius and Schur. They further show, in [ZZ20, Theorem 1.3], the eventual positivity of a series $H^*_{L,s,r,k_1,k_2}(q)$ that generalizes the series $H_{L,s,k}(q)$. In this case also, their results are asymptotic. The chief advantage of our methods is that they produce explicit bounds, and they also lead to a proof of Conjecture 3.4. Indeed, in [ZZ20, Page 12], the authors state that techniques that produce explicit bounds on when $H_{L,s,k}(q)$ is eventually positive may lead to a proof of Conjecture 3.4. Our methods confirm this.

A final remark about our results is that while we find explicit bounds throughout this chapter, we make no claims about the optimality of those bounds. The question of finding the minimal bounds for when these results hold remains open.

3.2 The case when L is large for Theorem 3.1

In this section, we build to proving Theorem 3.1 when L is relatively large, by which we mean L is larger than a constant times s. In each case, our lower bound on L is explicitly stated. We begin with a case pertaining to Conjecture 1.12, and then later generalize those arguments to prove the large L case of Theorem 3.1. The general technique used in this chapter will be illustrated in this section.

3.2.1 The case when the impermissible part k is $L+s-1$, and L is relatively large

In this section, we focus on the case $k = L+s-1$ in Theorem 3.1 with $L \geq s+3$. That is, we are considering the case $C_{L,s}$ (or equivalently $I_{L,s,L+s-1}$) when $L \geq s+3$. This corresponds to the case of $L \geq s+3$ in Conjecture 1.12.

For any $s \geq 1$, define the quantities:

- $F(s) = (10s-2)(15s-3) + 8s$;
- $\kappa(s) = (12s-1)((s+1)+(s+2)+\cdots(F(s)-1)) + 1$.

The number $\kappa(s)$ serves as M in our proof of Conjecture 1.12 when $L \geq s+3$.

Theorem 3.6. *If s and L are positive integers with $L \geq s+3$ and $N \geq \kappa(s)$, then*

$$|\{\pi \in C_{L,s} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|.$$

Proof. We construct an injective map

$$\phi : \{\pi \in D_{L,s} : |\pi| = N\} \to \{\pi \in C_{L,s} : |\pi| = N\}.$$

To show strict inequality, at the end of the proof we show that there is an element in the codomain of ϕ not in its range.

For $\pi \in D_{L,s}$, the image of π under ϕ is given in cases depending on the frequency of $L+s-1$ in π. Hence, for brevity, we set $f = f_{L+s-1}$, so any $\pi \in D_{L,s}$ has the form

$$\pi = ((s+1)^{f_{s+1}}, \ldots, (L+s-1)^{f}, (L+s)^{f_{L+s}}).$$

Our definition of the image of π under ϕ is given in two cases, when $f = 0$ and when $f \geq 1$, and each case contains several subcases. Our strategy for ensuring ϕ is injective is to have images of partitions under ϕ from different cases have different frequencies of s, while ensuring that in each case itself ϕ is injective. To make our strategy and arguments on injectivity clear, we summarize in Table 3.1 the frequencies of s in partitions in the image of ϕ for each case. As the right hand column of Table 3.1 contains disjoint sets, the frequency of s in a partition in the image immediately determines from what case its preimage comes. Then we only need to ensure that ϕ is injective in each case.

Case	Possible frequencies of s
1(a)	Multiples of 12
1(b)(i)	15
1(b)(ii)	20
1(b)(iii)	2,4,6,8
2(a)	Odd numbers other than 15
2(b)	14

Table 3.1: The frequency of s in the image of a partition under the function ϕ in the different cases for Theorem 3.6.

Case 1: Suppose $f = 0$ in π. In this case, we obtain $\phi(\pi)$ by inserting some number of parts equal to s into the partition π; as $f = 0$, we do not need to remove the parts of size $L + s - 1$, but must compensate by altering the other parts of π. The number of parts equal to s to be inserted into π is given by the subcases below.

Case 1(a): Suppose that there exists m such that $s + 1 \leq m \leq F(s) - 1$ and $f_m \geq 12s$. Let m_0 be the least such number. Then define

$$\phi(\pi) = (s^{12m_0}, (s+1)^{f_{s+1}}, \ldots m_0^{f_{m_0}-12s}, \ldots).$$

We can see that ϕ is injective in this case because from the frequency of s in $\phi(\pi)$ we can easily determine m_0, and from this π can be recovered.

Case 1(b): Suppose that the condition of Case 1(a) does not hold. That is, for every m such that $s + 1 \leq m \leq F(s) - 1$, we have $f_m \leq 12s - 1$. Note that if such partitions do not exist, then Case 1(b) does not arise and there is no need to construct an injection. Since $N \geq \kappa(s)$, we must have $L + s \geq F(s)$, and also there must exist an $h \geq F(s)$ such that $f_h > 0$. Let l be the least such number. Thus, we can write π as

$$\pi = ((s+1)^{f_{s+1}}, \ldots, (F(s)-1)^{f_{F(s)-1}}, \ldots, l^{f_l}, \ldots).$$

We have some further subcases.

Case 1(b)(i): If $f_{5s+1} \geq 1$ and $f_{10s-1} \geq 1$, then define

$$\phi(\pi) = (s^{15}, (s+1)^{f_{s+1}}, \ldots, (5s+1)^{f_{5s+1}-1}, \ldots, (10s-1)^{f_{10s-1}-1}, \ldots).$$

The injectivity of ϕ is clear in this case.

Case 1(b)(ii): If $f_{5s+1} = 0$ or $f_{10s-1} = 0$ and $f_{5s+2} \geq 1$ and $f_{15s-2} \geq 1$, then define

$$\phi(\pi) = (s^{20}, (s+1)^{f_{s+1}}, \ldots, (5s+2)^{f_{5s+2}-1}, \ldots, (15s-2)^{f_{15s-2}-1}, \ldots).$$

The injectivity of ϕ is also clear in this case.

Case 1(b)(iii): If $f_{5s+1} = 0$ or $f_{10s-1} = 0$ and $f_{5s+2} = 0$ or $f_{15s-2} = 0$. Then at least one of the following statements is true:

- T_1: $f_{5s+1} = 0$ and $f_{5s+2} = 0$;
- T_2: $f_{5s+1} = 0$ and $f_{15s-2} = 0$;
- T_3: $f_{10s-1} = 0$ and $f_{5s+2} = 0$;
- T_4: $f_{10s-1} = 0$ and $f_{15s-2} = 0$.

The indices in each of the statements are intentionally chosen to be coprime with each other. For example, let us show that $5s+1$ and $15s-2$ are coprime. If $g = \gcd(5s+1, 15s-2)$, then $g \mid (5s+1)$ and $g \mid (15s-2)$. But then $g \mid 3(5s+1) - (15s-2) = 5$. Therefore, $g = 1$ or $g = 5$, but $g \neq 5$ since $5 \nmid (5s+1)$. The other pairs can be shown to be coprime with similar ease.

Since $F(s) - 8s = (10s-2)(15s-3)$ and the aforementioned indices in each statement are coprime, by Sylvester's lemma the following equations have nonnegative integer solutions for all $m \geq F(s)$:

- $(5s+1)x_m + (5s+2)y_m = m - 2s$;
- $(5s+1)z_m + (15s-2)w_m = m - 4s$;
- $(10s-1)u_m + (5s+2)v_m = m - 6s$;
- $(10s-1)p_m + (15s-2)q_m = m - 8s$.

That the lower bound on m is sufficient for all the equations to have nonnegative integer solutions follows from the lower bound being sufficient for the last equation to have such solutions; there the lower bound on m is the one specified by Sylvester's lemma. For each $m \geq F(s)$, fix some values of x_m, y_m, z_m, w_m, u_m, v_m, p_m and q_m that satisfy the equations, and keep these values fixed throughout the proof. Recall that l was defined to be the least number greater than or equal to $F(s)$ that appears with nonzero frequency in the partition π. Then we have the following cases:

- if T_1 is true, define

$$\phi(\pi) = (s^2, (s+1)^{f_{s+1}}, \ldots, (5s+1)^{x_l}, (5s+2)^{y_l}, \ldots, (F(s)-1)^{f_{F(s)-1}}, \ldots, l^{f_l - 1}, \ldots);$$

- if T_1 is false and T_2 is true, define

$$\phi(\pi) = (s^4, (s{+}1)^{f_{s+1}}, \ldots, (5s{+}1)^{z_l}, \ldots, (15s{-}2)^{w_l}, \ldots, (F(s){-}1)^{f_{F(s)-1}}, \ldots, l^{f_l-1}, \ldots);$$

- if T_1 and T_2 are false and T_3 is true, define

$$\phi(\pi) = (s^6, (s{+}1)^{f_{s+1}}, \ldots, (5s{+}2)^{v_l}, \ldots, (10s{-}1)^{u_l}, \ldots, (F(s){-}1)^{f_{F(s)-1}}, \ldots, l^{f_l-1}, \ldots);$$

- if T_1, T_2 and T_3 are false and T_4 is true, define

$$\phi(\pi) = (s^8, (s{+}1)^{f_{s+1}}, \ldots, (10s{-}1)^{p_l}, \ldots, (15s{-}2)^{q_l}, \ldots, (F(s){-}1)^{f_{F(s)-1}}, \ldots, l^{f_l-1}, \ldots).$$

The function ϕ is injective in Case 1(b). To see why, given a partition $\hat{\pi} = \phi(\pi)$ whose frequency of s is 2, 4, 6 or 8, we can recognize π as coming from this case. Then if, for example, the frequency of s in $\hat{\pi}$ is 2, then T_1 is true, and the frequencies of $5s+1$ and $5s+2$ in $\hat{\pi}$ give the values of x_l and y_l, respectively. Then, from the defining equation for x_l and y_l, given by

$$(5s+1)x_l + (5s+2)y_l = l - 2s,$$

we can recover l. From there it is easy to find π. A similar argument applies if the frequency of s in $\hat{\pi}$ is 4, 6 or 8. Thus, in all of Case 1(b), ϕ is injective.

This completes the description of ϕ for the case $f = 0$. Note, in aggregate, the function ϕ is injective in Case 1. If $\hat{\pi} = \phi(\pi)$, then the frequency of s in $\hat{\pi}$ indicates from which subcase π comes and, as shown above, π is then recoverable.

Case 2: Suppose $f \geq 1$. Hence, to produce the image of π under ϕ in this case, we must remove all parts of size $L + s - 1$. Recall, π has the form

$$\pi = ((s+1)^{f_{s+1}}, \ldots, (L+s-1)^f, (L+s)^{f_{L+s}}).$$

Since $L \geq s + 3$, we have $(L - s - 1)j \geq 1$ for all $j \geq 1$, and therefore

$$(L+s-1)j - s(2j-1) \geq s+1. \tag{3.1}$$

Then, by Lemma 3.5, for all $j \geq 1$, the equation

$$(L+s-1)j = s(2j-1) + (s+1)r_{s+1,j} + (s+2)r_{s+2,j} + \cdots + (2s+1)r_{2s+1,j} \tag{3.2}$$

has nonnegative integer solutions $r_{s+1,j}, r_{s+2,j}, \ldots, r_{2s+1,j}$. For each $j \geq 1$, fix a solution $r_{s+1,j}, r_{s+2,j}, \ldots, r_{2s+1,j}$.

Case 2(a): Suppose $f \neq 8$. Since $L \geq s+3$, we have $L+s-1 > 2s+1$. Define

$$\begin{aligned}\phi(\pi) = (s^{2f-1}, (s+1)^{f_{s+1}+r_{s+1,f}}, \ldots, (2s+1)^{f_{2s+1}+r_{2s+1,f}}, (2s+2)^{f_{2s+2}}, \\ \ldots, (L+s-2)^{f_{L+s-2}}, (L+s-1)^0, (L+s)^{f_{L+s}}).\end{aligned} \tag{3.3}$$

The case $f = 8$ is dealt with separately below to ensure injectivity of ϕ since then $2f-1 = 15$, and the frequency of 15 for s in partitions in the image of ϕ has already been used in Case 1(b)(i).

Case 2(b): Suppose $f = 8$. It follows from (3.1) that

$$8(L+s-1) - 15s \geq s+1,$$

and thus

$$8(L+s-1) - 14s \geq s+1.$$

Therefore, by Lemma 3.5, the equation

$$8(L+s-1) = 14s + (s+1)t_{s+1} + (s+2)t_{s+2} + \cdots + (2s+1)t_{2s+1} \tag{3.4}$$

has a nonnegative integer solution. Fix a solution $t_{s+1}, t_{s+2}, \ldots, t_{2s+1}$ of this equation. Thus, for

$$\pi = ((s+1)^{f_{s+1}}, \ldots, (L+s-1)^8, (L+s)^{f_{L+s}}),$$

we define

$$\begin{aligned}\phi(\pi) = (s^{14}, (s+1)^{f_{s+1}+t_{s+1}}, \ldots, (2s+1)^{f_{2s+1}+t_{2s+1}}, (2s+2)^{f_{2s+2}}, \\ \ldots, (L+s-2)^{f_{L+s-2}}, (L+s-1)^0, (L+s)^{f_{L+s}}).\end{aligned}$$

To see why ϕ is injective in Case 2, suppose $\hat{\pi} = \phi(\pi)$ and the frequency of s in $\hat{\pi}$ is either an odd number not equal to 15 or 14. In the former case, from the frequency of s, we can determine $f = f_{L+s-1}$ from (3.3) for its preimage; then from (3.2), one can determine the constants $r_{s+1,f}, \ldots, r_{2s+1,f}$. From this, it is clear the partition π can be reconstructed, so ϕ is injective in Case 2(a). In the latter case when the frequency of s is 14, we can apply a similar argument using (3.4). We conclude ϕ is injective in Case 2.

We refer the reader back to Table 3.1 to note that the map ϕ is injective overall. If $\hat{\pi} = \phi(\pi)$, then the frequency of s in $\hat{\pi}$ gives the case from which π came, and in each case itself ϕ was shown to be injective.

The injection above shows that

$$|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for every $N \geq \kappa(s)$. To complete the proof of Theorem 3.6, we prove that the inequality is in fact strict. To show this, we find a partition of N that is in $C_{L,s}$ but not in the image of ϕ. Since $N \geq \kappa(s)$ is large enough, by Sylvester's lemma, there exist nonnegative integers x_0 and y_0 such that

$$N = 10s + (s+1)x_0 + (s+2)y_0.$$

Consider the partition $\lambda_N = (s^{10}, (s+1)^{x_0}, (s+2)^{y_0})$ of N. Since $L \geq 3$ we have $L+s-1 > s+2$, so λ_N is in $C_{L,s}$. However, the partition λ_N is not in the image of ϕ since the frequency of s in λ_N is 10, and 10 does not occur as a frequency of s in a partition in the image of ϕ by Table 3.1.

□

3.2.2 Generalizing Theorem 3.6 if the impermissible part k is large

A careful analysis of the proof of Theorem 3.6 in the previous section shows that we have not used the fact that the impermissible part is one less than the largest allowable part. Therefore, the proof can be extended for a general impermissible part k under some restrictions. We presented the proof for $k = L+s-1$ in the previous section first to keep the case analysis simpler, and illustrate our techniques for the later proofs. In the proof below, we explain how the proof of Theorem 3.6 can be easily generalized.

We modify the definitions of $F(s)$ and $\kappa(s)$. For $s \geq 1$, define the quantities:

- $F'(s) = (21s-2)(35s-3) + 8s$;
- $\kappa'(s) = (12s-1)((s+1) + (s+2) + \cdots (F'(s)-1)) + 1$.

Theorem 3.7. *Suppose L, s and k are positive integers such that $L \geq s+2$ and $2s+2 \leq k \leq L+s$. Then*

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for any $N \geq \kappa'(s)$.

Proof. The proof of Theorem 3.7 is the same as the proof of Theorem 3.6 with $L+s-1$ being replaced with k everywhere, with some minor modifications. Note that we do not replace $L+s$ with $k+1$; we only change the impermissible part from $L+s-1$ to k. Since $k \geq 2s+1$, when $L+s-1$ is replaced by k, the crucial equation (3.1) becomes

$$kj - s(2j-1) \geq s+1,$$

and holds for all $j \geq 1$. The condition $k \geq 2s+2$ is required to ensure that the impermissible part k is different from $2s+1$, which may have been added as a part in the analogue of (3.2) given by

$$kj = s(2j-1) + (s+1)r_{s+1,j} + (s+2)r_{s+2,j} + \cdots + (2s+1)r_{2s+1,j}.$$

Finally, we observe that the proof of Theorem 3.6 requires modification if the impermissible part k is one of the numbers $5s+1$, $5s+2$, $10s-1$ or $15s-2$ because, to produce the image of a partition under ϕ, these numbers were added as parts in Case 1(b). If k is one these numbers, then we can repeat the same proof with the numbers $5s+1$, $5s+2$, $10s-1$ and $15s-2$ replaced with $7s+1$, $7s+2$, $21s-1$ and $35s-2$, respectively, and it is this modification that necessitates changing the constants $F(s)$ and $\kappa(s)$ to $F'(s)$ and $\kappa'(s)$, respectively. Note that for $s=1$, the numbers $10s-1$ and $7s+2$ coincide and are equal to 9. Thus, for $s=1$ and $k=9$, we choose a set of numbers different from $5s+1$, $5s+2$, $10s-1$ and $15s-2$ (6, 7, 9 and 13 when $s=1$). The choice of 7, 13, 21 and 29 (instead of 6, 7, 9 and 13) solves the problem for $s=1$ and $k=9$. These are all the necessary modifications needed to obtain a proof of the theorem. $\square$

3.2.3 An analogue of Theorem 3.6 if the impermissible part k is small

Theorem 3.7 requires the condition $k \geq 2s+2$. The next theorem focuses on when $s+1 \leq k \leq 2s+1$. For $s \geq 1$, we modify $F(s)$ and $\kappa(s)$ as follows:

- $F''(s) = (120s(s+1)-2)(180s(s+1)-3) + 420s(s+1)$;
- $\kappa''(s) = (300s(s+1)-1)((s+1)+(s+2)+\cdots(F''(s)-1)) + 1$.

Theorem 3.8. *Suppose L, s and k are positive integers such that $L \geq 3s+3$ and $s+1 \leq k \leq 2s+1$. Then*

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for any $N \geq \kappa''(s)$.

Proof. Although the proof of Theorem 3.8 is similar in style and essence to that of Theorem 3.6, it requires several more substantial modifications, so we explain them in detail.

We again construct an injective map

$$\phi : \{\pi \in D_{L,s} : |\pi| = N\} \to \{\pi \in I_{L,s,k} : |\pi| = N\}.$$

To show strict inequality, at the end of the proof we show that there is an element in the codomain of ϕ not in its range.

For $\pi \in D_{L,s}$, the image of π under ϕ is given in cases depending on the frequency of k in π. Hence, for brevity, we set $f = f_k$. So any $\pi \in D_{L,s}$ has the form

$$\pi = ((s+1)^{f_{s+1}}, \ldots, k^f, \ldots, (L+s)^{f_{L+s}}).$$

Our definition of the image of π under ϕ is given in two cases, when $f \geq 3$ and when $f \leq 2$, and each case contains several subcases. Our strategy for ensuring ϕ is injective is, like in

Theorem 3.6, to have the image of partitions under ϕ from different cases have different frequencies of s.

Case 1: Suppose $f \geq 3$. Then there exists a unique $j(f) \geq 0$ such that

$$(60(s+1)-3)j(f)+3 \leq f \leq (60(s+1)-3)(j(f)+1)+2.$$

We regard $j(\cdot)$ as a function from $\{3, 4, ...\}$ to the set of nonnegative integers. Because $k \geq s+1$, for any $i \geq 3$, we have

$$ki - s(i+3j(i)-2) \geq 2s+2.$$

Then, by Lemma 3.5 (applied with s replaced by $2s+1$ there), for all $i \geq 3$, the equation

$$ki = s(i+3j(i)-2) + (2s+2)r_{2s+2,i} + (2s+3)r_{2s+3,i} + \cdots + (4s+3)r_{4s+3,i} \qquad (3.5)$$

has nonnegative integer solutions $r_{2s+2,i}, r_{2s+3,i}, \ldots, r_{4s+3,i}$. For each $i \geq 3$, fix a solution $r_{2s+2,i}, r_{2s+3,i}, \ldots, r_{4s+3,i}$. Since $L \geq 3s+3$, we have $L+s \geq 4s+3$, and so $2s+2, \ldots, 4s+3$ are valid parts for partitions in $D_{L,s}$. Define

$$\phi(\pi) = (s^{f+3j(f)-2}, \ldots, k^0, \ldots, (2s+2)^{f_{2s+2}+r_{2s+2,f}}, \ldots, (4s+3)^{f_{4s+3}+r_{4s+3,f}}, \ldots).$$

Note that because $s+1 \leq k \leq 2s+1$, the frequency of k in $\phi(\pi)$ is genuinely 0, as it is not one of the parts whose frequency has increased.

To see why ϕ is injective in Case 1, note that each member of the set

$$V = \{i+3j(i)-2 : i \geq 3\}$$

is uniquely determined by its defining value of i; if $i > i'$, then $j(i) \geq j(i')$, so $i+3j(i)-2 > i'+3j(i')-2$. Thus, if $\hat{\pi} = \phi(\pi)$, and the frequency of s in $\hat{\pi}$ is in V, we can reverse the process above. From the frequency of s in $\hat{\pi}$, we can recover f; from there we can use f, $j(f)$ and (3.5) to determine the constants $r_{2s+2,f}, r_{2s+3,f}, \ldots, r_{4s+3,f}$. From this point, determining π is straightforward. We note here, for showing that ϕ is injective overall later, that the members of V are congruent to $1, 2, 3, 4, \ldots, -3$ modulo $60(s+1)$. That is, no member of V is congruent to $0, -1$ or -2 modulo $60(s+1)$.

Case 2: Suppose $f \leq 2$. As in Case 1, to obtain the image of a partition under ϕ, we must remove the parts of size k (if any) and insert parts of s. To ensure $|\phi(\pi)| = |\pi|$, we must alter the frequencies of other parts to compensate. The number of parts equal to s to be inserted into π is given by the subcases below. We describe all subcases of Case 2, and then discuss why ϕ is injective in Case 2.

Case 2(a): Suppose that there exists m such that $s+1 \leq m \leq F''(s)-1$ and $f_m \geq 300s(s+1)$. Let m_0 be the least such number. Notice that $m_0 \neq k$ because $f \leq 2$. Then define

$$\phi(\pi) = (s^{300(s+1)m_0 - f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, m_0^{f_{m_0} - 300s(s+1)}, \ldots).$$

Case 2(b): Suppose that the condition of Case 2(a) does not hold. That is, for every m such that $s+1 \leq m \leq F''(s) - 1$, we have $f_m \leq 300s(s+1) - 1$. Note that if such partitions do not exist, then Case 2(b) does not arise and there is no need to construct an injection. Since $N \geq \kappa''(s)$, we must have $L + s \geq F''(s)$, and also there must exist an $h \geq F''(s)$ such that $f_h > 0$. Let l be the least such number. Thus, we can write π as

$$\pi = ((s+1)^{f_{s+1}}, \ldots, k^f, \ldots, (F''(s) - 1)^{f_{F''(s)-1}}, \ldots, l^{f_l}, \ldots).$$

We have some further subcases. To ease notation, we define the following quantities:

- $\alpha = 60s(s+1) + 1$;
- $\beta = 60s(s+1) + 2$;
- $\gamma = 120s(s+1) - 1$;
- $\delta = 180s(s+1) - 2$.

Note that the quantities α, β, γ and δ are chosen such that they are distinct from k and

- $\alpha, \beta, \gamma, \delta < F''(s)$,
- $\gcd(\alpha, \beta) = \gcd(\alpha, \delta) = 1$,
- $\gcd(\gamma, \beta) = \gcd(\gamma, \delta) = 1$,
- $\alpha + \gamma = 180s(s+1)$,
- $\beta + \delta = 240s(s+1)$.

Case 2(b)(i): If $f_\alpha \geq 1$ and $f_\gamma \geq 1$, then define

$$\phi(\pi) = (s^{180(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \alpha^{f_\alpha - 1}, \ldots, \gamma^{f_\gamma - 1}, \ldots).$$

Case 2(b)(ii): If $f_\alpha = 0$ or $f_\gamma = 0$ and $f_\beta \geq 1$ and $f_\delta \geq 1$, then define

$$\phi(\pi) = (s^{240(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \beta^{f_\beta - 1}, \ldots, \delta^{f_\delta - 1}, \ldots).$$

Case 2(b)(iii): If $f_\alpha = 0$ or $f_\gamma = 0$ and $f_\beta = 0$ or $f_\delta = 0$, then at least one of the following statements is true:

- T_1: $f_\alpha = 0$ and $f_\beta = 0$;
- T_2: $f_\alpha = 0$ and $f_\delta = 0$;

- T_3: $f_\gamma = 0$ and $f_\beta = 0$;
- T_4: $f_\gamma = 0$ and $f_\delta = 0$.

Since $F''(s) - 420s(s+1) = (\gamma - 1)(\delta - 1)$, and since the relevant numbers are coprime, by Sylvester's lemma the following equations have nonnegative integer solutions for all $m \geq F''(s)$:

- $\alpha x_m + \beta y_m = m - 60s(s+1)$;
- $\alpha z_m + \delta w_m = m - 120s(s+1)$;
- $\gamma u_m + \beta v_m = m - 360s(s+1)$; (3.6)
- $\gamma p_m + \delta q_m = m - 420s(s+1)$.

That the lower bound on m is sufficient for all the equations to have nonnegative integer solutions follows from the lower bound being sufficient for the last equation to have such solutions; there the lower bound on m is the one specified by Sylvester's lemma. For each $m \geq F''(s)$, fix some values of x_m, y_m, z_m, w_m, u_m, v_m, p_m and q_m that satisfy the equations, and keep these values fixed throughout the proof. Recall that l was defined to be the least number greater than or equal to $F''(s)$ that appears with nonzero frequency in the partition π. Then we have the following cases:

- if T_1 is true, define

$$\phi(\pi) = (s^{60(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \alpha^{x_l}, \beta^{y_l}, \\ \ldots, (F''(s)-1)^{f_{F''(s)-1}}, \ldots, l^{f_l - 1}, \ldots);$$

- if T_1 is false and T_2 is true, define

$$\phi(\pi) = (s^{120(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \alpha^{z_l}, \ldots, \delta^{w_l}, \\ \ldots, (F''(s)-1)^{f_{F''(s)-1}}, \ldots, l^{f_l - 1}, \ldots);$$

- if T_1 and T_2 are false and T_3 is true, define

$$\phi(\pi) = (s^{360(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \beta^{v_l}, \ldots, \gamma^{u_l}, \\ \ldots, (F''(s)-1)^{f_{F''(s)-1}}, \ldots, l^{f_l - 1}, \ldots);$$

- if T_1, T_2 and T_3 are false and T_4 is true, define

$$\phi(\pi) = (s^{420(s+1)-f}, (s+1)^{f_{s+1}}, \ldots, k^0, \ldots, (s+k)^{f_{s+k}+f}, \ldots, \gamma^{p_l}, \ldots, \delta^{q_l}, \\ \ldots, (F''(s)-1)^{f_{F''(s)-1}}, \ldots, l^{f_l - 1}, \ldots).$$

Note in all cases $\phi(\pi)$ has at least one part of size s, no parts of size k, and all parts are $\leq L+s$, so $\phi(\pi) \in I_{L,s,k}$. Define the following sets:

- $V_1 = \{60(s+1)i : i \geq 1\}$;
- $V_2 = \{60(s+1)i - 1 : i \geq 1\}$;
- $V_3 = \{60(s+1)i - 2 : i \geq 1\}$.

Note that the frequency of s in a partition in the image of ϕ in Case 2 lies in V_1, V_2 or V_3 according to whether f is 0, 1 or 2, respectively. To see why ϕ is injective in Case 2, suppose $\hat{\pi} = \phi(\pi)$ and frequency of s in $\hat{\pi}$ lies in one of those three sets. This frequency can be a member of one of two sets:

$$\{300i(s+1) - h : i \geq 2,\ h = 0,1,2\} \quad \text{or} \quad \{60i(s+1) - h : i = 1,2,3,4,6,7,\ h = 0,1,2\}. \tag{3.7}$$

The former set of values pertains to Case 2(a), while the latter to Case 2(b). These possible values for the frequency of s in $\hat{\pi}$ distinguish which subcase π comes from. Say, for example, the frequency of s in $\hat{\pi}$ is $120(s+1) - h$ for some $h = 0,1,2$. From this, we recover f as h. We also know from the frequency of s in $\hat{\pi}$ that for π the condition T_1 above is false, but T_2 is true. Hence, the frequencies of α and δ in $\hat{\pi}$ give the values of z_l and w_l, respectively. We can then use the second equation in (3.6) to find the value of l. Once the value of l is known, we can reconstruct π. When the frequency of s in $\hat{\pi}$ is some other value in the sets given in (3.7), we can similarly reconstruct π.

Finally, we note that ϕ is injective overall. As discussed in both Cases 1 and 2 separately, ϕ is injective. However, the sets V, V_1, V_2 and V_3 are all pairwise disjoint. Indeed, members of V_1, V_2 and V_3 are congruent to 0, -1 and -2 modulo $60(s+1)$, respectively, whereas members of V, as noted in Case 1, are not congruent to 0, -1 or -2 modulo $60(s+1)$. As these are the possible values of the frequency of s in a partition in the image of ϕ, they distinguish the different cases for preimages under ϕ, and we conclude that ϕ is injective.

The injection above shows that

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for every $N \geq \kappa''(s)$. To complete the proof of Theorem 3.6, we prove that the inequality is in fact strict by finding a partition of N that is in $I_{L,s,k}$ but not in the image of ϕ. Since $N \geq \kappa''(s)$ is large enough, by Sylvester's lemma, there exist nonnegative integers x_0 and y_0 such that

$$N = 480s(s+1) + (2s+2)x_0 + (2s+3)y_0.$$

This gives a partition $\lambda_N = (s^{480(s+1)}, (2s+2)^{x_0}, (2s+3)^{y_0})$ of N that is in $I_{L,s,k}$ (because $s+1 \leq k \leq 2s+1$), but not in the image of ϕ since the frequency of s in λ_N is $480(s+1)$. This is different from the frequencies of s in partitions in the image of ϕ. $\square$

3.3 Proofs of Theorems 3.1 and 3.3

In Section 3.3.1, we prove $H_{L,s,k}(q)$ is eventually positive for $s+1 \le k \le L+s$ (Theorem 3.9). The bound M for which $N \ge M$ guarantees the coefficient of q^N in $H_{L,s,k}(q)$ is positive is initially given as depending on L and s, so we do not immediately obtain a proof of Theorem 3.1. However, from this result, and results in Section 3.2, we are able to obtain a quick proof of Theorem 3.1 in Section 3.3.2. As noted in Section 3.1, Conjectures 1.12 and 1.13 can be obtained as corollaries of Theorem 3.1. In Section 3.3.3, we prove Theorem 3.3.

3.3.1 The case $s+1 \le k \le L+s$ for $H_{L,s,k}(q)$

For positive integers $L \ge 3$ and s, define:

- $P_{L,s} = (s+1)(s+2)\dots(s+L)$;
- $$\gamma(L,s) = ((s+1)+(s+2)+\cdots+(s+L)) \cdot \left(P_{L,s}^{\left(P_{L,s}^2-1\right)L+2} + \left(\left(P_{L,s}^2-1\right)L-2\right)P_{L,s}\right). \tag{3.8}$$

The number $\gamma(L,s)$, which only depends on L and s, serves as our bound M in the next theorem.

Theorem 3.9. *For positive integers L, s and k, with $L \ge 3$ and $s+1 \le k \le L+s$, the inequality*

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|$$

holds for every $N \ge \gamma(L,s)$.

Proof. For $N \ge \gamma(L,s)$, we construct an injective map

$$\psi : \{\pi \in D_{L,s} : |\pi| = N\} \longrightarrow \{\pi \in I_{L,s,k} : |\pi| = N\}.$$

To show strict inequality, at the end of the proof we show that there is an element in the codomain of ψ not in its range.

We shall separate our argument into cases and subcases depending on the frequencies of parts of $\pi = ((s+1)^{f_{s+1}}, \dots, k^{f_k}, \dots, (L+s)^{f_{L+s}})$ in the domain. The part whose frequency defines the cases is k; hence, to simplify notation, we set $f = f_k$, so a partition in the domain has the form

$$\pi = ((s+1)^{f_{s+1}}, \dots, k^{f}, \dots, (L+s)^{f_{L+s}}).$$

Case 1: Suppose $f = 0$. Since $N \ge \gamma(L,s)$ is large enough, there is an m such that $s+1 \le m \le L+s$ and $f_m \ge s$. Let m_0 be the least such number. Clearly, the number m_0 cannot be k, since $f = 0$. Then define

$$\psi(\pi) = (s^{m_0}, (s+1)^{f_{s+1}}, \dots, m_0^{f_{m_0}-s}, \dots).$$

As $f_k = f = 0$, and the frequency of $s > 0$ in $\psi(\pi)$, we see that $\psi(\pi) \in I_{L,s,k}$.

It is clear that ψ is injective on the domain in this case, and the frequency of s in a partition in its image is in the set

$$U_1 = \{s+1, \dots, L+s\}.$$

Case 2: Suppose $f \neq 0$ in π. We have some subcases. As the partitions of $I_{L,s,k}$ have no part equal to k but must have a part of size s, to obtain $\psi(\pi)$ from π we remove the parts of size k and add parts of size s, and compensate in some way so that $|\psi(\pi)| = |\pi|$. Choose α and β as follows:

- if $k \neq s+1$ and $k \neq s+2$, then choose $\alpha = 1$ and $\beta = 2$;
- if $k = s+1$, choose $\alpha = 2$ and $\beta = 3$;
- if $k = s+2$, choose $\alpha = 1$ and $\beta = 3$.

Since $L \geq 3$, we have $s+1 \leq s+\alpha < s+\beta \leq L+s$. Importantly, α and β are chosen so that $k \neq s+\alpha$ and $k \neq s+\beta$. Furthermore, the numbers $s+\alpha$ and $s+\beta$ are coprime except possibly when $k = s+2$; in that case, if s is odd, the greatest common divisor of $s+\alpha$ and $s+\beta$ is 2. The case $k = s+2$ with s is odd will require special treatment below because of this.

Case 2(a): Suppose π has $f \geq P_{L,s}^2$. For any $j \geq P_{L,s}^2$, let $h(j)$ be the integer satisfying $P_{L,s}^{h(j)} \leq j < P_{L,s}^{h(j)+1}$, and consider the set

$$U_a = \{j + (h(j)-3)P_{L,s} - 2 : j \geq P_{L,s}^2\}.$$

Clearly every $j \geq P_{L,s}^2$ gives a unique member of U_a. Conversely, from each member of U_a, its defining value of j can be recovered; if $j > j'$, then $h(j) \geq h'(j)$, so $j+(h(j)-3)P_{L,s}-2 > j' + (h(j')-3)P_{L,s} - 2$.

As $N \geq \gamma(L,s)$, it is easy to verify, for any $j \geq P_{L,s}^2$, that

$$kj \geq s(j + (h(j)-3)P_{L,s} - 2) + (s+\alpha-1)(s+\beta-1).$$

Therefore, if $k \neq s+2$, or $k = s+2$ and s is even, by Sylvester's lemma, since $s+\alpha$ and $s+\beta$ are coprime, for any $j \geq P_{L,s}^2$, the equation

$$kj = s(j + (h(j)-3)P_{L,s} - 2) + (s+\alpha)x_j + (s+\beta)y_j \tag{3.9}$$

has nonnegative integer solutions x_j and y_j.

We noted earlier that when $k = s+2$ and s is odd then $s+\alpha = s+1$ and $s+\beta = s+3$ have a greatest common factor of 2, so we cannot immediately apply Sylvester's lemma to obtain x_j and y_j in (3.9). We can however overcome this difficulty as follows. We note that since $P_{L,s}$ is even, $kj - s(j + (h(j) - 3)P_{L,s} - 2)$ is also even for all j and is greater than or equal to $(s+1)(s+3)$. Thus, it follows that

$$Q = \frac{kj - s(j + (h(j) - 3)P_{L,s} - 2)}{2}$$

is an integer greater than the Frobenius number of the coprime integers $\frac{s+1}{2}$ and $\frac{s+3}{2}$. Thus, by Sylvester's lemma, the quantity Q can be expressed as a nonnegative integer combination of $\frac{s+1}{2}$ and $\frac{s+3}{2}$, which can be used to find solutions x_j and y_j to (3.9). So, even in the case when $k = s+2$ and s is odd, (3.9) has nonnegative integer solutions.

In any case, for each $j \geq P_{L,s}^2$, fix a solution x_j and y_j to (3.9) and keep it fixed throughout the entire proof.

Define

$$\psi(\pi) = (s^{f+(h(f)-3)P_{L,s}-2}, \ldots, (s+\alpha)^{f_{s+\alpha}+x_f}, \ldots, (s+\beta)^{f_{s+\beta}+y_f}, \ldots, k^0, \ldots),$$

where it is understood that the part k is not precisely placed (it may, for example, be the case that $k < s+\beta$).

To see that ψ is injective in this case, suppose that $\hat{\pi} = \psi(\pi)$ and the frequency of s in $\hat{\pi}$ is in U_a. As the defining values of members of U_a can be recovered, we can recover f from the frequency of s. Once f is found, we can use (3.9) to find x_f and y_f. From here, we can easily recover π.

We repeat the above strategy for the remaining cases. To that end, define the set

$$U_b = \{P_{L,s}^h + (h-4)P_{L,s} : h \geq 3\} \cup \{P_{L,s}^h + (h-4)P_{L,s} + 1 : h \geq 3\},$$

noting that the union is disjoint. The reader is invited to confirm that 1) each member of U_b uniquely determines its defining value of h, and 2) the sets U_1, U_a and U_b are pairwise disjoint. The image of π under ψ in each of the remaining cases has its frequency of s lie in U_b. Hence it suffices to show that when ψ is restricted to the domain of the remaining cases, it is injective.

The remaining case is when $f < P_{L,s}^2$.

Case 2(b): Suppose that $0 < f < P_{L,s}^2$ in π. Since $N \geq \gamma(s)$ is large enough, there exists an h such that $1 \leq h \leq L$ and

$$f_{s+h} \geq P_{L,s}^{(f-1)L+(h+2)} + ((f-1)L + (h-2))P_{L,s}. \tag{3.10}$$

For brevity of notation, for any $0 < i < P_{L,s}^2$ and $1 \leq h \leq L$ set

$$m_{i,h} = P_{L,s}^{(i-1)L+(h+2)} + ((i-1)L + (h-2))P_{L,s}.$$

We make a few key observations about the numbers $m_{i,h}$, the first of which is that $m_{i,h} \in U_b$ for any valid i and h. Second, each number $m_{i,h}$ is determined uniquely by its defining value of (i,h). To see why, if $m_{i,h} = m_{i',h'}$ then, as the exponent in the first term of $m_{i,h}$ is at least three, we have

$$(i-1)L + h + 2 = (i'-1)L + h' + 2,$$

and thus $h - h' = (i' - i)L$. But since $1 \leq h, h' \leq L$, this implies that $h = h'$, so $i' = i$.

Let p be the least integer $1 \leq h \leq L$ for which (3.10) is satisfied. Notice that the restriction on f prevents $s + p$ from being k. By definition,

$$f_{s+p} \geq m_{f,p}. \tag{3.11}$$

Notice that $m_{f,p}$ is divisible by $P_{L,s}$, and thus also by $(s+p)$; hence, we can define $j_{f,p}$ by

$$(s+p) j_{f,p} = s m_{f,p}. \tag{3.12}$$

From (3.11) and (3.12), it is easy to verify that

$$f_{s+p} \geq j_{f,p} + 2s.$$

For any integer u, we define η_u to be 1 if u is odd and 0 otherwise. One can easily verify that for any $1 \leq t \leq P_{L,s}^2$ and any $1 \leq i \leq L$, we have

$$2s(s+i) + tk - s\left(\delta_{k,s+2}\right)\eta_s \eta_t \geq (s+\alpha-1)(s+\beta-1), \tag{3.13}$$

We explain the peculiar term $s\left(\delta_{k,s+2}\right)\eta_s\eta_t$. As before, when $k \neq s+2$, or $k = s+2$ and s is even, the numbers $s+\alpha$ and $s+\beta$ are coprime, so by Sylvester's lemma, there exist nonnegative integer solutions $z_{t,i}$ and $w_{t,i}$ such that

$$2s(s+i) + tk = s\left(\delta_{k,s+2}\right)\eta_s\eta_t + (s+\alpha)z_{t,i} + (s+\beta)w_{t,i}. \tag{3.14}$$

When $k = s+2$ and s is odd, recall that $s+\alpha = s+1$ and $s+\beta = s+3$ are not coprime, but we can remedy this issue as before. The term $s\left(\delta_{k,s+2}\right)\eta_s\eta_t$ guarantees the left hand side of (3.13) is even. We then apply the same fix as earlier: we divide the left hand side of (3.13) by 2, and the result is greater than the Frobenius number of the coprime integers $\frac{s+1}{2}$ and $\frac{s+3}{2}$. Then applying Sylvester's lemma gives us nonnegative integer solutions to (3.14) in this case as well.

In any case, for each $1 \leq t \leq P_{L,s}^2$ and $1 \leq i \leq L$, fix a solution $z_{t,i}$ and $w_{t,i}$ to (3.14).

Let $n_{f,p}$ be defined as $m_{f,p} + \left(\delta_{k,s+2}\right)\eta_s\eta_f$. Notice that $n_{f,p} \in U_b$. To obtain $\psi(\pi)$ from π, we add to π a part s with frequency $n_{f,p}$, reduce the frequency of $s+p$ by $j_{f,p} + 2s$, remove the f parts of k and add $(s+\alpha)$ and $(s+\beta)$ with frequencies of $z_{f,p}$ and $w_{f,p}$, respectively.

Case of π	Value of f	Possible frequencies of s in $\psi(\pi)$.
1	$f = 0$	U_1
2(a)	$f \geq P_{L,s}^2$	U_a
2(b)	$0 < f < P_{L,s}^2$	U_b

Table 3.2: The possible frequencies of s in partitions in the image of ψ.

Thus, if $p \neq \alpha$, $p \neq \beta$, we define

$$\psi(\pi) = \left(s^{n_{f,p}}, \ldots, (s+\alpha)^{f_{s+\alpha}+z_{f,p}}, \ldots, (s+\beta)^{f_{s+\beta}+w_{f,p}}, \ldots (s+p)^{f_{s+p}-j_{f,p}-2s}, \ldots k^0, \ldots\right).$$

If $p = \alpha$, we define

$$\psi(\pi) = \left(s^{n_{f,\alpha}}, \ldots, (s+\alpha)^{f_{s+\alpha}+z_{f,\alpha}-j_{f,\alpha}-2s}, \ldots, (s+\beta)^{f_{s+\beta}+w_{f,\alpha}}, \ldots k^0, \ldots\right).$$

If $p = \beta$, we define

$$\psi(\pi) = \left(s^{n_{f,\beta}}, \ldots, (s+\alpha)^{f_{s+\alpha}+z_{f,\beta}}, \ldots, (s+\beta)^{f_{s+\beta}+w_{f,\beta}-j_{f,\beta}-2s}, \ldots k^0, \ldots\right).$$

We first note that it follows from (3.12) and (3.14) that $|\pi| = |\psi(\pi)|$. As the frequency of s in $\psi(\pi)$ is $n_{f,p}$, it lies in U_b, as noted earlier. To see why ψ is injective in this case, suppose that $\hat{\pi} = \psi(\pi)$ and its frequency of s has the form $n_{f,p}$. Recall, both f and p can be recovered from $m_{f,p}$. However, from $n_{f,p}$ it is clear that we can recover $m_{f,p}$; if $n_{f,p}$ is divisible by $P_{L,s}$, then $m_{f,p} = n_{f,p}$, and otherwise $m_{f,p} = n_{f,p} - 1$. Thus, f and p are recoverable from $n_{f,p}$ as well. From there, using (3.12), we can determine $j_{f,p}$. Furthermore, from f and p, we can use (3.14) to determine $z_{f,p}$ and $w_{f,p}$. From this point, we can easily recover π.

We summarize the possible frequencies of s in a partition in the image of ψ and the case to which it pertains in Table 3.2. As discussed earlier, as U_1, U_a and U_b are pairwise disjoint, the frequency of s in a partition in the image of ψ characterizes the case from which its preimage comes. The injectivity of ψ over all cases follows from ψ being injective in each case, which has been demonstrated.

The injection above shows that

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for every $N \geq \gamma(L,s)$. To complete the proof of Theorem 3.9, we must prove that the inequality is in fact strict, so we find a partition of N that is in $I_{L,s,k}$ but not in the image of ψ. Since $N \geq \gamma(L,s)$ is large enough, by Sylvester's lemma, there exist nonnegative integers x_0 and y_0 such that

$$N = s\left(L + s + 1 + (\delta_{k,s+2})\,\eta_s \eta_{N-L}\right) + (s+\alpha)x_0 + (s+\beta)y_0. \tag{3.15}$$

This gives a partition

$$\lambda_N = (s^{L+s+1+(\delta_{k,s+2})\eta_s\eta_{N-L}}, (s+\alpha)^{x_0}, (s+\beta)^{y_0})$$

of N. Again, the term $(\delta_{k,s+2})\,\eta_s\eta_{N-L}$ is to ensure that (3.15) has a solution even in the case $k = s+2$ and s is odd, as before. It is easy to see λ_N has the desired properties. Its frequency of s is either $L+s+1$ or $L+s+2$ and its frequency of k is 0, so it is in $I_{L,s,k}$. Furthermore, it is easy to see its frequency of s is not a member of U_1, U_a or U_b, so λ_N is not in the range of ψ. $\square$

3.3.2 Proofs of Theorem 3.1 and Conjectures 1.12 and 1.13

As noted earlier, while the bound in Theorem 3.9 depends on both L and s, we can use that theorem in tandem with Theorems 3.7 and 3.8 to give a proof of Theorem 3.1. For that, define

$$\Gamma(s) = \gamma(3s+2, s), \tag{3.16}$$

where $\gamma(\cdot,\cdot)$ is defined as in (3.8).

Proof of Theorem 3.1. We show that if $N \geq \Gamma(s)$, the inequality

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|$$

holds.

If $L \geq 3s+3$ and $2s+2 \leq k \leq L+s$, then by Theorem 3.7 the inequality holds for all $N \geq \kappa'(s)$.

However, if $L \geq 3s+3$ and $s+1 \leq k \leq 2s+1$, then by Theorem 3.8 the inequality holds for all $N \geq \kappa''(s)$. Thus, combining these two results, we see that for $L \geq 3s+3$ and $s+1 \leq k \leq L+s$, if $N \geq \max(\kappa'(s), \kappa''(s))$ then the inequality holds.

Finally, if $3 \leq L \leq 3s+2$ and $s+1 \leq k \leq L+s$, then by Theorem 3.9 the inequality holds for all $N \geq \gamma(3s+2, s) = \Gamma(s)$. It follows that for all $L \geq 3$ and $s+1 \leq k \leq L+s$, the inequality holds for all N larger than the three constants $\kappa'(s), \kappa''(s)$ and $\Gamma(s)$. A simple comparison reveals that $\Gamma(s)$ is the largest of the three constants, and we give a short proof of this. First, an easy calculation shows that $\kappa'(s) < 10^7(s+1)^5$ and $\kappa''(s) < 10^{11}(s+1)^{10}$. Next, we show that $\Gamma(s)$ is always larger than these numbers. Observe that $P_{3s+2,s} > (s+1)^{3s+2} \geq 32$ and thus $(P^2_{3s+2,s} - 1) > 1000$. Using this, we find

$$\begin{aligned}
\Gamma(s) = \gamma(3s+2, s) &> P_{3s+2,s}^{P^2_{3s+2}-1} \\
&> P_{3s+2,s}^{1000} \\
&> (s+1)^{1000(3s+2)} \\
&\geq (s+1)^{5000}
\end{aligned}$$

$$
\begin{aligned}
&\geq 2^{4990}(s+1)^{10} \\
&> 10^{11}(s+1)^{10} \\
&> \max(\kappa'(s), \kappa''(s)),
\end{aligned}
$$

completing the proof. $\square$

Remark 10. If $s+2 \leq L \leq 3s+2$ and $k \geq 2s+2$, then we can use the bound $\kappa'(s)$ in Theorem 3.7 instead of the larger bound $\gamma(3s+2, s)$ suggested by the proof of Theorem 3.1.

As remarked in Section 1.2, setting $k = L+s-1$ and $k = L$ in Theorem 3.1 proves Conjectures 1.12 and 1.13, respectively. We state these separately, however, in the next corollaries so that the bound M for when the partition inequalities hold is explicit.

Corollary 3.10. *If $L \geq 3$ and s are positive integers, then*

$$|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}| \quad \text{(Conjecture 1)}$$

and

$$|\{\pi \in C^*_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}| \quad \text{(Conjecture 2)}$$

for all $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (3.16).

3.3.3 The proof of Theorem 3.3

In the previous section, we proved Theorem 3.1, and, as remarked in Section 1.2, this proves Theorem 3.3 in the cases $s+1 \leq k \leq L+s$. We will, however, be able to use Theorem 3.1 to prove Theorem 3.3 in general. We first prove some lemmas.

Lemma 3.11. *For positive integers $L \geq 3$, s and $k \geq s+1$, the difference $H_{L,s,k+s}(q) - H_{L,s,k}(q)$ is nonnegative.*

Proof. The lemma is stated in the simplest form that we need. We shall, however, prove a stronger statement: for L, s and k as in the lemma, the difference $H_{L,s,k+i}(q) - H_{L,s,k}(q)$ is nonnegative for all $s \leq i \leq L+s$.

We have

$$H_{L,s,k+i}(q) - H_{L,s,k}(q) = \frac{q^{s+k}(1-q^i)}{(1-q^s)(1-q^{s+1})\cdots(1-q^{L+s})}.$$

For $s \leq i \leq L+s$, the factor $(1-q^i)$ in the numerator is also present in the denominator, and so it cancels. Hence the difference $H_{L,s,k+i}(q) - H_{L,s,k}(q)$ is nonnegative. $\square$

From Theorem 3.1 and Lemma 3.11, it follows that $H_{L,s,k}(q)$ is eventually positive for all $k \geq s+1$ whenever $L \geq s$. However, we are still left with various cases when $L < s$. For example, if $s = 10$ and $L = 3$, then Theorem 3.1 shows that $H_{L,s,k}(q)$ is eventually positive whenever k is $11, 12$ or 13. Then, according to Lemma 3.11, the series $H_{L,s,k}(q)$ is eventually positive whenever k is $21, 22$ or 23, which leaves the gap $14 \leq k \leq 20$. To complete the proof of Theorem 3.3, a close analysis of the cases covered by a combination of Theorem 3.1 and Lemma 3.11 gives that when $L < s$ it suffices to prove that $H_{L,s,k}(q)$ is eventually positive whenever $L + s < k \leq 2s$. We do so in Lemma 3.13, but before that we need another lemma.

Lemma 3.12. *For positive integers $L \geq 3$ and s, the coefficient of q^N in the series*

$$\frac{q^s - q^{L+s-1}}{(q^s; q)_{L+1}}$$

is positive whenever $N \geq \gamma(s, s)$, where $\gamma(\cdot, \cdot)$ is defined in (3.8)*.*

Proof. Given a natural number N,

- A_N is the set of partitions of N with parts in $\{s, \ldots, L + s\}$, and s appears as a part at least once;
- B_N is the set of partitions of N with parts in $\{s, \ldots, L + s\}$, and $L + s - 1$ appears as a part at least once.

Proving the lemma is equivalent to showing that $|A_N| \geq |B_N|$ for all $N \geq \gamma(s, s)$. When $L \geq s+1$, this is easy to show for all $N \geq 1$; if B_N is nonempty and $\pi \in B_N$, remove a part of size $L + s - 1$ from π and insert parts s and t, where $t = L - 1 \geq s$. This process clearly defines an injective function from B_N to A_N. We must therefore deal with the case when $L \leq s$. For that, given a natural number N,

- C_N is the set of partitions of N where the smallest part is s, all parts are $\leq L + s$, and $L + s - 1$ does not appear as a part;
- D_N is the set of partitions of N with parts in the set $\{s + 1, \ldots, L + s\}$;
- E_N is the set of partitions of N with parts in the set $\{s, \ldots, L + s\}$, and $L + s - 1$ does not appear as a part;
- F_N is the set of partitions of N with parts in the set $\{s, \ldots, L + s\}$.

Notice that $C_N = \{\pi \in C_{L,s} : |\pi| = N\}$ and $D_N = \{\pi \in D_{L,s} : |\pi| = N\}$. We could use Corollary 3.10 here, but since $L \leq s$, we can use Theorem 3.9 to obtain a stronger bound by setting $k = L + s - 1$ there; the inequality $|C_N| > |D_N|$ then holds for all $N \geq \gamma(s, s)$. Since $C_N \subset E_N$, we also have $|E_N| \geq |D_N|$ for all $N \geq \gamma(s, s)$. Notice that $B_N = F_N \setminus E_N$ and $A_N = F_N \setminus D_N$. Hence $|A_N| \geq |B_N|$ for all $N \geq \gamma(s, s)$. $\square$

As noted above, the following result completes the proof of Theorem 3.3.

Lemma 3.13. *For positive integers L, s and k such that $3 \leq L < s$ and $L + s \leq k \leq 2s$, the coefficient of q^N in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$.*

Proof. Our proof is by strong induction. The base case ($k = L + s$) has already been proven in Theorem 3.1. Next, assume that for some i such that $L + s \leq i < 2s$, the coefficient of q^N in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $L + s \leq j \leq i$. We shall prove that the coefficient of q^N in $H_{L,s,i+1}(q)$ is also positive whenever $N \geq \Gamma(s)$. Consider the difference

$$\begin{aligned} H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q) &= \frac{q^s(q^{i-L+2} - q^{i+1})}{(q^s;q)_{L+1}} \\ &= \frac{q^{i-L+2}(q^s - q^{L+s-1})}{(q^s;q)_{L+1}}. \end{aligned}$$

Thus, from Lemma 3.12, the coefficient of q^N in the difference $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \gamma(s,s) + 2s$ (because $i < 2s$). It is easy to verify, for any positive integer s, that $\Gamma(s) \geq \gamma(s,s) + 2s$. Thus, the coefficient of q^N in $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \Gamma(s)$.

The induction hypothesis states that the coefficient of q^N in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $L + s \leq j \leq i$. Combining this with Theorem 3.1, the coefficient of q^N in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $s + 1 \leq j \leq i$. Since $L \geq 3$ and $i \geq L + s$, we have $s + 1 \leq i - L + 2 \leq i$, and thus the coefficient of q^N in $H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \Gamma(s)$.

Thus, we have shown whenever $N \geq \Gamma(s)$ that the coefficient of q^N in both the series $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ and $H_{L,s,i-L+2}(q)$ is positive. This shows that the coefficient of q^N in $H_{L,s,i+1}(q)$ is positive whenever $N \geq \Gamma(s)$, completing the induction argument. □

We collect all of these results together to complete the proof of Theorem 3.3, which, as noted in Section 1.2, generalizes Conjecture 3.2 and Theorem 3.1.

Proof of Theorem 3.3. Suppose $N \geq \Gamma(s)$. If $L \geq s$, then Theorem 3.1 and Lemma 3.11 immediately prove Theorem 3.3.

If $L < s$ and $s + 1 \leq k \leq L + s$, then Theorem 3.9 completes the proof of Theorem 3.3. If $L < s$ and $L + s < k \leq 2s$, then Lemma 3.13 completes the proof of Theorem 3.3. Thus, combining these two cases, the theorem holds for $L < s$ and $s + 1 \leq k \leq 2s$. Since the result holds for $L < s$ and $s + 1 \leq k \leq 2s$, an application of Lemma 3.11 covers the cases $L < s$ and $k > 2s$. This covers all cases and completes the proof. □

3.4 Proof of Conjecture 3.4

In this section, we prove Conjecture 3.4, which pertains to the series $G_{L,2}(q)$. In [BU19], Berkovich and Uncu found an alternative expression for $G_{L,2}(q)$.

Theorem 3.14 (Berkovich and Uncu (2019))**.** *For $L \geq 3$,*

$$G_{L,2}(q) = \frac{H_{L,2,L}(q)}{1-q^L}.$$

From Theorem 3.3, we know that $H_{L,2,L}(q)$ is eventually positive, as $H_{L,2,L}(q)$ is the particular case of $s = 2$ and $k = L$ in that theorem, and from there it can be shown that $G_{L,2}(q)$ is also eventually positive. However, to prove Conjecture 3.4, we need to prove that the required series is not merely eventually positive, but that all its coefficients, with the exception of a few small terms, are nonnegative. We therefore need a method for the particular case $s = 2$ in the series $H_{L,s,L}(q)$ that analyzes the coefficients of q^n for small n. As for the previous conjectures, our methods highly depend on Sylvester's lemma and Lemma 3.5.

Recall the following notation from Section 3.1, which we require throughout this section. For a positive integer $L \geq 3$,

- $C^*_{L,2} = I_{L,2,L}$ denotes the set of partitions such that the smallest part is 2, all parts are $\leq L+2$, and L is not a part;
- $D_{L,2}$ denotes the set of nonempty partitions with parts in the set $\{3, 4, \ldots, L+2\}$.

From (1.1), for $N \geq 0$, the coefficient of q^N in $H_{L,2,L}(q)$ is

$$|\{\pi \in C^*_{L,2} : |\pi| = N\}| - |\{\pi \in D_{L,2} : |\pi| = N\}|.$$

We use this combinatorial interpretation along with Theorem 3.14 to obtain information about $G_{L,2}(q)$. We prove the last part of Conjecture 3.4 by first proving it for large L (i.e. $L \geq 11$), and then we prove it for smaller values (i.e. $5 \leq L \leq 10$). The cases $L = 3$ and $L = 4$ are done afterwards.

Theorem 3.15. *For $L \geq 11$,*

$$G_{L,2}(q) + q^3 \succeq 0.$$

Proof. For $N > 3$, we construct an injective map

$$\phi : \{\pi \in D_{L,2} : |\pi| = N\} \to \{\pi \in C^*_{L,2} : |\pi| = N\}.$$

For a partition π in the domain, we let f be the frequency of L, so π has the form $(3^{f_3}, \ldots, L^f, \ldots, (L+2)^{f_{L+2}})$. Our definition of the image of π under ϕ is given in two cases, when $f > 0$ and $f = 0$, with the latter case containing several subcases. We describe all the cases first, and show that ϕ is injective later. For the reader wishing to look ahead, Table 3.3 contains a summary of the cases.

Case 1: Suppose $f > 0$. Since $L \geq 11$, for any $i \geq 1$, we have $(L-8)i \geq 3$, and thus by Lemma 3.5 (applied with $s = 2$ there), there are nonnegative integers x_i, y_i and z_i such that

$$Li = 8i + 3x_i + 4y_i + 5z_i.$$

For each $i \geq 1$, fix some values of x_i, y_i and z_i and keep them fixed throughout the proof. Define

$$\phi(\pi) = \left(2^{4f}, 3^{x_f+f_3}, 4^{y_f+f_4}, 5^{z_f+f_5}, 6^{f_6}, \dots, L^0, \dots (L+2)^{f_{L+2}}\right).$$

Case 2: When $f = 0$, to obtain $\phi(\pi)$ from π, there are no parts of L to remove. Thus, to obtain $\phi(\pi)$, we must insert parts of size 2 into π and compensate in some way. For this, we must consider several subcases. We denote the smallest part of π by $s(\pi)$.

Case 2(A): When $s(\pi) \geq 5$, we define

$$\phi(\pi) = \left(2^1, (s(\pi)-2)^1, (s(\pi))^{f_{s(\pi)}-1}, \dots\right).$$

Case 2(B): Suppose $s(\pi) \leq 4$, so $s(\pi)$ is either 3 or 4.

Case 2(B)(i): If $f_4 \geq 1$, we define

$$\phi(\pi) = \left(2^2, 3^{f_3}, 4^{f_4-1}, \dots\right).$$

Case 2(B)(ii): Suppose $f_4 = 0$, so $s(\pi) = 3$. We have further subcases.

Case 2(B)(ii)(a): If $f_3 \geq 2$, we define

$$\phi(\pi) = \left(2^3, 3^{f_3-2}, 5^{f_5}, \dots\right).$$

Case 2(B)(ii)(b): Suppose $f_3 = 1$. Then $\pi = \left(3, 5^{f_5}, \dots\right)$. Since $N > 3$, there exists an $m \geq 5$ such that $f_m \geq 1$. Let m_0 be the least such number. We have further subcases depending on whether m_0 is odd or even.

Case 2(B)(ii)(b)(α): If m_0 is odd, we define

$$\begin{aligned} \phi(\pi) &= \left(2^1, 3^0, \dots, \left(\frac{m_0+1}{2}\right)^2, \dots, m_0^{f_{m_0}-1}, \dots\right) && \text{if } m_0 > 5, \\ \phi(\pi) &= \left(2^1, 3^2, 4^0, 5^{f_5-1}, \dots\right) && \text{if } m_0 = 5. \end{aligned}$$

In both cases, the part 3 and a part m_0 were removed from π, and a part 2 and two parts $\frac{m_0+1}{2}$ were inserted into π to obtain $\phi(\pi)$. This ensures $|\phi(\pi)| = |\pi|$ regardless of the value of m_0.

Case 2(B)(ii)(b)(β): If m_0 is even, we define

$$\begin{aligned} \phi(\pi) &= \left(2^1, 3^0, \dots, \left(\frac{m_0}{2}\right)^1, \left(\frac{m_0}{2}+1\right)^1, \dots, m_0^{f_{m_0}-1}, \dots\right) && \text{if } m_0 > 6, \\ \phi(\pi) &= \left(2^1, 3, 4, 5^0, 6^{f_6-1}, \dots, \right) && \text{if } m_0 = 6. \end{aligned}$$

In both cases, the part 3 and a part m_0 were removed from π, and a part 2, a part $\frac{m_0}{2}$ and a part $\frac{m_0}{2}+1$ were inserted into π to obtain $\phi(\pi)$. This ensures $|\phi(\pi)| = |\pi|$ regardless of the value of m_0.

Case	Description of case					f_2 in $\phi(\pi)$	Next parts
1	$f > 0$					mult. of 4	$*$
2(A)	$f = 0$	$s(\pi) \geq 5$				1	$s(\pi) - 2, s(\pi)^{\dagger}$
2(B)(i)	$f = 0$	$s(\pi) \leq 4$	$f_4 \geq 1$			2	$*$
2(B)(ii)(a)	$f = 0$	$s(\pi) \leq 4$	$f_4 = 0$	$f_3 \geq 2$		3	$*$
2(B)(ii)(b)(α)	$f = 0$	$s(\pi) \leq 4$	$f_4 = 0$	$f_3 = 1$	m_0 odd	1	$\left(\frac{m_0+1}{2}\right)^2$
2(B)(ii)(b)(β)	$f = 0$	$s(\pi) \leq 4$	$f_4 = 0$	$f_3 = 1$	m_0 even	1	$\frac{m_0}{2}, \frac{m_0}{2}+1$

† $s(\pi)$ may appear with frequency 0.

Table 3.3: The frequency of 2 in the image of a partition under the function ϕ in the different cases for Theorem 3.15. The quantity m_0 is defined in Case 2(B)(ii)(b). The column "Next parts" indicate the second and third smallest parts, which are equal in the Case 2(B)(ii)(b)(α).

The map ϕ is easily seen to be injective in each case separately. To see it is injective overall, observe that images under ϕ in most of the cases are separated by the frequency of 2; the only cases in which images have the same frequency of 2 are Cases 2(A), 2(B)(ii)(b)(α) and 2(B)(ii)(b)(β). The Case 2(B)(ii)(b)(α) is separated from the other two cases by the frequency, 2, of the second smallest part. Finally, the Cases 2(A) and 2(B)(ii)(b)(β) are separated by the difference between the second smallest and the third smallest parts; this is 1 for Case 2(B)(ii)(b)(β) and at least 2 for Case 2(A). These differences are listed in Table 3.3.

Thus we have shown that for $N > 3$,

$$|\{\pi \in C^*_{L,2} : |\pi| = N\}| - |\{\pi \in D_{L,2} : |\pi| = N\}| \geq 0.$$

However, if $N = 3$, this is not true; it is easy to see that $|\{\pi \in D_{L,2} : |\pi| = 3\}| = 1$ and $|\{\pi \in C^*_{L,2} : |\pi| = 3\}| = 0$. Furthermore, if $N = 1, 2$, we easily find that $|\{\pi \in C^*_{L,2} : |\pi| = N\}| \geq |\{\pi \in D_{L,2} : |\pi| = N\}|$.

Let $H_{L,2,L}(q) = \sum_{n \geq 0} a_n q^n$. Then the above combinatorial results imply that $a_n \geq 0$ for all $n \neq 3$, and $a_n = -1$ for $n = 3$, whence $H_{L,2,L}(q) + q^3 \succeq 0$. We can in fact make a stronger claim about the coefficients a_n when $n \geq 14$; we have $a_n \geq 1$ for all $n \geq 14$. To see this, we find a partition of n in $C^*_{L,2}$ not in the image of ϕ. Recall $L \geq 11$. For $n = 14$, the partition $\pi_{14} = (2^1, 3^4)$ is in $C^*_{L,2}$ but not in the image of ϕ. For $n \geq 15$, we have $n - 11 \geq 4$, and thus, by Lemma 3.5 (with $s = 3$ there), there are nonnegative integers x_n, y_n, z_n and u_n such that

$$n = 11 + 4x_n + 5y_n + 6z_n + 7u_n.$$

For each $n \geq 15$, fix some choice of x_n, y_n, z_n and u_n. Thus $\pi_n = (2^1, 3^3, 4^{x_n}, 5^{y_n}, 6^{z_n}, 7^{u_n})$ is a partition of n in $C^*_{L,2}$ not in the image of ϕ.

Let $G_{L,2}(q) = \sum_{n\geq 0} b_n q^n$. To prove the theorem, we are required to show $b_n \geq 0$ whenever $n \neq 3$, and $b_3 \geq -1$. By Theorem 3.14, for any $n \geq 0$,

$$b_n = a_n + a_{n-L} + a_{n-2L} + \cdots .$$

Using the division algorithm, we have $n = Lq + r$ where $0 \leq r < L$, and thus we can rewrite b_n as

$$b_n = \sum_{i=0}^{q} a_{Li+r}. \tag{3.17}$$

If $r \neq 3$, then none of the terms on the right hand side of (3.17) are negative and thus $b_n \geq 0$ as required. If $r = 3$ and $q \geq 1$, then from (3.17), we have $b_n \geq a_{L+3} + a_3$. Since $L \geq 11$, we have $L + 3 \geq 14$, and thus $a_{L+3} \geq 1$, which implies $b_n \geq 0$. Finally, if $r = 3$ and $q = 0$, then $b_n = b_3 = a_3 = -1$. $\square$

Next, we prove Conjecture 3.4 for $5 \leq L \leq 10$.

Theorem 3.16. *For $5 \leq L \leq 10$,*

$$G_{L,2}(q) + q^3 \succeq 0.$$

Proof. Let $N_L = \frac{L(L+3)}{2} + 2$; we give the values of N_L in Table 3.4. For $5 \leq L \leq 10$ and $N \geq N_L$, we construct an injective map

$$\psi : \{\pi \in D_{L,2} : |\pi| = N\} \to \{\pi \in C^*_{L,2} : |\pi| = N\}.$$

L	5	6	7	8	9	10
N_L	22	29	37	46	56	67

Table 3.4: The table of values of N_L versus L for $5 \leq L \leq 10$.

We again let f be the frequency of L in $\pi \in D_{L,2}$; so, $\pi = (3^{f_3}, 4^{f_4}, \ldots, L^f \ldots, (L+2)^{f_{L+2}})$. To define the image of π under the map ψ, we consider several cases depending on f. We describe ψ first and explain why it is injective later.

Case 1: Suppose that f is a positive even number. Then define

$$\psi(\pi) = \left(2^{\frac{Lf}{2}}, 3^{f_3}, \ldots, L^0, \ldots, (L+2)^{f_{L+2}}\right).$$

Case 2: Suppose that f is a positive odd number. Then define

$$\psi(\pi) = \left(2^{L(\frac{f-1}{2})+1}, 3^{f_3}, \ldots, (L-2)^{f_{L-2}+1}, \ldots, L^0, \ldots, (L+2)^{f_{L+2}}\right).$$

Case 3: Suppose $f = 0$. Since $N \geq N_L$ is large enough, there exists an i such that $3 \leq i \leq L+2$ and $f_i \geq 2$. Let i_0 be the least such number. Note that $i_0 \neq L$ since $f = 0$. We have further subcases depending on whether $i_0 = L+1$ or not.

Case 3(i): Suppose $i_0 \neq L+1$. Then define

$$\psi(\pi) = \left(2^{i_0}, 3^{f_3}, \ldots, i_0^{f_{i_0}-2}, \ldots L^0, \ldots, (L+2)^{f_{L+2}}\right).$$

Case 3(ii): Suppose $i_0 = L+1$. Then define

$$\psi(\pi) = \left(2^2, 3^{f_3}, \ldots, (L-1)^{f_{L-1}+2}, L^0, (L+1)^{f_{L+1}-2}, \ldots, (L+2)^{f_{L+2}}\right).$$

It is easy to see that ψ is injective in each case. To see that ψ is injective overall, note that the frequency of 2 modulo L in the image distinguishes the cases with one exception: when $i_0 = L+1$ and $i_0 = L+2$. The frequencies of 2 in the image in those cases are 2 and $L+2$, respectively. However, while those frequencies are the same modulo L, they are different numbers, so distinguish the cases. Hence the map ψ is injective.

Let $H_{L,2,L}(q) = \sum_{n \geq 0} a_{L,n} q^n$ and $G_{L,2}(q) = \sum_{n \geq 0} b_{L,n} q^n$. Then, from Theorem 3.14, for all $n \geq 0$,

$$b_{L,n} = a_{L,n} + a_{L,n-L} + a_{L,n-2L} + \cdots. \tag{3.18}$$

From the injectivity of ψ, we have $a_{L,N} \geq 0$ for all $5 \leq L \leq 10$ and $N \geq N_L$. In fact, we can show $a_{L,N} \geq 1$ for all $5 \leq L \leq 10$ and $N \geq N_L$. To see this, we find a partition in $C^*_{L,2}$ that is not in the image of ψ. For $L \geq 5$, note that $N_L \geq 2L+12$. But from $N \geq N_L$, we conclude $N - 2(L+3) \geq 6$. Hence, by Sylvester's lemma, there are nonnegative integers x_L and y_L such that

$$N = 2(L+3) + 3x_L + 4y_L.$$

For each $5 \leq L \leq 10$, fix some values of x_L and y_L. Thus, there is a partition λ_L of N given by

$$\lambda_L = (2^{L+3}, 3^{x_L}, 4^{y_L}).$$

Note that λ_L is not in the image of ψ since the frequency of 2 is $L+3$, which is not possible for any partition in the image of ψ.

We are therefore left with determining the nonnegativity of $a_{L,N}$ when $N \leq N_L$. Since $5 \leq L \leq 10$, the numbers $a_{L,N}$ for $N \leq N_L$ can easily be calculated by, for example, a short Magma program. The negative values of $a_{L,N}$ for $5 \leq L \leq 10$ and $N \leq N_L$ are given in Table 3.5. In that table, we have also given the value of $a_{L,N+L}$ when $a_{L,N}$ is negative.

Using (3.18) and Table 3.5, along with the facts that $a_{5,2} = 1$ and $a_{7,2} = 1$, we conclude that $b_{L,N} \geq 0$ for $5 \leq L \leq 10$ if $N \neq 3$, and $b_{L,3} = -1$. Hence, for all $5 \leq L \leq 10$, we see that $G_{L,2}(q) + q^3 \succeq 0$. □

Finally, we prove Conjecture 3.4 for the cases $L = 3$ and $L = 4$ in the next two theorems.

L	N	$a_{L,N}$	$a_{L,N+L}$
5	3	-1	2
5	7	-1	2
6	3	-1	1
7	3	-1	3
7	9	-1	10
8	3	-1	3
9	3	-1	4
10	3	-1	5

Table 3.5: The values $5 \leq L \leq 10$ and $0 \leq N \leq N_L$ where $a_{L,N}$ is negative. Also given is $a_{L,N+L}$ in those cases.

Theorem 3.17. *For $L = 3$,*

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0.$$

Proof. For $N > 43$, we construct an injective map

$$\phi : \{\pi \in D_{3,2} : |\pi| = N\} \to \{\pi \in C^*_{3,2} : |\pi| = N\}.$$

Recall that each $\pi \in D_{3,2}$ has the form $\pi = (3^{f_3}, 4^{f_4}, 5^{f_5})$, and each partition in $C^*_{3,2}$ has 2 as a part, but cannot have 3 as a part. We have cases depending on the frequency of 3 in π. We fully describe ϕ and later show it is injective.

Case 1: Suppose that f_3 is a positive even number. Then define

$$\phi(\pi) = \left(2^{\frac{3f_3}{2}}, 3^0, 4^{f_4}, 5^{f_5}\right).$$

Case 2: Suppose that $f_3 \geq 3$ is an odd number. Then define

$$\phi(\pi) = \left(2^{3\left(\frac{f_3-3}{2}\right)+2}, 3^0, 4^{f_4}, 5^{f_5+1}\right).$$

Case 3: Suppose $f_3 = 1$. Then $\pi = (3^1, 4^{f_4}, 5^{f_5})$. Since $N > 43$, either $f_4 \geq 1$ or $f_5 \geq 1$. We have further subcases based on these conditions.

Case 3(i): Suppose $f_4 \geq 1$. Then we define

$$\phi(\pi) = \left(2^1, 3^0, 4^{f_4-1}, 5^{f_5+1}\right).$$

Case 3(ii): Suppose $f_4 = 0$ and $f_5 \geq 1$. Then we define

$$\phi(\pi) = \left(2^4, 3^0, 5^{f_4-1}\right).$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a_n	0	0	1	-1	0	-1	1	0	0	-1	1	0	1	-1	2	-1	2	0	2

Table 3.6: The values a_n for $0 \leq n \leq 18$ in Theorem 3.17.

Case 4: Suppose $f_3 = 0$. Then $\pi = (4^{f_4}, 5^{f_5})$. Since $N > 43$, either $f_4 \geq 8$ or $f_5 \geq 4$. We have further subcases based on these conditions.

Case 4(i): If $f_4 \geq 8$, then define

$$\phi(\pi) = \left(2^{16}, 3^0, 4^{f_4-8}, 5^{f_5}\right).$$

Case 4(ii): If $f_4 \leq 7$ and $f_5 \geq 4$, then define

$$\phi(\pi) = \left(2^{10}, 3^0, 4^{f_4}, 5^{f_5-4}\right).$$

In each case it is clear that we can recover π from $\phi(\pi)$. To see ϕ is injective overall, the frequency of 2 in partitions in the image distinguishes the different cases; in Cases 1 and 2, the frequency of 2 is congruent to 0 or 2 modulo 3, while in the other cases the frequency of 2 is either 1, 4, 16 or 10. Thus, the map ϕ is injective overall whenever $N > 43$.

Let $H_{3,2,3}(q) = \sum_{n\geq 0} a_n q^n$ and $G_{3,2}(q) = \sum_{n\geq 0} b_n q^n$. From Theorem 3.14,

$$b_n = a_n + a_{n-3} + a_{n-6} + \cdots.$$

The injectivity of ϕ shows that $a_n \geq 0$ for all $n > 43$. For $n \leq 43$, a_n can be calculated easily using a computer. It can be verified that a_n is negative only when n is either 3, 5, 9, 13 or 15 and in all of these cases, $a_n = -1$. Table 3.6 contains the values of a_n for $n \leq 18$. From Table 3.6 and the fact that a_n is negative only when n is either 3, 5, 9, 13 or 15 (and in all of these cases, $a_n = -1$), we obtain $b_n \geq 0$ whenever $n \neq 3$, $n \neq 9$ and $n \neq 15$. Also apparent is that $b_3 = -1$, $b_9 = -1$ and $b_{15} = -1$. This proves that $G_{3,2}(q) + q^3 + q^9 + q^{15} \succeq 0$. ☐

Theorem 3.18. *For $L = 4$,*

$$G_{L,2}(q) + q^3 + q^9 \succeq 0.$$

Proof. For $N > 20$, we construct an injective map

$$\psi : \{\pi \in D_{4,2} : |\pi| = N\} \to \{\pi \in C^*_{4,2} : |\pi| = N\}.$$

Recall that partitions $\pi \in D_{4,2}$ have the form $\pi = (3^{f_3}, 4^{f_4}, 5^{f_5}, 6^{f_6})$, whereas partitions in $C^*_{4,2}$ must have a part of size 2 and cannot have a part of size 4. We have cases depending on the frequency of 4 in π. We describe the map ψ fully and then show it is injective later.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	0	0	1	-1	0	0	-1	1	1	-1	1	1	0	2

Table 3.7: The values a_n in Theorem 3.18 for $0 \le n \le 13$.

Case 1: Suppose $f_4 \ge 1$. Then we define

$$\psi(\pi) = (2^{2f_4}, 3^{f_3}, 4^0, 5^{f_5}, 6^{f_6}).$$

Case 2: Suppose $f_4 = 0$. Then $\pi = (3^{f_3}, 5^{f_5}, 6^{f_6})$. Since $N > 20$, either $f_3 \ge 2$, $f_5 \ge 2$ or $f_6 \ge 3$. We consider subcases based on these conditions.

Case 2(i): Suppose $f_3 \ge 2$. Then we define

$$\psi(\pi) = (2^3, 3^{f_3-2}, 4^0, 5^{f_5}, 6^{f_6}).$$

Case 2(ii): Suppose $f_3 \le 1$ and $f_5 \ge 2$. Then we define

$$\psi(\pi) = (2^5, 3^{f_3}, 4^0, 5^{f_5-2}, 6^{f_6}).$$

Case 2(iii): Suppose $f_3 \le 1$, $f_5 \le 1$ and $f_6 \ge 3$. Then we define

$$\psi(\pi) = (2^9, 3^{f_3}, 4^0, 5^{f_5}, 6^{f_6-3}).$$

It is easy to see that ψ is injective in each case. To see that ψ is injective overall, clearly the frequency of 2 in partitions in the image distinguishes cases. In Case 1, the frequency of 2 is even, and in the other cases the frequency of 2 is 3, 5 or 9. Thus, ψ is injective.

Let $H_{4,2,4}(q) = \sum_{n\ge 0} a_n q^n$ and $G_{4,2}(q) = \sum_{n\ge 0} b_n q^n$. Then

$$b_n = a_n + a_{n-4} + a_{n-8} + \cdots .$$

The injectivity of ψ shows that $a_n \ge 0$ for all $n > 20$. For $n \le 20$, a_n can be calculated easily using a computer (or by hand). It can be verified that a_n is negative only when n is either 3, 6 or 9, and in all of these cases, $a_n = -1$. Table 3.7 contains the values of a_n for $n \le 13$.

From Table 3.7 and the fact that a_n is negative only when n is either 3, 6 or 9 (and in all of these cases, $a_n = -1$), we obtain $b_n \ge 0$ whenever $n \ne 3$ and $n \ne 9$. Also, $b_3 = -1$ and $b_9 = -1$. This proves that $G_{4,2}(q) + q^3 + q^9 \succeq 0$.

□

Chapter 4

A Comparison of Integer Partitions Based on Smallest Part

In this chapter, we conduct a study of the nonnegativity of the difference between two closely related generating series of integer partitions having bounded difference between largest and smallest parts. The main goal is to prove Theorems 4.1 and 4.2. We begin by restating some definitions and results, described in Chapter 1. For positive integers L and s,

- $G_{L,s}(q)$ is the generating series

$$G_{L,s}(q) = \sum_{\substack{\pi\in\mho,\\ s(\pi)=s,\\ l(\pi)-s(\pi)\le L}} q^{|\pi|} - \sum_{\substack{\pi\in\mho,\\ s(\pi)\ge s+1,\\ l(\pi)-s(\pi)\le L}} q^{|\pi|}. \tag{4.1}$$

Berkovich and Uncu proved that $G_{L,1}(q)$ is nonnegative for any L. Recall the result for $G_{L,2}(q)$. For $L=3$,

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0;$$

for $L=4$,

$$G_{L,2}(q) + q^3 + q^9 \succeq 0;$$

and for $L \ge 5$,

$$G_{L,2}(q) + q^3 \succeq 0.$$

This was Conjecture 3.4 (by Berkovich and Uncu), which we proved in Chapter 3.

Next, we explore the nonnegativity properties of $G_{L,s}(q)$. Define the following quantities:

$$\begin{aligned} \delta(s) &:= e^{3\Gamma(s)}, \\ \delta'(s) &:= 10s + (s+2)(s+3)(\delta(s)+1). \end{aligned} \tag{4.2}$$

We state Theorem 4.1 and prove it in Section 4.1.

Theorem 4.1. *If s and $L \geq s+1$ are given positive integers, then the coefficient of q^n in $G_{L,s}(q)$ is positive whenever $n \geq \delta'(s)$, where $\delta'(s)$ is as defined in* (4.2).

Then we focus on the case $s = 3$ and obtain an extension of Conjecture 3.4. We prove Theorem 4.2 below in Section 4.2.

Theorem 4.2. *For $L \geq 10$,*

$$G_{L,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + q^{16} \succeq 0.$$

For $5 \leq L \leq 9$, we have the following:

$$\begin{gathered}
G_{9,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} \succeq 0, \\
G_{8,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} + q^{20} \succeq 0, \\
G_{7,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + 2q^{14} + q^{16} + q^{20} \succeq 0, \\
G_{6,3}(q) + q^4 + q^5 + q^8 + q^{10} + q^{12} + q^{13} + 2q^{14} + 2q^{16} + q^{18} + 2q^{20} + q^{22} \succeq 0, \\
G_{5,3}(q) + q^4 + q^5 + q^8 + q^{10} + 2q^{12} + q^{13} + q^{14} + 2q^{16} + q^{17} + q^{18} + 3q^{20} + q^{22} + q^{24} \succeq 0,
\end{gathered}$$

and for $L = 4$,

$$\begin{aligned}
G_{4,3}(q) &+ q^4 + q^5 + q^8 + q^{10} + q^{11} + 2q^{12} + 2q^{14} + 3q^{16} + q^{17} \\
&+ 2q^{18} + q^{19} + 4q^{20} + 3q^{22} + q^{23} + 4q^{24} + q^{25} + 4q^{26} + 5q^{28} \\
&+ q^{29} + 3q^{30} + 6q^{32} + 3q^{34} + 4q^{36} + 2q^{38} + 4q^{40} + 2q^{44} \succeq 0.
\end{aligned}$$

4.1 Proof of Theorem 4.1

We restate Theorem 4.3 from Chapter 1, which is a generalization of Theorem 1.14.

Theorem 4.3. *For $L \geq 1$,*

$$G_{L,s}(q) = \frac{H_{L,s,L}(q)}{1-q^L}.$$

Proof. Though the proof of Theorem 4.3 is a direct generalization of the proof of Theorem 1.14 in [BU19, Theorem 5.1], we provide all the details here for the sake of completeness.

We begin by simplifying the first generating function in the definition of $G_{L,s}(q)$ in (4.1),

$$\sum_{\substack{\pi\in\mho,\\ s(\pi)=s,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|}.$$

All the partitions counted by this generating function have smallest part equal to s and largest part less than or equal to $L+s$. Therefore,

$$\sum_{\substack{\pi\in\mho,\\ s(\pi)=s,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} = \frac{q^s}{(1-q^s)(1-q^{s+1})\cdots(1-q^{L+s})} = \frac{q^s}{(q^s;q)_{L+1}}. \tag{4.3}$$

For the second generating function of (4.1), we fix the number of parts of the partition to be n and then sum over all n. Suppose π is a partition into n parts where each part is at least $s+1$. Then, thinking about the Ferrers diagram of π, the whole set of columns over the smallest part of π is generated by the q-factor

$$\frac{q^{(s+1)n}}{1-q^n}.$$

Stripping the columns over the smallest part from the far left of the Ferrers diagram of π, we are left with a new partition that has at most $n-1$ parts and largest part bounded above by L. It is well known (see [Aig07, Proposition 1.1]) that these partitions are generated by the q-binomial coefficient

$$\begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q = \frac{(q;q)_{L+n-1}}{(q;q)_L(q;q)_{n-1}}.$$

Thus, we have

$$\sum_{\substack{\pi\in\mho,\\ s(\pi)\geq s+1,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} = \sum_{n=1}^{\infty} \frac{q^{(s+1)n}}{1-q^n} \begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q.$$

By simple calculation, we find

$$\begin{aligned} \frac{1}{1-q^n} \begin{bmatrix} L+n-1 \\ n-1 \end{bmatrix}_q &= \frac{1}{1-q^L} \begin{bmatrix} L+n-1 \\ n \end{bmatrix}_q \\ &= \frac{1}{1-q^L} \frac{(q^L;q)_n}{(q;q)_n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\substack{\pi\in\mho,\\ s(\pi)\geq s+1,\\ l(\pi)-s(\pi)\leq L}} q^{|\pi|} &= \frac{1}{1-q^L} \sum_{n=1}^{\infty} q^{(s+1)n} \frac{(q^L;q)_n}{(q;q)_n} \\ &= \frac{1}{1-q^L} \left(-1 + \sum_{n=0}^{\infty} q^{(s+1)n} \frac{(q^L;q)_n}{(q;q)_n} \right) \\ &= \frac{1}{1-q^L} \left(\frac{1}{(q^{s+1};q)_L} - 1 \right), \end{aligned} \tag{4.4}$$

where the last step follows from the q-binomial theorem (see [BU19, Equation (2.1)]):

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Note that in (4.4), we used the q-binomial theorem with $a = q^L$ and $z = q^{s+1}$. Substituting (4.3) and (4.4) in the definition of $G_{L,s}(q)$ gives us

$$\begin{aligned} G_{L,s}(q) &= \frac{q^s}{(q^s;q)_{L+1}} - \frac{1}{1-q^L}\left(\frac{1}{(q^{s+1};q)_L} - 1\right) \\ &= \frac{1}{1-q^L}\left(\frac{q^s(1-q^L)}{(q^s;q)_{L+1}} - \left(\frac{1}{(q^{s+1};q)_L} - 1\right)\right) \\ &= \frac{1}{1-q^L} H_{L,s,L}(q), \end{aligned}$$

as required. □

Conjecture 3.4 asserts that $H_{L,s,L}(q)$ is eventually positive, and Theorem 3.3 gives a bound $\Gamma(s)$, which can be written explicitly in terms of s, after which the series is positive. Thus the natural question that arises here is the eventual nonnegativity of $G_{L,s}(q)$ and to find a bound M, which depends on L and s, such that the coefficient of q^n in the series $G_{L,s}(q)$ is nonnegative whenever $n \geq M$.

Suppose s and $L \geq s+1$ are given positive integers. Let $H_{L,s,L}(q) = \sum_{n\geq 0} a_{L,n} q^n$ and let $G_{L,s}(q) = \sum_{n \geq 0} b_{L,n} q^n$. Then Theorem 4.3 shows that

$$b_{L,n} = a_{L,n} + a_{L,n-L} + a_{L,n-2L} + \cdots = \sum_{\substack{0 \leq m \leq n \\ m \equiv n (\mathrm{mod} L)}} a_{L,m}. \tag{4.5}$$

We introduce some more notation. Define

- $\eta_1(L,s) = \sum_{n<\Gamma(s)} |a_{L,n}|$,
- $\eta_2(L,s) = \max(\eta_1(L,s), \Gamma(s))$,
- $\eta_3(L,s) = (L+1)\eta_2(L,s)$.

Theorem 4.4. *If s and $L \geq s+1$ are given positive integers, then the coefficient of q^n in $G_{L,s}(q)$ is nonnegative whenever $n \geq \eta_3(L,s)$.*

Proof. Suppose $n \geq \eta_3(L,s)$. We can rewrite (4.5) as

$$b_{L,n} = \sum_{\substack{\eta_2(L,s) \leq m \leq n \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} + \sum_{\substack{\Gamma(s) \leq m < \eta_2(L,s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} + \sum_{\substack{m < \Gamma(s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m}. \tag{4.6}$$

Note that the second sum may be empty. The first sum in the right hand side of (4.6) contains at least $\eta_2(L,s)$ terms, all of which are positive by Theorem 3.3. Thus

$$\sum_{\substack{\eta_2(L,s)\leq m\leq n\\ m\equiv n(\mathrm{mod} L)}} a_{L,m} \geq \eta_2(L,s). \tag{4.7}$$

For the second sum in the right hand side of (4.6), by Theorem 3.3 we have

$$\sum_{\substack{\Gamma(s)\leq m<\eta_2(L,s)\\ m\equiv n(\mathrm{mod} L)}} a_{L,m} \geq 0. \tag{4.8}$$

For the third sum in the right hand side of (4.6), using the triangle inequality, we obtain

$$\left|\sum_{\substack{m<\Gamma(s)\\ m\equiv n(\mathrm{mod} L)}} a_{L,m}\right| \leq \sum_{\substack{m<\Gamma(s)\\ m\equiv n(\mathrm{mod} L)}} |a_{L,m}| \leq \sum_{m<\Gamma(s)} |a_{L,m}| = \eta_1(L,s) \leq \eta_2(L,s),$$

and thus

$$\sum_{\substack{m<\Gamma(s)\\ m\equiv n(\mathrm{mod} L)}} a_{L,m} \geq -\eta_2(L,s). \tag{4.9}$$

The theorem now follows immediately from (4.6), (4.7), (4.8) and (4.9).

$\square$

The bound $\eta_3(L,s)$ depends on both L and s. Our next goal is to find a bound M which depends only on s, and such that the coefficient of q^n in $G_{L,s}(q)$ is nonnegative whenever $n \geq M$. For this, we need the following two standard results.

1. For given positive integers a, b and n with $\gcd(a,b) = 1$, the number of solutions of $ax + by = n$ in nonnegative integer pairs (x,y) is either $\lfloor \frac{n}{ab} \rfloor$ or $\lfloor \frac{n}{ab} \rfloor + 1$ (see [Tri00]).
2. The number of integer partitions $p(m)$ is bounded above by $e^{3\sqrt{m}}$. That is $p(m) \leq e^{3\sqrt{m}}$ (see [dAP09]).

Using Fact 1 above and a careful analysis of the proof of Theorem 3.3 in Chapter 3, we can show that the coefficients of q^N in $H_{L,s,L}(q)$ are not only positive, but quite large whenever $N \geq \Gamma(s)$. To be precise, we obtain the following strengthening of Theorem 3.3 in the case $k = L$.

Theorem 4.5. *For positive integers $L \geq 3$ and s, with $L \geq s+1$, the coefficient of q^N in $H_{L,s,L}(q)$ is greater than or equal to $\left\lfloor \frac{N-10s}{(s+2)(s+3)} \right\rfloor$ whenever $N \geq \Gamma(s)$.*

Proof. From Theorem 3.3, we know that the coefficient of q^N in $H_{L,s,L}(q)$ is positive whenever $N \geq \Gamma(s)$. However, in the proof of Theorem 3.3, we observe that several partitions are in the codomain but not in the image of the relevant injective map.

Firstly suppose $L \geq 2s+3$. We use the proof of Theorem 3.7 in the case $k = L$. Note that in the proof of Theorem 3.7, there is no partition in the image for which s appears as a part with frequency 10. Thus any partition of the form $(s^{10}, (s+1)^x, (s+2)^y)$ is not in the range of the map. From Fact 1 above, the number of such partitions is at least $\left\lfloor \frac{N-10s}{(s+1)(s+2)} \right\rfloor$.

Secondly suppose $L \leq 2s+2$. We use the proof of Theorem 3.9 in the case $k = L$.

Suppose $s+3 \leq L \leq 2s+2$, then in the proof of Theorem 3.9, there is no partition in the image for which s appears as a part with frequency 1. Thus any partition of the form $(s^1, (s+1)^x, (s+2)^y)$ is not in the range of the map. From Fact 1 above, the number of such partitions is at least $\left\lfloor \frac{N-s}{(s+1)(s+2)} \right\rfloor$.

Thirdly suppose $L = s+1$, then in the proof of Theorem 3.9, any partition of the form $(s^1, (s+2)^x, (s+3)^y)$ is not in the range of the map. From Fact 1 above, the number of such partitions is at least $\left\lfloor \frac{N-s}{(s+2)(s+3)} \right\rfloor$.

Fourthly suppose $L = s+2$ and s is even, then in the proof of Theorem 3.9, any partition of the form $(s^1, (s+1)^x, (s+3)^y)$ is not in the range of the map. Since s is even, $\gcd(s+1, s+3) = 1$, and thus from Fact 1 above, the number of such partitions is at least $\left\lfloor \frac{N-s}{(s+1)(s+3)} \right\rfloor$.

Finally suppose $L = s+2$ and s is odd, we have three further subcases. If N is odd, then in the proof of Theorem 3.9, any partition of the form $(s^1, (s+1)^x, (s+3)^y)$ is not in the range of the map. Note that $N-s, s+1$ and $s+3$ are all even numbers. Since $\gcd\left(\frac{s+1}{2}, \frac{s+3}{2}\right) = 1$, from Fact 1 above (applied with $a = \frac{s+1}{2}, b = \frac{s+3}{2}$ and $n = \frac{N-s}{2}$), the number of such partitions is at least $\left\lfloor \frac{2(N-s)}{(s+1)(s+3)} \right\rfloor$.

If N is even and $s \neq 1$, then in the proof of Theorem 3.9, any partition of the form $(s^2, (s+1)^x, (s+3)^y)$ is not in the range of the map. Note that $N-2s, s+1$ and $s+3$ are all even numbers. From Fact 1 above (applied with $a = \frac{s+1}{2}, b = \frac{s+3}{2}$ and $n = \frac{N-2s}{2}$), the number of such partitions is at least $\left\lfloor \frac{2(N-2s)}{(s+1)(s+3)} \right\rfloor$.

If N is even and $s = 1$, then $L = 3$ (since $L = s+2$). Therefore, in the proof of Theorem 3.9, any partition of the form $(1^6, 2^x, 4^y)$ is not in the range of the map. Note that $N-6, 2$ and 4 are all even numbers. From Fact 1 above (applied with $a = 1$, $b = 2$ and $n = \frac{N-6}{2}$), the number of such partitions is at least $\left\lfloor \frac{(N-6)}{4} \right\rfloor$.

It is easy to verify that the expressions we obtain in each of the cases above are at least $\left\lfloor \frac{N-10s}{(s+2)(s+3)} \right\rfloor$. □

Recall that $a_{L,n}$ denotes the coefficient of q^n in $H_{L,s,L}(q)$, and $b_{L,n}$ denotes the coefficient of q^n in $G_{L,s}(q)$. Recall their relationship from (4.5). From the combinatorial interpretation of $H_{L,s,L}(q)$, it is clear that for any $m \in \mathbb{N}$, $-p(m) \leq a_{L,m} \leq p(m)$.

Proof of Theorem 4.1. We use the definitions of $\delta(s)$ and $\delta'(s)$ in (4.2).

Suppose $n \geq \delta'(s)$. We have from (4.5)

$$b_{L,n} = \sum_{\substack{\delta'(s) \leq m \leq n \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} + \sum_{\substack{\Gamma(s) \leq m < \delta'(s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} + \sum_{\substack{m < \Gamma(s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} \tag{4.10}$$

For $m \geq \delta'(s)$, we have $m \geq \Gamma(s)$ and $\lfloor \frac{m-10s}{(s+2)(s+3)} \rfloor \geq \delta(s)$. Thus, by Theorem 4.5, we have $a_{L,m} \geq \delta(s)$ whenever $m \geq \delta'(s)$. The first sum in the right hand side of (4.10) contains at least 1 term (corresponding to $m = n$) and each term in the sum is greater than or equal to $\delta(s)$. Thus

$$\sum_{\substack{\delta'(s) \leq m \leq n \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} \geq \delta(s). \tag{4.11}$$

For the second sum in the right hand side of (4.10), by Theorem 3.3

$$\sum_{\substack{\Gamma(s) \leq m < \delta'(s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} \geq 0. \tag{4.12}$$

For the third sum in the right hand side of (4.10), note that for any $m \in \mathbb{N}$, $a_{L,m} \geq -p(m)$ and by Fact 2, Page 87, we have

$$a_{L,m} \geq -p(m) \geq -e^{3\sqrt{m}} \geq -e^{3m}.$$

Therefore,

$$\sum_{\substack{m < \Gamma(s) \\ m \equiv n (\mathrm{mod} L)}} a_{L,m} \geq - \sum_{\substack{m < \Gamma(s) \\ m \equiv n (\mathrm{mod} L)}} e^{3m} \geq - \sum_{m < \Gamma(s)} e^{3m} > -\delta(s). \tag{4.13}$$

The theorem now follows immediately from (4.10), (4.11), (4.12) and (4.13).

□

4.2 Proof of Theorem 4.2

Our strategy is to write general proofs to show that the coefficients of q^N in $H_{L,3,L}(q)$ are nonnegative when N is larger than a small bound; and then analyze the coefficients of q^N for small N using machine computation (Sage in our case). Once we know when these coefficients are negative, we can complete the proof of Theorem 4.2 using Theorem 4.3. We need several lemmas.

Lemma 4.6. *Suppose n is a positive integer such that $n \geq 4$ and $n \neq 7$. Then the equation $4x + 5y + 6z = n$ has a solution in nonnegative integer triples (x, y, z).*

Lemma 4.7. *Suppose n is a positive integer such that $n \geq 5$ and $n \neq 8, 9$. Then the equation $5x + 6y + 7z = n$ has a solution in nonnegative integer triples (x, y, z).*

Lemma 4.8. *Suppose n is a positive integer such that $n \geq 4$. Then the equation $4x+5y+6z+7u=n$ has a solution in nonnegative integer tuples (x,y,z,u).*

There are several elementary ways of proving the above lemmas. For example, for Lemma 4.6, we can apply Sylvester's lemma with $a=4$ and $b=5$ to obtain a solution for all $n \geq 12$ and do the remaining cases manually. One could also use the division algorithm.

Lemma 4.9. *Suppose the equation $4x+5y+6z=n$ has a solution (α,β,γ) in nonnegative integer triples. Then the equation $4x+5y+6z=n+6$ has a solution different from $(\alpha,\beta,\gamma+1)$ whenever $n \geq 4$ and $n \neq 5$.*

Proof. First suppose $\alpha \geq 1$, then $(\alpha-1,\beta+2,\gamma)$ is a required solution. Thus assume $\alpha=0$. Next assume $\gamma \geq 1$, so $(\alpha+3,\beta,\gamma-1)$ is a required solution. Thus also assume $\gamma=0$, so $\beta \geq 2$. Then $(\alpha+4,\beta-2,\gamma)$ is a solution. ☐

Lemma 4.10. *For $L \geq 22$ and $N \geq 21$, the coefficient of q^n in $H_{L,3,L}(q)$ is nonnegative.*

Proof. Recall the following notation from Chapter 1.

- $D_{L,s}$ denotes the set of nonempty partitions with parts in the set $\{s+1,\ldots,L+s\}$.
- $I_{L,s,k}$ is the set of partitions where the smallest part is s, all parts are $\leq L+s$, and k does not appear as a part.

Further, recall from (1.1) in Chapter 1 that when $s+1 \leq k \leq L+s$, the coefficient of q^N in $H_{L,s,k}(q)$ is given as

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| - |\{\pi \in D_{L,s} : |\pi| = N\}|.$$

Thus it suffices to show that for $L \geq 22$ and $N \geq 21$, there is an injective map

$$\phi : \{\pi \in D_{L,3} : |\pi| = N\} \to \{\pi \in I_{L,3,L} : |\pi| = N\}.$$

Let $\pi = \left(4^{f_4},\ldots,L^{f_L},\ldots,(L+3)^{f_{L+3}}\right)$ be an element of $D_{L,3}$ with $|\pi|=N$. We define $\phi(\pi)$ depending on the following cases. Let f denotes f_L.

Case 1: Suppose $f \geq 1$. For all $i \geq 1$, $(L-18)i \geq 4$, so the equation

$$(L-18)i = 4x_i + 5y_i + 6z_i + 7u_i$$

has a nonnegative integer solution by Lemma 4.8. For each $i \geq 1$, fix such a solution x_i, y_i, z_i and u_i. Define

$$\phi(\pi) = \left(3^{6f}, 4^{f_4+x_f}, 5^{f_5+y_f}, 6^{f_6+z_f}, 7^{f_7+u_f}, \ldots, L^0, \ldots\right).$$

Case 2: If $f = 0$, we have the following subcases. Recall that the smallest part of π is denoted by $s(\pi)$.

Case 2(a): If $s(\pi) \geq 7$, define

$$\phi(\pi) = \left(3^1, (s(\pi) - 3)^1, (s(\pi)^{f_{s(\pi)}-1}), \ldots, \right).$$

Case 2(b): Suppose $s(\pi) \leq 6$. We have the following subcases.

Case 2(b)(i): If $f_4 \geq 1$ and $f_5 \geq 1$, define

$$\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^{f_5-1}, 6^{f_6}, \ldots\right).$$

Case 2(b)(ii): Suppose $f_4 = 0$ or $f_5 = 0$. We have the following subcases.

Case 2(b)(ii)(α): Suppose $f_6 \geq 1$. Define

$$\phi(\pi) = (3^2, 4^{f_4}, 5^{f_5}, 6^{f_6-1}, \ldots).$$

Case 2(b)(ii)(β): Suppose $f_6 = 0$. Thus, in this subcase either $f_4 = f_6 = 0$ or $f_5 = f_6 = 0$. We have further subcases.

Case 2(b)(ii)(β)(I): Suppose $f_4 = f_6 = 0$. Then $\pi = \left(5^{f_5}, 7^{f_7}, \ldots\right)$.

Case 2(b)(ii)(β)(I)(A): Suppose $f_5 \geq 3$. Define

$$\phi(\pi) = (3^5, 5^{f_5-3}, 7^{f_7}, \ldots).$$

Case 2(b)(ii)(β)(I)(B): Suppose $f_5 = 1$, so $\pi = \left(5^1, 7^{f_7}, \ldots, \right)$. Let $m_1 \geq 7$ be the least number with a non-zero frequency in π.

Case 2(b)(ii)(β)(I)(B)(i): Suppose $m_1 \neq 7, 11, 12$. Then $m_1 - 3 \geq 5$ and $m_1 - 3 \neq 8, 9$. Thus, by Lemma 4.7, there exist some nonnegative integers u_{m_1-3}, v_{m_1-3} and w_{m_1-3} such that

$$m_1 - 3 = 5u_{m_1-3} + 6v_{m_1-3} + 7w_{m_1-3}.$$

Define

$$\phi(\pi) = \left(3^1, 5^{1+u_{m_1-3}}, 6^{v_{m_1-3}}, 7^{w_{m_1-3}}, m_1^{f_{m_1}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(ii): Suppose $m_1 = 7$.

Case 2(b)(ii)(β)(I)(B)(ii)(I): Suppose $f_7 \geq 2$. Then define

$$\phi(\pi) = \left(3^5, 4^1, 5^0, 7^{f_7-2}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(ii)(II): Suppose $f_7 = 1$. Then $\pi = \left(5^1, 7^1, 8^{f_8}, \ldots\right)$. Let $m_2 \geq 8$ be the least number with a non-zero frequency in π.

Case 2(b)(ii)(β)(I)(B)(ii)(II)(a): Suppose $m_2 = 8$. Define

$$\phi(\pi) = \left(3^2, 4^1, 5^2, 7^0, 8^{f_8-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(ii)(II)(b): Suppose $m_2 \geq 9$. Define

$$\phi(\pi) = \left(3^2, 4^1, 5^1, (m_2-3)^1, m_2^{f_{m_2}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(iii): Suppose $m_1 = 11$. Then $\pi = \left(5^1, 11^{f_{11}}, \ldots\right)$. We have further subcases.

Case 2(b)(ii)(β)(I)(B)(iii)(a): Suppose $f_{11} \geq 2$. Define

$$\phi(\pi) = \left(3^9, 5^0, 11^{f_{11}-2}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(iii)(b): Suppose $f_{11} = 1$. Then $\pi = \left(5^1, 11^1, 12^{f_{12}}, \ldots\right)$. Let $m_3 \geq 12$ be the least number with a non-zero frequency in π. Then define

$$\phi(\pi) = \left(3^8, (m_3-8)^1, m_3^{f_{m_3}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(iv): Suppose $m_1 = 12$. Then $\pi = \left(5^1, 12^{f_{12}}, \ldots\right)$.

Case 2(b)(ii)(β)(I)(B)(iv)(I): Suppose $f_{12} \geq 2$. Define

$$\phi(\pi) = \left(3^7, 4^2, 5^0, 12^{f_{12}-2}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(iv)(II): Suppose $f_{12} = 1$. Then $\pi = \left(5^1, 12^1, 13^{f_{13}}, \ldots\right)$. Let $m_4 \geq 13$ be the least number with a non-zero frequency in π. Then $\pi = \left(5^1, 12^1, m_4^{f_{m_4}}, \ldots\right)$.

Case 2(b)(ii)(β)(I)(B)(iv)(II)(a): Suppose $m_4 = 13$. Define

$$\phi(\pi) = \left(3^4, 6^3, 13^{f_{13}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(B)(iv)(II)(b): Suppose $m_4 \geq 14$. Then $m_4 - 10 \geq 4$, and thus by Lemma 4.8 there exist nonnegative integers $X_{m_4-10}, Y_{m_4-10}, Z_{m_4-10}$ and U_{m_4-10} such that

$$m_4 - 10 = 4X_{m_4-10} + 5Y_{m_4-10} + 6Z_{m_4-10} + 7U_{m_4-10}.$$

For each $m_4 \geq 14$, fix a solution to the above equation and define

$$\phi(\pi) = \left(3^9, 4^{X_{m_4-10}}, 5^{Y_{m_4-10}}, 6^{Z_{m_4-10}}, 7^{U_{m_4-10}}, m_4^{f_{m_4}-1}, \ldots,\right).$$

Case 2(b)(ii)(β)(I)(C): Suppose $f_5 = 2$, and thus $\pi = \left(5^2, 7^{f_7}, \ldots,\right)$. Let $m_5 \geq 7$ be the least number with a non-zero frequency in π.

Case 2(b)(ii)(β)(I)(C)(i): Suppose $m_5 \neq 10$. Then $m_5 - 3 \geq 4$ and $m_5 - 3 \neq 7$. Thus, by Lemma 4.6 there are nonnegative integers x_{m_5-3}, y_{m_5-3} and z_{m_5-3} of the equation

$$m_5 - 3 = 4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3}.$$

For each $m_5 \geq 7$ such that $m_5 \neq 10$, fix a solution to the above equation and define

$$\phi(\pi) = \left(3^1, 4^{1+x_{m_5-3}}, 5^{y_{m_5-3}}, 6^{1+z_{m_5-3}}, m_5^{f_{m_5}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(C)(ii): Suppose $m_5 = 10$. Then $\pi = (5^2, 10^{f_{10}}, \ldots)$.

Case 2(b)(ii)(β)(I)(C)(ii)(a): Suppose $f_{10} \geq 2$. Then define

$$\phi(\pi) = \left(3^{10}, 10^{f_{10}-2}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(C)(ii)(b): Suppose $f_{10} = 1$. Then $\pi = (5^2, 10^1, 11^{f_{11}}, \ldots)$. Let $m_6 \geq 11$ be the least number with a non-zero frequency in π.

Case 2(b)(ii)(β)(I)(C)(ii)(b)(i): Suppose m_6 is odd. Then define

$$\phi(\pi) = \left(3^7, \left(\frac{m_6 - 1}{2}\right)^2, m_6^{f_{m_6}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(I)(C)(ii)(b)(ii): Suppose m_6 is even. Then define

$$\phi(\pi) = \left(3^7, \left(\frac{m_6}{2} - 1\right)^1, \left(\frac{m_6}{2}\right)^1, m_6^{f_{m_6}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(II): Suppose $f_5 = f_6 = 0$. Since $s(\pi) \leq 6$, we have $f_4 \geq 1$. Thus $\pi = \left(4^{f_4}, 7^{f_7}, \ldots\right)$.

Case 2(b)(ii)(β)(II)(A): Suppose $f_4 \geq 3$. Define

$$\phi(\pi) = \left(3^4, 4^{f_4-3}, 7^{f_7}, \ldots\right)$$

Case 2(b)(ii)(β)(II)(B): Suppose $f_4 = 1$, and thus $\pi = \left(4^1, 7^{f_7}, \ldots\right)$. Let $m_7 \geq 7$ be the least number with a non-zero frequency in π. So $\pi = \left(4^1, m_7^{f_{m_7}}, \ldots\right)$.

Case 2(b)(ii)(β)(II)(B)(i): Suppose $m_7 \neq 10, 14$. Then $m_7 - 3 \geq 4$ and $m_7 - 3 \neq 7, 11$. We know by Lemma 4.6 that there is a triple $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3})$ such that

$$m_7 - 3 = 4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3}.$$

As we will define $\phi(\pi)$ to have frequency of 3 equal to 1 in this case, to avoid injectivity problems with the Case 2(b)(ii)(β)(I)(C)(i), if $m_7 = m_5 + 6$, using Lemma 4.9, we choose a solution such that $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1+z_{m_5-3})$. Note that here we have used $m_7 \neq 14$ to ensure that $m_5 - 3 \neq 5$, which is required to use Lemma 4.9.

Define

$$\phi(\pi) = \left(3^1, 4^{1+x_{m_7-3}}, 5^{y_{m_7-3}}, 6^{z_{m_7-3}}, \ldots, m_7^{f_{m_7}-1}, \ldots\right)$$

Case 2(b)(ii)(β)(II)(B)(ii): Suppose $m_7 = 10$. Then $\pi = \left(4^1, 10^{f_{10}}, \ldots\right)$.

Case 2(b)(ii)(β)(II)(B)(ii)(a): Suppose $f_{10} \geq 2$. Define

$$\phi(\pi) = \left(3^4, 6^2, 10^{f_{10}-2}, \ldots\right).$$

Case 2(b)(ii)(β)(II)(B)(ii)(b): Suppose $f_{10} = 1$. Thus $\pi = \left(4^1, 10^1, 11^{f_{11}}, \ldots\right)$. Let $m_8 \geq 11$ be the least number with a non-zero frequency in π. Then define

$$\phi(\pi) = \left(3^7, (m_8 - 7)^1, m_8^{f_{m_8}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(II)(B)(iii): Suppose $m_7 = 14$, so $\pi = \left(4^1, 14^{f_{14}}, \ldots\right)$. Define

$$\phi(\pi) = \left(3^6, 4^0, 14^{f_{14}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(II)(C): Suppose $f_4 = 2$, and thus $\pi = \left(4^2, 7^{f_7}, \ldots\right)$. Let $m_9 \geq 7$ be the least number with a non-zero frequency in π.

Case 2(b)(ii)(β)(II)(C)(i): Suppose m_9 is odd. Define

$$\phi(\pi) = \left(3^1, \left(\frac{m_9+5}{2}\right)^2, m_9^{f_{m_9}-1}, \ldots\right).$$

Case 2(b)(ii)(β)(II)(C)(ii): Suppose m_9 is even. Define

$$\phi(\pi) = \left(3^1, \left(\frac{m_9}{2}+2\right)^1, \left(\frac{m_9}{2}+3\right)^1, m_9^{f_{m_9}-1}, \ldots\right).$$

To prove the injectivity of the map ϕ, we organize the cases based on the various frequencies of 3 in $\phi(\pi)$.

First, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 1.

A1. Case 2(a): Then $\phi(\pi) = \left(3^1, (s(\pi)-3)^1, (s(\pi)^{f_{s(\pi)}-1}), \ldots,\right)$, where $s(\pi) \geq 7$.

A2. Case 2(b)(ii)(β)(I)(B)(i): Then $\phi(\pi) = \left(3^1, 5^{1+A}, 6^B, 7^C, m_1^{f_{m_1}-1}, \ldots\right)$, where $m_1 \geq 8$, $m_1 \neq 11, 12$, and A, B and C are some nonnegative integers such that at least one of these is positive.

A3. Case 2(b)(ii)(β)(I)(C)(i): Then $\phi(\pi) = \left(3^1, 4^{1+x_{m_5-3}}, 5^{y_{m_5-3}}, 6^{1+z_{m_5-3}}, m_5^{f_{m_5}-1}, \ldots\right)$, where $m_5 \geq 7, m_5 \neq 10$, and $4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3} = m_5 - 3$.

A4. Case 2(b)(ii)(β)(II)(B)(i): Then $\phi(\pi) = \left(3^1, 4^{1+x_{m_7-3}}, 5^{y_{m_7-3}}, 6^{z_{m_7-3}}, \ldots, m_7^{f_{m_7}-1}, \ldots\right)$, where $m_7 \geq 7, m_7 \neq 10, 14$, and $4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3} = m_7 - 3$. Moreover, if $m_7 = m_5 + 6$, then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$.

A5. Case 2(b)(ii)(β)(II)(C)(i): Then $\phi(\pi) = \left(3^1, \left(\frac{m_9+5}{2}\right)^2, m_9^{f_{m_9}-1}, \ldots\right)$, where $m_9 \geq 7$ is odd.

A6. Case 2(b)(ii)(β)(II)(C)(ii): Then $\phi(\pi) = \left(3^1, \left(\frac{m_9}{2}+2\right)^1, \left(\frac{m_9}{2}+3\right)^1, m_9^{f_{m_9}-1}, \ldots\right)$, where $m_9 \geq 7$ is even.

We explain why the map ϕ is injective so far. In Case A1, the second smallest and the third smallest parts differ by at least 3, and the frequency of the second smallest part is 1. This separates it from all the other cases. In Case A2, the number 4 is not present as a part, which separates it from Cases A3 and A4, and it contains 5 as a part, which separates it from Cases A5 and A6. In Cases A3 and A4, the number 4 is present as a part, which separates it from Cases A5 and A6. Cases A5 and A6 are separated by the frequency of the second smallest part.

Thus, we only need to show that Cases A3 and A4 are also separated from each other. That is, they have no common element in $\phi(\pi)$. Suppose to the contrary that Cases A3 and A4 have a common element. But then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) = (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$. From the given relations $4x_{m_5-3} + 5y_{m_5-3} + 6z_{m_5-3} = m_5 - 3$ and $4x_{m_7-3} + 5y_{m_7-3} + 6z_{m_7-3} = m_7 - 3$, we obtain $m_7 = m_5 + 6$. But, we also know that if $m_7 = m_5 + 6$, then $(x_{m_7-3}, y_{m_7-3}, z_{m_7-3}) \neq (x_{m_5-3}, y_{m_5-3}, 1 + z_{m_5-3})$, giving the required contradiction.

Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 2.

- Case 2(b)(ii)(α): Then $\phi(\pi) = (3^2, 4^{f_4}, 5^{f_5}, \ldots)$, where $f_4 = 0$ or $f_5 = 0$.
- Case 2(b)(ii)(β)(I)(B)(ii)(II)(a): Then $\phi(\pi) = (3^2, 4^1, 5^2, \ldots)$.
- Case 2(b)(ii)(β)(I)(B)(ii)(II)(b): Then $\phi(\pi) = \left(3^2, 4^1, 5^1, (m_2-3)^1, m_2^{f_{m_2}-1}, \ldots\right)$, where $m_2 \geq 9$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 2, these cases are distinguishable.

Note that the only case in which the frequency of 3 in $\phi(\pi)$ is 3 is Case 2(b)(i), where $\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^{f_5-1}, 6^{f_6}, \ldots\right)$. Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 4.

- Case 2(b)(ii)(β)(I)(B)(iv)(II)(a): Then $\phi(\pi) = \left(3^4, 6^3, 13^{f_{13}-1}, \ldots\right)$.
- Case 2(b)(ii)(β)(II)(A): Then $\phi(\pi) = \left(3^4, 4^{f_4-3}, 7^{f_7}, \ldots\right)$, where $f_4 \geq 3$.
- Case 2(b)(ii)(β)(II)(B)(ii)(a): Then $\phi(\pi) = \left(3^4, 6^2, 10^{f_{10}-2}, \ldots\right)$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 4, these cases are distinguishable. Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 5.

- Case 2(b)(ii)(β)(I)(A): Then $\phi(\pi) = (3^5, 5^{f_5-3}, 7^{f_7}, \ldots)$, where $f_5 \geq 3$.
- Case 2(b)(ii)(β)(I)(B)(ii)(I): Then $\phi(\pi) = \left(3^5, 4^1, 7^{f_7-2}, \ldots\right)$, where $f_7 \geq 2$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 5, these cases are distinguishable. Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 6.

- Case 1 with $f = 1$: Then $\phi(\pi) = \left(3^6, 4^\alpha, 5^\beta, 6^\gamma, 7^\delta, \ldots\right)$, where α, β, γ and δ are some nonnegative integers such that at least one of these is positive.
- Case 2(b)(ii)(β)(II)(B)(iii): Then $\phi(\pi) = \left(3^6, 14^{f_{14}-1}, \ldots\right)$, where $f_{14} \geq 1$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 6, these cases are distinguishable. Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 7.

- Case 2(b)(ii)(β)(I)(B)(iv)(I): Then $\phi(\pi) = \left(3^7, 4^2, 12^{f_{12}-2}, \ldots\right)$, where $f_{12} \geq 2$.
- Case 2(b)(ii)(β)(I)(C)(ii)(b)(i): Then $\phi(\pi) = \left(3^7, \left(\frac{m_6-1}{2}\right)^2, m_6^{f_{m_6}-1}, \ldots\right)$, where $m_6 \geq 11$ is odd.
- Case 2(b)(ii)(β)(I)(C)(ii)(b)(ii): Then $\phi(\pi) = \left(3^7, \left(\frac{m_6}{2} - 1\right)^1, \left(\frac{m_6}{2}\right)^1, m_6^{f_{m_6}-1}, \ldots\right)$, where $m_6 \geq 11$ is even.
- Case 2(b)(ii)(β)(II)(B)(ii)(b): Then $\phi(\pi) = (3^7, (m_8 - 7)^1, m_8^{f_{m_8}-1}, \ldots)$, where $m_8 \geq 11$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 7, the next parts after 3 and their frequencies distinguish the various cases.

Note that the only case in which the frequency of 3 in $\phi(\pi)$ is 8 is Case 2(b)(ii)(β)(I)(B)(iii)(b), where $\phi(\pi) = \left(3^8, (m_3 - 8)^1, m_3^{f_{m_3}-1}, \ldots\right)$. Next, we organize the cases in which the frequency of 3 in $\phi(\pi)$ is 9.

- Case 2(b)(ii)(β)(I)(B)(iii)(a): Then $\phi(\pi) = \left(3^9, 11^{f_{11}-2}, \ldots\right)$, where $f_{11} \geq 2$
- Case 2(b)(ii)(β)(I)(B)(iv)(II)(b): Then $\phi(\pi) = \left(3^9, 4^\alpha, 5^\beta, 6^\gamma, 7^\delta, m_4^{f_{m_4}-1}, \ldots,\right)$, where α, β, γ and δ are some nonnegative integers such that at least one of these is positive, and $m_4 \geq 14$.

Clearly, when the frequency of 3 in $\phi(\pi)$ is 9, these cases are distinguishable.

Note that the only case in which the frequency of 3 in $\phi(\pi)$ is 10 is Case 2(b)(ii)(β)(I)(C)(ii)(a), where $\phi(\pi) = \left(3^{10}, 10^{f_{10}-2}, \ldots\right)$, where $f_{10} \geq 2$.

Finally, the only case in which the frequency of 3 in $\phi(\pi)$ is $6f$ for some $f \geq 2$ is Case 1. Hence all the cases are distinguishable and the map ϕ is injective. $\square$

Remark 11. Let $H_{L,3,L}(q) = \sum_{n\geq 0} a_{L,n}q^n$. Then the above combinatorial results imply that $a_{L,n} \geq 0$ for all $n \geq 21$. We can in fact make a stronger claim about the coefficients $a_{L,n}$ when $n \geq 21$; we have $a_{L.n} \geq 1$ for all $n \geq 21$. To see this, we find a partition of n in $I_{L,3,L}$ not in the image of ϕ. For $n \geq 23$, the partition $\pi_n = (3^4, 5^1, (n-17)^1)$ is such a partition of n. For $n = 22$ and $n = 21$, $(3^4, 5^2)$ and $(3^4, 4^1, 5^1)$ are such partitions, respectively.

We are left with the cases $4 \leq L \leq 21$. We deal with $7 \leq L \leq 21$ in Lemma 4.11 below. The proof of this lemma is similar in spirit to the proof of Theorem 3.16 in Chapter 3.

Lemma 4.11. *For $7 \leq L \leq 21$ and $N \geq N'_L$, the coefficient of q^N in $H_{L,3,L}(q)$ is nonnegative whenever $N \geq N'_L$, where $N'_L = L^2 + 10L + 7$.*

Proof. It suffices to show that for $7 \leq L \leq 21$ and $N \geq N'_L$, there is an injective map

$$\psi : \{\pi \in D_{L,3} : |\pi| = N\} \to \{\pi \in I_{L,3,L} : |\pi| = N\}.$$

Let $\pi = \left(4^{f_4}, \ldots, L^{f_L}, \ldots, (L+3)^{f_{L+3}}\right)$ be an element of $D_{L,3}$. We define $\psi(\pi)$ depending on the following cases. Let f denote f_L, so $\pi = (4^{f_4}, 5^{f_5}, \ldots, L^f \ldots, (L+3)^{f_{L+3}})$. To define the image of π under the map ψ, we consider several cases depending on f.

Case 1: Suppose that f is a positive number such that $f \equiv 0 \pmod 3$. Then define

$$\psi(\pi) = \left(3^{\frac{Lf}{3}}, 4^{f_4}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 2: Suppose $f \equiv 1 \pmod 3$. Then define

$$\psi(\pi) = \left(3^{L(\frac{f-1}{3})+1}, 4^{f_4}, \ldots, (L-3)^{f_{L-3}+1}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 3: Suppose $f \equiv 2 \pmod 3$. Then define

$$\psi(\pi) = \left(3^{L(\frac{f-2}{3})+2}, 4^{f_4}, \ldots, (L-3)^{f_{L-3}+2}, \ldots, L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 4: Suppose $f = 0$. Since $N \geq N'_L$ is large enough, either $f_{L+2} \geq 6$ or there exists an $i \neq L+2$ such that $4 \leq i \leq L+3$ and $f_i \geq 3$. Note that the condition on N is in fact tight to guarantee this. We have further subcases.

Case 4(i): Suppose $f_{L+2} \geq 6$. Then define

$$\psi(\pi) = \left(3^{2L+4}, 4^{f_4}, \ldots L^0, (L+2)^{f_{L+2}-6}, (L+3)^{f_{L+3}}\right).$$

Case 4(ii): Suppose that $f_{L+2} \leq 5$ and that there exists an $i \neq L+2$ such that $4 \leq i \leq L+3$ and $f_i \geq 3$. Let i_0 be the least such number. Note that $i_0 \neq L$, since $f = 0$. We have further subcases depending on whether $i_0 = L+1$ or not.

Case 4(ii)(a): Suppose $i_0 \neq L+1$. Then define

$$\psi(\pi) = \left(3^{i_0}, 4^{f_4}, \ldots, i_0^{f_{i_0}-3}, \ldots L^0, \ldots, (L+3)^{f_{L+3}}\right).$$

Case 4(ii)(b): Suppose $i_0 = L+1$. Then define

$$\psi(\pi) = \left(3^3, 4^{f_4}, \ldots, (L-2)^{f_{L-2}+3}, L^0, (L+1)^{f_{L+1}-3}, \ldots, (L+3)^{f_{L+3}}\right).$$

It is easy to see that ψ is injective in each case. To see that ψ is injective overall, note that the frequency of 3 modulo L in the image distinguishes all the cases, except possibly the cases 4(i) and 4(ii)(a), for which the frequencies of 3 might be the same modulo L but clearly different as numbers. Hence the map ψ is injective.

□

Next, we handle the cases $4 \leq L \leq 6$.

Lemma 4.12. *The coefficient of q^N in $H_{6,3,6}(q)$ is nonnegative whenever $N \geq 67$.*

Proof. It suffices to show that for $N \geq 67$, there is an injective map

$$\phi : \{\pi \in D_{6,3} : |\pi| = N\} \to \{\pi \in I_{6,3,6} : |\pi| = N\}.$$

Let $\pi = \left(4^{f_4}, \ldots, 6^{f_6}, \ldots, 9^{f_9}\right)$ be an element of $D_{6,3}$. Note that a partition in the image should have smallest part 3 and should not have 6 as a part. We define $\phi(\pi)$ depending on the following cases. Let f denote f_6.

Case 1: Suppose $f > 0$. Define

$$\phi(\pi) = \left(3^{2f}, 4^{f_4}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9}\right).$$

Case 2: Suppose $f = 0$. Then $\pi = (4^{f_4}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9})$. Since $N \geq 67$, there exists $4 \leq i \leq 9$ such that $i \neq 6$ and $f_i \geq 3$. Let i_0 be the least such number. Note that the condition on N is in fact tight to guarantee this.

Case 2(i): Suppose i_0 is odd, i.e. i_0 is $5, 7$ or 9. Define

$$\phi(\pi) = \left(3^{i_0}, \ldots, 6^0, \ldots i_0^{f_{i_0}-3} \ldots\right).$$

Case 2(ii): Suppose $i_0 = 4$. Define

$$\phi(\pi) = \left(3^1, 4^{f_4-3}, 5^{f_5}, 6^0, 7^{f_7}, 8^{f_8}, 9^{f_9+1}\right).$$

Case 2(iii): Suppose $i_0 = 8$. Define

$$\phi(\pi) = \left(3^3, 4^{f_4}, 5^{f_5+3}, 6^0, 7^{f_7}, 8^{f_8-3}, 9^{f_9}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. Hence the map ϕ is injective. $\square$

Lemma 4.13. *The coefficient of q^N in $H_{5,3,5}(q)$ is nonnegative whenever $N \geq 159$.*

Proof. It suffices to show that for $N \geq 159$, there is an injective map

$$\phi : \{\pi \in D_{5,3} : |\pi| = N\} \to \{\pi \in I_{5,3,5} : |\pi| = N\}.$$

Let $\pi = \left(4^{f_4}, 5^{f_5}, \ldots, 8^{f_8}\right)$ be an element of $D_{5,3}$. Note that a partition in the image should have smallest part 3 and should not have 5 as a part. We define $\phi(\pi)$ depending on the following cases. Let f denote f_5.

Case 1: Suppose f is a positive number with $f \equiv 0 \pmod 3$. Define

$$\phi(\pi) = \left(3^{\frac{5f}{3}}, 4^{f_4}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 2: Suppose $f > 1$ and $f \equiv 1 \pmod 3$. Define

$$\phi(\pi) = \left(3^{5\left(\frac{f-4}{3}\right)+4}, 4^{f_4+2}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 3: Suppose $f \equiv 2 \pmod 3$. Define

$$\phi(\pi) = \left(3^{5\left(\frac{f-2}{3}\right)+1}, 4^{f_4}, 5^0, 6^{f_6}, 7^{f_7+1}, 8^{f_8}\right).$$

We are left with the cases $f = 0$ and $f = 1$.

Case 4: Suppose $f = 0$. Then $\pi = \left(4^{f_4}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right)$. Since $N \geq 159$ is large enough, at least one of the following conditions is true:

- $f_4 \geq 6$.
- $f_6 \geq 1$.
- $f_7 \geq 3$.
- $f_8 \geq 12$.

Note that the condition on N is not tight. The bound of 159 will be required in Case 5 below.

Case 4(i): Suppose $f_4 \geq 6$. Define

$$\phi(\pi) = \left(3^8, 4^{f_4-6}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 4(ii): Suppose $f_4 \leq 5$ and $f_6 \geq 1$. Define

$$\phi(\pi) = \left(3^2, 4^{f_4}, 5^0, 6^{f_6-1}, 7^{f_7}, 8^{f_8}\right).$$

Case 4(iii): Suppose $f_4 \leq 5$, $f_6 = 0$ and $f_7 \geq 3$. Define

$$\phi(\pi) = \left(3^7, 4^{f_4}, 5^0, 7^{f_7-3}, 8^{f_8}\right).$$

Case 4(iv): Suppose $f_4 \leq 5$, $f_6 = 0$, $f_7 \leq 2$ and $f_8 \geq 12$. Define

$$\phi(\pi) = \left(3^{32}, 4^{f_4}, 5^0, 7^{f_7}, 8^{f_8-12}\right).$$

Case 5: Suppose $f = 1$. Then $\pi = \left(4^{f_4}, 5^1, 6^{f_6}, 7^{f_7}, 8^{f_8}\right)$. Since $N > 158$ is large enough, at least one of the following conditions is true:

- $f_4 \geq 1$.
- $f_6 \geq 11$.
- $f_7 \geq 7$.
- $f_8 \geq 8$.

Note that the condition on N is tight to guarantee this.

Case 5(i): Suppose $f_4 \geq 1$. Define

$$\phi(\pi) = \left(3^3, 4^{f_4-1}, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8}\right).$$

Case 5(ii): Suppose $f_4 = 0$ and $f_6 \geq 11$. Define

$$\phi(\pi) = \left(3^{13}, 4^8, 5^0, 6^{f_6-11}, 7^{f_7}, 8^{f_8}\right).$$

Case 5(iii): Suppose $f_4 = 0$, $f_6 \leq 10$ and $f_7 \geq 7$. Define

$$\phi(\pi) = \left(3^{18}, 4^0, 5^0, 6^{f_6}, 7^{f_7-7}, 8^{f_8}\right).$$

Case 5(iv): Suppose $f_4 = 0$, $f_6 \leq 10$, $f_7 \leq 6$ and $f_8 \geq 8$. Define

$$\phi(\pi) = \left(3^{23}, 4^0, 5^0, 6^{f_6}, 7^{f_7}, 8^{f_8-8}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. In Cases 1, 2 and 3, the frequency of 3 is $0, 4$ and 1 modulo 5, respectively. In Cases 4 and 5, it is always 2 or 3 modulo 5 and different for each subcase. Hence the map ϕ is injective.

□

Lemma 4.14. *The coefficient of q^N in $H_{4,3,4}(q)$ is nonnegative whenever $N \geq 1006$.*

Proof. It suffices to show that for $N \geq 1006$, there is an injective map

$$\phi : \{\pi \in D_{4,3} : |\pi| = N\} \to \{\pi \in I_{4,3,4} : |\pi| = N\}.$$

Let $\pi = (4^{f_4}, 5^{f_5}, 6^{f_6}, 7^{f_7})$ be an element of $D_{4,3}$. Note that a partition in the image should have smallest part 3 and should not have 4 as a part. We define $\phi(\pi)$ depending on the following cases. Let f denotes f_4.

Recall from Lemma 4.7, that for $n \geq 10$, there exist nonnegative integer solutions (x_n, y_n, z_n) of the equation

$$n = 5x_n + 6y_n + 7z_n.$$

For each n, fix such a nonnegative integer solution (x_n, y_n, z_n).

Case 1: Suppose $10 \leq f < 100$. Define

$$\phi(\pi) = (3^f, 4^0, 5^{f_5+x_f}, 6^{f_6+y_f}, 7^{f_7+z_f}).$$

Case 2: Suppose $f \geq 100$. Define

$$\phi(\pi) = (3^{f+30}, 4^0, 5^{f_5+x_{f-90}}, 6^{f_6+y_{f-90}}, 7^{f_7+z_{f-90}}).$$

Case 3: Suppose $0 \leq f \leq 9$. Since $N \geq 1006$ is large enough, at least one of the following conditions is true:

- $f_5 \geq 62$.
- $f_6 \geq 57$.
- $f_7 \geq 53$.

Note that the condition on N is in fact tight to guarantee this.

Case 3(i): Suppose $f_5 \geq 62$. Define

$$\phi(\pi) = \left(3^{f+100}, 4^0, 5^{f_5-62+x_{f+10}}, 6^{f_6+y_{f+10}}, 7^{f_7+z_{f+10}}\right).$$

Case 3(ii): Suppose $f_5 \leq 61$ and $f_6 \geq 57$. Define

$$\phi(\pi) = \left(3^{f+110}, 4^0, 5^{f_5+x_{f+12}}, 6^{f_6-57+y_{f+12}}, 7^{f_7+z_{f+12}}\right).$$

Case 3(iii): Suppose $f_5 \leq 61$, $f_6 \leq 56$ and $f_7 \geq 53$. Define

$$\phi(\pi) = \left(3^{f+120}, 4^0, 5^{f_5+x_{f+11}}, 6^{f_6+y_{f+11}}, 7^{f_7-53+z_{f+11}}\right).$$

It is easy to see that ϕ is injective in each case. To see that ϕ is injective overall, note that the frequency of 3 in the image distinguishes the cases. Hence the map ϕ is injective. $\square$

Lemmas 4.10, 4.11, 4.12, 4.13 and 4.14 show that for N larger than a small number, the coefficient of q^N in $H_{L,s,L}(q)$ is nonnegative. Finally, we use all these lemmas, along with some computation to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. First, we focus on the case $L \geq 22$. Let $H_{L,3,L}(q) = \sum_{N\geq 0} a_{L,N}q^N$. Then by the remark following the proof of Lemma 4.10, $a_{L,N} \geq 1$ whenever $N \geq 21$. For $N \leq 20$, we first observe that $a_{L,N}$ is independent of L. To see why, note that from the combinatorial interpretation of $H_{L,3,L}(q)$, for $L \geq N+1$, we have $a_{L,N} = a_{N+1,N}$, which is equal to the difference between the number of partitions of N with smallest part 3 and the number of partitions of N with smallest part at least 4.

Then for $N \leq 20$, an easy computer program shows that $a_{L,N}$ is negative only when N is one of $4, 5, 8, 10, 12, 14$ or 16, and in each case $a_{L,N}$ is exactly -1. Let $G_{L,3}(q) = \sum_{N\geq 0} b_{L,N}q^N$. Then, by Theorem 4.3, we know that

$$b_{L,N} = a_{L,N} + a_{L,N-L} + a_{L,N-2L} + \cdots$$

Thus, we find that $b_{L,N}$ is negative only when N is one of $4, 5, 8, 10, 12, 14$ or 16, and in each case $b_{L,N}$ is exactly -1. This gives us Theorem 4.2 for $L \geq 22$.

Next, for $7 \leq L \leq 21$, Lemma 4.11 along with a computer program for $N \leq N'_L$ prove Theorem 4.2. Similarly, for $4 \leq L \leq 6$, the Lemmas 4.12, 4.13 and 4.14 along with a computer program for small values of N complete the proof of Theorem 4.2. $\square$

Remark 12. The programming for $7 \leq L \leq 21$ and $N \leq N'_L$ turned out to be a difficult task in Magma. For example, it is hard to calculate the number of partitions of 250 with all parts in the set {4,5, ..., 17} using Magma. The command $Partitions(250, min\_part = 4, max\_part = 17).cardinality()$ in Sage also does not work (it takes too long and ultimately stops working). We overcame this problem through another related command and some mathematics. In Sage, we noticed that the command $Partitions(n, max\_part = 17).cardinality()$ is very fast even for large n (even until $n = 1000000$, it is fast!) Thus, we calculate the number of partitions with all parts in the set $\{4, 5, \ldots, 17\}$ in terms of the number of partitions $p_{17}(n)$ of n with maximum part at most 17. We do this by viewing partitions with all parts in the set $\{4, 5, \ldots, 17\}$ as partitions with maximum part 17 and no part 1, 2 and 3. Let A, B and C denote the set of partitions of n with maximum part 17 and also having 1, 2 and 3 as a part, respectively. Then we need to find the cardinality of the set $A^{\complement} \cap B^{\complement} \cap C^{\complement}$. Using inclusion and exclusion principle, we find that the number of partitions of n with all parts in the set $\{4, 5, \ldots, 17\}$ is given by

$p_{17}(n) - p_{17}(n-1) - p_{17}(n-2) + p_{17}(n-4) + p_{17}(n-5) - p_{17}(n-6)$, and thus can be easily computed.

4.3 Future directions

The following questions may lead to interesting answers.

Question 4.15. *Is there a nice polynomial $p_s(q)$ such that $G_{L,s}(q) + p_s(q) \succeq 0$?*

Question 4.16. *If the answer to Question 4.15 is yes, what is the least possible degree of $p_s(q)$?*

Moreover, it may be interesting to study a series analogous to $G_{L,s}(q)$ for partitions with further restrictions on parts, such as for partitions with only odd parts or for self conjugate partitions.

Chapter 5

Cumulative Recurrence Relations for Integer Partitions

Recall from Chapter 1, Euler's famous recurrence relation for integer partitions:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

It can be shown that the above recurrence can be written succinctly as

$$p(n) = \sum_{k \geq 1} (-1)^{k-1} \left(p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right)\right). \tag{5.1}$$

For details, see [And76, Corollary 1.8] or [Aig07, Page 130]. In the next section, we give a natural combinatorial proof, using the principle of inclusion and exclusion (PIE), for Euler's recurrence relation for integer partitions.

5.1 A bijective proof of Euler's recurrence (5.1) using PIE

Let n be a given positive integer. Let X_n denote the set of partitions of n that do not have 1 as a part, and let Y_n denote the set of partitions of n that have 1 as a part. Define the map $\phi : X_n \to Y_n$ as follows. Suppose $\pi = (2^{f_2}, 3^{f_3}, \ldots) \in X_n$. Let $i \geq 2$ be the least number having nonzero frequency in π. Define

$$\phi(\pi) = (1^i, 2^0, 3^0, \ldots, (i-1)^0, i^{f_i - 1}, \ldots).$$

Clearly ϕ is an injective map. An easy bijection $Par(n-1) \cup X_n \to Par(n)$ gives $|X_n| = p(n) - p(n-1)$. Additionally, we use PIE to find $|\phi(X_n)|$. Note that $\phi(X_n)$ consists of the set of partitions of n having i parts of 1 and no parts of $2, 3, \cdots (i-1)$ for some $i \geq 2$. Define the following sets:

- $A_{j,k}(n)$ is the set of partitions of n having exactly k parts of size j;

- $B_{j,k}(n)$ is the set of partitions of n having at least k parts of size j.

A simple bijection $B_{j,k}(n) \to Par(n-jk)$ shows that $|B_{j,k}(n)| = p(n-jk)$. Moreover, if $j \neq j'$, then $|B_{j,k}(n) \cap B_{j',k'}(n)| = p(n-jk-j'k')$. Further, notice that for all j and n, $A_{j,0}(n)$ and $B_{j,1}(n)$ are disjoint sets and $A_{j,0}(n) \cup B_{j,1}(n) = Par(n)$. Thus, the sets $A_{j,0}(n)$ and $B_{j,1}(n)$ are complements of each other in the set $Par(n)$. That is $A_{j,0}(n) = B_{j,1}^{\complement}(n)$. We have

$$|\phi(X_n)| = \sum_{i\geq 2} |A_{1,i}(n) \cap A_{2,0}(n) \cap \cdots \cap A_{i-1,0}(n)|.$$

Note that

$$\begin{aligned}
|A_{1,i}(n) \cap A_{2,0}(n) \cap \cdots \cap A_{i-1,0}(n)| &= |A_{1,0}(n-i) \cap A_{2,0}(n-i) \cap \cdots \cap A_{i-1,0}(n-i)| \\
&= |B_{1,1}^{\complement}(n-i) \cap B_{2,1}^{\complement}(n-i) \cap \cdots \cap B_{i-1,1}^{\complement}(n-i)| \\
&= |(B_{1,1}(n-i) \cup B_{2,1}(n-i) \cup \cdots \cup B_{i-1,1}(n-i))^{\complement}| \\
&= \sum_{s=0}^{i-1} (-1)^s \sum_{(i_1,i_2,\ldots,i_s)\in T_{s,i}} p(n-i-i_1-i_2\cdots-i_s).
\end{aligned}$$

where $T_{s,i}$ denotes the set of distinct s-tuples $(i_1, i_2, \ldots, i_s)$ such that $1 \leq i_1, i_2, \ldots, i_s \leq i-1$. Note that the last equality above is obtained by PIE. Hence we obtain

$$|\phi(X_n)| = \sum_{i\geq 2} \sum_{s=0}^{i-1} (-1)^s \sum_{(i_1,i_2,\ldots,i_s)\in T_{s,i}} p(n-i-i_1-i_2\cdots-i_s),$$

and thus

$$\begin{aligned}
p(n) &= p(n-1) + \sum_{i\geq 2} \sum_{s=0}^{i-1} (-1)^s \sum_{(i_1,i_2,\ldots,i_s)\in T_{s,i}} p(n-i-i_1-i_2\cdots-i_s) \\
&= \sum_{i\geq 1} \sum_{s=0}^{i-1} (-1)^s \sum_{(i_1,i_2,\ldots,i_s)\in T_{s,i}} p(n-i-i_1-i_2\cdots-i_s) \\
&= \sum_{l\geq 1} a_l p(n-l).
\end{aligned}$$

where a_l denotes the number of partitions of l into an odd number of distinct parts minus the number of partitions of l into an even number of distinct parts. Note that the condition $s \leq i-1$ is redundant because for a partition of l into distinct parts with largest part i and number of parts $s+1$, it is obvious that $s+1 \leq i$, so $s \leq i-1$.

By Euler's Pentagonal Number Theorem, $a_l = (-1)^{k-1}$ if $l = k(3k-1)/2$ or $l = k(3k+1)/2$, and 0 otherwise. There is an excellent combinatorial proof for this via involution. For details, see [And76, Thm 1.6, Page 10].

5.2 Cumulative recurrence relations for $k = 5$

For a given k, we describe a procedure to find a recurrence that, for sufficiently large n, expresses $p(n)$ in terms of $p(n-k)$, $p(n-k-1)$, $p(n-k-2)$ etc. To be clear, we seek an expression for $p(n)$ that does not involve $p(n-1), p(n-2), \cdots, p(n-(k-1))$ but does involve $p(n-k)$. We describe this in detail for $k = 5$ in this section. The procedure for any other k is similar and is described in Section 5.3.

Let X_n denote the set of partitions of n that do not have 5 as a part and let Y_n denote the set of partitions of n that have 5 as a part. For $n \geq 132241$, we define the map $\phi : X_n \to Y_n$ as follows. Let $\pi = (1^{f_1}, 2^{f_2}, \cdots 5^0, \cdots) \in X_n$. We define the image of π under the map ϕ based on the following cases.

Case 1: Suppose there exists m such that $1 \leq m \leq 95$ and $f_m \geq 30$. Let m_0 be the least such number. Define

$$\phi(\pi) = (1^{f_1}, \ldots 5^{6m_0}, \ldots m_0^{f_{m_0}-30}, \ldots).$$

Case 2: Suppose $f_m \leq 29$ for all $1 \leq m \leq 95$. Since $n \geq 132241$, there exists a $p \geq 96$ such that $f_p \geq 1$. Let n_0 be the least such number.

Case 2(A): Suppose $f_1 \geq 1$ and $f_4 \geq 1$. Define

$$\phi(\pi) = (1^{f_1-1}, \ldots 4^{f_4-1}, 5^1, \ldots).$$

Case 2(B): Suppose $f_1 = 0$, Define

$$\phi(\pi) = (1^{n_0-55}, \ldots 5^{11}, \ldots 95^{f_{95}}, \ldots n_0^{f_{n_0}-1}, \ldots).$$

Case 2(C): Suppose $f_1 > 0$ and $f_4 = 0$. We have the following subcases.

Case 2(C)(i): Suppose $n_0 \equiv 0 \pmod 4$. Define

$$\phi(\pi) = (1^{f_1-1}, \ldots 4^{\frac{n_0-24}{4}}, 5^5, \ldots 95^{f_{95}}, \ldots n_0^{f_{n_0}-1}, \ldots).$$

Case 2(C)(ii): Suppose $n_0 \equiv 1 \pmod 4$. Define

$$\phi(\pi) = (1^{f_1-1}, \ldots 4^{\frac{n_0-49}{4}}, 5^{10}, \ldots 95^{f_{95}}, \ldots n_0^{f_{n_0}-1}, \ldots).$$

Case 2(C)(iii): Suppose $n_0 \equiv 2 \pmod 4$. Define

$$\phi(\pi) = (1^{f_1-1}, \ldots 4^{\frac{n_0-74}{4}}, 5^{15}, \ldots 95^{f_{95}}, \ldots n_0^{f_{n_0}-1}, \ldots).$$

Case 2(C)(iv): Suppose $n_0 \equiv 3 \pmod 4$. Define

$$\phi(\pi) = (1^{f_1-1}, \ldots 4^{\frac{n_0-99}{4}}, 5^{20}, \ldots 95^{f_{95}}, \ldots n_0^{f_{n_0}-1}, \ldots).$$

Since the map ϕ is injective within each case and the frequency of 5 in the partition in the image is different for different cases, the map ϕ is injective, and thus $|X_n| \leq |Y_n|$. Moreover, since $|X_n| = p(n) - p(n-5)$ and $|Y_n| = p(n-5)$, we obtain $p(n) \leq 2p(n-5)$ whenever $n \geq 132241$. We refine the inequality obtained to an equality by measuring the size of $\phi(X_n)$ exactly in each of the cases by using PIE. We recall the sets $A_{j,k}(n)$ and $B_{j,k}(n)$ from Section 5.1 and also define the set $C_{j,k}(n)$ as follows.

- $A_{j,k}(n)$ is the set of partitions of n having exactly k parts of size j.
- $B_{j,k}(n)$ is the set of partitions of n having at least k parts of size j.
- $C_{j,k}(n)$ is the set of partitions of n having at most k parts of size j.

Recall that $|B_{j,k}(n)| = p(n - jk)$. Further, note that $C_{j,k}(n) = B^{\complement}_{j,k+1}(n)$ and thus

$$|C_{j,k}(n)| = p(n) - p(n - j(k+1)).$$

First we describe the image of X_n under ϕ in each of the cases and then we will measure their sizes. The image in the cases are given as follows.

- Case 1:

$$\bigsqcup_{\substack{m_0=1,\\ m_0\neq 5}}^{95} \left(A_{5,6m_0}(n) \bigcap_{\substack{i=1,\\ i\neq 5}}^{m_0-1} C_{i,29}(n) \right).$$

- Case 2(A):

$$C_{1,28}(n) \cap C_{4,28}(n) \cap A_{5,1}(n) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n).$$

- Case 2(B):

$$\bigsqcup_{n_0\geq 96} \left(A_{1,n_0-55}(n) \cap A_{5,11}(n) \bigcap_{\substack{i=2,\\ i\neq 5}}^{95} C_{i,29}(n) \bigcap_{i=96}^{n_0-1} A_{i,0}(n) \right).$$

When $n_0 = 96$, the last intersection in the above expression is vacuous and can be omitted.

- Case 2(C)(i):

$$\bigsqcup_{\substack{n_0\geq 96,\\ n_0\equiv 0(4)}} \left(C_{1,28}(n) \cap A_{4,\frac{n_0-24}{4}}(n) \cap A_{5,5}(n) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n) \bigcap_{i=96}^{n_0-1} A_{i,0}(n) \right).$$

- Case 2(C)(ii):

$$\bigsqcup_{\substack{n_0\geq 96,\\ n_0\equiv 1(4)}} \left(C_{1,28}(n) \cap A_{4,\frac{n_0-49}{4}}(n) \cap A_{5,10}(n) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n) \bigcap_{i=96}^{n_0-1} A_{i,0}(n) \right).$$

- Case 2(C)(iii):

$$\bigsqcup_{\substack{n_0\geq 96,\\ n_0\equiv 2(4)}} \left(C_{1,28}(n) \cap A_{4,\frac{n_0-74}{4}}(n) \cap A_{5,15}(n) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n) \bigcap_{i=96}^{n_0-1} A_{i,0}(n) \right).$$

- Case 2(C)(iv):

$$\bigsqcup_{\substack{n_0\geq 96,\\ n_0\equiv 3(4)}} \left(C_{1,28}(n) \cap A_{4,\frac{n_0-99}{4}}(n) \cap A_{5,20}(n) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n) \bigcap_{i=96}^{n_0-1} A_{i,0}(n) \right).$$

Next, we measure the size of image in each of these cases. We begin with Case 2(A) since it is the most illustrative.

5.2.1 Size of the image in Case 2(A)

Since $A_{5,1}(n)$ can be identified with $A_{5,0}(n-5)$, which is same as $C_{5,0}(n-5)$, the image in Case 2(A) has size

$$\begin{aligned}
&\left| C_{1,28}(n-5) \cap C_{4,28}(n-5) \cap C_{5,0}(n-5) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n-5) \right| \\
&= \left| B^{\complement}_{1,29}(n-5) \cap B^{\complement}_{4,29}(n-5) \cap B^{\complement}_{5,1}(n-5) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} B^{\complement}_{i,30}(n-5) \right| \\
&= \left| \left(B_{1,29}(n-5) \cup B_{4,29}(n-5) \cup B_{5,1}(n-5) \bigcup_{\substack{i=2,\\ i\neq 4,5}}^{95} B_{i,30}(n-5) \right)^{\complement} \right| \\
&= p(n-5) - \left| B_{1,29}(n-5) \cup B_{4,29}(n-5) \cup B_{5,1}(n-5) \bigcup_{\substack{i=2,\\ i\neq 4,5}}^{95} B_{i,30}(n-5) \right|. \qquad (5.2)
\end{aligned}$$

We evaluate the size of the set in (5.2) and our strategy will be to use PIE. We rewrite the set in (5.2) as

$$\left|\bigcup_{i=1}^{95} H_i\right|,$$

where

$$H_i = \begin{cases} B_{i,30}(n-5) & \text{if} \quad i \neq 1,4,5 \\ B_{i,29}(n-5) & \text{if} \quad i = 1,4 \\ B_{5,1}(n-5) & \text{if} \quad i = 5. \end{cases}$$

We need the following notation. Let $n, s, a_1, \cdots a_k, b_1, \cdots b_l$ be positive integers.

- $T_{n,s}$ denotes the set of partitions into s distinct parts with largest part at most n.
- $T_{n,s}(a_1, \cdots a_k; b_1, \cdots b_l)$ denotes the set of partitions into s distinct parts with largest part at most n that contain $a_1, \cdots, a_k$ as a part and do not contain $b_1, \cdots, b_l$ as a part.

By PIE, we have

$$\left|\bigcup_{i=1}^{95} H_i\right| = \sum_{s=1}^{95} (-1)^{s-1} \sum_{(i_1,i_2,\cdots i_s)\in T_{95,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}|.$$

If none of the numbers i_j in the inner sum is $1, 4$ or 5, then $H_{i_j} = B_{i_j,30}(n-5)$ for all j, and thus

$$|H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = p(n - 5 - 30i_1 - 30i_2 - \cdots - 30i_s).$$

If one the numbers i_j is 1, then $i_s = 1$ (since partitions are defined to be a weakly decreasing list), but none of these is 4 or 5, then $H_1 = B_{1,29}(n-5)$ and all others are $B_{i,30}(n-5)$. Thus

$$|H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = p(n-5-29-30i_1-30i_2-\cdots-30i_{s-1}) = p(n-4-30(i_1+i_2+\cdots i_s)).$$

Similarly going through the other cases, we obtain

$$\begin{aligned} \sum_{(i_1,i_2,\cdots i_s)\in T_{95,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = & \sum_{\pi\in T_{95,s}(;1,4,5)} p(n-5-30|\pi|) \\ & + \sum_{\pi\in T_{95,s}(1;4,5)} p(n-4-30|\pi|) \\ & + \sum_{\pi\in T_{95,s}(4;1,5)} p(n-1-30|\pi|) \\ & + \sum_{\pi\in T_{95,s}(5;1,4)} p(n+140-30|\pi|) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\pi\in T_{95,s}(1,4;5)} p(n-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(1,5;4)} p(n+141-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(4,5;1)} p(n+144-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(1,4,5;)} p(n+145-30|\pi|).
\end{aligned}
$$

Hence, the size of the image in Case 2(A) is given by

$$
\begin{aligned}
\sum_{s=0}^{95}(-1)^s\Bigg(&\sum_{\pi\in T_{95,s}(;1,4,5)} p(n-5-30|\pi|) + \sum_{\pi\in T_{95,s}(1;4,5)} p(n-4-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(4;1,5)} p(n-1-30|\pi|) + \sum_{\pi\in T_{95,s}(5;1,4)} p(n+140-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(1,4;5)} p(n-30|\pi|) + \sum_{\pi\in T_{95,s}(1,5;4)} p(n+141-30|\pi|)\\
&+ \sum_{\pi\in T_{95,s}(4,5;1)} p(n+144-30|\pi|) + \sum_{\pi\in T_{95,s}(1,4,5;)} p(n+145-30|\pi|)\Bigg). \qquad (5.3)
\end{aligned}
$$

We use a similar approach to find the size of the image in other cases.

5.2.2 Size of the image in Case 1

The size of the image of Case 1 is given by

$$
\begin{aligned}
&\sum_{\substack{m_0=1,\\ m_0\neq 5}}^{95} \left| C_{5,0}(n-30m_0) \bigcap_{\substack{i=1,\\ i\neq 5}}^{m_0-1} C_{i,29}(n-30m_0) \right|\\
&= \sum_{\substack{m_0=1,\\ m_0\neq 5}}^{95} \left| B^{\complement}_{5,1}(n-30m_0) \bigcap_{\substack{i=1,\\ i\neq 5}}^{m_0-1} B^{\complement}_{i,30}(n-30m_0) \right|\\
&= \sum_{\substack{m_0=1,\\ m_0\neq 5}}^{95} \left| \left(B_{5,1}(n-30m_0) \bigcup_{\substack{i=1,\\ i\neq 5}}^{m_0-1} B_{i,30}(n-30m_0) \right)^{\complement} \right|\\
&= \sum_{\substack{m_0=1,\\ m_0\neq 5}}^{95} p(n-30m_0) - \left| B_{5,1}(n-30m_0) \bigcup_{\substack{i=1,\\ i\neq 5}}^{m_0-1} B_{i,30}(n-30m_0) \right|.
\end{aligned}
$$

First suppose $m_0 \geq 6$. Define

$$H_i = \begin{cases} B_{i,30}(n - 30m_0) & \text{if} \quad i \neq 5 \\ B_{5,1}(n - 30m_0) & \text{if} \quad i = 5. \end{cases}$$

We have

$$\left| \bigcup_{i=1}^{m_0-1} H_i \right| = \sum_{s=1}^{m_0-1} (-1)^{s-1} \sum_{(i_1,i_2,\cdots i_s) \in T_{m_0-1,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| \, .$$

If none of the numbers i_j is 5, then $H_{i_j} = B_{i_j,30}(n - 30m_0)$ for all j, and thus

$$|H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = p(n - 30m_0 - 30i_1 - 30i_2 - \cdots - 30i_s).$$

If one of the numbers i_j is 5 (say $i_t = 5$), then $H_1 = B_{5,1}(n - 30m_0)$ and all others are $B_{i,30}(n - 30m_0)$, and thus

$$\begin{aligned} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| &= p(n - 30m_0 - 5 - 30i_1 - \cdots 30i_{t-1} - 30i_{t+1} \cdots - 30i_s) \\ &= p(n + 145 - 30(m_0 + i_1 + i_2 + \cdots i_s)). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \sum_{(i_1,i_2,\cdots i_s) \in T_{m_0-1,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = &\sum_{\pi \in S_{m_0,s+1}(;5)} p(n - 30|\pi|) \\ &+ \sum_{\pi \in S_{m_0,s+1}(5;)} p(n + 145 - 30|\pi|). \end{aligned}$$

Hence, the size of image in Case 1 when $m_0 \geq 6$ is given by

$$\sum_{m_0=6}^{95} \sum_{s=0}^{m_0-1} (-1)^s \left(\sum_{\pi \in S_{m_0,s+1}(;5)} p(n - 30|\pi|) + \sum_{\pi \in S_{m_0,s+1}(5;)} p(n + 145 - 30|\pi|) \right).$$

Next suppose $1 \leq m_0 \leq 4$. Define

$$H_i = \begin{cases} B_{i,30}(n - 30m_0) & \text{if} \quad 1 \leq i \leq m_0 - 1 \\ B_{5,1}(n - 30m_0) & \text{if} \qquad i = m_0. \end{cases}$$

We have

$$\left| \bigcup_{i=1}^{m_0} H_i \right| = \sum_{s=1}^{m_0} (-1)^{s-1} \sum_{(i_1,i_2,\cdots i_s) \in T_{m_0,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| \, .$$

If none of the numbers i_j is m_0, then $H_{i_j} = B_{i_j,30}(n - 30m_0)$, and thus

$$|H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = p(n - 30m_0 - 30i_1 - 30i_2 - \cdots - 30i_s).$$

If one of the numbers i_j is m_0, then $i_1 = m_0$ (since partitions are defined to be a weakly decreasing list). Therefore, $H_{i_1} = B_{5,1}(n - 30m_0)$ and all others are $B_{i,30}(n - 30m_0)$. Thus,

$$\begin{aligned}|H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| &= p(n - 30m_0 - 5 - 30i_2 - 30i_3 - \cdots 30i_s)\\ &= p(n - 5 - 30(i_1 + i_2 + \cdots i_s)).\end{aligned}$$

We obtain

$$\sum_{(i_1,i_2,\cdots i_s)\in T_{m_0,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}| = \sum_{\pi\in S_{m_0,s+1}} p(n - 30|\pi|) + \sum_{\pi\in S_{m_0,s}} p(n - 5 - 30|\pi|).$$

Thus the total size of the image in Case 1 is given by

$$\begin{aligned}&\sum_{m_0=1}^{4}\sum_{s=0}^{m_0}(-1)^s \left(\sum_{\pi\in S_{m_0,s+1}} p(n - 30|\pi|) + \sum_{\pi\in S_{m_0,s}} p(n - 5 - 30|\pi|)\right)\\ &+ \sum_{m_0=6}^{95}\sum_{s=0}^{m_0-1}(-1)^s \left(\sum_{\pi\in S_{m_0,s+1}(;5)} p(n - 30|\pi|) + \sum_{\pi\in S_{m_0,s+1}(5;)} p(n + 145 - 30|\pi|)\right). \quad (5.4)\end{aligned}$$

5.2.3 Size of the image in Case 2(B)

The size of image of Case 2(B) is given by

$$\begin{aligned}&\sum_{n_0\geq 96}\left|\left(C_{1,0}(n - n_0) \cap C_{5,0}(n - n_0) \bigcap_{\substack{i=2,\\ i\neq 5}}^{95} C_{i,29}(n - n_0) \bigcap_{i=96}^{n_0-1} C_{i,0}(n - n_0)\right)\right|\\ &= \sum_{n_0\geq 96} p(n - n_0) - \left|\left(B_{1,1}(n - n_0) \cup B_{5,1}(n - n_0) \bigcup_{\substack{i=2,\\ i\neq 5}}^{95} B_{i,30}(n - n_0) \bigcup_{i=96}^{n_0-1} B_{i,1}(n - n_0)\right)\right|.\end{aligned}$$

Define

$$H_i = \begin{cases} B_{i,30}(n - n_0) & \text{if} \quad i \leq 95 \quad \text{and} \quad i \neq 1,5 \\ B_{i,1}(n - n_0) & \text{if} \quad i > 95 \quad \text{or} \quad i = 1,5. \end{cases}$$

We have

$$\left|\bigcup_{i=1}^{n_0-1} H_i\right| = \sum_{s=1}^{n_0-1}(-1)^{s-1} \sum_{(i_1,i_2,\cdots i_s)\in T_{n_0-1,s}} |H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_s}|\,.$$

In the interior sum, suppose t of the numbers i_j are less than or equal to 95 and different from 1 or 5 (say $i_{j_1}, i_{j_2}, \cdots i_{j_t}$). Then $H_{i_{j_l}} = B_{i_{j_l},30}(n - n_0)$ for $l \leq t$ and $H_{i_j} = B_{i_j,1}(n - n_0)$ for all other i_j.

We need some more notation.

- Set $T'_t := T_{95,t}(;1,5)$. That is, T'_t denotes the set of partitions into t distinct parts with largest part at most 95, with no part 1 or 5. Note that any partition in T'_t has at most 93 parts.
- Set $T''_{n,t} := T_{n,t}(;2,3,4,6,7,\cdots 95)$. That is, $T''_{n,t}$ denotes the set of partitions into t distinct parts with largest part at most n, and that do not contain $2,3,4,6,7,\cdots 95$ as a part. Notice that $T'_t \cap T''_{n,t} = \emptyset$.

We express any partition of $T_{n_0-1,s}$ as a partition of T'_t and a partition of $T''_{n_0-1,s-t}$. Then

$$\sum_{(i_1,i_2,\cdots i_s)\in T_{n_0-1,s}} |H_{i_1}\cap H_{i_2}\cap\cdots\cap H_{i_s}| = \sum_{t=0}^{\min(s,93)} \sum_{\pi'\in T'_t} \sum_{\pi''\in T''_{n_0-1,s-t}} p(n-n_0-30|\pi'|-|\pi''|).$$

Hence the size of image of Case 2(B) is given by

$$\sum_{n_0\geq 96} \sum_{s=0}^{n_0-1} \sum_{t=0}^{\min(s,93)} \sum_{\pi'\in T'_t} \sum_{\pi''\in T''_{n_0-1,s-t}} (-1)^s p(n-n_0-30|\pi'|-|\pi''|). \tag{5.5}$$

5.2.4 Size of the image in Case 2(C)

Case 2(C)(i): In the partitions in the image in Case 2(C)(i), 4 appears as a part $\frac{n_0-24}{4}$ times and 5 appears as a part 5 times. Removing all these parts of 4 and 5 gives a partition of $n-n_0-1$ with no 4 and 5. Thus, the image of Case 2(C)(i) is given by

$$\sum_{\substack{n_0\geq 96,\\ n_0\equiv 0(4)}} \Bigg| C_{1,28}(n-1-n_0)\cap C_{4,0}(n-1-n_0)\cap C_{5,0}(n-1-n_0) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n-1-n_0) \bigcap_{i=96}^{n_0-1} C_{i,0}(n-1-n_0)\Bigg|.$$

Similar arguments show that the size of the image in Cases 2(C)(ii), 2(C)(iii) and 2(C)(iv) is also the same, with the only change being in the congruence for n_0 in the index of summation. Thus the total size of the image in Case 2(C) (sum of all the four sizes of images in Cases 2(C)(i), 2(C)(ii), 2(C)(iii) and 2(C)(iv)) is given as

$$\sum_{n_0\geq 96} \Bigg| C_{1,28}(n-1-n_0)\cap C_{4,0}(n-1-n_0)\cap C_{5,0}(n-1-n_0) \bigcap_{\substack{i=2,\\ i\neq 4,5}}^{95} C_{i,29}(n-1-n_0) \bigcap_{i=96}^{n_0-1} C_{i,0}(n-1-n_0)\Bigg|,$$

which can be rewritten as

$$\sum_{n_0\geq 96} p(n-1-n_0) - \sum_{n_0\geq 96}\Bigg| B_{1,29}(n-1-n_0)\cup B_{4,1}(n-1-n_0)\cup B_{5,1}(n-1-n_0)$$
$$\bigcup_{\substack{i=2,\\ i\neq 4,5}}^{95} B_{i,30}(n-1-n_0)\bigcup_{i=96}^{n_0-1} B_{i,1}(n-1-n_0)\Bigg|.$$

Define

$$H_i = \begin{cases} B_{i,30}(n-1-n_0) & \text{if} \quad i\leq 95 \quad \text{and} \quad i\neq 1,4,5\\ B_{i,1}(n-1-n_0) & \text{if} \quad i>95 \quad \text{or} \quad i=4,5\\ B_{1,29}(n-1-n_0) & \text{if} \quad i=1\end{cases}$$

We have

$$\left|\bigcup_{i=1}^{n_0-1} H_i\right| = \sum_{s=1}^{n_0-1}(-1)^{s-1}\sum_{(i_1,i_2,\cdots i_s)\in T_{n_0-1,s}} \left|H_{i_1}\cap H_{i_2}\cap\cdots\cap H_{i_s}\right|.$$

First, in the interior sum, suppose that none of the numbers i_j is equal to 1. Further suppose t of the numbers i_j are less than or equal to 95 and different than 4 or 5 (say $i_{j_1}, i_{j_2}, \cdots i_{j_t}$). Then $H_{i_{j_l}} = B_{i_{j_l},30}(n-1-n_0)$ for $l\leq t$ and $H_{i_j} = B_{i_j,1}(n-1-n_0)$ for all other i_j. We need some more notation.

- Set $U'_t := T_{95,t}(;1,4,5)$. That is, U'_t denotes the set of partitions into t distinct parts with largest part at most 95, and no part $1,4$ or 5. Note that any partition in U'_t has at most 92 parts.

- Set $U''_{n,t} := T_{n,t}(;1,2,3,6,7,\cdots 95)$. That is, $U''_{n,t}$ denotes the set of partitions into t distinct parts with largest part at most n that do not contain $1,2,3,6,7,\cdots 95$ as a part. Notice that $U'_t\cap U''_{n,t}=\emptyset$.

We express any partition of $T_{n_0-1,s}(;1)$ as a partition of U'_t and a partition of $U''_{n_0-1,s-t}$. Thus

$$\sum_{(i_1,i_2,\cdots i_s)\in T_{n_0-1,s}(;1)} |H_{i_1}\cap H_{i_2}\cap\cdots\cap H_{i_s}| = \sum_{t=0}^{\min(s,92)}\sum_{\substack{\pi'\in U'_t,\\ \pi''\in U''_{n_0-1,s-t}}} p(n-1-n_0-30|\pi'|-|\pi''|).$$

Next, suppose that one of the numbers i_j is equal to 1. Then $i_s=1$ (since partitions are defined to be a weakly decreasing list). Further suppose t of the numbers i_j are less than or equal to 95 and different than 4 or 5 (say $i_{j_1}, i_{j_2}, \cdots i_{j_t}$). Then $H_{i_s} = B_{1,29}(n-1-n_0)$, $H_{i_{j_l}} = B_{i_{j_l},30}(n-1-n_0)$ for $1\leq l\leq t$ and $H_{i_j} = B_{i_j,1}(n-1-n_0)$ for all other i_j. For any partition of $T_{n_0-1,s}(1;)$, we remove the part of 1 and express the remaining partition as a partition of U'_t and a partition of $U''_{n_0-1,s-t-1}$. Thus

$$\sum_{(i_1,i_2,\cdots i_s)\in T_{n_0-1,s}(1;)} |H_{i_1}\cap H_{i_2}\cap\cdots\cap H_{i_s}|$$
$$= \sum_{t=0}^{\min(s-1,92)} \sum_{\pi'\in U'_t} \sum_{\pi''\in U''_{n_0-1,s-t-1}} p(n-30-n_0-30|\pi'|-|\pi''|).$$

Hence, the size of the image in Case 2(C) is given by

$$\sum_{n_0\geq 96}\sum_{s=0}^{n_0-1}\sum_{t=0}^{\min(s,92)}\sum_{\pi'\in U'_t}\sum_{\pi''\in U''_{n_0-1,s-t}} (-1)^s p(n-1-n_0-30|\pi'|-|\pi''|)$$
$$+\sum_{n_0\geq 96}\sum_{s=1}^{n_0-1}\sum_{t=0}^{\min(s-1,92)}\sum_{\pi'\in U'_t}\sum_{\pi''\in U''_{n_0-1,s-t-1}} (-1)^s p(n-30-n_0-30|\pi'|-|\pi''|). \quad (5.6)$$

We combine (5.3), (5.4), (5.5) and (5.6) to get an expression for $|X_n| = p(n) - p(n-5)$. Thus, for any $n \geq 132241$,

$$\begin{aligned}
p(n) = p(n-5) &+ \left\{ \sum_{m_0=1}^{4}\sum_{s=0}^{m_0}(-1)^s \left(\sum_{\pi\in S_{m_0,s+1}} p(n-30|\pi|) + \sum_{\pi\in S_{m_0,s}} p(n-5-30|\pi|)\right)\right.\\
&\left. + \sum_{m_0=6}^{95}\sum_{s=0}^{m_0-1}(-1)^s \left(\sum_{\pi\in S_{m_0,s+1}(;5)} p(n-30|\pi|) + \sum_{\pi\in S_{m_0,s+1}(5;)} p(n+145-30|\pi|)\right)\right\}1\\
&+ \left\{ \sum_{s=0}^{95}(-1)^s \left(\sum_{\pi\in T_{95,s}(;1,4,5)} p(n-5-30|\pi|) + \sum_{\pi\in T_{95,s}(1;4,5)} p(n-4-30|\pi|)\right)\right.\\
&+ \sum_{s=0}^{95}(-1)^s \left(\sum_{\pi\in T_{95,s}(4;1,5)} p(n-1-30|\pi|) + \sum_{\pi\in T_{95,s}(5;1,4)} p(n+140-30|\pi|)\right)\\
&+ \sum_{s=0}^{95}(-1)^s \left(\sum_{\pi\in T_{95,s}(1,4;5)} p(n-30|\pi|) + \sum_{\pi\in T_{95,s}(1,5;4)} p(n+141-30|\pi|)\right)\\
&\left. + \sum_{s=0}^{95}(-1)^s \left(\sum_{\pi\in T_{95,s}(4,5;1)} p(n+144-30|\pi|) + \sum_{\pi\in T_{95,s}(1,4,5;)} p(n+145-30|\pi|)\right)\right\}2(\mathrm{A})\\
&+ \left\{ \sum_{n_0\geq 96}\sum_{s=0}^{n_0-1}\sum_{t=0}^{\min(s,93)}\sum_{\pi'\in T'_t}\sum_{\pi''\in T''_{n_0-1,s-t}} (-1)^s p(n-n_0-30|\pi'|-|\pi''|)\right\}2(\mathrm{B})\\
&+ \left\{ \sum_{n_0\geq 96}\sum_{s=0}^{n_0-1}\sum_{t=0}^{\min(s,92)}\sum_{\pi'\in U'_t}\sum_{\pi''\in U''_{n_0-1,s-t}} (-1)^s p(n-1-n_0-30|\pi'|-|\pi''|)\right.\\
&\left. + \sum_{n_0\geq 96}\sum_{s=1}^{n_0-1}\sum_{t=0}^{\min(s-1,92)}\sum_{\pi'\in U'_t}\sum_{\pi''\in U''_{n_0-1,s-t-1}} (-1)^s p(n-30-n_0-30|\pi'|-|\pi''|)\right\}2(\mathrm{C})
\end{aligned}$$

To aide the reader, we recall the notation used above:

- $S_{n,s}$ denotes the set of partitions into s distinct parts with largest part n.
- $S_{n,s}(a_1, \cdots a_k; b_1, \cdots b_l)$ denotes the set of partitions into s distinct parts with largest part n that contain $a_1, \cdots, a_k$ as a part and do not contain $b_1, \cdots, b_l$ as a part.
- $T_{n,s}(a_1, \cdots a_k; b_1, \cdots b_l)$ denotes the set of partitions into s distinct parts with largest part at most n that contain $a_1, \cdots, a_k$ as a part and do not contain $b_1, \cdots, b_l$ as a part.
- $T'_t := T_{95,t}(; 1, 5)$.
- $T''_{n,t} := T_{n,t}(; 2, 3, 4, 6, 7, \cdots 95)$.
- $U'_t := T_{95,t}(; 1, 4, 5)$.
- $U''_{n,t} := T_{n,t}(; 2, 3, 6, 7, \cdots 95)$.

5.3 Cumulative recurrence relations for a general k

We use the same approach to get a recurrence for a general value of $k \geq 3$ by making some modifications in the map. Our aim again is to find a formula for $p(n)$ in terms of $p(n-k)$, $p(n-k-1)$, $p(n-k-2)$ etc. Let $z_k = \max(k(2k+1), (k^2-1)(k-1))$. Note that

$$z_k = \begin{cases} 21 & \text{if } \ k = 3 \\ (k^2-1)(k-1) & \text{if } \ k \geq 4. \end{cases}$$

Further, let

$$y_k = \frac{(k^2+k-1)z_k(z_k-1)}{2} + 1.$$

It is easy to see that $y_k < k^8$.

Let X_n denote the set of partitions of n that do not have k as a part and let Y_n denote the set of partitions of n that have k as a part. As before, we easily see that $|X_n| = p(n) - p(n-k)$. For a π in X_n, let f_i be the frequency of i in π. For $n \geq y_k$, we define the map $\psi : X_n \to Y_n$ as follows.

Case 1: Suppose there exists m such that $1 \leq m \leq z_k - 1$ and $f_m \geq k(k+1)$. Let m_0 be the least such number. Define

$$\psi(\pi) = (1^{f_1}, \ldots k^{(k+1)m_0}, \ldots m_0^{f_{m_0} - k(k+1)}, \ldots).$$

Case 2: Suppose $f_m \leq k(k+1) - 1$ for all $1 \leq m \leq z_k - 1$. Since $n \geq y_k$, there exists a $p \geq z_k$ such that $f_p \geq 1$. Let n_0 be the least such number.

Case 2(A): Suppose $f_1 \geq 1$ and $f_{k-1} \geq 1$. Define

$$\psi(\pi) = (1^{f_1-1}, \dots (k-1)^{f_{k-1}-1}, k^1, \dots).$$

Case 2(B): Suppose $f_1 = 0$. Define

$$\psi(\pi) = (1^{n_0-k(2k+1)}, \dots k^{2k+1}, \dots (z_k-1)^{f_{z_k-1}}, n_0^{f_{n_0}-1}, \dots).$$

Case 2(C): Suppose $f_1 > 0$ and $f_{k-1} = 0$. Let $n_0 \equiv r \pmod{k-1}$ for some $0 \leq r \leq k-2$. Define

$$\psi(\pi) = (1^{f_1-1}, \dots, (k-1)^{\frac{n_0-k^2(r+1)+1}{k-1}}, k^{k(r+1)}, \dots (z_k-1)^{f_{z_k-1}}, n_0^{f_{n_0}-1}, \dots).$$

Since the map ϕ is injective within each case and the frequency of k in the partition in the image is different for different cases, the map ϕ is injective, and thus $|X_n| \leq |Y_n|$. Since $|X_n| = p(n) - p(n-k)$ and $|Y_n| = p(n-k)$, we obtain $p(n) \leq 2p(n-k)$ whenever $n \geq y_k$. We refine the inequality obtained to an equality by measuring the size of $\phi(X_n)$ in each of the cases by using PIE. The image in the cases are given as follows.

- Case 1:
$$\bigsqcup_{m_0=1}^{z_k-1} \left(A_{k,(k+1)m_0}(n) \bigcap_{\substack{i=1,\\ i\neq k}}^{m_0-1} C_{i,k(k+1)-1}(n) \right).$$

- Case 2(A):
$$C_{1,k(k+1)-2}(n) \cap C_{k-1,k(k+1)-2}(n) \cap A_{k,1}(n) \bigcap_{\substack{i=2,\\ i\neq k,k-1}}^{z_k-1} C_{i,k(k+1)-1}(n).$$

- Case 2(B):
$$\bigsqcup_{n_0 \geq z_k} \left(A_{1,n_0-k(2k+1)}(n) \cap A_{k,2k+1}(n) \bigcap_{\substack{i=2,\\ i\neq k}}^{z_k-1} C_{i,k(k+1)-1}(n) \bigcap_{i=z_k}^{n_0-1} A_{i,0}(n) \right).$$

When $n_0 = z_k$, the last intersection in the above expression is vacuous and can be omitted.

- Case 2(C):

$$\bigsqcup_{r=0}^{k-2}\bigsqcup_{\substack{n_0\geq z_k,\\ n_0\equiv r(k-1)}}\Bigg(C_{1,k(k+1)-2}(n)\cap A_{k-1,\frac{n_0-k^2(r+1)+1}{k-1}}(n)\cap A_{k,k(r+1)}(n)$$
$$\bigcap_{\substack{i=2,\\ i\neq k,k-1}}^{z_k-1}C_{i,k(k+1)-1}(n)\bigcap_{i=z_k}^{n_0-1}A_{i,0}(n)\Bigg).$$

Proceeding as for the case $k=5$, the size of the image in Case 1 is equal to

$$\sum_{m_0=1}^{k-1}\sum_{s=0}^{m_0}(-1)^s\Bigg(\sum_{\pi\in S_{m_0,s+1}}p(n-k(k+1)|\pi|)+\sum_{\pi\in S_{m_0,s}}p(n-k-k(k+1)|\pi|)\Bigg)$$
$$+\sum_{m_0=k+1}^{z_k-1}\sum_{s=0}^{m_0-1}(-1)^s\Bigg(\sum_{\pi\in S_{m_0,s+1}(;k)}p(n-k(k+1)|\pi|)$$
$$+\sum_{\pi\in S_{m_0,s+1}(k;)}p(n-k+k^2(k+1)-k(k+1)|\pi|)\Bigg),\quad (5.7)$$

where

- $S_{n,s}$ denotes the set of partitions into s distinct parts with largest part n.
- $S_{n,s}(a_1,\cdots a_p;b_1,\cdots b_l)$ denotes the set of partitions into s distinct parts with largest part n that contain $a_1,\cdots,a_p$ as a part and do not contain $b_1,\cdots,b_l$ as a part.

Similarly, the size of image in Case 2(A) is given by

$$\sum_{s=0}^{z_k-1}(-1)^s\Bigg(\sum_{\pi\in T_{z_k-1,s}(;1,k-1,k)}p(n-k-k(k+1)|\pi|)+\sum_{\pi\in T_{z_k-1,s}(1;k-1,k)}p(n-k+1-k(k+1)|\pi|)$$
$$+\sum_{\pi\in T_{z_k-1,s}(k-1;1,k)}p(n-1-k(k+1)|\pi|)+\sum_{\pi\in T_{z_k-1,s}(k;1,k-1)}p(n+k(k(k+1)-2)-k(k+1)|\pi|)$$
$$+\sum_{\pi\in T_{z_k-1,s}(1,k-1;k)}p(n-k(k+1)|\pi|)+\sum_{\pi\in T_{z_k-1,s}(1,k;k-1)}p(n+k(k(k+1)-2)+1-k(k+1)|\pi|)$$
$$+\sum_{\pi\in T_{z_k-1,s}(k-1,k;1)}p(n+k(k(k+1)-1)-1-k(k+1)|\pi|)$$
$$+\sum_{\pi\in T_{z_k-1,s}(1,k-1,k;)}p(n+k(k(k+1)-1)-k(k+1)|\pi|)\Bigg),\quad (5.8)$$

where

- $T_{n,s}(a_1, \cdots a_p; b_1, \cdots b_l)$ denotes the set of partitions into s distinct parts with largest part at most n that contain $a_1, \cdots, a_p$ as a part and do not contain $b_1, \cdots, b_l$ as a part.

Similarly, the size of image in Case 2(B) is given by

$$\sum_{n_0 \geq z_k} \sum_{s=0}^{n_0-1} \sum_{t=0}^{\min(s,z_k-3)} \sum_{\pi' \in T'_t} \sum_{\pi'' \in T''_{n_0-1,s-t}} (-1)^s p(n - n_0 - k(k+1)|\pi'| - |\pi''|), \tag{5.9}$$

where

- $T'_t := T_{z_k-1,t}(;1,k)$.
- $T''_{n,t} := T_{n,t}(;2,3,\cdots k-1,k+1,k+2,\cdots,z_k-1)$. Note that here also we express any partition of $T_{n_0-1,s}$ as a partition of T'_t and a partition of $T''_{n_0-1,s-t}$.

Similarly, the size of the image in Case 2(C) is given by

$$\begin{aligned}&\sum_{n_0 \geq z_k} \sum_{s=0}^{n_0-1} \sum_{t=0}^{\min(s,z_k-4)} \sum_{\pi' \in U'_t} \sum_{\pi'' \in U''_{n_0-1,s-t}} (-1)^s p(n - 1 - n_0 - k(k+1)|\pi'| - |\pi''|) \\ &+ \sum_{n_0 \geq z_k} \sum_{s=1}^{n_0-1} \sum_{t=0}^{\min(s-1,z_k-4)} \sum_{\pi' \in U'_t} \sum_{\pi'' \in U''_{n_0-1,s-t-1}} (-1)^s p(n - k(k+1) - n_0 - k(k+1)|\pi'| - |\pi''|),\end{aligned} \tag{5.10}$$

where

- $U'_t := T_{z_k-1,t}(;1,k-1,k)$.
- $U''_{n,t} := T_{n,t}(;1,2,3,\cdots k-2,k+1,k+2,\cdots,z_k-1)$. Note that as for $k=5$, we express any partition of $T_{n_0-1,s}(;1)$ as a partition of U'_t and a partition of $U''_{n_0-1,s-t}$. Moreover, for any partition of $T_{n_0-1,s}(1;)$, we remove the part of 1 and express the remaining partition as a partition of U'_t and a partition of $U''_{n_0-1,s-t-1}$.

We combine (5.7), (5.8), (5.9) and (5.10) to get an expression for $|X_n| = p(n) - p(n-k)$. Thus, for any $k \geq 3$ and any $n \geq y_k$, we obtain

$$\begin{aligned}p(n) = p(n-k) + \Bigg\{ &\sum_{m_0=1}^{k-1} \sum_{s=0}^{m_0} \sum_{\pi \in S_{m_0,s+1}} (-1)^s p(n - k(k+1)|\pi|) \\ &+ \sum_{m_0=1}^{k-1} \sum_{s=0}^{m_0} \sum_{\pi \in S_{m_0,s}} (-1)^s p(n - k - k(k+1)|\pi|) \\ &+ \sum_{m_0=k+1}^{z_k-1} \sum_{s=0}^{m_0-1} \sum_{\pi \in S_{m_0,s+1}(;k)} (-1)^s p(n - k(k+1)|\pi|)\end{aligned}$$

$$
\left.+\sum_{m_0=k+1}^{z_k-1}\sum_{s=0}^{m_0-1}\sum_{\pi\in S_{m_0,s+1}(k;)}(-1)^s p(n-k+k^2(k+1)-k(k+1)|\pi|)\right\}\text{Case 1}
$$

$$
\begin{aligned}
2(\mathrm{A})\Bigg\{ &+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(;1,k-1,k)}(-1)^s p(n-k-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(1;k-1,k)}(-1)^s p(n-k+1-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(k-1;1,k)}(-1)^s p(n-1-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(k;1,k-1)}(-1)^s p(n+k(k(k+1)-2)-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(1,k-1;k)}(-1)^s p(n-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(1,k;k-1)}(-1)^s p(n+k(k(k+1)-2)+1-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(k-1,k;1)}(-1)^s p(n+k(k(k+1)-1)-1-k(k+1)|\pi|)\\
&+\sum_{s=0}^{z_k-1}\sum_{\pi\in T_{z_k-1,s}(1,k-1,k;)}(-1)^s p(n+k(k(k+1)-1)-k(k+1)|\pi|)\Bigg\}
\end{aligned}
$$

$$
2(\mathrm{B})\left\{+\sum_{n_0\ge z_k}\sum_{s=0}^{n_0-1}\sum_{t=0}^{\min(s,z_k-3)}\sum_{\substack{\pi'\in T'_t,\\ \pi''\in T''_{n_0-1,s-t}}}(-1)^s p(n-n_0-k(k+1)|\pi'|-|\pi''|)\right\}
$$

$$
\begin{aligned}
2(\mathrm{C})\Bigg\{ &+\sum_{n_0\ge z_k}\sum_{s=0}^{n_0-1}\sum_{t=0}^{\min(s,z_k-4)}\sum_{\substack{\pi'\in U'_t,\\ \pi''\in U''_{n_0-1,s-t}}}(-1)^s p(n-1-n_0-k(k+1)|\pi'|-|\pi''|)\\
&+\sum_{n_0\ge z_k}\sum_{s=1}^{n_0-1}\sum_{t=0}^{\min(s-1,z_k-4)}\sum_{\substack{\pi'\in U'_t,\\ \pi''\in U''_{n_0-1,s-t-1}}}(-1)^s p(n-n_0-k(k+1)(1+|\pi'|)-|\pi''|).\Bigg\}
\end{aligned}
$$

5.4 Recurrences for partitions with bounded maximum part

Recall from Chapter 1 that for a given natural number m, the quantity $p_m(n)$ denotes the number of partitions of n with largest part at most m. The quantity $p_m(n)$ is well known and the simple conjugation map for Ferrers diagrams provides a bijective proof showing that $p_m(n)$ is also equal to the number of partitions of n with at most m parts. Our goal is to find some recurrences for $p_m(n)$ similar to those for $p(n)$ above. However, here the case analysis is much simpler than for the recurrences above. The recurrence for $p_m(n)$ also helps us to prove a nice result related to Ramanujan's congruences for integer partitions.

Analogous to our notation before, we require the following notation. For fixed m, we define the following sets.

- $A'_{j,k}(n)$ denotes the set of partitions of n with largest part at most m, having exactly k parts of size j.
- $B'_{j,k}(n)$ denotes the set of partitions of n with largest part at most m, having at least k parts of size j.
- $C'_{j,k}(n)$ denotes the set of partitions of n with largest part at most m, having at most k parts of size j.
- $T_{n,s}$ denotes the set of partitions into s distinct parts with largest part at most n.
- $S_{n,s}$ denotes the set of partitions into s distinct parts with largest part n.
- $S_{n,s}(a_1, \cdots a_k; b_1, \cdots b_l)$ denotes the set of partitions into s distinct parts with largest part n that contain $a_1, \cdots, a_k$ as a part and do not contain $b_1, \cdots, b_l$ as a part.

Theorem 5.1. *Let $m > 5$ and $n > 2m(m+1) - 20$ be positive integers. Then the following recurrence relation holds for $p_m(n)$:*

$$\begin{aligned} p_m(n) = p_m(n-5) &+ \sum_{i_0=1}^{4} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s+1}} (-1)^s p_m(n - 5|\pi|) \\ &+ \sum_{i_0=1}^{4} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s}} (-1)^s p_m(n - 5 - 5|\pi|) \\ &+ \sum_{i_0=6}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(;5)} (-1)^s p_m(n - 5|\pi|) \\ &+ \sum_{i_0=6}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(5;)} (-1)^s p_m(n + 20 - 5|\pi|). \end{aligned}$$

Proof. Let X_n denote the set of partitions of n with largest part at most m and no part equal to 5, and let Y_n denote the set of partitions of n with largest part at most m and with 5 as a part. We define an injective map $\phi : X_n \to Y_n$ and find the size of the image.

Let $\pi = (1^{f_1}, 2^{f_2}, \ldots 5^0, \ldots m^{f_m}) \in X_n$. Since n is large enough, there exists an $1 \le i \le m$ and $i \ne 5$ such that $f_i \ge 5$. Let i_0 be the least such number. Then define

$$\phi(\pi) = (1^{f_1}, \ldots 5^{i_0}, \ldots i_0^{f_{i_0}-5}, \ldots).$$

The image in this case can be described as

$$\bigsqcup_{\substack{i_0=1,\\ i_0\neq 5}}^{m} \left(A'_{5,i_0}(n) \bigcap_{\substack{j=1,\\ j\neq 5}}^{i_0-1} C'_{j,4}(n) \right).$$

The size of the image is equal to

$$\begin{aligned}
&\sum_{\substack{i_0=1,\\ i_0\neq 5}}^{m} \left| A'_{5,i_0}(n) \bigcap_{\substack{j=1,\\ j\neq 5}}^{i_0-1} C'_{j,4}(n) \right| \\
&= \sum_{\substack{i_0=1,\\ i_0\neq 5}}^{m} \left| A'_{5,0}(n-5i_0) \bigcap_{\substack{j=1,\\ j\neq 5}}^{i_0-1} C'_{j,4}(n-5i_0) \right| \\
&= \sum_{\substack{i_0=1,\\ i_0\neq 5}}^{m} p_m(n-5i_0) - \left| B'_{5,1}(n-5i_0) \bigcup_{\substack{j=1,\\ j\neq 5}}^{i_0-1} B'_{j,5}(n-5i_0) \right|.
\end{aligned}$$

First suppose $i_0 \geq 6$. Define H_j as

$$H_j = \begin{cases} B'_{j,5}(n-5i_0) & \text{if} \quad j \neq 5 \\ B'_{5,1}(n-5i_0) & \text{if} \quad j = 5. \end{cases}$$

We have

$$\left| \bigcup_{j=1}^{i_0-1} H_j \right| = \sum_{s=1}^{i_0-1} (-1)^{s-1} \sum_{(j_1,j_2,\cdots j_s)\in T_{i_0-1,s}} \left| H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s} \right|.$$

If none of the numbers j_t's in the sum is equal to 5, then

$$|H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}| = p_m(n - 5i_0 - 5j_1 - 5j_2 \cdots - 5j_s).$$

If one of the numbers j_t in the sum is 5, say $j_l = 5$, then

$$\begin{aligned}
|H_{j_1} \cap H_{j_2} \cap \cdots \cap H_{j_s}| &= p_m(n - 5i_0 - 5 - 5j_2 - 5j_3 \cdots - 5j_{l-1} - 5j_{l+1} \cdots - 5j_s) \\
&= p_m(n + 20 - 5i_0 - 5j_1 - 5j_2 \cdots - 5j_s).
\end{aligned}$$

Hence

$$\left| \bigcup_{i=1}^{i_0-1} H_j \right| = \sum_{s=1}^{i_0-1} (-1)^{s-1} \left(\sum_{\pi\in S_{i_0,s+1}(;5)} p_m(n - 5|\pi|) + \sum_{\pi\in S_{i_0,s+1}(5;)} p_m(n + 20 - 5|\pi|) \right).$$

Thus, the size of the image in the case $i_0 \geq 6$ is given by

$$\sum_{i_0=6}^{m}\sum_{s=0}^{i_0-1}\sum_{\pi\in S_{i_0,s+1}(;5)}(-1)^s p_m(n-5|\pi|)+\sum_{i_0=6}^{m}\sum_{s=0}^{i_0-1}\sum_{\pi\in S_{i_0,s+1}(5;)}(-1)^s p_m(n+20-5|\pi|). \quad (5.11)$$

Next suppose $1 \leq i_0 \leq 4$. Define

$$H_j = \begin{cases} B_{j,5}(n-5i_0) & \text{if} \quad 1 \leq j \leq i_0 - 1 \\ B_{5,1}(n-5i_0) & \text{if} \quad j = i_0. \end{cases}$$

We have

$$\left|\bigcup_{j=1}^{i_0} H_j\right| = \sum_{s=1}^{i_0}(-1)^{s-1}\sum_{(j_1,j_2,\cdots j_s)\in T_{i_0,s}}|H_{j_1}\cap H_{j_2}\cap\cdots\cap H_{j_s}|.$$

If none of the numbers j_t's is equal to i_0, then

$$|H_{j_1}\cap H_{j_2}\cap\cdots\cap H_{j_s}| = p_m(n-5i_0-5j_1-5j_2\cdots-5j_s).$$

If one of the numbers j_t is equal to i_0, then $j_1 = i_0$, and thus

$$|H_{j_1}\cap H_{j_2}\cap\cdots\cap H_{j_s}| = p_m(n-5i_0-5j_2\cdots-5j_s-5).$$

Hence

$$\left|\bigcup_{j=1}^{i_0} H_j\right| = \sum_{s=1}^{i_0}(-1)^{s-1}\left(\sum_{\pi\in S_{i_0,s+1}} p_m(n-5|\pi|)+\sum_{\pi\in S_{i_0,s}} p_m(n-5-5|\pi|)\right).$$

Thus, the size of the image in the case $i_0 \leq 4$ is given by

$$\sum_{i_0=1}^{4}\sum_{s=0}^{i_0}\sum_{\pi\in S_{i_0,s+1}}(-1)^s p_m(n-5|\pi|)+\sum_{i_0=1}^{4}\sum_{s=0}^{i_0}\sum_{\pi\in S_{i_0,s}}(-1)^s p_m(n-5-5|\pi|). \quad (5.12)$$

Since the map ϕ is injective, $|X_n| = |\phi(X_n)|$. As before, we easily see that $|X_n| = p_m(n) - p_m(n-5)$. Summing (5.11) and (5.12), we obtain the size of $|\phi(X_n)|$. This completes the proof of Theorem 5.1. □

For brevity, for $n > 2m(m+1) - 20$, we rewrite the recurrence relation in Theorem 5.1 as follows.

$$p_m(n) = \sum_{l=1}^{L(m)} a_l p_m(n-5l), \quad (5.13)$$

where $L(m)$ is a fixed number depending on m, and a_l is some integer. First, we estimate the value of $L(m)$. Using Theorem 5.1, it is clear that

$$L(m) = \max(M_1, M_2, M_3, M_4),$$

where

$$\begin{aligned}
M_1 &= \max\{|\pi| : 1 \le i_0 \le 4, 0 \le s \le i_0, \pi \in S_{i_0,s+1}\},\\
M_2 &= \max\{|\pi| + 1 : 1 \le i_0 \le 4, 0 \le s \le i_0, \pi \in S_{i_0,s}\},\\
M_3 &= \max\{|\pi| : 6 \le i_0 \le m, 0 \le s \le (i_0 - 1), \pi \in S_{i_0,s+1}(;5)\},\\
M_4 &= \max\{|\pi| - 4 : 6 \le i_0 \le m, 0 \le s \le (i_0 - 1), \pi \in S_{i_0,s+1}(5;)\}.
\end{aligned}$$

It is easy to verify that $M_1 = 10$ and $M_2 = 11$. To calculate M_3, observe that the maximum is achieved when $i_0 = m$ and $s = m - 2$, and $\pi = \{m, (m-1), \dots, 6, 4, 3, 2, 1\}$. Thus, $M_3 = \frac{m(m+1)}{2} - 5$. Similarly for M_4, the maximum is achieved when $i_0 = m$ and $s = m - 1$, and $\pi = \{m, (m-1), \dots, 1\}$. Thus $M_4 = \frac{m(m+1)}{2} - 4$. Since $m \ge 6$, it is clear that

$$L(m) = M_4 = \frac{m(m+1)}{2} - 4.$$

Since the maximum is achieved when $s = m - 1$, we have $a_{L(m)} = (-1)^{m-1}$. Next, we use our recurrence in Theorem 5.1 to prove Theorem 5.2.

Theorem 5.2. *Let $m \ge 6$ be a fixed positive integer. Then $p_m(5n+4)$ is not divisible by 5 for infinitely many positive integers n.*

Proof. First, we show that $p_m(5n+4)$ is not divisible by 5 for some n. Let m' be the smallest number strictly larger than m that is congruent to 4 modulo 5. Suppose that $m \equiv 0 \pmod 5$, so $m' = m+4$, and from there we see $p_m(m') = p(m') - 7$. This follows from the observation that the partitions of $m+4$ that have a part larger than m are given by $m+4$, $(m+3)+1$, $(m+2)+2$, $(m+2)+1+1$, $(m+1)+3$, $(m+1)+2+1$, and $(m+1)+1+1+1$. We obtain similar results when m is from some other congruence class modulo 5. To summarize, we obtain:

- If $m \equiv 0 \pmod 5$, then $m' = m + 4$, then $p_m(m') = p(m') - 7$.
- If $m \equiv 1 \pmod 5$, then $m' = m + 3$, then $p_m(m') = p(m') - 4$.
- If $m \equiv 2 \pmod 5$, then $m' = m + 2$, then $p_m(m') = p(m') - 2$.
- If $m \equiv 3 \pmod 5$, then $m' = m + 1$, then $p_m(m') = p(m') - 1$.
- If $m \equiv 4 \pmod 5$, then $m' = m + 5$, then $p_m(m') = p(m') - 12$.

By Ramanujan's congruence, we know that $p(m')$ is divisible by 5, and thus $p_m(m')$ is not divisible by 5. Next, suppose to the contrary that there are only finitely many values of n such that $p_m(5n+4)$ is not divisible by 5. Let n' be the largest n of the form 4 modulo 5 such that $p_m(n')$ is not divisible by 5. Note that $5L(m) = 5^{\frac{m(m+1)}{2}} - 20 > 2m(m+1) - 20$. Therefore, using (5.13), we obtain

$$\begin{aligned} p_m(n' + 5L(m)) &= \sum_{l=1}^{L(m)} a_l p_m(n' + 5L(m) - 5l) \\ &= \left(\sum_{l=1}^{L(m)-1} a_l p_m(n' + 5L(m) - 5l) \right) + (-1)^{m-1} p_m(n'). \end{aligned}$$

We rewrite this as:

$$(-1)^{m-1} p_m(n') = p_m(n' + 5L(m)) - \sum_{l=1}^{L(m)-1} a_l p_m(n' + 5(L(m) - l)).$$

Since n' is the largest number of the form 4 modulo 5 for which $p(n')$ is not divisible by 5, the left hand side of the above equation is not divisible by 5, whereas the right hand side is divisible by 5, giving a contradiction. Hence, there are infinitely many values of n for which $p_m(5n+4)$ is not divisible by 5.

□

We note that it is easy to generalize these recurrences and congruences for any natural number k instead of 5. First, we note that Theorem 5.1 generalizes to the following result.

Theorem 5.3. *Let k and $m > k$ be positive integers, and $n > \binom{k-1}{2} m(m+1) - k(k-1)$ be a positive integer. Then the following recurrence relation holds for $p_m(n)$:*

$$\begin{aligned} p_m(n) = p_m(n-k) &+ \sum_{i_0=1}^{k-1} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s+1}} (-1)^s p_m(n - k|\pi|) \\ &+ \sum_{i_0=1}^{k-1} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s}} (-1)^s p_m(n - k - k|\pi|) \\ &+ \sum_{i_0=k+1}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(;k)} (-1)^s p_m(n - k|\pi|) \\ &+ \sum_{i_0=k+1}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(k;)} (-1)^s p_m(n - k + k^2 - k|\pi|). \end{aligned}$$

We can generalize the proof of Theorem 5.2, and using the recurrence in Theorem 5.3, we obtain the following general result.

Theorem 5.4. *For given positive integers k, l, r and m such that $m > k$ and $r < k$, either $p_m(kn+r)$ is divisible by l for all integers n, or $p_m(kn+r)$ is not divisible by l for infinitely many positive integers n.*

Recall the following notation from Chapter 1. For a given natural number m:

- Let $p_m(n)$ denote the number of partitions of n with largest part at most m;
- Let $p_{=m}(n)$ denote the number of partitions of n with largest part exactly m;
- Let $p(n, \leq m)$ denote the number of partitions of n with at most m parts;
- Let $p(n, m)$ denote the number of partitions of n with exactly m parts.

Then, by the conjugation map (for Ferrers diagram), we have $p(n, \leq m) = p_m(n)$, and thus $p(n, \leq m)$ also satisfies Theorem 5.3 and Theorem 5.4. Moreover, by the conjugation map (for Ferrers diagram), we have $p(n, m) = p_{=m}(n)$. Further, an easy bijection ([Aig07, Page 32]) shows that $p(n, m) = p_m(n - m)$. Thus $p(n, m)$ and $p_{=m}(n)$ also satisfy results similar to Theorems 5.3 and 5.4. These results are stated below.

Theorem 5.5. *Let k and $m > k$ be positive integers, and $n > \binom{k-1}{2} m(m+1) - k(k-1) + m$ be a positive integer. Then the following recurrence relation holds for $p(n, m)$:*

$$\begin{aligned}
p(n,m) = p(n-k,m) &+ \sum_{i_0=1}^{k-1} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s+1}} (-1)^s p(n-k|\pi|, m) \\
&+ \sum_{i_0=1}^{k-1} \sum_{s=0}^{i_0} \sum_{\pi \in S_{i_0,s}} (-1)^s p(n-k-k|\pi|, m) \\
&+ \sum_{i_0=k+1}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(;k)} (-1)^s p(n-k|\pi|, m) \\
&+ \sum_{i_0=k+1}^{m} \sum_{s=0}^{i_0-1} \sum_{\pi \in S_{i_0,s+1}(k;)} (-1)^s p(n-k+k^2-k|\pi|, m).
\end{aligned}$$

Theorem 5.6. *For given positive integers k, l, r and m such that $m > k$ and $r < k$, either $p(kn+r, m)$ is divisible by l for all integers n, or $p(kn+r, m)$ is not divisible by l for infinitely many positive integers n.*

5.5 Future directions

It would be interesting to classify those values of k, l, r and m, where $m > k$ and $r < k$, for which $p_m(kn+r)$ is divisible by l for all positive integers n. Then, by Theorem 5.4, for the remaining values of k, l, r and m, we find that $p_m(kn+r)$ is not divisible by l for infinitely many positive integers n. The same question can also be asked about $p(kn+r, m)$. However, the answers might be related because of the relation $p(n, m) = p_m(n - m)$.

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