Informal Spoken Language in a Mathematics Classroom: How High-School Students Talk About Solving for x

by

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Abstract

In secondary mathematics education in British Columbia, written communication is recognized as the dominant form of mathematical language, while little emphasis has been placed on spontaneous, spoken, peer-to-peer language. This prioritizing misses out on the opportunity to see student thinking through their informal speech. The purpose of this thesis is to attend to what students say to each other when they talk about the doing of mathematics in small groups. In particular, I seek to respond to the question, "What informal terms do students use in their spoken language while solving algebra equations in small groups together?" I recorded student conversations as they solved algebra questions, transcribed their discussions, and categorized aspects of their language. I found that students used a variety of terms outside of the mathematics register, terms that demonstrated different implications of mathematics and exhibited singular features of language. Furthermore, I discovered that when students worked with one another, they consistently used metaphorical language to express mathematical operations and objects.

Keywords: high-school mathematics; spontaneous discourse; mathematics register; elements of language; metaphor; algebra equations

To my family, who inspired me to love learning.

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Chapter 1. Introduction

One day, during my early years as a secondary mathematics teacher, I overheard one student say to another, ‘What you do to one side you *have* to do to the other.’ Normally, I would have ignored this seemingly common, even innocuous, statement, but that day this student said a phrase that I had used earlier in the week with my exact tone and intonation. It was like having a parrot echo back my words. At the time, I found this both amusing and a little unnerving as I recognized the power that I had to influence my students’ thinking and the manner in which they communicated mathematical concepts. However, beyond that brief noticing, I thought little of it and went back to helping other students in the class.

I remembered this incident a few years later at the start of a new school year. During a discussion in a grade 12 class, one of my students paused. ‘Ms. Richards,’ she said, ‘I just realized that you will be my only math teacher through my whole time in high-school!’ Two other students in the class chimed in with the same realization. I smiled at them, made a passing remark, and continued with the lesson. But, after they left, I thought about the ramifications of these students not experiencing any other high-school mathematics teacher. Furthermore, as I remembered the parroted words from that student years before, I realized that these three had not heard any other high-school teacher’s words for mathematical processes, notions, and ideas. The language that I used and the ways that I talked about mathematical concepts had been predominant in shaping their formative mathematical understanding in the last four years. How had that impacted the way they talk about mathematics? And how had my words affected their comprehension of mathematical concepts?

Amidst these two incidents, I regularly encountered frustrated students in my classes when it came to my teaching of algebra. Whether I was working with grade 9 students, fresh out of middle school, or Pre-Calculus 12 students, I could not get away from the forgetfulness that happened with them whenever we started a new unit involving algebra. Whether there was a basic mathematical operation (like addition or subtraction) or more complex concepts (like dealing with rational or exponential functions) there seemed to be a collective memory loss for many of my students.

Early on in a new unit, I overheard questions like, ‘Do we move this over there now?’ or ‘Do you remember how we can get rid of that?’ or ‘Where do we even start?’ Students were frustrated as they had to relearn how to apply basic algebra operations, and I wondered how they had managed to forget these actions so quickly. As I looked closer at their questions, I wondered how their language around algebra was impacting their ability to understand the equations. Furthermore, I pondered if those words and expressions were helping them or inhibiting them from being able to tackle algebra questions with confidence and to remember how to solve these mysterious equations.

It was with some of these questions and others in mind that I started the master's program at Simon Fraser University. Although I quickly targeted *student language-use in algebra* as a topic I wanted to explore, it was not until we studied John Mason's (2002) book *Researching your own practice: The discipline of noticing* that I started to notice both my students' words in algebra and my own. When I paid attention, I heard them use words I had never used before. Some brought in metaphorical language that surprised me, while others used colloquial terms that I would have never associated with algebra. I noticed that many students were not using mathematical language (in the sense of the mathematics register) when they talked to each other. Furthermore, I realized that the words I often used were not explicitly mathematical either. Through my observations, I recognized that both my students and I relied heavily on metaphor to talk about the doing of high-school-level mathematics, especially as the concepts and their operations became increasingly complex or abstract.

What drew my interest the most was trying to identify the variety of metaphors that my students used as they worked to solve algebraic equations, specifically when they spoke with one another as they solved equations in small groups. As pointed out to me by my supervisor Dr. David Pimm, students seemed to talk *around* mathematical language in the mathematics register when they solved equations instead of speaking directly about the mathematical operations they were performing. The more I listened to student conversations, the more I heard this practice happening as they worked on algebra equations. Although mathematical language was not absent from their discussions, it was not central. In wanting to look further into this topic and learn more about what terms my students used to discuss mathematics, I started to develop a question to guide my study: what oral terms do students use to discuss mathematics?

These various questions connected to my studies as an undergraduate student where I was drawn both to mathematics and to English. People would comment in surprise when they found out I was studying both subjects, because they viewed the two fields as significantly different. However, as I continue to study mathematics and English, I am increasingly convinced of the deep connections between them. When I was hired as a teacher, I predominantly taught mathematics, but I had not forgotten my interests in English. I was curious about how language affected student understanding of mathematics and my students' abilities to communicate their practices and learning. After coming to the realization that metaphor was the means so many students and teachers were drawing upon to talk about algebra, I got excited. It had never occurred to me that a pillar of my English studies – metaphorical language – could be applicable to my work as a mathematics teacher. I wanted to learn more to uncover the extent of my students' use of metaphor and try to make sense of what their language was 'saying' about mathematics.

Once I started exploring the topic of metaphor in mathematics, I quickly discovered that I was not the only person interested in the intersection of these two ideas. David Pimm leant me a copy of his 1987 book *Speaking mathematically* shortly after I determined my thesis question, and I spent the summer of 2018 immersing myself in the differences between spoken and written forms of communication, ideas about the mathematics register, and the role of metaphor in mathematical discourse. Reading statements such as, "Metaphor and analogy are figures of speech which make natural language powerful, and I suggest that there are comparable processes at work in mathematics itself, as well as metaphor being commonly employed in its teaching" (p. 93) felt like they were being spoken directly to me and the work that I wanted to do for my thesis. The scope of this field came alive for me even more when I was invited to attend a festschrift for David Pimm in early March 2020, where I became aware of how important *Speaking mathematically* was for the many scholars and academics who attended the event.

Another significant influence that started to impact my understanding of metaphor deeply, both within and beyond mathematics, was George Lakoff and Mark Johnson's (2003) book *Metaphors we live by*. Originally published in 1980 (and referenced throughout Pimm's work), Lakoff and Johnson make the claim that metaphor is inescapable in our daily realities, not just through language, but also through our thinking and actions. Three of their key ideas regarding metaphor opened my eyes to the potential that their work had in the field of mathematics:

1. Abstract thinking is primarily metaphorical (though not completely).
2. Metaphorical thought is universal, it is inevitable, and it happens without us being aware of it.
3. Without metaphor, abstract concepts are incomplete. (p. 272)

At the upper levels of high-school mathematics, the concepts become more and more abstract, so Lakoff and Johnson's thesis had a significant impact on how I viewed not just my students' discourse, but also the way that they were thinking about the concepts they were learning. Although I was not able to explore the depth of student mathematical thinking in this study, Lakoff and Johnson's ideas became a lens through which I listened to my students' conversations.

This thesis is a result of my explorations into the terms my students used while solving algebraic equations. Chapter 2 is my literature review where I look at research focusing on different forms of mathematical communication, the mathematics register, and mathematical metaphors. Chapter 3 describes the methodology I used to respond to my research question and how I developed this study with a group of Pre-Calculus 11 students. Chapter 4 addresses the analysis of four specific student groups (with three or four students in each one), paying attention to the mathematical conversations that took place among them and singling out interesting language that they used. Chapter 5 explores the different categories of words I found throughout all student conversations and discusses some specific metaphors regularly used in student discourse. Finally, Chapter 6 addresses my conclusions and further considerations.

Chapter 2. Literature Review

The relationship between language and mathematics is complex and intimate. Like any field of study, mathematics has its own disciplinary literacy with “discursive norms” (Kleve & Penn, 2016, p. 287) for reading, writing, and speaking that are specific to the content area (Lemke, 1990; Halliday & Martin, 1993; Martin & Veel, 1998; Schleppegrell, 2004; Coffin, 2006; Kleve & Penn, 2016). To navigate properly through and appropriately communicate within the field, learners must develop their ability to make effective sense of and use of mathematical discourse. However, this subject-specific literacy is not innately known and must be taught (Heath, 1983; Temple & Doerr, 2012; Kleve & Penn, 2016), unlike the effortlessly occurring student-language that mimics the ways students communicate outside the classroom. For both the learner and teacher, there is significant tension between the subject literacy and student-language, along with the role each form of discourse plays in learning mathematics.

What, then, is all involved in the intricate process of developing knowledge of new concepts while also acquiring the language used to teach those concepts? Language is both the medium through which ideas are known, as well as the path by which they are communicated. It is through language that knowledge is constructed, so literacy development within a subject area is a way of knowing the concepts particular to that field, as is the case in mathematics (Gee, 1996; Cope & Kalantzis, 1997; Kleve & Penn, 2016). Thus, through this chapter, I look at the various ways that different forms of language and communication interact with one another in the field of mathematics and mathematical learning. While the relationships between linguistic forms and contexts is complex, and cannot be fully explored here, I introduce some significant voices and influences working in these specific fields. Lastly, I take time to discuss the varied and multifaceted relationship between metaphor and the language of mathematics as it pertains to this study.

2.1. Language and mathematics

Although several scholars argue that mathematics is not a language in its own right, it has been compared to other recognized languages because it has its own meaning-making system (Halliday, 1978; Pimm, 1987; Temple & Doerr, 2012). It is also a field that requires learning language in a discourse-specific way to articulate it fully and communicate effectively with others. As involving a language system, mathematics incorporates a variety of subject-specific discourse only found in mathematics along with other more common terminology. Temple and

Doerr (2012) state that mathematical language involves “the use of specialized terms, the use of everyday words with specialized meanings, specialized expressions, and particular sentence constructions less common in everyday speech” (p. 289). Thus, mathematics blends a variety of terms and structures only found within the field along with many terms and structures found outside of it.

As in any subject area, students need to learn the language of mathematics alongside development of new concepts. However, some studies have shown that when teachers use the specific vocabulary and language of mathematics, it can inhibit students’ ability to understand new concepts because they do not understand the words being used (Adler, 1999; Temple & Doerr, 2012). Furthermore, one of the greatest difficulties is for students to engage with the multi-semiotic nature of the mathematical language, since there are so many ways in which meaning is made within the discipline (Schleppegrell, 2007; Temple & Doerr, 2012). Some of those semiotic systems form dichotomies between informal and formal language, spoken and written communication, and teacher-centred and peer-to-peer discourse. In reality, all of these relationships are on a spectrum, and I will explore them below, along with the mathematics register and the language of algebra.

Informal and formal language

In mathematics class, student development of the language of mathematics is generally seen as a movement from expressing mathematical ideas in their everyday jargon to communicating with the more formal and conventional mathematical language (Barwell, 2015). An explicit example of this expected shift comes from the 2005 Ontario elementary mathematics curriculum where students are expected to use “everyday language” and “basic” vocabulary in younger grades, which is replaced by “mathematical vocabulary” in later elementary grades, suggesting movement from one form to the other. Comparatively, the 2017 British Columbia mathematics curriculum lists as a curricular competency for all grades, “Use mathematical vocabulary and language to contribute to mathematical discussions” (https://curriculum.gov.bc.ca, 2017). While this phrasing is not grounded in the shift that the Ontario curriculum articulates, it still indicates that there is a difference between mathematical and non-mathematical language. Furthermore, these examples indicate a clear distinction between informal (or basic) and formal (or mathematical) language, as well as a valuing of formal language with its (expected) rigour and regulatory aims that seek to remove ambiguity (Caspi & Sfard, 2012). Students see the relationship between these two forms of language as anywhere from a dichotomy to a continuum, both with hierarchical implications (Gutiérrez et al., 2010; Barwell, 2015) since they are expected to move from one linguistic form to another throughout their time in school.

Informal language is generally considered less articulate and precise (both descriptors more often associated with spoken communication). Furthermore, it is part of the learning process as students work to understand new concepts and map what they already know onto new things they are seeking to understand. In such situations, informal language provides a way for students to process new concepts using language that is familiar to them, even if it does not fit within the mathematics register or is somewhat imprecise or even vague (Lakoff & Núñez, 1997).

Conversely, formal mathematical language has been viewed by many as the idealized form of communication in mathematics, and, thus, students are expected to learn how to use it, going beyond their every day, informal language. Barwell (2015) states his view of what formal mathematical language is: “I interpret formal mathematical language as referring to the standard language used to talk about mathematics, which encodes the meanings of mathematics (and which is sometimes referred to as the mathematics register)” (p. 333). I will talk more about the mathematics register in the next sub-section, but an important point that Barwell makes is that formal mathematical language has undergone some kind of standardization which includes words, symbolic representation, and supports certain kinds of structures. This formalized code should enable more mathematicians to communicate their ideas with one another across different natural languages and times.

Lakoff and Núñez (1997) look at a brief history of formalized mathematical language and note that many nineteenth- and twentieth-century mathematicians and philosophers were looking at the way proofs – which have historically been considered the highest form of formal mathematical communication – were constructed. They saw the “string of symbols” (p. 23) as a formal code for communication that could be understood by anyone who knew the code. They also paraphrase Gottlob Frege (1848-1925), a mathematician, logician, and philosopher, as saying that, “meaning could be reduced to truth and reference, and that truth and reference could be modeled using abstract symbols and sets” (p. 28). So, formalized mathematical language was largely viewed as symbolic notation that pointed to mathematical truths. We now have an awareness of the notion of the mathematics register to add to our understanding of formal language in the discipline.

The mathematics register

The most widely accepted form of formal language in mathematics is called the mathematics register. In his chapter 'Some aspects of sociolinguistics', Halliday (1975) presents the idea of the mathematics register. He defines what a register is and, therefore, what the mathematics register is:

> A register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. We can refer to a 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is used for mathematical purposes. (p. 65)

While the mathematics register is largely based within natural language, it does not exclusively consist of words; it is also made up of styles, structures, and modes of arguments (though I am only focusing on terms, words, and phrases in this thesis due to the nature of the data I collected). The emphasis of the register is the meanings it holds within its domain (i.e. mathematics), as it takes a particular collection of words or phrases and gives them discourse-specific meanings. Halliday also proposes the possibility of inventing words to add new meanings to the mathematics register but affirms that everyday words can also be given specialized meanings (some of which can be seen in the section below, entitled *Mathematical metaphors*).

While the mathematics register is considered a formalization of the language used to communicate and make meaning of mathematical concepts, it is not an unambiguous, sanitized language. As Pimm (1987) states, it has appropriated many words from common English which leaves space for nebulous meaning and misconception. He lists several examples where students indicated confusion between the mathematical meaning of a word and its meaning outside of the mathematics register (for example the *oddness* of a number or the *difference* between two numbers). Regardless of these possible misunderstandings, for students, "part of learning mathematics is learning to speak like a mathematician, that is, acquiring control over the mathematics register" (ibid., p. 76).

Spoken and written communication

Another dichotomy in mathematical language is found in the differences between spoken and written communication in mathematics classrooms. In his book *Speaking mathematically*, Pimm (1987) specifically addresses student talk in his chapters "Pupils' mathematical talk" and "Overt and covert classroom communication”. In the first chapter, he says that spoken language provides an opportunity for students to articulate, clarify, and organize their thoughts (p. 23). Furthermore, he notes that there are two primary reasons for students to speak in a mathematics class: to talk to communicate with others (peers or teacher) and to talk (to themselves) for their own reasoning and sense-making. The first refers to using spoken language as a way of passing on information or trying to make another person understand something; the second is a when a student speaks aloud to themselves as a means of processing and organizing their own thoughts (p. 38). (It is for this and other reasons that spoken communication is often associated with informal language.)

Through examples, Pimm explores various contexts in which it is valuable to develop students’ ability to orally communicate mathematical knowledge orally, especially when the purpose is what Brown (1982) calls “message-oriented speech” (p. 38), a form of speech that teachers need to teach explicitly. Message-oriented speech is when the speaker has a targeted message they are seeking to communicate to the listener, generally in an attempt to change the listener’s knowledge or understanding of that topic. At the end of his account, Pimm still asks in what ways spoken language may be a better vehicle (or not) for certain contexts and forms of mathematical communication.

In contrast to spoken communication, the written form is generally more structured and formal. In their article ‘Learning and assessing mathematics through reading and writing’, Bossé and Faulconer (2008) report that often the mathematical writing in classroom settings seeks to emulate styles and structures of the textbooks that students use. Unlike when speaking to communicate mathematical ideas, there is often the expectation that written communication “requires a solid understanding of numeric, symbolic, graphical, and verbal representations” (p. 9) in order to communicate accurately and precisely. Students need to be taught how to write mathematically as it requires specific instructions and a strong grasp of particular vocabulary and symbolization. However, even with the barriers that students experience around mathematical writing, Bossé and Faulconer quote numerous studies that show the value of mathematical writing as it helps to deepen mathematical understanding for students when they engage in writing mathematics. It can lead to increased knowledge of content, and also provide a foundation for “recognizing and generating connections” (p. 9).

Language of algebra

Due to its symbolic nature, algebra is naturally considered formalized written language; Caspi and Sfard (2012) refer to symbolic notation as the “hallmark of algebra” (p. 46). They also highlight two typical forms of algebra tasks: $a(b + c) = ab + ac$ and $2x + 1 = 13$; the first example uses general letters to indicate the distributive property for any number a multiplied inside a binomial term $(b + c)$, while the second is a simple algebraic equation where students might be asked to solve the equation for x. Using words (instead of symbols), one of the ways the second example can be read is ‘one added to some unknown number multiplied by two is equal to thirteen’. The symbolic version of this statement physically and visually takes up less space and is (arguably) more precise. This is just one example that shows how algebra is a predominantly formalized and symbolic form of communication that seeks to remove ambiguity and clarify what is communicated.

Throughout this study, students worked on algebra tasks, given in symbolic form, while I listened to their recordings and transcribed what they said, using words. This occurrence was singular within my experience as a mathematics teacher, in that there was a movement from formalized written mathematics (the task provided to students) to informal spoken communication (the student recordings) and then further still to informal written language (my transcriptions of the students’ conversations). However, the language of algebra is present in each step.

Peer-to-peer and teacher-centred discourse

Peer-to-peer discourse and teacher discourse share many similarities, but I want to highlight their differences to reinforce why I chose to exclusively record peer-to-peer interactions for my study. Group interactions in a mathematics class provide opportunities for students to learn mathematical concepts as they compare their processes and approaches to those of their peers and seek to explain their thinking with one another (Yackel & Cobb, 1996; Boukafri et al., 2018). Caspi and Sfard (2012) undertook a study with a similar focus to mine, where they researched the development of student language through peer-to-peer discussions over time. The students were given a set of algebra equations and asked to solve them with a partner. Unlike my study, however, the students were in the seventh grade and had not yet been given any formal training regarding algebra nor the register surrounding it, so their conversation largely consisted of informal, everyday language.

Before beginning the first session, the researchers commented that the peer-to-peer informal discourse was valuable as a lens through which to see student learning and understanding. Furthermore, they noted that whenever students had the opportunity to choose between formal or informal discourse, which was the case when they worked without teacher oversight, their preference was for “informal algebraic treatment” (p. 52). One of the conclusions of the first part of this study was that the communication between students as they worked together was predominantly spoken and contained a lot of vague, imprecise language.

According to Caspi and Sfard’s study, along with the research of many others (Stein et al., 2008; Temple & Doerr, 2012; Boukafri et al., 2018), the teacher played a critical role in student language development as they moved from informal to formal discourse. When teachers are at the centre of – or, at the very least, participants in – mathematical discussion, they take on the role of expert, encouraging students to refine their ideas and practice appropriate use of the mathematics register (Temple & Doerr, 2012; Boukafri et al., 2018). Teachers employ different strategies to guide students as they shift from informal, everyday language to more formalized and precise modes of communication. Temple and Doerr (2012) argue that, in order for a classroom discussion to be mathematically meaningful, teachers need to engage feedback strategies that create opportunities for “students to practice using the mathematical register in conversations with an experienced interlocutor - the teacher” (p. 302). This strategic awareness is contrary to the spontaneous nature of peer-to-peer discourse; however it provides an intentional foundation upon which students can learn formal mathematical language.

2.2. Metaphor and mathematics

Amidst these varied forms of communication, metaphor and metaphorical language is always present. Before exploring its role in the teaching and learning of mathematics, it is important to understand how metaphor functions in the development and construction of knowledge in general.

Metaphor as a way of knowing

According to English (1997), metaphor is a means of constructing knowledge, and humans use metaphor in a fundamental way to reason, think, and communicate. Many scholars and researchers have done work that shows how important metaphors are as a mechanism for understanding ideas, especially as the ideas become increasingly abstract (Black, 1979; Kuhn, 1979; Reddy, 1979; Lakoff & Johnson, 1980; Davis, 1984; Pimm, 1987; Gentner, 1989; Silva & Moses, 1990; Presmeg, 1992; Petrie & Oschlag, 1993; Chiu, 2011). In the school setting, Pimm (1987) states that metaphors are used as a “conceptual bridge” (p. 98) for students to make meaning, especially when they are introduced to new concepts. (I will explain this idea later in this sub-section.)

Presmeg (1992) says that the Greek word *metaphora* (from which *metaphor* is derived) means to “transfer or carry over” (p. 268), and the Concise Oxford Dictionary defines *metaphor* as an “application of name or descriptive term to an object to which it is not literally applicable” (“Metaphor”, 1997). These two partial definitions emphasize the way that, within metaphor, meaning from one object is put onto another. Lakoff (1994) describes this occurrence as a mapping of one domain onto a different domain, which is when an idea or concept is taken from an already existing area of knowledge (known as the source domain) and transferred onto an idea or concept that is not yet known (the target domain). This conceptual mapping is what makes metaphor such a powerful device and is a significant focus of metaphor theory. Some of the significant results of the work that Lakoff and Núñez (1997) have produced in this branch of cognitive linguistics are:

- Metaphors are cross-domain conceptual mappings. That is, they *project* the structure of a source domain onto a target domain. Such projections or mappings can be stated precisely.
- Metaphorical mappings are not arbitrary, but are motivated by our everyday experiences – especially bodily experiences.
- As with the rest of our conceptual systems, our systems of conventional conceptual metaphors is effortless and below the level of conscious awareness.
- Metaphor does not reside in words; it is a matter of thought. Metaphorical linguistic expressions are surface manifestations of metaphorical thought. (p. 32)

To rephrase, metaphors shift the idea or construction of one thing onto another, they are rooted in ordinary experiences, they are often subconscious, and they begin in the mind before moving to words. Lakoff and Núñez’s findings reveal the deeply embedded nature of metaphors in how humans interact with and make sense of the world. Other researchers support this idea and argue that metaphor may exist implicitly in all fields of human knowledge and understanding (Johnson, 1987; Lakoff, 1987; Presmeg, 1997; Lakoff & Johnson, 2003).

If metaphor is so significant in both language and knowledge development, then the “metaphoric process” (Pimm, 1987, p. 95), where meaning is constructed from one domain and extended to another, is essential to learning. So what role does metaphor play in the specific development of mathematical knowledge and knowing?

Metaphor as a way of knowing mathematics

As English (1997) says at the beginning of her book of collected essays *Mathematical reasoning: Analogies, metaphors, and images*, "metaphor is central to the structure of mathematics and to our reasoning with mathematical ideas" (p. 7). Mathematicians, educators, and students alike use metaphorical reasoning to understand and communicate mathematical concepts (Pimm, 1987; English, 1997; Lakoff & Núñez, 1997; Sfard, 1997). Because of the abstract nature of some mathematical ideas and processes, metaphor is particularly helpful as it provides a way to develop understanding through cross-domain mappings (Lakoff, 1994; English, 1997). As described in the previous sub-section, metaphor provides a foundation upon which to base new knowledge and understanding because there is something known (source domain) that is being transferred to something unknown (target domain); in the case of mathematical learning, the target domain may be a new mathematical concept, but the source domain can be any previously understood idea (whether from inside mathematics or not). As concepts become more abstract in mathematics, metaphor is increasingly helpful as it facilitates bridging the knowledge gap for learners (much like Pimm's *conceptual bridge*). This practice can be especially valuable for teachers introducing new ideas to their students, because it provides a point of connection for students to develop new knowledge in terms of old experiences (Goldin, 1992; Presmeg, 1997).

However, one of the challenges of having such a strong presence of metaphor in mathematical understanding is that students do not often see the metaphor as a tool for understanding mathematical operations or processes; they see the mathematics *as* that metaphor without separation between the two (Pirie & Kieren, 1994). This experience is common within the use of mathematical metaphors, especially with those that have become a part of the mathematics register.

Mathematical metaphors

The field of mathematics uses metaphor to structure areas of abstract reasoning and understanding. There are several mathematical metaphors that are so deeply embedded in the discipline that they have become a part of the mathematics register (Pimm, 1987; English, 1997; Presmeg, 1997). Some examples include mapping:

- an equation to a balance or set of scales (Presmeg, 1997; English, 1997);
- a function to a machine (Matos, 1991; English, 1997);
- a number system to a point on a line (van Dormolen, 199l);

- a vector to an arrow (Lakoff & Núñez, 1997).

For mathematical metaphors to be effective, they must be transferrable across different times and cultures. Thus, the images found in the mathematics register often make use of commonplace experiences and involve concepts like “motion, spatial relations, object manipulation, space, time, and so on” (Lakoff & Núñez, 1997, p. 30). This connection to everyday and commonplace experiences makes mathematical metaphors structured and gives them some stability. Teachers may also develop local, school-based metaphors with their classes, which are less transferrable beyond a specific group but may provide more relatable connections for those students (Johnson, 1987; Voigt, 1994; Presmeg, 1997).

Different scholars have found ways of differentiating these mathematical metaphors that are ubiquitous in student mathematical learning and knowledge development. Lakoff and Núñez (1997) distinguish between what they call *grounding* and *linking metaphors*. *Grounding metaphors* establish mathematical ideas in familiar, ordinary experiences, while *linking metaphors* connect different branches of mathematics to each other. One example of a *grounding metaphor* that I draw on in Chapter 5 is *arithmetic is motion*. In this example, the familiar idea of motion is transferred onto the new concept of arithmetic (or, in the case of this study, algebra). Alternately, *linking metaphors* build metaphorical connections within and between mathematical concepts. These authors share the example of understanding number as represented by a point on a line. In this case, arithmetic and geometry are being linked and “we project our knowledge of geometry onto arithmetic in a precise way via metaphor” (p. 34).

Pimm (1987) uses different labels and distinguishes between two types of metaphors within mathematics education: *extra-mathematical metaphors* and *structural metaphors*. Much like Lakoff and Núñez’s *grounding metaphor*, for Pimm *extra-mathematical metaphors* “attempt to explain or interpret mathematical ideas and processes in terms of real-world events” (p. 95) and can involve commonplace objects and experiences. Some examples are *an equation is a balance* or *a graph is a picture*. *Structural metaphors* are similar to *linking metaphors* as they involve extending metaphorical ideas within mathematics and from one mathematical concept to another. Pimm provides the example of “the *slope* of a curve at a point” (p. 99), which is borrowing from the understanding of a *slope* of a line. These different forms of metaphor allow for increased comprehension and depth of understanding as connections are formed; however, he argues that, when possible, it is essential for students to eventually recognize that the metaphorical images and examples are not the mathematics themselves but simply representations of abstract mathematical concepts.

Analogy and metonymy

While I do not spend much a lot of time looking at analogy and metonymy in my students' conversations, it is worth mentioning them here because they are tools connected to metaphor and used for mathematical thinking and reasoning. Some scholars state that metaphor is a form of analogy (Pimm, 1987; Presmeg, 1997), where metaphor is a "condensed analogy" (Pimm, p. 100). Analogy communicates a lot with little information and tries to transfer knowledge of a new idea by drawing from the learner's experience (Rattermann, 1997). Much like metaphor, it is a way that knowledge can be mapped and transferred.

Metonymy is defined as a device used to understand the whole of something in terms of its parts (or some of its parts). Pimm (1988) looks at the relationship between metaphor and metonymy in his article 'Mathematical metaphor'. In it he addresses how both metaphor and metonymy are embedded into the mathematics register. Metonymy is especially present when there is a substitution of names, one thing *for* another, unlike a metaphor, which is one thing seen *as* another. An example of metonymy outside of the mathematics register is *table three needs more water,* where the people at table three are being labelled as and substituted by their table number. Through the examples Pimm uses, he shows how different metaphors and metonyms within the mathematics register have a referent, a connection to the non-mathematical use of the word, but they take the meaning of the term beyond the referent when used in a mathematical context. Although this use of language can make mathematical communication more accessible, it can also lead to confusion or misconception, especially in cases of "concept-stretching" (Pimm, 1988, p. 33), when a connection between two objects is pushed beyond the actuality of the source domain. However, when used appropriately, metaphor and metonymy can draw deeper connections between the mathematical object or concept and its referent, creating the opportunity for fuller understanding.

2.3. Research question

There are many ideas regarding mathematics, language, and metaphor examined above. Much like Caspi and Sfard (2012), I feel that informal, spoken, peer-to-peer language has not been given much time in the larger research area, especially when compared with formal language, written communication, and teacher-centred discourse. While these latter forms of language and communication are necessary elements of learning and knowing mathematics, I wonder what can be illuminated about the student experience in mathematics class by focusing on the former. I agree with Lakoff & Núñez (1997) when they claim that informal language is part of the learning process and gives students an opportunity to use the words that are most comfortable for them, even if their language is ambiguous. Informal language is especially evident when it takes the form of spoken language, particularly as students are talking with one another and working to create meaning of new concepts with their peers. Furthermore, embedded in their language is metaphorical reasoning and mathematical metaphors that they have knowingly or subconsciously mapped from other areas, some of which may belong to the mathematics register and others that may be more personal or local. Thus, my research question is **what informal terms do students use in their spoken language when solving algebraic equations in small groups together?**

Chapter 3. Methodology

This chapter provides information about the setting, the participants, and the form my exploration took. To engage with my research question, I collected qualitative data by recording student conversations while they worked on different algebra tasks in small groups. In what follows, I provide: an account of the context in which my exploration took place; a description of the mathematics courses and its participants; a defense for the process of my exploration and the data taken from it; an explanation of how the data was analyzed and how the results will be organized.

3.1. Setting

For the last six years, I have been a secondary mathematics teacher at an independent school in British Columbia. In general, the student population comes from upper-middle class homes where their education is valued and supported. There is a small international program in the high-school that makes up 7-12% of each grade, with most students coming from South Korea and China. The High-School follows a mostly semester-based schedule and students take four courses every day for half a year. The semesters run September to January and February to June. This arrangement has often impacted my students' mathematical learning, because it is possible that there has been a year or more since their last mathematics class, especially for those who have their mathematics class in semester one of one year and semester two of the following year.

In the time since I started my master's program, and became interested in this topic, there were two courses that I taught. Each had a different makeup of students with varying mathematical backgrounds: the first was Foundations of Mathematics 12 and the second was Pre-Calculus 11. In my Pre-Calculus 11 class, students were between the ages of sixteen and eighteen. I chose these courses for my research because they were the only courses I had been assigned; I was also teaching an International Baccalaureate mathematics class, but I did not include that in this study due to time restrictions in the course.

The way that I organized the class time for both groups was similar. The students sat in table groups, which were randomly designated when the students selected cards as they entered the room. I shuffled a deck of cards that matched to the number of students in the class, and students would select a card from my hand upon entering the classroom. They would then sit at the table with the corresponding card. Each table group would be made up of whichever students selected the same numbered card (i.e. all students who selected a card with a Queen would sit at the same table). I used this method to organize students in random groups throughout the whole semester, so that students became comfortable working with any of their peers and would not get stuck in complacent group dynamics or roles by sitting with the same people each class. Frequently, I used a short instructional time, and then the students would be given questions to work on with their table groups at one of the eight whiteboards on the walls of the classroom. I encouraged students to work collaboratively within their table group and discuss the mathematics whenever they were solving questions; if there were students isolating themselves and working alone at this point in the lesson, I would ask them to re-engage with their group and work independently at a later point.

3.2. Participants: Pre-Calculus 11 class

While teaching Foundations of Mathematics 12, I finalized the idea for my thesis topic. Working with the students helped me develop and formalize the ideas I wanted to address for my thesis, but I did not do any of my actual research with them. Instead, I focused on my Pre-Calculus 11 class, which I taught during the Fall of 2019 and who became the focus group for my work. It was a large class of twenty-eight students, twenty-six in grade 11 and two in grade 12. There were four students from our international program, and it was clear that all four of them had worked on some of the course content before, unlike most of the local students. All were students bound for Pre-Calculus 12 and possibly Calculus 12 before graduating from high-school.

Most of the students were taking this course as a requirement for mathematics, sciences, or engineering in university; many of them were motivated to ask questions and take educational risks. There were still students who struggled to understand the content, but their peers would often help them. Working in groups came naturally for this class, and they frequently talked with one another and explained their thinking to each other without any prompting from me.

3.3. Exploration: The algebra tasks

To find out what kinds of words and phrases students used when talking to each other while solving algebraic equations, I developed several different algebra tasks to give to my students. I planned to have at least six questions for the students to solve, with each question containing a different mathematical operation from the previous questions. Operations in the first task included addition, subtraction, multiplication, and division. In the later tasks, I wanted the questions to increase in mathematical complexity to incorporate the concepts students were currently learning in class and to see how the students' language would accommodate the increasingly abstract ideas; the operations in these tasks included creating and canceling out common denominators (for rational functions), square-rooting, and squaring (for radical functions).

To work on the algebra tasks, students worked with their randomly selected table groups. Only three of the seven groups stayed in the classroom in order to spread out the locations of the groups and decrease the chance that recording devices would pick-up conversations from more than one group. All students recorded their discussions, whether they were in the classroom or in the hallways. As a class, we would pick which groups would go somewhere else in the school and which ones would stay in the classroom. Next, I would hand out the papers with the instructions and algebra tasks written on them. Students staying in the classroom took the paper and went to one of the large classroom whiteboards with their recording device. Groups working at other locations in the school took a small whiteboard, a whiteboard marker, the paper with the algebra task, and their recording devices before heading to their respective places.

Task #1

For the first session, I created an algebra task that involved mostly one- or two-step equations to set a baseline for the research. The main goal of this task was to help students adjust to the process of working in small groups and using the recording devices, so I did not include any questions that involved concepts from the Pre-Calculus 11 content. The task and instructions were:

Record your conversation while solving the algebra problems:

1. $-4x = -20$
2. $3x + 6 = 18$
3. $-3 + \frac{x}{5} = -12$
4. $\frac{6}{x} - (-8) = 12$
5. $-40 = 8(x - 9)$
6. $\frac{1}{3}(x - 2) = \frac{5}{6}$

At the end of the first task, I had students listen to the first two to three minutes of their conversation and make note of the words they used the most, as well as what surprised them about their conversation. I also asked them for feedback on the setting, since for this task I had all of them stay in the classroom and record their conversations. At this time, I had not yet received approval from my School Board to run the project. Thus, without parental or student consent to keep the recordings, students practiced recording themselves and then deleted the file before the end of class.

Task #2

For the second task, my rationale was to give the students an algebra task with questions that involved a more complex topic (rational functions). I gave this task to my Pre-Calculus 11 students shortly after their summative assessment for our unit on rational functions, hoping that they would remember how to solve the equations.

During the trial, I noticed that the second and fifth question contained two different variables, which were typographical errors. In that moment, I ran around to as many groups as I could to point out the error, which was captured in the group recordings. Below are the instructions and the task questions. I have included the original version and the adjusted questions for #2 and #5:

Record your conversation while solving the algebra problems:

1. $\frac{x}{2}+\frac{x}{3}=5$

2. $\frac{3(x-1)}{2}=z-2$

   $\left[\textit{should have been}: \frac{3(x-1)}{2}=x-2\right]$

3. $\frac{x}{x-2}=\frac{2}{x-2}+2$

4. $\frac{3x}{5}-\frac{x-5}{7}=3$

5. $\frac{2}{x+5}+\frac{20}{z^2-25}=\frac{3}{z-5}$

   $\left[\textit{should have been}: \frac{2}{x+5}+\frac{20}{x^2-25}=\frac{3}{x-5}\right]$

6. $\frac{5}{4z-2}-\frac{1}{1-2z}=\frac{7}{3z+6}$

Task #3

The third task was given in December, about four school weeks before the end of the semester and the end of the course. The students had learned the majority of the new content for the course and would be doing a unit on finances and reviewing the course material after the winter break; thus they were expected to have learned and understood most of the Pre-Calculus 11 content at that point. For completing the task, students followed the same procedure as Task #2: in their randomly selected group, they chose where they would work in the school, and they made sure they had access to a whiteboard, markers, the paper instructions, and a recording device.

The rationale for these questions was that they covered a range of concepts, getting increasingly more complex. The sixth question incorporated multiple concepts from the course. Below are the instructions and the task questions:

Record your conversation while solving the algebra problems:

1. $6 - \frac{8}{x} = 10$
2. $\frac{1}{2x} + \frac{4}{x} = \frac{9}{2x}$
3. $12 = 4\left(x + \frac{1}{2}\right)^2 + 3$
4. $\sqrt{5x - 5} = \sqrt{4x - 1}$
5. $\sqrt{2x + 11} + \sqrt{x + 6} = 2$
6. $\sqrt{x + 15} + \sqrt{x + 7} = \frac{4}{\sqrt{x+7}}$

3.4. Data

There were two ways that I collected data: through student recordings and field notes. The field notes did not play a significant role in my work, but it was nevertheless helpful for me to formulate my understanding of my research process as the students worked on the tasks. The student recordings were the most significant data for this project since I was focusing on the spoken words students used while working together. Since I had multiple groups working on the algebra tasks at the same time, it was not possible for me to be present and make notes while each of them recorded their conversations. Thus, I had to depend on the digital recordings that the students submitted in order to hear all of their discussions while solving the algebraic equations.

Pre-Calculus 11 student recordings

My Pre-Calculus 11 students completed all three trials, and I had them submit their recordings for the second and third tasks. I received seven recordings for Task #2 and five recordings from Task #3, but one of the five recordings from Task #3 did not work. Overall, I received eleven working recordings from the two tasks. I planned to transcribe each recorded conversation, but I found there was too much non-mathematical data in the recordings (such as conversations about evening plans or social media posts); most of the submissions were between fifteen and twenty minutes, and the students would have long pauses or go off topic. At the advice of my supervisors, I listened to all the recordings and transcribed the parts of the conversations that directly related to solving the algebra equations or that had interesting words or phrases that I wanted to explore for this project. It is reasonable to accept that this selection process left out certain elements of language that I would want to address upon listening to the recordings in the future, but I chose the words and phrases that were the most relevant at the time.

I imported the transcripts into NVivo, a data management software, and read through them line-by-line, highlighting and categorizing the different kinds of words that I noticed. I did not have any pre-determined categories before starting the process, and I created them as I went through each transcript and identified interesting language. By the end of the process, I had twenty-six categories, which I will explore in Chapter 5.

Field notes

During the second and third tasks with my Pre-Calculus 11 students, I walked around to the different groups and wrote down anything I thought was note-worthy. Although the focus of my research was the words students use when they talk about algebra – and, thus, the most important data was their conversations – it was also interesting to take note of environmental and contextual elements. For instance, I noted that five minutes into Task #2, all groups were engaged in their activity, though one group included two students who were more interested in looking at their phones than actively participating in the algebra task. As I continued to walk by the different groups, I noticed that some switched who was writing the work on the whiteboard while others had the same student in that role throughout the task. Often the student who was writing led the conversation and influenced the direction of the task; generally, in consequence, the writer had more air-time in the discussion.

For the third task, question #2, which was $\frac{1}{2x} + \frac{4}{x} = \frac{9}{2x}$, came to the solution $9 = 9$ or $1 = 1$. Two different groups had a difficult time understanding what that meant, especially since the variable x had 'disappeared'. I noticed that another mathematics teacher from my school walked past one of the groups in the hallway and was called in to help with that question. I did not hear much of their conversation at the time, but I was able to hear how the teacher guided them to understand the final statement when I listened to their group's recording.

3.5. Analysis

Throughout my process of analysis, I focused on the student conversations. As highlighted in my Chapter 2 Literature Review, I was interested in metaphor within informal mathematical conversations. There were four metaphors I had noticed my students using while solving algebra equations which had inspired me to write this thesis, and they were that of MOVEMENT, ELIMINATION, TRANSFORMATION, and BALANCE. In my Chapter 5 *Thematic analysis*, I took a deeper look at each metaphor within my students' recordings. In my chapter before that, *Analysis by groups*, I looked at a variety of different terms and types of language that came out of individual group conversations.

Analysis by groups

My first step in analyzing the data was to take an in-depth look into four of the eleven transcribed conversations by focusing on specific words or phrases used by the groups and looking at the singular features of language and implications of the mathematics present in their discussions. Since I had created the different categories while reading through the eleven transcriptions of the different student recordings and wanted to address themes that came up throughout them in Chapter 5, the specific group analysis gave me an opportunity to look closer at some of the singular statements that students would make and talk about them in more detail. This analysis provided me the opportunity to address interesting features of language that I would not discuss anywhere else in the thesis.

Of the four different groups, I looked at two conversations from Task #2 and two from Task #3 to address language that was used for the varied mathematical operations in the two tasks. There are portions of the transcriptions shared from each group, and all student names are pseudonyms. The reasons I selected the groups I did are as follows: the first group (John, Tony, Greyson, and Stephen) included terms with onomatopoeia as well as some unique colloquial phrases that I wanted to explore; the second group (Sara, Lim, and Roxie) used some phrases that commonly came up in a mathematics classroom, but they used it in a unique way; for the two groups working through Task #3, the first group (Stephanie, Sara, and Lydia) used language around verbs that I wanted to address, while the second group (Greyson, Madelyn, and Carol) had conversations that included elements of the mathematics register that were noteworthy.

After finding three or four examples of discourse that I wanted to highlight in the different groups, I listened to each recording to get the context for each case. Once I had sufficient information to understand the students' work or calculations, I looked at the different mathematical implications or features of the language the students were using. I included the relevant transcripts, using ellipses in the middle of a line when a student changed the topic of their conversation to one that was not relevant to this study or at the end of a line when a student did not complete their thought.

Thematic analysis

After looking at some of the groups and the language that was specific to their conversations, I wanted to take a broader view and look at the categories that came out of analyzing all eleven transcriptions. The categories I developed are listed in Table 1 in Chapter 5. The table lists the twenty-six categories I put the students' language into, but I removed a category I called Z-

UNSURE, which was where I would temporarily leave cases that did not fit under any existing heading. I would revisit the heading regularly to see if a new category could be developed by the examples in there. By the end of my analysis of the transcriptions, I still had some cases under the Z-UNSURE heading, but they were singular instances or did not have a clear direction for a category. I would be interested to come back to those examples and look at them further in the future. Lastly, while reviewing the categories that came out of the transcriptions, I created meta-categories to organize further the headings that had organically come through the process of reading the transcripts.

For the second half of my thematic analysis, I focused on metaphors that I had previously identified in my students' spontaneous discourse. In this study, they were some of the most frequently used metaphors that my students employed while working through the assigned tasks. I explored the ways students used different examples within each metaphor and what the source domain said about the target domain. I wanted to show as much variety as possible and see some of the nuances and similarities surrounding how students used their language across groups. Like the *Analysis by groups* sub-section, I included transcripts to give context to the mathematical work students were talking about.

Chapter 4. Analysis of Group Conversations

In this chapter, I describe and discuss four different student group recordings, two from Task #2 and two from Task #3. Furthermore, I explore the student-language employed that was specific to these conversations. I seek to outline some of the implications of the words and phrases that the students used and highlight the layered meanings of mathematics that were present in their word-choice. In the transcripts in the following examples, I have bolded the words or phrases I focus on and discuss.

4.1. Task #2: John, Tony, Greyson, and Stephen

One of the randomly created groups was made up of four friends who chose to work together in class whenever they could. Several times throughout their recording they came back to a phrase similar to one I often use when teaching algebra: "what you do to one side you have to do to the other"; however, they were using it in a different way. As they worked on the second question in the task, $\frac{3(x-1)}{2} = z - 2$ (uncorrected version), two of the students had the following conversation while they were trying to create a common denominator for the terms on both sides of the equation:

Tony: Do you add two **to the top and the bottom**?

Greyson: **What you do to the bottom, you have to do to the top**; I learned that once.

Tony: Alrighty.

When Tony said, "you add", it was unclear if he was using the term *add* literally, which would mean that he was not employing the correct technique for changing the form of the fraction. Regardless, Greyson used the phrase, "What you do to the bottom, you have to do to the top", to communicate a vague approach to creating a common denominator. A version of the phrase was repeated when the students worked on the fourth question in the task, which was $\frac{3x}{5} - \frac{x-5}{7} = 3$:

John: Alright, I'm going to work on question four. So first, find the common denominator, and **what you do to the bottom, you do to the top.**

Stephen used the same phrasing later in the question while he was creating a common denominator between the two fractions on the left side of the equation:

Stephen: And **what you do to the bottom, you do to the top**, so common denominator five and seven is what? Going to be thirty-five, boys. Like that, bop, so five times seven is thirty-five, and then **what you do to the bottom, you do to the top**, so then that would be twenty-one *x*.

I want first to address the implications that come with the words *top* and *bottom* within the context of a fraction. This language created the image of containers where there is a top and a bottom to the mathematical object. This view further implies that the fraction has two separate items – a top and a bottom – as opposed to it being a whole and single mathematical object. When seen this way, the elements of *top* and *bottom* can take precedent over the whole. Another implication is that there was an innate relationship between the numerator and the denominator of the fraction; if there is a top, there must be a bottom, and vice versa.

For students, the words *numerator* and *denominator* do not explicitly communicate connection or relationship to the other, as each can be linguistically understood as a separate object without inherent reference to the other; for many of them, a *numerator* is an independent object, as is a *denominator*. Their labels are identifiers of pieces within the fraction, instead of one object in relationship with another object which makes up the fraction. A fraction could be seen either as a whole or as two things that are related, but when the words *top* and *bottom* are used, neither of those images are necessarily called upon.

Another interesting element of the students' language occurred when it was first used by Greyson, and he said, "what you do to the bottom, you **have to** do to the top". None of the other students repeated the modal verb *have to*, though it seemed to be suggested in their subsequent simplified statements that, "what you do to the bottom, you do to the top". Using *have to* implied necessity, that there was a required action. The verb lends itself to the sense that there are mathematical rules in place that must be followed to correctly solve the equation. In this context, *have* also implied a required follow-through: once one step was done, the next step *must* be done.

The last piece of this statement that stood out was that all the students used the phrase, "what you do". This language was active and assumed there was a subject performing the verb of *doing* to solve the equation. Furthermore, the students were the subject in that active role. This language implied agency, since there was a 'you' that could 'do' something to the mathematical objects in the equation. The equation and its mathematical objects are not in control; the student as the *doer* is. I will further look at student agency implied by language in the next chapter, under the metaphor of TRANSFORMATION.

Another linguistic element that I want to highlight comes from Stephen's earlier statement when he used the word 'bop', an onomatopoeia, within the context of other mathematical processes during question four:

Stephen: Going to be thirty-five, boys. Like that, **bop**, so five times seven is thirty-five, and then what you do to the bottom you do to the top, so then that would be twenty-one *x*. Alright, and then minus, since you can't do that you just go **bop** five brackets *x* five minus [student rearranged the order of *x* minus five] over thirty-five.

There are two distinct ways that the student might be using onomatopoeia in this situation: either as an auditory placeholder, indicating that something is happening, or to replace the language of mathematical operations being done. If the 'bop' is removed from his statement, his mathematical process is still apparent, which indicated that the onomatopoeia, in this instance, is not necessary to understand his mathematical process. This explanation mirrors what Ma (2018) says about onomatopoeia acting as "quasi-language" (p. 43) while the subject works towards communicating with real or precise language. Thus, 'bop' could indicate that the student did not yet have the appropriate language to precisely communicate his work and the onomatopoeia acted as a temporary placeholder for a formal mathematical operation.

However, if 'bop' is intentionally replacing a mathematical word, there are some interesting linguistic features. *Bop* is an onomatopoeia for lightly hitting or tapping something, so it could imply that one term is *hitting* another, which is a plausible reading of the second use of 'bop' in his utterance. The onomatopoeia could also infer the seemingly magical appearance of an object within the equation; in this case, the appearance of the 5 in the numerator and 35 in the denominator, as the second fraction in the question changes from $\frac{x-5}{7}$ to $\frac{5(x-5)}{35}$. With both conjectures, 'bop' indicated that something had suddenly happened.

The third linguistic feature that came out of this group's recording was the use of colloquialisms amidst mathematical language. The earliest example occurred during the first question, $\frac{x}{2}+\frac{x}{3}=5$:

Stephen: x equals six. Because five times six equals thirty, thirty over six is the same as five over one. Correct?

John: Correct.

Greyson: Also, um, **simple mode:** if x was six, then you can just cancel. I think.

John: Atta boy!

Preceding Greyson's use of the phrase 'simple mode', the group was checking if the answer they got for x was correct. Greyson did not speak up till the end of the conversation when he decided to replace x with 6. The use of the phrase 'simple mode' suggested that there were other *modes* of solving for x, some of which were more complex, or at least not simple. Based on the context, the implication was that inputting a specific value for x (in this case, the number 6) was easier or simpler than doing something else.

The second colloquialism occurred during the fourth question, $\frac{3x}{5} - \frac{x-5}{7} = 3$, as the group was trying to determine how to solve the equation. At this point in the conversation, the equation was in the state of $\frac{21x}{35} - \frac{5x-25}{35} - 3 = 0$:

Stephen: Wait. Explain to me the steps you took to get there.

Tony: I just . . . subtracted it [the 3 from the right side of the equation]. . .

John: Anyway, so **I'm going to go down my path with you**, times three by thirty-five [trying to create a common denominator], and then you get one-oh-five and then what you gotta do after is collect like terms, so twenty-one *x* minus five *x* equals what? Sixteen *x*.

Greyson: Can we restart this?

John: No.

There are several implications worth noting in John's phrase, beginning with the word *path*. Mapping the image of a path onto the mathematics implied that in the student's process there was a direction, end-point or destination, and steps to get there. A path is also outlined or defined in some way and has indications that the person walking along it has continued down the correct route and not strayed. Furthermore, there is the indication that solving the equation is a journey where the pacing can be adjusted. Using this metaphor set up the interpretation that whatever the student said afterwards were steps down the path to the destination of determining the correct solution(s) to the equation.

Another interesting part of John's statement was that he called it "**my** path", which indicated a sense of personalization while also implying that there was more than one path or way of solving the problem. Thirdly, he ended the phrase by adding, "with you", which inferred that this is a mathematical trek that the students can go on together. It does not have to be done in isolation; the path is wide enough for more than one person to walk down at a time, and they can come to the destination as a group.

Lastly, the metaphor of *solving an equation is a path* is one rooted in a sense of time. The present is *the now* of the students solving the problem, and the narrative they were working within was sequential where time was the *thing* structuring their story. They cannot make it further down the path without taking the preceding steps to get there. This sense of time was further implied later in John's statement by his two uses of 'then' to indicate that a step had been taken – a mathematical operation had been performed – and *now* the next thing could happen.

The discussion between John, Tony, Greyson, and Stephen involved language that demonstrated several things: it implied their agency as the 'doers' of the mathematics, it used onomatopoeia as a filler for language, and it included colloquialisms that established a narrative in time and provided a sense of personalization.

4.2. Task #2: Sara, Lim, and Roxie

Unlike the previous group, these three students were not close friends and were not as comfortable with each other at the start of the task. However, they quickly started working through the assignment, trying different methods and asking each other questions. After only a minute and a half, they made the following exchange while working on the first question in the task, $\frac{x}{2}+\frac{x}{3}=5$. They had created common denominators for the terms on the left side of the equation and then changed the 5 on the right side to $\frac{30}{6}$. Below is their exchange when they started to simplify the work they had done on the right side:

Roxie: So that has to be five over one, would it not?

Sara: Yeah, okay . . .

Lim: Wait . . .

Roxie: Wait, what?

Lim: Mmm . . . Keep going, keep going.

Roxie: Wait, what? Tell me!

Sara: Wait, that wouldn't make sense because this has to **be the same.**

Roxie: Yeah.

Lim: **It is the same** though.

Roxie: But, I mean, like.

Lim: Now **it is the same**. I don't think you simplify it yet. You don't simplify it because you already **made it the same**.

Sara: Okay, so let's erase that part.

When Sara first said, "the same", it was a reference to how the group had made 5 into $\frac{30}{6}$, so that it had a common denominator with the two terms on the left side of the equal sign. However, after changing it to a fraction, they then simplified it back to 5, which was what was *the same* as $\frac{30}{6}$ in Sara's comment. But Lim's statement at the end that, "now it is the same . . . because you already made it the same", was communicating that $\frac{30}{6}$ had *the same* denominator as the two terms on the other side, which they had changed from the original expression to $\frac{3x}{6}$ and $\frac{2x}{6}$.

In this interaction, the term 'the same' was being used to indicate two different mathematical relationships: the first was the same-ness of an integer, 5, and an equivalent fraction, $\frac{30}{6}$, while the second was communicating the same-ness of the denominators for all three terms in the equation. With the equivalent fractions (or equating a fraction with an integer), the student referred to two terms that were equal in value but not in representation. Using the language of 'the same' loses the distinction between the whole number and the fraction; while they are equal in value, they are not *the same* type of number, and the insinuation that they are can create confusion around which articulation of the number is more helpful or meaningful in a given context. Similarly, when the students used the phrase 'the same' to refer to the common denominators of two (or more) different fractions, the metonymy similarly does not leave room for differentiating between different fractions. Even if the denominators are *the same*, there is a loss of other mathematical information that comes with simplifying the language in a way that only focuses on the similarity in the denominators.

Lastly, when the students used the language of 'the same' for both equivalent fractions and common denominators, it made the conversation difficult to follow. Even the students seemed to experience confusion over what mathematical object was being referred to as *the same*. This is either because the language of 'the same' is not robust enough or because the students did not include any additional information about what was *the same*: the phrase lost meaning throughout the conversation.

Another linguistic feature that this group used to communicate a variety of operations was the metaphor of MOVEMENT. I will talk about this theme further in the next chapter, but here I will highlight three different ways that these students used the metaphor. In question two, $\frac{3(x-1)}{2} = x - 2$ (corrected version), they had the following interaction as they tried to create a common denominator for the terms on both sides of the equation:

Roxie: Then I guess you have to put that [the $x - 2$] over two too, to make it, like, the same. And then times this one [the, now, numerator of $x - 2$] too.

Sara: Okay. *[sound of whiteboard marker on the whiteboard]*

Lim: Now we put it [the equation] to zero, right? And then **move to the other side.**

Sara: Can I bring all of this over?

Lim: Yeah

Sara: Then this would be negative, right? And this would be positive?

Roxie: Yeah. Okay.

A similar phrase came up while the group was working on the fourth question, $\frac{3x}{5} - \frac{x-5}{7} = 3$. They had worked through the following steps of the equation:

(1) $\frac{3x}{5} - \frac{x-5}{7} = 3$

(2) $\frac{16x-25}{35} = \frac{105}{35}$

(3) $16x - 25 = 105$

(4) $16x = 130$

After step (4), the below interaction occurred:

Sara: Wait, so you did one-thirty, and then you divided by sixteen?

Roxie: I **moved this over** [dividing the 16 on both sides], so that it could be over sixteen and then I simplified it.

Sara: Okay.

Roxie: What do you think?

There are several implications that come from the language of MOVEMENT, regardless of the mathematical operation the movement is indicating. First, this metaphor suggests that the equation is made up of three distinct parts: the equal sign, the left side (of it), and the right side (of it). Within these parts, the equal sign plays a distinct role in being the only means through which the two sides of the equation can relate to one another. This relationship draws on a further metaphoric consequence: that the equal sign is a bridge or divider over (and not under) which mathematical objects can travel; without the equal sign, the movement that the students referred to could not happen. Furthermore, if the equal sign is seen as a bridge (rather than a tunnel), then the implication is that the numbers and variables are the objects that get moved or transported over that bridge. Lastly, based on Sara's comment at the end of the first interaction, the signs or symbols of mathematical operations undergo some sort of transformation when crossing the equal sign, but the numbers and variables, as independent objects, do not.

A final implication from this group's MOVEMENT-language addresses the numbers or variables that were on one side of the equation and now appear on the other side of the equation; however, the phrases the students used like "move to the other side" or "bring all of this over" do not provide any information about the operations attached to those numbers. When Sara and Lim spoke in the first example, they set the equation to equal zero by collecting all terms on one side of the equation; based on Sara's questions that, "Then this would be negative, right? And this would be positive?", the mathematical operations they deployed were addition and subtraction. However, in Roxie's comment later in the discussion, it appears that her MOVEMENT-language referred to the operation of division. Without the context of the other statements the students made, it is not clear which mathematical processes the language of *move* or *bring over* referred to.

In the final question, which was $\frac{5}{4z-2} - \frac{1}{1-2z} = \frac{7}{3z+6}$, another interesting use of language came up which involved the acronym FOIL. The students had the conversation below as they tried to figure out their next step after creating a common denominator with binomial factors for all three fractions:

Sara: I don't actually know if this is right, but would you simplify, like, I mean, **FOIL** first and then move it over [the fraction on the right side of the equation]? Or move it over right away?

Roxie: **FOIL** which one?

Sara: Like, you know how at some point you're supposed to **FOIL in?**

Roxie: Oh yeah, you'd **FOIL** first cause if you . . . okay, just watch. *[sound of marker on the whiteboard]*

The students used the acronym FOIL (which stands for 'First, Outside, Inside, Last') as a verb to indicate the distributive property between two binomials, a common algorithm for both high-school mathematics teachers and students alike. The students' use of the word stood out to me here both because it is used several times in succession and because of Sara's comment at the start: "but would you simplify, like, I mean, FOIL first?" Since *FOIL* is meant to point to the distributive property, it is an operation of expansion, not simplification, which directly contrasts Sara's statement. Furthermore, it appears that the word has become an operation in its own right, not just an acronym to remind students how to apply the distributive property between two binomials, which implies that *FOIL* is a mathematical operation. *FOIL* also was used as a transitive verb. Secondly, it is unclear if the students are discussing two binomials or if they are using the word *FOIL* to indicate any kind of distribution, which is not uncommon for students to do. Thus, within these students' discussion, *FOIL* had not only become a mathematical operation, but acted as a replacement for the distributive law. And a 'law' is not a verb, but it is highly metaphoric inside the mathematics register.

Throughout their conversation for this question in the task, there were also several instances when they used the language of TIME. Some examples are shown below outside of the context of the equation the students were working on:

Sara: Could we do it, like, **later**?

Roxie: I guess.

Later in the problem:

Roxie: What I think is easier just put this into here and **then** you can cross all that off and **then** like terms and **then** find x.

And near the end of the discussion:

Lim: Wait, but don't you have to put this in here **now**?

Even without a clear understanding of which mathematical operations the students were doing, all three of these examples make it clear that the group was solving the problem in time. Much like the first group, the narrative of them solving the equations was structured and based in time, which is especially evident in Roxie's comment when she used 'then' three times in the same sentence to indicate what she was doing next. Furthermore, Sara and Lim's questions during this task showed some interesting features about the language of TIME. First, when Sara asked, "Could we do it, like, later?", she indicated that there were some steps or operations that do not have a necessary order or timing to them; whatever she referred to could be done *now* or it could be done *later*. Lim's statement signifies the opposite idea, because there was something that must be done *now*. Looking at both of their statements together, there is a recognition that some operations are more rigidly based in time and (consequently) order than others.

In summary, Sara, Roxie, and Lim's conversation highlights the confusion created by using the phrase *the same*, shows implications of using the metaphor of MOVEMENT, and uncovers *FOIL* as a mathematical operation (and as a verb).

4.3. Task #3: Stephanie, Sara, and Lydia

The third task was the last one I did with this class. During Stephanie, Sara, and Lydia's work on the first question, $6 - \frac{8}{x} = 10$, and based on their conversation, the state of the equation was either in this form:

$$\frac{6x}{x} - \frac{8}{x} = \frac{10x}{x}$$

or in this form:

$$\frac{6x}{x} - \frac{8}{x} - \frac{10x}{x} = 0$$

As they worked through this question, they had the following interaction:

Lydia:	And then you can cancel out the denominators, right?
Stephanie:	We have to move the ten over to this side first, I'm pretty sure.
Sara:	It's not cancelled out, right? It's just that **all the *x*'s are one combined thing**, so the whole equation is just under *x*, right?

(Within the context of their work, Sara would have meant "the whole equation is just [over] *x*" instead of "under".)

There are some interesting implications that come from Sara's phrase, "all the x's are one combined thing". First, this statement implies that there is a relationship among the individual xs in the three denominators, even though they are each an independent object in the equation in the forms shown above. Sara's language indicated that the mathematical statement

$\frac{6x}{x} - \frac{8}{x} - \frac{10x}{x} = 0$ is equivalent to $\frac{6x-8-10x}{x} = 0$, without having to write out that step. In the rewritten equation, the xs in the denominator are now a single object as a common denominator instead of three independent objects. However, they could subsequently be written as independent objects again. Furthermore, the expression 'one combined thing' acknowledged that the xs have a *coming together* when they become a common denominator. But her phrase focused on the denominators as the *things* in relationship with one another and lost the relationship between the numerators and the denominators of the individual fractions, especially if changes needed to be made to any of the fractions in order for them to have a common denominator in the first place.

To give context for another interesting feature of language, this group of students chose to work at one of the tables in an open hallway of the school, which provided the opportunity for another teacher to join their conversation while they were working through the second question, $\frac{1}{2x} + \frac{4}{x} = \frac{9}{2x}$. The conversation below happened after the teacher came by and the students asked for some help solving the equation:

> Teacher: Is there something you could multiply each of these [fractions] by – the same thing – that would cancel off the denominators?
>
> Lydia: We multiplied this [the fraction $\frac{4}{x}$] by two, so that they [the denominators] were all two x and then we crossed out the [denominators]. . . *[tapping sound followed by the sound of a whiteboard marker writing on the whiteboard].* Usually **we solve for *x*, but you can't really solve for *x* here**.

The first noteworthy word in this exchange was 'solve', which is defined within the mathematics register by the Oxford English Dictionary as: "*Mathematics*. To find the answer or solution to (a problem, etc.); to work out" ("Solve", 2020). When Lydia said, "usually we solve for x", her words implied that the overarching goal of manipulating an equation was to solve for x or to find a solution (or solutions) for the variable. Moreover, there was the suggestion that finding the value(s) of the variable were what made the equation work. Using the language of *solve* is some of the more mathematically precise language that was used by these students in their conversation.

The second part of Lydia's statement, "but you can't really solve for x here", also had some further linguistic features. To give some context, the solution to the equation is that x can be any real number other than zero; the equation is not dependent on the value of x, so when the students found a common denominator for the fractions and simplified, the xs *disappeared*. Thus, their language suggested that if x were no longer present, then there would be no solutions and the equation would not be solvable. A further view is that it is possible to have an equation that is unsolvable.

As the conversation progressed, the teacher recognized the students' confusion and saw that the equation was not dependent on the value of the variable, so he asked the questions below:

Teacher:	So, what happens if you put in a one for *x*?
Lydia:	For *x*?
Sara:	Like, what do you mean?
Stephanie:	Like, make *x* one? Like two times one, two times one? So, it would just be over two, over two, over two.
Sara:	Oh!
Teacher:	Yeah.
Stephanie:	So, you'd simplify that. Well, I feel like we'd still want to add them together and then simplify, right?
Teacher:	So, if *x* was equal to one, you'd have one half plus four plus nine halves. **Is that true?**
Stephanie:	Yes?
Teacher:	Yeah, it is.

Throughout all the transcripts that I recorded for this project, this teacher was the only person who used the language of *truth* when looking at an equation. One of the linguistic features of asking if the equation were *true* is that there is an imbedded assumption that an equation is something that is, can be, or must be true. Crucial to the view of *an equation as a truth claim* is the meaning of the equal sign. Vermeulen and Meyer (2017) state that there are two ways an equal sign is typically viewed in school mathematics: as a symbol of operation or a symbol of relationship. In this example, the teacher communicated that the equal sign was indicating a relationship between the two sides of the equation. Within the "relational conception view" (p. 137), the equal sign signifies that both sides of the equation are equal, which lends itself to the view of *an equation as a truth claim*, and that idea is what the teacher was trying to point out to the students here. Lastly, if an equation is a truth claim, there is an implied possibility that it can be made false.

Later in the discussion, the students worked on the third question, $12 = 4\left(x + \frac{1}{2}\right)^2 + 3$. After the students had applied the distributive property to $4\left(x + \frac{1}{2}\right)^2$ and had their equation in the following state, $12 = 4x^2 + 4x + 4$, after simplifying, they had the following interaction:

Sara:	Oh, can I just bring the four over here [to the left side of the equation] so that I leave all the [terms with] *x*s over here?
Lydia:	I thought we were going to try and factor it.
Stephanie:	We can try factoring it after. So, if you bring this [the twelve] over, you know how you have to have squared, *x* and then a number? To factor?
Sara:	Yeah.
Stephanie:	So, if we do that, then we could try to factor.
Sara:	Factoring first?
Stephanie:	Well, after we move the twelve over.
Sara:	Oh, I see, I see. So, what's four . . . minus eight . . .
Lydia:	So, two times four . . . No that isn't . . . You can't factor that, right?
Sara:	We could, just, **pull out four**.
Lydia:	Oh right, yeah.
Stephanie:	Ohhhh. Look at you go!

When Sara said the phrase, "pull out four", she referred to factoring something in the equation. One of the first implications that come with using the language of *pulling out* a number is that the given number is something that can be removed from or taken out of another number/term. It creates the image of two distinct objects being held together. Thus, the phrase also gave an indication that the original number/term and the factor were connected or have a relationship with one another, like there was a force drawing them together and the student (or whomever) was pulling them apart from one another. Lastly, if the factor was getting pulled *out*, then it must have been *inside* or *within* the original term and was now being separated from it.

A final notable part of the students' conversation occurred while the group was working on question #5, $\sqrt{2x + 11} + \sqrt{x + 6} = 2$, after it was in this state:

$$\sqrt{2x + 11} = 2 - \sqrt{x + 6}$$

They began by talking about a similar question on a test they had recently written that involved radicals (though they incorrectly use the word *fractions* instead of *radicals*):

Stephanie:	Oh man. I messed that up on my test. I multiplied the numbers and I multiplied the fractions [meant radicals]. I didn't FOIL everything.
Sara:	What? Oh!
Stephanie:	Oh, but it didn't have a minus there [between an integer and a radical term], so maybe that?
Lydia:	Yeah, if it didn't have a minus, then it's fine.
Stephanie:	Okay. *[laughs]*
Sara:	Okay, no, no. I understand. I understand. I was just like, what? I didn't do that either!
Lydia:	Okay, now, um, [back to the original question] plus *x* plus six.
Sara:	Ah, see, like literally. Like, I didn't even know that because there's a minus sign you can't.
Stephanie:	Well, it just means they're **two separate things**. But **if they're together** and it was just a two on the outside of it, then I think you just multiply the two twos and the two radicals. Because that's like . . . Well, I don't know. I was really confused about multiplying fractions [meant radicals] on the test.

What Sara was confused by was that she thought she could make the following steps:

$$\left(\sqrt{2x+11}\right)^2 = \left(2-\sqrt{x+6}\right)^2$$

$$2x+11 = 4-(x+6)$$

Stephanie tried to explain to Sara that the subtraction symbol (or 'minus sign') between the 2 and $\sqrt{x+6}$ meant that she could not square both sides and get the outcome she had intended.

The way that Stephanie explained this process to Lydia was by saying that the 2 and $\sqrt{x+6}$ are 'two separate things'. This language suggested that the two terms were distinctly different mathematical objects, and the subtraction symbol/minus sign was a third object that created the separation between the two terms. Furthermore, she followed up this statement by describing a contrasting example, where the two terms were 'together', which looked like $2\sqrt{x+6}$, so there was no subtraction symbol and the 2 was being multiplied with the $\sqrt{x+6}$. The implication here was that the operation of multiplication brings two terms 'together', unlike the subtraction symbol, which made the terms 'two separate things'. This idea proposed that there were different rules that applied when terms were together (multiplied) as opposed to when they were separate (subtracted). Multiplication suggested a *togetherness*, whereas subtraction suggested an *apartness*.

Stephanie, Sara, and Lydia's conversation included several noteworthy uses of language which provided the following opportunities: addressing the relationship between individual fractions and a single fraction with a common denominator; introducing the implications of the word *solve*; indicating the relationship between numbers that is implied in the phrase *to pull out*; looking at suggested ways that numbers/terms interact with one another depending on the operation between them.

4.4. Task #3: Greyson, Madelyn, and Carol

Throughout much of this last group's recording, the three students worked on the task individually while occasionally asking questions of one another and interjecting in their peer's work at different points. While Greyson mostly worked on his own through this task, Carol and Madelyn worked together more often, trying to solve the equations.

One noteworthy interaction happened while Greyson and Madelyn started to work on question #2, $\frac{1}{2x} + \frac{4}{x} = \frac{9}{2x}$:

> Madelyn: Okay, Greyson, do you wanna do number two?
>
> Greyson: No, I don't like math. I mean, no, I do like math, I just ... I'll just stop talking.
>
> Madelyn: Haha! Okay, so we wanna get the same denominators, so we'll multiply the four by two so that we have the same denominator and then we can cancel them all out. Actually, I'm going to write it **already simplified**. So, we get one plus four x equals nine. And then we'll go four x equals, we move the one over?

In this exchange, Madelyn was articulating the mathematical steps to create a common denominator between the three fractions then to be able to simplify the equation to a linear function. In this context, her statement, "I'm going to write it already simplified", had a few implications. First, this expression communicated that there were steps that could be written out or demonstrated but would not be. Her process did not need to be written out in full, or at all, to continue solving the equation. Another implication was that simplification is something that is *done to* the equation by the student/mathematician; it is not inherent in the mathematics. Lastly, there was the suggestion that mathematical steps or processes can take place unseen. The product is shown, but the process is not visible, or it does not have to be.

Another notable interaction between Carol and Madelyn happened while they were working on the third question, $12 = 4\left(x + \frac{1}{2}\right)^2 + 3$, once they had gotten it to the state $12 = \ 4\left(x^2 + x + \frac{1}{4}\right) + 3$:

Madelyn: So, then, times them [terms inside the brackets] by four.

Carol: Then that would be one. . . Then simplify it. I'm pretty sure you can **drop the brackets**, can't you? Like, the brackets here, **you don't have the keep them**.

Madelyn: Oh, so you would get four *x* squared. . .

Carol: Yeah, you can just move them over.

Greyson: Oh my. We're getting somewhere, everyone!

The students had already been working through the equation for a few minutes before having this conversation. Carol's statement, "drop the brackets", has some interesting implications, the first being that the brackets are an expendable symbol at this point in the process and removing them is as easy as *dropping them* or making them disappear. The language indicated that removing them could be done on a whim, especially since the student ended her comment with the phrase, "you don't have to keep them". This second statement indicated that the brackets could still be present, but they were unnecessary now. Taking this idea further, there were mathematical implications that there are objects and/or symbols that are present in equations which may not be necessary or become unnecessary (over time). Lastly, based on the comments made both in this interaction and in the one these students had during the second question, there was the inference that simplification is valued in mathematics, so if something is unnecessary, it should be removed.

Carol's starts to work on question #5, $\sqrt{2x + 11} + \sqrt{x + 6} = 2$, on her own, and she has an interesting moment as she speaks aloud to herself:

Carol: [Question] Five. Hmmm . . . Plus eleven *[mumbling and sound of whiteboard marker]* **Um de dum dum dum**. Square root them both, and two x plus eleven equals . . . **bum de bum bum bum**. Two times two is four. Oh. Negative . . . *[sound of whiteboard marker on the whiteboard].*

While onomatopoeia was already introduced in Task #2, when Stephen used the word *bop*, Carol's words in this task applied this device in a different way. Instead of the sounds being a possible indicator of a mathematical process occurring, as in Stephen's case, here it appeared to be sounds the student made in conjunction with working through the equation. Although these sounds seem to have little meaning for understanding the mathematics Carol was doing, they still implied that something was being done and emphasized the human element of the verbal language used in solving mathematical equations (an instance of self-talking).

Although the students in these examples did not always use precise language from within the mathematics register – or words at all, like this example where Carol was using onomatopoeia instead – there was still reason to believe that work was happening and she was making progress in solving the equation. There was the possibility that making sounds was part of the individual working out the problem. Just because the words were not mathematical, does not mean that they could not point to mathematical work.

A final interesting interaction happened later in the conversation around question #5. After Greyson had tried to solve it on his own, without success, Carol's voice got louder in the recording, and she started the question over from the beginning. For approximately three and a half minutes, she verbalized her process of working through the equation and almost exclusively used the language of precise mathematical operations. Just before their time ran out, and students were expected to return to class, the recording ended with this interaction:

> Carol: Okay, I'll do mine [her work] now. **Plus** eleven. **Equals** two. *x* **plus** six. Cause, like, **this should be a negative**, right? There we go. Then you're **squaring this side**.
>
> Greyson: I'm pretty, I'm gonna bring them all to this, so, that's negative . . .
>
> Carol: Let's just, like, see how this goes.

Although it is only a small part of the conversation, Carol was using mathematical language like 'be negative' and 'squaring this side', which made her steps clear as she solved the problem. Through the process of analyzing these transcripts, I have noticed my assumption as the teacher that students who used mathematically precise language comprehend the mathematics. Thus, the implication is that the use of language found in the mathematics register is an indicator of mathematical knowledge and comprehension.

Throughout Greyson, Carol, and Madelyn's conversation, there were examples that emphasized the value of simplification, suggested onomatopoeia as an indicator of human processing, and revealed the assumption that the use of language in the mathematics register was a sign of mathematical knowledge.

4.5. Summary

Spontaneous student discourse is full of interesting terms, phrases, and images, and I was surprised by many of them. The previous excerpts from the data included instances of onomatopoeia, language of TIME, the use of the algorithm *FOIL* as a mathematical verb, the metaphor of MOVEMENT, the significance of simplification, and other diverse phrases and terms. Throughout their conversations, the students' informal language blended in effortlessly with the mathematics register or superseded it entirely. After looking at some of the phrases that independent groups used, I wanted to take a step back and see some of the themes that emerged across the groups, common language, or types of language, that all or many groups used, and look more deeply at the metaphors that came out of student conversations. This I will do in the following chapter.

Chapter 5. Thematic Analysis

This chapter begins by addressing the language that students used across multiple groups, contexts, and tasks, and the categories into which I put their terms. First, I explain the reasoning behind the labels I created for my categories and meta-categories (that is, categories of categories). Secondly, I focus on my analysis of some themes that I drew from the category of METAPHOR.

5.1. Explanation of categories

While reading through transcripts, I noticed similar words used by various students and wanted to see what kinds of language the students most commonly used in general throughout the class. I also wanted to generalize more specific words or phrases that I noticed students using. For example, one group used the language of *cancel*, while another used the language of *get rid of*, which I felt were trying to communicate a similar concept in different ways, so I made them into one category: ELIMINATION.

After I transcribed the student recordings, I read through the conversations and created twenty-six categories to group the different kinds of language I found. I subsequently noticed that the categories I had created could be grouped under similar meta-categories which allowed me to see the larger themes that came out of the words and phrases that students used. While developing the meta-categories, I explored with different headings and ways of ordering the categories before settling on the final version below. While I still feel that there are other ways of breaking up the various categories, or other lenses through which to organize the different kinds of language, these labels are the ones that make the most sense to me based on the focus of this study, with an emphasis on the meta-category of **METAPHOR**.

Table 1. **Categories of Terms**

TYPES OF LANGUAGE	
Colloquial Phrases	Onomatopoeia
Correct Mathematical Terms	Quasi-Mathematical Terms
Incorrect Mathematical Terms	'Verbified' Nouns
VERBS	
Be/Is	Put
Do	Need (or modal verb Must)
Find	Solve
Get	Want
Keep	
METAPHORS	
Balance	Movement
Elimination	Relationship/Connection
Insertion and Expansion	Sameness
Isolation	Transformation
EXPERIENCE OF MATHEMATICS	
Seeking Correctness	Time

In the following sections, there are explanations of each of the meta-categories and their inside categories. When indicating a meta-category, I use small caps with bolding (e.g., **TYPES OF LANGUAGE**) and for the contained categories – which may be referred to as a 'sub-category' or simply 'category' – I use small caps (e.g., COLLOQUIAL PHRASES).

Types of language

While all the meta-categories are *types of language*, the sub-categories listed under **TYPES OF LANGUAGE** were broader than the others and did not have the same kind of repeated, specific examples like those that fit within **VERBS** and **METAPHORS**. For example, the first sub-category, COLLOQUIAL PHRASES, consisted of phrases that may have also been included in the **METAPHORS** meta-category, but the terminology that the students used was more informal and common in their everyday language, especially compared with the mathematical discourse more often seen in teacher–student interactions in a mathematics class (Temple & Doerr, 2012). Two instances were, "**Simple mode**" (as addressed in Chapter 4) and, "I mean, it's kinda like, **it's**

basic”. The phrases in this category were highly ambiguous, and it was often difficult to understand what operation or object, if any, was being referenced, if any.

The three categories CORRECT MATHEMATICAL TERMS, INCORRECT MATHEMATICAL TERMS, and QUASI-MATHEMATICAL TERMS related to one another. Not surprisingly, CORRECT MATHEMATICAL TERMS was the category referenced most often, as it signified any time a student used language within the mathematics register. This included nouns such as *like terms*, *the distributive property*, *difference of squares*, and *trinomial*, as well as verbs for mathematical operations like *add*, *multiply*, *divide*, and *factor*. The category of INCORRECT MATHEMATICAL TERMS was populated with the same language, but only when students used a term incorrectly. One example is when a student was moving from the step $3(x-1)$ to $3x-3$ and said, “**Factor** that first: three x minus three”. He used the word ‘factor’ but applied the distributive property, which is the inverse operation to factoring. Lastly, the QUASI-MATHEMATICAL TERMS referred to language that the students used interchangeably with the more formal CORRECT MATHEMATICAL TERMS and had been accepted within the school-based mathematics register, but not necessarily the general mathematics register (Presmeg, 1997; Voigt, 1994). Some examples were *plus* (instead of *add*), *times* (instead of *multiply*), *flip* (instead of *reciprocal*), and *FOIL* (to indicate the distributive property).

The ONOMATOPOEIA category is self-explanatory and included instances when students used ‘sound-words’ as they were working. I showed two examples in Chapter 4.

The last category, ‘VERBIFIED’ NOUNS, was a heading I invented to classify cases when students used nouns to indicate an action or mathematical operation. Some examples were, “So, **common denominator** five and seven” and “Let’s **like terms** that”. In both situations listed, students were using nouns to communicate that they were doing something like creating a common denominator or combining like terms. They bypassed verbs or formal operations and simply used the nouns to communicate an operation.

Verbs

Unlike the meta-category **TYPES OF LANGUAGE**, **VERBS** does not require the same level of explanation. The sub-categories under this heading consist of verbs that were frequently used by students as they communicated their activity or action. Sometimes their use of the different verbs was passive, sometimes active. There were several verbs that students used that were included in the **METAPHORS** meta-category, because I indicated that they belonged to a large metaphor instead of being considered as a singularly used verb. An example of this is when students used the word 'make', which is not listed here, because it is included in the **METAPHORS** meta-category under the sub-category of TRANSFORMATION.

Within **VERBS**, the only two sub-categories that I want to comment on are BE/IS and NEED (OR MODAL VERB MUST). For BE/IS, I originally called it *Identity* to include all the conjugations and tenses of the verb *to be*, but it fitted better in the **VERBS** meta-category after I went back and looked at the transcripts. For the NEED (OR MODAL VERB MUST) category, I had initially called it *Necessary/Mandatory* because of the students' frequent use of *need*, *must*, *have to*, *gotta*, and other similar terms. While there are a variety of ways *necessity* was communicated, the word NEED seemed appropriate as an umbrella term for all the phrases students used, especially since they were all in some verb-form or an informal version of it (unlike the word *necessary* which is a noun or adjective).

Metaphors

Even with all the work I have been doing with metaphors, I still had a difficult time determining which categories should go under this heading. Some of the sub-categories were conventional metaphors (like BALANCE and MOVEMENT), but others seemed to me less sure, especially those that fit well within the mathematics register (like INSERTION AND EXPANSION and ISOLATION). However, as many scholars have noted, there are conventional metaphors that are so deeply embedded in the language of the mathematics classroom that we do not see them as metaphors anymore (English, 1997; Pimm 1987; Presmeg, 1997), which may explain part of my struggle.

I listed the INSERTION and EXPANSION metaphors as a single category because students used the two images interchangeably, predominantly when applying the distributive property. Some of the phrases that fell under this category were, "Let's multiply the brackets **in**", "Then you **expand** that", and "Does [a term] **go into** [another term]?" I also included the phrase, "plug it **in**", which students used to indicate they were inputting a solution for the variable.

While the ISOLATION category could have gone under the **VERBS** meta-category, since students used the term 'isolate' to talk about a given variable, they also used the phrase, "get [variable] **by itself**". While there is a literal understanding of isolation, the phrase "get x by itself", had metaphorical implications associated with human relationships and interactions, which is why I placed it under **METAPHORS**.

For the RELATIONSHIP/CONNECTION category, I put in student language that indicated an association between two objects or ideas. I included any phrase with words like ‘together’, ‘each other’, ‘all the things’, ‘everything’, and ‘the whole thing’, because they were nouns, adjectives, and adverbs that indicated relationship within the equation. I also included terms like ‘combine’ and ‘in common’. One of my favourite lines from this category, one that is rife with relational language, was when a student expressed loudly, “They’re **so close** but they’re **so unrelated** at [the] exact same time!”

Within the SAMENESS category, much of the language could have fit under the heading of RELATIONSHIP/CONNECTION (and several of the student examples started in that category); however, the word ‘same’ was used so often that I decided to create a separate classification. The word was used in the context of ‘the **same** side’, ‘the **same** denominators’, and ‘the **same** thing’. It was also used to articulate that students had ‘made [some objects] **the same**’ and to ask, ‘should [something] be **the same**?’. The word was indiscriminately attached to mathematical objects and operations alike, and it did not always indicate that two or more things were identical, which is why I placed it under **METAPHOR**.

Lastly, I go more in depth with the metaphors of BALANCE, ELIMINATION, MOVEMENT, and TRANSFORMATION later in this chapter, so I will not explain them here.

Experience of mathematics

The two categories under this heading, SEEKING CORRECTNESS and TIME, were ubiquitous throughout all eleven recordings, and I felt that they were representations of how the students experienced their lived reality of the mathematical process within the assigned tasks. For the category of SEEKING CORRECTNESS, there were numerous instances of students asking questions like, “Does this [solution] work?” and “Can we do [this operation or action]?” They were continually verifying steps with one another and asking questions that indicated they wanted to know if the work they were doing was correct (whatever that might mean to the student asking). This language did not indicate anything about the actual mathematics the students were doing, but it did reveal the experience the students were having while they worked through the tasks.

Within the TIME category, I included any terms that indicated the students were doing the task in a sense of time, not like a static mathematical equation (which is often how it is viewed on the page). Words included in this category were ‘first’, ‘then’, ‘now’, ‘afterwards’, ‘later’, ‘before’, as well as other similar terms that indicated a sense of present, past, and future. Much like the phrases under SEEKING CORRECTNESS, the language in this category did not highlight anything specific about the mathematical operations or objects but was an indicator of the student experiencing while trying to solve the equations.

All four of these meta-categories had interesting features of language found in the implications of mathematics itself and in the student experience while engaging with the mathematics. While I could have chosen any of the meta-categories and their sub-categories to investigate further, my starting point for this project had been a curiosity about the kind of metaphors students used while talking about mathematics. Furthermore, in looking at the language that students used most often throughout the transcripts, several of the metaphors were near the top of the list. Thus, the rest of this chapter will be spent addressing four of the categories I put under **METAPHORS**.

5.2. Focus on metaphor

I have long been interested in the intersection between mathematics and language, and some of the questions I had at the start of my master's degree were:

1. What metaphors do students use in mathematics class?
2. How do they use them?

As I paid more attention to my students' spoken language in class, there were four dominant metaphors that I heard them using: BALANCE, TRANSFORMATION, ELIMINATION, and MOVEMENT. After formalizing the plan for my thesis and then going through the transcripts, these four metaphors continued to play a dominant role in student discourse. Throughout my transcriptions of the student conversations BALANCE was referenced 18 times, TRANSFORMATION 24 times, ELIMINATION 54 times, and MOVEMENT 62 times. The rest of this chapter is an exploration of the language students used to communicate each metaphor and different features of their language.

Movement

As Lakoff and Núñez (1997) highlighted, conceptual metaphors in mathematics often begin with spatial reasoning, and the metaphor of MOVEMENT in algebra is a fundamental example of this idea. When students learn arithmetic, they are given the grounding metaphors of "Arithmetic Is Object Construction" (p. 35) and "Arithmetic Is Motion" (p. 37), both of which have direct connections to student language of MOVEMENT in algebra, where they communicated the mathematical terms as objects that could be moved across 'the bridge' (itself a metaphor) of the equal sign. Within my students' conversations, the most common words they used were 'move' and 'side', and there were a variety of ideas and operations that they indicated through these two words.

The first example came from Amy, Jennifer, and Casey's conversation as they started to work on the third question of Task 2, $\frac{x}{x-2} = \frac{2}{x-2} + 2$:

Jennifer: So, I **move this one** [the fraction $\frac{2}{x-2}$] **to the left**?

Amy: Uh.

Casey: Oh, do you times this [the equation], like, for both **sides**? *x* minus two?

Jennifer: I can do that after.

In algebra, there is an arbitrary nature regarding which side mathematical objects are on, in the sense that objects can freely move from one side to another while still maintaining the truth claim of the equation. For this equation, the students used a spatial awareness to label the sides of their equation; in branding one as the *left* side, they implied that the label for the other was the *right* side. Within that set-up, the students communicated that the mathematical objects present, in this case the fraction $\frac{2}{x-2}$, could *be moved* from one side to another; however, they gave no indication of what operation *move* was substituting. (In this instance, *move* was said in place of subtraction.)

Another example came from David, Stephen, and Jennifer's exchange at the start of their work on Task 3, question #5, $\sqrt{2x+11} + \sqrt{x+6} = 2$:

David: You **move that, move that** [$\sqrt{x+6}$].

Stephen: **Move that bad boy over** [same term].

David: **Move that boy. Over** to the two. Oh, wait, is it minus now?

Jennifer: Yeah, minus.

Stephen: Yeah.

In this recording, the students were speaking quickly and with high-pitched tones as they repeated the imperative of *move*, almost like a chant. Like the first group, their use of 'move' did not indicate what operation was happening, and the students then had to clarify the sign of the term they moved: "Oh, wait, is it minus now?" Later examples show how students used the language of MOVEMENT but did not clarify the operation involved, which caused some groups to use the incorrect sign moving forward as they solved the equation.

While Inan, Millie, and Stephanie worked on Task 2, question #3, $\frac{x}{x-2} = \frac{2}{x-2} + 2$, the group decided to give all three terms a common denominator as their first step. They then had this exchange:

Millie:	So, maybe it's like, you times *x* minus two [to the 2], right?
Inan:	Yup, so then it would be two *x* minus four over *x* minus two [changing form of second term on right side].
Millie:	Okay, and then minus two, *x*, uh, two *x* minus four, and then over *x* minus two.
Inan:	Plus.
Millie:	Plus? Why is it plus? But you **move to the other side**, right?
Inan:	Did we **move it** [second term on right side in the form $\frac{2x-4}{x-2}$]?
Millie:	Should we **move it**?
Inan:	Well, I'm saying that it should be, we should keep the equal sign **here**, and then just do the work over **there**.
Millie:	Oh, okay, so you add them together, so it's like, *x* over *x* minus two equals *x* minus two [under] two *x* minus two, right?
Inan:	Yes.
Millie:	Because that's two. And then . . . what should . . .
Stephanie:	Then you can subtract that [term on left side of equation] from **this side**.
Millie:	Yeah, yeah.
Inan:	And make it equal zero.

As I stated in the previous example, the use of *move* does not carry any information about the term or the operation; for this group of students, that meant that they needed to discuss what the sign of their moved term was when Millie asked, “Plus? Why is it plus?”. It is not until later in the conversation that Stephanie explicitly stated, “you can **subtract** that [term]” that an operation is applied to the movement. Furthermore, this conversation has some interesting features because Inan and Millie went back and forth trying to determine if a term *has been* or *should be* moved to the other side of the equation; their exchange indicated that there are rules or expectations around movement because they *should* or *should not* move something, though the students did not communicate what might necessitate a move. Lastly, the students designated labels to differentiate one side from the other, but instead of using *left* and *right*, they used the deictic 'here' and 'there'. While the labels were vague and even more arbitrary than labelling them *left* and *right*, they gave the group temporary placeholders for the two sides and allowed them to indicate when certain actions were being done in one designated area and not the other.

Along with 'move', another significant word within the movement metaphor was 'side'. Sometimes students used 'side' in conjunction with 'move', and sometimes they used it on its own. One example came from Rose, Madelyn, and Kate's group when they worked on Task 2, question #6, $\frac{5}{4z-2} - \frac{1}{1-2z} = \frac{7}{3z+6}$. They began the question by trying to find a common denominator for all the fractions. They had decided that a common denominator for all three fractions would consist of multiplying all three denominator terms together, but they first wanted to see if it would make a difference to rearrange the existing fractions:

Rose: Should I just **put this one** [fraction on right side] **on the other side**? Oh! I could just **switch this one** [fraction with denominator of $1 - 2z$] **on the other side** and it would plus - no. Okay, I'll just do. . . I'll just **subtract this** [fraction on right side] **all from this side.** So, five [over four z] minus two, [one] over negative two z plus one, subtract seven over [three z plus] six, equals zero. And then I have to do it [create a common denominator] to every single one.

Kate: Yup.

The group tried a few ideas for creating a common denominator for the fractions in the equation and came to the plan of writing their equation in the form $\frac{5}{4z-2} - \frac{1}{-2z+1} - \frac{7}{3z+6} = 0$ before moving forward with common denominators. In their conversation, they did not use the word *move*; their verb choices instead were 'put', 'switch', and 'subtract'. The first two verbs did not communicate any specific operation, but Rose eventually stated an explicit operation with the word *subtract*. Furthermore, Rose used a lot of *side*-language that indicated freedom of movement from one side to another.

Ryan, Casey, Jennifer, and Amy also used the word *side* several times while working on Task 2, question #2, $\frac{3(x-1)}{2} = z - 2$ (uncorrected version):

Casey: I would **times two to the other side.**

Ryan:	Times two?
Casey:	**Move this one** [the two in the denominator] **there** [to the right side].
Jennifer:	Yeah, **times two to the other side**.
Casey:	Like, **times two for both sides**, so you can cancel this one [denominator of the fraction], and put it, like, yeah.
Ryan:	Okay, so you're saying to do this? Make it . . .
Amy:	Three x minus one and [equals] z minus one.
Jennifer:	Minus two. In brackets.
Ryan:	Wait, how do you divide? How does dividing make it. . .
Amy:	No, cause you multiply.
Casey:	Oh, you don't need to make it [the 2 in $z-2$] four yet. You just. Do it like this. There's a bracket here. You need a bracket here [between the multiplied two and the $z-2$].
Ryan:	I'm confused about -
Jennifer:	It's just . . . And then you, like, three x minus three [using the distributive property on the left side of the equation].
Casey:	And then two z minus four. And then, do you **move it to the other side**?
Ryan:	Wait, how'd you get? Why'd you multiply these [the $2(z-2)$]? Why is this . . .
Casey:	Oh you just. How do you say it?
Ryan:	Oh, you mul- you FOIL'd it? Oh, you FOIL'd it. That makes sense.
Casey:	Yeah. And you **move negative three to the other side. It's going to be plus three**. So it's going to be negative one.

The students' conversation indicated the steps below:

(1) $\frac{3(x-1)}{2} = z - 2$

(2) $3(x-1) = 2(z-2)$

(3) $3x - 3 = 2z - 4$

(4) $3x = 2z - 1$

In the first five lines among Ryan, Casey, and Jennifer, the initial direction that included the word 'side' stated an explicit mathematical operation: "times two to the other side". However, the follow-up when Ryan asked for clarification was vague: "move this one there." Casey's third comment, "Like, times two for both sides, so you can cancel this one, and put it, like, yeah", appeared to be trying a different approach, one where she tried to explain why the number 2 needed to be multiplied and that it was an operation being done to 'both sides', unlike before when it was only stated as being done to one side. Like the previous example, this exchange showed the flexibility within using the word *side*.

While working on Task 3, Lydia, Sara, and Stephanie used the terms 'bring' and 'leave' several times throughout their conversation, which were also forms of MOVEMENT-language. The first time the words came up was while the students worked on the third question, $12 = 4\left(x + \frac{1}{2}\right)^2 + 3$. They had worked through the following steps:

(1) $12 = 4\left(x + \frac{1}{2}\right)^2 + 3$

(2) $12 = 4\left(x^2 + x + \frac{1}{4}\right) + 3$

(3) $12 = 4x^2 + 4x + 1 + 3$

(4) $12 = 4x^2 + 4x + 4$

Their conversation began with Stephanie considering what would happen if she had all of the terms on the same side of the equation:

Stephanie: So then that would be zero [what the equation is equal to].

Sara: Oh, can I just **bring the four over here**? So, that I **leave all the *x*s here**?

Lydia: Oh, I thought, uh, why don't we try factoring?

Stephanie: Yeah, we can factor after. So, if we **bring this** [the twelve] **over**. Because, you know how you have to have squared *x* and then a number? To factor?

Sara: Yeah.

Stephanie: So, if we could do that, then we could try factoring.

Sara: Factoring first?

Stephanie: Well, after we **move the twelve over**.

Sara: Oh, I see. I see.

Unlike the use of 'move', the words 'bring' and 'leave' assume that one side of the equation is a focal point to which objects are being brought or away from which they are being left. This language places greater emphasis on one side of the equation over the other. However, like 'move', they give no indication of what operation is being done.

To summarize, students communicated MOVEMENT predominantly through the words 'move' and 'side'. For both terms, there was a loss of information regarding the mathematical objects and operations involved in the movement. Furthermore, the language of *side* conveyed a fluidity and flexibility within the metaphor.

Elimination

ELIMINATION was a common metaphor for students to use when they were referring to something that they were removing from a term or equation. There were a variety of ways that students used the language of ELIMINATION, and, although there were nuanced differences between each of their words or phrases, each of them indicated removal or deletion of object in some way. The examples that I will focus on in this sub-section are *cancel*, *get rid of*, *cross out*, and *take out*.

Cancel was the word that most often occurred within the ELIMINATION category, and I found it in ten of the eleven group transcripts. While it fits within the metaphor of ELIMINATION, it is also a word that is within the mathematics register. In the *Oxford English Dictionary*, there are two definitions for cancel underneath the heading of "Arithmetic":

1. "To strike out (a figure) by drawing a line through it; esp. in removing a common factor, e.g. from the numerator and denominator of a fraction;
2. "To remove equivalent quantities of opposite signs, or on opposite sides of an equation, account, etc.; to balance a quantity of opposite sign, so that the sum is zero" ("Cancel", 2020).

Based on the work my students did for Tasks 2 and 3, they most frequently used the word *cancel* to refer to removing or canceling out denominators, which falls under the first definition listed above, as students were removing a common factor from the whole equation. Lydia and Carol's conversation during Task 2, question #1, $\frac{x}{2}+\frac{x}{3}=5$, demonstrated this use of *cancel*:

Lydia: Okay, the first thing we should probably do is make them [the denominators] all equal so we can **cancel all the denominators**.

Carol: Wait, what are we supposed to be doing in this section?

Lydia: We're supposed to solve for *x*.

Later in their process, the students determined how to turn the whole number 5 on the right side of the equation into the fraction $\frac{5}{1}$ and were multiplying both the numerator and the denominator by six to get a common denominator with the two fractions on the left side:

Lydia: What's five times six? Five times six is thirty, right?

Carol: Yeah, I'm gunna go with thirty.

Lydia: Okay, so thirty over six. And then because all this we can **cancel all of the** [denominators]

Carol: Oh, yes!

Lydia: And so now we have three *x* plus two *x* equals thirty.

Both times Lydia said 'cancel', she referred to removing the denominators of the fractions from the equation. In this case, she and Carol had found a common denominator for each term in the equation and were able to *cancel* out all of the denominators to get a linear equation. Algebraically, the work that they did was:

(1) $\frac{x}{2}+\frac{x}{3}=5$

(2) $\frac{3x}{6}+\frac{2x}{6}=\frac{30}{6}$

(3) $\left[\frac{3x}{6}+\frac{2x}{6}=\frac{30}{6}\right]\times 6$

(4) $3x+2x=30$

So, in this context, *cancelling out* the denominators was the action of multiplying the entire equation by the common denominator term in order to remove all denominators from the equation.

Amy, Ryan, Jennifer, and Casey's group used *cancel* in a similar way during Task 2, question #5, $\frac{2}{x+5}+\frac{20}{z^2-25}=\frac{3}{z-5}$ (uncorrected version):

Jennifer: How do we do this? Oh, yeah, yeah, yeah. We factor.

Casey: Oh! Okay

Jennifer: So that's *z* minus five times *z* plus five [factoring z^2-25]

Amy: And twenty!

Jennifer: Then z minus five times z plus five [creating a common denominator for $\frac{3}{z-5}$]

Ryan: And make it a common denominator here too, right? And then you can **cancel all of the denominators out**.

Amy: Oh! Yeah.

These students were trying to do the same operation as Lydia and Carol, where they created a common denominator for each term in the equation and then multiplied the entire equation by the common denominator in order to remove all denominators. The only difference between the previous example is that this group said, "cancel . . . out", explicitly indicating a removal from the equation.

Rose, Casey, Tony, and Ryan's group ran into some confusion that involved their *cancellation*-language while working on question #2 in Task 3, $\frac{1}{2x} + \frac{4}{x} = \frac{9}{2x}$:

Rose: Okay, next one. One over two x plus four over x equals nine over two x.

Casey: Mhm.

Rose: Okay, so the same denominator. So, we can multiply by two. We have one over two x plus eight over two x equals nine over two x. And then we can just **cancel** out the denominators.

Casey: Yup.

Rose: So, then we have nine equals nine. Wait, we can't do that, right?

Casey: *[giggles]*

Rose: Maybe I can **cancel** out these denominators, but not everything, right? **Cancel** out those? Or should I **cancel** out everything?

Casey: Yeah, you can **cancel** everything at the same time.

Rose: Then we have no x left.

Casey: Yeah. So, one plus eight equals nine. Hmmm . . . Oh, you probably cannot **cancel** them yet.

This question was the same one that the teacher helped Stephanie, Lydia, and Sara with in Chapter 4, where the 'truth' of the equation was not dependent on the value of x. Rose and Casey did not realize what it meant when they got the equation to the state $9 = 9$, and, instead, they went back through their steps to see if they could *cancel* different denominators in the equation without *cancelling* all denominators (removing the x-variable for which they were trying to solve the equation). Thus, as seen in the examples listed, the most common use of the word 'cancel' was to address removing common denominators from an equation by multiplying the whole equation by the common denominator term.

Along with *cancel*, students used the phrase *cross out* for a similar purpose. This action also falls under the *Oxford English Dictionary*'s definition of *cancel* because it includes the language, "To strike out (a figure) by drawing a line through it" ("Cancel", 2020), which is synonymous to the act of crossing something out. A clear example of this usage came from Lydia, Sara, and Stephanie's conversation as they began question #1 in Task 3, $6-\frac{8}{x}=10$:

Lydia: 'Kay, so, make it a common denominator.

Stephanie: Yeah, so, multiply six by *x* which is six *x* over *x*.

Sara: What about ten?

Stephanie: Oh yeah! Should we multiply ten by *x*?

Sara: Or do we do that after? Cause I know sometimes . . .

Lydia: Well, if you do, you could make it equal to zero and then you can **cross out the denominators.**

Sara: Yeah.

Stephanie: Yeah, so six *x* over *x* minus eight [over] *x* is equal to ten *x* over *x*.

Lydia: And then . . .

Stephanie: And then you **cancel out the denominators**, right?

Lydia said, "cross out the denominators", while Stephanie later used the phrase, "cancel out the denominators". They both referred to the same process addressed in the previous examples with *cancelling*, but Lydia used different terminology to indicate eliminating the denominators.

Howie, David, and Diana had a similar exchange while working on question #3 in Task 2, $\frac{x}{x-2}=\frac{2}{x-2}+2$:

David: [Question] Three. *x* over *x* minus two equals two over *x* minus two. *x minus* two, buddy . . . *x* minus two.

Howie: Oh, so what we can do is divide this [the two on the right side] by x minus two.

David: Wait, why would you do that?

Diana: So, like, make it common denominator?

Howie: So that you can **cross out** the whole bottom.

David: Oh!

Howie: So, x equals two plus two . . . *[mumbling].* X equals four.

When Howie said, "you can cross out the whole bottom", he was using the phrase like other students had used 'cancel out the denominators'. Although his language is informal, it communicates the same operation as *cancel*.

Unlike the language of *cancel*, some of the students used the phrase 'cross out' more generally than just removing the denominators of fractions from the equation. As Howie, David, and Elizabeth continued to work on question #3 in Task 2 (once they realized that x did not equal four), they had the interaction below:

David: So, *x* minus two equals *x* minus two.

Diana: So, wouldn't you just . . . okay . . .

Howie: And the answer is what?

David: Isn't this one? Essentially?

Diana: But that's not equal to . . .

David: No, no, no, it **crosses out**, one equals two . . .

Howie: Yeah, one equals two. That doesn't make sense.

Diana: One doesn't equal two.

David: We're trying to solve for *x* though, so you can't **cross out** the *x*.

Diana: What is *x* then? Well, like, just, what divided by what?

Howie: You can **cross out** this whole thing and then it's just two.

David: Okay, if you multiply by *x* times two. Okay, so *x* minus two equals two *x* minus two. And now you have what? *x* minus two equals two *x* minus four?

When David first said, "crosses out", he referred to the situation of $x - 2 = x - 2$, where the xs both got removed by combining like terms, and the group somehow came to the conclusion that, by *crossing out* the xs, the equation became $1 = 2$. When he used it again, "you can't cross out the x", he referred to the removal of x, though he indicated that action was one that could not be done. Lastly, when Howie said, "cross out", it was unclear what he meant. At this point in their equation, the group had removed any fractions, so he seemed to use the phrase as a general statement of elimination within the equation. Therefore, *cross out* was a phrase that students used similarly to *cancel*, but there was also greater flexibility in using it to demonstrate general elimination within an equation.

Another common phrase that students used within the ELIMINATION category was *get rid of*. Much like *cancel* and *cross out*, students used this phrase to indicate removing denominators from fractions, but it had an even greater variety of uses for general elimination than *cross out* did. Unlike *cancel*, *get rid of* is not a phrase within the mathematics register and does not have connections to specific mathematical operations. The example below shows how Inan, Millie, and Stephanie's group used *get rid of* much like others had used *cancel* in indicating that denominators were being removed from the equation in Task 2, question #2, $\frac{3(x-1)}{2} = z - 2$ (uncorrected version):

Stephanie:	Okay, so, first, that?
Millie:	Yeah.
Stephanie:	So that's three *x* minus . . .
Millie:	Three.
Stephanie:	Three. Thank-you! Over two. Equals *z* minus two over one. And then after that
Millie:	You can times two on each side?
Stephanie:	Pardon?
Millie:	Like, times two.
Stephanie:	Oh, yeah, to **get rid of** the denominators.
Millie:	Mhm.
Stephanie:	Let's just take rid of that. Take? Hm. I can't talk. Three *x* minus three equals *z* minus two. Oh wait. Times two.
Inan:	No, two *z*.
Stephanie:	So, two *z* minus four.
Inan:	Yes.

Clearly Stephanie said, “get rid of”, to indicate that she had cancelled out or eliminated the denominators from the equation. Madelyn also communicated the same use of ‘get rid of’ when she was working on the first question of Task #3, $6 - \frac{8}{x} = 10$, with Carol:

> Madelyn: First we should try to **get rid of** the denominator, right? Cause we don't want a denominator in the algebra equation; so, I think we multiply everything by *x*.
>
> Carol: Yup.

Although Madelyn was not starting with a common denominator for each term before addressing the need to *get rid of the denominator* – which is what many of the students did in the examples where they used the language of *cancelling* the denominators – her language showed that her process was similar; she multiplied all of the terms in the equation by x, which was the same operation students performed (though, often without showing it) when they *cancelled* or *crossed out* denominators.

There are several other examples of students using the phrase *get rid of* to indicate removing single terms from the equation. While working on Task 2, question #6, $\frac{5}{4x-2} - \frac{1}{1-2x} = \frac{7}{3x+6}$, Lydia had created the set-up below:

$$\frac{5}{2(2x-1)} + \frac{1}{2\text{x}-1} = \frac{7}{3x+6}$$

Carol had worked further ahead, and the two students had the following interaction:

> Lydia: Okay, for this one [$\frac{5}{2(2x-1)}$], how did you **get rid of** the [factor of] two here [in the denominator]?
>
> Carol: Uh, I times’d the two by the five, so I went times two here and then times two up there. Isn’t that how you **get rid of** it?
>
> Lydia: No, you divide it by two.
>
> Carol: Shoot! Then I totally messed it up. Okay, fine. Let’s go back.

In this example, the phrase ‘get rid of’ referred to a single factor instead of addressing the elimination of a set of common denominators. While the students determined that the steps Carol took did not help them achieve their goal to *get rid of* the number 2, their language demonstrated that the factor of 2 had been eliminated in some way and that it was no longer present in the equation.

While working on Task 3, question #5, $\sqrt{2x+11}+\sqrt{x+6}=2$, Sara, Stephanie, and Lydia also used the phrase to refer to eliminating a factor. Throughout their conversation, the steps they took looked like:

(1) $\sqrt{2x+11}=2-\sqrt{x+6}$

(2) $\left(\sqrt{2x+11}\right)^2=\left(2-\sqrt{x+6}\right)^2$

(3) $2x+11=4-4\sqrt{x+6}+x+6$

(4) $x+1=-4\sqrt{x+6}$

(5) $x+1=\left(-4\sqrt{x+6}\right)^2$ (an error here – the students did not square both sides of the equation)

(6) $x+1=16(x+6)$

(7) $x+1=16x+96$

(8) $15x+95=0$

(9) $5(3x+19)=0$

(10) $3x+19=0$

Their exchange through steps (8), (9), and (10) was as follows:

Sara: Just get, just **take five out** of there.

Lydia: Yeah, I'll **take out** five. So three x plus, uhhh, what's ninety-five divided by five?

Stephanie: Ninety-five divided by five? It's nineteen.

Lydia: Equals nineteen. And nineteen is a prime number, right?

Stephanie: Yup.

Sara: Yeah, that's our answer . . .

Lydia: So then you do, divide by five, right? And then zero is equal to . . .

Sara: Wait, how come you're dividing it by five again?

Lydia: To **get rid of** the [five]. . .

Sara: The one in the front [the factor of five]?

Lydia: Yeah.

Sara: Oh, I see.

Much like the earlier conversation, Lydia used the phrase ‘get rid of’ to indicate elimination of the factor of five. In this case, she was using the phrase to communicate division. I also bolded Sara’s statement, “take five out”, which was another form of ELIMINATION-language. Lydia repeated the phrase in her response to Sara to communicate that she was factoring 5 out of $15x + 95$. The language of *take out* has different implications than *get rid of*, based on how the students used each. *Get rid of* implied a complete removal from the equation, as the factor of five *disappeared* from the equation in step (10). However, when the students said “take out”, they referred to factoring out a number, which removed it from the $15x$ and 95, but did not eliminate it from the equation.

The last conversation I want to address within the ELIMINATION metaphor was between Carol and Lydia. While they were at the beginning of Task 2, question #6, $\frac{5}{4x-2} - \frac{1}{1-2x} = \frac{7}{3x+6}$, and discussing how to make a common denominator for the left side of the equation, they had the following interaction:

Carol: Man, this is hard. This is weird because this one $[1 - 2x]$ is backwards. Umm. . .

Lydia: Wouldn’t it be easier if we kept it cause this two is, like, it’s harder to make the same denominator if . . .

Carol: No, but, like . . .

Lydia: To make it the same

Carol: Maybe see, um, if you **switch this one out** [factor $4x - 2$], then it will look like this $[2(2x - 1)]$, and then we can then **get rid of this** two [factor in $2(2x - 1)$], by timesing it by five, right? Then times this one here $[1 - 2x]$ by negative one then that should **cancel** it out [create a common denominator between both fractions on the left side so that they can be cancelled out on that side]. Cause if we work on this one side of the equal side first, it will make the other side easier. Right? I’m gonna try it.

In this exchange, Carol had three different examples of ELIMINATION-language. She used the terms ‘get rid of’ and ‘cancel’ much like the other students had; ‘get rid of’ indicated that a number was being factored, while ‘cancel’ referred to the elimination of common denominators. However, she also used a new phrase: “switch this one out”. While the word ‘switch’ did not directly suggest elimination, the way that Carol used it by including ‘out’ in the phrase mimicked Sara’s use of ‘take out’ in the previous conversation; both Carol in this example and Sara in the previous one were indicating removal of a factor from a set of terms.

In summary, ELIMINATION-language like *cancel* falls within the mathematics register, but students also used phrases like *cross out* and *get rid of* in ways that communicated the same metaphor but were outside of the formal register. Furthermore, students used ELIMINATION-language both to refer to something being removed both from the whole equation and from individual terms.

Transformation

Within the context of algebra, transformation can simply be seen as one object changing to another object as a form of mathematical metamorphosis. The mathematical definition of "Transformation" in the *Oxford English Dictionary* includes the points:

- "change of form without alteration of quantity or value;
- "substitution of one . . . algebraical expression or equation for another of the same value;
- "also, a change of any mathematical entity in accordance with some definite rule or rules" ("Transformation", 2020)

In mathematics, *transformation* can also refer to "a new set of co-ordinates, involving a transformation of the equation of the locus" (*ibid*.), but I am not using this meaning of the word in the context of algebra. For the purpose of this thesis, the metaphor of TRANSFORMATION means that the value of an object or the truth claim of an equation is maintained, but the form is altered.

Furthermore, within this metaphor more than the other three, the students communicated an inherent sense of agency when they said that 'I', 'we', 'us', or 'you' were *doing* the transforming of the mathematical objects in the equation. In the mathematics classroom, *agency* is described as the sense students have to exert influence in their context (Boaler, 2002; Brown, 2020) and their capacity to determine a goal, plan how they will attain it, and evaluate their process (Hernandez & Iyengar, 2001; Brown, 2020). For many students in my class, their TRANSFORMATION-language indicated that they had a sense of agency that identified themselves as the active participant making decisions and enacting them.

One of the predominant ways that students communicated transformation and agency was with the word 'make'. It came up in Lydia and Carol's conversation within the first few seconds of them working on Task 2, question #1, $\frac{x}{2}+\frac{x}{3}=5$:

Lydia: The first thing **we** should probably do is **make** them [the denominators] all equal so that we can cancel all the denominators.

Carol: Wait, what are we supposed to be doing in this section [of the assignment]?

Lydia: We're supposed to solve for *x*.

Lydia stated that what "we should . . . do is make", which indicated that she and her peers were active members in the process by using 'we' and then following it with 'make'. They were choosing what should be done in the equation and then enacting that decision; in this case, they created a common denominator for the terms in the equation. Creating a common denominator was not a necessary first step, but it was the process that they chose to start with in order to solve for x. Furthermore, creating a common denominator for all three terms in the equation changed their form but maintained the value that they had in their existing format.

Casey, Rose, and Ryan also used 'make' in their exchange while they worked on the second question of Task #3, $\frac{1}{2x}+\frac{4}{x}=\frac{9}{2x}$, an equation for which the truth claim of the equation is not dependent on x. At this point in their conversation, Casey and Rose had already worked to solve the question and found that $9=9$, which caused them to go back and start again, thinking they had done something incorrectly. At this point, Ryan joined the conversation to see if he could solve it differently, by creating a common denominator for all three fractions:

Ryan: First, **we want to make this** [the denominator of the second fraction, $\frac{4}{x}$] **two *x*** so multiply it by two so this [numerator of second fraction] **becomes eight**. Okay. Then cross all this out [all denominators]. One plus eight is nine. And nine equals nine. One equals one. Wait, what are we supposed to be solving?

Casey: For *x*.

Ryan communicated agency in his first line when he said, "first we want to make this two x", referring to the denominator of the fraction $\frac{4}{x}$. Like in the previous example with Lydia and Carol, creating a common denominator is not a necessary first step, but Ryan vocalized his (and the group's) agency when he made a decision and executed it.

However, Ryan also used TRANSFORMATION-language when he said, "so this becomes eight", referring to the numerator of the fraction $\frac{4}{x}$ changing to 8 when he created a common denominator by multiplying both the numerator and the denominator by 2. The word 'becomes' is passive for the students, since the numerator of the fraction *becomes* 8; the students were not the ones doing the transforming in this case (even if their actively chosen path of creating a common denominator was what led to the creation of the 8). Thus, while language signifying transformation may lend itself to placing students in the active role, it does not necessitate that position.

Millie, Inan, and Stephanie also used ‘make’ to indicate transformation while they worked on Task 2, question #6, $\frac{5}{4z-2} - \frac{1}{1-2z} = \frac{7}{3z+6}$. They took the steps below while working on the question:

(1) $\frac{5}{4z-2} - \frac{1}{1-2z} = \frac{7}{3z+6}$

(2) $\frac{5(1-2z)}{(4z-2)(1-2z)} - \frac{1(4z-2)}{(1-2z)(4z-2)} = \frac{7}{3(z+2)}$

(3) $\frac{5-10z}{4z-8z^2-2+4z} - \frac{4z-2}{4z-8z^2-2+4z} = \frac{7}{3(z+2)}$

(4) $\frac{5-10z-4z-2}{-8z^2-2+8z} = \frac{7}{3(z+2)}$ (an error here – students did not make the 2 in the numerator positive when subtracting the two fractions on the left side of the equation)

(5) $\frac{-14z+3}{-8z^2+8z-2} = \frac{7}{3(z+2)}$

(6) $\frac{-14z+3}{-8z^2+8z-2} - \frac{7}{3(z+2)} = 0$

After getting to step (6) the students had the following conversation:

Millie: But can we simplify that one [the denominator of the first fraction]?

Inan: Probably.

Millie: Like, divide it by two?

Inan: Yup.

Millie: Four *z* squared plus four *z* minus one. Minus seven over three *z* plus two. Equals zero. Right, what should we do next?

Inan: I guess **the course of action would be take this and make it the same like term as this** ["this" referring to the two different denominators].

Stephanie: Yeah, the same denominators.

Inan: And I feel like that would take a long time.

Millie: Yeah *[laughs]*.

Inan: Okay, we know what to do, it's just that, we just don't have the energy to do it. Cool.

Inan’s statement above indicated a sense of recognition that he was in an active role because he was choosing the *course of action*, where he could *take* something and *make it* what he wanted. Although he does not use a pronoun like ‘our’ or ‘my’, and instead says “**the** course of action”, his language still suggests agency because he articulated a goal and the steps needed to achieve it.

Another word that students used to communicate transformation was ‘change’. Earlier in Millie, Inan, and Stephanie’s work for the previous question, $\frac{5}{4z-2}-\frac{1}{1-2z}=\frac{7}{3z+6}$, they were working to manipulate the original state of the equation. They had put it in the form $\frac{5}{2(2z-1)}-\frac{1}{1-2z}=\frac{7}{3(z+2)}$ when they had the following interaction:

Stephanie: Well, these two [denominators] are the same. That [first denominator] just has the two at the front, so you could just multiply the whole thing [second fraction] by two.

Millie: Are they [the denominators] the same?

Stephanie: Yeah, cause two z minus negative one, so if you move that to . . . Oh, wait, no, that's negative two z plus one!

Millie: Yeah, that's negative two z plus one. So . . .

Stephanie: Um.

Millie: I really don't know how to do this one.

Stephanie: It's a little confusing, not going to lie.

Inan: So, wait, what was our process?

Millie: Oh, how about, **we can change the negative two**, like, **change to negative two *z* plus one.** And then we can see what else we have.

Millie suggested that they change the denominator of the second fraction from $1-2z$ to $-2z+1$. This was a transformational move because it altered the form of the second fraction to create a denominator that more closely resembled that of the first fraction on the left side of the equation. As with the previous examples, this change was not something the students had to do, but it was a choice they made in order to simplify the equation and solve for the variable. While the students used ‘change’ similarly to ‘make’, the word ‘change’ has a direct connection to transformation as it specifies that one thing is being substituted for another.

Lydia and Carol also used ‘change’ when they began to write out the equation for question #6 in Task 2, $\frac{5}{4z-2} - \frac{1}{1-2z} = \frac{7}{3z+6}$:

Lydia: Okay, five over four – Oh, I like using x better than z cause, like, I don't like writing zs.

Carol: I know, zs look like twos. Like, that's what I hate about it.

Lydia: Yeah, **I'm just going to change it to *x*.**

In this case, Lydia and Carol demonstrated agency by identifying and deciding to change the z-variable to an x-variable. The students clearly recognized their control in the situation because they made an arbitrary change, but it was one that they wanted to make in order to solve the problem using their preferred variable. Much like Boaler (2002) and Brown (2020) stated, these students demonstrated that they had influence in their context and they were exerting that power.

The last phrase that was associated with the TRANSFORMATION metaphor was ‘turn into’. It came up a few times while Mike, Kate, Madelyn, and Rose were working through Task 2. The first time they said it was when they began question #3, $\frac{x}{x-2} = \frac{2}{x-2} + 2$, and they decided to create a common denominator for all three terms in the equation:

Rose: Okay, we can cancel, or do we have to . . .

Madelyn: Wait, **we have to turn this** [the 2 in the equation] **into** . . . x over x minus two.

Kate: That's two over x minus two.

Rose: Two bracket x [minus two].

Mike: Two x plus four.

Although there is some ambiguity around the signs of the different terms and which terms they were addressing at any given time, Madelyn’s phrase, “we have to **turn this into**”, suggested a transformation of the number 2 into something else, specifically $\frac{2(x-2)}{x-2}$; a metamorphosis of 2 had occurred where its value was the same, but its form had changed. She was using the phrase much like the other groups used ‘make’ and ‘change’.

The phrase came up again while this group worked on the next question, $\frac{3x}{5} - \frac{x-5}{7} = 3$. The group had created a common denominator of 35 for all three fractions/terms, cancelled out the denominators, and were combining the like terms for their equation, which now looked like $21x - 5x + 25 = 105$:

> Kate: Minus five x . . . twenty-five . . .
>
> Madelyn: Let's just actually, **let's just turn that into** sixteen x [the like terms of $21x$ and $-5x$].
>
> Kate: Check [correct], and twenty-five equals one-hundred and five.

In this case, Madelyn used the phrase 'turn that into' to indicate that the students were combining the like terms of $21x$ and $-5x$ to simplify the terms with xs in the numerator(s) to $16x$. The group was *turning them into* a single term. While Madelyn used the phrase previously to indicate the same concept of change of form, but not value, she and her peers applied it to a different operation here: in the first example, their language was used to communicate transformation of a 2 into an equivalent fraction and, and in this example, they were combining like terms. In both examples, there was a sense of student agency because either 'us' or 'we' were the active members turning the form of one object into another.

In summary, students communicated TRANSFORMATION in several ways: through the language of *make*, *change*, and *turn into*. While these terms indicated that the appearance of a mathematical object varied, the value of that object was invariant. Furthermore, within each of these examples the students demonstrated a sense of agency and communicated that they were the ones deciding on and performing the suggested changes.

Balance

For the metaphor of BALANCE, the central image is that of a scale or balance, traditionally a mechanical weighing device used to measure weight. Comparing this object with a mathematical equation enforces the idea that an equation is something that must *remain balanced* as it is manipulated in the process of solving for x. However, in his book *Speaking mathematically*, Pimm (1987) acknowledges that *an equation is a balance* is an extra-mathematical metaphor that has become "institutionalized" (p. 98) into a convention within school mathematics, so much so that most students and many teachers no longer see it as an image of comparison.

Throughout their conversations, there were two ways that my students communicated the metaphor of BALANCE: either they indicated that an operation would be performed on *both sides* of the equation or they verbally repeated the operation that they were doing. While working on the first question in Task 2, $\frac{x}{2}+\frac{x}{3}=5$, Inan, Millie, and Stephanie had an instance of both forms of communicating balance. At this point in their interaction, they had created a common denominator on the left side of the equation and had their equation in the form $\frac{3x}{6}+\frac{2x}{6}=5$:

Inan: Okay, so then we can add them [the two fractions]. Five *x* over six equals five. Then we need to, like. . .

Stephanie: What?

Millie: Uh, **times six on each side**.

Inan: Mm. I was about to go the long way. Okay! And then we just **divide five, divide five.** Bop. Beep. *x* equals six. There we go.

Stephanie: Yay!

When Millie said, "times six on each side", she indicated balance because the same operation was being done to *each side*. Inan also indicated balance when he repeated the operation, "divide five, divide five". Here balance was implied, because he did not explicitly state that the operation of dividing was occurring on both sides of the equation, but those listening in on the conversation could infer that was what he was doing, especially since he quickly solved the equation after performing the operation.

Later in their conversation, these three students used similar BALANCE-language while working on question #4, $\frac{3x}{5}-\frac{x-5}{7}=3$. They had done the following steps:

(1) $\frac{3x}{5}-\frac{x-5}{7}=3$

(2) $\frac{21x}{35}-\frac{5x-25}{35}=3$

(3) $\frac{21x-5x-25}{35}=3$ (an error here – the students did not apply the negative sign to the -25 when subtracting the two fractions on the left side of the equation)

(4) $\frac{16x-25}{35}-3=0$

(5) $\frac{16x-25}{35}-\frac{105}{35}=0$

(6) $\frac{16x-25-105}{35}=0$

Stephanie started their conversation at step (6):

Stephanie:	And then **multiply each side by thirty-five**?
Millie:	Yeah, yeah.
Stephanie:	That's the only thing I remember how to do from all this [solving algebra equations].
Inan:	It goes zero . . .
Millie:	And then move the . . .
Inan:	Could you see if one-thirty is divisible by sixteen? It is not. Okay.
Stephanie:	Um.
Inan:	Well, let's. Divisible by four, it must be. No! Not by four, but by two.
Stephanie:	Oh, simplify it?
Inan:	Um, so we got it *x* minus.
Millie:	But why you don't just move the negative one hundred and thirty to the other side?
Stephanie:	Oh yeah.
Inan:	Oh yeah! That's a thing you're allowed to do. Alright, so **plus one-thirty, plus one-thirty**. Sixteen *x* uhhh. . .
Stephanie:	Equals one-thirty?
Millie:	Yeah. And then divide it by sixteen.
Inan:	**Divide by sixteen, divide by sixteen**.
Stephanie:	I don't think one-thirty is divisible by sixteen.
Inan:	Well, that doesn't matter cause now *x* equals . . .
Stephanie:	Eight point one two five?
Millie:	Yeah.
Inan:	Epic style.

Here Stephanie used the language of doing an operation to *each side* of the equation to maintain balance, while Inan again implied balance by repeating operations, when he said both, “plus one-thirty” and also “divide by sixteen”. Based on the exchanges in this group, it was irrelevant which mathematical operation was being done to which object, as long as balance was preserved.

Many of the other students' conversations used similar language, either saying that something was being done to *both sides* or repeatedly mentioning an operation, but there were instances of altered versions of these phrasings. One example came from John, Greyson, Tony, and Stephen while they were working on the second question in Task 2, $\frac{3(x-1)}{2} = z - 2$ (uncorrected version). They were unsure of how to deal with two different variables and talked about trying another question instead but decided to input zero for the x-variable so that they could focus on isolating and solving for the z-variable. Their equation looked like $\frac{-3}{2} = z - 2$ at this point in their conversation:

Greyson: Yeah, it's just three. Negative three [in the numerator].

John: Negative three. Yeah.

Stephen: Now, we can **add two to both** [sides].

Tony: Yup. What do you do now?

Stephen: Um. Negative one over four equals z.

Greyson: Is that it though?

John: That's literally your answer. That can't be simplified.

While this group was having difficulty working through this question (as indicated by their conclusion of $2 + \frac{-3}{2}$ which does not equal $\frac{1}{4}$, like they said), they still used the language of BALANCE with the phrase, "Now, we can add two **to both**". The contextual assumption is that two was being added to both sides of the equation, but the metaphor of BALANCE was implied instead of explicitly stated. The assumption could be made that Stephen expected the rest of his group members to know what 'both' stood for, so he did not need to include the language of 'sides'.

The final example of BALANCE came from Howie, David, and Diana's conversation while working on Task 2, question #3, $\frac{x}{x-2} = \frac{2}{x-2} + 2$. The group had worked through the question and found that $x = 4$, which is incorrect, so at this point in the conversation, they tried to input the value to see if it worked:

David: Two over two plus two *[laughs]*. Four over two equals two over two plus two *[laughs]*. Two equals one plus two. Two does not equal three *[laughs]*! What?

Diana: Hm. Alright, well . . .

Howie: So maybe we don't cross this [the common denominator of $x - 2$] out.

David: Yeah, maybe we, uh, wait, that works. I don't know why we . . .

Diana: Don't you have to *[quiet mumbling]*, like, **do to one side you do to the other**?

David: Wait, I think if you divide this by, **if you divide this by negative two you also divide this one by negative two too**.

Diana: And there you have to do something with it, like, **what you do to one side you, you do to the other**, or no?

David: Oh! Yeah, yeah, yeah!

Howie: Shh. Dude, you're so loud!

David: Yeah, okay, cause *x* equal, yeah, cause **if you do this, you have to do this one too**, I'm pretty sure.

Diana: I think so.

David: Yeah, so, *x* is not equal to four.

I do not have any visuals of the work these students were doing, so it is unclear what they were referring to when they talked about *dividing by negative two* and said, "If you do **this**, you have to do **this** one too", but they were clearly using several examples of BALANCE-language. Diana made the explicit statement that, "what you do to one side you, you do to the other", which was the clearest articulation of the metaphor *equation as balance* that any of my students made.

To summarize, when students were engaging the metaphor of BALANCE, their examples either demonstrated balance by doing something *to both sides* or by repeating the operation to indicate that it was being done to both sides. While many scholars recognize that a lot of teachers and students no longer see BALANCE as a metaphor (Pimm, 1987; English, 1997; Presmeg, 1997), I still found it a helpful metaphor for students. The base structure of a balance or weights is an image that is easily transferable to more complex mathematical operations. Nothing about the metaphor inherently creates a loss of meaning (unlike MOVEMENT) in whichever operation is being referenced. It is a powerful and helpful image for the relationship between the two sides of an equation while solving for a variable.

5.3. Summary

At the beginning of this chapter, I addressed the categories I had created from listening to my students' conversations, and the meta-categories I later developed to provide a further level of organization. I then explained the reasoning behind each category and meta-category, recognizing that there are other ways of organizing the headings based on my description of each of them. However, these made the most sense to me within the context of this study, especially with metaphor being a key focus of my work. The remainder of the chapter focused on the **METAPHOR** meta-category and four of the dominant metaphors found in my student conversations: BALANCE, ELIMINATION, MOVEMENT, and TRANSFORMATION. I looked at a variety of student examples within each metaphor to see the different ways they were communicated and the inferences and implications that come with the students' spontaneous informal language.

Chapter 6. Conclusions

The question I was seeking to answer in this thesis was **what informal terms do students use in their spoken language when solving algebraic equations in small groups together?** Throughout my investigation of informal, spoken, peer-to-peer interactions, I found a variety of words and phrases that my students used while solving different algebraic equations. Some of the most commonly used terms fell into the meta-category of **METAPHOR** where they applied images and ideas outside of mathematics to help them build connections to the work they were doing with algebra. My conclusions below seek to place informal, spoken, peer-to-peer language within the context of mathematical learning in the high-school setting and determine what kinds of metaphor are most helpful for students.

6.1. What is the ideal language?

In comparing informal and formal language, spoken and written communication, and peer-to-peer and teacher-centred discourse, I see that there is value in each when it comes to student learning and comprehension of mathematics. Thus, I do not see that there is a singularly *ideal* language for learning; each form has its place throughout the learning process. However, when it comes to asking *what is the goal for student development as they mature in their ability to communicate mathematically?*, many researchers have expressed that there is a trajectory in education for students to move from informal to formal and from spoken to written communication (Caspi & Sfard, 2012; Temple & Doerr, 2012; Barwell, 2015).

I believe there is still a place for informal and spoken language. I agree with Temple and Doerr's statement that content specific language can become a distraction or barrier for students as they are learning new mathematical concepts (2012, p. 288); thus, the early stages of learning a new concept can be a place where informal and spoken language can provide a bridge to comprehending new ideas. When students are given the opportunity to practice informal, spoken language with their peers, they appear to have a greater level of comfort and are willing to take risks, using words, images, and metaphors that they might not express with their larger class or in front of a teacher-expert.

Robust metaphors compared with weak metaphors

Thinking of an ideal language in mathematics, I believe that there are some metaphors that support mathematical understanding and some that do not. As I listened for metaphors in my students' recordings, I got a sense of which terms caused confusion within groups and which terms provided clarity and greater comprehension of what the mathematics was doing. I differentiated between the two types of metaphors by labelling them *weak* and *robust*. I describe a weak metaphor like Zwicky (2010) in her article 'Mathematical analogy and metaphorical insight', when she says that weak or poor metaphors, much like weak mathematical analogies, can lead to misunderstanding when the thing a concept is seen *as* is misleading. These include metaphors that have some form of transfer from the source domain (as all metaphors do), but the transfer does not reinforce the mathematical concept as it becomes more complex or it carries with it some aspects or baggage that is not pertinent to the concept the student is learning.

The first metaphor that I put in under the *weak* label is MOVEMENT. As I listened to students use it, they frequently lost mathematical information, like which operation was involved in the process of *moving*. Losing that information was not as significant for the one- and two-step questions in Task #1, since the necessary operations fell under addition, subtraction, multiplication, and division. However, as the algebra tasks became more complex and higher-order operations were required (like square-rooting and squaring), the language of MOVEMENT and its loss of information became difficult for students in the groups to follow. An example of the confusion caused by MOVEMENT-language was during Inan, Millie, and Stephanie's conversation (on page 49) when they had to figure out what sign their *moving* term had because the operational information had been lost in their language.

Another metaphor that fell under the *weak* label was that of SAMENESS. When I explored Roxie, Sara, and Lim's discussion in Chapter 4, there was significant confusion created around which objects were *the same* and how that 'sameness' was relevant to the work they were doing. *Similarity* is an important concept in mathematics and is applied to many different operations and relationships between mathematical objects in algebra, but the language within SAMENESS loses meaning and can lead to misunderstanding.

Compared with weak metaphors, I define a robust metaphor as one that has greater points of connection to its source domain and is more flexible as the mathematics becomes increasingly abstract. In her same article, Zwicky states that a good (or robust) metaphor brings a person to greater levels of insight and enlightenment. One example of a robust metaphor is BALANCE. First, the metaphor of *an equation is a balance* is deeply embedded in the mathematics register, which shows its transferability over time and across cultures (a sign of robustness). Furthermore, when students used it, the language that they attached to it was either indicating that some operation had to be done to *both sides* of the equation or by repeating the operation to indicate that it was being applied each side. Unlike MOVEMENT-language, BALANCE-language did not force the students to lose mathematical information; if anything, it created a space for them to reinforce what mathematical operation was being done.

TRANSFORMATION was also a robust metaphor. While students were using TRANSFORMATION-language, not only did they have to specify what they were *making* into another thing, but they also used language of agency. This phenomenon was powerful because it provided an opportunity for students to make their own decisions, plan a course of action, and follow it through. The language they used within the robust metaphor of TRANSFORMATION provided a foundation for them to see themselves as active agents in the mathematics.

Language and student understanding

Throughout this research project I wondered *where is 'student understanding' in all of these words?* The first difficulty in responding to this question is that the idea of 'student understanding' proved a nebulous construct. What does it mean to confirm that a student *understands* a mathematical concept? How does a teacher measure or assess when a student has moved from *not understanding* or *misunderstanding* to *understanding*? And how does language play a role in determining 'student understanding', since it is the vehicle by which students communicate their learning? I had initially wanted to respond to these questions, but in working with my supervisors, we recognized the methodological challenge of such an undertaking.

However, I think there is a place for using spontaneous, spoken, peer-to-peer discourse as a means for assessing student understanding (however it may be defined). In looking at spontaneous conversations among my students, when students use language from the mathematics register correctly, repeatedly, and without any prompting from the teacher-expert, I felt confident in saying that that student had comprehended the associated concept and correctly made sense of it. As stated earlier, there is more to students demonstrating their mathematical learning than spoken language can provide (especially as it becomes more formal and symbolic), but this observation recognizes that there is a place for peer-to-peer, spoken language that offers teachers information about their students' learning.

6.2. Further considerations

As I look ahead to where this research could go, there are several things I want to address: the limitations of this study, what I learned as a teacher, and my further questions.

Limitations of the study

There are several limitations to this study, the first being that I only looked at one class of students in a single course over a five-month semester. Nevertheless, there was still a lot that I was able to learn from working with them, even though I could only listen to the words and phrases that were spoken within that specific class of students. While this meant that I could not reliably draw conclusions about the greater body of mathematics, I was able to collect meaningful data from my students' informal, spoken language.

Secondly, I was only able to collect data over a period of four months and with three different tasks, so there was not a lot of opportunity to see how additional phrases might have been used by students over a longer period of time. Thirdly, I did not change anything about my teaching as I went throughout the year; students were learning new concepts in the curriculum, but I did not intentionally work to help them develop their language, at least, not beyond what I would have done with students in the past before I had an active awareness of student language use. By the time I started reviewing the recordings, my semester with these students had already ended, and I was not able to provide any support for them to change the incorrect or weak terms they were using.

Lastly, while I do not see this specifically as a limitation to the study, I noticed that, throughout many of the recordings, students asked if they had found the correct solution. I had not provided an 'answer key' of any kind for them (which I might have done had it been a regular lesson where they were learning a new concept), and I found that the absence of that information created some anxiety and apprehension for many students. Were I to do this study again, I wonder if I would want to give them the solutions to ease some of that tension. Alternately, I could continue to leave the answer key out, because the focus of this work was student language, and it was irrelevant to me as a researcher if they got the correct answer or not.

What I have learned as a teacher

I want briefly to reflect on what I have learned as a teacher and how this study has and will continue to impact my practice. First off, I have an increased awareness of the variety of ways students communicate their learning and development in mathematics class. Much of my focus as a mathematics teacher has been around formal, written mathematics, especially for assessments. After this research, I want to explore how I can be more intentional about engaging both informal spoken language and formal spoken language for student learning and assessment.

Secondly, I recognize that, as the teacher, I play a central role in creating opportunities for my students to develop their knowledge of and ability to use the mathematics register. Now that I know the importance of moving between peer-to-peer conversations and teacher-centred discourse, I want to take advantage of this dynamic to support my students' growth in their language development. This realization has also inspired me to become more aware of the language I use while teaching.

Lastly, I have a greater understanding of the significance of moving from informal to formal language in mathematics. Even though I am not convinced this transition is of primary significance for all high-school mathematics students, I believe it is important for their language to develop as their mathematical knowledge increases in complexity.

Questions for further research

As I come to the end of this study, I find that I still have many questions that I felt working on this thesis raised but did not help me respond. My first set of questions are regarding student language:

- How would spontaneous student language have been different had I worked with more than one class of students? What commonalities might exist between the language of another group and the class in this study? What singular words, phrases, or terms would another group have?
- How would the language change were I to record students in a different geographical location or from a different socio-economic status? What similarities and differences would there be in a public-school setting? What about a different independent school?
- How would the terms change with a different age group? Is there a noticeable progression of language use? What if I started with students who were just learning algebra and had had little formal mathematical language training?

Some researchers are already working to answer some of these questions (Caspi & Sfard, 2012; Barwell, 2015), and I see valuable enhancement to student learning when informal language and spoken discourse is given more time and consideration in the classroom.

Regarding formal mathematical language and the mathematics register, I also have several questions, some of which are:

- When and how should elements of the mathematics register be taught? How does it help or hinder student learning of new concepts and their ability to communicate their thinking?
- In what ways does the register need to be changed or updated? How could students play a role in that evolution?
- How do metaphors fit into the register as students move from informal language to the more formal register? Are there metaphors that cannot or should not be replaced with more ‘precise’ language? If so, which ones?

As I continue to be drawn to mathematical metaphors, I find myself asking many of the same questions that Chiu (2011) has asked regarding metaphors in mathematics and how students use them:

> How and where do students learn these metaphors—from teachers, textbooks, or sources outside of the classroom? How do they put these metaphors together? Are there intermediate phases of metaphorical reasoning? Do people’s uses of standard mathematical metaphors continue changing as they develop expertise (e.g., mathematicians)? (p. 114)

Metaphor is a powerful tool as students develop their mathematical comprehension and learn new concepts. Working towards finding answers to these questions can help teachers provide greater opportunity for student understanding in their classes. I hope that I will be able to do further research in this field to explore and respond to some of these questions.

When I began this project, my goal had been to see how students used metaphors when they talked about mathematics with one another. What I did not anticipate was finding so many other categories and elements of language with such varied features and implications of mathematics. As I listened to the recordings, I found myself surprised by my students and the different ways they communicated with each other, sometimes bringing in images or words I myself have never associated with mathematics. There is a lot I hope to continue to learn from my students, both as a teacher and as a researcher, as I pay attention to the ways they speak when engaging with mathematics.

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