The minimax linear fractional programming problem with binary variables

by

Chloe Chen Li

B.Sc., Kwantlen Polytechnic University, 2018
Associate of Science, Kwantlen Polytechnic University, 2017

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Mathematics
Faculty of Science

Declaration of Committee

Name:	**Chloe Chen Li**
Degree:	**Master of Science**
Thesis title:	**The minimax linear fractional programming problem with binary variables**
Committee:	**Chair:** Natalia Kouzniak Senior Lecturer, Mathematics
	Abraham Punnen Supervisor Professor, Mathematics
	George Zhang Committee Member Adjunct Professor, Business
	Tamon Stephen Committee Member Associate Professor, Mathematics
	Binay Bhattacharya Examiner Professor, Computing Science

Abstract

The single ratio linear fractional programming problem when the denominator of the objective function ratio is non-negative is solvable in polynomial time. This result extends to several classes of optimization problems with binary variables. However, when the denominator is allowed to take both positive and negative values, even the unconstrained problem with binary variables is NP-hard. Generalization of this problem where a sum of ratios is also well studied. Experimental results on this however are restricted to non-negative denominators. In this thesis we consider the minimax version of the multi-ratio problem with binary variables. The continuous version of the minimax problem is also well-studied but to the best of our knowledge not much research work has been done on the binary version. We present various mathematical programming formulations of the problem for the case of non-negative denominators as well as arbitrary denominators. We also present algorithms to overcome some of the computational difficulties. Extensive computational analysis has been carried under the scenarios of unconstrained problems, problems with a knapsack constraint, and problems with assignment constraints to assess the relative merits of the models. The analysis disclosed some interesting insights into the structure of the problems.

Keywords: Minimax fractional programming; minimax fractional assignment problem; linear fractional binary programming; Operations Research;

Dedication

I dedicate this work to my beloved nana.

Acknowledgements

I would like to first thank my senior supervisor Professor Abraham Punnen. He introduced me to the operations research program and provided ongoing understanding, guidance, inspiration and support. Even during the special time of COVID-19 pandemic, he tried his best reaching out to help. He has given me many valuable inputs in my thesis. It is an honor to have chance to be his student. I want to thank Professor Zhang to be the member of my supervisory committee, Professor Natalia Kouzniak to be the chair of the examining committee, Professor Bhattacharya to be my examiner, and Professor Stephen to be the member of my examining committee. I would also like to show my gratitude to Professor Monagan, Professor Adcock, and Professor Randall Pyke for their support during my program. I want to thank my peer Nav Kaur, Jas Dhahan and Sajeththa Thavayogarajah for looking out for me throughout the program. Finally, I would like to thank my family for everything they have done to accommodate me through the past two years. I am grateful for all the support I received.

Table of Contents

List of Figures

List of abbreviations

LP	Linear programming
LFP	Linear fractional programming with either continuous or discrete decision variables
CP	Combinatorial programming
CFP	Combinatorial fractional programming
SLFP	Sum of linear fractions programming problem
MMFP	Minimax linear fractional programming in which all denominators are positive
MMFPU	Unconstrained linear fractional programming in which all denominators are positive
MMFPK	Knapsack constrained linear fractional programming in which all denominators are positive
MMFPA	assignment constrained linear fractional programming in which all denominators are positive
GMMFP	General minimax linear fractional programming in which all denominators are not restricted in sign
GMMFPU	General unconstrained minimax linear fractional programming in which all denominators are not restricted in sign
GMMFPK	General knapsack constrained minimax linear fractional programming in which all denominators are not restricted in sign
GMMFPA	General assignment constrained minimax linear fractional programming in which all denominators are not restricted in sign
MILP	Mixed integer linear programming
MMFP1	Linear formulation of MMFP problem using valid bounds of y defined in chapter 2

MMFP2	Linear formulation of MMFP problem using valid bounds of y defined in chapter 2 and employing binary expansion
MMFP1M	MMFP1 with y-bounds updated by a pre-solve plan for better performance
MMFP2M	MMFP2 with y-bounds updated by a pre-solve plan for better performance
GMMFP1	Linear formulation of GMMFP problem using valid bounds of y defined in chapter 3
GMMFP2	Linear formulation of GMMFP problem using valid bounds of y defined in chapter 3 and employing binary expansion
GMMFP1M	GMMFP1 with y-bounds updated by a pre-solve plan for better performance
GMMFP2M	GMMFP2 with y-bounds updated by a pre-solve plan for better performance

List of notations

X	General feasible region defined by $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, domain of a function
$\mathbf{x}$	Either the binary decision variable in $n \times n$ dimensional space as in $[x_{ij}]$ of a $n \times n$ matrix found in assignment constrained problems, or the binary/continuous decision variable in n-dimensional space found in all the other constrained types of problems
$\mathbf{x}^*$	Either the true optimal point or the optimal point produced by a solver
$\mathbb{R}_+$	$x \in \mathbb{R} : x \geq 0$
α^k	The constant term in the numerator of k^{th} objective ratio
β^k	The constant term in the denominator of k^{th} objective ratio
c_j^k	The j^{th} element of the coefficient vector $\mathbf{c}^k$ in the numerator of k^{th} objective ratio
c_{ij}^k	The $(ij)^{th}$ element of the coefficient matrix $[c_{ij}^k]$ in the numerator of k^{th} objective found in assignment problems
d_j^k	The j^{th} element of the coefficient vector $\mathbf{d}^k$ in the denominator of k^{th} objective ratio
d_{ij}^k	The $(ij)^{th}$ element of the coefficient matrix $[d_{ij}^k]$ in the denominator of k^{th} objective found in assignment problems
w_j	Weight of item j in the problem with knapsack constraints
W	Weight limit of the problem with knapsack constraints
$\mathbf{A}$	$r \times n$ matrix of real entries. In this thesis we assume that all entries are integers
$\mathbf{b}$	$r \times 1$ real vector
v_i^k	The i^{th} element of the binary vector $\mathbf{v}^k$ of k^{th} objective ratio used in binary expansion
$f_k(\mathbf{x})$	The k^{th} objective ratio: $(\alpha^k + \mathbf{c}^k\mathbf{x})/(\beta^k + \mathbf{d}^k\mathbf{x})$

y	Aggregated upper bound of all objective ratios such that $y \geq f_k(\mathbf{x}),\ \forall k \in K$
y^*	Either the true optimal value or the optimal value produced by a solver
y^l, y^u	Lower and upper bounds of y
y^k	$f_k(\mathbf{x})$
y^{lk},y^{uk}	Lower and upper bounds of y^k
z_j	Bilinear term yx_j
z_j^k	Bilinear term yv_j^k or $y^k v_j^k$
δ	Integrality tolerance of a MILP solver

Chapter 1

Introduction

The well-known *linear programming* (LP) problem is to minimize or maximize a linear objective function over a polyhedral set. An important property exploited in numerical algorithms on LP is that a local optimal solution is also a global optimal solution. The second important property of LP is that optimal solution is found at an extreme point of the polyhedral set representing the set of feasible solutions. Another function that shares these two properties is the linear fractional function with positive denominator. Formally the *linear fractional programming* (LFP) in the maximization sense can be defined as

$$\text{(LFP)} \quad \text{Maximize} \quad f(\mathbf{x}) = \frac{\alpha + \sum_{j=1}^{n} c_j x_j}{\beta + \sum_{j=1}^{n} d_j x_j} \tag{1.1}$$
$$\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$$

where $\mathbf{A}$ is $r \times n$ real matrix, $\mathbf{b}$ is $r \times 1$ real vector. For the objective function to have finite maximum value, it is required that the denominator is non-zero for all $\mathbf{x} \in X$. It is customary to assume that the denominator satisfies the condition:

$$\beta + \sum_{j=1}^{n} d_j x_j > 0 \quad \forall \mathbf{x} \in X. \tag{1.2}$$

[45][22][44]. This is known as the *denominator condition* which will be referred to in discrete case as well.

A real-valued function g defined on a convex subset $X \subseteq \mathbb{R}^n$ is *quasi-concave* over X if for any two distinct points $\mathbf{x}'$, $\mathbf{x}'' \in X$ and any real number $t \in (0, 1)$, we have

$$g(t\mathbf{x}' + (1-t)\mathbf{x}'') \geq \min\{g(\mathbf{x}'), g(\mathbf{x}'')\}.$$

If the inequality is strict, then g is *strictly quasi-concave* over X. Similarly a real function g defined on a convex subset $X \subseteq \mathbb{R}^n$ is *quasi-convex* over X if for any two distinct points $\mathbf{x}'$, $\mathbf{x}'' \in X$ and any real number $t \in (0,1)$, we have

$$g(t\mathbf{x}' + (1-t)\mathbf{x}'') \leq \max\{g(\mathbf{x}'), g(\mathbf{x}'')\}.$$

If the inequality is strict, then g is *strictly quasi-convex* over X.

The objective function f in LFP, if the denominator is strictly positive over the feasible set X, has following well-known properties:

1. f is both quasi-concave and quasi-convex [11].
2. The maximizer or minimizer, if it exists, is an extreme point of the polyhedral set defined by X [18].
3. f is monotonic on any line segment in the set X [6].

Exploiting these properties of LFP, linear reformulations have been developed so that it can be solved by linear solvers such as simplex [6] [12]. Large number of studies have been done on LFP [6]. A more general version of LFP is to have the sum of ratios as the objective function (SLFP). It can be formally written as

$$\begin{aligned} \text{(SLFP)} \quad \text{Maximize} \quad & \sum_{k=1}^{p} f_k(\mathbf{x}) = \sum_{k=1}^{p} \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j} \\ \text{Subject to} \quad & \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}. \end{aligned} \tag{1.3}$$

If the number of ratio $p \geq 2$, even if all denominators satisfy condition (1.2), SLFP is NP-hard [42]. Note that in the sum of ratio case, the property of linear fractional objective function being both quasi-concave and quasi-convex is no longer valid.

One closely related problem is the LFP with combinatorial restriction on the decision variable $\mathbf{x}$. In particular, the integer LFP is a well-studied programming problem [40]. Its applications include fractional assignment problem, fractional spanning tree problem, fractional binary problem, etc.[19] [39] [9]. A special class of combinatorial LFP problem is the single ratio 0-1 LFP. It can be formally defined as

$$\begin{aligned} \text{(0-1 LFP)} \quad \text{Maximize} \quad & f(\mathbf{x}) = \frac{\alpha + \sum_{j=1}^{n} c_j x_j}{\beta + \sum_{j=1}^{n} d_j x_j} \\ \text{Subject to} \quad & \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}. \end{aligned} \tag{1.4}$$

Majority of the studies on 0-1 LFP are on the problems where denominator condition is assumed. Checking that denominator of f in (1.4) is not zero is NP-hard [35]. In the 0-1 LFP problem satisfying (1.2), any local optimal solution is also a global optimal solution [21]. It is based on the convention that two distinct points in $\{0,1\}^n$ space are neighbour of each other if their coordinates are differed by only one element. Generally, single ratio 0-1 LFP is NP-hard [22][35]. With denominator condition (1.2) satisfied, the 0-1 LFP problem is solvable in polynomial time [22][8][9]. Among the techniques of solving 0-1 LFP, reformulation into mixed integer linear programming (MILP) model, parametrization algorithms such as Newton's method, and binary search algorithms are some common approaches [19][14]. A more general version of 0-1 LFP is to have the sum of ratios as the objective function. It can be formally written as

$$\begin{aligned} &\text{(0-1 SLFP)} \quad \text{Maximize} \quad \sum_{k=1}^{p} f_k(\mathbf{x}) = \sum_{k=1}^{p} \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j} \\ &\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}. \end{aligned} \tag{1.5}$$

Note that 0-1 SLFP is NP-hard even if denominator condition is satisfied when $p \geq 2$ [35]. The 0-1 SLFP are studied in various contexts such as scheduling, set covering, facility location, query optimization in data bases and information retrieval [5][4][3][45][41][22].

Another class of objective functions is the one expressed as the maximum (or minimum) of the given number of linear fractional functions. This type of objective function can be written as

$$\max\{f_k(\mathbf{x}), \quad k = 1, \cdots, p\}, \tag{1.6}$$

where

$$\begin{aligned} f_k(\mathbf{x}) &= \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j}, \quad k \in \{1, \cdots, p\} \\ \mathbf{x} &\in X = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}. \end{aligned}$$

Notice that the objective function itself is a result of a maximization of the p number of objective ratios at each feasible point in a linearly constrained set. If we minimize the function given by (1.6), we obtain the minimax fractional programming problem (MMFP) which can be written as

$$\begin{aligned} &\text{(MMFP)} \quad \text{Minimize} \quad \max\left\{ f_k(\mathbf{x}) = \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j}, \quad k \in \{1, \cdots, p\} \right\} \\ &\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}.\} \end{aligned} \tag{1.7}$$

This programming problem is well-studied [24][15][43]. MMFP problems pose significant theoretical and computational challenges due to its property of possessing multiple local optima that are not globally optimal[24]. Special case of MMFP is when the objective function is the maximum of a number of linear functions. The resulting programming problem can be written as

$$\text{(MMLP)} \quad \text{Minimize} \quad \max\left\{ f_k(\mathbf{x}) = \alpha^k + \sum_{j=1}^{n} c_j^k x_j, \quad k \in \{1, \cdots, p\} \right\} \tag{1.8}$$
$$\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

Note that the MMLP can be solved in polynomial time using concepts of the ellipsoid algorithm for linear programming [36]. MMLP with integrality restriction on decision variable $\mathbf{x}$ is generally NP-hard. However, there are polynomial time algorithms to solve integer MMLP [38]. Minimax linear programming problems are studied in many papers [20][1]. One of the variations is minimax combinatorial programming problem [37]. Furthermore, binary requirement can be imposed on the decision variable to obtain 0-1 MMLP which can be written as

$$\text{(0-1 MMLP)} \quad \text{Minimize} \quad \max\left\{ f_k(\mathbf{x}) = \alpha^k + \sum_{j=1}^{n} c_j^k x_j, \quad k \in \{1, \cdots, p\} \right\} \tag{1.9}$$
$$\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

Again, this programming problem is a special class of more general 0-1 linear fractional minimax programming problem (0-1 MMFP). The 0-1 MMFP can be written as

$$\text{(0-1 MMFP)} \quad \text{Minimize} \quad \max\left\{ f_k(\mathbf{x}) = \left\{ \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j}, \quad k \in \{1, \cdots, p\} \right\} \right. \tag{1.10}$$
$$\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

This thesis focuses on 0-1 MMFP problems. The 0-1 minimax fractional programming problems can be found in various fields such as engineering, economics, and applied mathematics [15]. We present two examples where 0-1 MMFP can be applied.

Example 1. (Construction business) A newly setup construction contractor needs to purchase some equipment out of a list of n different equipment. The company plans to take a number of construction jobs all of which can be grouped into p-different types based on equipment needs. The company has the following information.

$$\begin{aligned} c_j^k &= \text{ cost when using equipment } j \text{ in the job type } k \\ d_j^k &= \text{ revenue from using equipment } j \text{ in the job type } k \end{aligned}$$

where $j = 1, \cdots, n$, and $k = 1, \cdots, p$. The constraints are given by inequalities $\mathbf{Ax} \leq \mathbf{b}$. Before beginning to send proposals for the jobs, the contractor needs to select the equipment to purchase.

$$x_j = 1 \text{ if equipment } j \text{ is selected}$$
$$x_j = 0 \text{ if equipment } j \text{ is not selected.}$$

The contractor wants to minimize the worst case cost per revenue ratio as selection criteria. The optimization problem for this case can be modelled in the form:

$$\begin{aligned} &\text{Minimize} \quad \max\left\{\frac{\sum_{j=1}^{n} c_j^k x_j}{\sum_{j=1}^{n} d_j^k x_j}, \quad k \in \{1, \cdots, p\}\right\} \\ &\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{Ax} \leq \mathbf{b}\}. \end{aligned}$$

Example 2. (Uncertainty problem in business) A trading company has their cost vector $\mathbf{c}$ and profit vector $\mathbf{d}$, for n different number of products, that come with uncertainty expressible in discrete values. There are p number of scenarios. The available data includes the cost matrix

$$\mathbf{c} = (\mathbf{c}^1, \cdots, \mathbf{c}^p) \text{ where } \mathbf{c}^k = (c_1^k, \cdots, c_n^k) \text{ for } k \in \{1, \cdots, p\},$$

the profit matrix

$$\mathbf{d} = (\mathbf{d}^1, \cdots, \mathbf{d}^p) \text{ where } \mathbf{d}^k = (d_1^k, \cdots, d_n^k) \text{ for } k \in \{1, \cdots, p\},$$

and a linear set of constraints on their binary decision variable

$$\mathbf{x} \in \{0,1\}^n : \mathbf{Ax} \leq \mathbf{b}.$$

As the management of the company is made of conservative thinkers, the company chooses to make decision that minimizes the worst case scenario. Hence, their problem is written as

$$\begin{aligned} &\text{Minimize} \quad \max\left\{\frac{\sum_{j=1}^{n} c_j^k x_j}{\sum_{j=1}^{n} d_j^k x_j}, \quad k \in \{1, \cdots, p\}\right\} \\ &\text{Subject to} \quad \mathbf{x} \in X = \{\mathbf{x} \in \{0,1\}^n : \mathbf{Ax} \leq \mathbf{b}\}. \end{aligned}$$

In example 1, the objective ratios correspond to distinct job types. In example 2, the objective ratios represent distinct scenarios. A common characteristic of all MMFP problems is that one single decision variable $\mathbf{x}$ applies to all objective ratios. Example 2 demonstrates that the problem can be viewed as a single ratio robust fractional 0-1 programming in which uncertainty in coefficients is expressed in discrete model. This discrete model is made of p number of scenarios. Therefore, from the programming with uncertain coefficients point of view, the 0-1 MMFP problem is a special case of robust fractional 0-1 programming problem in which the objective is the sum of a number of ratios and the coefficients are uncertain. This more general programming can be written as

$$\underset{\mathbf{x}\in X}{\text{Minimize}} \quad \max_{(c_j^k, d_j^k)\in U} \sum_{k=1}^{p} \frac{\alpha^k + \sum_{j=1}^{n} c_j^k x_j}{\beta^k + \sum_{j=1}^{n} d_j^k x_j} \tag{1.11}$$

where U is the uncertainty set containing all possible values of the coefficients $c_j^k, d_j^k\ \forall k \in \{1, \cdots, p\}$, $\forall j \in \{1, \cdots, n\}$. Recently published paper by Mehmanchi et al. focuses on problem (1.11) in which some of the coefficients are uncertain, and the uncertainty is modelled as intervals [30]. In the work of Mehmanchi et al., all numerators are assumed non-negative, all denominators are assumed positive, and the sense of optimization is maximin.

Contribution of this thesis

To the best of our knowledge, the 0-1 MMFP problem with unrestricted numerators and denominators is not yet studied. In our investigation of 0-1 MMFP problems, numerators are unrestricted in sign. Denominators are restricted in sign in chapter 2, and unrestricted in sign in chapter 3. In each of these two chapters we present two basic MILP formulations. In chapter 2, we investigate the basic formulations and give a proposition that describes an environment in which a type of floating point rounding errors can occur. This proposition can be expanded to cover a general structure of environments in which two types of floating point rounding errors can occur. Such environment is dependent on the formulation, problem data, and bounds of the objective. We propose data-dependent bound tightening techniques to avoid these environments. Basic formulations are equipped with these techniques to generate modified formulations. We test these formulations on extensive problem instances covering three types of constraints, and comments on the results are included in the thesis. The proposed modified formulations for positive denominator problems are found to be effective if the modifications are done appropriate to the problem data. Meanwhile, unrestricted denominator problems are found to be less prone to floating point rounding errors.

The rest of the thesis is organized as follows:

Chapter 2 is devoted to the 0-1 MMFP problem where all denominators are positive at all feasible points. The chapter begins with verification of this denominator condition and its computational complexity. Two basic equivalent MILP formulations are discussed for three types of constraints: no constraints, knapsack constraints, and assignment constraints. In each type, we discuss the computations of the bounds of continuous variable y which is the aggregated upper bound of all objective ratios. The variable y is also the objective of the problem. These basic formulations are investigated for possible shortcomings. We observe the structural weakness of these formulations under certain data instances in relaxed feasible set. This combined with the nature of digital computing could result in floating point rounding errors. We present an environment that is prone to these errors in a proposition. This proposition can be extended to cover a general environment in which the errors can occur. These environments depend on the formulation, data and the y-bounds used. We give some modified formulations based on our investigation. These are basic formulations equipped with a suitable bound-tightening pre-solve plan to avoid the errors as well as to increase the computing speed. Modifications depend on the data instance. Even the modified formulations do not always avoid computational errors. Hence, we give a heuristic built on recursion which uses the information extracted from the failed computing session for the next session. For each of these three types of constraints, we perform experiments and remarks on the results are presented.

Chapter 3 is on the general linear fractional minimax programming (GMMFP) problem. In this type of programming problems, we do not assume that denominators are positive. Denominators can take any value. We begin the chapter with discussion on big-M method which deals with unrestricted denominators. Equivalent MILP formulations are developed. In this case, we have two continuous variable substitutions due to unrestricted denominators. They are the aggregated upper bound y of all objective ratios and y^k representing the k^{th} objective ratio. As in chapter 2, we discuss two basic MILP formulations and computations of the bounds for y and y^k for the same three types of constraints. Floating point arithmetic errors are not found as often as in the chapter 2. Nevertheless, modified formulations are presented because there are still chances for errors. Unlike in the case of positive denominators, the modified formulations in this chapter are not significant improvements over the basic ones. We also give the recursion heuristic which can be used when we encounter a floating point rounding error. For each of three types of constraints, we perform experiments and remarks on the results are presented.

Chapter 4 is the last chapter in which we summarize the thesis and propose some recommendations for future research.

Chapter 2

Minimax 0-1 linear fractional programming problem

2.1 Introduction

In this chapter, we study the 0-1 MMFP problem and its MILP reformulations. Recall that 0-1 MMFP problem can be written as

$$\begin{aligned} &\text{(0-1 MMFP)} \quad \text{Minimize} \quad \underset{1\le k\le p}{\text{Max}} \left\{ \frac{\mathbf{c}^k\mathbf{x}+\alpha^k}{\mathbf{d}^k\mathbf{x}+\beta^k} \right\} \\ &\qquad\qquad\qquad \text{Subject to} \quad \mathbf{x}\in X=\{\mathbf{x}\in\{0,1\}^n : \mathbf{A}\mathbf{x}\le\mathbf{b}\}. \end{aligned} \tag{2.1}$$

Throughout this chapter we assume that

$$\mathbf{d}^k\mathbf{x}+\beta^k>0 \quad \forall k\in\{1,\cdots,p\},\ \mathbf{x}\in X. \tag{2.2}$$

Verification of condition (2.2) can be done by solving the linear program

$$\begin{aligned} (\text{LP}^k) \qquad \text{Minimize} \quad & \mathbf{d}^k\mathbf{x} \\ \text{Subject to} \quad & \mathbf{A}\mathbf{x}\le\mathbf{b} \\ & \mathbf{x}\in\{0,1\}^n \\ & k=1,\cdots,p. \end{aligned}$$

Suppose $\mathbf{x}^{k*}$ is the optimal point of the problem LP^k, where $k=1,\cdots,p$. Then, condition (2.2) is satisfied if and only if

$$\mathbf{d}^k\mathbf{x}^{k*}+\beta^k>0 \quad \forall k=1,\cdots,p.$$

Complexity of this verification depends on the structure of the solution space of the underlying linear program LP^k. In general, the 0-1 linear programming problem is NP-hard, therefore this verification is also NP-hard [13]. For some special cases, each LP^k can be done in polynomial time or pseudo-polynomial time. For example, the 0-1 linear assignment problem can be solved in polynomial time [25]. Hence, the corresponding verification can also be done in polynomial time if p is at most polynomial. The 0-1 knapsack constrained linear programming problem can be solved in pseudo-polynomial time [34][28]. Therefore, the corresponding verification can be done in pseudo-polynomial time if p is at most pseudo-polynomial.

Remark: The unconstrained 0-1 MMFP problem satisfying the condition (2.2) is NP-hard for any $p \geq 2$. Note that 0-1 linear programming is NP-hard and it is a special case of 0-1 linear minimax programming when $p = 1$ [13]. The 0-1 linear minimax programming is a special case of 0-1 MMFP problem satisfying the condition (2.2). The desired result follows.

Our primary focus in this chapter is to present mixed integer linear programming (MILP) formulations of 0-1 MMFP and assess the computational efficacy of these formulations. We present two basic formulations and their modifications for general constraints. After presenting the general formulations, we consider special cases when feasible solutions are obtained by choosing

1. only 0-1 restriction (unconstrained 0-1 MMFP)

2. only one linear constraint along with the 0-1 restriction (knapsack 0-1 MMFP)

3. the assignment problem constraints along with the 0-1 restriction (assignment 0-1 MMFP).

2.2 MILP formulations of MMFP

Minimax problem can be converted into linear objective form by introducing an additional variable y, an aggregated upper bound of all objective ratios. Problem (2.1) can be rewritten as

$$
\begin{aligned}
\text{(P1)} \quad & \text{Minimize} \quad && y \\
& \text{Subject to} \quad && \frac{\mathbf{c}^k\mathbf{x} + \alpha^k}{\mathbf{d}^k\mathbf{x} + \beta^k} \leq y \quad \forall k = 1, \cdots, p \\
& && \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& && \mathbf{x} \in \{0,1\}^n.
\end{aligned}
$$

Because of condition (2.2), we can rewrite P1 as

$$
\begin{aligned}
(\mathrm{P1}') \quad & \text{Minimize} \quad y \\
& \text{Subject to} \quad \sum_{j=1}^{n} c_j^k x_j + \alpha^k \le y \left(\sum_{j=1}^{n} d_j^k x_j + \beta^k \right) \quad \forall k = 1, \cdots, p \\
& \qquad \mathbf{Ax} \le \mathbf{b} \\
& \qquad \mathbf{x} \in \{0,1\}^n.
\end{aligned}
$$

Note that P1$'$ contains n number of bilinear terms yx_j. The strategy for linearizing these terms can be found in Glover and Woolsey's work [16][17]. We use the same idea to linearize these terms by substituting $z_j = yx_j$. We then employ the well-known *McCormick relaxation* to define z_j in linear expressions[29].

$$
\begin{aligned}
& z_j \le y^u x_j && \forall j = 1, \cdots, n \\
& z_j \le y + y^l (x_j - 1) && \forall j = 1, \cdots, n \\
& z_j \ge y^l x_j && \forall j = 1, \cdots, n \\
& z_j \ge y + y^u (x_j - 1) && \forall j = 1, \cdots, n.
\end{aligned}
$$

where y^l and y^u are lower bound and upper bound respectively of y. Since $x_j \in \{0, 1\}$, the McCormick constraints guarantee that $z_j = y$ when $x_j = 1$ and $z_j = 0$ when $x_j = 0$. Our first mixed 0-1 linear programming formulation of MMFP is given by

$$
\begin{aligned}
(\mathrm{MMFP1}) \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \qquad \sum_{j=1}^{n} c_j^k x_j + \alpha^k \le \sum_{j=1}^{n} d_j^k z_j + y\beta^k \quad \forall k = 1, \cdots, p \\
& \qquad z_j \le y^u x_j \qquad \forall j = 1, \cdots, n \\
& \qquad z_j \le y + y^l (x_j - 1) \quad \forall j = 1, \cdots, n \\
& \qquad z_j \ge y^l x_j \qquad \forall j = 1, \cdots, n \\
& \qquad z_j \ge y + y^u (x_j - 1) \quad \forall j = 1, \cdots, n \\
& \qquad \mathbf{Ax} \le \mathbf{b} \\
& \qquad \mathbf{x} \in \{0,1\}^n \\
& \qquad y \in [y^l, y^u].
\end{aligned}
$$

To implement this formulation, we need to calculate a pair of reasonable upper and lower bounds y^u and y^l, respectively, on the variable y. Let's first discuss the upper bound of y. Let θ^k denotes the optimal objective value of the unconstrained 0-1 linear fractional program satisfying condition (2.2).

$$\theta^k = \max\left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \mathbf{x} \in \{0,1\}^n\right\}, \quad k = 1, \cdots, p.$$

Lemma 2.1. $\theta = \max\{\theta^k,\ k = 1, \cdots, p\}$ is an upper bound of y. Furthermore, θ can be identified in polynomial time if p is at most polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1. Then,

$$y^* = \max_{1\le k\le p}\left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}\right\} \le \max_{1\le k\le p}\{\theta^k\} = \theta.$$

To prove the complexity, notice that we can identify each θ^k in polynomial time since condition (2.2) is satisfied [22]. Therefore, θ can be identified in polynomial time if p is at most polynomial. The linear time algorithm given in [22] uses repeated median finding technique. This could lead to some computational overhead even though it is a linear time bound.

■

We now present a simple formula for an upper bound.

Lemma 2.2. Let

$$\alpha = \max_{1\le k\le p}\left\{\alpha^k + \sum_{j=1}^n \{c_j^k : c_j^k > 0\}\right\}$$

$$\beta = \min_{1\le k\le p}\left\{\beta^k + \sum_{j=1}^n \{d_j^k : d_j^k < 0\}\right\}.$$

Then, $\frac{\alpha}{\beta}$ is an upper bound of y.

Proof.

$$\alpha^k + \sum_{j=1}^n \{c_j^k : c_j^k > 0\} \ge \alpha^k + \sum_{j=1}^n c_j^k x_j \quad \forall \mathbf{x} \in \{0,1\}^n, \quad \forall k \in \{1, \cdots, p\}.$$

Therefore,

$$\alpha \ge \max_{1\le k\le p}\left\{\alpha^k + \sum_{j=1}^n c_j^k x_j : \mathbf{x} \in \{0,1\}^n\right\}.$$

By condition (2.2), $\beta > 0$. Hence,

$$\frac{\alpha}{\beta} \ge \frac{\max_{1\le k\le p}\left\{\alpha^k + \sum_{j=1}^n c_j^k x_j : \mathbf{x} \in \{0,1\}^n\right\}}{\beta} \ge \max_{1\le k\le p}\left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \mathbf{x} \in \{0,1\}^n\right\}.$$

Therefore, $\frac{\alpha}{\beta}$ is an upper bound of y. ■

Corollary 2.2.1. When β^k and d_j^k are integers $\forall\ k \in \{1, \cdots, p\}$, $\forall\ j \in \{1, \cdots, n\}$ and condition (2.2) is satisfied, then α is an upper bound of y.

Proof. In this case, $\beta \geq 1$. Using the result of previous lemma, we obtain

$$y \leq \frac{\alpha}{\beta} \leq \alpha.$$

■

Note that α can be negative. If we want a non-negative upper bound for integer data, we could choose $\max\{0, \alpha\}$ as an upper bound of y. Notice that θ found in lemma 2.1 is a globally valid upper bound (or simply global upper bound) of y. The value $\frac{\alpha}{\beta}$ is also a global upper bound of y because $\frac{\alpha}{\beta} \geq \theta$. However, for computational purpose, we only need an upper bound which is bigger than or equal to the optimal objective value. We call this locally valid upper bound (or simply upper bound) of y. We will later discuss ways to tighten the y-bounds for computational advantage.

To find a lower bound of y, let γ^k be the optimal objective function value of the 0-1 linear fractional program satisfying the condition (2.2).

$$\gamma^k = \min\left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \mathbf{x} \in \{0,1\}^n\right\}, \quad k = 1, \cdots, p.$$

Lemma 2.3. $\gamma = \min\{\gamma^k, k = 1, \cdots, p\}$ is a lower bound of y. Furthermore, γ can be identified in polynomial time if p is at most polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1. Since

$$\gamma^k \leq \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}, \quad \forall k = 1, \cdots, p.$$

Then

$$y^* = \max_{1 \leq k \leq p} \left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}\right\} \geq \min_{1 \leq k \leq p} \{\gamma^k\} = \gamma.$$

To prove the complexity, notice that we can identify γ^k in polynomial time since condition (2.2) is satisfied [22]. Therefore, γ can be identified in polynomial time if p is at most polynomial. ■

Notice that γ found in lemma 2.3 is a globally valid lower bound (or simply a global lower bound) of y. We now present a simple formula for a lower bound of y. Let

$$T^k = \alpha^k + \sum_{j=1}^n \{c_j^k : c_j^k < 0\}, \quad k = 1, \cdots, p$$

$$B^k = \begin{cases} \beta^k + \sum_{j=1}^n \{d_j^k : d_j^k < 0\}, & \text{if } T^k < 0 \\ \beta^k + \sum_{j=1}^n \{d_j^k : d_j^k > 0\}, & \text{if } T^k \geq 0 \end{cases}, \quad k = 1, \cdots, p.$$

Then,

$$\frac{T^k}{B^k} \leq \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} \quad \forall \mathbf{x} \in \{0,1\}^n, \quad k = 1, \cdots, p.$$

That is,

$$\frac{T^k}{B^k} \leq \min\left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \mathbf{x} \in \{0,1\}^n\right\}, \quad k = 1, \cdots, p.$$

Hence,

$$\max_{1 \leq k \leq p}\left\{\frac{T^k}{B^k}\right\} \leq y^*.$$

Note that $\min\limits_{1 \leq k \leq p}\left\{\frac{T^k}{B^k}\right\}$ is a global lower bound of y since it is at most γ. For computations, we only need a lower bound which is smaller than or equal to the optimal objective function value. We call this locally valid lower bound (or simply a lower bound) of y.

Lemma 2.4. $\max\limits_{1 \leq k \leq p}\{\frac{T^k}{B^k}\}$ is a lower bound of y.

In the computational experiments in this chapter, we assume that all numerator and denominator coefficients are integers. By condition (2.2), we assume that all denominator coefficients are positive integers. Moreover, we use the following formulas for the global y-bounds.

$$y^u = \max\left\{0, \max_{1 \leq k \leq p}\left\{\alpha^k + \sum_{j=1}^n \{c_j^k : c_j^k > 0\}\right\}\right\} \tag{2.3}$$

$$y^l = \min_{1 \leq k \leq p}\left\{\frac{T^k}{B^k}\right\}. \tag{2.4}$$

Let us now give another formulation of MMFP using the binary expansion representation of integers [10]. For this formulation, we assume that d_j^k and β^k are all integers for $k = 1, \cdots, p$ and $j = 1, \cdots, n$. The idea is that any non-negative integer in the set $\{0, \cdots, t\}$ can be represented as a sum of some integers from the set $\{2^{1-1}, 2^{2-1}, \cdots, 2^{q-1}\}$, with $q = \lfloor \log_2(t) \rfloor + 1$. Let

$$D^k = \sum_{j=1}^n \{|d_j^k| : d_j^k < 0\} \quad \forall k = 1, \cdots, p.$$

This makes

$$\sum_{j=1}^n d_j^k x_j + D^k \geq 0 \quad \forall k = 1, \cdots, p.$$

Therefore we let

$$\sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p$$

where

$$q^k = \lfloor \log_2 (\sum_{j=1}^{n} |d_j^k|) \rfloor + 1 \quad \forall k = 1, \cdots, p,$$

$$v_i^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p, \quad i \in \{1, \cdots, q^k\}.$$

Now the constraint

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq y \sum_{j=1}^{n} d_j^k x_j + y\beta^k \quad \forall k = 1, \cdots, p$$

becomes

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq y \Bigg(\sum_{i=1}^{q^k} 2^{i-1} v_i^k - D^k \Bigg) + y\beta^k, \quad \forall k = 1, \cdots, p.$$

That is,

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq \sum_{i=1}^{q^k} 2^{i-1} y v_i^k - y D^k + y\beta^k \quad \forall k = 1, \cdots, p.$$

For each of the bilinear terms yv_i^k, we do the substitution $z_i^k = yv_i^k$. The number of z_i^k is the same as the number of v_i^k which is $\sum_{k=1}^{p} q^k$. Then, we use McCormick relaxation to linearize these bilinear terms. The total number of McCormick constraints is $4(\sum_{k=1}^{p} q^k)$. We obtain the following formulation MMFP2.

$$
\begin{aligned}
\text{(MMFP2)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& z_i^k \leq y^u v_i^k \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \leq y + y^l (v_i^k - 1), \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \geq y^l v_i^k \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \geq y + y^u (v_i^k - 1) \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \in \{0, 1\}^n \\
& y \in [y^l, y^u] \\
& v_i^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\}.
\end{aligned}
$$

The bounds y^l and y^u are computed the same as in MMFP1.

	MMFP1	MMFP2
No. of binary variables	n	$n + \sum_{k=1}^{p} q^k$
No. of continuous variables	$n + 1$	$1 + \sum_{k=1}^{p} q^k$
No. of constraints	$4n + p + r$	$2p + 4\sum_{k=1}^{p} q^k + r$

Table 2.1: Table of summary for counts of variables and constraints

To verify our models experimentally, we build an enumeration algorithm. We call this the *brute force algorithm*. Note that there are at most 2^n feasible points. We use the notation $f_k(\mathbf{x})$ to represent the k^{th} objective ratio computed at $\mathbf{x}$. At each point $\mathbf{x} \in \{0, 1\}^n$, the algorithm checks if $\mathbf{x}$ satisfies the given constraints. If it satisfies the constraints including condition (2.2), the algorithm computes values of all p ratios at $\mathbf{x}$ and returns the maximum value. We denote this maximum value as $f_{\max}(\mathbf{x})$ and add it into a list. At the end of complete enumeration, the algorithm finds the overall minimum value from the list. Note that the complexity of this brute force algorithm is $O(2^n(np + nr))$. The term np is for evaluations of p number of objective ratios, and the term nr is for checking if all r constraints in $\mathbf{Ax} \leq \mathbf{b}$ are satisfied. We use the brute force algorithm to verify the solutions produced by our models on relatively small data instances.

Definition 2.1. (Neighbouring points) Two points in the binary space $\{0,1\}^n$ are said to be neighbour of each other if their coordinates are differed by only one element.

Definition 2.2. (Local optimality) Let $X \subseteq \{0,1\}^n$ be the feasible set of an 0-1 program with objective function f. Let $\mathbf{x}^* \in X$ and $N(\mathbf{x}^*)$ be the set of all neighbouring binary points of $\mathbf{x}^*$. We say that $\mathbf{x}^*$ is a local optimal point if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \ \forall \mathbf{x} \in X \cap N(\mathbf{x}^*).$$

Note that single ratio linear fractional objective function with positive denominator is both quasi-convex and quasi-concave [11]. Therefore, local optimality is global optimality and optimal point is obtained at an extreme point of the convex feasible set [19] [18]. Using the above definition of neighbourhood in binary space, in the single ratio 0-1 LFP with positive denominator, local optimality is also global optimality [21]. This is not true for our 0-1 MMFP. Even the MMFP problems with continuous decision variables are known to have multiple local optima that are not global optima [23]. Example 1 below demonstrates the possible existence of a local optimal solution that is not a global solution in a 0-1 MMFP. Example 2 demonstrates the possible existence of multiple global optimal solutions.

Example 1. (A 0-1 MMFP problem with a local optimal solution). We are given four objective ratios:

$$f_1(x) = \frac{3 + 27x_1 + 28x_2 - 46x_3 - 23x_4}{25 + 38x_1 + 44x_2 + 41x_3 + 40x_4}$$

$$f_2(x) = \frac{-17 + 41x_1 - 32x_2 + 36x_3 - 49x_4}{46 + 46x_1 + 6x_2 + 18x_3 + 28x_4}$$

$$f_3(x) = \frac{27 + 13x_1 - 29x_2 + 23x_3 + 41x_4}{36 + 7x_1 + 14x_2 + 13x_3 + 38x_4}$$

$$f_4(x) = \frac{-43 - 33x_1 + 36x_2 + 41x_3 + 32x_4}{43 + 49x_1 + 3x_2 + 19x_3 + 14x_4}.$$

Suppose this is an unconstrained problem. The maximum objective values computed by the brute-force algorithm at all 16 points of the feasible set are listed in the table below. All denominators are positive at all points.

No.	$\mathbf{x}$	$f_{\max}(\mathbf{x})$	
1	(0,0,0,0)	0.75	
2	(1,0,0,0)	0.93	
3	(0,1,0,0)	0.45	
4	(1,1,0,0)	0.54	neighbour of (8)
5	(0,0,1,0)	1.02	
6	(1,0,1,0)	1.13	neighbour of (8)
7	(0,1,1,0)	0.52	neighbour of (8)
8	(1,1,1,0)	0.49	**local optimal**
9	(0,0,0,1)	0.92	
10	(1,0,0,1)	1.00	
11	(0,1,0,1)	0.44	**global optimal**
12	(1,1,0,1)	0.55	
13	(0,0,1,1)	1.05	
14	(1,0,1,1)	1.11	
15	(0,1,1,1)	0.84	
16	(1,1,1,1)	0.69	neighbour of (8)

In this example instance, both MMFP1 and MMFP2 identify the global optimal point correctly.

Example 2.

$$\begin{aligned} &\text{Minimize} \quad \text{Max}\left\{ f_1(\mathbf{x}) = \frac{1 - x_1 - x_2 + x_3 + x_4}{1 + x_1 + x_2 + x_3 - x_4}, \quad f_2(\mathbf{x}) = \frac{1 + x_1 + x_2 - x_3 - x_4}{1 + x_1 + x_2 + x_3 - x_4} \right\} \\ &\text{Subject to} \quad \mathbf{x} \in \{0,1\}^4 \end{aligned}$$

The three optimal points are $\mathbf{x} = [1,0,1,0], \mathbf{x} = [0,1,1,0]$ and $\mathbf{x} = [1,1,1,1]$ with the same optimal objective value of 1/3. We notice that $\mathbf{x} = [0,0,0,1]$ is not included in the search because this choice produces non-finite objective values. Also notice that the optimal points are not neighbours of each other.

2.2.1 Improving the formulations

In this section, we attempt to find better bounds of the continuous variable y in MMFP formulations. The goal is to improve the performance in terms of rate of convergence and reduction of floating point rounding errors. The bounds of a continuous variable are used in constructing McCormick relaxation for the bilinear terms involving that variable. Tight relaxations are important in deterministic global optimization [31]. Use of inexact floating point arithmetic in mixed integer linear programming is a source of some incorrect results [32]. Pre-solving and recursive solving are well-known procedures in optimization. We use these strategies in searching for better bounds. First, we argue that the globally valid bounds of y are not necessary. Note that y is an aggregated upper bound of all objective ratios. Therefore, if $\mathbf{x}^0$ is any feasible point, then

$$\text{optimal value of MMFP problem } = \min\ y \ \leq\ \max\{f_k(\mathbf{x}^0) : k = 1, \cdots, p\}.$$

Definition 2.3. [27] In mixed integer linear programming solvers, *integrality tolerance* is a small positive number δ. If the value of an integer variable in the linear programming relaxation is less than δ away from a nearest integer, then it is considered that x satisfies integrality requirement.

Definition 2.4. *Type I error* is the numerical error produced due to the rounding in the floating point arithmetic when one or more elements of a binary variable, during relaxation, is within integrality tolerance from zero but not zero. The resulting binary decision variable is produced as an optimal point by the solver.

Definition 2.5. *Type II error* is the numerical error produced due to the rounding in the floating point arithmetic when one or more elements of a binary variable, during relaxation, is within integrality tolerance from one, but not one. The resulting binary decision variable is produced as an optimal point by the solver.

In the following examples, the values of numerators and denominators are computed after obtaining the integral values of the optimal decision variable $\mathbf{x}$. We use the notation $\mathbf{x}^*$ to denote either the true optimal point or an optimal point produced by the solver. Similarly, we use y^* to denote either the true optimal value or an optimal value produced by the solver. For simplicity, we also introduce following notations:

$$f_k(\mathbf{x}) = \frac{\mathbf{c}^k\mathbf{x} + \alpha^k}{\mathbf{d}^k\mathbf{x} + \beta^k}$$

$$f_{\max}(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_p(\mathbf{x})\} \text{ at any } \mathbf{x} \in X$$

$$f_{\min}(\mathbf{x}) = \min\{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_p(\mathbf{x})\} \text{ at any } \mathbf{x} \in X$$

$$\left(\frac{\alpha}{\beta}\right)_{\min} = \min\left\{\frac{\alpha^k}{\beta^k}, k = 1, \cdots, p\right\}$$

$$\left(\frac{\alpha}{\beta}\right)_{\max} = \max\left\{\frac{\alpha^k}{\beta^k}, k = 1, \cdots, p\right\}.$$

Example 3. We set up an unconstrained 0-1 MMFP problem instance with $p = 2$, $n = 200$; c_j^k, α^k values are random uniformly distributed integers between -10 and 100; d_j^k, β^k are random uniformly distributed integers between 1 and 100. We use Gurobi solver with default integrality tolerance of 1×10^{-5}. We run MMFP1 formulation which returns the incorrect result shown in the table below.

Program	Obj.Val	Numerators at optimality	Denominators at optimality	y^l	y^u
MMFP1	-4.47553	[15,45]	[51,18]	-4.5	9112

The optimal point produced is $\mathbf{x}^* = \mathbf{0}$. At this point the actual objective values are $\frac{15}{51} \approx 0.294117$ and $\frac{45}{18} = 2.5$ with the larger value being 2.5 which is not the optimal value -4.47553 in the result. This is due to a type I error. The variable values are not exactly zeros but small positive numbers in the order of 10^{-6}. Due to these floating point values, McCormick constraints produce the values of $z_j = yx_j$ where $j = 1, \cdots, 200$ as positive numbers in the order of 10^{-2} instead of zeros for $\mathbf{x} = \mathbf{0}$. As a consequence, the program constraints

$$\alpha^k + \sum_{j=1}^{200} c_j^k x_j \leq y\beta^k + \sum_{j=1}^{200} d_j^k z_j, \quad k = 1, 2$$

have positive sum values $\sum_{j=1}^{200} d_j^k z_j$ on their right hand sides, which are

$$\sum_{j=1}^{200} d_j^1 z_j \approx 243.275$$

$$\sum_{j=1}^{200} d_j^2 z_j \approx 234.593$$

instead of zeros. Meanwhile, the left hand sides of the above inequalities are

$$\alpha^1 + \sum_{j=1}^{200} c_j^1 x_j \approx 15.0229$$

$$\alpha^2 + \sum_{j=1}^{200} c_j^2 x_j \approx 45.0239$$

with $y^l = -4.5$, $\beta^1 = 51$, $\beta^2 = 18$. These constraints become

$$15.0229 \leq 51y + 243.275$$

$$45.0239 \leq 18y + 234.593$$

resulting in the minimum y-value -4.47553 as we see in the result table. The correct optimal value is 0.014195 and the optimal point is a non-zero point. Note that in this example,

$$y^l < 0 < y^* < \left(\frac{\alpha}{\beta}\right)_{\min} < \left(\frac{\alpha}{\beta}\right)_{\max} < y^u.$$

Example 4. This time, we have $p = 2$, $n = 600$; c_j^k, α^k and d_j^k, β^k are random uniformly distributed integers between 1 and 100. We solve unconstrained 0-1 MMFP problem using the same Gurobi solver. The result is displayed in the below table.

Program	Obj.Val	Numerators at optimality	Denominators at optimality	y^l	y^u
MMFP1	0.0003662	[3,8]	[64,28]	9.848×10^{-5}	31080

The optimal point produced is $\mathbf{x}^* = \mathbf{0}$. This solution is incorrect and it is due to a type I error. The correct optimal value is $y^* = 0.064748$ and the optimal point is a non-zero point. Note that in this case,

$$0 < y^l < \left(\frac{\alpha}{\beta}\right)_{\min} < y^* < \left(\frac{\alpha}{\beta}\right)_{\max} < y^u.$$

Proposition 2.5. Assume that y^l and y^u are valid bounds of the aggregated upper bound y of MMFP1 computed by the formula given in (2.4) and (2.3) respectively, and all objectives have positive denominator coefficients in an unconstrained MMFP with integral data. Let δ denotes the integrality tolerance of the mixed integer linear programming solver, and y^* denotes the true optimal value of MMFP1. We assume that

$$y^u > 0$$

$$c_j^k < d_j^k y^u, \quad \forall k = 1, \cdots, p, \quad \forall j = 1, \cdots, n$$

$$y^l < \left(\frac{\alpha^k}{\beta^k}\right) < y^u \quad \forall k = 1, \cdots, p$$

and,

$$y^l < y^* < y^u.$$

Then there exists $\mathbf{x} = (x_1, \cdots, x_n)^T$ such that $0 < x_j < \delta \;\; \forall j \in \{1, \cdots, n\}$, and a n-value such that the $\min y < y^*$. In the following proof, we assume that the denseness of binary machine numbers is more than sufficient to support all our precision requirements.

Proof. In theory, $z_j = yx_j \ \forall j \in \{1, \cdots, n\}$. However, the solver instead receives the following McCormick constraints from the formulation MMFP1. For each $k = 1, \cdots, p$,

$$\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j - \sum_{j=1}^n d_j^k z_j}{\beta^k} \leq y \tag{1}$$

$$z_j \leq x_j y^u \tag{2}$$

$$z_j \leq y - y^l(1 - x_j) \tag{3}$$

$$z_j \geq x_j y^l \tag{4}$$

$$z_j \geq y - y^u(1 - x_j). \tag{5}$$

When $0 < x_j < \delta$, we can assume that $y < y^u(1 - x_j) \approx y^u$. Therefore right hand side of (5) is a negative value. Also right hand side of (4) is either very small positive value or a negative value. According to (1), to minimize y-value, z_j needs to be positive and as large as possible. We now discuss the constraints in more details. Let us look at the simple situation in which all x_j are assigned the same value (ie. $x_1 = x_2 = \cdots = x_n$), and let (2) and (3) take equality. We have

$$z_j = x_j y^u = y - y^l(1 - x_j), \quad \forall j = 1, \cdots, n.$$

This makes inequalities (4) satisfied. We continue to obtain

$$0 < x_j = \frac{y - y^l}{y^u - y^l} < \delta.$$

Due to the assumed denseness of machine numbers, there are sufficiently many y-values that satisfy both

$$y < y^u(1 - x_j) \quad \text{and} \quad 0 < \frac{y - y^l}{y^u - y^l} < \delta$$

where $y < y^u(1 - x_j)$ makes inequality (5) satisfied. We now fix the value $0 < x_j < \delta$ small enough so that $y^l < y < y^*$. We still need to satisfy the inequality (1) without violating $y^l < y < y^* < y^u$. By (1), for each $k \in \{1, \cdots, p\}$, we need

$$\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j - \sum_{j=1}^n d_j^k x_j y^u}{\beta^k} \leq y$$

and because x_j are equal to each other,

$$\frac{\alpha^k + x_j \sum_{j=1}^n (c_j^k - d_j^k y^u)}{\beta^k} \leq y.$$

By assumption, $c_j^k - d_j^k y^u < 0$ for all $j = 1, \cdots, n$ and for all $k = 1, \cdots, p$. Let

$$A^k(n) = -\frac{\sum_{j=1}^n (c_j^k - d_j^k y^u)}{n} > 0, \quad k = 1, \cdots, p.$$

Then, for each $k = 1, \cdots, p$, we need

$$\frac{\alpha^k - x_j n A^k(n)}{\beta^k} \leq y.$$

Notice that $nA^k(n)$ value grows with increasing n. We choose n large enough such that the largest value of the left hand sides of above p inequalities is less than y^l. ■

Generalization of proposition 2.5

Note that proposition 2.5 also applies to MMFP1 model on knapsack constrained 0-1 MMFP in which $\mathbf{x} = \mathbf{0}$ remains a feasible point. Suppose the second condition of the proposition is satisfied by only a number of indices k and j. Then, if these indices happen to be the indices that decide y value, type I error can still hit the corresponding elements of $\mathbf{x}$. From the McCormick relaxation, it can be observed that an environment for type II error is when y^l takes a large negative value. In this case, when a number of $x_j \in (1-\delta, 1)$, McCormick constraints can make corresponding $z_j = y + \epsilon$ (where ϵ is a small positive number) instead of $z_j = y$. Sufficiently many such z_j would give the first constraint of MMFP1 a room to minimize y-value too low. Since all $d_j^k > 0$, $\beta^k > 0$, we expect that type II error would not press the y-value too far below the true optimal value unless y^l is a huge negative number and d_j^k are small positive numbers.

In MMFP2 model the binary variable $\mathbf{v}^k$ needs attention. Since all denominator coefficients are positive, an environment for type I error shall be when y^u is a large positive number, and some q^k are sufficiently large (implying significantly large n-value). An environment for type II error shall be when y^l is a large negative number, and some q^k are sufficiently large.

A common characteristic of an error prone environment is that it gives the MILP solver a room to minimize the objective y below true y^* at a point $\mathbf{u} \in [0,1]^n$ which is too close to a feasible binary point. These environments often have loose y-bounds and larger n-values.

Idea for a set of tighter y-bounds

From proposition 2.5, we observe three computationally critical objective values in the structure of 0-1 MMFP where $\mathbf{x} = \mathbf{0}$ is a feasible solution. They are zero, $f_{\min}(\mathbf{0}) = \left(\frac{\alpha}{\beta}\right)_{\min}$, and $f_{\max}(\mathbf{0}) = \left(\frac{\alpha}{\beta}\right)_{\max}$. Zero is a natural integer; $\frac{\alpha^k}{\beta^k}$ is the value of $f_k(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$. We propose such critical objective values as candidates for improved y-bounds. Below diagrams explain the possible regions where the optimal value can fall into for the case where α^k/β^k values are positive for all $k = 1, \cdots, p$.

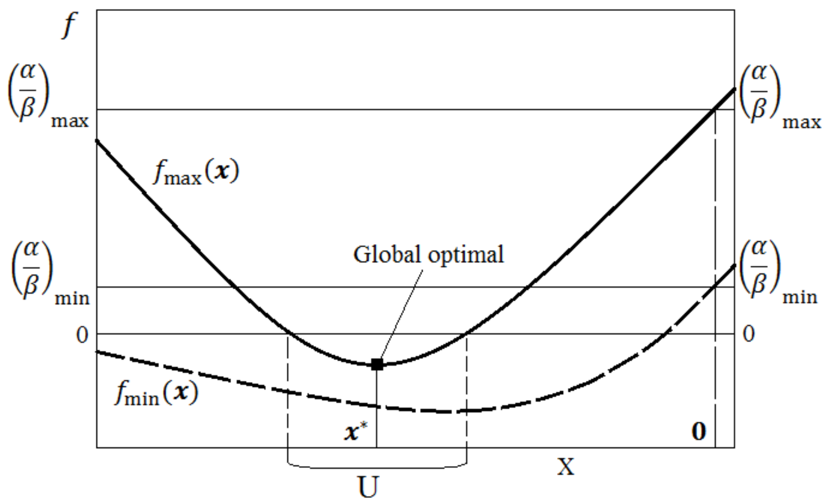


Figure 2.1: Optimal value is less than zero. Set $y^u = 0$

If the optimal value is smaller than zero, we can verify this by pre-solving minimization of a dummy objective function (a constant, for example) with the constraints requiring that $f_k(\mathbf{x}) \leq -\epsilon,\ \epsilon > 0$, for all $k = 1, \cdots, p$. We add knapsack constraint in pre-solve if it is the problem constraint. If the pre-solve problem is feasible then there is a subset $U \subseteq X$ where all objectives including $f_{\max}(\mathbf{x})$ are negative. Then we can set $y^u = 0$ in main program model so that the main program restricts its search to the sub-feasible set U (see figure 2.1) where the optimal point is. If the pre-solve is not feasible, then $f_{\max}(\mathbf{x}) \geq 0\ \forall \mathbf{x} \in X$. In this case we can set $y^l = 0$.

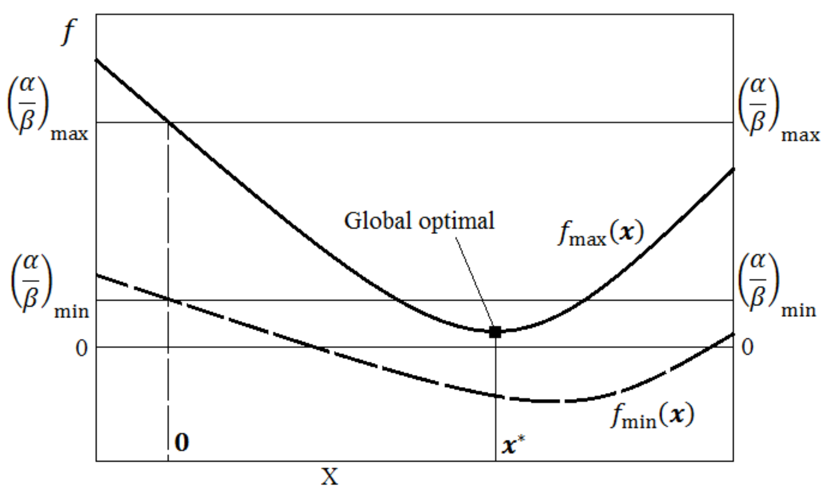


Figure 2.2: Optimal value is less than $(\frac{\alpha}{\beta})_{\min}$. Set $y^u = (\frac{\alpha}{\beta})_{\min}$

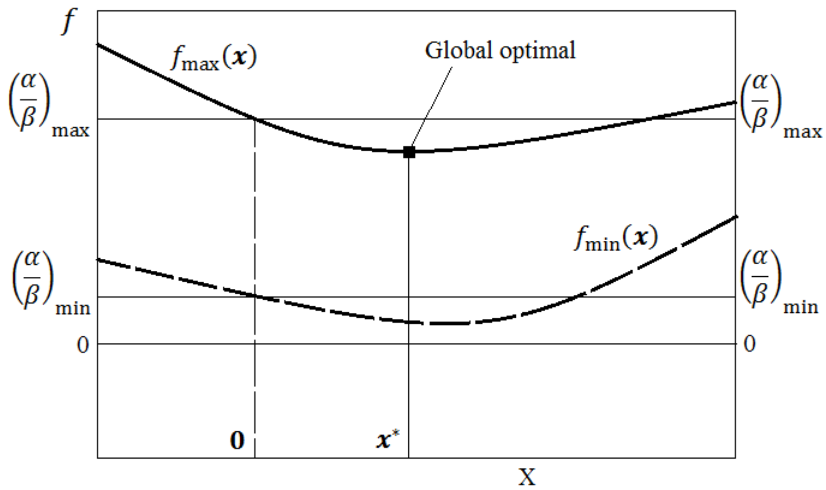


Figure 2.3: Optimal value is less than $(\frac{\alpha}{\beta})_{\max}$. Set $y^u = (\frac{\alpha}{\beta})_{\max}$

We wish to bound the optimal value from both above and below, especially from below as we are minimizing the objective. For example, the interval where the true y^* is captured can be between zero and $(\frac{\alpha}{\beta})_{\min}$ or between $(\frac{\alpha}{\beta})_{\min}$ and $(\frac{\alpha}{\beta})_{\max}$. We can do the pre-solve twice with appropriate constraints. Below are three basic pre-solve formulations targeting three aforementioned critical objective values. We assume all denominator coefficients are positive integers.

$$\text{pre-solve (1)}: \quad \{\min 0 : \alpha^k + \sum_{j \in J} c_j^k x_j \leq -1 \quad \forall k = 1, \cdots, p,\ \mathbf{x} \in X\}$$
$$\text{pre-solve (2)}: \quad \left\{\min 0 : f_k(\mathbf{x}) \leq \left(\frac{\alpha}{\beta}\right)_{\min} \quad \forall k = 1, \cdots, p\ ,\ \mathbf{x} \in X\right\}$$
$$\text{pre-solve (3)}: \quad \left\{\min 0 : f_k(\mathbf{x}) \leq \left(\frac{\alpha}{\beta}\right)_{\max} \quad \forall k = 1, \cdots, p\ ,\ \mathbf{x} \in X\right\}.$$

Number of constraints in these pre-solve formulations are generally fixed and they do not grow with n-value as in the main programs MMFP1 and MMFP2. Hence, their computing time is expected to be significantly lower than the main models. Depending on the values of $(\frac{\alpha}{\beta})_{\min}$ and $(\frac{\alpha}{\beta})_{\max}$, we present three sample pre-solve plans. We assume that we already have the valid bounds y^l and y^u obtained from (2.4) and (2.3) respectively.

Pre-solve plan A for y bounds for the case
$y^l < (\frac{\alpha}{\beta})_{\min}, \quad 0 < (\frac{\alpha}{\beta})_{\min} < (\frac{\alpha}{\beta})_{\max} < y^u$

Input: Program data, valid y^u, y^l

Output: Tightened y^u and y^l

if $y^u > 0$ and $y^l < 0$, then

  pre-solve (1)

  if pre-solve (1) is feasible, then set $y^u = 0$

  if pre-solve (1) is not feasible, then set $y^l = 0$

    if $y^l \leq (\frac{\alpha}{\beta})_{\min}$ and $y^u \geq (\frac{\alpha}{\beta})_{\max}$, then

      pre-solve (2)

      if pre-solve (2) is feasible, then set $y^u = (\frac{\alpha}{\beta})_{\min}$

      if pre-solve (2) is infeasible, then

        pre-solve (3)

        if pre-solve (3) is feasible, then

          set $y^u = (\frac{\alpha}{\beta})_{\max}$, $y^l = (\frac{\alpha}{\beta})_{\min}$

        if pre-solve (3) is not feasible, then set $y^l = (\frac{\alpha}{\beta})_{\max}$

else if $y^l \geq 0$, $y^l \leq (\frac{\alpha}{\beta})_{\min}$ and $y^u \geq (\frac{\alpha}{\beta})_{\max}$, then

  pre-solve (2)

  if pre-solve (2) is feasible, then set $y^u = (\frac{\alpha}{\beta})_{\min}$

  if pre-solve (2) is not feasible, then

    pre-solve (3)

    if pre-solve (3) is feasible, then

      set $y^u = (\frac{\alpha}{\beta})_{\max}$, $y^l = (\frac{\alpha}{\beta})_{\min}$

    if pre-solve (3) is not feasible, then set $y^l = (\frac{\alpha}{\beta})_{\max}$.

end.

Note: This plan is suitable if $0 \leq$ true y^*

Pre-solve plan B for y bounds for the case $y^l < (\frac{\alpha}{\beta})_{\min} < 0 < (\frac{\alpha}{\beta})_{\max} < y^u$

Input: Program data, valid y^u, y^l

Output: Tightened y^u and y^l

1. pre-solve (2)
2. **if** pre-solve (2) is feasible, then
3.   set $y^u = (\frac{\alpha}{\beta})_{\min}$
4. **if** pre-solve (2) is not feasible, then
5.   pre-solve (1)
6.   **if** pre-solve (1) is feasible, then
7.   set $y^u = 0$, $y^l = (\frac{\alpha}{\beta})_{\min}$
8.   **if** pre-solve (1) not feasible, then
9.   pre-solve (3)
10.   **if** pre-solve (3) is feasible, then
11.   set $y^u = (\frac{\alpha}{\beta})_{\max}$, $y^l = 0$
12.   **if** pre-solve (3) is not feasible than
13.   set $y^l = (\frac{\alpha}{\beta})_{\max}$
14. **end**.

Note: This plan is suitable if $(\frac{\alpha}{\beta})_{\min} \leq$ true y^*

Pre-solve plan C for y bounds for the case $y^l < (\frac{\alpha}{\beta})_{\min} < (\frac{\alpha}{\beta})_{\max} < 0 < y^u$

Input: Program data, valid y^u, y^l

Output: Tightened y^u and y^l

1. pre-solve (2)
2. **if** pre-solve (2) is feasible, then
3.   set $y^u = (\frac{\alpha}{\beta})_{\min}$
4. **if** pre-solve (2) is not feasible, then
5.   pre-solve (3)
6.   **if** pre-solve (3) is feasible, then
7.   set $y^u = (\frac{\alpha}{\beta})_{\max}$, $y^l = (\frac{\alpha}{\beta})_{\min}$
8.   **if** pre-solve (3) not feasible, then
9.   pre-solve (1)
10.   **if** pre-solve (1) is feasible, then
11.   set $y^u = 0$, $y^l = (\frac{\alpha}{\beta})_{\max}$
12.   **if** pre-solve (1) is not feasible than
13.   set $y^l = 0$
14. **end**.

Note: This plan is suitable if $(\frac{\alpha}{\beta})_{\min} \leq$ true y^*

In assignment problem, $\mathbf{x} = \mathbf{0}$ is not a feasible point. Therefore, in assignment problem, we only have pre-solve (1). Depending on the data at hand, and other critical objective values we may have, customized pre-solve plans can be constructed. For the computational experiments in this thesis, regardless of the data type, we use pre-solve plan A to modify MMFP1 and MMFP2 as follows:

Modified formulations: MMFP1M & MMFP2M
Input: Program data
Output: An optimal solution
1. compute valid y^l, y^u
2. execute pre-solve plan A to update y^l, y^u
3. solve main model (MMFP1 or MMFP2)
4. **end**.

We will refer to MMFP1 and MMFP2 as parent models of their respective modified models. In the computational results section, we separately demonstrate pre-solve plans B and C with data that are suitable. Here we present another modified formulation equipped with a customizable pre-solve plan. A demonstration of this modified formulation is also included in the computational results section.

A generalized modified formulation equipped with a customized pre-solve plan
Input: Program data
Output: An optimal solution
1. compute valid $y^l, y^u, (\frac{\alpha}{\beta})_{\min}, (\frac{\alpha}{\beta})_{\max}$
2. select a critical objective value ω
3. pre-solve: $\min\{0: \ f_k(\mathbf{x}) \leq \omega \ \ \forall k = 1, \cdots, p, \ \mathbf{x} \in X\}$
4. read the result of pre-solve
5. **if** another pre-solve is required, then
6. go to step (2)
7. **if** no more pre-solve is required, then
8. set y^l, y^u according to the pre-solve results
9. run main model
10.**end**.

In this modified formulation, the critical objective values are selected by the user. Suppose we begin with a pair of valid bounds y^l and y^u such that $y^l < y^* < y^u$. We use the binary search method to tighten the interval that contains y^*. If we want to capture y^* in an interval of length l

(as known as interval of capture), then the number of pre-solves needed is

$$\left\lceil \log_2 \left(\frac{y^u - y^l}{l} \right) \right\rceil .$$

Now let us consider the pre-solve plan A. Notice that the pre-solve plan A could produce the bounds

$$y^l < y^* < y^u = 0, \quad \text{or} \quad 0 < y^l = \left(\frac{\alpha}{\beta}\right)_{\max} < y^* < y^u.$$

In these cases, either the lower bound or the upper bound can be too loose. Since we are working on minimization problem, we present an example on the first case in which a type II error occurs.

Example 5. We have $p = 2$, $n = 1500$; $-100 \le c_j^k \le 100$, $1 \le d_j^k \le 100$ and $1 \le w_j \le 100$ where $j = 1, \cdots, 1500$ and $k = 1, 2$ are generated as random uniformly distributed integers. The values c_j^k are rearranged in increasing order, and the values d_j^k are rearranged in decreasing order. The values of α^k and β^k are all one for $k = 1, 2$. The knapsack weight limit is $W = 10000$. MMFP1M formulation using Gurobi solver with integrality tolerance $\delta = 1 \times 10^{-5}$ produces the following incorrect result.

Program	Obj.Val	Numerators at optimality	Denominators at optimality	y^l	y^u
MMFP1M	-1.0727	[-69,-73]	[84,84]	-39672	0

The optimal point produced is: $x_{219} = 1$, and all other $x_j = 0$. This is incorrect because

$$\max\left\{ -\frac{69}{84}, -\frac{73}{84} \right\} = -0.821428$$

is larger than the produced optimal value of -1.0727. Actual value of x_{219} is 0.999993589824491 which is about $6.41 \times 10^{-6} < \delta$ away from one, so the solver takes it as one in final solution. This affects the value of z_{219} in relationship with y. The McCormick constraints

$$\begin{aligned} z_{219} &\le x_{219} y^u \\ z_{219} &\le y + y^l (x_{219} - 1) \\ z_{219} &\ge x_{219} y^l \\ z_{219} &\ge y + y^u (x_{219} - 1) \end{aligned}$$

becomes

$$\begin{aligned}
z_{219} &\leq 0 \\
z_{219} &\leq y + 0.25430448279075 \\
z_{219} &\geq -39671.74569551721 \\
z_{219} &\geq y.
\end{aligned}$$

The second McCormick constraint allows y to be about 0.2543 less than z_{219} without violating the other three constraints. This is due to y^l being large negative number. According to the theory $z_{219} = yx_{219} = y$. Thus, in this minimization problem, MMFP1M produces a type II error accompanied by the mentioned incorrect result. However, both MMFP2 and MMFP2M produce correct optimal value $y^* = -1.007874$.

The pre-solve plans we present above or any other similar pre-solve plan may not entirely avoid type I and type II errors. Suppose the optimal solution produced is $(\mathbf{x}^*, y^*)$ and it comes with a type I error or a type II error or both. Then, since the solver is in minimization sense, we can assume that

$$\max\{f_k(\mathbf{x}^*) : k = 1, \cdots, p\} - y^* > \epsilon, \text{ where } \epsilon > 0.$$

We can verify the errors by discovering that at least one element of a binary variable is not in $\{0, 1\}$ but close to either zero or one within the integrality tolerance. We note that in MMFP2 formulation, we have more than one binary variable. We now turn to the recursion heuristic in which the result of the completed unsuccessful solve is used to compute better bounds for the next solve. Below heuristic assumes that

$$y^* \leq \text{ true optimal value } \leq \max\{f_k(\mathbf{x}^*) : k = 1, \cdots, p\}$$

with at least one of the two inequalities being strict.

The recursion heuristic

Input: Given data, precision level ϵ, non-negative integers λ_1, λ_2.

Output: An optimal solution.

1. compute valid bounds y^l, y^u
2. solve the program which returns optimal solution $\mathbf{x}^*, y^*$
3. **if** a type I or a type II error is discovered, i.e. $\max\{f_k(\mathbf{x}^*) : k = 1, \cdots, p\} - y^* > \epsilon$ update y^l, y^u as follows:
4. **if** tight bound is not required
5. $y^l = \lfloor y^* \rfloor - \lambda_1$
6. $y^u = \lceil \max\{f_k(\mathbf{x}^*) : k = 1, \cdots, p\} \rceil + \lambda_2$
7. **if** tight bound is required
8. $y^l = y^*$
9. $y^u = \max\{f_k(\mathbf{x}^*) : k = 1, \cdots, p\}$
10. go to step (2)
11. **end**.

Note: Choice of y^l, y^u updates depend on the tightness needed. The tightness is adjustable using appropriate λ_1, λ_2 values. If the incorrect optimal value y^* is too far below the true optimal value (which can be judged by a large ϵ value), then the new bounds may still be too loose. In this case, solving with an appropriate modified formulation should be done first.

In the next three sections, we discuss three special cases of MMFP. They are unconstrained 0-1 MMFP, knapsack constrained 0-1 MMFP, and assignment constrained 0-1 MMFP.

2.3 Unconstrained 0-1 MMFP problem

When there is no constraint on the decision variable $\mathbf{x}$, the MMFP problem is called unconstrained MMFP problem. Note that in this case, $\mathbf{x} = \mathbf{0}$ is a feasible solution. We introduce MMFP1 formulation for the unconstrained case as below.

$$
\begin{aligned}
\text{(MMFP1U)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq \sum_{j=1}^{n} d_j^k z_j + y\beta^k \quad \forall k = 1, \cdots, p \\
& z_j \leq y^u x_j \quad \forall j = 1, \cdots, n \\
& z_j \leq y + y^l (x_j - 1) \quad \forall j = 1, \cdots, n \\
& z_j \geq y^l x_j \quad \forall j = 1, \cdots, n \\
& z_j \geq y + y^u (x_j - 1) \quad \forall j = 1, \cdots, n \\
& \mathbf{x} \in \{0, 1\}^n \\
& y \in [y^l, y^u].
\end{aligned}
$$

Similarly, MMFP2 formulation in the unconstrained case is

$$
\begin{aligned}
\text{(MMFP2U)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& z_i^k \leq y^u v_i^k \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \leq y + y^l (v_i^k - 1) \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \geq y^l v_i^k \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \geq y + y^u (v_i^k - 1) \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& \mathbf{x} \in \{0, 1\}^n \\
& y \in [y^l, y^u] \\
& v_i^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\}.
\end{aligned}
$$

For experiments in this thesis, y^u and y^l for both MMFP1U and MMFP2U are computed by (2.3) and (2.4). These are global bounds if all denominator coefficients are positive integers.

We now present the modified formulations MMFP1UM and MMFP2UM to be tested in the computational experiments. These are MMFP1U and MMFP2U equipped with the pre-solve plan A which is designed for the case: $0 < (\frac{\alpha}{\beta})_{\min} < (\frac{\alpha}{\beta})_{\max}$ and $y^l < (\frac{\alpha}{\beta})_{\min}$. This plan is suitable when $0 \leq y^*$.

Modified formulations: MMFP1UM & MMFP2UM
Input: Given data
Output: An optimal solution
1. compute valid y^l, y^u
2. execute pre-solve plan A to update y^l, y^u
3. solve main model (MMFP1U or MMFP2U)
4. **end**.

Note that we can also equip MMFP1U and MMFP2U with pre-solve plan B or C for corresponding suitable data. We name them MMFP1U-B, MMFP1U-C, MMFP2U-B and MMFP2U-C in the computational experiments. We can also equip MMFP1U and MMFP2U with customizable pre-solve plan. In the experiment, we test MMFP2UM-Custom which is MMFP2U equipped with customizable pre-solve plan.

2.3.1 Experimental analysis on unconstrained 0-1 MMFP formulations

In this section, we discuss results of extensive computational experiments carried out using the formulations MMFP1U, MMFP1UM, MMFP2U, and MMFP2UM on various problem instances to assess the performance and to gain some insight. The machines used have the specifications: PC with Intel i7-4770 CPU and 16 GB RAM running 64-bit Windows 10 Enterprise operating system. Gurobi solver version 9.0.2 is called on Python 3.7 environment. We set the run time limit for all experiments as 3600 seconds. Gurobi integrality tolerance default value is 1×10^{-5}. We use the default Gurobi parameter settings. For random integers, we used 'randrange' function from random library in Python.

Test problems generation

The following are the categories of problem instances selected for the tests. In each category we ran the experiments for different n-values. We created instance reference codes as follows:

signs - constraint type - data type - p number

$$\text{signs} = \begin{cases} PP, & \text{Coefficients for both numerators and denominators are positive} \\ NP, & \text{Coefficients for numerators unrestricted and denominators positive} \\ NN, & \text{Coefficients for both numerators and denominators are unrestricted} \end{cases}$$

$$
\text{constraint type} = \begin{cases} U, & \text{Unconstrained} \\ K, & \text{Knapsack constrained} \\ A, & \text{Assignment constrained} \end{cases}
$$

$$
\text{data type} = \begin{cases} R, & \text{Random data} \\ P, & \text{Positively correlated data} \\ N, & \text{Negatively correlated data} \end{cases}
$$

For example, the instance reference code for positive coefficients on both numerators and denominators and unconstrained problem with random data and $p = 5$ would be PP-U-R-5. In this section, we have 6 instances to test.

(PP-U-R)	Random uniformly distributed integers: All data values $c_j^k, \alpha^k, d_j^k, \beta^k$ are randomly generated in the range [1,100].
(PP-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(PP-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100]. c_j^k are rearranged in increasing order and d_j^k are rearranged in decreasing order in each objective.
(NP-U-R)	Random uniformly distributed integers: Data values c_j^k, α^k are randomly generated in the range [-100,100]. Data values d_j^k, β^k are randomly generated in the range [1,100].
(NP-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(NP-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in decreasing order in each objective.

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	0.00059	0.0	0.04687	0.33427	0.0	**25.09384**
4000	0.00027	0.0	0.07812	0.3152	0.0	**7.73437**
6000	0.00015	0.0	0.09384	0.31217	0.0	**16.32814**
8000	0.00012	0.0	0.10943	0.27426	0.0	**11.93734**
10000	0.00016	0.0	0.16552	0.25691	0.0	**8.82821**
12000	8e-05	0.0	0.17208	0.27995	0.0	**88.91491**
14000	9e-05	0.0	0.18755	0.24796	7e-05	**17.52132**
16000	6e-05	0.0	0.23442	0.22173	0.0	**21.80101**
18000	9e-05	0.0	0.23438	0.24718	0.0	**34.79701**
20000	7e-05	0.0	0.26566	0.2333	0.0	**48.41484**

Table 2.3: PP-U-R-5

Above table presents the computational results of MMFP2U and MMFP2UM running with n size from 2000 to 20000. Under each program are objective value found by the solver, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors. For example, in the above table, MMFP2U finished fastest in every run, but it ran into type I error and produced the wrong solutions. Whereas MMFP2UM completed with no encountering of errors, therefore, it's time are noted in boldface. Full set of result tables are listed in appendix B. Summary of frequencies of least computing time among different formulations are presented in below graphs.

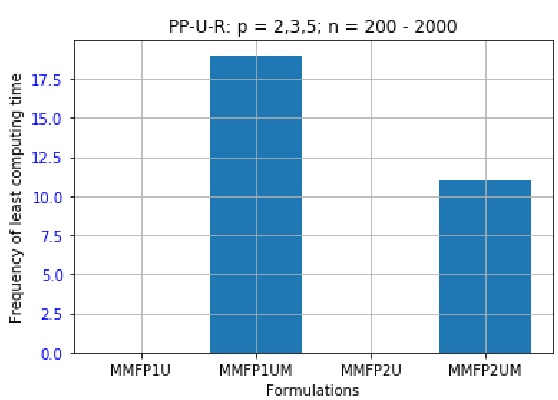


Figure 2.4: PP-U-R (small data)

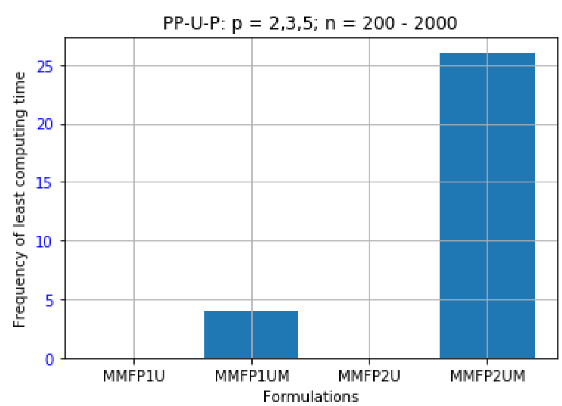


Figure 2.5: PP-U-P (small data)

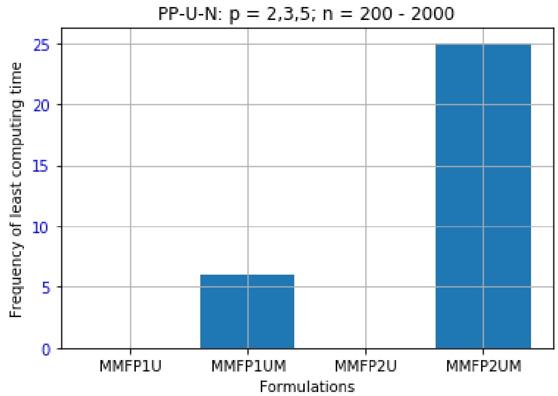


Figure 2.6: PP-U-N (small data)

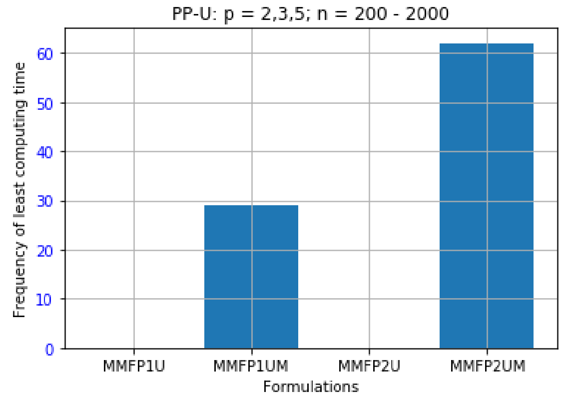


Figure 2.7: PP-U (small data)

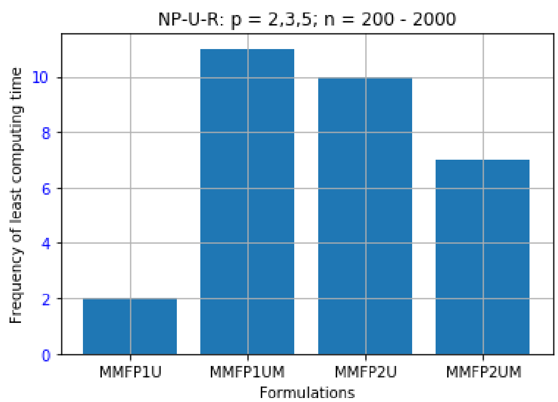


Figure 2.8: NP-U-R (small data)

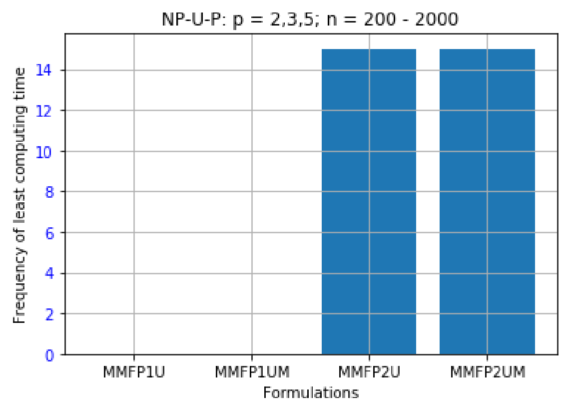


Figure 2.9: NP-U-P (small data)

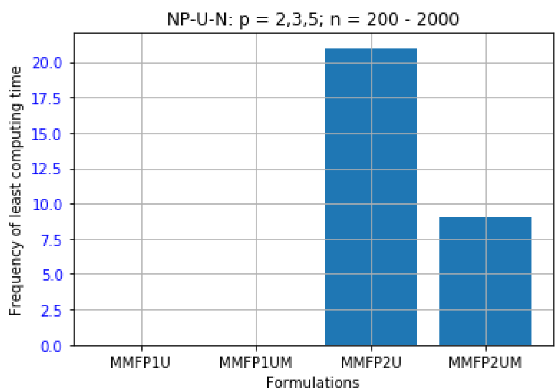


Figure 2.10: NP-U-N (small data)

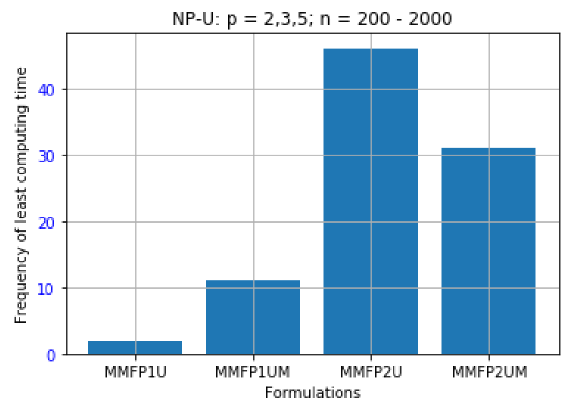


Figure 2.11: NP-U (small data)

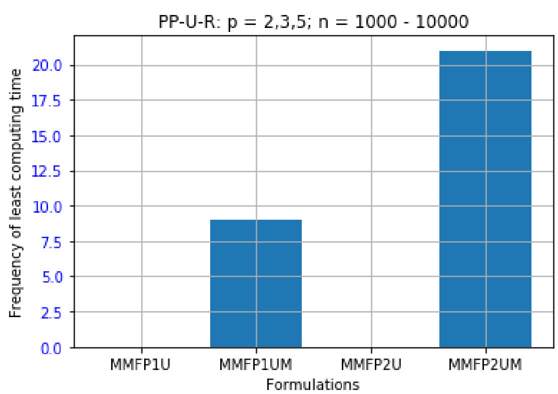


Figure 2.12: PP-U-R (medium data)

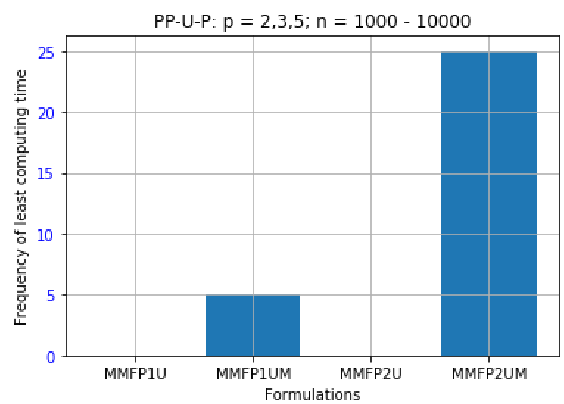


Figure 2.13: PP-U-P (medium data)

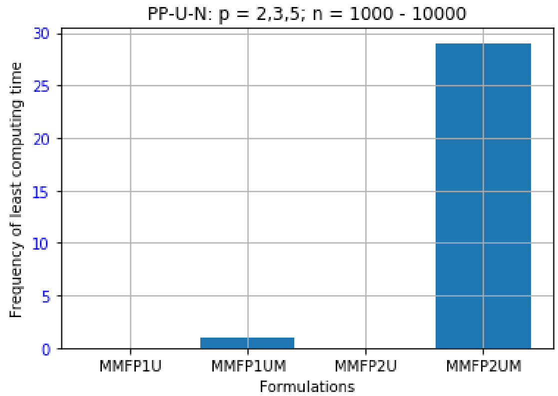


Figure 2.14: PP-U-N (medium data)

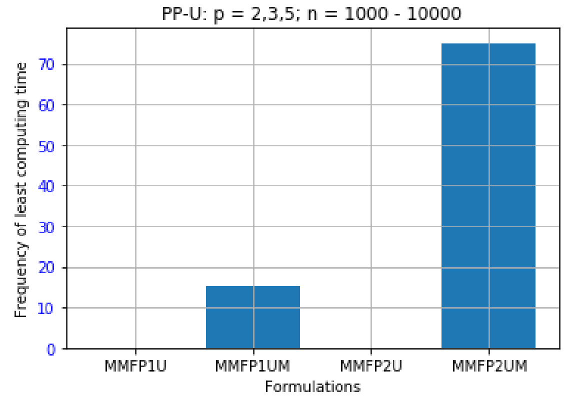


Figure 2.15: PP-U (medium data)

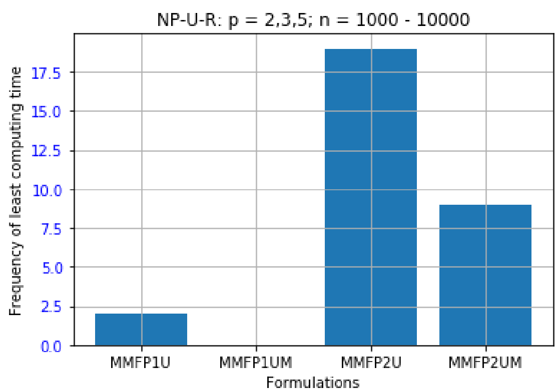


Figure 2.16: NP-U-R (medium data)

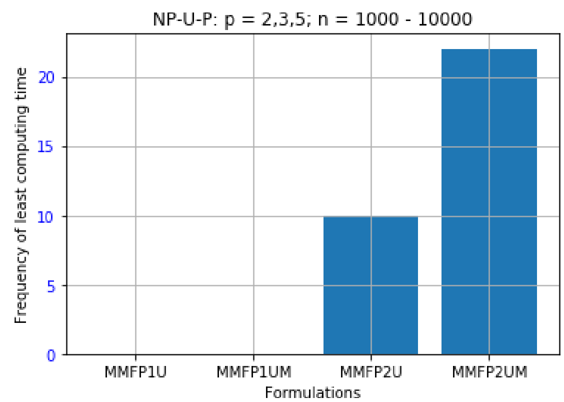


Figure 2.17: NP-U-P (medium data)

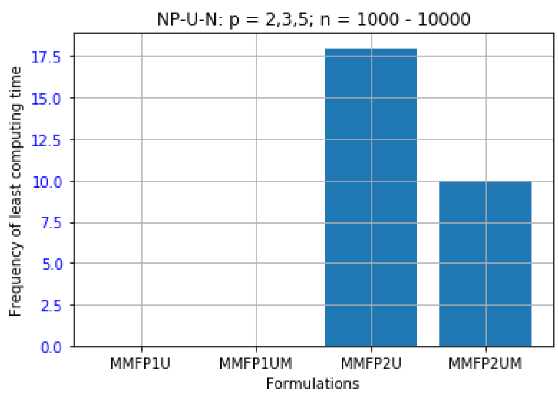


Figure 2.18: NP-U-N (medium data)

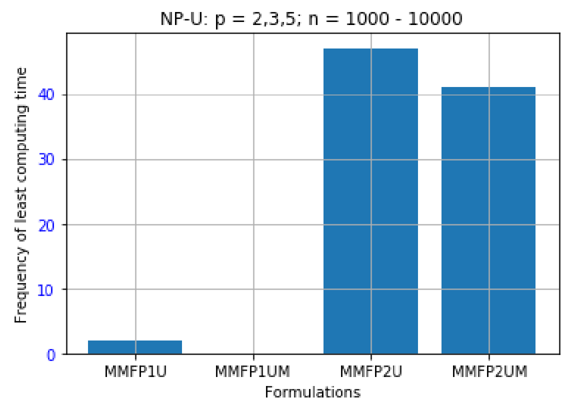


Figure 2.19: NP-U (medium data)

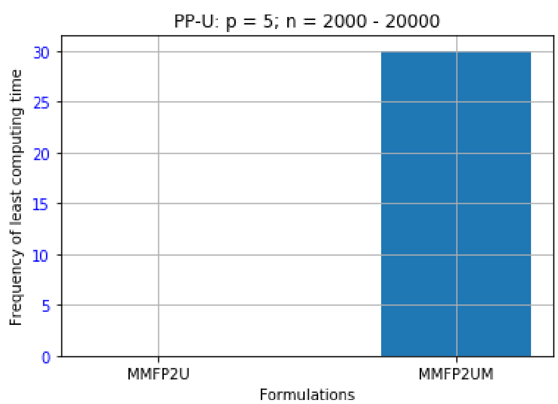


Figure 2.20: PP-U (large data)

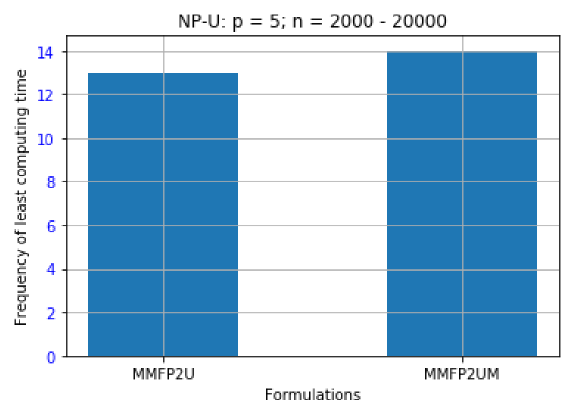


Figure 2.21: NP-U (large data)

In addition, we did some experiments to check the performance of the modified formulations equipped with pre-solve plan B and pre-solve plan C. We used NP-U-R class data for $p = 5$ and $n = 2000, \cdots, 10000$. We set the signs of α^k such that

$$\left(\frac{\alpha}{\beta}\right)_{\min} < 0 < \left(\frac{\alpha}{\beta}\right)_{\max} \qquad \text{for pre-solve plan B}$$

$$\left(\frac{\alpha}{\beta}\right)_{\min} < \left(\frac{\alpha}{\beta}\right)_{\max} < 0 \qquad \text{for pre-solve plan C.}$$

MMFP1UM-B is MMFP1U equipped with pre-solve plan B. MMFP1UM-C is MMFP1U equipped with pre-solve plan C. MMFP2UM-B and MMFP2UM-C are similarly defined. Pre-solve times were included in the time counts. In each test, the four modified formulations spent less than 0.1 seconds for pre-solvings.

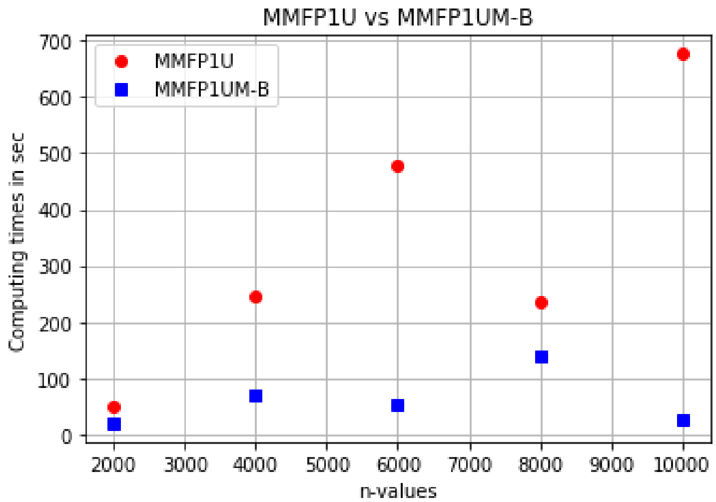


Figure 2.22: MMFP1U vs. MMFP1UM-B

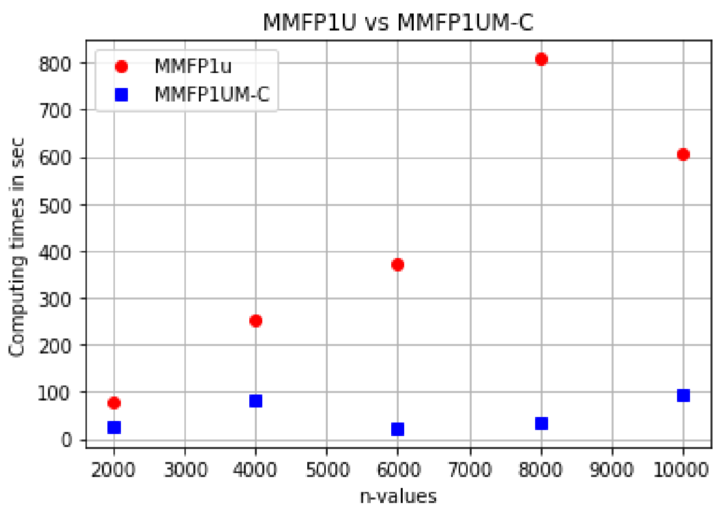


Figure 2.23: MMFP1U vs. MMFP1UM-C

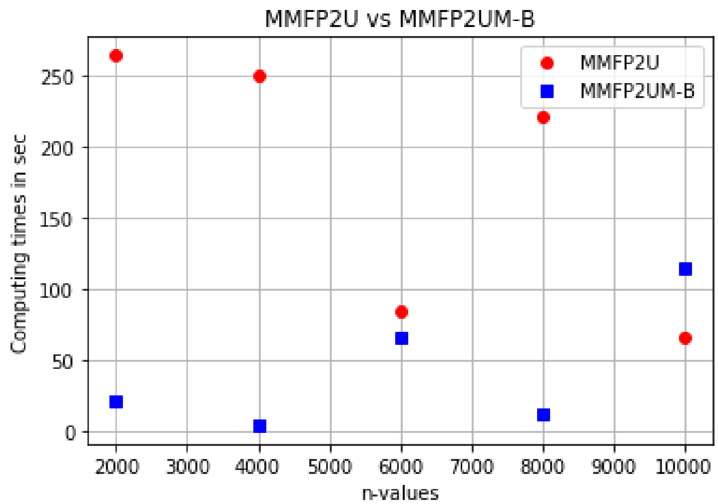


Figure 2.24: MMFP2U vs. MMFP2UM-B

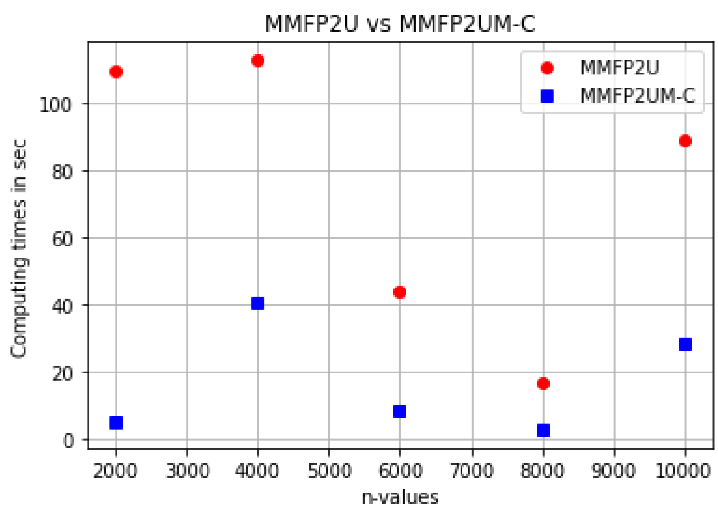


Figure 2.25: MMFP2U vs. MMFP2UM-C

For below MMFP2U vs. MMFP2UM-Custom plot, we used the NP-U-R data with $p = 5$ and $n = 2000, \cdots, 10000$. The selection of critical objective values followed the binary search process until y^* was captured within the interval of length ≤ 6. Total time of pre-solvings at each computation were found to be less than 0.3 seconds.

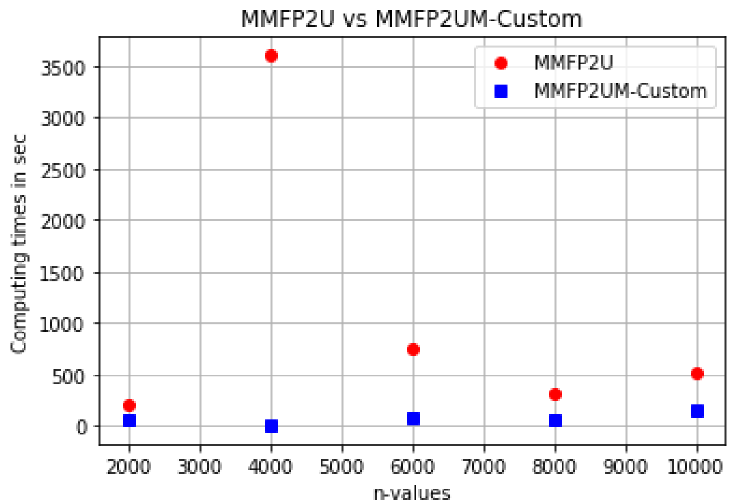


Figure 2.26: MMFP2U vs. MMFP2UM-customized

2.3.2 Remark on computational results

The modified formulations MMFP1UM and MMFP2UM are MMFP1U and MMFP2U equipped with the pre-solve plan A which is suitable if optimal value is non-negative. The pre-solve plan A is designed for the case:

$$y^l < \left(\frac{\alpha}{\beta}\right)_{\min},\ 0 < \left(\frac{\alpha}{\beta}\right)_{\min} < \left(\frac{\alpha}{\beta}\right)_{\max} <\ y^u \ \text{ and } y^* \geq 0.$$

The problem instance classes PP-U-R, PP-U-P and PP-U-N are where these modified formulations are expected to perform well for not only speed but also against computational errors. In fact these instance classes also fit the conditions stated in the proposition 2.5 if the y-bounds are loose and y^u is a large positive number. In the tests, MMFP1U failed to produce feasible solution almost all of the times. MMFP2U did produce feasible solutions for small problems in PP-U-R class. However in PP-U-P and PP-U-N classes, MMFP2U also failed to get feasible solutions most of the times. Modified formulations MMFP1UM and MMFP2UM produced feasible solutions in almost all the tests. They failed only a few times at larger problems in PP-U-P and PP-U-N classes. Most computational errors encountered are type I. The results verifies proposition 2.5 and the performance of pre-solve plan A.

The problem instance classes NP-U-P and NP-UN also fit the condition of proposition 2.5 in most cases when y-bounds are loose and y^u is a large positive number. MMFP1 produced infeasible solutions in most of the large problems. Other three formulations also produced incorrect solutions in smaller but noticeable numbers in large problems. Most of the errors found are type I errors. Since optimal values were negative in most of these problems, the formulations MMFP1UM and MMFP2UM did not perform better than their parent formulations on average. These results reflect the statement of proposition 2.5. and remind us that pre-solves should be planned according to the data.

Among the three sub classes -R, -P, -N, fewer errors are observed in the sub class -R in both PP- and NP- parent classes. This implies that correlations between the coefficients of numerators and denominators is a factor in occurrence of floating point rounding errors.

The additional tests in which we custom generated the data suitable for modified formulations equipped with pre-solve plans B and C confirm the performance of these pre-solves. In most tests these modified formulations outperformed their parent models. The final additional test on the customizable pre-solve plan in which y-bounds were tightened recursively using binary search pattern also outperformed the parent model. These results verify the nature of pre-solve that they need to be designed according to the data.

2.4 Knapsack constrained 0-1 MMFP problem

In this section, we present MILP formulations for 0-1 MMFP problem with knapsack constraint. We use the notation w_j for the weight value assigned to item j, and W for the total weight limit. Notice that $\mathbf{x} = \mathbf{0}$ is a feasible solution. The knapsack constraint is written as

$$\sum_{j=1}^{n} w_j x_j \leq W.$$

Remark: Discrete (including 0-1 binary case) linear minimax programming with knapsack constraint, when the number of objective ratio p is non-constant, is strongly NP-hard [2][26]. The 0-1 linear minimax programming problem is a special case of 0-1 MMFP problem satisfying condition (2.2). Therefore 0-1 MMFP knapsack programming with non-constant p is also strongly NP-hard.

We introduce MMFP1 formulation in the knapsack constrained case as follows:

$$
\begin{aligned}
\text{(MMFP1K)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k \le \sum_{j=1}^{n} d_j^k z_j + y\beta^k \quad \forall k = 1, \cdots, p \\
& z_j \le y^u x_j \qquad \forall j = 1, \cdots, n \\
& z_j \le y + y^l(x_j - 1) \quad \forall j = 1, \cdots, n \\
& z_j \ge y^l x_j \qquad \forall j = 1, \cdots, n \\
& z_j \ge y + y^u(x_j - 1) \quad \forall j = 1, \cdots, n \\
& \sum_{j=1}^{n} w_j x_j \le W \\
& \mathbf{x} \in \{0,1\}^n \\
& y \in [y^l, y^u].
\end{aligned}
$$

To implement this formulation, we need a pair of valid upper and lower bounds, y^u and y^l, on y. At first we discuss the upper bound of y. Let θ^k denotes the optimal objective value of the knapsack constrained 0-1 linear fractional program satisfying condition (2.2).

$$\theta^k = \max\Big\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \sum_{j=1}^{n} w_j x_j \le W,\ \mathbf{x} \in \{0,1\}^n\Big\}, \quad k = 1, \cdots, p.$$

Lemma 2.6. $\theta = \max\{\theta^k, k = 1, \cdots, p\}$ is an upper bound of y. Furthermore, θ can be identified in pseudo-polynomial time if p is at most pseudo-polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1K. Then,

$$y^* = \max_{1\le k\le p} \left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}\right\} \le \max_{1\le k\le p} \{\theta^k\} = \theta,$$

where $\mathbf{x}^*$ satisfies the knapsack constraint. To prove the complexity, notice that we can identify each θ^k in pseudo-polynomial time since condition (2.2) is satisfied [33]. Therefore, θ can be identified in pseudo-polynomial time if p is at most pseudo-polynomial. ■

Notice that θ found in lemma 2.6 is a global upper bound of y. The feasible region of knapsack constrained 0-1 MMFP is a subset of the feasible region of unconstrained 0-1 MMFP. Therefore, the global upper bound of y we have discussed in unconstrained 0-1 MMFP case is still a global upper bound for knapsack constrained 0-1 MMFP case.

To find a global lower bound of y, let γ^k be the optimal objective function value of the knapsack constrained 0-1 linear fractional program satisfying (2.2).

$$\gamma^k = \min\Big\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} : \sum_{j=1}^n w_j x_j \leq W,\ \mathbf{x} \in \{0,1\}^n\Big\}, \quad k = 1, \cdots, p.$$

Lemma 2.7. $\gamma = \min\{\gamma^k, k = 1, \cdots, p\}$ is a lower bound of y. Furthermore, γ can be identified in pseudo-polynomial time if p is at most pseudo-polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1. Since

$$\gamma \leq \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}, \quad \forall k = 1, \cdots, p.$$

Then

$$y^* = \max_{1\leq k\leq p} \left\{\frac{\alpha^k + \sum_{j=1}^n c_j^k x_j^*}{\beta^k + \sum_{j=1}^n d_j^k x_j^*}\right\} \geq \min_{1\leq k\leq p} \{\gamma^k\} = \gamma,$$

where $\mathbf{x}^*$ satisfies the knapsack constraint. To prove the complexity, notice that we can identify γ^k in pseudo-polynomial time since condition (2.2) is satisfied [33]. Therefore, γ can be identified in pseudo-polynomial time if p is at most pseudo-polynomial. ■

With the same reason, the global lower bound of y we have discussed in unconstrained 0-1 MMFP case is a global lower bound for knapsack constrained 0-1 MMFP case. Further more, lemma 2.4 is also applicable in this case.

Now we introduce MMFP2 formulation for the knapsack constrained case as follows:

$$
\begin{aligned}
\text{(MMFP2K)} \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k \leq \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& z_i^k \leq y^u v_i^k \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \leq y + y^l (v_i^k - 1) \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \geq y^l v_i^k \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& z_i^k \geq y + y^u (v_i^k - 1) \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\} \\
& \sum_{j=1}^{n} w_j x_j \leq W \\
& \mathbf{x} \in \{0, 1\}^n \\
& y \in [y^l, y^u] \\
& v_i^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p, \ \forall i = \{1, \cdots, q^k\}.
\end{aligned}
$$

For the experiments in this thesis, y^u and y^l for both of these formulations are computed by (2.3) and (2.4). These are global bounds if all denominator coefficients are positive integers. In addition, we have modified formulations MMFP1KM and MMFP2KM to be used in computational experiments. These formulations are MMFP1K and MMFP2K equipped with pre-solve plan A.

2.4.1 Experimental analysis on knapsack constrained 0-1 MMFP formulations

We ran MMFP1K, MMFP1KM, MMFP2K, and MMFP2KM with knapsack constraint on six problem instance classes. Weight values w_j were generated as random uniformly distributed integers in the range [1,100]. We calculated the size dependent weight limit W as follows: $W = \frac{(n/4)(w^l + w^u)}{2}$, where $w^l = 1$, $w^u = 100$. Problem instance classes were generated as follows:

(PP-K-R)	Random uniformly distributed integers: All data values $c_j^k, \alpha^k, d_j^k, \beta^k$ are randomly generated in the range [1,100].
(PP-K-P)	Positively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.

(PP-K-N)	Negatively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100]. c_j^k are rearranged in increasing order and d_j^k are rearranged in decreasing order in each objective.
(NP-K-R)	Random uniformly distributed integers: Data values c_j^k, α^k are randomly generated in the range [-100,100]. Data values d_j^k, β^k are randomly generated in the range [1,100].
(NP-K-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(NP-K-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in decreasing order in each objective.

Below table presents objective values found by the formulations, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors.

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-92.28571	0.0	0.31257	-92.28571	0.0	**0.25021**
4000	-96.3871	0.0	0.46891	-96.3871	0.0	**0.34371**
6000	-97.52941	0.0	0.60938	-97.52941	0.0	**0.48446**
8000	-98.02899	1e-05	0.93756	-98.02899	1e-05	**0.62507**
10000	-98.275	0.0	1.1407	-98.275	7e-05	**0.71874**
12000	-98.6055	0.0	**1.35893**	-98.60185	5e-05	0.73435
14000	-98.70958	0.0	1.67166	-98.69355	0.0	**0.81294**
16000	-98.83108	0.0	**2.24998**	-98.82759	4e-05	1.07807
18000	-455661.26197	0.0	1.12512	-98.93125	1e-05	**1.12492**
20000	-98.95676	8e-05	2.17047	-98.95789	3e-05	**1.18737**

Table 2.5: NP-K-P-5

Full set of result tables are listed in appendix B. Summary of frequencies of least computing time among different formulations are presented in below graphs.

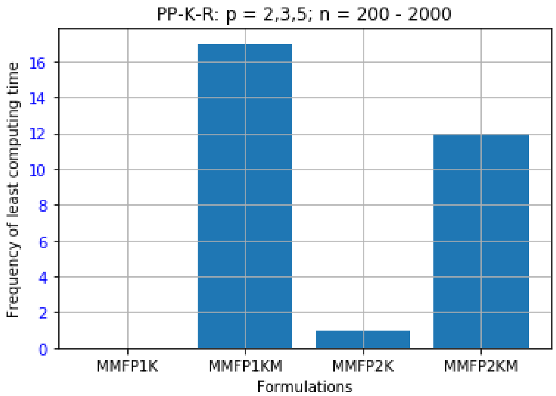


Figure 2.27: PP-K-R (small data)

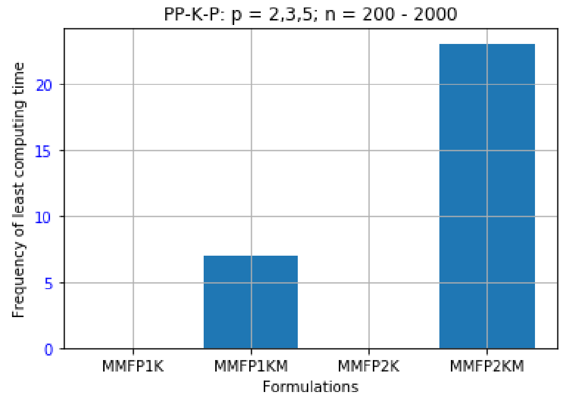


Figure 2.28: PP-K-P (small data)

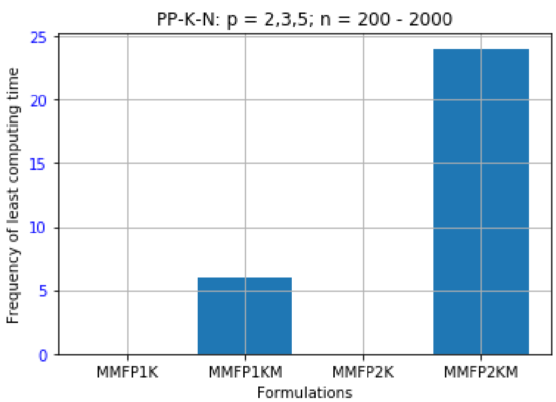


Figure 2.29: PP-K-N (small data)

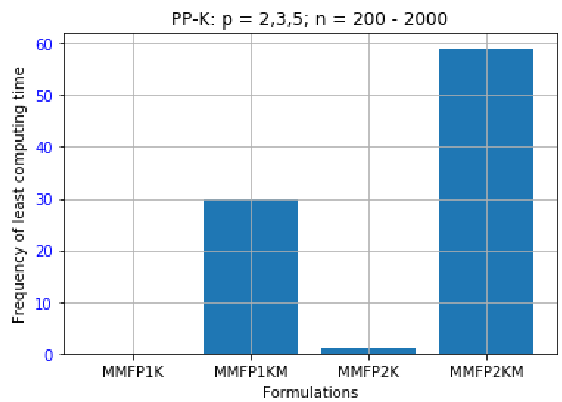
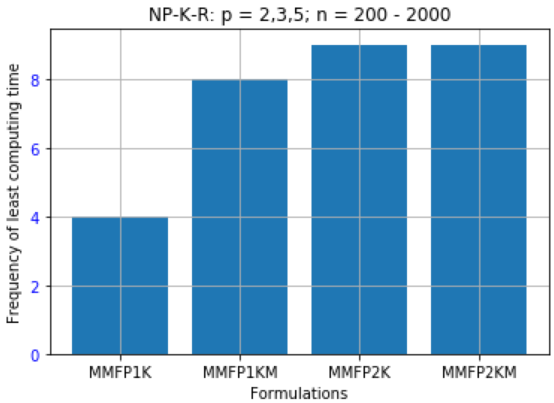


Figure 2.30: PP-K (small data)



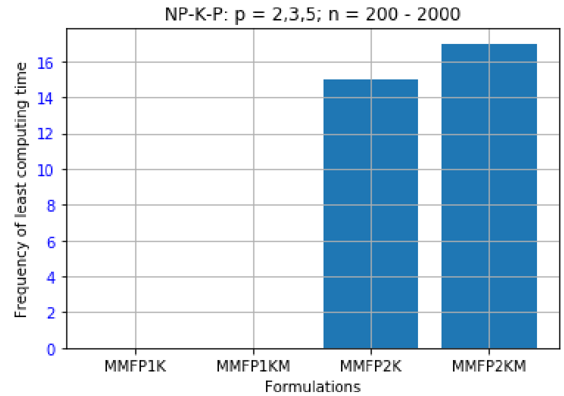
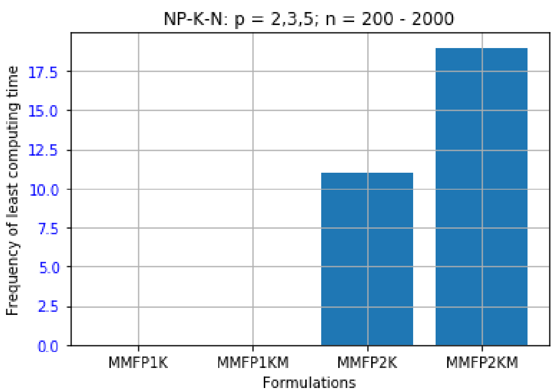


Figure 2.31: NP-K-R (small data)

Figure 2.32: NP-K-P (small data)



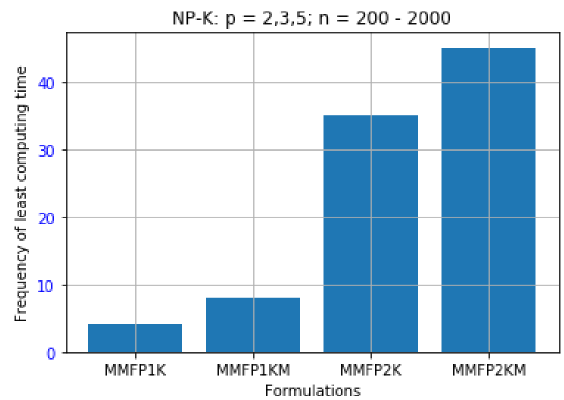


Figure 2.33: NP-K-N (small data)

Figure 2.34: NP-K (small data)

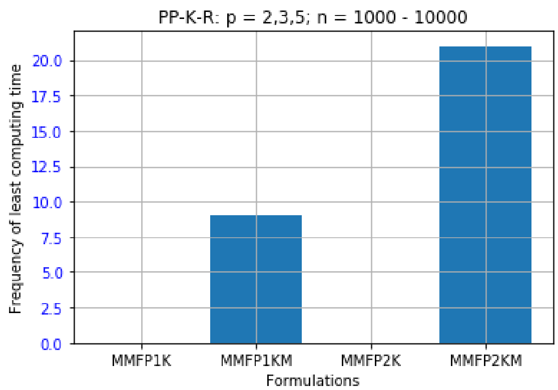


Figure 2.35: PP-K-R (medium data)

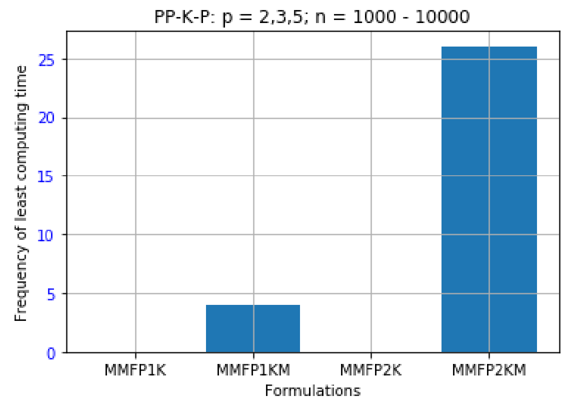


Figure 2.36: PP-K-P (medium data)

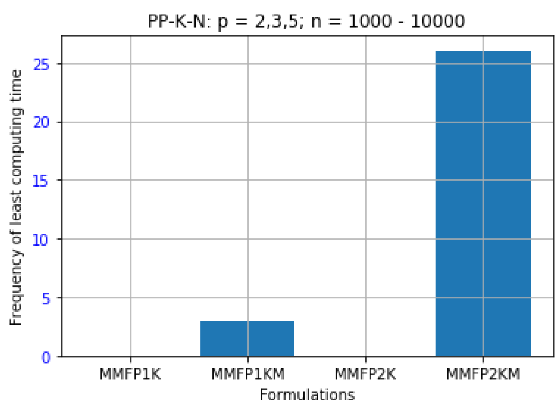


Figure 2.37: PP-K-N (medium data)

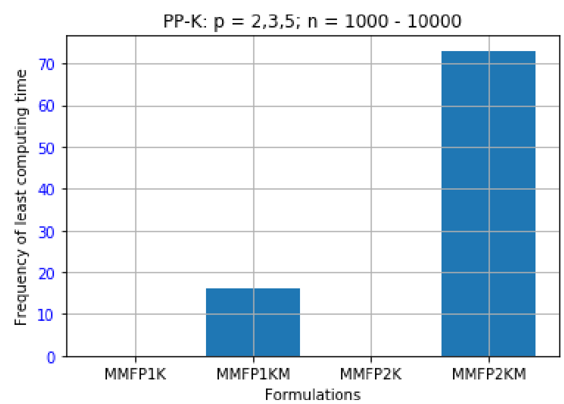


Figure 2.38: PP-K (medium data)

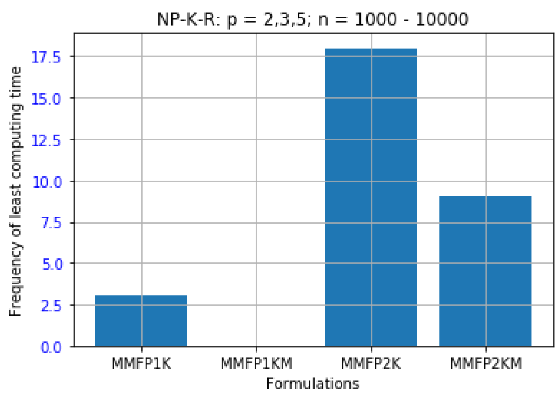


Figure 2.39: NP-K-R (medium data)

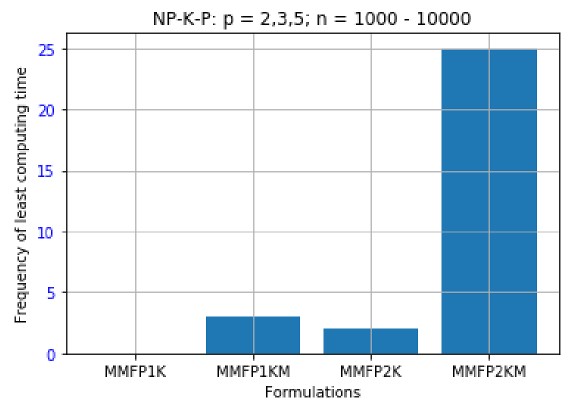


Figure 2.40: NP-K-P (medium data)

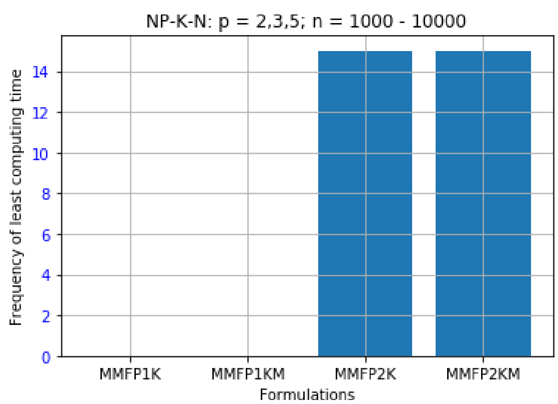


Figure 2.41: NP-K-N (medium data)

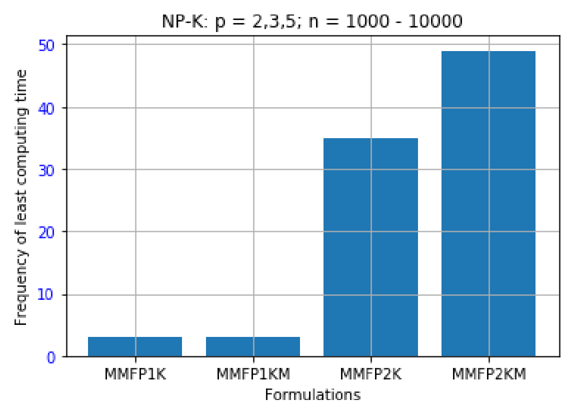


Figure 2.42: NP-K (medium data)

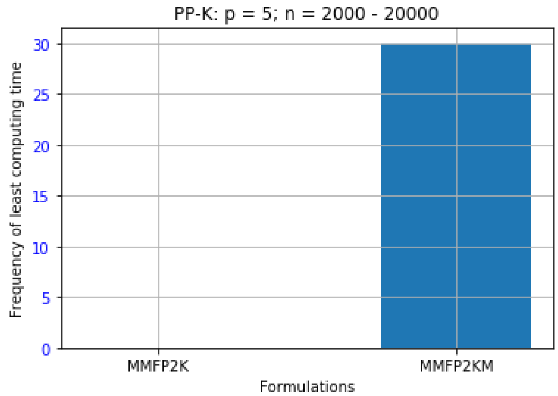


Figure 2.43: PP-K (large data)

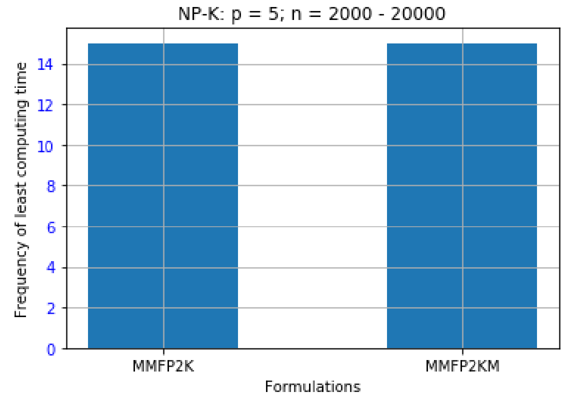


Figure 2.44: NP-K (large data)

Below plot displays the computing speed vs. n-values for 3 different p-values in the knapsack constraint problems. Pre-solve time for each computation was found to be less than 0.1 seconds.

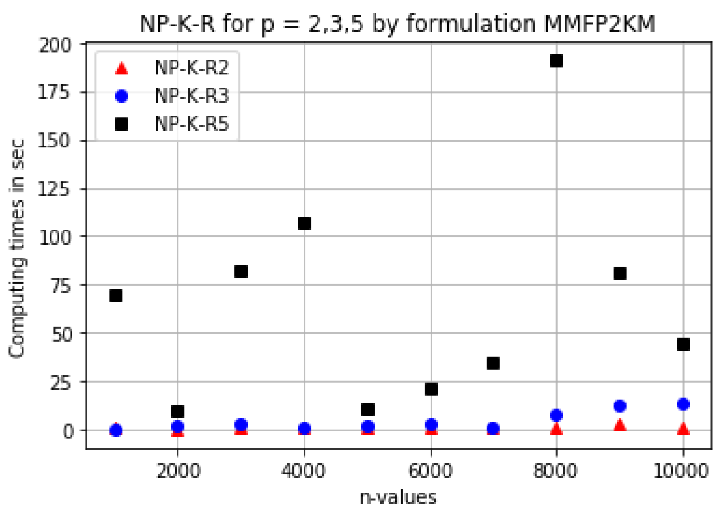


Figure 2.45: NP-K-R-2, NP-K-R-3, NP-K-R-5

In addition, we did some experiments to check the performance of the modified formulations equipped with pre-solve plans B and C. We use the NP-K-R class data for $p = 5$, and $n = 2000, 4000, \cdots, 10000$. We set the signs of α^k such that

$$\left(\frac{\alpha}{\beta}\right)_{\min} < 0 < \left(\frac{\alpha}{\beta}\right)_{\max} \quad \text{for pre-solve plan B}$$

$$\left(\frac{\alpha}{\beta}\right)_{\min} < \left(\frac{\alpha}{\beta}\right)_{\max} < 0 \quad \text{for pre-solve plan C.}$$

MMFP1KM-B is MMFP1K equipped with pre-solve plan B. MMFP1KM-C is MMFP1K equipped with pre-solve plan C. Pre-solve times were included in the time counts. In each test, the two modified formulations spent less than 0.1 seconds for pre-solvings.

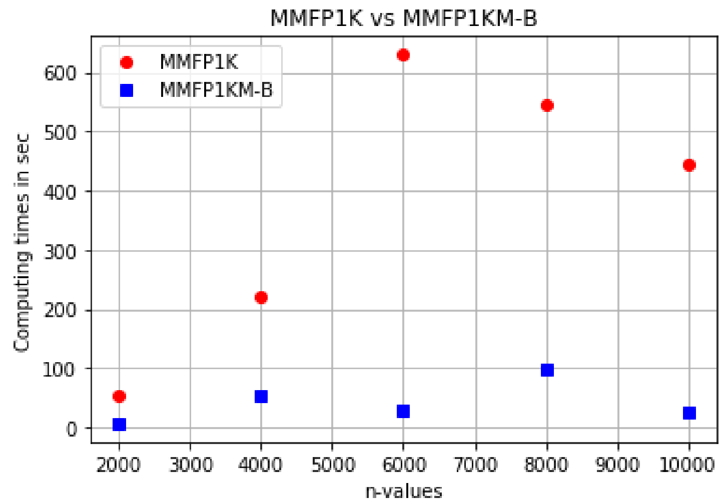


Figure 2.46: MMFP1K vs. MMFP1KM-B

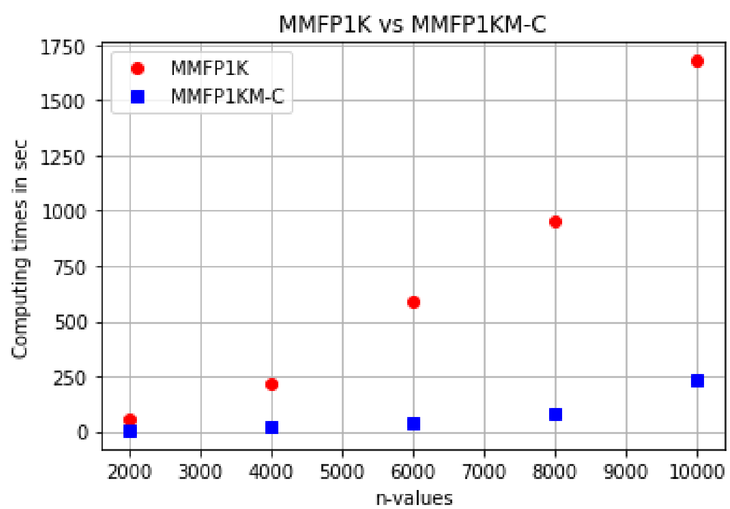


Figure 2.47: MMFP1K vs. MMFP1KM-C

For below MMFP2K vs. MMFP2KM-Custom plot, we used the NP-K-R data with $p = 5$ and $n = 2000, \cdots, 10000$. The selection of critical objective values followed the binary search process until y^* was captured within the interval of length ≤ 6. Total time of pre-solvings at each computation were found to be less than 0.4 seconds.

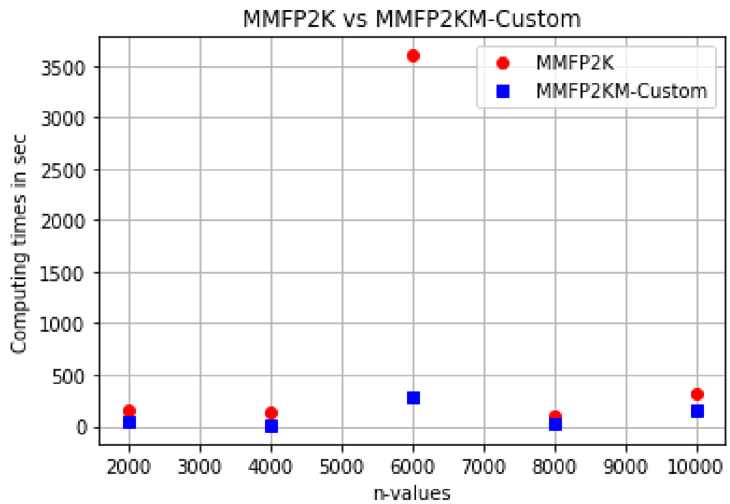


Figure 2.48: MMFP2K vs. MMFP2KM-Custom

2.4.2 Remark on computational results

Since in knapsack constrained problems, $\mathbf{x} = \mathbf{0}$ is a feasible point, we expect to see very similar results as in unconstrained problems with respect to the type I error. The modified formulations MMFP1M and MMFP2M produced only a few type I or type II errors on problem instance classes PP-K-P and PP-K-N. This is expected since pre-solve plan A used in MMFP1M and MMFP2M is designed for these problem instances. MMFP1 and MMFP2 produced type I errors in most of these problem instances. Also, as in unconstrained problems, fewer errors were produced in PP-K-R class compared to the other two classes.

Test results from the problem instance classes NP-K-R, NP-K-P and NP-K-N also showed close similarity with unconstrained problems. The modified formulations performed at least as good as or better than the parent formulations on average. In these problem instance classes, only a few computational errors were found. In the problem instance class NP-K-R, modified formulations generally did not perform as good as the parent formulations. Almost no computational errors were found in this problem instance class.

Results of the additional tests in which we customized the data suitable for the modified formulations equipped with pre-solve plans B and C confirm the performance of these pre-solves. In all the tests, these modified formulations outperformed the parent models. The final additional test on the customizable pre-solve plan in which y-bounds were tightened recursively using binary search also outperformed the parent model. These results verify the nature of pre-solve that they need to be designed according to the data.

2.5 Assignment constrained 0-1 MMFP problem

Assignment problem was proposed as a linear programming problem in 1950s, and its general linear fractional form appeared in 1970s [6]. The 0-1 *minimax linear fractional assignment programming* (MMFPA) problem can occur in the following real life scenario. Suppose we have n number of workers. In each objective ratio $k = 1, \cdots, p$, there are n number of jobs. We have cost values and benefit values stored in the objects c_{ij}^k and d_{ij}^k where $k = 1, \cdots, p$ and $i, j = 1, \cdots, n$. The binary decision variables are x_{ij}. Their definitions are as follows:

$$c_{ij}^k = \text{cost of worker } i \text{ taking the job } j \text{ from objective } k$$
$$d_{ij}^k = \text{benefit of worker } i \text{ doing job } j \text{ from objective } k$$

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ takes the job } j \\ 0, & \text{otherwise} \end{cases}, \quad \forall k = 1, \cdots, p.$$

We have p number of linear fractional objectives. For each $k = 1, \cdots, p$, we write

$$f_k(\mathbf{x}) = \frac{\alpha^k + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^k x_{ij}}{\beta^k + \sum_{i=1}^n \sum_{j=1}^n d_{ij}^k x_{ij}}.$$

The constraints are, for each $k = 1, \cdots, p$, each worker takes exactly one job and each job is assigned to exactly one worker. The minimax problem can be written as

$$\begin{aligned} \text{Minimize} \quad & \text{Max}\{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_p(\mathbf{x})\} \\ \text{Subject to} \quad & \sum_{j=1}^n x_{ij} = 1, \quad \forall i = 1, \cdots, n \\ & \sum_{i=1}^n x_{ij} = 1, \quad \forall j = 1, \cdots, n \\ & x_{ij} \in \{0, 1\}, \quad \forall i, j = 1, \cdots, n. \end{aligned}$$

Remark: Discrete (including 0-1 binary case) linear minimax assignment programming, when the number of objective ratio p is non-constant, is strongly NP-hard [2][26]. The 0-1 linear minimax programming problem is a special case of 0-1 MMFP problem satisfying condition (2.2). Therefore 0-1 MMFP assignment programming with non-constant p is also strongly NP-hard.

Formulation MMFP1 can be applied for MMFP1A if the denominators of all objectives are positive over the feasible region. As in MMFP1, we can linearize the problem by introducing the continuous variable $y \geq f_k(x)$ for all $k = 1, \cdots, p$. The resulting n^2 number of bilinear terms yx_{ij}

can be represented by continuous variables z_{ij} and are linearized by McCormick constraints using the bounds on y. The minimax linear fractional assignment problem can be formulated as follows:

$$
\begin{aligned}
\text{(MMFP1A)} \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} \le y\beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k z_{ij} \quad \forall k = 1, \cdots, p \\
& z_{ij} \le y^u x_{ij} \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n \\
& z_{ij} \le y + y^l (x_{ij} - 1) \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n \\
& z_{ij} \ge y^l x_{ij} \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n \\
& z_{ij} \ge y + y^u (x_{ij} - 1) \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1, \cdots, n \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1, \cdots, n \\
& x_{ij} \in \{0, 1\}, \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n \\
& y \in [y^l, y^u].
\end{aligned}
$$

Before implementing this formulation, we need a pair of reasonable upper and lower bounds, y^u and y^l, on y. Let θ^k denotes the optimal objective value of the assignment constrained 0-1 linear fractional program satisfying condition (2.2).

$$
\begin{aligned}
\theta^k = \quad & \text{Maximize} \quad \frac{\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij}}{\beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij}} \\
& \text{Subject to} \quad \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1, \cdots, n \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1, \cdots, n \\
& x_{ij} \in \{0, 1\}, \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n,
\end{aligned}
$$

where $k = 1, \cdots, p$.

Lemma 2.8. $\theta = \max\{\theta^k, k = 1, \cdots, p\}$ is an upper bound of y. Furthermore, θ can be identified in polynomial time if p is at most polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1A. Then,

$$y^* = \max_{1\leq k\leq p} \left\{ \frac{\alpha^k + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^k x_{ij}^*}{\beta^k + \sum_{i=1}^n \sum_{j=1}^n d_{ij}^k x_{ij}^*} \right\} \leq \max_{1\leq k\leq p} \{\theta^k\} = \theta,$$

where $\mathbf{x}^*$ satisfies the assignment constraints. To prove the complexity, notice that we can identify each θ^k in polynomial time since condition (2.2) is satisfied [33]. Therefore, θ can be identified in polynomial time if p is at most polynomial [19]. ■

Note that θ found in lemma 2.8 is a global upper bound of y. To find a global lower bound of y, let γ^k be the optimal objective function value of the assignment constrained 0-1 linear fractional program satisfying (2.2).

$$\begin{aligned} \gamma^k = \quad \text{Minimize} \quad & \frac{\alpha^k + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^k x_{ij}}{\beta^k + \sum_{i=1}^n \sum_{j=1}^n d_{ij}^k x_{ij}} \\ \text{Subject to} \quad & \sum_{j=1}^n x_{ij} = 1 \quad \forall i = 1, \cdots, n \\ & \sum_{i=1}^n x_{ij} = 1 \quad \forall j = 1, \cdots, n \\ & x_{ij} \in \{0,1\}, \quad \forall i = 1, \cdots, n \quad \forall j = 1, \cdots, n, \end{aligned}$$

where $k = 1, \cdots, p$.

Lemma 2.9. $\gamma = \min\{\gamma^k, k = 1, \cdots, p\}$ is a lower bound of y. Furthermore, γ can be identified in polynomial time if p is at most polynomial.

Proof. Let $(\mathbf{x}^*, y^*)$ be the optimal solution to MMFP1A. Since

$$\gamma \leq \frac{\alpha^k + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^k x_{ij}^*}{\beta^k + \sum_{i=1}^n \sum_{j=1}^n d_{ij}^k x_{ij}^*}, \quad \forall k = 1, \cdots, p.$$

Then

$$y^* = \max_{1\leq k\leq p} \left\{ \frac{\alpha^k + \sum_{i=1}^n \sum_{j=1}^n c_{ij}^k x_{ij}^*}{\beta^k + \sum_{i=1}^n \sum_{j=1}^n d_{ij}^k x_{ij}^*} \right\} \geq \min_{1\leq k\leq p} \{\gamma^k\} = \gamma,$$

where $\mathbf{x}^*$ satisfies the assignment constraints. To prove the complexity, notice that we can identify γ^k in polynomial time since condition (2.2) is satisfied [19]. Therefore, γ can be identified in polynomial time if p is at most polynomial. ■

We now present ways to generate simpler upper and lower bounds. Below lemma is a well-known proven fact which we use to construct the simple bounds of the continuous variable in the bilinear terms in the program formulations. Proof is presented for completion.

Lemma 2.10. Let $\mathbf{c} = [c_{ij}] \in \mathbb{R}^{n\times n}$ be a square matrix, $\mathbf{x} \in \{0,1\}^n$ be a binary variable. Let

$$g(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

be a function satisfying the linear assignment constraints. Define maximum row sum ($RS_{\max}$), minimum row sum ($RS_{\min}$), maximum column sum ($CS_{\max}$), and minimum column sum ($CS_{\min}$) of $\mathbf{c}$ as follows:

$$RS_{\max}(\mathbf{c}) = \sum_{i=1}^{n} \max_{1\le j\le n}\{c_{ij}\}, \quad RS_{\min}(\mathbf{c}) = \sum_{i=1}^{n} \min_{1\le j\le n}\{c_{ij}\}$$
$$CS_{\max}(\mathbf{c}) = \sum_{j=1}^{n} \max_{1\le i\le n}\{c_{ij}\}, \quad CS_{\min}(\mathbf{c}) = \sum_{j=1}^{n} \min_{1\le i\le n}\{c_{ij}\}.$$

Then

$$\max\{RS_{\min}(\mathbf{c}), CS_{\min}(\mathbf{c})\} \le g(\mathbf{x}) \le \min\{RS_{\max}(\mathbf{c}), CS_{\max}(\mathbf{c})\}.$$

Proof. By definition of assignment problem constraint, g takes exactly one element from each row of $\mathbf{c}$ and sum them up. A possible largest value of $g(\mathbf{x})$ is when g takes the largest element from each of the rows. Hence,

$$g(\mathbf{x}) \le RS_{\max}(\mathbf{c}).$$

Similarly, we can reason that

$$g(\mathbf{x}) \le CS_{\max}(\mathbf{c}).$$

Since we need to satisfy both above inequalities,

$$g(\mathbf{x}) \le \min\{RS_{\max}(\mathbf{c}), CS_{\max}(\mathbf{c})\}.$$

Again using the definition of assignment problem constraint, a possible smallest value of $g(\mathbf{x})$ is when g takes the minimum element from each of the rows. That means,

$$g(\mathbf{x}) \ge RS_{\min}(\mathbf{c}).$$

Similarly, we can reason that

$$g(\mathbf{x}) \ge CS_{\min}(\mathbf{c}).$$

Since we need to satisfy both above inequalities,

$$g(\mathbf{x}) \ge \max\{RS_{\min}(\mathbf{c}), CS_{\min}(\mathbf{c})\}.$$

The desired result follows.

■

Assuming that denominator coefficients are positive integers for all objective ratios, we derive simple formulas for the y bounds. Note that in this derivation the decision variable $\mathbf{x}$ satisfies the assignment constraints. By lemma 2.10, for all $k = 1, \cdots, p$,

$$\text{numerator of } f_k(\mathbf{x}) \leq \min\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\} + \alpha^k$$

$$\max\{RS_{\min}(\mathbf{c}^k), CS_{\min}(\mathbf{c}^k)\} + \alpha^k \leq \text{numerator of } f_k(\mathbf{x}).$$

By the assumption we stated earlier,

$$f_k(\mathbf{x}) \leq \min\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\} + \alpha^k, \quad k = 1, \cdots, p.$$

Therefore,

$$\max_{1\leq k\leq p}\{f_k(\mathbf{x})\} \leq \max_{1\leq k\leq p}\{\min\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\} + \alpha^k\}$$

and

$$y^* \leq \max_{1\leq k\leq p}\{\min\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\} + \alpha^k\}.$$

If the numerators of all objective ratios are non-negative for all feasible points, then zero is a valid lower bound of y. If at least one numerator is negative for some feasible points, then

$$\min_{1\leq k\leq p}\{\max\{RS_{\min}(\mathbf{c}^k), CS_{\min}(\mathbf{c}^k)\} + \alpha^k\}$$

is a valid lower bound of y. The results of above derivation give a pair of simple bounds of y as follows:

$$y^u = \max\Big\{0, \max_{1\leq k\leq p}\{\min\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\} + \alpha^k\}\Big\} \tag{2.5}$$

$$y^l = \min\Big\{0, \min_{1\leq k\leq p}\{\max\{RS_{\min}(\mathbf{c}^k), CS_{\min}(\mathbf{c}^k)\} + \alpha^k\}\Big\}. \tag{2.6}$$

MMFP1A can be modified using the binary expansion to form MMFP2A in the same way as in MMFP2 formulation. We introduce

$$D^k = \sum_{i=1}^{n}\sum_{j=1}^{n}\{|d_{ij}^k| : d_{ij}^k < 0\}, \quad k = 1, \cdots, p.$$

And let

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} + D^k = \sum_{s=1}^{q^k} 2^{s-1} v_s^k, \quad k = 1, \cdots, p$$

where

$$v_s^k \in \{0,1\} \quad s \in \{1,\cdots,q^k\},\ k=1,\cdots,p$$

$$q^k = \left\lfloor \log_2\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{ij}^k|\Big)\right\rfloor + 1, \quad k=1,\cdots,p.$$

The constraint

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} \le y\beta^k + y\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij}, \quad k=1,\cdots,p$$

becomes

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} \le y\beta^k + y\Big(\sum_{s=1}^{q^k} 2^{s-1}v_s^k - D^k\Big), \quad k=1,\cdots,p.$$

By using substitutions $z_s^k = yv_s^k$ for the bilinear terms, we get

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} \le \sum_{s=1}^{q^k} 2^{s-1}z_s^k + y(\beta^k - D^k), \quad k=1,\cdots,p.$$

The final formulation is

(MMFP2A) Minimize y

Subject to

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} \le \sum_{s=1}^{q^k} 2^{s-1}z_s^k + y(\beta^k - D^k) \quad k=1,\cdots,p$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} + D^k = \sum_{s=1}^{q^k} 2^{s-1}v_s^k \quad k=1,\cdots,p$$

$$z_s^k \le y^u v_s^k \quad \forall k=1,\cdots,p \quad \forall s \in \{1,\cdots,q^k\}$$

$$z_s^k \le y + y^l(v_s^k - 1) \quad \forall k=1,\cdots,p \quad \forall s \in \{1,\cdots,q^k\}$$

$$z_s^k \ge y^l v_s^k \quad \forall k=1,\cdots,p \quad \forall s \in \{1,\cdots,q^k\}$$

$$z_s^k \ge y + y^u(v_s^k - 1) \quad \forall k=1,\cdots,p \quad \forall s \in \{1,\cdots,q^k\}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i=1,\cdots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j=1,\cdots,n$$

$$x_{ij} \in \{0,1\}, \quad \forall i=1,\cdots,n \quad \forall j=1,\cdots,n$$

$$y \in [y^l, y^u].$$

For the experiments in this thesis, y^u and y^l for both MMFP1A and MMFP2A are computed by (2.5) and (2.6). These are valid bounds if all denominator coefficients are positive integers. In addition, we have modified formulations MMFP1AM and MMFP2AM. These are MMFP1A and MMFP2A equipped with pre-solve plan A. Notice that due to $\mathbf{x} = \mathbf{0}$ being not feasible, pre-solve plan A contains only pre-solve (1). That is, pre-solve plan A checks if zero can be used as an lower bound or upper bound of y.

2.5.1 Experimental analysis on assignment constrained 0-1 MMFP formulations

We ran MMFP1A, MMFP1AM, MMFP2A, and MMFP2AM on six problem instance classes. The two 2-dimensional array (matrices), namely $[c_{ij}^k]_{n\times n}$, $[d_{ij}^k]_{n\times n}$ were generated for each objective ratio $k = 1, \cdots, p$. Two 1-dimensional arrays α^k and β^k were generated for constant terms in the objectives. Problem instance classes were generated as follows:

(PP-A-R)	Random uniformly distributed integers: All data values $c_j^k, \alpha^k, d_j^k, \beta^k$ are randomly generated in the range [1,100].
(PP-A-P)	Random uniformly distributed integers: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(PP-A-N)	Random uniformly distributed integers: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100]. c_j^k are rearranged in increasing order and d_j^k are rearranged in decreasing order in each objective.
(NP-A-R)	Random uniformly distributed integers: Data values c_j^k, α^k are randomly generated in the range [-100,100]. Data values d_j^k, β^k are randomly generated in the range [1,100].
(NP-A-P)	Random uniformly distributed integers: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(NP-A-N)	Random uniformly distributed integers: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in decreasing order in each objective.

Below table presented objective value found by the solver, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors.

n	MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.15193	0.0	4.87503	-0.15193	0.0	**4.46409**
60	-0.12563	0.0	**6.26836**	-0.12563	0.0	13.58492
90	-0.09783	0.0	35.86444	-0.09783	0.0	**26.35222**
120	-0.09724	4e-05	73.78809	-0.09724	0.0	**69.81655**
150	-0.09505	9e-05	1338.57992	-0.09505	0.0001	**675.22626**
180	-0.08534	9e-05	251.40497	-0.08534	6e-05	**98.13707**
210	-0.0774	7e-05	156.68369	-0.0774	4e-05	**119.69937**
240	-0.07678	0.0001	**210.46307**	-0.07678	0.0001	492.80779
270	-0.07079	0.0001	165.34285	-0.07079	0.0001	**72.13409**
300	-0.07061	0.0001	**644.21453**	-0.07061	0.0001	670.90898

Table 2.7: NP-A-N-5

Full set of result tables are listed in appendix B. Summary of frequencies of least computing time among different formulations are presented in below graphs.

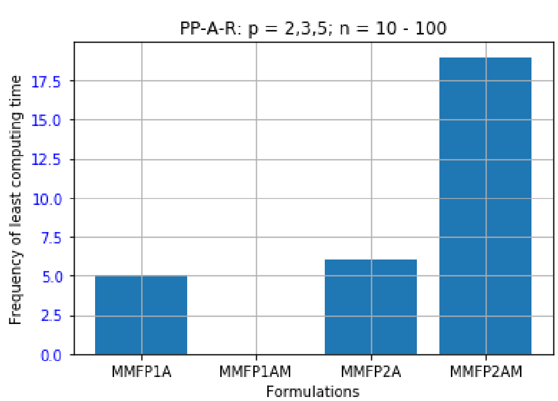


Figure 2.49: PP-A-R (small data)

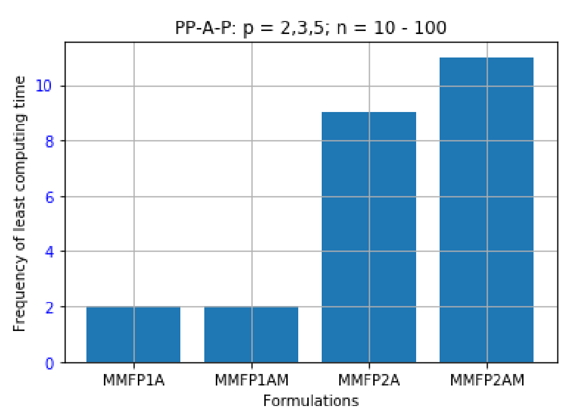


Figure 2.50: PP-A-P (small data)

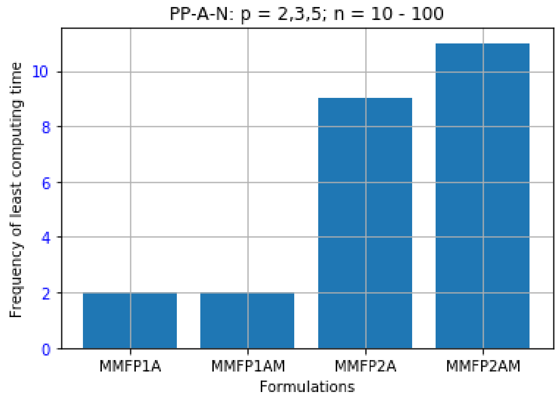


Figure 2.51: PP-A-N (small data)

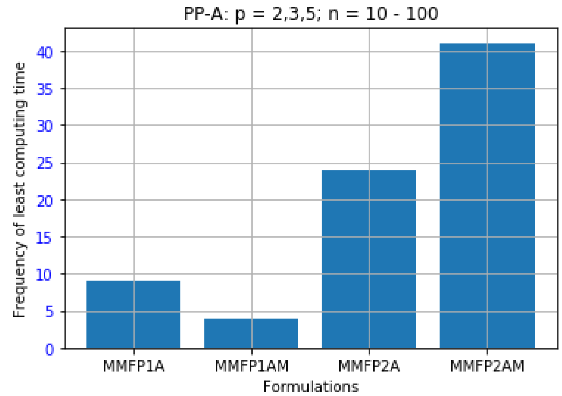


Figure 2.52: PP-A (small data)

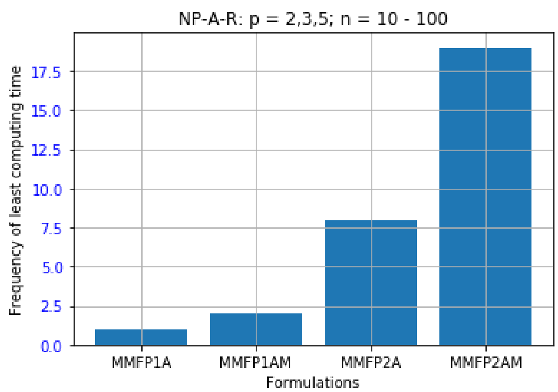


Figure 2.53: NP-A-R (small data)

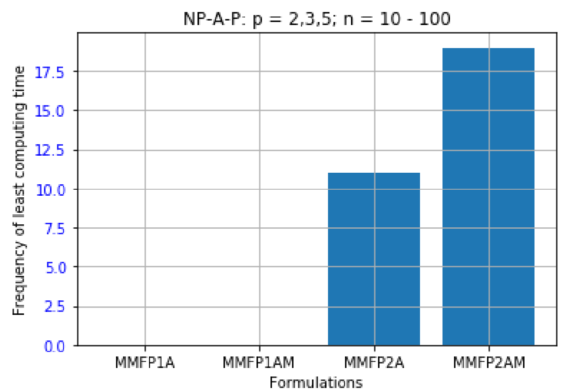


Figure 2.54: NP-A-P (small data)

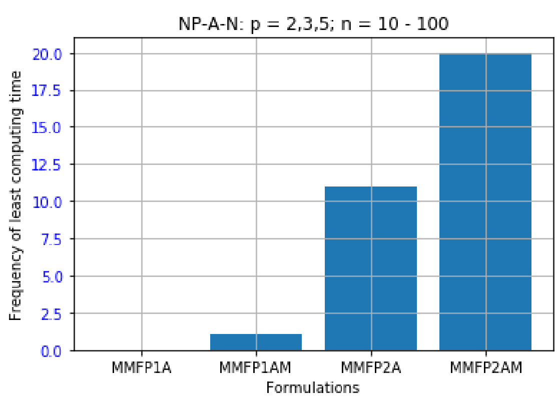


Figure 2.55: NP-A-N (small data)

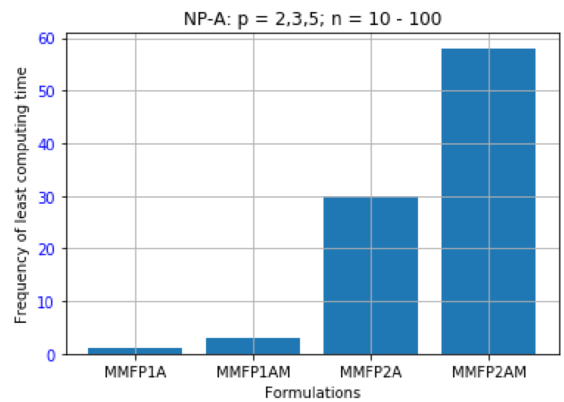


Figure 2.56: NP-A (small data)

2.5.2 Remark on computational results

Assignment problems do not allow $\mathbf{x} = \mathbf{0}$ as a feasible solution. We have not developed proper pre-solve plan in this section except the one that targets zero as possible bound. Hence, we do not expect that modified formulations would perform better than their parent models. Even in the problem instance classes PP-A-R, PP-A-P and PP-A-N we did not expect much since the best possible lower bound for y is zero.

The results seem to verify our expectation. However the modified models performed at least as good or better than their parent models in problem instance classes PP-A-R, PP-A-P and PP-A-N on average. Even in the problem instance classes NP-A-R, NP-A-P and NP-A-N, the modified formulation did at least as good as their parent models on average. We did find very few computational errors.

2.6 Conclusion

From the test results, in general, the computing time increased with the problem size p and n, but not always. Also in general, the model using binary expansion performed faster than the one without it. This could be explained by the reduction in the number of McCormick constraints. In few tests we found different optimal points produced by different formulations with equal or slightly different optimal values. We only tested with integral coefficients. If we allow rational coefficients we would need to re-scale them to become integers because the y-bounds used in the tests were designed to work with integer coefficients.

Performance of modified formulations equipped with pre-solve plan

We found that these formulations, when dealing with problem instances for which the pre-solve is designed for, performed better in both avoiding computational errors and speed compared to the parent formulations using loose valid y-bounds. However they were also not entirely error-free. At some problems with relatively higher n-values they also produced computational errors.

Recursion heuristic

From the experiments with type I or type II errors, we randomly selected some and applied recursion heuristic. In all these recursions we were able to find the correct solutions which were feasible. However, we were unable to test if the correct solutions found were in fact true optimal.

Type I and type II errors

In all results with errors, the incorrect optimal values produced were better (smaller) than the correct solution values found by either a modified formulation or recursion heuristic. When a type I error hit all elements of the decision variable, then the correct solution point was found to be a non-zero point. When a type I or type II error occurred at some elements of a binary variable, the incorrect optimal point produced and the correct solution point could be the same or different.

Chapter 3

The generalized 0-1 MMFP

3.1 Introduction

In chapter 2, we studied 0-1 MMFP where the denominators in all objective ratios are restricted to be positive. This obviously limits the applicability of the model, but with the advantage of having relatively simpler MILP formulations. Let us now discuss the general problem where the denominators are allowed to take both positive and negative values. Let $\{K1, K2\}$ be a partition of index set $K = \{1, \cdots, p\}$ of the objective function ratios such that denominators of the ratios in $K1$ are allowed to be positive or negative. The denominators of the ratios in $K2$ are restricted to be positive. This generalization is called *general minimax 0-1 linear fractional programming problem* (GMMFP). Note that when $K1$ is empty, GMMFP reduces to MMFP.

Let us now consider two mixed integer linear programming formulations of GMMFP. We need to treat the ratios corresponding to $K1$ separately while the ratios corresponding to $K2$ can be handled the same way as we did for MMFP. Thus, GMMFP can be written as

$$
\begin{aligned}
\text{Minimize} \quad & y \\
\text{Subject to} \quad & \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} = y^k \quad \forall k \in K1 && (3.1) \\
& y^k \leq y \quad \forall k \in K1 && (3.2) \\
& \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j} \leq y \quad \forall k \in K2 && (3.3) \\
& \mathbf{Ax} \leq \mathbf{b} \\
& \mathbf{x} \in \{0,1\}^n.
\end{aligned}
$$

The equality used in (3.1) allows us to deal with unrestricted nature of denominator. Notice that there is a cost for $|K1|$ number of additional constraints in (3.2). Now we can rewrite the formulation as

$$
\begin{aligned}
\text{Minimize} \quad & y \\
\text{Subject to} \quad & \alpha^k + \sum_{j=1}^{n} c_j^k x_j = y^k(\beta^k + \sum_{j=1}^{n} d_j^k x_j) \quad \forall k \in K1 \\
& y^k \le y \qquad\qquad\qquad\qquad\qquad\qquad\quad \forall k \in K1 \\
& \alpha^k + \sum_{j=1}^{n} c_j^k x_j \le y^k(\beta^k + \sum_{j=1}^{n} d_j^k x_j) \quad \forall k \in K2 \\
& \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \in \{0,1\}^n.
\end{aligned}
$$

Now we can linearize the bilinear terms in the constraints using McCormick relaxation as discussed in chapter 2. Since the denominator restricted case has been already discussed in chapter 2, for simplicity of presentation, we assume $K2$ to be empty, (i.e. $K1 = K$) for the rest of this chapter. Our basic MILP formulation of GMMFP is as follows:

$$
\begin{aligned}
\text{(P2)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & y^k = \frac{\mathbf{c}^k\mathbf{x} + \alpha^k}{\mathbf{d}^k\mathbf{x} + \beta^k} \quad \forall k = 1, \cdots, p \\
& y^k \le y \qquad\qquad \forall k = 1, \cdots, p \\
& y \in [y^{lk}, y^{uk}] \quad\ \ \forall k = 1, \cdots, p \\
& \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \in \{0,1\}^n.
\end{aligned}
$$

Recall that in chapter 2, we used an integer linear program to test if the condition (2.2) is satisfied. This verification is polynomially bounded for unconstrained and assignment constrained cases, and pseudo-polynomially bounded for the knapsack constrained case. Moreover, testing if there exists a solution that produce denominator zero is NP-hard even for the unconstrained case since it is precisely the partition problem. To deal with the possibility of solution having zero values and to make the problem well defined, we need to exclude solutions that make any of the denominators zero. In fact, even if any of the 0-1 solutions do not produce a zero denominator, we could still have a zero denominator in the continuous relaxation. Hence this needs to be addressed. For demonstration, we now consider the following example.

Example 1.

$$
\text{Minimize} \left\{ \ \max\left\{ \frac{1 + x_1 + x_2}{3 + x_1 - 5x_2}, \frac{3 - x_1 + x_2}{4 + x_1 - 7x_2} \right\} : \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \{0,1\}^2 \right\}.
$$

Note that no binary variable $\mathbf{x}$ makes the denominators zero. However in the relaxed feasible set $[0,1]^2$ there are points where a denominator takes zero value (see figure below). In the sub-region A, both denominators are negative. In B first denominator is positive and second denominator is negative. In C both denominators are positive. The two dashed lines represent the points where at least one denominator is zero. Therefore, they are excluded from the feasible set.

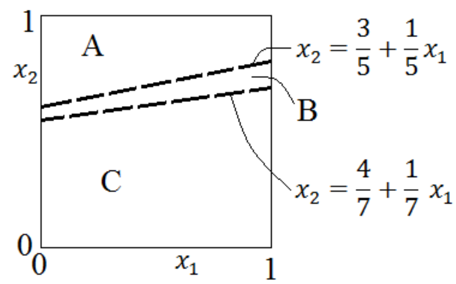


Figure 3.1: Relaxed feasible set

The true optimal solution occurs at $\mathbf{x} = (1,1)^T$ with the optimal value of -1.5. Since MMFP1 is developed assuming the denominators are positive for all variable values, it does not produce correct optimal solution.

Therefore we need to think about the possibility that some of the decision variables make a denominator zero in the relaxed feasible set. In this case, after excluding these variables, the relaxed feasible set becomes a disjunctive set as demonstrated in above example. One way of rebuilding the MILP with a convex relaxed feasible set is to use the well-known *big-M method* [7][19]. Big-M method adds extra binary variable(s) so that the new relaxed feasible set in the higher dimensional space is convex. This is explained in the following simple example.

Example 2. Suppose the objective function of an optimization program is

$$f(x) = \frac{3+x}{1-2x}, \quad x \in \{0,1\}.$$

The relaxed feasible set is $[0,1]$. This contains $x = \frac{1}{2}$ where the denominator of f is zero. If we exclude this variable value, then the relaxed feasible set becomes $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ which is not convex. Hence, we employ the big-M method. We add additional constraints with a binary variable t.

$$1 - 2x \geq \epsilon - Mt$$

$$1 - 2x \leq -\epsilon + M(1-t)$$

where M is a positive value larger than largest possible denominator. So we set $M = 10$. And ϵ is small positive value. In this case we set $\epsilon = 1$. The constraints of the optimization become

$$1 - 2x \geq 1 - 10t$$

$$1 - 2x \leq -1 + 10(1 - t)$$

$$x \in \{0, 1\}, \quad t \in \{0, 1\}.$$

If $1 - 2x \geq 1$, then t is forced to be zero. If $1 - 2x \leq -1$, then t is forced to be one. Hence, the case $x = \frac{1}{2}$ is not feasible. The relaxed feasible set is constructed by

$$t \geq \frac{1}{5}x$$

$$t \leq \frac{4}{5} + \frac{1}{5}x$$

$$x \in [0, 1], \quad t \in [0, 1]$$

which is a convex set in $\mathbb{R}^2$ space as presented in below diagram.

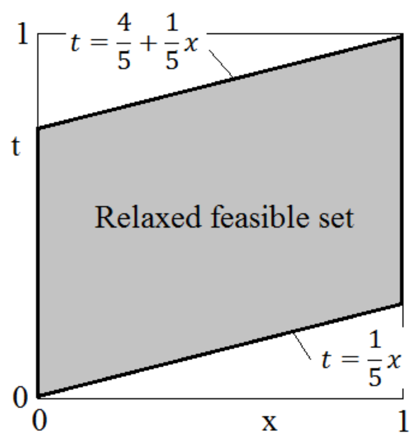


Figure 3.2: relaxed feasible set

Remark: The 0-1 MMFP problem satisfying the condition (2.2) is a special case of 0-1 GMMFP problem. Therefore, 0-1 GMMFP is at least as hard as 0-1 MMFP. Hence,

- Unconstrained 0-1 GMMFP with $p \geq 2$ is NP-hard
- Knapsack constrained 0-1 GMMFP with non-constant p is strongly NP-hard
- Assignment constrained 0-1 GMMFP with non-constant p is strongly NP-hard

3.2 MILP formulations for 0-1 GMMFP

Let us now give the first MILP formulation for 0-1 GMMFP. Using problem P2, each objective ratio is

$$y^k = \frac{\alpha^k + \sum_{j=1}^n c_j^k x_j}{\beta^k + \sum_{j=1}^n d_j^k x_j}, \quad k = 1, \cdots, p.$$

Assuming denominator is not zero,

$$\alpha^k + \sum_{j=1}^n c_j^k x_j = y^k \beta^k + \sum_{j=1}^n d_j^k y^k x_j, \quad k = 1, \cdots, p.$$

To linearize the pn number of bilinear terms $y^k x_j$ we use the substitution:

$$z_j^k = y^k x_j, \quad k = 1, \cdots, p.$$

There are pn number of continuous variables z_j^k. Suppose we have lower and upper bounds y^{lk} and y^{uk} respectively for the continuous variable y^k for each $k \in \{1, \cdots, p\}$, we use the McCormick relaxation for z_j^k as in MMFP formulations. To exclude the decision variable values that make a denominator zero in the relaxed feasible set, we add big-M constraints. Let $\epsilon > 0$ be a small positive number and $M > 0$ be a large number. For each $k = 1, \cdots, p$ we add the following constraints together with a binary variable t^k.

$$\beta^k + \sum_{j=1}^n d_j^k x_j \geq \epsilon - M t^k$$

$$\beta^k + \sum_{j=1}^n d_j^k x_j \leq -\epsilon + M(1 - t^k)$$

$$t^k \in \{0, 1\}.$$

When the k^{th} denominator takes positive value, the binary variable t^k is forced to be zero. When the k^{th} denominator takes negative value, the binary variable t^k is forced to be one. There is no other choice. Hence the zero denominators are not allowed. The formulation GMMFP1 is

$$
\begin{aligned}
\text{(GMMFP1)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{j=1}^{n} d_j^k z_j^k + y^k \beta^k \quad \forall k = 1, \cdots, p \\
& y^k \le y \quad \forall k = 1, \cdots, p \\
& z_j^k \le y^{uk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \le y^k + y^{lk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \ge y^{lk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \ge y^k + y^{uk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \ge \epsilon - M t^k \quad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \le -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \in \{0, 1\}^n \\
& t^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& y \in [y^l, y^u].
\end{aligned}
$$

If all denominator coefficients are integers, we take $\epsilon = 1$ and we assign M a value larger than the largest possible denominators over the feasible set. For example, we can set individual M^k as

$$M^k = |\beta^k| + \sum_{j=1}^{n} |d_j^k| + 2.$$

Before applying this formulation, we need to compute a pair of valid bounds for $y^k,\ k \in \{1, \cdots, p\}$, and y. A globally valid upper bound of y^k needs to be at least

$$\max \left\{ \frac{\mathbf{c}^k \mathbf{x} + \alpha^k}{\mathbf{d}^k \mathbf{x} + \beta^k} :\ \mathbf{x} \in X \right\}, \quad \forall k = 1, \cdots, p.$$

A globally valid lower bound of y^k needs to be at most

$$\min \left\{ \frac{\mathbf{c}^k \mathbf{x} + \alpha^k}{\mathbf{d}^k \mathbf{x} + \beta^k} :\ \mathbf{x} \in X \right\} \quad \forall k = 1, \cdots, p.$$

To compute the bounds as above, we have to solve $2p$ number of 0-1 single ratio linear fractional programming (LFP) problems with unrestricted denominators. Note that 0-1 single ratio LFP with unrestricted denominator is known to be NP-hard [22]. We use following computationally inexpensive procedure to find a set of global bounds. This procedure assumes that all denominator coefficients are integers. For each $k = 1, \cdots, p$,

$$\text{topMax}^k = \alpha^k + \sum_{j=1}^{n} \{c_j^k : c_j^k > 0\}$$

$$\text{topMin}^k = \alpha^k + \sum_{j=1}^{n} \{c_j^k : c_j^k < 0\}.$$

$$y^{uk} = \begin{cases} |\alpha^k|, & \text{if topMin}^k = \text{ topMax}^k \\ \text{topMax}^k, & \text{if topMin}^k \geq 0 \\ -\text{topMin}^k, & \text{if topMax}^k \leq 0 \\ \max\{|\text{topMin}^k|, |\text{topMax}^k|\}, & \text{otherwise} \end{cases}, \tag{3.4}$$

$$y^{lk} = \begin{cases} -|\alpha^k|, & \text{if topMin}^k = \text{ topMax}^k \\ -\text{topMax}^k, & \text{if topMin}^k \geq 0 \\ \quad \text{topMin}^k, & \text{if topMax}^k \leq 0 \\ -\max\{|\text{topMin}^k|, |\text{topMax}^k|\}, & \text{otherwise.} \end{cases}, \tag{3.5}$$

A pair of globally valid upper and lower bounds of the aggregated upper bound y can be computed as follows:

$$y^l = \min\{y^{lk}, k = 1, \cdots, p\} \tag{3.6}$$

$$y^u = \infty. \tag{3.7}$$

Now, we modify GMMFP1 in the same way we did in chapter 2 using binary expansion method to reduce the number of McCormick constraints [10]. We let

$$D^k = \sum_{j=1}^{n} \{|d_j^k| : d_j^k < 0\}, \quad \forall k = 1, \cdots, p.$$

Then

$$\sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k, \quad \forall k = 1, \cdots, p$$

where

$$q^k = \lfloor \log_2 (\sum_{j \in J} |d_j^k|) \rfloor + 1, \quad \forall k = 1, \cdots, p, \; i \in \{1, \cdots, q^k\}$$

and

$$v_i^k \in \{0, 1\}, \quad \forall \; k = 1, \cdots, p, \; i \in \{1, \cdots, q^k\}.$$

Now the constraint

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k = y^k \sum_{j=1}^{n} d_j^k x_j + y^k \beta^k, \quad \forall k = 1, \cdots, p$$

becomes

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k = y^k \left(\sum_{i=1}^{q^k} 2^{i-1} v_i^k - D^k \right) + y^k \beta^k, \quad \forall k = 1, \cdots, p.$$

That is,

$$\sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{i=1}^{q^k} 2^{i-1} y^k v_i^k + y^k (\beta^k - D^k), \quad \forall k = 1, \cdots, p.$$

For each of the bilinear term $y^k v_i^k$ we do the substitution $z_i^k = y^k v_i^k$. The number of z_i^k is the same as the number of v_i^k which is $\sum_{k=1}^{p} q^k$. We use McCormick relaxation to linearize these bilinear terms. The total number of McCormick constraints is $4(\sum_{k=1}^{p} q^k)$.

The final GMMFP formulation is

$$
\begin{aligned}
\text{(GMMFP2)} \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y^k(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& y^k \le y \qquad \forall k = 1, \cdots, p \\
& z_i^k \le y^{uk} v_i^k \qquad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \le y^k + y^{lk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \ge y^{lk} v_i^k \qquad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \ge y^k + y^{uk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \ge \epsilon - M t^k \qquad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \le -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& y \in [y^l, y^u] \\
& \mathbf{x} \in \{0,1\}^n \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& v_i^k \in \{0,1\}, \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& t^k \in \{0,1\}, \quad \forall k = 1, \cdots, p
\end{aligned}
$$

where

$$
D^k = \sum_{j=1}^{n} \{|d_j^k| : d_j^k < 0\} \qquad \forall k = 1, \cdots, p
$$

$$
q^k = \lfloor \log_2 \big(\sum_{j=1}^{n} |d_j^k| \big) \rfloor + 1 \quad \forall k = 1, \cdots, p.
$$

Bounds for y^k and y are computed the same as in GMMFP1.

	GMMFP1	GMMFP2
No. of binary variables	$n+p$	$n+\sum_{k=1}^{p} q^k + p$
No. of continuous variables	$1+p+np$	$1+p+\sum_{k=1}^{p} q^k$
No. of constraints	$4p+4np+r$	$5p+4\sum_{k=1}^{p} q^k + r$

Table 3.1: Table of summary for counts of variables and constraints

3.2.1 Improving the formulations

In GMMFP1, we have $z_j^k = y^k x_j$. Instead of this exact bilinear equation, the MILP solver receives four McCormick linear inequalities along with the upper and lower bounds y^{uk} and y^{lk} respectively for y^k. When these bounds are too loose and x_j is within the integrality tolerance δ from zero or one, z_j^k can take an incorrect value. This can give the solver a room to minimize the objective y-value too low. However, the values of c_j^k, d_j^k, α^k, and β^k are unrestricted in sign. That means, even with the incorrect z_j^k value, the first constraint of the model does not give much chance to the solver to minimize the y^k-value too low. The p number of constraints: $y \geq y^k$ also serve like an additional layer of defence against floating point rounding error. Similar observation can be explored for GMMFP2 model. Therefore, in general, 0-1 GMMFP formulations are more robust against floating point rounding errors compared to 0-1 MMFP formulations. Nevertheless, the loose bounds and large n-value are still the characteristics of the environment in which floating point rounding errors can occur. Notice that the unrestricted coefficients contribute in strengthening the robustness of 0-1 GMMFP models against the computational errors. During the investigation, 0-1 GMMFP models have been tested on some problems where denominator coefficients are positive integers. On average, they do not perform better than corresponding 0-1 MMFP models.

In this section, we study a few ways to find better bounds for y^k and y for improved performance. We begin with a pre-solve algorithm to tighten the upper bounds y^{uk} of the objective ratios y^k, where $k = 1, \cdots, p$, and the lower bound of the aggregated upper bound y of all objective ratios. Because of unrestricted denominators, we only have zero as a critical objective value for general problems. Also we cannot generally tighten the lower bounds y^{lk} of the objective ratios. This is explained by the following two diagrams.

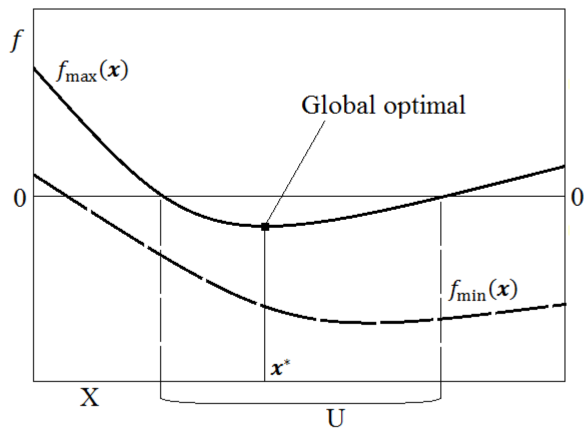


Figure 3.3: There exists a subset U of X such that $f_{\max}(\mathbf{x}) < 0 \ \forall \mathbf{x} \in U$

In the case described by figure 3.3, we find a subset U of the feasible set X such that $f_{\max}(\mathbf{x}) < 0$ for all $\mathbf{x} \in U$. In this case we can set upper bounds $y^{uk} = 0$ for all $k = 1, \cdots, p$ since we can locate the optimal point $\mathbf{x}^*$ in U. No improvement is made for the lower bounds y^{lk} however.

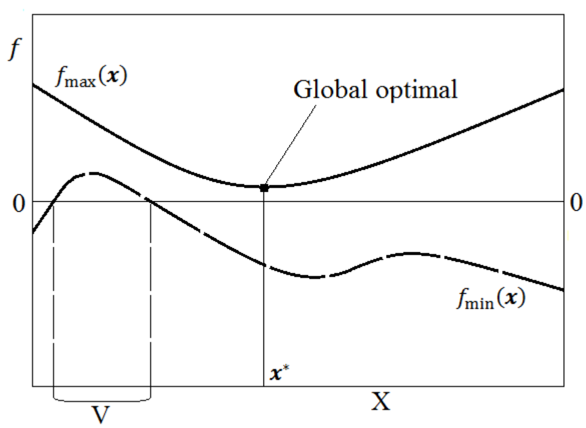


Figure 3.4: $f_{\max}(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in X$

In the case described by figure 3.4, we find that at every point $\mathbf{x}$ in the feasible set X, at least one objective ratio is non-negative. That is, $f_{\max}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$. We can set the lower bound of the aggregated upper bound y of all objective ratios to be zero. We cannot set the lower bounds $y^{lk} = 0$ for all $k = 1, \cdots, p$. It is because if we do this, we could be searching the solution in a subset, such as the subset V in figure 3.4, which does not contain the optimal point $\mathbf{x}^*$. Before we build a pre-solve plan, we argue that the lower bound y^l of the aggregated upper bound y does not need to be the minimum of y^{lk}, for all $k = 1, \cdots, p$. Using simple logical steps, it can be shown that $\max\{y^{lk} : k = 1, \cdots, p\}$ is at most the true optimal value.

Proposition 3.1. In a GMMFP problem with number of objective ratios p, number of decision variables n, suppose

$$y^k = f_k(\mathbf{x}) = k^{th} \text{ objective ratio}$$

where $y^{lk} \leq y^k \leq y^{uk}\ \forall \mathbf{x} \in X,\ \forall k = 1, \cdots, p$. Suppose GMMMP problem has the global optimal solution $(\mathbf{x}^*, y^*)$. Then,

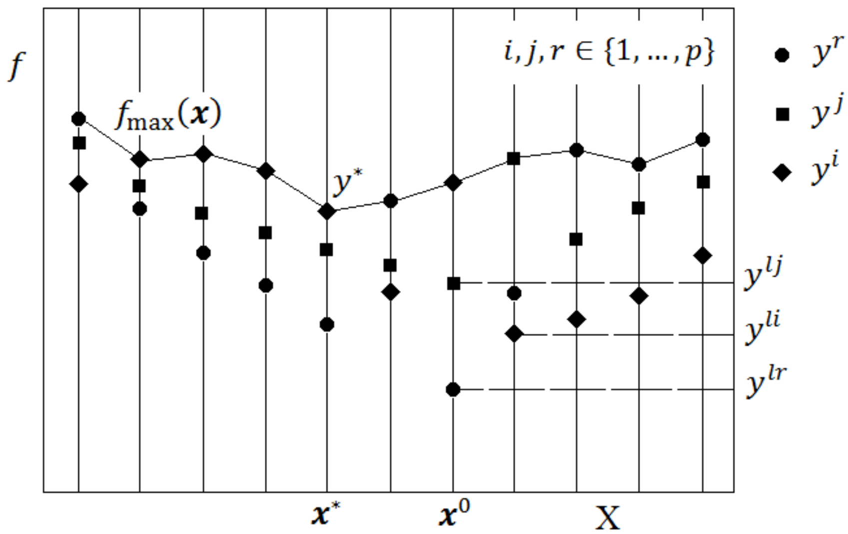
$$\max\{y^{lk}, k = 1, \cdots, p\} \leq y^*.$$

Proof. Let

$$f_{\max}(\mathbf{x}) = \max\{f_k(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then

$$y^* = \min\{f_{\max}(\mathbf{x}) : \mathbf{x} \in X\}.$$



Let $i, j \in \{1, \cdots, p\}$, and $\mathbf{x}^0 \in X$ be such that

$$f_i(\mathbf{x}^*) = y^* \text{ and } f_j(\mathbf{x}^0) = y^{lj} = \max\{y^{lk}, k = 1, \cdots, p\}.$$

We will prove using contradiction. Suppose $y^* < y^{lj}$. Since $y^* = f_{\max}(\mathbf{x}^*) \geq f_j(\mathbf{x}^*)$. Hence we arrive at

$$y^{lj} = \min\{f_j(\mathbf{x}) : \mathbf{x} \in X\} > f_j(\mathbf{x}^*).$$

This is a contradiction. Therefore, we must have $y^{lj} \leq y^*$. That is,

$$\max\{y^{lk}, k = 1, \cdots, p\} \leq y^*.$$

■

This proposition tells us that we can set the lower bound of the aggregated upper bound y to be

$$y^l = \max\{y^{lk}, k = 1, \cdots, p\}.$$

Pre-solve plan for the bounds of y, y^k
Input: Given GMMFP problem instance, and a large value $M > 0$
Output: y^{uk}, y^l
1. minimize {0: Numerator and denominator are non-zero and with opposite signs $\forall k = 1, \cdots, p$}
2. **if** (1) is feasible, then
3. set $y^{uk} = 0 \quad \forall k = 1, \cdots, p$
4. compute y^{lk} the same way as in previous sections
5. set $y^l = \max\{y^{lk}, k = 1, \cdots, p\}$
6. **if** (1) is not feasible, then
7. set $y^l = \max\left\{0,\ \max\{y^{lk}, k = 1, \cdots, p\}\right\}$
8. compute y^{lk}, y^{uk} the same as in previous sections
9. **end**.
Constraint construction for optimization step (2), assuming integral data
1. **for** each $k = 1, \cdots, p$
2. denominator of k^{th} ratio $\geq 1 - Ms^k$
3. denominator of k^{th} ratio $\leq -1 + M(1 - s^k)$
4. numerator of k^{th} ratio $\leq -1 + Ms^k$
5. numerator of k^{th} ratio $\geq 1 - M(1 - s^k)$
6. $s^k \in \{0, 1\}\ \forall k = 1, \cdots, p$

This pre-solve plan targets only zero as a candidate bound. Due to the use of four big-M constraints and an additional binary variable for each objective ratio, this pre-solve plan is not as good as those in chapter 2 in terms of computational cost. If we use a non-zero bound candidate or recursive pre-solving, the cost would go up significantly. Therefore, we do not do recursive pre-solving for GMMFP. The modified formulations (GMMFP1M, GMMFP2M) can be written as follows:

Modified formulations: GMMFP1M & GMMFP2M
Input: Given data
Output: An optimal solution
1. execute the pre-solve algorithm for y^{lk}, y^{uk} where $k = 1, \cdots, p$ and lower bound of y
2. solve main model (GMMFP1 or GMMFP2)
3. **end**.

The floating point arithmetic errors type I and II can come in GMMFP probblems as well. According to what we learned in chapter 2, a type I or type II error could occur when the bounds of the continuous variables used in McCormick relaxation are too loose and the n-value is large. When using the formulations in this chapter we cannot tighten the lower bounds y^{lk} as discussed above. However we can tighten the upper bounds y^{uk} and the lower bound of aggregated upper bound y. Above modified formulations may not always avoid the errors. Suppose we find the optimal solution $(\mathbf{x}^*, y^*)$ produced by the solver is incorrect. Then,

$$\max\{f_k(\mathbf{x}^*), \quad k = 1, \cdots, p\} - y^* > \epsilon, \text{ where } \epsilon > 0.$$

More precisely, we would find that at least one element of a binary variable is not in $\{0, 1\}$ but close to either zero or one within the integrality tolerance. In below algorithm we assume that

$$y^* \leq \text{ true optimal value } \leq \max\{f_k(\mathbf{x}^*), \quad k = 1, \cdots, p\}$$

with at least one of the two inequalities being strict.

The recursion heuristic

Input: Given data, precision level ϵ, non-negative integers λ_1, λ_2

Output: An optimal solution

1. compute the bounds y^{lk}, y^{uk}, and y^l = lower bound of y
2. solve the main model
3. return optimal solution $\mathbf{x}^*, y^*$
4. **if** $(\mathbf{x}^*, y^*)$ are found to have type I or II errors, then update y^l, y^{uk} as follows:
5. **if** tight bound is not required
6. $y^l = \lfloor y^* \rfloor - \lambda_1$
7. $y^{uk} = \lceil \max\{f_k(\mathbf{x}^*),\ k = 1, \cdots, p\} \rceil + \lambda_2$
8. **if** tight bound is required
9. $y^l = y^*$
10. $y^{uk} = \max\{f_k(\mathbf{x}^*),\ k = 1, \cdots, p\}$
11. go to step (2)
12. **end**.

The next example demonstrates the occurrence of type I and type II errors on a binary vector and application of the recursion heuristic.

Example 3. (Demonstration of computational errors) We set up an unconstrained 0-1 GMMFP problem instance with $p = 5, n = 120$; c_j^k, d_j^k are random uniformly distributed integers between -100 and 100; $\alpha^k = \beta^k = 1$. We use Gurobi solver with integrality tolerance $\delta = 1 \times 10^{-5}$. We run GMMFP2 formulation which returns the below incorrect result.

Program	Obj.Val	y^{lk}	y^{uk}	y^l
GMMFP2	-.1.27272	[-3023, -3547, -3194, -3026, -2717]	[3023, 3547, 3194, 3026, 2717]	-3547

The optimal solution $(\mathbf{x}^*, y^*)$ is

$$x_1^* = x_2^* = x_{64}^* = x_{87}^* = x_{92}^* = x_{93}^* = 1$$

$$x_j^* = 0 \ \ \forall j = 1, \cdots, n, \ \forall j \notin \{1, 2, 64, 87, 92, 93\}$$

$$y^* = -1.27272727.$$

However

$$f_{\max}(\mathbf{x}^*) = \max\{f_1(\mathbf{x}^*), \cdots, f_5(\mathbf{x}^*)\} = -1.25806452 > y^*$$

The decision variable $\mathbf{x}$ is not hit by any of the two types of computational errors. Nevertheless, type I or type II errors hit some elements of the binary variable, $\mathbf{v}^4$, for the 4^{th} objective ratio. We have

$$q^4 = 13, \text{ number of elements of } \mathbf{v}^4$$

$$v_4^4 \approx v_5^4 \approx 2.35 \times 10^{-6}, \ v_6^4 \approx 2.93 \times 10^{-6}, \ v_7^4 \approx 0.999997652$$

McCormick constraints for $z_i^k = y^k v_i^k$ are:

$$\begin{aligned} z_i^k &\leq y^{uk} v_i^k \\ z_i^k &\leq y^k + y^{lk}(v_i^k - 1) \\ z_i^k &\geq y^{lk} v_i^k \\ z_i^k &\geq y^k + y^{uk}(v_i^k - 1). \end{aligned}$$

Type I error: If $v_i^k = 0$ we must have $z_i^k = 0$. However, three of v_i^4 are positive but within δ away from zero during relaxation. For $k = 4$ and $i = 4, 5, 6$ the McCormick constraints are

$$\begin{aligned}
z_i^4 &\le 3026 \times 2.35 \times 10^{-6} = 0.0071111 \\
z_i^4 &\le y^k - 3026(2.35 \times 10^{-6} - 1) = y^k + 3025.9929 \\
z_i^4 &\ge -3026 \times 2.35 \times 10^{-6} = -0.0071111 \\
z_i^4 &\ge y^k + 3026(2.35 \times 10^{-6} - 1) = y^k - 3025.9929.
\end{aligned}$$

That is,

$$-0.0071111 \le z_i^4 \le 0.0071111.$$

Meanwhile the constraint

$$\sum_{j=1}^{120} c_j^4 x_j + \alpha^4 = \sum_{i=1}^{13} 2^{i-1} z_i^4 + y^4(\beta^4 - D^4)$$

where $\beta^4 - D^4 = -2850$, becomes

$$y^4 = \left(\sum_{i=1}^{13} 2^{i-1} z_i^4 - \sum_{j=1}^{120} c_j^4 x_j - 1 \right) / 2850$$

with $-0.0071111 \le z_i^4 \le 0.0071111$. Since we are minimizing y^4, the solver pushes $z_i^4,\ i \in \{4, 5, 6\}$, below zero as much as possible while also satisfying other constraints. The result is

$$z_4^4 = 0, \quad z_5^4 \approx -2.95 \times 10^{-6}, \quad z_6^4 \approx -3.7 \times 10^{-6}$$

instead of

$$z_4^4 = z_5^4 = z_6^4 = 0.$$

Type II error: If $v_i^k = 1$, we must have $z_i^k = y^k$. However, $v_7^4 = 0.99999765$ is less than one but within δ away from one during relaxation. McCormick constraints for $z_7^4 = y^4 v_7^4$ are

$$\begin{aligned}
z_7^4 &\le y^{u4} v_7^4 = 3025.9929 \\
z_7^4 &\le y^4 + y^{l4}(v_7^4 - 1) = y^4 + 0.0071111 \\
z_7^4 &\ge y^{l4} v_7^4 = -3025.9929 \\
z_7^4 &\ge y^4 + y^{u4}(v_7^4 - 1) = y^4 - 0.0071111.
\end{aligned}$$

That is,

$$-3025.9929 \leq z_7^4 \leq 3025.9929$$

$$y^4 - 0.0071111 \leq z_7^4 \leq y^4 + 0.0071111.$$

Meanwhile the constraint

$$\sum_{j=1}^{120} c_j^4 x_j + \alpha^4 = \sum_{i=1}^{13} 2^{i-1} z_i^4 + y^4(\beta^4 - D^4)$$

where $\beta^4 - D^4 = -2850$, becomes

$$y^4 = \left(\sum_{\substack{i=1,\cdots,13 \\ i\neq 7}} 2^{i-1} z_i^4 + 2^6 z_7^4 - \sum_{j=1}^{120} c_j^4 x_j - 1 \right) / 2850$$

with $y^4 - 0.0071111 \leq z_7^4 \leq y^4 + 0.0071111$. Since we are minimizing y^4 while also satisfying other constraints, the solver pushes y^4 below z_7^4. Combined with the type I error, the result is

$$z_7^4 = -1.26562241, \quad y^4 = -1.27272727, \quad \mathbf{x} = \mathbf{x}^*$$

This y^4 happens to be $\max\{y^1, \cdots, y^4\}$ at $\mathbf{x}^*$ and the solver does not find any better answer.

Recursion heuristic

After knowing that the error occurs, we employ the recursion heuristic. We update the bounds as follows:

$$y^l = \lfloor y^* \rfloor = -2, \quad y^{uk} = \lceil f_{\max}(\mathbf{x}^*) \rceil + 1 = 0, \quad \forall k = 1, \cdots, p$$

We run GMMFP2 the second time using these bounds. This time, the solver produces the correct solution. That is, it is a feasible solution. We get the same optimal point $\mathbf{x}^*$ with a optimal value $y^* = -1.25806452$ which agrees with the value of $f_{\max}(\mathbf{x}^*)$. We note that the computing time of the second solve is about 2.65% of the computing time of first solve.

In the following three sections, we discuss about the special types of 0-1 GMMFP problems. They are unconstrained 0-1 GMMFP, knapsack constrained 0-1 GMMFP and assignment constrained 0-1 GMMFP.

3.3 Unconstrained 0-1 GMMFP problem

In this section, we present unconstrained 0-1 GMMFP formulations together with experimental analysis. The first formulation is given as follows:

$$
\begin{aligned}
\text{(GMMFP1U)} \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{j=1}^{n} d_j^k z_j^k + y^k \beta^k \quad \forall k = 1, \cdots, p \\
& y^k \leq y \quad \forall k = 1, \cdots, p \\
& z_j^k \leq y^{uk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \leq y^k + y^{lk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \geq y^{lk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \geq y^k + y^{uk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \geq \epsilon - M t^k \quad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \leq -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& \mathbf{x} \in \{0, 1\}^n \\
& t^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& y \in [y^l, y^u].
\end{aligned}
$$

By using binary expansion, we give the second formulation of unconstrained 0-1 GMMFP.

$$
\begin{aligned}
\text{(GMMFP2U)} \quad \text{Minimize} \quad & y \\
\text{Subject to} \quad & \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y^k(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& y^k \le y \quad \forall k = 1, \cdots, p \\
& z_i^k \le y^{uk} v_i^k \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \le y^k + y^{lk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \ge y^{lk} v_i^k \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \ge y^k + y^{uk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \ge \epsilon - M t^k \quad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \le -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& y \in [y^l, y^u] \\
& \mathbf{x} \in \{0, 1\}^n \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& v_i^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& t^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p.
\end{aligned}
$$

For the experiments in this thesis, y^{uk}, y^{lk}, y^l, and y^u are computed using the formulas (3.4),(3.5),(3.6) and (3.7) for both GMMFP1U and GMMFP2U. These are the globally valid bounds if all denominators coefficients are integers. In addition, we have modified formulations GMMFP1UM and GMMFP2UM. These are GMMFP1U and GMMFP2U equipped with the pre-solve plan we have discussed.

3.3.1 Experimental analysis on unconstrained 0-1 GMMFP formulations

We followed the same instance reference notation style from Chapter 2. We ran GMMFP1, GMMFP1M, GMMFP2, GMMFP2M on three problem instance classes. Problem instance classes were generated as follows:

(NN-U-R)	Random uniformly distributed integers: All data value c_j^k, α^k, d_j^k and β^k are randomly generated in the range [-100,100]
(NN-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c_j^k, d_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective.
(NN-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in decreasing order in each objective.

Below table presented objective value found by the solver, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors.

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-156.0	0.0	39.48325	-156.0	0.0	**34.75879**
60	-820.0	0.0	434.80065	-820.0	0.0	**246.44262**
90	-1288.0	4e-05	1179.29206	-1288.0	8e-05	**661.35201**
120	-1861.0	7e-05	1247.61128	-1861.0	0.0	**525.44709**
150	-2332.0	7e-05	**406.06239**	-2332.0	7e-05	810.16993
180	-2746.0	0.00814	3600.24221	-2748.0	0.00749	3600.98591
210	-3193.0	0.00356	3600.2236	-3194.0	0.0001	**2768.16935**
240	-3686.0	0.0102	3601.30657	-3694.0	0.00763	3601.99866
270	-3977.0	0.00597	3600.80268	-3989.0	0.00229	3600.04732
300	-4121.0	0.00904	3601.70424	-4141.0	0.00341	3600.09961

Table 3.3: NN-U-R-3

The formulations GMMFP1U and GMMFP1UM did not complete most of the problem instances within the time limit of 3600 seconds. Full set of result tables are listed in appendix C. Summary of frequencies of least computing time among GMMFP2U and GMMFP2UM are presented in below graphs.

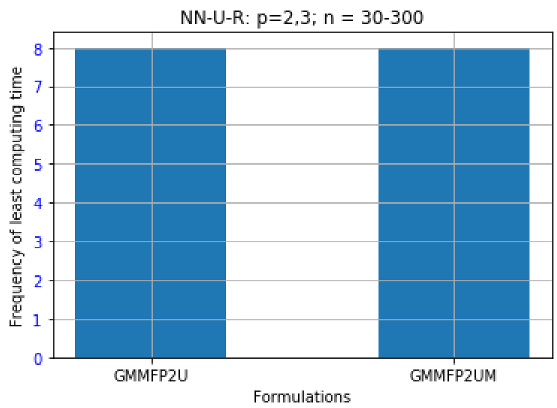


Figure 3.5: NN-U-R

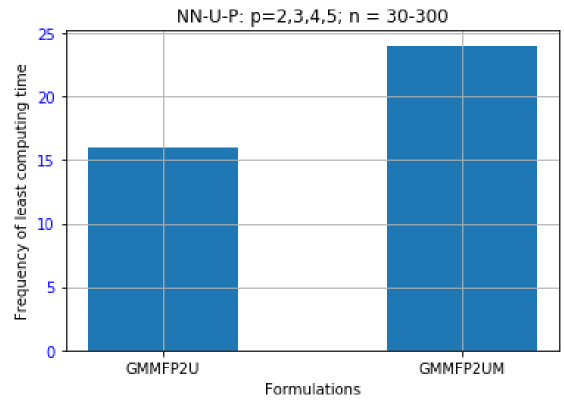


Figure 3.6: NN-U-P

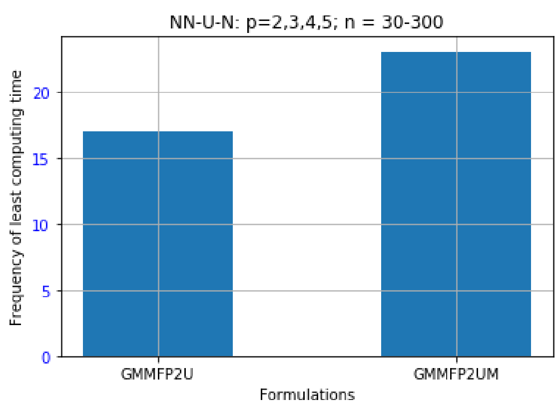


Figure 3.7: NN-U-N

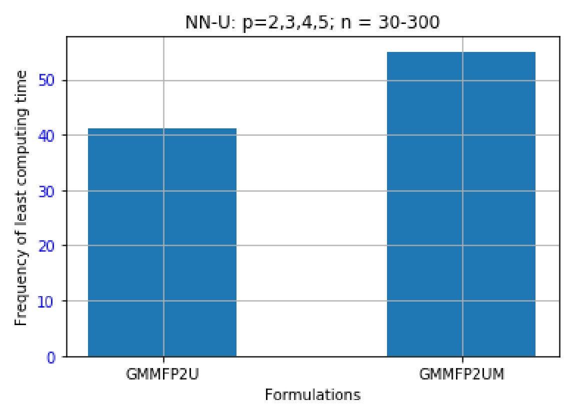


Figure 3.8: NN-U

3.3.2 Remark on computational results

As we discussed in section 3.2.1, the structure of 0-1 GMMFP (both formulation and the problem data) is more robust against floating point rounding errors compared to 0-1 MMFP. The only presolve we have targets zero as candidate bound. Hence, we did not have much expectation on the modified formulations especially in terms computing speed. However, GMMFP2UM performed at least as good as GMMFP2U on average. Only a few computational errors were found.

3.4 Knapsack constrained 0-1 GMMFP problem

The first MILP formulation for knapsack constrained 0-1 GMMFP problem is given as follows:

$$
\begin{aligned}
\text{(GMMFP1K)}\quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{j=1}^{n} d_j^k z_j^k + y^k \beta^k \quad \forall k = 1, \cdots, p \\
& y^k \leq y \quad \forall k = 1, \cdots, p \\
& z_j^k \leq y^{uk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \leq y^k + y^{lk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \geq y^{lk} x_j \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& z_j^k \geq y^k + y^{uk}(x_j - 1) \quad \forall k = 1, \cdots, p \quad \forall j = 1, \cdots, n \\
& \sum_{j=1}^{n} w_j x_j \leq W \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \geq \epsilon - M t^k \quad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j=1}^{n} d_j^k x_j \leq -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& \mathbf{x} \in \{0, 1\}^n \\
& t^k \in \{0, 1\}, \quad \forall k = 1, \cdots, p \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& y \in [y^l, y^u].
\end{aligned}
$$

Second MILP formulation for knapsack constrained 0-1 GMMFP problem is given as follows:

$$
\begin{aligned}
\text{(GMMFP2K)} \quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& \sum_{j=1}^{n} c_j^k x_j + \alpha^k = \sum_{i=1}^{q^k} 2^{i-1} z_i^k + y^k(\beta^k - D^k) \quad \forall k = 1, \cdots, p \\
& \sum_{j=1}^{n} d_j^k x_j + D^k = \sum_{i=1}^{q^k} 2^{i-1} v_i^k \quad \forall k = 1, \cdots, p \\
& y^k \leq y \quad \forall k = 1, \cdots, p \\
& z_i^k \leq y^{uk} v_i^k \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \leq y^k + y^{lk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \geq y^{lk} v_i^k \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& z_i^k \geq y^k + y^{uk}(v_i^k - 1) \quad \forall k = 1, \cdots, p, \quad \forall i \in \{1, \cdots, q^k\} \\
& \sum_{j=1}^{n} w_j x_j \leq W \\
& \beta^k + \sum_{j \in J} d_j^k x_j \geq \epsilon - M t^k \quad \forall k = 1, \cdots, p \\
& \beta^k + \sum_{j \in J} d_j^k x_j \leq -\epsilon + M(1 - t^k) \quad \forall k = 1, \cdots, p \\
& y \in [y^l, y^u] \\
& \mathbf{x} \in \{0,1\}^n \\
& y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1, \cdots, p \\
& v_i^k \in \{0,1\}, \quad \forall k = 1, \cdots, p, \ \forall i \in \{1, \cdots, q^k\} \\
& t^k \in \{0,1\}, \quad \forall k = 1, \cdots, p.
\end{aligned}
$$

The feasible set of knapsack constrained 0-1 GMMFP is a subset of the feasible set of unconstrained 0-1 GMMFP. Therefore, valid bounds of unconstrained 0-1 GMMFP are still valid in the knapsack constrained case. For the experiments in this thesis, the bounds y^{uk}, y^{lk}, y^l, and y^u for both GMMFP1K and GMMFP2K are computed using the formulas (3.4),(3.5),(3.6) and (3.7). These bounds are globally valid if all denominator coefficients are integers. In addition, we have modified formulations GMMFP1KM and GMMFP2KM. These are GMMFP1K and GMMFP2K equipped with the pre-solve plan we have discussed.

3.4.1 Experimental analysis on knapsack constrained 0-1 GMMFP formulations

We ran GMMFP1, GMMFP1M, GMMFP2, and GMMFP2M on three problem instance classes. Weight values w_j were generated as random uniformly distributed integers in the range [1,100]. The weight limit W was computed by the formula: $W = \frac{(n/4)(w^l + w^u)}{2}$, where $w^l = 1$, $w^u = 100$. Three problem instance classes were generated as follows:

(NN-K-R)	Random uniformly distributed integers: All data value c_j^k, α^k, d_j^k and β^k are randomly generated in the range [-100,100].
(NN-K-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c_j^k, d_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective.
(NN-K-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in decreasing order in each objective.

Below table presents objective value found by the solver, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors.

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-239.0	0.0	5.32929	-239.0	0.0	**2.53921**
60	-1017.0	0.0	**23.8003**	-1017.0	0.0	71.43476
90	-1257.0	0.0	**33.8336**	-1257.0	0.0	61.72672
120	-1717.0	1e-05	**161.45424**	-1717.0	0.0	225.02123
150	-2313.0	0.0	31.63259	-2313.0	6e-05	**31.60454**
180	-2651.0	0.0	41.67385	-2651.0	2e-05	**38.90288**
210	-3006.0	0.0	228.81245	-3006.0	4e-05	**195.94204**
240	-3309.0	0.0001	**438.92369**	-3309.0	9e-05	550.55311
270	-3863.0	9e-05	558.6511	-3863.0	0.0001	**424.4108**
300	-4313.0	9e-05	267.92885	-4313.0	7e-05	**156.351**

Table 3.5: NN-K-R-2

The formulations GMMFP1K and GMMFP1KM did not complete most of the problem instances within the time limit of 3600 seconds. Full set of result tables are listed in appendix C.

Summary of frequencies of least computing time among GMMFP2K and GMMFP2KM are presented in below graphs.

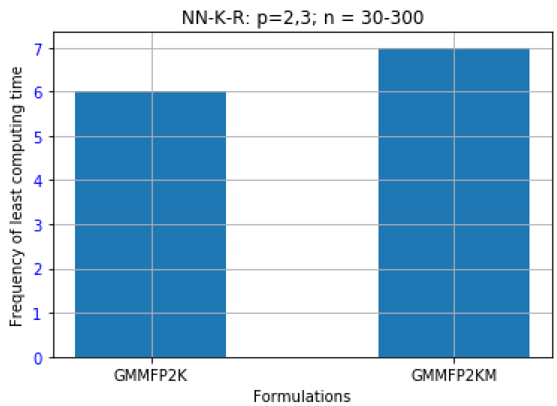


Figure 3.9: NN-K-R

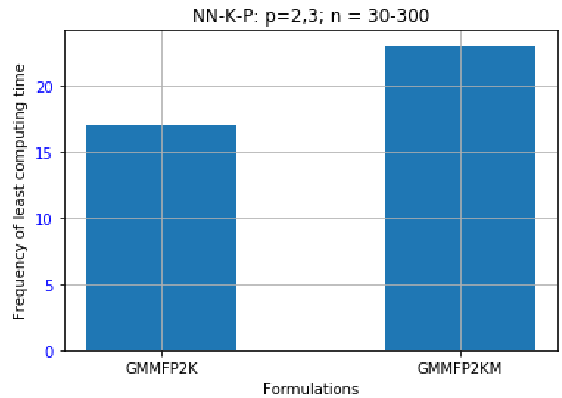


Figure 3.10: NN-K-P

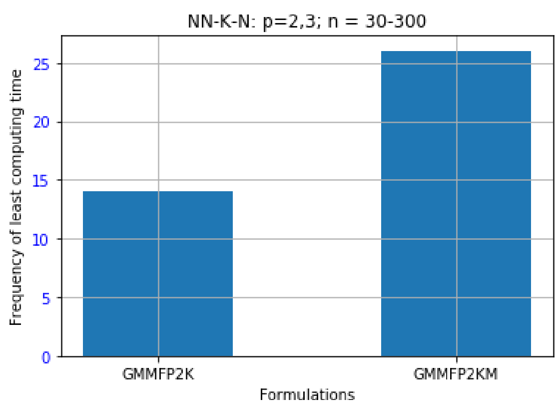


Figure 3.11: NN-K-N

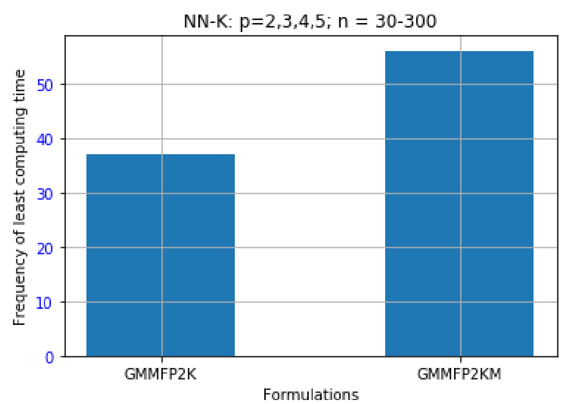


Figure 3.12: NN-K

3.4.2 Remark on computational results

The only difference between unconstrained model and knapsack constrained model is that the latter has one more constraint $\sum_{j=1}^{n} w_j x_j \leq W$. Since the weight limit W chosen is not tight, this additional constraint does not influence much from the point of view of floating point rounding error. Therefore, we expect that the results would look similar to the results in the unconstrained case. Almost no computational errors were found in the test results. As our expectation, results are very similar to those in unconstrained problems. On average, GMMFP2KM perform at least as good as GMMFP2K.

3.5 Assignment constrained 0-1 GMMFP problem

If the denominators are unrestricted in signs, we can adjust GMMFP1 formulation to obtain its corresponding assignment problem. We introduce the continuous variables y, $y^k = f_k(x)$, and $z^k_{ij} = y^k x_{ij}$. Also, we add binary variables t^k for big-M method along with a large positive number M and a small positive number ϵ to filter out the points in the relaxed feasible region that produce zero denominators. The pn^2 number of bilinear terms $y^k x_{ij}$ are linearized by using McCormick constraints. We present the first MILP formulation for assignment constrained 0-1 GMMFP as follows:

$$
\begin{aligned}
&\text{(GMMFP1A)}\quad \text{Minimize}\quad y\\
&\text{Subject to}\\
&\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c^k_{ij}x_{ij} = y^k\beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d^k_{ij}z^k_{ij} \quad \forall k = 1,\cdots,p\\
&y^k \leq y \qquad \forall k = 1,\cdots,p\\
&z^k_{ij} \leq y^{uk}x_{ij} \qquad \forall k = 1,\cdots,p,\quad \forall i = 1,\cdots,n,\quad \forall j = 1,\cdots,n\\
&z^k_{ij} \leq y^k + y^{lk}(x_{ij}-1) \qquad \forall k = 1,\cdots,p,\quad \forall i = 1,\cdots,n,\quad \forall j = 1,\cdots,n\\
&z^k_{ij} \geq y^{lk}x_{ij} \qquad \forall k = 1,\cdots,p,\quad \forall i = 1,\cdots,n,\quad \forall j = 1,\cdots,n\\
&z^k_{ij} \geq y^k + y^{uk}(x_{ij}-1) \qquad \forall k = 1,\cdots,p,\quad \forall i = 1,\cdots,n,\quad \forall j = 1,\cdots,n\\
&\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1,\cdots,n\\
&\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1,\cdots,n\\
&\beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d^k_{ij}x_{ij} \geq \epsilon - Mt^k \qquad \forall k = 1,\cdots,p\\
&\beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d^k_{ij}x_{ij} \leq -\epsilon + M(1-t^k) \quad \forall k = 1,\cdots,p\\
&x_{ij} \in \{0,1\}, \qquad \forall i = 1,\cdots,n,\quad \forall j = 1,\cdots,n\\
&y^k \in [y^{lk}, y^{uk}], \quad \forall k = 1,\cdots,p\\
&t^k \in \{0,1\}, \qquad \forall k = 1,\cdots,p.\\
&y \in [y^l, y^u].
\end{aligned}
$$

Since the denominators are unrestricted in signs, the computation of the bounds of each y^k is more involved. In the following procedure, we assume that all denominator coefficients are integers. We begin with the upper and lower bounds of numerators and denominators. Using simple notations we write,

$$\text{topMax}^k = \min\left\{RS_{\max}(\mathbf{c}^k), CS_{\max}(\mathbf{c}^k)\right\} + \alpha^k, \text{ as numerator upper bound}$$

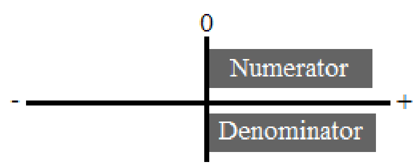
$$\text{topMin}^k = \max\left\{RS_{\min}(\mathbf{c}^k), CS_{\min}(\mathbf{c}^k)\right\} + \alpha^k, \text{ as numerator lower bound}$$

$$\text{botMax}^k = \min\left\{RS_{\max}(\mathbf{d}^k), CS_{\max}(\mathbf{d}^k)\right\} + \beta^k, \text{ as denominator upper bound}$$

$$\text{topMin}^k = \max\left\{RS_{\min}(\mathbf{d}^k), CS_{\min}(\mathbf{d}^k)\right\} + \beta^k, \text{ as denominator lower bound.}$$

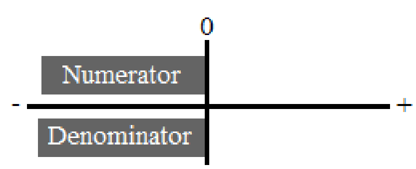
We pick four cases in which zero can be used as a valid lower or upper bound, one additional case in which bounds contain zero. We then put all other cases into one more case in which loose valid bounds contain zero. Note that this procedure gives a set of global bounds for y^k.

Case I: $\text{topMin}^k \geq 0$ and $\text{botMin}^k \geq 0$



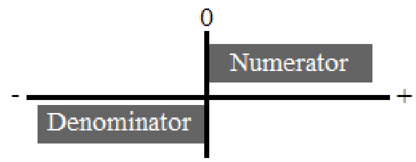
$$y^{uk} = \text{topMax}^k, \quad y^{lk} = 0. \tag{3.8}$$

Case II: $\text{topMax}^k \leq 0$ and $\text{botMax}^k \leq 0$



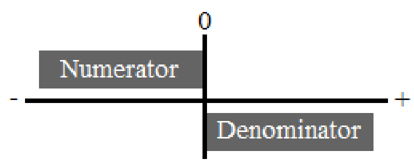
$$y^{uk} = -\text{topMin}^k, \quad y^{lk} = 0. \tag{3.9}$$

Case III: $\text{topMin}^k \geq 0$ and $\text{botMax}^k \leq 0$



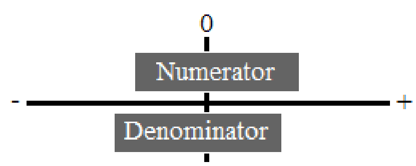
$$y^{uk} = 0, \quad y^{lk} = -\text{topMax}^k. \tag{3.10}$$

Case IV: $\text{topMax}^k \leq 0$ and $\text{botMin}^k \geq 0$



$$y^{uk} = 0, \quad y^{lk} = \text{topMin}^k. \tag{3.11}$$

Case V: $\text{topMax}^k > 0$, $\text{topMin}^k < 0$ and $\text{botMax}^k > 0$, $\text{botMin}^k < 0$



$$y^{uk} = \max\{-\text{topMin}^k, \text{topMax}^k\}, \quad y^{lk} = \min\{\text{topMin}^k, -\text{topMax}^k\}. \tag{3.12}$$

All other cases:

$$y^{uk} = \max\{|\text{topMax}^k|, |\text{topMin}^k|\}, \quad y^{lk} = -y^{uk}. \tag{3.13}$$

A pair of global bounds for the aggregated y can be computed as follows:

$$y^l = \min\{y^{lk} : k = 1, \cdots, p\} \tag{3.14}$$

$$y^u = \infty. \tag{3.15}$$

Similar to MMFP problems, we can employ the binary expansion on GMMFP1A. This gives us another formulation GMMFP2A. We introduce

$$D^k = \sum_{i=1}^{n}\sum_{j=1}^{n}\{|d_{ij}^k| : d_{ij}^k < 0\}, \quad k = 1, \cdots, p.$$

And let

$$\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} + D^k = \sum_{s=1}^{q^k} 2^{s-1} v_s^k, \quad k = 1, \cdots, p$$

where

$$v_s^k \in \{0, 1\}, \quad s \in \{1, \cdots, q^k\},\ k = 1, \cdots, p$$

$$q^k = \left\lfloor \log_2 \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}^k| \Big) \right\rfloor + 1, \quad k = 1, \cdots, p.$$

The constraint

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} = y^k \beta^k + y^k \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij}, \quad k = 1, \cdots, p$$

becomes

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} = y^k \beta^k + y^k \Big(\sum_{s=1}^{q^k} 2^{s-1} v_s^k - D^k \Big), \quad k = 1, \cdots, p.$$

By using substitutions $z_s^k = y^k v_s^k$ for the bilinear terms, we get

$$\alpha^k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^k x_{ij} = \sum_{s=1}^{q^k} 2^{s-1} z_s^k + y^k(\beta^k - D^k), \quad k = 1, \cdots, p.$$

The final formulation is

$$
\begin{aligned}
\text{(GMMFP2A)}\quad & \text{Minimize} \quad y \\
& \text{Subject to} \\
& y^k \le y & \forall k = 1, \cdots, p \\
& \alpha^k + \sum_{i=1}^{n} v \sum_{j=1}^{n} c_{ij}^k x_{ij} = \sum_{s=1}^{q^k} 2^{s-1} z_s^k + y^k(\beta^k - D^k) & \forall k = 1, \cdots, p \\
& \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} + D^k = \sum_{s=1}^{q^k} 2^{s-1} v_s^k & \forall k = 1, \cdots, p \\
& z_s^k \le y^{uk} v_s^k & \forall k = 1, \cdots, p, \quad \forall s \in \{1, \cdots, q^k\} \\
& z_s^k \le y^k + y^{lk}(v_s^k - 1) & \forall k = 1, \cdots, p, \quad \forall s \in \{1, \cdots, q^k\} \\
& z_s^k \ge y^{lk} v_s^k & \forall k = 1, \cdots, p, \quad \forall s \in \{1, \cdots, q^k\} \\
& z_s^k \ge y^k + y^{uk}(v_s^k - 1) & \forall k = 1, \cdots, p, \quad \forall s \in \{1, \cdots, q^k\} \\
& \sum_{j=1}^{n} x_{ij} = 1 & \forall i = 1, \cdots, n \\
& \sum_{i=1}^{n} x_{ij} = 1 & \forall j = 1, \cdots, n \\
& \beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} \ge \epsilon - M t^k & \forall k = 1, \cdots, p \\
& \beta^k + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^k x_{ij} \le -\epsilon + M(1 - t^k) & \forall k = 1, \cdots, p \\
& x_{ij} \in \{0, 1\}, & \forall i = 1, \cdots, n, \quad \forall j = 1, \cdots, n \\
& y^k \in [y^{lk}, y^{uk}], & \forall k = 1, \cdots, p \\
& v_s^k \in \{0, 1\}, & \forall k = 1, \cdots, p, \quad \forall s \in \{1, \cdots, q^k\} \\
& t^k \in \{0, 1\}, & \forall k = 1, \cdots, p \\
& y \in [y^l, y^u].
\end{aligned}
$$

For the experiments in this thesis, bounds of y^k, y for both GMMFP1A and GMMFP2A are computed by the set of formulas (3.8) to (3.15). These bounds are globally valid if all denominator coefficients are integers. In addition, we have modified formulations GMMFP1AM and GMMFP2AM. These are GMMFP1A and GMMFP2A equipped with the pre-solve plan we have discussed before.

	GMMFP1A	GMMFP2A
No. of binary variables	$n^2 + p$	$n^2 + \sum_{k=1}^{p} q^k + p$
No. of continuous variables	$pn^2 + p + 1$	$1 + p + \sum_{k=1}^{p} q^k$
No. of constraints	$4pn^2 + 4p + 4p$	$5p + 4\sum_{k=1}^{p} q^k + 2n$

Table 3.6: Table of summary for counts of variables and constraints

3.5.1 Experimental analysis on assignment constrained 0-1 GMMFP formulations

We ran GMMFP1A, GMMFP1AM, GMMFP2A, GMMFP2AM on three problem instance classes. Two 2-dimensional array (matrices), namely $[c^k_{ij}]_{n\times n}$, $[d^k_{ij}]_{n\times n}$ were generated for each objective ratio $k = 1, \cdots, p$. And two 1-dimensional arrays α^k and β^k were generated for constant terms in the objectives. Three problem instance classes were generated as follows:

(NN-A-R)	Random uniformly distributed integers: All data value c^k_j, α^k, d^k_j and β^k are randomly generated in the range [-100,100]
(NN-A-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c^k_j, d^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective.
(NN-A-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in decreasing order in each objective.

Below table presented objective value found by the solver, relative MIP optimality gap in percentage, and computational time in second (including the computational time for pre-solves, if any). The time value in boldface indicates that it is the fastest among all formulations that completed before the time limit of 3600 seconds without encountering any computational errors.

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-97.0	0.0	2.96839	-97.0	0.0	**2.69057**
20	-328.0	0.0	**5.62509**	-328.0	0.0	7.16625
30	-378.0	0.0	21.14833	-378.0	0.0	**20.81516**
40	-472.0	0.0	**54.73302**	-472.0	0.0	64.15143
50	-536.0	0.0	**31.30791**	-536.0	0.0	32.9594
60	-596.0	0.0	268.78734	-596.0	0.0	**51.63477**
70	-641.0	0.0	278.62422	-641.0	5e-05	**48.73853**
80	-706.0	1e-05	133.72353	-706.0	0.0	**115.3706**
90	-734.0	0.0	121.36211	-734.0	6e-05	**121.07237**
100	-843.0	0.37259	3600.10739	-844.0	0.0	**174.53628**

Table 3.8: NN-A-P-2

The formulations GMMFP1A and GMMFP1AM did not complete most of the problem instances within the time limit of 3600 seconds. Full set of result tables are listed in appendix C. Summary of frequencies of least computing time among GMMFP2A and GMMFP2AM are presented in below graphs.

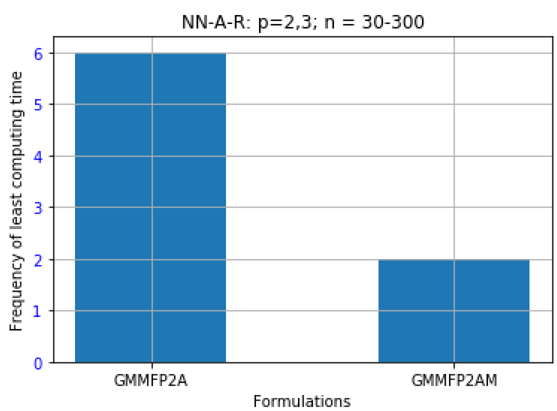


Figure 3.13: NN-A-R

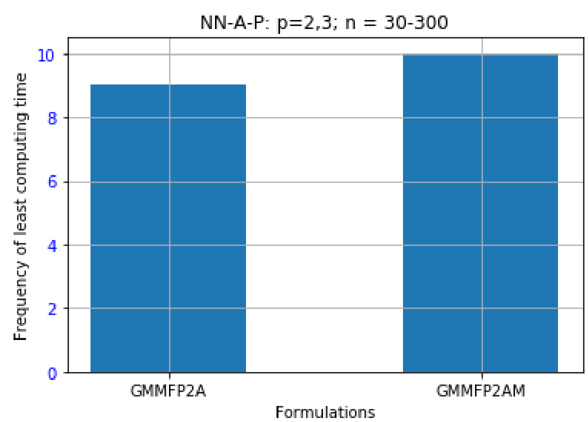


Figure 3.14: NN-A-P

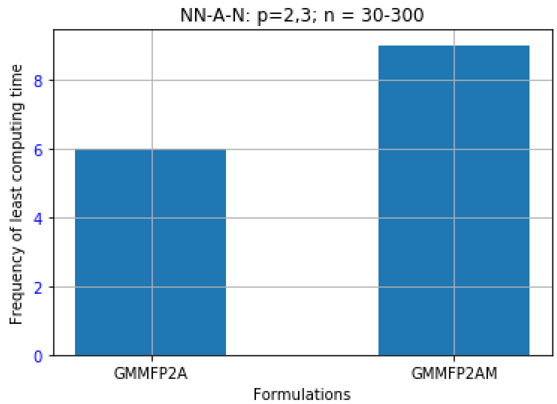


Figure 3.15: NN-A-N

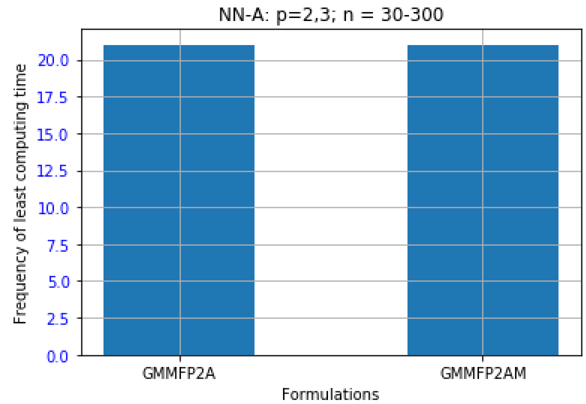


Figure 3.16: NN-A

3.5.2 Remark on computational results

In assignment problems, $\mathbf{x} = \mathbf{0}$ is not a feasible solution. GMMFPA is a sub-class of GMMFP, therefore, we do not have much expectation on the modified formulations. On average, GMMFP2AM did not perform better than GMMFP2A. In sub class -R, GMMFP2A did better than GMMFP2AM on average.

3.6 Conclusion

Compared with the restricted denominator case (in chapter 2), the unrestricted denominator case required much longer computing time for the same n and p values. This was observed in all three types of constraints. This is mainly due to having additional p number of equality constraints, $2p$ number of big-M constraints, p number of continuous variables and p number of binary variables.

Performance of modified formulations equipped with pre-solve plan

As we discussed in section 3.2.1, 0-1 GMMFP models are more resistant against floating point rounding errors compared to the 0-1 MMFP models due to the model design and the nature of the problem data it handles. Computational errors of both type I and type II still occurred in the experiments. However, the number of error occurrences was relatively very small. The pre-solve plan used in the modified formulations are built on only one critical objective value (zero) and therefore in the experiments they did not show better performance over the formulations without pre-solve plan in general.

Recursion heuristic

From the experiments with type I or type II errors we randomly selected some and applied recursion heuristic. In all these recursions, we were able to find the feasible solutions. However, we were unable to prove that these were in fact true optimal values.

Chapter 4

Conclusion

This thesis considers 0-1 MMFP, a sub-class of minimax programming problem in which objectives are linear fractional functions and decision variable is a binary vector. The 0-1 MMFP can be found in many applications.

In chapter 1, literature was reviewed on various types of linear and fractional programming problems with both continuous and discrete decision variables. Complexity issues on both linear and linear fractional optimizations were discussed. We then considered the maximum of a set of some functions over a feasible set as an objective function. In minimax programming, such objective function is minimized. That is, the program searches for the least worst-case solution. Note that the 0-1 MMFP is a NP-hard problem.

In chapter 2, we presented two basic or parent linear MILP formulations for the 0-1 MMFP in which all denominators are positive. One of the formulations uses binary expansion technique to reduce the number of constraints. Reduction in the number of constraints occurs when n is relatively large. The test results demonstrated two advantages of reduction in the number of constraints, namely, lower chance of occurrence of floating point rounding errors and faster computational speed in general. We discussed the inspection of the denominator condition and its complexity. The development of the formulations along with the computation of upper and lower bounds for the substituting continuous variable y (aggregated upper bound of all objective ratios) was presented. These derivations were done for three different cases of feasible sets: unconstrained case, knapsack constrained case and assignment constrained case. Before putting these basic models on our extensive tests, we investigated the structure of the programming and discovered some weakness against the floating point arithmetic rounding errors.

Such errors could occur in any digital computing environment. The case with all denominator coefficients being positive and where $\mathbf{x} = \mathbf{0}$ is a feasible solution attracted our special attention. This was where we first discovered a structural weakness against the floating point arithmetic errors especially when the bounds of y were loose and n was large. Reducing the integrality tolerance is a way to reduce such risk but it comes with significant increase in resource utilization. We gave proposition 2.5 which describes an environment in MMFP1U model where type I error hits all components of a binary variable. This proposition can be expended to cover all formulations in this thesis. A common characteristic of such environments is that it gives room for a MILP solver to minimize the objective below the true optimal value at a point too close to a feasible binary point. Extension of proposition 2.5 provides a general construction of such environment which is dependent on the problem data, objective y-bounds, and the formulation used. Therefore, we proposed various pre-solve plans to tighten the bounds of y together with the concept of critical objective values. Out of our investigation, we expected that, in certain problem types, these critical values could be used as candidate bounds. Design of these bound tightening plans depends on the data however. We added a customizable pre-solve plan which could be used with any strategy such as binary search to capture the optimal value in a desired interval. We equipped few of these plans to the parent models to obtain modified formulations. If the error persists to occur, we prepared a recursion heuristic which utilizes some information out of failed optimization session for the next attempt.

In our computational tests, we used integer coefficients for all objective ratios. In the results of the tests we did, we found the effectiveness of pre-solve plans against the errors as well as for speed if they were properly designed depending on the data. We also found that if these pre-solve plans were not designed according to the data, the modified formulations usually would not produce any improvement over their parent models. The extra times used in bound tightening were found to be relatively very small.

In the computational experiments, MMFP2 performed better than MMFP1 in terms of speed and error reduction in general. These results confirmed the effectiveness of reduction in number of constraints by using binary expansion. However when the problem size is significantly large, the number of constraints in MMFP2 also grow and occurrence of error were observed. For example, in PP-U-R class, MMFP1 produced incorrect solutions for all the n and p values we tested. In the same data class, MMFP2 avoided errors up to $n = 600$ and failed thereafter. In NP- class, MMFP1 failed after $n = 2000$ for all p values, meanwhile, MMFP2 managed to produce correct answers for all n values we tested with $p = 2$ and 3. However, MMFP2 produced errors after $n = 3000$ in $p = 5$ case. Also, we observed that the correlations between numerator and denominator coefficients are factors for both computing speed and error occurrence. In general, random (non-correlated) data is less vulnerable to errors but it requires more time to complete. On the other hand, the correlated data (positive and negative) is more error prone but it requires less time to complete.

In chapter 3 we studied 0-1 GMMFP, a general version, in which positive denominator assumption was dropped. Due to the possibility of a zero denominator, we discussed the use of big-M method in the formulations. As in chapter 2, we developed two basic linear formulations, one of them uses the binary expansion technique to reduce number of constraints. We discussed computation of bounds of two substituting continuous variables: y (aggregated upper bound of all objective ratios) and y^k which is simply the k^{th} objective ratio. Two basic formulations were developed for three constraints types: unconstrained case, knapsack constrained case and assignment constrained case.

It was discovered that 0-1 GMMFP is more robust against computational errors compared to 0-1 MMFP. We argue that this is due to the fact that numerator and denominator coefficients are not restricted in sign as well as the formulation design. However, a pre-solve plan that only checks if zero could be a bound for y was presented for possible improvement. Parent models were equipped with this pre-solve to generate the modified formulations. We also gave a recursion heuristic which could be used in case of the occurrence of floating point arithmetic errors.

In the computational tests, all the numerator and denominator coefficients were integers. From the results, as we expected, we found little advantage of modified formulations over their parent models. Compared to the experiments in chapter 2, unrestricted denominator problems in chapter 3 took significantly longer computational time due to the additional constraints and variables in the model design. We also observed that formulations with binary expansion performed better in terms of computing time. Correlated data classes finished in these experiments significantly faster than the class with random data.

In both chapter 2 and chapter 3, due to the resource constraints, majority of the experiments were done on the modified formulations which were designed for certain data but not for all data types. In the tests we found a few cases where different models produced different optimal solutions. We would like to recommend below topics for the future works on 0-1 MMFP problems.

- More extensive study can be done on pre-solving to tighten the bounds for 0-1 MMFP with positive denominators. The study can include the critical objective values and interval of capture l on diverse problem instances.

- Study can be done on the relationship between interval of capture l and data type in 0-1 MMFP. Ground truth data can be created on a diverse range of problem instances and use of machine learning to decide on the interval of capture.

- On the 0-1 GMMFP case, recursive pre-solving could consume more resources as the data size increases. Therefore, this problem can be divided into specialised sub-classes and research on each sub-class could produce some break through.

- Generalization/extension of proposition 2.5 can be developed to cover more formulations and problem data type.

- Theoretical exploration of floating point rounding errors in 0-1 fractional programming especially with respect to the McCormick constraints is also recommended.

- Since the use of McCormick relaxation is one of the sources for computational errors, we can explore alternative linearization method which is more robust against such error.

- Research can be done on a possible method of certifying if a solution is optimal.
- Study on existence of conditions that guarantee that a local optimal solution is a global optimal solution can be done.

Bibliography

[1] R. Ahuja. Minimax linear programming problem. *Operations Research Letters*, 4 (1985) 131–134.

[2] H. Aissi, C. Bazgan, and D. Vanderpooten. Min–max and min–max regret versions of combinatorial optimization problems: A survey. *European journal of operational research*, 197 (2009) 427–438.

[3] E. Amaldi, S. Bosio, and F. Malucelli. Hyperbolic set covering problems with competing ground-set elements. *Mathematical programming*, 134 (2012) 323–348.

[4] E. Amaldi, S. Bosio, F. Malucelli, and D. Yuan. Solving nonlinear covering problems arising in wlan design. *Operations research*, 59 (2011) 173–187.

[5] S. Arora, M. Puri, and K. Swarup. The set covering problem with linear fractional functional. *Indian Journal of Pure and Applied Mathematics*, 8 (1977) 578–588.

[6] E. B. Bajalinov. *Linear-fractional programming theory, methods, applications and software*, volume 84. Springer Science & Business Media (2013).

[7] J. Björkqvist. *Discrete and disjunctive optimization: Parallel strategies and applications in industrial scheduling.* Åbo Akademi University (2001).

[8] E. Boros and P. L. Hammer. Pseudo-boolean optimization. *Discrete applied mathematics*, 123 (2002) 155–225.

[9] J. Borrero, C. Gillen, and O. Prokopyev. Fractional 0-1 programming: applications and algorithms. *Journal of Global Optimization*, 69 (2017) 255–282.

[10] J. S. Borrero, C. Gillen, and O. A. Prokopyev. A simple technique to improve linearized reformulations of fractional (hyperbolic) 0–1 programming problems. *Operations Research Letters*, 44 (2016) 479–486.

[11] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge university press (2004).

[12] A. Charnes and W. W. Cooper. Programming with linear fractional functionals. *Naval Research logistics quarterly*, 9 (1962) 181–186.

[13] M. Conforti, G. Cornuéjols, and G. Zambelli. *Integer programming*, volume 271. Springer (2014).

[14] W. Dinkelbach. On nonlinear fractional programming. *Management science*, 13 (1967) 492–498.

[15] Q. Feng, H. Mao, H. Jiao, and J. Yin. On minimax general linear fractional programming. *2009 International Conference on Information Engineering and Computer Science*. IEEE (2009) 1–4.

[16] F. Glover and E. Woolsey. Further reduction of zero-one polynomial programming problems to zero-one linear programming problems. *Operations Research*, 21 (1973) 156–161.

[17] F. Glover and E. Woolsey. Converting the 0-1 polynomial programming problem to a 0-1 linear program. *Operations research*, 22 (1974) 180–182.

[18] N. N. Goshu. Fractional programming (2014).

[19] J. Guo. *The unrestricted linear fractional assignment problem.* Master's thesis, Simon Fraser University (2018).

[20] S. Gupta and A. Punnen. Minmax linear programs with grouped variables. *Opsearch*, 26 (1989) 117–186.

[21] P. L. Hammer and S. Rudeanu. *Boolean methods in operations research and related areas.* Springer (1968).

[22] P. Hansen, M. V. P. De Aragão, and C. C. Ribeiro. Hyperbolic 0–1 programming and query optimization in information retrieval. *Mathematical Programming*, 52 (1991) 255–263.

[23] H. Jiao and S. Liu. A new linearization technique for minimax linear fractional programming. *International Journal of Computer Mathematics*, 91 (2014) 1730–1743.

[24] H.-W. Jiao, F.-H. Wang, and Y.-Q. Chen. An effective branch and bound algorithm for minimax linear fractional programming. *Journal of Applied Mathematics*, 2014 (2014).

[25] S. N. Kabadi and A. P. Punnen. A strongly polynomial simplex method for the linear fractional assignment problem. *Operations Research Letters*, 36 (2008) 402–407.

[26] A. Kasperski and P. Zieliński. Robust discrete optimization under discrete and interval uncertainty: A survey. *Robustness analysis in decision aiding, optimization, and analytics*, 113–143. Springer (2016).

[27] E. Klotz and A. M. Newman. Practical guidelines for solving difficult mixed integer linear programs. *Surveys in Operations Research and Management Science*, 18 (2013) 18–32.

[28] M. J. Magazine and O. Oguz. A fully polynomial approximation algorithm for the 0–1 knapsack problem. *European Journal of Operational Research*, 8 (1981) 270–273.

[29] G. P. McCormick. Nonlinear programming: theory, algorithms, and applications (1983).

[30] E. Mehmanchi, C. P. Gillen, A. Gómez, and O. A. Prokopyev. On robust fractional 0-1 programming. *INFORMS Journal on Optimization*, 2 (2020) 96–133.

[31] J. Najman and A. Mitsos. Tighter mccormick relaxations through subgradient propagation. *Journal of Global Optimization*, 75 (2019) 565–593.

[32] A. Neumaier and O. Shcherbina. Safe bounds in linear and mixed-integer linear programming. *Mathematical Programming*, 99 (2004) 283–296.

[33] C. H. Papadimitriou. On the complexity of integer programming. *Journal of the ACM (JACM)*, 28 (1981) 765–768.

[34] D. Pisinger. Where are the hard knapsack problems? *Computers & Operations Research*, 32 (2005) 2271–2284.

[35] O. A. Prokopyev, C. N. de Meneses, C. A. S. Oliveira, and P. M. Pardalos. On multiple-ratio hyperbolic 0-1 programming problems (2005).

[36] A. P. Punnen. K-sum linear programming. *Journal of the Operational Research Society*, 43 (1992) 359–363.

[37] A. P. Punnen and Y. Aneja. Minmax combinatorial optimization. *European Journal of Operational Research*, 81 (1995) 634–643.

[38] A. P. Punnen and K. Nair. Polynomial algorithms for a class of discrete minmax linear programming problems. *Journal of the Operational Research Society*, 46 (1995) 499–506.

[39] T. Radzik. Newton's method for fractional combinatorial optimization. *Proceedings., 33rd Annual Symposium on Foundations of Computer Science*. IEEE (1992) 659–669.

[40] T. Radzik. Fractional combinatorial optimization. *Handbook of combinatorial optimization*, 429–478. Springer (1998).

[41] A. L. Saipe. Solving a (0,1) hyperbolic program by branch and bound. *Naval Research Logistics Quarterly*, 22 (1975) 497–515.

[42] S. Schaible and J. Shi. Fractional programming: the sum-of-ratios case. *Optimization Methods and Software*, 18 (2003) 219–229.

[43] S. Schaible and J. Shi. Recent developments in fractional programming: single-ratio and max-min case. *Nonlinear analysis and convex analysis*, 493506 (2004).

[44] H. A. Taha. An algorithm for zero-one fractional programming. *AIIE Transactions*, 7 (1975) 29–34.

[45] M. Tawarmalani, S. Ahmed, and N. V. Sahinidis. Global optimization of 0-1 hyperbolic programs. *Journal of Global Optimization*, 24 (2002) 385–416.

Appendices

Appendix A

Guidelines for problem instances

The following are the categories of problem instances selected for the tests. In each category we run the experiment for different n-values. We create instance reference codes as follow:

signs - constraint type - data type - p number

$$\text{signs} = \begin{cases} PP, & \text{Coefficients for both numerators and denominators are positive} \\ NP, & \text{Coefficients for numerators unrestricted and denominators positive} \\ NN, & \text{Coefficients for both numerators and denominators are unrestricted} \end{cases}$$

$$\text{constraint type} = \begin{cases} U, & \text{Unconstrained} \\ K, & \text{Knapsack constrained} \\ A, & \text{Assignment constrained} \end{cases}$$

$$\text{data type} = \begin{cases} R, & \text{Random data} \\ P, & \text{Positively correlated data} \\ N, & \text{Negatively correlated data} \end{cases}$$

For example, the instance reference code for positive coefficients on both numerators and denominators and unconstrained problem with random data and $p = 5$ would be PP-U-R-5.

(PP-U)	**Unconstrained**: There is no constraints on the decision variable x_j
(PP-U-R)	Random uniformly distributed integers: All data values $c_j^k, \alpha^k, d_j^k, \beta^k$ are randomly generated in the range [1,100].
(PP-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(PP-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Other coefficients c_j^k, d_j^k are generated as random uniformly distributed integers in the range [1,100]. c_j^k are rearranged in increasing order and d_j^k are rearranged in decreasing order in each objective.

(PP-K)	**Knapsack constraint**: knapsack constraint $\sum_{j\in J} w_j x_j \leq W$ is applied, where the weights w_j are generated as random uniformly distributed integers in the range [1,100]. The weight limit W is computed by the formula: $W = \frac{(n/4)(w^l+w^u)}{2}$, where $w^l = 1$, $w^u = 100$
(PP-K-R)	Random uniformly distributed integers: The same as in (PP-U-R)
(PP-K-P)	Positively correlated data: The same as in (PP-U-P)
(PP-K-N)	Negatively correlated data: The same as in (PP-U-N)
(PP-A)	**Assignment constraint**: Two 2-dimensional array (matrices), namely $[c^k_{ij}]_{n\times n}$, $[d^k_{ij}]_{n\times n}$ are generated for each objective ratio $k \in K$. And two 1-dimensional arrays α^k and β^k are generated for constant terms in the objectives.
(PP-A-R)	Random uniformly distributed integers: The same as in (PP-U-R)
(PP-A-P)	Random uniformly distributed integers: The same as in (PP-U-P)
(PP-A-N)	Random uniformly distributed integers: The same as in (PP-U-N)
(NP-U)	**Unconstrained**: There is no constraints on the decision variable x_j
(NP-U-R)	Random uniformly distributed integers: Data values c^k_j, α^k are randomly generated in the range [-100,100]. Data values d^k_j, β^k are randomly generated in the range [1,100].
(NP-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d^k_j are generated as random uniformly distributed integers in the range [1,100] and rearranged in increasing order in each objective.
(NP-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d^k_j are generated as random uniformly distributed integers in the range [1,100] and rearranged in decreasing order in each objective.
(NP-K)	**Knapsack constraint**: knapsack constraint $\sum_{j\in J} w_j x_j \leq W$ is applied, where the weights w_j are generated as random uniformly distributed integers in the range [1,100]. The weight limit W is computed by the formula: $W = \frac{(n/4)(w^l+w^u)}{2}$, where $w^l = 1$, $w^u = 100$
(NP-K-R)	Random uniformly distributed integers: The same as in (NP-U-R)
(NP-K-P)	Positively correlated data: The same as in (NP-U-P)
(NP-K-N)	Negatively correlated data: The same as in (NP-U-N)
(NP-A)	**Assignment constraint**: Two 2-dimensional array (matrices), namely $[c^k_{ij}]_{n\times n}$, $[d^k_{ij}]_{n\times n}$ are generated for each objective ratio $k \in K$. And two 1-dimensional arrays α^k and β^k are generated for constant terms in the objectives.
(NP-A-R)	Random uniformly distributed integers: The same as in (NP-U-R)
(NP-A-P)	Random uniformly distributed integers: The same as in (NP-U-P)
(NP-A-N)	Random uniformly distributed integers: The same as in (NP-U-N)
(NN-U)	**Unconstrained**: There is no constraints on the decision variable x_j
(NN-U-R)	Random uniformly distributed integers: All data value c^k_j, α^k, d^k_j and β^k are randomly generated in the range [-100,100]
(NN-U-P)	Positively correlated data: All α^k and β^k are taken as 1. Data value c^k_j, d^k_j are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective.

(NN-U-N)	Negatively correlated data: All α^k and β^k are taken as 1. Data value c_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in increasing order in each objective. Data values d_j^k are generated as random uniformly distributed integers in the range [-100,100] and rearranged in decreasing order in each objective.
(NN-K)	**Knapsack constraint**: knapsack constraint $\sum_{j\in J} w_j x_j \leq W$ is applied, where the weights w_j are generated as random uniformly distributed integers in the range [1,100]. The weight limit W is computed by the formula: $W = \frac{(n/4)(w^l+w^u)}{2}$, where $w^l = 1$, $w^u = 100$
(NN-K-R)	Random uniformly distributed integers: The same as in (NN-U-R)
(NN-K-P)	Positively correlated data: The same as in (NN-U-P)
(NN-K-N)	Negatively correlated data: The same as in (NN-U-N)
(NN-A)	**Assignment constraint**: Two 2-dimensional array (matrices), namely $[c_{ij}^k]_{n\times n}$, $[d_{ij}^k]_{n\times n}$ are generated for each objective ratio $k \in K$. And two 1-dimensional arrays α^k and β^k are generated for constant terms in the objectives.
(NN-A-R)	Random uniformly distributed integers: The same as in (NN-U-R)
(NN-A-P)	Random uniformly distributed integers: The same as in (NN-U-P)
(NN-A-N)	Random uniformly distributed integers: The same as in (NN-U-N)

Appendix B

Computational results for chapter 2

B.1 Computational results with small size data

B.1.1 Positive numerators and denominators with no constraint

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.01068	0.0	0.03125	0.27236	0.0	**0.10936**	0.27236	0.0	0.37522	0.27236	3e-05	0.14062
400	0.00672	0.0	0.06246	0.27851	9e-05	0.47789	0.27851	0.0	0.8907	0.27851	0.0	**0.26703**
600	0.00318	0.0	0.10935	0.14599	0.0	0.56249	0.14599	0.0	0.31249	0.14599	0.0	**0.28372**
800	0.00227	0.0	0.20311	0.17224	0.0	0.57808	0.00181	0.0	0.15624	0.17224	0.0	**0.31091**
1000	0.00183	0.0	0.26567	0.1433	0.0	0.94259	0.00154	0.0	0.17186	0.1433	0.0	**0.4531**
1200	0.00162	0.0	0.32812	0.15228	0.0	**0.34373**	0.15228	0.0	0.85945	0.15228	0.0	0.37498
1400	0.00233	0.0	0.4997	0.15231	0.0	1.78361	0.00175	0.0	0.18761	0.15231	0.0	**0.43764**
1600	0.00091	0.0	0.60941	0.10069	0.0	0.51558	0.00071	0.0	0.03125	0.10069	0.0	**0.39061**
1800	0.00137	0.0	0.78105	0.1493	0.0	3.25017	0.00111	0.0	0.01564	0.1493	0.0	**0.57045**
2000	0.00088	0.0	0.95317	0.11576	0.0	1.10943	0.00076	0.0	0.01564	0.11576	0.0	**0.6562**

Table B.1: PP-U-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.01213	0.0	0.03125	0.36697	0.0	**0.15624**	0.36697	0.0	1.06263	0.36697	0.0	0.3906
400	0.00562	0.0	0.06248	0.35972	0.0	**0.18749**	0.35972	0.0	1.23662	0.35972	0.0	0.52835
600	0.00426	0.0	0.10937	0.33653	6e-05	**0.28124**	0.33653	0.0	1.48445	0.33653	0.0	0.85933
800	0.00149	0.0	0.18749	0.21747	0.0	1.56262	0.21747	0.0	1.71889	0.21747	0.0	**0.3125**
1000	0.00183	0.0	0.32809	0.22738	7e-05	**0.5156**	0.00144	0.0	0.09375	0.22738	0.0	0.76567
1200	0.00133	0.0	0.40623	0.22674	4e-05	**0.7813**	0.00124	0.0	0.03125	0.22674	0.0	0.97217
1400	0.00206	0.0	0.5314	0.26907	0.0	**1.23456**	0.00151	0.0	0.03123	0.26907	0.0	1.25212
1600	0.00121	0.0	0.67196	0.21499	0.0	**0.56247**	0.00082	0.0	0.03123	0.21499	0.0	0.79683
1800	0.00094	0.0	0.82819	0.20984	0.0	**0.43748**	0.00072	0.0	0.03123	0.20984	0.0	0.70323
2000	0.00106	0.0	1.0157	0.21793	0.0	0.78131	0.00086	0.0	0.03123	0.21793	0.0	**0.59382**

Table B.2: PP-U-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.00753	0.0	0.04688	0.57506	0.0	**0.71871**	0.57506	0.0	9.26804	0.57506	0.0	9.55924
400	0.00323	0.0	0.0625	0.4	0.0	**0.39058**	0.4	0.0	2.27518	0.4	0.0	1.00006
600	0.00313	0.0	0.12499	0.47755	0.0	**2.2212**	0.47755	3e-05	44.08086	0.47755	0.0	12.80711
800	0.0034	0.0	0.23436	0.4494	0.0	**2.20326**	0.0025	0.0	0.21872	0.4494	0.0	20.38341
1000	0.003	0.0	0.28122	0.43625	0.0	**4.82913**	0.00252	0.0	0.10936	0.43625	0.0	19.4428
1200	0.00128	0.0	0.37509	0.37044	0.0	**2.42197**	0.00168	0.0	0.14062	0.37044	0.0	16.7514
1400	0.00113	0.0	0.51558	0.38889	0.0	**3.33028**	0.00063	0.0	0.04685	0.38889	0.0	15.98853
1600	0.00125	0.0	0.62497	0.39149	0.0	**2.56277**	0.0008	0.0	0.04685	0.39149	0.0	60.17139
1800	0.00131	0.0	0.8282	0.33943	0.0	61.69999	0.00099	0.0	0.15646	0.33943	0.0	**4.28149**
2000	0.00122	0.0	1.12532	0.39589	0.0	**2.91091**	0.001	0.0	0.04685	0.39589	0.0	28.31922

Table B.3: PP-U-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.00021	0.0	0.04687	0.99048	0.0	**0.0479**	0.00016	0.0	0.01562	0.99048	0.0	0.07812
400	0.0001	0.0	0.06651	0.90698	0.0	0.11457	8e-05	0.0	0.01561	0.90698	0.0	**0.09375**
600	7e-05	0.0	0.11993	0.86667	0.0	0.2339	6e-05	0.0	0.01561	0.86667	0.0	**0.15624**
800	5e-05	0.0	0.17195	0.89286	0.0	0.85933	4e-05	0.0	0.01561	0.89286	0.0	**0.18764**
1000	4e-05	0.0	0.24973	0.95402	0.0	2.40634	4e-05	0.0	0.03125	0.95402	0.0	**0.18749**
1200	3e-05	0.0	0.32813	0.86364	0.0	0.76564	3e-05	0.0	0.03123	0.86364	0.0	**0.0781**
1400	3e-05	0.0	0.4845	0.92453	0.0	1.43751	2e-05	0.0	0.03123	0.92453	0.0	**0.18747**
1600	2e-05	0.0	0.59355	0.6	0.0	1.21876	2e-05	0.0	0.03124	0.6	0.0	**0.0625**
1800	2e-05	0.0	0.71857	0.76	0.0	8.93851	2e-05	0.0	0.01563	0.76	0.0	**0.10936**
2000	2e-05	0.0	0.92426	0.95238	0.0	10.30235	2e-05	0.0	0.03123	0.95238	0.0	**0.20311**

Table B.4: PP-U-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.98113	0.0	0.20323	0.00016	0.0	0.01562	0.98113	0.0	**0.17712**
400	0.0001	0.0	0.0625	0.97333	0.0	0.98442	8e-05	0.0	0.01561	0.97333	0.0	**0.40126**
600	7e-05	0.0	0.10936	0.99679	1e-05	**0.18749**	6e-05	0.0	0.03124	0.99679	0.0	0.37496
800	5e-05	0.0	0.18749	0.97059	0.0	0.85945	4e-05	0.0	0.03124	0.97059	0.0	**0.2812**
1000	4e-05	0.0	0.26579	0.8	0.0	0.24997	4e-05	0.0	0.01559	0.8	0.0	**0.20333**
1200	3e-05	0.0	0.34373	0.9	0.0	0.5937	3e-05	0.0	0.03123	0.9	0.0	**0.2031**
1400	3e-05	0.0	0.4386	0.98964	5e-05	3.00023	2e-05	0.0	0.01561	0.98964	9e-05	**0.37496**
1600	2e-05	0.0	0.57808	0.88889	0.0	4.39086	2e-05	0.0	0.03124	0.88889	0.0	**0.29937**
1800	2e-05	0.0	0.7187	0.7931	0.0	2.93838	2e-05	0.0	0.01562	0.7931	0.0	**0.31247**
2000	2e-05	0.0	0.87514	0.88889	0.0	5.5317	2e-05	0.0	0.03123	0.88889	0.0	**0.18746**

Table B.5: PP-U-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03124	0.92593	0.0	**0.29684**	0.00016	0.0	0.03125	0.92593	0.0	0.29686
400	0.0001	0.0	0.0625	1.0	0.0	0.04687	8e-05	0.0	0.01562	1.0	0.0	**0.03123**
600	7e-05	0.0	0.12498	0.98901	0.0	**0.73431**	6e-05	0.0	0.01562	0.98901	0.0	1.047
800	5e-05	0.0	0.18749	0.92308	0.0	2.14085	4e-05	0.0	0.03124	0.92308	0.0	**0.29684**
1000	4e-05	0.0	0.2656	0.98425	0.0	1.95325	3e-05	0.0	0.02399	0.98425	0.0	**0.42197**
1200	3e-05	0.0	0.35802	0.99438	5e-05	3.7034	3e-05	0.0	0.03123	0.9944	8e-05	**2.20321**
1400	3e-05	0.0	0.4841	0.9798	8e-05	4.45335	2e-05	0.0	0.03124	0.9798	0.0	**0.79696**
1600	3e-05	0.0	0.59412	0.99248	6e-05	0.98444	2e-05	0.0	0.01562	0.99248	0.0	**0.53138**
1800	2e-05	0.0	0.73448	0.9	0.0	4.42464	2e-05	0.0	0.01563	0.9	0.0	**0.42196**
2000	2e-05	0.0	0.98449	0.9604	0.0	5.51577	2e-05	0.0	0.03124	0.9604	0.0	**0.73433**

Table B.6: PP-U-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03124	0.02513	0.0	**0.06229**	0.00016	0.0	0.01544	0.02513	0.0	0.10926
400	0.0001	0.0	0.0625	0.01259	0.0	0.15635	8e-05	0.0	0.01562	0.01259	0.0	**0.09387**
600	7e-05	0.0	0.12384	0.0122	0.0	**0.16784**	6e-05	0.0	0.01561	0.0122	0.0	0.1874
800	5e-05	0.0	0.21088	0.01178	0.0	0.26608	4e-05	0.0	0.01562	0.01178	0.0	**0.14039**
1000	4e-05	0.0	0.25686	0.01176	0.0	0.42188	4e-05	0.0	0.01564	0.01176	0.0	**0.15624**
1200	3e-05	0.0	0.35049	0.01176	0.0	1.3908	3e-05	0.0	0.01563	0.01176	0.0	**0.12513**
1400	3e-05	0.0	0.46863	0.01075	0.0	2.90637	2e-05	0.0	0.01564	0.01075	0.0	**0.17179**
1600	3e-05	0.0	0.57819	0.01101	0.0	3.25014	2e-05	0.0	0.01595	0.01101	0.0	**0.10911**
1800	2e-05	0.0	0.71873	0.01086	0.0	7.29691	2e-05	0.0	0.01564	0.01086	0.0	**0.21871**
2000	2e-05	0.0	0.89086	0.01062	0.0	1.62509	2e-05	0.0	0.03115	0.01062	0.0	**0.21838**

Table B.7: PP-U-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.00021	0.0	0.03125	0.02513	0.0	**0.09371**	0.00016	0.0	0.01562	0.02513	0.0	0.1876
400	0.0001	0.0	0.07812	0.02	0.0	0.17187	8e-05	0.0	0.03125	0.02	0.0	**0.10937**
600	6e-05	0.0	0.10927	0.02	0.0	0.25018	6e-05	0.0	0.01562	0.02	0.0	**0.125**
800	5e-05	0.0	0.18749	0.01263	0.0	0.50612	4e-05	0.0	0.01561	0.01263	0.0	**0.15623**
1000	4e-05	0.0	0.37498	0.01114	0.0	0.4531	3e-05	0.0	0.03124	0.01114	0.0	**0.17187**
1200	3e-05	0.0	0.40623	0.01153	0.0	0.53138	3e-05	0.0	0.0156	0.01153	0.0	**0.17187**
1400	3e-05	0.0	0.5383	0.01081	0.0	0.77104	2e-05	0.0	0.03123	0.01081	0.0	**0.26559**
1600	2e-05	0.0	0.64071	0.01093	0.0	1.29451	2e-05	0.0	0.03123	0.01093	0.0	**0.24997**
1800	2e-05	0.0	0.79683	0.01068	0.0	6.26949	2e-05	0.0	0.01561	0.01068	0.0	**0.56247**
2000	2e-05	0.0	0.95325	0.01081	0.0	1.6719	2e-05	0.0	0.01562	0.01081	0.0	**0.21881**

Table B.8: PP-U-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.02513	0.0	**0.10939**	0.00015	0.0	0.01562	0.02513	0.0	0.20331
400	0.0001	0.0	0.07812	0.01508	0.0	**0.14062**	8e-05	0.0	0.01562	0.01508	0.0	**0.14062**
600	7e-05	0.0	0.1406	0.01342	0.0	**0.21872**	6e-05	0.0	0.03123	0.01342	0.0	0.24998
800	5e-05	0.0	0.20322	0.01126	0.0	0.40623	4e-05	0.0	0.03123	0.01126	0.0	**0.17185**
1000	4e-05	0.0	0.29686	0.01176	0.0	0.78132	4e-05	0.0	0.03123	0.01176	0.0	**0.21982**
1200	3e-05	0.0	0.40633	0.01176	0.0	2.95335	3e-05	0.0	0.03125	0.01176	0.0	**0.28105**
1400	3e-05	0.0	0.54685	0.01112	0.0	0.93758	2e-05	0.0	0.03125	0.01112	0.0	**0.34372**
1600	3e-05	0.0	0.70309	0.01087	0.0	4.3765	2e-05	0.0	0.0625	0.01087	0.0	**0.92195**
1800	2e-05	0.0	0.82808	0.01093	0.0	5.64115	2e-05	0.0	0.03125	0.01093	0.0	**0.32809**
2000	2e-05	0.0	1.04694	0.01093	0.0	1.98793	2e-05	0.0	0.03125	0.01093	0.0	**0.2992**

Table B.9: PP-U-N-5

B.1.2 Positive numerators and denominators with knapsack constraints

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.00603	0.0	0.0465	0.25612	0.0	**0.15472**	0.25612	0.0	0.29918	0.25612	0.0	0.23436
400	0.00734	0.0	0.09373	0.22515	0.0	**0.24997**	0.22515	0.0	0.29686	0.22515	0.0	0.26561
600	0.00334	0.0	0.15598	0.16693	0.0	0.3906	0.16693	0.0	**0.21874**	0.16693	0.0	0.32823
800	0.00173	0.0	0.21885	0.14487	0.0	0.37471	0.00195	0.0	0.12499	0.14487	0.0	**0.3281**
1000	0.0022	0.0	0.31256	0.18645	0.0	1.04659	0.18645	0.0	1.04584	0.18645	0.0	**0.42195**
1200	0.00298	0.0	0.43735	0.19251	0.0	1.59399	0.19251	0.0	1.09369	0.19251	0.0	**0.31244**
1400	0.00072	0.0	0.54656	0.12848	0.0	0.31208	0.00075	0.0	0.03125	0.12848	0.0	**0.29686**
1600	0.0017	0.0	0.70319	0.16045	0.0	1.8208	0.00191	0.0	0.03124	0.16045	0.0	**0.65622**
1800	0.00111	0.0	0.86203	0.12986	0.0	2.5369	0.00136	0.0	0.03125	0.12986	0.0	**0.60947**
2000	0.00094	0.0	1.03123	0.12322	0.0	1.46241	0.00133	0.0	0.03123	0.12322	0.0	**0.61572**

Table B.10: PP-K-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0087	0.0	0.04687	0.38431	0.0	**0.32811**	0.38431	0.0	1.83204	0.38431	0.0	0.40621
400	0.00448	0.0	0.09374	0.28371	0.0	**0.4531**	0.28371	0.0	0.79094	0.28371	0.0	1.01575
600	0.00404	0.0	0.12499	0.30702	0.0	**0.34373**	0.30702	0.0	1.06257	0.30702	0.0	0.81258
800	0.00238	0.0	0.23436	0.31414	0.0	1.67203	0.31414	0.0	1.80409	0.31414	0.0	**0.92919**
1000	0.00101	0.0	0.31248	0.21987	0.0	**0.62498**	0.00154	0.0	0.09374	0.21987	0.0	0.73448
1200	0.00227	0.0	0.51575	0.26275	5e-05	1.34388	0.02192	0.0	0.10937	0.26275	0.0	**0.90632**
1400	0.00136	0.0	0.67197	0.26788	0.0	**1.18766**	0.00144	0.0	0.12499	0.26788	0.0	1.81266
1600	0.00107	0.0	0.71883	0.22235	3e-05	**0.55829**	0.00124	0.0	0.01875	0.22235	0.0	0.96604
1800	0.00113	0.0	0.9687	0.20805	0.0	**1.34906**	0.00143	0.0	0.03124	0.20805	0.0	1.84381
2000	0.00191	0.0	1.28528	0.27366	5e-05	4.5317	0.00242	0.0	0.04687	0.27366	0.0	**2.37249**

Table B.11: PP-K-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.01024	0.0	0.04687	0.54935	0.0	**0.81259**	0.54935	9e-05	17.10492	0.54935	0.0	6.97727
400	0.00389	0.0	0.07812	0.40819	0.0	**1.04694**	0.40819	0.0	9.24248	0.40819	0.0	22.56253
600	0.00387	0.0	0.1564	0.40663	0.0	**1.40712**	0.40663	0.0	10.50041	0.40663	0.0	10.42372
800	0.00262	0.0	0.20312	0.3687	0.0	7.61221	0.3687	0.0	8.54794	0.3687	0.0	**2.1329**
1000	0.00187	0.0	0.34373	0.39549	0.0	**2.20317**	0.39549	0.0	24.66254	0.39549	0.0	25.52228
1200	0.00155	0.0	0.5156	0.33624	0.0	26.33257	0.00143	0.0	0.14062	0.33624	0.0	**1.57829**
1400	0.00142	0.0	0.57825	0.33971	0.0	**3.90982**	0.0015	0.0	0.17186	0.33971	0.0	8.08485
1600	0.00141	0.0	0.85944	0.36407	0.0	**5.01655**	0.00166	0.0	0.18749	0.36407	0.0	37.56785
1800	0.00115	0.0	0.93758	0.3456	0.0	**2.75025**	0.00154	0.0	0.06248	0.3456	0.0	32.96035
2000	0.00101	0.0	1.06256	0.31373	0.0	**5.70232**	0.00149	0.0	0.0625	0.31373	0.0	9.87938

Table B.12: PP-K-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.58621	0.0	**0.09375**	0.00022	0.0	0.12499	0.58621	0.0	0.125
400	0.0001	0.0	0.07845	0.89474	0.0	0.50018	0.00011	0.0	0.03136	0.89474	0.0	**0.25011**
600	7e-05	0.0	0.14051	0.83333	0.0	0.37527	6e-05	0.0	0.03125	0.83333	0.0	**0.20311**
800	5e-05	0.0	0.21903	1.0	0.0	0.03114	6e-05	0.0	0.01562	1.0	0.0	**0.01562**
1000	4e-05	0.0	0.27452	0.98437	0.0	**0.57773**	5e-05	0.0	0.01562	0.98438	9e-05	0.96881
1200	3e-05	0.0	0.38055	0.875	0.0	0.29702	3e-05	0.0	0.03125	0.875	0.0	**0.10936**
1400	3e-05	0.0	0.5	0.75	0.0	10.64277	3e-05	0.0	0.01562	0.75	0.0	**0.28135**
1600	3e-05	0.0	0.65582	0.88235	0.0	0.9688	3e-05	0.0	0.03125	0.88235	0.0	**0.43481**
1800	2e-05	0.0	0.75003	0.97872	0.0	2.10944	3e-05	0.0	0.01562	0.97872	0.0	**0.42185**
2000	2e-05	0.0	0.23448	0.75	0.0	6.00048	3e-05	0.0	0.03125	0.75	0.0	**0.28124**

Table B.13: PP-K-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.99659	0.0	**0.09375**	0.00022	0.0	0.01563	0.99659	5e-05	0.20311
400	0.0001	0.0	0.06268	0.9602	0.0	1.26568	0.00011	0.0	0.01562	0.9602	0.0	**0.46872**
600	7e-05	0.0	0.12499	0.99107	0.0	**0.15624**	6e-05	0.0	0.01573	0.99107	0.0	0.28123
800	5e-05	0.0	0.18749	0.9871	0.0	1.04712	5e-05	0.0	0.01564	0.9871	0.0	**0.28133**
1000	4e-05	0.0	0.28124	0.97674	0.0	1.34379	5e-05	0.0	0.03123	0.97674	0.0	**0.29684**
1200	3e-05	0.0	0.35935	0.9883	0.0	1.59387	3e-05	0.0	0.03125	0.9883	0.0	**0.34387**
1400	3e-05	0.0	0.5001	0.86325	0.0	2.047	3e-05	0.0	0.03124	0.86325	0.0	**0.62497**
1600	3e-05	0.0	0.60936	0.81818	0.0	1.50018	3e-05	0.0	0.04687	0.81818	0.0	**0.30655**
1800	2e-05	0.0	0.78119	0.99488	0.0	5.51605	3e-05	0.0	0.03125	0.99488	0.0	**0.40622**
2000	2e-05	0.0	0.98444	0.98851	0.0	0.51562	3e-05	0.0	0.04685	0.98851	0.0	**0.48433**

Table B.14: PP-K-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.96703	0.0	1.32824	0.00022	0.0	0.01562	0.96703	0.0	**1.14602**
400	0.0001	0.0	0.07812	0.9	0.0	**0.23436**	0.00011	0.0	0.01562	0.9	0.0	0.32809
600	7e-05	0.0	0.10937	0.89474	0.0	**0.14062**	6e-05	0.0	0.03125	0.89474	0.0	0.20329
800	5e-05	0.0	0.20311	0.83333	0.0	**0.23434**	5e-05	0.0	0.03124	0.83333	0.0	0.24996
1000	4e-05	0.0	0.29686	0.93333	0.0	0.26561	5e-05	0.0	0.03123	0.93333	0.0	**0.18748**
1200	3e-05	0.0	0.39062	0.98174	9e-05	3.90743	3e-05	0.0	0.03124	0.98174	0.0	**2.08837**
1400	3e-05	0.0	0.50016	0.92857	0.0	4.4847	3e-05	0.0	0.03123	0.92857	0.0	**0.5156**
1600	3e-05	0.0	0.65632	0.98947	5e-05	2.48465	3e-05	0.0	0.04686	0.98947	0.0001	**1.2112**
1800	2e-05	0.0	0.82822	0.95238	0.0	2.35948	3e-05	0.0	0.04686	0.95238	0.0	**0.7657**
2000	2e-05	0.0	1.01569	0.9952	5e-05	3.03243	3e-05	0.0	0.04688	0.9952	0.0	**0.83036**

Table B.15: PP-K-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.03	0.0	**0.09374**	0.00022	0.0	0.01562	0.03	0.0	0.09375
400	0.0001	0.0	0.0625	0.01508	0.0	**0.12499**	0.00011	0.0	0.01562	0.01508	0.0	0.15624
600	7e-05	0.0	0.10935	0.0121	0.0	0.25017	9e-05	0.0	0.01562	0.0121	0.0	**0.21889**
800	5e-05	0.0	0.18747	0.01176	0.0	0.34373	5e-05	0.0	0.01562	0.01176	0.0	**0.17184**
1000	4e-05	0.0	0.26559	0.01176	0.0	0.46885	5e-05	0.0	0.01562	0.01176	0.0	**0.21872**
1200	3e-05	0.0	0.35947	0.01111	0.0	0.75008	3e-05	0.0	0.03125	0.01111	0.0	**0.29698**
1400	3e-05	0.0	0.46874	0.01064	0.0	2.09458	3e-05	0.0	0.03125	0.01064	0.0	**0.21876**
1600	2e-05	0.0	0.59382	0.01121	0.0	2.4199	3e-05	0.0	0.03125	0.01121	0.0	**0.28121**
1800	2e-05	0.0	0.75008	0.0105	0.0	3.28488	3e-05	0.0	0.03123	0.0105	0.0	**0.14061**
2000	2e-05	0.0	0.93762	0.01053	0.0	3.86108	3e-05	0.0	0.03124	0.01053	0.0	**0.18747**

Table B.16: PP-K-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.03518	0.0	**0.04685**	0.00022	0.0	0.01562	0.03518	0.0	0.10936
400	0.0001	0.0	0.0625	0.01508	0.0	**0.12499**	0.00011	0.0	0.01562	0.01508	0.0	0.14062
600	7e-05	0.0	0.10937	0.01508	0.0	0.23437	6e-05	0.0	0.03125	0.01508	0.0	**0.20324**
800	5e-05	0.0	0.18749	0.01153	0.0	0.35936	5e-05	0.0	0.01561	0.01153	0.0	**0.3437**
1000	4e-05	0.0	0.26561	0.0121	0.0	0.43748	5e-05	0.0	0.03123	0.0121	0.0	**0.24998**
1200	3e-05	0.0	0.37507	0.01153	0.0	0.40635	3e-05	0.0	0.03123	0.01153	0.0	**0.28134**
1400	3e-05	0.0	0.46871	0.01111	0.0	2.68759	3e-05	0.0	0.03125	0.01111	0.0	**0.31247**
1600	3e-05	0.0	0.60949	0.0121	0.0	2.54698	3e-05	0.0	0.03125	0.0121	0.0	**0.26573**
1800	2e-05	0.0	0.77905	0.01093	0.0	1.31254	3e-05	0.0	0.03124	0.01093	0.0	**0.31249**
2000	2e-05	0.0	0.95651	0.01069	0.0	1.90641	3e-05	0.0	0.03124	0.01069	0.0	**0.21884**

Table B.17: PP-K-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	0.0002	0.0	0.03125	0.02041	0.0	**0.07824**	0.00022	0.0	0.03125	0.02041	0.0	0.09373
400	0.0001	0.0	0.0625	0.01269	0.0	**0.10937**	0.00011	0.0	0.03125	0.01269	0.0	0.18748
600	7e-05	0.0	0.12499	0.01508	0.0	0.29686	9e-05	0.0	0.03125	0.01508	0.0	**0.20311**
800	5e-05	0.0	0.20358	0.01176	0.0	0.32829	6e-05	0.0	0.03125	0.01176	0.0	**0.2031**
1000	4e-05	0.0	0.28123	0.0121	0.0	0.55682	5e-05	0.0	0.03124	0.0121	0.0	**0.38249**
1200	3e-05	0.0	0.3751	0.01153	0.0	0.56247	3e-05	0.0	0.04687	0.01153	0.0	**0.51544**
1400	3e-05	0.0	0.5156	0.01121	0.0	0.92184	3e-05	0.0	0.04699	0.01121	0.0	**0.64056**
1600	3e-05	0.0	0.64071	0.01089	0.0	2.90648	3e-05	0.0	0.04687	0.01089	0.0	**0.50012**
1800	2e-05	0.0	0.81263	0.01089	0.0	4.28156	3e-05	0.0	0.04084	0.01089	0.0	**0.65635**
2000	2e-05	0.0	0.98443	0.01066	0.0	2.21887	3e-05	0.0	0.04687	0.01066	0.0	**0.71869**

Table B.18: PP-K-N-5

B.1.3 Positive numerators and denominators with assignment constraints

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.49729	0.0	**0.11494**	0.49729	0.0	2.48705	0.49729	0.0	0.60928	0.49729	0.0	0.53138
20	0.41348	0.0	**0.56229**	0.41348	0.0	0.93744	0.41348	0.0	0.86315	0.41348	0.0	0.76571
30	0.25198	0.0	3.03141	0.25198	0.0	3.56281	0.25198	0.0	1.28135	0.25198	0.0	**1.10944**
40	0.22247	0.0	8.64093	0.22247	0.0	9.69053	0.22247	0.0	2.14078	0.22247	0.0	**1.97265**
50	0.22975	0.0	19.95818	0.22975	0.0	21.73422	0.22975	0.0	2.12548	0.22975	0.0	**2.04703**
60	0.23251	0.0	55.49156	0.23251	0.0	69.51491	0.23251	0.0	3.01587	0.23251	0.0	**2.79808**
70	0.17565	0.0	126.47973	0.17565	0.0	175.50801	0.17565	0.0	4.04719	0.17565	0.0	**4.00031**
80	0.18306	0.0	205.49144	0.18306	0.0	321.24519	0.18306	0.0	4.6881	0.18306	0.0	**4.54742**
90	0.16718	7e-05	610.63039	0.16718	7e-05	750.40657	0.16743	2e-05	**6.16193**	0.16743	2e-05	6.20081
100	0.18954	0.98364	3600.00441	0.18954	0.98364	3600.00229	0.15939	0.0	6.84463	0.15939	0.0	**6.80058**

Table B.19: PP-A-R-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.5536	0.0	**0.48446**	0.5536	0.0	0.76567	0.5536	0.0	1.17202	0.5536	0.0	0.92195
20	0.47279	0.0	3.26689	0.47279	0.0	3.90992	0.47279	0.0	2.73818	0.47279	0.0	**2.62575**
30	0.40469	0.0	7.6725	0.40469	0.0	8.75085	0.40469	0.0	3.04727	0.40469	0.0	**2.841**
40	0.33746	0.0	12.57602	0.33746	0.0	13.81504	0.33746	0.0	5.48227	0.33746	0.0	**5.11298**
50	0.32419	0.0	33.27454	0.32419	0.0	35.13172	0.32419	0.0	11.62653	0.32419	0.0	**11.42452**
60	0.32097	0.0	101.71931	0.32097	0.0	109.09478	0.32097	0.0	13.21811	0.32097	0.0	**12.85662**
70	0.26983	0.0	244.87133	0.26983	0.0	261.43729	0.26983	1e-05	42.24631	0.26983	1e-05	**41.02788**
80	0.26772	0.0	657.30105	0.26772	0.0	761.65379	0.26772	0.0	11.84906	0.26772	0.0	**11.53681**
90	0.25848	0.0	1114.41415	0.25848	0.0	1213.44581	0.25898	0.0	21.51863	0.25898	0.0	**20.92289**
100	0.29494	0.98916	3600.00225	0.29494	0.98916	3600.01854	0.25609	0.0	117.3485	0.25609	0.0	**112.38099**

Table B.20: PP-A-R-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.67185	0.0	**1.98783**	0.67185	0.0	2.1417	0.67185	0.0	2.63862	0.67185	0.0	2.21876
20	0.62032	0.0	36.227	0.62032	0.0	41.90304	0.62032	0.0	26.5026	0.62032	0.0	**25.73559**
30	0.49378	0.0	53.36462	0.49378	0.0	58.89845	0.49378	0.0	**27.90501**	0.49378	0.0	28.2037
40	0.48606	0.0	111.60184	0.48606	0.0	120.04823	0.48606	0.0	54.59799	0.48606	0.0	**51.31367**
50	0.45655	0.0	274.12571	0.45655	0.0	284.66594	0.45655	0.0	93.59609	0.45655	0.0	**91.95592**
60	0.42031	0.0	501.21548	0.42031	0.0	512.96158	0.42031	0.0	**267.58441**	0.42031	0.0	267.80269
70	0.42397	0.0	**649.98936**	0.42397	0.0	678.03295	0.42397	0.0	703.92707	0.42397	0.0	698.03065
80	0.3994	0.0	2888.66134	0.3994	0.0	2872.99022	0.39979	0.0	**374.28359**	0.39979	0.0	376.69583
90	0.38986	0.0	3030.74855	0.38986	0.0	3094.84518	0.38986	0.0	**354.02**	0.38986	0.0	354.32023
100	0.46159	0.98988	3600.00812	0.46159	0.98988	3600.00964	0.38538	8e-05	**1405.78971**	0.38538	8e-05	1410.96325

Table B.21: PP-A-R-5

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.90458	0.0	0.15603	0.90458	0.0	**0.12499**	0.90458	7e-05	0.24999	0.90458	7e-05	0.25009
20	0.79947	1e-05	0.76585	0.79947	1e-05	0.78132	0.79947	0.0	**0.31246**	0.79947	0.0	0.3281
30	0.84772	0.0	3.4533	0.84772	0.0	3.31428	0.84772	0.0	**0.39063**	0.84772	0.0	0.39107
40	0.87535	9e-05	12.19602	0.87535	9e-05	11.91055	0.87535	0.0	**0.76558**	0.87535	0.0	0.81241
50	0.88396	7e-05	34.8839	0.88396	7e-05	34.6296	0.88396	0.0	1.07783	0.88396	0.0	**0.95287**
60	0.86872	9e-05	76.62814	0.86872	9e-05	77.3666	0.86872	0.0	1.10927	0.86872	0.0	**1.03112**
70	0.89126	9e-05	144.93636	0.89126	9e-05	146.13111	0.89126	0.0	**1.5625**	0.89126	0.0	1.59348
80	0.8881	8e-05	317.76134	0.8881	8e-05	325.84155	0.8881	2e-05	**4.74969**	0.8881	2e-05	4.82818

Table B.22: PP-A-P-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.91871	0.0	**0.09373**	0.91871	0.0	0.10947	0.91871	0.0	0.23437	0.91871	0.0	0.23436
20	0.85925	0.0	1.43741	0.85925	0.0	1.37506	0.85925	0.0	1.3909	0.85925	0.0	**1.35949**
30	0.87087	0.0	5.28129	0.87087	0.0	5.67338	0.87087	0.0	0.90625	0.87087	0.0	**0.82815**
40	0.89575	0.0	20.17195	0.89575	0.0	20.05313	0.89575	0.0	4.35979	0.89575	0.0	**4.1405**
50	0.90247	5e-05	35.17202	0.90247	5e-05	34.86328	0.90247	0.0	1.49977	0.90247	0.0	**1.43767**
60	0.89603	0.0001	106.19762	0.89603	0.0001	107.8144	0.89603	3e-05	5.54684	0.89603	3e-05	**5.27265**
70	0.91102	0.0001	201.03017	0.91102	0.0001	206.38809	0.91102	0.0001	3.67183	0.91102	0.0001	**3.64075**
80	0.90333	9e-05	375.50339	0.90333	9e-05	385.6348	0.90333	9e-05	**4.56301**	0.90333	9e-05	4.5939

Table B.23: PP-A-P-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.89126	0.0	**0.50004**	0.89126	0.0	0.56269	0.89126	0.0	0.59372	0.89126	0.0	0.62509
20	0.88359	0.0	6.71926	0.88359	0.0	**6.19269**	0.88359	0.0	11.23021	0.88359	0.0	11.09945
30	0.90762	0.0	36.10197	0.90762	0.0	35.82763	0.90762	0.0	**5.42314**	0.90762	0.0	5.45352
40	0.90456	9e-05	55.42901	0.90456	9e-05	54.82398	0.90456	0.0	8.32718	0.90456	0.0	**8.31144**
50	0.91849	9e-05	89.47576	0.91849	9e-05	89.30244	0.91849	0.0	**12.43012**	0.91849	0.0	12.45742
60	0.92573	0.0001	664.94138	0.92573	0.0001	656.54227	0.92573	0.0001	**38.51207**	0.92573	0.0001	38.82798
70	0.92859	0.0001	368.45614	0.92859	0.0001	395.46978	0.92859	8e-05	14.20018	0.92859	8e-05	**14.11959**
80	0.93166	0.0001	1244.71823	0.93166	0.0001	1243.29053	0.93166	9e-05	30.86147	0.93166	9e-05	**30.60484**

Table B.24: PP-A-P-5

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.94778	0.0	0.28133	0.94778	0.0	**0.28123**	0.94778	0.0	0.38834	0.94778	0.0	0.3906
20	0.82283	7e-05	1.46745	0.82283	7e-05	1.31255	0.82283	9e-05	**1.12493**	0.82283	9e-05	1.16987
30	0.89457	0.0001	4.53106	0.89457	0.0001	4.50949	0.89457	9e-05	0.72014	0.89457	9e-05	**0.70307**
40	0.86137	0.0001	13.06267	0.86137	0.0001	12.87685	0.86137	0.0	1.95319	0.86137	0.0	**1.87506**
50	0.89718	8e-05	45.88402	0.89718	8e-05	45.69266	0.89718	0.0	1.45554	0.89718	0.0	**1.43764**
60	0.88854	7e-05	87.46415	0.88854	7e-05	87.3749	0.88854	0.0001	**2.18589**	0.88854	0.0001	2.2657
70	0.89416	0.0001	214.07421	0.89416	0.0001	215.76727	0.89416	1e-05	**2.0938**	0.89416	1e-05	2.12523
80	0.89107	7e-05	683.23057	0.89107	7e-05	681.74628	0.89107	3e-05	**4.37381**	0.89107	3e-05	4.43772

Table B.25: PP-A-N-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.85243	0.0	**0.54695**	0.85243	0.0	0.59372	0.85243	0.0	0.57811	0.85243	0.0	0.56261
20	0.80349	7e-05	2.28136	0.80349	7e-05	2.25471	0.80349	0.0	1.31275	0.80349	0.0	**1.21884**
30	0.88077	0.0001	3.95755	0.88077	0.0001	4.09522	0.88077	0.0	1.28134	0.88077	0.0	**1.16655**
40	0.87606	6e-05	17.04799	0.87606	6e-05	17.39296	0.87606	0.0	2.59409	0.87606	0.0	**2.47503**
50	0.87467	0.0	43.98729	0.87467	0.0	44.08938	0.87467	0.0	**2.81279**	0.87467	0.0	2.82923
60	0.90869	7e-05	128.72845	0.90869	7e-05	123.65446	0.90869	0.0	4.42234	0.90869	0.0	**4.3283**
70	0.90665	8e-05	332.12463	0.90665	8e-05	330.08577	0.90665	9e-05	**9.90714**	0.90665	9e-05	9.90719
80	0.89817	8e-05	529.95576	0.89817	8e-05	547.98453	0.89817	8e-05	**14.20473**	0.89817	8e-05	14.36602

Table B.26: PP-A-N-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	0.92383	0.0	0.76573	0.92383	0.0	**0.7657**	0.92383	0.0	0.85932	0.92383	0.0	1.01556
20	0.89215	0.0	6.87303	0.89215	0.0	6.45375	0.89215	0.0	**4.92557**	0.89215	0.0	5.01069
30	0.91925	0.0	6.17335	0.91925	0.0	6.0005	0.91925	0.0	1.62512	0.91925	0.0	**1.60942**
40	0.91302	0.0	**105.52942**	0.91302	0.0	109.75483	0.91302	3e-05	135.8912	0.91302	3e-05	134.6243
50	0.9288	0.0	74.10362	0.9288	0.0	76.40065	0.9288	8e-05	**13.50264**	0.9288	8e-05	13.6608
60	0.91747	0.0001	206.51218	0.91747	0.0001	209.0129	0.91747	5e-05	16.0512	0.91747	5e-05	**15.59102**
70	0.9155	0.0001	815.9926	0.9155	0.0001	832.94981	0.9155	0.0001	48.58167	0.9155	0.0001	**48.19758**
80	0.92869	0.0001	2504.13467	0.92869	0.0001	2503.27823	0.92869	0.0001	90.46728	0.92869	0.0001	**89.61439**

Table B.27: PP-A-N-5

B.1.4 Unrestricted numerators and positive denominators with no constraint

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-1.72973	0.0	0.53133	-1.72973	0.0	**0.18745**	-1.72973	0.0	0.56258	-1.72973	0.0	0.34383
400	-3.03015	0.0	1.21848	-3.03015	0.0	**0.42187**	-3.03015	0.0	0.64058	-3.03015	0.0	0.48433
600	-2.79322	0.0	2.57801	-2.79322	0.0	1.93785	-2.79322	0.0	**0.45323**	-2.79322	0.0	0.60946
800	-3.96491	0.0	5.59345	-3.96491	0.0	1.12488	-3.96491	0.0	0.77833	-3.96491	0.0	**0.40622**
1000	-5.29762	0.0	2.11171	-5.29762	0.0	7.28185	-5.29762	0.0	0.37498	-5.29762	0.0	**0.25011**
1200	-3.61301	0.0	2.40633	-3.61301	0.0	3.28127	-3.61301	1e-05	**0.34373**	-3.61301	0.0	2.37308
1400	-4.53659	0.0	3.12509	-4.53659	0.0	3.37585	-4.53659	0.0	**0.31248**	-4.53659	0.0	0.58649
1600	-4.13063	0.0	3.7657	-4.13063	0.0	9.29705	-4.13063	0.0	**0.45308**	-4.13063	0.0	0.79681
1800	-4.31004	0.0	8.40634	-4.31004	0.0	11.39659	-4.31004	0.0	**0.42201**	-4.31004	0.0	1.17194
2000	-3.96013	0.0	15.57814	-3.96013	0.0	12.12571	-3.96013	0.0	0.56247	-3.96013	0.0	**0.50009**

Table B.28: NP-U-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-1.24838	0.0	0.85945	-1.24838	0.0	**0.40623**	-1.24838	0.0	1.87501	-1.24838	0.0	3.26796
400	-1.74627	0.0	1.45315	-1.74627	0.0	**0.52883**	-1.74627	0.0	2.5482	-1.74627	0.0	1.05287
600	-1.82022	0.0	3.09393	-1.82022	0.0	3.23454	-1.82023	0.0	1.69747	-1.82022	0.0	**1.50002**
800	-2.07246	0.0	4.03152	-2.07246	0.0	3.39079	-2.07246	0.0	**1.63095**	-2.07246	0.0	1.94792
1000	-3.09009	0.0	12.03453	-3.09009	0.0	2.92214	-3.09009	0.0	**0.51571**	-3.09009	0.0	0.57808
1200	-2.36	0.0	7.88194	-2.36	0.0	7.09419	-2.36	0.0	2.35946	-2.36	0.0	**2.06269**
1400	-2.45238	0.0	5.04714	-2.45238	0.0	8.76436	-2.45238	0.0	**1.68763**	-2.45238	0.0	4.31438
1600	-2.50185	0.0	34.20547	-2.50185	0.0	10.64242	-2.50185	0.0	2.43887	-2.50185	0.0	**2.35945**
1800	-2.62009	0.0	19.1097	-2.62009	0.0	13.01145	-2.62009	0.0	**2.27423**	-2.62009	0.0	5.09439
2000	-2.82468	0.0	8.61139	-2.82468	0.0	14.5166	-2.82468	0.0	8.20769	-2.82468	0.0	**2.95261**

Table B.29: NP-U-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.96124	0.0	1.096	-0.96124	0.0	**0.48433**	-0.96124	0.0001	24.17887	-0.96124	9e-05	26.97143
400	-1.024	0.0	2.56275	-1.024	0.0	**1.57825**	-1.024	0.0	29.22811	-1.024	0.0	38.71036
600	-1.299	0.0	**3.67208**	-1.299	0.0	3.87523	-1.299	0.0	26.70737	-1.299	0.0	32.6277
800	-1.54452	0.0	14.1267	-1.54452	0.0	**4.94009**	-1.54452	0.0	39.09338	-1.54452	0.0	63.71294
1000	-1.30653	0.0	17.66281	-1.30653	0.0	**5.17215**	-1.30653	0.0001	42.32833	-1.30653	9e-05	58.01286
1200	-1.42007	0.0	10.84655	-1.42007	0.0	**8.75087**	-1.42007	8e-05	26.49112	-1.42007	6e-05	42.59148
1400	-1.58503	0.0	**5.42252**	-1.58503	0.0	11.07895	-1.58503	0.0	56.8474	-1.58503	0.0	55.99419
1600	-1.38613	0.0	48.62629	-1.38613	0.0	**12.65615**	-1.38613	0.0	99.4728	-1.38613	0.0	181.75936
1800	-1.39296	0.0	32.94801	-1.39296	0.0	**13.7546**	-1.39296	0.0001	73.89872	-1.39296	0.0001	110.76994
2000	-1.71072	0.0	48.79516	-1.71072	0.0	36.93701	-1.71072	1e-05	**35.75921**	-1.71072	8e-05	96.08897

Table B.30: NP-U-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-33.0	0.0	0.87482	-33.0	0.0	0.14062	-33.0	0.0	**0.04687**	-33.0	0.0	0.06248
400	-49.75	0.0	2.2031	-49.75	0.0	0.39072	-49.75	0.0	0.04686	-49.75	0.0	**0.03125**
600	-82.83333	0.0	3.57784	-82.83333	0.0	0.68758	-82.83333	0.0	**0.04687**	-82.83333	0.0	0.07812
800	-84.71429	0.0	2.15586	-84.71429	0.0	1.27917	-84.71429	0.0	**0.06248**	-84.71429	0.0	0.0781
1000	-88.33333	0.0	8.59381	-88.33333	0.0	1.93766	-88.33333	0.0	**0.0625**	-88.33333	0.0	0.07812
1200	-74.5	0.0	15.06267	-74.5	0.0	2.21883	-74.5	0.0	**0.0781**	-74.5	0.0	0.07811
1400	-89.6	0.0	6.76559	-89.6	0.0	2.73455	-89.6	0.0	0.10949	-89.6	0.0	**0.06248**
1600	-92.35715	0.0	9.59977	-92.35714	0.0	3.23455	-92.35714	0.0	0.07812	-92.35714	0.0	**0.0625**
1800	-93.72222	0.0	45.08106	-93.72222	0.0	4.469	-93.72222	0.0	0.10937	-93.72222	0.0	**0.08819**
2000	-94.0	0.0	10.86007	-94.0	0.0	4.18771	-94.0	0.0	0.09374	-94.0	0.0	**0.07808**

Table B.31: NP-U-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-49.25	0.0	0.15624	-49.25	0.0	0.12499	-49.25	0.0	**0.03124**	-49.25	0.0	0.03125
400	-66.33333	0.0	2.48453	-66.33333	0.0	0.29686	-66.33333	0.0	**0.0625**	-66.33333	0.0	0.09372
600	-79.8	0.0	3.81263	-79.8	0.0	0.50008	-79.8	0.0	**0.0781**	-79.8	0.0	0.09374
800	-74.5	0.0	2.32832	-74.5	0.0	0.89069	-74.5	0.0	**0.0781**	-74.5	0.0	0.10938
1000	-85.0	0.0	10.12558	-85.0	0.0	1.48437	-85.0	0.0	0.07812	-85.0	0.0	**0.06248**
1200	-92.25	0.0	17.89372	-92.25	3e-05	2.50669	-92.25	0.0	0.09385	-92.25	0.0	**0.09373**
1400	-90.18184	0.0	8.04849	-90.18182	0.0	2.68771	-35781.45477	0.0	0.09373	-90.18182	0.0	**0.10936**
1600	-92.26667	0.0	34.86891	-92.26667	0.0	3.52506	-92.26667	0.0	0.10937	-92.26667	0.0	**0.10936**
1800	-92.73333	0.0	37.74886	-92.73333	0.0	4.03145	-92.73333	0.0	0.10937	-92.73333	0.0	**0.10935**
2000	-94.0	0.0	46.10933	-94.0	0.0	5.75041	-94.0	0.0	0.14183	-94.0	0.0	**0.14061**

Table B.32: NP-U-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-41.57143	0.0	0.79683	-41.57143	0.0	0.14062	-41.57143	0.0	0.07812	-41.57143	0.0	**0.0625**
400	-49.75	0.0	2.40648	-49.75	0.0	0.32811	-49.75	0.0	**0.0781**	-49.75	0.0	0.09373
600	-74.5	0.0	0.76571	-74.5	0.0	0.62875	-74.5	0.0	**0.09374**	-74.5	0.0	0.12498
800	-74.75	0.0	9.78237	-74.75	0.0	0.8595	-74.75	0.0	**0.10937**	-74.75	0.0	0.12498
1000	-83.0	0.0	2.05157	-83.0	0.0	1.10943	-83.0	0.0	**0.15638**	-83.0	0.0	0.17186
1200	-89.3	0.0	16.84491	-89.3	0.0	1.82822	-89.3	0.0	**0.14062**	-89.3	0.0	0.15621
1400	-90.09091	0.0	5.29927	-90.09091	0.0	1.95321	-90.09091	0.0	0.23438	-90.09091	0.0	**0.18747**
1600	-92.86667	0.0	27.23675	-92.86667	0.0	3.75179	-92.86667	0.0	0.18748	-92.86667	0.0	**0.18745**
1800	-92.9375	0.0	46.57008	-92.9375	0.0	4.78155	-92.9375	0.0	**0.18747**	-92.9375	0.0	0.18762
2000	-92.9375	0.0	51.06255	-92.9375	0.0	4.75035	-92.9375	0.0	0.24997	-92.9375	0.0	**0.17185**

Table B.33: NP-U-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.97973	0.0	0.12498	-0.97973	0.0	0.1873	-0.97973	0.0	**0.07812**	-0.97973	0.0	0.09395
400	-0.99935	0.0	2.2031	-0.99831	0.0	0.32797	-0.99831	0.0	**0.0625**	-0.99831	0.0	0.06264
600	-0.99497	0.0	0.65626	-0.99497	0.0	0.76564	-0.99497	0.0	**0.06248**	-0.99497	0.0	0.07791
800	-1.00504	0.0	7.75011	-1.00504	0.0	3.57805	-1.00504	0.0	**0.0781**	-1.00504	0.0	0.09376
1000	-1.01003	0.0	7.65739	-1.00506	1e-05	1.29685	-1.00505	4e-05	**0.07839**	-1.00506	0.0	0.15625
1200	-1.00504	0.0	1.84383	-1.00504	0.0	1.5624	-1.00504	0.0	**0.125**	-1.00504	0.0	0.17186
1400	-1.00672	0.0	19.31263	-1.00672	0.0	2.71852	-1.00672	0.0	**0.12503**	-1.00672	0.0	0.16639
1600	-1.34116	0.0	25.39087	-1.0072	0.0	3.35939	-1.0072	0.0	0.12521	-1.0072	0.0	**0.125**
1800	-1.00672	0.0	4.28131	-1.00672	0.0	3.76564	-1.00672	0.0	**0.10938**	-1.00672	0.0	0.12489
2000	-1.00504	0.0	5.48464	-1.00504	0.0	6.43737	-1.00504	0.0	**0.12535**	-1.00504	0.0	0.14068

Table B.34: NP-U-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.98985	0.0	0.57803	-0.98985	0.0	0.18759	-0.98985	0.0	0.09374	-0.98985	0.0	**0.06249**
400	-1.0	0.0	0.35903	-1.0	0.0	0.34371	-1.0	0.0	**0.07812**	-1.0	0.0	0.09374
600	-0.99995	0.0	4.7654	-0.99664	0.0	0.60943	-0.99737	0.0	0.0781	-0.99665	0.0	**0.12498**
800	-1.00336	0.0	8.18719	-1.00336	0.0	0.94403	-1.00336	0.0	**0.09374**	-1.00336	0.0	0.14061
1000	-1.0057	0.0	9.59437	-1.00336	0.0	1.3751	-1.00336	9e-05	**0.12498**	-1.00336	0.0	0.2031
1200	-1.00504	5e-05	1.87493	-1.00504	0.0	1.73682	-1.00504	0.0	0.32819	-1.00504	5e-05	**0.21873**
1400	-1.01501	0.0	18.79605	-1.00504	0.0	2.3908	-1.00504	0.0	**0.18746**	-1.00504	0.0	0.18758
1600	-1.00504	0.0	22.62465	-1.00504	1e-05	3.7126	-1.00504	4e-05	0.34371	-1.00504	7e-05	**0.23433**
1800	-1.00826	0.0	3.98782	-1.00826	0.0	3.96227	-1.00826	0.0	**0.25023**	-1.00826	0.0	0.26558
2000	-1.01005	0.0	40.75447	-1.00807	0.0	4.64069	-1.00807	0.0	**0.18748**	-1.00807	5e-05	0.85069

Table B.35: NP-U-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.98995	0.0	0.65607	-0.98995	0.0	0.17181	-0.98995	0.0	0.172	-0.98995	0.0	**0.10937**
400	-0.98995	0.0	0.4531	-0.99482	0.0	0.39061	-0.98995	0.0	**0.14063**	-0.98995	0.0	0.15602
600	-1.0	0.0	4.20279	-1.00005	0.0	0.59365	-1.0	0.0	0.15646	-1.0	0.0	**0.15625**
800	-1.00336	0.0	6.78148	-1.00336	0.0	0.96886	-1.00336	0.0	**0.14083**	-1.00336	0.0	0.23406
1000	-1.01661	0.0	11.1875	-1.0	0.0	1.60942	-1.0	0.0	0.28122	-1.0	0.0	**0.23453**
1200	-1.00336	0.0	13.57825	-1.00336	0.0	1.9688	-1.00336	0.0	**0.21835**	-1.00336	0.0	0.26559
1400	-1.00336	0.0	3.37489	-1.00336	0.0	7.53149	-1.00336	0.0	0.3907	-1.00336	0.0	**0.2812**
1600	-1.00605	0.0	25.48471	-1.00605	0.0	3.42207	-1.00605	0.0	**0.24997**	-1.00605	0.0	0.31264
1800	-1.00672	0.0	3.7813	-1.00672	0.0	4.59354	-1.00672	0.0	**0.53118**	-1.00672	0.0	2.28133
2000	-1.00672	0.0	39.39153	-1.00672	0.0	12.64082	-1.00672	0.0	**0.28135**	-1.00672	0.0	0.32809

Table B.36: NP-U-N-5

B.1.5 Unrestricted numerators and positive denominators with knapsack constraints

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-2.39303	0.0	0.46854	-2.39303	0.0	0.23441	-2.39303	0.0	0.54684	-2.39303	0.0	**0.18749**
400	-2.86364	0.0	1.26568	-2.86364	2e-05	0.3753	-2.86364	0.0	0.23436	-2.86364	0.0	**0.10937**
600	-2.99585	0.0	3.42172	-2.99585	0.0	1.6252	-2.99585	0.0	**0.64072**	-2.99585	0.0	1.09947
800	-3.78986	0.0	1.54694	-3.78986	0.0	1.03103	-3.78986	0.0	0.39075	-3.78986	0.0	**0.31247**
1000	-4.25568	0.0	2.4845	-4.25568	0.0	4.63401	-4.25568	0.0	**0.31247**	-4.25568	0.0	0.90489
1200	-4.64122	0.0	3.92151	-4.64122	9e-05	2.92215	-4.64122	0.0	**0.34373**	-4.64122	0.0	0.40621
1400	-4.64336	5e-05	3.28102	-4.64336	0.0	2.01554	-4.64336	0.0	**0.28136**	-4.64336	0.0	1.57832
1600	-4.35669	0.0	5.70331	-4.35669	0.0	4.28568	-4.35669	0.0	**0.46871**	-4.35669	0.0	1.37501
1800	-4.32727	0.0	6.23461	-4.32727	0.0	7.6045	-4.32727	0.0	**0.39072**	-4.32727	0.0	0.42197
2000	-5.67403	0.0	9.95327	-5.67403	0.0	9.22639	-5.67403	0.0	0.31247	-5.67403	0.0	**0.26558**

Table B.37: NP-K-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-1.23374	0.0	0.75076	-1.23374	4e-05	**0.42185**	-1.23374	0.0	1.14729	-1.23374	0.0	1.30341
400	-1.34204	0.0	1.89067	-1.34204	0.0	1.3159	-1.34204	0.0	1.50293	-1.34204	0.0	**1.0626**
600	-2.19444	0.0	**0.80655**	-2.19444	0.0	1.28142	-2.19444	0.0	1.99163	-2.19444	0.0	1.66793
800	-2.58173	0.0	5.38124	-2.58173	0.0	1.81262	-2.58173	0.0	**0.89088**	-2.58173	5e-05	1.40369
1000	-2.50355	0.0	8.65677	-2.50355	0.0	4.40649	-2.50355	0.0	**0.59372**	-2.50355	0.0	1.25352
1200	-2.13415	0.0	16.99916	-2.13415	0.0	4.40692	-2.13415	0.0	**1.85661**	-2.13415	0.0	1.95318
1400	-2.64228	0.0	3.98994	-2.64228	0.0	2.46885	-2.64228	0.0	3.70661	-2.64228	0.0	**1.81252**
1600	-2.85542	0.0	18.28461	-2.85542	0.0	16.25442	-2.85542	0.0	4.69064	-2.85542	0.0	**2.20336**
1800	-2.85978	0.0	6.81374	-2.85978	0.0	14.42178	-2.85978	0.0	2.16178	-2.85978	5e-05	**1.92102**
2000	-2.32509	0.0	8.36033	-2.32509	0.0	10.40278	-2.32509	0.0	2.23442	-2.32509	0.0	**2.12727**

Table B.38: NP-K-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-1.12121	0.0	1.07823	-1.12121	0.0	**0.68541**	-1.12121	9e-05	24.57309	-1.12121	0.0	30.99156
400	-1.21196	0.0	5.51611	-1.21196	0.0	**3.06275**	-1.21196	0.0	38.11552	-1.21196	0.0	19.82621
600	-1.43636	0.0	7.67033	-1.43636	0.0	**4.54723**	-1.43636	7e-05	43.9865	-1.43636	1e-05	26.41365
800	-1.34853	0.0	8.75251	-1.34853	0.0	**7.82922**	-1.34853	0.0	45.46991	-1.34853	0.0	32.04828
1000	-1.38696	0.0	18.84903	-1.38696	0.0	**8.48277**	-1.38696	0.0001	54.18291	-1.38696	0.0	33.9997
1200	-1.74016	0.0	15.97358	-1.74016	0.0	**12.4446**	-1.74016	0.0	40.63054	-1.74016	0.0	39.79319
1400	-1.66422	0.0	**4.7378**	-1.66422	0.0	18.17317	-1.66422	9e-05	67.45115	-1.66422	1e-05	34.98367
1600	-1.55263	0.0	**10.26339**	-1.55263	0.0	26.81938	-1.55263	0.0	21.94715	-1.55263	0.0	78.0833
1800	-1.69072	0.0	43.28977	-1.69072	0.0	**34.58394**	-1.69072	5e-05	60.36028	-1.69072	8e-05	43.96473
2000	-1.78067	0.0	**7.52545**	-1.78067	0.0	41.79992	-1.78067	2e-05	60.531	-1.78067	6e-05	52.87587

Table B.39: NP-K-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-73.25	0.0	0.10937	-73.25	0.0	0.10914	-73.25	0.0	**0.03125**	-73.25	0.0	0.047
400	-79.0	0.0	0.40627	-79.0	0.0	0.21874	-79.0	0.0	0.04687	-79.0	0.0	**0.03125**
600	-79.0	0.0	1.2969	-79.0	0.0	0.37459	-79.0	0.0	0.12499	-79.0	0.0	**0.04687**
800	-82.16667	0.0	1.82902	-82.16667	0.0	0.67387	-82.16667	0.0	0.09376	-82.16667	0.0	**0.07812**
1000	-89.3	0.0	3.56267	-89.3	0.0	0.89209	-89.3	0.0	0.07813	-89.3	0.0	**0.06248**
1200	-89.90909	0.0	1.78125	-89.90909	0.0	1.35939	-89.90909	0.0	**0.07812**	-89.90909	0.0	0.09374
1400	-92.86667	0.0	2.26561	-92.86667	0.0	2.01561	-92.86667	0.0	0.14062	-92.86667	0.0	**0.07811**
1600	-93.63158	0.0	6.98394	-93.63158	0.0	2.46881	-93.63158	0.0	0.10937	-93.63158	0.0	**0.0782**
1800	-94.10526	0.0	6.18826	-94.10526	0.0	3.14171	-94.10526	0.0	0.14062	-94.10526	0.0	**0.10935**
2000	-93.06667	0.0	8.40577	-93.06667	0.0	9.84956	-93.06667	0.0	0.10937	-93.06667	0.0	**0.10935**

Table B.40: NP-K-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-33.16667	0.0	0.15624	-33.16667	0.0	0.12512	-33.16667	0.0	**0.03125**	-33.16667	0.0	0.0625
400	-65.66667	0.0	0.29686	-65.66667	0.0	0.24999	-65.66667	0.0	**0.06248**	-65.66667	0.0	0.07812
600	-79.2	0.0	0.64068	-79.2	0.0	0.3906	-79.2	0.0	**0.0781**	-79.2	0.0	0.07812
800	-66.33333	0.0	0.93757	-66.33333	0.0	0.70308	-66.33333	0.0	**0.09373**	-66.33333	0.0	**0.09373**
1000	-84.71429	0.0	3.12642	-84.71429	0.0	0.95317	-84.71429	0.0	**0.07812**	-84.71429	0.0	0.10939
1200	-90.75	0.0	4.07837	-90.75	0.0	1.40639	-90.75	0.0	**0.10935**	-90.75	0.0	0.10947
1400	-91.0	0.0	5.74629	-91.0	0.0	2.28133	-91.0	0.0	0.15624	-91.0	0.0	**0.14058**
1600	-88.55556	0.0	4.70339	-88.55556	0.0	2.49762	-88.55556	0.0	**0.1406**	-88.55556	0.0	**0.1406**
1800	-93.29412	0.0	7.4691	-93.29412	0.0	2.92199	-93.29412	0.0	0.14064	-93.29412	0.0	**0.14059**
2000	-93.17647	0.0	8.82857	-93.17647	0.0	3.48457	-93.17647	0.0	0.18759	-93.17647	0.0	**0.17198**

Table B.41: NP-K-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-48.5	0.0	0.21873	-48.5	0.0	0.12515	-48.5	0.0	**0.03123**	-48.5	0.0	0.04687
400	-49.5	0.0	0.57821	-49.5	0.0	0.24999	-49.5	0.0	**0.04686**	-49.5	0.0	0.0625
600	-66.0	0.0	0.56258	-66.0	0.0	0.37496	-66.0	0.0	0.09386	-66.0	0.0	**0.09373**
800	-85.14286	0.0	1.12704	-85.14286	0.0	0.68748	-85.14286	0.0	**0.1406**	-85.14286	0.0	0.14062
1000	-87.77778	0.0	3.06271	-87.77778	0.0	0.9063	-87.77778	0.0	0.15624	-87.77778	0.0	**0.15623**
1200	-88.00001	0.0	2.26572	-88.0	0.0	1.40626	-88.0	0.0	**0.14062**	-88.0	0.0	0.18748
1400	-91.69231	0.0	6.15768	-91.69231	0.0	2.31262	-91.69231	0.0	0.20311	-91.69231	0.0	**0.17058**
1600	-93.44444	0.0	6.07844	-93.44444	0.0	2.28133	-93.44444	0.0	**0.20309**	-93.44444	0.0	0.24979
1800	-92.66667	0.0	5.58829	-92.66667	0.0	2.56276	-92.66667	0.0	**0.23435**	-92.66667	0.0	0.24974
2000	-90.45455	0.0	6.32269	-90.45455	0.0	4.0182	-90.63636	0.0	0.25009	-90.45455	0.0	**0.24995**

Table B.42: NP-K-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.98	0.0	0.17181	-0.98	0.0	0.10928	-0.98	0.0	**0.03125**	-0.98	0.0	0.04685
400	-1.00336	0.0	0.32802	-1.00336	0.0	0.29678	-1.00336	0.0	**0.06249**	-1.00336	0.0	0.09374
600	-0.9999	0.0	0.54676	-0.99797	0.0	0.6131	-0.99797	0.0	**0.09374**	-0.99797	0.0	0.12499
800	-1.00336	0.0	0.90625	-1.00336	0.0	1.00007	-1.00336	0.0	0.10935	-1.00336	0.0	**0.07823**
1000	-1.00605	0.0	1.40617	-1.00605	0.0	1.2969	-1.00605	0.0	0.10937	-1.00605	0.0	**0.09373**
1200	-1.00802	0.0	1.82852	-1.00504	0.0	1.46893	-1.00504	0.0	**0.12498**	-1.00504	0.0	0.23434
1400	-1.00722	0.0	2.15629	-1.00722	0.0	1.88125	-1.00722	0.0	**0.12497**	-1.00722	0.0	0.1406
1600	-1.00923	0.0	2.46835	-1.00672	0.0	2.23449	-1.00672	0.0	**0.15635**	-1.00672	0.0	0.23448
1800	-1.00672	0.0	2.99999	-1.00672	0.0	2.87515	-1.00672	0.0	**0.17186**	-1.00672	0.0	0.2656
2000	-1.00605	0.0	3.70289	-1.00605	0.0	4.03605	-1.00605	0.0	0.15623	-1.00605	0.0	**0.15622**

Table B.43: NP-K-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.96482	0.0	0.60932	-0.96482	0.0	0.10936	-0.96482	0.0	0.07812	-0.96482	0.0	**0.0625**
400	-0.99745	0.0	1.54707	-0.99745	0.0	0.28123	-0.99745	0.0	0.12499	-0.99745	0.0	**0.10948**
600	-1.0	0.0	0.53122	-1.0	0.0	0.60945	-1.0	0.0	0.14062	-1.0	0.0	**0.12499**
800	-1.0	0.0	0.85942	-1.0	0.0	0.9063	-1.0	0.0	0.10937	-1.0	0.0	**0.10935**
1000	-1.00025	0.0	1.15652	-1.0	0.0	1.42188	-1.0	0.0	0.12498	-1.0	0.0	**0.10936**
1200	-1.0072	0.0	1.99175	-1.0072	1e-05	1.71894	-1.0072	0.0	0.42197	-1.0072	0.0	**0.34384**
1400	-1.00827	0.0	1.59376	-1.01944	0.0	1.82834	-1.00504	0.0	0.31246	-1.00403	0.0	**0.29685**
1600	-1.00504	0.0	2.48445	-1.00504	0.0	2.64698	-1.00505	0.0	0.29695	-1.00504	0.0	**0.21874**
1800	-1.00672	0.0	3.35962	-1.00672	0.0	3.39096	-1.00672	0.0	**0.17186**	-1.00672	0.0	0.21885
2000	-1.0072	0.0	4.70306	-1.01007	0.0	3.79713	-1.0072	0.0	0.31246	-1.0072	0.0	**0.29683**

Table B.44: NP-K-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
200	-0.99497	0.0	0.65634	-0.99497	0.0	0.15624	-0.99497	0.0	0.10937	-0.99497	0.0	**0.0625**
400	-0.99727	0.0	2.15505	-0.9966	0.0	0.31249	-0.9966	0.0	0.22701	-0.9966	0.0	**0.17215**
600	-1.0	0.0	3.48683	-1.0	0.0	0.54697	-1.0	0.0	0.20312	-1.0	0.0	**0.1406**
800	-1.0005	0.0	6.03401	-0.99855	0.0	1.06256	-0.99855	1e-05	**0.20923**	-0.99856	0.0	0.29685
1000	-1.00336	0.0	1.35944	-1.00336	0.0	1.17194	-1.00336	0.0	0.24998	-1.00336	0.0	**0.23448**
1200	-1.01006	0.0	1.84394	-1.00672	0.0	1.73527	-1.00672	0.0	**0.26562**	-1.00672	0.0	0.3281
1400	-1.00336	0.0	2.39088	-1.00336	0.0	1.96895	-1.00336	0.0	**0.29684**	-1.00336	0.0	0.51571
1600	-1.00336	0.0	2.67204	-1.00336	0.0	2.6876	-1.00336	0.0	0.53137	-1.00336	0.0	**0.4531**
1800	-1.00504	0.0	3.32837	-1.00504	0.0	3.45332	-1.00504	0.0	0.42207	-1.00504	0.0	**0.29698**
2000	-1.0072	0.0	4.15654	-1.0072	0.0	3.78957	-1.0072	0.0	0.53121	-1.0072	0.0	**0.43746**

Table B.45: NP-K-N-5

B.1.6 Unrestricted numerators and positive denominators with assignment constraints

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-1.04272	0.0	0.28133	-1.04272	0.0	0.14062	-1.04272	0.0	0.12499	-1.04272	0.0	**0.07812**
20	-1.75559	0.0	0.74978	-1.75559	0.0	0.64061	-1.75559	0.0	0.21875	-1.75559	0.0	**0.17187**
30	-1.85352	0.0	12.17256	-1.85352	0.0	6.9052	-1.85352	0.0	**1.2813**	-1.85352	0.0	1.5158
40	-2.28706	0.0	26.4387	-2.28706	0.0	8.81413	-2.28706	0.0	2.51577	-2.28706	0.0	**0.84704**
50	-2.37241	0.0	70.18395	-2.37241	0.0	46.13737	-2.37241	0.0	2.13114	-2.37241	0.0	**1.26291**
60	-3.21772	0.0	119.79224	-3.21772	0.0	42.61254	-3.21772	0.0	**6.54322**	-3.21772	0.0	6.59635
70	-2.99915	0.0	265.47182	-2.99915	0.0	58.48044	-2.99915	0.0	6.03187	-2.99915	7e-05	**1.53467**
80	-3.27042	0.0	401.43212	-3.27042	9e-05	330.87484	-3.27042	0.0	10.89264	-3.27042	0.0	**2.9355**
90	-3.3488	5e-05	780.28406	-3.3488	0.0	388.63035	-3.3488	0.0	**2.76342**	-3.3488	3e-05	4.29858
100	-3.6385	0.0	1497.6182	-3.6385	0.0	245.11086	-3.6385	0.0	14.57216	-3.6385	0.0	**11.39129**

Table B.46: NP-A-R-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.9517	0.0	**0.144**	-0.9517	0.0	0.20312	-0.9517	0.0	0.15623	-0.9517	0.0	0.32809
20	-1.24091	0.0	2.00462	-1.24091	0.0	3.37227	-1.24091	0.0	**0.96889**	-1.24091	0.0	1.3751
30	-1.27395	0.0	16.38304	-1.27395	0.0	16.65661	-1.27395	0.0	2.87728	-1.27395	0.0	**1.35673**
40	-1.55688	0.0	39.72523	-1.55688	0.0	7.47329	-1.55688	0.0	**4.77763**	-1.55688	0.0	5.66018
50	-1.80862	0.0	89.31535	-1.80862	0.0	21.60875	-1.80862	0.0	9.69127	-1.80862	0.0	**5.52861**
60	-1.85783	0.0	130.2358	-1.85783	0.0	51.63634	-1.85783	0.0	53.63122	-1.85783	0.0	**9.76837**
70	-2.00786	0.0	270.827	-2.00786	0.0	229.37109	-2.00786	0.0	**33.19504**	-2.00786	0.0	47.58847
80	-2.00052	0.0	519.7262	-2.00052	0.0	397.47479	-2.00052	0.0	57.85277	-2.00052	0.0	**39.74735**
90	-2.12679	0.0	1067.59871	-2.12679	0.0	656.27009	-2.12679	0.0	104.74345	-2.12679	0.0	**36.93153**
100	-2.21116	0.0	1448.45551	-2.21116	0.0	855.00165	-2.21116	0.0	57.61204	-2.21116	0.0	**41.91465**

Table B.47: NP-A-R-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.47759	0.0	0.29686	-0.47759	0.0	**0.23436**	-0.47759	0.0	0.72103	-0.47759	0.0	0.35935
20	-0.96475	0.0	10.21087	-0.96475	0.0	3.33334	-0.96475	0.0	3.34954	-0.96475	0.0	**0.74996**
30	-0.97467	0.0	28.30772	-0.97467	0.0	25.61121	-0.97467	0.0	56.38408	-0.97467	0.0	**4.9589**
40	-1.05712	0.0	89.90487	-1.05712	0.0	**50.07132**	-1.05712	0.0	133.21933	-1.05712	0.0	113.72934
50	-1.19621	0.0	248.0784	-1.19621	0.0	311.89973	-1.19621	7e-05	2907.08889	-1.19621	0.0	**181.72586**
60	-1.2812	0.0	192.45008	-1.2812	0.0	222.58795	-1.2812	0.0	156.89793	-1.2812	0.0	**79.9188**
70	-1.28577	0.0	976.07709	-1.28577	0.0001	1240.77545	-1.28577	0.0	**210.5805**	-1.28577	8e-05	532.66817
80	-1.32815	0.0	1333.1155	-1.32815	0.0	1540.50968	-1.32815	6e-05	850.25209	-1.32815	0.0	**265.40909**
90	-1.4313	0.00279	3600.06138	-1.4313	0.0	2854.14114	-1.4313	3e-05	372.11575	-1.4313	0.0	**302.05169**
100	-1.50729	0.00238	3600.05742	-1.50707	0.00439	3600.19028	-1.50729	0.0	**332.56986**	-1.50729	9e-05	1976.95718

Table B.48: NP-A-R-5

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.24345	0.0	0.65634	-0.24345	0.0	0.17187	-0.24345	0.0	**0.12512**	-0.24345	0.0	0.14062
20	-0.23585	0.0	1.08127	-0.23585	0.0	0.67184	-0.23585	0.0	0.68747	-0.23585	4e-05	**0.23437**
30	-0.22105	5e-05	3.92842	-0.22105	0.0	2.59493	-0.22105	0.0	**0.18759**	-0.22105	0.0	0.21704
40	-0.2177	0.0	8.38653	-0.2177	0.0	5.4097	-0.2177	5e-05	0.42185	-0.2177	0.0	**0.37498**
50	-0.19801	0.0	71.50269	-0.19784	0.0	12.10487	-0.19784	1e-05	1.05149	-0.19784	0.0	**0.56265**
60	-0.1794	0.0	74.9543	-0.1794	0.0	26.33419	-0.1794	0.0	**0.50012**	-0.1794	0.0	0.68757
70	-0.18712	0.0	128.59785	-0.18712	0.0	56.21704	-0.18712	0.0	5.76844	-0.18712	0.0	**1.3928**
80	-0.18426	7e-05	1438.12505	-0.18426	0.0	84.70686	-0.18426	0.0	8.26219	-0.18426	7e-05	**2.85805**
90	-0.17259	0.0	2473.42567	-0.17259	0.0	145.53532	-0.17259	0.0	7.59175	-0.17259	8e-05	**1.68874**
100	-0.15434	0.0001	2738.01751	-0.15434	7e-05	219.34765	-0.15434	9e-05	4.48423	-0.15434	0.0	**2.96899**

Table B.49: NP-A-P-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.28215	0.0	0.54685	-0.28215	0.0	0.14062	-0.28215	0.0	0.26562	-0.28215	0.0	**0.12498**
20	-0.28231	0.0	1.26567	-0.28231	0.0	1.03144	-0.28231	0.0	0.54359	-0.28231	0.0	**0.35935**
30	-0.19268	0.0	3.14082	-0.19268	0.0	2.37541	-0.19268	0.0	**0.43748**	-0.19268	0.0	0.45308
40	-0.17647	3e-05	13.83109	-0.17647	0.0	7.54819	-0.17647	0.0	5.57955	-0.17647	0.0	**0.8439**
50	-0.2028	0.0	21.55356	-0.2028	0.0	13.36093	-0.2028	0.0	3.69208	-0.2028	0.0	**1.49002**
60	-0.17018	0.0	390.41041	-0.17018	0.0	33.36522	-0.17018	0.0	1.9064	-0.17018	0.0	**1.46889**
70	-0.15192	3e-05	128.80842	-0.15192	0.0	57.17258	-0.15192	2e-05	6.43967	-0.15192	8e-05	**2.97204**
80	-0.15258	0.0	1303.66458	-0.15258	0.0	88.85984	-0.15258	0.0	**2.01701**	-0.15258	0.0	14.26487
90	-0.15025	8e-05	3326.86736	-0.15025	0.0	121.94046	-0.15025	0.0001	12.48602	-0.15025	0.0	**3.03477**
100	-0.12417	0.04131	3600.05505	-0.12467	0.0	187.87219	-0.12467	0.0	**3.00037**	-0.12467	0.0	20.21062

Table B.50: NP-A-P-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.0998	0.0	0.82822	-0.0998	0.0	0.28123	-0.0998	0.0	**0.23435**	-0.0998	0.0	0.24996
20	-0.16412	0.0	2.79885	-0.16412	0.0	3.03514	-0.16412	0.0	2.52031	-0.16412	0.0	**1.50019**
30	-0.17812	0.0	9.00234	-0.17812	0.0	6.42164	-0.17812	6e-05	2.56276	-0.17812	0.0	**1.50335**
40	-0.14672	0.0	29.42144	-0.14672	0.0	17.39205	-0.14672	0.0	9.29722	-0.14672	0.0	**6.13508**
50	-0.13423	0.0	139.13559	-0.13423	0.0	79.80641	-0.13423	0.0	**7.65233**	-0.13423	0.0	12.2052
60	-0.10811	0.0	1199.05828	-0.10811	0.0	115.11461	-0.10811	0.0	13.85311	-0.10811	0.0	**8.4892**
70	-0.1245	0.0	1791.4321	-0.1245	2e-05	164.48256	-0.1245	0.0	26.45008	-0.1245	0.0	**16.50525**
80	-0.09388	0.02789	3600.04368	-0.09419	0.0	1348.02686	-0.09419	0.0	**54.82455**	-0.09419	0.0	108.9658
90	-0.1094	1.15659	3600.05158	-0.10995	0.0	152.91714	-0.10995	0.0	**8.43289**	-0.10995	0.0	20.75057
100	-0.10133	0.0	1470.71743	-0.10133	0.0001	479.6912	-0.10133	0.0	**72.994**	-0.10133	0.0	99.99221

Table B.51: NP-A-P-5

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.3	0.0	0.31988	-0.3	0.0	0.07813	-0.3	0.0	0.09373	-0.3	0.0	**0.07812**
20	-0.23897	9e-05	1.09371	-0.23897	0.0	0.62515	-0.23897	7e-05	0.21887	-0.23897	0.0	**0.20312**
30	-0.21745	0.0	18.77794	-0.21745	0.0	1.89417	-0.21745	0.0	0.42185	-0.21745	0.0	**0.25009**
40	-0.22849	0.0	8.7809	-0.22849	0.0	4.80794	-0.22849	7e-05	**0.34385**	-0.22849	0.0	0.46872
50	-0.23015	0.0	21.59934	-0.23015	0.0	10.95363	-0.23015	0.0	1.39491	-0.23015	0.0	**0.51572**
60	-0.22278	9e-05	226.09156	-0.22278	0.0	26.95747	-0.22278	0.0	0.797	-0.22278	9e-05	**0.63505**
70	-0.18597	7e-05	119.70304	-0.18597	0.0	42.53595	-0.18597	0.0	**1.71889**	-0.18597	0.0	4.74885
80	-0.1698	8e-05	278.67669	-0.1698	8e-05	86.23666	-0.1698	0.0	**2.4849**	-0.1698	0.0	5.17226
90	-0.15976	0.0	2223.1365	-0.15976	9e-05	90.18063	-0.15976	1e-05	2.68736	-0.15976	0.0	**1.59172**
100	-0.15834	0.0	1138.55656	-0.15834	0.0	172.06176	-0.15834	0.0	3.01585	-0.15834	0.0	**2.76932**

Table B.52: NP-A-N-2

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.23864	0.0	0.43777	-0.23864	0.0	**0.07812**	-0.23864	0.0	0.10937	-0.23864	0.0	**0.07812**
20	-0.21875	0.0	1.70298	-0.21875	0.0	1.15631	-0.21875	0.0	0.46884	-0.21875	0.0	**0.42185**
30	-0.1857	0.0	5.37513	-0.1857	0.0	3.2816	-0.1857	0.0	1.25006	-0.1857	0.0	**0.71885**
40	-0.17954	0.0	52.79486	-0.17954	0.0	6.5794	-0.17954	0.0	**2.99663**	-0.17954	0.0001	3.02054
50	-0.16993	0.0	40.385	-0.16993	0.0	14.9223	-0.16993	0.0	1.87515	-0.16993	0.0	**1.57832**
60	-0.18678	0.0	57.50706	-0.18678	0.0	32.521	-0.18678	0.0	1.85976	-0.18678	0.0	**1.14068**
70	-0.16774	0.0	1073.53408	-0.16773	0.0	64.7952	-0.16773	8e-05	6.23608	-0.16773	0.0	**2.34385**
80	-0.13642	0.0	269.90587	-0.13642	0.0	124.38364	-0.13642	0.0	**4.95447**	-0.13642	0.0	15.2508
90	-0.1446	0.0	2659.46326	-0.1446	0.0	125.64351	-0.1446	3e-05	**3.34168**	-0.1446	0.0	3.60972
100	-0.14329	0.01145	3600.04362	-0.14391	0.0	276.47672	-0.14391	0.0001	17.69091	-0.14391	0.0001	**14.21179**

Table B.53: NP-A-N-3

n	MMFP1A			MMFP1AM			MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-0.16319	0.0	0.59089	-0.16319	0.0	0.29686	-0.16319	0.0	0.40623	-0.16319	0.0	**0.23436**
20	-0.13467	0.0	10.36451	-0.13467	0.0	2.72281	-0.13467	0.0	2.87809	-0.13467	0.0	**1.26567**
30	-0.15337	0.0	6.85047	-0.15337	0.0	5.95405	-0.15337	0.0	2.1853	-0.15337	0.0	**1.83856**
40	-0.12725	0.0	14.60908	-0.12725	4e-05	11.73421	-0.12725	0.0	**4.51436**	-0.12725	0.0	7.0186
50	-0.14709	0.0	41.91742	-0.14709	0.0	31.43031	-0.14709	0.0	**7.11418**	-0.14709	0.0	8.9268
60	-0.12744	0.0	1185.4983	-0.12744	0.0	150.98681	-0.12744	0.0	**16.2039**	-0.12744	0.0	22.95086
70	-0.11819	3e-05	1998.33915	-0.11819	0.0	183.82984	-0.11819	0.0	16.59167	-0.11819	0.0	**10.82664**
80	-0.10279	0.00652	3600.04708	-0.10289	0.0	159.16635	-0.10289	0.0001	23.45895	-0.10289	0.0	**21.15439**
90	-0.10687	0.01904	3600.0715	-0.10696	0.0	1139.29631	-0.10696	0.0	130.29472	-0.10696	0.0	**58.78077**
100	-0.1065	0.0	1362.28087	-0.1065	0.0	974.86936	-0.1065	0.0	**32.48302**	-0.1065	0.0001	42.7884

Table B.54: NP-A-N-5

B.2 Computational results with medium size data

B.2.1 Positive numerators and denominators with no constraint

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.0013	0.0	0.28093	0.13945	0.0	**0.26567**	0.00096	0.0	0.01562	0.13945	0.0	0.37495
2000	0.00066	0.0	1.00353	0.14164	0.0	6.09351	0.0005	0.0	0.03125	0.14164	7e-05	**0.76571**
3000	0.00027	0.0	1.95302	0.08541	0.0	0.78131	0.00015	0.0	0.04711	0.08541	0.0	**0.62495**
4000	0.00029	0.0	3.78184	0.10487	0.0	4.39061	0.00023	0.0	0.03125	0.10487	0.0	**1.00005**
5000	0.00032	0.0	9.89075	0.11149	0.0	14.9844	0.00031	0.0	0.04687	0.11149	0.0	**1.17189**
6000	0.0014	0.0	35.79723	0.08805	0.0	88.81282	0.00016	0.0	0.04687	0.08805	0.0	**2.87515**
7000	0.0026	0.0	148.61025	0.08036	0.0	2.53129	0.00011	0.0	0.0625	0.08036	0.0	**1.93788**
8000	0.00266	0.0	258.61019	0.07711	0.0	4.35699	8e-05	0.0	0.06249	0.07711	0.0	**2.09398**
9000	0.00372	0.0	29.26593	0.08133	0.0	98.50837	0.00013	0.0	0.06248	0.08133	0.0	**3.25011**
10000	0.00018	0.0	0.64058	0.08434	0.0	9.91388	0.00017	0.0	0.0625	0.08434	0.0	**2.59115**

Table B.55: PP-U-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.00197	0.0	0.26562	0.26788	3e-05	1.28132	0.26788	0.0	3.58292	0.26788	0.0	**1.12512**
2000	0.00095	0.0	0.92193	0.23325	0.0	4.09406	0.00074	0.0	0.03125	0.23325	0.0	**2.95428**
3000	0.00055	0.0	2.03123	0.16873	9e-05	2.62563	0.00031	0.0	0.03124	0.16873	0.0	**1.23971**
4000	0.00048	0.0	3.62519	0.19457	0.0	32.74448	0.00037	0.0	0.04686	0.19457	0.0	**2.19852**
5000	0.00427	0.0	49.43543	0.2075	0.0	66.70677	0.00035	0.0	0.06249	0.2075	0.0	**3.25548**
6000	0.00141	0.0	40.56343	0.15326	8e-05	2.51584	0.00015	0.0	0.06248	0.15326	0.0	**2.21417**
7000	0.01022	0.0	250.29245	0.14205	3e-05	27.05769	0.00014	0.0	0.06249	0.14205	0.0	**2.06262**
8000	0.00015	0.0	0.53135	0.14765	0.0	**3.17699**	0.00011	0.0	0.06248	0.14765	0.0	5.661
9000	0.00012	0.0	0.59371	0.138	0.0	283.30667	0.0001	0.0	0.09374	0.15772	0.0	**4.27337**
10000	0.0001	0.0	0.68764	0.14221	0.0	39.6398	0.0001	0.0	0.06262	0.14221	0.0	**3.61802**

Table B.56: PP-U-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.00107	0.0	0.28125	0.38679	0.0	**1.82822**	0.00081	0.0	0.01562	0.38679	0.0	3.60007
2000	0.00092	0.0	0.93734	0.32881	0.0	**3.89087**	0.00069	0.0	0.17186	0.32881	0.0	34.06578
3000	0.00037	0.0	4.547	0.31313	0.0	28.55349	0.00021	0.0	0.04687	0.31313	0.0	**6.99004**
4000	0.00037	0.0	3.83117	0.34404	0.0	**18.4381**	0.00028	0.0	0.06259	0.34404	0.0	57.04054
5000	0.00356	0.0	10.51974	0.29554	0.0	**3.08255**	0.00031	0.0	0.0625	0.29554	0.0	16.229
6000	0.0003	0.0	0.54696	0.29252	0.0	65.50305	0.00017	0.0	0.07812	0.29252	0.0	**28.63689**
7000	0.00025	0.0	0.68757	0.25797	0.0	**3.58118**	0.00017	0.0	0.10937	0.25797	0.0	15.16712
8000	0.0002	0.0	0.87506	0.27581	0.0	8.57486	0.00015	0.0	0.10937	0.27581	0.0	**7.70491**
9000	0.00021	0.0	1.04695	0.24795	0.0	**3.06694**	0.00018	0.0	0.15016	0.24795	0.0	9.27976
10000	0.00017	0.0	1.28108	0.26007	0.0	**2.15646**	0.00016	0.0	0.14062	0.26007	0.0	11.69065

Table B.57: PP-U-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.2348	0.55556	0.0	**0.04686**	2e-05	0.0	0.01562	0.55556	0.0	0.08174
2000	1e-05	0.0	0.84284	0.66667	0.0	2.14067	1e-05	0.0	0.01562	0.66667	0.0	**0.1406**
3000	1e-05	0.0	0.45292	0.71429	0.0	0.71871	0.0	0.0	0.01563	0.71429	0.0	**0.10937**
4000	0.0	0.0	0.7812	0.92	0.0	2.5626	0.0	0.0	0.03125	0.92	0.0	**0.14062**
5000	0.0	0.0	1.15621	0.53846	0.0	**0.7343**	0.0	0.0	0.03125	1e-05	0.0	0.01562
6000	0.0	0.0	1.64224	0.52	0.0	**3.76644**	0.0	0.0	0.03125	0.0	0.0	0.01562
7000	0.0	0.0	2.43724	0.91935	0.0	14.72533	0.0	0.0	0.03125	0.91935	0.0	**0.23435**
8000	0.0	0.0	3.12502	0.91176	0.0	0.47198	0.0	0.0	0.03125	0.91176	0.0	**0.18747**
9000	0.0	0.0	4.26552	0.98917	0.0	5.04568	0.0	0.0	0.03123	0.98917	1e-05	**0.15634**
10000	0.0	0.0	0.79329	0.68	0.0	**5.4674**	0.0	0.0	0.03125	1e-05	0.0	0.03125

Table B.58: PP-U-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.23436	0.92308	0.0	1.03129	2e-05	0.0	0.01561	0.92308	0.0	**0.17526**
2000	1e-05	0.0	0.84369	0.9	0.0	0.57821	1e-05	0.0	0.01562	0.9	0.0	**0.15624**
3000	1e-05	0.0	0.45311	0.98704	0.0	0.40621	0.0	0.0	0.03123	0.98704	0.0	**0.21874**
4000	1e-05	0.0	0.76629	1.0	0.0	0.37496	0.0	0.0	0.03123	1.0	0.0	**0.03125**
5000	0.0	0.0	1.1407	0.98824	6e-05	2.84496	0.0	0.0	0.03125	0.98824	4e-05	**0.3437**
6000	0.0	0.0	1.65646	0.92	0.0	19.88156	0.0	0.0	0.03125	0.92	0.0	**0.39059**
7000	0.0	0.0	2.23443	0.95238	0.0	7.88258	0.0	0.0	0.03125	0.95238	0.0	**0.31906**
8000	0.0	0.0	0.64057	0.95714	0.0	19.38171	0.0	0.0	0.03126	0.95714	0.0	**0.24997**
9000	0.0	0.0	0.70318	0.55556	0.0	**0.50005**	0.0	0.0	0.03549	0.0	0.0	0.04687
10000	0.0	0.0	0.75005	0.98972	3e-05	30.22834	0.0	0.0	0.04687	0.98972	0.0	**0.31245**

Table B.59: PP-U-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.26559	0.98969	0.0	0.17186	2e-05	0.0	0.01562	0.98969	0.0	**0.15639**
2000	1e-05	0.0	0.87716	0.8	0.0	1.20303	1e-05	0.0	0.03125	0.8	0.0	**0.26559**
3000	1e-05	0.0	0.45309	0.93548	0.0	3.73446	0.0	0.0	0.03125	0.93548	0.0	**0.54693**
4000	1e-05	0.0	0.76117	0.99043	0.0001	1.81264	0.0	0.0	0.03123	0.99043	0.0001	**0.30417**
5000	0.0	0.0	1.15478	0.99054	7e-05	14.03832	0.0	0.0	0.04687	0.99054	9e-05	**1.35964**
6000	0.0	0.0	1.40615	0.98901	0.0	23.13519	0.0	0.0	0.03124	0.98901	0.0	**0.56257**
7000	0.0	0.0	2.05404	1.0	0.0	0.45321	0.0	0.0	0.03123	1.0	0.0	**0.06249**
8000	0.0	0.0	2.62528	0.83929	0.0	26.84349	0.0	0.0	0.03123	0.83929	0.0	**0.6095**
9000	0.0	0.0	3.53204	0.98925	0.0	52.92787	0.0	0.0	0.0625	0.98925	0.0	**0.56258**
10000	0.0	0.0	4.92332	0.99471	0.0001	35.04663	0.0	0.0	0.06248	0.99471	4e-05	**0.34373**

Table B.60: PP-U-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.23441	0.0121	0.0	0.23435	2e-05	0.0	0.01562	0.0121	0.0	**0.14059**
2000	1e-05	0.0	0.82819	0.01052	0.0	0.54682	1e-05	0.0	0.01562	0.01052	0.0	**0.21872**
3000	1e-05	0.0	0.43761	0.01049	0.0	1.15638	0.0	0.0	0.0156	0.01049	0.0	**0.19088**
4000	1e-05	0.0	0.73457	0.01039	0.0	2.84391	0.0	0.0	0.03123	0.01039	0.0	**0.17187**
5000	0.0	0.0	1.10941	1e-05	0.0	0.82819	0.0	0.0	0.01564	0.01032	0.0	**0.21872**
6000	0.0	0.0	1.60926	1e-05	0.0	1.23442	0.0	0.0	0.03123	0.01027	0.0	**0.25019**
7000	0.0	0.0	2.29008	1e-05	0.0	1.71874	0.0	0.0	0.01562	0.01025	0.0	**0.31247**
8000	0.0	0.0	3.09395	1e-05	0.0	2.21892	0.0	0.0	0.01562	0.01023	0.0	**0.23446**
9000	0.0	0.0	3.89105	0.0	0.0	2.68772	0.0	0.0	0.03123	0.01023	0.0	**0.2187**
10000	0.0	0.0	0.60944	0.0	0.0	3.6097	0.0	0.0	0.01564	0.01021	0.0	**0.23435**

Table B.61: PP-U-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.24998	0.0121	0.0	0.15622	2e-05	0.0	0.01564	0.0121	0.0	**0.12497**
2000	1e-05	0.0	0.79694	0.01077	0.0	0.60951	1e-05	0.0	0.03125	0.01077	0.0	**0.15624**
3000	1e-05	0.0	0.43759	0.0106	0.0	0.90633	0.0	0.0	0.03123	0.0106	0.0	**0.28122**
4000	1e-05	0.0	0.71889	0.0104	0.0	3.76643	0.0	0.0	0.01562	0.0104	0.0	**0.21884**
5000	0.0	0.0	1.0938	1e-05	0.0	0.89071	0.0	0.0	0.01564	0.01035	0.0	**0.26561**
6000	0.0	0.0	1.59389	1e-05	0.0	1.25025	0.0	0.0	0.01562	0.01032	0.0	**0.35945**
7000	0.0	0.0	2.19807	1e-05	0.0	1.67204	0.0	0.0	0.01564	0.01027	0.0	**0.39072**
8000	0.0	0.0	0.48433	1e-05	0.0	2.18767	0.0	0.0	0.03125	0.01025	0.0	**0.3437**
9000	0.0	0.0	0.59385	0.0	0.0	2.7346	0.0	0.0	0.03123	0.01022	0.0	**0.37494**
10000	0.0	0.0	0.65637	0.0	0.0	3.50183	0.0	0.0	0.03125	0.01022	0.0	**0.35934**

Table B.62: PP-U-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.2501	0.0121	0.0	**0.23436**	2e-05	0.0	0.01562	0.0121	0.0	0.26559
2000	1e-05	0.0	0.82704	0.0107	0.0	0.79696	1e-05	0.0	0.03126	0.0107	0.0	**0.34371**
3000	1e-05	0.0	0.43747	0.0105	0.0	1.21877	0.0	0.0	0.01564	0.0105	0.0	**0.50007**
4000	1e-05	0.0	0.76578	0.01041	0.0	1.64083	0.0	0.0	0.03125	0.01041	8e-05	**0.40621**
5000	0.0	0.0	1.14065	1e-05	0.0	0.90631	0.0	0.0	0.04686	0.01034	0.0	**0.40619**
6000	0.0	0.0	0.53133	1e-05	0.0	1.34378	0.0	0.0	0.03123	0.01029	0.0	**0.62495**
7000	0.0	0.0	0.68759	1e-05	0.0	1.73531	0.0	0.0	0.04686	0.01027	0.0	**0.53139**
8000	0.0	0.0	0.81678	1e-05	0.0	2.31281	0.0	0.0	0.03124	0.01025	0.0	**0.65633**
9000	0.0	0.0	0.98441	0.0	0.0	2.8909	0.0	0.0	0.04686	0.01023	0.0	**0.73447**
10000	0.0	0.0	1.17194	0.0	0.0	3.75044	0.0	0.0	0.04686	0.01022	0.0	**0.65631**

Table B.63: PP-U-N-5

B.2.2 Positive numerator and denominator with knapsack constraints

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.00123	0.0	0.2655	0.16649	0.0	0.68761	0.00186	0.0	0.2501	0.16649	0.0	**0.3906**
2000	0.00097	0.0	0.89099	0.13617	0.0	1.01331	0.00151	0.0	0.54684	0.13617	0.0	**0.60934**
3000	0.00057	0.0	1.95335	0.14871	0.0	6.89223	0.00066	0.0	0.03125	0.14871	0.0	**0.82821**
4000	0.00019	0.0	3.68754	0.09694	0.0	2.29891	0.0003	0.0	0.0625	0.09694	0.0	**0.90629**
5000	0.00031	0.0	8.78924	0.12038	8e-05	8.21932	0.00026	0.0	0.04689	0.12038	0.0	**1.78135**
6000	0.00128	0.0	11.93979	0.08762	0.0	8.7536	0.00024	0.0	0.06674	0.08762	0.0	**1.26566**
7000	0.00352	0.0	22.42692	0.09263	0.0	19.65821	0.00024	0.0	0.0625	0.09263	0.0	**1.52092**
8000	0.02229	0.0	253.19789	0.08802	0.0	32.52606	0.00029	0.0	0.07812	0.08802	0.0	**1.66521**
9000	6e-05	0.0	0.67712	0.06567	0.0	8.95584	0.00011	0.0	0.07811	0.06567	0.0	**2.3512**
10000	0.00011	0.0	0.67183	0.07299	3e-05	**1.60936**	0.00011	0.0	0.09373	0.07299	0.0	1.86798

Table B.64: PP-K-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.00148	0.0	0.2656	0.28266	0.0	**0.71882**	0.00229	0.0	0.15625	0.28266	0.0	1.28462
2000	0.0008	0.0	0.90631	0.19906	0.0	15.19249	0.00124	0.0	0.03125	0.19906	0.0	**1.3438**
3000	0.00063	0.0	1.97483	0.19802	0.0	7.25109	0.00072	0.0	0.04688	0.19802	0.0	**2.0782**
4000	0.00037	0.0	4.42258	0.18444	0.0	1.48104	0.00056	0.0	0.06248	0.18444	0.0	**1.04911**
5000	0.00682	0.0	53.04759	0.20755	0.0	37.10051	0.00038	0.0	0.06248	0.20755	0.0	**2.79708**
6000	0.00096	0.0	14.22022	0.18683	0.0	45.7975	0.00035	0.0	0.07813	0.18683	0.0	**5.59402**
7000	0.0015	0.0	65.59889	0.16884	0.0	3.43833	0.00037	0.0	0.10937	0.16884	0.0	**2.68843**
8000	0.00011	0.0	0.57808	0.11792	0.0	190.29952	0.00016	0.0	0.10936	0.11792	0.0	**4.06288**
9000	7e-05	0.0	0.62509	0.14325	0.0	3.64147	0.00012	0.0	0.10937	0.14325	0.0	**2.99751**
10000	0.00016	0.0	0.73449	0.12684	9e-05	3.26948	0.00015	0.0	0.10936	0.12684	0.0	**3.1251**

Table B.65: PP-K-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	0.00191	0.0	0.29466	0.43899	0.0	**2.39069**	0.00292	0.0	0.4062	0.43899	0.0	65.13313
2000	0.0008	0.0	1.0346	0.31603	0.0	**0.75007**	0.00123	0.0	0.06248	0.31603	0.0	4.47712
3000	0.00063	0.0	2.06801	0.32357	0.0	**18.32017**	0.00073	0.0	0.0625	0.32357	0.0	32.6014
4000	0.00049	0.0	3.84397	0.30902	0.0	**17.51679**	0.00075	0.0	0.06251	0.30902	0.0	31.56918
5000	0.00299	0.0	75.46686	0.2646	0.0	15.34818	0.00031	0.0	0.0781	0.2646	0.0	**7.4584**
6000	0.00024	0.0	0.60932	0.27075	0.0	8.9994	0.00027	0.0	0.11334	0.27075	0.0	**7.8416**
7000	0.00023	0.0	0.74993	0.25708	0.0	**3.19096**	0.00031	0.0	0.12499	0.25708	0.0	5.80604
8000	0.00017	0.0	1.04344	0.29294	0.0	**6.09958**	0.00026	0.0	0.18748	0.29294	0.0	37.63628
9000	0.00021	0.0	1.09919	0.26709	0.0	**6.12696**	0.00036	0.0	0.17186	0.26709	0.0	22.41182
10000	0.00018	0.0	1.29697	0.23932	0.0	151.86054	0.00017	0.0	0.17198	0.23932	0.0	**16.60526**

Table B.66: PP-K-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.23447	0.81818	0.0	0.29684	3e-05	0.0	0.03125	0.81818	0.0	**0.26575**
2000	1e-05	0.0	0.875	0.96667	0.0	0.45322	2e-05	0.0	0.03126	0.96667	0.0	**0.23436**
3000	1e-05	0.0	0.49985	0.95082	0.0	0.79695	1e-05	0.0	0.03124	0.95082	0.0	**0.34372**
4000	1e-05	0.0	0.75006	0.55556	0.0	0.21884	1e-05	0.0	0.03135	0.55556	0.0	**0.21872**
5000	0.0	0.0	1.20311	0.87671	0.0	6.58979	0.0	0.0	0.04687	0.87671	0.0	**0.89066**
6000	0.0	0.0	1.7345	0.91176	0.0	9.67846	0.0	0.0	0.06248	0.91176	0.0	**0.53119**
7000	0.0	0.0	2.33165	0.75	0.0	26.29088	0.0	0.0	0.04687	0.75	0.0	**0.62267**
8000	0.0	0.0	3.0472	0.80645	0.0	3.79454	0.0	0.0	0.06264	0.80645	0.0	**0.48767**
9000	0.0	0.0	0.68764	0.88971	0.0	13.98882	0.0	0.0	0.07812	0.88971	0.0	**0.74996**
10000	0.0	0.0	0.6876	0.88889	0.0	6.26731	0.0	0.0	0.0781	0.88889	0.0	**0.43747**

Table B.67: PP-K-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.24999	0.97917	0.0	2.29696	3e-05	0.0	0.03124	0.97917	0.0	**0.5626**
2000	1e-05	0.0	0.87246	0.88095	0.0	4.38694	2e-05	0.0	0.04687	0.88095	0.0	**0.61369**
3000	1e-05	0.0	0.45646	0.86667	0.0	1.12494	1e-05	0.0	0.03125	0.86667	0.0	**0.56246**
4000	1e-05	0.0	0.76559	0.98246	0.0	8.12351	1e-05	0.0	0.04688	0.98246	0.0	**1.20317**
5000	0.0	0.0	1.14067	0.89474	0.0	2.29711	0.0	0.0	0.0625	0.89474	0.0	**1.68762**
6000	0.0	0.0	1.70024	0.52941	0.0	**23.13318**	0.0	0.0	0.0781	1e-05	0.0	0.09373
7000	0.0	0.0	2.32123	1.0	0.0	0.32824	0.0	0.0	0.07811	1.0	0.0	**0.07811**
8000	0.0	0.0	0.62508	0.96825	0.0	28.75343	0.0	0.0	0.09374	0.96825	0.0	**2.10964**
9000	0.0	0.0	0.67182	0.57143	0.0	**34.24859**	0.0	0.0	0.10936	1e-05	0.0	0.12498
10000	0.0	0.0	0.82821	1.0	0.0	1.37924	0.0	0.0	0.09375	1.0	0.0	**0.09372**

Table B.68: PP-K-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.28122	1.0	0.0	0.06249	3e-05	0.0	0.03125	1.0	0.0	**0.04691**
2000	1e-05	0.0	0.96887	0.9	0.0	3.00028	2e-05	0.0	0.04688	0.9	0.0	**0.85944**
3000	1e-05	0.0	0.49998	1.0	0.0	0.26559	1e-05	0.0	0.04688	1.0	0.0	**0.04688**
4000	1e-05	0.0	0.76884	0.95312	0.0	13.09545	1e-05	0.0	0.0625	0.95312	0.0	**2.25099**
5000	0.0	0.0	1.15514	0.8	0.0	2.1475	0.0	0.0	0.09375	0.8	0.0	**1.07821**
6000	0.0	0.0	1.46893	0.97959	0.0001	20.34141	0.0	0.0	0.09374	0.97959	0.0	**4.23466**
7000	0.0	0.0	1.90638	0.8	0.0	**0.6562**	0.0	0.0	0.125	0.8	0.0	0.92198
8000	0.0	0.0	2.65652	0.98125	5e-05	18.24009	0.0	0.0	0.12498	0.98125	9e-05	**2.26583**
9000	0.0	0.0	3.56682	0.98077	0.0	**1.46911**	0.0	0.0	0.125	0.98077	0.0	3.73452
10000	0.0	0.0	4.65657	0.97368	0.0	33.981	0.0	0.0	0.17187	0.97368	0.0	**3.78158**

Table B.69: PP-K-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.23429	0.01135	0.0	0.34383	3e-05	0.0	0.01561	0.01135	0.0	**0.23666**
2000	1e-05	0.0	0.81241	0.01069	0.0	0.62508	2e-05	0.0	0.03127	0.01069	0.0	**0.3439**
3000	1e-05	0.0	0.46875	0.01045	0.0	1.64051	1e-05	0.0	0.03123	0.01045	0.0	**0.70323**
4000	1e-05	0.0	0.76575	0.01034	0.0	1.71886	1e-05	0.0	0.04686	0.01034	0.0	**0.93755**
5000	0.0	0.0	1.17189	1e-05	0.0	0.85946	0.0	0.0	0.04686	0.01032	0.0	**0.24998**
6000	0.0	0.0	1.68718	1e-05	0.0	1.21879	0.0	0.0	0.04685	0.01028	1e-05	**0.29683**
7000	0.0	0.0	2.4534	1e-05	0.0	1.73875	0.0	0.0	0.0625	0.01026	0.0	**0.34384**
8000	0.0	0.0	3.05322	1e-05	0.0	2.32815	0.0	0.0	0.06248	0.01024	2e-05	**0.34373**
9000	0.0	0.0	0.57906	0.0	0.0	2.84404	0.0	0.0	0.07812	0.01023	0.0	**0.81335**
10000	0.0	0.0	0.71883	0.0	0.0	3.49031	0.0	0.0	0.07195	0.01021	0.0	**1.04697**

Table B.70: PP-K-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.23436	0.01139	0.0	**0.26561**	3e-05	0.0	0.03123	0.01139	0.0	0.41547
2000	1e-05	0.0	0.8437	0.01062	0.0	0.51573	2e-05	0.0	0.03125	0.01062	0.0	**0.35934**
3000	1e-05	0.0	0.50016	0.01047	0.0	2.51995	1e-05	0.0	0.04686	0.01047	0.0	**1.40617**
4000	1e-05	0.0	0.76557	0.01043	0.0	1.625	1e-05	0.0	0.04688	0.01043	0.0	**1.13765**
5000	0.0	0.0	1.10984	1e-05	0.0	0.89067	0.0	0.0	0.06248	0.01036	0.0	**0.29683**
6000	0.0	0.0	1.64082	1e-05	0.0	1.28134	0.0	0.0	0.0625	0.01028	0.0	**1.62873**
7000	0.0	0.0	2.25024	1e-05	0.0	1.73444	0.0	0.0	0.06248	0.01025	0.0	**1.12513**
8000	0.0	0.0	0.54683	1e-05	0.0	2.31283	0.0	0.0	0.07811	0.01024	0.0	**1.71897**
9000	0.0	0.0	0.62506	0.0	0.0	2.92215	0.0	0.0	0.09377	0.01023	7e-05	**0.96887**
10000	0.0	0.0	0.71871	0.0	0.0	3.66609	0.0	0.0	0.09375	0.0	0.0	1.10942

Table B.71: PP-K-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	2e-05	0.0	0.28122	0.01113	0.0	**0.29685**	3e-05	0.0	0.03124	0.01113	0.0	0.47296
2000	1e-05	0.0	0.8439	0.01066	0.0	**0.6896**	2e-05	0.0	0.04262	0.01066	0.0	0.93744
3000	1e-05	0.0	0.46891	0.01043	0.0	1.40632	1e-05	0.0	0.04687	0.01043	0.0	**0.73433**
4000	1e-05	0.0	0.73449	0.01042	0.0	2.45318	1e-05	0.0	0.0625	0.01042	0.0	**1.05235**
5000	0.0	0.0	1.1251	1e-05	0.0	0.96886	0.0	0.0	0.07812	0.01033	0.0	**1.60958**
6000	0.0	0.0	0.57825	1e-05	0.0	1.37509	0.0	0.0	0.09374	0.01036	7e-05	**1.26564**
7000	0.0	0.0	0.71869	1e-05	0.0	1.85958	0.0	0.0	0.10954	0.01029	0.0	**1.83029**
8000	0.0	0.0	0.87511	1e-05	0.0	2.45462	0.0	0.0	0.10936	0.01026	0.0	**3.23332**
9000	0.0	0.0	1.01576	0.0	0.0	2.98466	0.0	0.0	0.12501	0.01022	0.0	**0.67182**
10000	0.0	0.0	1.27929	0.0	0.0	3.61489	0.0	0.0	0.14074	0.01021	0.0	**2.37507**

Table B.72: PP-K-N-5

B.2.3 Unrestricted numerators and positive denominators with no constraint

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-7.49462	0.0	4.78143	-7.49462	0.0	5.53166	-7.49462	0.0	0.39058	-7.49462	0.0	**0.28123**
2000	-7.60976	0.0	16.53324	-7.60976	0.0	10.97045	-7.60976	0.0	**0.34385**	-7.60976	0.0	0.45321
3000	-7.24907	0.0	30.25303	-7.24907	0.0	47.33584	-7.24907	0.0	**0.42184**	-7.24907	0.0	0.73444
4000	-8.14146	0.0	48.93225	-8.14146	0.0	88.50022	-8.14146	0.0	**0.61321**	-8.14146	0.0	2.00011
5000	-9.58081	0.0	471.56225	-9.58081	0.0	125.67219	-9.58081	0.0	**0.77795**	-9.58081	0.0	0.87504
6000	-11.89247	0.58504	3600.26412	-11.89247	0.0	82.78316	-11.89247	0.0	**0.79683**	-11.89247	0.0	0.9063
7000	-11.7594	0.0	687.37774	-11.7594	4e-05	92.39122	-11.7594	0.0	0.93764	-11.7594	0.0	**0.76569**
8000	-12.15385	0.0	936.63872	-12.15385	0.0	207.07584	-12.15385	0.0	1.12729	-12.15385	0.0	**1.05256**
9000	-11.96667	0.0	1884.46532	-11.96667	0.0	466.74485	-11.96667	0.0	**1.38941**	-11.96667	0.0	1.48444
10000	-11.415	0.0	2183.01057	-11.415	0.0	136.76928	-11.415	0.0	**1.46882**	-11.415	0.0	1.50015

Table B.73: NP-U-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-3.10762	0.0	5.26669	-3.10762	0.0	5.22361	-3.10762	7e-05	**1.85949**	-3.10762	0.0	12.8486
2000	-4.19545	0.0	19.40997	-4.19545	0.0	20.64283	-4.19545	0.0	**1.01556**	-4.19546	0.0	11.89489
3000	-5.79866	0.0	33.76391	-5.79866	0.0	22.14294	-5.79866	0.0	**2.29686**	-5.79866	0.0	2.32383
4000	-5.08664	0.0	54.75796	-5.08664	0.0	101.08021	-5.08664	0.0	**3.54807**	-5.08664	0.0	4.22112
5000	-5.50949	0.0	650.52796	-5.50909	0.0	132.92833	-5.50909	0.0	**1.37507**	-5.50909	0.0	4.29732
6000	-6.1962	0.0	689.07506	-6.1962	0.0	184.63166	-6.1962	0.0	**1.05006**	-6.1962	0.0	5.87271
7000	-5.31229	0.0	635.3004	-5.31229	0.0	261.54195	-5.31229	0.0	7.33561	-5.31229	0.0	**7.32231**
8000	-8.03	0.0	1494.53193	-8.03	0.0	1887.93635	-8.03	0.0	**1.76566**	-8.03	0.0	1.82322
9000	-5.60784	0.0	1675.62769	-5.60784	0.0	743.64441	-5.60784	0.0	**2.24176**	-5.60784	0.0	7.34936
10000	-5.7395	0.0	2400.25702	-5.7395	0.0	983.15054	-5.7395	4e-05	8.92881	-5.7395	0.0	**8.56302**

Table B.74: NP-U-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-2.44709	0.0	6.59433	-2.44709	0.0	11.93111	-2.44709	0.0	23.13557	-2.44709	0.0	**7.20976**
2000	-2.56496	0.0	**17.6645**	-2.56496	0.0	114.01979	-2.56496	0.0	38.88455	-2.56496	3e-05	59.54014
3000	-2.91419	0.0	**38.34989**	-2.91419	0.0	150.59695	-2.91419	6e-05	107.97778	-2.91419	0.0	51.46427
4000	-3.07692	0.0	56.98562	-3.07692	0.0	30.22585	-3.07692	0.0	23.00293	-3.07692	0.0	**8.81426**
5000	-3.42983	0.0	357.84432	-3.42982	0.0	708.79561	-3.42982	0.0	**28.31901**	-3.42982	0.0	114.95238
6000	-3.31894	0.0	770.34969	-3.31894	0.0	195.92418	-3.31894	0.0	**37.74394**	-3.31894	0.0	60.5929
7000	-3.17778	0.0	1130.40042	-3.17778	0.0	279.73536	-3.17778	0.0	**38.18163**	-3.17778	0.0	52.12805
8000	-3.29771	0.0	3379.38807	-3.29771	0.0	2085.37333	-3.29771	0.0	78.96939	-3.29771	0.0	**56.5659**
9000	-3.76984	0.0	2815.41645	-3.76984	0.0	2358.49339	-3.76984	0.0	**21.15597**	-3.76984	0.0	29.7628
10000	-3.50784	744.36444	3600.10713	-3.50329	780.86098	3600.08409	-3.50784	0.0	46.08346	-3.50784	7e-05	**40.78256**

Table B.75: NP-U-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-88.33333	0.0	9.23629	-88.33333	0.0	1.90984	-88.33333	0.0	**0.06248**	-88.33333	0.0	0.07812
2000	-93.82353	0.0	39.35525	-93.82353	0.0	6.37573	-93.82353	0.0	**0.09374**	-93.82353	0.0	0.09375
3000	-77680.9471	0.0	1.54702	-95.72414	0.0	11.04908	-95.72414	0.0	**0.07812**	-95.72414	0.0	**0.07812**
4000	-103808.93236	0.0	2.40659	-97.17073	0.0	580.76682	-97.23232	0.0	0.10948	-97.17073	0.0	**0.10933**
5000	-127063.96645	0.0	4.4226	-97.50943	0.0	739.36604	-97.50943	0.0	0.10937	-97.50943	0.0	**0.09373**
6000	-152175.97156	0.0	5.18949	-97.3617	0.0	972.72854	-97.3617	7e-05	**0.12499**	-97.3617	0.0	**0.12499**
7000	-177901.96044	0.0	7.17825	-97.86364	0.0	1387.95587	-97.86364	0.0	**0.10937**	-97.85938	4e-05	0.09374
8000	-204273.97603	0.0	5.54366	-98.19444	0.0	1557.92273	-98.19444	0.0	0.12499	-98.19444	0.0	**0.09373**
9000	-231622.93772	0.0	6.53226	-98.27586	0.0	2955.37628	-98.27738	0.0	0.1406	-98.27586	4e-05	**0.12497**
10000	-253141.96611	0.0	4.71688	-98.37975	0.0	76.79213	-98.37975	0.0	0.14062	-98.37975	0.0	**0.11202**

Table B.76: NP-U-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-85.14286	0.0	4.28153	-85.14286	0.0	1.42211	-85.14286	0.0	0.07813	-85.14286	0.0	**0.0781**
2000	-92.86667	0.0	11.59881	-92.86667	0.0	4.89113	-92.86667	0.0	**0.10938**	-92.86667	0.0	0.14062
3000	-78807.93173	0.0	1.28146	-95.25	0.0	222.42761	-95.25	0.0	0.15642	-95.25	0.0	**0.14062**
4000	-102284.97088	0.0	2.61918	-96.58333	0.0	415.98486	-96.58333	0.0	0.15624	-96.58333	0.0	**0.14063**
5000	-126282.98702	0.0	3.59561	-97.16667	0.0	40.86667	-126283.03691	0.0	0.17187	-97.16667	0.0	**0.20311**
6000	-151986.97985	0.0	6.86025	-97.47458	0.0	27.84743	-97.4864	0.0	0.21886	-97.47458	1e-05	**0.14086**
7000	-178549.96219	0.0	5.46935	-97.88462	0.0	36.84789	-97.88462	0.0	0.18749	-97.88462	5e-05	**0.14061**
8000	-205639.96297	0.0	5.40686	-99.48157	0.0	48.47397	-98.22785	0.0	**0.17199**	-98.22785	0.0	0.20311
9000	-231541.96279	0.0	3.42244	-98.13514	0.0	62.08832	-98.13514	1e-05	**0.15284**	-98.13514	7e-05	0.25
10000	-253666.97863	0.0	5.56397	-98.33333	0.0	76.24667	-98.33343	0.0	0.21875	-98.33333	0.0	**0.17186**

Table B.77: NP-U-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-82.5	0.0	1.79677	-82.5	0.0	1.21869	-82.5	0.0	**0.1406**	-82.5	0.0	0.17204
2000	-92.42857	0.0	52.73674	-92.42857	0.0	5.71696	-92.42857	0.0	**0.15623**	-92.42857	0.0	0.17184
3000	-77423.94609	0.0	1.75646	-95.25926	0.0	12.96898	-95.25926	0.0	0.32809	-95.25926	0.0	**0.26558**
4000	-102873.9641	0.0	2.9066	-96.45946	0.0	18.75051	-102873.82306	0.0	0.10936	-96.45946	0.0	**0.2656**
5000	-129911.93114	0.0	3.84403	-97.45098	0.0	48.77548	-97.45098	0.0	0.64069	-97.45098	0.0	**0.24998**
6000	-152459.9807	0.0	5.14162	-97.58621	0.0	685.19417	-152460.41845	0.0	0.18749	-97.58621	7e-05	**0.34372**
7000	-181085.95387	0.0	5.48468	-97.74576	0.0	809.37743	-97.74576	0.0	0.45309	-97.74576	0.0	**0.39058**
8000	-205657.96641	0.0	7.08824	-97.9375	0.0	75.85115	-97.9375	0.0	0.37497	-97.9375	0.0	**0.28122**
9000	-227810.976	0.0	6.9776	-98.13333	0.0	1516.72412	-227810.67614	0.0	0.4531	-98.13333	0.0	**0.28124**
10000	-256018.96438	0.0	5.00246	-98.3913	0.0	78.49823	-98.3913	0.0	0.54683	-98.3913	0.0	**0.43744**

Table B.78: NP-U-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-1.01085	0.0	1.50415	-1.00762	0.0	1.15633	-1.00336	0.0	**0.17184**	-1.00336	0.0	0.18745
2000	-1.04501	0.0	37.32584	-1.00757	0.0	14.51676	-1.00757	0.0	**0.14062**	-1.00757	0.0	0.17203
3000	-77744.9314	0.0	1.23452	-1.02339	0.0	302.33257	-1.00934	0.0	0.12497	-1.00826	0.0	**0.76557**
4000	-103343.93723	0.0	2.27015	-1.02	0.0	18.63416	-1.00932	0.0	**0.15622**	-1.08959	0.0	4.45343
5000	-127741.95427	0.0	3.67734	-1.25213	0.0	595.00718	-1.00901	0.0	0.125	-1.05382	0.0	0.60944
6000	-150887.99034	0.0	4.96932	-1.49	0.0	54.88818	-1.00913	0.0	**0.15623**	-1.12346	0.0	0.64068
7000	-182886.92871	0.0	6.39302	-1.81005	0.0	1567.61417	-1.0095	0.0	**0.15994**	-1.15528	0.0	0.4687
8000	-200832.98508	0.0	4.94571	-1.97999	0.0	163.97149	-1.00943	0.0001	0.1406	-1.00949	8e-05	**0.57516**
9000	-230412.96812	0.0	9.18056	-2.22999	0.0	187.89503	-1.00977	0.0	0.15623	-1.4	0.0	0.3149
10000	-256050.95879	0.0	10.46773	-2.55922	536.57154	3600.11742	-1.00962	0.0	**0.17184**	-1.26896	0.0	0.87494

Table B.79: NP-U-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-1.00336	0.0	1.65735	-1.00336	0.0	1.34385	-1.00336	0.0	**0.12513**	-1.00336	0.0	0.17187
2000	-1.00807	0.0	4.7667	-1.00807	0.0	4.4409	-1.00807	0.0	0.23436	-1.00807	6e-05	**0.21873**
3000	-78449.929	0.0	1.31258	-1.00854	0.0	18.92412	-1.00854	0.0	**0.28121**	-1.00854	0.0	1.06253
4000	-100525.00447	0.0	3.17224	-1.0844	0.0	18.46074	-1.00913	0.0	**0.28135**	-1.00913	6e-05	0.36251
5000	-127358.96998	0.0	3.75027	-1.24	0.0	32.50662	-1.0092	0.0	0.21872	-1.00918	1e-05	**0.34372**
6000	-152216.97073	0.0	6.07601	-1.51975	0.0	733.06157	-1.00922	0.0	**0.21874**	-1.00922	0.0	0.31247
7000	-177206.98942	0.0	4.57844	-1.74999	0.0	135.15094	-1.00942	7e-05	**0.17185**	-1.00942	0.0	0.18748
8000	-206645.9438	0.0	7.41152	-1.99513	0.0	1888.395	-1.00935	0.0	**0.28136**	-1.4	0.0	0.79681
9000	-230194.95863	0.0	5.26629	-2.21999	0.0	236.48386	-1.00945	6e-05	**0.24997**	-1.43814	0.0	0.49231
10000	-250099.99645	0.0	5.14099	-2.47999	0.0	242.85594	-1.00967	2e-05	0.23435	-1.00969	0.0	**0.70539**

Table B.80: NP-U-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-1.0	0.0	1.78105	-1.0	0.0	1.4063	-1.0	0.0	0.32811	-1.0	0.0	**0.26559**
2000	-1.01006	0.0	33.46885	-1.00605	0.0	3.98442	-1.00647	0.0	0.28132	-1.00605	0.0	**0.48447**
3000	-76546.95342	0.0	1.35975	-1.00841	0.0	18.20317	-1.00841	8e-05	0.96869	-1.00841	0.0	**0.40594**
4000	-102448.95422	0.0	2.67186	-1.13412	0.0	19.0159	-1.00891	0.0	**0.49709**	-1.00891	0.0	0.98433
5000	-128547.95254	0.0	3.43753	-1.2481	0.0	436.99195	-1.00897	0.0	0.67183	-1.00891	7e-05	**0.57808**
6000	-154141.96126	0.0	3.14072	-1.46	0.0	65.18737	-1.00932	0.0	**0.76572**	-1.00932	0.0	0.81254
7000	-176585.98489	0.0	4.37502	-1.71999	0.0	100.22474	-1.00942	0.0	**2.17188**	-1.00942	4e-05	3.32842
8000	-206440.94325	0.0	9.2034	-1.98503	0.0	1228.51148	-1.00949	3e-05	1.3102	-1.00949	8e-05	**1.26565**
9000	-231306.95665	0.0	5.32784	-2.22999	0.0	186.0067	-1.00955	7e-05	**0.50322**	-1.14907	0.0	11.1534
10000	-257676.9557	0.0	7.65613	-2.48065	0.0	2609.64418	-1.00961	0.0	**0.73434**	-1.00961	0.0	4.06274

Table B.81: NP-U-N-5

B.2.4 Unrestricted numerators and positive denominators with knapsack constraints

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-6.41964	0.0	5.11615	-6.41964	0.0	4.40825	-6.41964	0.0	**0.81261**	-6.41964	0.0	1.18033
2000	-7.42917	0.0	18.71182	-7.42917	0.0	29.78726	-7.42917	0.0	**0.40636**	-7.42917	0.0	0.59373
3000	-8.0	0.0	24.92816	-8.0	0.0	90.63149	-8.0	0.0	**0.56248**	-8.0	0.0	1.48461
4000	-8.3907	0.0	67.36577	-8.3907	0.0	27.01912	-8.3907	0.0	**0.70321**	-8.3907	7e-05	0.74177
5000	-9.62722	0.0	89.17191	-9.62722	0.0	78.15391	-9.62722	0.0	**0.76573**	-9.62722	0.0	0.84391
6000	-8.86175	0.0	167.0172	-8.86175	3e-05	66.37904	-8.86175	0.0	**1.12507**	-8.86175	0.0	1.30625
7000	-10.66518	0.0	234.19166	-10.66518	0.0	247.29992	-10.66518	0.0	1.32841	-10.66518	0.0	**1.12511**
8000	-10.87558	0.0	269.82032	-10.87558	0.0	202.53324	-10.87558	0.0	1.53144	-10.87558	0.0	**1.38185**
9000	-10.88144	0.0	356.30042	-10.88144	0.0	2153.93047	-10.88144	0.0	**2.08639**	-10.88144	6e-05	2.73467
10000	-11.56633	9e-05	362.36696	-11.56633	8e-05	234.25636	-11.56633	0.0	1.96175	-11.56633	0.0	**1.51581**

Table B.82: NP-K-R-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-6.57143	0.0	4.57851	-6.57143	0.0	3.38073	-6.57143	0.0	0.42183	-6.57143	0.0	**0.29684**
2000	-4.27755	0.0	11.48191	-4.27755	0.0	13.30931	-4.27755	0.0	2.65977	-4.27755	0.0	**2.23448**
3000	-5.89362	0.0	33.21065	-5.89362	0.0	135.24725	-5.89362	0.0	**0.73448**	-5.89362	0.0	2.93768
4000	-5.43919	0.0	48.88309	-5.43919	0.0	55.60717	-5.43919	0.0	**1.1719**	-5.43919	3e-05	1.21877
5000	-6.90722	0.0	91.51714	-6.90722	0.0	87.03232	-6.90722	0.0	**1.95437**	-6.90722	0.0	2.1408
6000	-6.47727	0.0	100.20608	-6.47727	0.0	158.59509	-6.47727	0.0	**1.57812**	-6.47727	0.0	3.32829
7000	-8.7069	0.0	151.81586	-8.7069	0.0	164.76631	-8.7069	0.0	**1.10943**	-8.7069	0.0	1.46877
8000	-6.21519	0.0	167.0901	-6.21519	0.0	1346.21857	-6.21519	0.0	**6.06285**	-6.21519	0.0	8.07858
9000	-5.8659	0.0	284.48727	-5.8659	0.0	3438.67792	-5.8659	0.0	**5.00126**	-5.8659	0.0	12.93949
10000	-6.15294	0.0	294.74994	-6.15294	0.0	275.24231	-6.15294	0.0	**2.89085**	-6.15294	0.0	13.51635

Table B.83: NP-K-R-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-2.15057	0.0	**7.04726**	-2.15057	0.0	13.81739	-2.15057	0.0	53.1118	-2.15057	1e-05	69.70219
2000	-3.1055	0.0	16.48646	-3.1055	0.0	27.59639	-3.1055	0.0	55.42957	-3.1055	0.0	**10.02034**
3000	-3.07184	0.0	**36.11282**	-3.07184	0.0	122.48213	-3.07184	0.0001	58.67834	-3.07184	0.0	82.20169
4000	-2.78026	0.0	75.26068	-2.78026	0.0	219.68811	-2.78026	1e-05	**55.55231**	-2.78026	0.0001	106.97944
5000	-3.48603	0.0	133.72211	-3.48603	0.0	398.75456	-3.48603	0.0	**10.08926**	-3.48603	0.0	10.51639
6000	-3.15591	0.0	120.16904	-3.15591	0.0	87.09889	-3.15591	0.0	127.98712	-3.15591	0.0	**21.08731**
7000	-3.24742	0.0	160.80197	-3.24742	0.0	1807.78817	-3.24742	0.0	40.90784	-3.24742	0.0	**35.23633**
8000	-3.43421	0.0	**166.27399**	-3.43421	0.0	2696.81407	-3.43421	9e-05	210.98642	-3.43421	0.0	190.81244
9000	-3.1949	0.0	322.61005	-3.18447	505.34965	3600.26791	-3.1949	3e-05	142.18078	-3.1949	4e-05	**81.49132**
10000	-3.59657	523.1877	3600.22926	-3.61694	0.0	256.62869	-3.61694	0.0	**20.62014**	-3.61694	0.0	44.86759

Table B.84: NP-K-R-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-85.28571	0.0	8.5313	-85.28571	0.0	1.00015	-85.28571	0.0	**0.09373**	-85.28571	0.0	0.12496
2000	-94.55	0.0	7.39082	-94.55	0.0	3.71864	-94.55	0.0	0.1876	-94.55	0.0	**0.15774**
3000	-77116.95315	0.0	1.47764	-95.61538	1e-05	6.32781	-95.61538	0.0	0.21872	-95.61538	0.0	**0.21871**
4000	-101817.96665	0.0	2.50042	-96.36667	0.0	9.01881	-96.36667	0.0	0.23434	-96.36667	0.0	**0.21874**
5000	-128209.94564	0.0	4.42188	-97.19149	0.0	24.12836	-97.19149	0.0	0.39059	-97.19149	0.0	**0.2033**
6000	-152243.95847	0.0	7.59358	-97.44828	0.0	20.49125	-97.44828	0.0	0.42072	-97.44828	0.0	**0.29685**
7000	-175855.99061	0.0	8.25031	-97.95161	0.0	31.05877	-97.95161	0.0	0.42183	-97.95161	0.0	**0.35964**
8000	-204160.97338	0.0	11.79704	-97.94186	0.0	**42.85198**	-97.94406	0.0	0.60951	-97.9375	6e-05	0.29684
9000	-228627.98265	0.0	6.31262	-98.29333	0.0	81.66433	-98.29333	5e-05	0.70318	-98.29333	0.0	**0.35934**
10000	-255237.9662	0.0	10.5626	-98.425	0.0	83.45949	-98.425	4e-05	0.73445	-98.425	0.0	**0.29683**

Table B.85: NP-K-P-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-83.16667	0.0	1.43752	-83.16667	0.0	0.96889	-83.16667	0.0	0.15624	-83.16667	0.0	**0.15621**
2000	-94.63636	0.0	4.70241	-94.63636	0.0	3.46666	-94.63636	0.0	0.2031	-94.63636	0.0	**0.15623**
3000	-76539.97614	0.0	1.70337	-95.0	0.0	8.8203	-76539.83282	0.0	0.32826	-95.0	0.0	**0.23446**
4000	-103288.94438	0.0	2.62747	-96.66667	3e-05	18.65835	-96.66667	0.0	0.43747	-96.66667	4e-05	**0.24997**
5000	-126864.96715	0.0	4.5348	-97.42308	0.0	12.89593	-97.42308	0.0	0.40945	-97.42308	0.0	**0.37508**
6000	-153964.95264	0.0	5.35989	-97.46	0.0	20.1455	-97.46	0.0	**0.41963**	-97.61151	0.0	0.35935
7000	-176280.99377	0.0	8.73743	-97.94366	0.0	**76.22441**	-176280.65537	0.0	0.42185	-97.93846	8e-05	0.39059
8000	-202645.97598	0.0	10.75196	-98.1875	0.0	61.37405	-98.1875	0.0	0.74994	-98.1875	0.0	**0.45434**
9000	-228049.97107	0.0	8.00537	-98.29348	0.0	82.21244	-98.29348	0.0	0.49995	-98.29348	1e-05	**0.42184**
10000	-257767.95463	0.0	9.54947	-98.51485	0.0	104.79632	-98.51609	0.0	0.84381	-98.51485	0.0	**0.45308**

Table B.86: NP-K-P-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-85.00001	0.0	2.96658	-85.0	0.0	1.06257	-85.0	0.0	0.23445	-85.0	0.0	**0.17187**
2000	-50708.98342	0.0	0.76558	-94.3	0.0	3.67213	-94.3	0.0	0.34373	-94.3	0.0	**0.28133**
3000	-78987.89894	0.0	1.59377	-96.0	0.0	9.20446	-96.0	0.0	0.43758	-96.0	0.0	**0.32809**
4000	-103665.92631	0.0	2.93844	-96.625	0.0	17.74962	-96.625	0.0	0.59797	-96.625	0.0	**0.43756**
5000	-131397.90881	0.0	2.61222	-97.38462	0.0	14.56351	-97.40422	0.0	0.6095	-97.38462	0.0	**0.62508**
6000	-156508.94324	0.0	5.84418	-97.47917	0.0	113.86906	-97.47917	4e-05	0.84043	-97.47917	0.0	**0.59381**
7000	-178944.95594	0.0	8.10992	-97.92424	0.0	29.58666	-97.92424	5e-05	0.84386	-97.92424	2e-05	**0.73171**
8000	-207624.93749	0.0	7.23839	-98.10667	0.0	59.32132	-98.10667	4e-05	0.92191	-98.10667	0.0	**0.58144**
9000	-231196.94264	0.0	8.50075	-98.06579	0.0	**88.92193**	-98.06329	3e-05	1.10943	-98.0641	2e-05	0.65635
10000	-258818.93827	0.0	9.45545	-98.207	0.0	106.17761	-98.20513	8e-05	0.98443	-98.20513	5e-05	**0.73432**

Table B.87: NP-K-P-5

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-1.00336	0.0	1.51014	-1.00336	0.0	1.25022	-1.00357	0.0	0.15626	-1.00336	0.0	**0.12498**
2000	-50879.00384	0.0	0.6875	-1.00757	0.0	3.36389	-1.00757	0.0	**0.34371**	-1.0124	0.0	0.24996
3000	-73598.02027	0.0	1.29672	-1.00605	0.0	8.98568	-1.00605	0.0	**0.28133**	-1.00605	0.0	0.41777
4000	-101642.96989	0.0	2.48415	-1.00826	0.0	18.0078	-1.01644	0.0	0.23436	-1.00826	0.0	**0.40622**
5000	-129643.93615	0.0	3.90602	-1.00918	0.0	27.20601	-1.00949	0.0	0.42194	-1.00918	0.0	**0.40621**
6000	-149064.01425	0.0	5.25773	-1.0095	0.0	44.94566	-1.0095	0.0	**0.37495**	-1.00945	9e-05	0.422
7000	-183365.91848	0.0	6.66128	-1.01751	0.0	73.21593	-1.00963	2e-05	3.98113	-1.00963	0.0	**0.67192**
8000	-207267.94331	0.0	11.25064	-1.17576	0.0	90.17715	-1.00967	0.0	0.56245	-1.00958	0.0	**0.37497**
9000	-229419.95671	0.0	14.04495	-1.00963	0.0	77.87795	-1.00963	0.0	0.48883	-1.00963	0.0	**0.42183**
10000	-258012.9625	0.0	18.91393	-1.96579	0.0	180.46128	-1.00965	0.0	**0.49996**	-1.00965	0.0	1.92057

Table B.88: NP-K-N-2

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-0.99997	0.0	1.38228	-0.99888	0.0	1.26565	-0.99888	0.0	**0.18749**	-0.99888	0.0	0.20308
2000	-1.00785	0.0	3.85717	-1.02066	0.0	3.52402	-1.00785	0.0	**0.28135**	-1.00785	6e-05	0.30122
3000	-78435.90631	0.0	1.37951	-1.00875	0.0	6.0317	-1.00875	0.0	**0.29684**	-1.00875	0.0	0.42184
4000	-101632.9855	0.0	2.93138	-1.00883	0.0	17.9991	-1.00883	0.0	0.43759	-1.00883	0.0	**0.42195**
5000	-130785.93312	0.0	4.4563	-1.00918	0.0	27.39864	-1.01204	0.0	0.39407	-1.00918	0.0	**0.49872**
6000	-154611.95836	0.0	4.21899	-1.0094	0.0	44.98095	-1.00954	0.0	0.34373	-1.0094	0.0	**0.93742**
7000	-179970.94836	0.0	4.55457	-1.47815	0.0	62.41885	-1.00947	0.0	**0.40637**	-1.00947	0.0	0.62315
8000	-206599.94148	0.0	11.89604	-1.00942	0.0	102.15307	-1.00942	0.0	**0.62496**	-1.00942	0.0	0.73431
9000	-229255.97734	0.0	11.06911	-1.21055	0.0	137.21087	-1.0096	0.0	0.60153	-1.00959	0.0	**0.58607**
10000	-255578.96276	0.0	15.08701	-1.0141	0.0	177.99787	-1.00957	0.0	0.79698	-1.00957	0.0	**0.59372**

Table B.89: NP-K-N-3

n	MMFP1			MMFP1M			MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
1000	-1.0	0.0	1.32815	-1.0	0.0	1.3594	-1.0	0.0	**0.28121**	-1.0	0.0	0.3126
2000	-1.01207	0.0	4.34492	-1.00605	0.0	3.7535	-1.00605	0.0	0.43747	-1.00605	0.0	**0.36477**
3000	-77813.91736	0.0	1.61376	-1.00826	0.0	9.61498	-1.00826	0.0	**0.53133**	-1.00826	0.0	0.54683
4000	-103115.96368	0.0	3.23547	-1.00865	0.0	17.72139	-1.00865	8e-05	**0.53478**	-1.00865	0.0	1.15401
5000	-128052.95616	0.0	3.66745	-1.00854	0.0	27.52429	-1.00854	0.0	1.03133	-1.00854	0.0	**0.73446**
6000	-151273.97421	0.0	5.20356	-1.00922	0.0	45.33736	-1.00918	4e-05	0.90406	-1.00922	0.0	**1.10646**
7000	-179508.95158	0.0	6.79734	-1.5958	0.0	71.56579	-1.00949	0.0	**0.69798**	-1.00949	0.0	0.73447
8000	-210046.91647	0.0	9.07817	-1.00929	0.0	97.80592	-1.00941	0.0	0.59369	-1.00929	0.0	**2.62636**
9000	-230435.95181	0.0	10.11576	-1.18176	0.0	139.72162	-1.00952	8e-05	**0.68216**	-1.00949	4e-05	2.39045
10000	-259081.9427	0.0	15.8498	-1.62389	0.0	181.84024	-1.00955	0.0	**0.92195**	-1.00955	7e-05	1.28119

Table B.90: NP-K-N-5

B.3 Computational results with large size data

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	0.00059	0.0	0.04687	0.33427	0.0	**25.09384**
4000	0.00027	0.0	0.07812	0.3152	0.0	**7.73437**
6000	0.00015	0.0	0.09384	0.31217	0.0	**16.32814**
8000	0.00012	0.0	0.10943	0.27426	0.0	**11.93734**
10000	0.00016	0.0	0.16552	0.25691	0.0	**8.82821**
12000	8e-05	0.0	0.17208	0.27995	0.0	**88.91491**
14000	9e-05	0.0	0.18755	0.24796	7e-05	**17.52132**
16000	6e-05	0.0	0.23442	0.22173	0.0	**21.80101**
18000	9e-05	0.0	0.23438	0.24718	0.0	**34.79701**
20000	7e-05	0.0	0.26566	0.2333	0.0	**48.41484**

Table B.91: PP-U-R-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	1e-05	0.0	0.01562	0.99655	1e-05	**0.34382**
4000	0.0	0.0	0.03125	0.97826	0.0	**0.34365**
6000	0.0	0.0	0.03125	0.94872	0.0	**0.35942**
8000	0.0	0.0	0.04688	0.95276	4e-05	**1.40621**
10000	0.0	0.0	0.04687	0.94444	0.0	**1.12501**
12000	0.0	0.0	0.04687	0.98438	0.0	**0.65615**
14000	0.0	0.0	0.06248	0.96667	0.0	**0.5312**
16000	0.0	0.0	0.06255	0.98039	0.0	**0.50013**
18000	0.0	0.0	0.07813	1.0	0.0	**0.06248**
20000	0.0	0.0	0.07813	0.98684	0.0	**0.46871**

Table B.92: PP-U-P-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	1e-05	0.0	0.04687	0.01071	0.0	**0.34389**
4000	0.0	0.0	0.03125	0.01037	0.0	**0.43744**
6000	0.0	0.0	0.04666	0.01028	0.0	**0.85942**
8000	0.0	0.0	0.03114	0.01024	0.0	**0.65637**
10000	0.0	0.0	0.04687	0.01022	0.0	**0.68707**
12000	0.0	0.0	0.0625	0.01022	0.0	**0.4532**
14000	0.0	0.0	0.0627	0.01019	0.0	**0.92193**
16000	0.0	0.0	0.06228	0.01017	0.0	**0.60926**
18000	0.0	0.0	0.0625	0.01016	0.0	**0.43754**
20000	0.0	0.0	0.07812	0.01016	0.0	**0.53126**

Table B.93: PP-U-N-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	0.00119	0.0	0.04686	0.33427	0.0	**19.14795**
4000	0.00055	0.0	0.07812	0.3152	0.0	**14.54802**
6000	0.0003	0.0	0.10937	0.31217	0.0	**14.05712**
8000	0.00024	0.0	0.1406	0.27426	0.0	**8.39432**
10000	0.00016	0.0	0.14061	0.25691	8e-05	**9.08988**
12000	0.00017	0.0	0.23453	0.27995	0.0	**48.68132**
14000	0.00019	0.0	0.23446	0.24796	0.0	**13.86344**
16000	0.00013	0.0	0.28121	0.22173	0.0	**19.48663**
18000	0.00018	0.0	0.29686	0.24718	0.0	**33.1143**
20000	7e-05	0.0	0.28122	0.2333	0.0	**28.34906**

Table B.94: PP-K-R-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	2e-05	0.0	0.03124	0.99655	7e-05	**1.29708**
4000	1e-05	0.0	0.06248	0.97826	0.0	**1.31252**
6000	0.0	0.0	0.07811	0.94872	0.0	**1.59381**
8000	0.0	0.0	0.09373	0.95276	0.0	**2.96894**
10000	0.0	0.0	0.12498	0.94444	0.0	**4.34513**
12000	0.0	0.0	0.12517	0.98438	0.0	**3.01579**
14000	0.0	0.0	0.18763	0.96667	0.0	**5.28166**
16000	0.0	0.0	0.21872	0.98039	5e-05	**4.8292**
18000	0.0	0.0	0.23435	1.0	0.0	**0.23438**
20000	0.0	0.0	0.24997	0.98684	0.0	**7.57558**

Table B.95: PP-K-P-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	2e-05	0.0	0.03144	0.01071	0.0	**0.54711**
4000	1e-05	0.0	0.06252	0.01037	0.0	**1.10942**
6000	0.0	0.0	0.07796	0.01028	0.0	**1.34417**
8000	0.0	0.0	0.09401	0.01024	1e-05	**3.07821**
10000	0.0	0.0	0.12494	0.01022	5e-05	**1.99999**
12000	0.0	0.0	0.15622	0.01022	0.0	**5.03087**
14000	0.0	0.0	0.1561	0.01018	0.0001	**2.39021**
16000	0.0	0.0	0.23027	0.01017	7e-05	**2.40626**
18000	0.0	0.0	0.20308	0.01016	0.0	**2.84364**
20000	0.0	0.0	0.21874	0.01016	0.0	**3.15631**

Table B.96: PP-K-N-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-1.75802	0.0	83.62413	-1.75802	0.0	**6.88158**
4000	-1.73321	0.0	**65.98105**	-1.73321	0.0	87.76125
6000	-1.94286	0.0	202.47717	-1.94286	0.0	**51.16532**
8000	-2.34328	0.0	74.79839	-2.34328	0.0	**23.55714**
10000	-2.11217	0.0	**95.21456**	-2.11217	0.0	179.67899
12000	-2.38968	7e-05	**44.70473**	-2.38968	0.0	146.08935
14000	-2.47761	0.0	**43.06535**	-2.47761	0.0	200.57985
16000	-2.41492	4e-05	125.5793	-2.41492	0.0	**83.97099**
18000	-2.46711	0.0	**82.23409**	-2.46711	0.0	90.21238
20000	-2.401	0.0	**85.98351**	-2.40101	0.0	93.9901

Table B.97: NP-U-R-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-92.28571	0.0	0.26569	-92.28571	0.0	**0.23498**
4000	-96.3871	0.0	0.26549	-96.3871	0.0	**0.23424**
6000	-97.52941	0.0	**0.5784**	-97.52941	0.0	1.65658
8000	-98.02899	0.0	0.92186	-98.02899	0.0	**0.5**
10000	-98.275	0.0	1.17183	-98.275	0.0	**0.67189**
12000	-306474.12336	0.0	0.3906	-98.6055	0.0	**0.67182**
14000	-98.68595	0.0001	0.5625	-98.69355	1e-05	**0.34378**
16000	-98.82517	7e-05	**0.40642**	-98.86207	0.0	0.62513
18000	-455662.13099	0.0	0.1406	-98.95062	3e-05	0.95314
20000	-507990.54539	0.0	0.54676	-99.0	0.0	0.73422

Table B.98: NP-U-P-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-1.00757	0.0	**0.32813**	-1.00757	0.0	1.06258
4000	-1.00854	4e-05	0.76522	-1.00854	0.0	**0.4373**
6000	-154844.11357	0.0	0.32786	-1.00926	0.0	**0.84377**
8000	-1.00952	0.0	1.87547	-1.0095	3e-05	**1.79691**
10000	-1.00959	0.0	**0.64033**	-1.00959	0.0	2.69963
12000	-1.0097	0.0	**3.49983**	-1.19015	0.0	1.49997
14000	-1.00977	0.0	**2.8282**	-1.00977	5e-05	3.88983
16000	-1.43411	0.0	4.92189	-1.00974	3e-05	**2.00001**
18000	-1.43191	0.0	2.40621	-1.03215	0.0	11.42143
20000	-1.00982	0.0	**0.46882**	-1.00977	4e-05	2.89061

Table B.99: NP-U-N-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-1.75802	0.0	**17.20204**	-1.75802	2e-05	85.1452
4000	-1.73321	0.0	**60.28024**	-1.73321	0.0	146.5776
6000	-1.94286	0.0	217.86386	-1.94286	0.0	**175.29366**
8000	-2.34328	0.0	**195.68488**	-2.34328	7e-05	200.33309
10000	-2.11217	5e-05	79.02519	-2.11217	5e-05	**31.6494**
12000	-2.38968	0.0	**52.31295**	-2.38968	0.0001	122.43334
14000	-2.47761	0.0	**46.98474**	-2.47761	9e-05	76.40245
16000	-2.41492	1e-05	102.04652	-2.41492	4e-05	**63.78644**
18000	-2.46711	0.0	222.19852	-2.46711	0.0	**92.12771**
20000	-2.401	0.0	**143.01814**	-2.401	0.0001	464.82919

Table B.100: NP-K-R-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-92.28571	0.0	0.31257	-92.28571	0.0	**0.25021**
4000	-96.3871	0.0	0.46891	-96.3871	0.0	**0.34371**
6000	-97.52941	0.0	0.60938	-97.52941	0.0	**0.48446**
8000	-98.02899	1e-05	0.93756	-98.02899	1e-05	**0.62507**
10000	-98.275	0.0	1.1407	-98.275	7e-05	**0.71874**
12000	-98.6055	0.0	**1.35893**	-98.60185	5e-05	0.73435
14000	-98.70958	0.0	1.67166	-98.69355	0.0	**0.81294**
16000	-98.83108	0.0	**2.24998**	-98.82759	4e-05	1.07807
18000	-455661.26197	0.0	1.12512	-98.93125	1e-05	**1.12492**
20000	-98.95676	8e-05	2.17047	-98.95789	3e-05	**1.18737**

Table B.101: NP-K-P-5

n	MMFP2			MMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
2000	-1.00757	0.0	**0.34417**	-1.00757	0.0	0.37511
4000	-1.00854	9e-05	**0.45316**	-1.00854	0.0	0.7031
6000	-1.00926	0.0	**0.57816**	-1.00926	5e-05	1.54689
8000	-1.00952	0.0	**0.719**	-1.00952	0.0	2.32824
10000	-1.00961	0.0	0.57809	-1.00959	5e-05	**2.48436**
12000	-1.0097	0.0	0.92183	-1.00963	4e-05	**3.81201**
14000	-1.00977	0.0	**5.57706**	-1.00977	0.0	9.04624
16000	-1.00976	6e-05	**1.04669**	-1.00976	9e-05	1.46853
18000	-1.00979	0.0	**1.56246**	-1.00979	9e-05	3.96876
20000	-1.00978	6e-05	2.04682	-1.00983	1e-05	**1.20212**

Table B.102: NP-K-N-5

n	MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.93631	0.0	**82.78966**	-0.93631	2e-05	582.27223
60	-1.27901	2e-05	**225.98732**	-1.27901	1e-05	645.00709
90	-1.30221	0.02297	3600.03831	-1.30907	**0.00947**	3600.13
120	-1.51389	0.0	**357.31366**	-1.51389	0.0	478.81716
150	-1.56362	0.00278	3600.11501	-1.56561	0.00138	3600.10353
180	-1.71429	0.0001	2232.66617	-1.71429	0.0	**759.47335**
210	-1.72034	2e-05	**1661.12503**	-1.71911	0.0017	3600.13548
240	-1.75646	0.00232	3600.0984	-1.75858	**0.00056**	3600.21718
270	-1.85142	0.00125	3600.11397	-1.85142	**0.00085**	3600.19662
300	-1.86465	0.00812	3600.22621	-1.86737	**0.00129**	3600.28482

Table B.103: NP-A-R-5

n	MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.15252	0.0	2.67196	-0.15252	0.0	**1.89539**
60	-0.1409	0.0	15.7676	-0.1409	0.0	**3.82171**
90	-0.10715	0.0	**107.18753**	-0.10715	0.0	153.87939
120	-0.09628	9e-05	**23.90911**	-0.09628	0.0	25.98723
150	-0.09145	0.0	**21.22586**	-0.09145	9e-05	92.82701
180	-0.08967	9e-05	**117.27769**	-0.08967	8e-05	149.69282
210	-0.079	0.0001	**89.01904**	-0.079	0.0001	289.11255
240	-0.07166	0.0001	**800.29806**	-0.07166	0.0001	1076.15333
270	-0.07012	0.0001	**240.72992**	-0.07012	0.0001	1078.71567
300	-0.06604	0.0001	**520.3185**	-0.06604	0.0001	1014.73613

Table B.104: NP-A-P-5

n	MMFP2A			MMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.15193	0.0	4.87503	-0.15193	0.0	**4.46409**
60	-0.12563	0.0	**6.26836**	-0.12563	0.0	13.58492
90	-0.09783	0.0	35.86444	-0.09783	0.0	**26.35222**
120	-0.09724	4e-05	73.78809	-0.09724	0.0	**69.81655**
150	-0.09505	9e-05	1338.57992	-0.09505	0.0001	**675.22626**
180	-0.08534	9e-05	251.40497	-0.08534	6e-05	**98.13707**
210	-0.0774	7e-05	156.68369	-0.0774	4e-05	**119.69937**
240	-0.07678	0.0001	**210.46307**	-0.07678	0.0001	492.80779
270	-0.07079	0.0001	165.34285	-0.07079	0.0001	**72.13409**
300	-0.07061	0.0001	**644.21453**	-0.07061	0.0001	670.90898

Table B.105: NP-A-N-5

Appendix C

Computational results for chapter 3

C.1 Unrestricted numerators and denominators with no constraint

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-242.0	0.0	31.01878	-242.0	0.0	55.5042	-242.0	0.0	17.84999	-242.0	0.0	**7.51641**
40	-572.00418	0.93705	3600.08559	-491.0	1.25662	3600.23464	-572.0	0.0	**8.3483**	-572.0	0.0	20.6145
60	-633.0	1.09479	3600.0899	-807.0	0.64312	3600.09659	-1028.0	0.0	**15.50369**	-1028.0	0.0	16.67496
80	-1317.0	0.11693	3600.0921	-782.01012	0.88105	3600.21686	-1471.0	0.0	37.79348	-1471.0	0.0	**25.8634**
100	-1332.00732	0.83332	3600.06836	-1139.01573	1.14396	3600.17674	-1686.0	0.0	16.43901	-1686.0	0.0	**14.27948**
120	-1232.0	1.17045	3600.21016	-1259.00891	1.12389	3600.44119	-2056.0	0.0	**19.54893**	-2056.0	0.0	25.41213
140	-1525.00795	1.19737	3600.08442	-1724.0	0.94374	3600.07573	-2362.0	3e-05	**19.48747**	-2362.0	0.0	31.12041
160	-2224.0	0.53552	3600.09936	-2010.0	0.699	3600.18568	-3123.0	0.0	34.21598	-3123.0	0.0	**23.3289**
180	-1509.0	1.95891	3600.10851	-1683.0	1.653	3600.11899	-3427.0	0.0	**31.31003**	-3427.0	0.0	31.80066
200	-1832.0061	1.45087	3600.07299	-2269.0	0.97885	3600.10355	-3616.0	5e-05	**20.27621**	-3616.0	3e-05	22.58253

Table C.1: NN-U-R-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-42.875	0.0	16.89112	-42.875	0.0	**13.91514**	-42.875	0.0	17.49303	-42.875	0.0	16.55334
40	-224.00925	3.77212	3600.09249	-152.0	6.03289	3600.20168	-280.0	0.0	**133.01383**	-280.0	0.0	286.56222
60	-272.0	3.85662	3600.08754	-184.0	6.17935	3600.11806	-743.0	2e-05	755.98162	-743.0	0.0	**383.70227**
80	-427.00683	2.7798	3600.1362	-424.00604	2.80655	3600.148	-1116.0	5e-05	**179.03069**	-1116.0	1e-05	322.3364
100	-721.0	1.9251	3600.10784	-329.01188	5.4101	3601.24405	-1403.0	8e-05	**1877.25834**	-1403.0	9e-05	2522.26938
120	-344.5	6.27721	3600.09655	-402.01478	5.23609	3600.12621	-1857.0	0.00774	3600.59966	-1855.0	0.0122	3602.8293
140	-340.0	9.65882	3600.11789	-579.5	5.25367	3600.11696	-2099.0	9e-05	3567.06174	-2099.0	9e-05	**1850.96395**
160	-893.0	2.91713	3600.09846	-871.0	3.01607	3600.43663	-2615.0	0.0001	1810.74691	-2615.0	9e-05	**1493.27756**
180	-581.0	6.33563	3600.13482	-548.0	6.77737	3600.18963	-2560.0	9e-05	778.65526	-2560.0	8e-05	**750.0015**
200	-285.51819	12.36167	3600.24392	-502.00346	6.59955	3600.1939	-3227.0	0.00427	3600.03284	-3199.0	0.01457	3600.10333

Table C.2: NN-U-R-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-7.29412	0.0	9.62505	-7.29412	0.0	**4.50079**	-7.29412	0.0	10.23554	-7.29412	0.0	6.14693
40	-201.5	4.54342	3600.0309	-134.00075	7.33577	3600.0296	-278.00525	0.0	**764.62159**	-278.0	2e-05	981.41031
60	-144.0	10.23611	3600.04059	-238.5	5.78407	3600.03591	-583.0	0.08422	3600.53175	-437.0	0.44249	3600.11189
80	-210.0	8.7381	3600.03921	-147.4	12.87381	3600.04139	-472.0	1.27071	3600.06007	-586.5	0.81425	3601.05954
100	-130.00486	18.28389	3600.05486	-123.0	19.38211	3600.04872	-812.0	0.58932	3600.79855	-1140.0	0.13079	3600.0953
120	-259.5	9.79769	3600.03943	-380.33333	6.36722	3600.07725	-1276.0	0.25452	3600.04817	-1104.0	0.45176	3600.91548
140	-292.50188	11.04437	3600.05187	-198.00429	16.79254	3600.12164	-902.0	0.96925	3600.07006	-1192.0	0.48633	3600.08673
160	-189.0	20.5291	3600.08147	-196.5	19.70738	3600.13599	-612.5	2.40444	3600.0972	-520.66667	3.0078	3600.12101
180	-101.25666	43.56003	3600.10505	-201.66667	21.37355	3600.1437	-683.75	2.2501	3603.20432	-312.0	6.12191	3602.6934
200	-332.0	13.88855	3600.2411	-397.0	11.45088	3600.0778	-2547.0	0.01857	3600.06958	-558.0	3.67974	3600.12746

Table C.3: NN-U-R-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-4.08333	0.0	25.00593	-4.08333	0.0	**19.7397**	-4.08333	0.0	29.68061	-4.08333	0.0	32.61759
40	-18.74074	43.82213	3600.06421	-18.3	44.90164	3600.15333	-23.4	0.0	2115.09812	-23.4	0.0	**1963.77175**
60	-33.16735	36.20527	3600.07725	-53.66667	21.99379	3600.09447	-19.82143	30.35726	3600.21259	-19.08163	31.43557	3600.61031
80	-45.46154	40.00169	3600.10235	-44.9	40.51448	3600.41138	-30.85715	26.84362	3601.72305	-11.41176	78.80185	3600.31612
100	-55.6	38.51439	3600.08955	-44.62683	48.23047	3600.19208	-16.9092	72.36193	3600.43653	-82.70588	14.00126	3601.80593
120	-52.60293	47.47639	3600.10579	-78.8	31.36041	3600.14404	-39.05128	32.17703	3600.89025	-43.47619	28.72504	3600.40556
140	-62.75	44.4502	3600.23895	-69.98768	39.75003	3600.16906	-16.61672	90.08264	3602.46277	-1.88275	807.92574	3600.72273
160	-73.8	54.50136	3600.19303	-56.09091	72.02431	3600.33212	-28.30769	71.68168	3600.10351	-8.38462	248.57905	3600.20491
180	-52.4375	78.17998	3600.10067	-33.54286	122.78194	3600.14555	-5.09677	384.93801	3600.19566	-11.38053	190.42796	3600.28812
200	-57.51383	82.26694	3600.24998	-46.17647	102.71083	3600.13027	-9.96774	237.46896	3603.82326	-76.81818	29.02886	3600.18899

Table C.4: NN-U-R-5

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-2.03333	0.0	3.65622	-2.03333	0.0	4.04437	-2.03333	0.0	1.45332	-2.03333	0.0	**1.26567**
40	-253.0	2.62026	3600.10851	-258.00652	2.55417	3600.06183	-258.0	0.0	3.70527	-258.0	0.0	**1.32816**
60	-5.05374	219.86581	3600.13544	-5.04867	220.12468	3600.16848	-5.04762	0.0	2.90629	-5.04762	0.0	**2.01586**
80	-44.0119	37.89857	3600.2647	-35.01043	47.87669	3600.149	-45.0	0.0	2.48456	-45.0	0.0	**1.2188**
100	-385.01806	4.46468	3600.2086	-390.00135	4.39485	3600.0854	-392.0	0.0	**2.87519**	-392.0	0.0	4.47106
120	-172.00906	16.29531	3600.12915	-175.99998	15.90909	3600.18312	-184.0	0.0	2.26571	-184.0	0.0	**1.4219**
140	-394.0	7.2335	3600.0847	-426.0	6.61502	3600.18315	-427.0	0.0	3.4846	-427.0	0.0	**3.29442**
160	-309.0	11.36938	3600.15309	-298.0	11.82599	3600.07085	-345.0	0.0	6.40265	-345.0	0.0	**5.84424**
180	-322.0	13.83851	3600.16043	-322.34647	13.82256	3600.15715	-330.0	0.0	2.89078	-330.0	0.0	**2.45413**
200	-38.00959	123.46858	3600.18721	-35.00006	134.14869	3600.2365	-38.0	0.0	2.54705	-38.0	0.0	**1.56255**

Table C.5: NN-U-P-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-0.14286	0.0	**2.84436**	-0.14286	0.0	2.95329	-0.14286	0.0	3.76582	-0.14286	0.0	3.5159
40	-87.00357	9.31724	3600.04454	-94.0	8.5479	3600.07962	-94.0	0.0	7.29751	-94.0	0.0	**6.78118**
60	-22.01298	63.68909	3600.0538	-21.00907	66.78025	3600.19163	-26.0	0.0	3.59601	-26.0	0.0	**2.25009**
80	-1.50102	1259.39178	3600.1332	-1.50823	1252.94011	3600.17696	-1.5	0.0	4.46758	-1.5	0.0	**3.48166**
100	-7.53628	321.04022	3600.17994	-7.53628	321.04365	3600.35895	-7.28571	0.0	**3.54841**	-7.28571	0.0	3.59386
120	-3.01351	952.56775	3600.0688	-2.0594	1394.24832	3600.18645	-2.05882	0.0	10.62441	-2.05882	0.0	**6.00727**
140	-32.00698	97.69721	3600.18369	-35.01724	89.21271	3600.33573	-44.0	0.0	11.14641	-44.0	0.0	**5.91342**
160	-37.01053	86.08333	3600.09045	-9.00983	356.72043	3600.32053	-46.0	0.0	5.94068	-46.0	0.0	**5.82585**
180	-63.01401	65.47728	3600.18829	-80.02019	51.34929	3600.18616	-126.0	0.0	4.22241	-126.0	0.0	**2.59423**
200	-194.0	23.03093	3600.06002	-187.0	23.93048	3600.15617	-277.0	0.0	**6.68836**	-277.0	0.0	8.25615

Table C.6: NN-U-P-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.0667	0.0	1.35913	-1.0667	0.0	2.15647	-1.06667	0.0	1.70654	-1.06667	0.0	**1.01557**
40	-4.0581	208.09129	3600.02433	-4.05263	209.79473	3600.04311	-4.0	0.0	**2.35213**	-4.0	0.0	4.73479
60	-4.63213	330.23234	3600.06607	-4.63403	331.20633	3600.08834	-4.63158	0.0	7.96098	-4.63158	0.0	**4.85994**
80	-0.96553	1972.53131	3600.08204	-0.96688	1972.40912	3600.12129	-1.0	0.0	**25.87766**	-1.0	0.0	480.70899
100	-53.80047	40.35652	3600.03702	-42.01334	51.95937	3600.08005	-99.0	0.0	29.74994	-99.0	0.0	**11.56583**
120	-32.01264	94.83716	3600.08534	-30.00988	101.23299	3600.15507	-41.0	0.0	**7.57932**	-41.0	0.0	15.14167
140	0.3957	8363.29881	3600.05396	-	0.0	-	-1.43333	0.0	20.47051	-1.43333	0.0	**4.51942**
160	-2.0019	2054.55223	3600.19356	-3.99997	1027.75754	3600.2348	-3.66667	0.0	68.99746	-3.66667	0.0	**57.98109**
180	-61.01152	72.82212	3600.09051	-56.0	79.42857	3600.17978	-111.0	0.0	**59.32443**	-111.0	0.0	61.66311
200	-6.51533	803.56381	3600.1813	-11.33333	461.52941	3600.1891	-11.0	0.0	**72.02655**	-11.0	0.0	168.53942

Table C.7: NN-U-P-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	0.45763	0.0	5.219	0.45763	0.0	5.40648	0.45763	0.0	**2.09511**	0.45763	0.0	2.172
40	-1.11222	6e-05	3598.52969	-1.11222	0.04059	3600.04675	-1.11111	0.0	**39.79588**	-1.11111	0.0	40.33147
60	-1.14294	963.24227	3600.03367	-1.14294	963.20575	3600.09911	-1.14286	0.0	55.791	-1.14286	0.0	**55.78563**
80	-0.80095	2439.49713	3600.1159	-0.80101	2431.52975	3600.14399	-0.96296	0.0	**68.8774**	-0.96296	0.0	270.7474
100	-2.13793	876.95041	3600.0377	-2.13793	876.95041	3600.13058	-2.33333	0.0	**260.36402**	-2.33333	0.0	262.59745
120	0.0813	32781.95466	3600.05666	0.0813	32781.95466	3600.18918	-0.03846	0.0	**86.81343**	-0.03846	0.0	86.82028
140	-1.05	2756.14384	3600.18575	-1.05	2756.14384	3600.11455	-1.72973	0.0	98.40276	-1.72973	0.0	**98.37066**
160	-0.65258	5897.24734	3600.07475	-0.65258	5897.24372	3600.34194	-0.65218	0.0	**695.23546**	-0.65218	0.0	695.99375
180	-36.25	111.27586	3600.16292	-39.99998	100.75005	3600.42785	-87.0	0.0	**40.36549**	-87.0	9e-05	178.67051
200	-1.92279	2235.33583	3600.2302	-1.92279	2235.33583	3600.17623	-1.8	1.65079	3600.04243	-1.8	1.65079	3600.04152

Table C.8: NN-U-P-5

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-225.0	0.0	2.53084	-225.0	0.0	6.90882	-225.0	0.0	0.29684	-225.0	0.0	**0.25013**
40	-8.33624	69.55757	3600.01128	-8.33704	68.00697	3600.07248	-8.33333	0.0	3.29414	-8.33333	0.0	**1.10942**
60	-51.00868	26.48552	3600.04932	-50.00907	27.03491	3600.06782	-51.0	0.0	**1.78455**	-51.0	0.0	2.40631
80	-167.00532	9.88588	3600.07481	-146.00785	11.45139	3600.10709	-167.0	0.0	2.92207	-167.0	0.0	**2.62516**
100	-295.0	6.89153	3600.08308	-303.0	6.68317	3600.03312	-306.0	0.0	2.047	-306.0	0.0	**1.37817**
120	-713.0	2.95091	3600.03595	-720.0	2.91138	3600.1154	-720.0	0.0	3.70339	-720.0	0.0	**1.87217**
140	-45.00752	63.37997	3600.09488	-45.01169	63.38634	3600.18853	-45.0	0.0	1.23441	-45.0	0.0	**1.12511**
160	-1170.0	1.64103	3600.04798	-1179.0	1.62087	3600.0721	-1188.0	9e-05	5.28563	-1188.0	0.0	**2.03134**
180	-477.0	8.54088	3600.08492	-481.00752	8.46139	3600.03709	-481.0	0.0	4.11147	-481.0	0.0	**1.90635**
200	-85.00898	57.91142	3601.12374	-81.0	60.82716	3600.1525	-96.0	0.0	6.52124	-96.0	0.0	**4.20817**

Table C.9: NN-U-N-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.26797	0.0	1.15631	-1.26797	0.0	**0.60933**	-1.26797	0.0	2.5844	-1.26797	0.0	2.32829
40	-38.00267	20.18272	3600.02977	-50.0	15.1	3600.02485	-50.0	0.0	**9.26631**	-50.0	0.0	10.46115
60	-1.75	848.01151	3600.07057	-1.66874	886.6113	3600.08951	-1.69231	0.0	5.27134	-1.69231	0.0	**4.90657**
80	-1.41024	1269.23219	3600.10611	-1.38925	1286.05994	3600.12477	-1.38889	0.0	16.27803	-1.38889	0.0	**3.79773**
100	-1.02634	1932.08984	3600.05018	-1.00218	1978.68687	3600.10069	-1.5	0.0	**3.03056**	-1.5	0.0	3.17542
120	-203.0	11.57635	3600.08953	-225.00969	10.34618	3600.0434	-281.0	0.0	**5.04532**	-281.0	0.0	6.06123
140	-111.0	24.57658	3600.11691	-34.0	82.5	3600.08777	-155.0	0.0	**6.88818**	-155.0	0.0	7.03227
160	-119.99757	26.28389	3600.15782	-86.0	37.06977	3600.04818	-216.0	0.0	**7.28469**	-216.0	0.0	27.61901
180	-520.0	6.38654	3600.02883	-515.0	6.45825	3600.07645	-558.0	0.0	**5.54703**	-558.0	0.0	5.66027
200	-4.50001	968.28647	3600.14946	-4.13327	1054.41845	3600.19096	-4.32	0.0	**12.75827**	-4.32	0.0	13.38556

Table C.10: NN-U-N-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.73611	0.0	5.07835	-1.73611	0.0	**3.00023**	-1.73611	0.0	3.438	-1.73611	0.0	5.33592
40	-1.14685	0.0	2669.87889	-1.14995	0.0	1914.35665	-1.14667	0.0	1.32841	-1.14667	0.0	**1.31265**
60	-1.28704	1107.17625	3600.11063	-1.28704	1180.56362	3600.05242	-1.30556	0.0	130.85754	-1.28704	0.0	**6.72681**
80	-26.49253	77.28623	3600.08784	-27.00744	75.79366	3600.14582	-37.0	0.0	19.87917	-37.0	0.0	**15.025**
100	-29.99196	83.18922	3600.13118	-21.01106	119.17478	3600.14373	-30.0	0.0	13.83257	-30.0	0.0	**11.84575**
120	-1.60752	1932.96103	3600.08538	-1.61176	1932.90746	3600.10418	-1.58824	0.0	31.87235	-1.74074	0.0	30.89272
140	-14.80001	234.80303	3600.12882	-20.01136	173.45089	3600.10596	-21.0	0.0	**3.14328**	-21.0	0.0	11.14129
160	-4.38561	906.24853	3600.22511	-4.3854	906.4204	3600.08406	-4.38462	0.0	**19.60896**	-4.38462	0.0	31.60792
180	-7.42834	607.88438	3600.09429	-7.51771	600.49575	3600.28909	-7.5	0.0	**4.43783**	-7.5	0.0	10.91125
200	-1.03164	4777.77756	3600.36516	-1.03861	4745.74579	3600.20083	-1.0101	0.0	6.05734	-1.0101	0.0	**3.37637**

Table C.11: NN-U-N-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.29885	0.0	1.60991	-1.29885	0.0	**1.26585**	-1.29885	0.0	2.39073	-1.29885	0.0	1.40643
40	-1.21455	1.87486	3600.0288	-1.21429	1.065	3600.03555	-1.21429	0.0	**2.42182**	-1.21429	0.0	4.22309
60	-1.16343	1092.30031	3600.04885	-1.16341	1076.22363	3600.07715	-1.17045	0.0	263.06997	-1.17045	0.0	**12.70528**
80	-1.51248	1055.42988	3600.08243	-1.52005	1050.70676	3600.05476	-1.52	0.0	30.83705	-1.52	0.0	**12.14161**
100	-2.12143	939.02715	3600.08866	-2.15294	925.6392	3600.05637	-2.30769	0.0	**170.09647**	-2.30769	0.0	179.60737
120	-1.28875	2105.441	3600.07898	-1.27385	2126.3163	3600.1197	-1.27273	0.0	115.6001	-1.25806	0.0	**11.12983**
140	-1.63359	1763.20999	3600.08515	-1.57574	1827.98119	3600.04127	-1.82759	0.0	116.24335	-1.76471	0.0	**18.56105**
160	-1.01055	3487.20195	3600.53662	-1.0302	3420.65736	3600.2371	-1.20588	9e-05	**112.57394**	-1.21053	0.0	212.76068
180	-7.41174	490.65257	3600.10307	-8.17782	444.59566	3600.07941	-26.0	0.0	**49.01124**	-26.0	0.0	148.56291
200	-1.03389	4022.64232	3600.38137	-0.98453	4224.3789	3600.2144	-3.5	0.0	**20.89238**	-3.5	0.0	66.14662

Table C.12: NN-U-N-5

C.2 Unrestricted numerators and denominators with Knapsack constraint

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-39.5	0.0	6.71889	-39.5	0.0	1.76214	-39.5	0.0	3.78518	-39.5	0.0	**1.0654**
40	-325.00057	1.59076	3600.2068	-455.00303	0.85054	3600.18468	-455.0	0.0	23.64771	-455.0	0.0	**14.11179**
60	-540.00576	1.8296	3600.23031	-676.0	1.26036	3600.47376	-786.0	0.0	45.10571	-786.0	0.0	**30.77928**
80	-894.0	1.14877	3600.09542	-841.0	1.28419	3600.18252	-1057.0	0.0	87.38702	-1057.0	0.0	**52.87195**
100	-1088.0	1.23989	3600.24081	-1168.99914	1.08469	3600.12331	-1570.0	0.0	37.02503	-1570.0	1e-05	**35.74179**
120	-1317.0	1.47684	3600.05715	-1353.70125	1.40969	3600.09509	-1784.0	0.0	**79.14208**	-1784.0	0.0	106.20249
140	-1248.0	2.13702	3600.0741	-1678.0	1.33313	3600.21577	-2243.0	0.0	**69.77341**	-2243.0	0.0	97.16236
160	-1125.03102	2.58746	3600.13715	-1130.0	2.57168	3600.2416	-2266.0	0.0	**140.5763**	-2266.0	0.0	174.0903
180	-1411.0	2.45429	3600.30046	-1790.0	1.72291	3600.11152	-2917.0	0.0	**113.52644**	-2917.0	3e-05	199.95904
200	-1601.0	2.15053	3600.07661	-1312.0	2.84451	3600.10219	-2948.0	6e-05	**56.87509**	-2948.0	3e-05	345.37207

Table C.13: NN-K-R-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-13.4	0.0	23.73969	-13.4	0.0	10.47465	-13.4	0.0	18.14383	-13.4	0.0	**7.1258**
40	-146.0035	6.70529	3600.0921	-146.0049	6.70522	3600.19242	-146.0	0.0	203.45527	-146.0	0.0	**191.80087**
60	-293.0	3.23208	3600.21803	-230.0	4.3913	3600.44626	-497.0	0.0	**645.28334**	-497.0	1e-05	1385.46334
80	-314.0	4.85987	3600.06	-326.0	4.64417	3600.61761	-723.0	0.0001	2014.01617	-723.0	0.0	**1751.21139**
100	-316.0	4.97468	3600.08375	-162.0	10.65432	3600.20489	-1164.0	0.04114	3600.13696	-1190.0	0.00035	3601.04631
120	-271.0	10.54244	3600.16936	-385.00783	7.12451	3600.24616	-1229.0	0.09761	3600.30526	-1268.0	0.06374	3603.78322
140	-422.0	6.80569	3600.1531	-227.0	13.51101	3600.13227	-1479.0	0.02764	3601.71127	-1473.0	0.03239	3601.1598
160	-373.01805	9.0156	3600.21891	-509.0	6.33988	3600.36621	-1974.0	0.06097	3600.26703	-2002.0	0.04465	3602.78327
180	-473.0	8.05497	3600.16503	-593.66667	6.21449	3600.20599	-2029.0	0.00738	3600.10346	-1981.0	0.03494	3600.18088
200	-750.0	4.69067	3600.12575	-474.0	8.00422	3600.24312	-2450.0	0.03002	3600.13753	-2380.0	0.06076	3600.32998

Table C.14: NN-K-R-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-9.68421	0.0	0.73436	-9.68421	0.0	**0.64973**	-9.68421	0.0	1.39043	-9.68421	0.0	1.39079
40	-80.33333	11.98808	3600.01277	-80.33333	0.0	3219.82133	-80.33529	0.0	**127.18291**	-80.33529	0.0	391.53275
60	-52.0	27.82692	3600.02774	-58.0	24.84483	3600.0379	-223.0	0.97341	3600.02622	-181.5	1.44681	3600.07146
80	-118.5046	16.58581	3600.02519	-90.5	22.02762	3600.02049	-482.0	0.73342	3600.53108	-484.0	0.72611	3600.11073
100	-88.0095	27.63327	3600.02095	-85.00522	28.64524	3600.04647	-479.0	0.93413	3600.02037	-513.0	0.80476	3600.32459
120	-223.00306	12.78905	3600.01605	-139.25019	21.08255	3600.01815	-799.0	0.45832	3601.37063	-357.0	2.29734	3600.72107
140	-98.66667	37.84797	3600.02917	-216.503	16.70414	3600.0367	-1360.0	0.16894	3600.06487	-138.7	10.49321	3600.21838
160	-104.60337	35.17474	3600.02824	-122.6	29.8646	3600.03589	-69.20001	21.24188	3600.06242	-936.0	0.6381	3601.87046
180	-153.14514	27.71786	3600.01624	-136.75	31.16088	3600.0926	-42.07143	44.85179	3600.60061	-63.09525	29.62303	3600.05286
200	-138.25111	34.77548	3600.08992	-158.33333	30.23789	3600.0415	-265.0	7.38051	3600.05286	-889.0	1.49645	3600.05578

Table C.15: NN-K-R-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-2.45652	0.0	8.34744	-2.45761	0.0	8.27095	-2.45652	0.0	57.99144	-2.45652	0.0	**9.57881**
40	-13.91667	62.80838	3600.0888	-18.0757	48.12672	3600.23755	-6.97529	11.66305	3600.08053	-13.91667	2.64633	3600.14506
60	-27.6	50.92028	3600.06472	-24.73333	56.93801	3600.14542	-19.06452	21.37523	3600.30777	-22.6	21.38633	3600.29392
80	-46.0	32.23913	3600.10173	-29.66667	50.53933	3600.14576	-35.77778	19.29988	3601.46783	-61.0	10.52332	3600.11612
100	-40.14286	52.43416	3600.06995	-42.4	49.58962	3600.793	-60.46155	14.75783	3600.34045	-16.25	63.7545	3600.82174
120	-46.5	61.06452	3600.09193	-53.21739	53.23039	3600.23259	-30.96774	36.76543	3600.14702	-11.64865	99.52706	3602.83128
140	-52.22222	61.52128	3600.09286	-34.33333	94.09709	3600.07784	2.80357	525.20518	3600.841	-27.46667	53.99381	3600.19981
160	-35.2	111.15909	3600.15494	-22.81818	172.01992	3600.13822	-8.59375	191.51711	3600.54732	-11.33333	146.30232	3600.59112
180	-57.81818	70.77673	3600.31173	-53.25	76.93427	3600.28704	-36.25641	48.53732	3600.11833	-23.87234	73.95089	3600.66668
200	-45.875	92.01362	3600.13444	-43.6875	96.67096	3601.16912	-5.31579	371.35868	3600.15611	-12.75001	167.11703	3604.67549

Table C.16: NN-K-R-5

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.72727	0.0	0.3903	-1.72727	0.0	0.46888	-1.72727	0.0	0.62496	-1.72727	0.0	**0.19249**
40	-47.01064	0.0	143.63328	-47.0	0.0	166.94384	-47.0	0.0	**1.70749**	-47.0	0.0	2.43634
60	-67.00531	20.12495	3600.08895	-67.01196	20.131	3600.09575	-67.0	0.0	1.34666	-67.0	0.0	**1.17216**
80	-90.01414	22.57407	3600.04786	-90.01145	22.57478	3600.10068	-90.0	0.0	1.2144	-90.0	0.0	**0.57809**
100	-244.0	10.13087	3600.04634	-241.01452	10.27318	3600.02509	-249.0	0.0	**3.81385**	-249.0	0.0	4.62405
120	-245.00761	10.55474	3600.04073	-237.0	10.94515	3600.10366	-245.0	0.0	2.28154	-245.0	0.0	**1.44063**
140	-40.01458	82.3946	3600.12069	-40.01636	82.36419	3600.09533	-40.0	0.0	1.75001	-40.0	0.0	**1.2343**
160	-521.0	7.24568	3600.05286	-540.0	6.95556	3600.06775	-547.0	0.0	17.61673	-547.0	0.0	**6.39501**
180	-2.80105	1651.86732	3600.05914	-2.80114	1648.99896	3600.12877	-2.8	0.0	2.27032	-2.8	0.0	**1.92175**
200	-174.0	27.83333	3600.03884	-162.0	29.96914	3600.03881	-180.0	0.0	**1.53134**	-180.0	0.0	2.93859

Table C.17: NN-K-P-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-0.48485	0.0	1.04796	-0.48485	0.0	0.56257	-0.48485	0.0	0.94672	-0.48485	0.0	**0.26561**
40	0.05196	0.0	117.31231	0.05196	0.0	118.90937	0.05882	0.0	2.78897	0.05882	0.0	**2.47189**
60	-19.00668	68.08099	3600.02492	-19.49958	66.33479	3600.12124	-19.0	0.0	2.50832	-19.0	0.0	**2.2228**
80	-47.00913	41.20457	3600.15029	-46.0	42.13043	3600.22417	-47.0	0.0	2.53971	-47.0	0.0	**2.094**
100	-1.01095	2451.2599	3600.05673	-1.00289	2470.61785	3600.05712	-1.0	0.0	17.88427	-1.0	0.0	**2.93456**
120	-123.0	24.87805	3600.04681	-117.0	26.20513	3600.06176	-161.0	9e-05	15.77258	-161.0	0.0	**14.39649**
140	-3.50636	1073.3988	3600.06806	-3.5102	1071.81252	3600.52715	-3.5	0.0	**3.36366**	-3.5	0.0	37.21447
160	-55.0	74.07273	3600.11507	-43.01208	94.99628	3600.10282	-90.0	0.0	**6.70022**	-90.0	0.0	8.93376
180	-52.01143	83.13535	3600.04281	-46.03023	94.06796	3600.28631	-76.0	0.0	9.25663	-76.0	0.0	**8.16113**
200	-46.0056	100.72675	3600.38738	-47.01667	98.53916	3600.03129	-91.0	0.0	**7.74773**	-91.0	0.0	11.60763

Table C.18: NN-K-P-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-0.34463	0.0	0.54685	-0.34463	0.0	**0.39059**	-0.34463	0.0	1.40636	-0.34463	0.0	3.06573
40	0.03296	0.0	925.79106	0.03333	0.0	147.49669	0.03333	0.0	**3.23817**	0.03333	0.0	6.75028
60	-1.06536	1321.37657	3600.0881	-1.06522	1305.40632	3600.03061	-1.06522	0.0	**5.0781**	-1.06522	0.0	10.08139
80	-2.50958	825.04814	3600.03976	-2.50113	827.93057	3600.06226	-2.5	0.0	**17.78694**	-2.5	0.0	24.4688
100	-0.89259	2687.96636	3600.09381	-1.0	2399.67309	3600.172	-0.88889	0.0	18.28363	-0.88889	0.0	**4.70439**
120	-73.00401	40.13472	3600.05967	-39.00751	75.98518	3600.08494	-98.0	0.0	**13.65224**	-98.0	0.0	43.15587
140	-4.50687	748.52231	3600.15692	-5.20353	648.17457	3600.30965	-5.2	0.0	17.00791	-5.2	0.0	**4.20565**
160	-3.33333	1286.81549	3600.14704	-3.42943	1250.79319	3600.1068	-3.25	0.0	**14.32729**	-3.25	0.0	19.32615
180	-69.02446	61.64446	3600.05124	-50.02717	85.43304	3600.07141	-99.0	6e-05	**219.48189**	-99.0	3e-05	259.66344
200	-0.12641	37557.98056	3600.28451	-0.10257	41603.37461	3600.0347	-1.4	0.0	**18.1053**	-1.4	0.0	69.96697

Table C.19: NN-K-P-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.06383	0.0	1.94403	-1.06383	0.0	**1.75637**	-1.06383	0.0	4.20862	-1.06383	0.0	3.42767
40	-0.43	0.52544	3600.01533	-0.43	0.0	2084.07195	-0.43	0.0	192.29546	-0.43	0.0	**117.80662**
60	-1.29232	1201.22132	3600.04271	-1.37255	1102.61534	3600.12173	-1.37255	0.0	164.80903	-1.29167	0.0	**2.31692**
80	-0.16279	10823.61185	3600.07998	-0.22218	7911.07076	3600.1776	-0.19231	0.0	**184.45606**	-0.19231	0.0	253.26476
100	-1.67948	1350.00958	3600.0441	-1.04059	2179.49664	3600.04583	-2.3	9e-05	355.04947	-2.3	0.0	**123.54564**
120	-1.09091	2650.2384	3600.06802	-1.09091	2654.05675	3600.1816	-1.09677	0.0	**116.40715**	-1.09677	0.0	122.01709
140	0.08701	40272.78622	3600.10704	0.08701	40274.93515	3600.26087	0.06897	2.19587	3600.03906	0.06897	2.19676	3600.15954
160	-0.1396	29274.88318	3600.13933	-0.54545	7238.82168	3600.17975	-0.16	1.74031	3600.04435	-0.20513	0.0	**87.32037**
180	-9.25001	491.53999	3600.17661	-10.49999	432.90504	3600.26241	-23.0	0.0	47.29327	-23.0	0.0	**26.37061**
200	-2.42892	1993.7142	3600.28256	-2.85714	1694.75034	3600.25976	-2.88462	0.0	843.59752	-2.88462	0.0	**153.8753**

Table C.20: NN-K-P-5

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-11.0	0.0	0.73524	-11.0	0.0	0.62509	-11.0	0.0	0.39058	-11.0	0.0	**0.21873**
40	-3.73333	0.0	199.52392	-3.73333	0.0	184.4114	-3.73333	0.0	**1.9689**	-3.73333	0.0	2.01572
60	-246.0	4.18639	3600.1017	-248.00984	4.1445	3600.05025	-248.0	0.0	5.23458	-248.0	0.0	**4.14194**
80	-115.99046	13.79432	3600.02784	-124.95617	12.73282	3600.03435	-134.0	0.0	**6.20086**	-134.0	0.0	6.48467
100	-249.00851	8.91934	3600.01882	-256.0	8.64844	3600.01915	-292.0	0.0	5.7189	-292.0	0.0	**3.04707**
120	-97.01098	27.2133	3600.08222	-117.01285	22.39059	3600.06722	-119.0	0.0	**1.8596**	-119.0	0.0	2.89889
140	-1.60651	1887.38506	3601.66613	-1.61629	1875.90852	3600.1707	-1.6	0.0	**1.71891**	-1.6	0.0	2.09384
160	-2.50681	1608.52263	3600.39678	-2.5273	1595.63837	3600.1194	-2.5	0.0	2.10733	-2.5	0.0	**2.01559**
180	-37.01452	123.46466	3600.13121	-37.01008	123.4796	3600.10448	-38.0	0.0	3.18756	-38.0	0.0	**1.75192**
200	-211.0	21.66825	3600.03315	-217.9089	20.94954	3600.03257	-258.0	0.0	**11.07841**	-258.0	0.0001	18.20319

Table C.21: NN-K-N-2

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-20.0	0.0	4.51023	-20.0	0.0	4.07586	-20.0	0.0	3.66774	-20.0	0.0	**0.51558**
40	-1.3211	0.0	28.47682	-1.3211	0.0	32.92088	-1.3211	0.0	**1.04734**	-1.3211	0.0	1.40626
60	-4.50391	230.99808	3600.07836	-4.50584	234.14988	3600.10673	-4.5	0.0	8.82575	-4.5	0.0	**3.26566**
80	-55.02221	36.58482	3600.04382	-45.01521	44.94003	3600.10373	-62.0	0.0	**5.59953**	-62.0	0.0	6.25525
100	-2.12846	981.40195	3600.20785	-2.12856	978.15058	3600.06574	-2.12821	0.0	**6.18799**	-2.12821	0.0	11.1705
120	-124.0	18.02419	3600.02666	-146.00158	15.15736	3600.04606	-212.0	0.0	**23.16989**	-212.0	2e-05	35.74117
140	-66.94845	50.1737	3600.23944	-59.98978	56.10973	3600.17729	-74.0	0.0	5.67192	-74.0	0.0	**3.99246**
160	-39.00506	86.78348	3600.09823	-20.02083	170.02189	3600.09706	-77.0	0.0	**2.89079**	-77.0	0.0	3.51392
180	-103.0	39.65049	3600.11939	-105.0	38.87619	3600.14025	-152.0	0.0	16.89505	-152.0	0.0	**10.20043**
200	-102.0	45.35294	3600.16053	-87.0081	53.33977	3600.15262	-172.0	0.0	**23.23541**	-172.0	0.0	50.34148

Table C.22: NN-K-N-3

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-1.4375	0.0	3.78129	-1.4375	0.0	2.44108	-1.4375	0.0	4.56293	-1.4375	0.0	3.95667
40	-9.54198	83.45333	3600.02293	-9.53849	0.0	3045.19659	-9.53846	0.0	9.59605	-9.53846	0.0	8.06775
60	-2.1697	692.4528	3600.08818	-2.35716	638.52532	3600.11304	-2.27273	0.0	25.55694	-2.27273	0.0	39.49377
80	-2.12577	947.74152	3600.09127	-2.12559	947.59554	3600.21054	-2.125	0.0	22.37404	-2.125	0.0	60.51135
100	-1.58824	1573.18641	3600.04352	-1.58928	1571.93729	3600.07938	-1.62857	0.0	428.18656	-1.62857	0.0	26.34602
120	-1.3684	2312.6537	3600.05123	-0.99652	3176.06175	3600.20604	-2.66667	0.0	19.45926	-2.66667	0.0	63.09585
140	-4.02115	883.57346	3600.04995	-3.62504	980.23083	3600.32433	-4.42857	0.0	217.07269	-4.5	0.0	79.30795
160	-2.15388	1851.31657	3600.19141	-2.00017	1994.4958	3600.35903	-2.13158	0.0	31.85434	-2.13158	0.0	73.61126
180	-8.00522	582.99389	3600.19604	-11.04514	422.26322	3600.14144	-12.0	0.0	550.42752	-12.0	0.0	109.0084
200	-1.61561	2953.91677	3600.27592	-1.42021	3360.475	3600.13792	-1.68421	8e-05	325.59045	-1.69388	0.0	470.94995

Table C.23: NN-K-N-4

n	GMMFP1			GMMFP1M			GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
20	-2.0	0.0	5.53024	-2.0	0.0	5.23652	-2.0	0.0	**4.84364**	-2.0	0.0	6.08389
40	-1.05637	0.0	761.9358	-1.05319	0.0	814.03345	-1.05217	0.0	204.71107	-1.05155	0.0	**150.60483**
60	-1.44613	798.46103	3600.04685	-1.44788	802.75095	3600.04393	-1.44444	0.0	**6.44913**	-1.44444	0.0	7.60456
80	-1.22747	1424.69715	3600.04461	-1.23087	1420.75838	3600.05246	-1.23214	0.0	**133.40099**	-1.2381	0.0	25.13011
100	-1.16838	1609.83625	3600.06644	-1.15899	1618.2717	3600.22593	-1.16667	0.0	**44.14446**	-1.16667	0.0	75.24497
120	-1.19217	2323.32952	3600.06067	-1.01923	2717.72992	3600.078	-1.06061	0.0	**84.78355**	-1.09375	0.0	177.54218
140	-1.00451	2993.50323	3600.18931	-1.02941	2921.05761	3600.11631	-1.0597	0.0	1707.23286	-1.02941	0.0	**159.50505**
160	-2.60084	1377.39934	3600.09716	-2.77042	1293.02671	3600.17816	-3.53846	0.0	994.09063	-3.5005	0.0	**2857.00108**
180	-2.11765	2006.41344	3600.17238	-2.11764	2006.4192	3600.32054	-2.2	0.0	**155.90955**	-2.23077	0.0	264.83092
200	-1.18605	4028.78371	3600.22385	-1.00029	4777.57366	3600.09914	-1.23913	3e-05	974.37234	-1.20513	0.0	**233.46082**

Table C.24: NN-K-N-5

C.3 Unrestricted numerators and denominators with assignment constraints

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-242.0	0.0	247.86528	-242.00124	0.0	283.59834	-242.0	0.0	10.7545	-242.0	0.0	**7.01905**
20	-430.0	3.21163	3600.04847	-477.0	2.79665	3600.02012	-1180.0	7e-05	**98.48184**	-1180.0	0.0	292.23321
30	-509.0	4.49509	3600.05037	-340.0	7.22647	3600.08413	-2059.0	6e-05	335.49038	-2059.0	0.0	**127.97091**
40	-476.0	6.97479	3600.025	-610.0	5.22295	3600.05696	-2964.0	9e-05	**804.09813**	-2964.0	5e-05	956.90053
50	-595.0	7.06723	3600.04336	-588.0	7.16327	3600.10604	-3924.0	9e-05	**2495.40121**	-3924.0	8e-05	2940.59075
60	-531.0	9.94915	3600.10293	-401.5	13.4807	3600.11624	-4762.0	8e-05	**1906.21556**	-4708.00146	0.01491	3600.03302
70	-315.0	20.64127	3600.1531	-515.0	12.23689	3600.12954	-5749.0	9e-05	2517.83827	-5749.0	8e-05	**2324.84402**
80	-289.0	26.0346	3600.09997	-553.0	13.12839	3600.2303	-6580.0	9e-05	3266.84288	-6580.0	9e-05	**1713.74517**
90	-502.0	16.60359	3600.12301	-417.0	20.19185	3600.19457	-7521.0	5e-05	**2344.37337**	-3149.00052	1.3926	3600.05292
100	-488.33333	19.16041	3600.4854	-443.5	21.19842	3600.2258	-8483.0	0.00319	3600.05751	-8169.00065	0.04326	3600.07144

Table C.25: NN-A-R-2

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-76.66667	0.0	343.4096	-76.66667	0.0	428.66574	-76.66667	0.0	**42.37692**	-76.66667	0.0	47.21038
20	-199.0	8.29648	3600.07221	-135.0	12.7037	3600.05665	-501.00021	0.96382	3600.07519	-709.00052	0.37932	3600.10762
30	-112.0	23.91071	3600.05133	-150.0	17.6	3600.07234	-635.50248	1.74611	3600.18197	-164.66723	11.79834	3600.14363
40	-66.88889	55.63123	3600.06985	-53.5	69.80374	3600.09221	-392.00087	6.35336	3600.13068	-1582.0	0.57459	3600.12096
50	-64.66667	73.2732	3600.06981	-85.4	55.24122	3600.12079	-255.00005	14.07588	3600.03537	-417.33334	8.21177	3600.07636
60	-62.16667	92.32976	3600.24942	-75.0	76.36	3600.18591	-429.00079	11.06202	3600.08452	-101.81383	53.16482	3600.12195
70	-28.5	238.40351	3600.13196	-212.0	31.18396	3600.14322	-164.25	39.42272	3600.08029	-97.40597	67.77158	3600.23155
80	-34.375	226.34545	3600.17999	-55.67993	139.35577	3600.64384	-470.66677	13.01775	3600.03799	-312.55714	20.46044	3600.07776
90	-39.09091	224.80698	3601.04347	-65.16667	134.45269	3600.15591	-400.60221	20.63164	3600.06141	-79.33334	108.28419	3600.10663
100	-115.66667	83.75216	3600.29947	-29.26471	333.97688	3600.3786	-262.50081	35.94221	3600.06283	-161.75063	57.87487	3600.08394

Table C.26: NN-A-R-3

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-13.91667	0.0	492.41228	-13.91667	0.0	570.74533	-13.91667	0.0	**90.8875**5	-13.91667	0.0	143.49652
20	-24.63636	70.4797	3600.09314	-31.71429	54.52703	3600.04645	-24.55032	39.18724	3600.05093	-28.001	28.74721	3600.0572
30	-30.05	91.31281	3600.0476	-24.78571	110.91931	3600.07587	-47.50017	40.25507	3600.14223	-35.21528	42.62454	3600.11613
40	-19.76923	192.73541	3600.09719	-10.54054	362.35897	3600.08836	-21.37047	114.2522	3600.0501	-33.89898	108.56482	3600.08729
50	-33.38462	142.3894	3600.09299	-20.8	229.14423	3600.14037	-10.40523	459.05717	3600.02638	-19.65297	219.51548	3600.0194
60	-24.16667	239.04138	3600.09035	-14.76045	392.0098	3600.18557	-17.56411	321.90232	3600.08135	-12.19515	474.67855	3600.07014
70	-19.34783	350.40899	3600.06269	-9.74026	697.03067	3600.4365	-15.39203	440.722	3600.05419	-8.38472	809.87937	3600.05271
80	-9.19718	848.82542	3600.20736	-15.42308	505.77307	3600.19293	-40.21176	191.94723	3600.04653	-43.69967	177.85718	3600.04517
90	-11.07071	795.51642	3600.2137	-17.81132	494.07839	3600.16235	-17.6667	498.13113	3600.08842	-36.10309	243.09497	3600.05333
100	-32.92857	296.8872	3600.20535	-20.44802	478.70415	3600.17431	-38.39029	250.57779	3600.04427	-9.63376	1017.19016	3600.09081

Table C.27: NN-A-R-4

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-103.0	0.0	364.59113	-103.00027	0.0	457.96017	-103.0	0.0	**5.58222**	-103.0	0.0	13.41126
20	-212.0	1.69811	3600.02715	-147.00337	2.89107	3600.04441	-263.0	0.0	20.74927	-263.0	0.0	**3.35931**
30	-118.99702	6.72288	3600.04963	-145.0	5.33793	3600.06527	-382.0	0.0	**7.47719**	-382.0	0.0	16.55328
40	-195.0	4.6	3600.05702	-205.0	4.32683	3600.11581	-472.0	0.0	**13.27991**	-472.0	0.0	111.96329
50	-188.0	4.77128	3600.06805	-107.0	9.14019	3600.17198	-513.0	0.0	40.56379	-513.0	0.0	**23.4092**
60	-191.0	6.18848	3600.11243	-184.0	6.46196	3600.21458	-592.0	0.0	**59.15006**	-592.0	0.0	113.49521
70	-261.0	4.57471	3600.14282	-217.0	5.70507	3600.28041	-732.0	0.0	173.80406	-732.0	0.0	**117.31094**
80	-140.0	11.71429	3600.08181	-94.0	17.93617	3600.15677	-695.0	0.0	91.63836	-695.0	3e-05	**54.88156**
90	-111.0	15.11712	3600.17641	-123.99761	13.4277	3600.28464	-738.0	0.0	99.7039	-738.0	0.0	**57.75758**
100	-115.0	16.6	3600.10459	-160.0	11.65	3600.2551	-832.0	0.0	1282.10221	-832.0	8e-05	**343.5678**

Table C.28: NN-A-P-2

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-22.5	0.0	55.66739	-22.5	0.0	53.49239	-22.5	0.0	**4.69256**	-22.5	0.0	7.32575
20	-86.0	6.51163	3600.12985	-101.0	5.39604	3600.05569	-224.0	0.0	716.57767	-224.0	0.0	**179.69077**
30	-91.0	8.82418	3600.03166	-99.5	7.98492	3600.07979	-337.0	6e-05	**1486.99353**	-337.0	5e-05	1872.12184
40	-107.0	7.72897	3600.14501	-57.0	15.38596	3600.11395	-375.0	0.0001	**1871.2697**	-375.0	6e-05	2798.59043
50	-50.0	22.34	3600.11853	-71.0	15.43662	3600.13336	-415.0	5e-05	**923.43846**	-402.00004	0.22277	3600.0598
60	-63.99983	22.25006	3600.07202	-52.33333	27.43312	3600.32096	-525.0	8e-05	2460.1486	-525.0	9e-05	**2117.23573**
70	-62.0	25.0	3600.42452	-34.2	46.1345	3600.25623	-507.0	0.25839	3600.05175	-554.0	9e-05	**1025.80606**
80	-26.0	63.11538	3600.07406	-68.99976	23.1595	3600.67193	-596.0	3e-05	**1615.52153**	-402.0	0.82915	3600.05642
90	-25.75	68.59223	3600.06447	-63.9997	27.00013	3600.52901	-621.0	7e-05	**2155.1969**	-621.0	8e-05	3280.68435
100	-52.0	35.53846	3600.21897	-51.0	36.2549	3600.40496	-639.00041	0.32519	3600.07162	-587.0	0.87205	3600.26646

Table C.29: NN-A-P-3

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-9.85714	0.0	1059.75901	-9.85714	0.0	669.74237	-9.85714	0.0	**73.17013**	-9.85714	0.0	106.19777
20	-21.5	22.95349	3600.26616	-25.25	19.39604	3600.07552	-129.00061	0.51267	3600.04844	-192.0	3e-05	**3150.3294**
30	-36.0	23.94444	3600.08549	-18.5	47.54054	3600.13695	-116.00494	1.28293	3600.03625	-138.0	0.91355	3600.14911
40	-35.5	27.53521	3600.08587	-35.0	27.94286	3600.144	-23.66694	14.30678	3600.04016	-45.50023	8.62424	3600.18068
50	-15.0	80.6	3600.14478	-17.54528	68.76234	3600.28142	-10.86004	51.90353	3600.2172	-108.01108	3.16053	3600.2032
60	-16.5	84.45455	3600.09786	-38.5	35.62338	3600.38469	-33.0	18.25191	3600.09594	-44.0	12.0357	3600.22093
70	-10.11111	149.42857	3600.13867	-26.5	56.39623	3600.3294	-58.0	10.38759	3600.13438	-104.00117	4.88455	3600.10697
80	-16.42857	92.25217	3600.30228	-14.90909	101.7561	3601.03393	-277.50246	1.41969	3600.0811	-97.00007	5.00102	3600.38778
90	-16.25	114.32308	3600.21092	-44.0	41.59091	3600.72023	-41.00027	44.70701	3600.04714	-120.80007	8.10285	3600.7044
100	-4.83333	387.34483	3600.05596	-12.75	146.21569	3600.82672	-249.0	1.71684	3600.06313	-32.4	33.90998	3600.69983

Table C.30: NN-A-P-4

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-139.0	0.0	665.86427	-139.0	0.0	453.64103	-139.0	0.0	**1.94192**	-139.0	0.0	2.70765
20	-100.00053	4.69997	3600.071	-176.00008	2.23863	3600.03554	-253.0	0.0	7.81454	-253.0	0.0	**7.30075**
30	-152.0	5.22368	3600.06262	-118.00061	7.01691	3600.04786	-393.0	0.0	19.43789	-393.0	0.0	**8.18196**
40	-163.0	4.99387	3600.07567	-165.0	4.92121	3600.05861	-462.0	0.0	40.13959	-462.0	7e-05	**25.40989**
50	-126.0	7.87302	3600.07043	-130.0	7.6	3600.15004	-493.0	0.0	142.46216	-493.0	0.0	**35.08954**
60	-154.0	7.23377	3600.24631	-152.0	7.34211	3600.1327	-613.0	7e-05	**66.47244**	-613.0	0.0	72.1235
70	-201.0	8.39303	3600.13035	-207.99977	8.07693	3600.22894	-710.0	0.0	87.80653	-710.0	0.0	**28.14323**
80	-132.0	12.19697	3600.11615	-188.0	8.26596	3600.23026	-755.0	8e-05	**53.72021**	-755.0	9e-05	166.25229
90	-84.0	19.71429	3600.11842	-73.0	22.83562	3600.36889	-729.0	0.0	126.12794	-729.0	0.0	**75.06625**
100	-135.0	12.65185	3601.00183	-228.0	7.08333	3600.68423	-824.0	3e-05	163.01107	-824.0	0.0	**124.43544**

Table C.31: NN-A-N-2

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-43.5	0.0	17.82106	-43.5	0.0	21.12523	-43.5	0.0	2.77018	-43.5	0.0	**1.81713**
20	-53.0	10.41509	3600.0763	-53.0	10.41509	3600.07436	-203.0	5e-05	818.22604	-203.0	9e-05	**812.33298**
30	-73.0	11.20548	3600.06183	-62.0	13.37097	3600.10817	-358.0	4e-05	1033.05289	-358.0	0.0	**764.05807**
40	-38.5	25.23377	3600.09369	-42.0	23.04762	3600.12114	-368.0	9e-05	**2542.06741**	-368.0	9e-05	2620.16117
50	-43.5	26.70115	3600.13831	-70.0	16.21429	3600.25158	-425.0	0.00657	3600.0356	-365.00001	0.40666	3600.10972
60	-29.25	49.11966	3600.13837	-61.49949	22.8376	3600.23982	-517.00005	0.00987	3600.03261	-511.00112	0.02405	3600.14021
70	-63.0	22.38095	3600.12592	-24.25	59.74227	3600.39783	-535.0	0.00232	3600.12538	-531.0	0.0874	3600.14671
80	-26.49997	59.03781	3600.23054	-50.66667	30.40132	3600.2721	-595.0	0.00046	3600.95608	-593.0	0.00475	3600.21489
90	-107.0	15.93458	3600.13868	-38.5	46.06494	3600.47897	-648.0	8e-05	1238.2214	-609.00001	0.44123	3600.22382
100	-56.0	32.16071	3600.31199	-44.0	41.20455	3600.50093	-688.0	0.00128	3600.02542	-566.00477	0.90564	3600.27916

Table C.32: NN-A-N-3

n	GMMFP1A			GMMFP1AM			GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-7.71429	0.0	303.70823	-7.71429	0.0	315.79603	-7.71429	0.0	43.45185	-7.71429	0.0	**36.76512**
20	-20.75	26.03614	3600.05416	-14.0	39.07143	3600.09841	-132.0	0.37761	3600.09573	-169.00027	0.07309	3600.06803
30	-26.16667	33.28025	3600.07376	-27.0	32.22222	3600.18332	-139.00076	1.07923	3600.09577	-252.00002	0.13891	3600.12695
40	-26.0	35.46154	3600.14025	-31.66737	28.93618	3600.18536	-18.00014	33.00474	3600.12155	-10.85744	43.29924	3600.15906
50	-33.5	33.41791	3600.11759	-26.66667	42.2375	3600.25151	-243.00027	0.70912	3600.03867	-33.67778	14.45712	3600.31131
60	-16.33333	83.79592	3600.33297	-16.4	83.45122	3600.79173	-23.43019	30.0766	3600.06412	-27.00005	28.13697	3600.36491
70	-21.4	70.54206	3600.20557	-24.2	62.26446	3601.19315	-137.00006	6.29908	3600.25307	-122.00005	4.479	3600.33285
80	-27.99997	55.07149	3600.6069	-23.0	67.26087	3600.84601	-154.00072	2.86633	3600.04673	-403.0	0.51729	3600.45182
90	-67.0	26.22388	3600.31837	-8.80006	206.2714	3600.61626	-32.95455	54.34897	3600.0469	-64.375	15.00448	3600.44441
100	-4.70833	401.47788	3600.19555	-27.66667	67.49398	3600.87253	-76.00315	9.07199	3600.14631	-42.77778	22.29776	3600.53554

Table C.33: NN-A-N-4

C.4 Additional computational results for GMMFP2 formulations

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-367.0	0.0	10.85942	-367.0	0.0	**8.74102**
60	-1406.0	0.0	**13.8127**	-1406.0	0.0	20.70632
90	-1524.0	7e-05	34.64089	-1524.0	0.0	**18.27037**
120	-2273.0	0.0	85.79514	-2273.0	0.0	**13.51938**
150	-2664.0	0.0	**25.14986**	-2664.0	1e-05	29.85202
180	-3313.0	0.0	**24.10841**	-3313.0	2e-05	30.57172
210	-3623.0	4e-05	**26.90973**	-3623.0	1e-05	35.88204
240	-4400.0	7e-05	**31.10726**	-4400.0	9e-05	50.51304
270	-4809.0	2e-05	**43.73691**	-4809.0	8e-05	46.18375
300	-5171.0	4e-05	**48.37354**	-5171.0	0.0	49.20291

Table C.34: NN-U-R-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-156.0	0.0	39.48325	-156.0	0.0	**34.75879**
60	-820.0	0.0	434.80065	-820.0	0.0	**246.44262**
90	-1288.0	4e-05	1179.29206	-1288.0	8e-05	**661.35201**
120	-1861.0	7e-05	1247.61128	-1861.0	0.0	**525.44709**
150	-2332.0	7e-05	**406.06239**	-2332.0	7e-05	810.16993
180	-2746.0	0.00814	3600.24221	-2748.0	0.00749	3600.98591
210	-3193.0	0.00356	3600.2236	-3194.0	0.0001	**2768.16935**
240	-3686.0	0.0102	3601.30657	-3694.0	0.00763	3601.99866
270	-3977.0	0.00597	3600.80268	-3989.0	0.00229	3600.04732
300	-4121.0	0.00904	3601.70424	-4141.0	0.00341	3600.09961

Table C.35: NN-U-R-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-52.0	0.0	0.48411	-52.0	0.0	**0.20304**
60	-341.0	0.0	0.48437	-341.0	0.0	**0.32796**
90	-108.0	0.0	0.60943	-108.0	0.0	**0.3281**
120	-4.0	0.0	0.57806	-4.0	0.0	**0.54688**
150	-407.0	0.0	**0.82821**	-407.0	0.0	1.21897
180	-177.0	0.0	**0.74996**	-177.0	0.0	0.79672
210	-251.0	0.0	**0.9688**	-251.0	0.0	1.37528
240	-269.0	0.0	**0.99992**	-269.0	5e-05	1.70297
270	-360.0	0.0	1.60937	-360.0	0.0	**1.53078**
300	-178.0	0.0	1.06283	-178.0	0.0	**1.04707**

Table C.36: NN-U-P-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.64	0.0	**1.56247**	-0.64	0.0	16.39234
60	-52.0	0.0	**2.51582**	-52.0	0.0	3.85003
90	-12.0	0.0	**1.57791**	-12.0	0.0	2.26388
120	-7.27778	0.0	2.29693	-7.27778	0.0	**1.68831**
150	-87.0	0.0	**2.28119**	-87.0	0.0	6.78496
180	-6.0	0.0	2.76569	-6.0	0.0	**1.17202**
210	-219.0	0.0	**2.60939**	-219.0	0.0	3.83988
240	-0.77778	0.0	8.06228	-0.77778	0.0	**5.89089**
270	-0.25	0.0	5.46774	-0.25	0.0	**1.45303**
300	0.5	0.0	3.98409	0.5	0.0	**1.77554**

Table C.37: NN-U-P-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-8.14286	0.0	**1.03123**	-8.14286	0.0	1.70312
60	-14.0	0.0	1.51567	-14.0	0.0	**1.42941**
90	-89.0	0.0	20.51008	-89.0	0.0	**7.62092**
120	-35.0	0.0	**14.12005**	-35.0	0.0	42.66652
150	-2.36364	0.0	**22.16133**	-2.36364	0.0	108.05917
180	-6.0	0.0	2.73554	-6.0	0.0	**7.70095**
210	-237.0	3e-05	178.82735	-237.0	0.0	**92.56919**
240	-127.0	0.0	60.33991	-127.0	0.0	**10.62976**
270	-138.0	0.0	**47.69332**	-138.0	5e-05	117.50314
300	-87.0	0.0	**47.25105**	-87.0	0.0	186.30164

Table C.38: NN-U-P-4 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.39241	0.0	16.61051	-0.39241	0.0	**7.17708**
60	-0.53846	0.0	44.22497	-0.53846	0.0	**24.04492**
90	-18.0	0.0	25.30915	-18.0	0.0	**10.99175**
120	-0.80952	0.0	**3000.95649**	-0.80952	0.08085	3600.02735
150	-0.28571	0.0	488.64886	-0.28261	0.0	**155.69752**
180	-0.90476	0.0	26.2587	-0.90476	0.0	**16.91089**
210	-121.0	6e-05	992.47478	-121.0	1e-05	**528.13312**
240	-26.0	0.0	249.44875	-26.0	0.0	**130.08951**
270	-1.0	0.0	789.25094	-0.94444	0.0	**6.75543**
300	-65.0	2e-05	**237.31099**	-65.0	5e-05	614.31928

Table C.39: NN-U-P-5(n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-423.0	0.0	1.93744	-423.0	0.0	**1.01568**
60	-2256.0	0.0	2.43728	-2256.0	0.0	**2.3751**
90	-119.0	0.0	3.18801	-119.0	0.0	**2.61707**
120	-4429.0	0.0	**1.95288**	-4429.0	0.0	2.68769
150	-1874.0	0.0	**2.84385**	-1874.0	0.0	3.87525
180	-4041.0	5e-05	**2.31209**	-4041.0	0.0	4.54834
210	-5375.0	0.0	82.6264	-5375.0	0.0	**53.01863**
240	-14.94444	0.0	**3.48673**	-14.94444	0.0	3.73247
270	-10875.0	0.0	7.15667	-10875.0	9e-05	**6.20993**
300	-10023.0	0.0	4.01694	-10023.0	2e-05	**1.71889**

Table C.40: NN-U-N-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-2.64835	0.0	1.79688	-2.64835	0.0	**1.34368**
60	-164.0	0.0	**3.95325**	-164.0	0.0	4.07838
90	-8.33333	0.0	3.45672	-8.33333	0.0	**2.98461**
120	-308.0	0.0	3.7034	-308.0	0.0	**1.768**
150	-348.0	0.0	**3.45329**	-348.0	0.0	4.55562
180	-1207.0	0.0	4.32847	-1207.0	7e-05	**3.04723**
210	-2571.0	0.0	**6.56301**	-2571.0	0.0	10.29751
240	-207.0	0.0	**5.67251**	-207.0	0.0	15.50231
270	-2688.0	0.0	5.20344	-2688.0	0.0	**4.30963**
300	-11217.0	0.0	8.50098	-11217.0	6e-05	**6.31036**

Table C.41: NN-U-N-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-7.33333	0.0	**1.54686**	-7.33333	0.0	1.71938
60	-704.0	0.0	9.27107	-704.0	0.0	**5.48478**
90	-15.58824	0.0	5.4714	-15.58824	0.0	**5.05151**
120	-861.0	0.0	10.38919	-861.0	0.0	**6.15296**
150	-22.69231	0.0	24.76917	-22.69231	0.0	**15.28262**
180	-61.0	0.0	**2.24876**	-61.0	0.0	4.3448
210	-809.0	0.0	**5.18782**	-809.0	0.0	12.56393
240	-2219.0	0.0	**31.73361**	-2219.0	0.0	46.77844
270	-1214.0	0.0	**19.02543**	-1214.0	0.0	21.20149
300	-88.66667	0.0	16.95431	-88.66667	0.0	**4.32943**

Table C.42: NN-U-N-4 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-6.46429	0.0	4.32573	-6.46429	0.0	**0.64071**
60	-13.64516	0.0	**18.52456**	-13.64516	0.0	73.93872
90	-27.2381	0.0	18.44706	-27.2381	0.0	**10.34521**
120	-9.09524	0.0	19.87641	-9.09524	0.0	**8.31427**
150	-26.25	0.0	**6.86357**	-26.25	0.0	20.90541
180	-21.16685	0.0	298.58288	-21.17674	0.0	**157.12812**
210	-1167.0	0.0	**15.57387**	-1167.0	0.0	37.66125
240	-731.0	0.0	53.11116	-731.0	0.0	**31.50033**
270	-489.0	0.0	**42.48937**	-489.0	0.0	50.97779
300	-90.0	0.0	40.49279	-90.0	0.0	**32.11266**

Table C.43: NN-U-N-5 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-239.0	0.0	5.32929	-239.0	0.0	**2.53921**
60	-1017.0	0.0	**23.8003**	-1017.0	0.0	71.43476
90	-1257.0	0.0	**33.8336**	-1257.0	0.0	61.72672
120	-1717.0	1e-05	**161.45424**	-1717.0	0.0	225.02123
150	-2313.0	0.0	31.63259	-2313.0	6e-05	**31.60454**
180	-2651.0	0.0	41.67385	-2651.0	2e-05	**38.90288**
210	-3006.0	0.0	228.81245	-3006.0	4e-05	**195.94204**
240	-3309.0	0.0001	**438.92369**	-3309.0	9e-05	550.55311
270	-3863.0	9e-05	558.6511	-3863.0	0.0001	**424.4108**
300	-4313.0	9e-05	267.92885	-4313.0	7e-05	**156.351**

Table C.44: NN-K-R-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-39.88889	0.0	34.9169	-39.88889	0.0	**28.04621**
60	-660.0	2e-05	**233.19192**	-660.0	5e-05	406.10095
90	-996.0	8e-05	**3502.04585**	-996.0	0.00423	3600.22525
120	-1441.0	0.02854	3600.0402	-1418.0	0.04709	3601.30547
150	-1698.0	0.02908	3600.06048	-1652.0	0.05908	3602.03444
180	-2280.0	0.00861	3600.69682	-2235.0	0.03304	3600.08931
210	-2613.0	0.02063	3600.264	-2627.0	0.01434	3600.08685
240	-3158.0	0.01418	3600.58891	-3147.0	0.01887	3600.84946
270	-3146.0	0.01932	3601.63798	-3147.0	0.01931	3600.09643
300	-3560.0	0.02384	3600.04945	-3525.0	0.03354	3601.90043

Table C.45: NN-K-R-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-27.0	0.0	0.90623	-27.0	0.0	**0.56248**
60	-223.0	0.0	0.56248	-223.0	0.0	**0.48435**
90	-100.0	0.0	**0.51538**	-100.0	0.0	0.68757
120	-4.0	0.0	**0.67189**	-4.0	0.0	0.7812
150	-338.0	0.0	**1.51562**	-338.0	0.0	1.71873
180	-177.0	0.0	**0.73456**	-177.0	0.0	1.32817
210	-220.0	0.0	**2.35937**	-220.0	0.0	4.10134
240	-226.0	4e-05	**6.81253**	-226.0	0.0	8.71891
270	-351.0	0.0	2.43763	-351.0	0.0	**2.21756**
300	-178.0	0.0	**0.82814**	-178.0	0.0	1.78124

Table C.46: NN-K-P-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.64	0.0	**1.09367**	-0.64	0.0	9.29396
60	-49.0	0.0	3.8477	-49.0	0.0	**3.64457**
90	-12.0	0.0	1.82824	-12.0	0.0	**1.3726**
120	-7.27778	0.0	**1.49989**	-7.27778	0.0	3.28918
150	-87.0	0.0	**2.9681**	-87.0	0.0	3.78676
180	-6.0	0.0	1.87498	-6.0	0.0	**1.29678**
210	-211.0	0.0	**6.56711**	-211.0	0.0	63.83262
240	-0.77778	0.0	**10.25102**	-0.77778	0.0	151.5213
270	-0.25	0.0	1.72	-0.25	0.0	**1.39019**
300	0.5	0.0	**5.00017**	0.0	0.0	0.90676

Table C.47: NN-K-P-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-8.14286	0.0	1.04692	-8.14286	0.0	**0.45309**
60	-14.0	0.0	3.10927	-14.0	0.0	**0.98447**
90	-82.0	0.0	92.33076	-82.0	0.0	**9.35966**
120	-35.0	0.0	26.18824	-35.0	0.0	**17.228**
150	-2.36364	0.0	371.3074	-2.36364	0.0	**107.57081**
180	-6.0	0.0	31.98722	-6.0	0.0	**5.75145**
210	-185.0	6e-05	131.88036	-185.0	0.0	**48.72227**
240	-126.0	0.0	62.09093	-126.0	0.0	**17.36845**
270	-125.0	0.0001	**277.59762**	-125.0	8e-05	390.15122
300	-87.0	0.0	**17.02431**	-87.0	0.0	33.52386

Table C.48: NN-K-P-4 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-0.39241	0.0	4.2186	-0.39241	0.0	**4.12494**
60	-0.53846	0.0	103.68813	-0.53846	0.0	**44.71535**
90	-18.0	0.0	84.27514	-18.0	0.0	**60.13914**
120	-0.80952	0.0	**790.76301**	-0.80952	0.0	2504.18675
150	-0.28261	0.0	**250.35018**	-0.28261	0.0	329.7177
180	-0.90476	0.0	54.30327	-0.90476	0.0	**30.44648**
210	-114.0	9e-05	259.57583	-114.0	3e-05	**169.8463**
240	-26.00005	0.0	238.34092	-26.0	0.0	**96.47234**
270	-1.0	8e-05	3552.70766	-0.94467	0.0	**51.55697**
300	-65.0	0.0	116.5611	-65.0	6e-05	**96.18923**

Table C.49: NN-K-P-5 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-259.0	0.0	3.25016	-259.0	0.0	**0.85944**
60	-1361.0	0.0	4.6722	-1361.0	0.0	**4.29709**
90	-119.0	0.0	4.78163	-119.0	0.0	**2.67412**
120	-2226.0	0.0	5.28308	-2226.0	0.0	**1.79708**
150	-1797.0	0.0	8.3641	-1797.0	0.0	**4.85973**
180	-3111.0	0.0	9.04751	-3111.0	0.0	**6.59093**
210	-3535.0	0.0	12.90746	-3535.0	0.0	**9.86013**
240	-14.94444	0.0	4.43777	-14.94444	0.0	**3.95337**
270	-5347.0	0.0	168.3986	-5347.0	0.0	**137.61606**
300	-6325.0	0.0	176.0672	-6325.0	0.0	**80.76029**

Table C.50: NN-K-N-2 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-2.64835	0.0	**1.93762**	-2.64835	0.0	3.53156
60	-164.0	0.0	**1.90638**	-164.0	0.0	2.70321
90	-8.33333	0.0	6.63856	-8.33333	0.0	**2.48463**
120	-308.0	0.0	3.29704	-308.0	0.0	**2.50258**
150	-348.0	0.0	**5.21915**	-348.0	0.0	6.78173
180	-1207.0	0.0	**4.53405**	-1207.0	0.0	15.40872
210	-2311.0	0.0	12.45563	-2311.0	0.0	**10.54762**
240	-207.0	0.0	**5.14094**	-207.0	0.0	5.21922
270	-2688.0	0.0	**4.67208**	-2688.0	0.0	20.9082
300	-6481.0	0.0	**30.60764**	-6481.0	0.0	34.10652

Table C.51: NN-K-N-3 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-7.33333	0.0	3.86462	-7.33333	0.0	**2.29716**
60	-627.0	0.0	15.43873	-627.0	0.0	**11.89124**
90	-15.58824	0.0	**8.70656**	-15.58824	0.0	9.31394
120	-861.0	0.0	6.22389	-861.0	0.0	**5.46908**
150	-22.69231	0.0	**7.99776**	-22.69231	0.0	28.11316
180	-61.0	0.0	3.23465	-61.0	0.0	**3.21889**
210	-809.0	0.0	18.31849	-809.0	0.0	**15.08364**
240	-2219.0	0.0	33.11942	-2219.0	0.0	**25.98723**
270	-1214.0	0.0	17.59538	-1214.0	0.0	**13.5537**
300	-88.66667	0.0	10.53252	-88.66667	0.0	**3.21955**

Table C.52: NN-K-N-4 (n=30-300)

n	GMMFP2			GMMFP2M		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
30	-6.46429	0.0	1.96887	-6.46429	0.0	**1.90655**
60	-13.64516	0.0	**18.08112**	-13.64516	0.0	138.67451
90	-27.2381	0.0	30.62464	-27.2381	0.0	**13.16627**
120	-9.09524	0.0	**13.25158**	-9.1	0.0	14.78847
150	-26.25	0.0	**8.85799**	-26.25	0.0	15.86397
180	-21.16667	0.0	176.08185	-21.16667	0.0	**72.03374**
210	-1167.0	0.0	**28.71662**	-1167.0	0.0	56.23266
240	-731.0	0.0	**35.74313**	-731.0	0.0	58.58073
270	-489.0	0.0	13.78493	-489.0	0.0	**10.67643**
300	-90.0	0.0	119.58242	-90.0	0.0	**54.09804**

Table C.53: NN-K-N-5 (n=30-300)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-202.33333	0.0	**8.67179**	-202.33333	0.0	15.45211
20	-1178.0	0.0	**260.78752**	-1178.0	0.0	395.44912
30	-2107.0	2e-05	**128.77585**	-2107.0	3e-05	196.20919
40	-3044.0	7e-05	**459.0712**	-3044.0	5e-05	727.76888
50	-3828.0	0.0	**1105.38963**	-3828.0	9e-05	2843.57632
60	-4732.0	7e-05	**2099.91305**	-4641.0	0.02343	3600.04303
70	-5650.0	0.00525	3600.17726	-5659.0	0.00262	3600.02258
80	-6515.0	0.03362	3600.09357	-6712.0	7e-05	**2804.48191**
90	-7452.0	0.00263	3600.07458	-7426.0	0.00694	3600.04219
100	-8395.0	0.0135	3600.05071	-6516.0	0.41327	3600.0979

Table C.54: NN-A-R-2 (n=10-100)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-29.0	0.0	80.36451	-29.0	0.0	**69.84198**
20	-613.00061	0.48913	3600.03325	-268.007	2.47611	3600.18528
30	-493.50101	2.53301	3600.11745	-948.00171	0.83411	3600.08751
40	-1670.00016	0.5071	3600.03703	-66.43753	43.67621	3600.04985
50	-431.28575	7.85085	3600.04481	-161.5385	23.36812	3600.07458
60	-187.0003	24.87379	3600.06208	-229.41177	16.97789	3600.05802
70	-54.15346	116.52393	3600.12195	-94.80958	64.52847	3600.11469
80	-210.2583	32.98043	3600.04874	-228.40006	32.69618	3600.08964
90	-273.00282	30.75351	3600.09402	-136.00041	62.73558	3600.08972
100	-213.91667	44.432	3600.10784	-53.63647	156.59626	3600.06944

Table C.55: NN-A-R-3 (n=10-100)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-97.0	0.0	2.96839	-97.0	0.0	**2.69057**
20	-328.0	0.0	**5.62509**	-328.0	0.0	7.16625
30	-378.0	0.0	21.14833	-378.0	0.0	**20.81516**
40	-472.0	0.0	**54.73302**	-472.0	0.0	64.15143
50	-536.0	0.0	**31.30791**	-536.0	0.0	32.9594
60	-596.0	0.0	268.78734	-596.0	0.0	**51.63477**
70	-641.0	0.0	278.62422	-641.0	5e-05	**48.73853**
80	-706.0	1e-05	133.72353	-706.0	0.0	**115.3706**
90	-734.0	0.0	121.36211	-734.0	6e-05	**121.07237**
100	-843.0	0.37259	3600.10739	-844.0	0.0	**174.53628**

Table C.56: NN-A-P-2 (n=10-100)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-24.66667	0.0	5.12497	-24.66667	0.0	**4.92728**
20	-235.0	0.0	**184.21108**	-235.0	0.0	249.12878
30	-333.0	6e-05	**1252.13128**	-333.0	8e-05	2066.47955
40	-365.0	0.0	**415.77698**	-365.0	0.0	996.96696
50	-463.0	0.0001	**2371.50216**	-463.0	8e-05	3109.2365
60	-467.00001	0.00497	3600.03839	-468.0	9e-05	**1723.27957**
70	-563.0	8e-05	**1190.96233**	-563.0	0.00074	3600.38309
80	-566.0	0.0001	2682.73308	-566.0	9e-05	**1819.18688**
90	-618.0	9e-05	**1943.60216**	-578.0	0.31153	3600.42789
100	-669.0	0.00149	3600.04275	-622.0	0.32	3600.44567

Table C.57: NN-A-P-3 (n=10-100)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-90.0	0.0	19.23883	-90.0	0.0	**13.70996**
20	-329.0	0.0	**16.86358**	-329.0	0.0	20.29918
30	-409.0	0.0	**44.35613**	-409.0	1e-05	56.18858
40	-516.0	0.0001	**55.6695**	-516.0	0.0	68.66658
50	-562.0	0.0	40.57149	-562.0	0.0	**32.72975**
60	-556.0	0.0	132.64892	-556.0	0.0	**70.89065**
70	-664.0	0.0	**209.56237**	-664.0	0.0	209.77427
80	-733.0	5e-05	252.75766	-733.0	7e-05	**130.21431**
90	-814.0	8e-05	205.12007	-814.0	0.0	**168.46659**
100	-821.0	0.0	432.69184	-821.0	0.0	**103.70438**

Table C.58: NN-A-N-2 (n=10-100)

n	GMMFP2A			GMMFP2AM		
	Obj.Val	Gap(%)	Time	Obj.Val	Gap(%)	Time
10	-43.0	0.0	38.3723	-43.0	0.0	**36.74291**
20	-203.00023	8e-05	1610.62736	-203.0	9e-05	**711.67187**
30	-301.0	8e-05	782.00706	-301.0	0.0	**761.70082**
40	-384.0	7e-05	**3134.01948**	-384.0	0.00888	3600.11105
50	-407.0	0.08846	3600.04123	-431.00001	0.00407	3600.2231
60	-512.0	0.00522	3600.08665	-512.0	0.00567	3600.0316
70	-498.0	0.21639	3600.06408	-431.0001	0.95237	3600.14213
80	-591.0	9e-05	**1521.16679**	-591.0	0.00286	3600.3614
90	-271.0	1.70993	3600.14667	-588.00001	0.2479	3600.59391
100	-323.00131	2.16827	3600.30522	-666.0	0.00612	3600.74159

Table C.59: NN-A-N-3 (n=10-100)