Cops and robbers on Cayley graphs and embedded graphs

by

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Abstract

We consider the game of *cops and robbers*, which is a game played on a finite graph G by two players, Alice and Bob. Alice controls a team of *cops*, and Bob controls a *robber*, both of which occupy vertices of G. On Alice's turn, she may move each cop to an adjacent vertex or leave it at its current position. Similarly, on Bob's turn, he may move the robber to an adjacent vertex or leave it at its current position. Traditionally, Alice wins the game when a cop occupies the same vertex as the robber—that is, when a cop *captures* the robber. Conversely, Bob wins the game by letting the robber avoid capture forever. In a variation of the game, Alice wins the game when each neighbor of the robber's vertex is occupied by a cop—that is, when cops *surround* the robber. We will consider both of these winning conditions.

The most fundamental graph invariant with regard to the game of cops and robbers is the *cop number* of a graph G, which denotes the minimum number of cops that Alice needs in order to have a winning strategy on G. We will introduce new techniques that may be used to calculate lower and upper bounds for the cop numbers of certain Cayley graphs. In particular, we will show that the well-known Meyniel's conjecture holds for both undirected and directed abelian Cayley graphs. We will also introduce new techniques for establishing upper bounds on the cop numbers of surface-embedded graphs bounded by the genus of the surface in the surrounding win condition.

Keywords: cops and robbers, Cayley graph, Meyniel's conjecture, planar graph, graph genus

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Introduction

1.1 Notation

We follow standard graph theoretic notation used by Bondy and Murty [7]. We say that a *graph* is a collection of *vertices* and *edges*, where vertices are elements of a set, and edges are sets consisting of exactly two distinct vertices. For a graph G, we often write $V(G)$ to denote the vertices of G, and we often write $E(G)$ to denote the edges of G. With this notation, $E(G) \subseteq \binom{V(G)}{2}$. We say that a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a graph G in which $u, v \in V(G)$, if $\{u, v\} \in E(G)$, we say that u and v are *adjacent*, and we also say that u and v are *endpoints* of the edge $\{u, v\}$. To briefly denote that u and v are *adjacent*, we often write $u \sim v$. Furthermore, we often write uv to denote the edge $\{u, v\}$.

For a graph G and a vertex $v \in V(G)$, we write $N(v)$ to denote the set of all vertices of G that are adjacent to v; that is $N(v) = \{u \in V(G) : uv \in E(G)\}$. We say that $N(v)$ is the *open neighborhood* of v, or simply the *neighborhood* of v. Furthermore, we write $N[v]$ to denote the set $N(v) \cup \{v\}$, and we say that $N[v]$ is the *closed neighborhood* of v. The *degree* of a vertex v denotes the number of vertices in $N(v)$, and we write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degree over all vertices in a graph G, respectively.

Given a graph G, we say that a *path* P in G is a subgraph of G whose vertices may be arranged into a non-repeating sequence $(v_1, \ldots, v_k)$ such that $v_i v_{i+1} \in E(G)$ for every $1 \leq i \leq k-1$, and such that P has exactly these $k-1$ edges. We say that v_1 and v_k are the *endpoints* of P. We say that the *length* of a path P is the number of edges in P. For a graph G and two vertices $u, v \in V(G)$, we say that the *distance* between u and v is the length of a shortest path with u and v as endpoints, and we write $\mathrm{dist}(u, v)$ for the distance between u and v. We say that a *geodesic path* (or just *geodesic*) from u to v is a path with u and v as endpoints and of length $\mathrm{dist}(u, v)$. For any vertex $v \in V(G)$, we have $\mathrm{dist}(v, v) = 0$. We say that a graph G is *connected* if for any $u, v \in V(G)$, there exists a path with u and v as endpoints. In this thesis, we will assume that all graphs are finite and connected.

Given a graph G, we say that a *cycle* C in G is a subgraph of G, containing at least three vertices, whose vertices may be arranged into a non-repeating sequence $(v_1, \ldots, v_k)$ such that $v_i v_{i+1} \in E(G)$ for every $1 \leq i \leq k-1$, and such that C contains exactly these $k-1$ edges, along with the edge $v_k v_1$. Informally, a cycle is a path of length at least 3 whose two endpoints are the same. We say that the *girth* of a graph G is the length of a shortest cycle in G.

Given a graph G and an orientable surface S, an *embedding* of G in S is a representation (often called a *drawing*) of G such that each vertex of G is represented by a distinct point in S, each edge of G is represented by a distinct simple arc in S, and such that the following properties hold:

- An arc representing an edge $e \in E(G)$ has endpoints in S corresponding to the endpoint vertices of e in G.
- An arc representing an edge $e \in E(G)$ does not contain any point in S corresponding to a vertex in G that is not an endpoint of e.
- No two arcs intersect in S except at their endpoints.

Given an orientable surface S, the *genus* of S is the maximum number of simple closed curves that may be removed from S without disconnecting S. Informally, a sphere has genus 0, and the genus of a general orientable surface S is the number of "holes" in S. The *genus* g of a graph G is the minimum genus g of a surface S in which G has an embedding. An orientable surface of genus 0 is homeomorphic to a sphere, and a graph that may be embedded in a genus 0 surface is called *planar*. An orientable surface of genus 1 is homeomorphic to a *torus*, and a graph that may be embedded in a surface of genus 1 is called *toroidal*.

We will also consider groups, and we will use standard group theoretic notation, such as that used by Herstein [22]. A *group* is a set A of elements paired with a binary operation $\times$ satisfying the following properties.

- For any elements $a, b \in A$, $a \times b \in A$.
- For any elements $a, b, c \in A$, $(a \times b) \times c = a \times (b \times c)$.
- There exists an *identity* element $e \in A$ such that for any $a \in A$, $a \times e = e \times a = a$.
- For any element $a \in A$, there exists an element $b \in A$ such that $a \times b = b \times a = e$. In this case, b is called the *inverse* of a, and we write $b = a^{-1}$.

We often omit the binary operation $\times$ and write ab for $a \times b$. We say that a group $(A, \times)$ is *abelian* if for any $a, b \in A$, $a \times b = b \times a$. For further definitions and basic results related to groups, such as subgroups, quotient groups, generating sets, and group homomorphisms, we refer the reader to Herstein [22].

1.2 Background

We consider the game of *cops and robbers*, a perfect information game played by two players on a finite graph G. The first player, whom we call Alice, controls a team of m cops. The second player, whom we call Bob, controls a robber. (Often, Alice is identified with the cops that she controls, and Bob is identified with the robber.) At the beginning of the game, Alice chooses a set of m (not necessarily distinct) vertices $v_1, \dots, v_m \in V(G)$ and places a cop at each of these vertices. Next, Bob chooses a vertex $w \in V(G)$ and places the robber at w. Alice and Bob then take turns moving the cops and robber throughout G. On Alice's turn, for each cop C, Alice may move C along an edge to an adjacent vertex or leave C at its current vertex. There is no restriction preventing Alice from moving two or more cops to a single vertex. On Bob's turn, Bob may move the robber along an edge to an adjacent vertex or leave the robber at its current vertex. Alice wins the game if the robber and any cop occupy the same vertex at any time, in which case we say that the robber is *captured*. Bob wins the game if the robber avoids capture forever. We may also give Bob a finite-time win condition by saying that Bob wins the game if the same game position ever appears twice. This change does not affect the strategy of either player.

The game of cops and robbers was first introduced by Quilliot in [30], and later independently by Nowakowski and Winkler [28]. These two papers both focused on games in which Alice uses only one cop. Later, Aigner and Fromme generalized the game so that Alice may control any number m of cops [1]. In Aigner and Fromme's paper, the authors introduce the concept of the *cop number* of a graph G, which is the minimum number m of cops that Alice needs in order to have a winning strategy on G. For a graph G, we write $c(G)$ for the cop number of G. Aigner and Fromme then give simple bounds for the cop numbers of planar graphs and graph of girth at least 5.

We will also consider the game of *surrounding cops and robbers*, first invented by Burgess et al. in [11]. The rules of the game are the same as those of cops and robbers, except that Alice wins the game only when the robber occupies a vertex whose neighbors are all occupied by cops—that is, Alice wins the game by surrounding the robber with cops. Furthermore, at the end of each of Bob's turns, the robber may not occupy the same vertex as a cop; this prevents Bob from leaving the robber at a vertex of high degree forever.

The slight change in rules between traditional cops and robbers and surrounding cops and robbers brings about some differences in gameplay. In the surrounding variant, Alice can no longer win the game simply by "capturing" the robber with a cop. Rather, whenever Alice moves a cop to the robber's vertex, the robber is simply forced to move away, since Bob may not leave the robber at a vertex occupied by a cop. Therefore, when Alice "captures" the robber with a cop in the surrounding variant, she accomplishes nothing other than forcing the robber to move to a new vertex. Additionally, in the surrounding variant, a cop may not "guard" all of the vertices in its neighborhood like in the traditional game, since the robber is no longer threatened by the prospect of being captured by a cop. Rather, if Alice wishes to prevent the robber from moving to a vertex

v, she must place a cop at v, as Bob is not allowed to move the robber to a vertex already occupied by a cop.

The game of surrounding cops and robbers is one of several variants of the game of cops and robbers. Another such variant is called *cops and attacking robber*, invented by Bonato et al. [6], in which the robber may capture a cop by moving to the cop's vertex, after which the captured cop is removed from the game. Yet another variant is *lazy cops and robbers*, invented by Offner and Ojakian [29], in which only one cop may move on each turn. When considering the game of surrounding cops and robbers, one natural parameter to consider is the *surrounding cop number* of a graph G, which is the minimum number of cops that Alice needs to have a winning strategy on G. The authors of [11] show several bounds on the surrounding cop number of certain graph classes, including grids and products of cycles.

The concept of cop number is a natural step in graph theory toward the study of dynamic graph parameters. The graph parameters traditionally considered by graph theorists, such as Hamiltonicity, chromatic number, genus, girth, or crossing number, are all based on the existence or nonexistence of a certain static structure in a graph. The cop number of a graph, however, is a certain characterization of how units move throughout a graph dynamically. In fact, cop number can be formally distinguished from traditional graph parameters by its complexity, as cop number decision problems are EXPTIME-complete [24], while decision problems for the traditional parameters listed above belong to NP. In the spirit of investigating a graph's dynamic properties, the game of cops and robbers has led to a field known as graph searching, which is a more general study of processes that move and spread through a graph, including graph sweeping, graph localization, and zero forcing (see [15], [12], and [32] for further discussion of these topics). While the graph theoretic game of cops and robbers has little application in real-life law enforcement, its importance and motivation lie in its description and characterization of how graphs behave in a dynamic sense.

1.3 Known results

We give some basic results and conjectures for the game of cops and robbers.

1.3.1 Standard win condition

We first consider the game of cops and robbers with the standard "capture" win condition. Recall that the game of cops and robbers is played on a finite graph G by two players, Alice and Bob. Recall that Alice controls m cops, while Bob controls a robber. If Alice has a winning strategy on G using m cops, then we say that G is *m-copwin*. The first result that we survey in cops and robbers characterizes graphs that are 1-copwin, or simply *copwin*.

Let G be a graph, and let $v \in V(G)$. If there exists a vertex $u \in V(G), u \neq v$ such that $N[v] \subseteq N[u]$, then we say that v is a *corner* of G. (Some authors also say that v is a *pitfall* [1] or an *irreducible vertex* [28].) We say that G has an *elimination ordering* $(v_1, \dots, v_n)$ if for each vertex

$v_i \in V(G), 1 \leq i \leq n-1$, v_i is a corner in the graph $G[v_i, v_{i+1}, \ldots, v_n]$. We then have the following theorem.

Theorem 1.3.1 ([28])**.** *Let G be a graph. Then G is copwin if and only if G has an elimination ordering.*

Theorem 1.3.1 gives a complete characterization of graphs with cop number equal to 1. For $m \geq 2$, m-copwin graphs do not admit such clean characterizations, but many graph classes still have well-understood bounds on cop number. For instance, the following theorems show that the cop number of a graph embedded on a surface of bounded genus is bounded above. Recall that for a graph G, we write $c(G)$ for the cop number of G. Recall further that we assume that all graphs are finite and connected.

Theorem 1.3.2 ([26])**.** *Let G be a graph with genus at most 1. Then $c(G) \leq 3$.*

Theorem 1.3.3 ([8])**.** *Let G be a graph with genus g. Then $c(G) \leq \frac{4}{3}g + \frac{10}{3}$.*

Additionally, for certain Cayley graphs, cop number may be bounded above by a function of the graph degree. A Cayley graph is defined as follows.

Definition 1.3.4. Let G be a group, and let $S \subseteq G$ be a generating set of G satisfying $S = S^{-1}$ that does not contain the identity of G. Then the *Cayley graph* of G and S, written $\text{Cay}(G, S)$, is a graph whose vertex set is give by G and whose edges are defined as follows. For each element $a \in G$ and each generator $s \in S$, $a \sim sa$ in $\text{Cay}(G, S)$; that is, a and sa are adjacent. Furthermore, a Cayley graph is called *normal* if $g^{-1}Sg = S$ for all $g \in G$.

Theorem 1.3.5 ([16])**.** *Let G be a d-regular Cayley graph on an abelian group. Then $c(G) \leq \lceil \frac{1}{2}(d+1) \rceil$.*

Theorem 1.3.6 ([17])**.** *Let G be a d-regular normal Cayley graph. Then $c(G) \leq d$.*

For graphs of large girth, on the other hand, cop number can be bounded below.

Theorem 1.3.7 ([1])**.** *Let G be a graph of girth at least 5. Then $c(G) \geq \delta(G)$.*

Theorem 1.3.8 ([10])**.** *Let $t \geq 1$, and let G be a graph of girth at least $4t+1$ and minimum degree δ. The $c(G) \geq \frac{1}{et}(\delta - 1)^t$.*

One question that has persisted throughout the study of cops and robbers is the following. What is the maximum cop number of a graph on n vertices? One conjecture that is widely suspected to be true, but that is still open, is Meyniel's conjecture, which suggests the following answer.

Conjecture 1.3.9 ([17])**.** *Let G be a graph on n vertices. Then $c(G) = O(\sqrt{n})$.*

Even the following weaker conjecture, which is implied by Meyniel's conjecture, is still widely open. This conjecture appears in [4] but has likely been considered since Meyniel's conjecture was posed.

Conjecture 1.3.10. *There exists a value $\epsilon > 0$ such that for all graphs G on n vertices, $c(G) \leq n^{1-\epsilon}$.*

Meyniel's conjecture, first posed in 1985, is perhaps the furthest reaching conjecture in the area of cops and robbers. It has driven a great deal of research, but since being posed 35 years ago, little progress has been made. Currently, the best general upper bound for the cop number of a graph on n vertices is as follows.

Theorem 1.3.11 ([27, 34])**.** *Let G be a graph on n vertices. Then*

$$c(G) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log n}}}\right).$$

If Conjecture 1.3.9 is true, then it would be best possible, as we will see that there exist several graph families in which a graph on n vertices has cop number $\Omega(\sqrt{n})$. However, the overall lack of progress even on Conjecture 1.3.10 shows how difficult Conjecture 1.3.9 is. On the other hand, during 35 years of study, no graph family in which a graph on n vertices has cop number greater than $O(\sqrt{n})$ has been found. We will conclude this section with a graph construction that shows that Conjecture 1.3.9, if true, is best possible.

Theorem 1.3.12 ([4])**.** *Let q be a prime power, and let P_q be the projective plane over the field $\mathbb{F}_q$, with $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Let G be a graph whose vertices are given by the points and lines of P_q such that each line of P_q is adjacent to the points that it contains. Then G is a $(q + 1)$-regular graph of girth 6, and hence $c(G) \geq q + 1$.*

For a prime power q, the graph G obtained from the projective plane P_q contains $2q^2 + 2q + 2$ vertices and has cop number $q+1$, implying that $c(G) \geq (1-o(1))\sqrt{\frac{|V(G)|}{2}} = \Omega(\sqrt{|V(G)|})$. Therefore, Conjecture 1.3.9 cannot be improved.

1.3.2 Surrounding win condition

Next, we consider the surrounding cops and robbers win condition. Given a graph G, we say that the *surrounding cop number* of G, written $s(G)$, is the minimum number m such that Alice has a winning strategy with m cops in the game of surrounding cops and robbers on G.

By the definition of the surrounding win condition, Alice can only win if she has enough cops to occupy every neighbor of the robber's vertex, which gives us a straightforward lower bound for $s(G)$:

Observation 1.3.13 ([11])**.** *Let G be a graph of minimum degree δ. Then $s(G) \geq \delta$.*

Furthermore, if a cop occupies every neighbor of the robber's vertex, then the robber clearly cannot avoid being captured in the traditional sense, giving another straightforward lower bound for $s(G)$:

Theorem 1.3.14 ([11])**.** *Let G be a graph with cop number $c(G)$. Then $s(G) \geq c(G)$.*

Bounds on the surrounding cop number of a graph may also be obtained from properties related to a graph's decomposition. One such property is the *treewidth* of a graph, which is defined as follows. For an integer $k \geq 1$, a *k-tree* is a graph that is formed from a k-clique by repeatedly adding vertices of degree exactly k whose neighbors induce a clique. Then, for a graph G, the treewidth of G is the minimum integer k for which G is a subgraph of some k-tree. A graph's treewidth gives the following bound on the surrounding cop number.

Theorem 1.3.15 ([11])**.** *Let G be a graph of treewidth k. Then $s(G) \leq k + 1$.*

Another parameter related to the surrounding cop number of a graph is *degeneracy*, which is defined as follows. For a graph G, the *degeneracy* of G is the minimum integer k for which every subgraph of G has a vertex of degree at most k. The degeneracy of a graph gives the following bound on the surrounding cop number.

Theorem 1.3.16. *Let G be a graph, and let k be the least integer for which G is k-degenerate. Then $s(G) \geq k$.*

Proof. This theorem is implied by the methods of [11] but is not stated explicitly, so we include a proof for completeness.

If every subgraph of G contains a vertex of degree at most $k-1$, then G is $(k-1)$-degenerate; hence, as G is not $(k-1)$-degenerate, G contains a subgraph $H \subseteq G$ of minimum degree k. Bob's strategy will be to begin the game with the robber at a vertex of H and to leave the robber at H for the entire game. If Alice has at most $k-1$ cops, then when the robber is at a vertex of H, the robber will always have an unoccupied neighbor in H. Furthermore, if a cop moves to the robber's vertex, then the robber will always have an available vertex of H to which to move. Therefore, Bob may leave the robber at H indefinitely and win the game. □

Surprisingly, apart from the bounds listed here and some straightforward bounds for simple graph classes, little else is known about the surrounding cop number. In particular, no analogues of Theorems 1.3.2 and 1.3.3 bounding the surrounding cop number of graphs of bounded genus are known.

1.4 Thesis structure

This thesis will be divided into two main parts. The first part of the thesis will consider the game of cops and robbers on Cayley graphs. We will first show a simple argument that proves Meyniel's conjecture for abelian Cayley graphs, showing that an abelian Cayley graph on n vertices has cop number at most $6\sqrt{n}$. Next, we will give a more technical argument that bounds the cop number of both directed and undirected abelian Cayley graphs on n vertices by approximately $1.33\sqrt{n}+10$ and $0.94\sqrt{n}+6$, respectively. Finally, we will construct several Cayley graph families on n vertices with cop number $\Omega(\sqrt{n})$.

In the second part of the thesis, we will consider the game of surrounding cops and robbers. We will prove upper bounds for the surrounding cop number of planar graphs, bipartite planar graphs, toroidal graphs, graphs of bounded genus, and graphs that exclude a minor. We will also extend Theorem 1.3.6 and show that for a d-regular normal Cayley graph, $s(G) = d$, which is best possible.

1.5 Submission disclosures

The results in Section 2.1 appear in [9].

The results in Sections 2.2 and Section 2.3.1 are joint work with Jérémie Turcotte and Seyyed Aliasghar Hosseini. These results have been submitted to the European Journal of Combinatorics and are currently in revision.

The results of Section 3.1 are joint work with Seyyed Aliasghar Hosseini. These results have been submitted to the Journal of Combinatorics and are awaiting review.

Chapter 2

Cops and robbers on Cayley graphs

2.1 Abelian Cayley graphs: a simple proof of Meyniel's conjecture

We will consider the game of cops and robbers on Cayley graphs over abelian groups. We have already defined Cayley graphs in the previous section, but as we wish to use notation that is specifically suited to abelian groups in this section, we will restate the definition of a Cayley graph over an abelian group. Given an abelian group $(G, +)$ and a set $S \subseteq G$ satisfying $0 \not\in S$ and $S = -S$, we define the graph $Cay(G, S)$ as follows. The elements of G become the vertices of $Cay(G, S)$, and two elements $u, v \in G$ are adjacent if $-u + v \in S$. We say that $Cay(G, S)$ is the *Cayley graph* on G generated by S. Whenever we consider a Cayley graph on an abelian group G generated by a set S, we will always assume that $S = -S$ and $0 \not\in S$. We note that the elements of S generate G if and only if $Cay(G, S)$ is connected, and the requirement that $S = -S$ ensures that vertex adjacency is symmetric; that is, $uv \in V(G)$ if and only if $vu \in V(G)$. We say that $Cay(G, S)$ is an *abelian Cayley graph* if G is an abelian group. In the definition above, we say that S is the *generating set* of $Cay(G, S)$, and we call the elements of S *generators*. Frankl considers the cop number of abelian Cayley graphs in [16].

We will outline Frankl's general approach to capturing a robber on an abelian Cayley graph. When playing cops and robbers on a Cayley graph on an abelian group G generated by $S \subseteq G$, we imagine that at each turn, the robber occupies some group element $r \in G$ and has a list of possible moves corresponding to the elements of S. The robber may choose any element $s \in S$ on his turn and move to the group element $r + s \in G$. We call this *playing the move* s. To capture the robber, we will let our cops follow a strategy that makes certain robber moves $s \in S$ unsafe for the robber. As we make certain robber moves unsafe, the robber's list of possible moves will become shorter, and the robber's movement options will become more limited. As the robber's movement becomes more limited, it will become easier for the cops to make even more robber moves unsafe, and the cops will be able to limit the robber's movement further. Eventually, the cops will make every move unsafe for the robber, and the robber will have no way to avoid capture. The precise meaning of an unsafe move will be discussed later.

Frankl shows in [16] that on an abelian Cayley graph, one cop can almost always make two robber moves unsafe, which gives the following theorem.

Theorem 2.1.1 ([16])**.** *Let* Γ *be a Cayley graph on an abelian group with a generating set* S *that satisfies* $S = -S$ *and* $0 \notin S$*. Then,*

$$c(\Gamma) \leq \left\lceil \frac{|S| + 1}{2} \right\rceil .$$

By applying the ideas of Frankl used in Theorem 2.1.1, we will prove the following theorem, which proves Meyniel's conjecture for abelian Cayley graphs.

Theorem 2.1.2. *Let* Γ *be a Cayley graph on an abelian group of* n *elements. Then* $c(\Gamma) \leq 6\sqrt{n}$*.*

Frankl's bound is linear to the size of the generating set of a Cayley graph, so for abelian Cayley graphs with large generating sets, our bound is an improvement over that of Frankl. The proof of Theorem 2.1.2 builds upon Frankl's idea of letting cops make certain robber moves unsafe. We use the fact that if a Cayley graph has distinct generators $a_1, a_2, \dots, a_j, b_1, b_2, \dots, b_j$ for which there exists an element $d \in G$ satisfying $d = a_1 - b_1 = a_2 - b_2 = \cdots = a_j - b_j \neq 0$, then we can execute the following strategy. If the robber is at vertex $r \in G$, then we show that we can place a cop at a vertex $r + \gamma d$ for some nonnegative integer γ. Then, if the robber uses a_i for any $1 \leq i \leq j$, then the cop responds with b_i. This way, if the robber uses any of $a_1, a_2, \dots, a_j$, then the difference between the cop and the robber's positions decreases by exactly d. Furthermore, if the robber uses any of these moves a_i a total of γ times, then the difference between the cop and robber becomes zero, and the robber is caught. Therefore, the robber must eventually stop using the moves $a_1, a_2, \dots, a_j$, and our cop essentially takes away all of the moves $a_1, a_2, \dots, a_j$ from the robber. We say that a cop that follows this strategy makes the moves $a_1, \dots, a_j$ *unsafe*. The difference between our method and that of Frankl is that while Frankl only allows one cop to take away two moves from the robber, we allow one cop to take away many moves from the robber.

Our proof also uses the following general idea. If the generating set of a Cayley graph is small, then a small number cops can capture the robber by using Frankl's strategy from [16]. If the generating set of a Cayley graph is large, however, then there will be many generators $a_1, a_2, \dots, a_j, b_1, b_2, \dots, b_j$ that can be paired into equal differences $a_1 - b_1 = a_2 - b_2 = \cdots = a_j - b_j \neq 0$, and we can find a strategy in which one cop takes away many moves from the robber. Thus, regardless of whether the generating set of a graph is small or large, we can capture the robber with a small number of cops.

For our proof, we will need the following definition and lemma.

Definition 2.1.3. Let $(G, +)$ be a finite abelian group, and let $S \subseteq G$. For $s \in S$, we say that a set $K \subseteq G$ *accounts for* s *with respect to* S if there exists an element $j \in S \cup \{0\}$ such that $s - j \in K$.

We say that K accounts for S if for each $s \in S$, K accounts for s with respect to S. When K is a singleton set $\{k\}$, we write that k accounts for s with respect to S.

This definition is essential for our strategy. A key idea that we will show in our proof is that given an abelian Cayley graph $Cay(G, S)$, we can construct a set $K = \{d_1, \ldots, d_l\}$ accounting for S; then if the robber is at vertex r, we will show that we can place l cops at vertices $r+k_1d_1, r+k_2d_2, \ldots, r+k_ld_l$ (for some nonnegative integers $k_1, \ldots, k_l$) and capture the robber in a finite number of moves. Hence the cardinality of K will be closely related to the number of cops needed to catch the robber, and bounding the cardinality of K will help us bound the cop number of $Cay(G, S)$. This brings us to the following lemma.

Lemma 2.1.4. *Let $(G, +)$ be an abelian group of order n, and let $S = \{a_1, \ldots, a_k\} \subseteq G$, $|S| \geq 2, S = -S$. Then there exists a set $K \subseteq G$ of order at most $\lfloor \frac{3}{2}\sqrt{n} + \frac{5}{2} \rfloor$ such that $0 \notin K$ and such that K accounts for S.*

Proof. Let $k = |S|$. If $k \leq \sqrt{n}$, then we may let $K = S$. Then, for any element $s \in S$, there exists an element $s \in K$ satisfying $s - 0 = s$, which implies that s accounts for s with respect to S. Hence, in this case, we have a set K of size at most $\sqrt{n}$ that accounts for S. Otherwise, $\sqrt{n} < k \leq n$, and there exists $0 < \epsilon \leq \frac{1}{2}$ such that $k = n^{1/2+\epsilon}$.

We will construct a set K that accounts for S by building K one element at a time. Whenever we add an element to K, we will simultaneously construct a set L of all elements $s \in S$ accounted for by K. We begin with $K = \emptyset$. As the empty set does not account for any elements, we also begin with $L = \emptyset$. We use the following algorithm, which we call the Pairing Algorithm.

1. Compute the multiset M of nonzero differences $a_i - a_j$ such that $a_i, a_j \in S$, and $a_i \notin L$.

2. Let y be an element in M that appears with greatest frequency. Add y to K, and for each $a_i, a_j \in S$ such that $a_i - a_j = y$, set $L \leftarrow L \cup \{a_i\}$.

3. Repeat the first two steps $\lceil \sqrt{n} \rceil$ times or until L contains all elements of S.

By construction, L records the elements of S accounted for by K. We claim that after the Pairing Algorithm terminates, L contains at least $k - \lceil \sqrt{n} \rceil$ elements. In proving this claim, if at any point L contains at least $k - \lceil \sqrt{n} \rceil$ elements, then we are done. Otherwise, we assume that each time the Pairing Algorithm executes Step 1, $|L| \leq k - \lceil \sqrt{n} \rceil$. We show that each time we add y to K in Step 2, we add at least $n^\epsilon - 1$ new distinct elements to L.

Consider some iteration of Step 1 of the Pairing Algorithm. Let z be the number of elements already in L during this iteration. We claim that when M is created at this iteration, M contains at least $k(k-1) - (k-1)z$ elements. Indeed, $k(k-1)$ counts the ordered pairs of distinct elements $a_i, a_j \in S$ and hence also counts the number of differences $a_i - a_j$. However, as M only contains differences of those pairs a_i, a_j such that a_i does not belong to L, $(k-1)z$ differences $a_i - a_j$ with a_i belonging to L are excluded from M. This gives us a lower bound for the cardinality of M:

$$\begin{aligned}|M| &\geq k(k-1) - (k-1)z \\ &> k^2 - k(z+1) \\ &= n^{1+2\epsilon} - n^{\frac{1}{2}+\epsilon}(z+1).\end{aligned}$$

We note that as each element in M belongs to the group G, there are at most n unique elements in M. Therefore, some element in M appears at least $\frac{|M|}{n}$ times: that is, at least

$$n^{2\epsilon} - n^{\epsilon-\frac{1}{2}}(z+1)$$

times. We know that $z \leq k - \lceil\sqrt{n}\rceil \leq k - \sqrt{n} = \sqrt{n}(n^\epsilon - 1)$. Therefore,

$$\frac{|M|}{n} \geq n^{2\epsilon} - n^{\epsilon-\frac{1}{2}}(\sqrt{n}(n^\epsilon - 1) + 1) \geq n^\epsilon - 1,$$

and hence, some element of M must appear at least $n^\epsilon - 1$ times. Therefore, the value y computed during Step 2 of the Pairing Algorithm must appear in M at least $n^\epsilon - 1$ times. Due to the group inverse property, each element $c \in S$ belongs to at most one difference $c - d = y$, where $d \in S$. Hence, each unaccounted element in S contributes to at most one appearance of y in M. Therefore, when we add y to K, we add at least $n^\epsilon - 1$ elements to L. By repeating this process $\lceil\sqrt{n}\rceil$ times, we add at least $n^{1/2+\epsilon} - \lceil\sqrt{n}\rceil$ new elements to L. Thus when the Pairing Algorithm terminates, $|L| \geq n^{1/2+\epsilon} - \lceil\sqrt{n}\rceil = |S| - \lceil\sqrt{n}\rceil$.

When the Pairing Algorithm terminates, the elements in S not yet accounted for by K are given by the set $S \setminus L$. If $S \setminus L = \emptyset$, then K accounts for S. In this case, $|K| \leq \lceil\sqrt{n}\rceil$, and hence we are done. Otherwise, $S \setminus L$ is nonempty, and K contains exactly $\lceil\sqrt{n}\rceil$ elements.

If $S \setminus L$ is nonempty, then we would like to add a few more elements to K in order to account for these remaining elements in $S \setminus L$. While $|S \setminus L| > 2$, we can find $b, b' \in S \setminus L$ such that $b + b' \neq 0$. We add $b + b'$ to K and add b and b' to L, as $b - (-b') = b + b'$, and $b' - (-b) = b + b'$. Similarly, if $L = \{b, b'\}$ with $b + b' \neq 0$, we then add $b + b'$ to K. If $L = \{b\}$, we add $a + b$ to K, where $a \in S$ and $a + b \neq 0$. If $L = \{b, -b\}$, we add $b + b$ and $(-b) + (-b)$ to K. After this process, K accounts for S.

Let r be the size of $S \setminus L$ when the Pairing Algorithm terminates. The process above adds at most $\frac{r}{2} + 1$ elements to K. As $|L| \geq |S| - \lceil\sqrt{n}\rceil$ when the Pairing Algorithm terminates, it follows that $r \leq \lceil\sqrt{n}\rceil$, and therefore the process above adds at most $\frac{1}{2}\lceil\sqrt{n}\rceil + 1$ elements to K. Furthermore, the Pairing Algorithm adds at most $\lceil\sqrt{n}\rceil$ elements to K. Therefore, we obtain the following bound:

$$|K| \leq \lceil\sqrt{n}\rceil + \frac{1}{2}\lceil\sqrt{n}\rceil + 1 \leq (\sqrt{n}+1) + \frac{1}{2}(\sqrt{n}+1) + 1 \leq \frac{3}{2}\sqrt{n} + \frac{5}{2}.$$

As $|K|$ is an integer, this implies that $|K| \leq \lfloor \frac{3}{2}\sqrt{n} + \frac{5}{2} \rfloor$. This completes the proof. □

We are now ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2: We show by induction on n that the cop number of a connected abelian Cayley graph on n vertices is at most $6\sqrt{n}$. For $1 \leq n \leq 99$, we consider an abelian group G of order n generated by a set $S \subseteq G$ satisfying $S = -S$ and $0 \notin S$, and we consider the graph $\Gamma = Cay(G, S)$. By Theorem 2.1.1, $c(\Gamma) \leq \lceil \frac{|S|+1}{2} \rceil \leq \frac{n}{2} + 1$. As $\frac{n}{2} + 1 < 6\sqrt{n}$ for $1 \leq n \leq 99$, the theorem holds. For the induction step, we assume that the statement is true for abelian Cayley graphs on fewer than n vertices for some $n \geq 100$. Again, we consider an abelian group G of order n generated by a set $S \subseteq G$ satisfying $S = -S$ and $0 \notin S$, and we consider the graph $\Gamma = Cay(G, S)$. We will show that $\lfloor 6\sqrt{n} \rfloor$ cops have a strategy to capture the robber on Γ.

Let S be the set of generators in Γ. If $|S| = 1$, then Γ is a cycle with cop number at most two, and the result follows. Otherwise, by Lemma 2.1.4, we can construct a set $K \subseteq G$ accounting for S such that $0 \notin K$ and $|K| \leq \frac{3}{2}\sqrt{n} + \frac{5}{2}$. We let $K = \{d_1, \dots, d_l\}$, and we denote the position of the robber with r. Our goal is to place l cops $c_1, \dots, c_l$ at vertices $r + k_1 d_1, r + k_2 d_2, \dots, r + k_l d_l$, where $k_1, \dots, k_l$ are nonnegative integers. Then, if the robber plays a move a for which there exists $b \in S$ such that $a - b = d_i$, then c_i will respond by playing b, decreasing the difference between c_i and the robber by exactly d_i, and all other cops c_j will play a, maintaining their original difference in position with the robber. If c_i moves closer to the robber k_i times, then the difference between c_i and the robber decreases to zero, and the robber is caught. However, as $\{d_1, \dots, d_l\}$ accounts for S, it follows that for any move $a \in S$ that the robber plays, there exists $b \in S$ such that $a - b = d_j$ for some $j \leq l$; thus, after any robber move, some cop c_j can play b and decrease its difference in position with the robber. Hence after the robber plays any $k_1 + k_2 + \cdots + k_l - l + 1$ moves, the robber will surely be caught. Furthermore, if the robber decides not to move at all, then one extra cop can catch the robber. Hence after placing l cops as described above, the number of cops required to catch the robber is at most $l + 1 = |K| + 1 \leq \frac{3}{2}\sqrt{n} + \frac{5}{2} + 1 < \lfloor 6\sqrt{n} \rfloor$; this last inequality holds for all positive integers n. Thus, it remains only to show that we can place l cops at vertices $r + k_1 d_1, r + k_2 d_2, \dots, r + k_l d_l$ for some nonnegative integers $k_1, \dots, k_l$, where r is the position of the robber.

If a cop c is at a vertex $r + k_i d_i$ following the strategy described above, we say that c is *busy*. If c is not busy, then we say that c is *free*. Using this language, the strategy above requires at most l cops to be busy. As $l = |K| \leq \frac{3}{2}\sqrt{n} + \frac{5}{2}$, as long as fewer than l cops are busy, we always have at least $\lfloor 6\sqrt{n} \rfloor - (\frac{3}{2}\sqrt{n} + \frac{5}{2}) + 1 > \frac{9}{2}\sqrt{n} - \frac{5}{2}$ free cops. As the number of free cops is an integer, we therefore always have at least $\lceil \frac{9}{2}\sqrt{n} - \frac{5}{2} \rceil$ free cops.

We show that for any $d_i \in K$, we can place a free cop at $r + k_i d_i$, where r is the position of the robber and k_i is some nonnegative integer. To show this, we choose an element $d_i \in K$, and we consider the game of cops and robbers played on $Cay(G/\langle d_i \rangle, \phi(S))$, where $\phi : G \to G/\langle d_i \rangle$ is the natural homomorphism $\phi : h \mapsto (h + \langle d_i \rangle)/\langle d_i \rangle$. As $|G/\langle d_i \rangle| \leq \frac{n}{2}$, there exists a strategy by which $\lfloor 6\sqrt{n/2} \rfloor$ cops can capture the robber on $G/\langle d_i \rangle$ by the induction hypothesis. For $n \geq 100$, $\lceil \frac{9}{2}\sqrt{n} - \frac{5}{2} \rceil > \lfloor 6\sqrt{n/2} \rfloor$; therefore, $\lceil \frac{9}{2}\sqrt{n} - \frac{5}{2} \rceil$ free cops can evaluate their positions on $G/\langle d_i \rangle$ using ϕ and follow a strategy that allows some free cop to capture the robber on $G/\langle d_i \rangle$. However, capturing the robber on $G/\langle d_i \rangle$ is equivalent to landing on a vertex $r + k_i d_i$, where k_i is a nonnegative integer and r is the position of the robber, which was our goal.

By repeating this process for each $d_i \in K$, we can place l cops at vertices $r+k_1d_1, r+k_2d_2, \dots, r+k_ld_l$ for some nonnegative integers $k_1, \dots, k_l$, where r is the position of the robber. Then by following the strategy above, the robber is captured after making at most $k_1 + k_2 + \cdots + k_l - l + 1$ moves. $\square$

2.2 Abelian Cayley graphs: a sharper upper bound

In this section, we will apply the ideas of the previous section in a more precise way in order to obtain a better upper bound for the cop number of Cayley graphs on abelian groups. We will make several changes from the previous section. First, we will consider directed Cayley graphs on abelian groups. A directed Cayley graph on an abelian group is defined as follows.

Definition 2.2.1. Let $(G, +)$ be an abelian group, and let $S \subseteq G$ be a generating set of G. The directed Cayley graph Γ generated by G and S is defined as follows:

- $V(\Gamma) = G$;
- For any $u, v \in V(G)$, $uv \in E(\Gamma)$ if and only if $v - u \in S$.

We write $\text{Cay}(G, S)$ to refer to the directed Cayley graph generated by G and S.

The only difference between this definition and our original definition for a Cayley graph on an abelian group is that we no longer require that $S = -S$. When $S = -S$, then this definition gives us a directed Cayley graph whose edges are all bidirectional, and we may consider such a graph to be undirected for the purposes of cops and robbers. Here, the requirement that S generate G ensures that the digraph $\text{Cay}(G, S)$ is strongly connected. Furthermore, in this section, when we consider a Cayley graph generated by an abelian group G and a generating set $S \subseteq G$, we will always assume that $0 \in S$.

We will use the same strategy of making robber moves unsafe, as defined in the previous section, until the robber has no safe move left and cannot avoid capture. As we will consider strategies in which certain robber moves unsafe, it will often be convenient to define a set $T \subseteq S$ consisting of safe moves for the robber and to assume that the robber may only play moves in T. When we consider such a set $T \subseteq S$ and assume that the robber may only play moves in T, we say that T is the *moveset* of the robber. The next definition gives a convenient way of writing an upper bound for the number of cops needed to capture the robber using such a strategy.

Definition 2.2.2. Let $0 \leq t \leq s \leq n$ and $s \geq 1$. We write $c(n, s, t)$ for the maximum number of cops required to capture the robber on a directed Cayley graph generated by an abelian group G of n elements and a generating set $S \subseteq G$ of s elements, when the robber's moveset is a set $T \subseteq S$ of t elements.

Since a graph with n vertices always has cop number at most n, $c(n, s, t)$ has an upper bound of n and thus is well defined for each triple (n, s, t). By this definition, if Γ is a Cayley graph over an abelian group G of n elements generated by a set $S \subseteq G$ of s elements, then $c(\Gamma) \leq c(n, s, s)$, since in the standard game of cops and robbers, the robber's moveset is the entire set S. For any integers $1 \leq s \leq n$, we write $c(n, s, 0) = 0$. The value $c(n, s, 0)$ corresponds to a game of cops and robbers in which the robber's moveset T is empty. We will see that the value $c(n, s, 0)$ arises in inductive arguments from situations in which all of the robber's moves are already guarded by cops and such that no additional cops are needed to capture the robber. As such, letting $c(n, s, 0) = 0$ is a natural definition.

This section will be organized as follows. First, we will define a general strategy for capturing a robber on a directed Cayley graph over an abelian group, and we will establish an upper bound for the cop number of directed Cayley graphs on abelian groups. Then, we will show that the same strategy also applies to undirected Cayley graphs on abelian groups, and we will establish a sharper bound for the cop number of undirected Cayley graphs on abelian groups.

2.2.1 An upper bound for directed abelian Cayley graphs

In this section, we will establish an upper bound for the cop number of directed Cayley graphs on abelian groups. As such, all graphs that we consider in this section will be directed graphs. We will use the strategy of guarding robber moves as discussed previously. Our main tool will be the following lemma, which essentially formalizes a general inductive strategy of capturing the robber by guarding robber moves until no robber move is safe.

Lemma 2.2.3. *Let $g(n, s, t) \geq 0$ be a real valued function defined for all integers $0 \leq t \leq s \leq n$ and $s \geq 1$, and let $h(n, s, t) \geq 1$ be a real-valued function defined for all integers $1 \leq t \leq s \leq n$. Suppose that g and h respect the following conditions for all $1 \leq t \leq s \leq n$:*

1. *For any abelian group G of n elements with a generating set $S \subseteq G$ of s elements and a subset $T \subseteq S$ of t elements, there exists an element $k \in G$ accounting for at least $h(n, s, t)$ elements of T with respect to S.*

2. *If* $n' \leq \frac{n}{2}$, $s' \leq s$, *and* $t' \leq t$, *then either* $g(n,s,t) \geq c(n',s',t')$ *or* $g(n,s,t) \geq g(n',s',t')$,.

3. *If* $t' \leq t - h(n,s,t)$, *then* $g(n,s,t) \geq g(n,s,t') + 1$.

Then $c(n,s,t) \leq g(n,s,t)$.

Proof. Let G be an abelian group generated by a set $S \subseteq G$, and let $T \subseteq S$. Let $|G| = n, |S| = s, |T| = t$. We show that $g(n,s,t)$ cops may capture the robber on the digraph $\mathrm{Cay}(G,S)$ when the robber's moveset is T.

We induct on n, and for fixed n, we induct on t. We show that the lemma holds when $n = 1$. When $n = 1$, we must have $t = s = n = 1$. As $h(n,s,t) \geq 1$ and $t = 1$, condition (1) implies that $h(n,s,t) = 1$, as an element $k \in G$ cannot account for more than t elements in T. Then, by condition (3), we have that $g(n,s,t) \geq 1$. Furthermore, as G is the trivial group generated by the identity, a single cop may capture the robber on $\mathrm{Cay}(G,S)$. Thus $c(n,s,t) = 1 \leq g(n,s,t)$. Furthermore, when $t = 0$, we have $c(n,s,0) = 0 \leq g(n,s,t)$, and the lemma holds.

Now suppose that $n \geq 2$ is fixed and that $t \geq 1$. By condition (1), we may choose an element $k \in G$ that accounts for at least $h(n,s,t)$ elements of T with respect to S. First, we attempt to move a cop to a vertex $r + \gamma k$, where γ is any nonnegative integer, and $r \in G$ is the vertex of the robber. Accomplishing this is equivalent to capturing the robber on the graph $\mathrm{Cay}(G/\langle k \rangle, \phi(S))$, where $\phi : G \to G/\langle k \rangle$ is the homomorphism $x \mapsto x + \langle k \rangle$, and letting the robber only use moves of $\phi(T)$. As $k \neq 0$, $n' = |G/\langle k \rangle| \leq n/2$, and clearly $s' = |\phi(S)| \leq s$, and $t' = |\phi(T)| \leq t$. Therefore, by condition (2), $g(n,s,t)$ cops suffice to capture the robber on $\mathrm{Cay}(G/\langle k \rangle, \phi(S))$ when the robber's moves are restricted to $\phi(T)$, either directly, or by the induction hypothesis if $g(n,s,t) \geq g(n',s',t')$. Hence, a cop C successfully reaches a vertex $r + \gamma k$ for some nonnegative integer γ, where $r \in G$ is the vertex of the robber.

Next, we show that at this point, C has a strategy to restrict the robber to a moveset of size at most $t - h(n,s,t)$. Let $A = \{a_1, \ldots, a_m\} \subseteq T$ be the set of robber moves accounted for by k with respect to S. If the robber plays a move $a' \notin A$, then C plays a', and C will stay at vertex of the form $r + \gamma k$, where r is the new position of the robber. If the robber plays a move $a_i \in A$, then C has a move $b_i \in S$ such that $a_i - b_i = k$. After C plays b_i, C now occupies a vertex $r + (\gamma - 1)k$, where r is the new position of the robber. Thus we see that whenever the robber plays a move $a_i \in A$ accounted for by k, the "difference" between the robber and C decreases by exactly k. Thus if the robber plays a move accounted for by k sufficiently many times (γ times), then the robber will be caught by C. Therefore, the robber must eventually stop playing all moves $a_i \in A$ accounted for by k. The number of moves $a_i \in A$ accounted for by k is at least $h(n,s,t)$, and hence C restricts the robber to a moveset $T \setminus A$ of size at most $t - h(n,s,t)$.

Once C has restricted the robber to a moveset of size $t' \leq t - h(n,s,t)$, then by condition (3), the remaining $g(n,s,t) - 1 \geq g(n,s,t')$ cops are enough to capture the robber. This completes the proof. □

Lemma 2.2.3 tells us that in order to bound the cop number of directed Cayley graphs on abelian groups, it is enough to find appropriate functions $g(n, s, t)$ and $h(n, s, t)$ that satisfy the conditions of Lemma 2.2.3 and such that $g(n, s, t)$ is not too large. In the following theorem, we give a simple example of a pair of functions $g(n, s, t)$ and $h(n, s, t)$ that satisfy the conditions of Lemma 2.2.3. We note that the following theorem is a generalization of a theorem of Hamidoune [20].

Theorem 2.2.4. *Let $0 \le t \le s \le n$ and $s \ge 1$. Then $c(n, s, t) \le t$.*

Proof. For $1 \le t \le s \le n$, we define $h(n, s, t) = 1$, and for $0 \le t \le s \le n$ and $s \ge 1$, we define $g(n, s, t) = t$. Then, for any $a \in T$, a accounts for a with respect to S, because there exists an element $0 \in S$ such that $a - 0 = a$. Therefore, $g(n, s, t)$ and $h(n, s, t)$ respect condition (1) of Lemma 2.2.3. It is easy to check that $g(n, s, t)$ and $h(n, s, t)$ also satisfy conditions (2) and (3) of Lemma 2.2.3. Thus the theorem holds. □

Now, we will attempt to find a pair $g(n, s, t)$ and $h(n, s, t)$ of functions that give us a better upper bound for $c(n, s, t)$. We will define a function $h(n, s, t)$ that satisfies condition (1) of the lemma, and we will continue to use this function $h(n, s, t)$ throughout the entire section.

Definition 2.2.5. For $1 \le t \le s \le n$, we define

$$h(n, s, t) = \begin{cases} 1 & t \le c\sqrt{n} + 9; \\ \frac{t(s-1)}{n} & t > c\sqrt{n} + 9, \end{cases}$$

where c is a fixed constant satisfying $0.7 \le c \le 1$ whose exact value we will choose later.

Lemma 2.2.6. *Let $1 \le t \le s \le n$, and let G be an abelian group on n elements generated by a set S of s elements, and let $T \subseteq S$ be a subset of t elements. Then there exists an element $k \in G$ accounting for at least $h(n, s, t)$ elements of T with respect to S.*

Proof. When $t \le c\sqrt{n} + 9$, let $a \in T$. Then a accounts for a with respect to S, because $0 \in S$, and $a - 0 = a$. Hence, if we let $k = a$, then k accounts for at least $1 = h(n, s, t)$ elements of T with respect to S.

When $t > c\sqrt{n} + 9$, let M be a multiset consisting of all differences in G of the form $a - b$, where $a \in T, b \in S$, $a \neq b$. The number of elements in M is equal to $t(s - 1)$, and hence, by the pigeonhole principle, some element of G appears in M at least $\frac{t(s-1)}{n}$ times. Let $k \in M$ be such an element. Note that $k \neq 0$. There exist at least $\frac{t(s-1)}{n}$ elements $a_i \in T$ for which some $b_i \in S$ satisfies $a_i - b_i = k$. Thus k accounts for at least $\frac{t(s-1)}{n}$ elements of T with respect to S. □

Next, we will define a function $g(n, s, t)$, and our goal for the remainder of the section will be to show that $g(n, s, t)$ and $h(n, s, t)$ satisfy the conditions of Lemma 2.2.3 and that $g(n, s, t)$ is not too large.

Definition 2.2.7. Let $0 \leq t \leq s \leq n$ and $s \geq 1$. We define

$$g(n,s,t) = \begin{cases} t & 1 \leq t \leq c\sqrt{n} + 9 \\ \gamma(n,s,t) + c\sqrt{n} + 10 & t > c\sqrt{n} + 9, \end{cases}$$

where $\gamma(n,s,t) = \log \frac{t}{c\sqrt{n}} \left(\log \frac{n}{n-s+1}\right)^{-1}$, and the value c is the same as in Definition 2.2.5.

This choice of $g(n,s,t)$ may not seem straightforward, so we present the intuition behind this definition of $g(n,s,t)$. We suppose that for integers $1 \leq t \leq s \leq n$, we have an abelian group G on n elements generated by a set $S \subseteq G$ of s elements, and a subset $T \subseteq S$ of t elements. We would like to estimate the number of elements of G needed to form a set K such that the elements of K altogether account for each element of T, since, as we have discussed, this will help us count the number of cops needed to make every robber move unsafe.

In order to estimate the number of elements needed in K, we may construct K iteratively. The iterative construction that we describe here is a refinement of the Pairing Algorithm from Section 2.1. If $t \leq c\sqrt{n}$, then it is enough just to let $K = T$, as each element $a \in T$ accounts for itself with respect to S. If $t > c\sqrt{n}$, we may choose one element $k \in G$ to account for at least $\frac{t(s-1)}{n}$ elements of T, as in the proof of Lemma 2.2.6. More generally, we can define a recursive process that adds elements to K, and we may run this process until at most $c\sqrt{n}$ elements of T are not accounted for by K. We define z_i to be the number of elements accounted for by K after i iterations of our process. We immediately see that $z_0 = 0$, and if we choose k as described earlier during the first iteration of our process, we may let $z_1 \geq \frac{t(s-1)}{n}$. Additionally, given z_{i-1}, there are $t - z_{i-1}$ elements of T not accounted for by K, and hence on the ith iteration of our procedure, we may add an element to K that accounts for $\frac{(s-1)(t-z_{i-1})}{n}$ new elements of T. Therefore, we obtain a recursive inequality for the number of elements in T accounted for by K after i iterations of our procedure:

$$z_i \geq z_{i-1} + \frac{(s-1)(t-z_{i-1})}{n} = \frac{n-s+1}{n} z_{i-1} + \frac{(s-1)t}{n},$$

which has a closed form of

$$z_i \geq t - t\left(\frac{n-s+1}{n}\right)^i,$$

from standard methods for solving recursions.

Hence, after i iterations, there are at most $t\left(\frac{n-s+1}{n}\right)^i$ elements of T not accounted for by K. As soon as the number of elements in T not accounted for by K is at most $c\sqrt{n}$, we may simply add the remaining unaccounted elements of T to K. Therefore, the recursive method we have described will run i times, where i is the smallest integer such that $t\left(\frac{n-s+1}{n}\right)^i \leq c\sqrt{n}$. We thus may calculate that

$$i = \lceil \gamma(n,s,t) \rceil,$$

and hence after the recursive method runs i times, at most $c\sqrt{n}$ elements of T will be left unaccounted for by K. At this point, the remaining $c\sqrt{n}$ unaccounted elements of T may be added to K, at which point the elements of K altogether account for all of T. In total, our count shows that our set K needs at most

$$\lceil \gamma(n,s,t) \rceil + c\sqrt{n}$$

elements. This counting method gives us an intuition with which we define the function $g(n,s,t)$. The extra additive constants of $g(n,s,t)$ are included for technical reasons that will become clear later.

In the following lemmas, we will bound $g(n,s,t)$ above, and we will show that $g(n,s,t)$ and $h(n,s,t)$ satisfy conditions (2) and (3) of Lemma 2.2.3.

Lemma 2.2.8. *Let $0 \le t \le s \le n$ and $s \ge 1$, and let $\epsilon = 10^{-6}$. If $d \ge \frac{1+\epsilon}{ce} + c$, then $g(n,s,t) \le d\sqrt{n} + 10$.*

Proof. We consider two cases :

1. If $t \le c\sqrt{n} + 9$, then

$$g(n,s,t) = t \le c\sqrt{n} + 9 < d\sqrt{n} + 9.$$

2. If $t > c\sqrt{n} + 9$, then $10 \le t \le n$. We first note that $g(n,s,t) \le g(n,s,s)$. We wish to find α such that $\gamma(n,s,s) \le \alpha\sqrt{n}$. This inequality can be rewritten as

$$1 \le \frac{c\sqrt{n}}{s}\left(\frac{n}{n-s+1}\right)^{\alpha\sqrt{n}} = r_{\alpha,c}(n,s).$$

One calculates that the derivative relative to s is

$$\frac{\partial r_{\alpha,c}}{\partial s} = \frac{1}{s}\left(cn^{\alpha\sqrt{n}+1/2}\right)(n-s+1)^{-\alpha\sqrt{n}}\left(\frac{-(n+1)+s(1+\alpha\sqrt{n})}{s(n-s+1)}\right).$$

We see that $r_{\alpha,c}(n,s)$ achieves a minimum at $s^* = \frac{n+1}{\alpha\sqrt{n}+1}$.

Hence, it suffices to find a value α such that

$$r_{\alpha,c}(n,s^*) = c\left(1-\frac{1}{n+1}\right)^{\alpha\sqrt{n}+1/2} \cdot \frac{1+\alpha\sqrt{n}}{\sqrt{n+1}} \cdot \left(1+\frac{1}{\alpha\sqrt{n}}\right)^{\alpha\sqrt{n}} \ge 1.$$

As $n \to \infty$, $r_{\alpha,c}(n,s^*) \sim c\alpha e$. Therefore, if we choose $\alpha = \frac{1+\epsilon}{ce}$, then as $n \to \infty$, $r_{\alpha,c}(n,s^*) \to 1+\epsilon$. Furthermore, one may verify computationally that for values $0.7 \le c \le 1$ and $n \ge 10$, $r_{\alpha,c}(n,s^*) - 1$ is positive and approaches ϵ.

Therefore, we choose $\alpha = \frac{1+\epsilon}{ce}$, and then $g(n,s,t) \le \left(\frac{1+\epsilon}{ce} + c\right)\sqrt{n} + 10 \le d\sqrt{n} + 10$.

□

Lemma 2.2.9. *Let $1 \leq t \leq s \leq n$, and let $\epsilon = 10^{-6}$. If there exists a real number d such that $d \geq \frac{1+\epsilon}{ce} + c$ and $c \geq \frac{d}{\sqrt{2}}$, then $g(n,s,t)$ respects condition (2) of Lemma 2.2.3.*

Proof. Let $n' \leq \frac{n}{2}$, $s' \leq s$, and $t' \leq t$.

1. If $t \leq c\sqrt{n} + 9$, then
$$g(n,s,t) = t \geq t' \geq c(n', s', t')$$
by Theorem 2.2.4.

2. If $t > c\sqrt{n} + 9$, then by Lemma 2.2.8 and our hypotheses on c, d,
$$g(n,s,t) > c\sqrt{n} + 10 \geq d\sqrt{n/2} + 10 > d\sqrt{n'} + 10 \geq g(n', s', t'),$$
when $t' > c\sqrt{n/2} + 9$, and
$$g(n,s,t) > c\sqrt{n} + 10 \geq t' \geq c(n', s', t'),$$
when $t' \leq c\sqrt{n/2} + 9$, by Theorem 2.2.4.

□

Lemma 2.2.10. *Let $1 \leq t \leq s \leq n$. Then $g(n,s,t)$ and $h(n,s,t)$ respect condition (3) of Lemma 2.2.3.*

Proof. Consider a value $t' \leq t - h(n,s,t)$.

We consider three cases :

1. If $2 \leq t \leq c\sqrt{n} + 9$, then $h(n,s,t) = 1$, and thus $t \geq t' + 1$. Then,
$$g(n,s,t) = t \geq t' + 1 = g(n,s,t') + 1$$

2. If $t > c\sqrt{n} + 9$ and $t' \leq c\sqrt{n} + 9$, then
$$g(n,s,t) = \gamma(n,s,t) + c\sqrt{n} + 10 \geq t' + 1 = g(n,s,t') + 1$$

3. If $t, t' > c\sqrt{n} + 1$, we know that $t' \leq t - \frac{t(s-1)}{n} = t\left(\frac{n-s+1}{n}\right)$.

   Thus,
$$g(n,s,t) = \gamma(n,s,t) + c\sqrt{n} + 10 \geq \log\left(\frac{t'}{c\sqrt{n}} \cdot \frac{n}{n-s+1}\right)\left(\log\frac{n}{n-s+1}\right)^{-1} + c\sqrt{n} + 10$$
$$= \gamma(n,s,t') + 1 + c\sqrt{n} + 10 = g(n,s,t') + 1.$$

□

We now can prove the following result.

Theorem 2.2.11. *The cop number of any directed abelian Cayley graph on n vertices is at most $1.3328\sqrt{n} + 10$.*

Proof. We let $\epsilon = 10^{-6}$. We first find values c and d such that $c \geq \frac{d}{\sqrt{2}}$ and $d \geq \frac{1+\epsilon}{ce} + c$ which minimize d. A computation shows that the optimal solution is $c = \sqrt{\frac{1+\epsilon}{(\sqrt{2}-1)e}} \approx 0.94$ and $d = \sqrt{\frac{2(1+\epsilon)}{(\sqrt{2}-1)e}} < 1.3328$. Then, by the lemmas of this section, $g(n,s,t)$ and $h(n,s,t)$ satisfy all three conditions of Lemma 2.2.3. Hence, by Lemmas 2.2.3 and 2.2.8, $c(\mathrm{Cay}(G,S)) \leq c(n,s,s) \leq g(n,s,s) \leq d\sqrt{n} + 10$. □

By considering only abelian groups of odd size, we may obtain a slightly better bound.

Theorem 2.2.12. *Let G be an abelian group with an odd number n of elements, and let $S \subseteq G$ be a generating set of G. Then $c(\mathrm{Cay}(G,S)) \leq 1.2279\sqrt{n} + 10$.*

Proof. In condition (2) of Lemma 2.2.3, we require $n' \leq \frac{n}{2}$ because of the bound $|G/\langle k \rangle| \leq n/2$ for any element $k \in G, k \neq 0$. However, if n is odd, then we know that $|G/\langle k \rangle| \leq n/3$, so we only need to require that $n' \leq \frac{n}{3}$ in this condition. Hence, we may relax the requirement $c \geq \frac{d}{\sqrt{2}}$ from Lemma 2.2.9 to $c \geq \frac{d}{\sqrt{3}}$.

Then, minimizing d with respect to $c \geq \frac{d}{\sqrt{3}}$ and $d \geq \frac{1+\epsilon}{ce} + c$ yields the solution $c = \sqrt{\frac{1+\epsilon}{(\sqrt{3}-1)e}} \approx 0.71, d = \sqrt{\frac{3(1+\epsilon)}{(\sqrt{3}-1)e}} < 1.2279$. Then the result follows as Theorem 2.2.11. □

2.2.2 An upper bound for undirected abelian Cayley graphs

In this section, we will establish an upper bound for the cop number of undirected Cayley graphs on abelian groups. We realize an undirected Cayley graph as a directed Cayley graph on an abelian group G generated by a set $S \subseteq G$ satisfying $S = -S$. We will define a value $c_u(n,s,t)$, which is a counterpart of $c(n,s,t)$ specifically suited to Cayley graphs $\mathrm{Cay}(G,S)$ on abelian groups G generated by sets S satsifying $S = -S$. By bounding $c_u(n,s,t)$, we will obtain a bound for the cop number of undirected abelian Cayley graphs.

Definition 2.2.13. Let $0 \leq t \leq s \leq n$ and $s \geq 1$. We write $c_u(n,s,t)$ for the maximum number of cops required to capture the robber on an undirected Cayley graph generated by an abelian group G of n elements and a generating set $S \subseteq G$ of s elements satisfying $S = -S$, when the robber's moveset is a set $T \subseteq S$ of t elements.

As before, for $1 \leq s \leq n$, we let $c_u(n,s,0) = 0$.

Our general approach in this section will be very similar to that of Section 2.2.1. We will establish a lemma analagous to Lemma 2.2.3 that bounds $c_u(n,s,t)$, and we will define functions g and h that satisfy our lemma and such that g is not too large. Note that the functions g and h that we will define in this section are not the same as the functions g and h from the previous section. As this section follows the same approach as Section 2.2.1, our presentation will be terser.

Lemma 2.2.14. *Let $g(n,s,t) \geq 0$ be a real valued function defined for all integers $0 \leq t \leq s \leq n$ and $s \geq 1$, and let $h(n,s,t) \geq 1$ be a real-valued function defined for all integers $3 \leq t \leq s \leq n$. Suppose that g and h respect the following conditions for all $3 \leq t \leq s \leq n$:*

1. *For any abelian group G of n elements with a generating set $S = -S$ of s elements and a subset $T \subseteq S$ of t elements, there exists an element $k \in G$ accounting for at least $h(n,s,t)$ elements of T with respect to S.*
2. *Either $g(n,s,t) \geq c_u(n',s',t')$, or $g(n,s,t) \geq g(n',s',t')$ for $n' \leq \frac{n}{2}$, $s' \leq s$, and $t' \leq t$.*
3. *$g(n,s,t) \geq g(n,s,t') + 1$ if $t' \leq t - h(n,s,t)$.*

Then $c_u(n,s,t) \leq g(n,s,t) + 2$.

Proof. We prove the statement by induction on n. For fixed n, we induct on t.

Suppose that $t \leq 2$. If $t = 0$, then $c_u(n,s,t) = 0$, and we are done. If $t = 1$, then the robber only has one legal move, and hence is restricted to a directed cycle. Thus, when $t = 1$, a single cop may capture the robber.

When $t = 2$, the robber is restricted to two moves a and b. If $a = -b$, then the robber is restricted to a cycle, and two cops may capture the robber. Otherwise, $a + b \neq 0$. In this case, we aim to place a cop at a vertex $r + \gamma(a+b)$, where γ is an integer, and $r \in G$ is the vertex of the robber. Accomplishing this is the same as capturing the robber on the Cayley graph $\mathrm{Cay}(G/\langle a+b \rangle, \phi(S))$, where $\phi : G \to G/\langle a+b \rangle$ is the homomorphism $x \mapsto x + \langle a+b \rangle$, and the robber is restricted to moves in $\phi(T)$. In the game on $\mathrm{Cay}(G/\langle a+b \rangle, \phi(S))$, $\phi(a) = -\phi(b)$, so the robber is restricted to a cycle, and two cops may capture the robber in this game. Therefore, a cop C successfully reaches a vertex $r + \gamma(a+b)$, where γ is an integer, and $r \in G$ is the vertex of the robber.

Now, whenever the robber plays the move a, C responds with $-b$, after which C occupies the vertex $r + (\gamma - 1)(a+b)$, where r is the new position of the robber. Similarly, if the robber plays b, then C responds with $-a$, after which C again occupies the vertex $r + (\gamma-1)(a+b)$. Therefore, after the robber plays any γ moves, the robber will be captured by C. Hence, in all cases, $c_u(g,s,t) \leq 2$. Note that these base cases of $t \leq 2$ also cover the base cases of $n \leq 2$.

When $t \geq 3$, we use the same inductive strategy as Lemma 2.2.3. □

Similarly to Lemma 2.2.3, we may define a pair $g(n,s,t)$ and $h(n,s,t)$ of functions from which Lemma 2.2.14 gives an upper bound for $c_u(n,s,t)$. We show this pair of functions in the following theorem, which is a generalization of Theorem 2.1.1 with a small additive error.

Theorem 2.2.15. *Let $0 \le t \le s \le n$ and $s \ge 1$. Then $c_u(n,s,t) \le \frac{t}{2} + 2$.*

Proof. For $0 \le t \le s \le n$ and $s \ge 1$, we let $g(n,s,t) = \frac{t}{2}$. For $3 \le t \le s \le n$, we let $h(n,s,t) = 2$. Whenever T contains at least three elements, we may always choose elements $a, b \in T$ such that $a + b \neq 0_G$. Then, letting $k = a + b$, we see that k accounts for a, as $-b \in S$, and $a - (-b) = k$. Furthermore, k accounts for b, as $-a \in S$, and $b - (-a) = k$. Therefore, k accounts for two elements of T with respect to S, and hence condition (1) of Lemma 2.2.14 is satisifed.

It is easy to check that conditions (2) and (3) of Lemma 2.2.14 hold with our choice of $g(n,s,t)$ and $h(n,s,t)$. Therefore, $c_u(n,s,t) \le \frac{t}{2} + 2$. □

We again define a function $h(n,s,t)$ that satisfies condition (1) of Lemma 2.2.14, and we will use this definition of $h(n,s,t)$ throughout the entire section. We note that the function h we define here is different from the function h of the previous section.

Definition 2.2.16. For $1 \le t \le s \le n$, we define

$$h(n,s,t) = \begin{cases} 2 & t \le c\sqrt{n} + 3 \\ \frac{t(s-1)}{n} & t > c\sqrt{n} + 3 \end{cases}$$

where c is a fixed constant satisfying $0.85 \le c \le 1.5$ whose exact value we will choose later.

Lemma 2.2.17. *Let $3 \le t \le s \le n$, and let G be an abelian group on n elements generated by a set $S = -S$ of s elements, and let $T \subseteq S$ be a subset of t elements. Then there exists an element $k \in G$ accounting for at least $h(n,s,t)$ elements of T with respect to S.*

Proof. When $3 \le t \le c\sqrt{n} + 3$, let $a, b \in T$ such that $k = a + b \neq 0$. Then for a, we have $-b \in S$, $a - (-b) = k$, and for b, we have $-a \in S$, and $b - (-a) = k$. Therefore, k accounts for at least two elements of T with respect to S.

When $t > c\sqrt{n} + 3$, the proof is the same as in Lemma 2.2.6. □

Now, we define a function g that we will use for the entire section. Again, the function g we define here is different from the function g of the previous section.

Definition 2.2.18. Let $0 \le t \le s \le n$ and $s \ge 1$. We define

$$g(n,s,t) = \begin{cases} \frac{t}{2} + 2 & 1 \le t \le c\sqrt{n} + 3 \\ \gamma(n,s,t) + \frac{c}{2}\sqrt{n} + 5 & t > c\sqrt{n} + 3, \end{cases}$$

where, again, $\gamma(n,s,t) = \log \frac{t}{c\sqrt{n}} \left(\log \frac{n}{n-s+1} \right)^{-1}$, and the value c is the same as in Definition 2.2.16.

Lemma 2.2.19. *Let $0 \le t \le s \le n$, $s \ge 1$, and let $\epsilon = 10^{-6}$. If $d \ge \frac{1+\epsilon}{ce} + \frac{c}{2}$, then $g(n,s,t) \le d\sqrt{n} + 5$.*

Proof. We consider two cases :

1. If $t \leq c\sqrt{n}+3$, then
$$g(n,s,t) = \frac{t}{2}+2 < \frac{c}{2}\sqrt{n}+5 < d\sqrt{n}+5.$$

2. If $t > c\sqrt{n}+3$, then $n \geq t \geq 4$. We again note that $g(n,s,t) \leq g(n,s,s)$. We wish to find α such that $\gamma(n,s,s) \leq \alpha\sqrt{n}$. This inequality can be rewritten as
$$1 \leq \frac{c\sqrt{n}}{s}\left(\frac{n}{n-s+1}\right)^{\alpha\sqrt{n}} = r_{\alpha,c}(n,s).$$
As in Lemma 2.2.8, $r_{\alpha,c}(n,s)$ achieves a minimum at $s^* = \frac{n+1}{\alpha\sqrt{n}+1}$.
Hence, it suffices to find a value α such that
$$r_{\alpha,c}(n,s^*) = c\left(1-\frac{1}{n+1}\right)^{\alpha\sqrt{n}+\frac{1}{2}} \cdot \frac{1+\alpha\sqrt{n}}{\sqrt{n+1}} \cdot \left(1+\frac{1}{\alpha\sqrt{n}}\right)^{\alpha\sqrt{n}} \geq 1.$$
As $n \to \infty$, $r_{\alpha,c}(n,s^*) \sim c\alpha e$. Therefore, if we choose $\alpha = \frac{1+\epsilon}{ce}$, then as $n \to \infty$, $r_{\alpha,c}(n,s^*) \to 1+\epsilon$. Furthermore, one may verify computationally that for values $0.85 \leq c \leq 1.5$ and $n \geq 4$, $r_{\alpha,c}(n,s^*)-1$ is positive and approaches ϵ.
Therefore, we choose $\alpha = \frac{1+\epsilon}{ce}$, and then $g(n,s,t) \leq \left(\frac{1+\epsilon}{ce}+\frac{c}{2}\right)\sqrt{n}+5 \leq d\sqrt{n}+5$.

□

Lemma 2.2.20. *Let $3 \leq t \leq s \leq n$, and let $\epsilon = 10^{-6}$. If there exists a real number d such that $d \geq \frac{1+\epsilon}{ce}+\frac{c}{2}$ and $\frac{c}{2} \geq \frac{d}{\sqrt{2}}$, then $g(n,s,t)$ respects condition (2) of Lemma 2.2.3.*

Proof. Let $n' \leq \frac{n}{2}$, $s' \leq s$, and $t' \leq t$.

1. If $t \leq c\sqrt{n}+3$, then
$$g(n,s,t) = \frac{t}{2}+2 \geq \frac{t'}{2}+2 \geq c_u(n',s',t')$$
by Theorem 2.2.15.

2. If $t > c\sqrt{n}+3$, then by Lemma 2.2.19 and our hypotheses on c and d,
$$g(n,s,t) > \frac{c}{2}\sqrt{n}+5 \geq d\sqrt{n/2}+5 \geq d\sqrt{n'}+5 \geq g(n',s',t'),$$
when $t' > c\sqrt{n/2}+3$, and
$$g(n,s,t) > \frac{c}{2}\sqrt{n}+5 \geq \frac{t'}{2}+2 \geq c_u(n',s',t'),$$
when $t' \leq c\sqrt{n/2}+3$, by Theorem 2.2.15.

□

Lemma 2.2.21. *Let $3 \leq t \leq s \leq n$. Then $g(n,s,t)$ and $h(n,s,t)$ respect condition (3) of Lemma 2.2.3.*

Proof. Consider a value $t' \leq t - h(n,s,t)$.

We consider three cases :

1. If $2 \leq t \leq c\sqrt{n}+3$, then $h(n,s,t)=2$, and thus $t \geq t'+2$. Then,

$$g(n,s,t) = \frac{t}{2} + 2 \geq \frac{t'}{2} + 3 = g(n,s,t') + 1$$

2. If $t > c\sqrt{n}+3$ and $t' \leq c\sqrt{n}+3$, then

$$g(n,s,t) = \gamma(n,s,t) + \frac{c}{2}\sqrt{n} + 5 > \frac{t'}{2} + 3 = g(n,s,t') + 1$$

3. If $t, t' > c\sqrt{n}+3$, we know that $t' \leq t - \frac{t(s-1)}{n} = t\left(\frac{n-s+1}{n}\right)$.

   Thus,

$$g(n,s,t) = \gamma(n,s,t) + \frac{c}{2}\sqrt{n} + 5 \geq \log\left(\frac{t'}{c\sqrt{n}} \cdot \frac{n}{n-s+1}\right)\left(\log\frac{n}{n-s+1}\right)^{-1} + \frac{c}{2}\sqrt{n} + 5$$

$$= \gamma(n,s,t') + 1 + \frac{c}{2}\sqrt{n} + 5 = g(n,s,t') + 1.$$

□

Theorem 2.2.22. *The cop number of an undirected Cayley graph on an abelian group of n elements is at most $0.9425\sqrt{n}+7$.*

Proof. Let G be an abelian group on n vertices generated by set $S \subseteq G, S = -S$ of s elements. We let $\epsilon = 10^{-6}$. We first find values c and d satisfying $\frac{c}{2} \geq \frac{d}{\sqrt{2}}$, $d \geq \frac{1+\epsilon}{ce} + \frac{c}{2}$, which minimize d. A computation of such values c, d yields $c = \sqrt{\frac{2(1+\epsilon)}{e(\sqrt{2}-1)}} \approx 1.33$ and $d = \sqrt{\frac{1+\epsilon}{e(\sqrt{2}-1)}} < 0.9425$. Then, by the lemmas of this section, $g(n,s,t)$ and $h(n,s,t)$ satisfy all three conditions of Lemma 2.2.14. Hence, by Lemmas 2.2.14 and 2.2.19, $c(\mathrm{Cay}(G,S)) \leq c(n,s,s) \leq g(n,s,s)+2 \leq d\sqrt{n}+7$. □

A strengthening similar to Theorem 2.2.12 is possible when n is odd.

Theorem 2.2.23. *Let G be an abelian group with an odd number n of elements. Let $S \subseteq G$ be a generating set of G, and let $S = -S$. Then $c(\mathrm{Cay}(G,S)) \leq 0.8683\sqrt{n}+7$.*

Proof. In condition (2) of Lemma 2.2.14, we require $n' \leq \frac{n}{2}$ because of the bound $|G/\langle k\rangle| \leq n/2$ for any element $k \in G, k \neq 0$. However, if n is odd, then we know that $|G/\langle k\rangle| \leq n/3$, so we only need to require that $n' \leq \frac{n}{3}$ in this condition. Hence, we may relax the requirement $\frac{c}{2} \geq \frac{d}{\sqrt{2}}$ from Lemma 2.2.20 to $\frac{c}{2} \geq \frac{d}{\sqrt{3}}$.

Then, minimizing d with respect to $\frac{c}{2} \geq \frac{d}{\sqrt{3}}$ and $d \geq \frac{1+\epsilon}{ce} + \frac{c}{2}$ yields the solution $c = \sqrt{\frac{1+\epsilon}{(\sqrt{3}-1)e}} \approx 1.00, d = \frac{1}{2}\sqrt{\frac{3(1+\epsilon)}{(\frac{e}{2}(\sqrt{3}-1))}} < 0.8683$. Then the result follows as in Theorem 2.2.22. □

Theorem 2.2.24. *Let G be an abelian group with n elements such that n is not a multiple of 2 or 3. Let $S \subseteq G$ be a generating set of G, and let $S = -S$. Then $c(\mathrm{Cay}(G, S)) \leq 0.8578\sqrt{n} + 7$.*

Proof. We let $\epsilon = 10^{-6}$. As 2 and 3 do not divide n, in condition (2) of Lemma 2.2.14, we only need to require $n' \leq \frac{n}{5}$. Hence, we may relax the requirement $\frac{c}{2} \geq \frac{d}{\sqrt{2}}$ from Lemma 2.2.20 to $\frac{c}{2} \geq \frac{d}{\sqrt{5}}$.

Then, minimizing d with respect to $\frac{c}{2} \geq \frac{d}{\sqrt{5}}$ and $d \geq \frac{1+\epsilon}{ce} + \frac{c}{2}$ yields the solution $c = d = \sqrt{\frac{2(1+\epsilon)}{e}} \approx 0.85776$. Then the result follows as in the Theorem 2.2.22. □

2.3 Meyniel extremal families

Meyniel's conjecture asserts that the cop number of a graph on n vertices is of the form $O(\sqrt{n})$. For a family of graphs $\mathcal{G}$, Meyniel's conjecture would imply that a lower bound of the form $c(G) = \Omega(\sqrt{n})$ for each graph $G \in \mathcal{G}, |G| = n$ is as large as possible. A family $\mathcal{G}$ of graphs for which such a lower bound holds for the cop number of graphs in $\mathcal{G}$ is called a *Meyniel extremal family*. Baird and Bonato show that there exists a family of projective plane incidence graphs on n vertices with cop number at least $\sqrt{\frac{n}{2}}$ [4], and Hasiri and Shinkar show that there exists a family of abelian Cayley graphs on n vertices with cop number at least $\sqrt{\frac{n}{5}}$ [21]. The result of Hasiri and Shinkar shows that the bound of Theorem 2.2.22 is best possible up to a constant factor.

We will give constructions for undirected and directed abelian Cayley graphs with cop number $\Theta(\sqrt{n})$. Our Cayley graph constructions, which use finite fields for groups, will show that the bounds given in Theorem 2.2.11 and Theorem 2.2.22 are best possible up to a constant factor. Our construction for undirected abelian Cayley graphs will improve the result of Hasiri and Shinkar. We will also give a construction of a Cayley graph on a generalized dihedral group whose lower bound for cop number asymptotically matches the currently best known lower bounds for cop number in terms of number of vertices.

2.3.1 Cayley graphs on abelian groups

We will consider an abelian group G. We let $0 \in S$ in order to simplify notation. When a cop or robber does not move during some turn, we consider that this cop or robber plays the move 0.

Let $p > 3$ be a prime, and let G be the additive group $(\mathbb{Z}/p\mathbb{Z})^2$. Note that $\mathbb{Z}/p\mathbb{Z}$ is in fact a field, and we will apply addition and multiplication on $\mathbb{Z}/p\mathbb{Z}$ in the standard way. Let S_1 and S_2

be defined as follows:

$$S_1 = \{(x, x^3) : x \in \mathbb{Z}/p\mathbb{Z}\},$$

$$S_2 = \{(x, x^2) : x \in \mathbb{Z}/p\mathbb{Z}\}.$$

We note that our sets S_1 and S_2 appear as examples of Sidon subsets for certain finite abelian groups in a paper by L. Babai and V. Sós [3].

It is straightforward to show that S_1 and S_2 are both generating sets of G. We note that S_1 is also closed under inverses, while S_2 is not closed under inverses in general. Therefore, we consider $\mathrm{Cay}(G, S_1)$ to be an undirected abelian Cayley graph, and we consider $\mathrm{Cay}(G, S_2)$ to be a directed abelian Cayley graph. We note that $|G| = p^2$. The next two theorems show that both $\mathrm{Cay}(G, S_1)$ and $\mathrm{Cay}(G, S_2)$ have a cop number of the form $\Theta(p)$.

Theorem 2.3.1. *Let G, S_1, and p be as in the construction above. Then the cop number of $\mathrm{Cay}(G, S_1)$ is exactly $\lceil \frac{1}{2}p \rceil = \lceil \frac{1}{2}\sqrt{|G|} \rceil$.*

Proof. We first give a lower bound for the cop number of $\mathrm{Cay}(G, S_1)$. Whenever a cop is able to capture the robber after the robber plays a move (x, x^3), we say that the cop *guards* the move (x, x^3). We show that a single cop cannot simultaneously guard more than two robber moves. Let $v \in G$ be a vertex occupied by a cop C, and let $r \in G$ be the vertex occupied by the robber. If the robber is not yet caught, then $v - r = (a, b)$, where a and b are not both zero. If C guards a move $(x, x^3) \in S_1$, then there must exist a move $(y, y^3) \in S_1$ by which C can capture the robber in reply to (x, x^3). It then follows that $(x, x^3) - (y, y^3) = (a, b)$. Thus x and y must satisfy

$$x - y = a$$

$$x^3 - y^3 = b.$$

By substitution, we obtain the equation

$$a^3 - 3a^2x + 3ax^2 = b.$$

We see that if $a \neq 0$, then the system of equations has at most two solutions; otherwise, $a = b = 0$. Therefore, for fixed a and b not both equal to 0, there exist at most two values x for which a solution to the system of equations exists. Hence C guards at most two robber moves $(x, x^3) \in S_1$.

The robber has a total number of moves equal to $|S_1| = p = \sqrt{|G|}$. If the total number of cops is less than $\frac{1}{2}p$, then the robber will always have some move that is not guarded by any cop. Then by naively moving to an unguarded vertex on each turn, the robber can evade capture forever. Hence the cop number of $\mathrm{Cay}(G, S_1)$ is at least $\frac{1}{2}p = \frac{1}{2}\sqrt{|G|}$. As cop number is an integer, the cop number of $\mathrm{Cay}(G, S_1)$ therefore is at least $\lceil \frac{1}{2}p \rceil$. It follows from Theorem 2.1.1 that the cop number of $\mathrm{Cay}(G, S_1)$ is exactly $\lceil \frac{1}{2}p \rceil$. $\square$

We now show an analoguous result for directed graphs.

Theorem 2.3.2. *Let G and S_2 be as in the construction above. Then the cop number of the directed graph* $\mathrm{Cay}(G, S_2)$ *is equal to* $|S_2| = p = \sqrt{|G|}$.

Proof. We first give a lower bound for the cop number of $\mathrm{Cay}(G, S_2)$. Whenever a cop is able to capture the robber after the robber plays a move (x, x^2), we say that the cop guards the move (x, x^2). We show that a single cop cannot guard more than one robber move. Let $v \in G$ be a vertex occupied by a cop C, and let $r \in G$ be the vertex occupied by the robber. If the robber is not yet caught, then $v - r = (a, b)$, where a and b are not both zero. If C guards a move (x, x^2), then there must exist a move (y, y^2) by which C can capture the robber in reply to (x, x^2). It then follows that $(x, x^2) - (y, y^2) = (a, b)$. Thus x and y must satisfy

$$x - y = a$$

$$x^2 - y^2 = b.$$

By substitution, we obtain the equation $a^2 - 2ax = b$, from which we see that whenever $a \neq 0$, x is uniquely determined; otherwise $a = b = 0$. Therefore, for fixed a and b not both equal to 0, there exists exactly one value x for which a solution to the system of equations exists. Hence the cop occupying C guards at most one robber move $(x, x^2) \in S_2$.

The robber has a total number of moves equal to $|S_2| = p = \sqrt{|G|}$. If the total number of cops is less than p, then the robber will always have some move that is not guarded by any cop. Then by naively moving to an unguarded vertex on each turn, the robber can evade capture forever. Hence the cop number of $\mathrm{Cay}(G, S_2)$ is at least $|S_2| = p = \sqrt{|G|}$.

For the upper bound, a theorem of Hamidoune states that a directed Cayley graph on an abelian group generated by a set $S \subseteq G$ satisfying $0 \in S$ has cop number at most $|S|$ [20, Lemma 3.3]. □

Our construction in Theorem 2.3.2 implies that if Meyniel's conjecture holds for strongly connected directed graphs, written as $c(G) \leq c\sqrt{n}$, then the constant must respect $c \geq 1$. Furthermore, from Theorem 2.3.2, we can construct a Meyniel extremal family of strongly connected directed graphs with cop number $(1 - o(1))\sqrt{n}$. It is shown in [23] and [4] that there exist graph families on n vertices with cop number $\Omega(\sqrt{n})$, but our multiplicative constant of $1 - o(1)$ is the largest constant of any known construction for directed graphs.

Corollary 2.3.3. *For n sufficiently large, there exists a strongly connected directed graph on n vertices with cop number at least* $\sqrt{n - 2n^{0.7625}} = (1 - o(1))\sqrt{n}$.

Proof. We borrow a lemma from number theory which tells us that for x sufficiently large, there exists a prime in the interval $[x - x^{0.525}, x]$ [5]. From this lemma it follows that for sufficiently large x, there exists a square of a prime in the interval $[x - 2x^{0.7625}, x]$.

For our construction, we let n be sufficiently large, and we choose a prime number p with $p^2 \in [n - 2n^{0.7625}, n]$. We let $G = (\mathbb{Z}/p\mathbb{Z})^2$, and we let S_2 be as in Theorem 2.3.2. We then attach a sufficiently long bidirectional path to one of the vertices of $\mathrm{Cay}(G, S_2)$ to obtain a strongly connected directed graph on n vertices with cop number equal to $c(G, S_2) = p \geq \sqrt{n - 2n^{0.7625}} = (1 - o(1))\sqrt{n}$. $\square$

We conjecture that the constructions given in Theorems 2.3.1 and 2.3.2 have greatest possible cop number in terms of n, up to an additive constant.

Conjecture 2.3.4. *The cop number of any undirected abelian Cayley graph on n vertices is at most $\frac{1}{2}\sqrt{n} + O(1)$.*

Conjecture 2.3.5. *The cop number of any directed abelian Cayley graph on n vertices is at most $\sqrt{n} + O(1)$.*

2.3.2 Cayley graphs on generalized dihedral groups

Finally, we will show that Cayley graphs may be used to construct graphs on n vertices with a cop number of at least $\sqrt{\frac{n}{2}}$, which is as large as the best known lower bound for cop number in terms of vertices. We will construct a family of Cayley graphs on a generalized dihedral group; however, for the sake of simplicity, rather than fully describing the group structure of our graph family, we will define our construction based on an abelian group structure.

Given a prime number $p \geq 3$, we construct a graph G_p as follows. For each element $(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2$, we define two vertices $u_{(x,y)}$ and $v_{(x,y)}$. Then, for each vertex $u_{(x,y)}$ and each element $a \in \mathbb{Z}/p\mathbb{Z}$, we add an edge $(u_{(x,y)}, v_{(x+a,y+a^2)})$ to G_p. With this construction, G_p is a bipartite p-regular graph on $2p^2$ vertices. One may show in fact that G is a Cayley graph on the generalized dihedral group $\mathrm{Dih}((\mathbb{Z}/p\mathbb{Z})^2)$.

Theorem 2.3.6. *Let $p \geq 3$ be a prime, and let G_p be given as above. Then $c(G_p) \geq p$. Letting $n = |G_p|$, $c(G_p) \geq \sqrt{\frac{n}{2}}$.*

Proof. We claim that the girth of G_p is at least 6. As G_p is bipartite, it suffices to check only that G_p has no 4-cycle. We may assume without loss of generality that if G_p contains a 4-cycle, then $u_{(0,0)}$ belongs to a 4-cycle.

Let C be a 4-cycle containing $u_{(0,0)}$, and let $u_{(a,b)}$ be the vertex of C at a distance of 2 from $u_{(0,0)}$. As $u_{(a,b)}$ belongs to C, $u_{(a,b)}$ must be reachable by two internally disjoint 2-paths from $u_{(0,0)}$.

Suppose $u_{(a,b)}$ is reached from $u_{(0,0)}$ by two edges $(u_{(0,0)}, v_{(x,x^2)})$ and $(u_{(a,b)}, v_{(a+y,a+y^2)})$. It must follow that $a = x - y$ and that $b = x^2 - y^2$. However, as before, this gives us the equation $a^2 - 2ax = b$, which tells us either that x and y are uniquely determined or that $a = b = 0$. If x and y are uniquely determined, then no two disjoint 2-paths from $u_{(0,0)}$ to $u_{(a,b)}$ can exist. If $a = b = 0$, then C is not a 4-cycle. In both cases, we have a contradiction.

Therefore, the girth of G_p is at least 6. Since G_p has girth at least 5, Theorem 1.3.7 then tells us that $c(G_p) \geq \delta(G_p) = p$. □

We note that G_p is a bipartite graph on n vertices with $\Theta(n^{3/2})$ edges and no $K_{2,2}$, which is best possible up to a constant factor by a theorem of Kővári, Sós, and Turán [25]. To the best of our knowledge, this is the first example of a construction of edge-extremal $K_{2,2}$-free bipartite graphs that uses Cayley graphs.

2.4 Acknowledgment

I am grateful to Matt DeVos for his suggestion to consider Sidon subsets for constructing abelian Cayley graphs of high cop number.

Chapter 3

Surrounding cops and robbers

3.1 Embedded and minor-free graphs

3.1.1 Introduction

We will consider the game of surrounding cops and robbers on graphs embedded in surfaces. We recall the rules for surrounding cops and robbers. The game of surrounding cops and robbers is played by two players: Alice and Bob. Again, Alice controls a team of cops, and Bob controls a robber. The players move their cops and robbers according to the same rules as the original game of cops and robbers. Unlike the original game of cops and robbers, in surrounding cops and robbers, Alice wins the game whenever every neighbor of the robber's vertex is occupied by a cop. Furthermore, Bob may not end a turn with the robber at the same vertex as a cop. For a graph G, we write $s(G)$ for the *surrounding cop number*, which is the minumum number of cops that Alice needs in order to have a winning strategy under these rules.

As stated before, these rule changes bring about some differences between the original game of cops and robbers and the surrounding variant of cops and robbers. Firstly, in the surrounding variant, Alice's task is more difficult than in the traditional version. Indeed, if Alice moves a cop to the same vertex as the robber, she does not win; rather, the robber is simply forced to move to a different vertex, as Bob may not leave the robber at the same vertex as a cop. Therefore, in the surrounding variant, “capturing" the robber with a cop does not win the game for Alice and serves only to force the robber to move. Secondly, unlike in the traditional game, it is safe for Bob to move the robber to a vertex that neighbors a cop, as being “captured" by a cop poses no threat. Therefore, if Alice wishes to prevent the robber from moving to a certain vertex v, she must place a cop at v, since Bob may not move the robber to a vertex that is already occupied by a cop. Finally, we see that any winning strategy for Alice in the surrounding variant also gives a winning strategy in the traditional version of cops and robbers, as a robber that is surrounded in the traditional version of the game may not avoid capture.

We will consider planar graphs, bipartite planar graphs, toroidal graphs, graphs of bounded genus, and in a more general flavor, graphs with an excluded minor. Our main tool will be the guarding of geodesic paths, introduced by Aigner and Fromme in [1] and used by many other authors (c.f. [27], [31], [33]). Given a graph G, we will choose certain geodesic paths in G to be *guarded*. We will refer to the robber's *region* or *territory* as the component occupied by the robber in the graph obtained by removing guarded paths from G. We will successively make the robber's territory smaller until the robber's territory contains a single vertex, at which point we will show that the robber is surrounded.

3.1.2 Planar graphs

In this section, we will consider the game of surrounding cops and robbers on planar graphs. We will show that for planar graphs G, $s(G) \leq 7$. We will need some preliminaries. We will say that a (u, v)-path is a path with endpoints u, v. We recall that if P is a (u, v)-path of length l in a graph G, then P is *geodesic* with respect to G if all (u, v)-paths in G have length at least l.

Definition 3.1.1. Let G be a graph with a subgraph H. We say that H is *geodesically closed* with respect to G if for any $u, v \in V(H)$, every geodesic (u, v)-path in G is a subgraph of H.

The following observation is a common tool in methods that use geodesic paths. We include a proof for completeness.

Observation 3.1.2. *Let G be a graph, and let $P = (v_0, v_1, \dots, v_k)$ be a geodesic path in G. Suppose that a vertex $w \in V(G)$ is adjacent to $v_j \in V(P)$. Then $j - 1 \leq dist(v_0, w) \leq j + 1$.*

Proof. The path $(v_0, \dots, v_j, w)$ is a path of length $j + 1$, so $\text{dist}(v_0, w) \leq j + 1$. Suppose that $\text{dist}(v_0, w) \leq j - 2$. Then $\text{dist}(v_0, v_j) \leq j - 1$, implying $\text{dist}(v_0, v_k) \leq k - 1$, which contradicts the assumption that P is a geodesic path. □

We will show that in the surrounding game, three cops may guard a geodesic path. In comparison, Aigner and Fromme show in [1] that in the traditional game of cops and robbers, a single cop may guard a geodesic path. Aigner and Fromme's strategy uses the concept of the robber's *shadow*, which is defined as follows.

Definition 3.1.3. Let G be a graph, and let $P = (v_0, \dots, v_k)$ be a geodesic path of G. Suppose that the robber occupies a vertex $w \in V(G)$ for which $\text{dist}(v_0, w) = d$. If $d \leq k$, then we say that the robber has a *P-shadow* at v_d. If $d > k$, then we say that the robber has a P-shadow at v_k.

The main idea behind Aigner and Fromme's proof is as follows. Suppose that there exists a geodesic path $P = (v_0, \dots, v_k)$ in a planar graph. On each turn, the robber's distance from v_0 changes by at most 1, so on each turn, the robber's P-shadow either does not move or moves to an adjacent vertex on P. Therefore, there exists a strategy using one cop C by which, after a finite number of moves, C may move to the vertex of P containing the robber's P-shadow on each turn. We say that this movement pattern of C is called *stalking the robber's P-shadow*. In the traditional game of cops and robbers, if the robber ever moves to a vertex on P, then the robber will occupy the same vertex as its P-shadow, and thus when C stalks the robber's P-shadow, C will capture the robber should the robber choose to move to P. Thus, by this strategy, the robber can never safely visit a vertex of P.

In the surrounding variant of cops and robbers, however, the robber is not threatened by the prospect of being "captured" by C. Therefore, we may modify Aigner and Fromme's strategy by adding up to two additional cops that follow C in P, occupying one or both neighbors of C in P at all times. By making this modification, we will see that whenever the robber attempts to move to a vertex $v \in V(P)$, C will either already occupy v, or one of C's "followers" will occupy v, which means that the robber will be blocked from moving to v. We first give a definition that concisely refers to this strategy.

Definition 3.1.4. Let $P = (v_0, \ldots, v_k)$ be a geodesic path in a graph G. Let $\mathcal{C} = \{C_1, \ldots, C_t\}$, where $t \in \{2, 3\}$, be a set of two or three cops. We say that the cops in $\mathcal{C}$ *stalk the robber's P-shadow* if they obey the following strategy:

- If $t = 2$, then C_2 moves to the vertex v_i of P containing the robber's P-shadow on each turn, and C_1 moves to v_{i-1} on each turn (except when $i = 0$, in which case C_1 moves to v_0).
- If $t = 3$, then C_2 moves to the vertex v_i of P containing the robber's P-shadow on each turn, C_1 moves to v_{i-1} on each turn, and C_3 moves to v_{i+1} on each turn (except when $i = 0$, in which case C_1 moves to v_0, and except when $i = k$, in which case C_3 moves to v_k).

In other words, when a set of cops stalks the robber's P-shadow, a single cop "captures" the robber's P-shadow on each turn, and one or two additional cops follow on both sides of the first cop. For a geodesic path P, the robber's P-shadow moves by at most one vertex on each turn, so a set of two or three cops can always stalk the robber's P-shadow after a finite number of moves. We illustrate three cops stalking a robber's shadow in Figure 3.1.

Lemma 3.1.5. *Let G be a graph. Let $P = (v_0, \ldots, v_k)$ be a geodesic path of G. If three cops stalk the robber's P-shadow, then the robber is unable to move to a vertex of P.*

Proof. We name our cops C_1, C_2, and C_3. We claim that when C_1, C_2, and C_3 begin stalking the robber's P-shadow, the robber must leave P for at least one turn. Indeed, if the robber occupies a vertex $v_j \in P$, then the robber's P-shadow occupies v_j, and C_1, C_2, and C_3 occupy v_j, along with both neighbors of v_j on P; therefore, as the robber may not end a move on the same vertex as a cop, the robber must move off of P.

Next, we show that after C_1, C_2, and C_3 begin stalking the robber's P-shadow, the robber cannot enter P from a vertex outside of P. Suppose that the robber occupies a vertex w that does not belong to P. If w is not adjacent to P, then the robber cannot move onto P. If w is adjacent to P and $\text{dist}(v_0, w) = j$, let $u \in P$ be a neighbor of w. By Observation 3.1.2, u is at a distance in

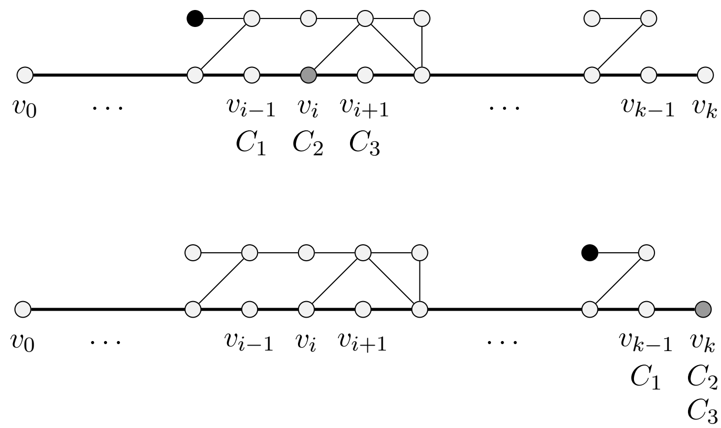

Figure 3.1: The figures show a geodesic path P drawn in bold. In each figure, the robber occupies the black vertex, and the robber's P-shadow occupies the grey vertex. In both figures, three cops C_1, C_2, and C_3 stalk the robber's P-shadow. In the top figure, the robber's P-shadow occupies an internal vertex v_i of P, so C_1, C_2, and C_3 occupy v_i and its neighbors. In the bottom figure, the robber's P-shadow occupies v_k, so C_1 occupies v_{k-1}, while C_2 and C_3 occupy v_k.

the set $\{j-1, j, j+1\}$ from v_0, so u is occupied by one of C_1, C_2, and C_3; therefore, the robber cannot move onto P.

We see that when C_1, C_2, and C_3 begin stalking the robber's P-shadow, the robber is forced to exit P, and then the robber is never again able to enter P. Thus the lemma is proven. □

Next, we show that if a graph G has a geodesic path P that is geodesically closed, then two cops stalking the robber's P-shadow can guard P.

Lemma 3.1.6. *Let G be a graph. Let $P = (v_0, \ldots, v_k) \subseteq G$ be a path. If P is geodesically closed, and if two cops stalk the robber's P-shadow, then the robber is unable to move to a vertex of P.*

Proof. We claim that by stalking the robber's P-shadow, C_1 and C_2 prevent the robber from entering P from outside of P. Suppose that the robber occupies a vertex $w \notin V(P)$ that is adjacent to $v_j \in V(P)$. By Observation 3.1.2, $\text{dist}(v_0, w) \in \{j-1, j, j+1\}$. Furthermore, as P is geodesically closed, $\text{dist}(v_0, w) \neq j-1$; otherwise, $(v_0, \ldots, w, v_j)$ is a geodesic path between two vertices in P that is not contained in P, a contradiction. Thus we see that $\text{dist}(v_0, w) \in \{j, j+1\}$, and thus the strategy of C_1 and C_2 dictates that a cop occupy the vertex v_j. Therefore, the robber is unable to enter P.

It remains to show that C_1 C_2 can force the robber to leave the path P by stalking the robber's P-shadow. Suppose that the robber occupies a vertex $v_j \in V(P)$ when C_1 and C_2 begin stalking the robber's P-shadow. As the robber occupies v_j, the robber's P-shadow also occupies v_j, and C_1 and C_2 occupy v_{j-1} and v_j. (If the robber occupies v_0, then C_1 and C_2 both occupy v_0.) Thus the robber must move away from v_j. If the robber leaves P, then the proof is complete; otherwise, the robber moves to v_{j+1}. Then C_1 C_2 move to v_j and v_{j+1}, and the robber must move off of P or move to v_{j+2}. By continuing to stalk the robber's P-shadow, the robber will either move off of P voluntarily, or the robber will reach v_k, at which point C_1 C_2 will occupy v_{k-1} and v_k. At this point, the robber will have no unoccupied neighbor in P, and the robber will be forced to leave P. This completes the proof. □

Our tools for guarding geodesic paths are in place. Now we will devise a strategy for using guarded geodesic paths to surround the robber. We will begin by enclosing the robber's region with two guarded paths. We will iteratively choose new paths to guard in order to make the robber's region smaller. Furthermore, we will always keep the robber's region enclosed by at least one path that requires only two cops to guard. Eventually, we will restrict the robber's region to a single vertex. When the robber's region consists a single vertex r, every neighbor of r in G will belong to a guarded path. Furthermore, by construction, a set of cops guards a geodesic path P by stalking the robber's P-shadow, so when the robber moves to a vertex r adjacent to P, all neighbors of r in P are occupied by cops. Therefore, if we say that each neighbor of r belongs to a guarded path, this will imply that each neighbor of r is occupied by a cop. Therefore, to show that we can surround the robber, we only need to show that we can reduce the robber's territory to a single vertex.

The following lemma shows that a path guarded by three cops can be exchanged for another guarded path using at most two extra cops. The lemma is quite technical in its statement, so we illustrate the maneuver that we are describing in Figure 3.2.

Lemma 3.1.7. *Let G be a planar graph with a fixed drawing in the plane. Let $P_1, P_2 \subseteq G$ be two (α, β)-paths in G, where $\alpha, \beta \in V(G)$. Let A be a component of $G \setminus (P_1 \cup P_2)$ enclosed by P_1 and P_2. Suppose the following hold:*

- *The robber occupies a vertex in A.*
- *For $i \in \{1, 2\}$, P_i is geodesic with respect to $P_i \cup A$.*
- *P_1 is not geodesically closed with respect to $P_1 \cup A$, and three cops C_1, C_2, and C_3 stalk the robber's P_1-shadow.*
- *P_2 is geodesically closed with respect to $P_2 \cup A$, and two cops C'_1 and C'_2 stalk the robber's P_2-shadow.*

Then, using at most two additional cops C_4 and C_5, the robber can be confined to a smaller region $B \subsetneq A$. Furthermore, the cops can keep the robber in B by guarding two (α, β)-paths, each requiring at most three cops, and at least one of which can be guarded with two cops.

Proof. Let $P_1 = (\alpha = v_0, \dots, v_k = \beta)$. We assume that the robber's P_1-shadow that C_1, C_2, and C_3 stalk is calculated with respect to v_0.

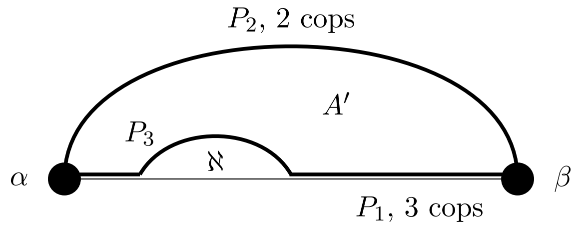

Figure 3.2: This figure illustrates the cop maneuver that is described in Lemma 3.1.7. We suppose that the robber is confined to the region enclosed by P_1 and P_2. Lemma 3.1.7 claims that there exists an (α, β)-path P_3 such that, when guarded by cops, the robber is confined to either the region A' or $\aleph$. If the robber is confined to A', we claim that the robber can be kept in A' by guarding P_2 with two cops and P_3 with three cops. If the robber is confined to $\aleph$, we claim that the robber can be kept in $\aleph$ by guarding P_1 with two cops and P_3 with two cops.

As P_1 is not geodesically closed with respect to $A \cup P_1$, we can choose a geodesic (w.r.t. $A \cup P_1$) path $S \not\subseteq P_1$, with endpoints $v_i, v_j \in P$ $(i < j)$, such that S is of shortest length out of all such geodesic paths. In Figure 3.3, such a path S is given by (v_i, x, v_l). Let $P_1(i,j) = (v_i, v_{i+1}, \dots, v_j)$, and let $S = (v_i, w_{i+1}, \dots, w_{j-1}, v_j)$. Let $\aleph$ be the region enclosed by (and not including) S and $P_1(i,j)$. Let S be chosen out of all (v_i, v_j)-geodesics to minimize the number of vertices in $\aleph$. By our choice of S, we may assume that $P_1 \cup \aleph$ does not contain any (v_i, v_j)-geodesic besides $P_1(i,j)$, as such a geodesic would allow for a smaller region $\aleph$. (Such a path S is called a *bypath* in [18], in which González and Mohar apply a similar technique to the lazy variant of cops and robbers.)

We claim that for any vertex $x \in A$, if x is adjacent to a vertex $v_l \in P(i,j) \setminus \{v_i, v_j\}$, then $\text{dist}(v_0, x) \in \{l, l+1\}$. We know from Observation 3.1.2 that $\text{dist}(v_0, x) \in \{l-1, l, l+1\}$. Suppose for the sake of contradiction that $\text{dist}(v_0, x) = l - 1$. Then x must belong to a geodesic path $S' = (v_m, \dots, x, v_l)$, where v_m is chosen to make S' as short as possible for this fixed value l. By the minimality of S, we must have $0 \leq m < i$, and so S' must cross S at some vertex w_p. As S and S' are geodesics, it then follows that $(v_i, \dots, w_p, \dots, v_l)$ is a geodesic shorter than S, a contradiction. We illustrate this contradiction in Figure 3.3, and in the particular case depicted in the figure, $x = w_p$. Thus the claim is proven.

We define $P_3 = (v_0, v_1, \dots, v_i, w_{i+1}, \dots, w_{j-1}, v_j, \dots, v_k)$. We calculate the robber's P_3-shadow with respect to y_0, and we let two additional cops C_4 and C_5 stalk the robber's P_3-shadow. (At this point we may assume that the robber's P_3-shadow does not occupy a vertex of P_1, as otherwise P_3 would already be guarded, and the argument would be complete.) Then, on each subsequent move after C_4 and C_5 begin stalking the robber's P_3-shadow, we move C_3 on P_1 toward the vertex v_j. We note that C_3 will reach v_j before the robber's P_1-shadow can reach v_j. We let C_1 and C_2 continue

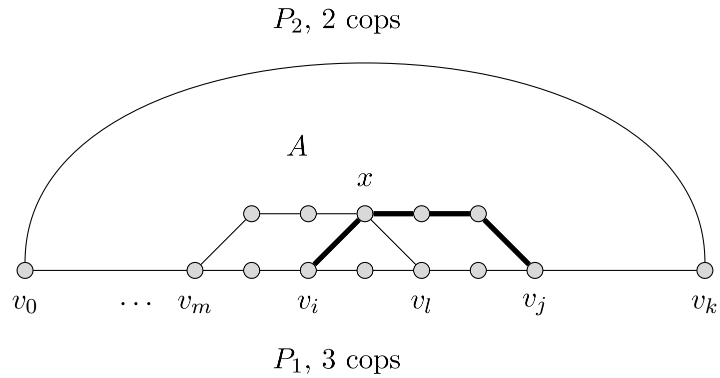

Figure 3.3: The figure shows a planar graph G in which the robber is restricted to a region A enclosed by two geodesic paths P_1 and P_2. In the figure, the shortest geodesic path S with respect to $P_1 \cup A$ with endpoints in P_1 and not belonging to P_1 is (v_i, x, v_l). If the bolded geodesic path beginning with v_i and ending with v_j is incorrectly chosen as S, then the shorter geodesic path (v_i, x, v_l) may be found, as described in the proof of Lemma 3.1.7.

to stalk the robber's P_1-shadow. We illustrate this point in the cops' strategy after C_3 reaches v_j in Figure 3.4.

We claim that if the robber's P_3-shadow ever occupies a vertex of P_1, then we have three cops in position to stalk the robber's P_3-shadow. If the robber's P_3 shadow occupies a vertex of P_1, then one of two cases must have occured.

Case 1: The robber's P_3-shadow moves from w_{i+1} to v_i. Then C_1 and C_2 move to v_{i-1}, v_i, and C_5 remains at w_{i+1}. The robber does not occupy a vertex of P_3, and C_1, C_2, C_5 stalk the robber's P_3-shadow.

Case 2: The robber's P_3-shadow moves from w_{j-1} to v_j. Then C_3 moves to v_{j+1}, and C_4, C_5 move to w_{j-1}, v_j. The robber does not occupy a vertex of P_3, and C_4, C_5, and C_3 stalk the robber's P_3-shadow.

In both cases, three cops successfully guard P_3. If the robber's region is enclosed by P_1 and P_3, then the robber's region is $\aleph$. P_3 is guarded by 3 cops, and P_1 is geodesically closed with respect to $P_1 \cup \aleph$; hence the robber's region is enclosed by two (α, β)-paths, one of which is guarded by only two cops C_1, C_2, and the lemma is proven. (In fact, in this case, P_3 is also geodesically closed with respect to the robber's territory, so we do not need all three cops to keep guarding P_3.) If the robber's region is enclosed by P_2 and P_3, then the robber if confined to a smaller region $B \subsetneq A$,

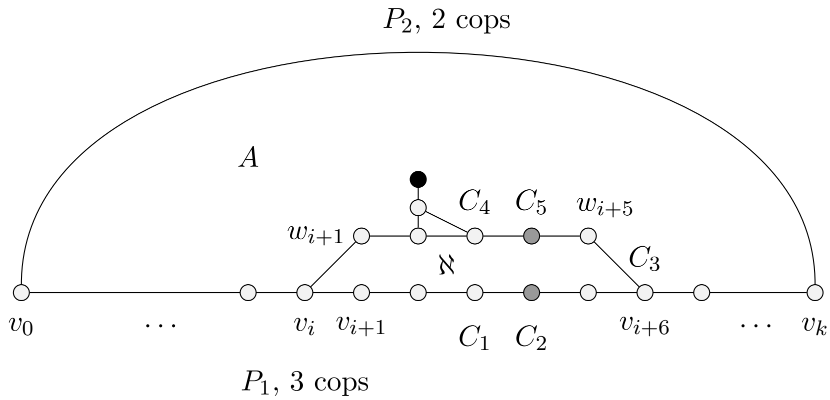

Figure 3.4: The figure shows a planar graph G in which the robber is restricted to a region A enclosed by two geodesic paths P_1 and P_2. The smaller region $\aleph$ is enclosed by $(v_i, \ldots, v_{i+6})$ and $(v_i, w_{i+1}, \ldots, w_{i+5}, v_{i+6})$. In the figure, the cops attempt either to remove the region $\aleph$ from the robber's territory or confine the robber to $\aleph$. To accomplish this, two cops C_4 and C_5 are stalking the robber's P_3 shadow, where $P_3 = (v_0, \ldots, v_i, w_{i+1}, \ldots, w_{i+5}, v_{i+6}, \ldots, v_k)$. Here, the robber's vertex is colored black, the robber's P_1-shadow and P_3-shadow are colored grey. As C_4 and C_5 are successfully stalking the robber's P_3-shadow, C_3 has stopped stalking the robber's P_1-shadow and has moved to v_{i+6}, where P_3 rejoins P_1. If the robber's P_3-shadow moves far enough left or right, then three cops will be stalking the robber's P_3 shadow, and the robber will not be able to move to P_3. This will either remove $\aleph$ from the robber's territory or confine the robber to $\aleph$.

namely one obtained from A by removing $\aleph$ and P_3. P_3 is guarded by 3 cops, and P_2 is guarded by two cops by assumption, and the lemma is proven.

Hence we arrive at a point in which the robber's P_3-shadow must lie on the subpath

$$(w_{i+1}, w_{i+2} \dots, w_{j-2}, w_{j-1}).$$

At this point, we move C_3 on P_3 toward the robber's P_3-shadow until C_3, C_4, and C_5 stalk the robber's P_3-shadow. At this point, one of two cases occurs:

Case 1: The robber occupies a vertex in the region enclosed by P_2 and P_3. In this case, P_2 is guarded by two cops, and we have three cops C_3, C_4, and C_5 guarding P_3. In this case, the proof is complete.

Case 2: The robber occupies a vertex in the region $\aleph$ enclosed by P_1 and P_3. In this case, we have three cops C_3, C_4, and C_5 guarding P_3. Additionally, P_1 is geodesically closed with respect to $P_1 \cup \aleph$, so C_1 and C_2 can guard P_1 by continuing to stalk the robber's P_1 shadow. In this case, the proof is complete. □

With this lemma in place, we can prove an upper bound for the surrounding cop number of planar graphs.

Theorem 3.1.8. *Let G be a planar graph. Then $s(G) \leq 7$.*

Proof. We will play the game in stages. In each Stage i of the game, we will have two paths P_1 and P_2 with common endpoints that are guarded by cops who stalk the robber's P_1-shadow and P_2-shadow. We will define the robber's region R_i at Stage i as the component of $G \setminus (P_1 \cup P_2)$ that the robber occupies. We will show that at each stage, we can reduce the robber's region to a new region $R_{i+1} \subsetneq R_i$.

If G is a tree, then by [11], $s(G) = 2$, and we are done. Otherwise, we begin the game at Stage 1. As G is not a tree, there exists an edge $(uv) \in E(G)$ such that a geodesic path exists from u to v in the graph $G \setminus (uv)$. We let P_2 be the path (u, v), and we let P_1 be any (u, v)-path geodesic with respect to $G \setminus (uv)$. P_2 is of length one and can clearly be guarded with two cops. P_1 is a geodesic path with respect to $G \setminus P_2$, and so P_1 can be guarded with three cops who stalk the robber's P_1-shadow. At this point, the component of the graph $G \setminus (P_1 \cup P_2)$ that the robber occupies is called R_1. We may redraw G so that R_1 is enclosed by P_1 and P_2

We describe the strategy that we follow at each Stage i of the game.

Case 1: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is not geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$.

Let P_1 be guarded by three cops C_1, C_2, and C_3. Let P_2 be guarded by cops C_1' and C_2'. Then by Lemma 3.1.7, we can use two extra cops C_4 and C_5 and replace P_1 with an (α, β)-path P_1' so that at most three cops guard P_1', and such that there exists an (α, β)-path P_2' guarded by at most two cops such that the robber is restricted to a component R_{i+1} of $G \setminus (P_1' \cup P_2')$ with $R_{i+1} \subsetneq R_i$. Depending on whether or not P_1' is geodesically closed with respect to $P_1' \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

Case 2: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$. Furthermore, we assume there exists an (α, β)-path in $P_1 \cup P_2 \cup R_i$ distinct from P_1 and P_2.

We choose a shortest (α, β)-path in $P_1 \cup P_2 \cup R_i$, and we call this path P_3. We guard P_3 with at most three cops. P_3 divides the region R_i into two parts, and hence the robber's region is either enclosed by P_1 and P_3, or the robber's region is enclosed by P_2 and P_3. In both cases, the robber's region is restricted to a region $R_{i+1} \subsetneq R_i$, and one of the paths enclosing R_{i+1} is guarded by at most two cops. Depending whether or not P_3 is geodesically closed with respect to $P_3 \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

Case 3: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$. There exists no (α, β)-path in $P_1 \cup P_2 \cup R_i$ distinct from P_1 and P_2.

In this case, without loss of generality, P_1 has only one vertex $x \in V(P_1)$ adjacent to the robber's region, and P_2 is not adjacent to the robber's region. Then the robber can be restricted to his region simply by placing a cop at x. We thus place two cops C_1 and C_2 at x and confine the robber to his region. If x has only one neighbor y in the robber's region, then we move C_1 to y and move C_2 to y on the next turn. We may continue this process until C_1 and C_2 occupy a vertex x that has at least two neighbors y, z in the robber's region R_i. Then we choose $P_2' = (x, y)$, and we let P_1' be an (x, y)-geodesic in $R_i \setminus (xy)$. We guard P_1' with at most three cops, and we redraw R_i so that the robber is enclosed by P_1' and P_2'. Then the robber is confined to a region $R_{i+1} \subsetneq R_i$, and depending on whether or not P_1' is geodesically closed with respect to $P_1' \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

If we continue this process, we will eventually reach a point in which the robber's region contains a single vertex, at which point the robber is surrounded. □

Theorem 3.1.9. *There exists a planar graph G with $s(G) \geq 6$.*

Proof. Let H be the truncated icosahedron, that is, the 3-regular polyhedron with 60 vertices and 90 edges whose shape resembles a soccer ball or a C_{60} molecule. Let G be the graph obtained by adding a vertex v to each face f of H and adding an edge from v to each vertex of f. We note that in G, all vertices have degree 5 or 6, and every degree 6 vertex has five neighbors of degree 6.

Suppose we have five cops. We let the robber execute the following strategy. The robber begins on a vertex r of degree 6 and waits for a cop to occupy r. As r has degree 6, and as we play with five cops, the robber will not be surrounded before a cop moves to r. If a cop moves to r, then at this point, at most four of the robber's neighboring vertices are occupied by cops. The robber then chooses an unoccupied neighbor of degree 6 and moves to this vertex. The robber repeats this strategy indefinitely and wins. Therefore, $s(G) \geq 6$. □

Although our methods give an upper bound of 7 for the surrounding cop number of planar graphs, we are currently unable to find examples of planar graphs with surrounding cop number equal to 7. This leaves the question of whether $s(G) \leq 6$ for planar graphs G, or whether a planar graph G with $s(G) = 7$ exists.

3.1.3 Bipartite planar graphs

In this section, we consider the game of surrounding cops and robbers played on bipartite planar graphs. We establish an upper bound of 4 for the surrounding cop number of bipartite planar graphs using the ideas of the previous section. Our method of proof will follow the previous section closely, and the main ideas of this section will essentially show that the methods of the previous section can be applied on bipartite planar graphs with fewer cops. We also show that our upper bound is tight.

First, we will need to establish specialized bipartite versions of the tools from the previous section. We will use the same notion of a robber's *shadow* on a geodesic path. However, the following observation will let us use the technique of stalking the robber's shadow on a geodesic path with fewer cops.

Lemma 3.1.10. *Let G be a bipartite graph, and let $P = (v_0, v_1, \dots, v_k)$ be a geodesic path in G. Suppose that a vertex $w \in V(G)$ is adjacent to $v_j \in V(P)$. Then either* $\operatorname{dist}(v_0, w) = j - 1$ *or* $\operatorname{dist}(v_0, w) = j + 1$.

Proof. By Observation 3.1.2, $\operatorname{dist}(v_0, w) \in \{j-1, j, j+1\}$. If $\operatorname{dist}(v_0, w) = j$, then there exists a closed walk $(v_0, \dots, v_j, w, \dots, v_0)$ of length $2j + 1$, which contradicts the assumption that G is bipartite. Hence $\operatorname{dist}(v_0, w) = j - 1$ or $\operatorname{dist}(v_0, w) = j + 1$. □

Lemma 3.1.10 essentially tells us that if the robber occupies a vertex x adjacent to a geodesic path P in a graph, then at most two vertices of P need to be occupied in order to prevent the robber from moving to a vertex on P. This is fewer than the three cops needed to prevent the robber from moving to a geodesic path in the general case. With this observation in hand, we will redefine the notion of stalking the robber's shadow for bipartite graphs. We emphasize that the definition of stalking the robber's shadow given here is different from the previous section, as the following definition only holds for bipartite graphs.

Definition 3.1.11. Let $P = (v_0, \dots, v_k)$ be a geodesic path in a bipartite graph G. Let $\mathcal{C} = \{C_1, \dots, C_t\}$, where $t \in \{1, 2\}$, be a set of one or two cops. We say that the cops in $\mathcal{C}$ *stalk the robber's P-shadow* if they obey the following strategy:

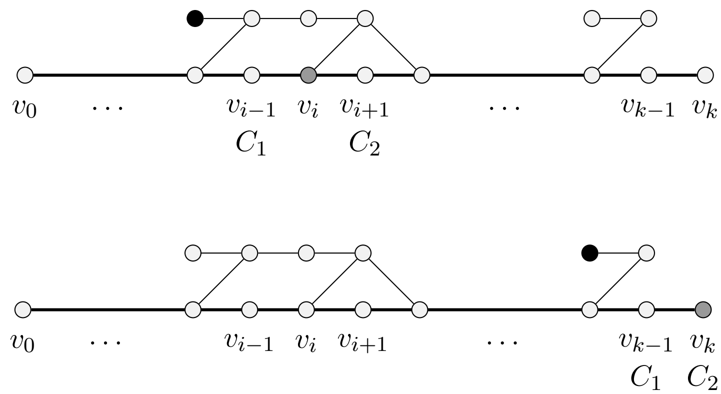

Figure 3.5: The figures show a geodesic path P, drawn in bold, of a bipartite graph. In each figure, the robber occupies the black vertex, and the robber's P-shadow occupies the grey vertex. In both figures, two cops C_1 and C_2 stalk the robber's P-shadow. In the top figure, the robber's shadow occupies an internal vertex v_i of P, so C_1 and C_2 occupy the neighbors of v_i. In the bottom figure, the robber's shadow occupies v_k, so C_1 occupies v_{k-1}, while C_2 occupies v_k.

- If $t = 1$, then letting v_i denote the vertex of P containing the robber's P-shadow, C_1 moves to v_{i-1} on each turn. If $i = 0$, then C_1 moves to v_0.
- If $t = 2$, then letting v_i denote the vertex of P containing the robber's P-shadow, C_1 moves to v_{i-1} on each turn, and C_2 moves to v_{i+1}. If $i = 0$, then C_1 moves to v_0, and if $i = k$, then C_2 moves to v_k.

We show two cops stalking the robber's shadow on a geodesic path of a bipartite graph in Figure 3.5. In the next lemmas, we show that in a bipartite graph, two cops stalking the robber's P-shadow on a geodesic path P prevent the robber from accessing P, and one cop stalking the robber's P-shadow on a geodesically closed path P prevents the robber from accessing P.

Lemma 3.1.12. *Let G be a bipartite graph. Let $P = (v_0, \dots, v_k)$ be a geodesic path of G. If two cops stalk the robber's P-shadow, then the robber is unable to move to a vertex of P from outside of P. Furthermore, there exists a strategy involving two cops by which the robber may be forced to leave P and never again move to a vertex of P.*

Proof. We name our cops C_1 and C_2. We first show that if the robber does not occupy a vertex of P when C_1 and C_2 begin stalking the robber's shadow on P, then the robber is unable to move to P. Suppose that the robber occupies a vertex w that does not belong to P. If w is not adjacent to P, then the robber cannot move onto P. If w is adjacent to P and $\text{dist}(v_0, w) = j$, let $u \in P$ be a neighbor of w. By Lemma 3.1.10, $u \in \{v_{j-1}, v_{j+1}\}$. At this point, the robber's shadow occupies v_j, and thus as C_1 and C_2 stalk the robber's P-shadow, u is occupied by a cop. Therefore, the robber cannot move onto P.

Now, we show that C_1 and C_2 can force the robber to leave P and never return. As the robber's shadow moves by at most one vertex on each turn, C_1 and C_2 may begin stalking the robber's P-shadow after a finite number of moves. If the robber occupies a vertex v_j of P when C_1 and C_2 begin stalking the robber's P-shadow, then the robber's shadow also occupies v_j, and C_1 and C_2 must occupy v_{j-1} and v_{j+1}. We let C_1 move to v_j and let C_2 stay put. If the robber's shadow moves to either v_j or v_{j+1}, then the robber must have left P, and C_1 and C_2 can stalk the robber's shadow using the original strategy, forcing the robber never to return to P. Otherwise, the robber's shadow moves to v_{j-1}. Then C_1 and C_2 move to v_{j-1} and v_j. If the robber's shadow moves to v_{j-1} or v_j, then by the same argument, C_1 and C_2 can stalk the robber's shadow on P and prevent the robber from accessing P. Otherwise, the robber's shadow again moves toward v_0. In this case, C_1 and C_2 can continue pushing the robber's shadow toward v_0 until either the robber's shadow can be stalked by C_1 and C_2 as in the previous discussion, or until the robber's shadow reaches v_0. If the robber's shadow reaches v_0, then this implies that the robber occupies v_0. Then C_1 and C_2 will occupy v_0 and v_1, and the robber's shadow will be forced to move to v_1, and the robber will be forced to leave P. Then C_2 can move to v_2, and C_1, C_2 can stalk the robber's shadow and prevent the robber from accessing P. □

Lemma 3.1.13. *Let G be a bipartite graph. Let $P = (v_0, \ldots, v_k) \subseteq G$ be a path that is geodesically closed. If one cop stalks the robber's P-shadow, then the robber is unable to move to a vertex of P from outside P.*

Proof. By the discussion in the proof of Lemma 3.1.5, a single cop can reach a vertex v_{j-1}, where v_j is the position of the robber's shadow, on every turn after a finite number of turns. We claim that by doing so, the cop prevents the robber from entering P from outside of P.

Suppose that the robber occupies a vertex w that is not in P and is adjacent to $v_j \in P$. As G is bipartite, either $\text{dist}(v_0, w) = j - 1$ or $\text{dist}(v_0, w) = j + 1$. Furthermore, as P is geodesically closed, $\text{dist}(v_0, w) = j + 1$; otherwise, $(v_0, \ldots, w, v_j)$ is a geodesic path between two vertices in P that is not contained in P, a contradiction. Thus we see that $\text{dist}(v_0, w) = j + 1$, and the cop's strategy dictates that the cop occupy the vertex v_j. Therefore, the robber is unable to move to v_j, and hence the robber is unable to enter P. □

We will establish a path-switching lemma that is similar to Lemma 3.1.7. We illustrate the maneuver described in this lemma in Figure 3.6.

Lemma 3.1.14. *Let G be a bipartite planar graph with a fixed drawing in the plane. Let $P_1, P_2 \subseteq G$ be two (α, β)-paths in G, where $\alpha, \beta \in V(G)$. Let A be a component of $G \setminus (P_1 \cup P_2)$ enclosed by P_1 and P_2. Suppose the following hold:*

- *The robber occupies a vertex in A.*
- *For $i \in \{1, 2\}$, P_i is geodesic with respect to $P_i \cup A$.*

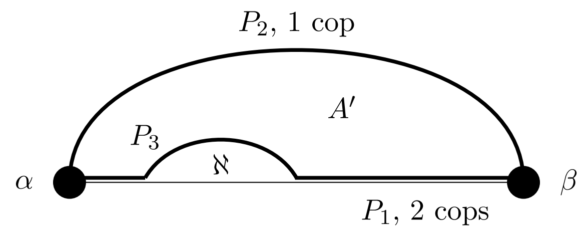

Figure 3.6: This figure illustrates the cop maneuver that is described in Lemma 3.1.14. We suppose that the robber is confined to the region enclosed by P_1 and P_2. Lemma 3.1.14 claims that there exists an (α, β)-path P_3 such that, when guarded by cops, the robber is confined to either the region A' or $\aleph$. If the robber is confined to A', we claim that the robber can be kept in A' by guarding P_2 with one cop and P_3 with two cops. If the robber is confined to $\aleph$, we claim that the robber can be kept in $\aleph$ by guarding P_1 with one cop and P_3 with one cop.

- *P_1 is not geodesically closed with respect to $P_1 \cup A$, and two cops C_1 and C_2 stalk the robber's P_1-shadow.*
- *P_2 is geodesically closed with respect to $P_2 \cup A$, and a single cop C_1' stalks the robber's P_2-shadow.*

Then, using at most one additional cop C_3, the robber can be confined to a smaller region $B \subsetneq A$. Furthermore, the cops can keep the robber in B by guarding two (α, β)-paths, each requiring at most three cops, and at least one of which can be guarded with two cops.

Proof. Let $P_1 = (\alpha = v_0, \ldots, v_k = \beta)$. We assume that the robber's P_1-shadow stalked by C_1 and C_2 is calculated with respect to v_0. As P_1 is not geodesically closed in $A \cup P_1$, we can choose a geodesic (w.r.t. $A \cup P_1$) path $S \not\subseteq P_1$, with endpoints $v_i, v_j \in P$ $(i < j)$ such that S is of shortest length out of all such geodesic paths. Let $P_1(i,j) = (v_i, v_{i+1}, \ldots, v_j)$, and let $S = (v_i, w_{i+1}, \ldots, w_{j-1}, v_j)$. Let $\aleph$ be the region enclosed by S and $P_1(i,j)$. We may let S be chosen out of all (v_i, v_j)-geodesics to minimize $\aleph$ and hence assume that $P_1 \cup \aleph$ does not contain any (v_i, v_j)-geodesic besides $P_1(i,j)$.

We claim that for any vertex $x \in A$, if x is adjacent to a vertex $v_l \in P(i,j) \setminus \{v_i, v_j\}$, then $\text{dist}(v_0, x) = l+1$. We know from Lemma 3.1.10 that $\text{dist}(v_0, x) \in \{l-1, l+1\}$. Suppose for the sake of contradiction that $\text{dist}(v_0, x) = l-1$. Then x must belong to a geodesic path $S' = (v_m, \ldots, x, v_l)$, where v_m is chosen to make S' as short as possible for this fixed value l. By the minimality of S, $m < i$, and so S' must cross S at some vertex w_p. As S and S' are geodesics, it then follows that $(v_i, \ldots, w_p, \ldots, v_l)$ is a geodesic shorter than S, a contradiction. Thus the claim is proven.

We define $P_3 = (v_0, v_1, \ldots, v_i, w_{i+1}, \ldots, w_{j-1}, v_j, \ldots, v_k)$, and we calculate the robber's P_3-shadow using v_0. We let an additional cop C_3 stalk the robber's P_3-shadow. (At this point we may

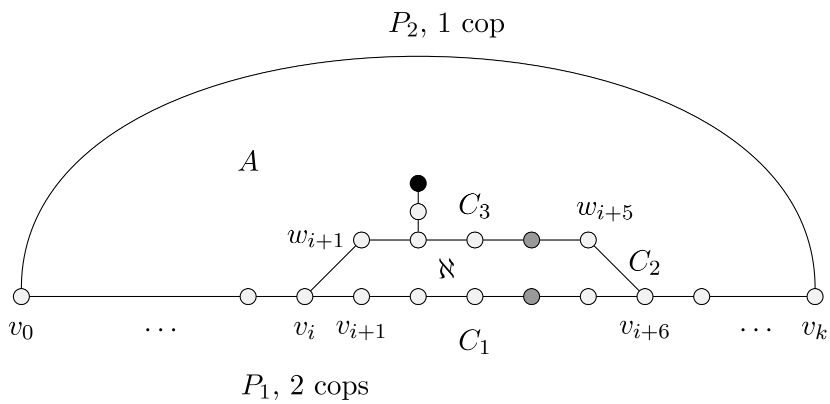

Figure 3.7: The figure shows a bipartite planar graph G in which the robber is restricted to a region A enclosed by two geodesic paths P_1 and P_2. The smaller region ℵ is enclosed by $(v_i, \ldots, v_{i+6})$ and $(v_i, w_{i+1}, \ldots, w_{i+5}, v_{i+6})$. In the figure, the cops attempt either to remove the region ℵ from the robber's territory or confine the robber to ℵ. To accomplish this, a cop C_3 is stalking the robber's P_3 shadow, where $P_3 = (v_0, \ldots, v_i, w_{i+1}, \ldots, w_{i+5}, v_{i+6}, \ldots, v_k)$. Here, the robber's vertex is colored black, the robber's P_1-shadow and P_3-shadow are colored grey. As C_3 is successfully stalking the robber's P_3-shadow, C_2 has stopped stalking the robber's P_1-shadow and has moved to v_{i+6}, where P_3 rejoins P_1. If the robber's shadow moves far enough left or right, then two cops will be stalking the robber's P_3 shadow, and the robber will not be able to move to P_3. This will either remove ℵ from the robber's territory or confine the robber to ℵ.

assume that the robber's P_3-shadow does not occupy a vertex of P_1, as otherwise P_3 would already be guarded, and the argument would be complete.) Then on each subsequent move after C_3 begins stalking the robber's P_3-shadow, we move C_2 on P_1 toward the vertex v_j. We note that C_2 will reach v_j before the robber's P_1-shadow can reach v_j. We let C_1 continue stalking the robber's P_1-shadow. We illustrate this point of the strategy in Figure 3.7.

We claim that if the robber's P_3-shadow ever occupies a vertex of P_1, then we have two cops in position to guard the path P_3 by stalking the robber's P_3-shadow. Indeed, if the robber's P_3 shadow occupies a vertex of P_1, then one of two cases must have occured.

Case 1: The robber's P_3-shadow moves from w_{i+1} to v_i. Then C_1 moves to v_{i-1}, and C_3 remains at w_{i+1}. The robber does not occupy a vertex of P_3, and C_1 and C_3 stalk the robber's P_3-shadow; hence C_1, C_3 are guarding P_3.

Case 2: The robber's P_3-shadow moves from w_{j-1} to v_j. Then C_2 moves to v_{j+1}, and C_3 remains at w_{j-1}. The robber does not occupy a vertex of P_3, and C_2 and C_3 stalk the robber's shadow on P_3; hence C_2 and C_3 are guarding P_3.

In both cases, two cops successfully guard P_3. If the robber's region is enclosed by P_1 and P_3, then the robber's region is $\aleph$. P_3 is guarded by two cops, and P_1 is geodesically closed with respect to $P_1 \cup \aleph$; hence the robber's region is enclosed by two paths, one of which is guarded by only one cop C_1, and the lemma is proven. (In fact, as before, P_3 is geodesically closed with respect to the robber's region, so we only need one cop to continue guarding P_3.) If the robber's region is enclosed by P_2 and P_3, then the robber is restricted to a smaller region $B \subsetneq A$, namely the region obtained by removing P_3 and $\aleph$ from A. P_3 is guarded by two cops, and P_2 is guarded by one cop by assumption, and the lemma is proven.

Hence we arrive at a point in which the robber's shadow must exist on the subpath

$$(w_{i+1}, w_{i+2} \ldots, w_{j-2}, w_{j-1}).$$

At this point, we move C_2 on P_3 toward the robber's shadow until C_2 and C_3 stalk the robber's P_3-shadow. At this point, one of three cases occurs:

Case 1: The robber occupies a vertex in the region enclosed by P_2 and P_3. In this case, P_2 is guarded by one cop, and we have two cops C_2 and C_3 guarding P_3. In this case, the proof is complete.

Case 2: The robber occupies a vertex in the region $\aleph$ enclosed by P_1 and P_3. In this case, we have two cops C_2 and C_3 guarding P_3. Additionally, P_1 is geodesically closed with respect to $P_1 \cup \aleph$, so C_1 can guard P_1 by continuing to stalk the robber's P_1-shadow. In this case, the proof is complete.

Case 3: The robber occupies a vertex w_l of P_3. In this case, C_2 occupies w_{l-1}, and C_3 occupies w_{l+1}. Then C_2 moves to w_l, and C_3 remains at w_{l+1}. C_1 continues stalking the robber's P_1-shadow. If the robber's P_3-shadow moves to w_l or w_{l+1}, then the robber moves off of P_3, and C_2 and C_3 can stalk the robber's P_3-shadow and bring us to Case 1 or Case 2. Otherwise, the robber's P_3-shadow moves toward v_i. In this case, C_2 and C_3 both move along P_3 toward v_i and push the robber's P_3-shadow toward v_i. By repeating this process, either the robber's P_3-shadow will make a movement that allows itself to be stalked, giving us Case 1 or Case 2, or the robber's P_3-shadow will reach w_{i+1}, with C_1 occupying v_i, C_3 occupying w_{i+1}, and C_2 occupying w_{i+2}. At this point, the robber must move to a vertex that is not on P_3, and the robber's P_3-shadow can be stalked on the next move, giving us Case 1 or Case 2. Thus in this case, the proof is complete. □

With this lemma in place, we can prove an upper bound for the surrounding cop number of bipartite planar graphs. As stated before, the proof of this theorem will follow the method of Theorem 3.1.8 very closely.

Theorem 3.1.15. *Let G be a planar bipartite graph. Then $s(G) \leq 4$.*

Proof. We will play the game in stages. In each Stage i of the game, we will have two paths P_1 and P_2 with common endpoints that are guarded by cops in such a way that the robber is unable to access the vertices of P_1 and P_2. We will define the robber's region R_i at Stage i as the component of $G \setminus (P_1 \cup P_2)$ that the robber occupies. We will show that at each stage, we can reduce the robber's region to a new region $R_{i+1} \subsetneq R_i$.

If G is a tree, then $s(G) = 2$, and we are done. Otherwise, we begin the game at Stage 1. As G is not a tree, there exists an edge $(uv) \in E(G)$ such that a geodesic path exists from u to v in the graph $G \setminus (uv)$. We let P_2 be the path (u, v), and we let P_1 be any (u, v)-path geodesic with respect to $G \setminus (uv)$. P_2 is geodesically closed with respect to G and thus can be guarded by one cop. P_1 is a geodesic path with respect to $G \setminus P_2$, and so P_1 can be guarded with two cops by stalking the robber's P_1-shadow. At this point, the component of the graph $G \setminus (P_1 \cup P_2)$ that the robber occupies is called R_1. We may redraw G so that R_1 is enclosed by P_1 and P_2.

We describe the strategy that we follow at each Stage i of the game.

Case 1: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is not geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$.

Let P_1 be guarded by two cops C_1, C_2. Let P_2 be guarded by a cop C_1'. Then by Lemma 3.1.14, we can use one extra cop C_3 and replace P_1 with an (α, β)-path P_1' so that at most two cops guard P_1', and such that there exists an (α, β)-path P_2' guarded by at most one cop such that the robber is restricted to a component R_{i+1} of $G \setminus (P_1' \cup P_2')$ with $R_{i+1} \subsetneq R_i$. Depending on whether or not P_1' is geodesically closed with respect to $P_1' \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

Case 2: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$. There exists an (α, β)-path in $P_1 \cup P_2 \cup R_i$ distinct from P_1 and P_2.

We choose a shortest (α, β)-path in $P_1 \cup P_2 \cup R_i$, and we call this path P_3. We guard P_3 with at most two cops. P_3 divides the region R_i into two parts, and hence the robber's region is either enclosed by P_1 and P_3, or the robber's region is enclosed by P_2 and P_3. In both cases, the robber's region is restricted to a region $R_{i+1} \subsetneq R_i$, and one of the paths enclosing R_{i+1} is guarded by at most one cop. Depending whether or not P_3 is geodesically closed with respect to $P_3 \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

Case 3: The robber's region R_i is enclosed by two guarded (α, β)-paths P_1 and P_2. P_1 is geodesically closed with respect to $P_1 \cup R_i$, and P_2 is geodesically closed with respect to $P_2 \cup R_i$. There exists no (α, β)-path in $P_1 \cup P_2 \cup R_i$ distinct from P_1 and P_2.

In this case, without loss of generality, P_1 has only one vertex $x \in P_1$ adjacent to the robber's region, and P_2 is not adjacent to the robber's region. Then the robber can be restricted to his region simply by placing a cop at x. We thus place two cops C_1 and C_2 at x and confine the robber to his region. If x has only one neighbor y in the robber's region, then we move C_1 to y and move C_2 to y on the next turn. We may continue this process until C_1 and C_2 occupy a vertex x that has at least two neighbors y, z in the robber's region R_i. Then we choose $P'_2 = (x, y)$, and we let P'_1 be an (x, y)-geodesic in $R_i \setminus (xy)$. We guard P'_1 with at most three cops and we may redraw R_i so that the robber is enclosed by P'_1 and P'_2. Then the robber is confined to a region $R_{i+1} \subsetneq R_i$, and depending on whether or not P'_1 is geodesically closed with respect to $P'_1 \cup R_{i+1}$, this brings us to Case 1, 2, or 3.

If we continue this process, we will eventually reach a point in which the robber's region consists of a single vertex, at which point the robber is surrounded. □

Unlike Theorem 3.1.8, we can show that the bound in Theorem 3.1.15 is tight.

Theorem 3.1.16. *There exists a planar bipartite graph G with $s(G) = 4$.*

Proof. Let $\mathcal{P}_5$ be the family of planar graphs with minimum degree 5. Let $H \in \mathcal{P}_5$, and let G be the graph obtained from H by subdividing each edge exactly once. We color vertices originally from H red, and we color vertices added as subdivisions blue. This is a proper coloring, and thus we see that G is bipartite.

We show that 3 cops are not sufficient to surround the robber on G. The robber uses the following strategy. The robber begins at a red vertex $r \in G$ and does not move until a cop occupies r. As red vertices have degree at least 5, the robber will not be surrounded before being captured. Suppose that a cop occupies r. As H has minimum degree 5, there are at least 5 red vertices within distance 2 of the robber. Let $v_1, \dots, v_5$ be 5 such red vertices. The cop occupying r does not have any of $v_1, \dots, v_5$ in its closed neighborhood. Furthermore, each of the other two cops does not have more than two of $v_1, \dots v_5$ in its closed neighborhood. Therefore, there exists a red vertex v_i within distance 2 of r that is not in the closed neighborhood of any cop. At this point, the robber uses the next two moves to move on the shortest path toward v_i. As v_i is not in the closed neighborhood of any cop, the robber will reach v_i before any cop reaches v_i. The robber then can repeat this strategy indefinitely and avoid being surrounded forever. Hence $s(G) \geq 4$, and by Theorem 3.1.15, $s(G) = 4$. □

3.1.4 Toroidal graphs

The aim of this section is to bound $s(G)$ for toroidal graphs G. We will mimic the strategy used by Lehner to show that every toroidal graph has cop number at most 3 [26]. The strategy of Lehner from [26] uses the notion of a *planar tiling* of a toroidal graph, which is informally defined as follows. Given a toroidal graph G, by definition, G has an embedding in the unit square S in the plane defined by $0 \leq x, y \leq 1$, with the lines $x = 0, x = 1$ identified, and with the lines $y = 0, y = 1$ identified. Then, a planar tiling G^T of G is obtained by copying the embedding of G in the unit square S into every square in the plane of the form $k_1 \leq x \leq k_1 + 1, k_2 \leq y \leq k_2 + 1$, for each

$k_1, k_2 \in \mathbb{Z}$. It will be useful to consider a planar tiling G^T of a toroidal graph G, as it will essentially allow us to circumvent the fact that G is not planar and apply strategies from planar graphs directly to G^T.

When we consider the game of surrounding cops and robbers on a toroidal graph G, rather than considering the graph G directly, we will consider an infinite planar tiling G^T of G, and we will attempt to capture the robber on G^T. When the robber chooses a vertex r of G, we will choose a corresponding vertex r_0 of G^T, and we will consider a large ball B around r_0. We will then guard geodesic paths from r_0 to the boundary of B, and we will divide the robber's region in B until the robber is restricted to a planar region of G^T. Then we will surround the robber by the planar strategy.

The following observation will be useful.

Observation 3.1.17. *In the proof of Theorem 3.1.8, if we allow P_2 to be guarded by 3 cops, then we have a strategy to capture the robber on a planar graph using 8 cops.*

We will establish some preliminaries.

Definition 3.1.18. We say that a cyclic order of integers in the form $(a, a+1, \ldots, a+m-1, a+m, a+m-1, \ldots, a+1)$ is called a *sawtooth order.*

Observation 3.1.19. *Let $A = (a, a+1, \ldots, a+m-1, a+m, a+m-1, \ldots, a+1)$ be a sawtooth order. Let $i, j \in A$, $i < j$. Then for any subsequence $B = (i, \ldots, j) \subseteq A$, replacing B with $(i, i+1, \ldots, j-1, j)$ gives a sawtooth order.*

We give an example of an application of the previous observation. Consider the sawtooth order $(1, 2, 3, 4, 5, 4, 3, 2)$. If we replace the subsequence $(1, 2, 3, 4, 5, 4)$ with the sequence $(1, 2, 3, 4)$, then we obtain a shorter sawtooth order $(1, 2, 3, 4, 3, 2)$.

Definition 3.1.20. Let H be a planar graph with an embedding in the plane, and let $v_0 \in V(H)$. Let $(h_1, h_2, h_3, \ldots, h_m)$ be a clockwise walk around the boundary of H. Then we define the following cyclic order:

$$(H, v_0)_{bd} := (dist(v_0, h_1), dist(v_0, h_2), dist(v_0, h_3), \ldots, dist(v_0, h_m)).$$

Lemma 3.1.21. *Let G be a planar graph with an embedding in the plane. Let $v_0, v_k \in V(G)$ be a pair of vertices at distance k, and let Π be the set of all geodesic paths from v_0 to v_k. There exist two geodesic paths P_1 and P_2 from v_0 to v_k enclosing an interior $\aleph$ such that $\bigcup \Pi \subseteq \aleph \cup P_1 \cup P_2$.*

The statement of Lemma 3.1.21 is illustrated in Figure 3.8.

Proof. We will prove the following statement $(*)$:

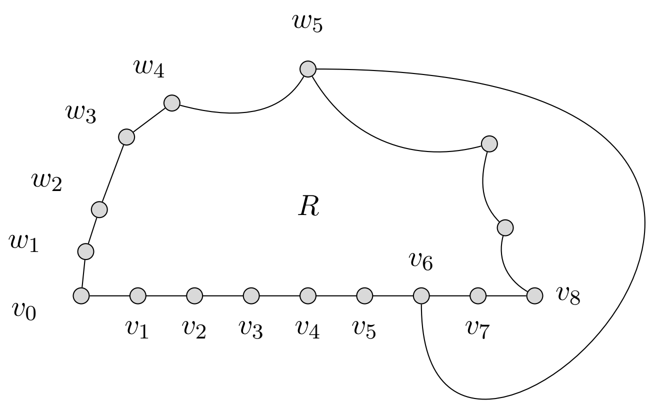

Figure 3.8: The figure shows part of a planar graph G embedded in the plane. Two vertices $v_0, v_8 \in V(G)$ at distance 8 are shown, and all (v_0, v_8)-geodesics belong to the region R, including its boundary. (Not all of these geodesics are shown in the figure.) In the figure, the two geodesics $P_1 = (v_0, w_1, w_2, w_3, w_4, w_5, v_6, v_7, v_8)$ and $P_2 = (v_0, v_1, \ldots, v_8)$ enclose all (v_0, v_8)-geodesics.

Let Π' be a proper subset of all geodesic paths from v_0 to v_k. Let $H \subseteq G$ be the union of all paths in Π', and let H inherit a planar embedding from G. Suppose that $(H, v_0)_{bd}$ is sawtooth. Then there exists a geodesic (v_0, v_k)-path $P' \notin \Pi'$ such that $(H \cup P', v_0)_{bd}$ is sawtooth. $(*)$

To show an example of the statement $(*)$, in Figure 3.8, we may take Π' to be all (v_0, v_8) geodesics contained in the region R, including its boundary. Then, we may find a path $P' = (v_0, w_1, w_2, w_3, w_4, w_5, v_6, v_7, v_8)$ such that $(H \cup P', v_0)_{bd}$ gives the sawtooth order

$$(0, 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1).$$

This statement $(*)$ implies that we can let Π' begin with a single geodesic path P_0, for which $(P_0, v_0)_{bd} = (0, 1, \ldots, k-1, k, k-1, \ldots, 1)$ is clearly sawtooth, and we can add new (v_0, v_k)-geodesics to Π' one at a time until Π' contains all (v_0, v_k)-geodesics—that is, until $\Pi' = \Pi$. Furthermore, in this way we can ensure that $(\bigcup \Pi, v_0)_{bd}$ is sawtooth. Then we let $(h_1, \ldots, h_l, \ldots, h_m)$ be a clockwise ordering of the boundary of $\bigcup \Pi$ such that $(dist(v_0, h_1), \ldots, dist(v_0, h_l))$ is increasing and such that $(dist(v_0, h_l), \ldots, dist(v_0, h_m))$ is decreasing. We let q_1 be a (v_0, h_1)-geodesic of $\bigcup \Pi$, and we let q_1' be an (h_m, v_k)-geodesic of $\bigcup \Pi$. We let q_2 be a (v_0, h_m)-geodesic of $\bigcup \Pi$, and we let q_2' be an (h_l, v_k)-geodesic of $\bigcup \Pi$. We see that $P_1 := q_1 \cup (h_1, \ldots, h_l) \cup q_1'$ and $P_2 := q_2 \cup (h_m, \ldots, h_l) \cup q_2'$ are (v_0, v_k)-geodesics of G that enclose a region $\aleph$ such that $\bigcup \Pi \subseteq \aleph \cup P_1 \cup P_2$. Then the lemma is proven. Thus we aim to prove the statement $(*)$.

Let Π' be any proper subset of all geodesic paths from v_0 to v_k. Let $H \subseteq G$ be the union of all paths in Π', and let H inherit a planar embedding from G. Suppose that $(H, v_0)_{bd}$ is sawtooth. Let $\aleph$ be the region including and enclosed by the boundary of H. Let $P^* \notin \Pi'$ be a geodesic path from v_0 to v_k. We write $P^* = (w_0, \ldots, w_k)$.

We make the following claim. Suppose that $w_i \in V(P^*)$ belongs to the boundary of H. Then $\text{dist}_H(v_0, w_i) = i$. To prove the lemma, suppose that $\text{dist}_H(v_0, w_i) < i$. Then there exists a path $P \subseteq H$ from v_0 to w_i of length $i' < i$, implying that $P \cup P^*(w_i, v_k)$ is a walk from v_0 to v_k of length less than k, a contradiction. Suppose, on the other hand, that $\text{dist}_H(v_0, w_i) > i$. Then there exists a path $P \subseteq H$ from w_i to v_k of length less than $k - i$, namely $(w_i, \ldots, w_k)$, implying that $P^*(v_0, w_i) \cup P$ is a walk from v_0 to v_k of length less than k, a contradiction. Thus we see that $\text{dist}_H(v_0, w_i) = i$.

If $P^* \subseteq \aleph$, then clearly $(H, v_0)_{bd}$ is sawtooth. Otherwise, P^* is not contained in $\aleph$. Let $w_{i+1} \in P^*$ be the first vertex of P^* that does not belong to $\aleph$, and let w_j be the first vertex in P^* after w_{i+1} that belongs to $\aleph$. By the previous discussion, $\text{dist}_H(v_0, w_i) = i, dist_H(v_0, w_j) = j$. As P^* is a geodesic, this implies that the subpath $q = (w_i, \ldots, w_j)$ is of length $j - i$ and that there exist paths $p = (v_0, \ldots, w_i) \subseteq H, p' = (w_j, \ldots, v_k)$ respectively of lengths i and $k - j$. Therefore, $P' := p \cup q \cup p'$ is a geodesic path in G that is not included in $\aleph$. We show that $(H \cup P', v_0)_{bd}$ is sawtooth.

As $(\text{dist}(v_0, w_i), \text{dist}(v_0, w_{i+1}), \ldots, \text{dist}(v_0, w_{j-1}), \text{dist}(v_0, w_j)) = (i, i+1, \ldots, j-1, j)$, the cyclic order $(H \cup P', v_0)_{bd}$ is obtained from $(H, v_0)_{bd}$ by replacing a subsequence $(i, \ldots, j)$ with $(i, i + 1, \ldots, j - 1, j)$. By Observation 3.1.19, this leaves us with another sawtooth order. Hence $(H \cup P', v_0)_{bd}$ is also sawtooth.

Thus the statement $(*)$ holds, and we see that there exist (v_0, v_k)-geodesic paths P_1 and P_2 that enclose a region $\aleph$ such that $\bigcup \Pi \subseteq \aleph \cup P_1 \cup P_2$. Thus the lemma is proven. □

We now have most of our tools in place for proving an upper bound on $s(G)$ for a toroidal graph G. As stated before, rather than working directly with an embedding of G on the torus, we will consider an infinite planar tiling G^T of G. A planar tiling of G is a type of *planar cover* of G, which is defined as a planar embedding of a graph G in the plane in which a vertex $v \in V(G)$ may be represented by multiple (and possible infinitely many) points in the plane, and an edge $e \in E(G)$ may be represented by multiple (and possibly infinitely many) points in the plane. In a planar cover of G, the standard rules for embeddings still apply; that is, two drawn edges may not cross, and a drawn edge may not include a point representing a vertex. The following definition and lemma will be essential to our strategy for toroidal graphs.

Definition 3.1.22. Let G be an infinite graph. We say that G has *polynomial growth* if there exists a polynomial f such that for any vertex $v \in G$, the number of vertices at distance exactly d from v is at most $f(d)$.

The following lemma is proven in [26].

Lemma 3.1.23. *Let G be a finite toroidal graph. Then there is an infinite planar cover of G with polynomial growth.*

We are now ready to prove an upper bound for the surrounding cop number of toroidal graphs.

Theorem 3.1.24. *Let G be a toroidal graph. Then $s(G) \leq 8$.*

Proof. Let $|G| = n$. We will essentially use the strategy of Lehner from [26] with some slight modifications. Rather than considering G directly, we will consider an infinite planar tiling G^T of G with polynomial growth (by Lemma 3.1.23). We note that there exists a natural projection function $\pi : V(G^T) \to V(G)$. If we play a game of cops and robbers on G, then at each point in the game, each cop C has an infinite number of preimages in G^T given by $\pi^{-1}(C)$. Furthermore, the robber r has an infinite number of preimages $\pi^{-1}(r)$. The robber r and a cop C occupy the same vertex in G if and only if some element of $\pi^{-1}(C)$ occupies the same vertex as some element of $\pi^{-1}(r)$ in G^T. Furthermore, the robber is surrounded on G if and only if some element of $\pi^{-1}(r)$ occupies a vertex $x \in G^T$ such that all neighbors of x are occupied by cop preimages.

In our strategy, rather than aiming to let the cop preimages on G^T surround any arbitrary robber preimage, we will focus on surrounding one predetermined robber preimage. This restriction can only make the game more difficult for the cops, and therefore any upper bound on the number of cops needed to surround a specific preimage of the robber on G^T also gives an upper bound for the number of cops needed to surround the robber on G.

From this point onward, we will identify the robber with the G^T preimage of the robber that we wish to surround. Let the robber begin the game at $r_0 \in G^T$. We let D be a large value that is to be determined later. We define $B = B_{r_0}(D)$ as the ball of radius D centered at r_0. Let Q be the circumference of this ball; that is, let Q be the set of vertices at distance exactly D from r_0. First, we will need a lemma. In the following lemma, we assume that the robber is never too close to the boundary of B, because in fact the cops will be able to capture the robber before the robber comes close to escaping B.

Lemma 3.1.25. *Let $v \in Q$. Let P be a geodesic path in B from r_0 to v. Suppose that the robber is at distance $d < D - 3n$ from r_0. Then P can be guarded by three cop preimages as in Lemma 3.1.5 before the robber reaches a distance of $d + 2n$ from r_0.*

Proof. Let $P = (r_0, v_1, \ldots, v_D = v)$.

As the diameter of G is at most n, three cops preimages C_1, C_2, and C_3 can reach v_{d+n+1}, v_{d+n+2}, and v_{d+n+3} within n moves. By assumption, when C_1, C_2, and C_3 reach their positions, the shadow of the robber on P is on the subpath $(r_0, v_1, \ldots, v_{d+n})$. Then C_1, C_2, and C_3 move along P toward the robber's P-shadow until they reach a position to stalk the robber's P-shadow. C_1, C_2 , and C_3 reach such a position before the robber reaches v_{d+2n}. □

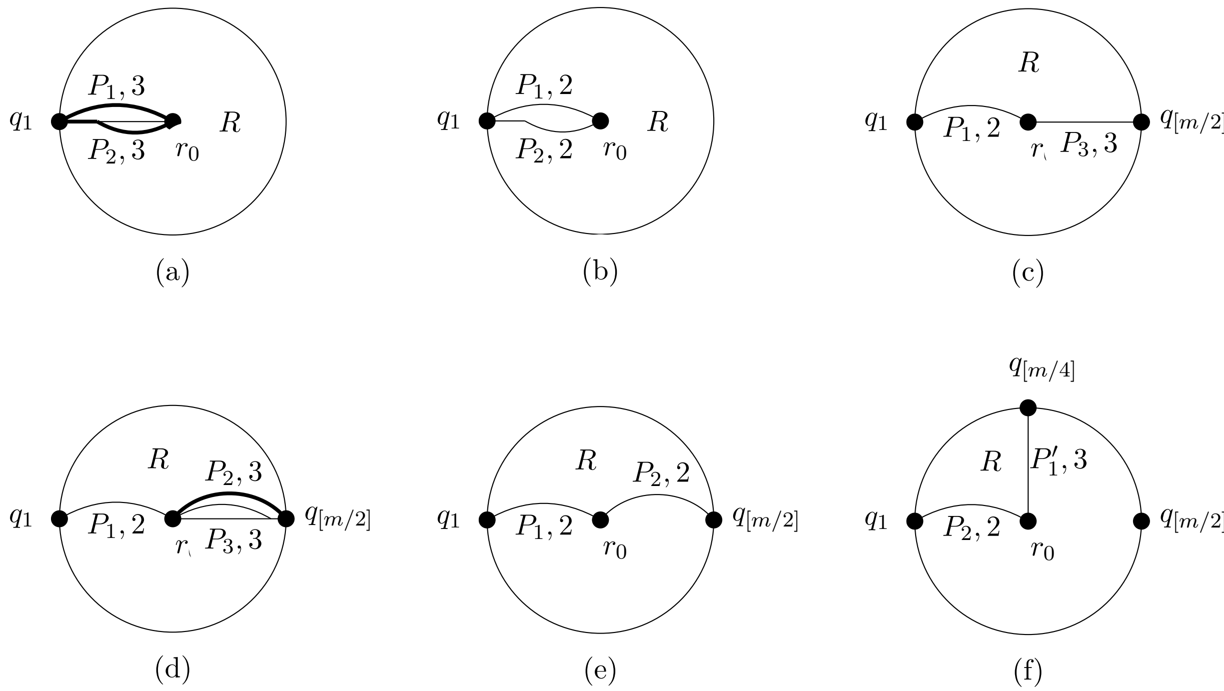

Figure 3.9: The figures show the initial maneuvers of the strategy of Theorem 3.1.24. In each figure, the circle represents the ball B around r_0, and R represents the robber's region in B. Figure (a) shows six cops guarding two (r_0, q_1)-geodesics P_1, P_2 that enclose all (r_0, q_1)-geodesics. Figure (b) shows that if R is not enclosed by P_1 and P_2, then P_1, P_2 can be guarded with four cops. Figure (c) shows R being divided by an $(r_0, q_{[m/2]})$-geodesic P_3. Figure (d) shows three additional cops guarding another $(r_0, q_{[m/2]})$-geodesic P_2 that separates R from all $(r_0, q_{[m/2]})$-geodesics. Figure (e) shows that the paths P_1, P_2 adjacent to R are geodesically closed with respect to R and thus need only four cops to guard. Figure (f) shows R being divided again by an $(r_0, q_{[m/4]})$-geodesic P_1'. The maneuvers shown in (d), (e), (f) can be repeated until the robber is contained in a planar region of G^T.

Now, let $|Q| = m$. Let $(q_1, \dots, q_m)$ be the cyclic ordering of the elements of Q according to the planar embedding of G^T. First, we use Lemma 3.1.21 to compute two geodesics P_1 and P_2 from r_0 to q_1 that enclose all (r_0, q_1)-geodesics. Then we use six cops to guard P_1 and P_2, as in Figure 3.9 (a). If the robber is enclosed by P_1 and P_2, then the robber is restricted to a planar region by six cops, and then we can win the game with eight cops by Observation 3.1.17. Otherwise, P_1 and P_2 are geodesically closed with respect to the robber's territory in B, and we only need four of our six cops to continue to guard P_1 and P_2 (see Figure 3.9 (b)).

Next, we choose a geodesic P_3 from r_0 to $q_{[m/2]}$ and guard P_3 with 3 cops (see Figure 3.9 (c)). Without loss of generality, the robber is restricted to a region of B enclosed by P_1 and P_3, which are guarded by a total of 5 cops.

From this point onward, we repeat the following procedure, which will recursively restrict the robber's territory until the robber is confined to a planar region of G^T.

Let the robber be confined to a region of B bounded by geodesics $P_1 = (r_0, \dots, q_a)$ and $P_3 = (r_0, \dots, q_b)$. Suppose further that P_1 is guarded by two cops. Using Lemma 3.1.21, we compute a geodesic P_2 from r_0 to q_a such that P_2 and P_3 enclose all (r_0, q_a)-geodesics in the robber's territory (see Figure 3.9 (d)). We then use three cops to guard P_2. If the robber is in the interior of $P_2 \cup P_3$, then the robber is confined to a planar region of B with six cops, and we win the game with eight cops by Observation 3.1.17. Otherwise, P_2 is geodesically closed with respect to the robber's territory and can be guarded by two cops. Now the robber is confined to a region of B that is enclosed by P_1 and P_2, each of which is guarded by two cops (see Figure 3.9 (e)). Next, we use three cops to guard a geodesic P_1' from r_0 to $q_{[(a+b)/2]}$ (see Figure 3.9 (f)). Now we see that, without loss of generality, the robber's territory in B is enclosed by a geodesic P_1' from r_0 to $q_{[(a+b)/2]}$ and a geodesic P_2 from r_0 to q_b. Furthermore, we see that P_2 is guarded by two cops. Thus the initial conditions of our procedure are satisfied again, and we can repeat our procedure.

We repeat this process until the robber's territory in B is enclosed by a geodesic P_1 from r_0 to q_a and a geodesic P_2 from r_0 to q_{a+1}. We see that reaching this point requires at most $\log m$ iterations of the procedure. As (q_i) are in cyclic order with respect to the drawing of G^T, once P_1 and P_2 are guarded as such, the robber is confined to a planar region of G^T and can be captured with eight cops by Observation 3.1.17.

As our procedure relies on Lemma 3.1.25, it remains only to show that the robber's distance from r_0 cannot reach $D - 3n$ before the procedure is complete. Any time a path is guarded, the robber is only able to move a distance of at most $2n$ away from r_0. To initialize our procedure, we require our cops to guard three paths. In each iteration of our procedure, we guard two paths. Furthermore, we execute at most $\log m = \log |Q|$ iterations of our procedure. Therefore, the total number of paths guarded in our strategy is at most $3 + 2 \log m$, and hence when our procedure is finished, the robber is at a distance of at most $6n + 4n \log m$ from r_0.

We now assign a value to D. We set $D = e^{kn}$, where k is a sufficiently large constant. Recall that m is bounded by the polynomial expression $f(m)$. We can bound the polynomial $f(m)$ by another polynomial $f^*(m)$ of the form Am^α such that $f(m) \leq f^*(m)$ for $m \geq 1$. Then the distance of the robber from r_0 is at most

$$6n + 4n\log m \leq 6n + 4n\log(AD^\alpha) \leq 6n + 4n(\log A + \alpha kn) < e^{kn} - 3n$$

for sufficiently large k. Therefore, we can choose D large enough that the robber stays sufficiently far from the boundary of B before being confined to a planar region of G^T. Finally, once the robber is confined to a planar region in G^T with at most six cops, we can surround the robber with at most eight cops by Observation 3.1.17. □

We show that the upper bound in Theorem 3.1.24 is close to best possible.

Theorem 3.1.26. *There exists a toroidal graph G such that $s(G) \geq 7$.*

Proof. Let H be a 6-regular tiling of equilateral triangles on the torus. Let G be a graph obtained by adding a vertex v at each face f of H and adding an edge from v to every vertex of f. Clearly G is toroidal. We call the vertices from H *original vertices*, and we call the additional vertices *face vertices*. We let the robber play on the original vertices, each of which has degree 12 in G.

Suppose there are six cops. We note that the robber cannot be surrounded on an original vertex. Suppose that a cop moves to occupy r. At this point, at most five neighboring original vertices of r are occupied by cops. Thus there exists an unoccupied original vertex adjacent to r, and the robber moves to this vertex. The robber repeats this process indefinitely and wins. Therefore, six cops are insufficient to surround the robber on G, and $s(G) \geq 7$. □

We can also use the methods in the proof of Theorem 3.1.24 to prove an upper bound on the surrounding cop number of bipartite toroidal graphs. Furthermore, we can show that this upper bound is tight.

Theorem 3.1.27. *Let G be a bipartite toroidal graph. Then $s(G) \leq 5$.*

Proof. The proof technique is nearly identical to that of Theorem 3.1.24. When G is bipartite, however, paths that are geodesic with respect to the robber's region can be guarded with two cops, and paths that are geodesically closed with respect to the robber's region can be guarded with one cop. In Figure 3.10, we give a sketch of how the ideas of Theorem 3.1.24 can be applied to a bipartite toroidal graph G to restrict the robber to a planar region of the planar tiling G^T using five cops. Then, by following the strategy of Theorem 3.1.15, the robber can be surrounded using five cops. □

Theorem 3.1.28. *There exists a bipartite toroidal graph G with $s(G) = 5$.*

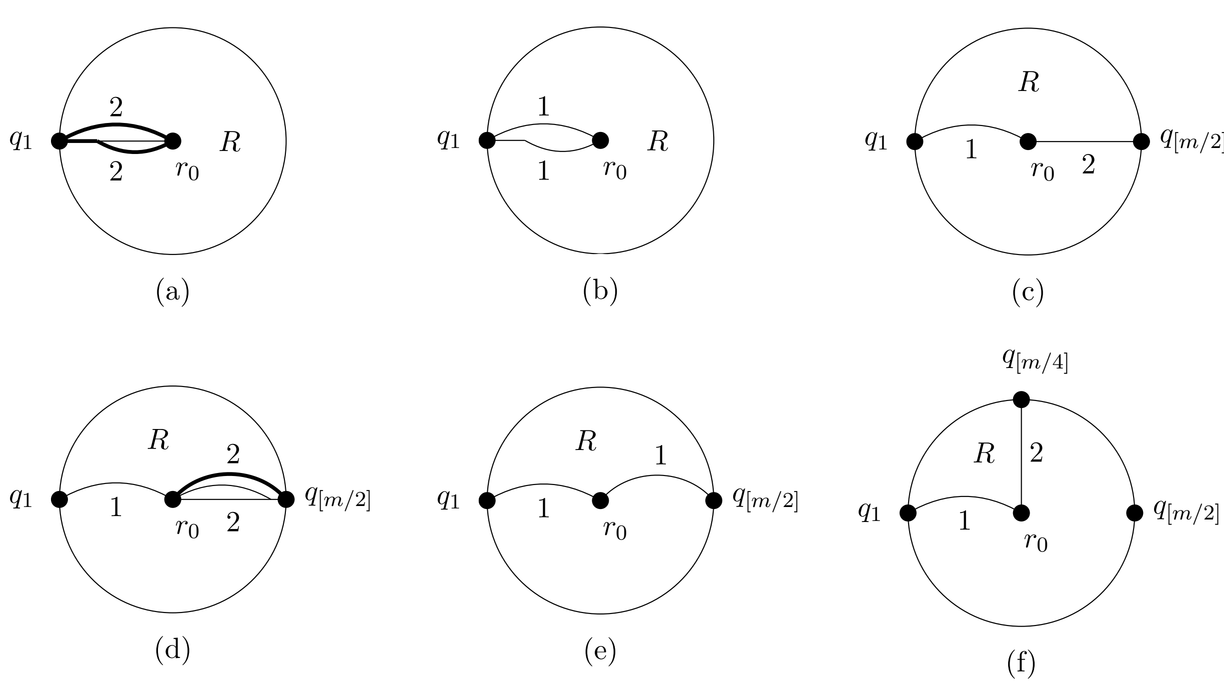

Figure 3.10: The figures show the initial maneuvers of the strategy of Theorem 3.1.27. The figures have the same meanings as those of Figure 3.9. In these figures, however, we let two cops guard a general path that is geodesic with respect to R, and we show one cop guarding a path that is geodesically closed with respect to R.

Proof. Let C_m, C_n be cycles respectively of length $m, n \geq 2$. Let $H = C_m \square C_n$—that is, the Cartesian product of an m-cycle and an n-cycle—and let H have a grid embedding on the torus. We construct a graph G as follows: at each face f of H, we add a quadrilateral, and we add an edge from each vertex of the quadrilateral to a vertex of f in a way that does not introduce crossings. G is clearly bipartite. We show that $s(G) = 5$.

By Theorem 3.1.27, $s(G) \leq 5$. We call the vertices of G that originate as vertices of H *original vertices.* We note that each original vertex is of degree 8 and has four neighbors that are original vertices. We show that four cops are not sufficient to surround the robber. We let the robber begin at an original vertex r. As r has degree 8, robber cannot be surrounded without a cop first occupying r. If a cop occupies r, then at most three neighboring original vertices of r are occupied by cops. Therefore, there exists an unoccupied neighboring original vertex of r, and the robber can move to this original vertex. The robber can repeat this process indefinitely. Therefore, exactly five cops are needed to surround the robber on G. □

3.1.5 Outerplanar graphs

In this section, we consider the game of surrounding cops and robbers on *outerplanar graphs*, which are defined as graphs that may be embedded in the plane so that all vertices are incident to a single face. Outerplanar graphs are typically drawn in the plane with all vertices incident to the outer face, which gives them this name. Clarke shows in [14] that the cop number of outerplanar graphs is at most 2. In this section, we will show that for outerplanar graphs G, $s(G) \leq 3$.

The following lemma is given by Chartrand and Harary in [13] and is fairly straightforward.

Lemma 3.1.29. *Let G be a two-connected outerplanar graph. Then G is Hamiltonian, and in any outerplanar embedding of G, the facial walk of the exterior face of G is a Hamiltonian cycle.*

We will prove that $s(G) \leq 3$ for two-connected outerplanar graphs G. Then we will use this to show that $s(G) \leq 3$ for all outerplanar graphs G.

For a graph G with an outerplanar embedding, we will say that an edge that is adjacent to the exterior face of G is called an *exterior edge.* We will say that any edge that is not an exterior edge is an *interior edge.*

Lemma 3.1.30. *Let G be a two-connected outerplanar graph with a fixed outerplanar embedding. Let $(v_0, \ldots, v_{n-1})$ be the Hamiltonian cycle given by the exterior facial walk of G. Let $v_i v_j$ be an interior edge of G. Then $G \setminus \{v_i, v_j\}$ has two components, which are induced by $(v_{i+1}, \ldots, v_{j-1})$ and $(v_{j+1}, \ldots, v_{i-1})$ (where addition is considered modulo n).*

Proof. Clearly the vertices $\{v_{i+1}, \ldots, v_{j-1}\}$ belong to a single component of $G \setminus \{v_i, v_j\}$, as do the vertices $\{v_{j+1}, \ldots, v_{i-1}\}$. Note that neither of these sets is empty, as v_i, v_j are not adjacent on the outer face of G. Furthermore, the vertices $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_i)$ form a cycle C. As G is outerplanar, it follows that any non-exterior edge adjacent to a vertex of $\{v_{i+1}, \ldots, v_{j-1}\}$ must

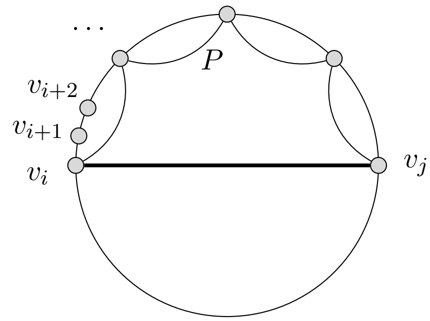

Figure 3.11: The figure shows a two-connected outerplanar graph. If two cops C_1, C_2 occupy the endpoints v_i and v_j of the bolded edge, then a robber occupying a vertex of $\{v_{i+1}, \ldots, v_{j-1}\}$ cannot cross the bold edge v_iv_j. Furthermore, there exists a strategy involving a third cop C_3 by which the geodesic path P may be guarded, further restricting the robber to a smaller region of the outerplanar graph.

be drawn in the interior of C and therefore cannot have an endpoint that does not belong to C. We hence see that $\{v_{i+1}, \ldots, v_{j-1}\}$ is a maximal connected component of $G \setminus \{v_i, v_j\}$, as is $\{v_{j+1}, \ldots, v_{i-1}\}$ by a similar argument. This proves the lemma. □

Corollary 3.1.31. *Let G be a two-connected outerplanar graph with a fixed outerplanar embedding. Let $(v_0, \ldots, v_{n-1})$ be the Hamiltonian cycle given by the exterior facial walk of G. Suppose that two cops occupy vertices v_i, v_j, where $v_iv_j \in E(G)$. Then the robber is restricted to either the vertex set $\{v_{i+1}, \ldots, v_{j-1}\}$ or the vertex set $\{v_{j+1}, \ldots, v_{i-1}\}$.*

Proof. If v_iv_j is an exterior edge of G, then without loss of generality, $\{v_{i+1}, \ldots, v_{j-1}\}$ contains all vertices of $G \setminus \{v_i, v_j\}$, and clearly the robber is restricted to $G \setminus \{v_i, v_j\}$. If v_iv_j is an interior edge of G, then by Lemma 3.1.30, the robber is restricted to a component of $G \setminus \{v_i, v_j\}$, and the result follows. □

Next, the following lemma shows that in a two-connected outerplanar graph, the robber's region can be reduced using three cops. Corollary 3.1.31 and Lemma 3.1.32 are illustrated in Figure 3.11

Lemma 3.1.32. *Let G be a two-connected outerplanar graph with a fixed outerplanar embedding. Let $(v_0, \ldots, v_{n-1})$ be the Hamiltonian cycle given by the exterior facial walk of G. Let v_iv_j be an edge of G, and suppose that two cops occupy v_i and v_j, restricting the robber to vertex set $X = \{v_{i+1}, \ldots, v_{j-1}\}$. Then there exists a strategy involving one additional cop by which two cops guard adjacent vertices v_k, v_l and restrict the robber to a vertex set $\{v_{k+1}, \ldots, v_{l-1}\} \subsetneq X$.*

Proof. We call the cop at v_i C_1, and we call the cop at v_j C_2. Let $H = G[v_i, v_{i+1}, \ldots, v_{j-1}, v_j] \setminus (v_iv_j)$. We note that the robber's territory in G is a subgraph of H. Let $P = (u_0, \ldots, u_q)$ be a (v_i, v_j)-geodesic in H. Note that $q \geq 2$. As before, the robber's P-shadow must either stay put in P on each turn or move to an adjacent vertex of P. Therefore, a cop C_3 can stalk the robber's P-shadow after a finite number of moves. We let C_3 execute such a strategy.

After C_3 begins stalking the robber's P-shadow, we let C_1 move toward the robber's P-shadow on each turn until reaching u_{k-1}, where u_k is the position of the robber's P-shadow. C_1 then moves to u_{k-1} on each subsequent turn, where u_k is a subsequent position of the robber's P-shadow. When C_1 executes this strategy, the robber will not be able to reach $v_i = u_0$; in order to reach u_0, the robber would first have to move to a vertex at distance one from u_0, at which point C_1 would occupy u_0. At the same time, we let C_2 move toward the robber's P-shadow until reaching u_{k+1}, where u_k is the position of the robber's shadow (or until reaching u_q if the robber's shadow occupies u_q). C_2 then moves to u_{k+1} on each subsequent turn, where u_k is a subsequent position of the robber's shadow (or to u_q if the robber's shadow occupies u_q). By a similar argument, the robber will not be able to reach v_j while C_2 executes this strategy. Therefore, when C_1 and C_2 execute this strategy, the robber's territory in G does not increase.

After C_1, C_2, and C_3, successfully execute their strategies, C_1, C_2, and C_3 together stalk the robber's P-shadow and guard the path P, and the robber is restricted to a component of $H \setminus \{u_0, \ldots, u_q\}$. For $0 \leq i \leq q-1$, let $P(u_i, u_{i+1})$ be the graph induced by the vertices of the exterior path from u_i to u_{i+1} that does not include any vertex u_j for $j \notin \{i, i+1\}$. As $q \geq 2$, $P(u_i, u_{i+1})$ is uniquely defined.

If $H \setminus \{u_0, \ldots, u_q\}$ is empty, then the robber has no territory, implying that the robber is surrounded by C_1, C_2, C_3. Otherwise, by Lemma 3.1.30, the components of the graph $H \setminus \{u_0, \ldots, u_q\}$ must be of the form $P(u_i, u_{i+1}) \setminus \{u_i, u_{i+1}\}$. Therefore, the robber is restricted to a region $P(u_i, u_{i+1}) \setminus \{u_i, u_{i+1}\}$. As P is guarded, we can move two cops to occupy u_i, u_{i+1} before the robber can reach either of u_i, u_{i+1}. We move two cops in such a way to u_i, u_{i+1}. The vertices u_i, u_{i+1} can be written as v_k, v_l. Furthermore, with v_k, v_l guarded, the robber is restricted to a vertex set $\{v_{k+1}, \ldots, v_{l-1}\}$, which is a proper subset of $\{v_{i+1}, \ldots, v_{j-1}\}$. Thus the lemma is proven. □

Corollary 3.1.33. *Let H be a two-connected subgraph of an outerplanar graph G. Suppose that two cops occupy adjacent vertices $v_i, v_j \in H$. Then there exists a strategy involving three cops that removes H from the territory of the robber.*

Proof. By following the strategy in Lemma 3.1.32, we can iteratively reduce the number of vertices in H that belong to the robber's territory. Note that each time we apply Lemma 3.1.32, the exterior of the robber's territory forms a cycle, so the robber's territory is a two-connected outerplanar graph. We can continue this process until no vertex of H belongs to the robber's territory. Then the robber is either surrounded or prevented from entering H. □

Corollary 3.1.33 gives us the tool that we need to devise a strategy that uses three cops to capture the robber on an outerplanar graph. Corollary 3.1.33 tells us that in an outerplanar graph G, three cops have a strategy to take away two-connected blocks from the robber repeatedly and "push"

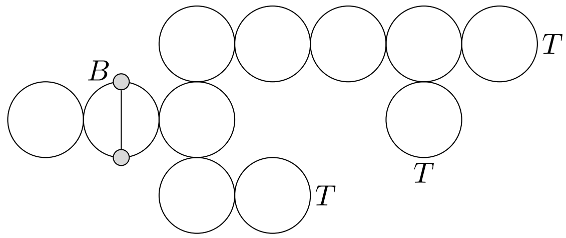

Figure 3.12: The figure shows an outerplanar graph G, with each circle in the figure representing a block of G. If two cops occupy the endpoints of the shown edge in the block B, and if the robber occupies a vertex to the right of this edge, then there exists a strategy involving three cops that forces the robber to the right out of the block B and pushes the robber through the block-cut tree of G until reaching a terminal block of G. In the figure, the terminal block that the robber might reach are labelled with the letter T. Once the robber reaches a terminal block of G, the three cops will have a strategy to surround the robber.

the robber toward a terminal block of the block-cut tree of G—that is, a maximal two-connected component of G with a single cut-vertex. (For a more detailed description of the block-cut tree of a graph, see [**?**, Section 5 what].) Once the robber reaches a terminal block of G, when the cops take away the robber's territory in this last block, the robber will be surrounded. We illustrate this idea in Figure 3.12.

Theorem 3.1.34. *Let G be an outerplanar graph. Then $s(G) \leq 3$.*

Proof. If G is a tree, then by [11], $s(G) \leq 2$. Otherwise, we name our cops C_1, C_2, and C_3. We begin the game by choosing a maximal two-connected subgraph $H \subseteq G$. We place two cops at two adjacent vertices $v_i, v_j \in H$. We show that at each point of the game, we can reduce the robber's territory R_i. We consider two cases:

Case 1: No two cops occupy the endpoints of an edge with both endpoints adjacent to the robber's territory, and a cop occupies a cut-vertex adjacent to the robber's territory.

Without loss of generality, let C_1 occupy a cut-vertex x adjacent to the robber's territory. We then let C_2 guard a neighbor y of x for which y belongs to the robber's territory. This reduces the robber's territory to $R_{i+1} \subsetneq R_i$ and depending on whether or not y is a cut-vertex of G, this brings us to Case 1 or 2.

Case 2: Two cops occupy the endpoints of an edge with endpoints u, v adjacent to the robber's territory.

Let $H \subseteq G$ be a maximal two-connected subgraph containing u, v. By Corollary 3.1.33, C_1, C_2, and C_3 have a strategy to remove all vertices of H from the robber's territory. Furthermore, as u, v are both adjacent to the robber's territory, the robber's territory has at least one vertex in

H. Therefore, by removing the vertices of H from the robber's territory, we reduce the robber's territory to a region $R_{i+1} \subsetneq R_i$. If the robber is not surrounded during the execution of such a strategy, then the robber is forced to leave H, and there exists a cut-vertex $x \in H$ adjacent to the robber's territory. As the vertices of H are guarded, the robber can be prevented from accessing x; that is, a cop can reach x before the robber. We let a cop move to x before the robber reaches x, and this brings us to Case 1.

By repeatedly reducing the robber's territory in this way, we will eventually reach a point in which the robber's territory is a single vertex. At this point, the robber is surrounded, and the cops win the game. □

Finally, we show that this bound is tight, even for bipartite outerplanar graphs.

Theorem 3.1.35. *There exists a bipartite outerplanar graph G with $s(G) = 3$.*

Proof. Let $G = P_1 \Box P_3$ be the grid with 8 vertices. We note that G is bipartite and outerplanar. We show that 2 cops cannot surround a robber on G.

We note that G has four degree 3 vertices that form a 4-cycle C. We let the robber begin the game at a vertex r of C. As the vertices of C have degree 3, the robber cannot be surrounded at r without a cop moving to occupy r. If a cop occupies r, then at most one of the robber's C-neighbors is occupied by a cop, and thus the robber can move to a neighboring vertex in C. The robber can repeat this strategy indefinitely. Therefore, $s(G) \geq 3$. By Theorem 3.1.34, it follows that $s(G) = 3$. □

3.1.6 Graphs of higher genus and graphs that exclude a minor

We will briefly consider the surrounding cop number bounds of graphs of higher genus and graphs that exclude a minor. In the traditional game of cops and robbers, strategies for capturing a robber on a graph of higher genus are similar to strategies for planar and toroidal graphs; that is, the cops capture the robber by guarding geodesic paths and iteratively reducing the robber's region, as in [31] and [33]. Currently, the best known general strategy for capturing the robber on a graph of genus g is given by Bowler et. al., who prove the following theorem.

Theorem 3.1.36 ([8])**.** *Let G be a graph of genus g. Then $c(G) \leq \lfloor \frac{4}{3}g + \frac{10}{3} \rfloor$.*

Theorem 3.1.36 is based on guarding geodesic paths in order to reduce the robber's region to a single vertex, and the strategy of Theorem 3.1.36 needs at most $\lfloor \frac{4}{3}g + \frac{10}{3} \rfloor$ geodesic paths to be guarded at one time. This immediately gives us the following result.

Theorem 3.1.37. *Let G be a graph of genus g. Then $s(G) \leq 4g+10$. Furthermore, if G is bipartite, then $s(G) \leq \lfloor \frac{8}{3}g + \frac{20}{3} \rfloor$.*

Proof. We apply the strategy of Theorem 3.1.36, using three cops to guard a geodesic path in the general case, and using two cops to guard a geodesic path if G is bipartite. □

More generally, we may also consider families of graphs that exclude a minor. A theorem of Andreae from [2] shows that if G is a graph that does not contain H as a minor, then for any vertex $h \in V(H)$ that is not adjacent to a leaf of H, $c(G) \leq |E(H-h)|$. Furthermore, the strategy of Andreae is carried out solely by guarding geodesic paths. Therefore, by using three cops to guard each path in Andreae's strategy (or two cops for the bipartite case), the strategy can be adapted to the surrounding variant of cops and robbers to give us the following theorem.

Theorem 3.1.38. *Let H be a graph, and let $h \in V(H)$ be a vertex of H that has no neighbor of degree one. If G is a graph that does not contain H as a minor, then $s(G) \leq 3|E(H-h)|$. If G is also bipartite, then $s(G) \leq 2|E(H-h)|$.*

3.1.7 Open questions

We have shown that for planar graphs G, $s(G) \leq 7$, and $s(G)$ may be as large as 6. Furthermore, for toroidal graphs G, we have shown that $s(G) \leq 8$, and $s(G)$ may be as large as 7. Thus we may naturally ask the following questions:

- Does there exist a planar graph G for which $s(G) = 7$?
- Does there exist a toroidal graph G for which $s(G) = 8$?

We conjecture that both of these questions have a negative answer.

Furthermore, González and Mohar show that three cops are enough to capture a robber on a planar graph even when only two cops are allowed to move on each turn [19]. These authors also show that four cops are sufficient to capture a robber on a planar graph even when each cop is required to move on every turn [18]. We may thus ask the following questions as well:

- How many cops are required to surround a robber on a planar (toroidal) graph when at most k cops may move on each turn?
- How many cops are required to surround a robber on a planar (toroidal) graph when all cops are required to move on each turn?

3.1.8 Acknowledgments

I thank Sebastián González Hermosillo de la Maza for fruitful discussions. I also thank Florian Lehner for making several helpful corrections to an earlier version of this section. Finally, I thank Bojan Mohar for several suggestions that improved the quality of the thesis.

3.2 Normal Cayley graphs

In this section, we will consider the game of surrounding cops and robbers on normal Cayley graphs. We will show that for a group G and a generating set $S \subseteq G$ closed under conjugation, the surrounding cop number of $\mathrm{Cay}(G, S)$ satisfies $s(\mathrm{Cay}(G, S)) = |S|$, which is equal to the minimum degree of $\mathrm{Cay}(G, S)$ and hence best possible.

As before, we describe walks on $\mathrm{Cay}(G, S)$ in terms of moves. Suppose that a cop or robber moves throughout the elements of G by traversing edges of $\mathrm{Cay}(G, S)$. We may imagine that whenever our cop or robber occupies an element $a \in G$, it may choose a generator $s \in S$ and then move to sa; that is, the cop or robber may transform its position using left multiplication by s. Hence any move that the cop or robber makes along an edge of $\mathrm{Cay}(G, S)$ may be interpreted as the selection of a generator $s \in S$ and a resulting group transformation. We will use this interpretation when we describe strategies of cops and robbers on Cayley graphs.

When we consider a game of cops and robbers on a normal Cayley graph $\mathrm{Cay}(G, S)$, whenever a cop or robber moves from a vertex $a \in G$ to a vertex sa for an element $s \in S$, we say that the cop or robber *plays the move* s. In other words, playing a move s is equivalent to applying left multiplication by s to the current position of the cop or robber. When considering a game of cops and robbers on a normal Cayley graph $\mathrm{Cay}(G, S)$, we will add the following modification. Suppose that the robber plays a move $s \in S$. Then, after each cop plays its subsequent move in response to the robber, we transform the position $a \in G$ of the robber and each cop by the transformation $a \mapsto x^{-1}ax$. As $\mathrm{Cay}(G, S)$ is a normal Cayley graph, the transformation $a \mapsto x^{-1}ax$ is an automorphism of game positions on $\mathrm{Cay}(G, S)$ and thus does not affect the strategy of either player. This transformation will make it easier to keep track of the "difference" between the position of the robber and a cop.

Recall that a *normal Cayley graph* is a Cayley graph constructed with a group G and a generating set S satisying $S = S^{-1}$ and $S = g^{-1}Sg$ for each $g \in G$. We will show that on a normal Cayley graph $\mathrm{Cay}(G, S)$, $|S|$ cops have a strategy to surround a robber. In fact, we will prove a stronger result. We will consider a nonempty set $T \subseteq S$, and we will only allow the robber to play moves in T, while still allowing cops to use all moves of S. We will show that in this modified game of surrounding cops and robbers, $|T|$ cops have a strategy to occupy all vertices tr, where $t \in T$ and r is the position of the robber. Letting $T = S$, this is equivalent to saying that $s(\mathrm{Cay}(G, S)) = |S|$.

Theorem 3.2.1. *Let G be a group, and let $S \subseteq G$ be a generating set of G that is closed under conjugation. Let $T \subseteq S$ be a nonempty set. Then in the modified game of surrounding cops and robbers in which the robber must play moves from T, there exists a winning strategy with $|T|$ cops.*

Proof. When $T = S$, the theorem tells us that $s(\mathrm{Cay}(G, S)) = |S|$, as the upper bound follows from the theorem statement, and the lower bound follows from the fact that the minimum degree of $\mathrm{Cay}(G, S)$ is equal to $|S|$.

We induct on $|T|$. When $|T| = 1$, the robber must move along a directed cycle D. A single cop may capture the robber by moving through D in the opposite direction that the robber moves.

Now, we consider a set $T \subseteq S$ with $|T| \geq 2$. We choose a cop C and first let C move to a vertex of $\langle T \rangle r_0$, where $r_0 \in G$ is initial the position of the robber. The cop C will reach a group element

$c \in G$ such that $c = t_k \dots t_1 r$, where r is the new position of the robber, and $t_1, \dots, t_k \in T$. We will show by induction on k that the $|T|$ cops have a strategy to capture the robber.

If $k = 1$, then $c = t_1 r$, and hence the robber may not play t_1. Furthermore, for any move $x \in T \setminus \{t_1\}$ that the robber plays, C may copy the robber and play x. Then, after applying the transformation $a \mapsto x^{-1}ax$ to the positions of C and the robber, and letting c' and r' respectively denote the new positions of C and the robber, $c'r'^{-1} = x^{-1}(xc)(xr)^{-1}x = cr^{-1} = t_1$, and hence $c' = t_1 r'$. This shows us that after C plays x, the position of C in relation to the robber does not change. Therefore, on each subsequent move, the robber must play a move of $T \setminus \{t_1\}$. Hence, by induction on $|T|$, the remaining $|T| - 1$ cops apart from C have a strategy to capture the robber.

Otherwise, suppose that $k \geq 2$. Recall that r represents the current position of the robber and that C occupies a vertex $c = t_k \dots t_1 r$. If the robber plays t_k, then we let C stay put at its current vertex. Then, after applying the transformation $a \mapsto t_k^{-1} a t_k$, and letting c' and r' respectively represent the new positions of the cop and the robber, $c'r'^{-1} = t_k^{-1} c (t_k r)^{-1} t_k = t_k^{-1} c r^{-1}$, and hence $c' = t_{k-1} \dots t_1 r'$. Then, the cops have a winning strategy by induction on k. On the other hand, if the robber chooses not to move, then C plays the move t_k^{-1}. Then, with c' representing the new position of C, $c' = t_{k-1} \dots t_1 r$, and the cops again win by induction on k.

Hence, we see that in order not to lose the game by induction immediately, the robber must play a move $x \in T, x \neq t_k$, in which case C plays x. Letting c' and r' respectively represent the new positions of the cop and the robber, then after applying the transformation $a \mapsto x^{-1}ax$, $c'r'^{-1} = x^{-1}(xc)(xr)^{-1}x = cr$, and hence $c' = t_k \dots t_1 r'$. Thus we see that if the robber plays a move $x \neq t_k$ and C copies the robber, then the position of C relative to the robber does not change. This implies that on each subsequent move, the robber must continue to play a move of $T \setminus \{t_k\}$. Therefore, C essentially forces the robber to play a strategy of the modified game of cops and robbers in which the robber may only use moves in $T \setminus \{t_k\}$. Hence, by induction on $|T|$, the remaining $|T| - 1$ cops apart from C have a strategy to win the game. $\square$

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