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THE LATTICES OF VARIETIES AND PSEUDOVARIETIES OF BAND MONOIDS

by

Shelly Luanne Wismath

B.Sc. Queen's University 1976

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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of

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ABSTRACT

The structure of the lattice of all varieties of bands has been completely determined (independently) by Birjukov, Fennemore and Gerhard. In this thesis the structure of this lattice is used to determine the structure of two related lattices: the lattice LBM of varieties of band monoids and the lattice LFBM of pseudovarieties of finite band monoids.

Chapters I and II provide an introduction and background to this problem. This includes a discussion of varieties and equational classes, semigroups and monoids, and pseudovarieties and generalized varieties of semigroups and monoids. Of special importance are three theorems of Ash which relate pseudovarieties, generalized varieties and varieties.

In Chapter III a function is defined from the lattice LB of varieties of bands to the lattice LBM. This function is shown to be a surjective lattice homomorphism, and so by determining exactly which varieties in LB are identified by the homomorphism, the shape of the image lattice LBM is determined. Finally a function from LBM to LFBM is defined, and shown to be a lattice isomorphism, thus establishing that LBM and LFBM have the same structure.

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I. Introduction

Varieties of bands have been studied by Kimura [10], Yamada [14], Petrich [12] and others. The structure of the lattice of all varieties of bands was completely determined by Birjukov [2], Fennemore [6] and Gerhard [8]. There are two other lattices closely related to this one: the lattice of varieties of band monoids, and the lattice of pseudovarieties of band monoids. The main result of this thesis is the determination of the structure of these two lattices.

We begin in Chapter II with a study of varieties, pseudovarieties and the related concept of generalized varieties. Section 1 gives a brief introduction to the area of universal algebra, leading up to a discussion of varieties and equational classes. Birkhoff's Theorem, stating that varieties are in fact the same as equational classes, is then quoted without proof. Since the only algebras to be studied here are semigroups and monoids, Section 2 gives definitions, examples and some facts about these two algebraic structures. The example of particular interest here is that of a band, a semigroup which satisfies the identity $x^2 = x$. The class $\underline{B}$ of all bands is a variety, and associated with it is the lattice $\underline{LB}$ of all varieties of bands.

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In Section 3 finiteness conditions are considered, leading to the definition of pseudovarieties of finite algebras. The interest in finite algebras comes from the field of automata theory, where there is a close correspondence between finite automata and finite monoids. This correspondence, and the resultant algebraic automata theory, led Eilenberg to define pseudovarieties of finite monoids and semigroups [5]. This was then extended to pseudovarieties of arbitrary algebras and to generalized varieties by Ash [1], who also proved several theorems relating these various concepts. These theorems are considered at the end of this section.

In Chapter III we turn to the specific question of varieties of bands and band monoids. Fennemore has shown in [6] that there are a countably infinite number of varieties of bands, and that each such variety is defined by one identity besides $x^2=x$; and he has given a complete picture of the lattice of such varieties. Since semigroups and monoids are so closely related, it is natural to try to use this lattice to obtain information about the lattice of varieties of band monoids. Given any variety $\underline{V}$ of semigroups (bands), the collection of monoids in $\underline{V}$ is a variety of (band) monoids. Thus we may define a function Mon on the class of varieties of bands by taking Mon($\underline{V}$) to be the set of monoids in $\underline{V}$, for any variety $\underline{V}$ of bands. In Section 1 it is shown that Mon is a lattice homomorphism from the lattice of varieties of bands (abbreviated

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as LB) onto the lattice of varieties of band monoids (LBM).
Therefore in order to study the structure of LBM, we look at the
congruence induced on LB by Mon. This is done in Sections 2 and
3, first for the base of the lattice LB and then for the
inductively defined part of the lattice. By showing which
varieties are identified under Mon and which are not, we obtain
a picture of the lattice of all varieties of band monoids.

In the final section of Chapter III this process is taken
one step further, and pseudovarieties are looked at. One of the
theorems of Ash mentioned earlier states that a collection of
finite algebras is a pseudovariety if and only if it consists of
the finite members of a generalized variety. Since generalized
varieties are varieties, this says in particular that if V is a
variety of (band) monoids, then the collection Fin(V) of finite
monoids in V is a pseudovariety of (band) monoids. This suggests
the definition of a function Fin from the lattice LBM of
varieties of band monoids to the lattice LPBM of pseudovarieties
of band monoids. It is shown that this function is a lattice
isomorphism, thus establishing the structure of the lattice of
pseudovarieties of band monoids.

II. Varieties, Pseudovarieties and Generalized Varieties

This chapter presents the background necessary for a study of the lattices of varieties and pseudovarieties of band monoids. It begins with a brief survey of the universal algebraic concepts needed to express Birkhoff's Theorem relating varieties and equational classes. This is done in general terms, for abstract algebras. The concepts needed are then looked at more specifically in terms of semigroups and monoids, the algebras to be considered here, and some examples and properties of these algebras are given. The final section then introduces pseudovarieties and generalized varieties, with theorems by Eilenberg and Ash relating pseudovarieties and ultimately, equational classes, and pseudovarieties, varieties and generalized varieties.

Section 1: Varieties and Equational Classes

This section presents some basic definitions and results from the area of universal algebra. Only enough background for later use in the discussion of lattices of varieties and pseudovarieties is given here, and all results are stated without proof. For a detailed study of this area, including proofs of the results here, the reader is referred to Burris and

Sankappanavar [3].

For any non-empty set A and any non-negative integer n, an n-ary operation on A is a function from A^n to A. An operation is said to be finitary if it is n-ary for some integer n. A type of algebras is a set $\mathcal{J}$ of function symbols, each of which has associated with it a non-negative integer called its arity. An algebra $\mathcal{A}$ of type $\mathcal{J}$ is a pair (A;F) consisting of a non-empty set A and a collection F of finitary operations on A indexed by $\mathcal{J}$. Thus for each n-ary function symbol f in $\mathcal{J}$, F contains an n-ary operation $f^{\mathcal{A}}$. The elements of F are called the fundamental operations of $\mathcal{A}$ while the set A is called the underlying set of $\mathcal{A}$. When no confusion can arise as to the underlying set involved, the fundamental operations are denoted by f rather than $f^{\mathcal{A}}$.

We now define the four important concepts of subalgebra, homomorphism, direct product and quotient algebra.

Let $\mathcal{A}$ = (A;F) be an algebra, and let B be a non-empty subset of A. Then $\mathcal{B}$ = (B;F) is called a subalgebra of $\mathcal{A}$ if each fundamental operation of $\mathcal{B}$ is the restriction to B of the corresponding operation of $\mathcal{A}$, and B is closed under each such operation.

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Let $\mathcal{A} = (A;F)$ and $\mathcal{B} = (B;F)$ be two algebras of the same type. A __homomorphism__ from $\mathcal{A}$ to $\mathcal{B}$ is a function $\emptyset$ from A to B, with the property that for any n-ary function symbol f in $\mathcal{F}$ and any $a_1,\ldots,a_n$ in A,

$$\emptyset(f^{\mathcal{A}}(a_1,\ldots,a_n)) = f^{\mathcal{B}}(\emptyset(a_1),\ldots, \emptyset(a_n)).$$

If the function $\emptyset$ is surjective, $\mathcal{B}$ is called a homomorphic image of $\mathcal{A}$.

Suppose that $\{\mathcal{A}_i = (A_i;F):i \in I\}$ is a family of algebras of the same type $\mathcal{F}$ for some index set I. The __direct__ __product__ of the $\mathcal{A}_i$'s is the algebra $\prod_{i \in I}\mathcal{A}_i = (\prod_{i \in I}A_i;F)$, with underlying set the cartesian product of the sets A_i. The operations on this set are defined co-ordinate-wise; that is, for f any n-ary function symbol in $\mathcal{F}$ and $a_1,\ldots,a_n$ in $\prod_{i \in I}A_i$, and for any i in I,

$$f^{\pi\mathcal{A}_i}(a_1,\ldots,a_n)(i) = f^{\mathcal{A}_i}(a_1(i),\ldots,a_n(i)).$$

A __congruence__ on an algebra $\mathcal{A} = (A;F)$ is an equivalence relation θ on A which satisfies the compatibility property: for any n-ary function f in F and for all a_i,b_i in A, if (a_i,b_i) is in θ for $1 \le i \le n$, then $(f(a_1,\ldots,a_n),f(b_1,\ldots,b_n))$ is in θ. The equivalence class of an element a of A under the equivalence relation θ will be denoted by a/θ, and the collection of all the equivalence classes on A by A/θ. When θ is a congruence, and f

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an n-ary function in F, the relation $f^{A/\theta}$ defined by $f^{A/\theta}(a_1/\theta,\ldots,a_n/\theta) = f^A(a_1,\ldots,a_n)/\theta$ is a function. This allows the definition of a new algebra $A/\theta = (A/\theta;F)$, called the _quotient algebra_ of A by θ, of the same type as A.

The concepts defined above can now be used to define the three-class operators H, S and P. For any class K of algebras of the same type,

H(K) = {A : A is a homomorphic image of an algebra in K},

S(K) = {A : A is a subalgebra of an algebra in K},

and

P(K) = {A : A is a direct product of algebras in K}.

Finally, a _variety_ is defined as any class of algebras of the same type which is closed under the three operators H, S and P. It will henceforth be assumed that any class of algebras under discussion contains only algebras of one type.

Proposition 2.1.1: Any intersection of varieties is a variety.

Proposition 2.1.2: For any given type of algebra, the collection of all algebras of that type is a variety.

From these two results it follows that for any class K of algebras, the intersection of all varieties containing K is the

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unique smallest variety which contains K. This variety is called the variety generated by K, and will be denoted by $\underline{V}(K)$. Another consequence is that for any variety $\underline{V}$, the family of all varieties of the same type which are contained in $\underline{V}$ forms a lattice, to be denoted by $\underline{LV}$. In particular, the family of all varieties of a given type forms a lattice, using $\underline{W} \cap \underline{U}$ and $\underline{V}(\underline{W} \cup \underline{U})$ as the meet and join respectively of any two varieties $\underline{W}$ and $\underline{U}$.

Theorem 2.1.3: For any class K of algebras, $\underline{V}(K) = HSP(K)$.

In order to develop an equivalent characterization of varieties, we now look at term and free algebras. Let X be a non-empty set of variables, and let $\mathcal{F}$ be a type of algebras. The set T(X) of terms of type $\mathcal{F}$ over X is defined as the smallest set containing X and any 0-ary function symbols from $\mathcal{F}$, and having the property that if f is an n-ary function symbol in $\mathcal{F}$ and $p_1, \ldots, p_n$ are in T(X), then $f(p_1, \ldots, p_n)$ is in T(X). The elements of T(X) are called terms. A term p is called an n-ary term if n or fewer variables appear in p; if the variables appearing in p are among $x_1, \ldots, x_n$, then p is written as $p(x_1, \ldots, x_n)$.

The term algebra of type $\mathcal{F}$ over X is $\mathcal{J}(X) = (T(X); F)$, where the operations in F satisfy $f^{\mathcal{J}(X)}(p_1, \ldots, p_n) = f(p_1, \ldots, p_n)$. This algebra $\mathcal{J}(X)$ is called the free algebra of type $\mathcal{F}$ over X; it has

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the universal mapping property over X for the class of all algebras of type $\mathcal{F}$.

If $\mathcal{A} = (A;F)$ is any algebra of type $\mathcal{F}$, then there is associated with each n-ary term $p = p(x_1,\ldots,x_n)$ of type $\mathcal{F}$ a term function p^A on A. This term function is the n-ary operation on A inductively defined as follows: if $p = x_i$, for some $1 \leq i \leq n$, then $p^A(a_1,\ldots,a_n) = a_i$; if $p = f(p_1(x_1,\ldots,x_n),\ldots,p_K(x_1,\ldots,x_n))$ for some k-ary function symbol f in $\mathcal{F}$, then $p^A(a_1,\ldots,a_n) = f(p_1^A(a_1,\ldots,a_n),\ldots,p_K^A(a_1,\ldots,a_n))$. Intuitively, the term function p^A may be thought of as producing from any $a_1,\ldots,a_n$ in A the element of A obtained by replacing the variables $x_1,\ldots,x_n$ of the term p by $a_1,\ldots,a_n$ respectively.

If p and q are n-ary terms of type $\mathcal{F}$ over a non-empty set X, the expression $p = q$ is called an <u>identity</u> of type $\mathcal{F}$ over X. An algebra $\mathcal{A} = (A;F)$ of type $\mathcal{F}$ is said to <u>satisfy</u> the identity $p = q$ if for any elements $a_1,\ldots,a_n$ of A,

$$p^A(a_1,\ldots,a_n) = q^A(a_1,\ldots,a_n).$$

<u>Proposition 2.1.4:</u> Let $\mathcal{A}$ be an algebra of type $\mathcal{F}$ and let $p = q$ be an identity of type $\mathcal{F}$ over X. Then $\mathcal{A}$ satisfies $p = q$ if and only if for any homomorphism $\emptyset$ from $\mathcal{J}(X)$ to $\mathcal{A}$, $\emptyset(p) = \emptyset(q)$.

Informally, a homomorphism φ from $J(X)$ to A may be thought of as picking out elements in A to be used as values for the variables in X. For n-ary terms p and q, the condition that A satisfies p=q then means that for any choice of elements $a_1,\ldots,a_n$ from A, the elements of A obtained by replacing x_i by a_i, for $1 \leq i \leq n$, in each of p and q are the same. In this sense we usually say that A satisfies the identity p=q if for any substitution $x_1 = a_1, \ldots, x_n = a_n$, we have $p(a_1,\ldots,a_n) = q(a_1,\ldots,a_n)$.

A class K of algebras is said to satisfy the identity p=q if every algebra in K does; K satisfies a set of identities if K satisfies every identity in the set. Given a set X of variables, we may define the set Id(K) of all identities over X which are satisfied by K. Conversely, given a set S of identities of type J over X, we let E(S) be the class of all algebras of type J which satisfy S. A class K is called an equational class if K=E(S) for some set S of identities. The relationship between equational classes and varieties is the content of Birkhoff's Theorem:

Theorem 2.1.5 (Birkhoff): A class K is an equational class if and only if it is a variety.

Section 2: Semigroups and Monoids

The algebras to be considered in the rest of this thesis are semigroups and monoids. These are defined, and some of their basic properties presented, in this section. Further details may be found in Clifford and Preston [4] or Petrich [11].

A semigroup is a set S with a single binary operation . on S which satisfies the associative law. Formally, this is denoted as $(S;\{.\})$, but usually it is written as $(S,.)$ and the underlying set S is itself called the semigroup. We allow the trivial semigroup, in which S is the empty set. A monoid is a semigroup M with an identity element 1_M for the operation .; we speak of the monoid M or $(M;.,1_m)$, and omit the subscript on the 1 when no confusion is possible. In fact the binary operation symbol . is usually omitted when referring to a product of elements in a semigroup or monoid: ab is used instead of a.b for the product of elements a and b. The notation for the product of an element a with itself n times is shortened to a^n.

Some of the universal algebraic concepts introduced in the previous section may now be interpreted for semigroups and monoids. A homomorphism between two semigroups S and T is a function $\emptyset$ from S to T such that $\emptyset(rs) = \emptyset(r)\emptyset(s)$, for all r and s in S. For a homomorphism of monoids, say $\emptyset:M\longrightarrow N$, it is also necessary that $\emptyset(1_M) = 1_N$. A subset N of a monoid M is a

submonoid of M if N is a monoid under the same binary operation and with the same identity element as M.

We now consider free semigroups and monoids. Given any non-empty set X, X^+ is defined to be the collection of all elements $x_1 \ldots x_n$, for $n > 0$ and $x_1, \ldots, x_n$ in X. The set X is then called an _alphabet_, and the elements of X^+ are called _words_ on X. Under the operation of concatenation of words, the set X^+ becomes a semigroup containing X. It is the free semigroup on X, and it has the universal mapping property for semigroups: for any semigroup S and any function f from X to S, there is a unique homomorphism g from X^+ to S which agrees with f on X. The unique word of length zero, denoted by 1, acts as an identity on words from X^+, so $X^* = X^+ \cup \{1\}$ is the free monoid on X.

An element s of a semigroup S which satisfies s.s=s is called an _idempotent_. A _band_ is a semigroup in which every element is an idempotent; that is, a semigroup which satisfies the identity $x^2 = x$. Idempotent elements are in abundant supply in finite semigroups, as the next proposition shows. A proof of this proposition may be found in Eilenberg [5].

Proposition _2.2.1:_ Let S be a finite semigroup, and let s be an element of S. Then there is a positive integer k such that s^k is an idempotent.

We may use the free semigroup X^+ on a set X to produce the free band on X. This is done by defining a relation B as the set of all pairs (x^2,x), for x in X^+, and then taking θ to be the smallest congruence on X^+ to contain B. The quotient semigroup X^+/θ is then a band, since for any x/θ in X^+/θ,

$$(x/\theta)^2 = x^2/\theta = x/\theta,$$

by the definition of θ. This band is called the <u>free</u> <u>band</u> on X. An important fact about the size of the free band is the following result, proven by Green and Rees [9]:

<u>Proposition</u> <u>2.2.2</u>: When X is a finite set, the free band on X is also finite.

We now give some examples of varieties of semigroups and monoids. From Proposition 2.1.2, the classes $\underline{S}$ of all semigroups and $\underline{M}$ of all monoids are each varieties. For any set T of identities for semigroups or monoids, the notation $\underline{V}(T)$ will be used for the variety of semigroups satisfying T, while $\underline{VM}(T)$ will denote the variety of monoids satisfying T. When T consists of a single identity $p=q$, this notation will be simplified to $\underline{V}(p=q)$ or $\underline{VM}(p=q)$. Thus $\underline{V}(x=y)$ is the trivial variety, consisting only of the semigroups $\{1\}$ and the empty set; $\underline{VM}(x=x)$ is the variety $\underline{M}$. Of particular importance in what follows are the varieties $\underline{B}=\underline{V}(x^2=x)$ of bands, and $\underline{BM}=\underline{VM}(x^2=x)$ of band

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monoids. We shall be interested in the associated lattices $\underline{LB}$
and $\underline{LBM}$.

It is obvious that semigroups and monoids are very similar.
A semigroup S may be a monoid; if it is not, a monoid may be
formed from it by adjoining to S a new element 1 which is not
already in S, and extending the operation of S to $S \cup \{1\}$ by
specifying that a.1=1.a=a for any a in $S \cup \{1\}$. Then $S \cup \{1\}$ will
be a monoid. Thus given any semigroup S, there is associated
with it the monoid S^1 defined as follows:

$$S^1 = \begin{cases} S, & \text{if S is a monoid} \\ S \cup \{1\}, & \text{if S is not a monoid.} \end{cases}$$

Another method of producing a monoid from a semigroup
involves the use of idempotents.

<u>Proposition</u> <u>2.2.3:</u> Let S be a semigroup and e an idempotent in
S. Then the set $eSe = \{ese: s \in S\}$ is a subsemigroup of S which is
a monoid.

Varieties of semigroups and monoids are also closely
related. Given a variety $\underline{V}$ of monoids, there is a least variety
of semigroups which contains it: this is the variety of
semigroups generated by the collection $\underline{V}$. Conversely, given a
variety $\underline{V}$ of semigroups, there are several associated varieties

of monoids.

Proposition 2.2.4: Let $\underline{V}$ be any variety of semigroups. Then $\underline{V} \cap \underline{M}$ is a variety of monoids. In particular, if $\underline{V}$ is a variety of bands, then $\underline{V} \cap \underline{M}$ is a variety of band monoids.

Proof: $\underline{V} \cap \underline{M}$ is the collection of all monoids in $\underline{V}$. Because $\underline{V}$ is a variety, a submonoid of a monoid in $\underline{V}$ is a monoid in $\underline{V}$; a homomorphic image of a monoid in $\underline{V}$ is a monoid in $\underline{V}$; and any direct product of monoids in $\underline{V}$ is a monoid in $\underline{V}$. Hence $\underline{V} \cap \underline{M}$ is a variety of monoids.

Proposition 2.2.5: Let T be any set of identities for semigroups or monoids. Then $\underline{V}(T) \cap \underline{M} = \underline{VM}(T)$.

Proof: This follows from the definitions of $\underline{V}(T)$ and $\underline{VM}(T)$.

Another method of going from a variety $\underline{V}$ of semigroups to a variety of monoids is to form the collection $A = \{S^1 : S \in \underline{V}\}$. This will not be a variety, but we may consider the variety of monoids generated by A, which will be denoted by $\underline{V}^1$. If a semigroup S is in a variety $\underline{V}$, it is not in general true that S^1 is also in $\underline{V}$. However it will be shown later that for certain varieties $\underline{V}$ of bands, S in $\underline{V}$ does imply that S^1 is also in $\underline{V}$.

Section 3: Pseudovarieties and Generalized Varieties

The motivation for the study of finite semigroups and monoids comes from the field of automata theory. The connection between finite automata and finite monoids, and an algebraic view of automata theory, is presented very completely by Eilenberg in [5]. Eilenberg has defined pseudovarieties of finite monoids and finite semigroups, and proved a parallel of Birkhoff's Theorem for pseudovarieties. Ash [1] has extended this to define pseudovarieties and generalized varieties of arbitrary algebras, and proved several theorems relating varieties, pseudovarieties and generalized varieties. In this section we examine some of these concepts and theorems. Detailed verifications of all the results in this section may be found in Eilenberg [5] and Ash [1].

Let X be any finite non-empty set, and let A be any subset of the free monoid X^*. The set A is called a recognizable subset of X^* if there is a finite automaton which recognizes A. Associated with A is a congruence on X^* called the syntactic congruence of A, and denoted by $\widehat{A}$. It is defined as follows: for any words s and t in X^*, $s \widehat{A} t$ if and only if for all u and v in X^*, usv is in A if and only if utv is in A. That this does define a congruence on X^* is easily verified. The quotient monoid $X^*/\widehat{A}$ is then called the syntactic monoid for A. The

relationship between recognizability of A and the syntactic monoid for A is given by the following proposition.

Proposition 2.3.1: A subset of X^* is recognizable if and only if its syntactic monoid is finite.

Similar definitions may be made, for a subset A of the free semigroup X^+, of the syntactic congruence and the syntactic semigroup of A; the analogue of Proposition 2.3.1 for semigroups will then hold. Thus from the field of automata theory comes the motivation for studying collections of finite semigroups and monoids.

The operators H and S used in defining varieties preserve finiteness, but the product operator P does not. This leads to the definition of the finite-product class operator P_f: for any class K of algebras, $P_f(K)$ is the class of algebras formed by taking direct products of finitely many members of K. Eilenberg has defined a pseudovariety of semigroups (or an S-variety) as a collection of finite semigroups closed under H, S and P_f. Similarly, a pseudovariety of monoids (or an M-variety) is a collection of finite monoids closed under H, S and P_f.

The basic properties of pseudovarieties are analogous to those of varieties. Any intersection of pseudovarieties of semigroups is a pseudovariety, and the class FS of all finite

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semigroups is a pseudovariety. The empty set is the smallest pseudovariety of semigroups, and $\underline{PS}$ is the largest. For any collection K of finite semigroups, there is a smallest $\underline{S}$-variety containing K, called the $\underline{S}$-variety generated by K and denoted by $(K)_S$. Using intersection and pseudovariety-generated-by as meet and join operations respectively, the family of all pseudovarieties of semigroups forms a lattice $\underline{LFS}$. Similarly for monoids, we have the lattice $\underline{LFM}$ of all pseudovarieties of finite monoids, with $\underline{FM}$ the largest and the empty set the smallest such pseudovarieties. Again, we will be especially interested in bands, looking at the pseudovarieties $\underline{FB}$ and $\underline{FBM}$ of finite bands and finite band monoids respectively, with their associated lattices $\underline{LFB}$ and $\underline{LFBM}$.

Pseudovarieties of semigroups and monoids are related much as varieties of these objects are. For any pseudovariety $\underline{W}$ of finite monoids, there is a least pseudovariety $(\underline{W})_S$ of finite semigroups which contains it; for any pseudovariety $\underline{V}$ of finite semigroups (bands), $\underline{V} \cap \underline{M}$ is a pseudovariety of finite (band) monoids. Another way of producing a pseudovariety of monoids from a pseudovariety $\underline{V}$ of semigroups is to form the pseudovariety $\underline{V}^1$ generated by the collection $\{S^1 : S \in \underline{V}\}$. As with varieties, a semigroup S may be in the pseudovariety $\underline{V}$ while the monoid S^1 is not. Investigation of $\underline{V}^1$ for various pseudovarieties $\underline{V}$ has recently been carried out by Pin [13].

Since Birkhoff's Theorem shows that varieties are in fact equational classes, it is natural to look for a way to relate pseudovarieties to equations. Let $X = \{x_1, x_2, \ldots\}$ be a set of variables. For any words p and q in X , we let <u>VPM</u>(p=q) be the set of all finite monoids which satisfy the identity p=q.

<u>Proposition 2.3.2:</u> VPM(p=q) is a pseudovariety of monoids.

Now consider a sequence $p_i = q_i$ of identities over X, for i≥1. The collection

$$\underline{V} = \bigcup_{\kappa=1}^{\infty} \bigcap_{i=\kappa}^{\infty} \underline{VPM}(p_i = q_i)$$

consists of finite monoids which satisfy the equations $p_i = q_i$ for all i greater than or equal to some integer κ. Then $\underline{V}$ is called an <u>ultimately equational class</u> and is said to be <u>ultimately defined</u> by the equations $p_i = q_i$, for i≥1.

<u>Proposition 2.3.3:</u> Any ultimately equational class is a pseudovariety of monoids.

<u>Theorem 2.3.4</u> (Eilenberg): Any non-empty pseudovariety of monoids is ultimately defined by a sequence of equations.

A similar discussion may be carried out for semigroups, with the following variation of the previous theorem as a result:

Theorem 2.3.5 (Eilenberg): Any pseudovariety of semigroups which contains the semigroup {1} is ultimately defined by a sequence of equations.

We note that the empty pseudovariety, of monoids or of semigroups, and the pseudovariety of semigroups containing only the empty semigroup cannot be defined by equations.

The concept of pseudovariety has been extended to any type of algebra, and a related notion defined, by Ash [1]. We first need some additional definitions and notation. The set of all identities of the type under consideration will be denoted by E. For any class K of algebras, POW(K) is the class of direct powers of members of K. A family of sets is said to be directed if for any two sets A and B in the family, there is a set C in the family with $A \subseteq C$ and $B \subseteq C$. Finally, a filter over a set I is a family of subsets of I closed under formation of finite intersections and supersets. With this background we look at three theorems of Ash, which are proved in [1].

Theorem 2.3.6 (Ash): For any class K of algebras, the following are equivalent conditions:

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(1)   K is closed under H, S, P_f and POW;

(2) $K = HSP_f POW(K)$;

(3)   K is the union of some directed family of varieties;

(4)   There exists a filter F over E such that, for all algebras A,

   A is in K iff $Id(A)$ is in F.

A _generalized_ _variety_ is therefore defined as any class of algebras satisfying any one of the conditions of Theorem 2.3.6. From condition (2) of this theorem, it also follows that any class K of algebras is contained in a _smallest_ generalized variety, which is then called the _generalized_ _variety_ _generated_ _by_ K, and denoted by $Gen(K) = HSP_f POW(K)$.

A _pseudovariety_ is defined to be any class of algebras closed under the operators H, S and P_f. The relationship between pseudovarieties and generalized varieties is given in the next theorem.

Theorem 2.3.7 (Ash): A class of algebras is a pseudovariety if and only if it consists of the finite members of some generalized variety. In fact, if _V_ is a pseudovariety, then _V_ consists of the finite members of Gen(_V_), the generalized variety generated by _V_.

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Theorem 2.3.8 (Ash): Let C be a countable class cf algebras. The following are equivalent conditions for any class L contained in C:

(1) There is a generalized variety K with $L = K \cap C$;

(2) There is a chain $\underline{V}_1 \subseteq \underline{V}_2 \subseteq \ldots$ of varieties with $L = (\bigcup_{i=1}^{\infty} \underline{V}_i) \cap C$;

(3) There is a sequence $e_1, e_2, \ldots$ of identities such that for any algebra A in C,

A is in L iff e_n is in $Id(A)$ for all but finitely many n.

Corollary 2.3.9 (Ash): If C is the class of all finite algebras of some finite type, then the three conditions of Theorem 2.3.8 are also equivalent to :

(4) $L = HSP_f(L)$.

In particular, Theorem 2.3.8 and its Corollary 2.3.9 generalize Theorems 2.3.4 and 2.3.5 of Eilenberg, in showing that pseudovarieties are equivalent to ultimately equational classes for any type of algebras. Theorem 2.3.7 relating pseudovarieties and generalized varieties will be of special importance in Chapter III, when the lattice of pseudovarieties of band monoids is discussed.

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III. Lattices of Varieties and Pseudovarieties of Band Monoids

In this chapter we study the lattice of varieties of band monoids and the lattice of pseudovarieties of band monoids. In the notation of Chapter II, the class of all bands is the variety $\underline{B} = \underline{V}(x^2=x)$, and $\underline{LB}$ denotes the lattice of all varieties of bands. Similarly we have the variety $\underline{BM} = \underline{VM}(x^2=x)$ of band monoids, with the lattice $\underline{LBM}$. We begin by showing that the mapping Mon taking $\underline{V}$ to $\underline{V} \cap \underline{M}$, for $\underline{V}$ in $\underline{LB}$, is a surjective lattice homomorphism from $\underline{LB}$ onto $\underline{LBM}$. Since the structure of the lattice $\underline{LB}$ is known, this allows us to study the structure of $\underline{LBM}$, by looking at the image of $\underline{LB}$ under Mon. This is done in Sections 2 and 3, with the result that the lattice of varieties of band monoids is determined. Finally in Section 4 Ash's theorems are applied, to obtain the lattice of pseudovarieties of band monoids.

Section 1: The Lattice Homomorphism

Let $\underline{V}$ be any variety of bands. From Proposition 2.2.4, $\underline{V} \cap \underline{M}$ is then a variety of band monoids. Thus the mapping Mon taking $\underline{V}$ to $\underline{V} \cap \underline{M}$, for $\underline{V}$ in $\underline{LB}$, is indeed a function from $\underline{LB}$ to $\underline{LBM}$. In this section we show that Mon is a surjective lattice homomorphism.

<u>Proposition</u> <u>3.1.1</u>: Mon is a lattice homomorphism.

<u>Proof</u>: Let $\underline{V}$ and $\underline{W}$ be varieties in the lattice $\underline{LB}$. Then

$$\text{Mon}(\underline{V} \cap \underline{W}) = (\underline{V} \cap \underline{W}) \cap \underline{M}$$
$$= (\underline{V} \cap \underline{M}) \cap (\underline{W} \cap \underline{M})$$
$$= \text{Mon}(V) \cap \text{Mon}(\underline{W}).$$

We must further show that

$$\text{Mon}(\underline{V} \vee \underline{W}) = \text{Mon}(\underline{V}) \vee \text{Mon}(\underline{W});$$

that is, that

$$(\underline{V} \vee \underline{W}) \cap \underline{M} = (\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M}).$$

Since $(\underline{V} \vee \underline{W}) \cap \underline{M}$ is a variety of monoids which contains all monoids in $\underline{V} \vee \underline{W}$, and $(\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M})$ is the least variety of monoids to contain all the monoids in $\underline{V} \cup \underline{W}$, we have

$$(\underline{V} \vee \underline{W}) \cap \underline{M} \supseteq (\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M}).$$

Now let M be any monoid in $(\underline{V} \vee \underline{W}) \cap \underline{M}$. By Theorem 2.1.3, $\underline{V} \vee \underline{W}$ is equal to HSP($\underline{V} \cup \underline{W}$). Thus there exist bands A in $\underline{V}$ and B in $\underline{W}$, a subsemigroup C of A x B, and a surjective homomorphism f from C onto M. Choose an element e of C such that f(e) is equal to

the identity 1 of M. Since C is a band, e is an idempotent.

Let D= eCe = {ece : c ∈ C}. From Proposition 2.2.3, D is a subsemigroup of C, and D is a monoid. Also for any m in M, there is a c in C such that f(c)=m; but then ece is in D, and

$$f(ece) \quad =f(e)f(c)f(e)$$
$$=1m1$$
$$=m.$$

Therefore the restriction of f to D is a surjective homomorphism from D onto M.

Let p_1 and p_2 be the projections of A x B onto A and B respectively. The images $I=p_1(D)$ and $J=p_2(D)$ are monoids in A and B respectively, so I is in $\underline{V} \cap \underline{M}$ and J is in $\underline{W} \cap \underline{M}$. Further, D is easily seen to be a submonoid of I x J, so that M is a homomorphic image of a submonoid of a product of monoids from $\underline{V} \cap \underline{M}$ and $\underline{W} \cap \underline{M}$. This establishes the inclusion

$$(\underline{V} \vee \underline{W}) \cap \underline{M} \subseteq (\underline{V} \cap \underline{M}) \vee (\underline{W} \cap \underline{M}).$$

We conclude that

$$Mon(\underline{V} \vee \underline{W}) = Mon(\underline{V}) \vee Mon(\underline{W}),$$

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and so Mon is a lattice homomorphism.

Proposition 3.1.2: The lattice homomorphism Mon from $\underline{LB}$ to $\underline{LBM}$ is surjective.

Proof: Let $\underline{U}$ be any variety of band monoids. Regarding $\underline{U}$ as a collection of semigroups, we may consider $\underline{V}(\underline{U})$, the least variety of semigroups to contain $\underline{U}$. This variety will contain only bands, so it will appear somewhere in the lattice $\underline{LB}$.

Clearly $\underline{U}$ is contained in $\underline{V}(\underline{U}) \cap \underline{M}$. For the opposite inclusion, suppose that M is any monoid in $\underline{V}(\underline{U})$. Since $\underline{V}(\underline{U})$ is equal to HSP$(\underline{U})$, M must be a homomorphic image of a subsemigroup of a product of members of $\underline{U}$. As a variety of monoids, $\underline{U}$ is closed under the formation of products, so we may take W in $\underline{U}$, T a subsemigroup of W, and g a surjective homomorphism from T to M. Then there is an element e in T such that g(e) = 1, and e is an idempotent. The set eTe then forms a monoid contained in T, as in Proposition 2.2.3, and the restriction of g to eTe is a surjective homomorphism from eTe to M. Thus M is a homomorphic image of a monoid eTe in W. Now eTe may not have the same identity element as W, so it may not be a submonoid of W. If it is, then eTe is also in $\underline{U}$, and so M is in H$(\underline{U}) = \underline{U}$. If not, then eTe $\cup$ $\{1_W\}$ is a submonoid of W, and then eTe is a homomorphic image of eTe $\cup$ $\{1_W\}$. In this case, M is a homomorphic image of a homomorphic image of a submonoid of W, so again M is in $\underline{U}$.

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Therefore $\underline{V}(\underline{U}) \cap \underline{M} = \underline{U}$, or $\text{Mon}(\underline{V}(\underline{U})) = \underline{U}$.

For the purposes of this discussion, we are interested in applying the mapping taking $\underline{V}$ to $\underline{V} \cap \underline{M}$ only to varieties $\underline{V}$ of bands. However, we note that there are other situations in which this mapping will define a lattice homomorphism. In the proof of Proposition 3.1.1, we used the fact that the semigroups involved were bands only in one key step, to enable us to produce an idempotent e. Any hypothesis about the domain of the mapping which enables us to produce the necessary idempotent at this stage will allow us to prove that the mapping $\underline{V} \longrightarrow \underline{V} \cap \underline{M}$ is a lattice homomorphism on that domain.

As an example of this, we may consider the lattice $\underline{LFS}$ of pseudovarieties of finite semigroups. Using Proposition 2.2.1 to guarantee the existence of the necessary idempotent, we may adapt the proof of Proposition 3.1.1 to show that the mapping FMon taking $\underline{V}$ to $\underline{V} \cap \underline{M}$, for all pseudovarieties $\underline{V}$ in $\underline{LFS}$, is also a lattice homomorphism on the domain $\underline{LFS}$.

Section 2: The Base of the Lattice of Varieties of Band Monoids

We now begin our examination of the lattice $\underline{LBM}$ of varieties of band monoids. We have seen that $\underline{LBM}$ is the image of the lattice $\underline{L\overset{*}{B}}$ of varieties of bands, under the lattice

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homomorphism Mon taking $\underline{V}$ to $\underline{V} \cap \underline{M}$, so our approach now is to look at the structure of the lattice $\underline{LB}$ and determine which varieties in it are identified under the action of Mon; that is, to determine the congruence induced on $\underline{LB}$ by Mon.

The structure of the lattice $\underline{LB}$ of varieties of bands has been established by Birjukov [2], Fennemore [6] and Gerhard [8]. In particular, Fennemore has shown in [7] that the varieties of bands in $\underline{LB}$ are each determined by one identity other than $x^2 = x$. Because henceforth we will be considering only varieties of bands, we will denote by $\underline{V}(p=q)$ the variety of bands satisfying the additional identity $p=q$, where p and q are words on the alphabet $X = \{a, d, x, y, x_1, x_2, \ldots\}$. From Proposition 2.2.3, the image of $\underline{V}(p=q)$ under Mon is then the variety of band monoids satisfying $p=q$, which we will denote from now on by $\underline{VM}(p=q)$. Following the notation of Fennemore [7], the words R_n, S_n and Q_n, for $n \geq 2$, are defined as follows:

$$R_2 = x_3 x_2 x_1$$
$$R_3 = x_1 x_2 x_3$$
$$R_n = \begin{cases} R_{n-1} x_n, & \text{for } n \text{ even, } n \geq 4 \\ x_n R_{n-1}, & \text{for } n \text{ odd, } n \geq 5 \end{cases}$$

$$Q_2 = x_2 x_3 x_1$$

$$Q_3 = x_1 x_2 x_3 x_1 x_3$$

$$Q_n = \begin{cases} Q_{n-1} x_n R_n, & \text{for } n \text{ even, } n \geq 4 \\ R_n x_n Q_{n-1}, & \text{for } n \text{ odd, } n \geq 5 \end{cases}$$

$$S_2 = x_3 x_1 x_2 x_1$$

$$S_3 = x_1 x_2 x_3 x_1 x_3 x_2 x_3$$

$$S_n = \begin{cases} S_{n-1} x_n R_n, & \text{for } n \text{ even, } n \geq 4 \\ R_n \overline{x}_n S_{n-1}, & \text{for } n \text{ odd, } n \geq 5 \end{cases}$$

For any word A, $\overline{A}$ will denote the mirror image of A; so for example, $\overline{R_3} = x_3 x_2 x_1$.

The structure of the lattice $\underline{LB}$ is shown in Figure 1. The portion of the lattice above the variety $\underline{V}(axya=axaya)$ will be referred to as the inductively defined part of the lattice; the portion below and including the variety $\underline{V}(R_3 d\overline{R}_3 = Q_3 d\overline{Q}_3)$ will be called the base of the lattice. Identities for the varieties not specifically labelled in Figure 1 may be found in Fennemore [7].

There are several easily verified facts about the words and identities involved in $\underline{LB}$ which will be useful for later analysis of $\underline{LBM}$. For $n \geq 3$, the words R_n, S_n and Q_n each have n variables. If a word R is formed by the concatenation of two words P and Q, that is R=PQ, then $\overline{R} = \overline{PQ} = \overline{Q}\overline{P}$; also $\overline{\overline{P}}=P$ for any word P, and $\overline{x}=x$ for any variable x.

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$\underline{B}$.

$\underline{V}(R_5 = S_5)$ $\underline{V}(\overline{R}_5 = \overline{S}_5)$

$\underline{V}(R_5 = Q_5)$ $\underline{V}(\overline{R}_5 = \overline{Q}_5)$

$\underline{V}(\overline{R}_4 = \overline{S}_4)$ $\underline{V}(R_4 = S_4)$

$\underline{V}(\overline{R}_4 = \overline{Q}_4)$ $\underline{V}(R_4 = Q_4)$

$\underline{V}(R_3 = S_3)$ $\underline{V}(R \ dR = Q \ dQ)$ $\underline{V}(\overline{R}_3 = \overline{S}_3)$

$\underline{V}(R_3 = Q_3)$ $\underline{V}(axya = axaya)$ $\underline{V}(\overline{R}_3 = \overline{Q}_3)$

$\underline{V}(\overline{R}_2 = \overline{S}_2)$ $\underline{V}(R_2 = S_2)$

$\underline{V}(ax = axa)$ $\underline{V}(axya = ayxa)$ $\underline{V}(xa = axa)$

$\underline{V}(\overline{R}_2 = \overline{Q}_2)$ $\underline{V}(a = axa)$ $\underline{V}(R_2 = Q_2)$

$\underline{V}(ax = a)$ $\underline{V}(xy = yx)$ $\underline{V}(xa = a)$
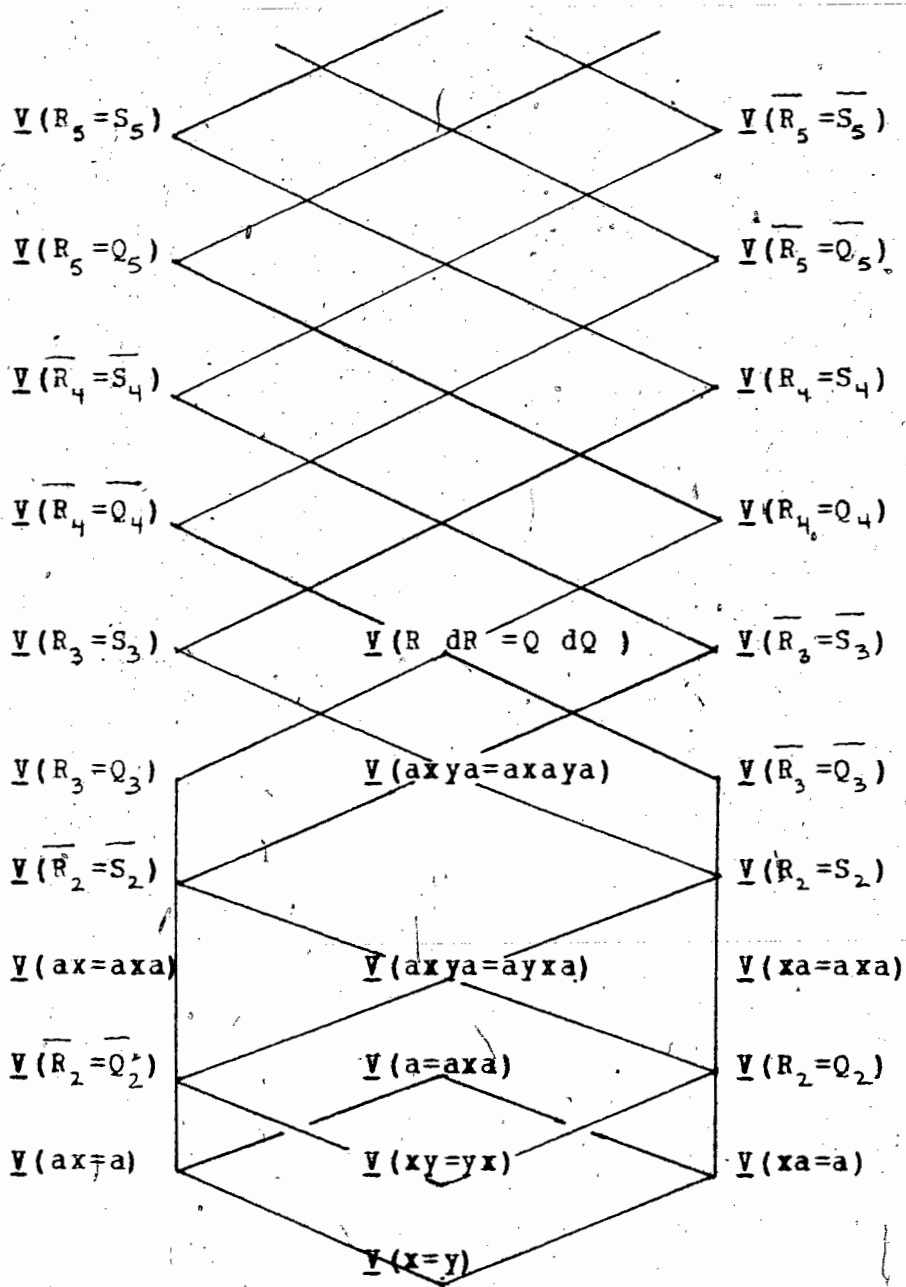
$\underline{V}(x = y)$

Figure 1

The Lattice of Varieties of Bands

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An important property of $\underline{LB}$ is its symmetry. The lattice is symmetric about a vertical line through $\underline{V}(x=y)$, in the sense that the corresponding varieties on either side of the line are $\underline{V}(P=Q)$ and $\underline{V}(\overline{P}=\overline{Q})$, for some identity $P=Q$. This symmetry means that many of the results to be obtained in the following sections may be "dualized": in any proof involving words $P,Q,\ldots$, replacing the words by their mirror images $\overline{P},\overline{Q},\ldots$ throughout will give a proof of the "mirror image" or dual result.

For the remainder of this section, the image under Mon of the base of the lattice $\underline{LB}$ is examined. The first proposition deals with the first two layers of the base.

Proposition 3.2.1:

(i) $\underline{VM}(ax=a) = \underline{VM}(xa=a) = \underline{VM}(a=axa) = \underline{VM}(x=1) = \underline{VM}(x=y)$.

(ii) $\underline{VM}(\overline{R_2}=\overline{Q_2}) = \underline{VM}(xy=yx) = \underline{VM}(R_2=Q_2) = \underline{VM}(axya=ayxa)$.

Proof: (i) Clearly any monoid in $\underline{V}(x=y)$ must be the trivial monoid $\{1\}$, so $\underline{VM}(x=y) = \underline{VM}(x=1)$. Suppose that M is a monoid in any of $\underline{V}(xa=a)$, $\underline{V}(ax=a)$ or $\underline{V}(a=axa)$. Then for any m in M, the substitution $x=m$ and $a=1$ results in each case in $m=1$, so that M satisfies the identity $x=1$. Therefore $\underline{VM}(xa=a) = \underline{VM}(ax=a) = \underline{VM}(a=axa) = \underline{VM}(x=y) = \underline{VM}(x=1)$, the trivial variety.

(ii) Since $\underline{V}(xy=yx)$ is contained in $\underline{V}(R_2=Q_2)$, it follows that

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$\underline{VM}(xy=yx)$ is contained in $\underline{VM}(R_2=Q_2)$. Conversely, any monoid in $\underline{VM}(R_2=Q_2)$ satisfies the identity $x_3x_2x_1 = x_2x_3x_1$. If m and n are any elements of M, then the substitution of $x_1=1$, $x_2=m$ and $x_3=n$ into this identity gives $nm=mn$. Thus M is in $\underline{VM}(xy=yx)$, and $\underline{VM}(R_2=Q_2) = \underline{VM}(xy=yx)$. A dual argument shows that $\underline{VM}(\overline{R_2}=\overline{Q_2}) = \underline{VM}(xy=yx)$.

If M is any monoid in $\underline{VM}(axya=ayxa)$, and m and n are any elements of M, then the substitution $x=m$, $y=n$ and $a=1$ in the identity $axya=ayxa$ gives $mn=nm$. Therefore $\underline{VM}(axya=ayxa)$ is contained in $\underline{VM}(xy=yx)$. Since the opposite inclusion follows from that for varieties of bands, we have that $\underline{VM}(axya=ayxa) = \underline{VM}(xy=yx)$.

Next we consider the varieties $\underline{V}(xa=axa)$, $\underline{V}(R_2=S_2)$ and $\underline{V}(\overline{R_3}=\overline{Q_3})$ which appear on the right side of the base of the lattice $\underline{LB}$, and their mirror images $\underline{V}(ax=axa)$, $\underline{V}(\overline{R_2}=\overline{S_2})$ and $\underline{V}(R_3=Q_3)$ on the left side of the base of the lattice.

Proposition 3.2.2:   (i) $\underline{VM}(xa=axa) = \underline{VM}(R_2=S_2) = \underline{VM}(\overline{R_3}=\overline{Q_3})$;

(ii) $\underline{VM}(ax=axa) = \underline{VM}(\overline{R_2}=\overline{S_2}) = \underline{VM}(R_3=Q_3)$.

Proof: (i) Because the corresponding inclusions are true for varieties of bands, we have

$$\underline{VM}(xa=axa) \subseteq \underline{VM}(R_2=S_2) \subseteq \underline{VM}(\overline{R_3}=\overline{Q_3}).$$

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Now let M be any monoid in $\underline{VM}(\overline{R}_3=\overline{Q}_3)$, so that the identity $x_3x_2x_1=x_3x_1x_3x_2x_1$ holds in M. Let m and n be any two elements of M. The substitution $x_1=n$, $x_2=m$ and $x_3=1$ produces $mn=nmn$ from this identity. Therefore M is in $\underline{VM}(xa=axa)$, and it follows that

$$\underline{VM}(xa=axa) \;=\; \underline{VM}(P_2=S_2) \;=\; \underline{VM}(\overline{R}_3=\overline{Q}_3).$$

(ii) The proof is dual to that of (i).

Because of the pattern which will arise when the inductively-defined portion of the lattice $\underline{LB}$ is considered, the three varieties identified as equal in (i) above will be referred to as $\underline{VM}(R_2=S_2)$, and the mirror image variety from (ii) as $\underline{VM}(\overline{R}_2=\overline{S}_2)$. From the above propositions, we know that the image of the base of $\underline{LB}$ under Mon is as shown in Figure 2.

It is clear that $\underline{VM}(x=1)$ is contained in but not equal to $\underline{VM}(xy=yx)$. The next proposition will show that neither of the varieties $\underline{VM}(R_2=S_2)$ and $\underline{VM}(\overline{R}_2=\overline{S}_2)$ is contained in the other. From this it will follow that these two varieties, their meet and their join are all distinct.

<u>Proposition</u> <u>3.2.3</u>: Neither of $\underline{VM}(R_2=S_2)$ and $\underline{VM}(\overline{R}_2=\overline{S}_2)$ is contained in the other.

<u>Proof</u>: Recall that $\underline{VM}(R_2=S_2) = \underline{VM}(xa=axa)$, and that $\underline{VM}(\overline{R}_2=\overline{S}_2) =$
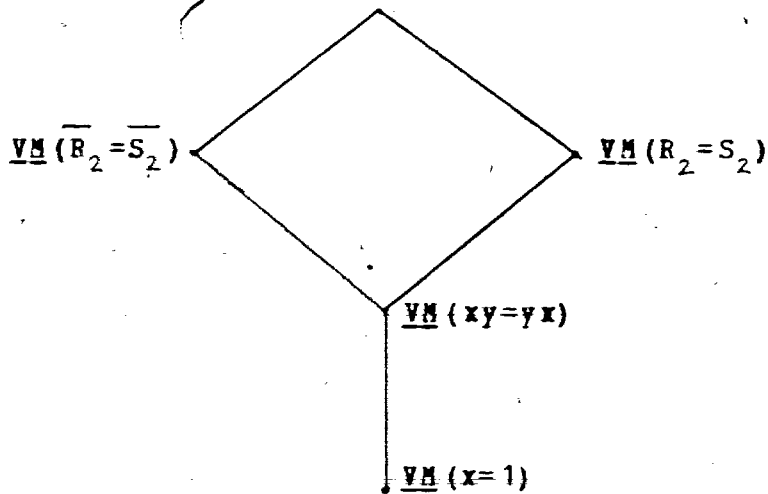
$$\underline{VM}(\overline{R_2=S_2}) \longleftarrow \quad \longrightarrow \underline{VM}(R_2=S_2)$$

$$\underline{VM}(xy=yx)$$

$$\underline{VM}(x=1)$$

Figure 2

The Image of the Base of **LB** under Mon

$\underline{VM}(ax=axa)$. Since $\underline{V}(xa=axa)$ is not contained in $\underline{V}(ax=axa)$, there is a semigroup A which satisfies xa=axa but not ax=axa. If A is a monoid, then A is in $\underline{VM}(xa=axa)$ but not in $\underline{VM}(ax=axa)$.

If A is not a monoid, let M be the monoid A¹. Then M will not satisfy the identity ax=axa. We must show that M does satisfy the identity xa=axa. Let m and n be any elements of M, and consider the substitution x=m and a=n. If neither m nor n is 1, then mn=nmn. If m=1, then mn = n = n1n =nmn; if n=1, then mn = m = 1m1 = nmn. Thus in each case mn = nmn, and the identity xa=axa is satisfied in M. Therefore M is in $\underline{VM}(xa=axa)$ but not in $\underline{VM}(ax=axa)$.

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This completes the proof that the base of the lattice $\underline{LBM}$ of varieties of band monoids is in fact as shown in Figure 2.

Section 3: The Lattice of Varieties of Band Monoids

In this section we will consider the image under Mon of the inductively defined part of the lattice $\underline{LB}$. In the right-hand edge of the lattice, above the base, the pattern $\underline{V}(\overline{R_n=S_n})$, $\underline{V}(R_{n+1}=Q_{n+1})$, $\underline{V}(R_{n+1}=S_{n+1})$ and $\underline{V}(\overline{R_{n+2}=Q_{n+2}})$ is repeated for n odd and $n \geq 3$. The left-hand edge of the lattice has the mirror image of this pattern.

Three main results will be proved in this section. The first is that for $n \geq 3$, $\underline{VM}(R_n=Q_n) = \underline{VM}(\overline{R_{n-1}=S_{n-1}})$; and dually, that $\underline{VM}(\overline{R_n=Q_n}) = \underline{VM}(R_{n-1}=S_{n-1})$. This shows that a certain amount of collapsing occurs in going from $\underline{LB}$ to $\underline{LBM}$ by Mon. Next we show that for $n \geq 3$, $\underline{VM}(\overline{R_{n-1}=S_{n-1}})$ is contained in but not equal to $\underline{VM}(R_n=S_n)$, and dually, that $\underline{VM}(R_{n-1}=S_{n-1})$ is contained in but not equal to $\underline{VM}(\overline{R_n=S_n})$. A series of propositions is needed to establish these results, and the proofs will use induction. Finally, it will be shown that for $n \geq 2$, the varieties $\underline{VM}(R_n=S_n)$, $\underline{VM}(\overline{R_n=S_n})$, together with their meet and their join are all distinct. Combined with the results of Section 2 for the base, these results will then be used to determine the structure of the complete lattice $\underline{LBM}$.

We begin now to show that $\underline{VM}(R_n = Q_n) = \underline{VM}(\overline{R_{n-1} = S_{n-1}})$ for $n \geq 3$. The following proposition establishes a rather technical result which will be the key to proving these varieties equal.

Proposition 3.3.1: Let $n \geq 4$. Let M be a monoid with identity element 1, and let $a_1, \ldots, a_{n-1}$ be any elements of M. Let $b_1 = a_3$, $b_2 = a_2$, $b_3 = 1$, $b_4 = a_1$, and let $b_t = a_{t-1}$, for $5 \leq t \leq n$. Then

$$R_n(b_1, \ldots, b_n) = \overline{R_{n-1}(a_1, \ldots, a_{n-1})},$$

and

$$Q_n(b_1, \ldots, b_n) = \overline{S_{n-1}(a_1, \ldots, a_{n-1})}.$$

Proof: We use induction on n. For $n=4$, we have

$$
\begin{aligned}
R_4(b_1, b_2, b_3, b_4) &= b_1 b_2 b_3 b_4 \\
&= a_3 a_2 1 a_1 \\
&= a_3 a_2 a_1 \\
&= \overline{R_3(a_1, a_2, a_3)}
\end{aligned}
$$

and

$$
\begin{aligned}
Q_4(b_1, b_2, b_3, b_4) &= b_1 b_2 b_3 b_1 b_3 b_4 b_1 b_2 b_3 b_4 \\
&= a_3 a_2 1 a_3 1 a_1 a_3 a_2 1 a_1 \\
&= a_3 a_2 a_3 a_1 a_3 a_2 a_1 \\
&= \overline{S_3(a_1, a_2, a_3)}.
\end{aligned}
$$

Thus the result holds for $n=4$.

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Now assume that the result of the proposition is true for all k such that $4 \leq k < n$, and consider $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_n$ as above. Then using the definitions for R_{n-1} and R_n, and the induction hypothesis, we get that for n odd,

$$
\begin{aligned}
R_n(b_1, \ldots, b_n) &= b_n R_{n-1}(b_1, \ldots, b_{n-1}) \\
&= a_{n-1} \overline{R_{n-2}(a_1, \ldots, a_{n-2})} \\
&= \overline{R_{n-2}(a_1, \ldots, a_{n-2}) a_{n-1}} \\
&= \overline{R_{n-1}(a_1, \ldots, a_{n-1})}
\end{aligned}
$$

and for n even,

$$
\begin{aligned}
R_n(b_1, \ldots, b_n) &= R_{n-1}(b_1, \ldots, b_{n-1}) b_n \\
&= \overline{R_{n-2}(a_1, \ldots, a_{n-2}) a_{n-1}} \\
&= \overline{a_{n-1} R_{n-2}(a_1, \ldots, a_{n-2})} \\
&= \overline{R_{n-1}(a_1, \ldots, a_{n-1})} .
\end{aligned}
$$

Now using the induction hypothesis again and the result just established for R_n, we get for n even

$$
\begin{aligned}
Q_n(b_1, \ldots, b_n) &= Q_{n-1}(b_1, \ldots, b_{n-1}) b_n R_n(b_1, \ldots, b_n) \\
&= \overline{S_{n-2}(a_1, \ldots, a_{n-2})} a_{n-1} \overline{R_{n-1}(a_1, \ldots, a_{n-1})} \\
&= \overline{R_{n-1}(a_1, \ldots, a_{n-1}) a_{n-1} S_{n-2}(a_1, \ldots, a_{n-2})} \\
&= \overline{S_{n-1}(a_1, \ldots, a_{n-1})} ,
\end{aligned}
$$

and for n odd,

$$Q_n(b_1, \ldots, b_n) = R_n(b_1, \ldots, b_n) b_n Q_{n-1}(b_1, \ldots, b_{n-1})$$

$$= \overline{R_{n-1}}(a_1, \ldots, a_{n-1}) a_{n-1} \overline{S_{n-2}}(a_1, \ldots, a_{n-2})$$

$$= \overline{S_{n-1}}(a_1, \ldots, a_{n-1}).$$

Thus the proposition holds for all $n \geq 4$.

Theorem 3.3.2: For all $n \geq 3$, $\underline{VM}(R_n = Q_n) = \underline{VM}(\overline{R_{n-1}} = \overline{S_{n-1}})$.

Proof: Since $\underline{VM}(\overline{R_{n-1}} = \overline{S_{n-1}})$ is contained in $\underline{VM}(R_n = Q_n)$, only the opposite inclusion need be proved. For $n = 3$, this was done in Proposition 3.2.2. Now let M be any monoid in $\underline{VM}(R_n = Q_n)$, where $n \geq 4$. We must show that if $a_1, \ldots, a_{n-1}$ are any $n-1$ elements of M, then

$$\overline{R_{n-1}}(a_1, \ldots, a_{n-1}) = \overline{S_{n-1}}(a_1, \ldots, a_{n-1}).$$

Now let $b_1 = a_3$, $b_2 = a_2$, $b_3 = 1$, $b_4 = a_1$, and $b_t = a_{t-1}$ for all $5 \leq t \leq n$. Since $n \geq 4$, Proposition 3.3.1 says that

$$\overline{R_{n-1}}(a_1, \ldots, a_{n-1}) = R_n(b_1, \ldots, b_n)$$

and

$$\overline{S_{n-1}}(a_1, \ldots, a_{n-1}) = Q_n(b_1, \ldots, b_n).$$

But since $b_1, \ldots, b_n$ are all in M which is in $\underline{V}(R_n = Q_n)$,

$$R_n(b_1, \ldots, b_n) = Q_n(b_1, \ldots, b_n),$$

and thus

$$\overline{R_{n-1}}(a_1, \ldots, a_{n-1}) = \overline{S_{n-1}}(a_1, \ldots, a_{n-1}).$$

Therefore for any $n \geq 3$, $\underline{VM}(R_n = Q_n) = \underline{VM}(\overline{R_{n-1} = S_{n-1}})$.

· The preceeding two proofs can be dualized to give the following dual of Theorem 3.3.2:

__Theorem 3.3.3:__ For any $n \geq 3$, $\underline{VM}(\overline{R_n} = \overline{Q_n}) = \underline{VM}(R_{n-1} = S_{n-1})$.

Because the corresponding inclusion is true for bands, it follows that $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$ is contained in $\underline{VM}(R_n = S_n)$, for $n \geq 3$. The goal of the next series of propositions is to show that $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$ is however not equal to $\underline{VM}(R_n = S_n)$, for $n \geq 3$. The problem now is thus to produce a monoid in $\underline{VM}(R_n = S_n)$ which is not in $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$, for each $n \geq 3$.

Since for $n \geq 3$ $\underline{V}(\overline{R_{n-1} = S_{n-1}})$ is contained in but not equal to $\underline{V}(R_n = S_n)$, there exists a semigroup A_n satisfying $R_n = S_n$ but not $\overline{R_{n-1} = S_{n-1}}$. Let $M_n = A_n{}^1$; that is,

$$M_n = \begin{cases} A_n, & \text{if } A_n \text{ is a monoid} \\ A_n \cup \{1\}, & \text{if } A_n \text{ is not a monoid,} \end{cases}$$

where in the latter case 1 will act as an identity for M_n. Then M_n does not satisfy $\overline{R_{n-1}} = \overline{S_{n-1}}$. We will show that M_n does satisfy $R_n = S_n$. Of course if $M_n = A_n$, this is obviously true. Therefore in what follows it will be assumed that $M_n = A_n \cup \{1\}$, where A_n is in $\underline{V}(R_n = S_n)$, for $n \geq 3$.

Let $a_1, \ldots, a_n$ be any elements of M_n. It must be shown that $R_n(a_1, \ldots, a_n) = S_n(a_1, \ldots, a_n)$. If $a_1 = a_2 = \ldots = a_n = 1$, then $R_n(a_1, \ldots, a_n) = 1 = S_n(a_1, \ldots, a_n)$, so we may henceforth assume that not all of $a_1, \ldots, a_n$ are equal to 1. We will examine two cases: first when not all of a_1, a_2 and a_3 are equal to 1, and second when $a_1 = a_2 = a_3 = 1$ but not all of $a_4, \ldots, a_n$ are equal to 1. Before stating the propositions which will deal with these two situations, we introduce some notation to shorten the expressions involved. We will denote $R_n(a_1, \ldots, a_n)$ by $R_n(\vec{a})$, and $R_n(f(a_1), \ldots, f(a_n))$ by $R_n(\overrightarrow{f(a)})$; $R_{n+1}(a_1, \ldots, a_{n+1})$ and $R_{n+1}(f(a_1), \ldots, f(a_{n+1}))$ will be represented by $R_{n+1}(\vec{a}, a_{n+1})$ and $R_{n+1}(\overrightarrow{f(a)}, f(a_{n+1}))$ respectively; and similarly for S_n and S_{n+1}.

<u>Proposition</u> <u>3.3.4</u>: Let $n \geq 3$, and let $a_1, \ldots, a_n$ be elements of a monoid M such that not all of a_1, a_2 and a_3 are equal to 1. Define a function f from $\{a_1, \ldots, a_n\}$ to $\{a_1, \ldots, a_n\} - \{1\}$ as follows:

$$f(a_1) = \begin{cases} a_1, & \text{if } a_1 \neq 1 \\ a_2, & \text{if } a_1 = 1, \ a_2 \neq 1 \\ a_3, & \text{if } a_1 = a_2 = 1, \ a_3 \neq 1 \end{cases}$$

$$f(a_2) = \begin{cases} a_2, & \text{if } a_2 \neq 1 \\ a_3, & \text{if } a_2 = 1, \ a_3 \neq 1 \\ a_1, & \text{if } a_2 = a_3 = 1, \ a_1 \neq 1 \end{cases}$$

$$f(a_3) = \begin{cases} a_3, & \text{if } a_3 \neq 1 \\ a_2, & \text{if } a_3 = 1, \ a_2 \neq 1 \\ a_1, & \text{if } a_3 = a_2 = 1, \ a_1 \neq 1 \end{cases}$$

$$f(a_4) = \begin{cases} a_4, & \text{if } a_4 \neq 1 \\ f(a_3), & \text{if } a_4 = 1 \end{cases}$$

$$f(a_5) = \begin{cases} a_5, & \text{if } a_5 \neq 1 \\ f(a_1), & \text{if } a_5 = 1 \end{cases}$$

and

$$f(a_i) = \begin{cases} a_i, & \text{if } a_i \neq 1 \\ f(a_{i-2}), & \text{if } a_i = 1. \end{cases} \qquad \text{for } 6 \leq i \leq n.$$

Then

$$R_n(a_1, \ldots, a_n) = R_n(f(a_1), \ldots, f(a_n))$$

and

$$S_n(a_1, \ldots, a_n) = S_n(f(a_1), \ldots, f(a_n)).$$

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<u>Proof:</u> The proof will be by induction on n. We first verify that the proposition holds for $n=3$. Since the value of the function f at a_i is determined according to whether or not $a_i=1$, our method is to check all possible combinations of which members of $\{a_1, a_2, a_3\}$ are equal to 1. We recall that

$$R_3 = x_1 x_2 x_3$$

and

$$S_3 = x_1 x_2 x_3 x_1 x_3 x_2 x_3.$$

There are three cases to consider.

Case 1: None of a_1, a_2 or a_3 is equal to 1.
Then $f(a_i) = a_i$ for $1 \le i \le 3$, and so

$$R_3 \overrightarrow{(a)} = R_3 \overrightarrow{(f(a))}$$

and

$$S_3 \overrightarrow{(a)} = S_3 \overrightarrow{(f(a))}.$$

Case 2: Exactly one of a_1, a_2 and a_3 is equal to 1.
Then there are three subcases to look at:

(i) $a_1=1$, $a_2 \ne 1$, $a_3 \ne 1$.
Then $f(a_1)=f(a_2)=a_2$, and $f(a_3)=a_3$, so

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$$R_3 \overrightarrow{(a)} \quad = \quad 1a_2a_3$$
$$= \quad a_2a_2a_3$$
$$= \quad f(a_1)f(a_2)f(a_3)$$
$$= \quad R_3\overrightarrow{(f(a))},$$

and

$$S_3 \overrightarrow{(a)} \quad = \quad 1a_2a_31a_3a_2a_3$$
$$= \quad a_2a_2a_3a_2a_3a_2a_3$$
$$= \quad f(a_1)f(a_2)f(a_3)f(a_1)f(a_3)f(a_2)f(a_3)$$
$$= \quad S_3\overrightarrow{(f(a))}.$$

(ii) $a_1 \neq 1$, $a_2 = 1$, $a_3 \neq 1$.

Then $f(a_1) = a_1$ and $f(a_2) = f(a_3) = a_3$, so

$$R_3 \overrightarrow{(a)} \quad = \quad a_11a_3$$
$$= \quad a_1a_3a_3$$
$$= \quad f(a_1)f(a_2)f(a_3)$$
$$= \quad R_3\overrightarrow{(f(a))},$$

and

$$S_3 \overrightarrow{(a)} \quad = \quad a_11a_3a_1a_31a_3$$
$$= \quad a_1a_3a_3a_1a_3a_3a_3$$
$$= \quad f(a_1)f(a_2)f(a_3)f(a_1)f(a_3)f(a_2)f(a_3)$$
$$= \quad S_3\overrightarrow{(f(a))}.$$

(iii) $a_1 \neq 1$, $a_2 \neq 1$, $a_3 = 1$.

Then $f(a_1) = a_1$ and $f(a_2) = f(a_3) = a_2$, so

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$$R_3 \overrightarrow{(a)} = a_1 a_2 1$$

$$= a_1 a_2 a_2$$

$$= f(a_1) f(a_2) f(a_3)$$

$$= R_3 \overrightarrow{(f(a))},$$

and

$$S_3 \overrightarrow{(a)} = a_1 a_2 1 a_1 1 a_2 1$$

$$= a_1 a_2 a_2 a_1 a_2 a_2 a_2$$

$$= f(a_1) f(a_2) f(a_3) f(a_1) f(a_3) f(a_2) f(a_3)$$

$$= S_3 \overrightarrow{(f(a))}.$$

Case 3: Exactly two of a_1, a_2 and a_3 are equal to 1. Suppose that $a_i = a_j = 1$, and $a_K \neq 1$. Then $f(a_i) = f(a_j) = f(a_K) = a_K$; therefore

$$R_3 \overrightarrow{(a)} = a_K$$

$$= a_K a_K a_K$$

$$= f(a_1) f(a_2) f(a_3)$$

$$= R_3 \overrightarrow{(f(a))},$$

and

$$S_3 \overrightarrow{(a)} = a_K$$

$$= a_K a_K a_K a_K a_K a_K a_K$$

$$= f(a_1) f(a_2) f(a_3) f(a_1) f(a_3) f(a_2) f(a_3)$$

$$= S_3 \overrightarrow{(f(a))}.$$

This completes the proof for the inductive base n=3. We now assume that $R_K \overrightarrow{(a)} = R_K \overrightarrow{(f(a))}$ and $S_K \overrightarrow{(a)} = S_K \overrightarrow{(f(a))}$ for any integer k

such that $3 \leq k \leq n$, and look at the words R_{n+1} and S_{n+1} under the substitution $x_1 = a_1, \ldots, x_{n+1} = a_{n+1}$.

For $n+1 \geq 4$, R_{n+1} and S_{n+1} are inductively defined as follows:

$$R_{n+1} = \begin{cases} R_n\, x_{n+1}, & n+1 \text{ even, } n+1 \geq 4 \\ x_{n+1}\, R_n, & n+1 \text{ odd, } n+1 \geq 5 \end{cases}$$

and

$$S_{n+1} = \begin{cases} S_n\, x_{n+1}\, R_{n+1}, & n+1 \text{ even, } n+1 \geq 4 \\ R_{n+1}\, x_{n+1}\, S_n, & n+1 \text{ odd, } n+1 \geq 5. \end{cases}$$

The cases $n+1$ even and $n+1$ odd must be dealt with separately.

Suppose that $n+1$ is odd and $n+1 \geq 5$. Then

$$\begin{aligned} R_{n+1}(\overrightarrow{a}, a_{n+1}) &= a_{n+1} R_n \overrightarrow{(a)} \\ &= a_{n+1} R_n \overrightarrow{(f(a))}, \end{aligned}$$

by the induction hypothesis. If $a_{n+1} \neq 1$, then $f(a_{n+1}) = a_{n+1}$, and so

$$\begin{aligned} R_{n+1}(\overrightarrow{a}, a_{n+1}) &= f(a_{n+1}) R_n \overrightarrow{(f(a))} \\ &= R_{n+1} \overrightarrow{(f(a), f(a_{n+1}))}. \end{aligned}$$

If however $a_{n+1} = 1$, then $R_{n+1}(\overrightarrow{a}, a_{n+1}) = R_n \overrightarrow{(f(a))}$. But if $n+1 = 5$, then $f(a_{n+1}) = f(a_5) = f(a_1)$, and $R_n \overrightarrow{(f(a))}$ begins with $f(a_1)$; while if

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$n+1>5$, then $f(a_{n+1})=f(a_{n-1})$, and $\overrightarrow{R_n(f(a))}$ begins with $f(a_{n-1})$. Thus for any $n+1$ odd and $n+1\geq 5$, when $a_{n+1}=1$ it follows that

$$
\begin{aligned}
R_{n+1}\overrightarrow{(a,a_{n+1})} &= \overrightarrow{R_n(f(a))} \\
&= f(a_{n+1})\overrightarrow{R_n(f(a))} \\
&= R_{n+1}\overrightarrow{(f(a),f(a_{n+1}))}.
\end{aligned}
$$

Still assuming that $n+1$ is odd and $n+1\geq 5$, we have that

$$
\begin{aligned}
S_{n+1}\overrightarrow{(a,a_{n+1})} &= R_{n+1}\overrightarrow{(a,a_{n})}a_{n+1}S_n\overrightarrow{(a)} \\
&= R_{n+1}\overrightarrow{(f(a),f(a_{n+1}))}a_{n+1}S_n\overrightarrow{(f(a))}, \qquad (*)
\end{aligned}
$$

using the above result for R_{n+1} and the induction hypothesis for S_n. If $a_{n+1}\neq 1$, then $f(a_{n+1})=a_{n+1}$, and $(*)$ becomes

$$
R_{n+1}\overrightarrow{(f(a),f(a_{n+1}))}f(a_{n+1})S_n\overrightarrow{(f(a))},
$$

which is just $S_{n+1}\overrightarrow{(f(a),f(a_{n+1}))}$.

If $a_{n+1}=1$, then $f(a_{n+1})$ is either $f(a_1)$, if $n+1=5$, or $f(a_{n-1})$ if $n+1>5$. But if $n+1=5$, then $S_n\overrightarrow{(f(a))}$ begins with $f(a_1)$; while if $n+1>5$ and $n+1$ is odd, $S_n\overrightarrow{(f(a))}$ begins with $f(a_{n-1})$. Thus when $a_{n+1}=1$, $f(a_{n+1})$ is always the same as the first element of $S_n\overrightarrow{(f(a))}$; so in this case $(*)$ becomes

$$S_{n+1}(\overrightarrow{a}, a_{n-1}) = R_{n+1}(\overrightarrow{f(a)}, f(a_{n-1})) \, 1S_n(\overrightarrow{f(a)})$$

$$= R_{n-1}(\overrightarrow{f(a)}, f(a_{n-1})) f(a_{n-1}) S_n(\overrightarrow{f(a)})$$

$$= S_{n+1}(\overrightarrow{f(a)}, f(a_{n-1})).$$

When $n+1$ is even, and $n+1 \geq 4$, a very similar argument shows first that $R_{n+1}(\overrightarrow{a}, a_{n-1}) = R_{n+1}(\overrightarrow{f(a)}, f(a_{n-1}))$, and then using this, that $S_{n+1}(\overrightarrow{a}, a_{n-1}) = S_{n+1}(\overrightarrow{f(a)}, f(a_{n-1}))$.

Therefore for any $n \geq 3$,

$$R_n(a_1, \ldots, a_n) = R_n(f(a_1), \ldots, f(a_n))$$

and

$$S_n(a_1, \ldots, a_n) = S_n(f(a_1), \ldots, f(a_n)).$$

Corollary 3.3.5: Let $a_1, \ldots, a_n$ be elements of $\overline{A}_n = A_n \cup \{1\}$, for $n \geq 3$. If not all of a_1, a_2 and a_3 are equal to 1, then

$$R_n(a_1, \ldots, a_n) = S_n(a_1, \ldots, a_n).$$

Proof: Let the function f be defined as in Proposition 3.3.4. Then $f(a_1), \ldots, f(a_n)$ are all in A_n, which satisfies $R_n = S_n$; therefore

$$R_n(a_1, \ldots, a_n) = R_n(f(a_1), \ldots, f(a_n))$$

$$= S_n(f(a_1), \ldots, f(a_n))$$

$$= S_n(a_1, \ldots, a_n).$$

The other situation which will occur in showing that the monoid M_n does satisfy $R_n = S_n$ for $n \geq 3$ is the following: there will be elements $a_1, \ldots, a_n$ of M_n, where $n \geq 4$, with the property that $a_1 = a_2 = a_3 = 1$, but not all of $a_4, \ldots, a_n$ are equal to 1. The next proposition provides a way to handle this situation.

Proposition 3.3.6: Let $a_1, \ldots, a_n$ be elements of a monoid M, with $n \geq 4$, such that $a_1 = a_2 = a_3 = 1$, but not all of $a_4, \ldots, a_n$ are equal to 1. Let $k(a_1, \ldots, a_n) = \min\{t : 4 \leq t \leq n, a_t \neq 1\}$. Define a function h_n from $\{a_1, \ldots, a_n\}$ to $\{a_4, \ldots, a_n\}$ as follows:

$$h_n(a_i) = \begin{cases} a_{k(\vec{a})}, & \text{if } i = 1, 2 \text{ or } 3 \\ a_i, & \text{if } 4 \leq i \leq n. \end{cases}$$

Then

$$R_n(a_1, \ldots, a_n) = R_n(h_n(a_1), \ldots, h_n(a_n))$$

and

$$S_n(a_1, \ldots, a_n) = S_n(h_n(a_1), \ldots, h_n(a_n)).$$

Proof: We use induction on n. When $n = 4$, the hypothesis means that $a_1 = a_2 = a_3 = 1$ and $a_4 \neq 1$, so $k(\vec{a}) = 4$, and $h_4(a_i) = a_4$, for $1 \leq i \leq 4$. Then

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$$R_4(\vec{a}) \quad = \quad 111a_4$$
$$= \quad a_4\,a_4\,a_4\,a_4$$
$$= \quad h_4(a_1)\,h_4(a_2)\,h_4(a_3)\,h_4(a_4)$$
$$= \quad R_4(\overrightarrow{h_4(a)}),$$

and similarly,

$$S_4(\vec{a}) \quad = \quad a_4$$
$$= \quad h_4(a_4)$$
$$= \quad S_4(\overrightarrow{h_4(a)}).$$

Thus the results hold for $n=4$.

We now assume that $R_k(\vec{a})=R_k(\overrightarrow{h_k(a)})$ and $S_k(\vec{a})=S_k(\overrightarrow{h_k(a)})$ for all k such that $4\leq k\leq n$, and look at R_{n+1} and S_{n+1}. Again the odd and even cases must be considered separately.

Suppose that $n+1$ is odd, and $n+1\geq 5$. By definition, $R_{n+1}(\vec{a},a_{n+1})=a_{n+1}R_n(\vec{a})$, and $S_{n+1}(\vec{a},a_{n+1})=R_{n+1}(\vec{a},a_{n+1})a_{n+1}S_n(\vec{a})$. If $k(\vec{a},a_{n+1})=n+1$, this means that $a_1=a_2=\ldots=a_n=1$, and $a_{n+1}\neq 1$; thus $h_{n+1}(a_i)=a_{n+1}$, for $i=1,2,3$ or $n+1$, and $h_{n+1}(a_j)=a_j=1$ for $4\leq j\leq n$. But then $R_n(\vec{a})=1$ and $S_n(\vec{a})=1$, so

$$R_{n+1}(\vec{a},a_{n+1}) \quad = \quad a_{n+1}$$
$$= \quad R_{n+1}(\overrightarrow{h_{n+1}(a)},h_{n+1}(a_{n+1})),$$

and

$$S_{n+1}(\overrightarrow{a}, a_{n+1}) = R_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})) h_{n+1}(a_{n+1}) S_n(\overrightarrow{h_{n+1}(a)})$$

$$= S_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})).$$

Now suppose that $k(\overrightarrow{a}, a_{n+1}) \neq n+1$. This means that at least one of $a_1, \ldots, a_n$ is not equal to 1, and therefore the function h_n is defined on $\#\{a_1, \ldots, a_n\}$. By the induction hypothesis, $R_n(\overrightarrow{a}) = R_n(\overrightarrow{h_n(a)})$. But $k(\overrightarrow{a})$ picks out the index of the first element of the list $(a_1, \ldots, a_n)$ which is not equal to 1, while $k(\overrightarrow{a}, a_{n+1})$ picks out the index of the first element of the list $(a_1, \ldots, a_{n+1})$ which is not equal to 1. Clearly $k(\overrightarrow{a}) = k(\overrightarrow{a}, a_{n+1})$. Therefore

$$h_n(a_i) = \begin{cases} a_{k(\overrightarrow{a})}, & \text{for } i=1, 2 \text{ or } 3 \\ a_i, & \text{for } 4 \leq i \leq n, \end{cases}$$

so that $h_n(a_i) = h_{n+1}(a_i)$ for $1 \leq i \leq n$. From this it follows that

$$R_{n+1}(\overrightarrow{a}, a_{n+1}) = a_{n+1} R_n(\overrightarrow{a})$$

$$= a_{n+1} R_n(\overrightarrow{h_n(a)})$$

$$= a_{n+1} R_n(\overrightarrow{h_{n+1}(a)})$$

$$= h_{n+1}(a_{n+1}) R_n(\overrightarrow{h_{n+1}(a)})$$

$$= R_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})).$$

Now using this result for R_{n+1} and the induction hypothesis that $S_n(\vec{a}) = S_n(\overrightarrow{h_n(a)})$, we get that

$$
\begin{aligned}
S_{n+1}(\vec{a}, a_{n+1}) &= R_{n+1}(\vec{a}, a_{n+1}) \cdot a_{n+1} S_n(\vec{a}) \\
&= R_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})) h_{n+1}(a_{n+1}) S_n(\overrightarrow{h_n(a)}) \\
&= R_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})) h_{n+1}(a_{n+1}) S_n(\overrightarrow{h_{n+1}(a)}) \\
&= S_{n+1}(\overrightarrow{h_{n+1}(a)}, h_{n+1}(a_{n+1})).
\end{aligned}
$$

When $n+1$ is even and $n+1 \geq 6$, we have $R_{n+1}(\vec{a}, a_{n+1}) = R_n(\vec{a}) a_{n+1}$ and $S_{n+1}(\vec{a}, a_{n+1}) = S_n(\vec{a}) a_{n+1} R_{n+1}(\vec{a}, a_{n+1})$, and the argument above for $n+1$ odd is easily adapted to prove this case too.

We are now able to prove

<u>Theorem 3.3.7</u>: For any $n \geq 3$, $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$ is contained in but not equal to $\underline{VM}(R_n = S_n)$.

<u>Proof</u>: As discussed earlier, it will suffice to prove that the monoid $M_n = A_n \cup \{1\}$, which is not in $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$, is in $\underline{VM}(R_n = S_n)$, for $n \geq 3$.

Let $a_1, \ldots, a_n$ be any elements of M_n. If $a_1 = a_2 = \ldots = a_n = 1$, then $R_n(a_1, \ldots, a_n) = 1 = S_n(a_1, \ldots, a_n)$. If not all of a_1, a_2 and a_3 are equal to 1, we can apply Corollary 3.3.5 to get $R_n(a_1, \ldots, a_n) = S_n(a_1, \ldots, a_n)$. For $n=3$, this proves that M_n is in $\underline{VM}(R_n = S_n)$, so we need now only consider $n \geq 4$. Finally, suppose

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that $a_1 = a_2 = a_3 = 1$, but not all of $a_4, \ldots, a_n$ are equal to 1. Defining the function h_n as in Proposition 3.3.6, we get $R_n(a_1, \ldots, a_n) = R_n(h_n(a_1), \ldots h_n(a_n))$ and $S_n(a_1, \ldots, a_n) = S_n(h_n(a_1), \ldots h_n(a_n))$. Now none of $h_n(a_1)$, $h_n(a_2)$ or $h_n(a_3)$ is equal to 1, so by Corollary 3.3.5 again, $R_n(h_n(a_1), \ldots, h_n(a_n)) = S_n(h_n(a_1), \ldots, h_n(a_n))$. Therefore $R_n(a_1, \ldots, a_n) = S_n(a_1, \ldots, a_n)$ in this situation too. We conclude that M_n does satisfy the identity $R_n = S_n$. Thus $\underline{VM}(\overline{R_{n-1} = S_{n-1}})$ is contained in but not equal to $\underline{VM}(R_n = S_n)$.

Starting with the proper containment of $\underline{V}(R_{n-1} = S_{n-1})$ in $\underline{V}(\overline{R_n = S_n})$ for bands, we could produce as before a semigroup $\overline{A_n}$ and a monoid $\overline{M_n} = \overline{A_n} \cup \{1\}$ which satisfy $\overline{R_n = S_n}$ but not $R_{n-1} = S_{n-1}$, for $n \geq 3$. By dualizing each of Propositions 3.3.4 and 3.3.6 and Corollary 3.3.5, we could prove the following dual of Theorem 3.3.7:

<u>Theorem 3.3.8:</u> For any $n \geq 3$, $\underline{VM}(R_{n-1} = S_{n-1})$ is contained in but not equal to $\underline{VM}(\overline{R_n = S_n})$.

The proof of Theorem 3.3.7 shows in fact that if A_n is a semigroup which satisfies $R_n = S_n$, for $n \geq 3$, then $M_n = A_n{}^1$ also satisfies $R_n = S_n$. This result can be used again to give us more information about the lattice of varieties of band monoids.

Proposition 3.3.9: For any $n \geq 2$, neither of the varieties $\underline{VM}(R_n=S_n)$ and $\underline{VM}(\overline{R_n}=\overline{S_n})$ is contained in the other.

Proof: For $n=2$, this was established in Proposition 3.2.3. For $n \geq 3$, there is a semigroup B_n which satisfies $R_n=S_n$ but not $\overline{R_n}=\overline{S_n}$. Let D_n be the monoid $B_n{}^1$. Just as in the proof of Theorem 3.3.7, D_n will satisfy $R_n=S_n$; but it does not satisfy $\overline{R_n}=\overline{S_n}$, since B_n does not. Therefore D_n is in $\underline{VM}(R_n=S_n)$ but not in $\underline{VM}(\overline{R_n}=\overline{S_n})$. Dually, we can produce a monoid which is in $\underline{VM}(\overline{R_n}=\overline{S_n})$ but not in $\underline{VM}(R_n=S_n)$.

Corollary 3.3.10: For $n \geq 2$, the varieties $\underline{VM}(R_n=S_n)$, $\underline{VM}(\overline{R_n}=\overline{S_n})$, together with their meet and their join are all distinct.

The results we have obtained in this section can now be used to determine the structure of the lattice $\underline{LBM}$ of all varieties of band monoids. We look at a portion of the lattice $\underline{LB}$, as shown in Figure 3. The symbols $\underline{V}_1,\dots,\underline{V}_6$ will denote various joins in the lattice, as indicated in Figure 3; and $\underline{VM}_i$ will be used for $\underline{V}_i \cap \underline{M}$.

Proposition 3.3.11:

(i) $\underline{VM}_1=\underline{VM}_2=\underline{VM}_3=\underline{VM}_4$

(ii) $\underline{VM}_5=\underline{VM}(\overline{R_n}=\overline{Q_n}) = \underline{VM}(R_{n-1}=S_{n-1})$

and

(iii) $\underline{VM}_6=\underline{VM}(R_n=Q_n) = \underline{VM}(\overline{R_{n-1}}=\overline{S_{n-1}})$.
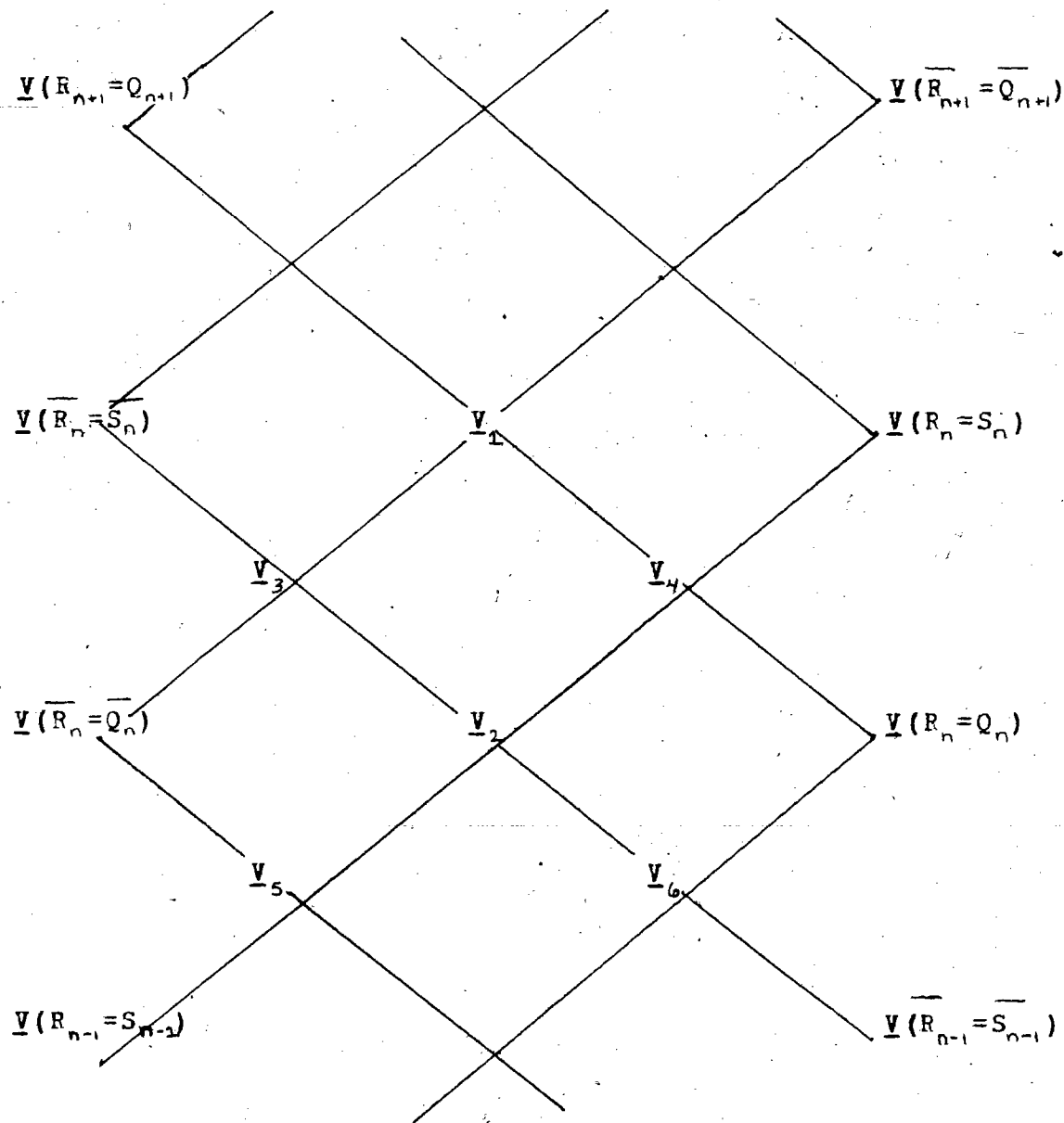
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Figure 3

A Portion of The Lattice LB

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Proof: (i) $\underline{VM}_1$ $=$ $\underline{VM}(\overline{R_n=Q_n}) \vee \underline{VM}(R_n=Q_n)$

$=$ $\underline{VM}(R_{n-1}=S_{n-1}) \vee \underline{VM}(\overline{R_{n-1}=S_{n-1}})$

$=$ $\underline{VM}_2$.

Since $\underline{VM}_2 \leq \underline{VM}_3 \leq \underline{VM}_1$, and $\underline{VM}_2 \leq \underline{VM}_4 \leq \underline{VM}_1$, therefore $\underline{VM}_1 = \underline{VM}_2 = \underline{VM}_3 = \underline{VM}_4$.

(ii) We know that $\underline{VM}(R_{n-1}=S_{n-1})$ is contained in $\underline{VM}_5$ which in turn is contained in $\underline{VM}(\overline{R_n=Q_n})$. By Theorem 3.3.3, $\underline{VM}(R_{n-1}=S_{n-1}) = \underline{VM}(\overline{R_n=Q_n})$. Therefore $\underline{VM}(R_{n-1}=S_{n-1}) = \underline{VM}_5 = \underline{VM}(\overline{R_n=Q_n})$.

(iii) This is just the dual of (ii).

For $n \geq 3$, therefore, the image of the portion of the lattice LB shown in Figure 3 is as shown in Figure 4.

From Corollary 3.3.10, we know that the four varieties shown in Figure 4 are all distinct. Of course, dual arguments show that if we start with the mirror image of the portion of the lattice shown in Figure 3, then we actually get the mirror image of the portion shown in Figure 4, where once again the four varieties are distinct.
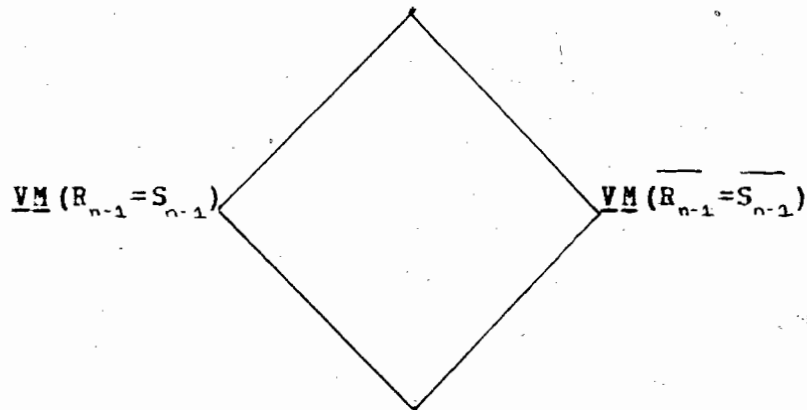
$$\underline{VM}(R_{n-1} = S_{n-1}) \qquad \underline{VM}(\overline{R_{n-1} = S_{n-1}})$$

Figure 4

A Portion of The Lattice L̲B̲M̲

Combining these results with those from Section 2 about the base of the lattice, we have proved the following result:

Theorem 3.3.12 The structure of the lattice L̲B̲M̲ of all varieties of band monoids is shown in Figure 5.

Section 4: The Lattice of Pseudovarieties of Band Monoids

Having determined the structure of the lattice L̲B̲M̲ of varieties of band monoids, we may now use Ash's results to relate this to pseudovarieties. From Theorem 2.3.7, we know that any pseudovariety is precisely the class of finite members of some generalized variety. In particular, if V̲ is a variety in

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$$\underline{BM}$$

$$\underline{VM}\,(\overline{R_4 = S_4}) \qquad\qquad \underline{VM}\,(R_4 = S_4)$$

$$\underline{VM}\,(R_3 = S_3) \qquad\qquad \underline{VM}\,(\overline{R_3 = S_3})$$

$$\underline{VM}\,(\overline{R_2 = S_2}) \qquad\qquad \underline{VM}\,(R_2 = S_2)$$
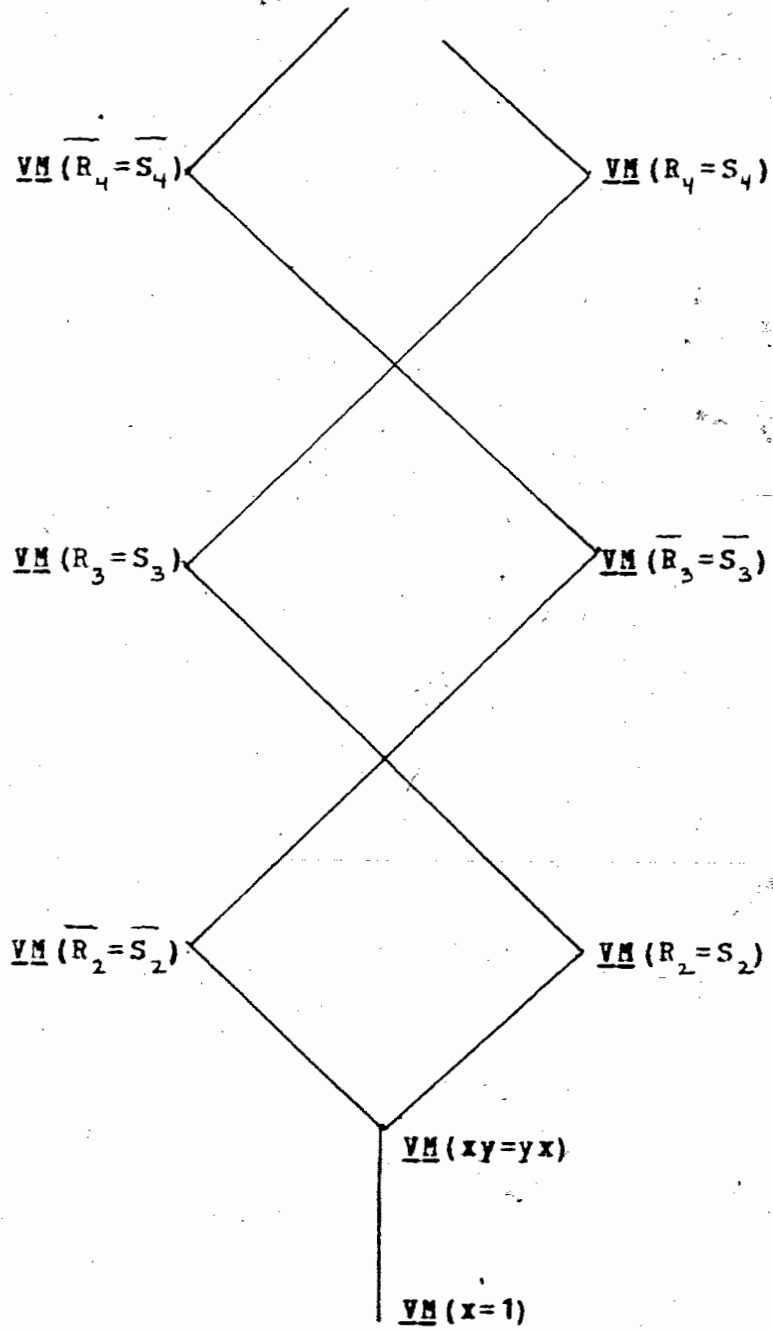
$$\underline{VM}\,(xy = yx)$$

$$\underline{VM}\,(x = 1)$$

Figure 5 .

The Lattice of Varieties of Band Monoids

LBM, then the collection Fin($\underline{V}$) of finite monoids in $\underline{V}$ is a pseudovariety. We denote by $\underline{LFBM}$ the lattice of pseudovarieties of finite band monoids. Then we may define a function Fin from $\underline{LBM}$ to $\underline{LFBM}$ by letting Fin take $\underline{V}$ to Fin($\underline{V}$), for any $\underline{V}$ in $\underline{LBM}$. In this section we show that Fin is in fact a lattice isomorphism, thus determining the structure of the lattice $\underline{LFBM}$. We do this by showing that Fin is a bijection with the property that both it and its inverse are order-preserving.

Proposition 3.4.1: The function Fin is injective.

Proof: Let $\underline{V}$ and $\underline{W}$ be varieties from $\underline{LBM}$, with $\underline{V} \neq \underline{W}$. If $\underline{V}$ is all of $\underline{BM}$, then $\underline{W}$ is properly contained in $\underline{BM}$, and it follows from Proposition 2.2.2 that Fin($\underline{W}$) is properly contained in Fin($\underline{V}$), which is all of $\underline{FBM}$. A similar argument holds if $\underline{W}$ is all of $\underline{BM}$ and $\underline{V}$ is not. Hence we may now assume that both $\underline{V}$ and $\underline{W}$ are proper subvarieties of $\underline{BM}$. Then there are distinct equations P=Q and H=K such that $\underline{V} = \underline{VM}(P=Q)$ and $\underline{W} = \underline{VM}(H=K)$, and without loss of generality we may choose a monoid M which satisfies P=Q but not H=K. Let n be the number of variables in the identity H=K. Then there exist $a_1, \ldots, a_n$ in M such that $H(a_1, \ldots, a_n) \neq K(a_1, \ldots, a_n)$. Let S be the free band semigroup generated by $\{a_1, \ldots, a_n\}$. By Proposition 2.2.2, S is a finite semigroup. Let N be the monoid S^1. This will be the free band monoid on $\{a_1, \ldots, a_n\}$, so the submonoid L of M generated by $\{a_1, \ldots, a_n\}$ will be a homomorphic image of N, and hence is also

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finite. Thus L will be a finite monoid which satisfies P=Q, but does not satisfy H=K. Therefore Fin($\underline{V}$) is not equal to Fin($\underline{W}$), and Fin is injective.

This argument in fact shows that if $\underline{V}$ is not contained in $\underline{W}$, then Fin($\underline{V}$) is not contained in Fin($\underline{W}$); that is, that if Fin($\underline{V}$) is contained in Fin($\underline{W}$), then $\underline{V}$ is contained in $\underline{W}$. The converse implication is obviously true, so

$$\text{Fin}(\underline{V}) \subseteq \text{Fin}(\underline{W}) \quad \text{if and only if} \quad \underline{V} \subseteq \underline{W}.$$

Proposition 3.4.2: The function Fin is surjective.

Proof: Let $\underline{V}$ be any pseudovariety of band monoids in $\underline{LFBM}$. By Theorem 2.3.7, $\underline{V}$ = Fin($\underline{W}$), where $\underline{W}$ is the generalized variety generated by $\underline{V}$. Since $\underline{W}$ = $HSP_f POW(\underline{V})$, $\underline{W}$ still satisfies $x^2=x$. By Theorem 2.3.6, $\underline{W}$ must be the union of some directed family D of varieties from the lattice $\underline{LBM}$.

Suppose that the directed family D is a finite one. Then the union W of members of D is just a variety $\underline{U}$ in $\underline{LBM}$, and we have $\underline{V}$ = Fin($\underline{U}$).

If D is not a finite directed family, there are only two possibilities for the union $\underline{W}$ of members of D. This union may be all of $\underline{BM}$; in this case we have $\underline{V}$ = Fin($\underline{W}$) = Fin($\underline{BM}$). Otherwise,

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W must be the class of all band monoids which are contained in some proper subvariety of BM. Clearly then Fin(W) is contained in Fin(BM). But also any finite band monoid is contained in some proper subvariety of BM (Gerhard, private communication), and so Fin(BM) is contained in Fin(W). Therefore V = Fin(W) = Fin(BM).

Thus for any pseudovariety V in LFBM, there is some variety U in LBM for which V = Fin(U), and Fin is surjective.

Theorems 3.4.1 and 3.4.2, together with the remarks following the proof of Theorem 3.4.1, give us the following theorem:

Theorem 3.4.3: The function Fin is a lattice isomorphism from the lattice LBM of varieties of band monoids onto the lattice LFBM of pseudovarieties of finite band monoids.

From this theorem we conclude that the structure of the lattice of pseudovarieties of band monoids is the same as that of the lattice of varieties of band monoids, as shown in Figure 5.

Bibliography

[1] C.J. Ash, Pseudovarieties and generalized varieties, (preprint).

[2] A.P. Birjukov, The lattice of varieties of idempotent semigroups, All-Union Collcquium on General Algebra, Riga, 1967, 16-18.

[3] S. Burris and H.P.Sankappanavar, A course in universal algebra, Springer-Verlag, New York, 1980.

[4] A.H. Clifford and J.B. Preston, The algebraic theory of semigroups, Vol.1, American Mathematical Society, Providence, R.I., 1961.

[5] S. Eilenberg, Automata, languages and machines, Vols. A and B, Academic Press, New York, 1976.

[6] C. Fennemore, All varieties of bands, Semigroup Forum, 1, 1970, 172-179.

[7] C. Fennemore, All varieties of bands, Math. Nachr. 48, 1971, 237-262.

[8] J.A. Gerhard, The lattice of equational classes of idempotent semigroups, Journal of Algebra, 15, 1970, 195-224.

[9] J.A. Green and D. Rees, On semigroups in which $x^r = x$, Proc. Cambridge Phil. Soc., 48, 1952, 35-40.

[10] N. Kimura, The structure of idempotent semigroups I, Pac. J. Math, 8, 1958, 257-275.

[11] M. Petrich, Introduction to semigroups, Charles E. Merrill Publishing Co., Columbus, Ohio, 1973.

[12] M. Petrich, Topics in semigroups, Lecture Notes, Pennsylvania State University, 1967.

[13] J.E. Pin, Semigroupe des parties et relations de Green, Laboratoire Informatique Theorique et Programmation, 82-53, October 1982.

[14] M. Yamada, The structure of separative bands, Doctoral Dissertation, University of Utah, 1962.