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THÈSES CANADIENNES SUR MICROFILME

46

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PATH AND CYCLE DECOMPOSITIONS OF COMPLETE MULTIGRAPHS

by

Gillian Marie Monay

B.Sc. (Hons.), Simon Fraser University, 1981

**THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
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ABSTRACT

In this thesis we show that if the edges of the complete multigraph on n vertices (in which each edge has multiplicity two) can be partitioned into n Hamiltonian paths having the property that any two paths intersect in exactly one edge, then the complete multigraph on $n' = n(5^a 13^b 17^c)$ vertices (each edge having multiplicity two) can be partitioned into n' Hamiltonian paths having the same intersection property. (Here a , b and c are natural numbers.)

We also show that if the edges of the complete multigraph on n vertices (in which each edge has multiplicity two) can be partitioned into n Hamiltonian paths having the property that any two paths intersect in exactly one edge, then the arcs of the complete symmetric directed graph on $4n$ vertices can be partitioned into $4n$ directed cycles of length $4n-1$ so that any two cycles intersect in exactly one edge (undirected arc).

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I. Introduction

In this thesis we will investigate the following three questions.

1. When can the edges of $2K_n$ be partitioned into Hamiltonian paths so that any two paths intersect in exactly one edge?
2. When can the edges of DK_n be partitioned into directed cycles of length $n-1$ so that any two cycles intersect in exactly one oppositely directed edge?

It is not always possible to direct the cycles, and so in this case we will look at the undirected counterpart to question 2. Clearly, any solution to question 2 will also be a solution to question 3.

3. When can the edges of $2K_n$ be partitioned into cycles of length $n-1$ so that any two cycles intersect in exactly one edge?

Here DK_n denotes the complete symmetric directed graph on n vertices, and $2K_n$ denotes the complete multigraph on n vertices in which there are two edges between every pair of distinct vertices. Similarly, we denote by mK_n the complete multigraph on n vertices in which there are m edges between every pair of distinct vertices. When $m=1$ we simply write K_n .

If the edges of a graph, G , can be partitioned into subgraphs belonging to a given family, say \mathcal{L} , of subgraphs of G then we say that G can be decomposed into subgraphs belonging to \mathcal{L} . Many people have looked at the problem of partitioning the edges of a graph into isomorphic copies of a fixed subgraph, the case where the family \mathcal{L} contains only one member. In particular, Bermond and Faber [6] have studied the problem of partitioning the edges of DK_n into directed cycles. From them we get the following theorem.

Theorem 1.1: For every n , the edges of DK_n can be partitioned into directed cycles (circuits) of length n .

In proving this theorem we need to look at the cases for n even and n odd separately. However, before doing so we need to give the following definition. Let $C = (v_0, v_1, \dots, v_r, v_0)$ be a circuit in DK_x and let S be some set of $x-n$ symbols. If the vertices of DK_x are labelled with elements of $Z_n \cup S$ then we define $C+k$ to be the circuit $(v_0+k, v_1+k, \dots, v_r+k, v_0+k)$, where addition is performed modulo n on those elements in Z_n only (elements in S remain unchanged).

Proof. (n even). Label the vertices of DK_{n+1} with the elements of $Z_n \cup \{\infty\}$, where ∞ is a new symbol satisfying $v+\infty=\infty$ for all $v \in Z$.

Define a circuit, D_0 , of length n by

$$D_0 = (\infty, 0, n-1, 1, n-2, \dots, n/2+1, n/2-1, \infty).$$

Define an additional $n-1$ circuits of length n by $D_k = D_0 + k$, for $k=1, 2, \dots, n-1$. Notice that in the above circuits, $D_k = (v_0, v_1, \dots, v_{n-1}, v_0)$, the differences $v_i - v_{i-1}$ ($1 \leq i \leq n-1$) and $v_0 - v_{n-1}$, taken modulo n , are precisely the integers $2, 3, \dots, n-1$ and ∞ . That is, no circuit contains an edge of the form $(i, i+1)$. Therefore the union of these n circuits is $DK_{n+1} - D_n$, where $D_n = (0, 1, 2, \dots, n-1, 0)$, and the $n+1$ circuits of length n , D_0, D_1, \dots, D_n , form a decomposition of DK_{n+1} .

(n odd). As before, label the vertices of DK_{n+1} with the elements of $Z_n \cup \{\infty\}$. Define a circuit, D_0 , of length n by

$$D_0 = (\infty, 0, n-1, 1, n-2, \dots, a-1, c+1, a, c-1, a+1, \dots, b-1, b, \infty)$$

where $a = \lfloor n/4 \rfloor$, $b = \lfloor n/2 \rfloor$ and $c = \lfloor 3n/4 \rfloor$ (where $\lfloor x \rfloor$ is the greatest integer not larger than x). Define an additional $n-1$ circuits of length n by $D_k = D_0 + k$, for $k=1, 2, \dots, n-1$. Notice that in the above circuits, $D_k = (v_0, v_1, \dots, v_{n-1}, v_0)$, the differences $v_i - v_{i-1}$ ($1 \leq i \leq n-1$) and $v_0 - v_{n-1}$, taken modulo n , are precisely the integers $1, 2, 3, \dots, n-1$, except $b+1$, and ∞ . That is, no circuit contains an edge of the form $(i, i+b+1)$. Therefore the union of these n circuits is $DK_{n+1} - D_n$, where $D_n = (0, b+1, 2(b+1), 3(b+1), \dots, (n-1)(b+1), 0)$, and the $n+1$ circuits of length n , D_0, D_1, \dots, D_n , form a decomposition of DK_{n+1} .

Alspach and Varma [5] have looked at the problem of partitioning the edges of K_n into cycles of length $n-1$, where $n=2p^e+1$, p is a prime and e is a positive integer. The following is a corollary to Theorem 1 of their paper.

Corollary 1.2: Let p be a prime and e be any positive integer. If $n=2p^e+1$ then K_n can be decomposed into cycles of length $n-1$ for every n .

Decompositions of the complete multipartite graph have also been studied. In particular, Sotteau [31] and Cockayne and Hartnell [11] have looked at partitioning the edges of the complete multipartite graph into cycles.

As it is not possible to mention all of the work done in this area, we refer the reader to 'Recent Results in Graph Decompositions', a comprehensive survey by Chung and Graham [10]. The authors discuss, among other things, partitioning the edges of complete graphs, multigraphs, multipartite graphs and hypergraphs. In addition to this, Chung and Graham list many conjectures as well as open problems and questions in this area.

While many authors have looked at the problem of partitioning the edges of a graph into isomorphic copies of a fixed subgraph, few have attempted to place further restrictions on these subgraphs. In 1972, Hell and Rosa [16] introduced (n, k, m) G -designs. These are partitions of mK_n into subgraphs on k vertices whose adjacency matrices are equivalent to the symmetric $(0, 1)$ -matrix, G . The design is balanced if every vertex appears in the same number, say r , of subgraphs. Note that if G is the adjacency matrix of a regular graph then the design is automatically balanced. Hell and Rosa were mainly concerned with balanced P -designs, the case where G is the

adjacency matrix of a simple path on k vertices. From them we get the following.

Theorem 1.3: A balanced P -design with $v=k$ and $m=2$ exists for every k .

Tarsi [32] was also interested in partitioning the edges of the complete multigraph into simple paths, however he only looked at the weaker unbalanced case. That is, where it is not necessary that every vertex appears in the same number of subgraphs.

Following the work done by Hell and Rosa, balanced G -designs received a lot of attention. Rosa and Huang [29] looked at balanced C -designs where C is the adjacency matrix of a simple cycle of length k . The easily derived necessary conditions for a balanced (n, k, m) C -design to exist are $n \geq k$, $mn(n-1) \equiv 0 \pmod{2k}$ and $m(n-1) \equiv 0 \pmod{2}$. Bermond, Huang and Sotteau [7] have conjectured that these necessary conditions are also sufficient.

Another interesting graph decomposition problem is the Oberwolfach problem which was first mentioned by Ringel. "Let $n = k_1 + k_2 + \dots + k_s$, be an odd integer, where $k_i \geq 3$ for $1 \leq i \leq s$. Is it possible to decompose K_n into 2-factors so that each 2-factor of the decomposition contains exactly one cycle of length k_i for every i , $1 \leq i \leq s$?" Since this problem was first put forth it has been generalized to include the case where n is even. "Let $n = k_1 + k_2 + \dots + k_s$, be an even integer, where $k_i \geq 3$ for $1 \leq i \leq s$. Is it

possible to decompose $K_n - F$, where F is a 1-factor of K_n , into 2-factors so that in the decomposition there is exactly one cycle of each length k_1, k_2, \dots, k_s ? Alspach and Haggkvist [2] have come up with the following results.

1. Let $n \equiv 2 \pmod{4}$ and let $n = k_1 + k_2 + \dots + k_s$ where each k_i is even. Then if F is a 1-factor of K_n we know that $K_n - F$ can be partitioned into 2-factors, each containing cycles of length k_i for $1 \leq i \leq s$.
2. Suppose k divides n , where both k and n are even. Then we can decompose $K_n - F$ into 2-factors each of which contains cycles of length k only.

For a survey of work done on the Oberwolfach problem the reader is referred to Rosa [27].

Recently, Hering [17] and Alspach, Heinrich and Rosenfeld [3] have looked at the problem of partitioning a complete graph into cycles, and they have imposed yet a different type of restriction on the subgraphs. Hering was the first to ask if the edges of $2K_n$ could be partitioned into cycles of length $n-1$ so that any two cycles intersect in exactly one edge. If such a partitioning exists we write $2K_n \rightarrow C_{n-1}$. In 1979, Hering and Rosenfeld [19] asked the same question, except this time for the directed case. That is, for which values of n can the edges of DK_n be partitioned into directed cycles of length $n-1$, DC_{n-1} , so that any two directed cycles intersect in exactly one oppositely directed edge. If such a partitioning exists we write $DK_n \rightarrow DC_{n-1}$. This problem has been studied by Alspach, Heinrich

and Rosenfeld. In addition, these authors have also investigated the problem of partitioning the edges of DK_n into antidirected cycles of length $n-1$ so that any two cycles intersect in exactly one oppositely directed edge. It is clear that in this case n must be odd. The following theorem appears in their paper, 'Edge Partitions of the Complete Symmetric Directed Graph and Related Designs'.

Theorem 1.4: If $n=p^e > 3$ where p is a prime and e is a positive integer then $DK_n \rightarrow DC_{n-1}$.

Proof. Label the vertices of DK_n with the elements of $GF(n)$, where $GF(n)$ is the Galois field having $n=p^e$ elements. Let b be a generator of the multiplicative cyclic group of $GF(n)$. Define a circuit D_0 by

$$D_0 = (1, b, b^2, \dots, b^{n-2}, b^{n-1} = 1).$$

Define an additional $n-1$ circuits of length $n-1$ by $D_k = D_0 + b^k$ for $k=1, 2, \dots, n-1$.

Suppose D_0 and D_k have two oppositely directed edges in common. Say (b^k, b^{k+1}) and (b^s, b^{s+1}) are the two edges of D_0 which intersect with two edges, say $(b^r + b^l, b^{r+1} + b^{l+1})$ and $(b^t + b^l, b^{t+1} + b^{l+1})$, of D_k . Then

$$b^k = b^{r+1} + b^l \quad \text{and} \quad b^{k+1} = b^r + b^l$$

and

$$b^s = b^{t+1} + b^l \quad \text{and} \quad b^{s+1} = b^t + b^l.$$

Thus, $b^k - b^s = b(b^r - b^t)$ and $b(b^k - b^s) = b^r - b^t$ and so we get

$b^2 (b^k - b^s) = b^k - b^s$. Thus either $b^2 = 1$ (which is impossible since b is a generator of the multiplicative cyclic group of $GF(a)$ and $n > 3$) or $b^k = b^s$. We therefore get $k = s$, which contradicts our assumption that D_0 and D_i have two distinct edges in common. Thus D_0 and D_i have at most one edge in common. Since we can generalize this argument to obtain the case where the two cycles in question are D_i and D_j , we see that any two cycles in DK_n have at most one edge in common. It therefore follows that any two cycles must have exactly one edge in common.

The following corollary to Theorem 1.4 clearly answers part of Hering's original question and is somewhat stronger than Proposition 9 in his paper, 'Block Designs with Cyclic Block Structure'. Here C_k is a simple cycle with k edges.

Corollary 1.5: If $n = p^e > 3$ where p is a prime and e is a positive integer then $2K_n \rightarrow C_{n-1}$.

In this thesis we will expand on the results found by Alspach, Heinrich and Rosenfeld, namely Theorem 1.4 and Corollary 1.5. Because of the method that will be used to do this, we will first require partitions of the edges of $2K_n$ into Hamiltonian paths so that any two paths have exactly one edge in common. If such a decomposition exists we write $2K_n \rightarrow P_n$, where P_k is a simple path with k vertices. We say this path has length $k-1$. Briefly, we have the following results in this thesis.

1. If $2K_n \rightarrow P_n$ then $2K_m \rightarrow P_{rn}$ for $r=5, 13$ and 17 .

2. If $2K_n \rightarrow P_n$ then $DK_n \rightarrow DC_{4n-1}$.

We also know that $2K_n \rightarrow P_n$ for $n=2, 3, 5, 6, \dots, 20$. (It is impossible to partition $2K_n$ into paths of length three so that any two paths have exactly one edge in common.) A computer was used to solve many of these cases. In [18] Hering showed that $2K_n \rightarrow C_n$ for $n=4, 5, \dots, 36$.

The basic plan of this thesis is as follows. We begin in Chapter II with a discussion on the relationship between graph decompositions and block designs, latin squares and graph embeddings. In Chapter III we introduce some useful definitions and we also present three lemmas which will be used throughout this thesis. Chapter IV contains the results on decompositions of $2K_n$ into intersecting Hamiltonian paths, and Chapter V contains the results on decompositions of DK_n and $2K_n$ into intersecting cycles of length $n-1$. In the Appendix we present a list of the paths which give use $2K_n \rightarrow P_n$ for $n=2, 3, 5, 6, \dots, 20$ and a few other isolated cases. We also give outlines of the computer programs used in this work.

II. Chapter 2

In this chapter we will discuss the relationships that graph decompositions have with three other areas of combinatorial mathematics. In each section we will introduce many new definitions, however only those definitions concerning latin squares will be used in later chapters of this thesis.

Block Designs

A design is a way of selecting subsets (blocks) from a non-empty finite set of objects so that certain conditions are satisfied. In particular, a (v, b, r, k, λ) -balanced incomplete block design (a BIBD) is an arrangement of a finite set of v objects into b blocks so that:

1. every object appears in r blocks
2. every block contains exactly k objects
3. every pair of distinct objects appears together in exactly λ blocks.

Simple counting arguments give the following two necessary conditions for the existence of a (v, b, r, k, λ) -BIBD.

1. $vr = bk$
2. $\lambda(v-1) = r(k-1)$

Thus if we know v, k and m , we can find b and r using the above necessary conditions. We will refer to such a design as a (v, k, m) -BIBD.

In fact, if we let the n paths in a decomposition $2K_n \rightarrow P_n$ be the points and think of the $\binom{n}{2}$ edges in K_n as the blocks then this decomposition defines an $(n, 2, 1)$ -BIBD since every two points (paths) appear in exactly one block (edge). Similarly, a decomposition $2K_n \rightarrow C_{n-1}$ also defines an $(n, 2, 1)$ -BIBD.

It has long been known that the problem of finding values of k for which the edges of mK_v can be partitioned into subgraphs, each of which is isomorphic to K_k , is equivalent to finding values of k for which a (v, k, m) -BIBD exists. Wilson [34] has shown that for a fixed m and k , mK_v can be decomposed into copies of K_k provided the necessary conditions are satisfied and v is sufficiently large.

In a BIBD the blocks of the design are simply subsets of a set and hence the order in which the elements of a block are listed is unimportant. Many authors have studied BIBD's in which the elements of block have been given a specific order (see, for example, [4], [13], [14], [21], [25] and [30]). These designs are called directed BIBD's, and a (v, k, m) directed BIBD is an arrangement of v elements into blocks of size k so that each ordered pair of elements appears in m blocks. For example, given a block $1\ 2\ \dots\ k-1\ k$, the cyclic order gives the ordered pairs $(1, 2), (2, 3), \dots, (k-1, k)$ and $(k, 1)$ and the transitive order gives the ordered pairs $(1, 2), (2, 3), \dots, (k-1, k)$ and $(1, k)$. It

is easy to see that the existence of a $(v, k, 1)$ directed BIBD in which the order assigned to the elements within a block is cyclic is equivalent to a decomposition of DK_v into directed cycles of length k . Thus by Theorem 1.1 we see that a cyclically directed $(v, v-1, 1)$ BIBD exists for all $v \geq 3$.

Hering [17] also looked at $(v, v-1, 1)$ directed BIBD's in which the elements of each block had been given a cyclic ordering and in which every pair of blocks in the design has the same number d of adjacent elements in common. If a directed BIBD with a cyclic ordering on the elements satisfied this property then he called it a harmonic design. Thus if $d=1$, the existence of a harmonic design on $n \geq 3$ elements implies $2K_n \rightarrow C_{n-1}$. In Proposition 9 of his paper [17] Hering displays a construction for harmonic designs on n elements, where $n=p^e$, p is an odd prime and e is a positive integer. This construction (attributed to L. Danzer) actually gives us $DK_n \rightarrow DC_{n-1}$ as proven in Theorem 1.4.

Hell and Rosa [16] have looked at (v, k, m) -BIBD's, and instead of restricting themselves to only cyclic or transitive orderings of the elements in these blocks, they allow any two elements of a block to be either "related" or "unrelated". (Thus it is not necessary that related elements occur beside each other in the cyclic representation of a block, as in the cases where the elements within the blocks had been given a cyclic or transitive ordering.) This relationship between certain pairs of elements in block defines an adjacency matrix $G=(g_{ij})$ of a graph,

where $g_{ij} = 1$ if elements i and j are related and 0 otherwise. A balanced (n, k, m) G -design is an (n, k, m) -BIBD in which every block defines an adjacency matrix equivalent to the $k \times k$ matrix G , every element in the design occurs in the same number of blocks (this is the balance requirement) and every two distinct objects are related in m blocks. Hell and Rosa looked at the case where the adjacency matrix, G , is the adjacency matrix of a path. They showed that a balanced P -design with $n=k$ exists if and only if k or m is even. This problem was later completely solved independently by Hung and Mendelsohn [22] and Huang [20] who showed that the necessary conditions for a balanced (n, k, m) P -design to exist are also sufficient. Their results can be summarized by saying that a balanced (n, k, m) P -design exists if and only if $nk(n-1) \equiv 0 \pmod{2(k-1)}$ (see [9]). Later, Huang and Rosa looked at balanced (n, k, m) G -designs in which G is the adjacency matrix of a cycle. These are called balanced circuit designs and are denoted by $BCD(n, k, m)$. In 1979 Alspach [1] asked the following two questions.

1. Is it true that there exists a $BCD(n, k, 1)$ if and only if n is odd, $n \geq k$ and $k \mid \binom{n}{2}$?
2. Is it true that there exists a $BCD(n, k, m)$ if and only if $n \geq k$, $mn(n-1) \equiv 0 \pmod{2k}$ and $m(n-1) \equiv 0 \pmod{2}$?

The first question was partially answered by Alspach and Varma [5] who showed that if $k=2p^e$ then the necessary conditions are also sufficient, and by Rosa [28] who showed that if $k=p^e$ then the necessary conditions are sufficient. Question 2 was looked

at by Bernard, Huang and Sotteau, [7] and Bernard and Sotteau [8] who showed, respectively, that the necessary conditions are sufficient for all even k between 2 and 16 and all n , and for $k=3,5$ and 7 and for all n .

Latin Squares

An $n \times n$ array $A=(a_{ij})$ with entries from $\{1,2,\dots,n\}$ is called a latin square of order n if it has the property that for $1 \leq j < k \leq n$ $a_{ij} \neq a_{ik}$ and $a_{ji} \neq a_{ki}$. We say that two latin squares $A=(a_{ij})$ and $B=(b_{ij})$, both of order n , are orthogonal if $\{(a_{ij}, b_{ij}) \mid 1 \leq i, j \leq n\} = \{(i, j) \mid 1 \leq i, j \leq n\}$. A latin square is called self-orthogonal if it is orthogonal to its transpose. Finally, a latin square is called horizontally-complete if every ordered pair of distinct elements appears exactly once as adjacent elements in some row of the square.

We begin with the observation that a latin square of order n defines a 1-factorization of $K_{n,n}$, the complete bipartite graph on $2n$ elements. Let the vertices of $K_{n,n}$ be partitioned into two subsets $X=\{x_1, x_2, \dots, x_n\}$ and $Y=\{y_1, y_2, \dots, y_n\}$ so that every edge of $K_{n,n}$ is incident with one vertex from X and one vertex from Y . The n 1-factors are defined as follows. Given a latin square $A=(a_{ij})$ let the edge $x_i y_j$ be in the k^{th} 1-factor, F_k , if and only if $a_{ij}=k$. For example, Figure 2.1 shows a latin square of order four, and Figure 2.2 shows the 1-factorization of $K_{4,4}$ defined by this latin square. In general, we note that any $n \times n$ array with elements from $\{1,2,\dots,k\}$ defines a decomposition of $K_{n,n}$ into k

subgraphs.

$$\Lambda = \begin{array}{cccc}
 1 & 3 & 4 & 2 \\
 4 & 2 & 1 & 3 \\
 2 & 4 & 3 & 1 \\
 3 & 1 & 2 & 4
 \end{array}$$

Figure 2.1

Latin square of order 4

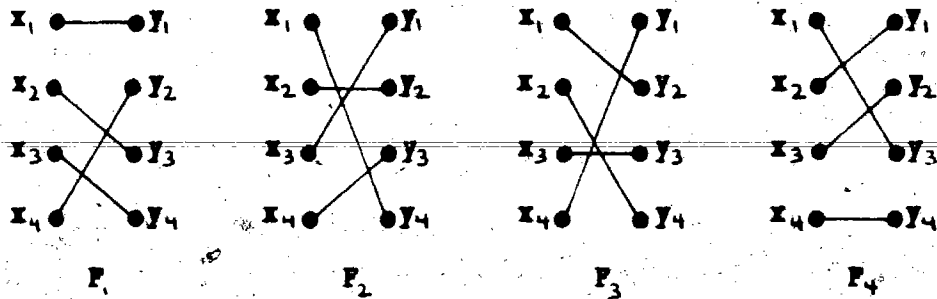


Figure 2.2

A 1-factorization of $K_{4,4}$

Alspach, Heinrich and Rosenfeld have also recognized the relationship between graph decompositions and latin squares [3]. The following theorem is due to them.

Theorem 2.1: If $DK_n \rightarrow DC_{n-1}$, then there exists a self-orthogonal latin square of order n .

The proof of this theorem is simple and is based on the fact that every vertex is missed by exactly one cycle. Label the vertices of DK_n with the integers $1, 2, \dots, n$. Let the latin

square defined by the decomposition of DK_n be $A=(a_{ij})$, and let $a_{ii}=i$. If ij is an edge of the cycle that misses vertex k then let $a_{ij}=k$. The latin square is self-orthogonal since every two cycles intersect along exactly one edge. For instance, $DK_5 \rightarrow DC_4$ (Figure 2.3) and thus there exists a self-orthogonal latin square of order five (Figure 2.4). The converse of this theorem is not always true as the latin square may define some cycles whose lengths are less than $n-1$. Figure 2.5 shows a self-orthogonal latin square of order seven while Figure 2.6 shows that this latin square defines two directed cycles of length three instead of one directed cycle of length six.

1	2	4	3
2	3	5	4
3	4	1	5
4	5	2	1
5	1	3	2

Figure 2.3

$DK_5 \rightarrow DC_4$

1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3
4	3	2	1	5

Figure 2.4

Latin square of order 5

1	3	5	7	2	4	6
7	2	4	6	1	3	5
6	1	3	5	7	2	4
5	7	2	4	6	1	3
4	6	1	3	5	7	2
3	5	7	2	4	6	1
2	4	6	1	3	5	7

Figure 2.5

Latin square of order 7

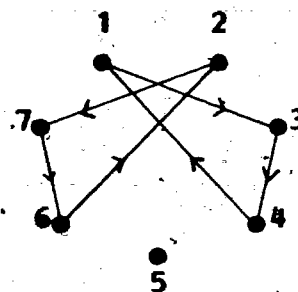


Figure 2.6

Two directed cycles in K_7

Although the converse to Theorem 2.1 does not, in general, hold, we can use the following theorem (see Denes and Keedwell [12]) to obtain, from horizontally complete latin squares, decompositions of the complete symmetric directed graph into Hamiltonian paths and cycles.

It is known that horizontally complete latin squares exist for all even n (see [12]), however not as much information is available for odd n . Nonetheless, horizontally complete latin squares of odd order are known to exist in at least one infinite family (see [23]) and a few isolated cases.

Theorem 2.2: If a horizontally complete latin square of order n exists then

1. DK_n can be partitioned into n Hamiltonian paths, and
2. DK_{n+1} can be partitioned into n Hamiltonian cycles.

Proof. (part one) Label the vertices of DK_n with the n elements of the latin square and let xy be an arc in the i^{th} Hamiltonian path if and only if the element x occurs immediately before the element y in the i^{th} row of the latin square. Since the latin square is horizontally complete the union of the Hamiltonian paths will give us DK_n .

(part two) Label the vertices of DK_{n+1} with the n elements of the latin square and one additional element, say v , and define n paths of length $n-1$ as above. For $i=1, \dots, n$, add the arcs va and bv to the i^{th} path if a and b are the elements in the

first and last columns, respectively, of the i^{th} row of the latin square. Thus these two arcs, together with the $n-1$ arcs in the path, form a directed Hamiltonian cycle in DK_{n+1} . Since the square is latin, we know that these additional $2n$ arcs are all distinct. Hence we have a decomposition of DK_{n+1} into Hamiltonian cycles.

Although horizontally complete latin squares of odd order are only known to exist in a few cases, Tillson [33], using a combination of direct construction and induction, has shown that the decompositions mentioned in Theorem 2.2 exist for all odd n . These decompositions, however, do not give rise to horizontally complete latin squares.

Embeddings

With the recent (1974) solution by Ringel and Youngs of Heawood's Conjecture [26] and the proof of the Four Colour Theorem by Appel and Haken in 1976 (see [35] for a survey of this problem), graph embeddings have been enjoying increased attention. It turns out that some graph embeddings provide decompositions of $2K_n$ into cycles of length $n-1$. We will briefly discuss this connection.

First we will give a few definitions. A rotation of a vertex v of a graph G is an oriented cyclic permutation of all vertices adjacent to v . This permutation assigns an orientation

(clockwise or counterclockwise) to the vertex v . If we are given a rotation for every vertex v in G then we have a rotation of the graph G . It is easy to see that if $G=K_n$ then every cyclic permutation in the rotation of G has length $n-1$. If the graph G can be drawn on the surface S with no edges crossing then we say that G can be embedded into the surface S .

If in an embedding of G into a surface S every region is a triangle, then this is a triangular embedding of G into S . If we have a triangular embedding of K_n in some surface then every pair of distinct vertices u and v are in exactly two triangles of the embedding, say uvx and uvy . Thus u and v are adjacent in the rotation of vertex x and they are also adjacent in the rotation of vertex y . Figure 2.7 is a triangular embedding of K_7 in the torus; identify the pairs of opposite edges as indicated by the vertex labellings. In this embedding every vertex has been assigned a counterclockwise orientation. Figure 2.8 gives us the rotation of K_7 defined by this embedding (where $a. hij\dots$ means that when we rotate counterclockwise around the vertex a , we meet the vertices h, i, j, \dots in that order).

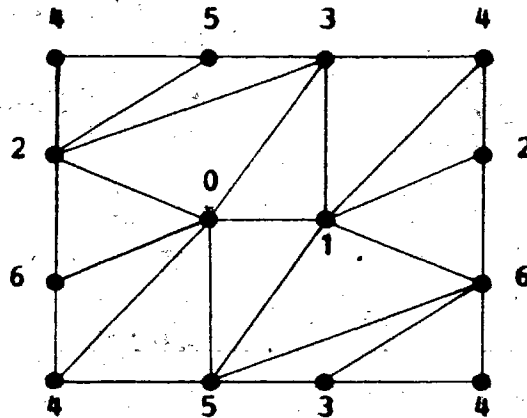


Figure 2.7

Triangular embedding of K_7 in the torus

0.	1	3	2	6	4	5
1.	2	4	3	0	5	6
2.	3	5	4	1	6	0
3.	4	6	5	2	0	1
4.	5	0	6	3	1	2
5.	6	1	0	4	2	3
6.	0	2	1	5	3	4

Figure 2.8

A rotation of K_7

If we let every two adjacent elements in a cyclic permutation define an edge, then a triangular embedding of K_n in some surface gives us a partition of the edges of $2K_n$ into cycles of length $n-1$. We have found in [26] two rotations of complete graphs which give $2K_n \rightarrow C_{n-1}$. One of them is $2K_7 \rightarrow C_6$ and is given in Figure 2.8, and the other is $2K_6 \rightarrow C_5$ and is given in Figure 2.9.

0.	3	1	5	4	2
1.	4	2	5	0	3
2.	0	3	5	1	4
3.	1	4	5	2	0
4.	2	0	5	3	1
5.	0	1	2	3	4

Figure 2.9

A rotation of K_6

In general, however, triangular embeddings of K_n do not give rise to $2K_n \rightarrow C_{n-1}$ as some of the cycles formed may have no edges in common while others may have more than one edge in common.

Ringel also introduced the chord problem as a means to simplify the calculation of a triangular embedding of a graph into a surface. Basically, the chord problem is to find a Hamiltonian path in K_{2s+1} such that for all edges ij in the Hamiltonian path

$$\{l(ij)\} = \{1, 1, 2, 2, \dots, s, s\}$$

where $l(ij) = \min\{|i-j|, 2s+1-|i-j|\}$. His solution to this problem gives a decomposition of K_{2s+1} into Hamiltonian paths, because if hi and jk are two edges in the path such that $l(hi) = l(jk)$ then either $|j-i| = |k-h| = s$ or $|j-h| = |k-i| = s$. Below we have the diagram for $s=6$. This is just another example of a particular path decomposition problem and it should be clear from Figure 2.10 how one obtains a solution to the chord problem for any value of s .

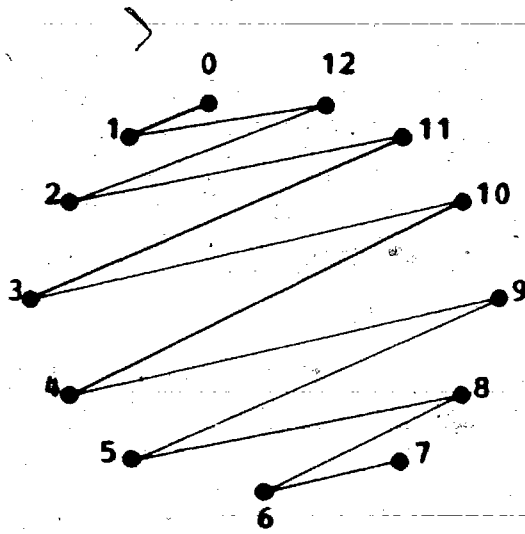


Figure 2.10

The chord problem

III. Definitions

Recall that $2K_n$ denotes the complete multigraph on n vertices in which every edge occurs twice. If e is an edge in $2K_n$ that is incident with the vertices x and y then we will often write xy instead of e . We will indicate a path by its sequence of vertices, and the path $v_0 v_1 \dots v_{k-1} v_k$ has length k . We write P_k to denote a path of length k . Likewise we will indicate a cycle by its sequence of vertices, and the cycle $v_1 v_2 \dots v_{k-1} v_k$ has length k . We write C_k to denote a cycle of length k .

If the vertices of $2K_n$ are labelled with the integers $1, 2, \dots, n$, then the length of an edge xy is

$$\min\{|x-y|, n-|x-y|\}.$$

Thus the lengths of the edges of $2K_n$ belong to $\{1, 2, \dots, m\}$ where $m = \lfloor n/2 \rfloor$.

We will now introduce the concept of rotating an edge. It must be noted here that this definition of rotation is different from the rotation described in the section on Embeddings in Chapter II. If xy is an edge in $2K_n$ with vertices labelled $1, 2, \dots, n$, then by rotating xy k times we mean that we increase each label by k , where arithmetic is done modulo n on the residues $1, 2, \dots, n$. The edge xy rotated k times gives us the edge uv , where $x+k \equiv u \pmod{n}$ and $y+k \equiv v \pmod{n}$. Clearly rotating an edge does not change its length. If the edge xy is rotated k times to give us the edge uv then the distance between xy and uv

is defined to be $\min\{k, n-k\}$.

As well as rotating edges, we can rotate cycles and paths. By rotating a cycle k times we mean that we simultaneously rotate each edge in the cycle k times (see Figure 3.1). Similarly, by rotating a path k times we mean that we simultaneously rotate each edge in the path k times (see Figure 3.2).

The rotation of a fixed cycle C in $2K_n$ is the set of cycles formed when C is rotated k times, for $k=0, 1, \dots, n-1$. Similarly, the rotation of a fixed path P in $2K_n$ is the set of paths formed when P is rotated k times, for $k=0, 1, \dots, n-1$. Thus a rotational decomposition of $2K_n$ into cycles of length $n-1$ is a decomposition obtained by the rotation of a fixed cycle, and the rotational decomposition of $2K_n$ into paths of length $n-1$ is a decomposition obtained by the rotation of a fixed path. Figures 3.1 and 3.2 give, respectively, rotational decompositions of $2K_5$ into cycles of length 4 and paths of length 4.

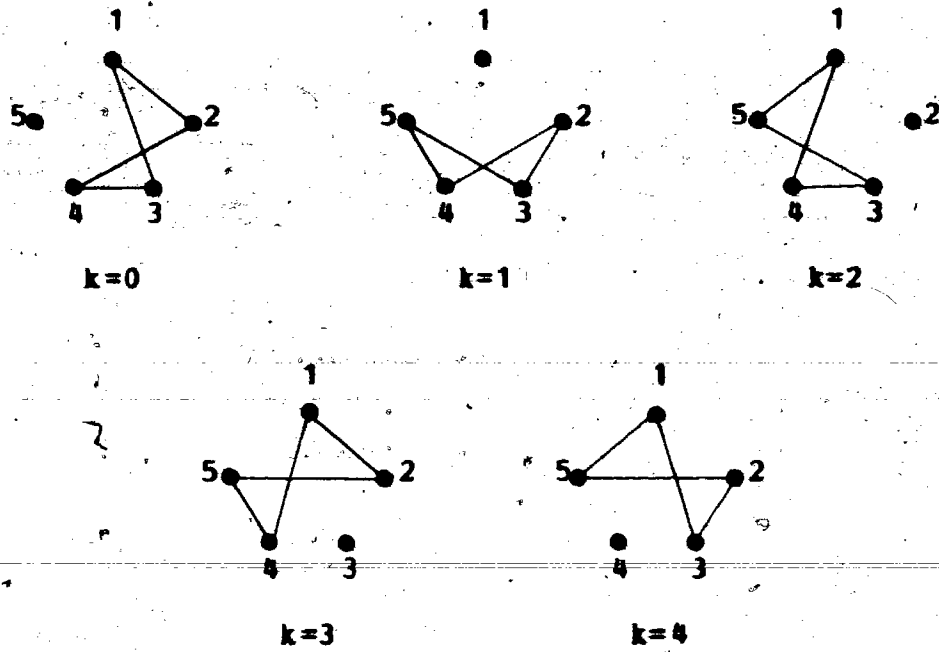


Figure 3.1

Rotational Decomposition of $2K_5 \rightarrow C_4$

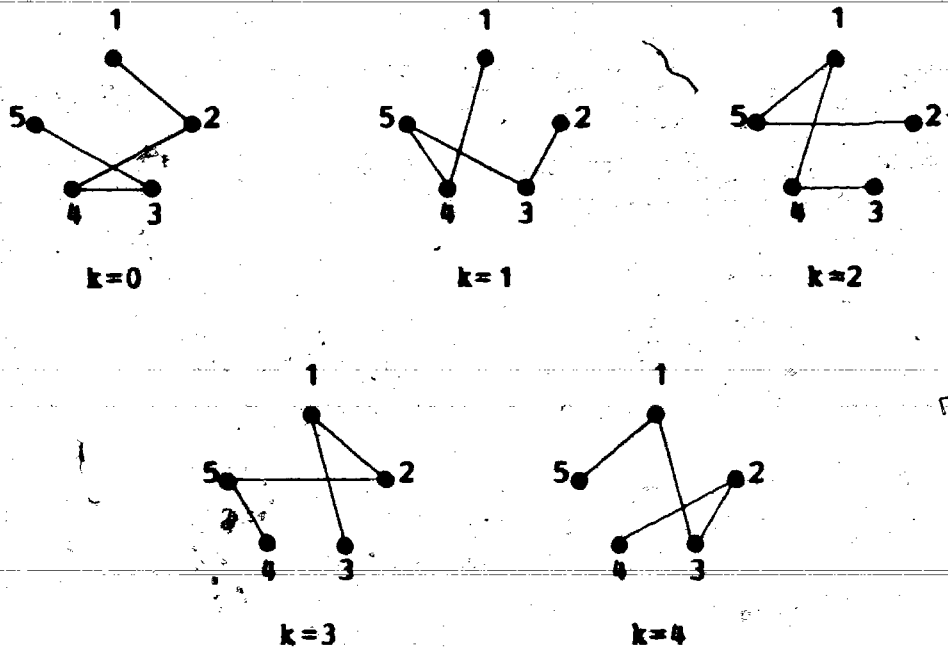


Figure 3.2

Rotational Decomposition of $2K_5 \rightarrow P_5$

We will now give a lemma which indicates when a rotational decomposition of $2K_n$ into cycles of length $n-1$ exists, with the additional property that any two cycles intersect in exactly one edge. This lemma is used in Theorem 5.4. We will then give an equivalent lemma for paths that is used extensively in the Appendix as a means of finding a rotational path decomposition of $2K_n$ with the aforementioned intersection property. First we note that if n is even then any edge of length $n/2$ in $2K_n$ will always coincide with itself after $n/2$ rotations. Thus we can omit this fact from the statement of this lemma and its subsequent applications.

Lemma 3.1a: Let n be even. Then $2K_n \rightarrow C_{n-1}$ by rotating a fixed cycle C if and only if

1. There are two edges of every length $1, 2, \dots, (n/2) - 1$ and one edge of length $n/2$ in C .
2. The integers $1, 2, \dots, (n/2) - 1$ each occur exactly once as the distance between two edges of the same length in C .

Proof. Let C be a cycle of length $n-1$ in $2K_n$, and suppose this cycle satisfies conditions 1 and 2 of the lemma. Let uv and xy be two edges of C that are distance k apart, and suppose that we can rotate uv k times to get xy (as opposed to rotating xy k times to get uv). If C' is the cycle obtained by rotating C k times then C and C' have exactly the edge xy in common.

Moreover, if we rotate C $n-k$ times to get the cycle C'' , then C and C'' have the edge uv in common. Thus for $1 \leq k \leq (n/2) - 1$, we see that for each k there are exactly 2 cycles (obtained by rotating C either k or $n-k$ times) which intersect C exactly once. From the comment preceding this lemma, we also know that the cycle obtained by rotating C $n/2$ times intersects C along the edge of length $n/2$. Thus, in its rotation, C intersects each other cycle exactly once. Since the choice of C was arbitrary (all other cycles in the rotation of C also satisfy the conditions of the lemma) we see that any two cycles in the rotation of C intersect in exactly one edge. Thus $2K_n \rightarrow C_{n-1}$ by rotating the fixed cycle C .

Now suppose that $2K_n \rightarrow C_{n-1}$ by rotating a fixed cycle C . Suppose there is some k , $1 \leq k \leq (n/2) - 1$, such that there are at least three edges of length k in C . Then when we look at the rotation of C , every edge of length k appears at least three times and this contradicts our assumption that this is a decomposition of $2K_n$. Using a similar argument it is easy to see that there is also no k , $1 \leq k \leq (n/2) - 1$, such that there is at most one edge of length k in C . Thus there must be exactly two edges of every length $1, 2, \dots, (n/2) - 1$ in C , and hence one edge of length $n/2$.

Now let uv and xy be two edges in C of length s that are distance k apart, and let $u'v'$ and $x'y'$ be two edges in C of length s' that are also distance k apart. Then the cycle obtained by rotating C k times will have two edges in common with C . Since this contradicts our assumption that $2K_n \rightarrow C_{n-1}$, we see that each possible distance occurs at most once as the distance between two edges of the same length. Since the edge of length $n/2$ intersects itself after $n/2$ rotations, all remaining possible distances each occur exactly once as the distance between two edges of the same length. Thus if $2K_n \rightarrow C_{n-1}$ by rotating a fixed cycle C , then this cycle satisfies conditions 1 and 2 of the lemma.

Lemma 3.1b: Let n be odd. Then $2K_n \rightarrow C_{n-1}$ by rotating a fixed cycle C if and only if

1. There are two edges of every length $1, 2, \dots, (n-1)/2$ in C .

2. The integers $1, 2, \dots, (n-1)/2$ each occur exactly once as the distance between two edges of the same length in C .

Lemma 3.2a: Let n be even. Then $2K_n \rightarrow P_n$ by rotating a fixed path P if and only if

1. There are two edges of every length $1, 2, \dots, (n/2) - 1$ and one edge of length $n/2$ in P .
2. The integers $1, 2, \dots, (n/2) - 1$ each occur exactly once as the distance between two edges of the same length in P .

Lemma 3.2b: Let n be odd. Then $2K_n \rightarrow P_n$ by rotating a fixed path P if and only if

1. There are two edges of every length $1, 2, \dots, (n-1)/2$ in P .
2. The integers $1, 2, \dots, (n-1)/2$ each occur exactly once as the distance between two edges of the same length in P .

The proofs of the above three lemmas are essentially the same as that of Lemma 3.1a and have therefore been omitted.

Using the proof of Theorem 1.4 we see that the decomposition $2K_n \rightarrow C_{n-1}$, where n is a power of a prime, is rotational over $GF(n)$. Some of the smaller values of n for which $2K_n \rightarrow P_n$ using a rotational decomposition are listed in the Appendix. It is interesting to note that there is no rotational

decomposition of $2K_7 \rightarrow P_7$. This can easily be seen by looking at all paths of length six on seven vertices that have two edges of length one, two edges of length two and two edges of length three and seeing that none of these paths satisfies condition 2 of Lemma 3.2b. We used a computer to search for a set of seven paths of length six in $2K_7$, any two of which intersect along exactly one edge. The program used actually found hundreds of such non-isomorphic decompositions, and the algorithm for this program is given in the Appendix, along with one of the decompositions found.

The following lemma gives us some information concerning the labelling of the end vertices of each path in a path decomposition and is used in the proofs of Theorems 4.1, 4.2, 4.3 and 5.1.

Lemma 3.3: If $2K_n$ has a path decomposition with the property that any two paths have exactly one edge in common, then each path must have length $n-1$. Moreover, it is possible to assign a direction to each path so that every vertex in $2K_n$ is the initial vertex of a path and the terminal vertex of a path.

Proof. In order for every path to have exactly one edge in common with each of the remaining paths it is necessary that the number of paths in the decomposition be one more than the number of edges in each path. Since there are $n(n-1)$ edges in $2K_n$ it follows that each path must have length $n-1$.

Since each vertex in $2K_n$ has degree $2n-2$ and each of the n paths in the decomposition contributes either 1 or 2 to the degree of each vertex we see that each vertex is an end vertex of exactly two paths. The following algorithm describes how to assign a direction to each path (by labelling its end vertices) so that every vertex in $2K_n$ is the initial vertex of a path and the terminal vertex of a path.

1. Let $k=1$.
2. Choose any path in the decomposition that has not yet been labelled and call it Q_k . Arbitrarily label its end vertices s_k (initial) and t_k (terminal). Let $i=k+1$.
3. Let Q_i be the path that has the vertex labelled t_{i-1} as an end vertex. Label this end vertex of Q_i with s_i . Now this vertex has two labels attached to it. Label the other end vertex of Q_i with t_i .
4. If the vertex labelled t_i has received only one label (namely t_i) then let $i=i+1$ and go to 3, otherwise go to 5.
5. The vertex labelled t_i has received two labels, s_k and t_i . If $i=n$ then the terminal vertex of Q_n coincides with the initial vertex of Q_k , and the labelling is complete. If $i < n$ then i vertices in $2K_n$ have received two labels and $n-i$ vertices have received no labels. In this case, let $k=i+1$ and go to 2.

It is easy to see that this process terminates and that it does, in fact, result in every vertex being the initial vertex of one path and the terminal vertex of another path.

IV. Path Decompositions

In this chapter we present three theorems and a corollary which, together with the decompositions $2K_n \rightarrow P_n$ that are listed explicitly in the Appendix, give several infinite families of complete multigraphs (having multiplicity two) which can be decomposed into Hamiltonian paths having the property that any two paths have exactly one edge in common.

All of the work in this chapter is original, and the multiplication method used in the proofs of the theorems in this chapter is again used in Chapter V.

Theorem 4.1: If $2K_n \rightarrow P_n$ then $2K_{5n} \rightarrow P_{5n}$.

Proof. Label the vertices of $2K_n$ with the integers $1, 2, \dots, n$ and let the paths in the decomposition be labelled Q_1, Q_2, \dots, Q_n . Thus for any i and j , with $1 \leq i < j \leq n$, Q_i and Q_j intersect in exactly one edge. Associated with each path Q_j there is an initial vertex s_j and a terminal vertex t_j , so that every vertex in $2K_n$ is the initial vertex of a path and the terminal vertex of a path. (This follows from Lemma 3.3.) Thus we have assigned a direction to each path Q_j (directed from s_j to t_j) so that it now consists of arcs ab . We will use the word edge instead of arc when we wish to ignore the direction assigned to the paths.

Label all arcs of Q_1 with 0. Now look at Q_j , for $j=2, 3, \dots, n$. If the arc ab of Q_j has already been labelled 0 in the labelling of Q_1, \dots, Q_{j-1} then now label it 1. If the arc ab has not yet been labelled then label it 0. In doing this we see that if the arc ab is on two paths then it is once labelled 0 and once labelled 1. If, however, the arc ab is on one path and the arc ba is on another path then they are both labelled 0.

Let $A=(a_{ij})$ be the self-orthogonal latin square of order 5 defined by $a_{ij}=2j-i$ (see Figure 4.1), where arithmetic is performed modulo 5 on the residues $1, 2, \dots, 5$.

1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1
2	4	1	3	5

Figure 4.1

Self-orthogonal latin square of order 5

This latin square defines five permutations, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 , on five symbols each by $\alpha_s(i)=j$ if and only if $a_{ij}=s$, $1 \leq i \leq 5$. For example, $\alpha_1=(2453)$. In addition, this latin square defines a 1-factorization of $K_{5,5}$. The five 1-factors, F_1, F_2, F_3, F_4 and F_5 are shown in Figure 4.2.

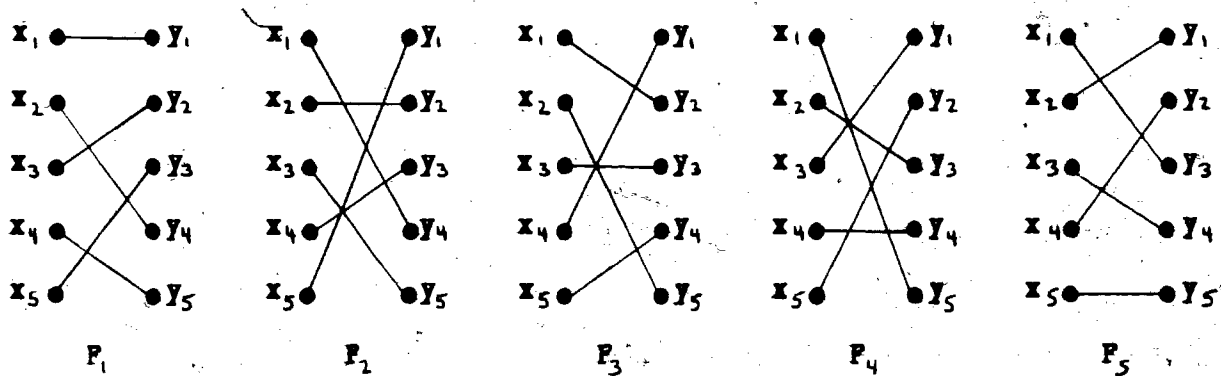


Figure 4.2

The 1-factorization of $K_{5,5}$ defined by A

Each 1-factor F_s is related to the permutation $\alpha_s, 1 \leq s \leq 5$, by $\alpha_s(i) = j$ if and only if in F_s there is an edge from x_i to y_j . The five 1-factors defined by the transpose of A , which we will denote by A' , are called F'_1, F'_2, F'_3, F'_4 , and F'_5 . The 1-factor F'_i is related to the permutation $\alpha_s, 1 \leq s \leq 5$, by $\alpha_s(j) = i$ if and only if in F'_i there is an edge from x_i to y_j . The 1-factors defined by A' are given in Figure 4.3.

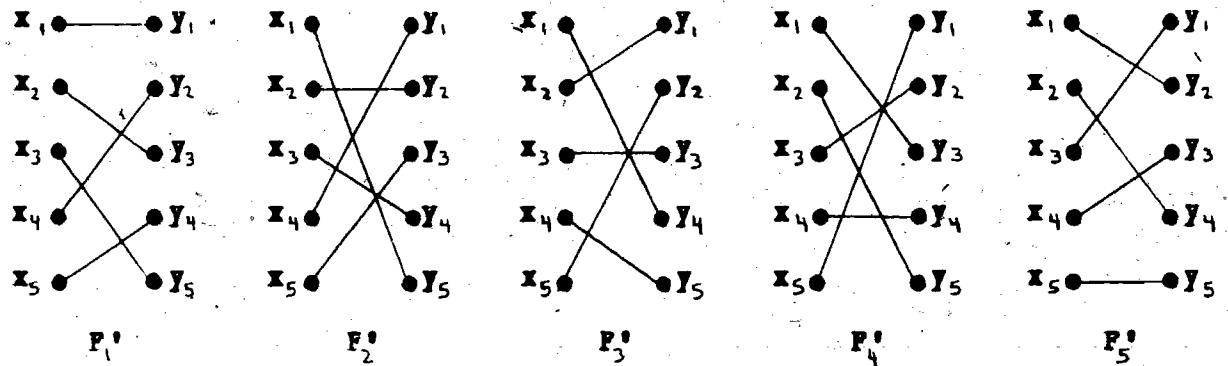


Figure 4.3

The 1-factorization of $K_{5,5}$ defined by A^*

Arrange the vertices of K_{5n} in a $5n \times n$ array called G , and let $G(i, j)$ be the vertex in the i^{th} row and j^{th} column of G . For each path Q_j we will define 5 paths, q_{1j} , q_{2j} , q_{3j} , q_{4j} and q_{5j} , each of length $5n-1$, with vertices in G .

To get the first path, q_{1j} , we will replace each arc of Q_j with one of the subgraphs of $K_{5,5}$ defined previously. We do this as follows. For each arc xy in Q_j :

- 1* If xy is labelled 0 then q_{1j} contains the edges $G(1, x)G(1, y)$, $G(2, x)G(4, y)$, $G(3, x)G(2, y)$, $G(4, x)G(5, y)$ and $G(5, x)G(3, y)$. These edges are those defined by F_1 . We say that the arc xy of Q_j is replaced by the 1-factor F_1 in G .
- 2* If xy is labelled 1 then q_{1j} contains the edges $G(1, x)G(1, y)$, $G(2, x)G(3, y)$, $G(3, x)G(5, y)$, $G(4, x)G(2, y)$ and $G(5, x)G(4, y)$. These edges are those defined by the 1-factor F_2 . We say that the arc xy of Q_j is replaced by the 1-factor F_2 in G .

This defines five subpaths, each of length $n-1$, of q_{ij} which must be connected by four edges to give us q_{ij} . First we note that 1* and 2* can be restated as follows. For each arc xy in Q_j :

1. If xy is labelled 0, then q_{ij} contains the edges $G(a,x)G(b,y)$ if and only if $\alpha_1(a)=b$, where $1 \leq a \leq 5$.
2. If xy is labelled 1, then q_{ij} contains the edges $G(a,x)G(b,y)$ if and only if $\alpha_1(b)=a$, where $1 \leq b \leq 5$.

Now we must look at the subpaths of q_{ij} defined above. Suppose the subpath of q_{ij} starting at vertex $G(a,s_j)$, $a \neq 1$, ends at vertex $G(b,t_j)$. Then for some r , $\alpha_1^r(a)=b$. Since $\alpha_1^4=e$, where e is the identity permutation, we may assume that $1 \leq r \leq 4$. This allows us to compute the terminal vertices of each of the subpaths in q_{ij} . Thus instead of looking at the subpaths, we need look only at the edges $G(a,s_j)G(b,t_j)$ of $K_{5,5}$, where $b = \alpha_1^r(a)$ and $1 \leq a \leq 5$. Figure 4.4 shows what these subgraphs look like for the various values of r . Note that the path starting at $G(1,s_j)$ always ends at $G(1,t_j)$.

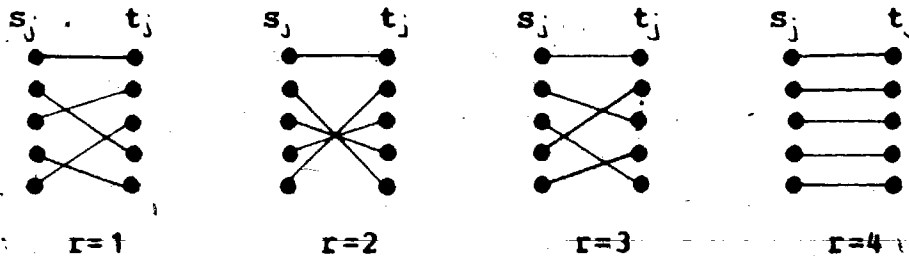


Figure 4.4

Possible start and end vertices of the subpaths of q_{ij}

If there is a set of edges which can be added to any of the subgraphs of K_{55} given in Figure 4.4 to form a path of length 9, then these edges can be added to the subpaths of q_{ij} to form a path of length $5n-1$.

Figure 4.5 below shows the subgraphs of Figure 4.4 with a set of four additional edges which form a path of length 9 in each case. (We comment that the additional edges form a path in a decomposition of $2K_5 \rightarrow P_5$.)

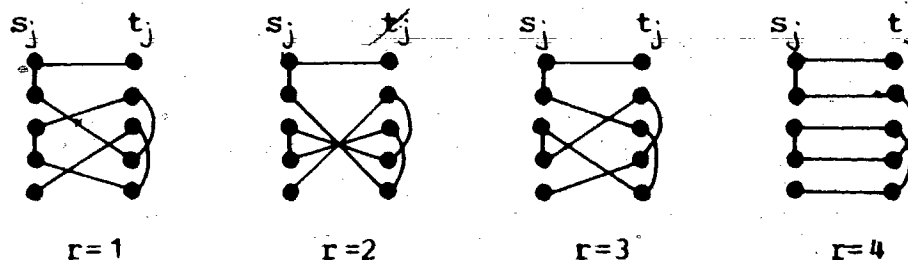


Figure 4.5

Closing edges to be used with graphs in Figure 4.4

It is clear that replacing each of the five edges between s_j and t_j with their corresponding subpaths will not form any cycles. Thus if the edges $G(1, s_j)G(2, s_j)$, $G(3, s_j)G(4, s_j)$, $G(2, t_j)G(4, t_j)$ and $G(3, t_j)G(5, t_j)$ are added to the five subpaths of q_{ij} we get a path of length $5(n-1)+4=5n-1$. This path is q_{ij} .

To get q_{ij} , $2 \leq i \leq 5$, we do the following. For each arc xy in Q_j :

1. If xy is labelled 0 then q_{ij} contains the edges $G(i, x)G(i, y)$, $G(i+1, x)G(i+3, y)$, $G(i+2, x)G(i+1, y)$.

$G(i+3,x)G(i+4,y)$ and $G(i+4,x)G(i+2,y)$. That is, we replace the arc xy of Q_j by the 1-factor F_i in G .

2. If xy is labelled 1 then q_{ij} contains the edges $G(i,x)G(i,y)$, $G(i+1,x)G(i+2,y)$, $G(i+2,x)G(i+4,y)$, $G(i+3,x)G(i+1,y)$ and $G(i+4,x)G(i+3,y)$. That is, we replace the arc xy of Q_j by the 1-factor F_i' in G .

This defines five subpaths of q_{ij} which must be connected by four edges to give us q_{ij}' . These four edges are $G(i,s_j)G(i+1,s_j)$, $G(i+2,s_j)G(i+3,s_j)$, $G(i+1,t_j)G(i+3,t_j)$ and $G(i+2,t_j)G(i+4,t_j)$. Since the edges of q_{ij}' are obtained from the edges of q_{ij} by relabelling the rows of G it follows that the four edges added above do indeed give us a path of length $5n-1$.

Thus for each path Q_j , $1 \leq j \leq n$, we have defined five paths, q_{1j} , q_{2j} , q_{3j} , q_{4j} and q_{5j} , of length $5n-1$ on the vertices of G . Moreover, all edges in $2K_{5n}$ have been used.

We must now check that any two paths have exactly one edge in common.

Suppose the two paths are q_{aj} and q_{bj} ($1 \leq a < b \leq 5$); these paths are both obtained from Q_j . If xy is an arc of Q_j then q_{aj} and q_{bj} do not intersect in any edge of the form $G(r,x)G(t,y)$. This is because the arc xy was replaced, in q_{aj} and q_{bj} respectively, by either F_a and F_b (xy is labelled 0) or by F_a' and F_b' (xy is labelled 1), and F_a and F_b are edge disjoint, as are F_a' and F_b' .

Since the "closing" edges come from the decomposition $2K_5 \rightarrow P_5$, and any two of these paths have exactly one edge in

common, we know that q_{aj} and q_{bj} have exactly one edge in common (Figure 4.6).

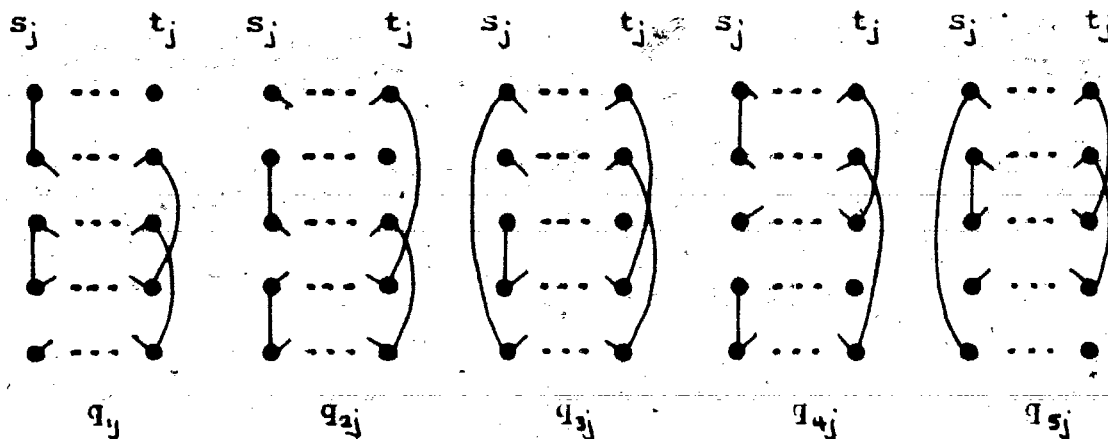


Figure 4.6

The different closing edges required for q_{ij} , $1 \leq i \leq 5$

Now suppose the two paths are q_{aj} and q_{bk} , where $1 \leq a, b \leq 5$. Since these paths were defined by Q_j and Q_k , respectively, we see that q_{aj} and q_{bk} do not intersect in any edge of the form $G(r, v)G(t, v)$, where $v = s_j, s_k, t_j$ or t_k . (These are the closing edges.) We know Q_j and Q_k have exactly one edge in common, say xy . Then either xy is an arc in both Q_j and Q_k , or xy is an arc of Q_j and yx is an arc of Q_k .

In the first case, assuming $j < k$, the arc xy is labelled 0 in Q_j and 1 in Q_k . Here we replaced xy with P_a in constructing q_{aj} and we replaced xy with P'_b in constructing q_{bk} . Since A is self-orthogonal, P_a and P'_b have exactly one edge in common. Hence q_{aj} and q_{bk} have exactly one edge in common.

In the second case, the arc xy is labelled 0 in Q_j and the arc yx is labelled 0 in Q_k . Here we replaced the arc xy with F_a in constructing q_{aj} and we replaced the arc yx with F_b in constructing q_{bk} . Since replacing the arc yx with F_b is equivalent to replacing the arc xy with F_b' , and since F_a and F_b' have exactly one edge in common, we see that q_{aj} and q_{bk} have exactly one edge in common.

Thus any two paths have exactly one edge in common. ■

Theorem 4.2: If $2K_n \rightarrow P_n$ then $2K_{3n} \rightarrow P_{13n}$.

Proof: The proof of this theorem is identical to the proof of Theorem 4.1, except

1. The matrix $A=(a_{ij})$ is of order 13 and is defined by $a_{ij} = 6j - 5i$, where arithmetic is done modulo 13 on the residues $1, 2, \dots, 13$. The permutations α_i defined by this latin square are such that $\alpha_i^3 = e$, $1 \leq i \leq 13$.
2. The twelve edges that are required to connect the 13 subpaths of q_{ij} are $G(1, s_j)G(2, s_j)$, $G(8, s_j)G(4, s_j)$, $G(9, s_j)G(5, s_j)$, $G(3, s_j)G(6, s_j)$, $G(11, s_j)G(12, s_j)$, $G(10, s_j)G(7, s_j)$, $G(2, t_j)G(8, t_j)$, $G(4, t_j)G(9, t_j)$, $G(5, t_j)G(3, t_j)$, $G(6, t_j)G(11, t_j)$, $G(12, t_j)G(10, t_j)$ and $G(7, t_j)G(13, t_j)$. Note that the union of these edges is the path mentioned in the Appendix that gives us $2K_n \rightarrow P_n$ under rotation.

Figure 4.7, below, is the latin square A that is defined by $a_{ij} = 6j - 5i$. Figure 4.8 shows three subgraphs of $K_{13,13}$ obtained by letting $(x,s)(y,t)$ be an edge in the subgraph $(r=b)$ if and only if $\alpha_r^b(x) = y$. Figure 4.9 shows that we do indeed get paths on adding the twelve closing edges.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	7	13	6	12	5	11	4	10	3	9	2	8
2	9	2	8	1	7	13	6	12	5	11	4	10	3
3	4	10	3	9	2	8	1	7	13	6	12	5	11
4	12	5	11	4	10	3	9	2	8	1	7	13	6
5	7	13	6	12	5	11	4	10	3	9	2	8	1
6	2	8	1	7	13	6	12	5	11	4	10	3	9
7	10	3	9	2	8	1	7	13	6	12	5	11	4
8	5	11	4	10	3	9	2	8	1	7	13	6	12
9	13	6	12	5	11	4	10	3	9	2	8	1	7
10	8	1	7	13	6	12	5	11	4	10	3	9	2
11	3	9	2	8	1	7	13	6	12	5	11	4	10
12	11	4	10	3	9	2	8	1	7	13	6	12	5
13	6	12	5	11	4	10	3	9	2	8	1	7	13

Figure 4.7

Self-orthogonal latin square of order 13

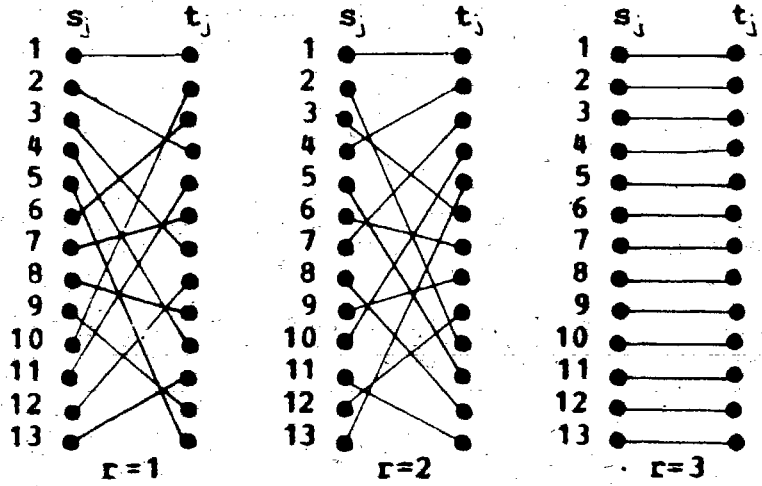


Figure 4.8

Possible start and end vertices of the subpaths of q_{ij}

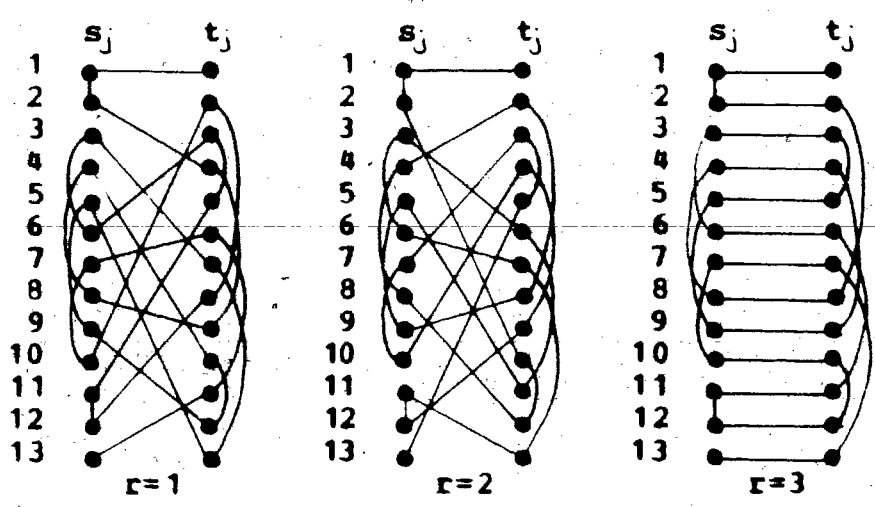


Figure 4.9

Closing edges to be used with graphs in Figure 4.8

Theorem 4.3: If $2K_n \rightarrow P_n$ then $2K_{nn} \rightarrow P_{nn}$.

Proof. Again, the proof of this theorem follows from the proof

of Theorem 4.1. However, we have the following changes to note.

1. The matrix $A=(a_{ij})$ is of order 17 and is defined by $a_{ij} = 7j - 6i$, where we are working modulo 17 on the residues $1, 2, \dots, 17$. The permutations, α_i , defined by this matrix are all of order four:
2. The sixteen edges that are required to connect the 17 subpaths of g_{ij} are given by $G(1, s_j)G(3, s_j)$, $G(13, s_j)G(5, s_j)$, $G(8, s_j)G(7, s_j)$, $G(2, s_j)G(6, s_j)$, $G(16, s_j)G(14, s_j)$, $G(11, s_j)G(15, s_j)$, $G(9, s_j)G(10, s_j)$, $G(4, s_j)G(12, s_j)$, $G(3, t_j)G(13, t_j)$, $G(5, t_j)G(8, t_j)$, $G(7, t_j)G(2, t_j)$, $G(6, t_j)G(16, t_j)$, $G(14, t_j)G(11, t_j)$, $G(15, t_j)G(9, t_j)$, $G(10, t_j)G(4, t_j)$ and $G(12, t_j)G(17, t_j)$. Note that the union of these edges is a path P of length 16 whose rotation gives $2K_{17} \rightsquigarrow P_{17}$, and this path appears in the Appendix.

Figure 4.10, below, gives the latin square A of order 17 defined previously. Figure 4.11 shows four subgraphs of $K_{17,17}$ where $(x, s)(y, t)$ is an edge in the subgraph ($r=b$) if and only if $\alpha_i^b(x) = y$. Figure 4.12 shows these same subgraphs together with the closing edges. It is easy to check that no cycles are formed.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	8	15	5	12	2	9	16	6	13	3	10	17	7	14	4	11
2	12	2	9	16	6	13	3	10	17	7	14	4	11	1	8	15	5
3	6	13	3	10	17	7	14	4	11	1	8	15	5	12	2	9	16
4	17	7	14	4	11	1	8	15	5	12	2	9	16	6	13	3	10
5	11	1	8	15	5	12	2	9	16	6	13	3	10	17	7	14	4
6	5	12	2	9	16	6	13	3	10	17	7	14	4	11	1	8	15
7	16	6	13	3	10	17	7	14	4	11	1	8	15	5	12	2	9
8	10	17	7	14	4	11	1	8	15	5	12	2	9	16	6	13	3
9	4	11	1	8	15	5	12	2	9	16	6	13	3	10	17	7	14
10	15	5	12	2	9	16	6	13	3	10	17	7	14	4	11	1	8
11	9	16	6	13	3	10	17	7	14	4	11	1	8	15	5	12	2
12	3	10	17	7	14	4	11	1	8	15	5	12	2	9	16	6	13
13	14	4	11	1	8	15	5	12	2	9	16	6	13	3	10	17	7
14	8	15	5	12	2	9	16	6	13	3	10	17	7	14	4	11	1
15	2	9	16	6	13	3	10	17	7	14	4	11	1	8	15	5	12
16	13	3	10	17	7	14	4	11	1	8	15	5	12	2	9	16	6
17	7	14	4	11	1	8	15	5	12	2	9	16	6	13	3	10	17

Figure 4.10

Self-orthogonal latin square of order 17

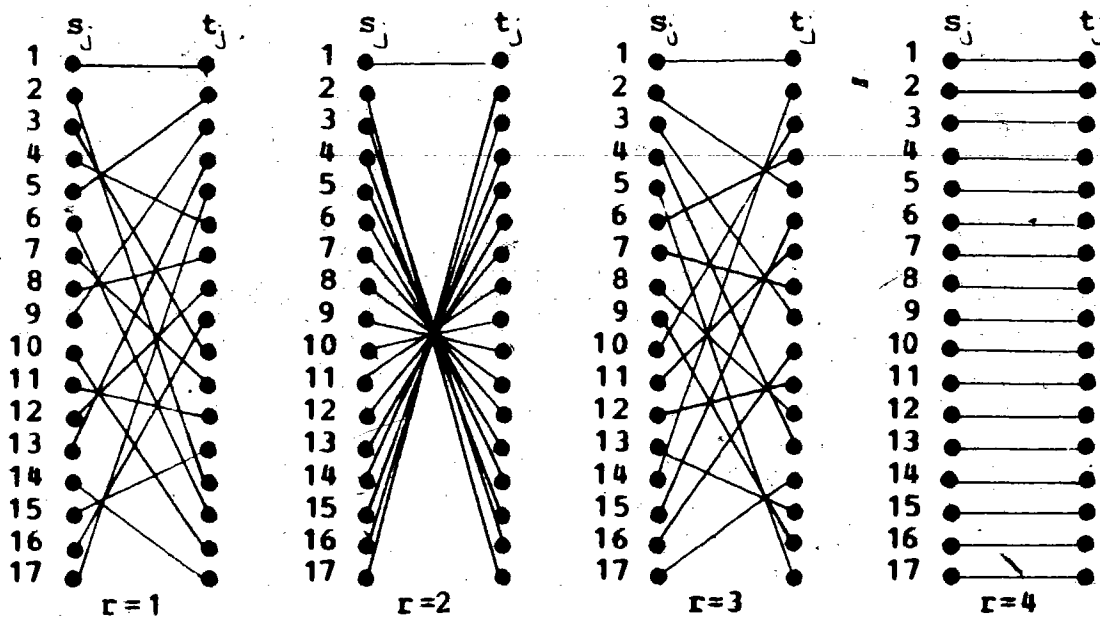


Figure 4.11

Possible start and end vertices of the subpaths of q_j

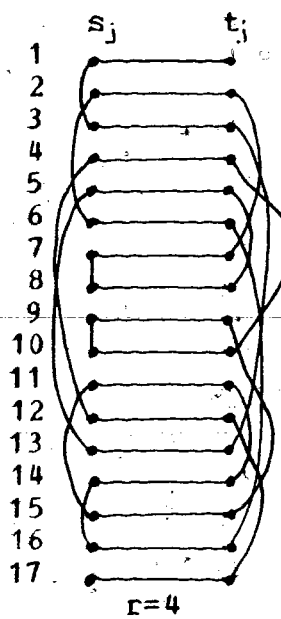
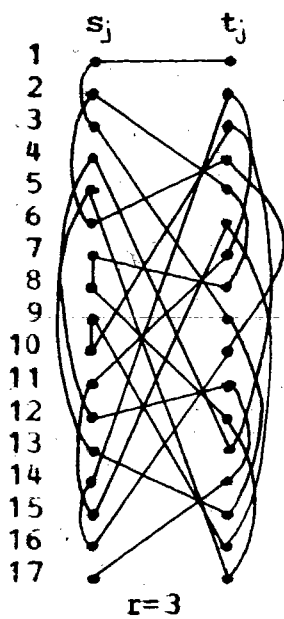
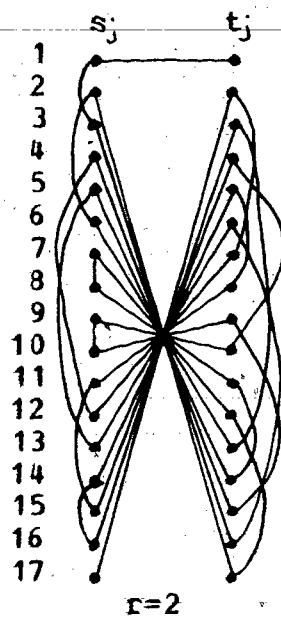
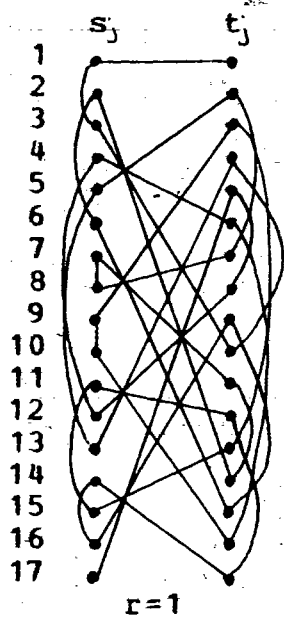


Figure 4.12

Closing edges to be used with graphs in Figure 4.11

Corollary 4.4: If $2K_n \rightarrow P_n$ then $2K_{dn} \rightarrow P_{dn}$, where $d=5^a 13^b 17^c$ and a, b and c are natural numbers.

Proof. This follows from Theorems 4.1, 4.2 and 4.3. ■

V. Cycle Decompositions

In this chapter we will expand on the work done by Alspach, Heinrich and Rosenfeld in [3] by using the multiplication method of the previous chapter. We will first show that the existence of a decomposition of $2K_n$ into Hamiltonian paths having the property that any two paths intersect in exactly one edge implies the existence of a decomposition of DK_{4n} into directed cycles of length $4n-1$ having the property that any two cycles intersect in exactly one oppositely directed edge. This gives us several infinite families of values of n so that DK_n can be decomposed into directed cycles having the required intersection property.

Theorem 5.1: If $2K_n \rightarrow P_n$ then $DK_{4n} \rightarrow DC_{4n-1}$.

We will prove this theorem in two parts. First we will show that if $2K_n \rightarrow P_n$ then $2K_{4n} \rightarrow C_{4n-1}$, and then we will show that it is possible to orient the edges in each cycle so that what we get is actually $DK_{4n} \rightarrow DC_{4n-1}$.

Proof (Part One). Label the vertices of $2K_n$ with the integers $1, 2, \dots, n$ and let the paths in the decomposition be labelled Q_1, Q_2, \dots, Q_n . Thus for any i and j , with $1 \leq i < j \leq n$, Q_i and Q_j intersect in exactly one edge. Associated with each path Q_i

there is an initial vertex s_j and a terminal vertex t_j , so that every vertex in $2K_n$ is the initial vertex of a path and the terminal vertex of a path. (This follows from Lemma 3.3.) Thus we have assigned a direction to each path Q_j (directed from s_j to t_j) so that it now consists of arcs ab . We will again use the word edge instead of arc when we wish to ignore the direction assigned to the paths.

Label all arcs of Q_1 with 0. Now look at Q_j , for $j=2, 3, \dots, n$. If the arc ab of Q_j has already been labelled 0 then now label it 1. If the arc ab has not yet been labelled then label it 0. In doing this we see that if the arc ab is on two paths then it is once labelled 0 and once labelled 1. If, instead, the arc ab is on one path and the arc ba is on another path then they are both labelled 0. This labelling scheme was also used in the proof of Theorem 4.1.

The self-orthogonal matrix $A=(a_{ij})$ (Figure 5.1) defines four subgraphs, P_1, P_2, P_3 and P_4 , which partition the edges of $K_{4,4}$ (Figure 5.2).

1	4	4	2
3	2	1	3
2	4	3	2
3	1	1	4

Figure 5.1

Self-orthogonal matrix of order 4

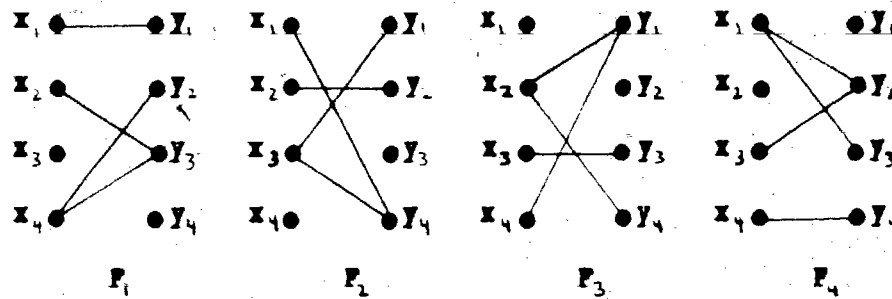


Figure 5.2

Subgraphs of $K_{4,4}$ defined by A

If F_1', F_2', F_3' and F_4' (Figure 5.3) are the four subgraphs of $K_{4,4}$ defined by A' (the transpose of A), then for $1 \leq j, k \leq 4$ we see that F_j and F_k' have exactly one edge in common.

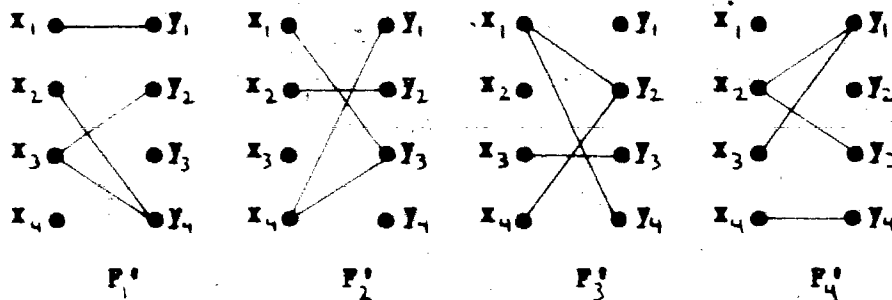


Figure 5.3

Subgraphs of $K_{4,4}$ defined by A'

Arrange the vertices of K_{4n} in a $4 \times n$ array, G , and let $G(i, j)$ be the vertex in the i^{th} row and j^{th} column of G . For each path Q_j we define four cycles, c_{1j} , c_{2j} , c_{3j} and c_{4j} , each of length $4n-1$ with vertices in G .

First we show how to get c_{ij} . Begin by replacing each arc of Q_j with one of the subgraphs of $K_{4,4}$ defined above. We do this as follows. For each arc xy in Q_j :

1. If xy is labelled 0 then c_{ij} contains the edges $G(1,x)G(1,y)$, $G(2,x)G(3,y)$, $G(4,x)G(2,y)$, and $G(4,x)G(3,y)$. These edges are those defined by F_1 . We say that the arc xy of Q_j is replaced by the subgraph F_1 in G .
2. If xy is labelled 1 then c_{ij} contains the edges $G(4,x)G(1,y)$, $G(2,x)G(2,y)$, $G(1,x)G(3,y)$, and $G(4,x)G(3,y)$. These edges are those defined by F_2' . and we say that the arc xy of Q_j is replaced by the subgraph F_2' in G .

This defines two subpaths of c_{ij} which must be connected by three edges to give us c_{ij} . Note that the subpath which starts at vertex $G(1,s_j)$ always ends at vertex $G(1,t_j)$, and the subpath that starts at vertex $G(2,s_j)$ always ends at vertex $G(2,t_j)$. Thus the initial and terminal vertices of each subpath do not depend on the length of Q_j or the labelling of the edges in Q_j .

In forming the cycle c_{ij} we will omit the vertex $G(4,t_j)$ and add the three edges $G(1,s_j)G(3,s_j)$, $G(3,s_j)G(2,s_j)$ and $G(1,t_j)G(2,t_j)$. These edges come from the decomposition $2K_4 \rightarrow C_3$, and the reason they were chosen will be explained later.

Thus c_{ij} is given as:

$$G(1,s_j) \dots G(1,t_j)G(2,t_j) \dots G(2,s_j)G(3,s_j)G(1,s_j).$$

To get c_{2j} we do the following. For each arc xy of Q_j :

1. If xy is labelled 0 then replace xy by F_2 in G .

2. If xy is labelled 1 then replace xy by P_1' in G .

This defines two subpaths of c_{2j} which have the same property as the two subpaths of c_{1j} . That is, the subpath that starts at $G(1, s_j)$ ends at $G(1, t_j)$ and the subpath that starts at $G(2, s_j)$ ends at $G(2, t_j)$.

In forming the cycle c_{2j} we will omit the vertex $G(3, t_j)$ and add the three edges $G(1, s_j)G(4, s_j)$, $G(2, s_j)G(4, s_j)$ and $G(1, t_j)G(2, t_j)$.

Thus c_{2j} is given as:

$$G(1, s_j) \dots G(1, t_j)G(2, t_j) \dots G(2, s_j)G(4, s_j)G(1, s_j).$$

To get c_{3j} we do the following. For each arc xy of Q_j :

1. If xy is labelled 0 then replace xy by P_2 in G .
2. If xy is labelled 1 then replace xy by P_2' in G .

This defines two subpaths as before. In completing the cycle c_{3j} we will omit the vertex $G(2, t_j)$ and add the three edges $G(1, s_j)G(3, s_j)$, $G(1, s_j)G(4, s_j)$ and $G(3, t_j)G(4, t_j)$.

Thus c_{3j} is given as:

$$G(3, s_j) \dots G(3, t_j)G(4, t_j) \dots G(4, s_j)G(1, s_j)G(3, s_j).$$

Finally, to get c_{4j} we do the following. For each arc xy of Q_j :

1. If xy is labelled 0 then replace xy by P_3 in G .
2. If xy is labelled 1 then replace xy by P_3' in G .

Again, this defines two subpaths as before. In completing the cycle c_{4j} we will omit the vertex $G(1, t_j)$ and add the three edges $G(2, s_j)G(3, s_j)$, $G(2, s_j)G(4, s_j)$ and $G(3, t_j)G(4, t_j)$.

Thus c_{4j} is given as:

$$G(3, s_j) \dots G(3, t_j) G(4, t_j) \dots G(4, s_j) G(2, s_j) G(3, s_j).$$

We now note that since every vertex is the terminal vertex of some path Q_j then every vertex of G is left out of exactly one cycle.

Now we must check to see that any two cycles have exactly one edge in common.

Suppose the two cycles are c_{aj} and c_{bj} ($1 \leq a < b \leq 4$); these cycles are obtained from the same path Q_j . If xy is an arc of Q_j then c_{aj} and c_{bj} do not intersect in any edge of the form $G(r, x)G(t, y)$. This is because the arc xy was replaced, in c_{aj} and c_{bj} respectively, by either F_a and F_b (xy is labelled 0), or F_{a-1}^1 and F_{b-1}^1 (xy is labelled 1, and both a and b are even), or F_{a-1}^0 and F_{b-1}^0 (xy is labelled 1, and both a and b are odd), or F_{a-1}^1 and F_{b-1}^0 (xy is labelled 1, a is even and b is odd), or F_{a-1}^0 and F_{b-1}^1 (xy is labelled 1, a is odd and b is even). In all five of these cases the pairs of subgraphs are edge disjoint.

However, since the "closing" edges come from the decomposition $2K_n \rightarrow C_3$, and any two of these triangles have exactly one edge in common, we know that c_{aj} and c_{bj} have exactly one edge in common (Figure 5.4).

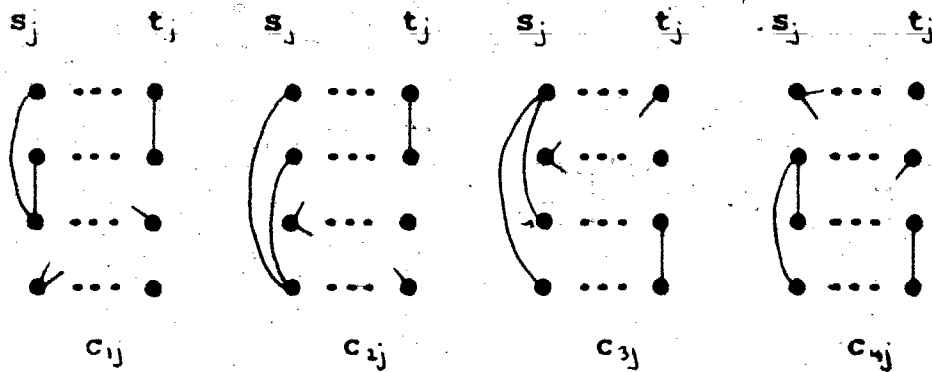


Figure 5.4

The closing edges for the cycles

If c_{a_j} and c_{b_k} were defined by two different paths, say Q_j and Q_k , then c_{a_j} and c_{b_k} do not intersect in any edge of the form $G(r, v)G(t, v)$, where $v = s_j, s_k, t_j,$ or t_k . (These are the closing edges.) We know that Q_j and Q_k have exactly one edge in common, say xy . Then either xy is an arc in both Q_j and Q_k or xy is an arc of Q_j and yx is an arc of Q_k .

In the first case, assuming $j < k$, the arc xy is labelled 0 in Q_j and 1 in Q_k . Here we replaced xy with F_a in constructing c_{a_j} and we replaced xy with either F_{b-1}^0 or F_{b+1}^0 (depending on whether b is even or odd) in constructing c_{b_k} . We know that F_a has exactly one edge in common with F_{b-1}^0 and exactly one edge in common with F_{b+1}^0 , so c_{a_j} and c_{b_k} have exactly one edge in common.

In the second case, the arc xy is labelled 0 in Q_j and the arc yx is labelled 0 in Q_k . Here we replaced the arc xy with F_a in constructing c_{a_j} and we replaced the arc yx with F_b in constructing c_{b_k} . It is easy to see that replacing the arc yx

with F_b is equivalent to replacing the arc xy with F_b' , and since F_a and F_b' have exactly one edge in common it follows that c_{aj} and c_{bk} have exactly one edge in common.

Thus any two cycles have exactly one edge in common.

(Part two). The above construction actually gives us $DK_{4n} \rightarrow DC_{4n-1}$ as follows. Assign a direction to each of the four subgraphs of $K_{4,4}$ as shown in Figure 5.5.

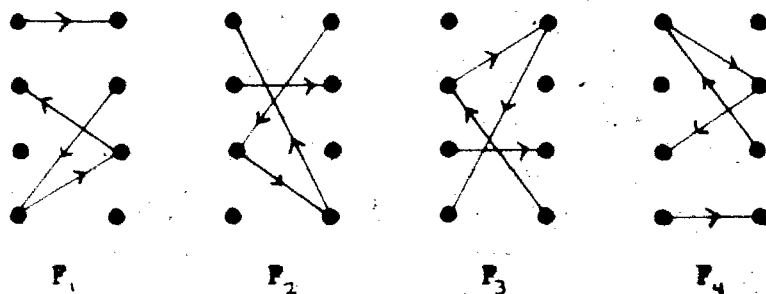


Figure 5.5

Directing the subgraphs of Figure 5.2

By assigning directions to F_1, F_2, F_3 and F_4 we also get directions for $F_1', F_2', F_3',$ and F_4' . These are shown in Figure 5.6.

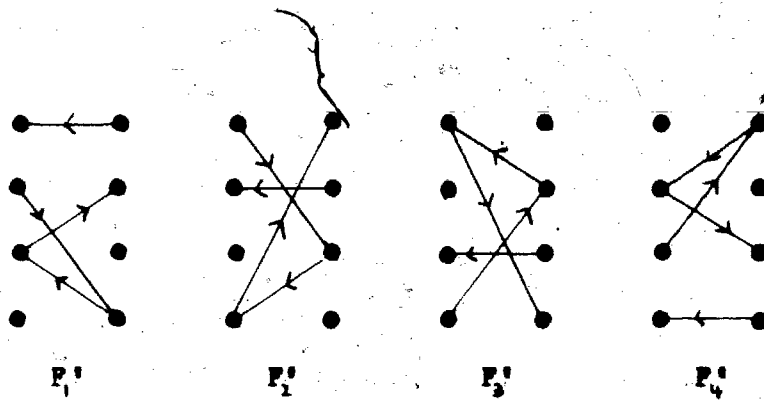


Figure 5.6

Directing the subgraphs of Figure 5.3

As in the undirected case, it is easy to see that the 1-factors F_a and F_{a-1} (a is even) and F_a and F_{a+1} (a is odd) can be joined together to create four subpaths of any length. Also note that if F_a and F'_b intersect then they intersect in one oppositely directed arc as required. Given the directions assigned to the edges in the 1-factors of Figures 5.5 and 5.6 we are forced to direct the closing edges as in Figures 5.7 and 5.8

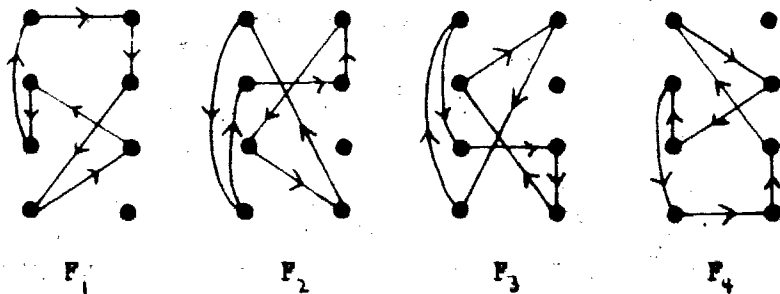


Figure 5.7

Directing the closing edges

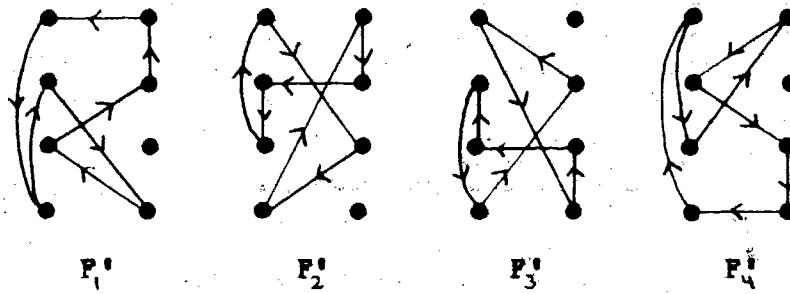


Figure 5.8

Directing the closing edges

Since the closing edges that were added to the 1-factors give directed cycles of length nine it is clear that, as in the undirected case, the subgraphs shown in Figures 5.5 and 5.6 can be joined together to get directed cycles of length $4n-1$.

Now we must check that any two of these directed cycles intersect in exactly one oppositely directed edge. If the two cycles are c_{aj} and c_{bj} , $1 \leq a < b \leq 4$, then these cycles were obtained from the same path Q_j and hence they must intersect in the closing edges. Since the first edge in the path Q_j is labelled either 0 or 1 (but not both) as is the last edge of this path we must only check that the closing edges shown for the 1-factors in Figure 5.7 intersect in exactly one oppositely directed edge as do the closing edges shown for the 1-factors in Figure 5.8. This can easily be checked by observation. By the preceding statement we see that it is not necessary that the closing edges for the 1-factors, F_a , intersect in exactly one oppositely directed edge with the closing edges for the 1-factors, F'_b .

If the two cycles are c_{aj} and c_{bk} then these were defined by Q_j and Q_k . Let xy be an edge in both Q_j and Q_k . Then we have two cases.

In the first case, suppose the arc xy is labelled 0 in Q_j and 1 in Q_k . Here we replace the arc xy with F_a in constructing c_{aj} and we replace the arc xy with F_{b-1}^0 (b is even) or with F_{b+1}^0 (b is odd) in constructing c_{bk} . A quick check of the graphs in Figures 5.7 and 5.8 show that c_{aj} and c_{bk} will, in fact, intersect in exactly one oppositely directed edge.

In the second case, suppose that the arc xy is labelled 0 in Q_j and the arc yx is labelled 0 in Q_k . Since replacing the arc yx with F_b is equivalent to replacing the arc xy with F_b' and since F_a and F_b' have exactly one oppositely directed edge in common, as can be seen from Figures 5.7 and 5.8, it follows that c_{aj} and c_{bk} intersect in exactly one oppositely directed edge.

Thus we see that if $2K_n \rightarrow P_n$ then $DK_{4n} \rightarrow DC_{4n-1}$. ■

The following example shows that we can find a decomposition of DK_{12} into directed cycles of length eleven (Figure 5.10) since we can find a decomposition of $2K_3$ into paths of length two (Figure 5.9).

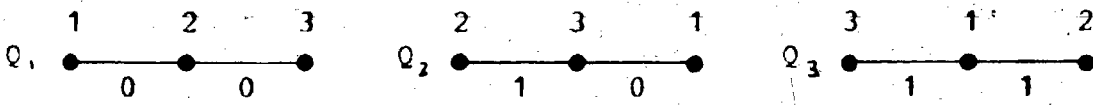


Figure 5.9

$$2K_3 \rightarrow P_3$$

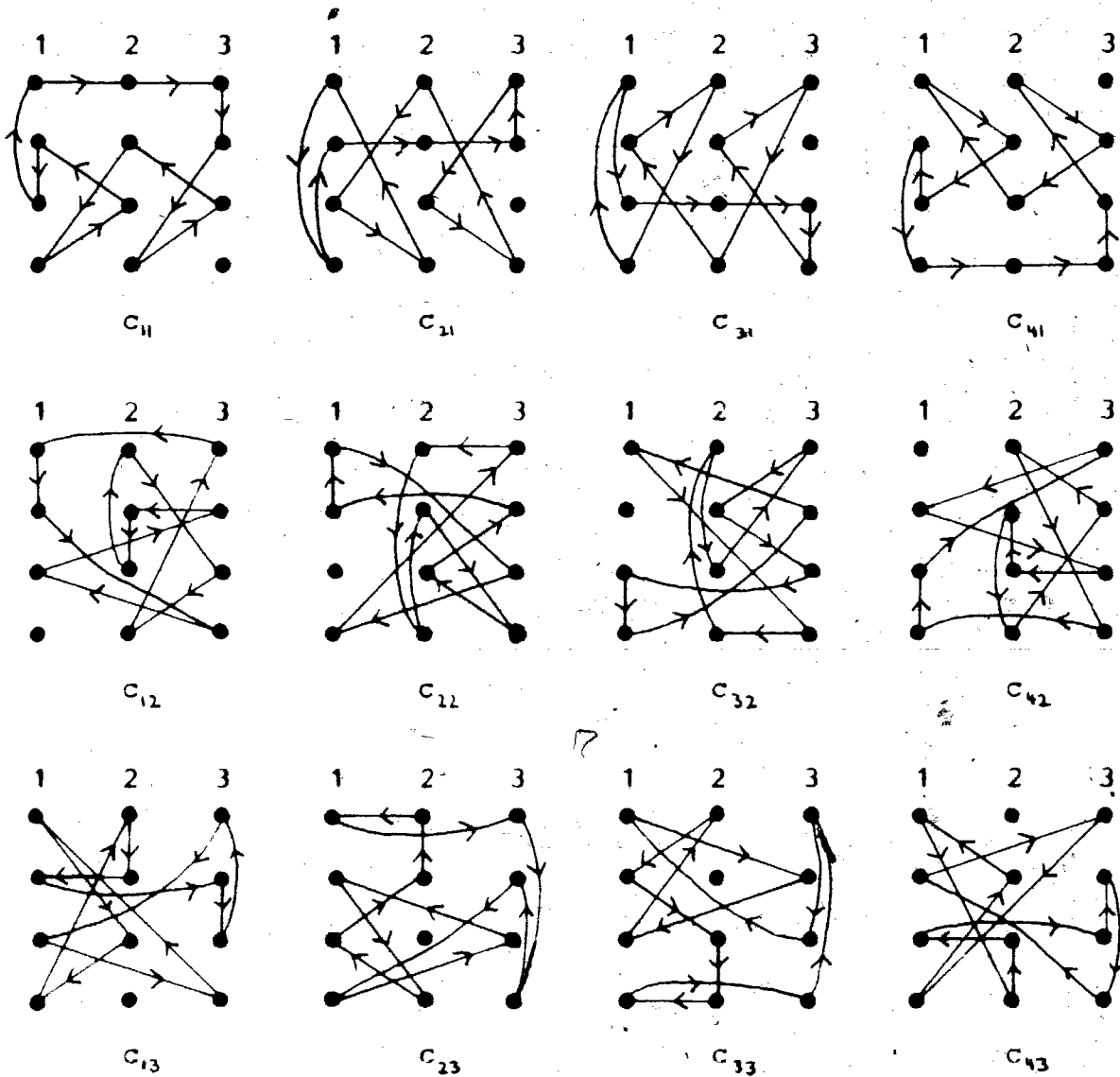


Figure 5.10

$$DK_{12} \rightarrow DC_{11}$$

Corollary 5.2: $DK_{dn} \rightarrow DC_{dn-1}$ where n is an integer such that we already have a decomposition of $2K_n \rightarrow P_n$, $d=5^a 13^b 17^c$ and a , b and c are any natural numbers.

Proof. This follows from Corollary 4.4, Theorem 5.1 and the path decompositions given in the appendix. ■

Corollary 5.3: $2K_{dn} \rightarrow C_{dn-1}$ where n is an integer such that we already have a decomposition of $2K_n \rightarrow P_n$, $d=5^a 13^b 17^c$ and a , b and c are any natural numbers.

Proof. Replace each directed edge in Corollary 5.2 with an undirected edge. ■

Thus Corollary 5.2 gives us several infinite families of complete symmetric directed graphs which can be decomposed into directed cycles having the property that any two of them intersect in exactly one oppositely directed edge. Similarly, Corollary 5.3 gives us several infinite families of complete graphs which can be decomposed into cycles having the property that any two cycles intersect in exactly one edge. This undirected case has also been looked at by Hering [18] who recently found that $2K_n \rightarrow C_{n-1}$ for $4 \leq n \leq 36$. The following Theorem can be used to supplement the work done on this problem

Theorem 5.4: Suppose $2K_n \rightarrow C_{n-1}$ by rotating a fixed cycle C . If

this cycle contains two edges of length k that are distance k apart, where k and n are relatively prime, then $2K_{n+1} \rightarrow C_n$.

Proof. Label the vertices of $2K_n$ with the integers $1, 2, \dots, n$ and let the edges $(1) (1+k)$ and $(1+k) (1+2k)$ be the edges of a cycle C satisfying the conditions of the theorem. Add a vertex labelled v^* to $2K_n$ and replace the edge $(1) (1+k)$ of C with the two edges $(1) (v^*)$ and $(1+k) (v^*)$ to get a new cycle C' of length n . We say that v^* is a fixed point in the rotation of C' . That is, when we rotate C' r times, the edge ij becomes $(i+r) (j+r)$ while the edge $(i) (v^*)$ becomes $(i+r) (v^*)$. (Here we are working modulo n on the residues $1, 2, \dots, n$.)

From Lemma 3.1 we know that the integers $1, 2, \dots, n$ each occur exactly once in C as the distance between two edges of the same length, where $m = \lfloor (n-1)/2 \rfloor$. Since C' contains two edges of the same length that are distance r apart, for $r=1, 2, \dots, m$, (recall that the edges $(i) (v^*)$ and $(i+k) (v^*)$ are distance k apart) we can see by Lemma 3.1 that the rotation of C' gives m cycles of length n that satisfy the property that any two cycles intersect in exactly one edge. Since each of these cycles has exactly one edge of length k , and since k and n are relatively prime, these edges form a cycle of length n that intersects each other cycle exactly once.

Although the method of using differences to find $DK_n \rightarrow DC_{n-1}$ (and hence $2K_n \rightarrow C_{n-1}$) in Theorem 1.4 will always give a

rotational decomposition, it is easy to see that these particular decompositions will never satisfy the conditions of Theorem 5.4. Thus this theorem is useful only if a rotational decomposition is obtained some other way. Figure 5.11 gives an example of this theorem when $n=7$ and $k=2$ (although we could just as well use $k=1$ or $k=3$ in this case).

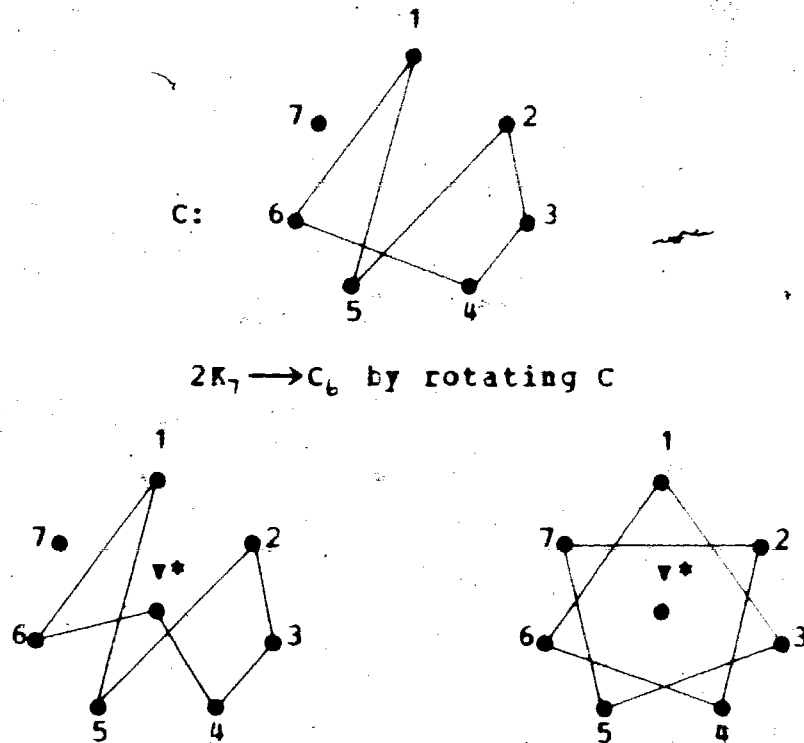


Figure 5.11

$2K_8 \rightarrow C_7$ using Theorem 5.4

It is also known that we can use Theorem 5.4 to get $2K_5 \rightarrow C_4$ from the decomposition $2K_4 \rightarrow C_3$.

APPENDIX

In this section of the thesis we will give explicitly the decompositions of $2K_n$ into Hamiltonian paths with the property that any two paths intersect in exactly one edge, for all values of n as known up to $n=26$. It is easy to see that such a decomposition is impossible for $n=4$. The problem of finding $2K_n \rightarrow P_n$ for n between 2 and 20, as well as $n=25$ and $n=26$, was relatively easy in all cases but one because all but this one are either rotational or are obtained from Theorem 4.1. For $n=7$ a rotational decomposition of $2K_7 \rightarrow P_7$ does not exist, and we cannot apply any of the theorems in Chapter IV. Thus it was necessary to find a set of seven paths of length six in $2K_7$ such that any two paths intersect along exactly one edge. This task proved to be too hard to do by hand, so a computer was used. (A computer was also used to find the paths which give a rotational decomposition for $2K_n \rightarrow P_n$ for $n=14, 16, 18, 19$ and 20 .) The algorithm used is given here for both the particular case seven and the five cases mentioned above.

n: $2K_n \rightarrow P_n$ by rotating the given path
 or by applying Theorem 4.1

2: 1 2

3: 1 2 3

5: 1 2 4 3 5

6: 1 3 4 2 5 6

8: 1 4 5 7 6 2 8 3

9: 1 2 5 7 3 4 9 6 8

10: 1 3 9 2 7 8 5 6 4 10

11: 1 11 4 6 9 10 2 8 3 7 5

12: 1 11 6 3 2 8 10 7 12 4 5 9

13: 1 2 8 4 9 5 3 6 11 12 10 7 13

14: 13 11 5 9 2 1 6 12 7 4 3 14 10 8

15: $2K_3 \rightarrow P_3$ and so by Theorem 4.1 $2K_{15} \rightarrow P_{15}$

16: 11 7 1 2 10 15 6 8 14 5 3 4 9 12 16 13

17: 1 3 13 5 8 7 2 6 16 14 11 15 9 10 4 12 17

18: 15 8 16 5 17 9 13 18 14 11 2 1 3 4 6 12 7 10

19: 17 12 16 5 2 1 10 8 14 7 15 19 13 3 4 18 6 9 11

20: 11 6 4 3 1 2 12 9 18 5 20 14 10 17 8 16 13 7 19 15

25: $2K_5 \rightarrow P_5$ and so by Theorem 4.1 $2K_{25} \rightarrow P_{25}$

26: 5 22 17 9 13 25 23 12 11 21 15 8 4 20 14 1 2 16 18 3

6 24 19 10 7 26

We will now describe the algorithm that was used to find the rotational decomposition $2K_n \rightarrow P_n$ for $n=14, 16, 18, 19$ and 20. First recall Lemma 3.2a which states that $2K_n \rightarrow P_n$ by rotating a fixed path P if and only if

1. There are two edges of every length $1, 2, \dots, (n/2)-1$ and one edge of length $n/2$ in P .
2. The integers $1, 2, \dots, (n/2)-1$ each occur exactly once as the distance between two edges of the same length in P .

This lemma was used in creating the algorithm for the even values of n , and Lemma 3.2b was used to create the algorithm for the case $n=19$. Since the two algorithms are very similar we will only give the algorithm for n even.

1. For $i=1, 2, \dots, (n/2)-1$ and for $j \in \{1, 2, \dots, (n/2)-1\}$ let the two edges of length i be distance j apart so that this function is a bijection. The assignment of a distance to two edges of the same length was not random, but was obtained from the observation of a pattern that existed for smaller values of n .
2. For each $i=1, 2, \dots, (n/2)-1$ let x_i vary from 1 to n . Let the edges of length i be $(x_i)(x_i+i)$ and $(x_i+j)(x_i+i+j)$. If using this value of x_i does not create any vertices of degree three then proceed with the next value of i . For $i=n/2$ also let x_i vary from 1 to n and let the edge of length i be $(x_i)(x_i+i)$.

3. If the union of these edges is a path of length $n-1$ then we have finished, otherwise the union of these edges contains a cycle. In this case, continue with step 2.

Since a rotational decomposition of $2K_7 \rightarrow P_7$ does not exist, and we could not apply any of the theorems in Chapter IV, it was natural to ask if a decomposition of $2K_7 \rightarrow P_7$ existed at all and if so, were there many such decompositions. Before trying the case for $n=7$, which required a computer program, we looked for non-rotational decompositions of $2K_5 \rightarrow P_5$ and $2K_6 \rightarrow P_6$.

The following gives five paths of length four that give a decomposition of $2K_5 \rightarrow P_5$ that is not rotational.

1	2	3	4	5
2	4	5	3	1
3	5	2	1	4
4	3	1	5	2
5	1	4	2	3

The following gives six paths of length five that give a decomposition of $2K_6 \rightarrow P_6$ that is not rotational.

1	2	3	4	5	6
2	5	1	6	3	4
3	5	4	1	6	2
4	2	6	5	1	3
5	3	6	4	2	1
6	4	1	3	2	5

The following is one of the many non-rotational decompositions of $2K_7 \rightarrow P_7$ that were found by computer, and the algorithm used is given below.

1	2	3	4	5	6	7
1	3	5	7	6	4	2
2	3	6	4	7	1	5
3	1	4	7	2	6	5
3	7	2	4	5	1	6
4	3	5	2	6	1	7
4	1	2	5	7	3	6

1. Label the vertices of $2K_7$ with the integers $1, \dots, 7$ and assume that 1234567 is a path in the decomposition of $2K_7 \rightarrow P_7$.
2. Find all paths of length six in $2K_7$ that have exactly one edge in common with the path 1234567 , and call them $Q(1), \dots, Q(x)$. (Since the paths are undirected, the path $v_1 v_2 \dots v_7$ is the same as the path $v_7 v_6 \dots v_1$.)
3. Let $i_1 = 1$.
4. Assume $Q(i_1)$ is a path in the decomposition and let $i_2 = i_1 + 1$.
5. If $Q(i_2)$ has exactly one edge in common with $Q(i_1)$ then let $Q(i_2)$ be a path in the decomposition and let $i_3 = i_2 + 1$. Otherwise let $i_2 = i_1 + 1$ and go to step 5.
6. If $Q(i_3)$ has exactly one edge in common with each of the previously chosen paths then let $Q(i_3)$ be a path in the decomposition and let $i_4 = i_3 + 1$. Otherwise let $i_3 = i_2 + 1$ and go to step 6.
- ...
7. If $Q(i_c)$ has exactly one edge in common with each of the previously chosen paths then we have a decomposition of $2K_7 \rightarrow P_7$. Otherwise let $i_c = i_{c-1} + 1$ and go to step 7.

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