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PERTURBATIONS ABOUT A FINITE ELASTIC INFLATION

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ABSTRACT

A general solution for a class of plane strain boundary value problems involving perturbations about a finite inflation of a slab containing a circular hole or inclusion is obtained. The governing equations of equilibrium for the perturbed state are derived in terms of a general strain-energy function. An exact general analytic solution is obtained for Mooney-Rivlin materials although the method is not restricted to this particular class of materials. Applications are made to the case where a perturbational uniaxial tension is acting at sections far from the cavity or the inclusion and to some cases where perturbational loadings are applied at the edge of the hole. The deformation, the stress field and the stress concentration around the hole are investigated in detail and the computational results are presented graphically.

The general solution obtained is also applicable to problems involving geometric perturbations of the boundaries of the original body. Specific analytic solutions are obtained taking into account both the perturbation due to an applied stress field and the perturbation in the geometry of the original body. We investigate the problem of a slab with a rough cavity and the case where the cross-section of the hole is elliptic, both in the context of a perturbational uniaxial tension.

The method can also be extended to materials with a strain-energy function that may be regarded as a perturbation of the Mooney-Rivlin form.

To my professor,

I.D. Mangeron

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INTRODUCTION

In view of the nonlinear character of the equations governing the Theory of Finite Elasticity, so far, only a few exact solutions valid for any elastic material are known. Further problems can be solved if a particular form is specified for the strain-energy function, but even with this assumption, the number of known exact solutions remains limited. Further progress, however, can be made with the aid of approximation theories.

In particular there is the theory of successive approximations, where the first approximation is the solution to a corresponding problem of linear elasticity. The method is limited to problems involving deformations that do not depart much from the infinitesimal deformation, so the classical linear theory gives a reasonable first approximation. It therefore, inevitably excludes some of the more interesting effects in finite elasticity.

Another method is based on the theory developed by Green, Rivlin and Shield [1], on the superposition of small deformations upon large elastic deformations for which the solution is available. This allows one to consider large deformations in contrast to the method of successive approximation, based on linear theory.

Related to the Green, Rivlin and Shield theory, other theories, involving perturbation of the properties of the original body, have been developed. Perturbation of the strain-energy function has been considered by Spencer [2], [3] and a method of finite deformation solutions of

problems with geometric perturbations on the boundary surfaces, has been formulated by Graham [4].

Many problems involving incompressible materials, have been solved by assuming a suitable deformation field and then determining the surface tractions required to maintain equilibrium of the deformed body. This inverse procedure is not applicable to most compressible materials. The assumed deformation can seldom be maintained without body forces, as the only controllable deformations for these materials are homogeneous deformations. Various approximation procedures have been proposed.

Spencer [3] has proposed the perturbation of the solution for an incompressible body with the volume changes calculated for the perturbed elastic constants. Haddow and Faulkner [5] have developed a method of solution to consider finite expansion of a thick compressible spherical elastic shell. Their procedure is applicable to any admissible strain-energy function and also to a material whose constitutive coefficients are not derivable from a strain-energy function.

Other methods are in a sense more restrictive. Most of them depend on some particular feature of the geometry of the body which suggests a method of simplifying approximation (such as the approximation of a thin membrane).

The method used in this work is based in part on the theory of Green Rivlin and Shield [1] of superposition of small deformations upon large elastic deformations. This theory has been applied to a variety of specific problems.

The problem of small torsion superimposed on finite extension of a

cylinder, investigated by Green and Shield [6], received a special attention at that time as it provides one of the most elegant and convincing experimental verifications of the theory. Small bending of a circular bar superimposed on a finite extension or compression has been considered by Fosdick and Shield [7], where, in addition a solution is given for the small bending under its own weight of a stretched horizontal cylinder. The gravity effects have also been considered by Vaughan [8] in the problem of finite axial compression of cylindrical blocks.

Green and Spencer [9] have obtained the solution to the problem of small deformations superimposed upon finite extension and torsion of a Neo-Hookean circular cylinder. In these investigations the finite deformation has been considered a simple extension or a compression.

Further progress has been made towards problems involving non-uniform large extension in investigating cases with a high degree of symmetry [10] or a special choice of radial displacement [11].

In this work we obtain an exact general analytic solution to a class of boundary value problems involving perturbations about a large non-uniform radial extension. This class of boundary value problems arises when a slab with a circular hole or inclusion, finitely deformed due to a uniform pressure applied at the opening, is further subjected to some perturbations. We obtain an exact general analytic solution for the plane-strain case concerning incompressible materials of the Mooney-Rivlin type. The method, however, is not restricted to this particular class of materials. Applications are made to a number of specific boundary value problems. We use the solutions to investigate the effect of the hole on the

stress and displacement fields.

The study is further extended to problems involving perturbations of properties of the original reference body.

The work is organized as follows. We briefly present the general theory of Green, Rivlin and Shield [1], for small deformations superimposed on large elastic deformations (Chapter 2). In addition, we give a summary of Spencer's theory of finite deformations with a perturbed strain-energy function [2] and the results of the work of Graham on finite elastic deformations of irregularly shaped bodies [4].

Having stated the general problem (Chapter 3, section 3.1) to be solved, we give the solution of the initial deformation problem (section 3.2).

The governing equilibrium equations for the perturbed state, formulated for an incompressible material in terms of a general strain-energy function (Chapter 4), are solved exactly for Mooney-Rivlin materials (Chapter 5).

Applications of the general solutions are made to a number of boundary value problems of interest. Firstly, we consider that the perturbation is caused by uniaxial tensile loadings acting at great distances from the cavity (Chapter 6, section 6.1.1). The cases where a rigid bonded inclusion (section 6.1.2) or an inserted frictionless inclusion (section 6.1.3) embedded into the slab are present, are also considered. The analytic expressions have been evaluated numerically for various parameters to allow detailed investigation of the deformation and stress fields.

In Linear Elasticity, the stress distribution caused by a load applied to

a slab weakened by a cut-out differ considerably from that in an unweakened body. We have found here that the stress concentration effect at the hole is magnified due to the fact that the slab has previously been finitely deformed.

Further we investigate the effect of small shearing forces, uniformly distributed at the edge of the hole (Chapter 7), and obtain the specific solution to the problem where a perturbational radial loading is applied at the opening (Chapter 8).

The general solution obtained is also applicable to problems involving perturbations of the boundary surfaces of the original reference body. Specific analytic solutions are given taking into account both perturbation due to an applied stress field and the geometry of the original body (Chapter 9). We examine two cases, namely, the problem of a slab with a rough cavity (section 9.2) and also the case where the cross-section of the hole is elliptic (section 9.1) where, in both cases, a perturbational uniaxial tension is applied. In addition to the analytic solution, a numerical solution to the later problem has also been obtained [20] using Colsys numerical procedure [12], [13]. The results obtained are in good agreement with those presented here.

Finally, we show how the study can be extended to materials with a strain-energy function which is a perturbation of Mooney-Rivlin form (Chapter 10).

Results related to the computational work that has been interpreted in each relevant chapter are given in the Appendix I and II.

Concluding remarks and topics for further research are outlined in the last chapter.

2. GENERAL THEORY

The method employed in this study is based in part upon the theory of a perturbational displacement field superimposed on a finite deformation. This theory has been developed by Green, Rivlin and Shield [1] and is also described in Green and Zerna [14]. The notation of Green and Zerna is adopted. A brief summary of the results and principal notations is given here.

Let B_0 , B and B' denote the body in its unstrained, finitely deformed and perturbed configuration, respectively. We shall represent by \bar{v} the displacement vector of a typical point in B and by $\bar{v} + \varepsilon \bar{w}$ in B' where ε is a nondimensional constant so small that squares and higher powers may be neglected compared with ε . For any curvilinear system $\{\theta^k\}$ moving with the body, we take the natural base vectors at a generic point to be \bar{g}_i and \bar{g}^i in B_0 , \bar{G}_i and \bar{G}^i in B and $\bar{G}_i + \varepsilon \bar{G}'_i$, $\bar{G}^i + \varepsilon \bar{G}'^i$ in B' . The corresponding metric tensor components are g_{ik} , g^{ik} in B_0 , G_{ik} , G^{ik} in B and, to first order in ε , $G_{ik} + \varepsilon G'_{ik}$, $G^{ik} + \varepsilon G'^{ik}$ in B' . Then,

$$G'_{ik} = w_{i||k} + w_{k||i}, \quad (2.1)$$

$$G'^{ik} = -G^{ir} G^{ks} G'_{rs}, \quad (2.2)$$

where a double line stands for covariant differentiations with respect to the coordinates in the body B .

The determinants of the metric tensor components g_{ik} and G_{ik} are

denoted by g and G , respectively and the determinant of $G_{ik} + \varepsilon G'_{ik}$ is denoted by $G + \varepsilon G'$. To the first order of ε we have:

$$G' = G G^{ik} G'_{ik}. \quad (2.3)$$

The strain invariants associated with the body B' are expressed by

$$\begin{aligned} I_1 + \varepsilon I'_1 &= g^{rs} G_{rs} + \varepsilon g^{rs} G'_{rs}, \\ I_2 + \varepsilon I'_2 &= g_{rs} G^{rs} I_3 + \varepsilon g_{rs} (G'^{rs} I_3 + G^{rs} I'_3), \\ I_3 + \varepsilon I'_3 &= G/g + \varepsilon G^{rs} G'_{rs} I_3. \end{aligned} \quad (2.4)$$

We shall assume the body is homogeneous and isotropic, so that the strain energy W is a function solely of the strain invariants. Thus, corresponding to the state of deformation B ,

$$W = W(I_1, I_2, I_3). \quad (2.5)$$

The relation between stress and strain can be written in the form

$$\tau^{ik} = \phi g^{ik} + \bar{\nu} B^{ik} + p G^{ik}. \quad (2.6)$$

The functions ϕ , $\bar{\nu}$ and p are defined by the expressions

$$\phi \equiv \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}, \quad \bar{\nu} \equiv \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}, \quad p \equiv 2\sqrt{I_3} \frac{\partial W}{\partial I_3} \quad (2.7)$$

and the tensor components B^{ik} by

$$B^{ik} \equiv (g^{ik} g^{rs} - g^{ir} g^{ks}) G_{rs} . \quad (2.8)$$

For the configuration B' , the strain energy W becomes

$$W = W(I_1 + \varepsilon I'_1, I_2 + \varepsilon I'_2, I_3 + \varepsilon I'_3) . \quad (2.9)$$

The scalar invariants Φ , Ψ and p , which, for the body B are functions of I_1 , I_2 , I_3 , become functions of $I_1 + \varepsilon I'_1$, $I_2 + \varepsilon I'_2$, $I_3 + \varepsilon I'_3$ and may be represented by $\Phi + \varepsilon \Phi'$, $\Psi + \varepsilon \Psi'$, $p + \varepsilon p'$. Up to order ε , we may show that

$$\begin{aligned} \Phi' &= AI'_1 + FI'_2 + EI'_3 - (\Phi/2I_3)I'_3 , \\ \Psi' &= FI'_1 + BI'_2 + DI'_3 - (\Psi/2I_3)I'_3 , \\ p' &= (EI'_1 + DI'_2 + CI'_3)I_3 + (p/2I_3)I'_3 , \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} A &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1^2} , & B &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2^2} , & C &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_3^2} , \\ D &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_2 \partial I_3} , & E &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_3 \partial I_1} , & F &\equiv \frac{2}{\sqrt{I_3}} \frac{\partial^2 W}{\partial I_1 \partial I_2} . \end{aligned} \quad (2.11)$$

The elastic potential W which appears in relations (2.11) may be expressed

in the form given by (2.5) and depends exclusively on I_1, I_2, I_3 .

The quantities A, B, \dots, F are evaluated for the finitely strained body \mathcal{B} and in turn depend only on I_1, I_2 and I_3 .

The tensor components B^{ik} in (2.8) become $B^{ik} + \varepsilon B'^{ik}$ for the body \mathcal{B}' where

$$B'^{ik} = (g^{ik} g^{rs} - g^{ir} g^{ks}) G'_{rs}. \quad (2.12)$$

Associated with the final state of strain \mathcal{B}' , the stress tensor field is modified to $\tau^{ik} + \varepsilon \tau'^{ik}$ in which

$$\tau'^{ik} = \Phi' g'^{ik} + \Psi' B'^{ik} + \Upsilon B'^{ik} + p' G'^{ik} + p G'^{ik}. \quad (2.13)$$

For the incompressible, homogeneous, isotropic solid, the elastic potential depends only on I_1 and I_2 since I_3 is unity in every deformation. The constitutive relations (2.6) and (2.13) retain the same form but now

$$\begin{aligned} \Phi &= 2 \frac{\partial W}{\partial I_1}, & \Psi &= 2 \frac{\partial W}{\partial I_2}, \\ A &= 2 \frac{\partial^2 W}{\partial I_1^2}, & B &= 2 \frac{\partial^2 W}{\partial I_2^2}, & F &= 2 \frac{\partial^2 W}{\partial I_1 \partial I_2}, \end{aligned} \quad (2.14)$$

where in this case

$$W = W(I_1, I_2), \quad I_3 = 1. \quad (2.15)$$

Also,

$$I'_3 = 0 \quad (2.16)$$

and in view of the conditions (2.15) and (2.16), the first two relations in (2.10) reduce to

$$\Phi' = AI'_1 + FI'_2, \quad \Psi' = FI'_1 + BI'_2. \quad (2.17)$$

The stress tensors τ^{ik} and τ'^{ik} are indeterminate up to an arbitrary pressure which arises as a consequence of the incompressibility constraint. The functions p and p' that depend on position and time can be determined only when an initial boundary value problem is posed.

In the absence of body forces the stress equations of equilibrium in B are

$$\tau^{ik}_{||i} = 0 \quad (2.18)$$

and in B'

$$[\tau'^{ik} + \tau^{ir k}_{||r} + \tau^{ik r}_{||r}]_{||i} = 0. \quad (2.19)$$

We have assumed a quasi-static loading and therefore inertia terms have been ignored.

On the boundary surface, the stress vector \bar{t} is required to equilibrate the surface force \bar{P} . If $\bar{P} + \epsilon\bar{P}'$ is an applied force at the boundary of B' , measured per unit area of the corresponding surface in B

with the outward normal \bar{n} , then the surface conditions are

$$\bar{t} = \bar{P} \quad , \quad \bar{t}' = \bar{P}' \quad . \quad (2.20)$$

If the surface forces \bar{P} are expressed in terms of their components then the boundary conditions lead to

$$n_i \tau^{ij} = P^j \quad (2.21)$$

and

$$n_i [\tau^{ij} + \tau_{w^j}^{ir} + \tau_{w^r}^{ij}] = P^j \quad (2.22)$$

Related to the Green, Rivlin and Shield theory of superposition of small deformations upon large elastic deformations, G. A. C. Graham [4] has developed a method of obtaining finite deformation solutions for bodies with irregular shapes. The undeformed shape of these bodies is a perturbation of the undeformed shape of another body, for which a finite deformation solution is already available.

Let

$$f(\theta^1, \theta^2, \theta^3) = 0 \quad (2.23)$$

be the surface that bounds the undeformed body. A finite deformation that is consistent with the given tractions on the surface (2.23) is assumed to be known throughout the body. Replacing the surface (2.23) by a perturbed surface

$$f(\theta^1, \theta^2, \theta^3) + \epsilon g(\theta^1, \theta^2, \theta^3) = 0 \quad , \quad (2.24)$$

the tractions acting across the surface $f = 0$ to maintain the finite deformation will not in general maintain the given deformation when acting on the perturbed surface $f + \varepsilon g = 0$. In order to satisfy the traction boundary condition on this surface, an additional deformation $\varepsilon \bar{w}$, assumed to be of order ε , is superimposed on the given deformation.

It is found that up to first order in ε , the value of the tractions acting across the surface (2.24) is given by

$$\bar{t} + \varepsilon \bar{t}' = \left[\chi \frac{\partial f}{\partial \theta^i} \tau^{ij} + \varepsilon \left(\chi \frac{\partial f}{\partial \theta^i} \lambda^{ij} + \chi \frac{\partial g}{\partial \theta^i} \tau^{ij} - \chi \frac{\partial f}{\partial \theta^i} \tau^{ij} \right) \right] \bar{G}_j \quad (2.25)$$

where

$$\lambda^{ij} \equiv \tau^{ij} + \tau^{ik} w_{||k}^j + \tau^{ij} w_{||k}^k, \quad (2.26)$$

$$\chi \equiv \left(\frac{\partial f}{\partial \theta^i} \frac{\partial f}{\partial \theta^j} G^{ij} \right)^{-1/2}, \quad (2.27)$$

$$\chi \equiv \left(\frac{\partial f}{\partial \theta^r} \frac{\partial g}{\partial \theta^s} \right) G^{rs} / \left[\left(\frac{\partial f}{\partial \theta^i} \frac{\partial f}{\partial \theta^j} \right) G^{ij} \right]^{3/2}. \quad (2.28)$$

Let us take the strain-energy function W^* to be a perturbation of a strain energy function W :

$$W^* = W + \varepsilon W' \quad (2.29)$$

where $\varepsilon W'$ is a small perturbing strain-energy function. Assume that an explicit solution can be found for the given problem where the material is that with strain-energy function W . Then, replacing W by W^* ,

an additional small deformation is superimposed on the existing finite deformation. The theory of finite elastic deformation with a perturbed strain-energy function has been formulated by A. J. M. Spencer [2] and it closely resembles the theory of small deformation superposed on finite elastic deformations. However, the functions Φ' , Ψ' and p' of Spencer's theory differ from the functions Φ , Ψ and p , which appear in the paper of Green, Rivlin and Shield [1], by the addition of terms involving the perturbing strain-energy function. In fact,

$$\begin{aligned}\Phi' &= AI'_1 + FI'_2 + EI'_3 - (\Phi/2I_3)I'_3 + (2/\sqrt{I_3}) \frac{\partial W'}{\partial I_1}, \\ \Psi' &= FI'_1 + BI'_2 + DI'_3 - (\Psi/2I_3)I'_3 + (2/\sqrt{I_3}) \frac{\partial W'}{\partial I_2}, \\ p' &= (EI'_1 + DI'_2 + CI'_3)I_3 + (p/2I_3)I'_3 + (2\sqrt{I_3}) \frac{\partial W'}{\partial I_3}.\end{aligned}\tag{2.30}$$

3. FINITE INFLATION

3.1. Introduction

In this and subsequent chapters we construct and solve the equations governing the equilibrium of a class of finitely inflated slabs on which a perturbation is superimposed. This class of boundary value problems arises when a slab which contains a circular hole or inclusion is finitely deformed due to an expanding pressure applied at the inner boundary and it is further subjected to some perturbations.

Rather to assume a certain displacement field and then verify that the prescribed deformation can be maintained by surface tractions only, we wish to obtain all combinations of two-dimensional displacement fields that can be superimposed on the initial inflation such that the equilibrium can be controlled by surface forces alone.

The equilibrium equations for the perturbed state are formulated for incompressible materials in terms of a general strain-energy function and solved exactly for Mooney-Rivlin materials.

The general solution is specialized for several boundary value problems of interest. The effect of the hole on deformation and stress fields is investigated in detail.

The general solution is further applied to problems involving irregularly shaped cross-sections of the cavity and it can be extended to materials for which the strain-energy function is a perturbation of Mooney-Rivlin form.

3.2. Solution of the Finite Deformation Problem

In this section we consider the radial deformation of an infinite slab of a homogeneous, incompressible material. In its unstrained and unstressed state, the slab has a circular hole of radius a_0 removed from its center and is inflated by a uniform pressure p applied to the curved surface of the hole. During this process the thickness l of the slab is maintained constant by rigid boundaries along its plane faces.

Let x_1, x_2 and x_3 be the cartesian coordinates of a generic particle in the undeformed reference configuration, and let y_1, y_2 and y_3 be the coordinates of the same particle in the deformed configuration B . Then, in terms of the cylindrical polar coordinates

$$\theta^1 = r, \quad \theta^2 = \theta, \quad \theta^3 = z, \quad (3.2.1)$$

of particles in the deformed state, the deformation is described by

$$y_1 = r \cos \theta, \quad y_2 = r \sin \theta, \quad y_3 = z, \quad (3.2.2)$$

$$x_1 = rQ(r) \cos \theta, \quad x_2 = rQ(r) \sin \theta, \quad x_3 = z.$$

The non-zero components of the metric matrix and its inverse are known to be

$$G_{11} = 1, \quad G_{22} = r^2, \quad G_{33} = 1 \quad (3.2.3)$$

and

$$G^{11} = 1, \quad G^{22} = 1/r^2, \quad G^{33} = 1 \quad (3.2.4)$$

Further computation reveals that

$$g_{11} = \left(Q + r \frac{dQ}{dr}\right)^2, \quad g_{22} = r^2 Q^2, \quad g_{33} = 1, \quad (3.2.5)$$

and

$$g^{11} = 1/\left(Q + r \frac{dQ}{dr}\right)^2, \quad g^{22} = 1/r^2 Q^2, \quad g^{33} = 1. \quad (3.2.6)$$

It follows from relations (3.2.5) that

$$g = r^2 Q^2 \left(Q + r \frac{dQ}{dr}\right)^2 \quad (3.2.7)$$

and from relations (3.2.3) that

$$G = r^2 \dots \quad (3.2.8)$$

We may use the relations (3.2.3) - (3.2.8) to express the strain invariants in the form

$$I_1 = 1 + 1/Q^2 + 1/\left(Q + r \frac{dQ}{dr}\right)^2, \quad (3.2.9)$$

$$I_2 = \left[\left(Q + r \frac{dQ}{dr}\right)^2 + Q^2 + 1\right] / Q^2 \left(Q + r \frac{dQ}{dr}\right)^2.$$

The incompressibility constraint (2.15) requires that

$$Q\left(Q + r \frac{dQ}{dr}\right) = 1 \quad (3.2.10)$$

and leads, on integration, to

$$Q(r) = (r^2 + K)^{1/2} / r. \quad (3.2.11)$$

The constant K may be either positive or negative, according as the ratio of the radius of the hole before and after the expansion, a_0/a , is less

or greater than unity. We replace relation (3.2.10) by

$$Q(r) = (1 - k^2/r^2)^{1/2} \quad (3.2.12)$$

where

$$k^2 \equiv -K = a^2 - a_0^2 \quad (3.2.13)$$

Moreover, under the condition of volume preservation, the relations (3.2.5), (3.2.6), (3.2.7) and (3.2.9) reduce to

$$g_{11} = 1/Q^2, \quad g_{22} = r^2 Q^2, \quad g_{33} = 1, \quad (3.2.14)$$

$$g^{11} = Q^2, \quad g^{22} = 1/r^2 Q^2, \quad g^{33} = 1, \quad (3.2.15)$$

$$g = r^2, \quad (3.2.16)$$

$$I_1 = I_2 = 1 + Q^2 + 1/Q^2. \quad (3.2.17)$$

and the tensor B^{ik} (relation (2.8)) may be expressed as

$$B^{11} = 1 + Q^2, \quad B^{22} = (1 + 1/Q^2)/r^2, \quad (3.2.18)$$

$$B^{33} = Q^2 + 1/Q^2, \quad B^{12} = B^{13} = B^{23} = 0.$$

Substituting relations (3.2.4), (3.2.15) and (3.2.18) into relation (2.6) we obtain the state of stress throughout the body \mathcal{B}

$$\tau^{11} = Q^2 \bar{\phi} + (1 + Q^2) \bar{\Psi} + p,$$

$$r^2 \tau^{22} = (1/Q^2) \bar{\phi} + (1 + 1/Q^2) \bar{\Psi} + p,$$

$$\tau^{33} = \bar{\phi} + (1 + 1/Q^2) \bar{\Psi} + p,$$

$$\tau^{12} = \tau^{13} = \tau^{23} = 0.$$

(3.2.19)

The functions Φ and Ψ are derived from relations (2.14).

The equations of equilibrium (2.18) may be written in the equivalent form

$$\tau^{ik}_{,i} + \Gamma^i_{ir} \tau^{rk} + \Gamma^k_{ir} \tau^{ir} = 0, \quad (3.2.20)$$

where, for the metric tensor of the strained body B , the only non-zero Christoffel symbols are

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}. \quad (3.2.21)$$

We can prove that the equilibrium equations can be expressed as

$$\frac{\partial}{\partial r} [p + Q^2 \Phi + (1 + Q^2) \Psi] + \frac{1}{r} (Q^2 - 1/Q^2) (\Phi + \Psi) = 0, \quad (3.2.22)$$

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0.$$

These equations are satisfied in the absence of body forces provided that the hydrostatic pressure p is such that

$$p = p(r) = H - L(r) - Q^2 \Phi - (1 + Q^2) \Psi \quad (3.2.23)$$

where H is a constant and

$$L(r) = \int_a^r [(Q^2 - 1/Q^2) (\Phi + \Psi)/r] dr. \quad (3.2.24)$$

We now substitute the relation (3.2.23) into (3.2.19) and obtain the stress field τ^{ik} in the form

$$\begin{aligned}
\tau^{11} &= H - L(r) \\
r^2 \tau^{22} &= H - L(r) - (Q^2 - 1/Q^2)(\Phi + \Psi) \\
\tau^{33} &= H - L(r) + (1 - Q^2)\Phi - (1 - 1/Q^2)\Psi \\
\tau^{12} &= \tau^{13} = \tau^{23} = 0
\end{aligned}
\tag{3.2.25}$$

In order to support the deformation, the following forces must be applied at the boundaries:

(i) a distribution of radial tractions p over the surface $r = a$

$$p \equiv (P^1)_{r=a} = -(\tau^{11})_{r=a} = -H
\tag{3.2.26}$$

(ii) a distribution of normal forces on the plane boundaries

$$(P^3)_{z=l/2} = (\tau^{33})_{z=l/2} = H - L(r) + (1 - Q^2)\Phi + (1 - 1/Q^2)\Psi
\tag{3.2.27}$$

A boundedness condition for $L(r)$ as $r \rightarrow \infty$ is required

$$L(r) \sim L^*
\tag{3.2.28}$$

where L is a constant.

If we choose a vanishing state of stress at infinity, then the constant H must take the value

$$H = L \equiv \int_a^\infty [(Q^2 - 1/Q^2)(\Phi + \Psi)/r] dr
\tag{3.2.29}$$

* In using the notation " \sim " we shall omit, throughout, an explicit reference to " $r \rightarrow \infty$ ", but this will always be understood.

4. SUPERPOSITION OF A SMALL DEFORMATION FIELD

ON A FINITE INFLATION

Consider that the slab strained as described in chapter 3 is further perturbed by a certain force distribution, thus reaching its final state of equilibrium B' .

The displacement field \bar{w} superposed on the previous finite deformation is unknown at this stage. We denote, respectively by $\langle w_1, w_2 \rangle$, $\langle w^1, w^2 \rangle$ and $\langle \hat{w}^1, \hat{w}^2 \rangle$ the covariant, contravariant and physical components of the displacement vector $\bar{w}(r, \theta)$, referred to the base vectors at points P in the body B .

In order to find the final stress field, some preliminary calculations are necessary.

The covariant derivatives of the components w^j and w_j are obtained in the form

$$\begin{aligned} w^1_{||1} &= w^1_{,1} , \\ w^1_{||2} &= w^1_{,2} - r w^2 , \\ w^2_{||1} &= w^2_{,1} + \frac{1}{r} w^2 , \\ w^2_{||2} &= w^2_{,2} + \frac{1}{r} w^1 , \\ w^1_{||3} &= w^2_{||3} = 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
w_{1||1} &= w_{1,1} \quad , \\
w_{2||2} &= w_{2,2} + rw_1 \quad , \\
w_{1||2} &= w_{1,2} - \frac{1}{r} w_2 \quad , \\
w_{2||1} &= w_{2,1} - \frac{1}{r} w_2 \quad , \\
w_{1||3} &= w_{2||3} = 0 \quad .
\end{aligned}
\tag{4.2}$$

If the relations (4.2), together with (3.2.4) and (3.2.8), are substituted into relations (2.1) and (2.2), we find that the contributions G'_{ik} and G'^{ik} to the covariant and contravariant components of the metric tensor in B' can be written as

$$\begin{aligned}
G'_{11} &= 2w_{1,1} \quad , \\
G'_{22} &= 2w_{2,2} + 2rw_1 \quad , \\
G'_{12} &= w_{1,2} + w_{2,1} - \frac{2}{r} w_2 \quad , \\
G'_{13} &= G'_{23} = G'_{33} = 0 \quad ,
\end{aligned}
\tag{4.3}$$

and

$$\begin{aligned}
G'^{11} &= -G'_{11} \quad , \\
G'^{22} &= -\frac{1}{r^4} G'_{22} \quad , \\
G'^{12} &= -\frac{1}{r^2} G'_{12} \quad , \\
G'^{13} &= G'^{22} = G'^{33} = 0 \quad .
\end{aligned}
\tag{4.4}$$

Relations (2.3) along with (3.2.4), (3.2.8) and (4.3) give

$$G' = 2r^2 [w_{1,1} + \frac{1}{r^2}(w_{2,2} + rw_1)] \quad (4.5)$$

We substitute the results (3.2.15) and (4.3) into (2.12) and verify that

$$B^{,11} = \frac{1}{r^2} G'_{22} ,$$

$$B^{,22} = \frac{1}{r^2} G'_{11} ,$$

$$B^{,33} = Q^2 G'_{11} + \frac{1}{r^2 Q^2} G'_{22} , \quad (4.6)$$

$$B^{,12} = -\frac{1}{r^2} G'_{12} ,$$

$$B^{,13} = B^{,23} = 0$$

Writing $\langle u, v \rangle$ for $\langle \hat{w}_1, \hat{w}_2 \rangle$ we have

$$w_1 = w^1 = u , \quad w_2 = rv , \quad w^2 = \frac{v}{r} . \quad (4.7)$$

Thus relations (4.3) - (4.6) become, respectively

$$G'_{11} = 2 \frac{\partial u}{\partial r} ,$$

$$G'_{22} = 2r \left(\frac{\partial v}{\partial \theta} + u \right) ,$$

$$G'_{12} = \frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v ,$$

$$G'_{13} = G'_{23} = G'_{33} = 0 ,$$

(4.8)

$$G^{,11} = -2 \frac{\partial u}{\partial r} ,$$

$$G^{,22} = -\frac{2}{r^3} \left(\frac{\partial v}{\partial \theta} + u \right) ,$$

$$G^{,12} = -\frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) ,$$
(4.9)

$$G^{,13} = G^{,23} = G^{,33} = 0 ,$$

$$G' = 2r^2 \left[\frac{\partial u}{\partial r} + \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right) \right] ,$$
(4.10)

and

$$B^{,11} = \frac{2}{r} \left(\frac{\partial v}{\partial \theta} + u \right) ,$$

$$B^{,12} = -\frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) ,$$

$$B^{,22} = \frac{2}{r^2} \frac{\partial u}{\partial r} ,$$
(4.11)

$$B^{,33} = 2 \left[Q^2 \frac{\partial u}{\partial r} + \frac{1}{rQ^2} \left(\frac{\partial v}{\partial \theta} + u \right) \right] ,$$

$$B^{,13} = B^{,23} = 0 .$$

Since the material is incompressible, we have $G' = 0$, and therefore

$$\frac{\partial u}{\partial r} + \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right) = 0 ,$$
(4.12)

for all $\langle r, \theta \rangle$. We may use the above results together with (3.2.14), (3.2.15), (4.8) and (4.9) to show that the invariants I'_1 and I'_2 can be expressed as

$$I'_1 = I'_2 = 2(Q^2 - 1/Q^2) \frac{\partial u}{\partial r}$$
(4.13)

As an immediate consequence of (4.13), relations (2.17) may be written in the form

$$\Phi' = 2(A + F) \left(Q^2 - 1/Q^2 \right) \frac{\partial u}{\partial r} \quad (4.14)$$

$$\Psi' = 2(B + F) \left(Q^2 - 1/Q^2 \right) \frac{\partial u}{\partial r}$$

On substituting (3.2.4), (3.2.15), (3.2.18), (4.9), (4.11) and (4.14) into (2.13) and then using the condition (4.12), we may conclude that the stress components τ'^{ik} are given by the expressions

$$\begin{aligned} \tau'^{11} &= p' - 2 \frac{\partial u}{\partial r} \left\{ p + \Psi - \left(Q^2 - \frac{1}{Q^2} \right) \left[(A + F) Q^2 + (B + F) \left(1 + Q^2 \right) \right] \right\} , \\ r^2 \tau'^{22} &= p' + 2 \frac{\partial u}{\partial r} \left\{ p + \Psi + \left(Q^2 - \frac{1}{Q^2} \right) \left[(A + F) \frac{1}{Q^2} + (B + F) \left(1 + \frac{1}{Q^2} \right) \right] \right\} , \\ \tau'^{33} &= p' + 2 \frac{\partial u}{\partial r} \left\{ \left(Q^2 - \frac{1}{Q^2} \right) \Psi + \left(Q^2 - \frac{1}{Q^2} \right) \left[(A + F) + (B + F) \left(Q^2 + \frac{1}{Q^2} \right) \right] \right\} , \\ r \tau'^{12} &= - \frac{1}{r} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) (p + \Psi) , \\ \tau'^{13} &= \tau'^{23} = 0 . \end{aligned} \quad (4.15)$$

The stress equations of equilibrium corresponding to the configuration B' are

$$\lambda^{ij}{}_{||i} = 0 , \quad (4.16)$$

or, in component form

$$\begin{aligned} \lambda^{11}{}_{,1} + \lambda^{21}{}_{,2} + \frac{1}{r} (\lambda^{11} - r^2 \lambda^{22}) &= 0 , \\ \lambda^{12}{}_{,1} + \lambda^{22}{}_{,2} + \frac{1}{r} (2\lambda^{12} + \lambda^{21}) &= 0 , \quad \lambda^{33}{}_{,3} = 0 . \end{aligned} \quad (4.17)$$

Let us recall from (2.19) that

$$\lambda^{ij} = \tau^{,ij} + \tau^{ik} w^j \parallel_k + \tau^{ij} w^k \parallel_k \quad (4.18)$$

When the required covariant differentiations are performed, the expressions for λ^{ij} become

$$\begin{aligned} \lambda^{11} &= \tau^{,11} + \tau^{11} \frac{\partial u}{\partial r} , \\ \lambda^{22} &= \tau^{,22} + \frac{1}{r} \tau^{22} \left(\frac{\partial v}{\partial \theta} + u \right) , \\ \lambda^{33} &= \tau^{,33} , \\ \lambda^{12} &= \tau^{,12} + \frac{1}{r} \tau^{11} \frac{\partial v}{\partial r} , \\ \lambda^{21} &= \tau^{,12} + \tau^{22} \left(\frac{\partial u}{\partial \theta} - v \right) , \\ \lambda^{13} &= \lambda^{31} = \lambda^{23} = \lambda^{32} = 0 \end{aligned} \quad (4.19)$$

using also the incompressibility constraint (4.12). The equilibrium equations (4.17) become

$$\begin{aligned} \frac{\partial}{\partial r} \left\{ \tau^{,11} + \tau^{11} \frac{\partial u}{\partial r} \right\} + \frac{\partial}{\partial \theta} \left\{ \tau^{,12} + \tau^{22} \left(\frac{\partial u}{\partial \theta} - v \right) \right\} \\ + \frac{1}{r} \left\{ \tau^{,11} - r^2 \tau^{,22} + (\tau^{11} + r^2 \tau^{22}) \frac{\partial u}{\partial r} \right\} = 0 , \\ \frac{\partial}{\partial r} \left\{ \tau^{,12} + \frac{1}{r} \tau^{11} \frac{\partial v}{\partial r} \right\} + \frac{\partial}{\partial \theta} \left\{ \tau^{,22} - \tau^{22} \frac{\partial u}{\partial r} \right\} \\ + \frac{1}{r} \left\{ 3\tau^{,12} + \frac{2}{r} \tau^{11} \frac{\partial v}{\partial r} + \tau^{22} \left(\frac{\partial u}{\partial \theta} - v \right) \right\} = 0 , \\ \frac{\partial \tau^{,33}}{\partial z} = 0 . \end{aligned} \quad (4.20)$$

We now substitute the stress fields determined in (3.2.25) and (4.15) into (4.20). The last equation of (4.20) implies that the contribution p' to the final hydrostatic pressure does not depend on z . We obtain

$$\begin{aligned} & \frac{\partial}{\partial r} \left\{ p' + \frac{\partial u}{\partial r} \left[H - L(r) - 2(p + \Psi) + 2Q^2 \left(Q^2 - \frac{1}{Q^2} \right) (A + F) \right. \right. \\ & \quad \left. \left. + 2 \left(1 + Q^2 \right) \left(Q^2 - \frac{1}{Q^2} \right) (B + F) \right] \right\} \\ & - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left\{ \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) (p + \Psi) - \left(\frac{\partial u}{\partial \theta} - v \right) \left[H - L(r) + \left(\frac{1}{Q^2} - Q^2 \right) (\Phi + \Psi) \right] \right\} \\ & + \frac{1}{r} \frac{\partial u}{\partial r} \left\{ 2H - 2L(r) - 4(p + \Psi) + 2 \left(Q^2 - \frac{1}{Q^2} \right)^2 (A + B + 2F) \right. \\ & \quad \left. - \left(Q^2 - \frac{1}{Q^2} \right) (\Phi + \Psi) \right\} = 0 \quad , \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial v}{\partial r} [H - L(r)] - \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) (p + \Psi) \right\} \\ & + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left\{ p' - \frac{\partial u}{\partial r} \left[H - L(r) - 2(p + \Psi) - \frac{2}{Q^2} \left(Q^2 - \frac{1}{Q^2} \right) (A + F) \right. \right. \\ & \quad \left. \left. - 2 \left(1 + \frac{1}{Q^2} \right) \left(Q^2 - \frac{1}{Q^2} \right) (B + F) + \left(\frac{1}{Q^2} - Q^2 \right) (\Phi + \Psi) \right] \right\} \\ & + \frac{1}{r} \left\{ \frac{2}{r} \frac{\partial v}{\partial r} [H - L(r)] - \frac{3}{r^2} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) (p + \Psi) \right. \\ & \quad \left. + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} - v \right) \left[H - L(r) + \left(\frac{1}{Q^2} - Q^2 \right) (\Phi + \Psi) \right] \right\} = 0 \quad , \end{aligned}$$

$$\frac{\partial p'}{\partial z} = 0$$

(4.21)

5. EXACT SOLUTION OF THE EQUILIBRIUM EQUATIONS

To solve the equations of equilibrium (4.21), we seek a solution in the form of a Fourier Series whose coefficients are themselves functions of r :

$$\begin{aligned} u(r, \theta) &= \sum_{n=0}^{\infty} \{u_n(r) \cos(n\theta) + \eta_n(r) \sin(n\theta)\} , \\ v(r, \theta) &= \sum_{n=0}^{\infty} \{v_n(r) \sin(n\theta) + \zeta_n(r) \cos(n\theta)\} , \\ p'(r, \theta) &= \sum_{n=0}^{\infty} \{p'_n(r) \cos(n\theta) + \xi'_n(r) \sin(n\theta)\} . \end{aligned} \quad (5.1)$$

The incompressible nature of the material requires that

$$\frac{du_n}{dr} + \frac{1}{r}(nv_n + u_n) = 0 \quad (5.2)$$

and

$$\frac{d\eta_n}{dr} - \frac{1}{r}(n\zeta_n - \eta_n) = 0 \quad (5.3)$$

must be fulfilled for all n . Substitution of (5.1) into the equilibrium equations (4.21) then gives a set of four ordinary differential equations which together with (5.2) and (5.3) determine u_n , η_n , v_n , ζ_n , p'_n and ξ'_n as functions of r .

In order to proceed with the solution, we assume a specific form for the strain-energy function. The material considered here is of the Mooney-Rivlin type with

$$W = c_1(I_1 - 3) + c_2(I_2 - 3) , \quad (5.4)$$

although the method used is not restricted to this particular form.

Relations (2.14) and (5.4) yield

$$\Phi = 2c_1, \quad \Psi = 2c_2, \quad A = B = F = 0. \quad (5.5)$$

Having the strain-energy function (5.4) and the radial deformation expressed by (3.2.12), we can evaluate the definite integral (3.2.24) to obtain

$$L(r) = \left(\frac{k^2}{r^2} - \frac{k^2}{a^2} + \ln \frac{r^2/a^2}{(r^2 - k^2)/a_0^2} \right) c \quad (5.6)$$

where

$$c \equiv c_1 + c_2. \quad (5.7)$$

Consequently, in view of (3.2.29), the constant H takes the value

$$H = -\left(\frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) c \quad (5.8)$$

and hence, the expanding pressure (3.2.26) is then

$$p = \left(\frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) c. \quad (5.9)$$

For the Mooney-Rivlin material, the stress fields τ^{ik} (3.2.25) and τ'^{ik} (4.15) take the form

$$\tau^{11} = \hat{\tau}_{11} = H - L(r),$$

$$r^2 \tau^{22} = \hat{\tau}_{22} = H - L(r) - 2(Q^2 - \frac{1}{Q^2})c, \quad (5.10)$$

$$\tau^{33} = \hat{\tau}_{33} = H - L(r) + 2(1 - Q^2)c_1 - 2(1 - \frac{1}{Q^2})c_2,$$

and

$$\begin{aligned}\tau'^{11} &= \hat{\tau}'_{11} = p' - 2(H - L(r) - 2Q^2 c) \frac{\partial u}{\partial r}, \\ r^2 \tau'^{22} &= \hat{\tau}'_{22} = p' + 2(H - L(r) - 2Q^2 c) \frac{\partial u}{\partial r}, \\ \tau'^{33} &= \hat{\tau}'_{33} = p' + 2c_2 (Q^2 - \frac{1}{Q^2}) \frac{\partial u}{\partial r},\end{aligned}\tag{5.11}$$

$$r\tau'^{12} = \hat{\tau}'_{12} = -\frac{1}{r}(H - L(r) - 2Q^2 c) \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right).$$

We may use the assumptions (5.4) and (4.12) in equations (4.21)

to obtain

$$\frac{1}{2c} \frac{\partial p'}{\partial r} + Q^2 \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2 Q^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} (4 - 2Q^2 + \frac{1}{Q^2}) \frac{\partial u}{\partial r} + \frac{1}{r^2 Q^2} u = 0,$$

$$\frac{1}{2c} \frac{\partial p'}{\partial \theta} + rQ^2 \frac{\partial^2 v}{\partial r^2} - \frac{1}{Q^2} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r} (2 - Q^2) \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) = 0,$$

$$\frac{\partial p'}{\partial z} = 0.\tag{5.12}$$

Seeking solutions of the form (5.1) we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \left\{ \left[\frac{1}{2c} \frac{dp'_n}{dr} + Q^2 \frac{d^2 u_n}{dr^2} + \frac{1}{r} (4 - 2Q^2 + \frac{1}{Q^2}) \frac{du_n}{dr} + \frac{1}{r^2 Q^2} (1 - n^2) u_n \right] \cos(n\theta) \right. \\ \left. + \left[\frac{1}{2c} \frac{d\xi'_n}{dr} + Q^2 \frac{d^2 \eta_n}{dr^2} + \frac{1}{r} (4 - 2Q^2 + \frac{1}{Q^2}) \frac{d\eta_n}{dr} + \frac{1}{r^2 Q^2} (1 - n^2) \eta_n \right] \sin(n\theta) \right\} = 0,\end{aligned}\tag{5.13}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \left\{ \left[-\frac{n}{2c} p'_n + rQ^2 \frac{d^2 v_n}{dr^2} + \frac{n}{Q^2} \frac{du_n}{dr} + \frac{1}{r} (2 - Q^2) \left(r \frac{dv_n}{dr} - nu_n - v_n \right) \right] \sin(n\theta) \right. \\ \left. + \left[\frac{n}{2c} \xi'_n + rQ^2 \frac{d^2 \zeta_n}{dr^2} - \frac{n}{Q^2} \frac{d\eta_n}{dr} + \frac{1}{r} (2 - Q^2) \left(r \frac{d\zeta_n}{dr} + m\eta_n - \zeta_n \right) \right] \cos(n\theta) \right\} = 0.\end{aligned}\tag{5.14}$$

The equations obtained from the relations (5.13) and (5.14) as well as the incompressibility conditions (5.2) and (5.3) take a particular form when $n=0$. Therefore, we shall investigate the case $n=0$ and $n \neq 0$ separately.

Case 1. Provided $n \neq 0$, $v_n(r)$ and $\xi_n(r)$ can be eliminated from conditions (5.2) and (5.3):

$$v_n = -\frac{1}{n} \left(r \frac{du_n}{dr} + u_n \right), \quad (5.15)$$

$$\xi_n = \frac{1}{n} \left(r \frac{d\eta_n}{dr} + \eta_n \right). \quad (5.16)$$

The relations (5.13) and (5.14) yield a system of four ordinary differential equations

$$\frac{1}{2c} \frac{dp'_n}{dr} + Q^2 \frac{d^2 u_n}{dr^2} + \frac{1}{r} (4 - 2Q^2 + \frac{1}{Q^2}) \frac{du_n}{dr} - \frac{n^2 - 1}{r^2 Q^2} u_n = 0, \quad (5.17)$$

$$-\frac{n}{2c} p'_n + rQ^2 \frac{d^2 v_n}{dr^2} + \frac{n}{Q^2} \frac{du_n}{dr} + \frac{2 - Q^2}{r} \left(r \frac{dv_n}{dr} - nu_n - v_n \right) = 0, \quad (5.18)$$

$$\frac{1}{2c} \frac{d\xi'_n}{dr} + Q^2 \frac{d^2 \eta_n}{dr^2} + \frac{1}{r} (4 - 2Q^2 + \frac{1}{Q^2}) \frac{d\eta_n}{dr} - \frac{n^2 - 1}{r^2 Q^2} \eta_n = 0, \quad (5.19)$$

$$\frac{n}{2c} \xi'_n + rQ^2 \frac{d^2 \zeta_n}{dr^2} - \frac{n}{Q^2} \frac{d\eta_n}{dr} + \frac{1}{r} (2 - Q^2) \left(r \frac{d\zeta_n}{dr} + m\eta_n - \zeta_n \right) = 0, \quad (5.20)$$

that uncouples into two independent sets: $\{(5.17), (5.18)\}$ and $\{(5.19), (5.20)\}$. In view of (5.15), the function p'_n can be deduced from equation (5.18) to be

$$\begin{aligned}
p'_n(r) = & -\frac{2Q}{n^2} \left\{ r^2 Q^2 \frac{d^3 u_n}{dr^3} + 2r(1+Q^2) \frac{d^2 u_n}{dr^2} \right. \\
& \left. + (2-Q^2 - \frac{n^2}{Q^2}) \frac{du_n}{dr} + \frac{n^2-1}{r} (2-Q^2) u_n \right\} \quad (5.21)
\end{aligned}$$

and then substitute into (5.17). Similarly, using relation (5.16);

ξ'_n can be eliminated from (5.19) and (5.20). It can be proved that

p'_n and ξ'_n have the same functional form in u_n and η_n , respectively

and that both $u_n(r)$ and $\eta_n(r)$ satisfy the fourth order differential

equation

$$\begin{aligned}
r^2 Q^2 \frac{d^4 u_n}{dr^4} + 2r(2+Q^2) \frac{d^3 u_n}{dr^3} + [8 - (3+n^2)Q^2 - \frac{n^2}{Q^2}] \frac{d^2 u_n}{dr^2} \\
- \frac{1}{r} [2(2+n^2) - (3+n^2)Q^2 + \frac{3n^2}{Q^2} - \frac{2n^2}{Q^2}] \frac{du_n}{dr} \\
+ \frac{1}{r^2} (1-n^2) (4-3Q^2 - \frac{n^2}{Q^2}) u_n = 0 \quad (5.22)
\end{aligned}$$

On substituting the expression (3.2.12) for $Q(r)$, equation (5.22) becomes

$$Lu_n(r) = b_0(r)r^2 \frac{d^4 u_n}{dr^4} + b_1(r)r \frac{d^3 u_n}{dr^3} + b_2(r) \frac{d^2 u_n}{dr^2} + b_3(r) \frac{1}{r} \frac{du_n}{dr} + b_4(r) \frac{1}{r^2} u_n = 0$$

where

$$b_0(r) \equiv 1 - 3\frac{k^2}{r^2} + 3\frac{k^4}{r^4} - \frac{k^6}{r^6}, \quad (5.23)$$

$$b_1(r) \equiv 6 - 14\frac{k^2}{r^2} + 10\frac{k^4}{r^4} - 2\frac{k^6}{r^6},$$

$$b_2(r) \equiv -(2n^2-5) + (4n^2-17)\frac{k^2}{r^2} - (3n^2+1)\frac{k^4}{r^4} + (n^2+3)\frac{k^6}{r^6}, \quad (5.24)$$

$$b_3(r) \equiv -(2n^2+1) + (4n^2-1)\frac{k^2}{r^2} + (n^2+5)\frac{k^4}{r^4} - (n^2+3)\frac{k^6}{r^6},$$

$$b_4(r) \equiv (n^2-1)^2 - (n^4-1)\frac{k^2}{r^2} + 5(n^2-1)\frac{k^4}{r^4} - 3(n^2-1)\frac{k^6}{r^6}.$$

To investigate the behaviour of the point at infinity and to determine the appropriate form of the solution, we map the point at infinity into the origin, using the transformation

$$r = 1/t. \quad (5.25)$$

Higher derivatives with respect to r transform as follows

$$\frac{du}{dr} = -t^2 \frac{du}{dt},$$

$$\frac{d^2u}{dr^2} = t^4 \frac{d^2u}{dt^2} + 2t^3 \frac{du}{dt}, \quad (5.26)$$

$$\frac{d^3u}{dr^3} = -t^6 \frac{d^3u}{dt^3} - 6t^5 \frac{d^2u}{dt^2} - 6t^4 \frac{du}{dt},$$

$$\frac{d^4u}{dr^4} = t^8 \frac{d^4u}{dt^4} + 12t^7 \frac{d^3u}{dt^3} + 36t^6 \frac{d^2u}{dt^2} + 24t^5 \frac{du}{dt}.$$

Thus, in terms of the parameter t , equation (5.23) becomes

$$Lu_n(t) = a_0(t)t^4 \frac{d^4u_n}{dt^4} + a_1(t)t^3 \frac{d^3u_n}{dt^3} + a_2(t)t^2 \frac{d^2u_n}{dt^2} + a_3(t)t \frac{du_n}{dt} + a_4(t)u_n = 0$$

where

$$a_0(t) \equiv 1 - 3k^2t^2 + 3k^4t^4 - k^6t^6,$$

$$a_1(t) \equiv 6 - 22k^2t^2 + 26k^4t^4 - 10k^6t^6, \quad (5.28)$$

$$a_2(t) \equiv -(2n^2-5) + (4n^2-31)k^2t^2 - (3n^2-47)k^4t^4 + (n^2-21)k^6t^6,$$

$$a_3(t) \equiv -(2n^2+1) + (4n^2-1)k^2t^2 - (7n^2-5)k^4t^4 + 3(n^2-1)k^6t^6,$$

$$a_4(t) \equiv (n^2-1)^2 - (n^4-1)k^2t^2 + 5(n^2-1)k^4t^4 - 3(n^2-1)k^6t^6.$$

Equation (5.27) is equivalent to

$$\frac{d^4 u_n}{dt^4} + p_3(t) \frac{d^3 u_n}{dt^3} + p_2(t) \frac{d^2 u_n}{dt^2} + p_1(t) \frac{du_n}{dt} + p_0(t) u_n = 0,$$

where

(5.29)

$$p_0(t) \equiv a_4(t)/t^4 a_0(t),$$

$$p_1(t) \equiv a_3(t)/t^3 a_0(t),$$

(5.30)

$$p_2(t) \equiv a_2(t)/t^2 a_0(t),$$

$$p_3(t) \equiv a_1(t)/t a_0(t).$$

The point $t=0$ is a regular singularity for (5.29), as $p_0(t), \dots, p_3(t)$ are not analytic but all of $t^4 p_0(t), t^3 p_1(t), t^2 p_2(t)$ and $t p_3(t)$ are analytic in a neighbourhood of $t=0$.

We shall construct the general solution of the equation (5.27) by finding four independent solutions.

If u_n is assumed to be a function of the form

$$u_n(t) = t^\lambda \sum_{j=0}^{\infty} a_{n,j} t^j, \quad (5.31)$$

then the equation (5.27) implies that

$$\begin{aligned}
L\varphi_n \equiv & f(\lambda)\alpha_{n,0}t^\lambda + f(\lambda+1)\alpha_{n,1}t^{\lambda+1} \\
& + [f(\lambda+2)\alpha_{n,2} + k^2g(\lambda)\alpha_{n,0}]t^{\lambda+2} + [f(\lambda+3)\alpha_{n,3} + k^2g(\lambda+1)\alpha_{n,1}]t^{\lambda+3} \\
& + [f(\lambda+4)\alpha_{n,4} + k^2g(\lambda+2)\alpha_{n,2} + k^4h(\lambda)\alpha_{n,0}]t^{\lambda+4} \\
& + [f(\lambda+5)\alpha_{n,5} + k^2g(\lambda+3)\alpha_{n,3} + k^4h(\lambda+1)\alpha_{n,1}]t^{\lambda+5} \\
& + [f(\lambda+6)\alpha_{n,6} + k^2g(\lambda+4)\alpha_{n,4} + k^4h(\lambda+2)\alpha_{n,2} + k^6j(\lambda)\alpha_{n,0}]t^{\lambda+6} \\
& + \dots \\
& + [f(\lambda+6+s)\alpha_{n,6+s} + k^2g(\lambda+4+s)\alpha_{n,4+s} + k^4h(\lambda+2+s)\alpha_{n,2+s} \\
& \quad + k^6j(\lambda+s)\alpha_{n,0}]t^{\lambda+6+s} + \dots = 0
\end{aligned}$$

(5.32)

where, for each n , it is convenient to define the functions

$f(\lambda)$, $g(\lambda)$, $h(\lambda)$, and $j(\lambda)$ by the expressions

$$f(\lambda) \equiv \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 6\lambda(\lambda-1)(\lambda-2) - (2n^2-5)\lambda(\lambda-1) - (2n^2+1)\lambda + (n^2-1)^2,$$

$$g(\lambda) \equiv -3\lambda(\lambda-1)(\lambda-2)(\lambda-3) - 22\lambda(\lambda-1)(\lambda-2) + (4n^2-31)\lambda(\lambda-1) + (4n^2-1)\lambda - (n^4-1),$$

$$h(\lambda) \equiv 3\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 26\lambda(\lambda-1)(\lambda-2) - (3n^2-47)\lambda(\lambda-1) - (7n^2-5)\lambda + (5n^2-5),$$

$$j(\lambda) \equiv -\lambda(\lambda-1)(\lambda-2)(\lambda-3) - 10\lambda(\lambda-1)(\lambda-2) + (n^2-21)\lambda(\lambda-1) + (3n^2-3)\lambda - (3n^2-3).$$

(5.33)

The functions (5.33) are related to the coefficients of u_n and its derivatives as they occur in the differential equation.

By equating to zero the coefficients of $t^{\lambda+j}$, $j = 1, 2, \dots$, a recursive system is obtained

$$\begin{aligned} f(\lambda+1)\alpha_{n,1} &= 0, \\ f(\lambda+2)\alpha_{n,2} + k^2g(\lambda)\alpha_{n,0} &= 0, \end{aligned} \quad (5.34)$$

$$\begin{aligned} f(\lambda+6+s)\alpha_{n,6+s} + k^2g(\lambda+4+s)\alpha_{n,4+s} + k^4h(\lambda+2+s)\alpha_{n,2+s} + k^6j(\lambda+s)\alpha_{n,s} &= 0, \\ \dots \dots \dots \end{aligned} \quad (5.34)$$

which can be solved for $\alpha_{n,1}$, $\alpha_{n,2}$, ... as functions of λ , except possibly at zeros of $f(\lambda+j)$. It can be shown that the indicial equation

$$f(\lambda) = 0 \quad (5.35)$$

has solutions

$$\lambda = \{\pm(1 \pm n)\}. \quad (5.36)$$

These form a set of four roots, all differing by integers and distinct provided $n \neq 1$. In case $n=1$, there is a double root and again all roots differ by integers.

Case 1a. Consider $n \neq 1$ and let

$$\lambda_1 \equiv n+1, \quad \lambda_2 \equiv n-1, \quad \lambda_3 \equiv -n+1, \quad \lambda_4 \equiv -n-1, \quad (5.37)$$

such that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.

If λ is set equal to λ_1 , then the quantities $f(\lambda_1+j) \neq 0$ for all $j \geq 1$. Thus, there is always a power series solution associated to the largest root:

$$\varphi_n^{(1)}(t) = t^{\lambda_1} \sum_{j=0}^{\infty} \alpha_{n,j}(\lambda_1) t^j. \quad (5.38)$$

Each of the coefficients $\alpha_{n,j}^*$ is uniquely determined in terms of a non-zero arbitrary constant $\alpha_{n,0}$ by the recurrence relations (5.34). We find that

$$\alpha_{n,1+2s} = 0, \quad s = 0, 1, 2, \dots$$

$$\alpha_{n,6+2s} = -[k^2 g(\lambda_1+4+2s) \alpha_{n,4+2s} + k^4 h(\lambda_1+2+2s) \alpha_{n,2+2s} + k^6 j(\lambda_1+2s) \alpha_{n,2s}] / f(\lambda_1+6+2s),$$

with

$$\alpha_{n,2} = -k^2 [g(\lambda_1) / f(\lambda_1+2)] \alpha_{n,0}$$

and

$$\alpha_{n,4} = -[k^2 g(\lambda_1+2) \alpha_{n,2} + k^4 h(\lambda_1) \alpha_{n,0}] / f(\lambda_1+4). \quad (5.39)$$

For the remaining roots, there is at least one N , specific to each λ , for which

$$f(\lambda+N) = 0, \quad \lambda = \lambda_2, \lambda_3, \lambda_4 \quad (5.40)$$

and a relation of the form

$$f(\lambda+N) \alpha_{n,N} = -[k^2 g(\lambda+N-2) \alpha_{n,N-2} + k^4 h(\lambda+N-4) \alpha_{n,N-4} + k^6 j(\lambda+N-6) \alpha_{n,N-6}]$$

(5.41)

* If no λ is specified, evaluation of $\alpha_{n,j}$ at $\lambda=\lambda_1$ will be understood.

cannot be fulfilled; consequently, there is no corresponding power series representation about $t=0$, unless, the right-hand side for that particular N happens to be zero.

We shall show that there is also, for all n , a power series solution, associated with the second largest root.

Regardless of the specific value of n , $f(\lambda_2+j)$ vanishes if

$$j = N \equiv \lambda_1 - \lambda_2 = 2 \quad (5.42)$$

and does not vanish for any other value of j . Corresponding to $j=2$, relations (5.34) give

$$f(\lambda+2)\alpha_{n,N} = -k^2 g(\lambda)\alpha_{n,0} \quad (5.43)$$

It turns out that $\lambda_2 = n-1$ is also a root of $g(\lambda)$, and therefore, $\alpha_{n,N}$ determined as a rational function of λ , does not have λ_2 as a pole.

The terms

$$\beta_{n,j} \equiv \alpha_{n,j}(\lambda_2) \quad , \quad j > 2 \quad (5.44)$$

are now readily obtained and also will not have λ_2 as a pole. Thus, the differential equation has a second solution of the form (5.31):

$$\varphi_n^{(2)}(t) = t^{\lambda_2} \sum_{j=0}^{\infty} \beta_{n,j}(\lambda_2) t^j \quad (5.45)$$

The leading term is t^{λ_2} and so, the corresponding solution is different from any associated with λ_1 . The coefficients $\beta_{n,j}$ satisfy the recursive

relations (5.34) where $\lambda = \lambda_2$. In this case

$$\beta_{n,1+2s} = 0,$$

$$\begin{aligned} \beta_{n,6+2s} = & -[k^2 g(\lambda_2 + 4 + 2s) \beta_{n,4+2s} + k^4 h(\lambda_2 + 2 + 2s) \beta_{n,2+2s} \\ & + k^6 j(\lambda_2 + 2s) \beta_{n,2s}] / f(\lambda_2 + 6 + 2s) \end{aligned}$$

such that

$$\begin{aligned} \beta_{n,2} &= -k^2 \left[\frac{\partial g(\lambda)}{\partial \lambda} / \frac{\partial f(\lambda+2)}{\partial \lambda} \right]_{\lambda=\lambda_2} \beta_{n,0}, \\ \beta_{n,4} &= -[k^2 g(\lambda_2 + 2) \beta_{n,2} + k^4 h(\lambda_2) \beta_{n,0}] / f(\lambda_2 + 4). \end{aligned} \quad (5.46)$$

When $\lambda = \lambda_3$, $f(\lambda+j) = 0$ for $j = \lambda_1 - \lambda_2 = 2n-2$ and also for $j = \lambda_1 - \lambda_3 = 2n$.

We cannot draw a general conclusion all n , regarding the form of the solution.

If the right-hand side of the relation (5.41) is not zero for both $j = 2n-2$ and $j = 2n$, no number $\alpha_{n,N}$ satisfies the equation and thus, no solution of the type (5.31) exists. We note that the functions $\gamma_{n,j}(\lambda)$ defined as

$$\gamma_{n,j}(\lambda) \equiv (\lambda - \lambda_3) \alpha_{n,j}(\lambda) \quad (5.47)$$

are analytic at $\lambda = \lambda_3$ and satisfy the recurrence relations (5.34) not only for λ near λ_3 but also for $\lambda = \lambda_3$. Let

$$\tilde{\varphi}_n^{(3)}(t, \lambda) \equiv (\lambda - \lambda_3) \varphi(t, \lambda) = t \sum_{j=0}^{\infty} \sum_{n,j} \gamma_{n,j}(\lambda) t^j. \quad (5.48)$$

Although the formal series (5.48) satisfies the differential equation for $\lambda = \lambda_3$, its first $N-1$ terms vanish, and ultimately the series is merely a multiple of the first solution $\varphi_n^{(1)}$. In fact,

$$\tilde{\varphi}_n^{(3)} = \frac{\gamma_{n,N}}{\alpha_{n,0}} \varphi_n^{(1)}. \quad (5.49)$$

However, the derivative of (5.47) with respect to λ leads to an independent solution $\varphi_n^{(3)}(t)$, associated with the indicial root λ_3 .

Calculations reveal that

$$\varphi_n^{(3)}(t) = \frac{\gamma_{n,N}}{\alpha_{n,0}} \varphi_n^{(1)}(t) \ln(t) + t^{\lambda_3} \sum_{j=0}^{\infty} \left(\frac{\partial \gamma_{n,j}}{\partial \lambda} \right)_{\lambda=\lambda_3} t^j \quad (5.50)$$

where

$$(\gamma_{n,j})_{\lambda=\lambda_3} = 0, \quad j = 0, 1, 2, \dots, N-1$$

$$(\gamma_{n,j})_{\lambda=\lambda_3} = \frac{\gamma_{n,N}}{\alpha_{n,0}} \alpha_{n,j-N}(\lambda_1), \quad j = N+1, N+2, \dots$$

and

$$\left(\frac{\partial \gamma_{n,0}}{\partial \lambda} \right)_{\lambda=\lambda_3} = \alpha_{n,0},$$

$$\left[\frac{\partial f(\lambda+2)}{\partial \lambda} \gamma_{n,2} + f(\lambda+2) \frac{\partial \gamma_{n,2}}{\partial \lambda} + k^2 \frac{\partial g(\lambda)}{\partial \lambda} \gamma_{n,0} + k^2 g(\lambda) \frac{\partial \gamma_{n,0}}{\partial \lambda} \right]_{\lambda=\lambda_3} = 0,$$

.....

$$\left[\frac{\partial f(\lambda+6+s)}{\partial \lambda} \gamma_{n,6+s} + f(\lambda+6+s) \frac{\partial \gamma_{n,6+s}}{\partial \lambda} + k^2 \frac{\partial g(\lambda+4+s)}{\partial \lambda} \gamma_{n,4+s} + \right.$$

$$\left. k^2 g(\lambda+4+s) \frac{\partial \gamma_{n,4+s}}{\partial \lambda} + k^4 \frac{\partial h(\lambda+2+s)}{\partial \lambda} \gamma_{n,2+s} + k^4 h(\lambda+2+s) \frac{\partial \gamma_{n,2+s}}{\partial \lambda} + \right.$$

$$\left. k^6 \frac{\partial j(\lambda+s)}{\partial \lambda} \gamma_{n,1} + k^6 j(\lambda+s) \frac{\partial \gamma_{n,1}}{\partial \lambda} \right]_{\lambda=\lambda_3} = 0, \quad s = 0, 1, 2, \dots$$

(5.51)

If the two critical coefficients can be determined from (5.41), then there is a power series solution associated also with the third root:

$$\varphi_n^{(3)}(t) = t^{\lambda_3} \sum_{j=0}^{\infty} \alpha_{n,j}(\lambda_3) t^j \quad (5.52)$$

The two types of the solutions (5.50) and (5.52) can be written in the same form as

$$\varphi_n^{(3)}(t) = a_n \varphi_n^{(1)}(t) \ln(t) + t^{\lambda_3} \sum_{j=0}^{\infty} \gamma_{n,j}^* t^j \quad (5.53)$$

where $a_n \equiv 0$ and $\gamma_{n,j}^* \equiv \alpha_{n,j}(\lambda_3)$ if for a certain n the solution has the structure (5.52), and $a_n \equiv \gamma_{n,N}/\alpha_{n,0}$ and $\gamma_{n,j}^* \equiv \left(\frac{\partial \gamma_{n,j}}{\partial \lambda}\right)_{\lambda=\lambda_3}$ if the solution is of the form (5.53).

Analogously,

$$\varphi_n^{(4)}(t) = b_n \varphi_n^{(1)}(t) \ln(t) + t^{\lambda_4} \sum_{j=0}^{\infty} \delta_{n,j}^* t^j \quad (5.54)$$

is the fourth independent solution where $b_n \equiv 0$ and $\delta_{n,j}^* \equiv \alpha_{n,j}(\lambda_4)$ if, for a given n , the solution is of a power series form, and $b_n \equiv \delta_{n,N}/\alpha_{n,0}$, $\delta_{n,j}^* \equiv \left(\frac{\partial \delta_{n,j}}{\partial \lambda}\right)_{\lambda=\lambda_4}$ if the solution involves logarithms. Similar recursive relations with (5.51) hold for $\delta_{n,j}^*$ and $\frac{\partial \delta_{n,j}}{\partial \lambda}$ evaluated, though, at $\lambda=\lambda_4$.

Case 1b. For the specific value $n=1$, the fourth order differential-equation (5.27), in $u_n(t)$, lowers its order and becomes an equation of order three in $u'(t) \equiv du_1/dt$:

$$Lu_1'(t) = c_0(t)t^3 \frac{d^3 u'}{dt^3} + c_1(t)t^2 \frac{d^2 u'}{dt^2} + c_2(t)t \frac{du'}{dt} + c_3(t)u' = 0,$$

where

(5.55)

$$c_0(t) \equiv 1 - 3k^2 t^2 + 3k^4 t^4 - k^6 t^6,$$

$$c_1(t) \equiv 6 - 22k^2 t^2 + 26k^4 t^4 - 10k^6 t^6,$$

(5.56)

$$c_2(t) \equiv 3 - 27k^2 t^2 + 44k^4 t^4 - 20k^6 t^6,$$

$$c_3(t) \equiv -3 + 3k^2 t^2 - 2k^4 t^4.$$

We follow the procedure used in the preceding case and seek a power series solution of the form

$$u_1'(t) = t^\lambda \sum_{j=0}^{\infty} \alpha_{1,j} t^j. \quad (5.57)$$

Equation (5.55) yields

$$\sum_{j=0}^{\infty} \{ f_1(\lambda+j) \alpha_{1,j} t^{\lambda+j} + k^2 g_1(\lambda+j) \alpha_{1,j} t^{\lambda+j+2} + k^4 h_1(\lambda+j) \alpha_{1,j} t^{\lambda+j+4} + k^6 j_1(\lambda+j) \alpha_{1,j} t^{\lambda+j+6} \} = 0 \quad (5.58)$$

where we have defined

$$f_1(\lambda) \equiv \lambda(\lambda-1)(\lambda-2) + 6\lambda(\lambda-1) + 3\lambda - 3,$$

$$g_1(\lambda) \equiv -3\lambda(\lambda-1)(\lambda-2) - 22\lambda(\lambda-1) - 27\lambda + 3,$$

(5.59)

$$h_1(\lambda) \equiv 3\lambda(\lambda-1)(\lambda-2) + 26\lambda(\lambda-1) + 44\lambda - 2,$$

$$j_1(\lambda) \equiv -\lambda(\lambda-1)(\lambda-2) - 10\lambda(\lambda-1) - 20\lambda.$$

The solutions of the indicial equation $f_1(\lambda) = 0$ are found to be

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = -3. \quad (5.60)$$

As has been shown, there is at least one independent solution expressible in a power series form, and this corresponds to the largest root. It will be shown that the solution associated with the second root involves logarithms and the solution corresponding to the last root is again expressible in a power series form.

Setting $\lambda=1$, we obtain

$$u_1^{(1)}(t) = t \sum_{j=0}^{\infty} \alpha_{1,j} t^j. \quad (5.61)$$

The coefficients $\alpha_{1,j} \equiv \alpha_{1,j}(\lambda_1)$ are to be determined from the recurrence relations

$$\begin{aligned} \alpha_{1,2} &= -k^2 [g_1(1)/f_1(3)] \alpha_{1,0}, \\ \alpha_{1,4} &= -[k^2 g_1(3) \alpha_{1,2} + k^4 h_1(1) \alpha_{1,0}] / f_1(5), \\ \alpha_{1,6+2s} &= -[k^2 g_1(5+2s) \alpha_{1,4+2s} + k^4 h_1(3+2s) \alpha_{1,2+2s} \\ &\quad + k^6 j_1(1+2s) \alpha_{1,2s}] / f_1(7+2s), \\ \alpha_{1,1+2s} &= 0, \quad s = 0, 1, 2, \dots \end{aligned} \quad (5.62)$$

By setting $\lambda = \lambda_2$, we obtain the function $f_1(\lambda_2 + j)$ which vanishes when j takes the value

$$j = \lambda_1 - \lambda_2 = 2 \quad (5.63)$$

The corresponding relation, obtained by equating to zero the coefficient of the first power of t in (5.59), is given by

$$f_1(\lambda_2+2)\alpha_{1,2}(\lambda_2) + k^2 g_1(\lambda_2)\alpha_{1,0}(\lambda_2) = 0 \quad (5.64)$$

Since the function $g_1(\lambda)$ does not vanish for $\lambda=\lambda_2$, the relation (5.64) cannot be satisfied for any choice of $\alpha_{1,2}(\lambda_2)$. It can be verified that a solution associated with $\lambda_2=-1$ is furnished by

$$u_1^{(2)}(t) = \frac{\beta_{1,2}}{\beta_{1,0}} u_1^{(1)}(t) \ln(t) + \sum_{j=0}^{\infty} \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} t^{-1+j} \quad (5.65)$$

where

$$\beta_{1,j}(\lambda) \equiv (\lambda+1)\alpha_{n,j}(\lambda) \quad (5.66)$$

The critical coefficient here is found to be

$$\beta_{1,2} = -k^2 [g_1(\lambda) / \left. \frac{\partial f_1(\lambda+2)}{\partial \lambda} \right]_{\lambda=-1} \alpha_{1,0} \quad (5.67)$$

and the derivatives $\beta'_{1,j} \equiv \frac{\partial \beta_{1,j}}{\partial \lambda}$ satisfy recursively

$$\begin{aligned} & [f_1'(\lambda+6+s)\beta_{1,6+s} + f_1(\lambda+6+s)\beta'_{1,6+s} + k^2 g_1'(\lambda+4+s)\beta_{1,4+s} + \\ & k^2 g_1(\lambda+4+s)\beta'_{1,4+s} + k^4 h_1'(\lambda+2+s)\beta_{1,2+s} + k^4 h_1(\lambda+2+s)\beta'_{1,2+s} + \\ & k^6 j_1'(\lambda+6)\beta_{1,s} + k^6 j_1(\lambda+6)\beta'_{1,s}]_{\lambda=-1} = 0, \quad s = 0, 1, 2, \dots \end{aligned} \quad (5.68)$$

such that

$$\beta_{1,0} = \beta_{1,1} = 0,$$

$$\beta_{1,j} = \frac{\beta_{1,2}}{\alpha_{1,0}} \alpha_{1,j-2}, \quad j = 3, 4, \dots$$

$$\left(\frac{\partial \beta_{1,0}}{\partial \lambda} \right)_{\lambda=-1} = \alpha_{1,0},$$

$$\left(\frac{\partial \beta_{1,2}}{\partial \lambda} \right)_{\lambda=-1} = - \left[\frac{\partial^2 f_1(\lambda+2)}{\partial \lambda^2} \beta_{1,2} + 2k^2 \frac{\partial g_1(\lambda)}{\partial \lambda} \frac{\partial \beta_{1,0}}{\partial \lambda} / 2 \frac{\partial f_1(\lambda+2)}{\partial \lambda} \right]_{\lambda=-1},$$

$$\begin{aligned} \left(\frac{\partial \beta_{1,4}}{\partial \lambda} \right)_{\lambda=-1} &= - \left[\frac{\partial f_1(\lambda+4)}{\partial \lambda} \beta_{1,4} + k^2 \left[\frac{\partial g_1(\lambda+2)}{\partial \lambda} \beta_{1,2} + g_1(\lambda+2) \frac{\partial \beta_{1,2}}{\partial \lambda} \right] \right. \\ &\quad \left. + k^4 h_1(\lambda) \frac{\partial \beta_{1,0}}{\partial \lambda} \right]_{\lambda=-1}. \end{aligned} \quad (5.69)$$

Considering now the smaller root λ_3 , we see that $f_1(\lambda_3+j)$ becomes zero for both

$$j = \lambda_2 - \lambda_3 = 2 \quad (5.70)$$

and

$$j = \lambda_1 - \lambda_3 = 4 \quad (5.71)$$

and correspondingly, the coefficients of t^{-1} and t are respectively given by

$$f_1(\lambda_3+2) \gamma_{1,2} + k^2 g_1(\lambda_3) \gamma_{1,0} = 0, \quad (5.72)$$

$$f_1(\lambda_3+4) \gamma_{1,4} + k^2 g_1(\lambda_3+2) \gamma_{1,2} + k^4 h_1(\lambda_3) \gamma_{1,0} = 0, \quad (5.73)$$

where

$$\gamma_{1,j} \equiv \alpha_{1,j}(\lambda_3). \quad (5.74)$$

It turns out that $g_1(\lambda_3) = 0$ and hence, relation (5.72) is fulfilled for any value of $\gamma_{1,2}$. The relation (5.73) can be satisfied if the sum of the last two terms can be made zero. This may be arranged if

$$\gamma_{1,2} = -k^2 \frac{h_1(\lambda_3)}{g_1(\lambda_3+2)} \gamma_{1,0} = \frac{k^2}{2} \gamma_{1,0} \quad (5.75)$$

If $\gamma_{1,2}$ satisfies (5.7), then relation (5.73) is fulfilled for any $\gamma_{1,4}$.

In particular, we can choose

$$\gamma_{1,4} = 0 \quad (5.76)$$

All other coefficients are obtained from the recurrence relations

$$\begin{aligned} \gamma_{1,6+2s} = & -[k^2 g_1(1+2s) \gamma_{1,4+2s} + k^4 h_1(-1+2s) \gamma_{1,2+2s} \\ & + k^6 j_1(-3+2s) \gamma_{1,2s}] / f_1(3+2s), \end{aligned} \quad (5.77)$$

$$\gamma_{1,1+2s} = 0, \quad s = 0, 1, 2, \dots$$

Writing

$$u_1^{(3)} = t^3 \sum_{j=0}^{\infty} \gamma_{1,j} t^j, \quad (5.78)$$

it follows from (5.61), (5.65) and (5.78) that, corresponding to $n=1$, the differential equation (5.27) has the solution $u_1(t)$ satisfying

$$\begin{aligned} \frac{du_1}{dt} = & A_1 \sum_{j=0}^{\infty} \alpha_{1,j} t^{1+j} + B_1 \left[\frac{\beta_{1,2}}{\alpha_{1,0}} \ln(t) \sum_{j=0}^{\infty} \alpha_{1,j} t^{1+j} + \right. \\ & \left. \sum_{j=0}^{\infty} \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} t^{-1+j} \right] + C_1 \sum_{j=0}^{\infty} \gamma_{1,j} t^{-3+j} \end{aligned} \quad (5.79)$$

Case 2. If $n=0$, the equilibrium equations (5.13), (5.14) and the conditions of incompressibility (5.2) furnish

$$\frac{du_0}{dr} + \frac{u_0}{r} = 0, \quad (5.80)$$

$$rQ^2 \frac{d^2 \zeta_0}{dr^2} + (2-Q^2) \left(\frac{\partial \zeta_0}{\partial r} - \frac{\zeta_0}{r} \right) = 0, \quad (5.81)$$

$$\frac{1}{2c} \frac{dp'_0}{dr} + Q^2 \frac{d^2 u_0}{dr^2} + \frac{2}{r} (2-Q^2) \frac{du_0}{dr} = 0. \quad (5.82)$$

In view of the expression (3.2.12) for $Q(r)$, the equation (5.82) may be readily integrated to yield

$$p'_0(r) = M_0 + 2c \frac{k^2}{r^2} \frac{du_0}{dr}. \quad (5.83)$$

Further, it follows from (5.80) that

$$u_0 = \frac{A_0}{r} \quad (5.84)$$

and hence,

$$p'_0(r) = M_0 - 2A_0 c \frac{k^2}{r^4}. \quad (5.85)$$

To integrate equation (5.81) we firstly apply the inverse transformation (5.25). Equation (5.81) becomes

$$(1-k^2 t^2) t^2 \frac{d^2 \zeta_0}{dt^2} + (1-3k^2 t^2) t \frac{d\zeta_0}{dt} - (1+k^2 t^2) \zeta_0 = 0. \quad (5.86)$$

It can be shown that the general solution of the equation (5.86)

is given by

$$\zeta_0(t) = \tilde{A}_0 \sum_{j=0}^{\infty} a_{0,j} t^{1+j} + \tilde{B}_0 \left[\frac{b_{0,2}}{a_{0,0}} \ln(t) \sum_{j=0}^{\infty} a_{0,j} t^{1+j} + \sum_{j=0}^{\infty} \left(\frac{\partial b_{0,j}}{\partial \lambda} \right)_{\lambda=-1} t^{-1-j} \right]. \quad (5.87)$$

The coefficients $a_{0,j}$ are determined by the recurrence relations

(5.87) in the form

$$a_{0,2+2s} = -k^2 [g_0(1+2s)/f_0(3+2s)] a_{0,2s}, \quad (5.88)$$

$$a_{0,1+2s} = 0, \quad s = 0, 1, 2, \dots$$

where the functions $f_0(\lambda)$ and $g_0(\lambda)$ have been defined as follows

$$f_0(\lambda) = \lambda(\lambda-1) + \lambda - 1, \quad (5.89)$$

$$g_0(\lambda) = -\lambda(\lambda-1) - 3\lambda + 1.$$

The values of λ for which $f_0(\lambda) = 0$ are

$$\lambda_1 = 1, \quad \lambda_2 = -1. \quad (5.90)$$

The coefficients $b_{0,j}$ are defined by

$$b_{0,j}(\lambda) \equiv (\lambda - \lambda_2) a_{0,j}(\lambda), \quad (5.91)$$

and, we obtain that

$$b_{0,j} = 0, \quad j = 0, 1.$$

$$b_{0,2} = -k^2 [g_0(\lambda)/(\lambda-1)]_{\lambda=-1} a_{0,0} = -k^2 a_{0,0}, \quad (5.92)$$

$$b_{0,j} = (b_{0,2}/a_{0,0}) a_{0,j-2}, \quad j = 3, 4, \dots$$

The derivatives $b'_{0,j} \equiv \frac{\partial b_{0,j}}{\partial \lambda}$ satisfies recursively

$$\begin{aligned} & \{f'_0(\lambda+2+j)b_{0,2+j} + f(\lambda+2+j)b'_{0,2+j} \\ & + k^2 [g'_0(\lambda+j)b_{0,j} + g_0(\lambda+j)b'_{0,j}]\}_{\lambda=-1} = 0, \\ & j = 0, 1, 2, \dots \quad (5.93) \end{aligned}$$

where it can be shown that

$$\left(\frac{\partial b_{0,1}}{\partial \lambda}\right)_{\lambda=-1} = a_{0,0} \quad (5.94)$$

The series involved in all solutions obtained here have radii of convergence at least as large as the distance to the nearest singularity of the coefficient functions which occurs at $t = 1/k$.

We reassemble the solution, firstly including expression (5.84) for $u_0(r)$ and integrating relation (5.79) to obtain $u_1(t)$. We use the partial results (5.38), (5.45), (5.52) and (5.54) which express $u_n(t)$ for every $n \neq 0, 1$ and apply, throughout, the inverse transformation (5.25). Further, we derive $v_n(r)$ and $\zeta_n(r)$, making use of the relation (5.87) and employing the incompressibility conditions (5.2) and (5.3) together with the corresponding results for $u_n(r)$. We obtain that any perturbational displacement field $\varepsilon \bar{w} = \langle \varepsilon u, \varepsilon v \rangle$, that can be superimposed on the finite inflation and ensures equilibrium by surface tractions alone, belongs to the general solution:

$$\begin{aligned}
u(r, \theta) = & \frac{A_0}{r} + \left(\sum_{j=0}^{\infty} \frac{\alpha_{1,j}}{2+j} r^{-2-j} \right) [A_1 \cos(\theta) + \tilde{A}_1 \sin(\theta)] \\
& + \left[\frac{\beta_{1,2}}{\alpha_{1,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \frac{\alpha_{1,j}}{2+j} r^{-2-j} - \frac{\beta_{1,2}}{\alpha_{1,0}} \sum_{j=0}^{\infty} \frac{\alpha_{1,j}}{(2+j)^2} r^{-2-j} \right. \\
& \left. + \left(\frac{\partial \beta_{1,0}}{\partial \lambda} \right)_{\lambda=-1} \ln \frac{a_0}{r} + \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} r^{-j} \right] [B_1 \cos(\theta) + \tilde{B}_1 \sin(\theta)] \\
& + \left(\sum_{\substack{j=0 \\ j \neq 2}}^{\infty} \frac{\gamma_{1,j}}{j-2} r^{2-j} + \gamma_{1,2} \ln \frac{a_0}{r} \right) [C_1 \cos(\theta) + \tilde{C}_1 \sin(\theta)] \\
& + \left(\sum_{n=2}^{\infty} \left\{ \sum_{j=0}^{\infty} \alpha_{n,j} r^{-n-1-j} \right\} [A_n \cos(n\theta) + \tilde{A}_n \sin(n\theta)] \right. \\
& \left. + \left(\sum_{j=0}^{\infty} \beta_{n,j} r^{-n+1-j} \right) [B_n \cos(n\theta) + \tilde{B}_n \sin(n\theta)] \right. \\
& \left. + \left(a_n \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \alpha_{n,j} r^{-n-1-j} + \sum_{j=0}^{\infty} \gamma_{n,j}^* r^{n-1-j} \right) [C_n \cos(n\theta) + \tilde{C}_n \sin(n\theta)] \right. \\
& \left. + \left(b_n \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \alpha_{n,j} r^{-n-1-j} + \sum_{j=0}^{\infty} \delta_{n,j}^* r^{n+1-j} \right) [D_n \cos(n\theta) + \tilde{D}_n \sin(n\theta)] \right\},
\end{aligned}$$

(5.95)

$$\begin{aligned}
v(r, \theta) = & \tilde{A}_0 \sum_{j=0}^{\infty} a_{0,j} r^{-1-j} + \tilde{B}_0 \left[\frac{b_{0,2}}{a_{0,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} a_{0,j} r^{-1-j} + \sum_{j=0}^{\infty} \left(\frac{\partial b_{0,j}}{\partial \lambda} \right)_{\lambda=-1} r^{1-j} \right] \\
& + \left(\sum_{j=0}^{\infty} \frac{1+j}{2+j} \alpha_{1,j} r^{-2-j} \right) [A_1 \sin(\theta) - \tilde{A}_1 \cos(\theta)] + \left[\frac{\beta_{1,2}}{\alpha_{1,0}} \left(\ln \frac{a_0}{r} \right) \sum_{j=0}^{\infty} \frac{1+j}{2+j} r^{-2-j} \right. \\
& \left. + \sum_{j=0}^{\infty} \frac{\alpha_{1,j}}{(2+j)^2} r^{-2-j} + \left(1 - \ln \frac{a_0}{r} \right) \left(\frac{\partial \beta_{1,0}}{\partial \lambda} \right)_{\lambda=-1} \right. \\
& \left. + \sum_{j=1}^{\infty} \frac{j-1}{j} \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} r^{-j} \right] [B_1 \sin(\theta) - \tilde{B}_1 \cos(\theta)] \\
& + \left[\sum_{\substack{j=0 \\ j \neq 2}}^{\infty} \frac{j-3}{j-2} \gamma_{1,j} r^{2-j} + \left(1 - \ln \frac{a_0}{r} \right) \gamma_{1,2} \right] [C_1 \sin(\theta) - \tilde{C}_1 \cos(\theta)]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \left[\sum_{j=0}^{\infty} (n+j) \alpha_{n,j} r^{-n-1-j} \right] [A_n \sin(n\theta) - \tilde{A}_n \cos(n\theta)] \right. \\
& + \left[\sum_{j=0}^{\infty} (n-2+j) \beta_{n,j} r^{-n+1-j} \right] [B_n \sin(n\theta) - \tilde{B}_n \cos(n\theta)] \\
& + \left[a_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} (n+j) \alpha_{n,j} r^{-n-1-j} + \sum_{j=0}^{\infty} \alpha_{n,j} r^{-n-1-j} \right) \right. \\
& - \left. \sum_{j=0}^{\infty} (n-j) \gamma_{n,j}^* r^{n+1-j} \right] [C_n \sin(n\theta) - \tilde{C}_n \cos(n\theta)] \\
& + \left[b_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} (n+j) \alpha_{n,j} r^{-n-1-j} + \sum_{j=0}^{\infty} \alpha_{n,j} r^{-n-1-j} \right) \right. \\
& - \left. \sum_{j=0}^{\infty} (n+1-j) \delta_{n,j}^* r^{n+1-j} \right] [D_n \sin(n\theta) - \tilde{D}_n \cos(n\theta)] \left. \right\}.
\end{aligned}$$

(5.96)

In order to calculate the additional stresses, we firstly obtain the hydrostatic pressure $p^*(r, \theta)$ making use of the results (5.85) and (5.21) and performing the required higher derivatives of the displacement components. Then, we substitute into relation (5.11) along with the displacement gradients derived from (5.95) and (5.96). Finally we obtain the following incremental stress components:

$$\begin{aligned}
\tau^{11}(r, \theta) = & \sigma_0 - 4\tau_0 c \left(1 + \frac{1}{2} \ln \frac{r^2}{r^2 - k^2} \right) / r^2 \\
& + 2c \left\{ \left(\sum_{j=0}^{\infty} F_{1,j}^1 \alpha_{1,j} r^{-3-j} \right) [A_1 \cos(\theta) + \tilde{A}_1 \sin(\theta)] \right. \\
& + \left[\frac{\beta_{1,2}}{\alpha_{1,0}} \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} F_{1,j}^1 \alpha_{1,j} r^{-3-j} - \sum_{j=0}^{\infty} F_{1,j}^2 \alpha_{1,j} r^{-3-j} \right) \right. \\
& + \left. \sum_{j=0}^{\infty} F_{1,j}^3 \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} r^{-1-j} \right] [B_1 \cos(\theta) + \tilde{B}_1 \sin(\theta)] \\
& + \left. \left(\sum_{j=0}^{\infty} F_{1,j}^4 \alpha_{1,j} r^{1-j} \right) [C_1 \cos(\theta) + \tilde{C}_1 \sin(\theta)] \right\}
\end{aligned}$$

$$\begin{aligned}
& - 2c \sum_{n=2}^{\infty} \frac{1}{n^2} \left\{ \left(\sum_{j=0}^{\infty} F_{n,j}^5 \alpha_{n,j} r^{-n-2-j} \right) [A_n \cos(n\theta) + \tilde{A}_n \sin(n\theta)] \right. \\
& \quad + \left[\left(\sum_{j=0}^{\infty} F_{n,j}^6 \beta_{n,j} r^{-n-j} \right) [B_n \cos(n\theta) + \tilde{B}_n \sin(n\theta)] \right. \\
& \quad + \left[a_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} F_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} F_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& \quad + \left. \left(\sum_{j=0}^{\infty} F_{n,j}^8 \gamma_{n,j}^* r^{n-2-j} \right) [C_n \cos(n\theta) + \tilde{C}_n \sin(n\theta)] \right. \\
& \quad + \left[b_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} F_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} F_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& \quad + \left. \left. \left(\sum_{j=0}^{\infty} F_{n,j}^9 \delta_{n,j}^* r^{n-j} \right) [D_n \cos(n\theta) + \tilde{D}_n \sin(n\theta)] \right] \right\},
\end{aligned}$$

(5.97)

where

$$F_{1,j}^1(k,r) \equiv (-1+3j+j^2) - (4+5j+j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right),$$

$$F_{1,j}^2(k,r) \equiv (-3-2j) + (5+2j) \frac{k^2}{r^2},$$

$$F_{1,j}^3(k,r) \equiv (-3-j-j^2) + (2-j-j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right),$$

$$F_{1,j}^4(k,r) \equiv (3-5j+j^2) + (3j-j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right),$$

$$\begin{aligned}
F_{n,j}^5(k,r) & \equiv [(n+1)(n^2-n+2) - n(n+4)j - (3n+2)j - j^3] \\
& \quad + [4n(n+1) + (2n^2+8n+4)j + (3n+4)j^2 + j^3] \frac{k^2}{r^2} \\
& \quad + n^2(n+1+j) \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right),
\end{aligned}$$

$$\begin{aligned}
F_{n,j}^6(k,r) & \equiv [n(n-1)(n+4) - (n^2-8n+4)j - (3n-4)j^2 - j^3] \\
& \quad + [2n(n-2)j + (3n-2)j^2 + j^3] \frac{k^2}{r^2} \\
& \quad + n^2(n-1+j) \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right),
\end{aligned}$$

$$F_{n,j}^7(k,r) \equiv [n(n+4) + 2(3n+2)j + 3j^2]$$

$$- [(2n^2+8n+4) + 2(3n+4)j + 3j^2] \frac{k^2}{r^2}$$

$$- n^2 \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right)$$

$$F_{n,j}^8(k,r) \equiv [-n^2 - n(n-4)j + (3n-2)j^2 - j^3]$$

$$+ [4n(n-1) + (2n^2-8n+4)j - (3n-4)j^2 + j^3] \frac{k^2}{r^2}$$

$$- n^2 (n-1-j) \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right)$$

$$F_{n,j}^9(k,r) \equiv [(n+1)(1-n^2-3n) - (n^2+8n+4)j + (3n+1)j^2 - j^3]$$

$$+ [(n+1)(1-n^2) + 2n(n+2)j - (3n-2)j^2 + j^3] \frac{k^2}{r^2}$$

$$- n^2 (n+1-j) \left(\frac{r^2}{r^2-k^2} + \ln \frac{r^2}{r^2-k^2} \right)$$

$$r^2 \tau^{22}(r,\theta) = M_0 + 4A_0 c \left(1 - \frac{k^2}{r^2} + \frac{1}{2} \ln \frac{r^2}{r^2-k^2} \right) / r^2$$

$$+ 2c \left\{ \left(\sum_{j=0}^{\infty} G_{1,j}^1 \alpha_{1,j} r^{-3-j} \right) [A_1 \cos(\theta) + \tilde{A}_1 \sin(\theta)] \right.$$

$$+ \left[\frac{\beta_{1,j}}{\alpha_{1,0}} \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} G_{1,j}^1 \alpha_{1,j} r^{-3-j} - \sum_{j=0}^{\infty} G_{1,j}^2 \alpha_{1,j} r^{-3-j} \right) \right.$$

$$+ \left. \sum_{j=0}^{\infty} G_{1,j}^3 \left(\frac{\partial \beta_{1,2}}{\partial \lambda} \right)_{\lambda=-1} r^{-1-j} \right] [B_1 \cos(\theta) + \tilde{B}_1 \sin(\theta)]$$

$$+ \left(\sum_{j=0}^{\infty} G_{1,j}^4 \gamma_{1,j} r^{-1-j} \right) [C_1 \cos(\theta) + \tilde{C}_1 \sin(\theta)]$$

$$\begin{aligned}
& - 2c \sum_{n=2}^{\infty} \frac{1}{n^2} \left\{ \left(\sum_{j=0}^{\infty} G_{n,j}^5 \alpha_{n,j} r^{-n-2-j} \right) [A_n \cos(n\theta) + \tilde{A}_n \sin(n\theta)] \right. \\
& \quad + \left(\sum_{j=0}^{\infty} G_{n,j}^6 \beta_{n,j} r^{-n-j} \right) [B_n \cos(n\theta) + \tilde{B}_n \sin(n\theta)] \\
& \quad + \left[a_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} G_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} G_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& \quad + \sum_{j=0}^{\infty} G_{n,j}^8 \gamma_{n,j}^* r^{n-2-j} \left. \right] [C_n \cos(n\theta) + \tilde{C}_n \sin(n\theta)] \\
& \quad + \left[b_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} G_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} G_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& \quad + \sum_{j=0}^{\infty} G_{n,j}^9 \delta_{n,j}^* r^{n-j} \left. \right] [D_n \cos(n\theta) + \tilde{D}_n \sin(n\theta)] \left. \right\} , \\
\end{aligned} \tag{5.98}$$

where

$$G_{1,j}^1(k,r) \equiv (3+3j+j^2) - (6+5j+j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right) ,$$

$$G_{1,j}^2(k,r) \equiv (-3-2j) + (5+2j) \frac{k^2}{r^2} ,$$

$$G_{1,j}^3(k,r) \equiv (1-j+j^2) - (j+j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right) ,$$

$$G_{1,j}^4(k,r) \equiv (7-5j+j^2) - (2-3j+j^2) \frac{k^2}{r^2} - \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right) ,$$

$$\begin{aligned}
G_{n,j}^5(k,r) & \equiv [(n+1)(-3n^2-n+2) - n(5n+4)j - (3n+2)j^2 - j^3] \\
& \quad + [2n(n+1)(2+n) + 4(n+1)^2j + (3n+4)j^2 + j^3] \frac{k^2}{r^2} \\
& \quad + n^2(n+1+j) \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right) , \\
\end{aligned}$$

$$\begin{aligned}
G_{n,j}^6(k,r) & \equiv [-n(n-1)(3n-4) - (5n^2-8n+4)j - (3n-4)j^2 - j^3] \\
& \quad + [2n^2(n-1) + 4n(n-1)j + (3n-2)j^2 + j^3] \frac{k^2}{r^2} \\
& \quad + n^2(n-1+j) \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right) , \\
\end{aligned}$$

$$G_{n,j}^7(k,r) \equiv [n(4n^2-3n+4) + 2(3n+2)j + 3j^2] \\ - [2(n^3+4n+2) + 2(3n+4)j + 3j^2] \frac{k^2}{r^2} \\ - n^2 \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right),$$

$$G_{n,j}^8(k,r) \equiv [n^2(4n-5) - n(5n-4)j + (3n-2)j^2 - j^3] \\ - [2n(n-1)(n-2) - 4(n-1)^2j + (3n-4)j^2 - j^3] \frac{k^2}{r^2} \\ - n^2(n-1-j) \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right),$$

$$G_{n,j}^9(k,r) \equiv [(3n^3+2n^2+4n-1) - (5n^2+8n+4)j + (3n+1)j^2 - j^3] \\ + [(n+1)(1-3n^2) + 4n(n+1)j - (3n-2)j^2 + j^3] \frac{k^2}{r^2} \\ - n^2(n+1-j) \left(\frac{r^2}{r^2-k^2} - \ln \frac{r^2}{r^2-k^2} \right),$$

$$r\tau^{12}(r,\theta) = c \left(2 - \frac{k^2}{r^2} + \ln \frac{r^2}{r^2-k^2} \right) < \tilde{A}_0 \sum_{j=0}^{\infty} j a_{0,j} r^{-2-j} \\ + \tilde{B}_0 \left[\frac{b_{0,2}}{a_{0,0}} \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} j a_{0,j} r^{-2-j} \right) \right. \\ \left. + \sum_{j=0}^{\infty} a_{0,j} r^{-2-j} \right] - \sum_{j=0}^{\infty} \left(\frac{\beta_{1,j}}{3\lambda} \right)_{\lambda=-1} (2-j) r^{-j} \\ + \left[\sum_{j=0}^{\infty} (2+j) \alpha_{1,j} r^{-3-j} \right] [A_1 \sin(\theta) - \tilde{A}_1 \cos(\theta)] \\ + \left[\frac{\beta_{1,2}}{\alpha_{1,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} (2+j) \alpha_{1,j} r^{-3-j} + \frac{\beta_{1,2}}{\alpha_{1,0}} \sum_{j=0}^{\infty} \alpha_{1,j} r^{-3-j} \right. \\ \left. - \sum_{j=0}^{\infty} j \left(\frac{\beta_{1,j}}{3\lambda} \right)_{\lambda=-1} r^{-1-j} \right] [B_1 \sin(\theta) - \tilde{B}_1 \cos(\theta)] \\ - \sum_{j=0}^{\infty} (2-j) \gamma_{1,j} r^{-1-j} [C_1 \sin(\theta) - \tilde{C}_1 \cos(\theta)]$$

$$\begin{aligned}
& - \sum_{n=2}^{\infty} \frac{1}{n} \left\{ \left(\sum_{j=0}^{\infty} H_{n,j}^1 \alpha_{n,j} r^{-n-2-j} \right) [A \sin(n\theta) - \tilde{A}_n \cos(n\theta)] \right. \\
& + \left(\sum_{j=0}^{\infty} H_{n,j}^2 \beta_{n,j} r^{-n-j} \right) [B \sin(n\theta) - \tilde{B}_n \cos(n\theta)] \\
& + \left[a_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} H_{n,j}^1 \alpha_{n,j} r^{-n-2-j} + \sum_{j=0}^{\infty} H_{n,j}^3 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& + \left. \sum_{j=0}^{\infty} H_{n,j}^4 \gamma_{n,j}^* r^{-n-2-j} \right] [C \sin(n\theta) - \tilde{C}_n \cos(n\theta)] \\
& + \left[b_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} H_{n,j}^1 \alpha_{n,j} r^{-n-2-j} + \sum_{j=0}^{\infty} H_{n,j}^3 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& + \left. \sum_{j=0}^{\infty} H_{n,j}^5 \delta_{n,j}^* r^{-n-j} \right] [D \sin(n\theta) - \tilde{D}_n \cos(n\theta)] \left. \right\} > ,
\end{aligned}$$

(5.99)

where

$$H_{n,j}^1 \equiv 2(n+1)(n+j) + j^2,$$

$$H_{n,j}^2 \equiv 2(n-1)(n+j) + j^2,$$

$$H_{n,j}^3 \equiv 2(n+1+j),$$

$$H_{n,j}^4 \equiv 2(n-1)(n-j) + j^2,$$

$$H_{n,j}^5 \equiv 2(n+1)(n-j) + j^2.$$

and

$$\begin{aligned}
\tau^{33}(r, \theta) &= M_0 - 2A_0 \left[c \frac{k^2}{r^2} + c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2 - k^2} \right) \right] / r^2 \\
&+ 2 \left\{ \left(\sum_{j=0}^{\infty} L_{1,j}^1 \alpha_{1,j} r^{-3-j} \right) [A_1 \cos(\theta) + \tilde{A}_1 \sin(\theta)] \right. \\
&+ \left[\frac{\beta_{1,2}}{\alpha_{1,2}} \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} L_{1,j}^1 \alpha_{1,j} r^{-3-j} - \sum_{j=0}^{\infty} L_{2,j}^2 \alpha_{1,j} r^{-3-j} \right) \right. \\
&+ \left. \sum_{j=0}^{\infty} L_{1,j}^3 \left(\frac{\partial \beta_{1,j}}{\partial \lambda} \right)_{\lambda=-1} r^{-1-j} \right] [B_1 \cos(\theta) + \tilde{B}_1 \sin(\theta)] \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{j=0}^{\infty} L_{1,j}^4 \alpha_{1,j} r^{1-j} \right) [C_1 \cos(\theta) + \tilde{C}_1 \sin(\theta)] \\
- 2 \sum_{n=2}^{\infty} & \left\{ \left(\sum_{j=0}^{\infty} L_{n,j}^5 \alpha_{n,j} r^{-n-2-j} \right) [A_n \cos(n\theta) + \tilde{A}_n \sin(n\theta)] \right. \\
& + \left(\sum_{j=0}^{\infty} L_{n,j}^6 \beta_{n,j} r^{-n-j} \right) [B_n \cos(n\theta) + \tilde{B}_n \sin(n\theta)] \\
& + \left[a_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} L_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} L_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& + \left. \sum_{j=0}^{\infty} L_{n,j}^8 \gamma_{n,j}^* r^{n-2-j} \right] [C_n \cos(n\theta) + \tilde{C}_n \sin(n\theta)] \\
& + \left[b_n \left(\ln \frac{a_0}{r} \sum_{j=0}^{\infty} L_{n,j}^5 \alpha_{n,j} r^{-n-2-j} - \sum_{j=0}^{\infty} L_{n,j}^7 \alpha_{n,j} r^{-n-2-j} \right) \right. \\
& + \left. \sum_{j=0}^{\infty} L_{n,j}^9 \delta_{n,j}^* r^{n-j} \right] [D_n \cos(n\theta) + \tilde{D}_n \sin(n\theta)] \left. \right\},
\end{aligned}$$

(5.100)

where

$$L_{1,j}^1(k,r) \equiv c \left[(1+3j+j^2) - (5+5j+j^2) \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right] / n^2$$

$$- c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right)$$

$$L_{1,j}^2(k,r) \equiv c \left[(-3-2j) + (5+2j) \frac{k^2}{r^2} \right] / n^2$$

$$L_{1,j}^3(k,r) \equiv c \left[(-1-j-j^2) + (1-j-j^2) \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right] / n^2$$

$$- c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right)$$

$$L_{1,n}^4(k,r) \equiv c \left[(5-5j+j^2) - (1-3j+j^2) \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right] / n^2$$

$$c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2} \right)$$

$$L_{n,j}^5(k,r) \equiv c \left[-(n+3)(n^2-1) + (3n^2-4n)j + (3n+2)j + j^3 \right]$$

$$+ \{ (n+1) [(n+1)(n+2) - n] + (3n^2+8n+4)j \}$$

$$+ (3n^2+2)j^2 + j^3 \frac{k^2}{r^2} + n^2(n+1+j) \frac{r^2}{r^2-k^2} >/n^2$$

$$+ c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2}\right) (n+1+j) ,$$

$$L_{n,j}^6(k,r) \equiv c \{ (n-1) [(n-2)^2 - n] + (3n^2 - 8n + 4)j + (3n-4)j^2 + j^3 \}$$

$$+ \{ (n-1)(n^2 - n - 1) + (3n^2 - 4n)j + (3n-2)j^2 + j^3 \} \frac{k^2}{r^2}$$

$$+ n^2(n-1+j) \frac{r^2}{r^2-k^2} >/n^2 + c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2}\right) (n-1+j) ,$$

$$L_{n,j}^7(k,r) \equiv c \{ (3n^2 + 4n) + (6n+4)j + 3j^2$$

$$- \{ (3n^2 + 4n + 4) + (6n+8)j + 3j^2 \} \frac{k^2}{r^2} - n^2 \frac{r^2}{r^2-k^2} \} /n^2$$

$$- c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2}\right) ,$$

$$L_{n,j}^8(k,r) \equiv c \{ (n-1) [(n-1)^2 + 1] + (3n^2 + 4n)j + (3n-2)j^2 - j^3$$

$$+ \{ [n^2 - (n-2)(n-1)^2] + (3n^2 - 8n + 4)j$$

$$+ (-3n+4)j^2 + j^3 \} \frac{k^2}{r^2} - n^2(n-1-j) \frac{r^2}{r^2-k^2} >/n^2$$

$$- c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2}\right) (n-1-j) ,$$

$$L_{n,j}^9(k,r) \equiv c \{ [n(n+1)(n+4) + (3n^2 + 8n)j + (3n+4)j - j^3]$$

$$+ [n(n+1)(-3n+1) + (3n^2 + 4n)j - (3n+2)j^2 + j^3] \frac{k^2}{r^2}$$

$$- n^2(n+1-j) \frac{r^2}{r^2-k^2} >/n^2 - c_2 \left(1 - \frac{k^2}{r^2} - \frac{r^2}{r^2-k^2}\right) (n+1-j) .$$

It is worth noting that the displacement field that constitutes the general solution of the problem corresponding to the Mooney-Rivlin

strain-energy function is also valid for neo-Hookean solids when the constant $c = c_1 + c_2$ of the Mooney-Rivlin material is identified with the constant c , ($c = c_1$, $c_2 = 0$) of the neo-Hookean solid. However, although the nonvanishing stress τ^{11} , τ^{22} and τ'^{11} , τ'^{22} and τ'^{12} corresponding to Mooney-Rivlin strain-energy function are the same as those corresponding to the neo-Hookean, the stress τ^{33} and τ'^{33} differ.

6. PERTURBATIONAL UNIAXIAL TENSION

In order to illustrate the method outlined in the preceding sections, we shall now investigate some specific boundary value problems.

In this chapter we consider that the perturbation, applied to a finitely deformed slab, is caused by uniform tensile loadings acting at sections far from the opening. We shall investigate the case of a slab with a circular hole and two cases where a rigid inclusion is embedded into the body.

6.1. Perturbational Uniaxial Tension Applied to a Finitely Deformed Slab with a Circular Hole

Let $P \equiv \varepsilon P$ denote the perturbational stress distribution which is applied in the x_1 direction. No additional forces, besides the expanding pressure, are to be applied at the hole. Then, the boundary conditions at infinity may be expressed as

$$\tau_{x_1 x_1} \sim P, \quad \tau_{x_2 x_2} \sim 0, \quad \tau_{x_1 x_2} \sim 0. \quad (6.1.1)$$

We perform the appropriate tensor transformation into polar coordinates and obtain

$$\begin{aligned} \tau^{11} + \varepsilon(\tau'^{11} + \tau^{11} \frac{\partial u}{\partial r}) &\sim P \cos^2 \theta, \\ r^2 [\tau^{22} + \varepsilon(\tau'^{22} - \tau^{22} \frac{\partial u}{\partial r})] &\sim P \sin^2 \theta, \\ r[\tau^{12} + \varepsilon(\tau'^{12} + \frac{1}{r} \tau^{11} \frac{\partial v}{\partial r})] &\sim -\frac{P}{2} \sin(2\theta). \end{aligned} \quad (6.1.2)$$

However, by (5.10), $\hat{\tau}_{ik} \sim 0$ and therefore

$$\begin{aligned}\tau'^{11} &\sim \frac{P}{2}[1 + \cos(2\theta)] , \\ r^2 \tau'^{22} &\sim \frac{P}{2}[1 - \cos(2\theta)] , \\ r \tau'^{12} &\sim -\frac{P}{2} \sin(2\theta) .\end{aligned}\tag{6.1.3}$$

In view of the expressions (5.97), (5.98) and (5.99) for the stress components τ'^{ik} , relations (6.1.3) are satisfied provided

$$\begin{aligned}M_0 &= \frac{P}{2} , \quad C_2 = \frac{P}{8c} , \\ C_n &= 0 , \quad \text{for all } n \neq 2 , \\ \tilde{C}_n &= D_n = \tilde{D}_n = 0 , \quad \text{for all } n .\end{aligned}\tag{6.1.4}$$

We also require that the small uniaxial loadings at infinity induce no changes in the applied force distribution at the hole. Hence, at the boundary surface $r=a$, we must have

$$\begin{aligned}[\tau'^{11} + \tau'^{11} \frac{\partial u}{\partial r}]_{r=a} &= 0 , \\ [\tau'^{12} + \frac{1}{r} \tau'^{11} \frac{\partial v}{\partial r}]_{r=a} &= 0 .\end{aligned}\tag{6.1.5}$$

After insertion of expressions (5.10), (5.11) for τ'^{11} , τ'^{12} and τ'^{22} , and further, using the representations (5.1) for $u(r,\theta)$ and $v(r,\theta)$ together with the incompressibility constraints (5.2) and (5.3), we find that the contribution of terms corresponding to $n=0$ yield

$$A_0 = \frac{P}{2} a^2 / (4 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2}) c . \quad (6.1.6)$$

Corresponding to $n \neq 0$, we obtain

$$\begin{aligned} & \left\{ \left(1 - \frac{k^2}{a^2}\right) r^2 \frac{d^3 \omega_n}{dr^3} + \left(4 - 2 \frac{k^2}{a^2}\right) r \frac{d^2 \omega_n}{dr^2} + \left[(1 - 2n^2) + \left(1 + \frac{3n^2}{2}\right) \frac{k^2}{a^2} - \frac{n^2}{1 - k^2/a^2} \right. \right. \\ & \quad \left. \left. + \frac{n^2}{2} \ln \frac{a_0^2}{a^2} \right] \frac{d\omega_n}{dr} + (n^2 - 1) \left(1 + \frac{k^2}{a^2}\right) \frac{1}{r} \omega_n \right\}_{r=a} = 0 , \\ & \left\{ 2 \left(1 - \frac{k^2}{a^2}\right) r^2 \frac{d^2 \omega_n}{dr^2} + \left(2 - 3 \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2}\right) r \frac{d\omega_n}{dr} + (n^2 - 1) \left(2 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2}\right) \omega_n \right\}_{r=a} = 0 \end{aligned} \quad (6.1.7)$$

where ω_n stands for either $u_n(r)$ or $\eta_n(r)$. Relations (6.1.7)

provide for each n , two sets of coupled equations for the unknown

constants $\{\tilde{A}_n, \tilde{B}_n\}$ and $\{A_n, B_n\}$. It can be shown that, except when

$n=2$, each system admits only the trivial solution

$$\tilde{A}_n = \tilde{B}_n = 0 , \quad n \neq 0, \quad (6.1.8)$$

$$A_n = B_n = 0 , \quad n \neq 0, n \neq 2.$$

In order to obtain the value of the constants A_2 and B_2 , we have to specify some results.

The functions f , g , h , and j involved in the recursive relations for the coefficients α , β , γ and δ are obtained from their general expressions (5.33) by setting $n=2$. The indicial roots (5.36) become for $n=2$

$$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \quad \lambda_4 = -3 . \quad (6.1.9)$$

In fact, only the first three roots contribute to the solution of the posed boundary value problem since the constants D_n and \tilde{D}_n must vanish to meet the stress conditions at infinity.

We have proven that for every $n \neq 0$, there are two independent power series solutions, corresponding to the largest and the second largest indicial root. These solutions, for $n=2$, have the form

$$\begin{aligned} u_2^{(1)}(r) &= \sum_{j=0}^{\infty} \alpha_{2,j} r^{-3-j}, \\ u_2^{(2)}(r) &= \sum_{j=0}^{\infty} \beta_{2,j} r^{-1-j}. \end{aligned} \quad (6.1.10)$$

The coefficients $\alpha_{2,j}$ and $\beta_{2,j}$ satisfy the recurrence relations

$$\begin{aligned} \left. \begin{array}{l} \alpha_{2,6+s} \\ \beta_{2,6+s} \end{array} \right\} &= -k^2 g \left(\begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} + 4+s \right) \left\{ \begin{array}{l} \alpha_{2,4+s} \\ \beta_{2,4+s} \end{array} \right\} + k^4 h \left(\begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} + 2+s \right) \left\{ \begin{array}{l} \alpha_{2,2+s} \\ \beta_{2,2+s} \end{array} \right\} + \\ &k^6 j \left(\begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} + s \right) \left\{ \begin{array}{l} \alpha_{2,s} \\ \beta_{2,s} \end{array} \right\} / f \left(\begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} + 6+s \right), \quad s = 0, 1, 2, \dots \end{aligned} \quad (6.1.11)$$

in which λ_1, λ_2 respectively take values 1 and 3. It can be shown that in this case the initial terms are given by

$$\begin{aligned} \alpha_{2,2} &= -k^2 \frac{g(3)}{f(5)} \alpha_{2,0} = \frac{1}{2} k^2 \alpha_{2,0}, \\ \alpha_{2,4} &= -\frac{1}{f(7)} [k^2 g(5) \alpha_{2,2} + k^4 h(3) \alpha_{2,0}], \end{aligned} \quad (6.1.12)$$

and

$$\beta_{2,2} = -k^2 \left[\frac{\partial g}{\partial \lambda} / \frac{\partial f}{\partial \lambda} \right]_{\lambda=1} \beta_{2,0} = -\frac{1}{3} k^2 \beta_{2,0}, \quad (6.1.13)$$

$$\beta_{2,4} = -\frac{1}{f(5)} [k^2 g(3) \beta_{2,2} + k^4 h(1) \beta_{2,0}].$$

To investigate the nature of the solution associated with the first root, we note that in case $\lambda = -1$, $f(\lambda+j) = 0$ for $j=2$ and $j=4$.

When $j=2$, the relations (5.41) yield

$$f(1) \alpha_{2,2}(\lambda_3) + k^2 g(-1) \alpha_{2,0}(\lambda_3) = 0 \quad (6.1.14)$$

Although $f(1) = f(\lambda_2) = 0$, it turns out that also $g(-1) = 0$ and thus $\alpha_{2,2}(\lambda_3)$ would not be an impediment in determining recursively the coefficients $\alpha_{2,j}(\lambda_3)$. However, when $j=4$, we have

$$f(3) \alpha_{2,4}(\lambda_3) = - [k^2 g(1) \alpha_{2,2}(\lambda_3) + k^4 h(-1) \alpha_{2,0}(\lambda_3)]. \quad (6.1.15)$$

We recall that $f(3) = f(\lambda_1) = 0$ and, it can be shown that, whereas $g(1) = 0$, $h(-1) \neq 0$ and the relation (6.1.15) cannot be satisfied for any choice of $\alpha_{2,4}$. Consequently there is no power series solution associated with $\lambda_3 = -1$; the corresponding solution involves logarithms and it follows from relation (5.47) that

$$u_2^{(3)}(r) = \frac{\gamma_{2,4}}{\alpha_{2,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \alpha_{2,j} r^{-3-j} + \sum_{j=0}^{\infty} \left(\frac{\partial \gamma_{2,j}}{\partial \lambda} \right)_{\lambda=-1} r^{-1-j}, \quad (6.1.16)$$

where

$$\gamma_{2,j}(\lambda) \equiv (\lambda+1) \alpha_{2,j}, \quad (6.1.17)$$

with

$$\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 0. \quad (6.1.18)$$

The calculations show that the critical coefficient $\gamma_{2,4}$ may be expressed by

$$\gamma_{2,4} = -k^4 [h(\lambda) / \frac{\partial f(\lambda+4)}{\partial \lambda}]_{\lambda=-1} \alpha_{2,0} = -\frac{1}{2} k^4 \alpha_{2,0} \quad (6.1.19)$$

The remaining γ coefficients are related to the α 's by

$$\gamma_{2,4+s} = -\frac{1}{2} k^4 \alpha_{2,s}, \quad s = 0, 1, 2, \dots \quad (6.1.20)$$

To obtain the derivatives of $\gamma_{2,j}(\lambda)$ involved in the solution (6.1.16), we use the property that the γ 's themselves, by definition, satisfy relations of the form (5.34). It can be shown that

$$\begin{aligned} \gamma'_{2,0} &= \alpha_{2,0}, \quad * \\ \gamma'_{2,2} &= -\frac{g'(-1)}{f'(1)} k^2 \alpha_{2,0} = -2k^2 \alpha_{2,0}, \end{aligned} \quad (6.1.21)$$

$$\gamma'_{2,4} = -[f''(3)\gamma_{2,4} + 2k^2 g(1)\gamma'_{2,2} + 2k^4 h'(-1)\gamma'_{2,0}] = \frac{23}{24} k^4 \alpha_{2,0}.$$

All other γ' are obtained recursively by

$$\gamma'_{2,1+2s} = 0, \quad s = 0, 1, 2, \dots$$

$$\begin{aligned} \gamma'_{2,6+2s} &= -\{f'(5+2s)\gamma_{2,6+2s} + k^2[g'(3+2s)\gamma_{2,4+2s} + g(3+2s)\gamma'_{2,4+2s}] \\ &\quad + k^4[h'(1+2s)\gamma_{2,2+2s} + h(1+2s)\gamma'_{2,2+2s}] \\ &\quad + k^6[j'(-1+2s)\gamma_{2,2s} + j(-1+2s)\gamma'_{2,2s}]\} / f(5+2s). \end{aligned} \quad (6.1.22)$$

* In this and subsequent chapters, in order to simplify the writing, we denote by " ' " derivatives with respect to λ . If no λ is mentioned, evaluation at the appropriate λ is understood, i.e. we evaluate $\alpha_{2,j}$ at $\lambda_1=3$, $\gamma_{2,j}$ at $\lambda_2=1$ and $\gamma'_{2,j}$ at $\lambda_3=-1$.

We conclude that the radial displacement $u(r, \theta)$ is a function of the form

$$u(r, \theta) = A_0 u_0(r) + [A_2 u_2^{(1)}(r) + B_2 u_2^{(2)}(r) + C_2 u_2^{(3)}(r)] \cos(2\theta). \quad (6.1.23)$$

The constant A_0 has the value given by (6.1.6) and the constant C_2 is given by (6.1.4). The constants A_2 and B_2 are still unknown and are to be determined from the boundary conditions imposed at $r=a$. Relations (6.1.7) yield

$$\begin{aligned} & < A_2 \sum_{j=0}^{\infty} \{ (12-12j-8j^2-8j^3) + (18+26j+10j^2+j^3) \frac{k^2}{a^2} + 4(3+j) (1 - \frac{k^2}{a^2})^{-1} \\ & \quad - 2(3+j) \ln \frac{a_0^2}{a^2} \} \alpha_{2,j} r^{-4-j} \\ & + B_2 \sum_{j=0}^{\infty} \{ (12+8j-2j^2-j^3) - (2+2j-4j^2-j^3) \frac{k^2}{a^2} + 4(1+j) (1 - \frac{k^2}{a^2})^{-1} \\ & \quad - 2(1+j) \ln \frac{a_0^2}{a^2} \} \beta_{2,j} r^{-2-j} \\ & + C_2 \{ \gamma_{2,4} \ln \frac{a_0}{a} \sum_{j=0}^{\infty} [(12-12j-8j^2-j^3) + (18+26j+10j^2+j^3) \frac{k^2}{a^2} + 4(3+j) (1 - \frac{k^2}{a^2})^{-1} \\ & \quad - 2(3+j) \ln \frac{a_0^2}{a^2}] \alpha_{2,j} r^{-4-j} \\ & - \gamma_{2,4} \sum_{j=0}^{\infty} [(12+16j+3j^2) - (26+20j+3j^3) \frac{k^2}{a^2} - 4(1 - \frac{k^2}{a^2})^{-1} \\ & \quad + 2 \ln \frac{a_0^2}{a^2}] \alpha_{2,j} r^{-4-j} \\ & + \sum_{j=0}^{\infty} [(-4+4j+4j^2-j^3) + (10-6j-2j^2+j^3) - 4(1-j) (1 - \frac{k^2}{a^2})^{-1} \\ & \quad + 2(1-j) \ln \frac{a_0^2}{a^2}] \gamma_{2,j}' r^{-j} \} \Big|_{r=a} = 0, \end{aligned}$$

(6.1.24)

and

$$\begin{aligned}
& < A_2 \sum_{j=0}^{\infty} \left\{ (24+12j+2j^2) - (18+11j+2j^2) \frac{k^2}{a^2} - (6+j) \ln \frac{a_0^2}{a^2} \right\} \alpha_{2,j} r^{-4-j} \\
& + B_2 \sum_{j=0}^{\infty} \left\{ (8+4j+2j^2) - (4+3j+2j^2) \frac{k^2}{a^2} - (4+j) \ln \frac{a_0^2}{a^2} \right\} \beta_{2,j} r^{-2-j} \\
& + C_2 \left\{ \gamma_{2,4} \ln \frac{a_0}{a} \sum_{j=0}^{\infty} \left[(24+12j+2j^2) - (18+11j+2j^2) \frac{k^2}{a^2} - (6+j) \ln \frac{a_0^2}{a^2} \right] \alpha_{2,j} r^{-4-j} \right. \\
& \quad - \gamma_{2,4} \sum_{j=0}^{\infty} \left[(-12-4j) + (11+4j) \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2} \right] \alpha_{2,j} r^{-4-j} \\
& \quad \left. + \sum_{j=0}^{\infty} \left[(8-4j+2j^2) - (6-5j+2j^2) \frac{k^2}{a^2} - (2+j) \ln \frac{a_0^2}{a^2} \right] \gamma'_{2,j} r^{-j} \right\}_{r=a} = 0.
\end{aligned}
\tag{6.1.25}$$

We shall evaluate the nondimensional constants A_0^* , A_2^* and B_2^* defined as

$$\frac{A_0}{C_2} \equiv a^2 A_0^*, \quad \frac{A_2}{C_2} \equiv a^4 A_2^*, \quad \frac{B_2}{C_2} \equiv a^2 B_2^*, \tag{6.1.26}$$

rather than A_0 , A_2 and B_2 .

We note that all odd coefficients α , β and γ' vanish and so the remaining terms are of even order. Furthermore, for simplification, we shall write α_{2i} instead of $\alpha_{2,2i}$, β_{2i} instead of $\beta_{2,2i}$, γ'_{2i} for $\gamma'_{2,2i}$ and set

$$\begin{aligned}
\alpha_{2i} &\equiv k^{2i} \hat{\alpha}_{2i}, & \beta_{2i} &\equiv k^{2i} \hat{\beta}_{2i}, & \gamma'_{2i} &\equiv k^{2i} \hat{\gamma}'_{2i}, \\
\sigma^2 &\equiv k^2/a^2, & \alpha_{2,0} &\equiv 1, & \beta_{2,0} &\equiv 1.
\end{aligned}
\tag{6.1.27}$$

Thus, equations (6.1.24) and (6.1.25), respectively become

$$\begin{aligned}
& A_2^* \sum_{i=0}^{\infty} [(12-24i-32i^2-8i^3) + (18+52i+40i^2+8i^3)q^2 + 4(3+2i)(1-q^2)^{-1} \\
& \quad - 2(3+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + B_2^* \sum_{i=0}^{\infty} [(12+16i-8i^2-i^3) - (2+4i-16i^2-8i^3)q^2 + 4(1+2i)(1-q^2)^{-1} \\
& \quad - 2(1+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\beta}_{2i} \\
& - \frac{1}{2} q^4 \ln \frac{a_0}{a} \sum_{i=0}^{\infty} [(12-24i-32i^2-8i^3) + (18+52i+40i^2+8i^3)q^2 + 4(3+2i)(1-q^2)^{-1} \\
& \quad - 2(3+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + \frac{1}{2} q^4 \sum_{i=0}^{\infty} [(12+32i+12i^2) - (26+40i+12i^2)q^2 - 4(1-q^2)^{-1} \\
& \quad + 2 \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + \sum_{i=0}^{\infty} [(-4+8i+16i^2-8i^3) + (10-12i-8i^2+8i^3)q^2 - 4(1-2i)(1-q^2)^{-1} \\
& \quad + 2(1-2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\gamma}_{2i} = 0, \tag{6.1.28}
\end{aligned}$$

$$\begin{aligned}
& A_2^* \sum_{i=0}^{\infty} [(24+24i+8i^2) - (18+22i+8i^2)q^2 - (6+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + B_2^* \sum_{i=0}^{\infty} [(8+8i+8i^2) - (4+6i+8i^2)q^2 - (4+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\beta}_{2i} \\
& - \frac{1}{2} q^4 \ln \frac{a_0}{a} \sum_{i=0}^{\infty} [(24+24i+8i^2) - (18+22i+8i^2)q^2 - (6+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + \frac{1}{2} q^4 \sum_{i=0}^{\infty} [(-12-8i) + (11+8i)q^2 + \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\alpha}_{2i} \\
& + \sum_{i=0}^{\infty} [(8-8i+8i^2) - (6-10i+8i^2)q^2 - (2+2i) \ln \frac{a_0^2}{a^2}] q^{2i} \hat{\gamma}_{2i} = 0 \tag{6.1.29}
\end{aligned}$$

The solution of the posed boundary value problem, given by the displacement field \bar{u} and the stress field $\bar{\sigma}^{ik}$ may be expressed in the form

$$\begin{aligned} \varepsilon u(r, \theta) = & \frac{P}{8c} \left\{ \frac{a^2}{r} A_0^* + [a^4 A_2^* \sum_{j=0}^{\infty} \alpha_{2,j} r^{-3-j} + a^2 B_2^* \sum_{j=0}^{\infty} \beta_{2,j} r^{-1-j} \right. \\ & \left. - \frac{k^4}{2} \ln \frac{a_0}{a} \sum_{j=0}^{\infty} \alpha_{2,j} r^{-3-j} + \sum_{j=0}^{\infty} \gamma'_{2,j} r^{1-j} \right\} \cos(2\theta), \quad (6.1.30) \end{aligned}$$

$$\begin{aligned} \varepsilon v(r, \theta) = & \frac{P}{16c} \left\{ a^4 A_2^* \sum_{j=0}^{\infty} (j+2) \alpha_{2,j} r^{-3-j} + a^2 B_2^* \sum_{j=0}^{\infty} j \beta_{2,j} r^{-1-j} \right. \\ & \left. - \frac{k^4}{2} \left[\ln \frac{a_0}{r} \sum_{j=0}^{\infty} (j+2) \alpha_{2,j} r^{-3-j} + \sum_{j=0}^{\infty} \alpha_{2,j} r^{-3-j} \right] \right. \\ & \left. + \sum_{j=0}^{\infty} (j-2) \gamma'_{2,j} r^{1-j} \right\} \sin(2\theta), \quad (6.1.31) \end{aligned}$$

$$\begin{aligned} \varepsilon \hat{r}'_{11}(r, \theta) = & \frac{P}{2} \left[1 - a^2 A_0^* \left(1 + \frac{1}{2} \ln \frac{r^2}{r^2 - k^2} \right) / r^2 \right] \\ & - \frac{P}{16} \left\{ a^4 A_2^* \sum_{j=0}^{\infty} \left[(12 - 12j - 8j^2 - j^3) + (24 + 28j + 10j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ & \left. \left. + 4(3+j) \left(\frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \alpha_{2,j} r^{-4-j} \right. \\ & \left. + a^2 B_2^* \sum_{j=0}^{\infty} \left[(12 + 8j - 2j^2 - j^3) + (4j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ & \left. \left. + 4(1+j) \left(\frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \beta_{2,j} r^{-2-j} \right. \\ & \left. - \frac{k^4}{2} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \left[(12 - 12j - 8j^2 - j^3) + (24 + 28j + 10j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ & \left. \left. + 4(3+j) \left(\frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \alpha_{2,j} r^{-4-j} \right. \\ & \left. + \frac{k^4}{2} \sum_{j=0}^{\infty} \left[(12 + 16j + 3j^2) - (28 + 20j + 3j^2) \frac{k^2}{r^2} \right. \right. \\ & \left. \left. + 4 \left(\frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \alpha_{2,j} r^{-4-j} \right. \\ & \left. + \sum_{j=0}^{\infty} \left[(-4 + 4j + 4j^2 - j^3) + (8 - 4j - 2j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ & \left. \left. - 4(1-j) \left(\frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \gamma'_{2,j} r^{-j} \right\} \cos(2\theta), \quad (6.1.32) \end{aligned}$$

$$\begin{aligned}
\hat{\varepsilon}\tau'_{22}(r, \theta) = & \frac{P}{2} \left[1 + a^2 A_0^* \left(1 - \frac{k^2}{r^2} + \frac{1}{2} \ln \frac{r^2}{r^2 - k^2} \right) / r^2 \right] \\
& - \frac{P}{16} \{ a^4 A_2^* \sum_{j=0}^{\infty} [-(36+28j+8j^2+j^3) + (48+36j+10j^2+j^3) \frac{k^2}{r^2} \\
& \quad + 4(3+j) \left(\frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right)] \alpha_{2,j} r^{-4-j} \right. \\
& + a^2 B_2^* \sum_{j=0}^{\infty} [-(4+8j+2j^2+j^3) + (8+8j+4j^2+j^3) \frac{k^2}{r^2} \\
& \quad - 4(1+j) \left(\frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right)] \beta_{2,j} r^{-2-j} \\
& - \frac{k^4}{2} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} [-(36+28j+8j^2+j^3) + (48+36j+10j^2+j^3) \frac{k^2}{r^2} \\
& \quad + 4(3+j) \left(\frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right)] \alpha_{2,j} r^{-4-j} \\
& + \frac{k^4}{2} \sum_{j=0}^{\infty} [(28+16j+3j^2) - (36+20j+3j^2) \frac{k^2}{r^2} \\
& \quad - 4 \left(\frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right)] \alpha_{2,j} r^{-4-j} \\
& + \sum_{j=0}^{\infty} [(12 - 12j + 4j^2 - j^3) + (4j - 2j^2 + j^3) \frac{k^2}{r^2} \\
& \quad - 4(1-j) \left(\frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right)] \gamma_{2,j} r^{-j} \} \cos(2\theta) ,
\end{aligned}$$

(6.1.33)

$$\begin{aligned}
\hat{\varepsilon}\tau'_{12}(r, \theta) = & - \frac{P}{16} \left(2 - \frac{k^2}{r^2} + \ln \frac{r^2}{r^2 - k^2} \right) [a^4 A_2^* \sum_{j=0}^{\infty} (12+6j+j^2) \alpha_{2,j} r^{-4-j} \\
& + a^2 B_2^* \sum_{j=0}^{\infty} (4+2j+j^2) \beta_{2,j} r^{-2-j} - \frac{k^4}{2} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} (12+6j+j^2) \alpha_{2,j} r^{-4-j} \\
& - \frac{k^4}{2} \sum_{j=0}^{\infty} (6+2j) \alpha_{2,j} r^{-4-j} + \sum_{j=0}^{\infty} (4-2j+j^2) \gamma_{2,j} r^{-j}] \sin(2\theta) .
\end{aligned}$$

(6.1.34)

In the limiting case, when the finite deformation vanishes (that is $k/a = 0$), Classical Elasticity results [16] are recovered;

$$v_r = \frac{P}{8c} \left[\frac{a^2}{r} + \left(2\frac{a^2}{r} + r - \frac{a^4}{r^3} \right) \cos(2\theta) \right],$$

$$v_\theta = -\frac{P}{8c} \left(r + \frac{a^4}{r^3} \right) \sin(2\theta),$$

$$\tau_{rr} = \frac{P}{2} \left[\left(1 - \frac{a^2}{r^2} \right) - \left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4} \right) \cos(2\theta) \right], \quad (6.1.35)$$

$$\tau_{\theta\theta} = \frac{P}{2} \left[\left(1 + \frac{a^2}{r^2} \right) - \left(1 + 3\frac{a^4}{r^4} \right) \cos(2\theta) \right],$$

$$\tau_{r\theta} = -\frac{P}{2} \left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4} \right) \sin(2\theta),$$

taking into account that $6(c_1 + c_2)$ of the Mooney-Rivlin material is associated with Young's modulus, E , for infinitesimal deformations.

Deformation field and stress distribution.

For similar problems in Linear Elasticity, the stress distribution caused by some load applied to a slab weakened by a cut-out, differ considerably from that in an unweakened body. The following question arises: is the stress concentration effect at the hole diminished or magnified by the fact that the slab has previously been finitely deformed? This leads us to investigate the effect of the hole on the deformation field and stress distribution.

We recall that the constants A_0^* , A_2^* and B_2^* , determined by the boundary conditions at $r=a$, are to be found from relation (6.1.6) and the coupled equations (6.1.28) - (6.1.29). The value of these constants

and correspondingly, the deformation field and the state of stress have been computed for a selection of various finite deformations in the form of an input a/a_0 . The numerical results and curves have been obtained on MTS Integrated Graphics System on an IBM370 with a PL1 programme. The specific values of a/a_0 will be taken within $1 \leq a/a_0 \leq 2$ which is the range of practical interest. We need an estimation of the sums of the form

$$\sum_{i=0}^{\infty} i^n \left(\frac{k}{r^2}\right)^i \hat{\alpha}_{2i}, \quad \sum_{i=0}^{\infty} i^n \left(\frac{k}{r^2}\right)^i \hat{\beta}_{2i}, \quad (6.1.36)$$

$$\sum_{i=0}^{\infty} i^n \left(\frac{k}{r^2}\right)^i \hat{\gamma}_{2i}, \quad n = 0, 1, 2, 3.$$

It emerges that the convergence of the series is rapid on the whole. As expected, the larger the finite deformation, the slower the convergence; for a fixed finite deformation, the convergence is better as r increases. The corresponding numerical results are presented in the Appendix I. The truncated sums along with the number of terms to be taken into account for 7-digit accuracy are tabulated (Table I, Appendix I), as function of the magnitude of the finite deformation and the distance. An error estimation is also given.

Since the analytical solution is valid as long as the additional displacement field gives rise to small strains, there is a limitation on the admissible axial loading applied at infinity. We choose $P/8c = 0.02$, (from Appendix II), to remain within the theory of small deformation superposed on finite deformation, for all $1 \leq a/a_0 \leq 2$.

Table 6.1.1 shows the computed values of the nondimensional constants A_0^* , A_2^* and B_2^* corresponding to a number of initial inflations.

Table 6.1.1

a/a_0	A_0^*	A_2^*	B_2^*
1.00	1.00000	-1.00000	2.00000
1.10	0.99575	-0.68102	2.07724
1.20	0.98544	-0.48028	2.12036
1.30	0.97171	-0.35378	2.14162
1.40	0.95522	-0.27541	2.14878
1.50	0.93999	-0.22896	2.14686
1.75	0.89973	-0.19288	2.12110
2.00	0.86276	-0.21295	2.08362

Once these constants are found, all incremental displacements and stresses may be computed after substitution in the formulae (6.1.30) - (6.1.34).

Regarding the displacement field, during the finite inflation, the material points move radially, and naturally, these displacements, after the equilibrium has been reached, are related to the expanding pressure that maintains the equilibrium. Once the axial tension is applied, the points undergo a further incremental displacement. We shall write the relations (6.1.30) and (6.1.31) in a more convenient form:

$$\begin{aligned}
\varepsilon_u(ma, \theta) = & \frac{P}{8c} a \{ A_0^{*m-1} + [A_2^{*m-3} \sum_{i=0}^{\infty} q^{2i} m^{-2i} \hat{\alpha}_{2i} + B_2^{*m-1} \sum_{i=0}^{\infty} q^{2i} m^{-2i} \hat{\beta}_{2i} \\
& - \frac{q^4}{2} \ln \frac{a_0}{am} \sum_{i=0}^{\infty} q^{2i} m^{-2i} \hat{\alpha}_{2i} + m \sum_{i=0}^{\infty} q^{2i} m^{-2i} \hat{\gamma}'_{2i}] \cos(2\theta) \},
\end{aligned}
\tag{6.1.37}$$

$$\begin{aligned}
\varepsilon_v(ma, \theta) = & \frac{P}{16c} a \{ A_2^{*m-3} \sum_{i=0}^{\infty} (2+2i) q^{2i} m^{-2i} \hat{\alpha}_{2i} + B_2^{*m-1} \sum_{i=0}^{\infty} 2i q^{2i} m^{-2i} \hat{\beta}_{2i} \\
& - \frac{q^4}{2} [\ln \frac{a_0}{am} \sum_{i=0}^{\infty} (2+2i) q^{2i} m^{-2i} \hat{\alpha}_{2i} + m^{-3} \sum_{i=0}^{\infty} q^{2i} m^{-2i} \hat{\alpha}_{2i}] \\
& + m \sum_{i=0}^{\infty} (-2-2i) q^{2i} m^{-2i} \hat{\gamma}'_{2i} \} \sin(2\theta),
\end{aligned}
\tag{6.1.38}$$

where we have set $r \equiv ma$. The additional displacements depend both on the expanding pressure and the tensile loading. It is interesting to note that while the expanding pressure is responsible for the direction of the incremental displacement, the intensity of the axial loading affects only the magnitude, and merely plays the role of a scale factor which is chosen here to be 0.2 (8c). This large value enhances the presentation of the data. Some significant results are presented in a few graphs.

It can be seen that the effect of the hole is of a very localized character. Typical deformations of a polar grid are presented in Figs. 6.1.1 - 6.1.4. In this instance, the finite inflation is such that $a/a_0 = \{1.25, 2.00\}$. Figs. 6.1.1 and 6.1.2 cover a zone in the vicinity of the hole, while Figs. 6.1.3 and 6.1.4 refer to a larger area. The comparison is made with the linear elasticity problem, obtained as a limiting case when the finite deformation vanishes keeping unaltered the perturbational forces. The investigation has been repeated for a number of values of the degree of inflation. It emerges that as the ratio k/a increases

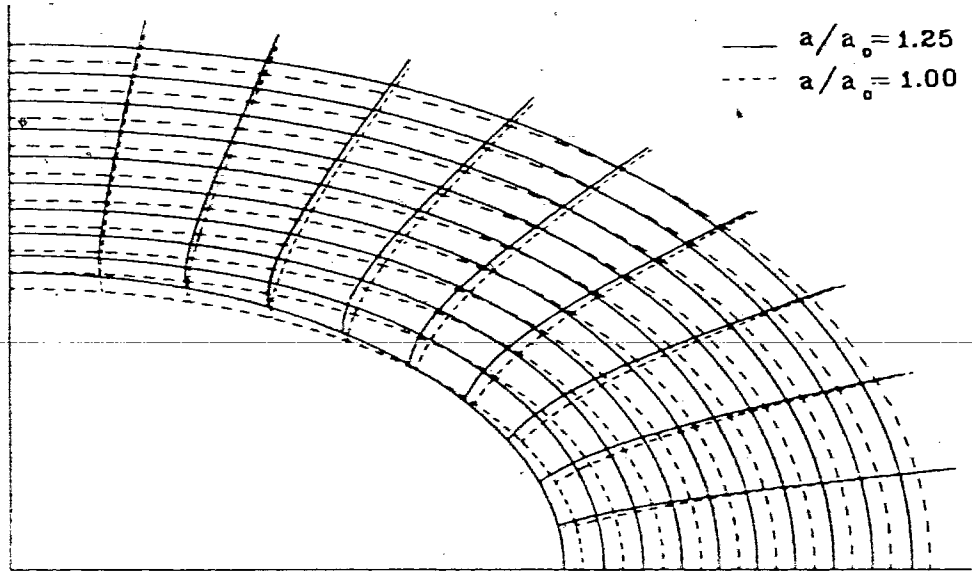


Fig. 6.1.1

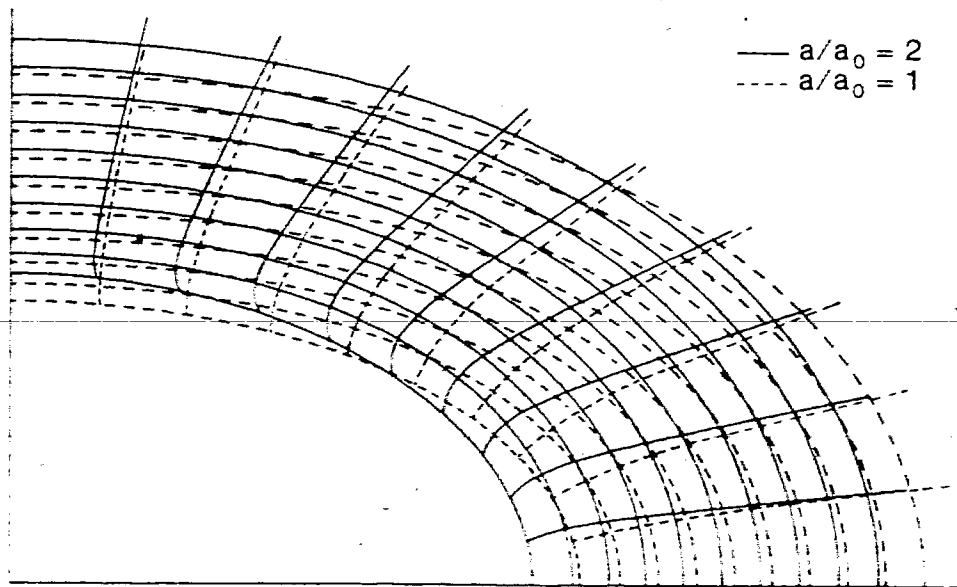


Fig. 6.1.2

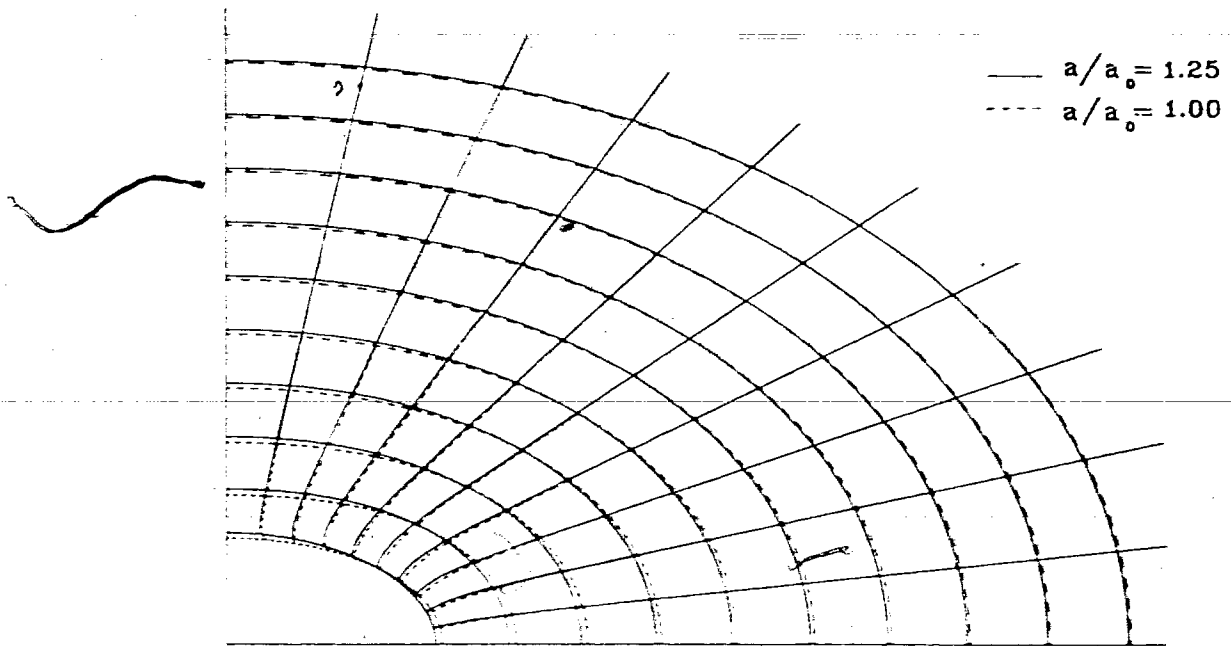


Fig. 8.1.3

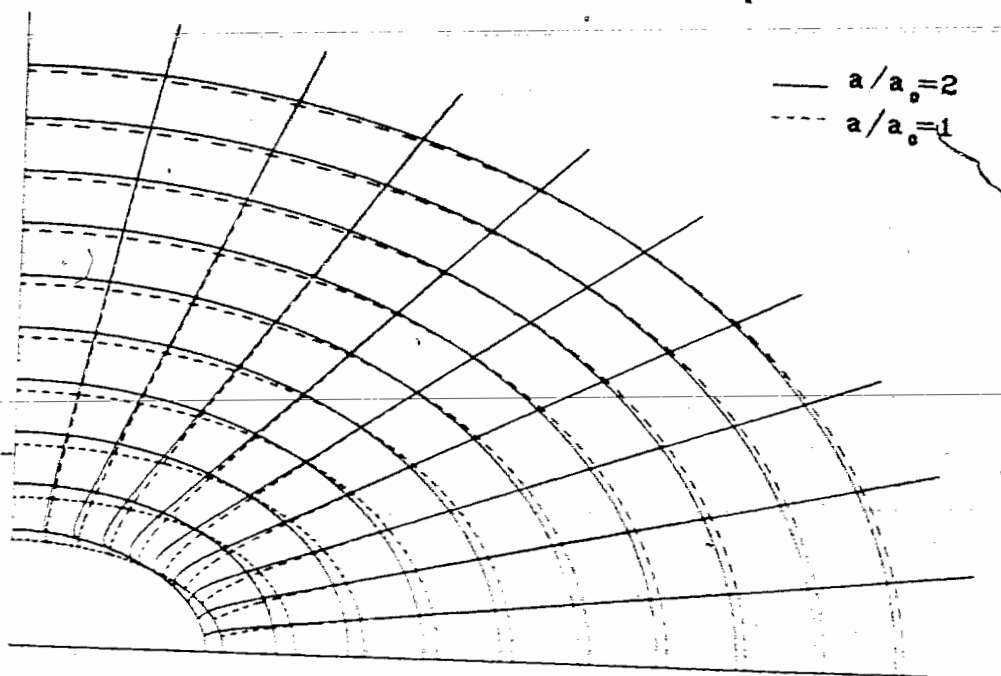


Fig. 6.1.4

(that is, as a/a_0 increases), the distortion of the radial lines is more pronounced and is also more dispersed. The deformation of the cavity and surrounding layers are now examined in more detail.

The deformation of the hole, corresponding to different values of the expanding pressure are given in Fig. 6.1.5. For a better comparison, the curves are drawn under the assumption that the various initial radii of the hole, denoted by a_0 , reach after the first deformation, the same value $r=a$, which is taken as a reference.

Fig. 6.1.6 describes how concentric layers enclosing the hole (the reference circle is taken here to be $r=ma$) are deformed when the body is only stretched at infinity and not subjected to any finite deformation. Figs. 6.1.7 and 6.1.8 presented here correspond to computational results for $a/a_0 = \{1.10, 2.00\}$, respectively. It emerges that the larger the finite deformation, the less is the deviation from its original circular contour.

The deformation spectrum, however, depends on the initial finite deformation only near the hole. Away from the hole (Fig. 6.1.9, 6.1.3 and 6.1.4), the radial lines and the concentric layers approach a definite shape which corresponds to the classical linear elasticity problem.

Regarding the stress field, we recall that a stress vector, $\bar{\sigma} + \delta\bar{\sigma}$, associated with a surface in \mathcal{B}' whose unit normal in its position in \mathcal{B} is \bar{n} , and measured per unit area of \mathcal{B} is given by the general expression

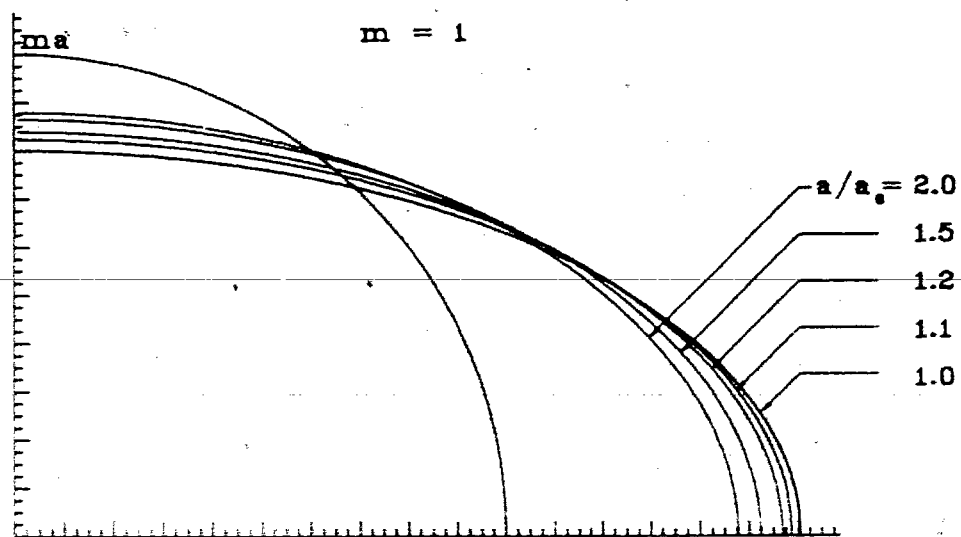


Fig. 8.1.5

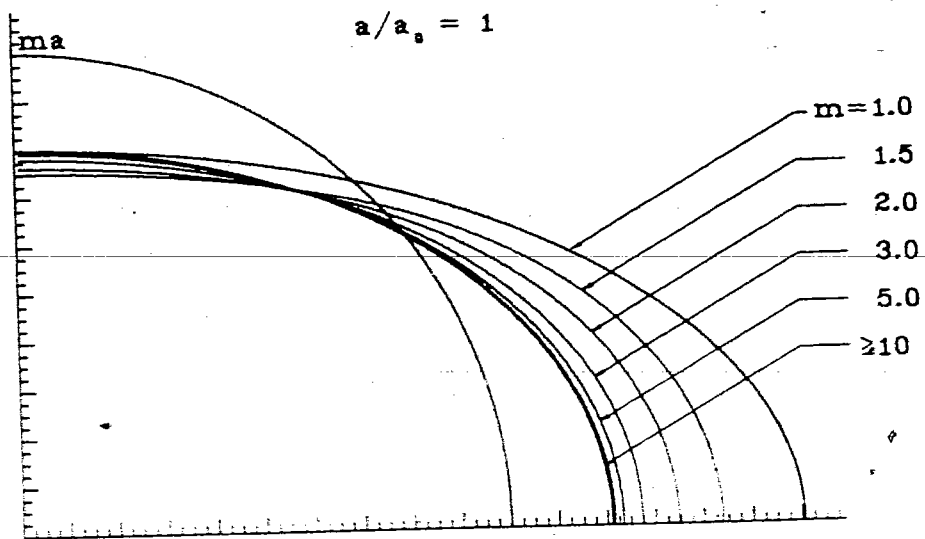


Fig. 8.1.6

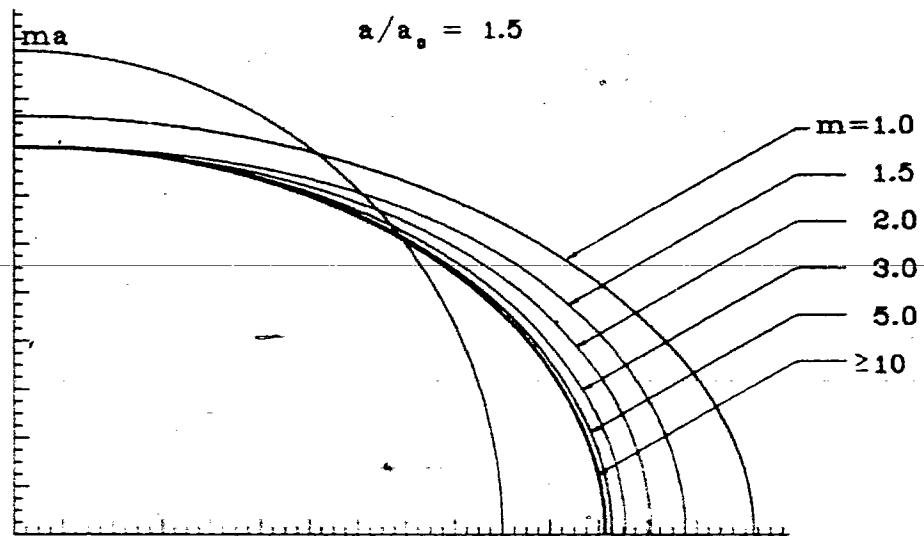


Fig. 8.1.7

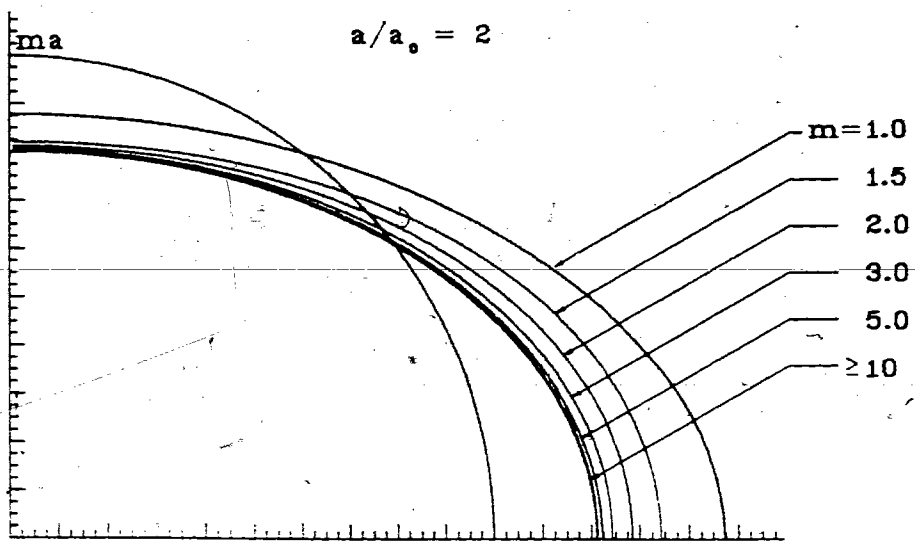


Fig. 6.1.8

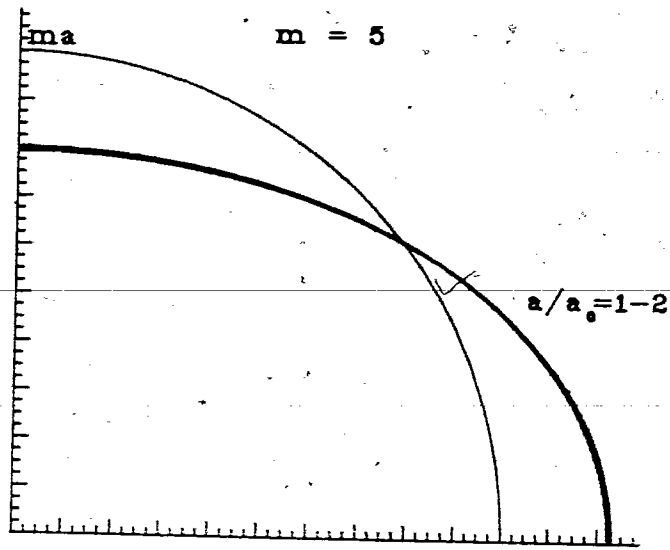


Fig. 6.1.9

$$\bar{t} + \epsilon \bar{t}' = n_i (\tau^{ij} + \epsilon \lambda^{ij}) \bar{G}_j \quad (6.1.39)$$

Consequently, the stress vector associated with a surface which a plane in \mathcal{B} , passing through the axis of the hole is deformed into, is given by

$$\bar{t} + \epsilon \bar{t}' = \epsilon [\tau'^{12} + \tau^{22} \left(\frac{\partial u}{\partial \theta} - v \right)] \bar{G}_1 + [\tau^{22} + \epsilon (\tau'^{22} - \tau^{22} \frac{\partial u}{\partial r})] \bar{G}_2 \quad (6.1.40)$$

We substitute into (6.1.40), relation (5.10) for τ^{22} , relation (6.1.33) for τ'^{22} and $\frac{\partial u}{\partial r}$ derived from (6.1.30). We obtain the following expression for the normal stress:

$$\begin{aligned} \tau_{\theta\theta}(r, \theta) &\equiv r^2 \left[\tau^{22} + \epsilon (\tau'^{22} - \tau^{22} \frac{\partial u}{\partial r}) \right] \\ &= \left(\frac{k^2}{r^2} - 2 + 2 \frac{r^2}{r^2 - k^2} - \ln \frac{r^2}{r^2 - k^2} \right) c \\ &\quad + \frac{P}{2} \left[1 + \frac{A_0^* a^2}{4r^2} \left(2 - 3 \frac{k^2}{r^2} + 2 \frac{r^2}{r^2 - k^2} + \ln \frac{r^2}{r^2 - k^2} \right) \right] \\ &\quad - \frac{P}{16} \left\{ a^4 A_2^* \sum_{j=0}^{\infty} \left[-(24 + 24j + 8j^2 + j^3) + (42 + 34j + 10j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ &\quad \quad \left. \left. - 2(3+j) \ln \frac{r^2}{r^2 - k^2} \right] \alpha_{2,j} r^{-4-j} \right. \\ &\quad \left. + a^2 B_2^* \sum_{j=0}^{\infty} \left[-(4j + 2j^2 + j^3) + (6 + 6j + 4j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ &\quad \quad \left. \left. - 2(1+j) \ln \frac{r^2}{r^2 - k^2} \right] \beta_{2,j} r^{-2-j} \right. \\ &\quad \left. - \frac{k^4}{2} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \left[-(24 + 24j - 8j^2 - j^3) + (42 + 34j + 10j^2 + j^3) \frac{k^2}{r^2} \right. \right. \\ &\quad \quad \left. \left. - 2(3+j) \ln \frac{r^2}{r^2 + k^2} \right] \alpha_{2,j} r^{-4-j} \right. \\ &\quad \left. + \frac{k^4}{2} \sum_{j=0}^{\infty} \left[(24 + 16j + 3j^2) - (34 + 20j + 3j^2) \frac{k^2}{r^2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& + 2 \ln \frac{r^2}{r^2 - k^2} \alpha_{2,j} r^{-4-j} \\
& + \sum_{j=0}^{\infty} [(8-8j+4j^2-j^3) + (2+2j-2j^2+j^3) \frac{k^2}{r^2} \\
& + 2(1-j) \ln \frac{r^2}{r^2 - k^2} \gamma'_{2,j} r^{-j}] \cos(2\theta). \quad (6.1.41)
\end{aligned}$$

For the purpose of the numerical computation, we use the relations (6.1.27)

and also set $r \equiv ma$. We may rewrite (6.1.41) in the form:

$$\begin{aligned}
\tau_{\theta\theta}(ma, \theta) = & c \langle (q^2 m^{-2} - 2 + 2 \frac{m^2}{m^2 - q^2} + \ln \frac{m^2}{m^2 - q^2}) \\
& + \frac{P}{8c} [4 + A_0^* m^{-2} (2 - 3q^2 m^{-2} + 2 \frac{m^2}{m^2 - q^2} + \ln \frac{m^2}{m^2 - q^2})] \\
& - \frac{P}{16c} \{ A_2^* m^{-4} \sum_{i=0}^{\infty} [-(24+48i+32i^2+8i^3) + (42+68i+40i^2+8i^3) q^2 m^{-2} \\
& - (6+4i) \ln \frac{m^2}{m^2 - q^2}] q^{2i} m^{-2i} \hat{\alpha}_{2i} \\
& + B_2^* m^{-2} \sum_{i=0}^{\infty} [-(8i+8i^2+8i^3) + (6+12i+16i^2+8i^3) q^2 m^{-2} \\
& - (2+4i) \ln \frac{m^2}{m^2 - q^2}] q^{2i} m^{-2i} \hat{\beta}_{2i} \\
& - \frac{q^2}{2} m^{-4} \ln \frac{a_0}{am} \sum_{i=0}^{\infty} [-(24+48i+32i^2+8i^3) + (42+68i+40i^2+8i^3) q^2 m^{-2} \\
& - (6+4i) \ln \frac{m^2}{m^2 - q^2}] q^{2i} m^{-2i} \hat{\alpha}_{2i} \\
& + \frac{q^2}{2} m^{-4} \sum_{i=0}^{\infty} [(24+32i+12i^2) - (34+40i+12i^2) q^2 m^{-2} \\
& + 2 \ln \frac{m^2}{m^2 - q^2}] q^{2i} m^{-2i} \hat{\alpha}_{2i} \\
& + \sum_{i=0}^{\infty} [(8-16i+16i^2-8i^3) + (2+4i-8i^2+8i^3) q^2 m^{-2} \\
& + (2-4i) \ln \frac{m^2}{m^2 - q^2}] q^{2i} m^{-2i} \gamma'_{2i} \rangle \cos(2\theta) \quad (6.1.42)
\end{aligned}$$

Letting $\theta = \pm\pi/2$ in (6.1.40), we obtain the stresses in a section normal to the direction of the uniaxial tension yielding a vanishing shear stress $\tau_{r\theta}$ and a maximum value for the normal stress $\tau_{\theta\theta}$. The corresponding results for $\tau_{\theta\theta}$ are plotted in Fig. 6.1.10. Although the effect of the hole is very localized and $\tau_{\theta\theta}$ approaches very rapidly the value at infinity as r increases, a magnifying effect on stress concentration, due to the finite deformation, has been ascertained.

We shall now investigate the stress $\tau_{\theta\theta}$ in more detail. If only the expanding pressure is applied, we have an axially symmetric finite deformation, and $\tau_{\theta\theta}$ is only r dependent. Varying the applied pressure, $\tau_{\theta\theta}$ increases everywhere with a/a_0 . The maximum value is attained at the hole and approaches zero very fast as we go away from the hole (Fig. 6.1.11). We consider now the complete problem, when the uniaxial tension is also applied and a small stretch is superimposed on the existing inflation. It emerges that there is a definite increase in the contribution of the uniaxial tension to the total stress $\tau_{\theta\theta}$ with the intensity of the inflation. At the hole, at $\theta = \pm\pi/2$ where the largest value of $\tau_{\theta\theta}$ occurs, the stress resulting from the uniaxial tension alone is 3.12 times the value at infinity for a previous inflation of $a/a_0 = 1.20$ and increases to more than 4 if $a/a_0 = 1.75$. These should be compared with the classical result 3.00 which is the stress concentration factor corresponding to the case of a slab free of expanding pressure. Some other values are given in Table 6.1.2.

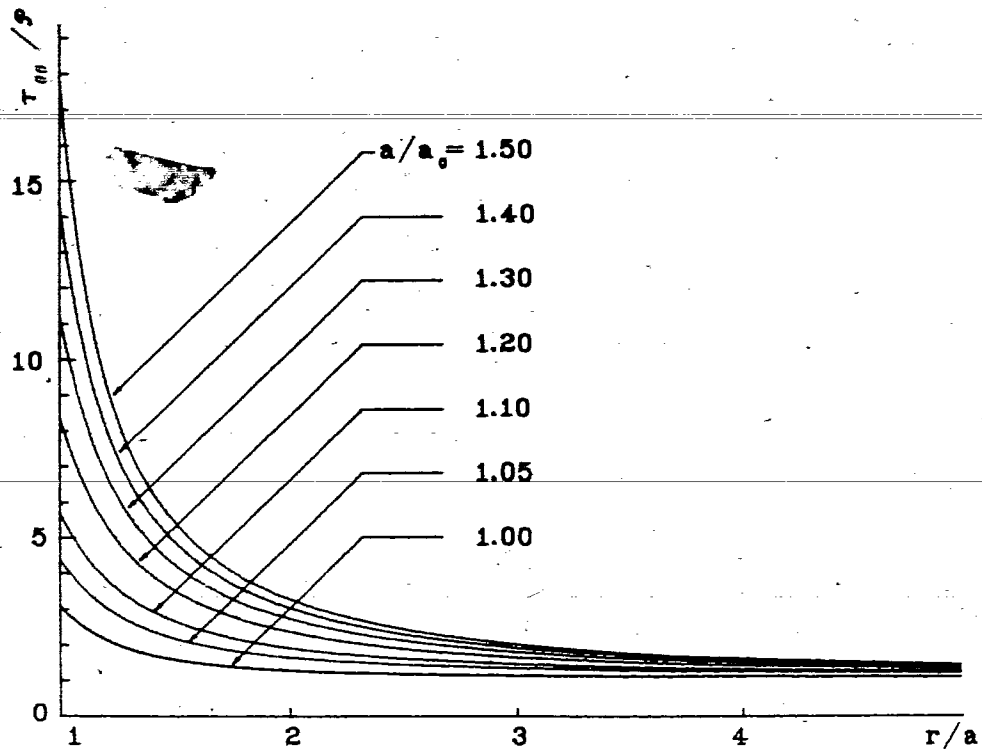


Fig. 6.1.10

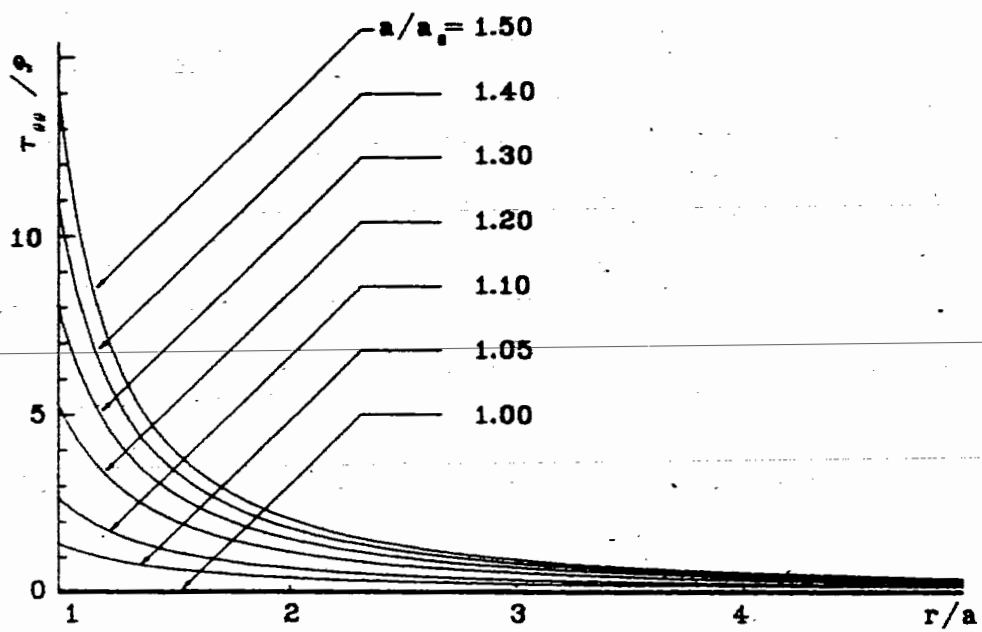


Fig. 6.1.11

Table 6.1.2

$a/a_0 _m$	1.00	1.20	1.40	1.60	1.80	2.00	3.00	4.00	5.00	10.00
1.00	3.00	2.07	1.65	1.42	1.30	1.22	1.08	1.04	1.02	1.01
1.10	3.03	2.05	1.63	1.41	1.29	1.22	1.08	1.04	1.02	1.01
1.20	3.12	2.04	1.61	1.40	1.28	1.21	1.08	1.04	1.02	1.01
1.30	3.24	2.03	1.60	1.39	1.28	1.21	1.08	1.04	1.02	1.01
1.40	3.40	2.02	1.58	1.38	1.27	1.20	1.08	1.04	1.02	1.01
1.50	3.58	2.02	1.57	1.37	1.26	1.20	1.07	1.04	1.02	1.01
1.75	4.12	2.00	1.53	1.35	1.25	1.19	1.07	1.04	1.02	1.01
2.00	4.77	1.97	1.51	1.33	1.23	1.18	1.07	1.04	1.02	1.01

The variation of $\tau_{\theta\theta}$ around the hole is shown in the diagram 6.1.12.

Compared with the classical solution, the area with compressive stress gradually diminishes as the finite deformation gets larger.

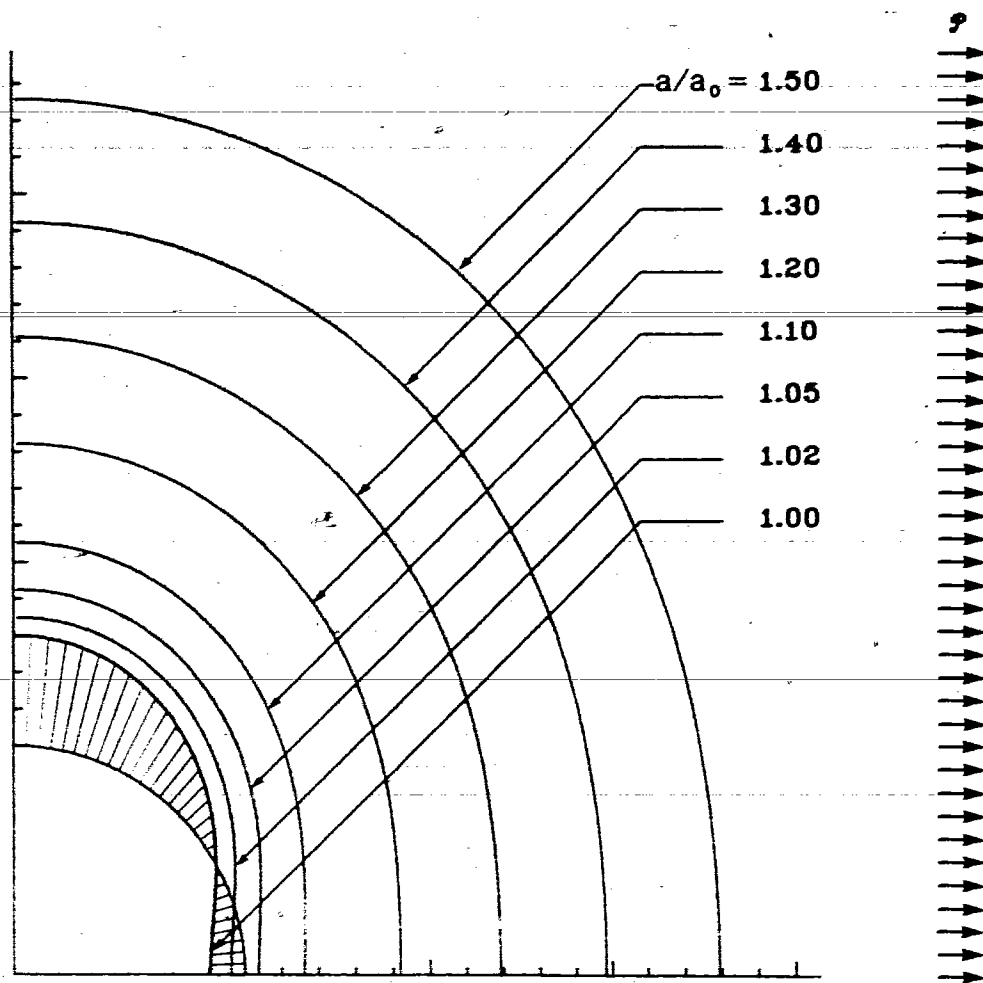


Fig. 6.1.12

6.2. Perturbational Uniaxial Tension Applied to a Finitely Deformed Slab with a Bonded Rigid Inclusion

In this section we consider that a rigid cylindrical inclusion is embedded into the body. We assume that the inclusion is joined to the surrounding slab along its edge and is not displaced during the stretching of the slab. Thus, while the boundary conditions at infinity remain as they have been stated in (6.1.1), we require that there are no displacements at the boundary $r=a$

$$u(a, \theta) = 0, \quad v(a, \theta) = 0. \quad (6.2.1)$$

Relations (6.2.1) yield

$$A_0^* = 0 \quad (6.2.2)$$

and the nonzero constants A_2^* and B_2^* must satisfy the coupled equations

$$A_2^* \sum_{i=0}^{\infty} q^{2i} \hat{\alpha}_{2i} + B_2^* \sum_{i=0}^{\infty} q^{2i} \hat{\beta}_{2i} - \frac{q^4}{2} \ln \frac{a_0}{a} \sum_{i=0}^{\infty} q^{2i} \hat{\alpha}_{2i} + \sum_{i=0}^{\infty} q^{2i} \hat{\gamma}_{2i} = 0 \quad (6.2.3)$$

$$A_2^* \sum_{i=0}^{\infty} (2+2i) q^{2i} \hat{\alpha}_{2i} + B_2^* \sum_{i=0}^{\infty} 2i q^{2i} \hat{\beta}_{2i} - \frac{q^4}{2} \ln \frac{a_0}{a} \sum_{i=0}^{\infty} (2+2i) q^{2i} \hat{\alpha}_{2i} - \frac{q^4}{2} \sum_{i=0}^{\infty} q^{2i} \hat{\alpha}_{2i} + \sum_{i=0}^{\infty} (2i-2) q^{2i} \hat{\gamma}_{2i} = 0 \quad (6.2.4)$$

where the notations given by (6.1.26) and (6.1.27) have also been employed.

The state of deformation and the stress field are formally described by relations (6.1.30) - (6.1.34). However, here, $A_0^* = 0$ and $\{A_2^*, B_2^*\}$, determined from different boundary conditions, are specific to each case.

It can be shown that, if the finite deformation is removed, the results reduce in this case to

$$\begin{aligned}
 v_r &= \frac{P}{8c} \left(r - 2\frac{a^2}{r} + \frac{a^4}{r^3} \right) \cos(2\theta) , \\
 v_\theta &= -\frac{P}{8c} \left(r - \frac{a^4}{r^3} \right) \sin(2\theta) , \\
 \tau_{rr} &= \frac{P}{2} \left[1 + \left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4} \right) \cos(2\theta) \right] , \\
 \tau_{\theta\theta} &= \frac{P}{2} \left[1 - \left(1 - 3\frac{a^4}{r^4} \right) \cos(2\theta) \right] , \\
 \tau_{r\theta} &= -\frac{P}{2} \left(1 - 2\frac{a^2}{r^2} + 3\frac{a^4}{r^4} \right) \sin(2\theta) ,
 \end{aligned} \tag{6.2.5}$$

which coincide with the classical solution [16].

Deformation field and stress distribution.

In order to determine the deformations and stresses throughout the body, we have to obtain the specific values for the constants A_2^* and B_2^* which are fixed by the boundary conditions. Following the procedure shown in the previous sections, numerical calculations have been carried out for various values of the ratio a/a_0 . Some results are given in the Table 6.2.1. These constants are then substituted into relations (6.1.37), (6.1.38) and (6.1.42). We vary the ratio k/a to compare the deformations and stresses for different initial finite inflations. Taking a sequence of values for m , we investigate the deformation and stresses in layers at various distances from the inclusion. We also vary θ from the line of the axial tension, ($\theta=0$), to the line

Table 6.2.1

a/a_0	A_0^*	A_2^*	B_2^*
1.00	0.00000	1.00000	-2.00000
1.10	0.00000	0.71568	-1.56997
1.20	0.00000	0.50318	-1.24905
1.30	0.00000	0.33687	-1.00406
1.40	0.00000	0.20193	-0.81340
1.50	0.00000	0.08932	-0.66255
1.75	0.00000	-0.12819	-0.40074
2.00	0.00000	-0.28871	-0.23898

perpendicular to it. The computational results are presented graphically in Figs. 6.2.1 - 6.2.5.

Near the bonded inclusion, the layers closely follow the shape of the inclusion whereas the radial distortion is very pronounced. The radial lines are bent in the reverse way to the inclusionless case, as the elastic material, bonded to the inclusion, is prevented from shifting (Figs. 6.2.1 and 6.2.2). The presence of the inclusion considerably changes the deformation and stress field, yet the effect is concentrated in the vicinity of the inclusion. As the ratio k/a increases the distortion of the radial lines is more pronounced and more widespread (Fig. 6.2.2).

Away from the inclusion, as has been observed in the case of a slab with a hole, the deformed layers and radial lines take the shape that cor-

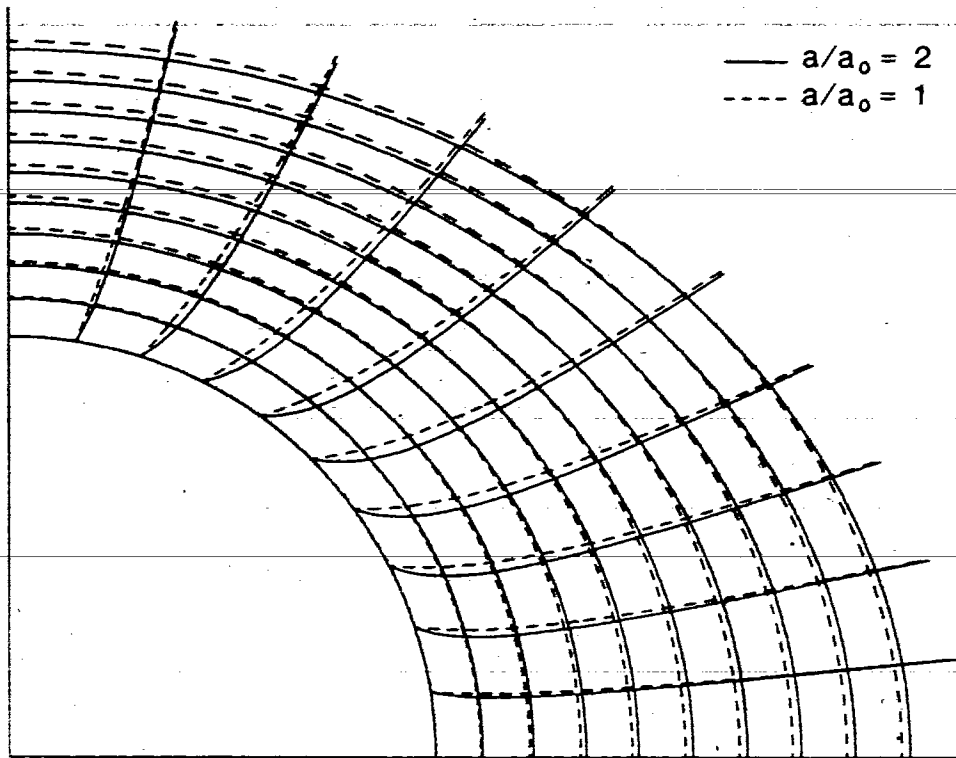


Fig. 6.2.1

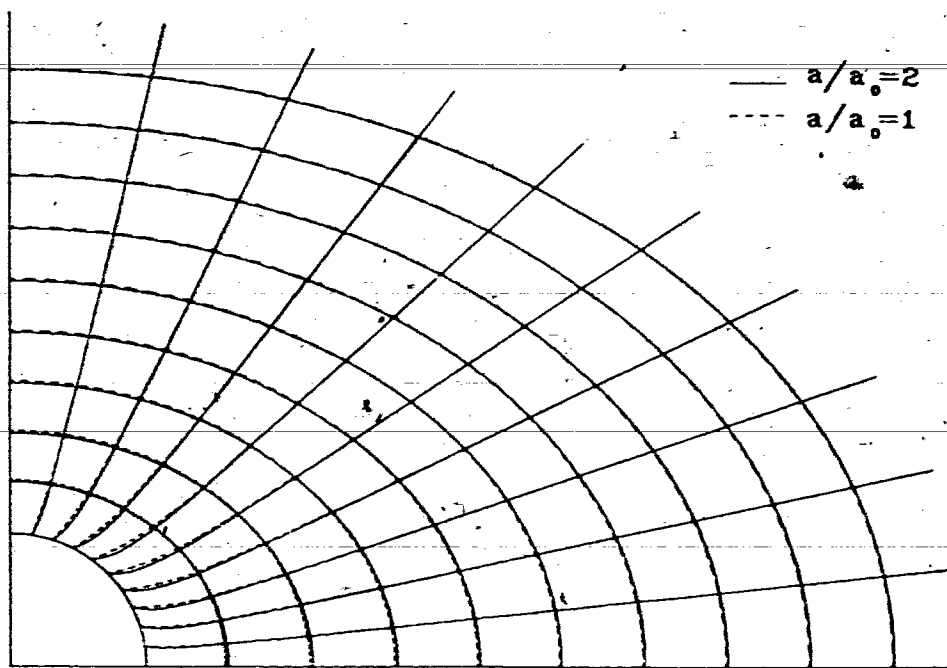


Fig. 8.2.2

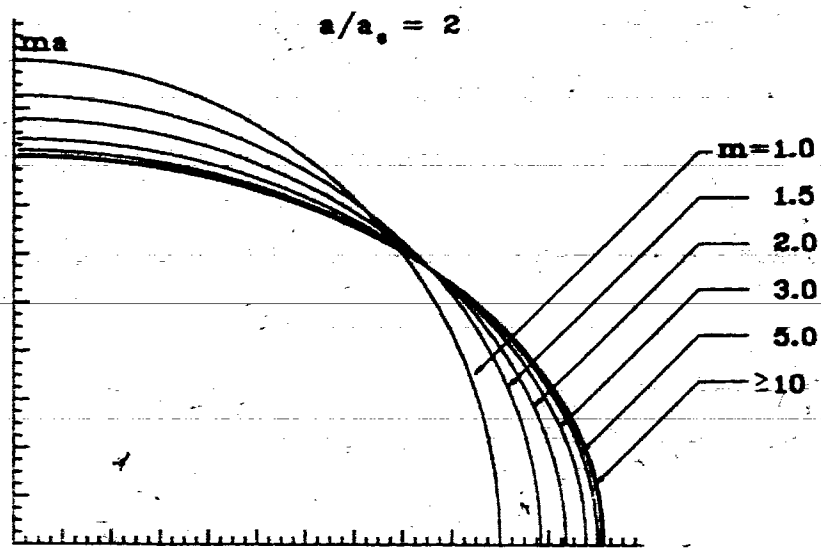


Fig. 6.2.3.

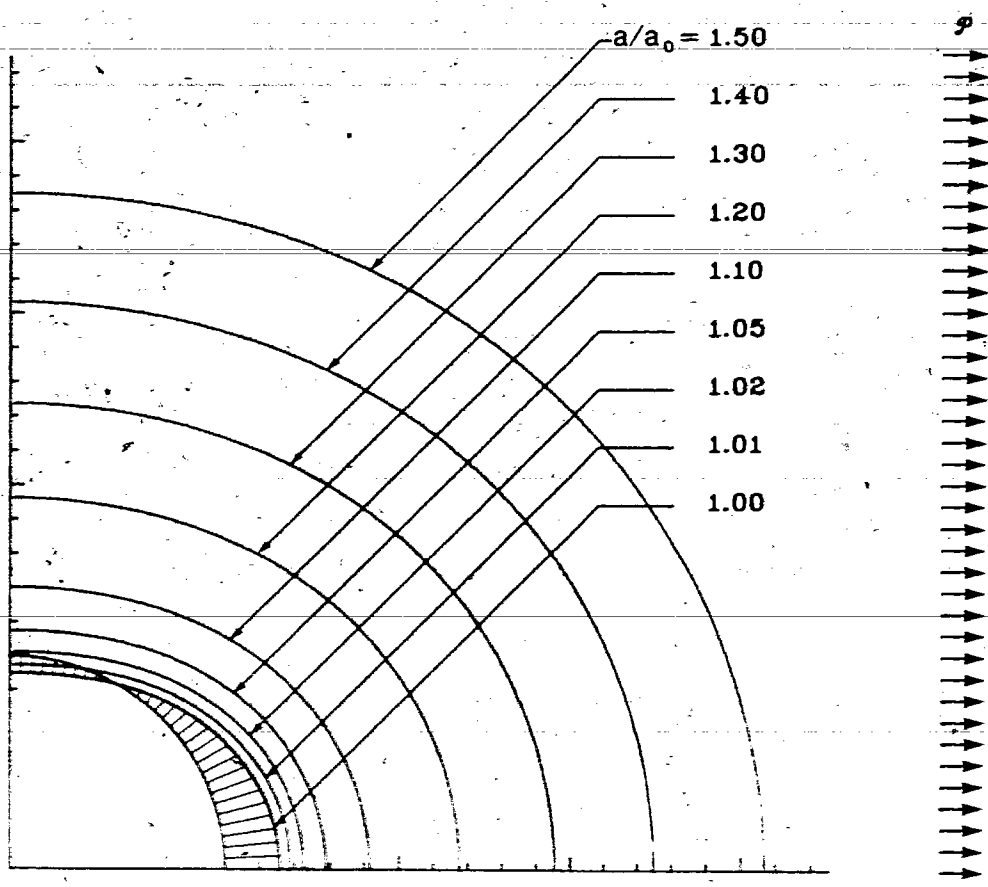


Fig. 8.2.4

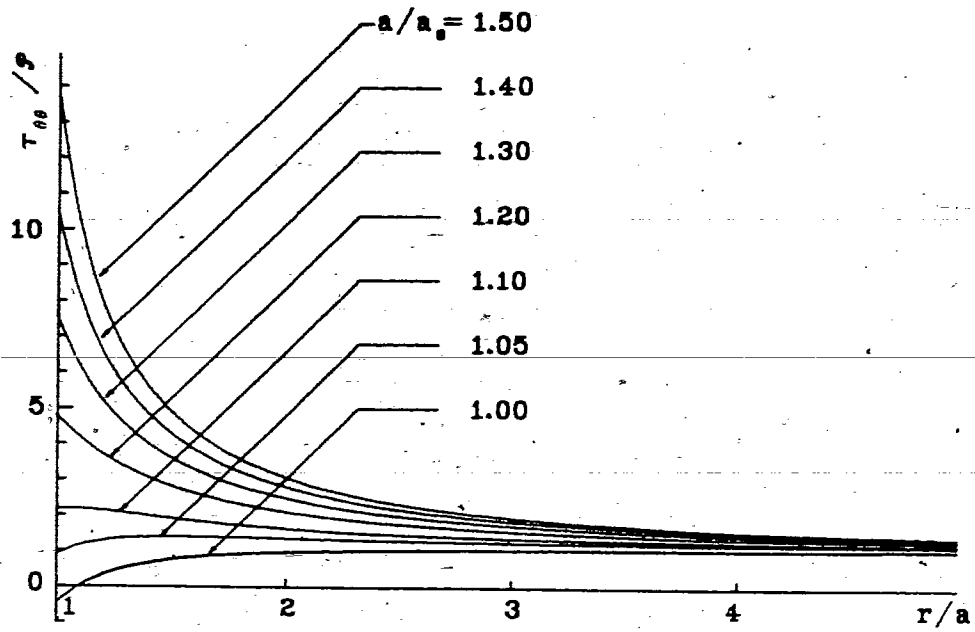


Fig. 6.2.5

responds to the classical linear elasticity solution (Figs. 6.2.2 and 6.2.3).

Regarding the stress field, at the edge of the inclusion, in the case of zero finite deformation, the normal stress $\tau_{\theta\theta}$ has negative values in the zone $\theta = \pi/2 - \pi/6, \pi/2 + \pi/6$. The stress $\tau_{\theta\theta}$ remains negative for small finite deformations but as k/a increases, this zone with compressive stress vanishes and $\tau_{\theta\theta}$ is tensile everywhere (Fig 6.2.4).

The variation of $\tau_{\theta\theta}$ with r at the section $\theta = \pi/2$, for various ratios a/a_0 is shown in the diagrams 6.2.5. As $k/a \rightarrow 0$, the diagrams approach the curve corresponding to the linear elasticity problem and, as r increases, the normal stress $\tau_{\theta\theta}$ approaches the value specified at infinity.

6.3. Perturbational Uniaxial Loading Applied to a Finitely Deformed Slab Containing an Inserted Inclusion

We shall now investigate a version of the boundary value problem considered in section 6.2. We assume that the inclusion is only inserted into the opening and no friction occurs between the inclusion and the surrounding slab. Thus, the mixed boundary conditions

$$u(a, \theta) = 0, \quad \left(\tau_{r\theta}^{12} + \frac{1}{r} \tau_{r\theta}^{11} \frac{\partial v}{\partial r} \right)_{r=a} = 0 \quad (6.3.1)$$

are to be satisfied at $r=a$.

The deformation field and stress distribution are described by (6.1.30) - (6.1.34), where, consistent with relations (6.3.1),

$$A_0^* = 0, \quad (6.3.2)$$

and $\{A_2^*, B_2^*\}$ are solutions of the system

$$A_2^* \sum_{i=0}^{\infty} q^{2i} \hat{\alpha}_{2i} + B_2^* \sum_{i=0}^{\infty} q^{2i} \hat{\beta}_{2i} - \frac{q^4}{2} \ln \frac{a_0}{a} \sum_{i=0}^{\infty} q^{2i} \hat{\alpha}_{2i} + \sum_{i=0}^{\infty} q^{2i} \hat{\gamma}_{2i} = 0, \quad (6.3.3)$$

$$\begin{aligned} & A_2^* \sum_{i=0}^{\infty} \left[(24+24i+8i^2) - (18+22i+8i^2)q^2 - (6+2i) \ln \frac{a_0}{a} \right] q^{2i} \hat{\alpha}_{2i} \\ & + B_2^* \sum_{i=0}^{\infty} \left[(8+8i+8i^2) - (4+6i+8i^2)q^2 - (4+2i) \ln \frac{a_0}{a} \right] q^{2i} \hat{\beta}_{2i} \\ & - \frac{q^4}{2} \ln \frac{a_0}{a} \sum_{i=0}^{\infty} \left[(24+24i+8i^2) - (18+22i+8i^2)q^2 - (6+2i) \ln \frac{a_0}{a} \right] q^{2i} \hat{\alpha}_{2i} \\ & + \frac{q^4}{2} \sum_{i=0}^{\infty} \left[(-12-8i) + (11+8i)q^2 + \ln \frac{a_0}{a} \right] q^{2i} \hat{\alpha}_{2i} \\ & + \sum_{i=0}^{\infty} \left[(8-8i-8i^2) - (6-10i+8i^2)q^2 - (2+2i) \ln \frac{a_0}{a} \right] q^{2i} \hat{\gamma}_{2i} = 0. \end{aligned} \quad (6.3.4)$$

In the special case of infinitesimal deformation only, once again the results coincide with those predicted by the Classical Elasticity

Theory:

$$\begin{aligned}
 v_r &= \frac{P}{8c} \left(r - \frac{a^2}{r} \right) \cos(2\theta) , \\
 v_\theta &= -\frac{P}{8c} r \sin(2\theta) , \\
 \tau_{rr} &= \frac{P}{2} \left[1 + \left(1 + 2\frac{a^2}{r^2} \right) \cos(2\theta) \right] , \\
 \tau_{\theta\theta} &= \frac{P}{2} [1 - \cos(2\theta)] , \\
 \tau_{r\theta} &= -\frac{P}{2} \left(1 - \frac{a^2}{r^2} \right) \sin(2\theta) .
 \end{aligned} \tag{6.3.5}$$

Deformation field and stress distribution.

The constants A_2^* and B_2^* , solutions of the coupled equations (6.3.3) and (6.3.4), are evaluated numerically for several values of ratio a/a_0 .

The results are given in Table 6.3.1.

Table 6.3.1

a/a_0	A_0^*	A_2^*	B_2^*
1.00	0.00000	0.00000	-1.00000
1.10	0.00000	-0.01841	-0.70954
1.20	0.00000	-0.04355	-0.50819
1.30	0.00000	-0.07583	-0.36475
1.40	0.00000	-0.11348	-0.26005
1.50	0.00000	-0.15451	-0.18191
1.75	0.00000	-0.26206	-0.05702
2.00	0.00000	-0.36618	0.01282

Investigating the displacement field, it is observed that the radial lines bend very little. They merely adjust to the tendency of points to accumulate about the stretching axis (Figs. 6.3.1 and 6.3.2). Comparing the results with those for the boundary value problems 6.1 and 6.2, we note that the displacements associated with a given layer in the vicinity of the inserted inclusion are larger than those that would correspond to the case of the bonded inclusion but not as large as those associated with the case where no inclusion is present (Figs. 6.3.3; 6.2.3 and 6.1.8). These results might have been anticipated by considering the nature of the constraint imposed by an inclusion that allows the surrounding material to slip.

The hoop stress $\tau_{\theta\theta}$ is given in Fig. 6.3.4. If the finite deformation is removed, the stress $\tau_{\theta\theta}$ is positive everywhere and no area with compressive stress occurs. For $\theta=0$, $\tau_{\theta\theta}$ vanishes and as θ increases, the value of $\tau_{\theta\theta}$ increases reaching at $\theta = \pi/2$ its maximum, which is the value of the load at infinity. If the slab was previously inflated this maximum value increases with the degree of inflation.

Fig. 6.3.5 illustrates the variation of $\tau_{\theta\theta}$ with r at the section corresponding to $\theta = \pi/2$, for a number of finite deformations. In all cases, as we go away from the inclusion, $\tau_{\theta\theta}$ decreases fast to the value specified at infinity.

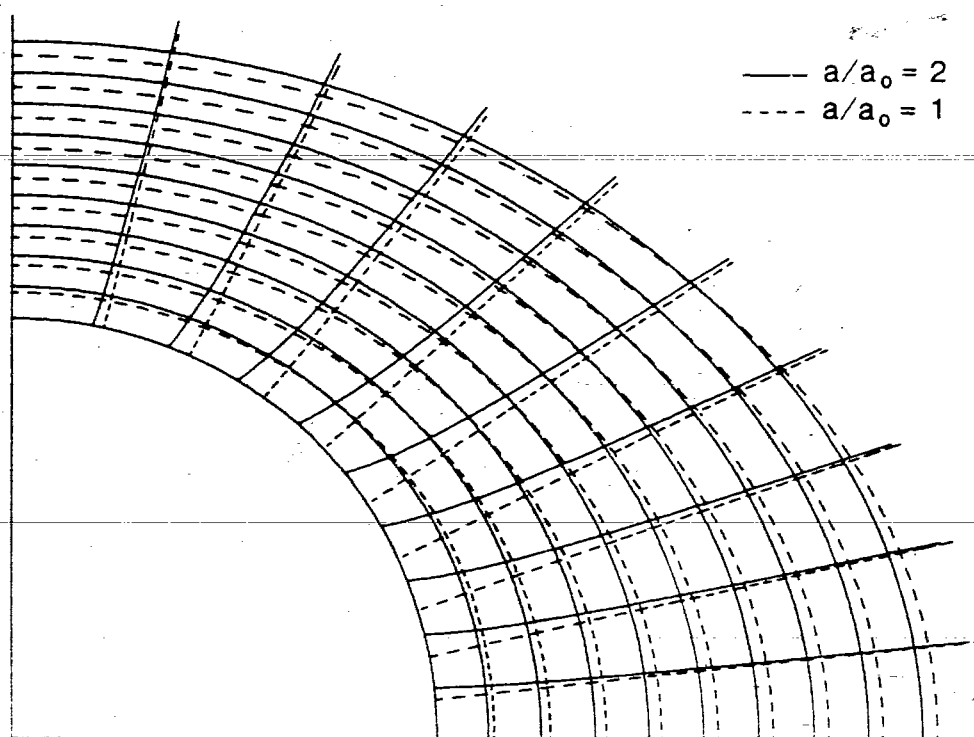


Fig. 6.3.1

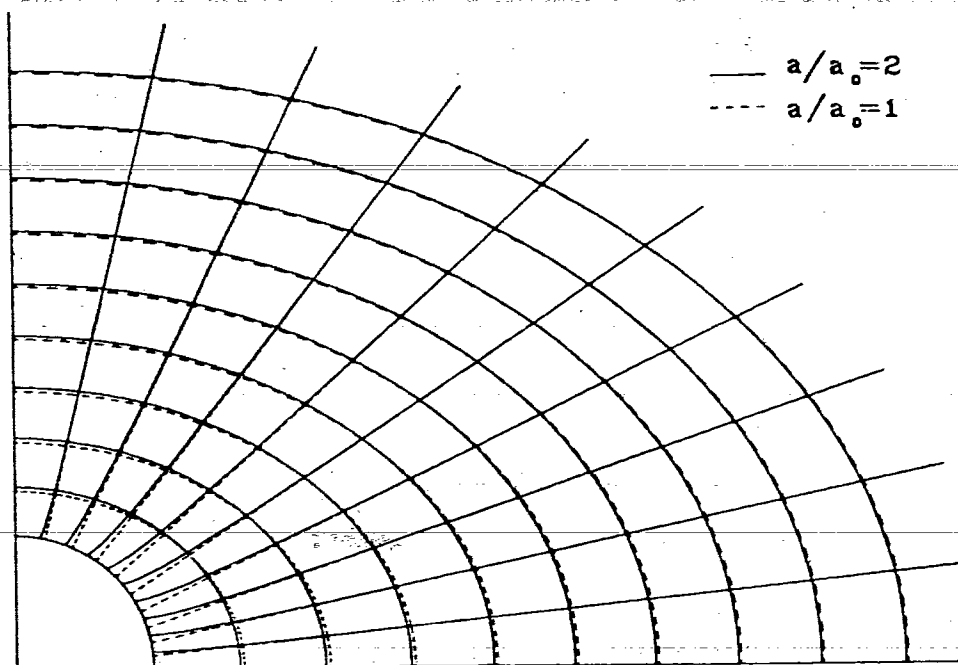


Fig. 6.3.2

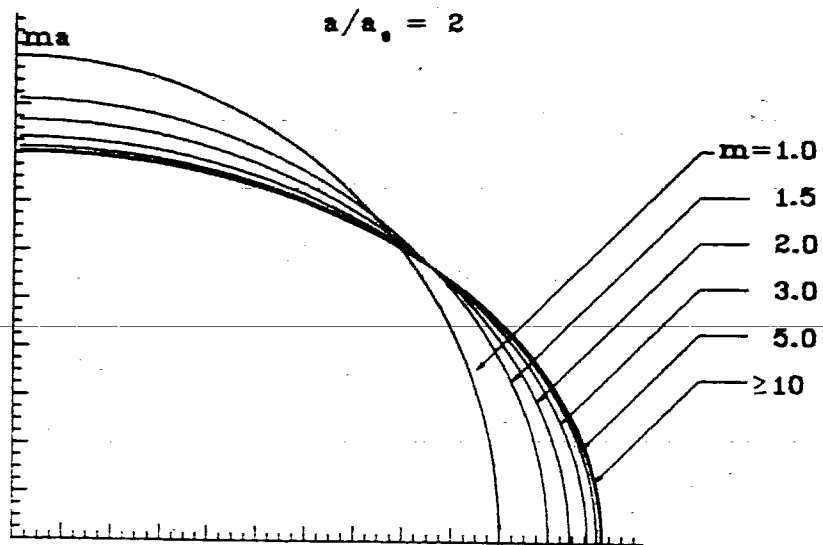


Fig. 6.3.3

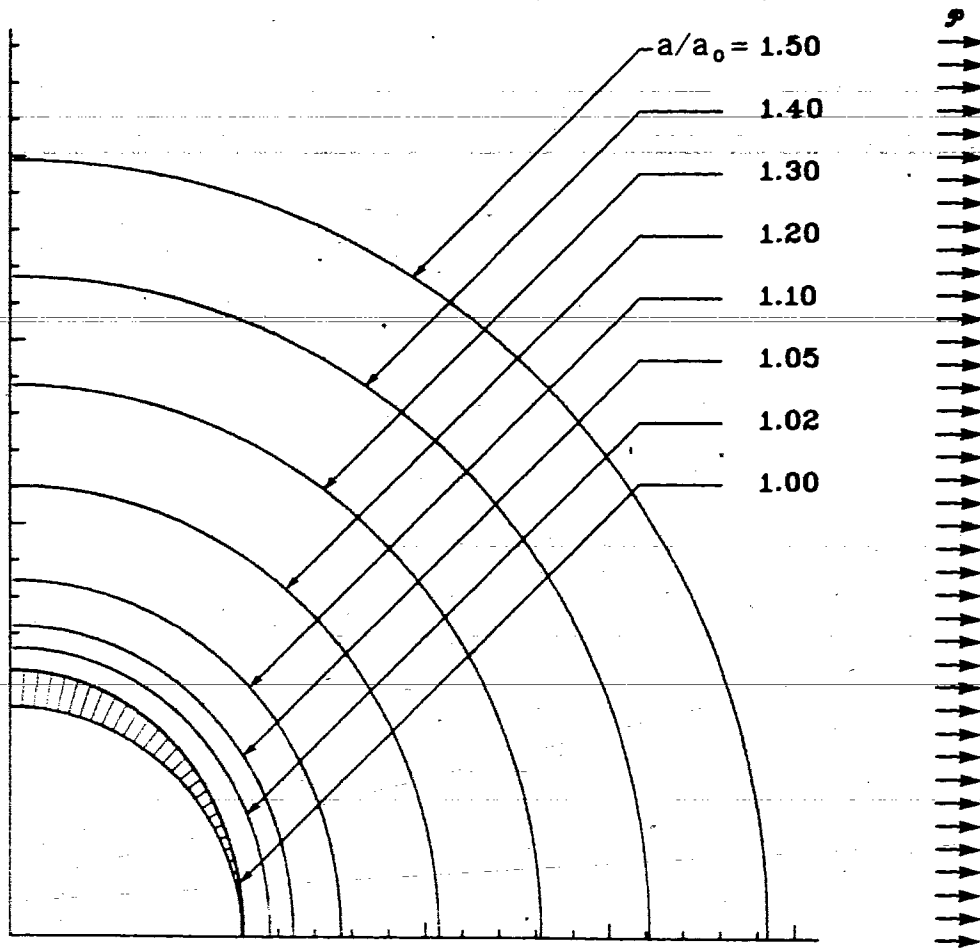


Fig. 6.3.4

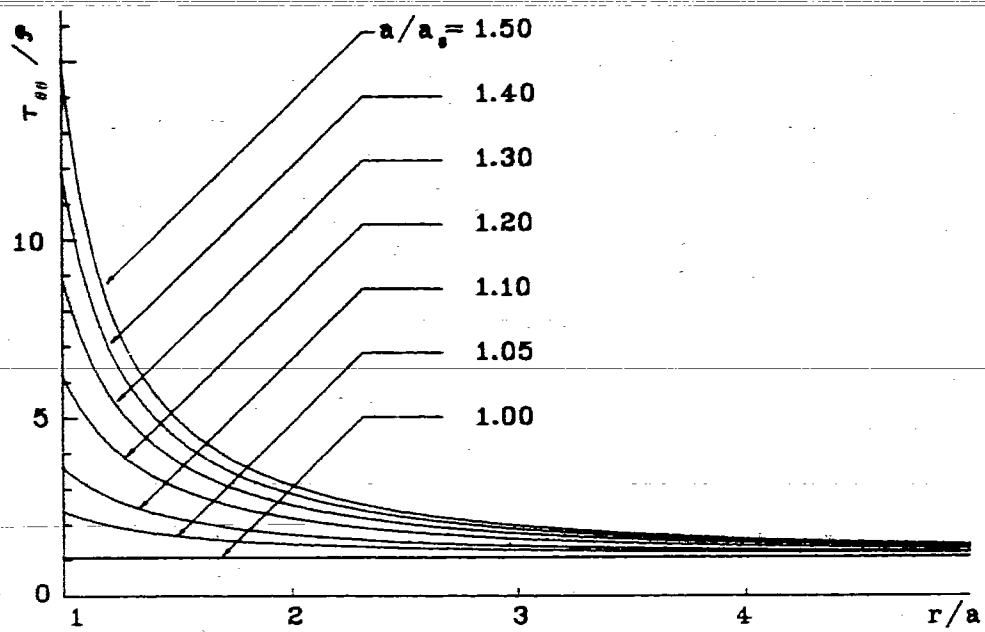


Fig. 6.3.5

7. PERTURBATIONAL SHEARING FORCES

In this chapter we specialize the general results to the case where the perturbation is caused by small shearing forces $T \equiv \epsilon T$, uniformly distributed along the edge of the hole. The corresponding boundary conditions at $r=a$ are given by

$$\left[\tau^{,11} + \tau^{11} \frac{\partial u}{\partial r} \right]_{r=a} = 0, \quad (7.1)$$

and

$$\left[\tau^{,12} + \frac{1}{r} \tau^{11} \frac{\partial v}{\partial r} \right]_{r=a} = -T. \quad (7.2)$$

We may assume that, by applying a small twist, no radial displacement should take place at $r=a$, that is,

$$u(r, \theta) = 0, \quad (7.3)$$

and we also may require a state of free stress at infinity

$$\hat{\tau}^{,ik} \sim 0. \quad (7.4)$$

It can be shown that the constants involved in the general solution (5.95) and (5.96), must be chosen such that

$$\begin{aligned} M_0 &= 0, \quad A_0 = B_0 = \tilde{B}_0 = 0, \\ A_n &= \tilde{A}_n = B_n = \tilde{B}_n = 0, \quad n \neq 0, \\ C_n &= \tilde{C}_n = D_n = \tilde{D}_n = 0, \quad \text{all } n. \end{aligned} \quad (7.5)$$

if the requirements (7.1), (7.3) and (7.4) are to be satisfied. The boundary conditions (7.2) yields

$$c \left[\left(-2 + \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2} \right) v + 2 \left(1 - \frac{k^2}{a^2} \right) \frac{\partial v}{\partial r} \right]_{r=a} = -T, \quad (7.6)$$

and, in view of (7.5), we obtain

$$\tilde{A}_0 = \frac{M}{2\pi c} / \sum_{j=0}^{\infty} \left[(4+2j) - (3+2j) \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right] a_{0,j} a^{-j}, \quad (7.7)$$

where $M \equiv 2\pi T a^2$.

The solution to the boundary value problem may be expressed by

$$\begin{aligned} u(r, \theta) &= 0, \\ v(r, \theta) &= \zeta_0(r) = \tilde{A}_0 \sum_{j=0}^{\infty} a_{0,j} r^{-1-j}, \\ \tau_{r1} &= r^2 \tau_{r2} = 0, \\ r \tau_{r12} &= -\tilde{A}_0 c \left(2 - \frac{k^2}{r^2} + \ln \frac{r^2}{r^2 - k^2} \right) \sum_{j=0}^{\infty} (2+j) a_{0,j} r^{-2-j}. \end{aligned} \quad (7.8)$$

In a limiting case, when the finite deformation is removed the solution (7.8) leads to

$$\begin{aligned} v_r &= 0, \\ v_\theta &= M/8\pi c r, \\ \tau_{rr} &= \tau_{\theta\theta} = 0, \\ \tau_{r\theta} &= -M/2\pi r^2 \end{aligned} \quad (7.9)$$

which are in agreement with the results of Classical Elasticity.

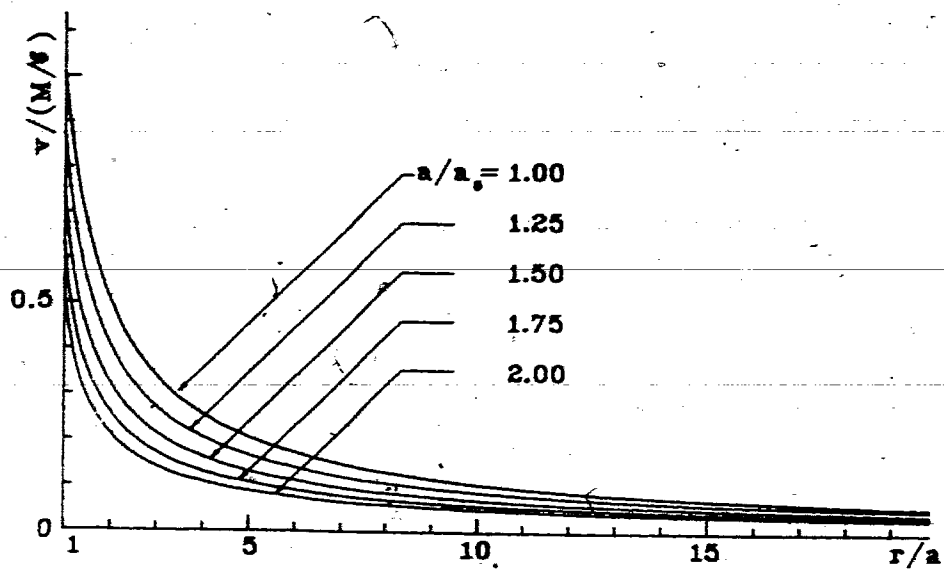


Fig. 7.1

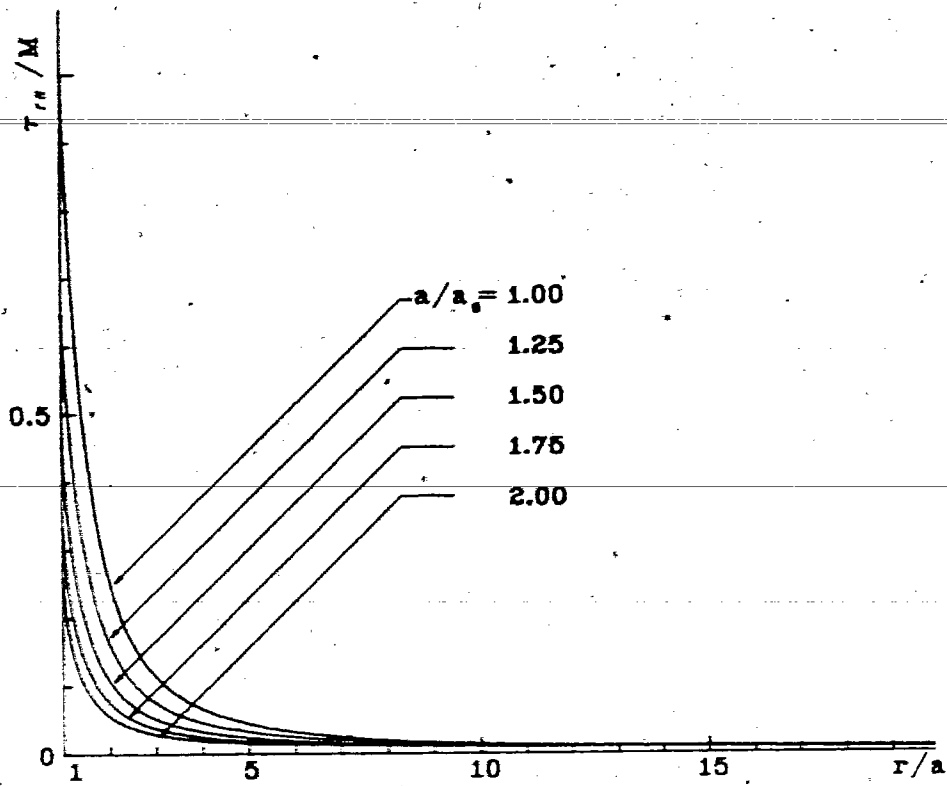


Fig. 7.2

This problem has axial symmetry. By applying a couple M at the opening, the incremental displacements that occur are only tangential and the body deforms like concentric rings sliding over one another. Once again we note that the displacements (Fig. 7.1) and stresses (Fig. 7.2) tend to the form corresponding to the linear elasticity solution as $k/a \rightarrow 0$.

8. PERTURBATIONAL RADIAL FORCES

We consider that the slab is subjected to the force distribution,

$$T \equiv \epsilon T,$$

$$T(a, \theta) \equiv \begin{cases} 0, & 0 < \theta \leq \pi \\ -T_0 \sin(\theta), & \pi < \theta \leq 2\pi \end{cases} \quad (8.1)$$

acting in the radial direction at the hole. The function $T(a, \theta)$ could model a small weight distribution supported by the lower half of the curved surface of the hole. Far from the opening, the uniaxial tension (6.1.1) remains unaltered.

We shall expand the boundary conditions in Fourier series and retain from the general solution those contributions satisfying the required conditions. It can be shown that $T(a, \theta)$ has the Fourier series representation

$$T_0 \left[\frac{1}{\pi} - \frac{1}{2} \sin(\theta) - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{4n^2 - 1} \cos(2n\theta) \right]. \quad (8.2)$$

We recall from (6.1.4) that the stress conditions at infinity are satisfied provided

$$\begin{aligned} M_0 &= \frac{P}{2}, \quad C_2 = \frac{P}{8c}, \\ C_n &= 0, \quad \text{for all } n \neq 2, \\ \tilde{C}_n &= D_n = \tilde{D}_n = 0, \quad \text{for all } n. \end{aligned} \quad (8.3)$$

The boundary conditions at $r=a$ require that

$$\begin{aligned} [\tau'^{11} + \tau^{11} \frac{\partial u}{\partial r}]_{r=a} &= -T, \\ [\tau'^{12} + \frac{1}{r} \tau^{11} \frac{\partial v}{\partial r}]_{r=a} &= 0, \end{aligned} \quad (8.4)$$

where T is given by (8.2). We substitute into (8.4) the expressions (5.10), (5.97) and (5.99) for τ'^{11} , τ^{11} and τ'^{12} and the displacement gradients $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ determined from (5.95) and (5.96). It can be shown that the contribution of terms corresponding to $n=0$ yield

$$A_0 = \frac{P}{2} - \frac{T_0}{\pi} a^2 / [4 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2}] c. \quad (8.5)$$

The contribution of terms regarding $n=1$ provides two coupled equations for the unknown constants \tilde{A}_1 and \tilde{B}_1

$$\begin{aligned} < \tilde{A}_1 \{ \sum_{j=0}^{\infty} [(-1+3j+j^2) - \frac{(7+5j+j^2)k^2}{a^2} - (1 - \frac{k^2}{a^2})^{-1} + \frac{1}{2} \ln \frac{a_0^2}{a^2}] \alpha_{1,j} r^{-3-j} \\ + \tilde{B}_1 \{ \frac{\beta_{1,2}}{\alpha_{1,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} [(-1+3j+j^2) - \frac{(7+5j+j^2)k^2}{a^2} - (1 - \frac{k^2}{a^2})^{-1} \\ + \frac{1}{2} \ln \frac{a_0^2}{a^2}] \alpha_{1,j} r^{-1-j} - \frac{\beta_{1,2}}{\alpha_{1,0}} \sum_{j=0}^{\infty} (-3-2j) + (5+2j) \frac{k^2}{a^2} \alpha_{1,j} r^{-3-j} \\ + \sum_{j=0}^{\infty} [(-3-j+j^2) + \frac{(5-j-j^2)k^2}{a^2} - (1 - \frac{k^2}{a^2})^{-1} \\ + \frac{1}{2} \ln \frac{a_0^2}{a^2}] (\frac{\partial \beta_{1,j}}{\partial \lambda})_{\lambda=-1} r^{-1-j} \} >_{r=a} = \frac{T_0}{4c}, \end{aligned} \quad (8.6)$$

$$\begin{aligned} < \tilde{A}_1 \{ \sum_{j=0}^{\infty} [(-4-2j) + (3+2j) \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2}] \alpha_{1,j} r^{-3-j} \\ + \tilde{B}_1 \{ \frac{\beta_{1,2}}{\alpha_{1,0}} \ln \frac{a_0}{r} \sum_{j=0}^{\infty} [(-4-2j) + (3+2j) \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2}] \alpha_{1,j} r^{-3-j} \\ - \frac{\beta_{1,2}}{\alpha_{1,0}} \sum_{j=0}^{\infty} (2-2\frac{k^2}{a^2}) \alpha_{1,j} r^{-3-j} + \sum_{j=0}^{\infty} [-2j - (1-2j) \frac{k^2}{a^2} \\ + \ln \frac{a_0^2}{a^2}] (\frac{\partial \beta_{1,j}}{\partial \lambda})_{\lambda=-1} r^{-1-j} \} >_{r=a} = 0, \end{aligned} \quad (8.7)$$

whereas the equations for A_1 and B_1 give

$$A_1 = B_1 = 0. \quad (8.8)$$

Further calculations reveal that corresponding to $n=2m$, $m = 1, 2, 3, \dots$

we have

$$\begin{aligned} < A_n \sum_{j=0}^{\infty} \{ [(n+1)(n^2-n+2) - n(n+4)j - (3n+2)j^2 - j^3] \\ & + [n(n+1)(-\frac{n}{2}+4) + (\frac{3}{2}n^2+8n+4)j + (3n+4)j^2 + j^3] \frac{k^2}{a^2} \\ & + (\frac{n^2}{1-k^2/a^2} - \frac{n^2}{2} \ln \frac{a_0^2}{a^2})(n+1+j) \} \alpha_{n,j} r^{-n-2-j} \\ & + B_n \sum_{j=0}^{\infty} \{ [n(n-1)(n+4) - (n^2-8n+4)j + (3n-4)j^2 - j^3] \\ & + [-\frac{n^2}{2}(n-1) + (\frac{3}{2}n^2-4n)j + (3n-2)j^2 + j^3] \frac{k^2}{a^2} \\ & + (\frac{n^2}{1-k^2/a^2} - \frac{n^2}{2} \ln \frac{a_0^2}{a^2})(n-1-j) \} \beta_{n,j} r^{-n-j} \\ & + C_n \{ a_n \ln \frac{a_0}{r} \sum_{j=0}^{\infty} \{ [(n+1)(n^2-n+2) - n(n+4)j - (3n+2)j^2 - j^3] \\ & + [n(n+1)(-\frac{n}{2}+4) + (\frac{3}{2}n^2+8n+4)j + (3n+4)j^2 + j^3] \frac{k^2}{a^2} \\ & + (\frac{n^2}{1-k^2/a^2} - \frac{n^2}{2} \ln \frac{a_0^2}{a^2})(n+1+j) \} \alpha_{n,j} r^{-n-2-j} \\ & - a_n \sum_{j=0}^{\infty} \{ [n(n+4) + 2(3n+2)j + 3j^2] - [(\frac{3}{2}n^2+8n+4) + 2(3n+4)j + 3j^2] \frac{k^2}{a^2} \\ & - (\frac{n^2}{1-k^2/a^2} - \frac{n^2}{2} \ln \frac{a_0^2}{a^2}) \} \alpha_{n,j} r^{-n-2-j} + \sum_{j=0}^{\infty} \{ [-n^2-n(n-4)j + (3n-2)j^2 - j^3] \\ & + [n(n+1)(\frac{n}{2}+4) + (\frac{3}{2}n^2-8n+4)j - (3n-4)j^2 + j^3] \frac{k^2}{a^2} \\ & - (\frac{n^2}{1-k^2/a^2} - \frac{n^2}{2} \ln \frac{a_0^2}{a^2})(n-1-j) \} \gamma_{n,j}^* r^{n-2-j} \} >_{r=a} = \frac{n^2 \pi_0}{\pi(4n^2-1)c} \end{aligned}$$

(8.9)

$$\begin{aligned}
& < A_n \sum_{j=0}^{\infty} \{ [4(n+1)(n+j) + 2j^2] - [3n(n+1) + (4n+3)j + 2j^2] \frac{k^2}{a^2} \\
& \quad - [n(n+1) + j] \ln \frac{a_0^2}{a^2} \} \alpha_{n,j} r^{-n-1-j} \\
& + B_n \sum_{j=0}^{\infty} \{ [4(n-1)(n+j) + 2j^2] - [(3n-1)(n-2) + (4n-5)j + 2j^2] \frac{k^2}{a^2} \\
& \quad - [n(n+1) - 2 + j] \ln \frac{a_0^2}{a^2} \} \beta_{n,j} r^{-n+1-j} \\
& + C_n \{ a_n \ln \frac{a_0}{a} \sum_{j=0}^{\infty} \{ [4(n+1)(n+j) + 2j^2] - [3n(n+1) + (4n+3)j + 2j^2] \frac{k^2}{a^2} \\
& \quad - [n(n+1) + j] \ln \frac{a_0^2}{a^2} \} \alpha_{n,j} r^{-n-1-j} \\
& - a_n \sum_{j=0}^{\infty} \{ -4(n+1+j) + [(4n+3) + 4j] \frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2} \} \alpha_{n,j} r^{-n-1-j} \\
& + \sum_{j=0}^{\infty} \{ [4(n-1)(n-j) + 2j^2] - [3n - (4n-3)j + 2j^2] \frac{k^2}{a^2} \\
& \quad - [n(n-1) + j] \ln \frac{a_0^2}{a^2} \} \gamma_{n,j}^* r^{n+1-j} \Big|_{r=a} > = 0, \tag{8.10}
\end{aligned}$$

which allows us to determine the constants A_{2m} and B_{2m} . Corresponding to the odd n 's we obtain

$$\begin{aligned}
A_n &= B_n = 0, \\
\tilde{A}_n &= \tilde{B}_n = 0, \quad n = 2m+1, \quad m = 1, 2, 3, \dots
\end{aligned} \tag{8.11}$$

Thus, the radial component of the displacement field, $u(r, \theta)$ reduces to

$$\begin{aligned}
u(r, \theta) &= A_0 u_0(r) + [\tilde{A}_1 u_1^{(1)}(r) + \tilde{B}_1 u_1^{(2)}(r)] \sin(\theta) + C_2 u_2^{(3)} \cos(2\theta) \\
&+ \sum_{m=1}^{\infty} [A_{2m} u_{2m}^{(1)}(r) + B_{2m} u_{2m}^{(2)}(r)] \cos(2m\theta), \tag{8.12}
\end{aligned}$$

and the corresponding relation for the tangential component $v(r, \theta)$, may be derived from (4.12) and (5.12). In terms of power series, the incremental displacements and stresses result from their general expressions (5.96) - (5.99) where the arbitrary constants are specified here by relations (8.3) and (8.5) - (8.11).

9. PERTURBATION DUE TO THE SHAPE OF THE OPENING

The general analytic solution (5.95) and (5.96) obtained for a slab with a circular opening is also applicable to cases involving geometric perturbations of the boundaries of the original body. Following from the work of Graham [4], the perturbation in shape is regarded as an additional displacement field that is superimposed on the finite deformation of the reference body, for which the solution is available.

In this chapter we shall obtain specific analytic solutions taking into account both perturbation due to an applied stress field and the perturbation in the geometry of the original body. We examine two cases, namely, the problem of a slab with a rough cavity and also the case where the cross-section of the hole is elliptic. In both cases, a perturbational uniaxial tension is applied.

9.1. Elliptic Boundary

Suppose that the slab has an elliptic opening S_{Γ_0} , in its undeformed state B_0 , given by the equation

$$\frac{x_1^2}{[a_0(1+\beta\varepsilon)]^2} + \frac{x_2^2}{a_0^2} = 1, \quad (9.1.1)$$

where β is a constant such that $0 \leq \beta \leq 1$. The body B_0 undergoes the finite deformation described by relation (3.2.2) reaching the state B . It can

be shown that the surface S_Γ in to which the surface S_{Γ_0} is finitely deformed, is given in terms of the convected coordinates (r, θ, z) by the expression

$$r^2 - a^2 + 2\beta\epsilon[(r^2 - k^2)\sin^2\theta - a_0^2] = 0, \quad (9.1.2)$$

up to the first order in ϵ . The body B is then further subjected to a small uniaxial tension at infinity reaching the final configuration B' . The equilibrium equations corresponding to B' are given by (4.20) where the stress components τ^{ik} and τ'^{ik} are expressed by relations (5.10) and (5.11), respectively.

Using the definitions (2.27) and (2.28), we compute the functions κ and χ with regard to equation (9.1.2). These functions are given by

$$\kappa = \frac{1}{2r}, \quad \chi = \frac{\beta}{r} \sin^2\theta. \quad (9.1.3)$$

On substituting (9.1.3) into (2.25), we obtain the traction across the surface (9.1.2) in the form

$$\bar{t} + \epsilon\bar{t}' = (\tau^{11} + \epsilon\lambda^{11})\bar{G}_1 + \epsilon\{\lambda^{12} + [\frac{\beta}{r}(r^2 - k^2)\sin(2\theta)]\tau^{22}\}\bar{G}_2. \quad (9.1.4)$$

Now at points of S_Γ , we find that to first order in ϵ , *

$$r = a[1 + \beta\epsilon(1 - k^2/a^2)\cos^2\theta] \quad (9.1.5)$$

* All calculations in this chapter are carried out to order ϵ , consistent with the theory.

and therefore, the radial deformation function $Q(r)$, (relation (3.2.12)), takes the value

$$Q(r) = \left[1 - \frac{k^2}{a^2} - 2\beta\epsilon \frac{k^2 a_0^2}{a^4} \cos^2 \theta \right]^{1/2}. \quad (9.1.6)$$

Consequently, it may be shown that the stress component τ^{11} at S_r is expressed by

$$\tau^{11} = \left[-\frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2} + 2\beta\epsilon \frac{k^2}{a^2} \left(2 - \frac{k^2}{a^2} \right) \cos^2 \theta \right] c. \quad (9.1.7)$$

If we require that the finitely inflated slab, subjected also to the uniaxial tension $P \equiv \epsilon P$ at infinity, is to be maintained in equilibrium by applying to S_r the same tractions as those given by (3.2.26), it is sufficient that

$$\lambda^{11} + 2\beta c \frac{k^2}{a^2} \left(2 - \frac{k^2}{a^2} \right) \cos^2 \theta = 0, \quad r = a, \quad (9.1.8)$$

$$\lambda^{12} + \left[\frac{\beta}{r} (r^2 - k^2) \sin(2\theta) \right] \tau^{22} = 0, \quad r = a,$$

in view of relation (9.1.4). We now substitute into (9.1.8) the stress components λ^{ij} given by relations (4.19) where τ^{ik} and $\tau^{,ik}$ are expressed by (5.10) and (5.11), respectively. The boundary conditions (9.1.8) yield

$$\left[\frac{1}{c} p'(r, \theta) + \left(4 - 3 \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) \frac{\partial u}{\partial r} \right]_{r=a} = -\beta \frac{k^2}{a^2} \left(2 - \frac{k^2}{a^2} \right) [1 + \cos(2\theta)],$$

$$\begin{aligned} \left[\frac{1}{r} \left(2 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) \left(\frac{\partial u}{\partial \theta} - v \right) - 2 \frac{a_0^2}{a^2} \frac{\partial v}{\partial r} \right]_{r=a} \\ = \beta \frac{a_0^2}{a^2} \left[2 - \frac{k^2}{a^2} - \frac{2}{1 - k^2/a^2} - \ln \frac{a_0^2}{a^2} \right] \sin(2\theta) \end{aligned}$$

$$(9.1.9)$$

The problem of superposition of a small stretch at infinity on a large radial deformation of the slab with an elliptic opening is reduced to searching for a solution to the equilibrium equations (4.20) that correspond to the circular case, satisfying the required conditions (6.1.1) at infinity and the appropriate inner boundary conditions (9.1.9). The boundary conditions at infinity are met provided relations (6.1.4) are satisfied. Regarding the boundary conditions at the hole, we note that the right hand side of the relations (9.1.9) account for the changes caused by the perturbation in shape. In fact, with $\beta=0$, these relations become the boundary conditions appropriate to the circular hole case.

It can be shown that the solution to the boundary value problem considered here is formally described by

$$u(r, \theta) = \frac{A_0}{r} + [A_2 u_2^{(1)}(r) + B_2 u_2^{(2)}(r) + C_2 u_2^{(3)}(r)] \cos(2\theta) \quad (9.1.10)$$

where the constants A_0 , A_2 and B_2 are derived from conditions (9.1.9) and C_2 is the same as given by (6.1.4). Relations (9.1.9) yield further that

$$A_0 = \left[\frac{P}{2c} + \beta \frac{k^2}{a^2} \left(2 - \frac{k^2}{a^2} \right) \right] a^2 / \left(4 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) \quad (9.1.11)$$

and also that A_2 and B_2 are solutions of the equations

$$\left\{ \left(1 - \frac{k^2}{a^2} \right) r^2 \frac{d^3 u_2}{dr^3} + \left(4 - 2 \frac{k^2}{a^2} \right) r \frac{d^2 u_2}{dr^2} + \left(-7 + 7 \frac{k^2}{a^2} - \frac{4}{1 - k^2/a^2} \right. \right. \\ \left. \left. + 2 \ln \frac{a_0^2}{a^2} \right) \frac{du_2}{dr} + 3 \left(1 + \frac{k^2}{a^2} \right) \frac{1}{r} u_2 \right\}_{r=a} = 2\beta \frac{k^2}{a^2} \left(2 - \frac{k^2}{a^2} \right),$$

$$\left\{ 2\left(1 - \frac{k^2}{a^2}\right)r \frac{d^2 u_2}{dr^2} + \left(2 - 3\frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2}\right) \frac{du_2}{dr} + 3\left(2 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2}\right) \frac{u_2}{r} \right\}_{r=a}$$

$$= -2\beta \frac{a_0^2}{a^2} \left[2 - \frac{k^2}{a^2} - \frac{2}{1 - k^2/a^2} - \ln \frac{a_0^2}{a^2} \right].$$

(9.1.12)

We shall follow the procedure outlined in Chapter 6 making the appropriate substitution of $u_2(r)$, du_2/dr and higher derivatives in terms of power series. We shall render the integrating constants non-dimensional and put the equations (9.1.12) in a more convenient form for numerical computations. Further, we compute the displacement and stress field.

A typical deformation field near the elliptic hole is given in Fig. 9.1.1, in terms of a distorted square grid against the reference grid. In this instance, the major-axis of the ellipse is taken perpendicular to the line of stretch where $\beta = .5$, $\varepsilon = .1$ and $a/a_0 = 1.5$. Experiments mentioned in the paper of Varley and Cumberbatch [17], although in a different context, produce configurations similar to those obtained here.

Investigating the stress field, we note that as a result of the finite inflation, the stress concentration effect is strengthened. Furthermore, the position of the elliptic hole with respect to the direction of the uniaxial tension has a significant impact on stress concentration. It can be shown that the hoop stress $\tau_{\theta\theta}(r, \theta)$ derived for two extremum positions of the ellipse, namely, when the major axis is perpendicular to the line of tension ($s=-1$), and when the major axis is parallel to it ($s=+1$), is given by the expression (6.1.41) evaluated at $r=a$, with the additional term

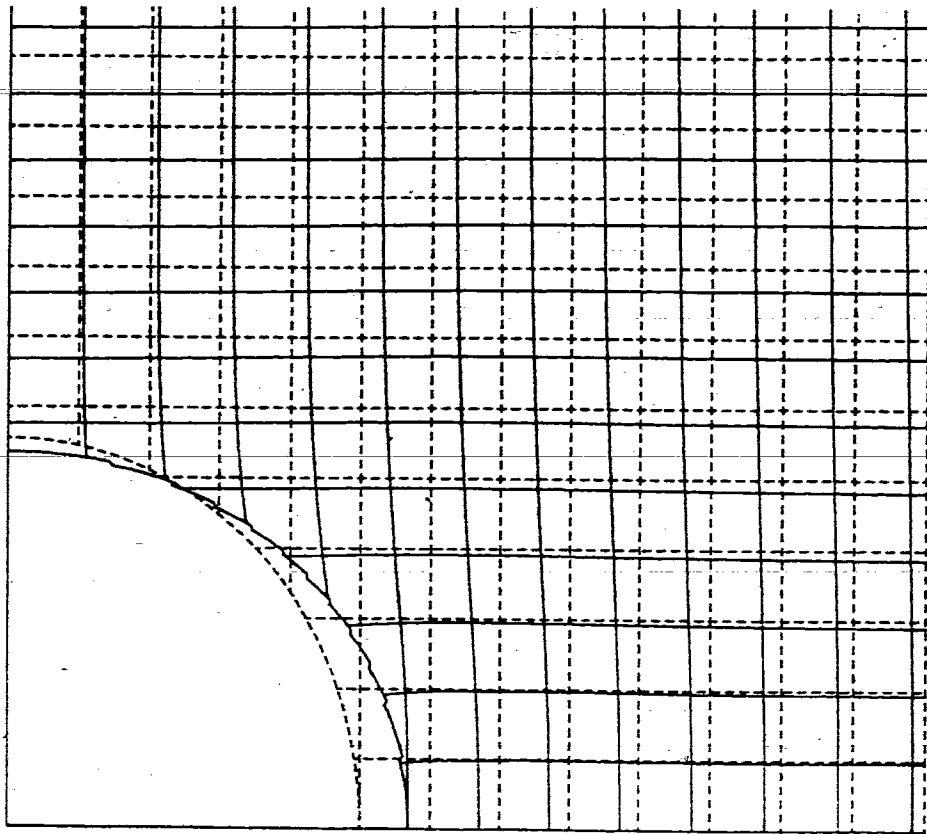


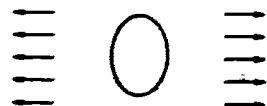


Fig. 9.1.1

$$\beta \varepsilon \frac{k^2}{a^2} \left(1 - \frac{k^2}{a^2}\right) \left[1 - \frac{1}{1 - k^2/a^2} - \frac{2}{(1 - k^2/a^2)^2}\right] [1 + s \cos(2\theta)] c$$

(9.1.13)

The results given in Table 9.1.1 show that, in agreement with Linear Elasticity, the stress concentration is more pronounced when the major axis of the ellipse is perpendicular to the direction of the uniaxial tension than when it is parallel to the line of loading.

Table 9.1.1

a/a_0	-	1.10	1.20	1.30	1.40	1.50	1.75	2.00
		3.17	3.39	3.67	3.98	4.45	5.31	6.46
		3.03	3.12	3.24	3.40	3.58	4.12	4.77
		3.02	3.11	3.22	3.36	3.53	4.02	4.61

These values should be compared with the Linear Elasticity results ¹⁸ [18].

Corresponding to a given imperfection, of an ellipse where $\varepsilon = .1$, $\beta = .5$,

the stress concentration factor is 3.10 when the axial tension is normal

to the major axis and is 2.81 when the line of tension is parallel to it,

¹⁸ The stress intensity factor corresponding to an elliptic hole with the axis a perpendicular to the uniaxial tension and the axis b parallel to it is given by $1 + 2a/b$.

compared with 3.00 which corresponds to the circular case. These values increase with the degree of inflation as is shown in Table 9.1.1.

9.2. Axisymmetric Imperfect Cylinder

In this section we assume that the surface of the cylindrical opening is rough, within a strip of order ϵ . Let the small variation of the radius with the angle be described, in the undeformed state B_0 , by a sine function, in the form

$$x_1^2 + x_2^2 = \{a_0 [1 + \epsilon h_m \sin(m\theta)]\}^2. \quad (9.2.1)$$

The body B_0 is finitely deformed into the state B consistent with (3.2.2), and correspondingly, it can be shown that the surface (9.2.1) deforms into the surface S_r given by the equation

$$r^2 - a^2 - 2\epsilon a_0 h_m \sin(m\theta) = 0. \quad (9.2.2)$$

As described in previous sections, we assume that the body is subjected to a small uniaxial loading at infinity. The stress-equilibrium equations associated with the final configuration, B' , are given by equations (4.20), where the stress components τ^{ik} and τ'^{ik} are, respectively expressed by (5.10) and (5.11).

It follows from relations (2.28), (2.29) and (9.2.2) that

$$x = \frac{1}{r}, \quad \chi = 0. \quad (9.2.3)$$

On substituting relation (9.2.3) into (2.25), the traction vector across the surface (9.2.2) becomes

$$\bar{t} + \varepsilon \bar{t}' = (\tau^{11} + \varepsilon \lambda^{11}) \bar{G}_1 + \varepsilon \left\{ \lambda^{12} - \frac{1}{r} [a_0 h_m \cos(m\theta)] \tau^{22} \right\} \bar{G}_2. \quad (9.2.4)$$

It can be shown that at the points of the surface (9.2.2) the stress component τ^{11} may be expressed by

$$\tau^{11} = \left[-\frac{k^2}{a^2} + \ln \frac{a_0^2}{a^2} + 2\varepsilon \frac{k^4 h_m}{a^4 a_0} \sin(m\theta) \right] c.$$

The deformation can be sustained by applying at the surface S_r , the same tractions as expressed by (3.2.26), provided

$$\begin{aligned} \lambda^{11} + \left[2\varepsilon \frac{k^4 h_m}{a^4 a_0} \sin(m\theta) \right] c &= 0, \quad r=a, \\ \lambda^{12} - \left[\frac{a_0 h_m}{a} \cos(m\theta) \right] \tau^{22} &= 0, \quad r=a. \end{aligned} \quad (9.2.5)$$

The boundary conditions (9.2.5) may be replaced by

$$\begin{aligned} \left[\frac{1}{c} p'(r, \theta) + \left(4 - 3\frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) \frac{\partial u}{\partial r} \right]_{r=a} &= -2 \frac{k^4 h_m}{a^4 a_0} \sin(m\theta), \\ \left[\frac{1}{r} \left(2 - \frac{k^2}{a^2} - \ln \frac{a_0^2}{a^2} \right) \left(\frac{\partial u}{\partial \theta} - v \right) - 2 \frac{a_0^2}{a^2} \frac{\partial v}{\partial r} \right]_{r=a} &= \frac{m a_0 h_m}{a^2} \left[2 - \frac{k^2}{a^2} - \frac{2}{1-k^2/a^2} - \ln \frac{a_0^2}{a^2} \right] \cos(m\theta) \end{aligned} \quad (9.2.6)$$

after substituting (4.19) for λ^{ik} and the corresponding expressions (5.10) and (5.11) for τ^{ik} and τ', ik .

It can be shown that, consistent with the boundary conditions (6.1.1) and (9.2.5), the radial displacement field may be written in the form

$$u(r, \theta) = A_0 u_0(r) + [A_2 u_2^{(1)}(r) + B_2 u_2^{(2)}(r) + C_2 u_2^{(3)}(r)] \cos(2\theta) \\ + [\tilde{A}_m u_m^{(1)}(r) + \tilde{B}_m u_m^{(2)}(r)] \sin(m\theta) \quad (9.2.7)$$

where the integration constants A_0 , A_2 , B_2 and C_2 are respectively given by (6.1.6), (6.1.24), (6.1.25) and (6.1.4) and \tilde{A}_m and \tilde{B}_m are determined by the two equations:

$$\begin{aligned} & \langle \tilde{A}_m \sum_{j=0}^{\infty} \{ [(m+1)(m^2-m-2) - m(m+4)j - (3m+2)j^2 - j^3] \\ & \quad + [m(m+1)(-\frac{m}{2} + 4) + (\frac{3}{2}m^2 + 8m+4)j + (3m+4)j^2 + j^3] \frac{k^2}{a^2} \\ & \quad + (\frac{m^2}{1-k^2/a^2} - \frac{m^2}{2} \ln \frac{a_0^2}{a^2}) (m+1+j) \} \alpha_{m,j} r^{-m-2-j} \\ & + \tilde{B}_m \sum_{j=0}^{\infty} \{ [m(m-1)(m+4) - (m^2-8m+4)j - (3m-4)j^2 - j^3] \\ & \quad + [-\frac{m^2}{2}(m-1) + (\frac{3m^2}{2} - 4m)j + (3m-2)j^2 + j^3] \frac{k^2}{a^2} \\ & \quad + (\frac{m^2}{1-k^2/a^2} - \frac{m^2}{2} \ln \frac{a_0^2}{a^2}) (m-1-j) \} \beta_{m,j} r^{-m-j} \rangle_{r=a} = \frac{m^2 k^4 h_m}{a^4 a_0} \end{aligned} \quad (9.2.8)$$

$$\begin{aligned} & \langle \tilde{A}_m \sum_{j=0}^{\infty} \{ [4(m+1)(m+j) + 2j^2] - [3m(m+1) + (4m+3)j + 2j^2] \frac{k^2}{a^2} \\ & \quad - [m(m+1) + j] \ln \frac{a_0^2}{a^2} \} \alpha_{m,j} r^{-m-1-j} \\ & + \tilde{B}_m \sum_{j=0}^{\infty} \{ [4(m-1)(m+j) + 2j^2] - [(3m-1)(m-2) + (4m-5)j + 2j^2] \frac{k^2}{a^2} \\ & \quad - [m(m+1) - 2 + j] \ln \frac{a_0^2}{a^2} \} \beta_{m,j} r^{-m+1-j} \rangle_{r=a} \\ & = -\frac{m^3 a_0 h_m}{2a^2} \left(2 - \frac{k^2}{a^2} - \frac{2}{1-k^2/a^2} - \ln \frac{a_0^2}{a^2} \right). \end{aligned} \quad (9.2.9)$$

10. PERTURBATION OF THE STRAIN ENERGY FUNCTION

The analytical solution obtained for problems involving materials with a specific strain energy function makes also possible to investigate cases corresponding to related materials, with a perturbed strain energy function. In view of the Spencer theory [2], a strain energy function W modified by a small perturbation to

$$W + \epsilon W' \quad (10.1)$$

will result in an additional small deformation superimposed on the existing finite deformation. We assume that W has the Mooney-Rivlin form

$$W = c_1(I_1 - 3) + c_2(I_2 - 3), \quad (10.2)$$

and we make no specific assumption about $\epsilon W'$, other than that ϵ is small.

It can be shown that the stress components τ^{ik} are given by (5.10) and remain unchanged whereas the stress components τ'^{ik} become

$$\begin{aligned} \tau'^{11} &= p' - 2[H-L(r)-2Q^2c] \frac{\partial u}{\partial r} + Q^2\Phi' + (1+Q^2)\Psi', \\ r^2 \tau'^{22} &= p' + 2[H-L(r)-2Q^2c] \frac{\partial u}{\partial r} + \frac{1}{Q^2}\Phi' + (1+\frac{1}{Q^2})\Psi', \\ r\tau'^{12} &= -\frac{1}{r}[H-L(r)-2Q^2c] \left[\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right], \end{aligned} \quad (10.3)$$

where the functions Φ' and Ψ' of the Spencer theory, given by relations

(2.30), reduce in this case to

$$\Phi' = 2 \frac{\partial W'}{\partial I_1}, \quad \Psi' = 2 \frac{\partial W'}{\partial I_2}. \quad (10.4)$$

Further calculations reveal that the stresses expressed by relations

(10.3) modify the first of the equilibrium equations (5.12) and bring

no change in the last two. We may show that

$$\begin{aligned} & \frac{1}{2c} \frac{\partial p'}{\partial r} + Q^2 \frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2 Q^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} [4 - 2Q^2 + \frac{1}{Q^2} \\ & \quad + (Q^2 + \frac{1}{Q^2})(\Phi' + \Psi')] \frac{\partial u}{\partial r} + \frac{1}{r^2 Q^2} u \\ & = - \left\{ \frac{1}{r} (Q^2 - \frac{1}{Q^2})(\Phi' + \Psi') + \frac{\partial}{\partial r} [Q^2 \Phi' + (1 + Q^2) \Psi'] \right\}, \\ & \frac{1}{2c} \frac{\partial p'}{\partial \theta} + r Q^2 \frac{\partial^2 v}{\partial r^2} - \frac{1}{Q^2} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r} (2 - Q^2) \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} - v \right) = 0, \\ & \frac{\partial p'}{\partial z} = 0. \end{aligned}$$

(10.5)

The effect of the terms containing Φ' and Ψ' is to increase the complexity of the equations which correspond to (5.27), but the procedure followed in chapter 5 will apply here as well.

CONCLUSION

In this work we have considered a class of boundary value problems involving perturbations about a finite inflation of a slab with a circular cavity. The equilibrium equations have been formulated for incompressible materials in terms of a general strain-energy function for the plane strain case. An exact general solution is obtained for materials with Mooney-Rivlin strain-energy, although the method is not restricted to this particular form.

Specific analytic solutions are obtained for a number of boundary value problems of interest. There are two categories: (a) problems where, far from the cavity or the inclusion embedded into the body, a perturbational uniaxial tension is applied and (b) problems where a perturbational load is acting at the hole. The analytic expressions have in many cases been evaluated numerically to allow detailed investigations of the deformation, the stress field and stress concentration effect around the hole.

The general solution has further been used to take into account both the perturbation due to an applied stress field and the perturbation in the geometry of the original body. We have examined the problem of a rough cavity and the case of an elliptic cross-section of the hole where, in both cases, a perturbational uniaxial tension is applied. In addition to the analytic solution, a numerical solution to the latter problem has also been obtained and the results are in good agreement.

Finally, we have shown how the study can be extended to materials with a strain-energy function which are perturbations of the Mooney-Rivlin form.

The work presented here allows further investigations. We note that the general solution derived for a slab of infinite extent is equally applicable to hollow cylinders that allow a stress distribution on the outer surface. Solutions to other boundary value problems, for a slab or a tube can be derived as special cases of the general solution. Moreover, solutions corresponding to other geometric perturbations of the boundary surfaces of the body may also be given. Further, we can seek specific solutions for materials with a strain-energy function that may be regarded as a perturbation of the Mooney-Rivlin type.

Apart from all these immediate extensions, a parallel problem to the one solved here, where we have assumed that the thickness of the slab is prevented from changing, one might explore the case where the thickness of the slab is allowed to change so that the resultant forces applied on the plane faces are zero. The solution of the problem is expected to be more complicated than that presented here.

APPENDIX I

In evaluating the coefficients $\hat{\alpha}_{2i}$, $\hat{\beta}_{2i}$, $\hat{\gamma}_{2i}$, and $\hat{\gamma}'_{2i}$ the results obtained in double precision (9 decimal digits) are within $\pm 0.1\%$ of the corresponding values obtained in extended precision (16 decimal digits) for all $i=1, 2, \dots, 10000$. This indicates that the recurrence formulas for the coefficients are numerically stable. Upon further investigation it also emerged that $|\hat{\alpha}_{2i}| \leq c/i, \dots, |\hat{\gamma}'_{2i}| \leq c/i$. All series (6.1.36) are of the form

$$\sum_{i=0}^{\infty} i^n g^i a_i$$

where $n=0, 1, 2, 3$, $0 \leq g \leq k^2/r^2 < 1$, $|a_i| \leq c/i$. This yields an error estimation for the truncated sums in the form

$$\left| \sum_{i=l}^{\infty} i^n g^i c_i \right| \leq \begin{cases} \frac{c l^{n-1} g^l (n-1)!}{(1-g)^n}, & n \geq 1 \\ \frac{c l^{n-1} g^l}{1-g}, & n = 0 \end{cases}$$

where c is a constant. Further, we include some tables with truncated sums along with the number of terms taken into account for a 7-digit accuracy.

Table A.I.1

SUB[(J+I)*((K/B)**2J)*GAMMA(2J) I = 0

A/A0	H - 1.00	1.20	1.40	1.60	1.80	2.00	3.00	4.00	5.00	10.00
1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.10	-0.01634	-0.00773	-0.00410	-0.00237	-0.00147	-0.00096	-0.00018	-0.00005	-0.00002	0.00000
1.20	-0.05374	-0.02530	-0.01321	-0.00758	-0.00467	-0.00103	-0.00058	-0.00018	-0.00007	0.00000
1.30	-0.10724	-0.04728	-0.02433	-0.01385	-0.00848	-0.00549	-0.00105	-0.00032	-0.00013	0.00000
1.40	-0.16510	-0.07075	-0.03594	-0.02031	-0.01238	-0.00799	-0.00152	-0.00047	-0.00019	-0.00001
1.50	-0.22586	-0.09411	-0.04725	-0.02655	-0.01612	-0.01038	-0.00196	-0.00061	-0.00024	-0.00001
1.75	-0.37876	-0.14768	-0.07238	-0.04016	-0.02422	-0.01552	-0.00291	-0.00090	-0.00036	-0.00002
2.00	-0.52374	-0.19201	-0.09237	-0.05080	-0.03048	-0.01946	-0.00362	-0.00112	-0.00045	-0.00002

NUMBER OF TERMS REQUIRED FOR 7-DIGITS ACCURACY

1.00	1	1	1	1	1	1	1	1	1	1
1.10	8	6	6	5	5	4	3	3	3	2
1.20	11	9	7	6	6	5	4	3	3	2
1.30	15	11	9	7	7	6	4	4	3	2
1.40	19	13	10	8	7	6	5	4	3	2
1.50	23	14	11	9	8	7	5	4	3	2
1.75	34	18	13	10	9	8	5	4	4	3
2.00	47	21	14	11	9	8	5	4	4	3

Table A.I.2

$SUM((J**I) * ((K/M)**2J) * GAMMA(2J)) \quad I = J$

K/M	1.00	1.20	1.40	1.60	1.80	2.00	3.00	4.00	5.00	10.00
1.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.10	-0.14888	-0.07268	-0.03681	-0.02074	-0.01261	-0.00812	-0.00453	-0.00087	-0.00019	-0.00001
1.20	-0.73315	-0.27620	-0.13104	-0.07111	-0.04222	-0.02676	-0.00888	-0.00150	-0.00061	-0.00003
1.30	-1.42773	-0.59304	-0.26371	-0.13823	-0.08036	-0.05023	-0.00891	-0.00272	-0.00109	-0.00006
1.40	-3.65366	-1.00811	-0.42096	-0.21374	-0.12196	-0.07532	-0.01303	-0.00395	-0.00158	-0.00009
1.50	-6.48349	-1.50687	-0.59255	-0.29230	-0.16407	-0.10028	-0.01700	-0.00512	-0.00205	-0.00012
1.75	-20.44720	-3.02666	-1.04004	-0.48296	-0.26230	-0.15710	-0.02560	-0.00762	-0.00304	-0.00018
2.00	-51.42759	-4.77354	-1.46638	-0.65013	-0.34466	-0.20347	-0.03227	-0.00954	-0.00379	-0.00022

NUMBER OF TERMS REQUIRED FOR 7-DIGITS ACCURACY

1.00	1	1	1	1	1	1	1	1	1	1
1.10	12	10	8	7	7	6	5	4	4	3
1.20	20	14	12	10	9	8	6	5	4	3
1.30	28	18	14	12	10	9	6	5	4	3
1.40	36	22	16	13	11	10	7	5	5	3
1.50	45	26	18	15	12	11	7	6	5	3
1.75	73	34	22	17	14	12	8	6	5	3
2.00	105	40	25	19	15	13	8	6	5	4

APPENDIX II

It can be shown that the physical components of the strain tensor γ_{ij} are given by

$$\gamma_{rr} = \frac{\partial u}{\partial r} - \frac{1}{2} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 \right],$$

$$\gamma_{\theta\theta} = -\frac{1}{r^2} \left\{ \frac{\partial u}{\partial r} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} - v \right)^2 \right] \right\},$$

$$\gamma_{r\theta} = \frac{1}{2r} \left[\frac{\partial u}{\partial \theta} - \frac{\partial u}{\partial r} \left(\frac{\partial u}{\partial \theta} - v \right) - \frac{\partial v}{\partial r} \left(\frac{\partial v}{\partial \theta} + u \right) \right].$$

where $u(r, \theta)$ and $v(r, \theta)$ are expressed by relation 6.1.30 and 6.1.31, respectively. The condition

$$|\gamma_{\max}| \ll 1$$

$$\left. \begin{array}{l} \gamma_{\max} \\ \gamma_{\min} \end{array} \right\} = \frac{1}{2} (\gamma_{rr} + \gamma_{\theta\theta}) \pm \left[\gamma_{r\theta}^2 + \frac{1}{2} (\gamma_{rr} - \gamma_{\theta\theta})^2 \right]^{1/2}$$

yields to a limitation on the admissible axial tension applied at infinity. Although the value $P/8c$ does not have any qualitative influence on the incremental deformation field as it merely plays the role of a scale factor for both u and v and neither does have an impact on the stress concentration effect, it gives however the upper limit for the axial loading such that the theory of small deformations superposed on finite deformations remains valid for the specified range $1 \leq a/a_0 \leq 2$.

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