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A STUDY OF UNIFORM BOUNDEDNESS

by

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ABSTRACT

The purpose of this thesis is to review some of the theorems concerning boundedness of linear operators and vector valued measures. Applications in the theory of topological vector spaces and summability are also discussed.

Chapter 1 is of introductory nature. In Chapter 2, by introducing the notion of K boundedness, we obtain a version of the uniform boundedness theorem which is valid for any arbitrary topological vector space. In Chapter 3 we employ a simple version of Rosenthal's lemma to give a proof of a result which is due to J. Diestel and B. Faires. We also establish the Vitali-Hahn-Saks-Nikodym theorem for a new class of rings of sets, namely the class of rings with property (QI). Among the other results obtained in this Chapter are generalized versions of the Phillips and Schur lemmas. In Chapter 4 the Nikodym Boundedness Theorem is proved in several settings. At the end of this Chapter we obtain an improvement of the Orlicz-Pettis theorem.

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CHAPTER 1

PRELIMINARIES

§1. Introduction.

There are several results concerning continuous linear functions and vector valued measures which derive a conclusion of uniform boundedness from a hypothesis concerning pointwise or setwise boundedness. Such results play a significant role in the theory of topological vector spaces, summability and integration. The purpose of this thesis is to review and discuss some of these results and their immediate applications in the theory of topological vector spaces and summability. Of particular interest are the following results.

1. The uniform boundedness principle for continuous linear functions.
2. The Vitali-Hahn-Saks-Nikodym theorem for finitely additive vector measures.
3. The Nikodym Boundedness theorem for finitely additive vector measures.

We prove each of the above results in a more general setting. In proving (1) and (2) we use primitive sliding hump arguments of the type originally used by Lebesgue, Hahn and Nikodym. In fact Baire category methods seem to be unsuitable here.

The conclusion of the Vitali-Hahn-Saks-Nikodym theorem is stronger than of the Nikodym Boundedness theorem. This indicates a possible existence of a more general type of ring than those for which the Nikodym Boundedness theorem holds. In Chapter 4 we introduce such a class of rings, namely the class of PQσ-rings. Interestingly this class contains the ring of ordinary density zero subsets of positive integers. Also we prove the Nikodym boundedness theorem for a ring generated by a full family (3.4 Definition 4) provided measures concerned are regular over finite sets (3.4 Definition 2).

In the remaining sections of this Chapter we list some results from the theory of topological vector spaces we are going to make use of in the next three chapters. All results (except results in section 5) are stated without proof and can be found in one of [4] [13] and [19].

§2. Topological vector spaces.

The following notation will be used throughout this thesis.

R - set of real numbers.

$\mathbb{C}$ - set of complex numbers.

N - set of positive integers.

$R+$ - set of non-negative real numbers.

2^X - power set of a given set X.

Definition 1. A subset A of a vector space X is said to be (i) absorbing if for each x in X there exists a scalar α with $x \in \alpha A$; (ii) balanced if $\lambda A \subseteq A$ for every λ with $|\lambda| \leq 1$; (iii) convex if for each pair $x,y \in A$, $\{\alpha x + (1-\alpha)y | 0 \leq \alpha \leq 1\} \subseteq A$; and (iv) absolutely convex if A is balanced and convex.

Definition 2. A vector space X with a topology T, which we write as (X,T), is called a topological vector space if the operations of vector addition and scalar multiplication are continuous.

Proposition 1. A vector space X with a topology T is a topological vector space if and only if there exists a fundamental neighbourhood system $\eta(0)$ at the origin of X such that:

(1) Each U in $\eta(0)$ is absorbing and balanced.

(2) For each U in $\eta(0)$ there exists V in $\eta(0)$ with $V + V \subseteq U$.

Definition 3. A topological vector space (X,T) is locally convex in case there exists a fundamental neighbourhood system $\eta(0)$ at the origin of X satisfying, in addition to condition (2) of Proposition 1, the condition

(1') Each U in $\eta(0)$ is absorbing, balanced, and convex.

Definition 4. Let X be a vector space. A function $p: X \to R+$ is called an F-seminorm, provided

(1) $p(0) = 0$.

(2) $p(x+y) \leq p(x)+p(y)$.

(3) $p(\lambda x) \leq p(x)$ whenever $|\lambda| \leq 1$, $x \in X$.

(4) $\lim_n p(\alpha_n x) = 0$ whenever (α_n) is a sequence of scalars with $\lim_n \alpha_n = 0$ and $x \in X$.

If, in addition, $p(x) > 0$ for every $x \neq 0$, then p is called an F-norm on X .

Proposition 2. A vector space X with a topology T is a topological vector space if and only if there exists a family F of F-seminorms on X generating the topology T on X ; also, T is Hausdorff if and only if F is total. i.e., for $x \in X$, $x \neq 0$, there exists $p \in F$ such that $p(x) \neq 0$.

Definition 5. Let X be a vector space. A function $p: X \to R+$ is called a seminorm, provided

(1) $p(0) = 0$.

(2) $p(x+y) \leq p(x)+p(y)$.

(3) $p(\lambda x) = \lambda p(x)$ for every scalar λ and $x \in X$.

If, in addition, $p(x) > 0$ for every $x \neq 0$, then p is called a norm on X.

Remark. A vector space X with a topology T generated by a seminorm (respectively, norm) on X is called a seminormed (respectively, normed) space. A complete normed space is called a Banach space.

Proposition 3. (X,T) is a locally convex topological vector space if and only if T is generated by a family of seminorms on X.

Definition 6. A subset B of a topological vector space X is called bounded if for each neighbourhood U of zero in X there exists $\lambda \in R+$ such that $B \subseteq \lambda U$.

Proposition 4. Let X be a topological vector space and let $A \subseteq X$. Then the following statements are equivalent.

(1) A is bounded.

(2) For every sequence (t_n) of positive numbers with $\lim_n t_n = 0$ and every sequence (x_n) in A, $\lim_n t_n x_n = 0$.

If X is a locally convex space, then statement (1) is also equivalent to (3) A is bounded with respect to each continuous seminorm $\|\ \|$ on X.

Definition 7. If z is any complex number, then sgnz is defined by,

$$\operatorname{sgn} z = \begin{cases} 0 & \text{if } z = 0 . \\ |z|/z & \text{if } z \neq 0 . \end{cases}$$

Remark. $|z| = z . \operatorname{sgn} z$ for any $z \in ¢$.

§3. Continuous linear maps.

Proposition 1. Let (X,T_1) and (Y,T_2) be two topological vector spaces over the same field. The set of all continuous linear functions from X to Y, denoted by $L(X,Y)$, is a vector space with the pointwise addition and the pointwise scalar multiplication.

Proposition 2. Let X,Y be seminormed spaces. A linear function f from X to Y is continuous if there exists $M > 0$ such that $\|f(x)\| \leq M\|x\|$ for every $x \in X$.

Proposition 3. If $\|f\| = \mathrm{Sup}\{\|f(x)\| \mid x \in X, \|x\| \leq 1\}$ for $f \in L(X,Y)$, then $\| \ \|$ is a seminorm on $L(X,Y)$; moreover, it is a norm if Y is a normed space.

Theorem 1. Let f be a continuous linear mapping from a subspace A of a topological vector space X into a complete Hausdorff topological vector space Y; then there exists a unique continuous linear map F from the closure $\bar{A}$ of A into Y such that $F|_A = f$.

Definition 1. Let $\mathcal{F} \subseteq L(X,Y)$ where X,Y are topological vector spaces. Then $\mathcal{F}$ is called

(a) pointwise bounded if $\{f(x) \mid f \in \mathcal{F}\}$ is a bounded subset of Y for each $x \in X$;

(b) uniformly bounded if $\{f(x) \mid f \in \mathcal{F}$ and $x \in A\}$ is a bounded subset of Y for each bounded subset A of X.

If f is a linear function from a topological vector space X into scalars, then f is called a linear functional. The set of all continuous linear functionals on X, which is denoted by X^*, is called the dual space of X. The topology generated on X by X^* is called the weak topology of X. In case X is a seminormed space X^* is a Banach space with the norm defined in Proposition 3; moreover for each $x \in X$ the linear functional $\hat{x}$ on X^*, defined by $\hat{x}(f) = f(x)$, belongs to X^{**}. The locally convex topology generated by $\{\hat{x} | x \in X\}$ on X^* is called the weak* topology on X^*.

Theorem 2. (Hahn-Banach). Let X be a vector space and p a seminorm on X. Suppose f is a linear functional defined on a vector subspace Y of X such that $|f(x)| \leq p(x)$ for every $x \in Y$. Then f can be extended to a linear functional F on X such that $|F(x)| \leq p(x)$ for every $x \in X$.

The following propositions are immediate consequences of the Hahn-Banach theorem.

Proposition 4. Let Y be a closed subspace of a locally convex space X, and $a \in X \setminus Y$. Then there exists $f \in X^*$ such that $f(a) = 1$ and $f(y) = 0$ for $y \in Y$.

Proposition 5. Let X be a seminormed space. If $x \in X$ with $\|x\| \neq 0$, then there exists $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$.

Proposition 6. Let X be a seminormed space. Then for every $x \in X$ $\|x\| = \mathrm{Sup}\{|f(x)| \,|\, f \in X^*, \|f\| \leq 1\}$.

Remark: Proposition 4 implies that if X is a locally convex Hausdorff space, then X^* is total over X. i.e., for each $x \in X$ with $x \neq 0$ there exists $f \in X^*$ such that $f(x) \neq 0$.

Theorem 3. Let X be a normed space and Γ a linear subspace of X^* which is also total. Then Γ is a weak * dense subset of X^*.

Theorem 4. (Banach-Alaoglu). If X is a normed space then the unit disc in X^*, i.e., $\{f \in X^* | \|f\| \le 1\}$, is weak * compact. If, in addition, X is separable, then it is weak * metrizable.

Theorem 5. (Banach-Dieudonne). Let X be a Banach space and S a subspace of X^*. Then S is weak * closed if and only if $\{f \in S | \|f\| \le 1\}$ is weak * closed.

We conclude this section stating some properties of the Banach spaces c_0, ℓ_∞ and ℓ_1. Let ω be the set of all scalar sequences. Then $c_0 = \{(x_n) \in \omega | \lim_n x_n = 0\}$ is a Banach space with the norm $\| \|_\infty$ defined by $\|(x_n)\|_\infty = \text{Sup}\{|x_n| \mid n \in N\}$. $c_{00} = \{(x_n) \in c_0 | x_n = 0$ for all but finitely many $n\}$ is a dense subspace of c_0.

$\ell_\infty = \{(x_n) \in \omega \mid (x_n)$ is bounded$\}$ is a Banach space with the same supremum norm. $m_0 = \{(x_n) \in \ell_\infty \mid \{x_n | n \in N\}$ is finite$\}$ is a dense subspace of ℓ_∞. Also note that c_0 is a closed subspace of ℓ_∞.

$\ell_1 = \{(x_n) \in \omega \mid \sum_{n=1}^{\infty} |x_n| < \infty\}$ is a Banach space with the norm $\| \|_1$, defined by $\|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n|$. c_{00} is a dense subspace of ℓ_1.

Remark: $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$.

§4. Convergence of series.

Definition 1. A formal infinite series $\sum_{i=1}^{\infty} x_i$ in a topological vector space (X,T) is said to be (1) convergent in (X,T) if $(\sum_{i=1}^{n} x_i)_{n \in N}$ converges in (X,T); (ii) weakly convergent in (X,T) if there exists an $x \in X$ such that $\sum_{i=1}^{\infty} f(x_i)$ converges to $f(x)$ for every $f \in X^*$; (ii) subseries convergent if for any increasing sequence (i_n) of positive integers, the series $\sum_{n=1}^{\infty} x_{i_n}$ converges in (X,T), and (iv) unconditionally convergent if for any permutation σ of N, the series $\sum_{i=1}^{\infty} x_{\sigma(i)}$ converges to the same element x in X.

Remark: Let $A \subseteq N$. $\sum_{i \in A} x_i$ converges in (X,T) means that $\lim_n \sum_{i \in A \cap [1,n]} x_i$ exists in (X,T). Therefore $\sum_{i=1}^{\infty} x_i$ is subseries convergent if and only if $\sum_{i \in A} x_i$ converges for every $A \subseteq N$.

Theorem 1. If a series $\sum_{i=1}^{\infty} x_i$ in a locally convex space X is subseries convergent, then it is unconditionally convergent.

The proof of the above theorem can be found in [9].

Theorem 2. Let (x_n) be a Cauchy sequence in a locally convex space X. If (x_n) converges weakly to x in X, then $\lim_n x_n = x$.

§5. Some families of sets.

Definition 1. Let Ω be an arbitrary set. A subfamily $\mathcal{R}$ of 2^{Ω} is said to be a ring in case $A \cup B,\ A \setminus B \in \mathcal{R}$ whenever $A, B \in \mathcal{R}$.

If, in addition, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$ for every sequence (A_i) in $\mathcal{R}$, then $\mathcal{R}$ is called a σ-ring.

Definition 2. A ring $\mathcal{R}$ of subsets of a set Ω is called an algebra if $\Omega \in \mathcal{R}$.

~~Remark: A ring $\mathcal{R}$ is closed under finite intersections.~~

Definition 3. A ring $\mathcal{R}$ is called a $Q\sigma$-ring (respectively, an $FQ\sigma$-ring) in case for every sequence (A_i) of pairwise disjoint sets (respectively, finite sets) in $\mathcal{R}$, there exists a subsequence (A_{i_j}) of (A_i) such that $\bigcup_{j=1}^{\infty} A_{i_j} \in \mathcal{R}$.

Definition 4. A ring $\mathcal{R}$ is called hereditary (or an ideal) if $\mathcal{R}$ is closed under subsets. If $\mathcal{R}$ is an ideal then $\mathcal{F} = \{A^c \mid A \in \mathcal{R}\}$ is called a filter. (Note that filters are closed under supersets.)

Example 1. A $Q\sigma$-ring $\mathcal{R}$ which is not a σ-ring.

Let $\mathcal{F}$ be a non-principal maximal filter of subsets of $\mathbb{N}$. This means that $\cap\mathcal{F} = \phi$ and if there exists a filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathcal{F}' = \mathcal{F}$. The existence of such filters is implied by the Zorn's lemma. Let $\mathcal{R} = \{A^c \mid A \in \mathcal{F}\}$. We claim that $\mathcal{R}$ is an ideal satisfying the following:

(1) $\mathcal{R}$ contains all finite subsets of N.

(2) $\mathcal{R}$ is a Qσ-ring .

(3) $\mathcal{R}$ is not a σ-ring.

By the definition of a filter it is clear that $\mathcal{R}$ is an ideal. Let $A \subseteq N$. Suppose $A, A^c \notin \mathcal{F}$, Since $\mathcal{F}$ is maximal, there exist $E, F \in \mathcal{F}$ such that $A \cap E = \phi$ and $A^c \cap F = \phi$; consequently $E \cap F = \phi$. This contradiction shows that $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. To prove (1) let $n \in N$. Since $\mathcal{F}$ is non-principal $\{n\} \notin \mathcal{F}$. Hence $N \setminus \{n\} \in \mathcal{F}$ so that $\{n\} \in \mathcal{R}$. This shows that every finite set is in $\mathcal{R}$. To prove (2) let (A_n) be a pairwise disjoint sequence of members of $\mathcal{R}$. Then $\bigcup_{n=1}^{\infty} A_{2n} \in \mathcal{F}$ or $(\bigcup_{n=1}^{\infty} A_{2n})^c \in \mathcal{F}$. If $\bigcup_{n=1}^{\infty} A_{2n} \in \mathcal{F}$, then

$$\bigcup_{n=1}^{\infty} A_{2n} \subseteq (\bigcup_{n=1}^{\infty} A_{2n-1})^c \in \mathcal{F} .$$ Therefore $\bigcup_{n=1}^{\infty} A_{2n} \in \mathcal{R}$ or $\bigcup_{n=1}^{\infty} A_{2n-1} \in \mathcal{R}$.

(3) follows from (1) and the fact that $N \notin \mathcal{R}$.

Lemma 1. Let $\mathcal{R}$ be a ring of subsets of a set Ω . Suppose $\alpha: \mathcal{R} \to R+$ is a unbounded function such that:

(1) $\alpha(A \cup B) \leq \alpha(A) + \alpha(B)$ for $A, B \in \mathcal{R}$ with $A \cap B = \phi$.

(2) $\alpha(A \setminus B) \geq |\alpha(A) - \alpha(B)|$ for $A, B \in \mathcal{R}$ with $B \subseteq A$.

Then there exists a disjoint sequence (A_n) of members of $\mathcal{R}$ such that $\lim_n \alpha(A_n) = \infty$.

Proof. Define $\bar{\alpha}: 2^{\Omega} \to [0,\infty]$ such that $\bar{\alpha}(A) = \text{Sup}\{\alpha(B) | B \in \mathcal{R}, B \subseteq A\}$.

Case 1. Suppose there exists a $k > 0$ and an $E \subseteq \Omega$ with $\bar{\alpha}(E) = \infty$ such that for every $F \subseteq E$ with $F \in R$ and $\alpha(F) \geq k$, $\bar{\alpha}(F) = \infty$.

Since $\bar{\alpha}(E) = \infty$, there exists $F_1 \subseteq E$ with $F_1 \in R$ such that $\alpha(F_1) > k$; hence $\bar{\alpha}(F_1) = \infty$. Let $F_2 \subseteq F_1$ with $F_2 \in R$ such that $\alpha(F_2) > \alpha(F_1) + 1$. Note that $\bar{\alpha}(F_2) = \infty$ since $\alpha(F_2) > k$ and $F_2 \subseteq E$; so by induction we can construct a decreasing sequence (F_n) of members of R such that $\alpha(F_{n+1}) > \alpha(F_n) + n$. Set $A_n = F_n \setminus F_{n+1}$. Then $\alpha(A_n) \geq |\alpha(F_n) - \alpha(F_{n+1})|$ by (2) and hence $\alpha(A_n) > n$. This implies that (A_n) is a disjoint sequence of members of R such that $\lim_n \alpha(A_n) = \infty$.

Case 2. Suppose for each $k > 0$ and each $E \subseteq X$ with $\bar{\alpha}(E) = \infty$, there exists $F \subseteq E$ with $E \in R$ and $\alpha(F) \geq k$ such that $\bar{\alpha}(F) < \infty$.

Since $\bar{\alpha}(\Omega) = \infty$, there exists an $A_1 \subseteq \Omega$ with $A_1 \in R$ such that $\alpha(A_1) \geq 1$ and $\bar{\alpha}(A_1) < \infty$. Now we show that $\bar{\alpha}(\Omega \setminus A_1) = \infty$. Let $P \in R$. Then $P = (P \setminus A_1) \cup (P \cap A_1)$ and $P \setminus A_1$, $P \cap A_1$ are disjoint members of R. Hence by 1, $\alpha(P) \leq \alpha(P \setminus A_1) + \alpha(P \cap A_1)$. Taking supremum over $P \in R$ we have

$$\bar{\alpha}(\Omega) = \sup_{p \in R} \alpha(P) \leq \sup_{p \in R} \alpha(P \setminus A_1) + \sup_{p \in R} \alpha(P \cap A_1)$$

$$\leq \bar{\alpha}(\Omega \setminus A_1) + \bar{\alpha}(A_1) .$$

Since $\bar{\alpha}(\Omega) = \infty$ and $\bar{\alpha}(A_1) < \infty$, this implies $\bar{\alpha}(\Omega \setminus A_1) = \infty$.

Choose $A_2 \subseteq \Omega \setminus A_1$ with $A_2 \in R$ such that $\alpha(A_2) \geq 2$ and $\bar{\alpha}(A_2) < \infty$. Using the same argument we can show that $\bar{\alpha}(\Omega \setminus A_1 \cup A_2) < \infty$; so by induction we can construct a disjoint sequence (A_n) of members of R such that $\alpha(A_n) \geq n$ for $n \in N$. Hence $\lim_n \alpha(A_n) = \infty$.

Definition 5. An algebra R of subsets of a set Ω is said to have the interpolation property in case for every pair of sequences (A_n), (B_n) of members of R such that $A_n \subseteq B_m$ for $n,m \in N$, there exists $C \in R$ such that $A_n \subseteq C \subseteq B_m$ for $m,n \in N$.

CHAPTER 2

THE UNIFORM BOUNDEDNESS PRINCIPLE

§1. Introduction.

The material in this chapter is essentially contained in the Antosik-Swartz paper [2] with the exception of corollary 1 of theorem 1 in section 3 which is a generalization of the Banach-Steinhaus theorem. The uniform boundedness principle, one of the most important theorems in Functional Analysis, is a result which derives a conclusion of uniform boundedness from a hypothesis concerning pointwise boundedness. In proving this theorem our use of a matrix method in place of the Baire category theorem paves the way for some generalization of the classical version of the theorem. As a preliminary, in section 2, we obtain a result concerning infinite matrices in a topological vector space which is somewhat in the spirit of the Antosik-Mikusinski diagonal theorem [1]. By introducing the notion of a K-bounded set, we obtain an analogous statement of the uniform boundedness theorem which is valid for any arbitrary topological vector space.

§2. Basic facts.

We start with a lemma which can be viewed as an elementary sliding hump type argument.

Lemma 1. Let (λ_{mn}) be an arbitrary infinite matrix of positive numbers. Suppose (x_{mn}) is a given infinite matrix of non-negative numbers such that $\lim_m x_{mn} = 0$ for each n and $\lim_n x_{mn} = 0$ for each m. Then there exists a subsequence (m_i) of positive integers such that $x_{m_i m_j} < \lambda_{ij}$ for $i \neq j$.

Proof. Set $m_1 = 1$. Suppose $m_1, m_2, \ldots, m_n$ have been chosen such that $x_{m_i m_j} < \lambda_{ij}$ for $i,j = 1,2,\ldots,n$ and $i \neq j$. Since $\lim_p x_{m_i p} = 0$ and $\lim_p x_{p m_i} = 0$ for $i = 1,2,\ldots,n$, we can choose $m_{n+1} > m_n$ such that $x_{m_i m_{n+1}} < \lambda_{i,n+1}$ and $x_{m_{n+1} m_i} < \lambda_{n+1,i}$ for $i = 1,2,\ldots,n$. By induction the result follows.

We use the above lemma to obtain our main result in this section.

Theorem 1. Let (x_{mn}) be an infinite matrix in a topological vector space X. Suppose (i) $\lim_m x_{mn} = 0$ for each n and (ii) each subsequence (n_j) of positive integers has a subsequence (n_{j_k}) such that $\lim_m \sum_{k=1}^{\infty}{}' x_{mn_{j_k}} = 0$. Then $\lim_m x_{mm} = 0$.

Proof. Since every topological vector space X is generated by the set of all continuous F-seminorms on X , it is sufficient to consider the case when X is an F-seminormed space. We show that $(x_{mm})_{m \in N}$ has a subsequence which converges to zero. Since the same argument can be applied to an arbitrary subsequence of $(x_{mm})_{m \in N}$, we will have that $\lim_m x_{mm} = 0$.

Let (λ_{ij}) be an infinite matrix of positive numbers such that $\sum_{i,j} \lambda_{ij} < \infty$. Condition (ii) implies that $\lim_n x_{mn} = 0$ for each m . Thus, by lemma 1, there exists a subsequence (n_i) of positive integers such that $\|x_{n_i n_j}\| < \lambda_{ij}$ for $i \neq j$. ($\| \ \|$ denotes the F-seminorm.) To avoid double subscripts assume $n_i = i$. Let (i_k) be the subsequence satisfying the conclusion of condition (ii). Then for every $k \in N$,

$$\|x_{i_k i_k}\| = \| \sum_{\ell=1}^{\infty} x_{i_k i_\ell} - \sum_{\substack{\ell=1 \\ \ell \neq k}}^{\infty} x_{i_k i_\ell} \|$$

$$\leq \| \sum_{\ell=1}^{\infty} x_{i_k i_\ell} \| + \sum_{\ell=1}^{\infty} \lambda_{i_k i_\ell} .$$

Now note that $\lim_k \| \sum_{\ell=1}^{\infty} x_{i_k i_\ell} \| = 0$ by (ii), and $\lim_k \sum_{\ell=1}^{\infty} \lambda_{i_k i_\ell} = 0$ by the fact that $\sum_{i,j} \lambda_{ij} < \infty$. Hence $\lim_k x_{i_k i_k} = 0$. This completes the proof.

Remark. In the above theorem it can be concluded that $\lim_n x_{mn} = 0$ uniformly for $m \in N$.

To verify this let (m_i) and (n_i) be two subsequences of positive integers. It is readily seen that the matrix $(x_{m_i n_j})$ satisfies conditions (i) and (ii). An application of theorem 1 shows that $\lim_i x_{m_i n_i} = 0$. This shows that $\lim_n x_{mn} = 0$ uniformly for $m \in N$.

§3. Main results.

The classical uniform boundedness theorem states that a pointwise bounded family of continuous linear operators on a Banach space is uniformly bounded on bounded subsets. By introducing the notion of a K-bounded set we give an analogous statement of the uniform boundedness theorem which is valid for arbitrary topological vector spaces.

Definition 1. Let B be a subset of a topological vector space X. B is said to be K-bounded if for each sequence (x_n) of elements of B and each sequence of scalars (t_n) which converges to zero, the sequence $(t_n x_n)$ has a subsequence $(t_{n_i} x_{n_i})$ such that $\sum_{i=1}^{\infty} t_{n_i} x_{n_i} \in X$.

Remark. It is easy to see that every K-bounded set is bounded. An example of a bounded subset of a normed space, which is not K-bounded, is given at the end of this section.

Definition 2. A topological vector space X is said to be a (K)-space if every bounded subset of X is K-bounded.

Proposition 1. Let X be an F-seminormed space. Then X is a (K)-space if and only if each sequence (x_n) in X, which converges to zero, has a subsequence (x_{n_i}) such that $\sum_{i=1}^{\infty} x_{n_i} \in X$.

Proof. To prove the necessary part suppose X is a (K)-space. Let (x_n) be a sequence in X with $\lim_n x_n = 0$. First we show that there exists a sequence (t_n) of positive numbers, which diverges to infinity, such that $\lim_n t_n x_n = 0$.

Let $(V_n)_{n \in N}$ be a local base at zero in X with $V_{n+1} \subseteq V_n$ for $n \in N$. Since $\lim_n x_n = 0$, we can construct a subsequence (n_i) of positive integers such that $x_n \in \frac{1}{i} V_i$ for $n \geq n_i$. Define the sequence (t_n) by,

$$t_n = \begin{cases} 1 & \text{if } 1 \leq n < n_1 \\ i & \text{if } n_i \leq n < n_{i+1} . \end{cases}$$

It is easy to check that $\lim_n t_n = \infty$ and that $\lim_n t_n x_n = 0$. Hence $\{t_n x_n | n \in N\}$ is bounded. Since X is a (K)-space, $\{t_n x_n | n \in N\}$ is K-bounded and hence there exists a subsequence (n_i) of positive integers such that $\sum_{i=1}^{\infty} \frac{1}{t_{n_i}} (t_{n_i} x_{n_i}) \in X$. i.e., $\sum_{i=1}^{\infty} x_{n_i} \in X$.

The sufficient part can be easily checked.

Corollary 1. Every complete F-seminormed space X is a (K)-space.

Proof. Let (x_n) be a sequence in X with $\lim_n x_n = 0$. Choose a subsequence (x_{n_i}) of (x_n) such that $\sum_{i=1}^{\infty} \|x_{n_i}\| < \infty$. Clearly $\sum_{i=1}^{\infty} x_{n_i}$ satisfies the Cauchy condition. Since X is complete, $\sum_{i=1}^{\infty} x_{n_i} \in X$. Hence the previous proposition implies that X is a (K)-space.

Theorem 1. (The uniform boundedness principle.)

Let $\mathcal{F}$ be a family of pointwise bounded continuous linear functions of a topological vector space X to a topological vector space Y. Then $\mathcal{F}$ is uniformly bounded on every K-bounded subset A of X.

Proof. Let $B = \{f(x) \mid f \in \mathcal{F}$ and $x \in A\}$. We want to show that B is a bounded subset of Y. Let $(f_n(x_n))_{n \in N}$ be a sequence in B and (t_n) a sequence of positive numbers with $\lim_n t_n = 0$.

Set $a_{nm} = t_n^{\frac{1}{2}} f_n(t_m^{\frac{1}{2}} x_m)$ for $n,m = 1,2,3,\ldots$. Since the sequence (f_n) of continuous linear functions is pointwise bounded and $\lim_n t_n^{\frac{1}{2}} = 0$,

$$(1) \quad \lim_n a_{nm} = \lim_n t_n^{\frac{1}{2}} f_n(t_m^{\frac{1}{2}} x_m) = 0 \quad \text{for } m = 1,2,\ldots .$$

Since A is K-bounded, each subsequence (m_i) of (m) has a subsequence (m_{i_j}) such that $\sum_{j=1}^{\infty} t_{m_{i_j}}^{\frac{1}{2}} x_{m_{i_j}} \in X$. Again by the facts that (f_n) is pointwise bounded and $\lim_n t_n^{\frac{1}{2}} = 0$ we have

$$(2) \quad \lim_n \sum_{j=1}^{\infty} a_{nm_{i_j}} = \lim_n t_n^{\frac{1}{2}} f_n\left(\sum_{j=1}^{\infty} t_{m_{i_j}}^{\frac{1}{2}} x_{m_{i_j}}\right) = 0 .$$

Therefore theorem 1 of the previous section implies that

$\lim_n t_n f_n(x_n) = \lim_n a_{nn} = 0$. This completes the proof.

Remark. If X is a (K)-space, then F is uniformly bounded on every bounded subset of X.

Corollary 1. (Banach-Steinhaus).

Let (f_n) be a sequence of continuous linear functions from an F-seminormed (K)-space X to a Hausdorff topological vector space Y. If $\lim_n f_n(x) = f(x)$ exists for every $x \in X$, then f is a continuous linear function from X to Y. Moreover, this convergence is uniform on every compact subset of X.

Proof. Since Y is Hausdorff, $f: X \to Y$ is well defined and evidently it is linear. First we show that f is continuous. Since X is first countable it suffices to show that for each sequence (x_n) in X with $\lim_n x_n = 0$, $\lim_n f(x_n) = 0$.

Construct, as in proposition 1, a sequence (t_n) of positive numbers with $\lim_n t_n = \infty$ such that $\lim_n t_n x_n = 0$. $\{t_n x_n | n \in N\}$ is a bounded subset of X and, moreover, since $\lim_n f_n(x) = f(x)$ for each $x \in X$, the sequence (f_n) is pointwise bounded. Therefore theorem 1 implies that $M = \{f_n(t_m x_m) | n,m \in N\}$ is a bounded subset of Y. Since $\lim_m f_m(t_n x_n) = f(t_n x_n)$ for each $n \in N$, $(f(t_n x_n))_{n \in N}$ is a sequence in $\bar{M}$ and moreover $\bar{M}$ is bounded. Hence $\lim_n f(x_n) = \lim_n \frac{1}{t_n} f(t_n x_n) = 0$.

Now we show this convergence is uniform on every compact subset K of X. Suppose $(f_n - f)_{n \in N}$ does not converge to zero uniformly on K. Then there exists a subsequence (n_i) of (n), a sequence (x_i) in K and a neighbourhood U of zero in Y such that

(1) $(f_{n_i} - f)(x_i) \notin U$ for $i = 1,2,\ldots$.

Since X is first countable, K is sequentially compact and hence, perhaps by passing to a subsequence, we may take (x_i) converging to a point x in K.

Set $a_{ij} = (f_{n_i} - f)(x_j - x)$. The pointwise convergence of (f_n) to f implies that (2) $\lim_i a_{ij} = 0$ for $j = 1,2,\ldots$.

Since X is an F-seminormed (K)-space, proposition 1 implies that every subsequence $(x_{j_k} - x)_{k \in N}$ of $(x_j - x)_{j \in N}$ has a subsequence $(x_{j_{k_\ell}} - x)_{\ell \in N}$ such that $\sum_{\ell=1}^{\infty} (x_{j_{k_\ell}} - x) \in X$ and hence

(3) $\lim_i \sum_{\ell=1}^{\infty} a_{ij_{k_\ell}} = \lim_i (f_{n_i} - f)(\sum_{\ell=1}^{\infty} x_{j_{k_\ell}} - x) = 0$.

Hence theorem 1 of the previous section implies that

$\lim_i (f_{n_i} - f)(x_i - x) = \lim_i a_{ii} = 0$. Since $\lim_i (f_{n_i} - f)(x) = 0$, we have $\lim_i (f_{n_i} - f)(x_i) = 0$ which is a contradiction to (1).

Therefore (f_n) converges to f uniformly on K.

Corollary 2. Let X be an F-seminormed space, Y an F-seminormed (K)-space, and Z a Hausdorff topological vector space. If the bilinear map $F: X \times Y \to Z$ is separately continuous, then F is jointly continuous.

Proof. Since $X \times Y$ is first countable, it suffices to show that $(F(x_n, y_n))_{n \in N}$ converges to zero whenever (x_n) and (y_n) converge to zero in X and Y respectively. Consider the sequence (f_n) of continuous linear functions of Y to Z given by $f_n(y) = F(x_n, y)$ for each n. The separate continuity of F implies that $\lim_n f_n(y) = 0$ for every y in Y. Since $\{0, y_1, y_2, \ldots\ldots\}$ is a sequentially compact subset of X, the last corollary implies that $\lim_n f_n(y) = 0$ uniformly on $\{0, y_1, y_2, \ldots\ldots\}$ and hence $\lim_n F(x_n y_n) = \lim_n f_n(y_n) = 0$.

Corollary 3. If E is a subset of a seminormed space X such that $f(E)$ is bounded for every $f \in X^*$, then E is bounded.

Proof. X^* is a Banach space with the usual norm topology. Since $f(E)$ is bounded for each $f \in X^*$, $\hat{E} = \{\hat{x} | x \in E\}$ is a family of point-wise bounded continuous linear functions on X^*. Therefore theorem 1 implies that $\sup\{|\hat{x}(f)| \,|\, \hat{x} \in \hat{E}, f \in X^*, \|f\| \leq 1\} < \infty$. Since for each $x \in X$ $\|x\| = \sup\{|f(x)| \,|\, f \in X^*, \|f\| \leq 1\}$, this implies that $\sup\{\|x\| \,|\, x \in E\} < \infty$.

The above results are usually derived by means of the Baire Category theorem (see [18]). The assumption of completeness or barreledness is needed there. The following is an example of a normed space for which the uniform boundedness principle does not hold.

Let c_{00} be the space of real sequences (t_n) such that $t_n = 0$ eventually and equip c_{00} with the sup norm. The dual of c_{00} is then ℓ_1. Let e_n be the real sequence which has value 1 in the n^{th} place and zero elsewhere. Then $(ne_n)_{n \in N}$ is pointwise bounded on c_{00} but is not norm bounded.

Note that the set $\{e_n | n \in N\}$ is bounded but is not K-bounded. Also note that c_{00} is neither complete nor a (K)-space. An interesting but complicated example of a non-complete normed (K)- space is given in [10]. The following is a simple example of non-complete (K)-space.

Let X be a Banach space. We show that X with the weak topology is a (K)-space. Let $A \subseteq X$ be weakly bounded. The last corollary implies that A is bounded and hence A is K-bounded by the fact that every Banach space is a (K)-space. This shows that X with the weak topology is a (K)-space. But in general this space is not complete.

CHAPTER 3

BOUNDED VECTOR MEASURES

§1. Introduction.

The theory of vector measures, in addition to its major role in integration theory, is also important in some areas of functional analysis and summability. In this chapter we study this secondary role of vector measure theory. In doing so our strategy is to begin with some basic set theoretic manipulations. This is in fact necessary because a number of fundamental theorems of vector measure theory are based on the set theoretic structure of the corresponding domain space, which is generally a ring of subsets of a given set Ω. In order to generalize some important results, we define vector measures taking values in an arbitrary topological vector space instead of a Banach space. Since every topological vector space is generated by a class of F-seminorms, in most cases the results obtained for F-seminormed spaces can be readily generalized to topological vector spaces.

In section 2 we obtain some basic straightforward properties of vector measures. Section 3 is started with a simple version of the Rosenthal's lemma. We use this lemma to establish a structural link between the Banach spaces c_0, ℓ_∞ and bounded vector measures. This in turn becomes a powerful tool to obtain some important results concerning topological vector spaces, including a generalization of the Orlicz-Pettis theorem for locally convex spaces. The materials in sections 2 and 3, although generalized to some extent, are essentially contained in Mathematical surveys - number 15 by J. Diestel and J.J. Uhl, Jr. [6]. Section 4 deals with convergence and boundedness of sequences of vector measures. The Vitali-Hahn-Saks-Nikodym theorem, which is proved in a more general setting, plays a vital role in this section. At the end of this section we introduce the notion of full classes and discuss several applications of the previous results in matrix summability.

§2. Elementary properties of vector measures.

As the title indicates, we group in this section all basic properties of vector measures which follow directly from definitions.

Definition 1. Let $\mathcal{R}$ be a ring of subsets of a set Ω and X a topological vector space. A function $\mu: \mathcal{R} \to X$ is called a vector measure if $\mu(E \cup F) = \mu(E) + \mu(F)$ for every $E, F \in \mathcal{R}$ with $E \cap F = \phi$. If, in addition, $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for every sequence (E_n) of pairwise disjoint members of $\mathcal{R}$ with $\bigcup_{n=1}^{\infty} E_n \in \mathcal{R}$, then μ is called countably additive. Moreover, if $\{\mu(E) | E \in \mathcal{R}\}$ is a bounded subset of X, then μ is called bounded.

In what follows, unless otherwise stated, $\mathcal{R}$ denotes a ring of subsets of a set Ω and X denotes a seminormed space.

Definition 2. Let $\mu: \mathcal{R} \to X$ be a vector measure. The variation of μ is the extended non-negative function $|\mu|$ whose value on a set $E \subseteq \Omega$ is given by,

$$|\mu|(E) = \sup\{\sum_{i=1}^{n} \|\mu(E_i)\| \,|\, n \in N,\ E_1, E_2, \ldots, E_n \text{ are pairwise disjoint members of } \mathcal{R} \text{ such that } \bigcup_{i=1}^{n} E_i \subseteq E\}.$$

If $|\mu|(\Omega) < \infty$, then μ is called a measure of bounded variation.

The semivariation of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $E \subseteq \Omega$ is given by,

$\|\mu\|(E) = \text{Sup}\{|x^*\mu|(E) \mid x^* \in X^*, \|x^*\| \le 1\}$, where $|x^*\mu|$ is the variation of the real valued measure $x^*\mu$. If $\|\mu\|(\Omega) < \infty$, then μ is called a measure of bounded semivariation.

The following proposition is stated without a proof since its verification involves only simple computations.

Proposition 1. a. $|\mu|(E) \ge \|\mu\|(E)$ for every $E \subseteq \Omega$ and $\|\mu\|(E) \ge \|\mu(E)\|$ for every $E \in \mathcal{R}$.

b. $|\mu|$ is finitely additive on $\mathcal{R}$ and $\|\mu\|$ is finitely subadditive on $\mathcal{R}$.

c. $|\mu|$ and $\|\mu\|$ are both monotone, i.e., $|\mu|(E) \le |\mu|(F)$ and $\|\mu\|(E) \le \|\mu\|(F)$ for $E \subseteq F \subseteq \Omega$.

We use the following lemma to obtain a few other results.

Lemma 1. If W is a finite set of complex numbers, then there exists $V \subseteq W$ such that $\sum_{z \in W} |z| \le 8 \left| \sum_{z \in V} z \right|$.

Proof. Divide W into four disjoint sets taking intersection with each quadrant of the complex plane. For at least one of these sets, call it V, we have

$$\sum_{z \in W} |z| \le 4 \sum_{z \in V} |z|$$

$$\le 4 \sum_{z \in V} (|\text{Re}z| + |\text{Im}z|)$$

$$= 4[|\sum_{z\in V} \operatorname{Re} z| + |\sum_{z\in V} \operatorname{Im} z|] .$$

(The last equality follows from the fact that all $z \in V$ are in the same quadrant.)

$$= 4[|\operatorname{Re} \sum_{z\in V} z| + |\operatorname{Im} \sum_{z\in V} z|]$$

$$\leq 4[|\sum_{z\in V} z| + |\sum_{z\in V} z|] = 8|\sum_{z\in V} z| .$$

Remark. As a direct consequence of this lemma, we have the following.

If $\operatorname{Sup}\{|\sum_{i\in F} z_i| \,|\, F$ is a finite subset of $N\} < \infty$, then $\sum_{i=1}^{\infty} |z_i| < \infty$.

Proposition 2. A vector measure $\mu: R \to X$ is of bounded semivariation if and only if μ is bounded.

Proof. Let $x^* \in X^*$, $\|x^*\| \leq 1$ and let $E_1, E_2, \ldots, E_n$ be pairwise disjoint members of R. Then the above lemma implies that there exists $V \subseteq \{1,2,\ldots,n\}$ such that

$$\sum_{i=1}^{n} |x^*\mu(E_i)| \leq 8|\sum_{i\in V} x^*\mu(E_i)|$$

$$= 8|x^*\mu(\bigcup_{i\in V} E_i)|$$

$$\leq 8|\mu(\bigcup_{i\in V} E_i)| .$$

Consequently, if μ is bounded, then μ is of bounded semivariation. The converse is obvious.

Remark. In view of Proposition 2 a vector measure of bounded semivariation is also called a bounded vector measure.

§3. Strongly bounded vector measures.

One obvious property of a countably additive vector measure μ defined on a σ-ring $\mathcal{R}$ is that if (E_n) is a sequence of pairwise disjoint members of $\mathcal{R}$, then $\sum_{n=1}^{\infty} \mu(E_n)$ is subseries (and unconditionally) convergent. Nonetheless this property is shared by many noncountably additive vector measures. For instance, every bounded scalar measure has this property. On the other hand the vector measure $\nu: \mathcal{A} \to c_0$, where $\mathcal{A}$ is the family of all finite subsets of N, defined by $\nu(A) = \chi_A$ is a bounded vector measure not satisfying the above property. Because of its importance in theory of vector measures we single out this property.

Definition 1. Let $\mathcal{R}$ be a ring of subsets of a set Ω and X a topological vector space. A vector measure $\mu: \mathcal{R} \to X$ is called:

(i) strongly additive in case $\sum_{n=1}^{\infty} \mu(E_n)$ converges for each sequence (E_n) of pairwise disjoint members of $\mathcal{R}$.

(ii) strongly bounded in case $\lim_n \mu(E_n) = 0$ for each sequence (E_n) of pairwise disjoint members of $\mathcal{R}$.

Proposition 1. Let $\mu: \mathcal{R} \to X$ be a vector measure.

(a) Suppose X is a locally convex space. Then if μ is strongly bounded, μ is bounded.

(b) If X is sequentially complete then statements (i) and (ii) are equivalent.

Proof. (a) Let $\|\ \|$ be a continuous seminorm on X. It is sufficient to show that μ is bounded with respect to $\|\ \|$. Define $\alpha: R \to R+$ by $\alpha(E) = \|\mu(E)\|$. Let $A, B \in R$. If $A \cap B = \phi$, then

(1) $\alpha(A \cup B) = \|\mu(A \cup B)\| \leq \|\mu(A)\| + \|\mu(B)\| = \alpha(A) + \alpha(B)$.

If $A \subseteq B$ then,

(2) $\alpha(B \setminus A) = \|\mu(B \setminus A)\| = \|\mu(B) - \mu(A)\| \geq \left|\|\mu(B)\| - \|\mu(A)\|\right|$

$= |\alpha(B) - \alpha(A)|$.

Since $\lim_n \alpha(E_n) = 0$ for each disjoint sequence (E_n) in R, 1.5 lemma 1 implies that α is bounded, i.e., μ is bounded.

(b) (i) always implies (ii). To show that (ii) implies (i) let (E_n) be a sequence of pairwise disjoint members of R. Suppose $\sum_{n=1}^{\infty} \mu(E_n)$ does not satisfy the Cauchy condition. Then there exists an increasing sequence (n_i) of positive integers such that $\lim_i \sum_{j=n_i}^{n_{i+1}-1} \mu(E_j) \neq 0$. Set $F_i = \bigcup_{j=n_i}^{n_{i+1}-1} E_j$. Then (F_i) is a sequence of pairwise disjoint members of R with $\lim_i \mu(F_i) \neq 0$. This contradiction shows that $\sum_{n=1}^{\infty} \mu(E_n)$ satisfies the Cauchy condition and hence $\sum_{n=1}^{\infty} \mu(E_n)$ is convergent.

Remark 1. Suppose X is a locally convex space. If X is weakly sequentially complete then (i) and (ii) are equivalent by virtue of 1.4. theorem 2.

2. In statement (i) the convergence of $\sum_{n=1}^{\infty} \mu(E_n)$ is subseries and unconditional.

3. The set of all X valued strongly bounded vector measures defined on R forms a linear space.

The following definition extends the earlier one to a sequence of bounded vector measures.

Definition 2. Let R be a ring of subsets of a set Ω and X a topological vector space. Further let $\mu_n: R \to X$ be a bounded vector measure for each $n \in N$. Then the sequence (μ_n) is called:

(i) uniformly strongly additive in case for any sequence (E_n) of pairwise disjoint members of R, $\sum_{n=1}^{\infty} \mu_m(E_n)$ converges uniformly for $m \in N$.

(ii) uniformly strongly bounded in case for any sequence (E_n) of pairwise disjoint members of R, $\lim_n \mu_m(E_n) = 0$ uniformly for $m \in N$.

Proposition 2. If X is sequentially complete then (i) and (ii) are equivalent.

Proof. Follow the proof of part (b) of proposition 1.

We need the following simplified version of the Rosenthal's lemma [14] to establish our main theorem of this section. Although the proof of this lemma is simple it represents one of the most important results in measure theory.

Lemma 1. Let (μ_n) be a sequence of uniformly bounded nonnegative real-valued measures defined on 2^N-the power set of positive integers. Then for each $\varepsilon > 0$, there exists an infinite subset P of N such that $\mu_p(P \setminus \{p\}) < \varepsilon$ for every $p \in P$.

Proof. Let $\varepsilon > 0$. Partition N into a sequence (M_n) of pairwise disjoint infinite subsets of N. If there exists $n \in N$ such that $\mu_p(M_n \setminus \{p\}) < \varepsilon$ for every $p \in M_n$, our goal is achieved by setting $M_n = P$. Suppose for each n, there exists $p_n \in M_n$ such that

$$(1) \quad \mu_{p_n}(M_n \setminus \{p_n\}) \geq \varepsilon .$$

Let $P_1 = \{p_n | n \in N\}$. Then $P_1 \cap (M_n \setminus \{p_n\}) = \phi$ for $n = 1,2,\ldots$ and hence

$$(2) \quad \mu_{p_n}(P_1) + \mu_{p_n}(M_n \setminus \{p_n\} = \mu_{p_n}(P_1 \cup (M_n \setminus \{p_n\})) \leq M , \text{ where}$$

$M = \operatorname{Sup}\{\mu_n(E) | n \in N, E \subseteq N\}$. By (1) and (2) we have

$$(3) \quad \mu_{p_n}(P_1) \leq M - \varepsilon \quad \text{for} \quad n = 1,2,\ldots .$$

Next apply the same argument to $(\mu_{p_n})_{n\in N}$ and P_1. If the process does not stop, there is an infinite subset P_2 of P_1 such that (4) $\mu_p(P_2) \le M - 2\varepsilon$ for every $p \in P_2$.

Thus the process must stop before n iterations where n is the smallest positive integer such that $M-n\varepsilon < 0$. This completes the proof.

Now we are in a position to prove our main theorem in this section. This theorem gives a characterization for vector measures which are not strongly bounded. Recall that c_{00}-the space of all finitely nonzero sequences with the sup norm, and m_0-the space of all finitely valued sequences also with the sup norm, are dense subspaces of c_0 and ℓ_∞ respectively.

Theorem 1. Let R be a ring of subsets of a set Ω and X a seminormed space. Suppose $\mu: R \to X$ is a bounded vector measure. Then μ is not strongly bounded if there exists a linear topological embedding $T: c_{00} \to X$ and a sequence (E_n) of pairwise disjoint members of R such that $T(e_n) = \mu(E_n)$ where e_n denotes the sequence, 1 in the n^{th} place and zero elsewhere.

If, in addition, R is a σ-ring then the above statement remains true if the space c_{00} is replaced by m_0.

Proof. Suppose $\mu: R \to X$ is not strongly bounded. Then there exists a disjoint sequence (E_n) in R and an $\varepsilon > 0$ such that

(1) $\|\mu(E_n)\| > \epsilon$ for $n \in N$.

By virtue of the Hahn-Banach theorem there is $f_n \in X^*$ for each $n \in N$ such that

(2) $\|f_n\| = 1$ and $f_n(\mu(E_n)) = \|\mu(E_n)\| > \varepsilon$.

For $n \in N$, consider the variation $|f_n \circ \mu|$ of the scalar valued measure $f_n \circ \mu$. Since $|f_n \circ \mu|(E) \leq \|\mu\|(\Omega)$ for $E \in \mathcal{R}$, $(|f_n \circ \mu|)_{n \in N}$ is a uniformly bounded sequence of nonnegative real valued measures.

For $n \in N$ define $\mu_n: 2^N \to R+$ by,

$$\mu_n(P) = \sum_{i \in P} |f_n \circ \mu|(E_i) .$$

The strong additivity of $|f_n \circ \mu|$ implies that μ_n is a measure. Since for $n \in N$ and $P \subseteq N$, $\mu_n(P) = \sum_{i \in P} |f_n \circ \mu|(E_i) \leq \|\mu\|(\Omega)$, (μ_n) is a uniformly bounded sequence of nonnegative real valued measures. By Lemma 1 there exists an infinite subset $P = \{p_1 < p_2 < \ldots\ldots\}$ of N such that

(3) $\mu_{p_n}(P \setminus \{p_n\}) < \varepsilon/2$ for every $p_n \in P$.

Define $T: c_{00} \to X$ by $T((x_n)) = \sum_{n=1}^{\infty} x_n \mu(E_{p_n})$. Since only finitely many terms are nonzero in the above series, it is readily seen that T is linear. Moreover if $f \in X^*$ with $\|f\| \leq 1$, then

$$|f\circ T((x_n))| = |f(\sum_{n=1}^{\infty} x_n \mu(E_{p_n}))|$$

$$= |\sum_{n=1}^{\infty} x_n f\circ\mu(E_{p_n})|$$

$$\le \sum_{n=1}^{\infty} |x_n||f\circ\mu(E_{p_n})|$$

$$\le \|(x_n)\|_\infty \sum_{n=1}^{\infty} |f\circ\mu(E_{p_n})|$$

$$\le \|(x_n)\|_\infty \|\mu\|(\Omega) .$$

Consequently (4) $\|T((x_n))\| \le \|(x_n)\|_\infty \|\mu\|(\Omega)$.

On the other hand for $m \in N$,

$$\|T((x_n))\| \ge |f_{p_m} \circ T((x_n))| \text{ (since } \|f_{p_m}\| = 1)$$

$$= |f_{p_m}(\sum_{n=1}^{\infty} x_n \mu(E_{p_n}))|$$

$$= |\sum_{n=1}^{\infty} x_n f_{p_m} \circ \mu(E_{p_n})|$$

$$\ge |x_m f_{p_m} \circ \mu(E_{p_m})| - |\sum_{\substack{n=1\\ n\ne m}}^{\infty} x_n f_{p_m} \circ \mu(E_{p_n})|$$

$$\ge |x_m||f_{p_m}(\mu(E_{p_m}))| - \|(x_n)\|_\infty \sum_{\substack{n=1\\ n\ne m}}^{\infty} |f_{p_m} \circ \mu(E_{p_n})|$$

$$\ge |x_m|\varepsilon - \|(x_n)\|_\infty \mu_{p_m}(P \setminus \{p_m\}) \text{ (by (2))}$$

$$\geq |x_m|\varepsilon - \|(x_n)\|_\infty \varepsilon/2 \quad \text{by (3)} .$$

Taking supremum over m on the right hand side we have

(5) $\|T(x_n))\| \geq \|(x_n)\|_\infty \varepsilon/2$.

(4) and (5) implies T is a linear topological embedding. Finally note that $T(e_n) = \mu(E_{p_n})$.

Moving to the case in which $\mathcal{R}$ is a σ-ring, we proceed as above to produce an $\varepsilon > 0$, a sequence (f_n) in X^* and a pairwise disjoint sequence (E_n) of members of $\mathcal{R}$ such that

(6) $\|f_n\| = 1$ and $|f_n \circ \mu(E_n)| > \varepsilon$ for $n \in N$.

Define $\mu_n : 2^N \to R+$ by

$$\mu_n(P) = |f_n \circ \mu|(\bigcup_{i \in P} E_i) .$$

It is readily seen that (μ_n) is a uniformly bounded sequence of nonnegative real valued measures. Again Lemma 1 implies that there exists an infinite subset $P = \{p_1 < p_2 < \ldots\}$ of N such that

(7) $\mu_{p_n}(P \setminus \{p_n\}) < \varepsilon/2$ for $p_n \in P$.

If $(x_n) \in m_0$, we can write $(x_n) = \sum_{m=1}^{k} \beta_m \chi_{A_m}$ where $A_1, A_2, \ldots, A_k$ are pairwise disjoint subsets of N such that

$\bigcup_{m=1}^{k} A_m = N$. Define $T: m_0 \to X$ by,

$$T((x_n)) = \sum_{m=1}^{k} \beta_m \mu(\bigcup_{i \in A_m} E_{p_i}) .$$

The linearity of T can be easily verified by using some elementary set algebra. Moreover if $f \in X^*$ with $\|f\| \le 1$, then

$$|f_o T((x_n))| = |f(\sum_{m=1}^{k} \beta_m \mu(\bigcup_{i \in A_m} E_{p_i}))|$$

$$= |\sum_{m=1}^{k} \beta_m f_o \mu(\bigcup_{i \in A_m} E_{p_i})|$$

$$\le \sum_{m=1}^{k} |\beta_m| |f_o \mu(\bigcup_{i \in A_m} E_{p_i})|$$

$$\le \|(x_n)\|_\infty . \sum_{m=1}^{k} |f_o \mu(\bigcup_{i \in A_m} E_{p_i})|$$

$$\le \|(x_n)\|_\infty \|\mu\|(\Omega) .$$

Consequently (8) $\|T((x_n))\| \le \|(x_n)\|_\infty \|\mu\|(\Omega)$.

On the other hand for $\ell \in N$,

$$\|T((x_n))\| \ge |f_{p_\ell} \circ T((x_n))|$$

$$= |(\sum_{m=1}^{k} \beta_m f_{p_\ell} \circ \mu(\bigcup_{i \in A_m} E_{p_i})|$$

$$= \left| x_\ell f_{p_\ell} \circ \mu(E_{p_\ell}) + \sum_{m=1}^{k} \beta_m f_{p_\ell} \circ \mu\Big(\bigcup_{\substack{i \in A_m \\ i \neq \ell}} E_{p_i} \Big) \right|$$

$$\geq \left| x_\ell f_{p_\ell} \circ \mu(E_{p_\ell}) \right| - \left| \sum_{m=1}^{k} \beta_m f_{p_\ell} \circ \mu\Big(\bigcup_{\substack{i \in A_m \\ i \neq \ell}} E_{p_i} \Big) \right|$$

$$\geq |x_\ell| \varepsilon - \|(x_n)\|_\infty \sum_{m=1}^{k} \left| f_{p_\ell} \circ \mu \right| \Big(\bigcup_{\substack{i \in A_m \\ i \neq \ell}} E_{p_i} \Big) \quad \text{(by 6)}$$

$$= |x_\ell| \varepsilon - \|(x_n)\|_\infty \left| f_{p_\ell} \circ \mu \right| \Big(\bigcup_{i \in N \setminus \{\ell\}} E_{p_i} \Big)$$

$$= |x_\ell| \varepsilon - \|(x_n)\|_\infty \, \mu_{p_\ell}(P \setminus \{p_\ell\})$$

$$\geq |x_\ell| \varepsilon - \|(x_n)\|_\infty \, \varepsilon/2 \quad \text{(by 7)}$$

Taking supremum over ℓ on the right hand side we have

$$(9) \quad \|T((x_n))\| \geq \|(x_n)\|_\infty \, \varepsilon/2$$

(8) and (9) implies that T is a linear topological embedding of m_0 to X.

Finally we note that $T(e_n) = \mu(E_{p_n})$.

Remark 1. If X is a Banach space in this theorem, then c_{00} and m_0 can be replaced by c_0 and ℓ_∞ respectively.

Remark 2. Let X, Y be topological vector spaces. The statement "Y contains a copy of X" means that there is a linear topological embedding $T: X \to Y$.

Corollary 1. Let X be a Banach space containing no copy of c_0. If the series $\sum_{n=1}^{\infty} x_n$ is unordered bounded, i.e. $\{\sum_{n\in A} x_n | A$ is a finite subset of $N\}$ is a bounded subset of X, then $\sum_{n=1}^{\infty} x_n$ is subseries convergent.

Proof. Let $\mathcal{A}$ be the ring of all finite subsets of N. Define $\mu: \mathcal{A} \to X$ by $\mu(A) = \sum_{n\in A} x_n$. Clearly μ is a bounded vector measure. Since X does not contain a copy of c_0, theorem 1 implies that μ is strongly additive. Hence $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \mu(\{n\})$ is subseries convergent.

Corollary 2. Let $\mathcal{R}$ be a ring of subsets of a set Ω and X a locally convex space. Suppose $\mu: \mathcal{R} \to X$ is a bounded vector measure. If $\lim_n \mu(E_n)$ exists weakly in X for every increasing sequence (E_n) of members of $\mathcal{R}$, then μ is strongly bounded.

Proof. We may assume that X is a seminormed space because of the following reasons.

(1) If $\| \ \|$ is a continuous seminorm on X and if the sequence (x_n) in X weakly converges with respect to the locally convex topology on X, then (x_n) converges weakly in $(X, \| \ \|)$.

(2) μ is strongly bounded with respect to the locally convex topology if and only if μ is strongly bounded with respect to each continuous seminorm $\| \ \|$ on X.

~~Suppose $\mu: \mathcal{R} \to X$ is not strongly bounded.~~ By theorem 1 there is a topological linear embedding $T: c_{00} \to X$ and a sequence (F_n) of disjoint members of $\mathcal{R}$ such that $T(e_n) = \mu(F_n)$. Set $E_m = \bigcup_{n=1}^{m} F_n$. Then $T(\sum_{n=1}^{m} e_n) = \mu(E_m)$.

First we show that $\lim_m \sum_{n=1}^{m} e_n$ does not exist weakly in c_0. Suppose $\lim_m \sum_{n=1}^{m} e_n = (a_n)$ weakly in c_0. For each $k, e_k \in \ell_1 = c_0^*$ and hence $\lim_m e_k(\sum_{n=1}^{m} e_n) = e_k((a_n))$. This means $a_k = 1$ for each k, which is a contradiction since $(a_n) \in c_0$.

Now let $f \in X^*$. Then $f_o T \in c_{00}^*$ and moreover $f_o T$ can be extended uniquely over c_0 to a member of c_0^*. We denote this extension by $\overline{f_o T}$. On the other hand if $g \in c_0^*$, then $g_o T^{-1}$ is a continuous linear functional on the subspace $T(c_{00})$ of X. By virtue of the Hahn-Banach theorem we can extend $g_o T^{-1}$ over X to a member of X^*. We denote this extension by f. It is easy to check that $\overline{f_o T} = g$. Therefore every member of c_0^* can be written in the form $\overline{f_o T}$ for some $f \in X^*$.

Since $\lim_n \mu(E_n)$ exists weakly, there exists $x \in X$ such that

(1) $\lim_n f(\mu(E_n)) = f(x)$ for $f \in X^*$.

Suppose x does not belong to the closure of $T(c_{00})$ in X. Then by virtue of the Hahn-Banach theorem there exists $g \in X^*$ such that $g(x) \neq 0$ and g vanishes on $\overline{T(c_{00})}^X$. This contradicts the fact that $\lim_n \mu(E_n) = x$ weakly in X. Hence $x \in \overline{T(c_{00})}^X$. Therefore there exists a sequence (a_n) in c_{00} such that $\lim_n T(a_n) = x$ in X. Let $f \in X^*$; then

$$(2) \quad \lim_n f_o T(a_n) = f(x) .$$

Also note that (a_n) is Cauchy in c_{00} since $(T(a_n))$ is Cauchy in X. Consequently there is $a \in c_0$ such that $\lim_n a_n = a$ in c_0. Since $\overline{f_o T} \in c_0^*$, (3) $\lim_n \overline{f_o T}(a_n) = \overline{f_o T}(a)$.

Now $$\left|\overline{f_o T}\left(\sum_{m=1}^{n} e_m\right) - \overline{f_o T}(a)\right| \leq \left|f(\mu(E_n)) - f(x)\right| + \left|f(x) - f_o T(a_n)\right|$$

$$+ \left|f_o T(a_n) - \overline{f_o T}(a)\right| .$$

By (1), (2) and (3) the right hand side tends to zero as n tends to infinity. Consequently $\lim_n \sum_{m=1}^{n} e_m$ exists weakly. This contradiction shows that μ is strongly bounded. This completes the proof.

Now we employ the above corollary to prove the Orlicz-Pettis theorem for locally convex spaces. This theorem was first proved by Orlicz for weakly sequentially complete Banach spaces. Kalton [8] recently obtained this theorem for separable topological groups and then derived the result for separable locally convex spaces. For an alternative proof of the locally convex version of this theorem, the reader is referred to McArthurs paper [11].

Corollary 3. (Orlicz-Pettis). Let X be a locally convex space. If $\sum_{n=1}^{\infty} x_n$ is a weakly subseries convergent series in X, then $\sum_{n=1}^{\infty} x_n$ is subseries convergent.

Proof. Let $\mathcal{A}$ be the ring of all finite subsets of N. Define $\mu: \mathcal{A} \to X$ by $\mu(A) = \sum_{n \in A} x_n$. Evidently μ is finitely additive. To show that μ is bounded, it suffices to prove that μ is bounded with respect to each continuous seminorm $\| \ \|$ on X. Consider the subset $\mathcal{F} = \{ \sum_{n \in A} \hat{x}_n | A \in \mathcal{A}\}$ of $(X, \| \ \|)^{**}$; the second dual of X with respect to the seminorm topology. For every $f \in (X, \| \ \|)^*$, $f \in X^*$; the dual space of X with respect to the locally convex topology, and hence $\sum_{n=1}^{\infty} f(x_n)$ is subseries convergent so that $\sum_{n=1}^{\infty} |f(x_n)| < \infty$. Consequently for $A \in \mathcal{A}$, $|(\sum_{n \in A} \hat{x}_n)(f)| = |\sum_{n \in A} f(x_n)| \leq \sum_{n=1}^{\infty} |f(x_n)| < \infty$. Since $(X, \| \ \|)^*$ is a Banach space, the uniform boundedness princple implies that:

$$\operatorname{Sup}\{ | \sum_{n \in A} f(x_n) | \, | A \in \mathcal{A}, f \in (X, \| \ \|)^*, \|f\| \leq 1\} < \infty .$$

Therefore $\operatorname{Sup}\{ \|\mu(A)\| \, | A \in \mathcal{A}\} = \operatorname{Sup}\{ \| \sum_{n \in A} x_n \| \, | A \in \mathcal{A}\} < \infty$. This shows that μ is bounded.

Let (A_n) be an increasing sequence in $\mathcal{A}$. Then by hypothesis, $\lim_n \mu(A_n) = \lim_n \sum_{m\in A_n} x_m$ exists weakly. Therefore the last corollary implies that μ is strongly bounded and hence $\sum_{n\in A} x_n$ satisfies the Cauchy condition for every $A \subseteq N$. Since $\sum_{n\in A} x_n$ exists weakly in X, 1.4 theorem 2 implies that $\sum_{n\in A} x_n$ is convergent in X. This completes the proof.

Remark. In chapter 4 we obtain another version of the Orlicz-Pettis theorem.

The following corollary establishes a characterization of complete seminormed spaces not containing a copy of c_{00}.

Corollary 4. A complete seminormed space X contains no copy of c_{00} if and only if every series $\sum_{n=1}^{\infty} x_n$ in X, with $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for every $f \in X^*$, is subseries convergent.

Proof. First suppose X contains no copy of c_{00}. Let $\sum_{n=1}^{\infty} x_n$ be a series in X with $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for $f \in X^*$. We define $\mu: \mathcal{A} \to X$ precisely as in the proof of the last corollary and follows the same argument to show that μ is bounded. Since X contains no copy of c_{00}, ~~Theorem 1 implies that μ is strongly bounded.~~ The completeness of X assures that μ is strongly additive. Hence $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \mu(\{n\})$ is subseries convergent.

To show that the converse is true, suppose X contains a copy of c_{00}. Then there are many nonconvergent series $\sum_{n=1}^{\infty} x_n$ in X such that $\sum_{n=1}^{\infty} |f(x)| < \infty$ for $f \in X^*$.

Corollary 5. Let X be a complete seminormed space. If X^* does not contain a copy of ℓ_∞, then X^* contains no copy of c_0.

Proof. Let $\sum_{n=1}^{\infty} f_n$ be a series in X^* such that $\sum_{n=1}^{\infty} |F(f_n)| < \infty$ for $F \in X^{**}$. If $E \subseteq N$, then $\sum_{n \in E} \hat{x}(f_n) (= \sum_{n \in E} f_n(x))$ exists for $x \in X$. By virtue of the Banach-Steinhaus theorem, $\sum_{n \in E} f_n$ converges with respect to the weak* topology on X^*. Define $\mu: 2^N \to X^*$ by $\mu(E) = \sum_{n \in E} f_n$ - weak* limit. Evidently μ is finitely additive. To show that μ is bounded consider the subset $\mathcal{F} = \{ \sum_{n \in E} f_n\text{-weak* limit} | E \subseteq N\}$ of X^*. By the fact that $\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} |\hat{x}(f_n)| < \infty$ for $x \in X$, we have that $\mathcal{F}$ is pointwise bounded. Hence the uniform boundedness principle implies that $\sup\{\| \sum_{n \in E} f_n\text{-weak* limit}\| | E \subseteq N\} < \infty$.

i.e., $\sup\{\|\mu(E)\| | E \subseteq N\} < \infty$.

Since X^* does not contain a copy of ℓ_∞, the last part of theorem 1 implies that μ is strongly bounded. Consequently $\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \mu(\{n\})$ is subseries convergent. Hence the last corollary implies that X^* contains no copy of c_0. This completes the proof.

The results we obtained so far demonstrate the utility of theorem 1 in the theory of topological vector spaces.

§4. Convergence and boundedness of a sequence of strongly bounded vector measures.

The main result we obtain in this section concerning sequences of strongly bounded vector measures is the Vitali-Hahn-Saks-Nikodym theorem. We prove this theorem for vector measures defined on a ring with a weaker structure than of a σ-ring. The proof is a modification of the proof given in [7] by Barbara Faires. We use this improved version of the theorem to obtain generalizations of both the Philips and Schur lemmas. We start with the following definition.

Definition 1. Let R be a ring of subsets of a set Ω. R is said to have property (QI) if for every disjoint sequence (A_n) in R and every sequence (B_n) in R with $A_m \cap B_n = \phi$ for $m,n \in N$, there exists a subsequence (A_{n_i}) of (A_n) and $C \in R$ such that:

$$\bigcup_{i=1}^{\infty} A_{n_i} \subseteq C,\ C \cap \left(\bigcup_{n=1}^{\infty} B_n\right) = \phi \quad \text{and} \quad C \cap A_n = \phi \text{ for } n \notin \{n_1, n_2, \ldots\ldots\}.$$

Remarks 1. The class of $Q\sigma$-rings and the class of algebras with the interpretation property both have property (QI).

2. Let R be a ring of subsets of N with property (QI). If R contains all finite subsets of N, then R is a $Q\sigma$-ring.

To verify (2) let (A_n) be a sequence of disjoint members of R. Then $N \setminus (\bigcup_{n=1}^{\infty} A_n)$ is countable. We write $N \setminus (\bigcup_{n=1}^{\infty} A_n) = \{k_1, k_2, \ldots\ldots\}$ and define $B_n = \{k_n\}$. Property (QI) of R implies that there is a subsequence (A_{n_i}) of (A_n) and $C \in \mathcal{R}$ such that

$$\bigcup_{i=1}^{\infty} A_{n_i} \subseteq C \text{ , } C \cap (N \setminus (\bigcup_{n=1}^{\infty} A_n) = \phi \text{ and } C \cap A_n = \phi \text{ for } n \notin \{n_1, n_2, \ldots\ldots\}.$$

This implies $\bigcup_{i=1}^{\infty} A_{n_i} = C \in \mathcal{R}$. Therefore $\mathcal{R}$ is a $Q\sigma$-ring.

Proposition 1. If $\mathcal{R}$ is a ring with property (QI), then $\mathcal{R}$ has the following property (we call this property $(QI)^1$).

For every disjoint sequence (A_n) in $\mathcal{R}$ and every sequence (B_n) in $\mathcal{R}$ with $A_n \subseteq B_m$ for $m,n \in N$, there exists a subsequence (A_{n_i}) of (A_n) and $C \in \mathcal{R}$ such that:

$$\bigcup_{i=1}^{\infty} A_{n_i} \subseteq C \subseteq \bigcap_{n=1}^{\infty} B_n \text{ and } C \cap A_n = \phi \text{ for } n \notin \{n_1, n_2, \ldots\ldots\} .$$

Proof. Let (A_n) be a disjoint sequence in $\mathcal{R}$ and (B_n) a sequence in $\mathcal{R}$. Suppose $A_n \subseteq B_m$ for $m,n \in N$. Set $D_n = B_1 \setminus B_n$ for $n \in N$. Since $A_n \subseteq B_m$ for $m,n \in N$, $A_n \cap D_m = \phi$ for $m,n \in N$. Since $\mathcal{R}$ has property (QI), there exists a subsequence (A_{n_i}) of (A_n) and $C \in \mathcal{R}$ such that:

$$(1) \quad \bigcup_{i=1}^{\infty} A_{n_i} \subseteq C, \; C \cap (\bigcup_{n=1}^{\infty} D_n) = \phi \text{ and } C \cap A_n = \phi \text{ for } n \notin \{n_1, n_2, \ldots\ldots\}.$$

In fact we can choose C such that $C \subseteq B_1$. By (1) $C \cap (B_1 \setminus B_n) = \phi$ for $n \in N$. Hence $C \subseteq B_n$ for $n \in N$.

$$\text{i.e., (2)} \quad C \subseteq \bigcap_{n=1}^{\infty} B_n .$$

The proposition follows from (1) and (2).

Theorem 1. (Vitali-Hahn-Saks-Nikodym).

Let X be a topological vector space and $\mathcal{R}$ a ring of subsets of a set Ω with property (QI). Suppose (μ_n) is a sequence of strongly bounded X valued measures on $\mathcal{R}$ with $\lim_n \mu_n(E) = 0$ for every $E \in \mathcal{R}$. Then the sequence (μ_n) is uniformly strongly bounded. i.e., for every disjoint sequence (E_n) in $\mathcal{R}$ $\lim_n \mu_m(E_n) = 0$ uniformly in m.

Proof. Since X is generated by a family of F-seminorms, we may assume that X is an F-seminormed space. Suppose the contrary. Then there exists a disjoint sequence (E_n) in $\mathcal{R}$, an $\varepsilon > 0$ and a subsequence (μ_{m_n}) of (μ_m) such that $\|\mu_{m_n}(E_n)\| > 3\varepsilon$ for $n \in N$. For simplicity we relabel (μ_{m_n}) by (μ_n) and write $\|\mu_n(E_n)\| > 3\varepsilon$ for $n \in N$.

Let $i_1 = 1$. Partition $N \setminus \{i_1\}$ into a sequence (Π_k^1) of pairwise disjoint infinite subsets of N. Consider the following two disjoint sequences in $\mathcal{R}$:

$$\{E_i \mid i \in \Pi_1^1\} \text{ and } \{E_i \mid i \in \{i_1\} \cup \bigcup_{k=2}^{\infty} \Pi_k^1\}.$$

The property (QI) of $\mathcal{R}$ implies that there exists an infinite subset Δ_1^1 of Π_1^1 and $F_1^1 \in \mathcal{R}$ such that:

$$\cup\{E_i \mid i \in \Delta_1^1\} \subseteq F_1^1 \text{ and } F_1^1 \cap (\cup\{E_i \mid i \in \{i_1\} \cup \bigcup_{k=2}^{\infty} \Pi_k^1\}) = \emptyset.$$

Suppose, for $1 \le k < n$, $F_k^1 \in R$ and an infinite $\Delta_k^1 (\subseteq \Pi_k^1)$ have been constructed such that:

(a$_1$) $\cup\{E_i | i \in \Delta_k^1\} \subseteq F_k^1$

(b$_1$) $F_k^1 \cap E_{i_1} = \phi$

(c$_1$) $F_k^1 \cap E_j = \phi$ for $j \in \bigcup_{p=k+1}^{\infty} \Pi_p^1$

(d$_1$) $F_1, F_2, \ldots, F_{n-1}$ are pairwise disjoint.

Consider the following two sequences in R.

$$\{E_i | i \in \Pi_n^1\} \text{ and } \{E_i | i \in \{i_1\} \cup \bigcup_{p=n+1}^{\infty} \Pi_p^1\} \cup \{F_1^1, F_2^1, \ldots, F_{n-1}^1\} .$$

By (c$_1$) the members of both sequences are pairwise disjoint. Again property (QI) implies that there exists an infinite subset Δ_n^1 of Π_n^1 and $F_n^1 \in R$ such that:

$$\cup\{E_i | i \in \Delta_n^1\} \subseteq F_n^1 \text{ and } F_n^1 \cap (\cup\{E_i | i \in \{i_1\} \cup \bigcup_{p=n+1}^{\infty} \Pi_p^1\} \cup F_1^1 \cup F_2^1 \cup \ldots \cup F_{n-1}^1) = \Phi .$$

Clearly F_n^1 satisfies (a$_1$), (b$_1$), (c$_1$) and (d$_1$). Therefore, by induction, we can construct a sequence (F_k^1) of disjoint members of R and a sequence (Δ_k^1) of disjoint subsets of N satisfying (a$_1$), (b$_1$), (c$_1$) and (d$_1$) for $k \in N$.

Define $\bar{\mu}_{i_1} : R \to R+$ by ,

$$\bar{\mu}_{i_1}(E) = \operatorname{Sup}\{\|\mu_{i_1}(F)\| \,|\, F \in R \text{ and } F \subseteq E\} .$$

To show that $\lim_k \bar{\mu}_{i_1}(F^1_k) = 0$, let $\varepsilon > 0$. Then for each k there is $A_k\ (\in R) \subseteq F^1_k$ such that $\|\mu_{i_1}(A_k)\| > \bar{\mu}_{i_1}(F^1_k) - \varepsilon$. Since μ_{i_1} is strongly bounded, $\lim_k \|\mu_{i_1}(A_k)\| = 0$. Consequently $\lim_k \bar{\mu}_{i_1}(F^1_k) = 0$.

Choose $k_1 \in N$ such that $\bar{\mu}_{i_1}(F^1_{k_1}) < \varepsilon$ and then $i_2 \in \Delta^1_{k_1}$ $(i_2 > i_1)$ such that $\|\mu_{i_2}(E_{i_1})\| < \varepsilon/2^2$. Note that, by (a_1), $E_{i_2} \subseteq F^1_{k_1}$. Partition $\Delta^1_{k_1} \setminus \{i_2\}$ into a sequence (Π^2_n) of disjoint subsets of $\Delta^1_{k_1} \setminus \{i_2\}$. Use the same induction procedure to construct a sequence (F^2_k) of disjoint members of R and a sequence (Δ^2_k) $(\Delta^2_k \subseteq \Pi^2_k)$ of disjoint subsets of N such that:

(a_2) $\cup\{E_i \,|\, i \in \Delta^2_k\} \subseteq F^2_k$ for $k \in N$.

(b_2) $F^2_k \cap (E_{i_1} \cup E_{i_2}) = \emptyset$ for $k \in N$.

(c_2) $F^2_k \cap E_j = \emptyset$ for $j \in \bigcup_{p=k+1}^{\infty} \Pi^2_p$.

Since $\lim_k \bar{\mu}_{i_2}(F^2_k) = 0$, $\lim_i \mu_i(E_{i_1}) = 0$ and $\lim_i \mu_i(E_{i_2}) = 0$, we can choose $k_2 \in N$ such that $\bar{\mu}_{i_2}(F^2_{k_2}) < \varepsilon$ and then $i_3 \in \Delta^2_{k_2}$ $(i_3 > i_2)$ such that $\|\mu_{i_3}(E_{i_1})\|$, $\|\mu_{i_3}(E_{i_2})\| < \varepsilon/2^3$. Note that $E_{i_3} \subseteq F^1_{k_1}, F^2_{k_2}$.

Proceeding in this manner we can construct inductively a sequence $(F^n_{k_n}) = (F_n)$ say, in $\mathcal{R}$ and an increasing sequence (i_n) of positive integers such that:

(1) $E_{i_k} \subseteq F_n$ for $n < k$

(2) $F_n \cap E_{i_k} = \phi$ for $1 \leq k \leq n$ (by (b_1) and (b_2)).

(3) $\bar{\mu}_{i_n}(F_n) < \varepsilon$ for $n \in N$.

(4) $\|\mu_{i_n}(E_{i_k})\| < \varepsilon/2^n$ for $1 \leq k \leq n$.

(5) $\|\mu_{i_n}(E_{i_n})\| > 3\varepsilon$ for $n \in N$.

Let $H_n = F_n \cup (\bigcup_{k=1}^{n} E_{i_k})$. Then (1) implies $E_{i_k} \subseteq H_n$ for $k,n \in N$. Since $\mathcal{R}$ has property $(IQ)^1$, there exists a subsequence (i_{k_ℓ}) of (i_k) and $C \in \mathcal{R}$ such that:

(6) $\bigcup_{\ell=1}^{\infty} E_{i_{k_\ell}} \subseteq C \subseteq \bigcap_{k=1}^{\infty} H_k$ and $C \cap E_{i_k} = \phi$ for $k \notin \{k_1, k_2, \ldots\ldots\}$.

Therefore, for each $p \in N$,

$$C = (C \setminus \bigcup_{\ell=1}^{p} E_{i_{k_\ell}}) \cup (\bigcup_{\ell=1}^{p-1} E_{i_{k_\ell}}) \cup (E_{i_{k_p}}) \text{ and hence}$$

$$\mu_{i_{k_p}}(C) = \mu_{i_{k_p}}(E_{i_{k_p}}) + \mu_{i_{k_p}}(C \setminus \bigcup_{\ell=1}^{p} E_{i_{k_\ell}}) + \mu_{i_{k_p}}(\bigcup_{\ell=1}^{p-1} E_{i_{k_\ell}}).$$

Consequently $\|\mu_{i_{k_p}}(C)\| \geq \|\mu_{i_{k_p}}(E_{i_{k_p}})\| - \|\mu_{i_{k_p}}(C \setminus \bigcup_{\ell=1}^{p} E_{i_{k_\ell}})\| - \|\mu_{i_{k_p}}(\bigcup_{\ell=1}^{p-1} E_{i_{k_\ell}})\|$

By (6) and the definition of H_{k_p}, $C \subseteq H_{k_p} = F_{k_p} \cup (\bigcup_{m=1}^{k_p} E_{i_m})$.

Since $F_{k_p} \cap (\bigcup_{m=1}^{k_p} E_{i_m}) = \phi$ by (2), $C \setminus \bigcup_{m=1}^{k_p} E_{i_m} \subseteq F_{k_p}$. Also

$C \setminus \bigcup_{m=1}^{k_p} E_{i_m} = C \setminus \bigcup_{\ell=1}^{p} E_{i_{k_\ell}}$ since $C \cap E_{i_m} = \phi$ for $m \notin \{k_1, k_2, \ldots\ldots\}$.

Hence (3) implies that $\|\mu_{i_{k_p}}(C \setminus \bigcup_{\ell=1}^{p} E_{i_{k_\ell}})\| \leq \bar{\mu}_{i_{k_p}}(F_{k_p}) < \varepsilon$.

Also $\|\mu_{i_{k_p}}(\bigcup_{\ell=1}^{p-1} E_{i_{k_\ell}})\| \leq \sum_{\ell=1}^{p-1} \|\mu_{i_{k_p}}(E_{i_{k_\ell}})\| \leq \sum_{\ell=1}^{p-1} \varepsilon/2^{k_p}$ (by (4))

$$\leq \varepsilon .$$

Therefore $\|\mu_{i_{k_p}}(C)\| > 3\varepsilon - \varepsilon - \varepsilon = \varepsilon$. This contradicts the fact that $\lim_i \mu_i(C) = 0$. Hence the sequence (μ_n) is uniformly strongly bounded.

Corollary 1. Let R, X be as in theorem 1. Suppose (μ_n) is a sequence of strongly bounded X-valued measures on R such that $\lim_n \mu_n(E) = \mu(E)$ exists for $E \in R$. Then μ is strongly bounded and, moreover, the sequence (μ_n) is uniformly strongly bounded.

If, in addition, X is complete, then for each disjoint sequence (E_m) in R $\lim_n \sum_{m \in A} \mu_n(E_m) = \sum_{m \in A} \mu(E_m)$ uniformly for $A \subseteq N$.

Proof. Again we may assume that X is an F-seminormed space. Let (E_i) be a disjoint sequence of members of R. First we show that $\lim_n \mu_n(E_i) = \mu(E_i)$ uniformly in i. Suppose $(\mu_n(E_i))_{n\in N}$ is not uniformly Cauchy in i. Then there exist two subsequences (n_k) and (i_k) of positive integers such that:

$$(1) \quad \lim_k \|(\mu_{n_{k+1}} - \mu_{n_k})(E_{i_k})\| \neq 0 .$$

Since $\mu_{n_{k+1}}$ and μ_{n_k} are both strongly bounded $\mu_{n_{k+1}} - \mu_{n_k}$ is also strongly bounded and, moreover, $\lim_k (\mu_{n_{k+1}} - \mu_{n_k})(E) = 0$ for $E \in R$. Thus theorem 1 implies that $\lim_i \|\mu_{n_{k+1}} - \mu_{n_k})(E_i)\| = 0$ uniformly in k. This contradicts (1). Therefore (2) $\lim_n \mu_n(E_i) = \mu(E_i)$ uniformly in i. For given $\varepsilon > 0$, there is $n_o \in N$ such that $\|\mu_{n_o}(E_i) - \mu(E_i)\| < \varepsilon/2$ for $i \in N$. Since $\lim_i \mu_{n_o}(E_i) = 0$, there is $i_o \in N$ such that $\|\mu_{n_o}(E_i)\| < \varepsilon/2$ for $i \geq i_o$. Thus $\|\mu(E_i)\| \leq \|\mu(E_i) - \mu_{n_o}(E_i)\| + \|\mu_{n_o}(E_i)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $i \geq i_o$ so that $\lim_i \mu(E_i) = 0$. Hence μ is strongly bounded.

An application of theorem 1 to the sequence $(\mu_n - \mu)_{n\in N}$ shows that the sequence (μ_n) is uniformly strongly bounded.

To prove the last part of the corollary let X be a complete space and (E_n) a disjoint sequence of members of $\mathcal{R}$. Then the sequence (μ_n) is uniformly strongly additive and μ is strongly additive. Now we show that (3) for $A \subseteq N$, $\lim_n \sum_{m\in A} \mu_n(E_m) = \sum_{m\in A} \mu(E_m)$. This is true when A is finite, so assume A is infinite. Let $A = \{m_1 < m_2 < \ldots\ldots\}$. Since (μ_n) is uniformly strongly additive, $\sum_{j=1}^{\infty} \mu_n(E_{m_j})$ $n = 1,2,\ldots$ are convergent uniformly in n. Therefore for given $\varepsilon > 0$, there exists $n_o \in N$ such that $\|\sum_{j=n_o}^{\infty} \mu_n(E_{m_j})\| < \frac{\varepsilon}{3}$ for $n \in N$. In fact we can choose n_o large enough to satisfy $\|\sum_{j=n_o}^{\infty} \mu(E_{m_j})\| < \frac{\varepsilon}{3}$. Since

$\lim_n \mu_n(E_{m_j}) = \mu(E_{m_j})$ for $j = 1,2,\ldots,n_o-1$, there exists $m_o \in N$ such that $\sum_{j=1}^{n_o-1} \|\mu_n(E_{m_j}) - \mu(E_{m_j})\| < \frac{\varepsilon}{3}$ for $n \geq m_o$. Therefore for $n \geq m_o$

$$\|\sum_{j=1}^{\infty} (\mu_n(E_{m_j}) - \mu(E_{m_j}))\| \leq \sum_{j=1}^{n_o-1} \|\mu_n(E_{m_j}) - \mu(E_{m_j})\| + \|\sum_{j=n_o}^{\infty} \mu_n(E_{m_j})\|$$

$$+ \|\sum_{j=n_o}^{\infty} \mu(E_{m_j})\|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

This proves (3) .

We define $\nu_n: 2^N \to X$ by $\nu_n(A) = \sum_{m \in A} \mu_n(E_m)$ for $n \in N$ and $\nu: 2^N \to X$ by $\nu(A) = \sum_{m \in A} \mu(E_m)$. Since $\sum_{m=1}^{\infty} \mu_n(E_m)$ is subseries convergent for $n \in N$, (ν_n) is a sequence of strongly bounded vector measures and, moreover, $\lim_n \nu_n(A) = \nu(A)$ for $A \subseteq N$ by (3). Therefore the first part of this corollary implies that $(\nu_n - \nu)_{n \in N}$ is uniformly strongly bounded. To show that $\lim_n \nu_n(A) = \nu(A)$ uniformly for $A \subseteq N$, suppose the contrary. Then there exists a subsequence $(\nu_{n_k} - \nu)_{k \in N}$ of $(\nu_n - \nu)_{n \in N}$ (for notational convenience we relabel $(\nu_{n_k} - \nu)$ by $(\nu_k - \nu)$), a sequence (A_k) of subsets of N and an $\varepsilon > 0$ such that $\|(\nu_k - \nu)(A_k)\| > \varepsilon$ for $k \in N$. By the definitions of ν_k and ν there is a finite subset F_k of A_k such that $\|(\nu_k - \nu)(F_k)\| > \varepsilon$ for $k \in N$. Now we use the induction to construct a sequence (G_i) of disjoint subsets of N and a subsequence $(\nu_{k_i} - \nu)$ of $(\nu_k - \nu)$ such that $\|(\nu_{k_i} - \nu)(G_i)\| > \varepsilon/2$. This leads to a contradiction since $(\nu_n - \nu)$ is uniformly strongly bounded.

Set $k_1 = 1$ and $G_1 = F_1$. Suppose $G_1, G_2, \ldots, G_n$ disjoint subsets of N, and $k_1 < k_2 < \ldots < k_n$ have been chosen such that

$\|(\nu_{k_i} - \nu)(G_i)\| > \varepsilon/2$ for $i = 1,2,\ldots,n$. Let $\overline{\nu_k-\nu}\,(G_1 \cup G_2 \cup \ldots \cup G_n)$

$= \text{Max}\,\{\|(\nu_k-\nu)(H)\| \,|\, H \subseteq G_1 \cup G_2 \cup \ldots\ldots \cup G_n\}$ for $k \in N$. Since

$\lim_k (\nu_k-\nu)(E) = 0$ for every $E \subseteq N$, $\lim_k \overline{\nu_k-\nu}\,(G_1 \cup G_2 \cup \ldots \cup G_n) = 0$.

(Note that $G_1 \cup G_2 \cup \ldots \cup G_n$ has only finitely many subsets.) Choose

$k_{n+1} > k_n$ such that $\overline{\nu_{k_{n+1}}-\nu}\,(G_1 \cup G_2 \cup \ldots \cup G_n) < \varepsilon/2$. Set

$G_{n+1} = F_{k_{n+1}} \setminus (G_1 \cup G_2 \cup \ldots \cup G_n)$. Then

$$\|(\nu_{k_{n+1}}-\nu)(G_{n+1})\| = \|(\nu_{k_{n+1}}-\nu)(F_{k_{n+1}}) - (\nu_{k_{n+1}}-\nu)(F_{k_{n+1}} \cap (G_1 \cup G_2 \cup \ldots \cup G_n)\|$$

(by the additivity of $\nu_{k_{n+1}}-\nu$)

$$\geq \|(\nu_{k_{n+1}}-\nu)(F_{k_{n+1}})\| - \overline{\nu_{k_{n+1}}-\nu}\,(G_1 \cup G_2 \cup \ldots \cup G_n)$$

$$> \varepsilon - \varepsilon/2 = \varepsilon/2 .$$

Therefore $\lim_n \nu_n(A) = \nu(A)$ uniformly for $A \subseteq N$.

i.e., $\lim_n \sum_{m\in A} \mu_n(E_m) = \sum_{m\in A} \mu(E_m)$ uniformly for $A \subseteq N$.

Remark 1. In the absence of the completeness assumption the last part of the corollary can be modified in the following way.

"If (E_m) is a disjoint sequence of members of R such that

$\lim_n \sum_{m\in A} \mu_n(E_m)$ exists for every $A \subseteq N$, then $\lim_n \sum_{m\in A} \mu_n(E_m) =$

$\sum_{m\in A} \mu(E_m)$ uniformly for $A \subseteq N$".

To see this it suffices to show that $\lim_n \sum_{m\in A} \mu_n(E_m) = \sum_{m\in A} \mu(E_m)$ for each $A \subseteq N$. The remaining part of the proof runs identically.

Let $A = \{m_1 < m_2 < \dots\}$. The convergence of $\sum_{j=1}^{\infty} \mu_n(E_{m_j})$ for $n \in N$, and the uniform strong boundedness of (μ_n) assure that $\sum_{j=1}^{\infty} \mu_n(E_{m_j})$ converges uniformly for $n \in N$. Therefore for given $\varepsilon > 0$, there exists $p_o \in N$ such that $\left|\sum_{j=p}^{\infty} \mu_n(E_{m_j})\right| < \varepsilon/3$ for $n \in N$ and $p \geq p_o$. Let $p \geq p_o$. Since $\lim_n \mu_n(E_{m_j}) = \mu(E_{m_j})$ for $j = 1,2,\dots,p$, there exists $n_o \in N$ such that $\sum_{j=1}^{p} |\mu_n(E_{m_j}) - \mu(E_{m_j})| < \varepsilon/3$ for $n \geq n_o$. Also we can choose $n_1 \geq n_o$ such that $\left|\sum_{j=1}^{\infty} \mu_{n_1}(E_{m_j}) - \lim_n \sum_{j=1}^{\infty} \mu_n(E_{m_j})\right| < \varepsilon/3$.

Now $\left|\sum_{j=1}^{p} \mu(E_{m_j}) - \lim_n \sum_{j=1}^{\infty} \mu_n(E_{m_j})\right| \leq \left|\sum_{j=1}^{p} \mu(E_{m_j}) - \sum_{j=1}^{\infty} \mu_{n_1}(E_{m_j})\right|$

$$+ \left|\sum_{j=1}^{\infty} \mu_{n_1}(E_{m_j}) - \lim_n \sum_{j=1}^{\infty} \mu_n(E_{m_j})\right| \leq \sum_{j=1}^{p} |\mu(E_{m_j}) - \mu_{n_1}(E_{m_j})|$$

$$+ \left|\sum_{j=p+1}^{\infty} \mu_{n_1}(E_{m_j})\right| + \varepsilon/3 < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Remark 2. The last part of the corollary 1 may be treated as a generalized version of the Phillip's lemma [12]. To verify this we derive the Phillip's lemma from the last corollary.

Corollary 2. (Phillip's lemma). Let (μ_n) be a sequence of bounded complex valued measures defined on 2^N. If $\lim_n \mu_n(E) = \mu(E)$ exists for each $E \subseteq N$, then $\lim_n \sum_{m=1}^{\infty} |\mu_n(\{m\}) - \mu(\{m\})| = 0$.

Proof. Since μ_n is bounded and scalar valued, it is strongly bounded. Letting $(E_m) = (\{m\})$ we have $\lim_n \sum_{m\in A} \mu_n(\{m\}) = \sum_{m\in A} \mu(\{m\})$ uniformly for $A \subseteq N$ by the last corollary. Thus for given $\varepsilon > 0$, there exists $n_o \in N$ such that:

$$\left| \sum_{m\in A} \mu_n(\{m\}) - \mu(\{m\}) \right| < \varepsilon/8 \quad \text{for } A \subseteq N \text{ and } n \geq n_o .$$

Therefore by 3.2 lemma 1, $\sum_{i=1}^{\infty} |\mu_n(\{m\}) - \mu(\{m\})| \leq \varepsilon$ for $n \geq n_o$.

Consequently $\lim_n \sum_{i=1}^{\infty} |\mu_n(\{m\}) - \mu(\{m\})| = 0$.

Corollary 3. Let R be a ring of subsets of Ω with property (QI) and X a complete Hausdorff topological vector space. Suppose $\mu_n: R \to X$, $n \in N$, is a countably additive and strongly bounded vector measure. If $\lim_n \mu_n(E) = \mu(E)$ exists for $E \in R$, then μ is countably additive and the sequence (μ_n) is uniformly countably additive.

Proof. Let (E_i) be a disjoint sequence of members of $\mathcal{R}$ such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$. Then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_n \mu_n(\bigcup_{i=1}^{\infty} E_i) = \lim_n \sum_{i=1}^{\infty} \mu_n(E_i)$ since μ_n is countably additive for $n \in N$. Since X is complete the last part of corollary 1 implies that $\lim_n \sum_{i=1}^{\infty} \mu_n(E_i) = \sum_{i=1}^{\infty} \mu(E_i)$. Hence μ is countably additive.

Uniform countable additivity of (μ_n) follows from the fact that (μ_n) is uniformly strongly additive.

Corollary 4. Let X be a separable Banach space and $\mathcal{R}$ a ring of subsets of a set Ω with property (QI). If the vector measure $\mu: \mathcal{R} \to X$ is bounded, then μ is strongly bounded.

Proof. Suppose μ is not strongly bounded. Then there exists a sequence (E_n) of disjoint members of $\mathcal{R}$ and an $\varepsilon > 0$ such that:

(1) $\|\mu(E_n)\| > \varepsilon$ for $n \in N$.

By virtue of the Hahn-Banach theorem, there is $f_n \in X^*$ with $\|f_n\| = 1$ such that (2) $|f_n \circ \mu(E_n)| > \varepsilon$ for $n \in N$.

By 1.3 theorem 4, the unit disc of X^* is weak* compact and since X is separable it is metrizable with respect to the weak* topology.

Therefore there exists a subsequence (f_{n_i}) of (f_n) and $f \in X^*$ with

$\|f\| \leq 1$ such that $\lim_i f_{n_i} = f$ (weak*). This implies

(3) $\lim_i f_{n_i} \circ \mu(E) = f_o \mu(E)$ for $E \in \mathcal{R}$.

For each $i \in N$, $f_{n_i} \circ \mu$ is strongly bounded since $f_{n_i} \circ \mu$ is a scalar valued bounded measure. Therefore corollary 1 implies that $(f_{n_i} \circ \mu)$ is uniformly strongly bounded. This contradicts (2). Hence μ is strongly bounded.

Definition 2. Let $\mathcal{R}$ be a ring of subsets of a set Ω and X a topological vector space. A vector measure $\mu: \mathcal{R} \to X$ is called regular over finite sets if for every $A \in \mathcal{R}$ and every neighbourhood U at zero in X there exists a finite set $B \in \mathcal{R}$ such that $\mu(B) - \mu(A) \in U$.

Definition 3. A ring $\mathcal{R}$ of subsets of a set Ω is said to have property (FQI) if for every disjoint sequence (A_n) of finite sets in $\mathcal{R}$ and every sequence (B_n) in $\mathcal{R}$ with $A_m \cap B_n = \emptyset$ for $m,n \in N$, there exists a subsequence (A_{n_i}) of (A_n) and $C \in \mathcal{R}$ such that:

$$\bigcup_{i=1}^{\infty} A_{n_i} \subseteq C, \quad C \cap (\bigcup_{n=1}^{\infty} B_n) = \emptyset \text{ and } C \cap A_n = \emptyset \text{ for } n \notin \{n_1, n_2, \ldots\}.$$

Remark. In theorem 1 and subsequent corollaries property (QI) can be replaced by property (FQI) provided that measures concerned are regular over finite sets.

The remainder of this section is devoted to discussing the implications of Theorem 1 and its corollaries in matrix summability theory.

Every series in a topological vector space gives rise naturally to a definition of a vector measure. The domain of this type of a vector measure is determined by the nature of convergence of the series. In this context we use the notion of full classes to obtain certain results concerning matrix summability. The notion of full classes was introduced by J.J. Sember and A. Freedman in their paper [17].

Definition 4. A ring $\mathcal{R}$ of subsets of N is called full in case whenever (x_n) is a sequence of real numbers for which $\sum_{n\in A} x_n$ exists for $A \in \mathcal{R}$, then $\sum_{n=1}^{\infty} |x_n| < \infty$.

The above definition is slightly different from the definition of a full class given in [17].

Remark. Let $\mathcal{R}$ be a full ring. If (x_n) is a sequence of complex numbers such that $\sum_{n\in A} x_n$ exists for $A \in \mathcal{R}$, then $\sum_{n=1}^{\infty} |x_n| < \infty$.

Proposition 1. Let $\mathcal{R}$ be a full ring and X a Banach space containing no copy of c_0. If (x_n) is a sequence in X such that $\sum_{n\in A} x_n \in X$ for $A \in \mathcal{R}$, then $\sum_{n=1}^{\infty} x_n$ is subseries convergent.

Proof. Suppose (x_n) is a sequence in X with $\sum_{n\in A} x_n \in X$ for $A \in \mathcal{R}$. Let $f \in X^*$. Then $\sum_{n\in A} f(x_n)$ converges for $A \in \mathcal{R}$. Since $\mathcal{R}$ is full, $\sum_{n=1}^{\infty} |f(x_n)| < \infty$. Therefore $\sum_{n=1}^{\infty} x_n$ is subseries convergent by Corollary 4 of 3.3 Theorem 1.

In the remainder of this section $\mathcal{R}$ denotes a ring of subsets of N .

Proposition 2. Let $\mathcal{R}$ be a $Q\sigma$-ring containing all finite subsets of N and X a complete topological vector space. If (x_n) is a sequence in X such that $\sum_{n \in A} x_n \in X$ for $A \in \mathcal{R}$, then $\sum_{n \in A} x_n$ is subseries convergent.

Proof. Let (x_n) be a sequence in X such that $\sum_{n \in A} x_n \in X$ for $A \in \mathcal{R}$. For each $n \in N$ define $\mu_n : \mathcal{R} \to X$ by

$$\mu_n(A) = \sum_{i \in A \cap [1,n]} x_i .$$

Clearly (μ_n) is a sequence of strongly bounded vector measures which converges setwise on $\mathcal{R}$ to μ defined by $\mu(A) = \sum_{n \in A} x_n$ for $A \in \mathcal{R}$. It follows by corollary 1 of theorem 1, that μ is also strongly bounded. Since X is complete, μ is strongly additive. Hence $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \mu(\{n\})$ is subseries convergent.

Next we establish our generalization of the Schur lemma.

Theorem 2. Let $\mathcal{R}$ be a $Q\sigma$-ring containing all finite subsets of N and (x_{mn}) a infinite matrix in a complete topological vector space X . Assume that $\sum_{n \in E} x_{mn}$ exists for $E \in \mathcal{R}$ and $m \in N$. If $\lim_m \sum_{n \in E} x_{mn}$

exists for $E \in \mathcal{R}$, then (i) $\lim_m x_{mn} = x_n$ exists for $n \in N$

(ii) $\lim_n \sum_{n \in E} x_{mn} = \sum_{n \in E} x_n$ uniformly for $E \subseteq N$.

Proof. (i) directly follows from the fact that $\lim_m \sum_{n \in E} x_{mn}$ exists for every $E \in \mathcal{R}$. Since $\sum_{n \in E} x_{mn}$ exists for every $E \in \mathcal{R}$, the previous proposition implies that $\sum_{n=1}^{\infty} x_{mn}$ is subseries convergent. Define $\mu_m: \mathcal{R} \to X$ by $\mu_m(A) = \sum_{n \in A} x_{mn}$ for $m \in N$, and $\mu: \mathcal{R} \to X$ by $\mu(A) = \lim_m \sum_{n \in A} x_{mn}$. Since $\sum_{n=1}^{\infty} x_{mn}$ is subseries convergent, μ_m is a strongly bounded vector measure. Now letting $(E_n) = (\{n\})$, we apply the last part of Corollary 1 of Theorem 1 to have $\lim_m \sum_{m \in A} \mu_m(\{n\}) = \sum_{n \in A} \mu(\{n\})$ uniformly for $A \subseteq N$.

i.e., $\lim_m \sum_{n \in A} x_{mn} = \sum_{n \in A} \lim_m x_{mn}$

$$= \sum_{n \in A} x_n \text{ uniformly for } A \subseteq N .$$

Remark. If $\mathcal{R} = 2^N$, in view of remark 1 after corollary 1, we can drop the completeness assumption in the above theorem.

The following example shows that $\mathcal{R}$ can not be replaced by any ring containing all finite sets.

Let $\mathcal{A}$ be the ring of all finite subsets of N and let

$$(x_{mn}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots\cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots\cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots\cdots \end{pmatrix}$$

Then $\lim_n \sum_{m\in A} x_{nm} = 0$ for every $A \in \mathcal{A}$, but clearly the conclusion of theorem 2 does not hold for the matrix (x_{mn}).

Corollary 1. Let (x_{mn}) be as in Theorem 2. The series $\sum_{n=1}^{\infty} x_{mn}$, $m = 1,2,\ldots$, are unordered uniformly convergent in the sense that if $\varepsilon > 0$, then there exists $n_o \in N$ such that $\|\sum_{n\in E} x_{mn}\| < \varepsilon$ for every m whenever $\text{Min } E \geq n_o$.

Proof. By Theorem 2, $\lim_m \sum_{n\in E} x_{mn} = \sum_{n\in E} x_n$ uniformly for $E \subseteq N$. Therefore the sequence $(\sum_{n\in E} x_{mn})_{m\in N}$ is uniformly Cauchy for $E \subseteq N$ and hence there exists $m_o \in N$ such that (1) $\|\sum_{n\in E} (x_{mn} - x_{kn})\| < \varepsilon/2$ for $m,k \geq m_o$ and $E \subseteq N$. Now we show that for each $m \in N$ there exists $p_m \in N$ such that:

(2) $\|\sum_{n\in E} x_{mn}\| < \varepsilon/2$ for $\text{Min } E \geq p_m$.

For, suppose the contrary. Then there exists a sequence (E_i) of subsets of N such that $\lim_i \operatorname{Min} E_i = \infty$ and $\|\sum_{n\in E_i} x_{mn}\| \geq \varepsilon/2$. For each i choose a finite subset F_i of E_i such that $\|\sum_{n\in F_i} x_{mn}\| > \varepsilon/3$ and notice that $\lim_i \operatorname{Min} F_i = \infty$. Set $G_1 = F_1$. Choose $i_2 \in N$ such that $\operatorname{Max} G_1 < \min F_{i_2}$ and set $G_2 = F_{i_2}$. Inductively we can construct a disjoint sequence (G_i) of finite sets such that $\operatorname{Max} G_i < \operatorname{Min} G_{i+1}$ and $\|\sum_{n\in G_i} x_{mn}\| > \varepsilon/3$ for $i \in N$. This contradicts the fact that $\sum_{n=1}^{\infty} x_{mn}$ is subseries convergent.

Now let $p = \operatorname{Max}\{p_1, p_2, \ldots\ldots, p_{m_o}\}$. Then, by (2),

(3) $\|\sum_{n\in E} x_{mn}\| < \varepsilon/2$ for $1 \leq m \leq m_o$ and $\operatorname{Min} E \geq p$.

For $m > m_o$ (4) $\|\sum_{n\in E} x_{mn}\| \leq \|\sum_{n\in E} (x_{mn} - x_{m_o n})\| + \|\sum_{n\in E} x_{m_o n}\| < \varepsilon/2 + \varepsilon/2$

for $\operatorname{Min} E \geq p$ by (1) and (3).

The result follows from (3) and (4).

We next show that theorem 2 can be viewed as a generalization of the classical version of the Schur lemma.

Corollary 2. Let (x_{mn}) be an infinite matrix of complex numbers.

Suppose $\sum_{n=1}^{\infty} |x_{mn}| < \infty$ for every $m \in N$. If $\lim_m \sum_{n \in E} x_{mn}$ exists, for each $E \subseteq N$ and if $\lim_m x_{mn} = x_n$ for each $n \in N$, then

(i) $\lim_m \sum_{n=1}^{\infty} |x_{mn} - x_n| = 0$ and

(ii) the series $\sum_{n=1}^{\infty} |x_{mn}|$, $m = 1,2,\ldots$, converge uniformly in m.

Proof. (i) Let $\varepsilon > 0$. By Theorem 2, there exists $m_o \in N$ such that $\left|\sum_{n \in E} (x_{mn} - x_n)\right| < \varepsilon/8$ for $m \geq m_o$ and $E \subseteq N$. Therefore by 3.2 Lemma 1, $\sum_{n=1}^{\infty} |x_{mn} - x_n| < \varepsilon$ for $m \geq m_o$.

(ii) By Corollary 1, there exists $p \in N$ such that $\left|\sum_{n \in E} x_{mn}\right| < \varepsilon/8$ for $m \in N$ and $\text{Min } E \geq p$. Hence $\sum_{n=p}^{\infty} |x_{mn}| < \varepsilon$ for $m \in N$. This implies (ii).

Remark. (ii) implies that $\sup_m \sum_{n=1}^{\infty} |x_{mn}| < \infty$.

CHAPTER 4

THE NIKODYM BOUNDEDNESS THEOREM

§1. Introduction.

The subject of this chapter is one of the truly impressive theorems of measure theory, the Nikodym Boundedness Theorem, which derives a conclusion of uniform boundedness from a hypothesis concerning setwise boundedness. It also has a strong impact on the theory of Banach spaces. ~~The validity of this theorem depends entirely on the structure of the~~ ring on which measures are defined. An algebraic characterization of such structures is still unknown. The recent developments in this area are largely contributed by the papers of G.L. Seever [16], Barbara Faires [7], R.B. Darst [5] and Corneliu Constantinescu [3]. Constantinescu obtained this theorem for measures defined on a $Q\sigma$-ring (see 1.5 Definition 2). Although a $Q\sigma$-ring has a nice algebraic structure, it is extremely difficult to construct such a ring explicitly. One aim in this chapter is to prove the Nikodym Boundedness Theorem for a more general class of rings, namely $PQ\sigma$-rings. Unlike the class of $Q\sigma$-rings, this class contains some well known examples of rings of sets. In this chapter we also deal with the measures defined on substructures of 2^N. These measures are especially important in summability theory. Some of the results in this chapter appear in the joint paper [15] by J.J. Sember and myself.

§2. Definitions and some examples.

The purpose of this section is to study a new class of rings of sets introduced below. It will be shown in the next section that the Nikodym Boundedness Theorem holds for measures defined on this type of ring. One of the important features of this class is that it contains some well-known examples of rings of sets. In what follows, unless signified otherwise, $\mathcal{R}$ denotes a ring of subsets of a set Ω.

Definition 1. A ring $\mathcal{R}$ is called a PQσ-ring (respectively, an FPQσ-ring) in case for every disjoint sequence (A_n) of sets (respectively, finite sets) in $\mathcal{R}$ and every sequence (t_n) of real numbers with $\lim_n t_n = \infty$ there exists a subsequence (A_{n_i}) of (A_n) satisfying the following:

For each i there is a partition $A_1^{n_i}, A_2^{n_i}, \ldots, A_{s_i}^{n_i}$ of A_{n_i} ($s_i \leq t_{n_i}$ and $A_1^{n_i}, A_2^{n_i}, \ldots, A_{s_i}^{n_i} \in \mathcal{R}$) such that $\bigcup_{i=1}^{\infty} A_{k_i}^{n_i} \in \mathcal{R}$ for every sequence (k_i) with $1 \leq k_i \leq s_i$.

Remark. It is easy to verify that every Qσ-ring is a PQσ-ring and that every FQσ-ring is an FPQσ-ring.

Example 1. An increasing sequence (p_n) of positive integers is called lacunary if $\lim_n (p_{n+1} - p_n) = \infty$. We show that the ring $\mathcal{L}$ of subsets of $\mathbb{N}$ generated by lacunary sequences is FPQσ but not PQσ. To this end let (A_n) be a sequence of pairwise disjoint finite subsets of $\mathbb{N}$ and (t_n) a sequence of positive integers with $\lim_n t_n = \infty$.

Choose a subsequence (A_{n_i}) of (A_n) such that $\text{Max } A_{n_i} + i < \text{Min } A_{n_{i+1}}$ and then partition each $A_{n_i} = \{p_1 < p_2 < \ldots < p_k\}$ in to $A_1^{n_i}, A_2^{n_i}, \ldots\ldots, A_{t_{n_i}}^{n_i}$ such that:

$$A_1^{n_i} = \{p_1, p_{1+t_{n_i}}, p_{1+2t_{n_i}}, \ldots\ldots\}$$

$$A_2^{n_i} = \{p_2, p_{2+t_{n_i}}, p_{2+2t_{n_i}}, \ldots\ldots\}$$

$$\vdots$$

$$A_{t_{n_i}}^{n_i} = \{p_{t_{n_i}}, p_{2t_{n_i}}, p_{3t_{n_i}}, \ldots\ldots\}.$$

It is readily seen that $\bigcup_{i=1}^{\infty} A_{k_i}^{n_i}$ is lacunary for every sequence (k_i) with $1 \le k_i \le t_{n_i}$. This shows that L is FPQC.

To show that L is not PQC, let $A_o = \{p_1 < p_2 < \ldots\ldots\}$ be an infinite lacunary sequence. Setting $A_n = (A_o+n) \setminus (A_o \cup A_1 \cup \ldots\ldots \cup A_{n-1})$, where $A_o + n = (p_i + n)_{i \in N}$, we can define inductively the disjoint sequence (A_n) in L.

Let (A_{n_i}) be a subsequence of (A_n). Further for each $i \in N$ let $A_1^{n_i}, A_2^{n_i}, \ldots\ldots, A_{s_i}^{n_i}$ be any finite partition of A_{n_i}. We show that there is a sequence (k_i), where $1 \le k_i \le s_i$, such that $\bigcup_{i=1}^{\infty} A_{k_i}^{n_i} \notin L$. Since A_{n_1} is infinite there exists $1 \le k_1 \le s_1$ such that $A_{k_1}^{n_1}$ is infinite. Consequently $A_{k_1}^{n_1} = F_1 + n_1$ for some infinite subset F_1 of A_0. A similar argument shows that there exists $1 \le k_2 \le s_2$ and an infinite subset F_2 of F_1 such that $F_2 + n_2 \subseteq A_{k_2}^{n_2}$. Inductively we can construct a decreasing sequence (F_i) of infinite subsets of A_0 and a sequence (k_i) of positive integers such that $F_i + n_i \subseteq A_{k_i}^{n_i}$.

Suppose $\bigcup_{i=1}^{\infty} A_{k_i}^{n_i} = N_1 \cup N_2 \cup \ldots\ldots \cup N_p$ where $N_i = (p_m^i)_{m\in N}$, $1 \le i \le p$, are lacunary sequences. Since $F_{p+1} + n_{p+1} \subseteq A_{k_{p+1}}^{n_{p+1}}$, there exists i_1 such that $N_{i_1} \cap (F_{p+1} + n_{p+1})$ is infinite. Consequently, there is an infinite $G_{p+1} \subseteq F_{p+1}$ such that $G_{p+1} + n_{p+1} \subseteq N_{i_1}$. Since (F_i) is a decreasing sequence of sets, $G_{p+1} \subseteq F_{p+1} \subseteq F_p$ and hence $G'_{p+1} + n_p \subseteq A_{k_p}^{n_p}$. Also since N_{i_1} is is lacunary and $G_{p+1} + n_{p+1} \subseteq N_{i_1}$, $(G_{p+1} + n_p) \cap N_{i_1}$ is finite. Therefore, there exists $i_2 (\neq i_1)$ such that $(G_{p+1} + n_p) \cap N_{i_2}$ is infinite. Consequently there is an infinite $G_p \subseteq G_{p+1}$ such that $G_p + n_p \subseteq N_{i_2}$. Proceeding in this manner we can find an infinite set $G_1 \subseteq F_{p+1} \subseteq F_1$ at the $(p+1)$th step such that $(G_1 + n_1) \cap N_i$ is finite for $1 \leq i \leq p$. This contradiction shows that L is not $PQ\sigma$.

Example 2. Let $A \subseteq N$. We denote by $A(n)$ the number of elements of $A \cap \{1,2,\ldots\ldots n\}$. A is said to be a set of zero density if $\lim_n \frac{A(n)}{n} = 0$. We show that the class of sets of zero density, denoted by η_δ^o, is a $PQ\sigma$-ring.

Let (A_n) be a disjoint sequence of members of η_δ^o and (t_n) a sequence of real numbers with $\lim_n t_n = \infty$. If (A_n) has a subsequence (A_{n_i}) consisting of finite sets, then it can be easily shown that the sequence (A_n) satisfies the condition given in the definition of a $PQ\sigma$-ring. So let us assume that all A_n's are infinite.

Set $n_1 = 1$. Suppose $n_1 < n_2 < \ldots\ldots < n_i$ have been chosen. Now choose $n_{i+1} > n_i$ such that:

1. $A_{n_k}(n)/n < 1/2^{i+1}$ for $1 \leq k \leq i$ and $n \geq n_{i+1}$.

2. $\operatorname{Min} A_{n_{i+1}} > n_i$.

3. $t_{n_{i+1}} > 2^{i+2}$.

4. $A_{n_i}(n_{i+1}) > 2^{i+1}$.

Such an n_{i+1} exists since (1)' $A_{n_1}, A_{n_2}, \ldots, A_{n_i}$ are of density zero, (2)' $\lim_n \operatorname{Min} A_n = \infty$, (3)' $\lim_n t_n = \infty$ and (4)' A_{n_i} is infinite.

Inductively we can construct a subsequence (A_{n_i}) of (A_n) satisfying conditions (1), (2), (3) and (4).

Partition each $A_{n_i} = \{p_1 < p_2 < \ldots\ldots\}$ into $A_1^{n_i}, A_2^{n_i}, \ldots\ldots, A_{2^{i+1}}^{n_i}$ in the following way.

$$A_1^{n_i} = \{p_1, p_{1+2^{i+1}}, p_{1+2.2^{i+1}}, \ldots\ldots\}$$

$$A_2^{n_i} = \{p_2, p_{2+2^{i+1}}, p_{2+2.2^{i+1}}, \ldots\ldots\}$$

$$\vdots$$

$$A_{2^{i+1}}^{n_i} = \{p_{2^{i+1}}, p_{2.2^{i+1}}, p_{3.2^{i+1}}, \ldots\ldots\} .$$

If $n \geq n_{i+1}$, (4) assures that $A_{n_i}(n) > 2^{i+1}$. Therefore, by the way A_{n_i} is partitioned, $A_k^{n_i}(n) \leq A_{n_i}(n)/_{2^i}$ for $1 \leq k \leq 2^{i+1}$.

Let i be a fixed positive integer and $j > i$. Then for

$1 \le k \le 2^{i+1}$ and for $n \ge n_j > n_i$ we have

$$\text{(i)} \quad A_k^{n_i}(n)/n \le A_{n_i}(n)/_{2^i.n} \le 1/_{2^i.2^j} .$$

The last inequality follows from (1). Also for any $n \in N$ and $1 \le k \le 2^{i+1}$ we have $A_k^{n_i}(n) \le 1$ or $A_k^{n_i}(n) \le A_{n_i}(n)/_{2^i}$ and hence

$$\text{(ii)} \quad A_k^{n_i}(n)/n \le \text{Max}\{1/n, A_{n_i}(n)/_{2^i n}\} \le \text{Max}\{1/n, \frac{1}{2^i}\} .$$

Let (k_i) be a sequence of positive integers such that $1 \le k_i \le 2^{i+1}$. We show that $\bigcup_{i=1}^{\infty} A_{k_i}^{n_i}$ is of density zero. Let $j \in N$ and $n_j < n \le n_{j+1}$. Then by (2) $A_{n_i} \cap [1,n] = \phi$ for $i > j+1$. Therefore,

$$\left(\bigcup_{i=1}^{\infty} A_{k_i}^{n_i}\right)(n)/n = \left(\bigcup_{i=1}^{j+1} A_{k_i}^{n_i}\right)(n)/n$$

$$= \sum_{i=1}^{j+1} A_{k_i}^{n_i}(n)/n \quad \text{(the } A_{k_i}^{n_i}\text{'s are disjoint)}$$

$$= \sum_{i=1}^{j-1} A_{k_i}^{n_i}(n)/n + A_{k_j}^{n_j}(n)/n + A_{k_{j+1}}^{n_{j+1}}(n)/n$$

$$\le \sum_{i=1}^{j-1} \frac{1}{2^{i+j}} + \text{Max}\{\frac{1}{n}, \frac{1}{2^j}\} + \text{Max}\{\frac{1}{n}, \frac{1}{2^{j+1}}\} .$$

The last unequality follows from (i) and (ii).

The right hand side of the inequality tends to zero as j goes to infinity. Hence $\lim_n \left(\bigcup_{i=1}^{\infty} A^{n_i}_{k_i} \right)(n)/n = 0$. This shows that η^{o}_{δ} is a PQσ-ring.

We conclude this section with the following proposition.

Proposition 1. Every FPQσ-ring $\mathcal{R}$ of subsets of N containing all finite sets is full.

Proof. Let (x_n) be a sequence of positive real numbers with $\sum_{n=1}^{\infty} x_n = \infty$. Choose positive integers $n_1 < n_2 < \ldots\ldots < n_i < \ldots$ such that

$$\sum_{n_i \le k < n_{i+1}} x_k > i \quad \text{for} \quad i = 1,2,\ldots\ldots .$$

Let $(t_i) = (i)$ and $A_i = \{n_i, n_i+1, \ldots\ldots, n_{i+1}-1\}$. Now for any partition $A^i_1, A^i_2, \ldots\ldots, A^i_{s_i}$ $(s_i \le i)$ of A_i there exists $1 \le k_i \le s_i$ such that $\sum_{k \in A^i_{k_i}} x_k \ge 1$. This completes the proof.

Remark. For subrings of 2^N containing all finite sets we have

$$\begin{array}{c} \{\text{Full rings}\} \supseteq \{\text{FPQ}\sigma\text{-rings}\} \underset{\neq}{\supset} \{\text{PQ}\sigma\text{-rings}\} \underset{\neq}{\supset} \{\text{Q}\sigma\text{-ring}\} \\ \cup\!\!\!\!\!/ \\ \{\text{FQ}\sigma\text{-rings}\} \end{array}$$

We have not come up with an example of a full ring which is not FPQσ .

§3. Main results.

Although the Nikodym Boundedness Theorem is subject to many generalizations, it is difficult to find one generalization that fits the others. We therefore consider here several situations for which the theorem holds.

Theorem 1. (Nikodym Boundedness Theorem).

Let $\mathcal{R}$ be a ring of subsets of a set Ω satisfying one of the following:

(a) $\mathcal{R}$ has property (QI).

(b) $\mathcal{R}$ is a PQσ-ring with the hereditary property.

Also let X be a locally convex space. Suppose $\mu_n: \mathcal{R} \to X$, $n = 1,2,\ldots$, are bounded vector measures such that $\{\mu_n(A) \mid n \in N\}$ is a bounded subset of X for every $A \in \mathcal{R}$. Then $\{\mu_n(A) \mid n \in N$ and $A \in \mathcal{R}\}$ is a bounded subset of X.

In addition, if the μ_n, $n = 1,2,\ldots$, are regular over finite sets, then (a) and (b) can be replaced by the following:

(a') $\mathcal{R}$ has property (FQI).

(b') $\mathcal{R}$ is an FPQσ-ring with the hereditary property.

(c') $\mathcal{R}$ is a full ring with the hereditary property and containing all finite sets. (In this case $\Omega = N$.).

Proof. First we establish the theorem for scalar valued measures; i.e., we assume that $X = \mathbb{C}$.

Suppose (1) $\sup\{|\mu_n(A)| \,|\, n \in N \text{ and } A \in \mathcal{R}\} = \infty$.

Define $\alpha: \mathcal{R} \to R+$ by $\alpha(A) = \sup_n |\mu_n(A)|$. Since the sequence (μ_n) is setwise bounded, α is defined and, moreover, by (1) α is unbounded. We also show that:

(2) $\alpha(A \cup B) \le \alpha(A) + \alpha(B)$ for $A,B \in \mathcal{R}$ with $A \cap B = \phi$.

(3) $|\alpha(B) - \alpha(A)| \le \alpha(B \setminus A)$ for $A,B \in \mathcal{R}$ with $A \subseteq B$.

Let $A,B \in \mathcal{R}$.

(2) If $A \cap B = \phi$, then $\alpha(A \cup B) = \sup_n |\mu_n(A \cup B)| \le \sup_n |\mu_n(A)| + \sup_n |\mu_n(B)|$ $= \alpha(A) + \alpha(B)$.

(3) Suppose $A \subseteq B$. For given $\varepsilon > 0$ there exists $n_o \in N$ such that $\alpha(B) - \varepsilon < |\mu_{n_o}(B)|$.

Hence $\alpha(B) - \alpha(A) - \varepsilon < |\mu_{n_o}(B)| - |\mu_{n_o}(A)| \le |\mu_{n_o}(B \setminus A)| \le \alpha(B \setminus A)$;

consequently $\alpha(B) - \alpha(A) \le \alpha(B \setminus A)$. Similarly $\alpha(A) - \alpha(B) \le \alpha(B \setminus A)$.

Now an application of 1.5 Lemma 1 to α shows that there exists a disjoint sequence (E_m) of members of $\mathcal{R}$ such that $\lim_m \alpha(E_m) = \infty$.

Thus by the definition of α we can find subsequences (μ_{n_i}) and (E_{m_i}) of (μ_n) and (E_m) respectively such that $\lim_i |\mu_{n_i}(E_{m_i})| = \infty$. For simplicity we relabel the sequence $(\mu_{n_i}(E_{m_i}))_{i \in N}$ by $(\mu_i(E_i))_{i \in N}$. Then we have

(4) $\lim_i |\mu_i(E_i)| = \infty$.

First we consider case (a) $\mathcal{R}$ is a ring with property (QI). Let (t_i) be a sequence of positive numbers with the limit zero such that:

(5) $\lim_i | t_i \mu_i(E_i) | = \infty$.

It is readily seen that $(t_i \mu_i)$ is a sequence of strongly bounded scalar valued measures (note that every bounded scalar valued measure is strongly bounded) with $\lim_i t_i \mu_i(E) = 0$ for every $E \in \mathcal{R}$. Therefore 3.3 Theorem 1 implies that $\lim_i t_j \mu_j(E_i) = 0$ uniformly for $j \in N$. This contradicts (5). Hence the Nikodym Boundedness Theorem holds when $\mathcal{R}$ is a ring with property (QI).

Now we consider case (b) $\mathcal{R}$ is a PQσ-ring with the hereditary property. Recall (4) $\lim_n |\mu_n(E_n)| = \infty$. Let $t_n = |\mu_n(E_n)|^{1/2}$. Since $\mathcal{R}$ is a PQσ-ring, there exists a subsequence (E_{n_i}) of (E_n) and a partition $E_1^{n_i}, E_2^{n_i}, \ldots\ldots, E_{s_i}^{n_i}$ $(s_i \leq t_{n_i})$ of each E_{n_i} such that $\bigcup_{i=1}^{\infty} E_{k_i}^{n_i} \in \mathcal{R}$ for every sequence (k_i) with $1 \leq k_i \leq s_i$. For each $i \in N$ we have $t_{n_i}^2 = |\mu_{n_i}(E_{n_i})| = |\mu_{n_i}(E_1^{n_i}) + \mu_{n_i}(E_2^{n_i}) + \ldots\ldots + \mu_{n_i}(E_{s_i}^{n_i})| \leq |\mu_{n_i}(E_1^{n_i})| + |\mu_{n_i}(E_2^{n_i})| + \ldots\ldots + |\mu_{n_i}(E_{s_i}^{n_i})|$. Since $s_i \leqq t_{n_i}$, there exists $1 \leq k_i \leq s_i$ such that $|\mu_{n_i}(E_{k_i}^{n_i})| \geq t_{n_i}$.

Let $E_{k_i}^{n_i} = A_i$. Then (6) $\bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$ and $\lim_i |\mu_{n_i}(A_i)| = \infty$.

Since $\mathcal{R}$ is hereditary, $\bigcup_{p \in P} A_p \in \mathcal{R}$ for $P \subseteq N$. Let $\nu_i : 2^N \to C$ be defined by $\nu_i(P) = \mu_{n_i}(\bigcup_{p \in P} A_p)$. Since (μ_{n_i}) is a sequence of bounded scalar valued measures with $\operatorname{Sup}_i |\mu_{n_i}(E)| < \infty$ for every $E \in \mathcal{R}$, it readily follows that (ν_i) is a sequence of bounded scalar valued measures with $\operatorname{Sup}_i |\nu_i(P)| < \infty$ for every $P \subseteq N$. Since 2^N is a σ-algebra (hence it is a ring with property (QI)), we have $\operatorname{Sup}\{|\nu_i(P)| \,|\, i \in N \text{ and } P \subseteq N\} < \infty$. This contradicts that $\lim_i |\nu_i(\{i\})| = \lim_i |\mu_{n_i}(A_i)| = \infty$. Hence the Nikodym Boundedness Theorem holds when $\mathcal{R}$ is a PQσ-ring with the hereditary property.

To prove the last part let us assume that the μ_n, $n = 1,2,\ldots\ldots$, are regular over finite sets. Then in (3) (E_n) can be replaced by a disjoint sequence (F_n) of finite sets in $\mathcal{R}$, so we have

(7) $\lim_n |\mu_n(F_n)| = \infty$.

Now cases (a') and (b') can be treated exactly the same way we treated cases (a) and (b). Therefore we only have to consider case (c') $\mathcal{R}$ is a full ring with the hereditary property and containing all finite sets. Perhaps by passing to a subsequence we can assume that in (7), $|\mu_n(F_n)| > 2^N$ and $\min F_{n+1} > \operatorname{Max} F_n$. Since $\lim_n |\mu_n(F_n)| = \infty$ implies that $\lim_n |\operatorname{Re} \mu_n(F_n)| = \infty$ or $\lim_n |\operatorname{Im} \mu_n(F_n)| = \infty$, we also can assume that the μ_n; $n = 1,2,\ldots\ldots$, are real valued measures.

Let $t_n = \frac{1}{n\,\mu(F_n)}$ for each n.

Then (8) $\sum_{n=1}^{\infty} |t_n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$ and $\sum_{n=1}^{\infty} t_n\, n\,\mu(F_n) = \infty$.

Since F_n is finite for $n \in N$, $\sum_{n=1}^{\infty} t_n\, n\,\mu(F_n) = \sum_{n=1}^{\infty} t_n \sum_{i \in F_n} n\,\mu(\{i\})$.

$$\text{Let } p_n = \begin{cases} t_m\, m\,\mu(\{n\}) & \text{if } n \in F_m \text{ for some } m . \\ 0 & \text{if } n \notin \bigcup_{m=1}^{\infty} F_m . \end{cases}$$

From (8) it is clear that $\sum_{n=1}^{\infty} |p_n| = \infty$. Without loss of generality we can assume that $\sum_{n=1}^{\infty} p_n^+ = \infty$ where $p_n^+ = \text{Max}\{p_n, 0\}$. Since $\mathcal{R}$ is full, there exists $A \in \mathcal{R}$ such that $\sum_{n \in A} p_n^+ = \infty$. Since $\mathcal{R}$ is hereditary we can choose A such that $p_n^+ > 0$ for every $n \in A$. Then clearly $A \subseteq \bigcup_{n=1}^{\infty} F_n$. Let $G_n = A \cap F_n$ for $n \in N$.

Then $\sum_{n \in A} p_n^+ = \sum_{n=1}^{\infty} t_n \sum_{i \in G_n} n\,\mu(\{i\}) = \sum_{n=1}^{\infty} t_n\, n\,\mu(G_n) = \infty$. Since $\sum_{n=1}^{\infty} |t_n| < \infty$, this implies that $\text{Sup}_n\, n\,|\mu(G_n)| = \infty$. Also notice that (G_n) is a disjoint sequence in $\mathcal{R}$ such that $\bigcup_{n=1}^{\infty} G_n = A \in \mathcal{R}$. To complete the proof one can follow the last portion of the proof for the case $\mathcal{R}$ is a PQσ-ring.

To extend the Nikodym Boundedness Theorem for locally convex spaces, let X be a locally convex space as stated in the theorem. Suppose $\| \ \|$ is a continuous seminorm on X. Now consider the following collection of bounded scalar valued measures defined on $\mathcal{R}$.

$$\mathcal{G} = \{f \circ \mu_n | f \in (X, \| \ \|)^*, \ \|f\| \le 1 \ \text{and} \ n \in N\} .$$

We show that $\mathcal{G}$ is uniformly bounded on $\mathcal{R}$. Let $(f_i \circ \mu_{n_i})_{i \in N}$ be a sequence in $\mathcal{G}$. Then for each $E \in \mathcal{R}$,

$$\sup_i |f_i \circ \mu_{n_i}(E)| \le \sup_i \|\mu_{n_i}(E)\| \quad \text{since} \quad \|f_i\| \le 1 .$$

$$< \infty \quad \text{since} \quad (\mu_{n_i}) \text{ is setwise bounded.}$$

Since the Nikodym Boundedness Theorem is true for scalar valued measures defined on $\mathcal{R}$, we have $(f_i \circ \mu_{n_i})$ is uniformly bounded on $\mathcal{R}$. Hence $\mathcal{G}$ is uniformly bounded on $\mathcal{R}$. This implies $\sup\{\|\mu_n(E)\| \,|\, n \in N$ and $E \in \mathcal{R}\} < \infty$ since $\|\mu_n(E)\| = \sup\{|f \circ \mu_n(E)| \,\big|\, f \in (X, \| \ \|)^*$ and $\|f\| \le 1\}$ by virtue of the Hahn-Banach theorem. Since $\| \ \|$ is an arbitrary continuous seminorm on X, the sequence $(\mu_n)_{n \in N}$ is uniformly bounded on $\mathcal{R}$.

Remark 1. If $\mathcal{R}$ is a δ-ring; i.e., closed under countable intersection, then the hereditary property in cases (b) and (b') may be dropped.

2. The Nikodym Boundedness Theorem is true for any sequence of vector measures for which the Vitali-Hahn-Saks-Nikodym theorem is true. The following example shows that the converse does not hold.

Let η_δ^o be the ring of sets of zero density. We have shown that, in section 2, η_δ^o is a PQσ-ring. Also it is easy to check that η_δ^o is hereditary. Let $\mu_n : \eta_\delta^o \to [0,1]$, $n = 1,2,\ldots\ldots$, be defined by $\mu_n(A) = \frac{A(n)}{n}$, where $A(n)$ is the number of elements of $A \cap [1,n]$. Then clearly (μ_n) is a sequence of strongly bounded measures such that $\lim_n \mu_n(A) = 0$. But it is easy to construct inductively a disjoint sequence (A_i) of finite sets and a subsequence (μ_{n_i}) of (μ_n) such that $\lim_i \mu_{n_i}(A_i) \neq 0$. Let $A_1 = \{1\}$ and $n_1 = 1$. Suppose disjoint finite sets $A_1, A_2, \ldots, A_i$ and positive integers $n_1 < n_2 < \ldots\ldots < n_i$ have been chosen such that $\mu_{n_j}(A_j) > \frac{1}{2}$ for $j = 1,2,\ldots,i$. Choose $n_{i+1} (> n_i)$ such that $\mu_{n_{i+1}}(A_1 \cup A_2 \cup \ldots \cup A_i) < \frac{1}{2}$. Set $A_{i+1} = \{1,2,\ldots\ldots,n_{i+1}\} \setminus (A_1 \cup A_2 \cup \ldots\ldots \cup A_i)$. Clearly $\mu_{n_{i+1}}(A_{i+1}) > \frac{1}{2}$.

The following corollary is useful in applications.

Corollary 1. Let $\mathcal{R}$ be a ring of sets as stated in (a) or (b) of Theorem 1 and X a Banach space. Suppose $\mu: \mathcal{R} \to X$ is a function such that $f \circ \mu$ is bounded and finitely additive for every f in some total subset Γ of X^*. Then μ is a bounded vector measure. In addition, if $f \circ \mu$ is regular over finite sets for $f \in \Gamma$ and if X is separable, then the conclusion remains true if $\mathcal{R}$ is as in (c') of Theorem 1.

Proof. To show that μ is finitely additive, let A,B be two disjoint members of $\mathcal{R}$. Since $f \circ \mu$ is finitely additive for $f \in \Gamma$, we have

$f(\mu(A \cup B)) = f\circ\mu(A \cup B) = f\circ\mu(A) + f\circ\mu(B) = f(\mu(A) + \mu(B))$. Since Γ is total this implies $\mu(A \cup B) = \mu(A) + \mu(B)$.

To show that μ is bounded, let $M = \{f \in X^* | f\circ\mu$ is bounded$\}$. Then M is a linear subspace of X^* containing the total set Γ; consequently M is a weak*-dense linear subspace of X^* by 1.3 Theorem 3. If it can be shown that $M_1 = \{f \in M | \|f\| \leq 1\}$ is weak* closed, then an appeal to 1.3 Theorem 5 (Banach-Dieudonne Theorem) establishes that M is a weak* closed subset of X^* and hence $M = X^*$. Let $(f_\alpha)_{\alpha\in\Lambda}$ be a net in M_1 such that $\lim_\alpha f_\alpha = f_1$ exists in the weak* topology on X^*. Then $\lim_\alpha f_\alpha(x) = f_1(x)$ for every $x \in X$. Since $\|f_\alpha\| \leq 1$ for each $\alpha \in \Lambda$, this implies $\|f_1\| \leq 1$.

To show that $f_1 \circ \mu$ is bounded we apply the Nikodym Boundedness Theorem to the collection $\{f_\alpha \circ \mu | \alpha \in \Lambda\}$ of bounded scalar valued measures on $\mathcal{R}$. First we observe that $\sup_\alpha |f_\alpha \circ \mu(E)| \leq \sup_\alpha \|f_\alpha\| \, \|\mu(E)\| \leq \|\mu(E)\|$, for every $E \in \mathcal{R}$. Therefore by the Nikodym Boundedness Theorem we have that $\sup\{|f_\alpha \circ \mu(E)| \,\big|\, \alpha \in \Lambda$ and $E \in \mathcal{R}\} < \infty$. Since $\lim_\alpha f_\alpha(\mu(E)) = f_1(\mu(E))$ for every $E \in \mathcal{R}$, this implies that $\sup\{|f_1(\mu(E))| \,\big|\, E \in \mathcal{R}\} < \infty$. Hence $f_1 \in M_1$ so that M_1 is weak* closed.

Now a similar application of the Nikodym Boundedness Theorem to the collection $\{f\circ\mu | f \in X^*$ and $\|f\| \leq 1\}$ of bounded scalar valued measures shows that $\sup\{\|\mu(E)\| \,|\, E \in \mathcal{R}\} = \sup\{|f\circ\mu(E)| \,\big|\, E \in \mathcal{R}, f \in X^*$ and $\|f\| \leq 1\} < \infty$.

To prove the last part, let $M = \{f \in X^* | f\circ\mu$ is bounded, and regular over finite sets$\}$. First we show that M is a linear subspace of X^*. Let $f,g \in M$ and let $A \in \mathcal{R}$. Since $f\circ\mu$ is regular over finite sets, for each $\varepsilon > 0$ there exists a finite subset B_1 of A such that $|f\circ\mu(A) - f\circ\mu(B_1)| < \varepsilon/4$. Since $g\circ\mu$ is regular over finite sets there exists a finite subset D of $A \setminus B_1$ such that $|g\circ\mu(A \setminus B_1) - g\circ\mu(D)| < \varepsilon/4$. i.e., $|g\circ\mu(A) - g\circ\mu(B_1 \cup D)| < \varepsilon/4$. Let $B_1 \cup D = C_1$. A similar application to $f\circ\mu$ and $A \setminus C_1$ shows that there exists a finite set $B_2 \supseteq C_1$ such that $|f\circ\mu(A) - f\circ\mu(B_2)| < \varepsilon/4$. So inductively we can construct sequences (B_i) and (C_i) of finite sets in $\mathcal{R}$ such that:

(1) $|f\circ\mu(A) - f\circ\mu(B_i)|$, $|g\circ\mu(A) - g\circ\mu(C_i)| < \varepsilon/4$ for $i \in N$.

(2) $B_1 \subseteq C_1 \subseteq B_2 \subseteq C_2 \subseteq \ldots \subseteq B_i \subseteq C_i \subseteq \ldots\ldots \subseteq A$.

Since $f\circ\mu$ is bounded and scalar valued, it is strongly bounded and hence $\lim_i f\circ\mu(C_i \setminus B_i) = 0$. Consequently there exists $i_1 \in N$ such that:

(3) $|f\circ\mu(C_{i_1} \setminus B_{i_1})| < \varepsilon/4$.

Now (4) $|f\circ\mu(A) - f\circ\mu(C_{i_1})| = |f\circ\mu(A) - f\circ\mu(B_{i_1}) - f\circ\mu(C_{i_1} \setminus B_{i_1})|$

$$\leq |f\circ\mu(A) - f\circ\mu(B_{i_1})| + |f\circ\mu(C_{i_1} \setminus B_{i_1})|$$

$$< \varepsilon/4 + \varepsilon/4 \text{ by (1) and (3)}.$$

Therefore $|(f\circ\mu + g\circ\mu)(A) - (f\circ\mu + g\circ\mu)(C_{i_1})| \le |f\circ\mu(A) - f\circ\mu(C_{i_1})|$

$$+ |g\circ\mu(A) - g\circ\mu(C_{i_1})|$$

$$< \varepsilon/2 + \varepsilon/4 \text{ by (4) and (1).}$$

This implies that $f\circ\mu + g\circ\mu$ is regular over finite sets and hence $f + g \in M$. It is clear that $\lambda f \in M$ for $\lambda \in \mathbb{C}$ and $f \in M$. Therefore M is a linear subspace of X^* containing the total set Γ. To show that $M = X^*$ again we claim that $M_1 = \{f \in M \mid \|f\| \le 1\}$ is weak* closed. Since X is separable, the unit disc in X^* is metrizable with respect to the weak* topology and it is also weak* closed. Therefore it suffices to show that if (f_n) is a sequence in M_1 such that $\lim_n f_n = f$ exists in weak* topology, then $f \in M_1$. First we claim that f is regular over finite sets. Let $A \in R$. Since R is hereditary, $2^A \subseteq R$. Now $(f_n \circ \mu|_{2^A})_{n\in N}$ is a sequence of scalar valued bounded vector measures defined on a σ-algebra. Also, since (f_n) weak* converges to f, $\lim_n f_n \circ \mu|_{2^A}(E) = f\circ\mu(E)$ for every $E \subseteq A$. Setting $E_k = \{n_k\}$, where $A = \{n_1, n_2, \ldots\ldots\}$, we apply the last part of Corollary 1 of 3.4 Theorem 1 to $(f_n \circ \mu|_{2^A})_{n\in N}$. Then we have $\lim_n \sum_{k\in P} f_n \circ \mu(\{n_k\}) = \sum_{k\in P} f\circ\mu(\{n_k\})$ uniformly for $P \subseteq N$. In particular taking P finite we have

(5) $\lim_n f_n \circ \mu(F) = f\circ\mu(F)$ uniformly on finite subsets F of A.

Therefore for given $\varepsilon > 0$, there exists n_o such that:

(6) $|f_n \circ \mu(F) - f\circ\mu(F)| < \varepsilon/3$ for every finite subset F of A and $n \geq n_o$.

Also since $\lim_n f_n \circ \mu(A) = f\circ\mu(A)$, there exists $n_1 > n_o$ such that:

(7) $|f_{n_1} \circ \mu(A) - f \circ \mu(A)| < \varepsilon/3$.

Since $f_{n_1} \circ \mu$ is regular over finite sets there exists a finite subset

F of A such that: (8) $|f_{n_1} \circ \mu(A) - f_{n_1} \circ \mu(F)| < \varepsilon/3$.

Now $|f\circ\mu(A) - f\circ\mu(F)| \leq |f\circ\mu(A) - f_{n_1} \circ \mu(A)| + |f_{n_1} \circ \mu(A) - f_{n_1} \circ \mu(F)| +$

$$|f_{n_1} \circ \mu(F) - f \circ \mu(F)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ by (7), (8) and (6).}$$

This shows that $f\circ\mu$ is regular over finite sets. As in the proof of the first part of this corollary we apply the Nikodym Boundedness Theorem to the sequence $(f_n \circ \mu)$ of bounded scalar valued measures to show that $f \circ \mu$ is bounded. By this we can conclude that $f \in M_1$ and hence M_1 is weak* closed.

Now a similar application of the Nikodym Boundedness Theorem to the collection $\{f \circ \mu | f \in X^* \text{ and } \|f\| \geq 1\}$ of bounded scalar valued measures shows that μ is bounded.

We use the above result to derive an Orlicz-Pettis type result for Banach spaces satisfying certain conditions.

Corollary 2. Let the ring $\mathcal{R}$ of subsets of N and the Banach space X satisfy one of the following :

(1) $\mathcal{R} = 2^N$. X contains no copy of l_∞ .

(2) $\mathcal{R}$ is a $Q\sigma$-ring containing all finite sets. X is separable.

(3) $\mathcal{R}$ is a $Q\sigma$-ring containing all finite sets. X contains no copy of c_o .

(4) $\mathcal{R}$ is a hereditary $PQ\sigma$-ring containing all finite sets. X contains no copy of c_0 .

(5) $\mathcal{R}$ is a full ring with the hereditary property and containing all finite sets. X is separable and contains no copy of c_0 .

Further let Γ be a total subset of X^* . Suppose $\sum_{n=1}^{\infty} x_n$ is a series in X such that $\sum_{n \in A} x_n$ is Γ-convergent for every $A \in \mathcal{R}$ in the sense that there exists $x_A \in X$ such that $\sum_{n \in A} f(x_n) = f(x_A)$ for every $f \in \Gamma$, then $\sum_{n=1}^{\infty} x_n$ is norm subseries convergent.

Proof. Define $\mu: \mathcal{R} \to X$ by $\mu(A) = x_A$ as above. Since Γ is total, μ is well defined and, moreover, $f_o\mu$ is finitely additive and regular over finite sets for every $f \in \Gamma$. Also since for each $f \in \Gamma$ $\left|\sum_{n \in A} f(x_n)\right| < \infty$ for every $A \in \mathcal{R}$, $\sum_{n \in A} |f(x_n)| < \infty$. (Note that $\mathcal{R}$ is full.) This implies $f_o\mu$ is bounded for every $f \in \Gamma$. By corollary 1,

μ is a bounded vector measure. If X is as in one of (1), (3), (4), (5) then 3.3 Theorem 1 implies that μ is strongly additive. If X is as in (2), then Corollary 4 of 3.4 Theorem 1 implies that μ is strongly additive. Hence $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \mu(\{n\})$ is subseries convergent in norm.

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