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A SURVEY OF 1-FACTORIZATIONS

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ABSTRACT

A survey of results on the existence of 1-factorizations or colourings of graphs is given. The first chapter deals with the existence of 1-factorizations of certain graphs. These graphs include complete graphs, bipartite graphs, circulants, line graphs of some graphs and products of some graphs. The latter includes cartesian, lexicographic, tensor and strong products of graphs.

The second chapter deals with the existence of 1-factorizations with certain properties. Perfect 1-factorizations, Kotzig factorizations and graphs with certain Q-indices are studied.

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TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

The definition of a graph which is used is that of an undirected graph with no loops or multiple edges.

Definition 0.1. A graph G is a set of vertices $V(G)$ and a set of edges $E(G)$ which are unordered pairs of elements of $V(G)$ such that if $v_i v_j \in E(G)$, then $v_i, v_j \in V(G)$ and $i \neq j$.

A 1-factor or perfect matching and a 1-factorization or a colouring are defined as follows.

Definition 0.2. A 1-factor F_i of a graph G has $F_i \subseteq E(G)$ such that each vertex of $V(G)$ has degree 1 in F_i. A 1-factorization $F = \{F_0, F_1, \ldots, F_n\}$ of a graph G is a partitioning of $E(G)$ into 1-factors F_i where $i \in \{0, 1, \ldots, n\}$.

Another way to view a 1-factorization of a graph G is as a colouring of the edges of G so that each vertex is incident with exactly one edge of each colour. In 1879, the problem of the existence of colourings of graphs was mentioned by Kempe [13]. The concept of factorizations of graphs was dealt with by Kőnig in a book on graph theory [14] which was published in 1936.

Note that necessary conditions for the existence of a 1-factorization of a graph G are that G be regular and $|V(G)|$ be even.

A near 1-factorization is defined on the complete graph K_n where n is odd.

Definition 0.3. A <u>near 1-factor</u> F_i of a graph G has $F_i \subseteq E(G)$ such that each vertex of $V(G)\backslash\{v_i\}$ has degree 1 and v_i has degree 0 in F_i. A <u>near 1-factorization</u> $F = \{F_0, F_1, \ldots, F_{n-1}\}$ of K_n is a partition of $E(K_n)$ into near 1-factors F_i, where $i \in \{0,1,\ldots,n-1\}$, for K_n a complete graph on the vertices $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$.

The first chapter deals with the existence of 1-factorizations of certain graphs including complete graphs, bipartite graphs, line graphs and certain products. The second chapter deals with the existence of 1-factorizations having certain properties including perfect 1-factorizations, a generalization called a Q-index and Kotzig factorizations.

CHAPTER 1

THE EXISTENCE OF 1-FACTORIZATIONS OF GRAPHS

Section 1. Basic Results

Although 1-factorizations of K_2, K_4 and K_6 are isomorphic all other complete graphs having an even number of vertices have more than one non-isomorphic 1-factorization. For each complete graph K_{2n} one of these 1-factorizations is the pyramidal 1-factorization described in definition 2.1.2 using the abelian group Z_{2n-1} with generator 1 and another is the bipyramidal 1-factorization described in definition 2.1.3 using the abelian group Z_{2n-3} with generator 1 .

A 1-factorization of every complete bipartite graph $K_{n,n}$ exists.

Definition 1.1.1. A bipartite graph is a graph having a vertex set $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_m\}$ such that each edge is of the form u_iv_j for some i in $\{1,2,\ldots,n\}$ and j in $\{1,2,\ldots,m\}$. A complete bipartite graph $K_{n,m}$ is a graph on the vertices $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_m\}$ with edges $\{u_iv_j \mid i \in \{1,2,\ldots,n\},$ $j \in \{1,2,\ldots,m\}\}$.

Note that the existence of a 1-factorization of $K_{n,m}$ implies that $n = m$. For $n \geq 4$ there is more than one 1-factorization of $K_{n,n}$. One of these is $F = \{F_1,F_2,\ldots,F_n\}$ where $F_i = \{v_1u_{i+1},v_2u_{i+2},\ldots,v_nu_i\}$.

A result of Stern and Lenz [5] leads to the existence of 1-factorizations of some circulants. The proof of this result uses Vizing's theorem [27].

Definition 1.1.2. A circulant is a graph $G(n,S)$ on n vertices $\{v_0, v_1, \ldots, v_{n-1}\}$ with symbol S such that $S \subseteq \{1,2,\ldots,n-1\}$, if $i \in S$ then $n-i \in S$, and $(i-j) \bmod n \in S$ if and only if $v_i v_j \in E(G(n,S))$.

Theorem 1.1.1. (Vizing [27]). The edges of a graph G with maximum degree k can be coloured in k or $k+1$ colours so that no two distinct edges incident with a vertex have the same colour.

Theorem 1.1.2. (Bolletino [5]). If a circulant $G(n,S)$ has an $s \in S$ such that the order of the subgroup of Z_n generated by s is even, then a 1-factorization of $G(n,S)$ exists.

Proof: If $s, s' \in S$, $s \neq s'$, with the order of the subgroup generated by each of s and s' even, then $G(n,\{s,-s\})$ consists of even length cycles and forms two 1-factors unless $s = -s$ and then $G(n,\{s\})$ itself is a 1-factor. Thus if $G(n,S\backslash\{s,-s\})$ has a 1-factorization then $G(n,S)$ has a 1-factorization. By induction, this leaves the case where S contains only one s which generates an even order subgroup of Z_n.

Suppose there is only one $s \in S$ such that the order of the subgroup of Z_n generated by s is even. As above the subgroup $G(n,\{s,-s\})$ consists of even length cycles and forms two 1-factors F_0 and F_0' unless $s = -s$ and then $G(n,\{s\})$ is a 1-factor F_0. The remaining edges $G(n,S)\setminus\{s,-s\})$ form two vertex disjoint isomorphic subgraphs on $\frac{n}{2}$ vertices. By Vizing's theorem stated in Theorem 1.1.1, each of these subgraphs can be coloured in $\frac{|S\setminus\{s,-s\}|}{2}$ or $\frac{|S\setminus\{s,-s\}|}{2}+1$ colours. Colour with corresponding colours in each subgraph so that vertices joined by edges F_0 have the same colour edges incident with them. If $\frac{|S\setminus\{s,-s\}|}{2}$ colours are used then a 1-factorization of $G(n,S)$ is formed. If $\frac{|S\setminus\{s,-s\}|}{2}+1$ colours are used then each pair of corresponding vertices is incident with edges of all but one colour. Colour the edges of F_0 with the corresponding missing colours. A 1-factorization of $G(n,S)$ is formed.□

This leaves circulants $G(n,S)$ where $G(n,\{s,-s\})$ consists of odd length cycles for each $s \in S$. Note that if n is even, say $n = 2^k n'$ where n' is odd, then 2^k divides each $s \in S$ and the components of $G(n,S)$ each contain an odd number of vertices. Therefore $G(n,S)$ does not have a 1-factorization if each $s \in S$ generates an odd order subgroup of Z_n. This leads to the following corollary.

Corollary 1.1.3. A circulant $G(n,S)$ has a 1-factorization if and only if there exists an $s \in S$ such that $n/\text{gcf}(n,s)$ is even.□

A Tait colouring is a 1-factorization of a regular graph of degree 3. Tait conjectured that other than a few specified exceptions all regular graphs of degree 3 have Tait colourings. Mark Watkins [29] and Castagna and Prins [7] prove the existence of Tait colourings for a class of regular graphs of degree 3 with one exception. The class of graphs is called generalized Petersen graphs and the exception is the Petersen graph.

Definition 1.1.3. The generalized Petersen graph $GP(n,k)$ is the graph on $2n$ vertices $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ with $E(GP(n,k)) = \{u_1v_1,u_2v_2,\ldots,u_nv_n,u_1u_2,u_2u_3,\ldots,u_nu_1,v_1v_{k+1}, v_2v_{k+2},\ldots,v_nv_k\}$.

Theorem 1.1.4 , (Castagna and Prins [7]). A 1-factorization exists for every generalized Petersen graph other than the Petersen graph $GP(5,2)$.□

Section 2. Line Graphs

A class of graphs where some results are known on the existence of 1-factorizations is line graphs of regular graphs.

Definition 1.2.1. A line graph $L(G)$ of a graph G has vertices $E(G)$ and edges $E(L(G)) = \{e_i e_j | e_i, e_j \in E(G)$ and e_i and e_j are adjacent in $G\}$.

Two theorems of Jaeger are useful in proving that the line ~~graphs of certain regular graphs have 1-factorizations.~~

Theorem 1.2.1. (F. Jaeger [11]). Given a connected, regular graph G with a 1-factorization and $|E(G)|$ even, then $L(G)$ has a 1-factorization.□

Theorem 1.2.2. (F. Jaeger [12]). Given a regular graph G with $|E(G)|$ even, then there exists a 1-factorization of $L(G)$ if there exists a partition of $E(G)$ into Hamiltonian cycles.

Proof. Suppose there are an odd number of Hamiltonian cycles in the partition of $E(G)$ then since $|E(G)|$ is even, each Hamiltonian cycle has even length and G has a 1-factorization. By theorem 1.2.1 $L(G)$ has a 1-factorization.

Suppose $E(G)$ is partitioned into an even number of Hamiltonian cycles $H_1, \ldots, H_{2k}$. Note that if $|V(G)|$ is even then the proof can be done as above. This is not the case if $|V(G)|$ is odd. Now $E(L(G))$ can be partitioned into $2k$ cycles each of length

$|V(G)|$ and corresponding to one of the cycles H_i with a 4-factor between each pair of cycles. In the original graph the edge v_iv_j is adjacent to two edges in the same Hamiltonian cycle and is adjacent to two edges at v_i and two edges at v_j in every other Hamiltonian cycle.

To pair off these cycles a 1-factorization $F = \{F_1, F_2, \ldots, F_{2k-1}\}$ of K_{2k} on the vertices $\{u_1, u_2, \ldots, u_{2k}\}$ is used. If u_iu_j is an edge of K_{2k} this corresponds to the pairing of H_i with H_j. F_1 corresponds to pairs of Hamiltonian cycles and the 4-factor between those pairs including the edges of the cycles. $F_2, F_3, \ldots, F_{2k-1}$ each pair off the cycles and correspond to the 4-factor between each pairing, not including edges of the cycles. Now $F_2, \ldots, F_{2k-1}$ each correspond to regular bipartite graphs of degree 4 each having a 1-factorization. This leaves the edges corresponding to F_1 which is a graph isomorphic to $L(H_i \cup H_j)$ where H_i and H_j are Hamiltonian cycles.

Now $L(H_i \cup H_j)$ is two cycles of length $|V(G)|$ with a 4-factor in between. $L(H_i \cup H_j)$ can be partitioned into three Hamiltonian cycles each of even length, giving a 1-factorization of $L(H_i \cup H_j)$. To do this the 4-factor is partitioned into two Hamiltonian cycles and then the cycles of length $|V(G)|$ and one of the Hamiltonian cycles are partitioned into two Hamiltonian cycles.

To partition the 4-factor into two Hamiltonian cycles direct the edges of H_i into a directed cycle and then define the following

Hamiltonian cycles B^+ and B^- in $L(G)$.

Let $B^+ = \{e_i e_j | e_i \in H_i, e_j \in H_j$ and $v_i \in e_i \cap e_j$ where v_i is the out-vertex of $e_i\}$

and $B^- = \{e_i e_j | e_i \in H_i, e_j \in H_j$ and $v_i \in e_i \cap e_j$ where v_i is the in-vertex of $e_i\}$.

Looking at B^+, suppose $e_j, e_{j'}$, and $e_{j''}$ are consecutive edges in H_j with e_i incident with and directed away from $e_j, e_{i'}$ incident with and directed away from e_j and $e_{j'}$ and $e_{i''}$ incident with and directed away from $e_{j'}$ and $e_{j''}$ where $e_i, e_{i'}, e_{i''} \in H_i$. A cycle is formed in $L(H_i \cup H_j)$ around the edges $e_j e_{i'}, e_{i'} e_{j'}, e_{j'} e_{i''}, e_{i''} e_{j''}$ around to $e_i e_j$ going around the edges of H_j. Note that each edge of each of the two cycles occurs twice, thus a Hamiltonian cycle is formed.

The reasoning is the same for B^-.

Now take B^+ and the two cycles of length $|V(G)|$ in $L(H_i \cup H_j)$ coming from H_i and H_j. Name these two cycles A_i and A_j. Choose an edge in H_j and label it e_0 with end vertices labelled v_1 and v_2. Label the edges of H_j adjacent at v_1 and v_2 by e_1 and e_2 and the edges of H_i coming from v_1 and v_2 by e_1' and e_2'.

In A_i there are two paths between e_1' and e_2'. Colour the vertices and edges of one of these with one colour and of the other with the other colour leaving the vertices e_1' and e_2' not coloured. If e_k and e_ℓ in A_j are adjacent to $e_i \neq e_1'$ or e_2', then colour e_ie_k and e_ie_ℓ with the colour with which e_i is not coloured and colour e_ke_ℓ with the same colour as e_i. Note that e_k and e_ℓ are adjacent to the same out going edge of H_i and are adjacent in H_j. Thus e_ke_ℓ is an edge of A_j.

In each colour there are two vertex disjoint chains. One in A_i ends at e_1', e_2' and one in $A_j \cup B^+$ ends at e_1, e_2. By colouring e_1e_1', e_2e_0, e_0e_2' in one colour and e_2e_2', e_1e_0, e_0e_1' in the other colour, two Hamiltonian cycles are formed.

These Hamiltonian cycles and B^- are of even length. Therefore $L(H_i \cup H_j)$ has a 1-factorization for any Hamiltonian cycles H_i and H_j.

Therefore, given that G can be partitioned into Hamiltonian cycles, $L(G)$ has a 1-factorization.□

The technique of this proof can be used to show that $L(G(n,\{1,k,n-k,n-1\}))$ for any n and $k \in \{2,\ldots,\lceil \frac{n}{2} \rceil\}$ has a 1-factorization using a decomposition into one Hamiltonian cycle and one 2-factor. This result can also be proved as a corollary of Theorem 1.2.2 and Theorem 1.1.4.

Corollary 1.2.3. Given any n and $k \in \{2,3,\ldots,\lceil \frac{n}{2} \rceil\}$ there exists a 1-factorization of the circulant $L(G(n,\{1,k,n-k,n-1\}))$.

Proof. The circulant $L(G(n,\{1,k,n-k,n-1\}))$ can be partitioned into a copy of $GP(n,k)$ and three 1-factors. By Theorem 1.1.4 $GP(n,k)$ has a 1-factorization unless $n = 5$ and $k = 2$. Thus if $G(n,\{1,k,n-k,n-1\}) \neq G(5,\{1,2,3,4\})$ then $L(G(n,\{1,k,n-k,n-1\}))$ has a 1-factorization. By Theorem 1.2.2, $L(G(5,\{1,2,3,4\}))$ has a 1-factorization as $G(5,\{1,2,3,4\})$ can be partitioned into two Hamiltonian cycles.□

Theorem 1.2.4. (B. Alspach [1]). A 1-factorization of $L(K_n)$ exists if and only if $n \equiv 0$ or $1 \pmod 4$.

Proof. The number of vertices of $L(K_n)$, $|V(L(K_n)| = \frac{n(n-1)}{2}$ is odd for $n \equiv 2$ or $3 \pmod 4$ and a 1-factorization cannot exist in these cases.

Now the number of vertices of $L(K_n)$ is even for $n \equiv 0$ or $1 \pmod 4$. For all $n \equiv 0 \pmod 4$, K_n has a 1-factorization. Therefore by Theorem 1.2.1 $L(K_n)$ has a 1-factorization for $n \equiv 0 \pmod 4$. For $n \equiv 1 \pmod 4$ a partitioning of the edges of K_n into Hamiltonian cycles exists. Therefore by Theorem 1.2.2 a 1-factorization of $L(K_n)$ exists for $n \equiv 1 \pmod 4$.□

Corollary 1.2.5. Given n and S so that every component of $G(n,S)$ has an even number of vertices, then $L(G(n,S))$ has a 1-factorization.

Proof. By Corollary 1.1.3 $G(n,S)$ has a 1-factorization. Thus by Theorem 1.2.1 a 1-factorization of $L(G(n,S))$ exists, since the number of edges in $G(n,S)$ is even.□

Corollary 1.2.6. The line graph of any generalized Petersen graph with an even number of edges has a 1-factorization.

Proof. This is a direct result of Theorem 1.2.1 and Theorem 1.1.4.□

Corollary 1.2.7. The line graph of the complete bipartite graph $K_{n,n}$ has a 1-factorization if and only if n is even.

Proof. If n is odd the number of edges is odd and $L(K_{n,n})$ does not have a 1-factorization. If n is even then the number of edges is even and since $K_{n,n}$ has a 1-factorization, $L(K_{n,n})$ has a 1-factorization by Theorem 1.2.1.□

Monar, Pisanski and Shawe-Taylor have two results dealing with line graphs of biregular graphs.

Definition 1.2.2. A biregular graph G with degrees ℓ and n is a bipartite graph with all vertices of degree ℓ or degree n; if $v_i, v_j \in V(G)$ have the same degree then $v_i v_j \notin E(G)$.

Theorem 1.2.8. (Monar, Pisanski and Shawe-Taylor [23]). Let G be a biregular graph with degrees 2ℓ and $2n$. Then a 1-factorization of $L(G)$ exists.□

The second result uses subdivision graphs.

Definition 1.2.3. Let G be a graph. Then a subdivision graph of G, denoted $S(G)$, replaces each edge $e = uv$ of G with a path ux_ev where x_e has degree 2.

Note that if G is a regular graph of degree d , then S(G) is a biregular graph with degrees 2 and d . Also if G is a biregular graph of degrees 2 and d , then a regular graph H of degree d exists such that G = S(H).

Theorem 1.2.8. (Monar, Pisanski and Shawe-Taylor [23]). Let G be a biregular graph with degrees 2 and d where d is odd. Then a 1-factorization of L(G) exists if and only if a 1-factorization of H exists where G = S(H).□

Section 3. Products of Graphs

In this section the problem of the existence of 1-factorizations of cartesian, lexicographic, tensor and strong products of graphs is considered.

Definition 1.3.1. For graphs G_1 and G_2 the cartesian product $G_1 \times G_2$ has $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1,v_1)(u_2,v_2) | u_1 = u_2$ and $v_1v_2 \in E(G_1)$ or $v_1 = v_2$ and $u_1u_2 \in E(G_2)\}$.

Definition 1.3.2. For graphs G_1 and G_2 the lexicographic product (wreath product) $G_1 \circ G_2$ $(G_1[G_2])$ has $V(G_1 \circ G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \circ G_2) = \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E(G_1)$ or $u_1 = u_2$ and $v_1v_2 \in E(G_2)\}$.

Definition 1.3.3. For a graph G and positive integer m, let $G(m) = G \circ \bar{K}_m$.

Definition 1.3.4. For graphs G_1 and G_2 the tensor product $G_1 \otimes G_2$ has $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1,v_1)(u_2,v_2) | u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

Definition 1.3.5. For a graph G and positive integer m, let $G\{m\} = G \otimes K_m$.

Definition 1.3.6. For graphs G_1 and G_2 the strong product $G_1 * G_2$

has $V(G_1 * G_2) = V(G_1) \times V(G_2)$ and $E(G_1 * G_2) = E(G_1 \times G_2) \cup E(G_1 \otimes G_2)$.

Kotzig's result, dealing with cartesian products of regular graphs, is given first.

<u>Theorem 1.3.1.</u> (Kotzig [15]). Let $G_1, G_2, \ldots, G_n$ be regular graphs such that G_i has a 1-factorization for some $i \in \{1,2,\ldots,n\}$ or G_i and G_j each have a 1-factor for some $i,j \in \{1,2,\ldots,n\}$, $i \neq j$, then a 1-factorization of $G_1 \times G_2 \times \ldots \times G_n$ exists.

<u>Proof.</u> If there is a $G_i \in \{G_1, G_2, \ldots, G_n\}$ such that G_i has a 1-factorization $F = \{F_1, F_2, \ldots, F_k\}$, then colour the edges of the cartesian product $G = G_1 \times G_2 \times \ldots \times G_{i-1} \times G_{i+1} \times \ldots \times G_n$ with $d + 1$ colours where the degree of each vertex in G is d. This existence is a direct result of Vizing's Theorem [27]. Then a $d + 1$ colouring or a 1-factorization of $F_1 \times G$ is formed, colouring the new edges with the missing colour at the corresponding vertices of G. Add a new colour for each F_i, $i \geqq 2$, colouring the edges of the form $(f,g)(f',g')$ with colour i for $f,f' \in F_i$, $i \geqq 2$.

If there are G_i and $G_j \in \{G_1, G_2, \ldots, G_n\}$ such that F_i and F_j are 1-factors of G_i and G_j, respectively, then partition the edges of $G_i \times G_j$ into $F_i \times H_j$ and $H_i \times F_j$ where H_i and H_j are graphs on the vertex sets of G_i and G_j, respectively such that $E(G_i) = E(F_i) \cup E(H_i)$ and $E(G_j) = E(F_j) \cup E(H_j)$. Now $F_i \times H_j$ and $H_i \times F_j$ are made up of disjoint graphs which form two copies of H_j or H_i with corresponding vertices joined by a 1-factor. Vizing's Theorem [27] is used again as above giving a 1-factorization of

$G_i \times G_j$. By the first result, a 1-factorization of $G_1 \times G_2 \times \ldots \times G_n$ may be formed.□

To show that the conditions given in the above theorem are not necessary, Kotzig goes on to prove the following result.

Theorem 1.3.2. (Kotzig [15]). For C a cycle of length greater than 3 and G a 3-regular graph, a 1-factorization of $C \times G$ exists.□

The existence of 1-factorizations of lexicographic, tensor and strong products have been studied by Pisanski, Shawe-Taylor and Monar. A certain lexicographic product is looked at first.

Theorem 1.3.3. (Monar and Pisanski [22]). Let G be a regular graph. Then a 1-factorization of $G(m)$ exists in each of the following cases:

a) A 1-factorization of G exists,

b) G is of even degree and m is even,

c) $m \equiv 0 \pmod 4$,

d) G has a 1-factor and m is even,

e) G is cubic and m is even, and

f) G is bipartite. □

Laskar and Hare [18] show that a 1-factorization of $K_n(m)$ exists if and only if mn is even. Parker [25] shows that if G is a cycle on n vertices, a 1-factorization of $G(m)$ exists if and only if mn is even.

Now a certain tensor product is looked at.

Theorem 1.3.4. (Pisanski, Shawe-Taylor and Monar [26]). Given a regular graph G and a positive integer m, then a 1-factorization of $G\{2m\}$ exists.□

Theorem 1.3.5. (Monar, Pisanski and Shawe-Taylor [23]). Let $G_1, G_2, \ldots, G_n$ be regular graphs such that G_i has a 1-factorization for some $i \in \{1,2,\ldots,n\}$ or G_i and G_j each have a 1-factor for some $i,j \in \{1,2,\ldots,n\}$, $i \neq j$. Then a 1-factorization of the lexicographic product $G_1 \circ G_2 \circ \ldots \circ G_n$ exists.

Proof. Their proof is reduced to proving the existence of a 1-factorization of $G \circ H$ where the edges of $G \circ H$ can be partitioned into $G \times H$ and $G\{|V(H)|\}$. The proof is completed using Theorem 1.3.1 and 1.3.4 where H has a 1-factorization or a 1-factor since $|V(H)|$ is even. This leaves the case where a 1-factorization of G, $F = \{F_1, F_2, \ldots, F_m\}$ exists, then $G\{|V(H)|\} = F_1\{|V(H)|\} \oplus F_2\{|V(H)|\} \oplus \ldots \oplus F_m\{|V(H)|\}$. Since each $F_i\{|V(H)|\}$ for $i \in \{1,\ldots,m\}$ can be reduced to copies of regular bipartite graphs each of which is 1-factorable, a 1-factorization exists. This completes the proof.□

To prove that the conditions given in the above Theorem are not necessary they go on to prove the following result.

Theorem 1.3.6. (Pisanski, Shawe-Taylor and Monar [26]). For C a cycle of length greater than 3 and G a 3-regular graph, a 1-factorization of $C[G]$ exists.

Theorem 1.3.7. (Monar, Pisanski and Shawe-Taylor [23]). Let $G_1, G_2, \ldots, G_n$ be regular graphs such that G_i has a 1-factorization for some $i \in \{1,2,\ldots,n\}$. Then a 1-factorization of $G_1 \otimes G_2 \otimes \ldots \otimes G_n$ exists.

Proof. Since the tensor product is commutative, assume that G_1 has the 1-factorization $F = \{F_1, F_2, \ldots, F_k\}$. Looking at the graph $G_1 \otimes G_2$, the set of edges $E(G_1 \otimes G_2)$ is made up of copies of $F_i \otimes G_2$ for $i \in \{1,2,\ldots,k\}$. Each of these can be partitioned into vertex disjoint copies of $H\{2\}$. By Theorem 1.3.4, the proof is complete.□

Theorem 1.3.8. (Monar, Pisanski and Shawe-Taylor [23]). Let $G_1, G_2, \ldots, G_n$ be regular graphs such that G_i has a 1-factorization for some $i \in \{1,2,\ldots,n\}$. Then a 1-factorization of the strong product $G_1 * G_2 * \ldots * G_n$ exists.

Proof. Now $E(G * H)$ can be partitioned into $G \times H$ and $G \otimes H$. Theorems 1.3.1 and 1.3.7 complete the proof.□

CHAPTER 2

THE EXISTENCE OF 1-FACTORIZATIONS WITH CERTAIN PROPERTIES

Section 1. Perfect 1-factorizations

Definition 2.1.1. A perfect 1-factorization is a 1-factorization $F = \{F_1, F_2, \ldots, F_n\}$ such that for every $i, j \in \{1,2,\ldots,n\}$, $i \neq j$, $E(F_i \cup F_j)$ is a Hamiltonian cycle.

A 1-factorization which is formed from a 1-factor by fixing a vertex and performing a cycle permutation on the other vertices is called a pyramidal 1-factorization by Mendelsohn and Rosa in [21].

Definition 2.1.2. Let $V(K_{2n}) = \{v_0, v_{a_1}, v_{a_2}, \ldots, v_{a_{2n-2}}, v_\infty\}$, where $0, a_1, a_2, \ldots, a_{2n-2}$ are the elements of an abelian group G of order $2n-1$. Additive notation is used and for $a_i \in G$, $\infty + a_i = \infty$. Let F_0 be a 1-factor of K_{2n}. Then $F_{a_i} = \{v_{a_j+a_i} v_{a_k+a_i}$ where $v_{a_j} v_{a_k} \in F_0\}$ is a 1-factor of K_{2n}. If the collection of 1-factors $F_0, F_a, \ldots, F_{a_{2n-2}}$ is a 1-factorization of K_{2n}, then it is called a pyramidal 1-factorization.

A pyramidal 1-factorization is used to construct a perfect 1-factorization of $K_{p+1}, K_{16}, K_{28}, K_{244}$ and K_{344}. A 1-factorization which gives a different perfect 1-factorization of K_8 and K_{24} is called a bipyramidal 1-factorization and is defined as follows.

Definition 2.1.3. Let $F = \{F_i | i \in Z_{2n-1}\}$ be the pyramidal 1-factorization of K_{2n} described in definition 2.1.2. Let

$$V(K_{2n+2}) = \{u_0,u_1,\ldots,u_{2n-1},u_\infty,u_\infty'\},\quad \tau(x) = \begin{cases} x & \text{if } x < \left\lfloor \frac{3n}{2} \right\rfloor \\ x+1 & \text{if } x \geq \left\lfloor \frac{3n}{2} \right\rfloor \end{cases} \quad \text{and}$$

$\tau(\infty) = \infty$. Additive notation in the group Z_{2n} is used and for $a \in Z_{2n}$, $a + \infty = \infty$. The 1-factors

$$F_i' = \{u_{\tau(x)+i}u_{\tau(y)+i} | v_xv_y \in F_0\} \cup \{u_{\left\lfloor \frac{3n}{2} \right\rfloor +i} u_\infty'\} \text{ for } i \in Z_{2n} \text{ and}$$

$F^* = \{u_iu_{i+n} | i = 0,1,\ldots,n-1\} \cup \{u_\infty,u_\infty'\}$ form a 1-factorization F' of K_{2n+2}. This 1-factorization is called a bipyramidal 1-factorization.

The following lemma reduces the number of subgraphs to be checked in proving that a pyramidal 1-factorization of K_{2n} is perfect, leaving $n-1$ cases to be checked.

Lemma 2.1.1. (B.A. Anderson [2]). Let G be an additive group of order $2n-1$ generated by a_1, with $a_i = a_{i-1} + a_1$. Let the vertices of K_{2n} be labelled v_∞ and v_{a_i} where $a_i \in G$. Let $F = \{F_{a_i} | a_i \in G\}$ be a pyramidal 1-factorization of K_{2n} with $v_{a_i}v_\infty \in F_{a_i}$ for $a_i \in G$ such that $F_{a_0} \cup F_{a_k}$ is a Hamiltonian cycle for k in $\{1,\ldots,n-1\}$. Then F is a perfect 1-factorization.

<u>Proof:</u> All arithmetic is done in G with $a_k + \infty = \infty$ for $a_k \in G$. Let σ be a permutation of the vertices of K_{2n} defined as $\sigma(v_{a_k}) = v_{a_k+a_1}$ for $v_{a_k} \in V(K_{2n})$, with the corresponding permutation on the edges $\sigma(v_{a_i}v_{a_j}) = \sigma(v_{a_i})\sigma(v_{a_j})$.

Given $i,j \in \{1,2,\ldots,2n-2\}$, $i < j$ then $j - i \le n-1$ or $i - j \equiv k \pmod{2n-1} < n-1$. If $j - i \le n-1$, then $\sigma^{-i}(F_{a_i}) = F_{a_0}$, $\sigma^{-i}(F_{a_j}) = F_{a_{(j-i)}}$ and $\sigma^{-i}(F_{a_i} \cup F_{a_j}) = F_{a_0} \cup F_{a_{(j-i)}}$. If $i - j \equiv k \pmod{2n-1} < n-1$, then $\sigma^{-j}(F_{a_i}) = F_{a_k}$, $\sigma^{-j}(F_{a_j}) = F_{a_0}$ and $\sigma^{-j}(F_{a_i} \cup F_{a_j}) = F_{a_k} \cup F_{a_0}$ for $k < n-1$. Thus for i,j in $\{1,2,\ldots,2n-2\}$ $F_{a_i} \cup F_{a_j} \cong F_{a_0} \cup F_{a_k}$ for some k in $\{1,2,\ldots,n-1\}$.

Therefore, if $F_{a_0} \cup F_{a_k}$ is a Hamiltonian cycle for k in $\{1,2,\ldots,n-1\}$, then F is a perfect 1-factorization.□

<u>Theorem 2.1.2.</u> (Kotzig [15]). For any odd prime p a perfect 1-factorization of K_{p+1} exists.

<u>Proof:</u> Using Lemma 2.1.1, with the group Z_p and generator 1 with addition modulo p on residues $0,1,\ldots,p-1$, a pyramidal 1-factorization of K_{p+1} is shown to be a perfect 1-factorization.

Let the vertex set of K_{p+1} be $\{v_0,v_1,\ldots,v_{p-1},v_\infty\}$. Consider the 1-factor F_0.

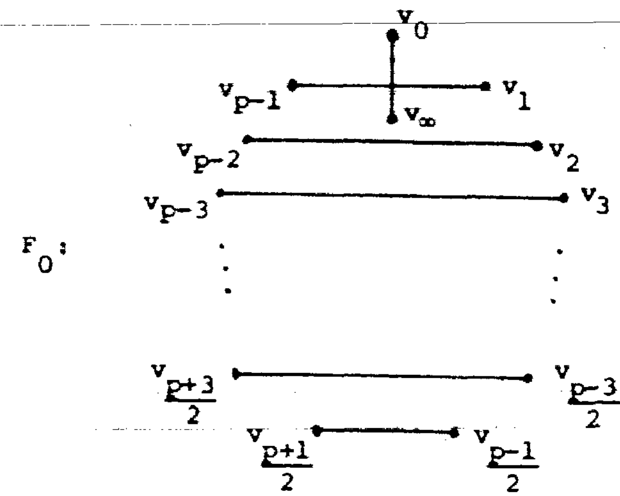

Figure 1

Let $F = \{F_0, F_1, \ldots, F_{p-1}\}$ be the set of 1-factors described in Lemma 2.1.1 using the permutation σ. The edge $v_\infty v_i$ for i in $\{0,1,\ldots,p-1\}$ occurs in exactly one 1-factor of F, namely F_i. Note that for $j,k \in \{1,\ldots,p-1\}$, $j \neq k$, $v_j v_k \in E(F_0)$ if and only if $k + j = 0$ and thus, in general, $v_k v_j \in E(F_i)$ if and only if $k + j - 2i = 0$. Since there is exactly one k such that $i + j = k$, the edge $v_i v_j$ occurs in exactly one 1-factor, namely F_k.

Therefore, F is a 1-factorization of K_{p+1} and by definition a pyramidal 1-factorization.

Given $k \in \{1,2,\ldots,p-1\}$, note that the edges of F_0 are in the form $v_\alpha v_{-\alpha}$ for $\alpha \in \{1,2,\ldots,\frac{p-1}{2}\}$ and $v_0 v_\infty$, while the edges of F_k are in the form $v_{k+\alpha} v_{k-\alpha}$ for $\alpha \in \{1,\ldots,\frac{p-1}{2}\}$ and $v_k v_\infty$. Thus $F_0 \cup F_k$ contains the cycle with the sequence of vertices

$$v_k, v_\infty, v_0, v_{2k}, v_{-2k}, v_{4k}, v_{-4k}, \ldots, v_{2(\frac{p-1}{2})k} = v_{-k}, v_{-(\frac{p-1}{2})k} = v_k .$$

For any $i,j \in \{0,1,\ldots,\frac{p-1}{2}\}$ any of $2ik = 2jk$, $-2ik = -2jk$ or $2ik = 2jk$ give $i = j$ or $i = -j$ since $k \neq 0$. If j is in $\{0,1,\ldots,\frac{p-1}{2}\}$ then $-j$ is not in $\{0,1,\ldots,\frac{p-1}{2}\}$ giving $i \neq -j$. Thus a cycle of length $p + 1$ is formed and $F_0 \cup F_k$ forms a Hamiltonian cycle for $k \in \{1,\ldots,p-1\}$.

Therefore, by Lemma 2.1.1, F is a perfect 1-factorization.□

Theorem 2.1.3. (B.A. Anderson [2]). A perfect 1-factorization of K_{16} exists.

Proof: Using Lemma 2.1.1 with the group Z_{15} and generator 1 with addition modulo 15 on residues 0,1,...,14, a pyramidal 1-factorization is shown to be a perfect 1-factorization.

Let the vertex set of K_{16} be $\{v_0, v_1, \ldots, v_{14}, v_\infty\}$. Consider the 1-factor F_0.

F_0:

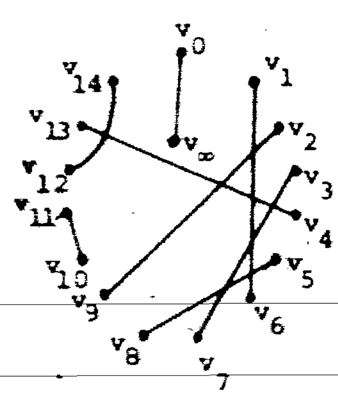

Figure 2

Let $F = \{F_0, F_1, \ldots, F_{14}\}$ be the set of 1-factors described in Lemma 2.1.1 using the permutation σ. The edge $v_\infty v_i$ for $i \in \{0,1,\ldots,14\}$ occurs in exactly one 1-factor of F, namely F_i. Note that $\{\pm(i + -j) | v_i v_j \in E(F_0),\ i,j \neq \infty\} = \{1,\ldots,14\}$. Thus F is a 1-factorization of K_{16} and by definition a pyramidal 1-factorization.

By checking that $F_0 \cup F_k$ forms a Hamiltonian cycle for $k \in \{1,\ldots,7\}$, by Lemma 2.1.1, it can be shown that F is a perfect 1-factorization.□

The following construction of Mullin and Nemeth for Room Squares gives a 1-factorization of K_{p^m+1} where p is an odd prime, m is an integer, $p^m > 3$ and $p^m \equiv 3 \pmod 4$. This construction is used to prove the existence of a perfect 1-factorization of K_{28}, K_{244} and K_{344}. The added structure allows for checking the union of only one pair of 1-factors to prove that the 1-factorization is perfect.

<u>Definition 2.1.4.</u> For an odd prime p and an integer m such that $p^m > 3$, $p^m \equiv 3 (\text{mod } 4)$, let x be a generator of the multiplicative subgroup of order $p^m - 1$ in $GF[p^m]$. Let $V(K_{p^m+1}) = \{v_\infty, v_a | a \in GF[p^m]\}$. Let F_0 be defined by $E(F_0) = \{v_0 v_\infty, v_1 v_x, v_{x^2} v_{x^3}, \ldots, v_{x^{p^m-3}} v_{x^{p^m-2}}\}$ and $E(F_k) = \sigma^k(F_0)$ where σ

is as defined in Lemma 2.1.1. The resulting collection of 1-factors is shown to be a pyramidal 1-factorization in the next result and is called a <u>Mullin-Nemeth 1-factorization</u>.

<u>Lemma 2.1.4.</u> (Mullin and Nemeth [24]). Let F be a Mullin-Nemeth 1-factorization, using the multiplicative generator x, of K_{p^m+1} where p is an odd prime and m is an integer such that $p^m > 3$, $p^m \equiv 3 \pmod 4$, then F is a pyramidal 1-factorization of K_{p^m+1}.

<u>Proof:</u> All arithmetic is done in $GF[p^m]$. Since x is a generator of the multiplicative subgroup of $GF[p^m]$, each of the vertices of $K^{*}_{p^m+1}$ occurs in F_0 exactly once.

Now look at the set $S = \{\pm x^0(1-x), \pm x^2(1-x), \ldots, \pm x^{p^m-3}(1-x)\}$. Note that $1-x \in GF[p^m]$ and $1-x \neq 0$ since $p^m > 3$. If $x^{2\alpha}(1-x) = x^{2\beta}(1-x)$ for some α,β with $0 \le \alpha,\beta \le \frac{p^m-3}{2}$, then $x^{2\alpha} = x^{2\beta}$, $2\alpha \equiv \alpha' \pmod{p^m-1} = \beta' \pmod{p^m-1} \equiv 2\beta$, since x generates $GF^*[p^m]$, and $\alpha = \beta$ since $0 \le \alpha,\beta \le \frac{p^m-3}{2}$. If $x^{2\alpha}(1-x) = -x^{2\beta}(1-x)$ for α,β such that $0 \le \alpha,\beta \le \frac{p^m-1}{2}$, then $x^{2\alpha} + x^{2\beta} = 0$. If $\alpha = \beta$ then $2x^{2\alpha} = 0$ and $GF[p^m]$ has characteristic 2 contradicting p an odd prime. If $\alpha \neq \beta$, say $\alpha < \beta$, then $x^{2\alpha}(1 + x^{2\beta-2\alpha}) = 0$ and since $x^{2\alpha} \neq 0$, $x^{2\beta-2\alpha} = -1$.

Thus $x^{2\beta-2\alpha} = -1$ since $0 < 2\beta - 2\alpha \le p^m-1$ which gives $2\beta - 2\alpha = \frac{p^m-1}{2}$. Thus $p^m-1 = 4(j-i)$ and $p^m \equiv 1 \pmod 4$ contradicting $p^m \equiv 3 \pmod 4$.

Therefore $S = GF[p^m]\setminus\{0\}$ and F is a pyramidal 1-factorization.□

<u>Lemma 2.1.5</u>. (B.A. Anderson [3]). Let $F = \{F_0,\ldots,F_{p^m-1}\}$ be a Mullin and Nemeth 1-factorization of K_{p^m+1} using the generator x where p is an odd prime, $p^m > 3$, $p^m \equiv 3 \pmod 4$, then F has the property that for $i,j,i',j' \in GF[p^m]$, $i \neq j$, $i' \neq j'$,

$F_i \cup F_j \cong F_{i'} \cup F_{j'}$.

<u>Proof</u>: All arithmetic is done in $GF[p^m]$.

For $\alpha \in \{x^0,x^2,\ldots,x^{p^m-3}\}$ define the permutation of the vertices of K_{p^m+1}, $\tau_\alpha(v_i) = v_{\alpha i}$ for $v_i \in V(K_{p^m+1})$. The corresponding permutation of the edges of K_{p^m+1} is defined by $\tau_\alpha(v_iv_j) = v_{\alpha i}v_{\alpha j}$ for $v_iv_j \in E(K_{p^m+1})$. Note that $\tau_\alpha F_0 = F_0$.

Given $k \in GF[p^m]\setminus\{0\}$ then either $k^{-1} \in \{x^0,x^2,\ldots,x^{p^m-3}\}$ or $-(k^{-1}) \in \{x^0,x^2,\ldots,x^{p^m-3}\}$ since $-k = k(x^{\frac{p^m-1}{2}})$ and $\frac{p^m-1}{2} \equiv 1 \pmod 4$. Let k' be k^{-1} or $-(k^{-1})$ such that $k' \in \{x^0,x^2,\ldots,x^{p^m-1}\}$. Now $\tau_{k'}(F_0) = F_0$ and

$$\tau_{k'}F_k = \begin{cases} F_1 & \text{if } k' = k^{-1} \\ F_{-1} & \text{if } k' = -k^{-1} \end{cases}$$ since if $v_{i+k}v_{j+k} \in F_k$ then

$v_iv_j \in F_0, v_{k'i}v_{k'j} \in F_0$ and thus $v_{k'(i+k)}v_{k'(j+k)} \in \begin{cases} F_1 & \text{if } k' = k^{-1} \\ F_{-1} & \text{if } k' = -k^{-1} \end{cases}$.

For $k \in GF[p^m]$ define the permutation σ_k as before on the vertices of K_{p^m+1}. Then $\sigma_{-k}(F_k) = F_0$ for $k \in GF[p^m]$. Thus for any $i,j \in GF[p^m]$, $\sigma_{-i}(F_i \cup F_j) = F_0 \cup F_{j-i}$. If $(j-i)^{-1} \in \{x^0, x^2, \ldots, x^{p^m-1}\}$ then $\tau_{(j-i)^{-1}}(\sigma_{-i}(F_i \cup F_j)) = F_0 \cup F_1$. If $(j-i)^{-1} \notin \{x^0, x^2, \ldots, x^{p^m-1}\}$ then $-(j-i)^{-1} \in \{x^0, x^2, \ldots, x^{p^m-1}\}$ and $\sigma_1(\tau_{-(j-i)^{-1}}(\sigma_{-i}(F_i \cup F_j))) = F_0 \cup F_1$.

Therefore $F_i \cup F_j \cong F_{i'} \cup F_{j'}$ for $i,i', j,j' \in GF[p^m]$.□

Theorem 2.1.6. (B.A. Anderson [3]). A perfect 1-factorization of K_{28} exists.

Proof: The polynomial $y^3 + 2y^2 + 1$ is irreducible over $GF[3]$. Thus the root x of $y^3 + 2y^2 + 1$ is a generator of $GF[3^3]$.

Using the Mullin-Nemeth 1-factorization let

$F_0 = \{v_0 v_\infty, v_1 v_x, v_{x^2} v_{x^2+2}, v_{x^2+2x+2} v_{2x+2}, v_{2x^2+2x} v_{x^2+1},$

$v_{x^2+x+2} v_{2x^2+2x+2}, v_{x^2+2x+1} v_{x+2}, v_{x^2+2x} v_2, v_{2x} v_{2x^2},$

$v_{2x^2+1} v_{2x^2+x+1}$, $v_{x+1} v_{x^2+x}$, $v_{2x^2+2} v_{2x^2+2x+1}$, $v_{x^2+x+1} v_{2x^2+x+2}$,

$v_{2x+1} v_{2x^2+x}\}$, let $F_k = \{v_{i+k} v_{j+k} \mid v_i v_j \in F_0\}$ for $k \in GF[3^3]$

and let $F = \{F_k | k \in GF[3^3]\}$. Now $F_0 \cup F_1$ gives the Hamiltonian cycle containing the sequence of vertices v_0, v_{x^2+2x+1}, v_{x+2},

v_{x^2+x+1}, v_{2x^2+x+2}, v_{2x^2+2}, v_{2x^2+2x+1}, v_{x^2+2}, v_{x^2}, v_{x^2+1}, v_{2x^2+2x}

v_{x^2+x}, v_{x+1}, v_2, v_{x^2+2x}, v_{2x}, v_{2x^2}, v_{2x^2+2x+2}, v_{x^2+x+2}, v_{2x^2+x},

v_{2x+1}, v_{2x^2+1}, v_{2x^2+x+1}, v_{2x+2}, v_{x^2+2x+2}, v_x, v_1, v_∞, v_0 .

By Lemma 2.1.5, F is a perfect 1-factorization.□

Theorem 2.1.7. (B.A. Anderson and D. Morse [4]). Perfect 1-factorizations of K_{244} and K_{344} exist.

Proof: For K_{244}, $244 = 3^5+1$, the polynomial $y^5 + 2y + 1$ is irreducible over $GF[3]$. The Mullin-Nemeth 1-factorization using x^{37} as a generator gives a perfect 1-factorization of K_{244} .

For K_{344}, $344 = 7^3+1$, the polynomial $y^3 + 6y + 2$ is irreducible over $GF[7]$. The Mullin-Nemeth 1-factorization using x^{67} as a generator gives a perfect 1-factorization of K_{344} .□

A construction for a perfect 1-factorization of K_{2p}, where p is an odd prime, uses a cyclic permutation of the vertices with corresponding permutation of the edges to partition the edges v_iv_j, where $|i-j|$ is even or $|i-j| = p$, into 1-factors. This leaves the edges of the circulants $G(2p,\{k,-k\})$ for odd $k \in \{0,1,\ldots,p-1\}$. Now for odd $k \in \{0,1,\ldots,p-1\}$, $G(2p,\{k,-k\})$ forms a Hamiltonian cycle on an even number of vertices, which has the obvious pair of 1-factors.

<u>Theorem 2.1.8.</u> (Kotzig [16]). For any prime p a perfect 1-factorization of K_{2p} exists.

<u>Proof</u>: All arithmetic is done modulo $2p$ on the residues $\{0,\ldots,2p-1\}$. Let the vertex set of K_{2p} be $\{v_0,v_1,\ldots,v_{2p-1}\}$. Consider the 1-factor F_0.

F_0:

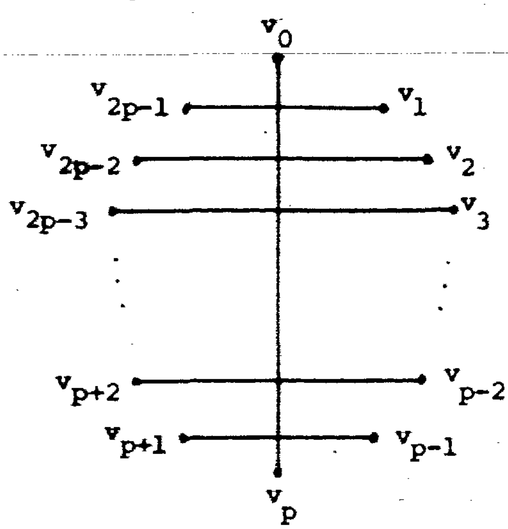

Figure 3

Rotate this configuration through $p-1$ rotations using the permutation $\rho = (v_0,v_1,\ldots,v_{2p-1})$. The corresponding permutation on the edges is $\rho(v_i v_j) = v_{i+1} v_{j+1}$. Thus p 1-factors F_i for $i \in \{0,\ldots,p-1\}$ are formed where $E(F_i) = \rho^i E(F_0)$. Note that for $j,k \in \{1,\ldots,2p-1\}$, $j \neq k$, $v_j v_k \in E(F_0)$ if and only if $k+j \equiv 0 \pmod{2p}$. Thus, $v_k v_j \in E(F_i)$ if and only if $\frac{k+j}{2} \equiv i \pmod{2p}$. For $i \in \{0,1,\ldots,p-1\}$ the 1-factor F_i also includes the edge $v_i v_{i+p}$.

This leaves the edges $v_i v_j$ where $|i-j| \in \{1,3,\ldots,p-2\}$ which are exactly the edges of the circulants $G(2p,\{k,-k\})$ for $k \in \{1,3,\ldots,p-2\}$, each of which is a Hamiltonian cycle. For $k \in \{1,3,\ldots,p-2\}$, let $E(F'_k) = \bigcup_{\alpha \in \{1,3,\ldots,2p-1\}} \{v_{\alpha k} v_{(\alpha+1)k}\}$ and $E(F''_k) = \bigcup_{\alpha \in \{0,2,\ldots,2p-2\}} \{v_{\alpha,k} v_{(\alpha+1)k}\}$. Note that $F'_k \cup F''_k \cong G(2p,\{k,-k\})$.

Thus, $F = \bigcup_{i \in \{0,\ldots,p-1\}} F_i \cup \left(\bigcup_{i \in \{1,3,\ldots,p-2\}} F'_i \right) \cup \left(\bigcup_{i \in \{1,3,\ldots,p-2\}} F''_i \right)$ is a 1-factorization of K_{2p} .

F is a perfect 1-factorization of K_{2p} if, for each $i,i' \in \{0,1,\ldots,p-1\}$ and $j,j', k,k' \in \{1,3,\ldots,p-2\}$, $F_i \cup F_{i'}$, $F'_j \cup F'_{j'}$, $F''_k \cup F''_{k'}$, $F_i \cup F'_j$, $F_i \cup F''_k$ and $F'_j \cup F''_k$ form Hamiltonian cycles for $i \neq i'$, $j \neq j'$, $k \neq k'$.

<u>Case 1</u>: A proof that for $i, i' \in \{0,1,\ldots,p-1\}$, $i \neq i'$, $F_i \cup F_{i'}$ is a Hamiltonian cycle is given.

Note that for $i > i'$ there is an $\alpha \in \{1,\ldots,p-1\}$ such that $\alpha = i-i'$, $\rho^{i'}(F_\alpha) = F_i$ and $\rho^{i'}(F_0) = F_{i'}$. Thus $F_i \cup F_{i'} \cong F_\alpha \cup F_0$.

The edges of F_0 are in the form $v_\beta v_{-\beta}$ for $\beta \in \{1,\ldots,p-1\}$ and $v_0 v_p$. The edges of F_α are in the form $v_{\alpha+\beta} v_{\alpha-\beta}$ for $\beta \in \{1,\ldots,p-1\}$ and $v_\alpha v_{\alpha+p}$. Thus, for each $\alpha \in \{1,\ldots,p-1\}$, $F_0 \cup F_\alpha$ contains the path given by the sequence of vertices $v_p, v_0, v_{2\alpha}, v_{-2\alpha}, v_{4\alpha}, v_{-4\alpha}, \ldots, v_{(p-1)\alpha}, v_{-(p-1)\alpha}$. If α is odd, then $-(p-1)\alpha \equiv p+\alpha \pmod{2p}$ and $p\alpha \equiv p \pmod{2p}$. Since $v_{p+\alpha} v_\alpha$ is an edge of F_α, the path continues with the sequence of vertices $v_\alpha, v_{-\alpha}, v_{3\alpha}, v_{-3\alpha}, \ldots, v_{p\alpha}$ forming a cycle with $2p$ vertices. If α is even, then $-(p-1)\alpha \equiv \alpha \pmod{2p}$ and $p+p\alpha \equiv p \pmod{2p}$. Since $v_\alpha v_{p+\alpha}$ is an edge of F_α, the path continues with the sequence of vertices $v_{p+\alpha}, v_{p-\alpha}, v_{p+3\alpha}, v_{p-3\alpha}, \ldots, v_p$ forming a cycle with $2p$ vertices.

Thus, for all $\alpha \in \{1,\ldots,p-1\}$, $F_0 \cup F_\alpha$ is a Hamiltonian cycle. Therefore, for $i, i' \in \{0,\ldots,p-1\}$, $i \neq i'$, $F_i \cup F_{i'}$ is a Hamiltonian cycle.

Case 2: A proof that for $j,j' \in \{1,3,\ldots,p-2\}$, $j \neq j'$, $F'_j \cup F'_{j'}$ forms a Hamiltonian cycle is given.

In the 1-factor F'_j, the vertex $v_{\alpha j}$ is a vertex of the edge $v_{\alpha j}v_{(\alpha-1)j}$ if α is even and of the edge $v_{\alpha j}v_{(\alpha+1)j}$ if α is odd. Thus, for each $i,i' \in \{1,3,\ldots,p-2\}$, $i \neq i'$, $F'_i \cup F'_{i'}$ contains the path represented by the sequence of vertices

$v_0, v_{-i}, v_{i'-i}, v_{i'-2i}, v_{2i'-2i}, \ldots, v_{(p-1)(i'-i)}, v_{-i'}, v_0$

since $|i'-i|$ even $(i,i'$ are odd).

Suppose $\alpha(i'-i) = \beta(i'-i)$ then $\alpha = \beta$ since $i \neq i'$. Now $\alpha(i'-i)$ is even and $\beta(i'-i)-i$ is odd so that $\alpha(i'-i) \neq \beta(i'-i)-i$. Thus there are $2p$ distinct vertices in the above path and $F'_i \cup F'_{i'}$ forms a Hamiltonian cycle.

Case 3: A proof that for $k,k' \in \{1,3,\ldots,p-2\}$, $k \neq k'$, $F''_k \cup F''_{k'}$ forms a Hamiltonian cycle is given.

Note that $\phi(F'_k) = F''_k$ and $\phi(F'_{k'}) = F''_{k'}$. Thus $F''_k \cup F''_{k'} \cong F'_k \cup F'_{k'}$.

Therefore, $F''_k \cup F''_{k'}$ forms a Hamiltonian cycle.

Case 4: A proof that for $i \in \{0,1,\ldots,p-1\}$ and $j \in \{1,3,\ldots,p-2\}$, $F_i \cup F'_j$ and $F_i \cup F''_j$ form Hamiltonian cycles is given.

Note that $\rho^{2p-i}(F_i) = F_0$, $\rho^{2p-i}(F'_j) = \begin{cases} F'_j & \text{if } i \text{ even} \\ F''_j & \text{if } i \text{ odd} \end{cases}$

and $\rho^{2p-i}(F''_j) = \begin{cases} F''_j & \text{if } i \text{ is even} \\ F'_j & \text{if } i \text{ is odd} \end{cases}$.

Thus $F_i \cup F'_j$ and $F_i \cup F''_j$ are each isomorphic to one of $F_0 \cup F'_j$ and $F_0 \cup F''_j$.

The edges of F_0 are of the form $v_k v_{k'}$ for $k \in \{1,\ldots,p-1\}$ and $v_0 v_p$. In the 1-factor $F_{j'}$, the vertex v_{aj} is a vertex of the edge $v_{aj(a-1)j}$ if a is even and of the edge $v_{aj}v_{(a+1)j}$ if a is odd. In the 1-factor F''_j the vertex v_{aj} is a vertex of the edge $v_{aj}\, v_{(a+1)j}$ if a is even and of the edge $v_{aj}\, v_{(a-1)j}$ if a is odd.

For each $j \in \{1,3,\ldots,p-2\}$, $F_0 \cup F'_j$ contains the path represented by the sequence of vertices $v_p, v_0, v_{-j}, v_j, v_{2j}, v_{-2j}, v_{-3j}, \ldots, v_{(p-1)j}, v_{-(p-1)j}, v_{-pj}$. For each $j \in \{1,3,\ldots,p-2\}$, $F_0 \cup F''_j$ contains the path represented by the sequence of vertices $v_p, v_0, v_j, v_{-j}, v_{-2j}, v_{2j}, v_{3j}, \ldots, v_{-(p-1)j}, v_{(p-1)j}, v_{pj}$.

As an earlier cases it is easy to verify that the vertices are distinct so that the cycles are indeed Hamiltonian.

Case 5: A proof that for $j,k \in \{1,3,\ldots,p-2\}$, $F'_j \cup F''_k$ forms a Hamiltonian cycle is given.

In the 1-factor F'_j the vertex v_{aj} is a vertex of the edge $v_{aj}\, v_{(a-1)j}$ if a is even and of the edge $v_{aj}\, v_{(a+1)j}$ if a

is odd and in the 1-factor F''_k the vertex $v_{\alpha k}$ is a vertex of the edge $v_{\alpha k} v_{(\alpha+1)k}$ if α is even and of the edge $v_{\alpha k} v_{(\alpha-1)k}$ if α is odd. For each $j,k \in \{1,3,\ldots,p-2\}$, $F'_j \cup F''_k$ contains the path represented by the sequence of vertices $v_0, v_k, v_{k+j}, v_{2k+j}, \ldots,$ $v_{pk+(p-1)j}, v_0$.

As in earlier cases it is easy to verify that the vertices are distinct so that the cycles are indeed Hamiltonian.

Therefore, F is a perfect 1-factorization.□

Another class of graphs which has been studied to determine the existence of perfect 1-factorizations is complete bipartite graphs $K_{n,m}$. Note that the existence of a 1-factorization of a bipartite graph requires that $n = m$. The following result of Kotzig implies that for the existence of a perfect 1-factorization of $K_{n,n}$, n must be odd.

Theorem 2.1.9. (Kotzig [15]). If G is a bipartite graph, regular of degree greater than 2 with a perfect 1-factorization then $|V(G)| \equiv 2 \pmod 4$.□

P.J. Laufer has proved the following result giving the existence of perfect 1-factorizations of complete bipartite graphs $K_{2n-1,2n-1}$ depending on the existence of a perfect 1-factorization of K_{2n} .

<u>Theorem 2.1.10</u>. (P. Laufer [19]). If a perfect 1-factorization of K_{2n} exists, then a perfect 1-factorization of $K_{2n-1,2n-1}$ exists.□

Section 2. Q-indices.

This section deals with a property of a 1-factorization F of a complete graph called a Q-index of F.

Definition 2.2.1. Given an integer n, let $F = \{F_1, \ldots, F_{2n-1}\}$ be a 1-factorization of K_{2n} and Q be a class of regular graphs of degree 2. The Q-index of F, denoted $Q(F)$, is the largest integer m such that there exists a partition of the 1-factors of F into classes $F^{(1)}, \ldots, F^{(r)}$ with $|F^{(i)}| \geq m$ for $i = 1,2,\ldots,r$ and if $F_i, F_j \in F^{(k)}$ then there is a graph $G \in Q$ such that $F_i \cup F_j \cong G$.

If Q is the class of graphs which are Hamiltonian cycles, then the 1-factorization F of K_{2n} is a perfect 1-factorization if $Q(F) = 2n-1$.

Definition 2.2.2. If Q is a class of graphs with at most one graph on $2n$ vertices, then Q_{2n} is the graph on $2n$ vertices.

Theorem 2.2.1. (E. Mendelsohn and A. Rosa [20]). Let Q and Q' be classes of graphs of degree 2 such that for each n, Q or Q' has at most one graph on $2n$ vertices and $Q_{2n} \not\cong Q'_{2n}$. Then for any 1-factorization F of K_{2n}, $Q(F) > (2n-1)/(2k+1)$ implies that $Q'(F) \leq 2k-1$ for k in $\{1,\ldots,n-1\}$.

Proof: Suppose $Q(F) > (2n-1)/(2k+1)$. By definition of the Q-index, there exists a partition of the 1-factors of F into classes $F^{(i)}$, i in $\{1,\ldots,r\}$, such that $F_j \cup F_{j'} \cong Q_{2n}$ for $F_j, F_{j'}$ in $F^{(i)}$

and $|F^{(i)}| \geq Q(F) > (2n-1)/(2k+1)$ for each i in $\{1,2,\ldots,r\}$. Thus $r < 2k+1$.

To find $Q'(F)$, look at any partition of the 1-factors of F into sets $G^{(1)},\ldots,G^{(s)}$, such that $G \cup G' \ncong Q'_{2n}$ for $G,G' \in G^j$.

Note that for any two 1-factors F and F' of $F^{(i)}$, F and F' must belong to distinct $G^{(j)}$'s. For any two 1-factors G and G' of G^j, G and G' must belong to distinct $F^{(i)}$'s. Thus $|G^{(j)}| \leq r < 2k+1$.

Therefore $Q'(F) < 2k+1$.□

In [21] E. Mendelsohn and A. Rosa give two results concerning the existence of 1-factorizations with certain Q-indices where Q is a certain class of regular graphs of degree 2.

The first of these requires a result on Steiner loops.

<u>Definition 2.2.2</u>. A <u>Steiner loop</u> G with the binary operation $\circ$ is defined by the following properties.

(1) For any a,b in G the equations $a \circ b = x$, $a \circ x = b$ and $x \circ a = b$ each have a unique solution.

(2) There exists an element 1 in G such that $a \circ 1 = a = 1 \circ a$ for every a in G.

(3) $a \circ a = 1$ for all a in G.

(4) $a \circ b = b \circ a$ for all a,b in G.

(5) $a \circ (a \circ b) = b$ for all a,b in G.

Lemma 2.2.2. (R. Bruck [6]). A Steiner loop of order $n+1$ exists if and only if a Steiner triple system of order n exists.

Proof: Suppose a Steiner triple system T of order n exists. If abc is any block of T, let $a \circ b = c$, $b \circ c = a$ and $a \circ c = b$. Since T is a Steiner triple system any pair of elements occurs exactly once and each of $a \circ b = x$, $a \circ x = c$ and $x \circ b = c$ would have a unique solution for x. Note that $a \circ b = c = b \circ a$. If $a \circ b = c$ then $a \circ (a \circ b) = a \circ c = b$ for any a, b, c in T. Add an element 1 to the set of elements of T and define $1 \circ a = a \circ 1 = a$ and $a \circ a = 1$ for all a in T. Thus, the elements of T with 1 and the above operation form a loop of order $n+1$. Therefore, there exists a Steiner loop of order $n+1$.

Suppose a Steiner loop T with operation $\circ$ and identity 1 of order $n+1$ exists. Let a, b, c be in T and $a,b,c \neq 1$. Let the block of a block design be $a\ b\ c$ when $a \circ b = c$. If $a \circ b = c$, then $a \circ (a \circ b) = a \circ c = b$, $c \circ a = b$, $b \circ a = c$, $b \circ (b \circ a) = b \circ c = a$ and $c \circ b = a$. Since $a \circ b = x$, $a \circ x = c$ and $x \circ b = c$ each have a unique solution for x, each pair of non-identity elements will occur together in exactly one block. Therefore a Steiner triple system of order n is formed.□

Theorem 2.2.3. (E. Mendelsohn and A. Rosa [21]). Let Q be a class of regular graphs of degree 2, so that for each n, Q contains at most one graph on $2n$ vertices, Q_{2n}. Then for any $n \geq 4$ there is a 1-factorization F of K_{2n} such that $Q(F) = 1$.

<u>Proof</u>: Let $n \geq 4$ be fixed.

<u>Case 1</u>: Let Q_{2n} be disconnected.

Look at the bipyramidal 1-factorization of K_{2n} described in Definition 2.1.3 coming from the pyramidal 1-factorization with F_0 as described in Theorem 2.1.2. The union of F^* and F_0' is the Hamiltonian cycle $(u_\infty, u_0, u_{n-1}, u_{n-2}, u_{2n-3}, u_1, u_n, u_{n-3}, u_{2n-4}, u_2, \ldots, u_{\lfloor \frac{3(n-1)}{2} \rfloor +1}, u_{\lfloor \frac{3(n-1)}{2} \rfloor +n-1}, u_{\lfloor \frac{3(n-1)}{2} \rfloor}, u_\infty', u_\infty)$. Now F^* is the same if a cyclic permutation is applied to the vertices $u_0, u_1, \ldots, u_{2n-3}$. Thus for any 1-factor F_i', $F^* \cup F_i'$ forms a Hamiltonian cycle. Therefore, since Q_{2n} is disconnected, $Q(F') = 1$.

<u>Case 2</u>: Q_{2n} is connected.

Let $n \equiv 1$ or $2 \pmod 3$. By Lemma 2.3.2, since a Steiner triple system of order n' exists if and only if $n' \equiv 1$ or $3 \pmod 6$ there is a Steiner loop T of order n'', if and only if $n'' \equiv 2$ or $4 \pmod 6$. Thus there exists a Steiner loop of order $2n$. Set up a 1-factorization F of K_{2n} as follows. For $a, b, a \neq b$, in T if $a \circ b = c$ is in T let the edge $a\,b$ be in the 1-factor F_c. From property 1 of a Steiner loop each edge is in one 1-factor and each vertex is an endpoint of an edge in each 1-factor exactly once. Let b, c be in T, then there is an a in T such that $a \circ b = c$. Thus $1c$ and $a\,b$ are edges in 1-factor F_c and $1\,b$ and $a\,c$ are edges in 1-factor F_b. Thus $F_c \cup F_b$ has a component which is a 4-cycle $(1\ b\ a\ c\ 1)$ and at least one more component since $n \geq 4$.

Let $n \equiv 3 \pmod 6$. Label the vertices of K_{2n} v_i, u_i where $i \in S$, $i \neq 1$, and S is a Steiner loop of order $n + 1 \equiv 4 \pmod 6$ with elements $\{1,2,\ldots,n+1\}$. Form near 1-factorizations $F^v = \{F_2^v, F_3^v, \ldots, F_{n+1}^v\}$ and $F^u = \{F_2^u, \ldots, F_{n+1}^u\}$ on K_n, on the vertex sets $\{v_2,\ldots,v_{n+1}\}$ and $\{u_2,\ldots,u_{n+1}\}$ as follows. For a, b, $a \neq b$ and $a \neq 1$, $b \neq 1$ then if $a \circ b = c$ let the edge $v_a v_b \in F_c^v$ and the edge $u_a u_b \in F_c^u$. From property 1 of a Steiner loop, each edge is in one near 1-factor and each vertex is in each near 1-factor F_i^v, F_i^u as an end vertex except v_i or u_i which stand alone. For the 1-factor F_i, take the two near 1-factorizations F_i^v, F_i^u and the edge $v_i u_i$. To complete the 1-factorization form any 1-factorization of the edges between the two sets of vertices deleting the 1-factor already used. By the same argument as above any two 1-factors of the first type will have at least three components, a 4-cycle in each set of vertices and at least one other component since $2n \geq 18$. There are n 1-factors of the first type and $n-1$ of the 1-factors formed from cross edges. None of the first can occur together. Therefore $Q(F) = 1$.

Let $n \equiv 0 \pmod 6$. Label the vertices $c_1, c_2, \ldots, c_n, r_1, r_2, \ldots, r_n$. Take a 1-factorization $H = \{H_1, \ldots, H_n\}$ of $K_{n,n}$ on the sets $\{c_1, c_2, \ldots, c_n\}$ and $\{r_1, r_2, \ldots, r_n\}$ corresponding to a unipotent (the element 1 down the main diagonal) latin square $C = \{c_{ij}\}$ formed from the latin square A with object set 1 through $\frac{n}{2}$ which is unipotent and the latin square B on

$$C: \left|\begin{array}{c|c} A & B \\ \hline B^+ & A \end{array}\right|$$

the object set $\frac{n}{2} + 1, \ldots, n$. Let the object set of A be relabelled so that there are all 1's down the main diagonal. Label the columns of $C, c_1, c_2, \ldots, c_n$ and the rows of $C, r_1, r_2, \ldots, r_n$ .. If $c_{ij} = k$, then let the edge $r_i c_j$ be in 1-factor k, H_k . Note that for each $k \in \{2,3,\ldots,n\}$ there is a proper subsquare containing k and 1 . If k in $\{1,2,\ldots,\frac{n}{2}\}$, then $k \in A$, a proper subsquare. If k in $\{\frac{n}{2} + 1, \ldots, n\}$ then there is a subsquare of order 2 containing k and 1 . Thus the union of H_1 and H_j forms a disconnected graph for j in $\{2,3,\ldots,n\}$. To complete the 1-factorization F , take a 1-factorization on n vertices and take two copies of it; one on the vertices c_i and one on the vertices r_i such that when one 1-factor is taken from each set to form a 1-factor of $K_{2n}, c_i c_j$ will be in the 1-factor if and only if $r_i r_j$ is. Thus H_1 with any of these new 1-factors will form many 4-cycles. Therefore H_1 must occur by itself in a partition used in finding the Q-index of F . Therefore $Q(F) = 1.\square$

Another result of E. Mendelsohn and A. Rosa deals with Q being a class of regular graphs of degree two where Q_{2n} is made up of 4-cycles with possibly one 6-cycle if n is odd. Here the Q-index is called the tightness index.

<u>Definition 2.2.3.</u> Let Q be a class of regular graphs of degree 2 such that for every integer $n \geq 3$, Q_{2n} is made up of 4-cycles with possibly one 6-cycle if n is odd. For any 1-factorization F, the Q-index of F, $Q(F)$ is called the <u>tightness index</u> of F, $TI(F)$.

<u>Theorem 2.2.4.</u> If $n \equiv 0 \pmod 2$, then there is a 1-factorization F of K_{2n} such that $TI(F) \geq 2$.

<u>Proof:</u> Let $n = 2k$. Label the vertices of K_{2k}, u_i and v_i for $i = 1,2,\ldots,k$. Let F be a 1-factorization of K_{2k} with $F = \{F_1,F_2,\ldots,F_{2k-1}\}$ where $F_1 = \{u_iv_i : i = 1,2,\ldots,k\}$. Label the vertices of the k-partite graph $K_{4,4,\ldots,4}$, u_i', u_i'', v_i', v_i'' for i in $\{1,2,\ldots,k\}$. Construct a partitioning of $K_{4,4,\ldots,4}$ into 4-cycles as follows. Let the 4-cycles $(u_\ell' \; v_m' \; u_\ell'' \; v_m'' \; u_\ell')$, $(u_\ell' \; u_m' \; u_\ell'' \; u_m'' \; u_\ell')$ or $(v_\ell' \; v_m' \; v_\ell'' \; v_m'' \; v_\ell')$ be in G_j, a 2-factor of $K_{4,4,\ldots,4}$, if and only if $u_\ell v_m$, $u_\ell u_m$ or $v_\ell v_m$ is in F_j for j in $\{2,3,\ldots,2k-1\}$. Partition each G_j into two 1-factors G_j' and G_j'' for j in $\{2,3,\ldots,2k-1\}$. By definition $G_j' \cup G_j''$ forms a graph whose components are 4-cycles.

The above 1-factors leave k disjoint copies of K_4 to partition into 1-factors. Let G_1' be the set of edges $v_i' \; v_i''$ and $u_i' \; u_i''$ for i in $\{1,2,\ldots,k\}$, let G_1'' be the set of edges $v_i' \; u_i'$ and $v_i'' \; u_i''$ for i in $\{1,2,\ldots,k\}$ and let G_1''' be the set of edges

$v_i' u_i''$ and $v_i'' u_i'$ for i in $\{1,2,\ldots,k\}$. Note that the union of any pair of these last three 1-factors forms a graph whose components are 4-cycles.

The 1-factors in $\{G_i', G_i'', G_1''' \mid i = 1,2,\ldots,2k-1\}$ form a 1-factorization, F of K_{4k} . The partition of the 1-factors $F^{(1)} = \{G_1', G_1'', G_1'''\}$, and $F^{(i)} = \{G_i', G_i''\}$ for i in $\{2,3,\ldots,2k-1\}$ shows that $TI(F) \geq 2$.□

Another Q-index defined for a particular class of graphs is called the Dundas index.

<u>Definition 2.2.4.</u> Let Q be a class of graphs such that Q_{2n} is a Hamiltonian cycle for each n, $n > 1$. For any 1-factorization F the Q-index of F is called the Dundas index of F and denoted $DI(F)$.

Note that for F a 1-factorization of K_{2n} , if $DI(F) = 2n-1$ then F is a perfect 1-factorization.

Section 3. Kotzig Factorizations.

A Kotzig factorization contains both a near 1-factorization and a Hamiltonian decomposition.

Definition 2.3.1. A Hamiltonian decomposition $H = \{H_1, \ldots, H_n\}$ is a partitioning of the edge-set of a graph into Hamiltonian cycles.

Definition 2.3.2. A Kotzig factorization $K(H,F)$ of K_{2n+1} is a Hamiltonian decomposition H of K_{2n+1} with 1-factorization F of K_{2n+1} such that each Hamiltonian cycle of H intersects each near 1-factor of F in exactly one edge.

A construction of E. Mendelsohn and C. Colbourn exhibits a Kotzig factorization of K_p where p is an odd prime. This construction is used by J. Horton in proving the existence of Kotzig factorization of K_{2n+1} for all integers n .

Theorem 2.3.1. (E. Mendelsohn and C. Colbourn [20]). A Kotzig factorization of K_{2n+1} exists for $2n+1$ a prime.

Proof. Let $V(K_{2n+1}) = \{v_0, v_1, \ldots, v_{2n}\}$, $F_i = \{v_j v_{i-j} \mid j = 0,1,\ldots,2n\}$ for $i \in \{1,2,\ldots,n\}$ and $H_i = \{v_j v_{i+j} \mid j = 0,1,\ldots,2n\}$ for i in $\{1,2,\ldots,n\}$. Now $F = \{F_0, F_1, \ldots, F_{2n}\}$ is a near 1-factorization of K_{2n+1} and $H = \{H_1, H_2, \ldots, H_{2n}\}$ is a Hamiltonian decomposition of K_{2n+1} . To prove that $K(H,F)$ is a Kotzig factorization of K_{2n+1} , for $\ell \in \{1,\ldots,n\}$ let $v_i v_j \in H_\ell$ and $i - j \equiv \ell \pmod{2n+1}$ where

$i + j \equiv k \pmod{2n+1}$. Then $v_i v_j$ is in F_k. Since

$$j \equiv \begin{cases} \frac{\ell+k}{2} \pmod{2n+1} & \text{for } \ell+k \text{ even} \\ \frac{(\ell+k) \pmod{2n+1}}{2} & \text{for } \ell+k \text{ odd} \end{cases} \quad \text{and} \quad i = \ell + j \pmod{2n},$$

each edge of H_ℓ is in a different 1-factor.□

Mendelsohn and Colbourn [20] also construct Kotzig factorizations of K_{2n+1} for n smaller than 21.

A construction of J. Horton [10] gives Kotzig factorizations of K_{2n+1} for all n. In this construction strong starters are used.

Definition 2.3.3. When considering abelian groups, additive notation is used. A strong starter of an abelian group G of order k is a set A of unordered pairs of elements from G with the following properties.

(a) For x in G, $x \neq 0$, there exists y in G, $y \neq 0$, such that $\{x,y\}$ in A.

(2) If $\{x,y\}$ and $\{x,z\}$ are in A, then $z = y$.

(3) $\{(x + -y) \mid \{x,y\} \text{ in } A\} = G \setminus \{0\}$.

(4) For $\{x,y\}$ in A, $(x+y) \neq 0$ and for any $\{x',y'\}$ in A, $\{x',y'\} \neq \{x,y\}$, then $(x+y) \neq (x'+y')$.

Strong starters in $GF[p^n]$ are known to exist ([8], [9], and [24]) where p is any odd prime and n is an integer except for $p^n = 3, 5$ or 9. For $p = 3$ the set $A = \{\{1,2\}\}$ is used and for $p = 5$ the set $A = \{\{1,2\}, \{2,3\}\}$ is used. In the first case A is not a strong starter since $(1+2) \equiv 0 \pmod 3$ which does not affect the construction, but in the second case the basic construction must be altered.

<u>Theorem 2.3.2.</u> (J. Horton [10]). Suppose a Kotzig factorization of K_{2n+1} exists, then a Kotzig factorization of $K_{p(2n+1)}$ exists where p is an odd prime.

<u>Proof.</u> Let $K(H,F)$ be a Kotzig factorization of K_{2n+1} on the vertex set $\{v_0,\ldots,v_{2n}\}$, labelled so that $H_1 = \{v_0v_1, v_1v_2, \ldots, v_{2n}v_0\}$ and where $F = \{F_0,\ldots,F_{2n}\}$, with v_m having degree 0 in F_m. Let $K(H",F")$ be a Kotzig factorization of K_p on the vertex set $\{u_0,\ldots,u_{p-1}\}$ described in Theorem 2.3.1. Let A be a strong starter of $GF(p)$ using the set $\{0,1,2,\ldots,p-1\}$. For $p = 3$, let $A = \{\{1,2\}\}$ and for $p = 5$, let $A = \{\{1,2\}, \{2,3\}\}$. Now relabel the vertices $\{u_0,\ldots,u_{p-1}\}$ so that for $p > 5$, $A = \{\{1,2\}, \{3,4\}, \ldots, \{p-2,p-1\}\}$. Let H' be the Hamiltonian decomposition corresponding to $H"$ and F' the near 1-factorization corresponding to $F"$.

Using H, p Hamiltonian cycles will be formed for each Hamiltonian cycle of K_{2n+1}. The edges of one of these Hamiltonian cycles, along with the edges of the $2n+1$ edge disjoint K_p's, are partitioned into $\frac{p+1}{2}$ Hamiltonian cycles. The partitioning into a near 1-factorization uses the near 1-factorizations F and F'. In order to ensure that a Kotzig factorization is formed, in using F' the latter $\frac{p+1}{2}$ Hamiltonian cycles are taken into account.

Let H_i be a Hamiltonian cycle of K_{2n+1} in H with edges $\{v_{h_0}v_{h_1}, v_{h_1}v_{h_2}, \ldots, v_{h_{2n}}v_{h_0}\}$. Then for $\ell \in \{0,1,\ldots,p-1\}$ define

$$H_i^{\ell} = \{0_{h_0}\ell_{h_1}, \ell_{h_1}0_{h_2}, 0_{h_2}\ell_{h_3}, \ldots, 0_{h_{2n-1}}\ell_{h_{2n-1}}, \ell_{h_{2n-1}}(2\ell)_{h_{2n}},$$

$$(2\ell)_{h_{2n}}1_{h_0}, 1_{h_0}(\ell+1)_{h_1}, (\ell+1)_{h_1}1_{h_2}, \ldots, (\ell+1)_{h_{2n-1}}(2\ell+1)_{h_{2n}},$$

$$\ldots, (p-1)_{h_0}(p-1+\ell)_{h_1}, \ldots, (p-1)_{h_{2n-2}}(p-1+\ell)_{h_{2n-1}},$$

$$(p-1+\ell)_{h_{2n-1}}(p-1+2\ell)_{h_{2n}}, (p-1+2\ell)_{h_{2n}}0_{h_0}\}.$$

$$= \begin{cases} k_{h_j}(k+\ell)_{h_{j+1}} & \text{for } j \in \{0,2,4,\ldots,2n-2\} \\ (k+\ell)_{h_j}k_{h_{j+1}} & \text{for } j \in \{1,3,5,\ldots,2n-3\} \end{cases} \quad \text{and for } k \text{ in } \{0,1,\ldots,p-1\}$$

$$\cup\{(k+\ell)_{h_{2n-1}}(k+2\ell)_{h_{2n}}, (k+2\ell)_{h_{2n}}(k+1)_{h_0} \text{ for } k \text{ in } \{0,1,\ldots,p-1\}\}.$$

Thus partition the edges corresponding to K_{2n+1} into Hamiltonian cycles H_i^{ℓ}, $\ell \in \{0,1,\ldots,p-1\}$ for each H_i in H.

This leaves the edges interior to $2n+1$ disjoint copies of K_p. The edges of $H_i^0 = \{0_0 0_1, 0_1 0_2, \ldots, 0_{2n}1_0, \ldots, (p-1)_{2n}0_0\}$

are used to connect these copies of K_p and to form another $\frac{p+1}{2}$ Hamiltonian cycles.

Now delete the edges $i_k j_k$ for $k \in \{0,1,\ldots,2n\}$, $\{i,j\} \in A$. In each copy of K_p this leaves $\frac{p-1}{2}$ Hamiltonian paths from H' since each difference occurs in a different Hamiltonian cycle of H''. Define $H_i''^{\ell}$ to be the Hamiltonian path on the vertices $0_\ell, 1_\ell, \ldots, p-1_\ell$ determined by the Hamiltonian cycle H_i' from H'.

Let $H_i''' = \{a_0a_1, b_1b_2, a_2a_3, \ldots, b_{2n-1}b_{2n}, a_{2n}b_0 \mid a_jb_j$ is the edge deleted from $H_j'^{i}\} \cup H_0'^{i} \cup H_1'^{i} \cup \ldots \cup H_{2n}'^{i}$ for $i \in \{1,2,\ldots,\frac{p-1}{2}\}$ and $H_0''' = \{b_kb_{k+1} \mid b_kb_{k+1} \notin H_i'''$ for $i \in \{1,2,\ldots,\frac{p-1}{2}\}$ and $b_kb_{k+1} \in H_1^0\} \cup \{j_k(j+1)_k \mid \{j,j+1\} \in A$, $k \in \{0,1,\ldots,2n\}\}$. Thus $H''' = \{H_i^{\ell}, H_j'''$ for $\ell \in \{0,1,\ldots,p-1\}$, $i \in \{1,\ldots,n\}$, $j \in \{0,1,\ldots,\frac{p-1}{2}\}\}$ is a partitioning of the edges into Hamiltonian cycles.

To construct a near 1-factorization of $K_{p(2n+1)}$ the edges not internal to the K_p's are partitioned as follows: $G_i'^{\ell} = \{j_k(\ell-j)_m$ where $v_kv_m \in F_i\}$ for $F_i \in F$ and $\ell \in \{0,1,\ldots,p-1\}$. The edges internal to the copies of K_p are partitioned into near 1-factors as follows: $G_i''^{\ell} = \{j_ik_i \mid u_ju_k \in F_\ell'\}$ for each $F_\ell' \in F'$ and $i \in \{0,1,\ldots,2n\}$. At this point, some care must be taken in choosing the $G_i''^{\ell}$ to go with $G_j'^{m}$ for given $j \in \{0,1,\ldots,2n\}$ and $m \in \{0,1,\ldots,p\}$, since both $G_i'^{\ell}$ and $G_i''^{\ell}$ may contain an edge of H_j'''. Relabel the 1-factors as $G_i'''^{m} = G_i''^{\ell}$ if $|E(H_j''') \cap E(G_i'^{m})| = 1$ and $E(G_i''^{\ell}) \cap E(H_j''') = \emptyset$ for all $j \in \{1,\ldots,\frac{p-1}{2}\}$ and $G_i'''^{m} = G_i''^{\ell}$ if $E(H_j''') \cap E(G_i'^{m}) = \emptyset$ for all $j \in \{1,\ldots,\frac{p-1}{2}\}$ and $G_i'''^{m}$ is not defined for m.

Let $G_i^{\ell} = G_i'^{\ell} \cup G_i'''^{\ell}$. Then $G = \{G_i^{\ell} \mid i \in \{0,1,\ldots,p-1\}, \ell \in \{0,1,\ldots,2n\}\}$ is a near 1-factorization.

Now, $K(G,H''')$ is a Kotzig factorization of $K_{p(2n+1)}$.□

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