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NEW METHODS TO GENERATE MINIMAL BROADCAST NETWORKS AND FAULT-TOLERANT MINIMAL BROADCAST NETWORKS

by

Siu-cheung Chau
B.Ed., University of Lethbridge, 1983

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New Methods to Generate Minimal Broadcast Networks and Fault-Tolerant Minimal Broadcast Networks

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To my wife, Lily

ABSTRACT

Broadcasting is the process of information dissemination in a communication network in which a message, originated by one member, is transmitted to all members of the network. A minimal broadcast network (mbn) is a communication network in which a message can be broadcast in minimum time regardless of originator. A minimum broadcast graph (mbg) is an mbn which has the fewest number of communication links. No technique is known for constructing mbgs of arbitrary size. We present new methods for constructing mbns which have approximately the minimum number of links possible. The resulting networks often have fewer links than previously described networks of this type. Fault-tolerant (ft) broadcasting is to broadcast with enough redundancy so that the broadcast can be completed even if links fail. We also present new methods to construct 1-ft and 2-ft mbns. The number of links of our 1-ft and 2-ft mbns is just a little more than half of the edges of the 1-ft and 2-ft mbns constructed by previous methods.

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Chapter 1

Definitions and Previous Results

1.1 Definitions

In a communication network, a member has a message which is to be disseminated to all other members. The series of calls to inform the other members is constrained by the following :

1. Each call requires one unit of time.
2. A member can only call an adjacent member.
3. A member can participate in at most one call per time unit.

This process is called broadcasting and the member that sends the message is called the originator [Mitchell 80].

Let $G = (V, E)$ be a graph that represents a communication network. The set of vertices V corresponds to the members of the network and the set of edges E corresponds to the communication links connecting pairs of members. Let n denote the number of vertices in G, e(G) denote the number of edges in G and t(u) denote the time required to broadcast using $u \in V$ as the originator. Let $t(G) = Max \{ t(u) : u \in V \}$. i.e. Every $u \in V$ can broadcast in less than or equal to t(G) units of time.

Let T(n) denote the minimum time required to broadcast a message in any communication network G with n members. T(n) is equal to ceiling of $\log_2 n$ because at each time unit, the number of informed vertices can be at most double the number of informed vertices in the previous time unit [Mitchell 80].

Definition 1: A minimal broadcast network (mbn) is a graph G such that [Farley 79]

$$t(G) = T(n)$$

$$= \lceil \log_2 n \rceil.$$

An mbn represents a communication network that can complete a broadcast regardless of originator, in minimum time.

Definition 2: A minimum broadcast graph (mbg) is a graph G such that G is an mbn and e(G) is minimum. An mbg is an mbn having the minimum number of edges [Mitchell 80].

An mbg represents a communication network with the fewest communication links between members that can complete a broadcast in minimum time regardless of originator.

Let B(n) denote the number of edges of an mbg of size n. The value of B(n) for arbitrary n is not known and it is conjectured that to determine B(n) is NP-Complete [Farley, et al. 79]. The value of B(n) is only known for $n <= 17$ [Mitchell 80] (see figure 1.1) or for $n = 2^k$ [Farley, et al. 79] where

$$B(n) = n/2 * \lceil \log_2 n \rceil.$$

The only known lower bound for B(n) is from the fact that the graph must be connected . Therefore, $B(n) >= n-1$. This is a very poor lower bound and it does not make use of any other properties of mbgs. Upper bounds for B(n) can be obtained from the size of known mbns.

The existing upper bounds for B(n) are [Farley 79]

$$B(n) <= n/2\lceil \log_2 n \rceil \quad \text{for } 3*2^{(\lceil \log_2 n \rceil - 2)} < n <= 2^{\lceil \log_2 n \rceil}$$

$$B(n) <= n/2\lceil \log_2 n \rceil - n/2 \quad \text{for } 2^{\lfloor \log_2 n \rfloor} < n <= 3*2^{(\lceil \log_2 n \rceil - 2)}$$

n	2	3	4	5	6	7	8	9	10
B(n)	1	2	4	5	6	8	12	10	12

n	11	12	13	14	15	16	17
B(n)	13	15	18	21	24	32	22

Figure 1-1: The value of B(n) for n <= 17

Definition 3: A <u>broadcasting</u> <u>scheme</u> is a sequence of calls between members of a communication network which complete a broadcast.

Mbgs represent the cheapest efficient communication networks. They may be used for message broadcasting in communication, parallel processing and distributed computing. No technique is known to generate an mbg of arbitrary size and the recognition problem for mbgs is NP-complete [Farley, et al. 79]. Only mbgs of size n <= 17 or n = 2^k are known. Heuristics can be used to generate mbns which have a small number of edges to approximate mbgs. In the following sections, existing algorithms to construct mbns are presented.

1.2 Farley's algorithm

Broadcasting can be accomplished in minimum time if there is a broadcast tree rooted at the originator. The most obvious graphs that satisfy this condition for each vertex are the complete graphs. Complete graphs have $n(n-1)/2$ edges and the problem becomes whether we can reduce the number of edges and still maintain the condition. Farley discovered that a subclass of star polygons also satisfies the above condition and they have fewer edges than complete graphs. The star polygons give an upper bound of

$$B(n) <= n\lceil \log_2 n \rceil$$

which is significantly less than the number of edges in complete graphs. Furthermore, Farley presented the first heuristic to generate mbns which have fewer edges than star polygons. The idea of his algorithm is to construct mbns by connecting two or three smaller size mbns together in a special way. By applying his algorithm recursively, mbns of arbitrary size can be constructed and the number of edges in the resulting mbns is no more than half of the edges of the star polygons.

1. Farley's two-way split

An mbn of size n can be formed by connecting 2 mbns S_1, S_2 each of size n_1 and n_2 respectively such that $n_1 + n_2 = n$ and $\lceil \log_2 n_1 \rceil = \lceil \log_2 n \rceil - 1$. Assume $n_1 >= n_2$. Connect every vertex in S_2 to a distinct vertex in S_1. The resulting graph G is an mbn of size n [Farley 79].

The broadcasting scheme for G is as follows: If the originator is in S_1 then start to broadcast within S_1. After S_1 has finished its own broadcasting, conduct calls between S_1 and S_2 through the links that connect them together. If the originator is in S_2 then in the first time unit the originator calls a vertex in S_1. After the first time unit, S_1

and S_2 each have an informed vertex. Starting from time unit two, both mbns can broadcast internally.

The time required to broadcast within either S_1 or S_2 is $\lceil \log_2 n \rceil - 1$ and one additional time unit is required to conduct calls between S_1 and S_2. The total time required for G to complete the broadcast is $\lceil \log_2 n \rceil$. Thus, the graph G constructed by the 2-way split is an mbn.

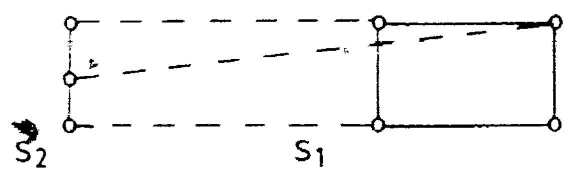


Figure 1-2: Example of an mbn constructed by Farley's 2-way split

2. Farley's three-way split

An mbn of size n in the range

$$2^{\lfloor \log_2 n \rfloor} < n <= 3*2^{(\lceil \log_2 n \rceil - 2)}$$

can be generated by connecting three mbns S_1, S_2 and S_3 of size n_1, n_2 and n_3 respectively such that $n_1 + n_2 + n_3 = n$ and $\lceil \log_2 n_i \rceil = \lceil \log_2 n \rceil - 2$.

If n is even then connect each member of the three components to a different member of a different component. If n is odd then do as above for n-1 of the members; then connect the remaining member to a member of a different component to which no member of its component is already connected. The resulting graph G is an mbn [Farley 79].

The calling scheme for G is as follows: Without loss of generality, assume the originator is in S_1 and it is connected to a vertex in S_2. In the first time unit the originator calls the vertex in S_2. Starting from the second time unit, S_1 and S_2 each contain an informed vertex and they can broadcast internally. After they have finished their own broadcasts, conduct calls from S_1 and S_2 to S_3. The time required for either S_1, S_2 or S_3 to broadcast internally is $\lceil \log_2 n \rceil - 2$. The calls between components require two extra time units. The total time required to complete a broadcasting in G is $\lceil \log_2 n \rceil$. Hence, the graph G constructed by the 3-way split is an mbn.

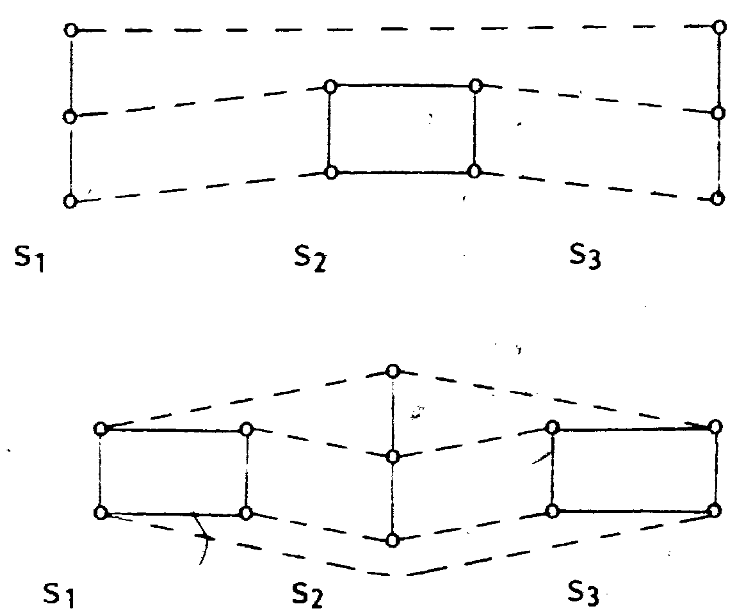


Figure 1-3: Examples of mbns constructed by Farley's three-way split

By applying Farley's two-way split and three-way split methods recursively, mbns of arbitrary size can be generated. In the two methods described above, no detail is given on how to split n into two or three parts. These methods work for any n_i which satisfy the conditions set forth. However, Farley chooses to split n under the following conditions :

1. For 2-way split, $|n_1-n_2| <= 1$.

2. For 3-way split, $|n_1-n_2| <= 1$, $|n_1-n_3| <= 1$ and $|n_2-n_3| <= 1$.

That is, he always splits as evenly as possible. Under these conditions, the mbns G generated by Farley's algorithm give

$$e(G) <= n/2\lceil \log_2 n \rceil$$

as an upper bound for the number of edges in G for the two-way split and

$$e(G) <= n/2\lceil \log_2 n \rceil - n/2$$

for the three-way split. Since the 3-way split has a better bound than the 2-way split, Farley always uses the 3-way split when it is possible. For certain sizes slight improvement on the above result can be obtained by using the best split for n recursively [Liestman 83]. This approach is computationally inefficient. In this thesis, we assume that splitting is done according to Farley's conditions. The result obtained by Farley is not bad if we compare mbns generated by Farley's algorithm to mbgs of size $n=2^k$. Actually, Farley's algorithm generates mbgs when $n=2^k$. However, it is not possible to judge whether other mbns generated by Farley's algorithm are good approximations of mbgs since we know neither the value of $B(n)$ nor a good lower bound for $B(n)$ when $n \neq 2^k$ and $n > 17$.

Chapter 2

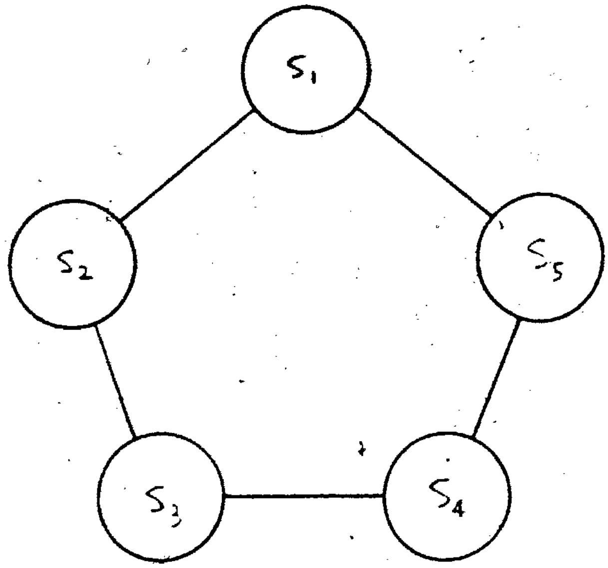
Algorithms to Approximate Minimum Broadcast Graphs

Using Farley's idea of generating mbns recursively, better approximations can be found. Instead of constructing mbns from two or three smaller mbns, we can use five, six or seven smaller mbns to construct larger mbns. In the following three sections, different heuristics to generate mbns based on Farley's idea are presented.

2.1 Five-way split method

The most straightforward way to extend Farley's algorithm is to construct mbns using more than three smaller mbns. However, it turns out that the simplest way does not work very well. Consider an mbn constructed by 5 smaller mbns. They are connected together in such a way that each mbn can be considered as a vertex in an mbg of size 5 (see figure 2-1). Every vertex in each small mbn is connected to two other vertices from different small mbns. Suppose that each smaller mbn is constructed by Farley's 2-way split and each requires $\lceil \log_2 n \rceil - 3$ units of time for broadcasting. We can send messages from the originator to a vertex in two other different smaller mbns in the first two time units. In the third time unit, the two just informed vertices can send messages to a vertex in the remaining two small mbns. Thus, after three time units each small mbn has one informed vertex and they can broadcast internally. The total time required for broadcasting in G is also $\lceil \log_2 n \rceil$. Hence, the graph G constructed by the straightforword 5-way split is an mbn.

The number of edges in G is



Each line represents that every vertex in one set is connected to at least one vertex of the other set.

Figure 2-1: Constructing mbns using straightforward 5-way split

$$
\begin{aligned}
e(G) &<= \sum_{i=1}^{5} (n_i/2\lceil \log_2 n_i \rceil) + \lceil n/5 \rceil * 5 \\
&<= \sum_{i=1}^{5} (n_i/2(\lceil \log_2 n \rceil - 3)) + n + 5 \\
&<= n/2(\lceil \log_2 n \rceil - 3) + n + 5 \\
&<= n/2\lceil \log_2 n \rceil - n/2 + 5
\end{aligned}
$$

Thus, no improvement on the bound is achieved by the straightforward 5-way split. Better results may be obtained by a less obvious approach.

Consider the graph constructed by the straightforward 5-way split. In the first three time units, some of the informed vertices are not involved in calls. We can make use of these unutilized but informed vertices to reduce the number of edges. The following is an improved version of the straightforward 5-way split.

Definition 1: Given a graph $G=(V, E)$, a partition of the set of vertices V into V_a and V_b is called an even adjacency split of V if

1. $| |V_a| - |V_b| | <= 1$

2. For every vertex $v \in V_a$, there is at least one vertex $u \in V_b$ that is adjacent to v.

3. For every vertex $u \in V_b$, there is at least one vertex $v \in V_a$ that is adjacent to u.

Definition 2: Given graphs $A = (V_a, E_a)$ and $B = (V_b, E_b)$ such that

$| |V_a| - |V_b| | <= 1.$

A graph G is formed by adding edges between V_a and V_b. A and B are said to be connected by a minimum adjacency connection if

1. The number of edges added is $Max(|V_a|, |V_b|)$.
2. V_a and V_b constitute an even adjacency split of G.

Note that if $|Va| = |Vb|$ the edges added are a perfect matching from V_a to V_b. If $|V_a| = |V_b|+1$, the edges added are a perfect matching from V_b to V_a plus an edge from the unmatched vertex of V_a to a vertex in V_b. Similarly, if $|V_b| = |V_a|+1$, the edges added are a perfect matching from V_a to V_b plus an edge from the unmatched vertex of V_b to a vertex in V_a.

Lemma 3: An mbn $G = (V, E)$ constructed by Farley's 2-way split algorithm has an even adjacency split.

Proof: Let $A = (V_a, E_a)$ and $B = (V_b, E_b)$ be the two smaller mbns used by the 2-way split algorithm to construct G. Assume that $|V_a| - |V_b| <= 1$.

Two cases :

1. If n is even then V_a and V_b are an even adjacency split of G.

2. If n is odd then let $v \in V_a$ be the vertex that is not connected to any vertex in V_b. Split V into two sets K_1 and K_2 such that

$$K_1 = V_a \setminus \{v\}$$

and

$$K_2 = V_b \cup \{v\}$$

Since v must be adjacent to at least one vertex in V_a, each vertex in K_1 must be adjacent to at least one vertex in K_2 and vice versa. K_1 and K_2 are an even adjacency split of G.

Hence, the graph G constructed by Farley's two-way split has an even adjacency split.

Five-way split method

Given n such that

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/5 \rceil \rceil = 3,$$

an mbn of size n can be constructed as follows :

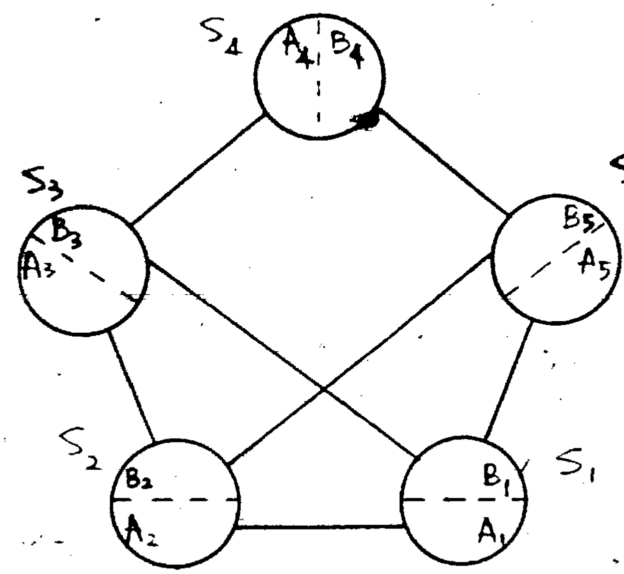
1. Partition n vertices into 5 sets S_i such that

$$\lceil n/5 \rceil = |S_1| >= |S_2| >= \dots >= |S_5| = \lfloor n/5 \rfloor$$

2. For each S_i, construct an mbn with an even adjacency split A_i, B_i such that $|A_i|-|B_i| <= 1$. This may be done recursively or using other heuristics such as Farley's 2-way

split.

3. Add edges to form minimum adjacency connections between the following pairs : (A_1, A_2), (B_2, A_3), (B_3, A_4), (B_4, B_5), (A_5, B_1), (B_1, B_3), (B_2, B_5).



Each line represents a minimum adjacency connection between the A_is' and the B_is'

Figure 2-2: Constructing mbns using 5-way split

Theorem 4: The graph G constructed by the 5-way split method is an mbn.

Proof: Refer to figure 2-3 and consider an originator in S_4. Without loss of generality, assume that the originator is in A_4. Consider the following calling scheme :

1. Time unit 1 : Conduct call between the pair $(v_4 \in A_4, u_4 \in B_4)$. This is possible because A_4 and B_4 are an even adjacency split of S_4.

2. Time unit 2 : Conduct call between the pairs ($v_4 \in A_4$, $u_3 \in B_3$), ($u_4 \in B_4$, $u_5 \in B_5$). This is possible because the sets A_4, B_3 and B_4, B_5 are connected by minimum adjacency connections.

3. Time unit 3 : Conduct call between the pairs ($u_3 \in B_3$, $u_1 \in B_1$), ($u_5 \in B_5$, $u_2 \in B_2$). Again, the sets are connected by minimum adjacency connections.

Each mbn S_i has at least one informed vertex after the first three time units. In time unit four, each S_i can start to broadcast internally. The time required for each small network is $\lceil \log_2 n/5 \rceil$.

The total time required to complete broadcasting in G is

$$t(G) = \lceil \log_2 \lceil n/5 \rceil \rceil + 3$$

Since

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/5 \rceil \rceil = 3$$

$$\therefore t(G) = \lceil \log_2 n \rceil - 3 + 3$$

$$\therefore \quad = \lceil \log_2 n \rceil.$$

Therefore, G can complete a broadcast in minimum time if the originator is in S_4.

Referring to figure 2-4 and 2-5 and using similar arguments, G can complete a broadcast in minimum time if the originator is in S_1, S_2, S_3 or S_5.

Therefore, G is an mbn.

Theorem 5: The graph G constructed by the 5-way split method has

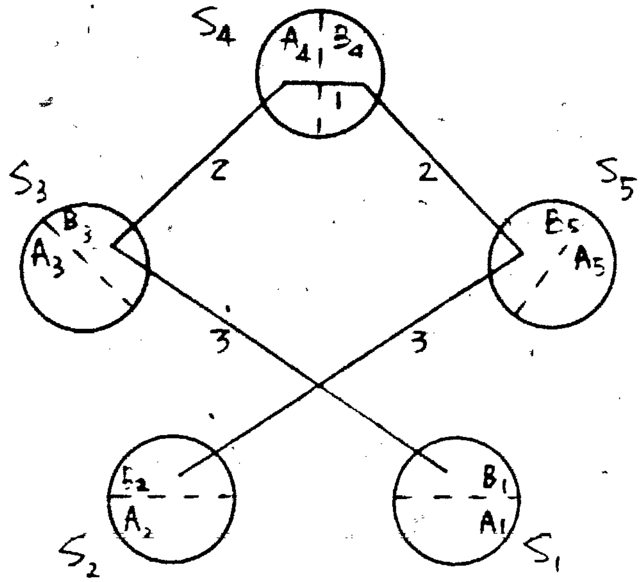


Figure 2-3: Calling scheme for the first three time units if the originator is in S_4 for mbns constructed by the 5-way split

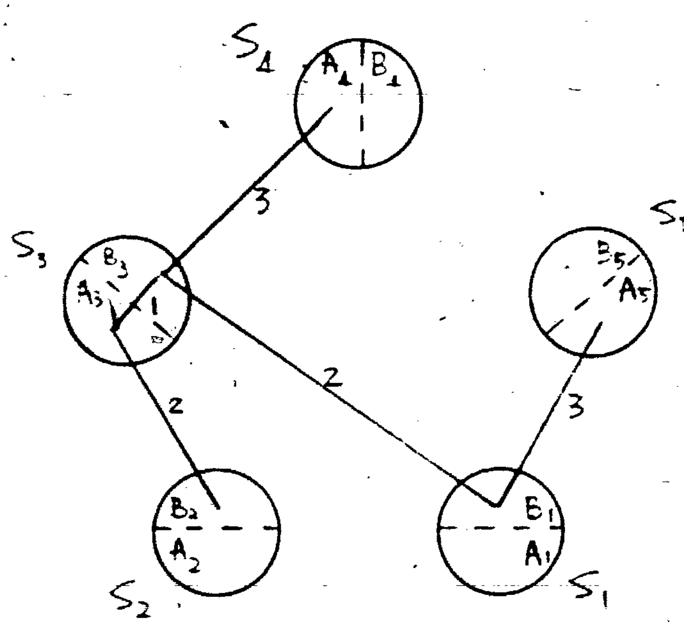


Figure 2-4: Calling scheme for the first three time units if the originator is in S_3 or S_5 for mbns constructed by the 5-way split

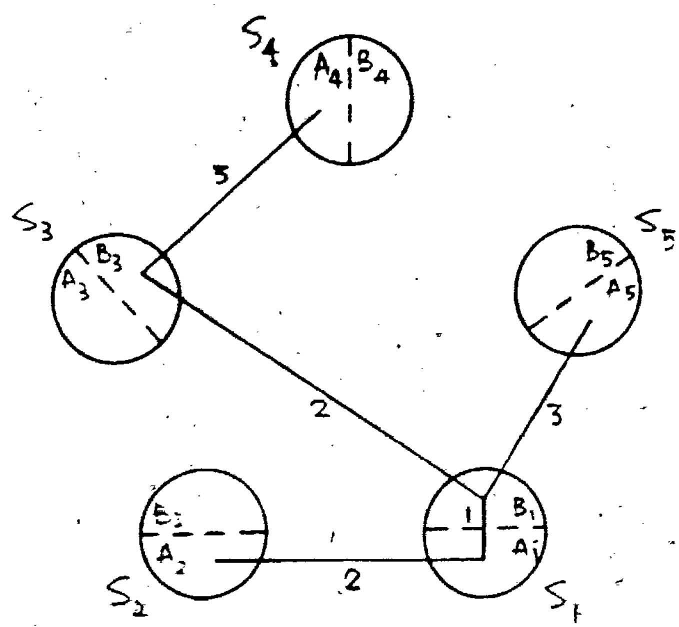


Figure 2-5: Calling scheme for the first three time units if the originator is in S_1 or S_2 for mbns constructed by the 5-way split

$e(G) <= n/2\lceil \log_2 n \rceil - 4n/5 + 7.$

Proof: Let n_i be the number of vertices in each S_i. Assume the S_i's are constructed by Farley's 2-way split method.

Therefore, the S_is will have a total of

$$\sum_{i=1}^{5} (n_i/2\lceil \log_2 n_i \rceil)$$

edges.

At most $\lceil n/10 \rceil * 7$ edges are needed to form minimum adjacency connections between pairs (A_1, A_2), (A_3, B_2), (A_4, B_3), (A_5, B_1), (B_1, B_3), (B_2, B_5), (B_4, B_5).

The total number of edges in G is

$$e(G) <= \sum_{i=1}^{5} (n_i/2\lceil \log_2 n_i \rceil) + \lceil n/10 \rceil * 7$$

$$<= \sum_{i=1}^{5} (n_i/2(\lceil \log_2 n \rceil - 3) + 7n/10 + 7$$

$$<= n/2(\lceil \log_2 n \rceil - 3) + 7n/10 + 7$$

$<= n/2\lceil \log_2 n \rceil - 4n/5 + 7$

Theorem 6: The graph $G = (V, E)$ constructed by the 5-way split method has an even adjacency split.

Proof: Split V into two sets K_1 and K_2 such that

$K_1 = A_1 \cup B_2 \cup A_3 \cup B_4 \cup A_5$

$K_2 = B_1 \cup A_2 \cup B_3 \cup A_4 \cup B_5$

Every $v \in K_1$ must be adjacent to at least one $u \in K_2$ and vice versa since the A_i's and the B_i's are even adjacency splits of the S_i's.

Let $j = n \bmod 5$. That is, j is the number of S_i's of size $\lfloor n/5 \rfloor + 1$.

Three cases :

1. If $|S_i| = \lfloor n/5 \rfloor$ for all i and $\lfloor n/5 \rfloor$ is even, then $|K_1|=|K_2|$. If $|S_i| = \lfloor n/5 \rfloor$ for all i and $\lfloor n/5 \rfloor$ is odd, then $|K_1|=|K_2|+1$.

2. If $j \neq 0$ and and ~~$\lfloor n/5 \rfloor$ is~~ even, then $|A_i| = |B_i|$ for $i>=j+1$ and $|A_i| = |B_i| + 1$ for $1<=i<=j$. Therefore, $|K_1| = |K_2|+1$ if j is odd and $|K_1| = |K_2|$ if j is even.

3. If $j \neq 0$ and and $\lfloor n/5 \rfloor$ is odd, then $|A_i| = |B_i| + 1$ for $i>=j+1$ and $|A_i| = |B_i|$ for $1<=i<=j$. Therefore, $|K_1| = |K_2|$ if j is odd and $|K_1| = |K_2|+1$ if j is even.

G has an even adjacency split K_1 and K_2.

From theorems 4, 5, and 6, the graph G constructed by the 5-way split method has the following properties :

1. G is an mbn.
2. $e(G) <= n/2\lceil \log_2 n \rceil - 4n/5 + 7$.
3. G has an even adjacency split.

2.2 Six-way split method

Given n such that

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/6 \rceil \rceil = 3,$$

an mbn of size n can be constructed as follows :

1. Partition n vertices into 6 sets S_i such that

   $$\lceil n/6 \rceil = |S_1| >= |S_2| >= \ldots >= |S_6| = \lfloor n/6 \rfloor.$$

2. For each S_i construct an mbn with an even adjacency split A_i, B_i such that $|A_i|-|B_i| <= 1$. This may be done recursively or using other heuristics such as Farley's 2-way split.

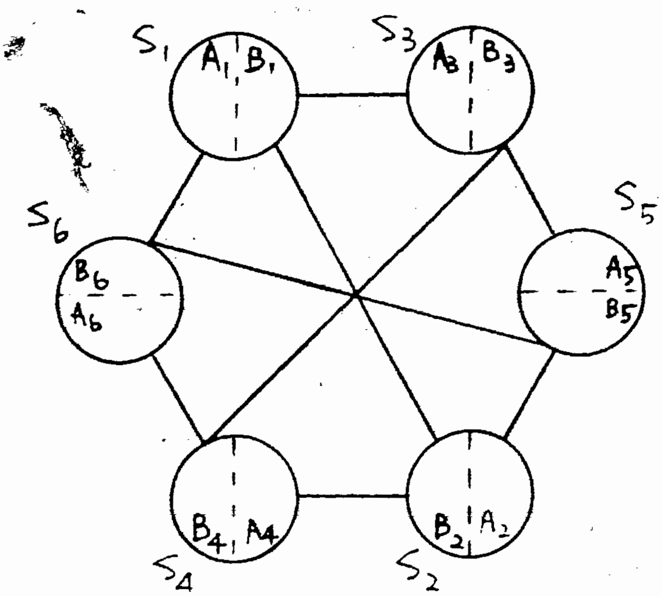
3. Add edges to form minimum adjacency connections between the following pairs : (B_1, A_3), (B_3, A_5), (B_5, A_2), (B_2, A_4), (B_4, A_6), (B_6, A_1), (B_1, B_2), (B_3, B_4), and (B_5, B_6).

Theorem 7: The graph G constructed by the 6-way split method is an mbn.

Proof: Referring to figure 2-7 and using similar arguments to those used in the proof of theorem 4, each mbn S_i has at least one informed vertex after the first three time units. In time unit four, each S_i can start to broadcast internally. The time required for each set is $\lceil \log_2 \lceil n/6 \rceil \rceil$.

The total time required to complete broadcasting in G is

$$t(G) = \lceil \log_2 \lceil n/6 \rceil \rceil + 3$$



Each line represents a minimum adjacency connection between the A_i's and the B_i's

Figure 2-6: Constructing mbns using 6-way split

Since

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/6 \rceil \rceil = 3$$

$$\therefore t(G) = \lceil \log_2 n \rceil - 3 + 3$$

$$\therefore \quad = \lceil \log_2 n \rceil$$

Therefore, G is an mbn.

Theorem 8: The graph G constructed by the 6-way split method has

$$e(G) <= n/2 \lceil \log_2 n \rceil - 3n/4 + 9$$

Proof: Let n_i be the number of vertices in each S_i. Assume the S_i's are constructed by Farley's 2-way split method. Therefore, the S_i's will have a total of

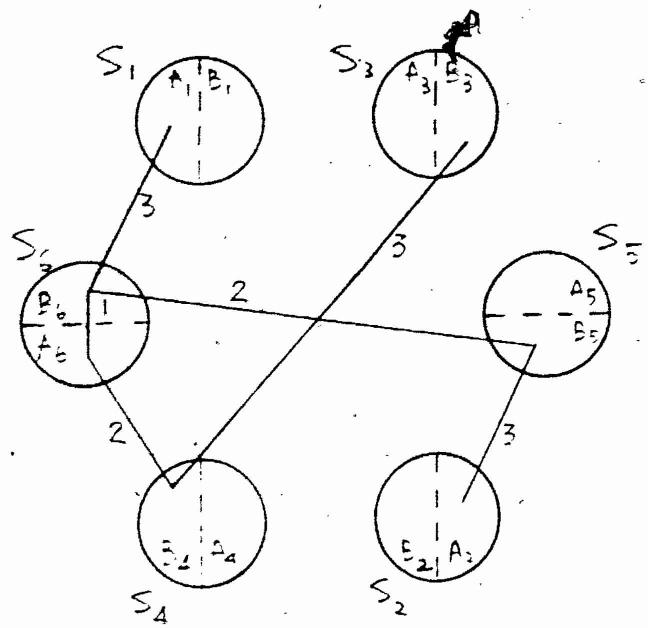


Figure 2-7: Calling scheme for the first three time units regardless of originator for mbns constructed by the 6-way split.

$$\sum_{i=1}^{6} (n_i/2\lceil \log_2 n_i \rceil)$$

edges.

At most $\lceil n/12 \rceil * 9$ edges are needed to form minimum adjacency connections between pairs (B_1, A_3), (B_3, A_5), (B_5, A_2), (B_2, A_4), (B_4, A_6), (B_6, A_1), (B_1, B_2), (B_3, B_4), and (B_5, B_6).

The total number of edges in G is

$$e(G) \le \sum_{i=1}^{6} (n_i/2\lceil \log_2 n_i \rceil) + \lceil n/12 \rceil * 9$$

$$\le \sum_{i=1}^{6} (n_i/2(\lceil \log_2 n \rceil - 3)) + 9n/12 + 9$$

$$\le n/2(\lceil \log_2 n \rceil - 3) + 9n/12 + 9$$

$$\le n/2\lceil \log_2 n \rceil - 3n/4 + 9$$

Theorem 9: The graph G = (V, E) constructed by the 6-way split method has an even adjacency split.

Proof: Split V into two groups K_1 and K_2 such that

$$K_1 = A_1 \cup B_2 \cup A_3 \cup B_4 \cup A_5 \cup B_6$$

$$K_2 = B_1 \cup A_2 \cup B_3 \cup A_4 \cup B_5 \cup A_6$$

Using similar arguments to those used in the proof of theorem 6, K_1 and K_2 are an even adjacency split of G.

From theorems 7, 8, and 9, the graph G constructed by the 6-way split method has the following properties :

1. G is an mbn.
2. $e(G) <= n/2\lceil \log_2 n \rceil - 3n/4 + 9$.
3. G has an even adjacency split.

2.3 Seven-way split method

Given n such that

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/7 \rceil \rceil = 3,$$

an mbn of size n can be constructed as follows :

1. Partition n vertices into 7 sets S_i. If $\lfloor n/7 \rfloor$ is even then

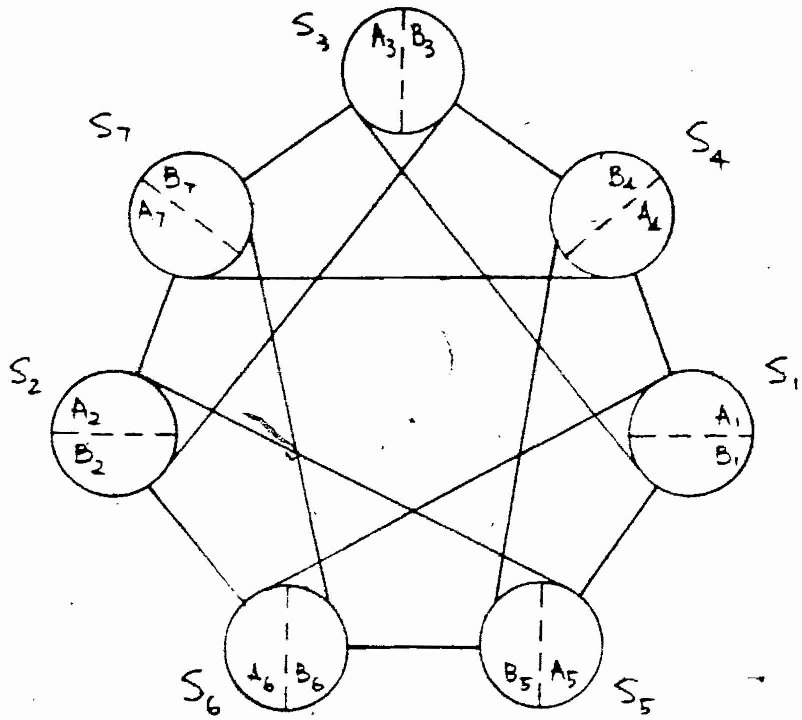
   $$\lceil n/7 \rceil = |S_1| >= |S_2| >= \ldots >= |S_7| = \lfloor n/7 \rfloor$$

   If $\lfloor n/7 \rfloor$ is odd then

   $$\lceil n/7 \rceil = |S_1| >= |S_4| >= |S_7| >= |S_2| >= |S_5| >= |S_6| >= |S_3| = \lfloor n/7 \rfloor$$

2. For each S_i construct an mbn with an even adjacency split A_i, B_i such that $0 <= |A_i|-|B_i| <= 1$. This may be done recursively or using other heuristics such as Farley's 2-way split.

3. Add edges to form minimum adjacency connections between the following pairs : (B_1, A_5), (B_5, B_6), (A_6, B_2), (A_2, A_7), (B_7, A_3), (B_3, B_4), (A_4, A_1), (A_1, A_6), (B_1, A_3), (A_2, A_5), (B_2, B_3), (B_6, B_7), (A_4, A_7), and (B_4, B_5).



Each line represents a minimum adjacency connection between the A_i's and the B_i's

Figure 2-8: Constructing mbns using 7-way split

Theorem 10: The graph G constructed by the 7-way split method is an mbn.

Proof: Referring to figure 2-9 and using similar arguments to those used in the proof of theorem 4, each mbn S_i has at least one informed vertex after the first three time units. In time unit four, each S_i can start to broadcast internally. The time required for each set is $\lceil \log_2 \lceil n/7 \rceil \rceil$.

The total time required to complete a broadcast in G is

$$t(G) = \lceil \log_2 \lceil n/7 \rceil \rceil + 3$$

Since

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/7 \rceil \rceil = 3$$

$$\therefore t(G) = \lceil \log_2 n \rceil - 3 + 3$$

$$\therefore \quad = \lceil \log_2 n \rceil$$

Therefore, G is an mbn.

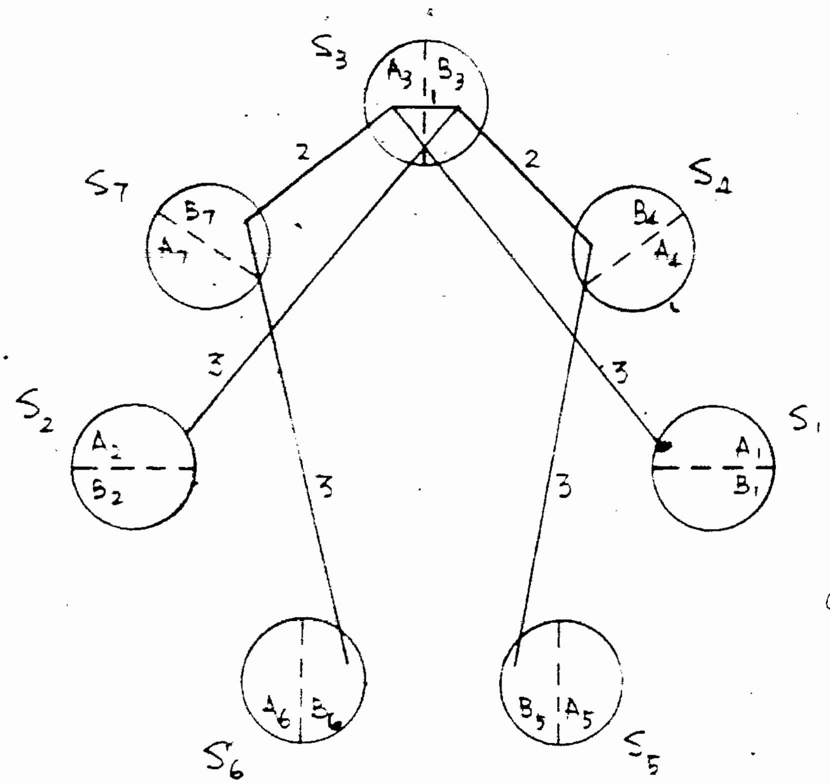


Figure 2-9: Calling scheme for the first three time units regardless of originator for mbns constructed by the 7-way split

Theorem 11: The graph G constructed by the 7-way split method has

$$e(G) <= n/2\lceil \log_2 n \rceil - n/2 + 14$$

Proof: Let n_i be the number of vertices in each S_i. Assume the S_i's are constructed by Farley's 2-way split method. Therefore, the S_i's will have a total of

$$\sum_{i=1}^{7} (n_i/2\lceil \log_2 n_i \rceil)$$

edges.

At most $\lceil n/14 \rceil * 14$ edges are needed to form a minimum adjacency connection between pairs (B_1, A_5), (B_5, B_6), (A_6, B_2), (A_2, A_7), (B_7, A_3), (B_3, B_4), (A_4, A_1), (A_1, A_6), (B_1, A_3), (A_2, A_5), (B_2, B_3), (B_6, B_7), (A_4, A_7), and (B_4, B_5).

The total number of edges in G is

$$\begin{aligned} e(G) &<= \sum_{i=1}^{7}(n_i/2\lceil \log_2 n_i \rceil + \lceil n/14 \rceil * 14 \\ &<= \sum_{i=1}^{7}(n_i/2(\lceil \log_2 n \rceil - 3)) + n + 14 \\ &<= n/2(\lceil \log_2 n \rceil - 3) + n + 14 \\ &<= n/2\lceil \log_2 n \rceil - n/2 + 14 \end{aligned}$$

Theorem 12: The graph G = (V, E) constructed by the 7-way split method has an even adjacency split.

Proof: Split V into two sets K_1 and K_2 such that

$K_1 = A_1 \cup B_2 \cup A_3 \cup B_4 \cup A_5 \cup B_6 \cup A_7$

$K_2 = B_1 \cup A_2 \cup B_3 \cup A_4 \cup B_5 \cup A_6 \cup B_7$

Using similar arguments to those used in the proof of theorem 6, K_1 and K_2 are an even adjacency split of G.

From theorems 10, 11, and 12, the graph G constructed by the 7-way split method has the following properties :

1. G is an mbn.
2. $e(G) <= n/2\lceil \log_2 n \rceil - n/2 + 14$.
3. G has an even adjacency split.

2.4 An algorithm to approximate mbgs

From the previous sections, mbns constructed by the 5, 6 and 7-way splits and Farley's 2-way split all have the property that they contain even adjacency splits. Because of this property, we can combine all these methods in an algorithm to approximate mbgs. Furthermore, this algorithm will be better if it can utilize the known mbgs as the basis for the algorithm.

Lemma 13: There is an mbg of size n with an even adjacency split for each n in the range 2<= n <=17.

Proof: The vertices of each mbg in figure 2-10 are divided into two sets A and B. Set A contains all the vertices that are marked with a "x" and set B contains all those marked with an "o". Clearly, A and B satisfy the conditions for even adjacency split in G. A and B are an even adjacency split of G.

An Algorithm to Approximate Mbgs

```
1. If n <= 17 then return the known mbg and stop.
      Else
        For m := 18 to ⌈n/2⌉ do
         begin
           Find the number of edges for mbn of size m
             constructed by Farley's 2-way split,
             5-way split, 6-way split and 7-way
             split if possible.
           Find and store the method that give the
             fewest edges.
         end
```

2. Find the number of edges for an mbn of size n constructed

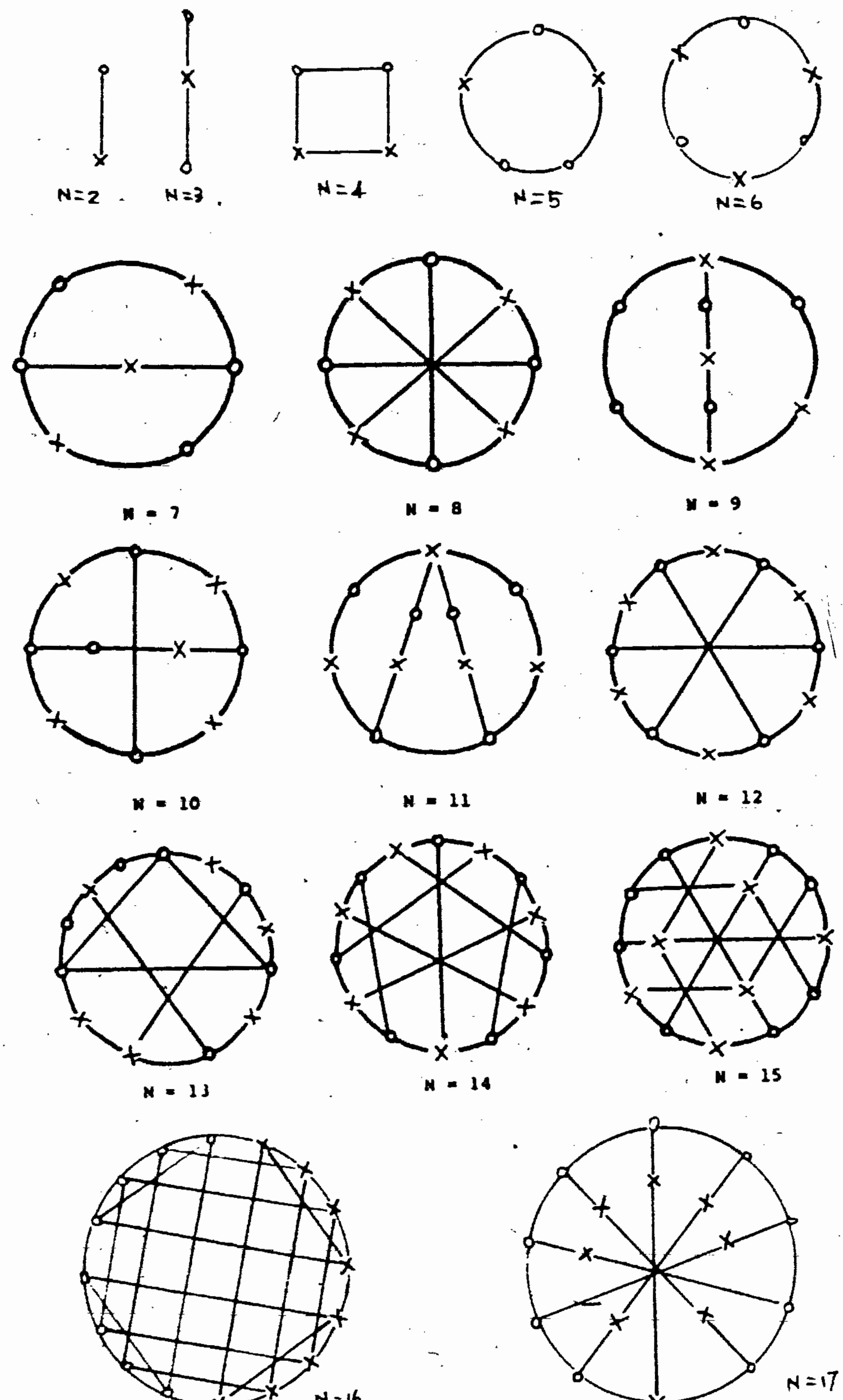


Figure 2-10: Known mbgs for n <= 17 which have an even adjacency split

by Farley's 2-way split, 5-way split, 6-way split and 7-way split. Construct the mbn using the method that give the fewest number of edges.

Lemma 14: If Farley's 3-way split can be used to construct an mbn of size n, then we can also use the 6-way split for the construction.

Proof: Farley's 3-way split method can be used when

$$2\lfloor \log_2 n \rfloor < n <= 3*2(\lceil \log_2 n \rceil - 2)$$

$$\therefore n <= 3*2(\lceil \log_2 n \rceil - 2)$$

$$\therefore n/6 <= 2(\lceil \log_2 n \rceil - 3)$$

Since $2(\lceil \log_2 n \rceil - 3)$ is an integer,

$$\therefore \lceil n/6 \rceil <= 2(\lceil \log_2 n \rceil - 3)$$

$$\therefore \lceil \log_2 \lceil n/6 \rceil \rceil <= \lceil \log_2 n \rceil - 3$$

Hence, the 6-way split method can be employed whenever the 3-way split method is applicable.

From the above lemma, this algorithm gives a better bound on B(n) than Farley's algorithm when

$$2\lfloor \log_2 n \rfloor < n <= 3*2(\lceil \log_2 n \rceil - 2)$$

because the 6-way split has a better bound for e(G) than the 3-way split. Furthermore, this algorithm also gives a better better bound on B(n) when

$$3*2(\lceil \log_2 n \rceil - 2) < n <= 7*2(\lceil \log_2 n \rceil - 3)$$

for we can use the 7-way split in this range instead of Farley's 2-way split. The bound of e(G) for the 7-way split is also better.

No improvement has been made for

$7*2^{(\lceil \log_2 n \rceil - 3)} < n <= 2^{\lceil \log_2 n \rceil}$.

It might be possible to proceed with the same technique as 5, 6 and 7-way split and split n vertices into 15, 31 or 63 sets to get improvement within this range. The method will become very complicated and no improvement can be made for all n within this range with this approach. This direction has not been investigated.

2.5 Summary

An algorithm to approximate mbgs is presented. This algorithm gives a better bound on B(n) than Farley's algorithm.

For $2^{\lfloor \log_2 n \rfloor} < n <= 5*2^{(\lceil \log_2 n \rceil - 3)}$

$B(n) <= n/2\lceil \log_2 n \rceil - 4n/5 + 7$.

For $5*2^{(\lceil \log_2 n \rceil - 3)} < n <= 3*2^{(\lceil \log_2 n \rceil - 2)}$

$B(n) <= n/2\lceil \log_2 n \rceil - 3n/4 + 9$.

For $3*2^{(\lceil \log_2 n \rceil - 2)} < n <= 7*2^{(\lceil \log_2 n \rceil - 3)}$

$B(n) <= n/2\lceil \log_2 n \rceil - n/2 + 14$.

The bound for B(n) is improved in the range

$2^{\lfloor \log_2 n \rfloor} < n <= 7*2^{(\lceil \log_2 n \rceil - 3)}$.

That is, the bound is improved three-quarter of the time. Further reduction in edges for constructing mbns of size

$7*2^{(\lceil \log_2 n \rceil - 3)} < n <= 2^{\lceil \log_2 n \rceil}$

may be possible by using the same technique to construct mbns with 15, 31 or 63 smaller mbns. The number of edges in mbns constructed by this algorithm and Farley's algorithm have been computed for 18<=n<=1024. The mbns generated by this algorithm have an average of approximately 8% fewer edges than the mbns generated by Farley's algorithm within the range where there is improvement and n>=36. A table for the comparison of e(G) between this algorithm and Farley's algorithm is given in Appendix A.

It is interesting to note in the table of Appendix A that the 6-way split always performs better than the 5-way split which seems to contradict the bounds given above. This is probably due to the fact that the proof of the bounds assume that the small mbns are constructed by the 2-way split. However, in practice the small mbns used will be the mbns having the fewest edges which may be constructed by the 2, 5, 6 or 7-way split or may be known mbgs. Consider the case n=160. The 5-way split will break the vertices into 5 sets of 32 vertices. Each small mbn is constructed by the 2-way split and contains 80 edges. Thus, the small mbns contain 400 edges and the total number of edges in the resulting mbn on 160 vertices is 512. The 6-way split will utilize 2 mbns on 26 vertices and 4 mbns on 27 vertices. These small mbns can be constructed by the 7-way split yielding a total of 308 edges in the 6 small mbns. Thus, the resulting mbn on 160 vertices has only 427 edges. This significant savings in the number of edges in the small mbns compensates for the extra edges added between the small mbns by the 6-way split method. Thus, the 6-way split may actually outperform the 5-way split in spite of the bounds. The behaviour of the 7-way split is similarly influenced by the actual method used to construct the small mbns.

Chapter 3

Algorithms to Approximate 1-ft Mbgs

3.1 Definitions

Let G = (V, E) be a graph that represents a communication network, and a subgraph G' = (V, E') with E' = E - E* where E* is a set of k edges in E. The set E* represents faulty communication links in the network. Broadcasting in G with enough redundancy so that broadcast can be completed with any set E* of faulty links is called k-fault-tolerant broadcasting (Fault-tolerant will be abbreviated ft below). A k-ft broadcasting scheme is a broadcasting scheme which contains k+1 mutually edge disjoint calling paths from the originator to each member of the network. Ft broadcasting is desirable if reliability is considered as an important factor in a communication network. The ft broadcasting scheme does not detect which communication link fails. The broadcast problem discussed in chapter 2 corresponds to 0-ft broadcasting.

Let n denote the number of vertices in G, e(G) denote the number of edges in G and $t_k(u)$ denote the time required to complete a k-ft broadcast using $u \in V$ as the originator. Let $t_k(G) = Max(t_k(u) : u \in V)$. That is, every $u \in V$ can complete a k-ft broadcast in less than or equal to $t_k(G)$ units of time.

In general, the minimum time $T_k(n)$ required to complete a k-ft broadcast in any communication network G of n members is not known. It has been shown that

$$T_k(n) >= \lceil \log_2 n \rceil + k$$

The only known values for $T_k(n)$ for $k > 0$ are those for $k=1,2$ [Liestman 81]. For $k = 0$ the result is given in the previous chapter.

For $k = 1$ and $n >= 3$,

$$T_1(n) = \lceil \log_2 n \rceil + 1.$$

For $k = 2$, $n >= 5$ and $n = 2^i - 1$,

$$T_2(n) = \lceil \log_2 n \rceil + 2.$$

For $k = 2$, $n >= 5$ and $n = 2^i - 1$,

$$T_2(n) = \lceil \log_2 n \rceil + 3.$$

Definition 1: A <u>k-fault-tolerant</u> <u>minimum</u> <u>broadcast</u> <u>network</u> (k-ft mbn) is a graph G such that a k-ft broadcast can be completed in minimum time [Liestman 81].

A k-ft mbn represents a communication network that can complete a k-ft broadcast regardless of originator in minimum time.

Definition 2: A <u>k-fault-tolerant</u> <u>minimum</u> <u>broadcast</u> <u>graph</u> (k-ft mbg) is a graph G such that G is a k-ft mbn and e(G) is minimum. A k-ft mbg is a k-ft mbn having the minimum number of edges. [Liestman 81].

A k-ft mbg represents a communication network with the fewest communication links between members that can complete a k-ft broadcast in minimum time regardless of originator.

Let $B_k(n)$ denote the number of edges of a k-ft mbg of size n. The value of $B_k(n)$ for arbitrary n is not known. The following upper bound for $B_k(n)$ where k = 1, 2 are from Liestman [Liestman 81].

For $n = 2^i$,

$$B_1(n) <= n\lceil \log_2 n \rceil - n/2.$$

$$B_2(n) <= n\lceil \log_2 n \rceil + n/2.$$

For $n = 2^{\lceil \log_2 n \rceil} + 2^i$,

$$B_1(n) <= n\lceil \log_2 n \rceil - n.$$

$$B_2(n) <= n\lceil \log_2 n \rceil.$$

For other n,

$$B_1(n) <= n\lceil \log_2 n \rceil.$$

$$B_2(n) <= n\lceil \log_2 n \rceil + n.$$

The above results are obtained from Liestman's heuristics to generate 1-ft mbns and 2-ft mbns. Other heuristics to generate 1-ft mbns and 2-ft mbns are presented in the following sections.

3.2 1-ft two-way split method

The method used in the 0-ft algorithms is not directly extendable to the k-ft cases for k=1,2. Let n be even. Suppose we partition n vertices into two sets of equal size and form two smaller k-ft mbns. Each of these can complete a k-ft broadcast in $T_k(n/2)$ units of time. As before, we add edges to connect the small 1-ft mbns together forming a graph G. If we use the first k+1 calls to ensure that each component has at least one informed vertex, we get

$$t_k(G) = T_k(n/2) + k + 1 > T_k(n)$$

for k=1,2 and even n. Thus, using the first k+1 calls in this fashion will not produce the desired results.

Let us consider the case for k=1.

$$T_1(n) = \lceil \log_2 n \rceil + 1 = \lceil \log_2 n/2 \rceil + 2 = T_0(n/2) + 2$$

We propose a scheme using two 0-ft mbns of size $n/2$ connected by additional edges. In the first time unit, the originator (in component A) calls a member of the other component (B). The two components then proceed with 0-ft broadcasts internally. This takes $T_0(n/2)$ units of time. During the last time unit, each member of one component calls a member of the other component. Thus, two calling paths from the originator to each vertex are completed. For a vertex in A, the two calling paths are :

1. The calling path within A.

2. The first call to B, a calling path within B and a call back to A.

Note that it is important that the originator calls two distinct members of B. The following heuristics to approximate 1-ft mbns are based on the above idea.

1-ft 2-way split for even n

Given n is even, a 1-ft mbn of size n can be constructed as follows :

1. Partition n vertices into two equal sets S_1 and S_2, and for each S_i construct a 0-ft mbn. This may be done by either the 2, 5, 6 or 7-way split.

2. Number the vertices in S_1 and S_2. Add edges to connect $v_j \in S_1$ to $u_j \in S_2$ for $j=1..n/2$.

3. Add edges to connect $v_j \in S_1$ to $u_{j+1} \in S_2$ for $j=1..(n/2-1)$ and $v_{n/2}$ to u_1.

Theorem 3: The graph G constructed by the 1-ft 2-way split algorithm for even n is a 1-ft mbn.

Proof: Without loss of generality, assume that the originator is

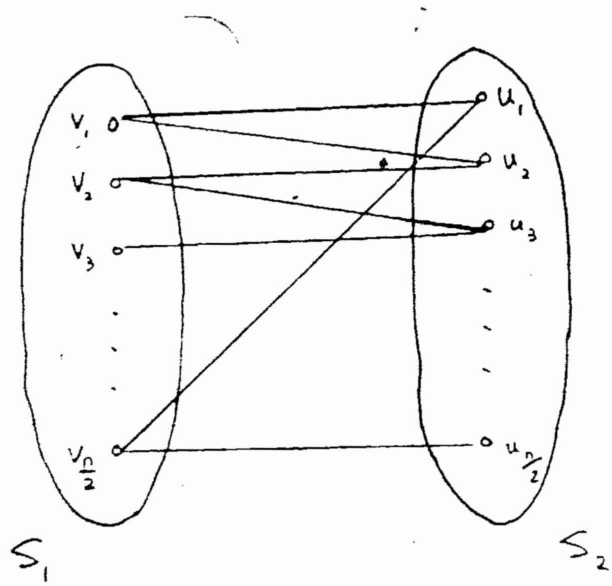


Figure 3-1: Constructing 1-ft mbns using 1-ft 2-way split

$v_j \in S_1$ and $j=n/2$. In the first time unit, v_j calls u_{j+1}. After the first time unit, each 0-ft mbn has an informed vertex. Starting from time unit two, each 0-ft mbn broadcasts internally. This takes $\lceil \log_2 n \rceil - 1$ time units. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between v_k and u_k for $k=1..n/2$. The total time required for this calling scheme is $\lceil \log_2 n \rceil + 1$. Thus, the time constraint is not violated.

For the vertices in S_1, each has one calling path which is obtained by broadcasting internally and another calling path which is from v_j to u_{j+1}, calls within S_2 and from u_k to v_k. Similarly, each vertex in S_2 has one calling path which is from v_j to u_{j+1} and calls within S_2. Moreover, each also has a second calling path from calls in S_1 and then from v_k to u_k.

Clearly, the two calling paths for each vertex in the S_is are disjoint. Hence, the graph G constructed by the 1-ft 2-way split for even n is a 1-ft mbn.

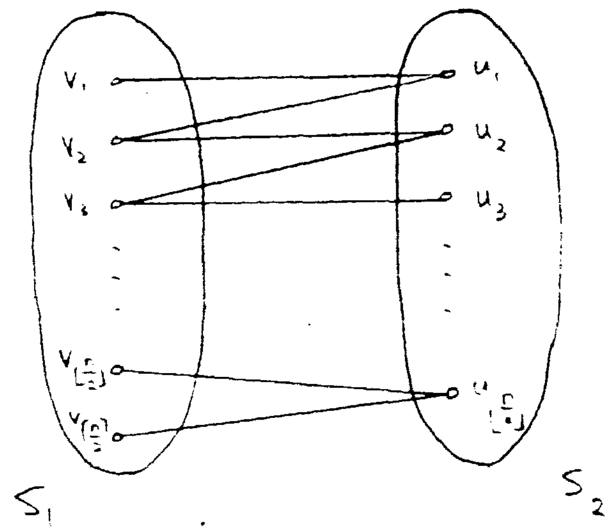


Figure 3-2: Possible method to construct 1-ft mbns when n is odd

If n is odd, the above scheme for the construction of 1-ft mbns does not work because not every vertex can participate in the calls at the last time unit. Consider the following scheme. Suppose n is odd and we partition n vertices into two sets S_1, S_2 such that $|S_1|=|S_2|+1$. Number the vertices and add edges to connect $u_i \in S_2$ to $v_i \in S_2$, $u_i \in S_2$ to $v_{i+1} \in S_2$, where $i=1..n/2$ (see figure 3-2). Let the originator be $v_k \in S_1$. In the first time unit, v_k calls u_k. In the second time unit, each S_i has one informed vertex and they can broadcast internally. This takes $\lceil \log_2 n \rceil - 1$ time units. In the last time unit, conduct calls between v_i and u_i if $i<k$, v_{i+1} and u_i if $i>k$. In this way, only the originator v_k does not participate in the calls at the last time unit. Thus, all vertices except the originator have two calling paths. However, this scheme does not work if the originator is in S_2. In fact, this scheme only works if the originator is in the larger set. If we can "force" the originator to be in the larger set all the time then this scheme will work for all odd n. The "forcing" of the originator to be in the larger set can be done by splitting n vertices into two equal sets S_1, S_2 and a single vertex w. Construct 0-ft mbns for $S_1 \cup \{w\}$ and $S_2 \cup \{w\}$ with the property that S_1, S_2 are also 0-ft mbns themselves. If the originator is in S_1 then we can consider $S_1 \cup \{w\}$ as the larger set and S_2 as the smaller set. Similarly, if the originator is in S_2 then we can consider $S_2 \cup \{w\}$ as the larger set and S_1 as the smaller set. Hence, our aim becomes to find a method to construct 0-ft mbns which have the desired property. If such a method exists, we can use the scheme described above to construct 1-ft mbns when n is odd. It turns out that a subset of 0-ft mbns constructed by Farley's 2-way split has this property.

Lemma 4: At least one 0-ft mbn G on n>=2 vertices constructable by Farley's 2-way split method contains a vertex v such that the graph G' obtained by deleting v and all edges incident with v from G is a 0-ft mbn on n-1 vertices.

Proof: Farley's 2-way split forms an mbn of size n by connecting two mbns S_1 and S_2 of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ respectively with $\lfloor n/2 \rfloor$ edges. These edges connect each vertex in S_2 to a distinct vertex in S_1. Any such set of connections is allowed by Farley's 2-way split method. Let us consider those mbns constructed by this method and also satisfy the following conditions :

1. If n=1, the vertex is a removable vertex.

2. If n=2, the mbn is a K_2. Choose either vertex and call it a removable vertex.

3. If n>=3, connect two small mbns S_1 and S_2 of size $\lceil n/2 \rceil$ and $\lceil n/2 \rceil$ respectively such that

   a. If n is even, then add a perfect matching between

S_1 and S_2. Choose one of the removable vertices from S_1 and S_2 to be the removable vertex of the new mbn of size n.

If n is odd, then add a perfect matching between $S_1 \setminus \{\text{the removable vertex in } S_1\}$ and S_2. The removable vertex from S_1 is designated to be the removable vertex of the new mbn of size n.

Let G be a 0-ft mbn constructed by Farley's 2-way split and satisfy the above condition at each step of the construction. G is constructed from two 0-ft mbns S_1 and S_2 of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ respectively. Let G' be the graph obtain from G by deleting the removable vertex v of G and its incident edges.

For n=2, the remaining graph G' is still a 0-ft mbn for it has only one vertex.

Assume that the lemma is true for n<=k and consider n=k+1.

Two cases :

1. Suppose k+1 is odd. The smaller mbn S_2 and the edges that connect the two mbns are not affected by the removal of v. Consider the larger set S_1. Condition 3 assures that v is also the removable vertex of S_1 and S_1 has less than k vertices. From our assumption the graph formed by deleting v and its incident edges from S_1 is also an mbn. Hence, the graph G' is an mbn.

2. Suppose k+1 is even and the removable vertex v is in S_1. S_2 is not affected by the removal of v. Consider the set S_1 that contains the removable vertex v. Condition 3 assures that v is also the removable vertex of S_1 and S_1 has less than k vertices. From our assumption the graph formed by deleting v and its incident edges from S_1 is also an mbn. Hence, the

graph G' is an mbn.

∴ The lemma is true for $n=k+1$.

∴ The lemma holds.

1-ft 2-way split for odd n

Given n such that n is odd, an 1-ft mbn of size n can be constructed as follows :

1. Partition n vertices into two equal sets S_1, S_2 and a single vertex w.

2. Construct 0-ft mbns for $S_1 \cup \{w\}$ and $S_2 \cup \{w\}$ such that S_1, S_2 are also 0-ft mbns. This can be done by using Farley's 2-way split (lemma 4).

3. Number the vertices in S_1 and S_2. Add edges between $v_j \in S_1$ and $u_j \in S_2$ for $j=1..\lfloor n/2 \rfloor$, $v_j \in S_1$ and $u_{j+1} \in S_2$ for $j=1..(\lfloor n/2 \rfloor-1)$, $v_{\lfloor n/2 \rfloor}$ and u_1. Note that the edges added form two disjoint perfect matchings from S_1 to S_2.

Theorem 5: The graph G constructed by the 1-ft 2-way split algorithm for odd n is a 1-ft mbn.

Proof: Without loss of generality, assume that the originator $v_j \in S_1$. Let $S_1 \cup \{w\}$ be the larger set and call it A. We can consider the graph G consisting of 0-ft mbns A and S_2. In the first time unit, v_j calls u_{j+1}. After the first time unit, each mbn has an informed vertex. Starting from time unit two, each mbn broadcasts internally. This takes $\lceil \log_2 n \rceil -1$ time units. In time unit $\lceil \log_2 n \rceil +1$, conduct calls between v_k and u_k for $k>j$, between v_k and u_{k+1} for $k<j$, and between w and u_1. The total time required for this calling scheme is $\lceil \log_2 n \rceil + 1$. Thus, the time constraint is not violated.

For the vertices in A, each has one calling path which is obtained by broadcasting internally and another calling path which is from v_j to u_{j+1}, calls within S_2 and from u_k to v_k for $k>j$, or from u_{k+1} to v_k for $k<j$, or from u_1 to w. Similarly, each vertex in S_2 has one calling path which is from v_j to u_{j+1} and calls within S_2. Moreover, each also has a second calling path from calls within A and then from v_k to u_k for $k>j$, or from v_k to u_{k+1} for $k<j$, or from w to u_1. Clearly, the two calling paths for each vertex in A and S_2 are disjoint.

Suppose w is the originator. We can let $S_1 \cup \{w\}$ be the larger set. We can use the same broadcasting scheme as described above for the first $\lceil \log_2 n \rceil$ time units. In the last time unit, conduct calls between the vertices in S_1 and S_2. Thus, every vertex except the originator w has two calling paths. Hence, the graph G constructed by the 1-ft 2-way split for odd n is a 1-ft mbn.

Theorem 6: The 1-ft mbns constructed by the 1-ft 2-way split algorithms give

$$e(G) <= \lceil n/2 \rceil * \lceil \log_2 n \rceil + n/2.$$

Proof: At most $2 * \lceil n/2 \rceil / 2 * \lceil \log_2 \lceil n/2 \rceil \rceil$ edges are needed to construct the two smaller 0-ft mbns of size $\lceil n/2 \rceil$. At most n edges are needed to connect v_j to u_j, v_j to u_{j+1}, and $v_{n/2}$ to u_1 for $j=1..n/2$.

$$e(G) <= \lceil n/2 \rceil * \lceil \log_2 n/2 \rceil + n$$

$$<= \lceil n/2 \rceil (\lceil \log_2 n \rceil - 1) + n$$

$$<= \lceil n/2 \rceil * \lceil \log_2 n \rceil + n/2.$$

Although lemma 4 does not hold directly for 0-ft mbns constructed by the 5, 6, and 7-way splits, a similar result can be obtained by adding a small number of edges to the 0-ft mbn. However, the bound of e(G) for the resulting 1-ft mbns will not be better than the above graph.

3.3 1-ft Six-way split method

As we have seen in chapter 2, better results can be obtained by using more small 0-ft mbns to construct large 0-ft mbns. It is natural to think that the same idea may work for 1-ft mbns. The following is a scheme based on the same idea.

1-ft 6-way split for even n

If n is even and

$$\lceil \log_2 n \rceil - \lceil \log_2 \lceil n/6 \rceil \rceil = 3,$$

a 1-ft mbn can be constructed as follows :

1. Partition n vertices into 6 sets S_i such that

   a. $\lfloor n/6 \rfloor <= |S_i| <= \lceil n/6 \rceil$.

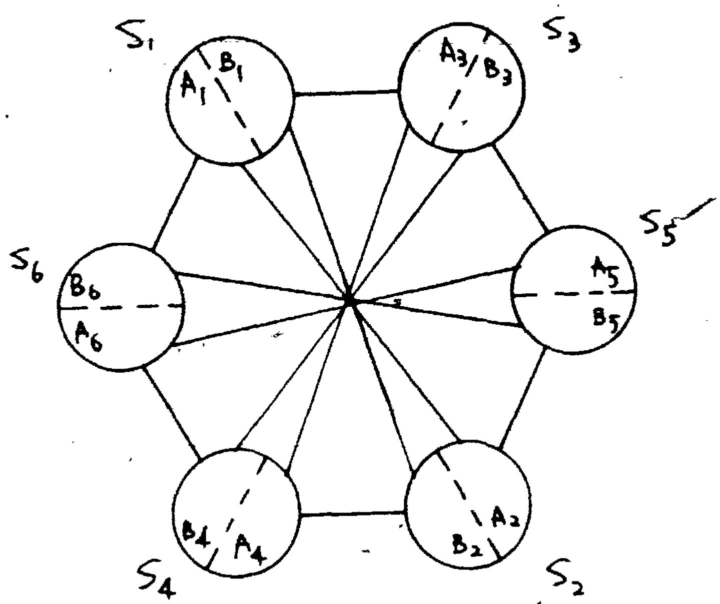
   b. $|S_1| = |S_2|$, $|S_3| = |S_4|$ and $|S_5| = |S_6|$.

2. Construct 0-ft mbns with even adjacency splits A_i and B_i for each S_i such that

   $$0 <= |A_i| - |B_i| <= 1.$$

3. Add edges to form minimum adjacency connections to connect the following pairs (B_1, B_6), (A_6, A_4), (B_4, B_2), (A_2, A_5), (B_5, B_3), and (A_3, A_1).

4. Add edges to form perfect matchings between S_1 and S_2, S_3



Each line represents an even adjacency connection between the A_i's and the B_i's

Figure 3-3: Constructing 1-ft mbns using 1-ft 6-way split

and S_4, and S_5 and S_6.

Theorem 7: The graph G constructed by the 1-ft 6-way split for even n is a 1-ft mbn.

Proof: Without loss of generality, we can assume that the originator is $v_1 \in A_1$. Consider the following broadcasting scheme :

1. In time unit 1, the originator v_1 calls $u_1 \in B_1$ such that u_1 is also the first vertex that v_1 calls when v_1 starts to broadcast internally within S_1. This is possible because A_1 and B_1 are an even adjacency split of S_1 and from theorems 6, 9, 12 and lemma 5 in chapter 2, a vertex always starts to broadcast by calling a vertex in the other set of an even adjacency split.

2. In time unit 2, v_1 and u_1 call elements $v_2 \in A_2$ and $u_2 \in B_2$

respectively. This is possible because there is a perfect matching between S_1 and S_2.

3. In time unit 3, conduct calls between the following pairs : ($v_1 \in A_1$, $v_3 \in A_3$), ($u_1 \in B_1$, $u_6 \in B_6$), ($v_2 \in A_2$, $v_5 \in A_5$), and ($u_2 \in B_2$, $u_4 \in B_4$). These calls are possible because they are connected by minimum adjacency connections.

4. In time unit 4, each mbn starts to broadcast internally which takes $\lceil \log_2 n \rceil - 3$ time units. No collision will occur in S_1 or S_2 for their first call in broadcasting internally are between the informed vertices v_1 and u_1, and v_2 and u_2.

5. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between the vertices of the following pairs : S_1 and S_2, S_3 and S_4, S_5 and S_6. These calls are possible for there are perfect matchings between them.

The time required for this scheme is $\lceil \log_2 n \rceil + 1$. Thus, the time constraint is not violated.

For the vertices in S_1, each has a calling path from broadcasting internally, another calling from v_1 to $v_2 \in S_2$, calls within S_2, and calls between S_1 and S_2.

For the vertices in S_2 except v_2, each has a calling path from v_1 to v_2, calls within S_2 and another calling path from calls within S_1, calls between S_1 and S_2. For v_2, it has a calling path from v_1 to v_2 and another calling path from v_1 to u_1, u_1 to u_2, and u_2 to v_2.

Clearly, the two calling paths for vertices in S_1 and S_2 are disjoint. Using similar arguments, each vertex in S_3, S_4, S_5, and S_6 also has two disjoint calling paths. Hence, G is a 1-ft mbn.

Theorem 8: The 1-ft mbn G constructed by the 1-ft 6-way split for even n has

$$e(G) <= n/2\lceil \log_2 n \rceil - n/2 + 6.$$

Proof: Let n_i be the number of edges for each S_i, and assume that the S_i's are constructed by Farley's 2-way split. Each small 0-ft mbns require at most $n_i/2\lceil \log_2 n_i \rceil$ edges. The total number of edges for all six 0-ft mbns is $\sum_{i=1}^{6}(n_i/2\lceil \log_2 n_i \rceil)$.

At most $\lceil n/12 \rceil * 6$ edges are required to form minimum adjacency connections between pairs (B_1, B_6), (A_6, A_4), (B_4, B_2), (A_2, A_5), (B_5, B_3), and (A_3, A_1).

At most n/2 edges are required to form perfect matchings between the pairs (S_1, S_2), (S_3, S_4), (S_5, S_6).

The total number of edges

$$\begin{aligned} e(G) &<= \sum_{i=1}^{6}(n_i/2\lceil \log_2 n_i \rceil) + n/2 + \lceil n/12 \rceil * 6 \\ &<= n/2(\lceil \log_2 n \rceil - 3) + n/2 + n/2 + 6 \\ &<= n/2\lceil \log_2 n \rceil - n/2 + 6. \end{aligned}$$

As with the 1-ft 2-way split for even n, if n is odd, not all vertices can participate in the calls at the last time unit. Thus, the 1-ft 6-way split for even n method cannot be used for odd n. The approach used in the 1-ft 2-way split for odd n is to let every vertex except the originator participate in calls during the last time unit completing the second calling path for every vertex. Another approach is to make sure that one vertex has two calling paths before the last time unit. The remaining vertices can all complete their second paths during the last time unit. The following structure is useful in constructing networks in which the second approach can be used.

Sink Structure

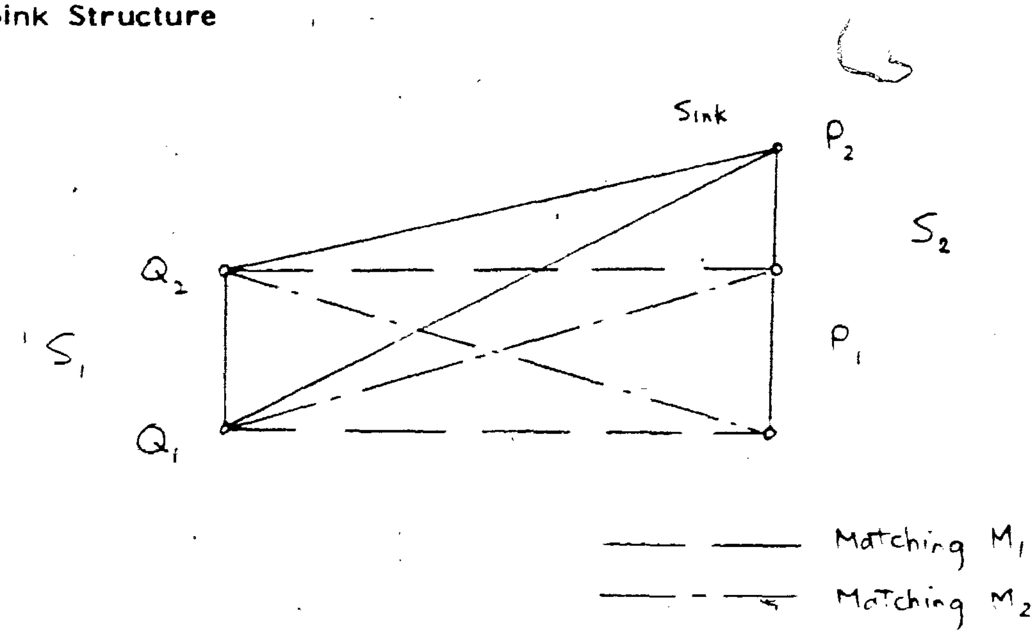


Figure 3-4: Example of an sink structure

A <u>sink</u> <u>structure</u> for odd n such that

$n <= 2^{\lceil \log_2 n \rceil} - 3$,

can be constructed as follows :

1. Partition n vertices into 2 sets A, B such that $|A|=|B|+1$.

2. Construct 0-ft mbns for A and B using Farley's 2-way split. Let A be constructed by mbns P_1 and P_2 with $|P_1|=|P_2|+1$. Let B be constructed by mbns Q_1 and Q_2 with $|Q_1|=|Q_2|<2^k$.

3. Choose a vertex s in P_2, called the sink and add edges to connect s to every vertex in B.

4. Add edges to construct two disjoint perfect matchings M_1 and M_2 from B to $A\setminus\{s\}$ such that each vertex in B has at least one edge that is connected to P_1.

Lemma 9: A sink structure is a 1-ft mbn.

Proof: 2 cases :

1. Assume that the originator is in Q_1. In time unit 1, the originator calls a vertex in P_1 through a perfect matching (say M_1). In time unit 2, P_1 and Q_1 start to broadcast internally which takes $\lceil \log_2 n \rceil - 2$ time units. Since $|Q_1| < 2^k$; at least one informed vertex must be idle at some time unit during the internal broadcast of Q_1. This idle informed vertex can call the sink s which is not participating in any call during this period of time. In time unit $\lceil \log_2 n \rceil$, conduct calls between the vertices in (P_1, P_2) and (Q_1, Q_2). Thus, before the last time unit, the sink s already has two disjoint calling paths. The first calling path is from calls within Q_1 and to s. The second calling path is from the originator to a vertex in P_1, calls within P_1 and to s. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between the vertices in $A \setminus \{s\}$ and B using the other perfect matching M_2. For vertices in $A \setminus \{s\}$, they have a calling path form the originator to a vertex in A and calls within A. They have another calling path from calls within B and then calls between B and $A \setminus \{s\}$. For vertices in B, they have a calling path from broadcasting internally. They have another calling path from the originator to a vertex in A, calls within A and then calls between B and $A \setminus \{s\}$. Hence, G is a 1-ft mbn. Similarly, we can use the same arguments if the originator is in either Q_2 or P_1.

2. Assume that the originator v is in P_2. In time unit 1,

the originator calls a vertex in A using an edge from a perfect matching (say M_1). In time unit 2, A and B start to broadcast internally which takes $\lceil \log_2 n \rceil - 1$ time units. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between the vertices in B and $A \setminus \{v\}$ through the perfect matching M_2 and an edge connected to the sink. Thus, every vertex except the originator has two disjoint calling paths.

Hence, G is a 1-ft mbn.

The sink structures have more edges than the 1-ft mbns constructed by the 1-ft 2-way split for odd n. However, the sink structure can be incorporated in schemes that use more than 2 small 0-ft mbns to build 1-ft mbns. The following scheme to construct 1-ft mbns for odd n uses a sink structure as part of the building blocks. The construction is similar to the one used in the 1-ft 6-way split for even n. The difference is that S_1 and S_2 are replaced by a sink structure. Extra edges are then added to "force" the first informed vertex of A to be in P_1 whenever the originator is not in A.

1-ft 6-way split for odd n

If n is odd and $n < 2^k - 1$ and

$$\lceil \log_2 n \rceil = \lceil \log_2 \lceil n/6 \rceil \rceil + 3,$$

a 1-ft mbn of size n can be constructed as follows :

1. Partition n vertices into 6 sets S_i as evenly as possible such that

   a. $|S_1|$ is odd and $|S_1| < 2^k - 1$.

   b. $|S_1| = |S_2| + 1$.

   c. $|S_3| = |S_4|$ and $|S_5| = |S_6|$.

2. Construct 0-ft mbns with even adjacency splits A_i, B_i for each S_i, i=3..6, using either 2, 5, 6, or 7-way splits such that

$$0 <= |A_i| - |B_i| <= 1.$$

Construct a sink structure for S_1 and S_2 where S_1 consists of P_1, P_2 and S_2 consists of Q_1, Q_2.

3. Add edges to connect the vertices in the following pairs (A_4, A_6), (B_3, B_5), (B_6, P_2), (A_3, P_1), (A_5, Q_1), (B_4, Q_2), and (B_6, P_1) such that every vertex in one set is connected to at least one vertex of the other set. The reason to connect the pair (B_6, P_1), is to "force" P_1 to have the first informed vertex in S_1 whenever the originator is not in S_1.

4. Add edges to form perfect matchings between the pairs (S_3, S_4), and (S_5, S_6).

Theorem 10: The graph G=(V, E) constructed by the 1-ft 6-way split for odd n is a 1-ft mbn.

Proof: Let s be the sink in the sink structure for S_1 and S_2. Using similar arguments to those used in the proof of theorem 7, the vertices in $V \setminus \{s\}$ all have two disjoint calling paths. Moreover, using similar arguments to those used in the proof of lemma 9, the sink s also has two disjoint calling paths. Hence, G is a 1-ft mbn.

Theorem 11: The 1-ft mbn G constructed by the 1-ft 6-way split for odd n has

$$e(G) <= n/2\lceil \log_2 n \rceil - n/12 + 9.$$

Proof: At most $\sum_{i=1}^{6}(n_i/2\lceil \log_2 n_i \rceil) = n/2(\lceil \log_2 n \rceil - 3)$ edges are needed to construct 0-ft mbns for the S_i's.

At most $\lceil n/12 \rceil * 7$ edges are needed to connect the pairs (A_4, A_6), (B_3, B_5), (B_6, P_2), (A_3, P_1), (A_5, O_1), (B_4, O_2), and (B_6, P_1).

At most $\lceil n/6 \rceil * 2$ edges are needed to form the perfect matching between pairs S_3 and S_4, and S_5 and S_6.

At most $\lfloor n/6 \rfloor * 3$ edges are needed to form the two perfect matchings and the edges connecting to the sink in the sink structure consisting of S_1 and S_2.

The total number of edges is

$$e(G) <= n/2(\lceil \log_2 n \rceil - 3) + \lceil n/12 \rceil * 7 + \lceil n/6 \rceil * 2 + \lfloor n/6 \rfloor * 3$$

$$<= n/2\lceil \log_2 n \rceil - 3n/2 + 17n/12 + 9$$

$$<= n/2\lceil \log_2 n \rceil - n/12 + 9$$

3.4 Summary

Four different methods to construct 1-ft mbns are presented. The methods greatly improve the upper bound for $B_1(n)$. In fact, the new bound is approximately one half of the old bound.

For n is even, n >= 12 and $2^{\lfloor \log_2 n \rfloor} < n <= 3*2^{(\lceil \log_2 n \rceil - 2)}$,

$$B_1(n) <= n/2\lceil \log_2 n \rceil - n/2 + 6.$$

For n is odd, n > 12 and $2^{\lfloor \log_2 n \rfloor} < n <= 3*2^{(\lceil \log_2 n \rceil - 2)} - 3$,

$$B_1(n) <= n/2\lceil \log_2 n \rceil - n/12 + 9.$$

For other n,

$$B_1(n) <= \lceil n/2 \rceil * \lceil \log_2 n \rceil + n/2.$$

It is possible that similar constructions using other numbers of small 0-ft mbns may result in further improvements.

Chapter 4

Algorithms to Approximate 2-ft Mbgs

As we have seen in chapter 3, for 1-ft mbns every vertex must have two edge disjoint calling paths. Similarly, for 2-ft mbns every vertex must have three edge disjoint calling paths. Furthermore, the approach used in constructing 1-ft mbns can also be extended to construct 2-ft mbns. That is, we can use 0-ft mbns as building blocks to construct 2-ft mbns. Using similar broadcasting schemes as in 1-ft mbns, we can use $T_2(n)-2$ time units to construct one calling path to every vertex and use the last two time units to complete two more edge disjoint calling paths to every vertex. The following methods to construct 2-ft mbns are based on this idea.

4.1 2-ft Four-way split for n mod 4 = 0

If n mod 4 = 0, a 2-ft mbn can be constructed as follows :

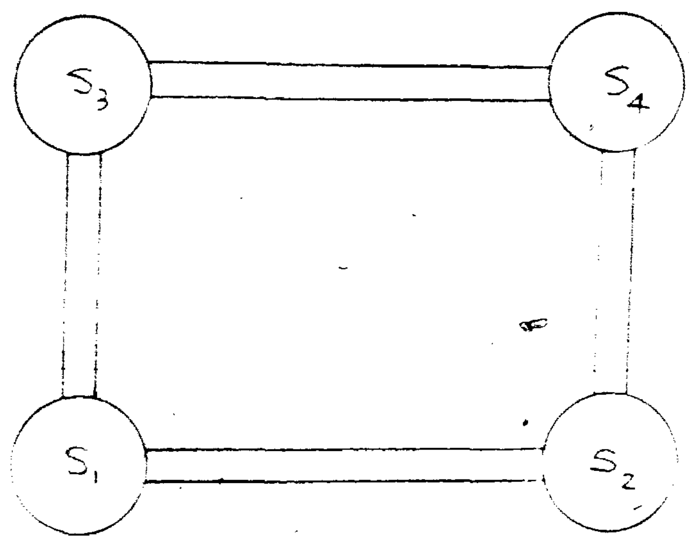
1. Partition n vertices into 4 sets S_i such that

   $$n/4 = |S_1| = |S_2| = |S_3| = |S_4|.$$

2. Construct 0-ft mbns for each S_i using any method.

3. Add edges to form two edge disjoint matchings between the vertices of each of the pairs (S_1, S_2), (S_3, S_4), (S_1, S_3), and (S_2, S_4).

Theorem 1: The graph G constructed by the 2-ft 4-way split



Each line represents an even adjacency connection between the two connected sets

Figure 4-1: Constructing 2-ft mbns using 2-ft 4-way split for n divisible by 4

for n mod 4=0, is a 2-ft mbn.

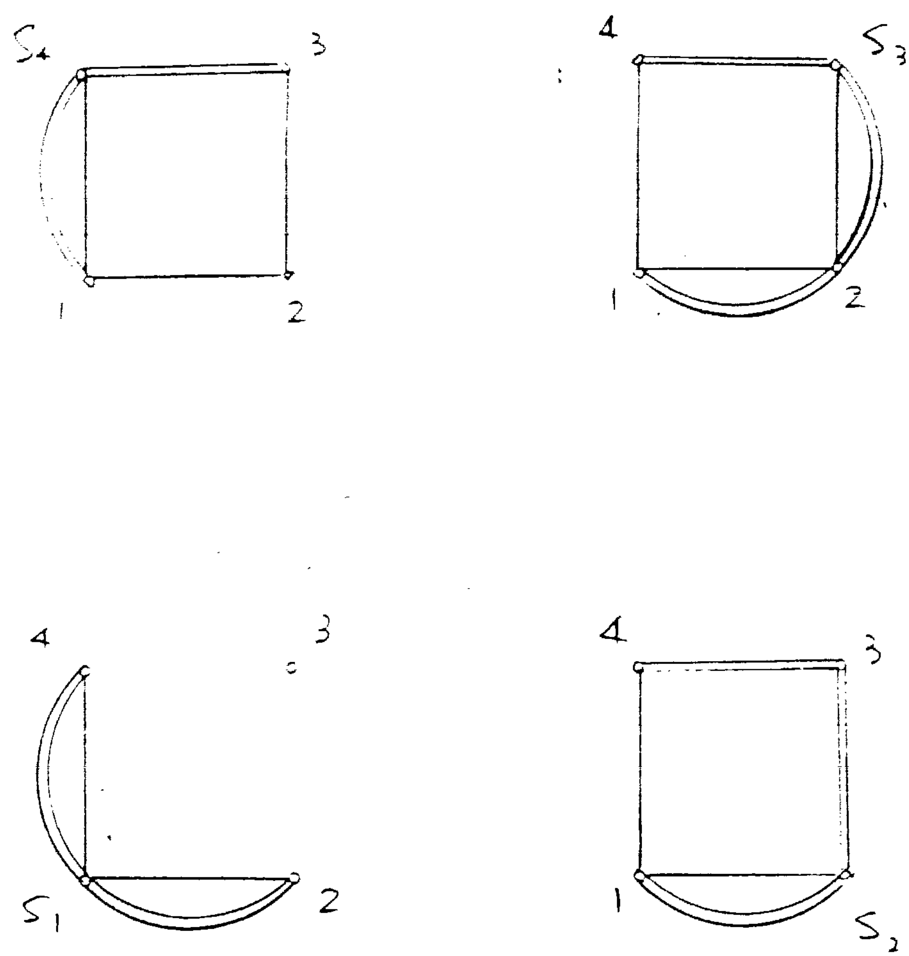
Proof: Without loss of generality, assume that the originator v_1 is in S_1. Consider the following broadcasting scheme :

1. In time unit 1, the originator v_1 calls a vertex v_2 in S_2.

2. In time unit 2, v_1 calls v_3 in S_3 and v_2 calls v_4 in S_4.

3. After 2 time units, each S_i has an informed vertex. They can broadcast internally during the next $\lceil \log_2 n \rceil - 2$ time units.

4. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between the vertices from the pairs (S_1, S_2), and (S_3, S_4) through the other perfect matchings between them.



A single line denotes that a vertex in one set calls a vertex in the other set

A double line denotes that each vertex in one set calls a distinct vertex in the other set

Figure 4-2: The three disjoint calling paths for the vertices in the S_i's in 2-ft mbns constructed by the 2-ft 4-way split method

5. In time unit $\lceil \log_2 n \rceil + 2$, conduct calls between the vertices from the pairs (S_1, S_3), and (S_2, S_4) through the other perfect matchings between them.

Consider the vertices in S_2. They have one calling path from the originator v_1 to v_2 and calls within S_2. They have another path from calls within S_1 and calls between the vertices in S_1 and S_2 through a different matching. Their third calling path is from v_1 to v_3 in S_3, calls within S_3, calls between the vertices in S_3 and S_4, and calls between vertices in S_4 and S_2. The three calling paths for each vertices in S_2 are clearly edge disjoint. Thus, every vertex in S_2 has three edge disjoint calling paths.

Referring to figure 4-2 and using similar arguments as for vertices in S_2, every vertex in each S_i has three disjoint calling paths. Thus, G is a 2-ft mbn.

Theorem 2: The 2-ft mbn G constructed by the 2-ft 4-way split for n mod 4=0 has

$$e(G) <= n/2\lceil \log_2 n \rceil + n.$$

Proof: At most $\sum_{i=1}^{4} (n_i/2\lceil \log_2 n_i \rceil)$ edges are needed to construct 0-ft mbns for each S_i.

At most n/4*8 edges are needed to form perfect matchings between pairs (S_1, S_2), (S_3, S_4), (S_1, S_3), and (S_2, S_4).

The total number of edges is

$$\begin{aligned} e(G) &<= \sum_{i=1}^{4} (n_i/2\lceil \log_2 n_i \rceil) + n/4*8 \\ &<= n/2(\lceil \log_2 n \rceil - 2) + 2n \\ &<= n/2\lceil \log_2 n \rceil + n. \end{aligned}$$

Since every vertex has to participate in the last two calls, the 2-ft 4-way split for n divisible by 4 cannot be used if n mod 4 > 0. However, as we have seen in chapter 3, we can use sink structures to overcome this difficulty. The following sections describe methods to construct 2-ft mbns for n not divisible by 4, using sink structures as part of the building blocks.

4.2 2-ft Four-way split for n mod 8 = 1

The construction of 2-ft mbns for n mod 8 = 1 is very similar to the one used in the 2-ft 4-way split method for n mod 4 = 0. The difference is that the connection between S_1 and S_2 is replaced by a sink structure with S_1 being the larger set. The connection between S_1 and S_3 is also replaced by a sink structure with S_1 being the larger set and having a different sink from the previous sink structure. The two sink structures ensure that the two sinks s_1 and s_2 have two edge disjoint calling paths in the first $T_0(n)$ time units. In the next to last time unit, every vertex except s_1 participates in the calls. Similarly, in the last time unit, every vertex except s_2 participates in the calls. Thus, every vertex can have three calling paths.

2-ft 4-way split for n mod 8 = 1

If n mod 8 = 1 and n >= 16, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 4 sets S_i such that

    a. $|S_2|$ is even

    b. $|S_2| = |S_3| = |S_4|$.

    c. $|S_1| = \lceil n/4 \rceil$

    d. $|S_1| = |S_2| + 1$.

2. Construct 0-ft mbns for S_4 using any method. Construct sink structures for S_1 and S_2, S_1 and S_3 having different sinks s_1 and s_2 respectively.

3. Add edges to form two edge disjoint matchings between the pairs (S_2, S_4), and (S_3, S_4).

Theorem 3: The graph G constructed by the 2-ft 4-way split for n mod 8 = 1 is a 2-ft mbn.

Proof: Using the same broadcast scheme and the same arguments to those used in the proof of theorem 1, every vertex except the two sinks has three edge disjoint calling paths. Using similar arguments to those used in the proof of lemma 9 of chapter 3, the two sinks have two edge disjoint calling paths in the first $T_0(n)$ time units. Furthermore, each sink participates in one of the calls in the last two time units which gives them another calling path. Hence, the two sinks also have three edge disjoint calling paths. Thus, G is a 2-ft mbn.

Theorem 4: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 8 = 1 has

$$e(G) <= n/2\lceil \log_2 n \rceil + 3n/2.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 8-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lfloor n/4 \rfloor * 2$ edges are needed to connect the sinks to vertices in S_2 and S_3.

The total number of edges is

$$e(G) <= n/2\lceil \log_2 n\rceil + n + \lfloor n/4\rfloor * 2$$
$$<= n/2\lceil \log_2 n\rceil + n + n/2$$
$$<= n/2\lceil \log_2 n\rceil + 3n/2.$$

4.3 2-ft Four-way split for n mod 4 = 2

The construction of 2-ft mbns for n mod 4 = 2 is very similar to the one used in the 2-ft 4-way split method for n mod 4 =0. The difference is that S_1 and S_2, S_3 and S_4 are replaced by sink structures with S_2 and S_4 being the larger set. The two sink structures ensure that the two sinks have two edge disjoint calling paths in the first $T_0(n)$ time units. In the last two time units, every vertex except the sinks participate in two calls but the two sinks only participate in one of the calls in the last two time units. Thus every vertex can have three calling paths.

2-ft 4-way split for n mod 4 = 2

If n mod 4 = 2 and $n < 2^k - 2$ and n >= 8, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 4 sets S_i such that

   a. $|S_1|$ is even.

   b. $|S_2| = |S_4| = \lceil n/4\rceil$.

   c. $|S_1| = |S_3| = \lfloor n/4\rfloor$.

2. Construct sink structures for S_1 and S_2, S_3 and S_4.

3. Add edges to form two edge disjoint matchings between the pairs (S_1, S_3), and (S_2, S_4).

Theorem 5: The graph G constructed by the 2-ft 4-way split for n mod 4 = 2, is a 2-ft mbn.

Proof: Using similar arguments to those used in the proof of theorem 3, G is a 2-ft mbn.

Theorem 6: The 2-ft mbn G constructed by the 2-ft 4-way split for n mod 4 = 2 has

$$e(G) <= n/2\lceil \log_2 n \rceil + 3n/2.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 4-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lfloor n/4 \rfloor * 2$ edges are needed to connect the vertices in S_1 and S_3 to their sinks.

The total number of edges is

$$\begin{aligned} e(G) &<= n/2\lceil \log_2 n \rceil + n + \lfloor n/4 \rfloor * 2 \\ &<= n/2\lceil \log_2 n \rceil + n + n/2 \\ &<= n/2\lceil \log_2 n \rceil + 3n/2. \end{aligned}$$

4.4 2-ft Four-way split for n mod 8 = 3

The construction of 2-ft mbns for n mod 8 = 3 is exactly the same as the construction used in constructing 2-ft mbns for n mod 4 = 2. The only difference is that S_1 is the smaller set in the sink structures consisting of S_1 and S_2, S_1 and S_3 instead of the larger set. Let S_1 consist of two sets Q_1 and Q_2 such that $Q_1=Q_2<= 2^i-2$. There will be at least two idle informed vertices when Q_1 or Q_2 broadcast internally. We can use these two idle informed vertices to call the sink in S_2 and the sink in S_3 giving two edge disjoint calling paths to the two sinks in the first $\lceil \log_2 n \rceil$ time units. In the last two time units, every vertex except the two sinks participate in two calls and each sink participates in only one call. Hence, every vertex can have three edge disjoint calling paths.

2-ft 4-way split for n mod 8 = 3

If n mod 8 = 3, n >= 16, and $n \neq 2^k - 5$, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 4 sets S_i such that

   a. $|S_2| = |S_3| = |S_4| = \lceil n/4 \rceil$.

   b. $|S_2| = |S_1| + 1$.

   c. $|S_1|$ is even.

   d. $|S_1| <= 2^k - 4$.

2. Construct a sink structure for S_1 and S_2, S_1 and S_3 such that S_1 is the smaller set in the sink structures. Let S_2 be constructed by two 0-ft mbns P_1, P_2 such that

   $|P_1| = |P_2| + 1$.

   Let S_3 be constructed by two 0-ft mbns F_1, F_2 such that

   $|F_1| = |F_2| + 1$.

3. Add edges to form two edge disjoint matchings between the pair (S_2, S_4), and (S_3, S_4) such that each vertex in S_4 is connected to at least one vertex in P_1 and F_1. This is to ensure that the first informed vertex is in P_1 if the originator is not in S_2 and the first informed vertex is in F_1 if the originator is not in S_3.

Theorem 7: The graph G constructed by the 2-ft 4-way split for n mod 8=3, is a 2-ft mbn.

Proof: Using similar arguments to those used in the proof of theorem 3, G is a 2-ft mbn.

Theorem 8: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 8 = 3 has

$$e(G) <= n/2\lceil \log_2 n \rceil + 3n/2.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 4-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lfloor n/4 \rfloor * 2$ edges are needed to connect the vertices in S_2, and S_3 to their sinks.

The total number of edges is

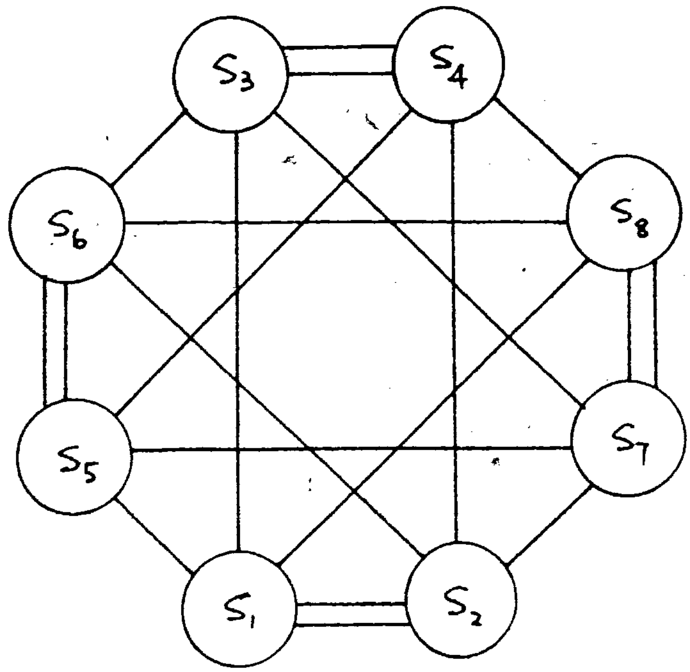
$$e(G) <= n/2\lceil \log_2 n \rceil + n + \lfloor n/4 \rfloor * 2$$

$$<= n/2\lceil \log_2 n \rceil + n + n/2$$

$$<= n/2\lceil \log_2 n \rceil + 3n/2.$$

As we have seen in previous chapters, we can use more 0-ft mbns as our building blocks to construct 1-ft mbns. The following methods to construct 2-ft mbns use eight 0-ft mbns to construct 2-ft mbns instead of four.

4.5 2-ft Eight-way split for n mod 4 = 0



Each line represents an even adjacency connection between the connected sets

Figure 4-3: Constructing 2-ft mbns using 2-ft 8-way split for n divisible by 4

If n mod 4 = 0, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 8 sets S_i such that

   $\lfloor n/8 \rfloor <= |S_1| = |S_2| = |S_3| = |S_4| <= \lceil n/8 \rceil$, and

   $\lfloor n/8 \rfloor <= |S_4| = |S_5| = |S_7| = |S_8| <= \lceil n/8 \rceil$.

2. Construct 0-ft mbns for each S_i using any method.

3. Add edges to form two edge disjoint matchings between the vertices of each of the pairs (S_1, S_2), (S_3, S_4), (S_5, S_6), and (S_7, S_8).

4. Add edges to form perfect matchings between the vertices of each of the pairs (S_1, S_3), (S_2, S_4), (S_5, S_7), and (S_6, S_8).

5. Add edges to form minimum adjacency connections between the pairs (S_1, S_5), (S_1, S_8), (S_2, S_7), (S_2, S_6), (S_3, S_6), (S_3, S_7), (S_4, S_5), and (S_4, S_8).

Theorem 9: The graph G constructed by the 2-ft 8-way split for n mod 4=0, is a 2-ft mbn.

Proof: Without loss of generality, assume that the originator v_1 is in S_1. Consider the following broadcasting scheme :

1. In time unit 1, the originator v_1 calls a vertex v_2 in S_2.

2. In time unit 2, v_1 calls v_8 in S_8 and V_2 calls v_6 in S_6.

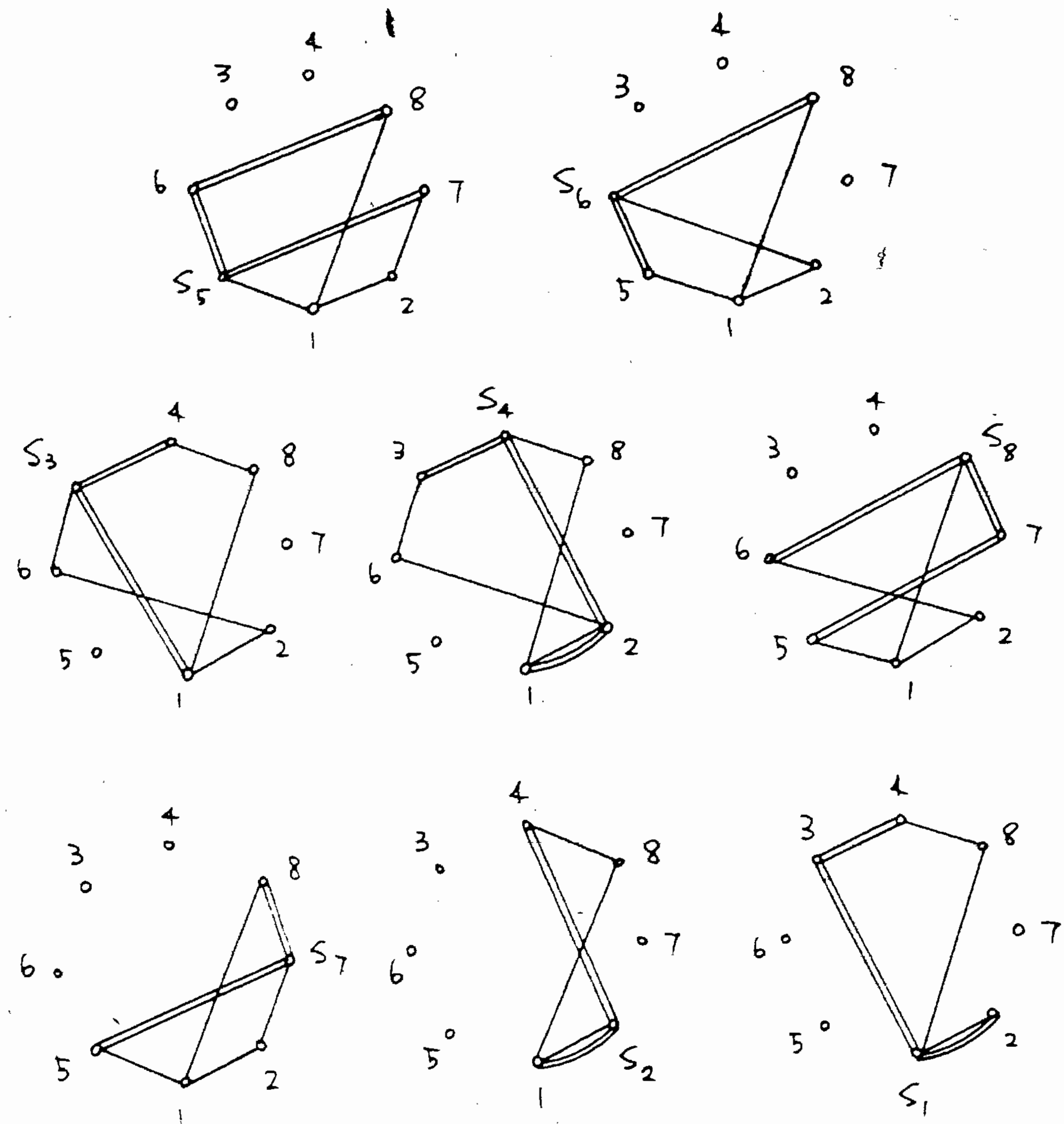
3. In time unit 3, v_1 calls v_5 in S_5, V_2 calls v_7 in S_7, v_6 calls v_3 in S_3, and v_8 calls v_4 in S_4.

4. After 3 time units, each S_i has an informed vertex. They can broadcast internally during the next $\lceil \log_2 n \rceil - 3$ time units.

5. In time unit $\lceil \log_2 n \rceil + 1$, conduct calls between the vertices from the pairs (S_1, S_2), (S_3, S_4), (S_5, S_7), and (S_6, S_8) through the perfect matchings between them. For the pair (S_1, S_2) use the perfect matching that has not been used in time unit 1.

6. In time unit $\lceil \log_2 n \rceil + 2$, conduct calls between the vertices from the pairs (S_1, S_3), (S_2, S_4), (S_5, S_6), and (S_7, S_8) through the perfect matchings between them.

Consider the vertices in S_5. They have one calling path from the originator v_1 to v_5 and calls within S_5. They have another path from v_1 to v_2 in S_2, v_2 to v_7 in S_7, calls within S_7 and



A single line represents that a vertex in one set calls a vertex in the other set

A double line represents that every vertex in one set calls a distinct vertex in the other set

Figure 4-4: The three disjoint calling paths for the vertices in the S_i's in 2-ft mbns constructed by the 2-ft 8-way split method calls between the vertices in S_7 and S_5. Their third calling path is from v_1 to v_8 in S_8, calls within S_8, calls between the vertices in S_8 and S_6, and calls between vertices in S_6 and S_5. The three calling paths for each vertices in S_5 are clearly edge disjoint. Thus, every vertex in S_5 has three edge disjoint calling paths.

Referring to figure 4-4 and using similar arguments as for vertices in S_5, every vertex in each S_i has three disjoint calling paths. Thus, G is a 2-ft mbn.

Theorem 10: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 4=0 has

$$e(G) <= n/2\lceil \log_2 n \rceil + n + 10.$$

Proof: At most $\sum_{i=1}^{8} (n_i/2\lceil \log_2 n_i \rceil)$ edges are needed to construct 0-ft mbns for each S_i.

At most $\lceil n/8 \rceil * 8$ edges are needed to form minimum adjacency connection between pairs (S_1, S_5), (S_1, S_8), (S_2, S_7), (S_2, S_6), (S_3, S_6), (S_3, S_7), (S_4, S_5), and (S_4, S_8).

At most $\lceil n/8 \rceil * 12$ edges are needed to form perfect matchings between pairs (S_1, S_2), (S_3, S_4), (S_5, S_6), (S_7, S_8), (S_1, S_3), (S_2, S_4), (S_5, S_7), and (S_6, S_8).

The total number of edges is

$$e(G) <= \sum_{i=1}^{8} (n_i/2\lceil \log_2 n_i \rceil) + \lceil n/8 \rceil * 20$$

Since $\lceil n/8 \rceil <= (n + 4)/8$,

$$e(G) <= n/2(\lceil \log_2 n \rceil - 3) + 5n/2 + 10$$

$$<= n/2\lceil \log_2 n \rceil + n + 10.$$

Since every vertex has to participate in the calls in the last two time units, the 2-ft 8-way split for n divisible by 4 cannot be used if n mod 4 > 0. Similarly, we can use sink structures to overcome this difficulty. The following sections describe methods to construct 2-ft mbns for n not divisible by 4, using sink structures as part of the building blocks.

4.6 2-ft Eight-way split for n mod 4 = 1

The construction of 2-ft mbns for n mod 4 = 1 is very similar to the one used in the 2-ft 8-way split method for n mod 4 = 0. The difference is that the connection between S_7 and S_8 is replaced by a sink structure with S_8 being the larger set. The connection between S_6 and S_8 is also replaced by a sink structure which has only one perfect matching between them and a different sink from the previous sink structure. Extra edges are then added to "force" the first informed vertex in S_8 to be in the larger mbn that constitutes S_8. The two sink structures ensure that the two sinks s_1 and s_2 have two edge disjoint calling paths in the first $T_g(n)$ time units. In the next to last time unit, every vertex except s_1 participates in the calls. Similarly, in the last time unit, every vertex except s_2 participates in the calls. Thus, every vertex can have three calling paths.

2-ft 8-way split for n mod 4 = 1

If n mod 4 = 1, n >=32 and $n \neq 2^K-3$, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 8 sets S_i such that

   a. $|S_5|$ is even and $|S_8| < 2^K$.

   b. $\lfloor n/8 \rfloor <= |S_1| = |S_2| = |S_3| = |S_4| <= \lceil n/8 \rceil$.

c. $|S_5| = |S_6| = |S_7|$.

d. $|S_8| = |S_5| + 1$.

2. Construct 0-ft mbns for each S_i, i=1..5 using any method. Construct a 0-ft mbn for S_6 using the 2-way split and construct a sink structure for S_7, S_8. Let S_8 be constructed by two 0-ft mbns P_1, P_2 such that

   $$|P_1| = |P_2| + 1.$$

   Choose a vertex w in P_2 other than the sink s in the sink structure.

3. Add edges to form two edge disjoint matchings between the pairs (S_1, S_2), (S_3, S_4), and (S_5, S_6).

4. Add edges to form perfect matchings between the pairs (S_1, S_3), (S_2, S_4), (S_5, S_7), and (S_6, S_8 - w).

5. Add edges to connect the following pairs : (S_1, S_5), (S_1, S_8), (S_2, S_7), (S_2, S_6), (S_3, S_6), (S_3, S_7), (S_4, S_5), and (S_4, S_8) such that every vertex in one set is connected to at least one vertex of the other set.

6. Add edges to connect the vertices in S_6 to w. These edges will be used to give w two disjoint calling paths in the first $T_0(n)$ calls.

7. Add edges to connect the vertices in P_1 to the vertices in S_1 and S_4 which are not connected to any vertices in P_1. These edges will be used to "force" the first informed vertex in S_8 to be in P_1 if the originator is not in S_8.

Theorem 11: The graph G constructed by the 2-ft 8-way split for n mod 4 = 1, is a 2-ft mbn.

Proof: Using the same broadcast scheme and the same arguments to those used in the proof of theorem 9, every vertex except the two sinks has three edge disjoint calling paths. Using similar arguments to those used in the proof of lemma 9 of chapter 3, the two sinks have two edge disjoint calling paths in the first $T_0(n)$ time units. Furthermore, each sink participates in one of the calls in the last two time units which gives them another calling path. Hence, the two sinks also have three edge disjoint calling paths. Thus, G is a 2-ft mbn.

Theorem 12: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 4 = 1 has

$$e(G) <= n/2\lceil \log_2 n\rceil + 11n/8 + 19.5.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 8-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lfloor n/8\rfloor*2$ edges are needed to connect the sinks to vertices in S_6 and S_7.

At most $\lceil n/16\rceil*2$ edges are needed to connect the vertices in P_1 to vertices that are not connected to P_1 in S_1 and S_4.

The total number of edges is

$$e(G) <= n/2(\lceil \log_2 n\rceil - 3) + \lceil n/8\rceil*20 + \lfloor n/8\rfloor*2 + \lceil n/16\rceil*2$$

Since $\lceil n/8\rceil <= (n + 7)/8$,

$$e(G) <= n/2\lceil \log_2 n\rceil + n + 19.5 + 3n/8$$

$$<= n/2\lceil \log_2 n\rceil + 11n/8 + 19.5.$$

4.7 2-ft Eight-way split for n mod 4 = 2

The construction of 2-ft mbns for n mod 4 = 2 is very similar to the one used in the 2-ft 8-way split method for n mod 4 =0. The difference is that S_1 and S_2, S_3 and S_4 are replaced by sink structures with S_2 and S_4 being the larger set. Extra edges are then added to "force" the first informed vertex in S_2 to be in the larger mbn that constitutes S_2, and the first informed vertex in S_4 to be in the larger mbn that constitutes S_4. The two sink structures ensure that the two sinks have two edge disjoint calling paths in the first $T_0(n)$ time units. In the last two time units, every vertex except the sinks participates in two calls but the two sinks only participate in one of the calls in the last two time units. Thus every vertex can have three calling paths.

2-ft 8-way split for n mod 4 = 2

If n mod 4 = 2, n >= 8, and $n \neq 2^k - 2$, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 8 sets S_i such that

   a. $|S_1|$ is even.

   b. $\lfloor n/8 \rfloor \le |S_5| = |S_6| = |S_7| = |S_8| \le \lceil n/8 \rceil$.

   c. $|S_1| = |S_3|$ and $|S_2| = |S_4|$.

   d. $|S_2| = |S_1| + 1$.

   e. $|S_2| < 2^k$.

2. Construct 0-mbns for each S_i, i=5..8 using any method. Construct a sink structure for S_1 and S_2, S_3 and S_4. Let S_2 be constructed by two 0-ft mbns P_1, P_2 such that

   $|P_1| = |P_2| + 1$.

   Let S_4 be constructed by two 0-ft mbns F_1, F_2 such that

   $|F_1| = |F_2| + 1$.

3. Add edges to form two edge disjoint matchings between the

pairs (S_5, S_6), and (S_7, S_8).

4. Add edges to form perfect matchings between the pairs (S_1, S_3), (S_2, S_4), (S_5, S_7), and (S_6, S_8).

5. Add edges to connect the pairs : (S_1, S_5), (S_1, S_8), (S_2, S_7), (S_2, S_6), (S_3, S_6), (S_3, S_7), (S_4, S_5), and (S_4, S_8) such that every vertex in one set is connected to at least one vertex of the other set.

6. Add edges to connect the vertices in P_1 to the vertices in S_6 and S_7 which are not connected to any vertices in P_1. ~~These edges will be used to "force" the first informed vertex~~ in S_2 to be in P_1 if the originator is not in S_2. Similarly, add edges to connect the vertices in F_1 to the vertices in S_5 and S_8 which are not connected to any vertices in F_1. These edges will be used to "force" the first informed vertex in S_4 to be in F_1 if the originator is not in S_4.

Theorem 13: The graph G constructed by the 2-ft 8-way split for n mod 4 = 2, is a 2-ft mbn.

Proof: Using the similar arguments to those used in the proof of theorem 11, G is a 2-ft mbn.

Theorem 14: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 4 = 2 has

$$e(G) <= n/2\lceil \log_2 n \rceil + 3n/2 + 19.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 8-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lceil n/16 \rceil * 2$ edges are needed to connect the vertices in

P_1 to vertices that are not connected to P_1 in S_6 and S_7, and at most $\lceil n/16 \rceil * 2$ edges are needed to connect the vertices in F_1 to vertices that are not connected to F_1 in S_5 and S_8.

At most $\lfloor n/8 \rfloor * 2$ edges are needed to connect the vertices in S_1 and S_3 to their sinks.

The total number of edges is

$$e(G) <= n/2(\lceil \log_2 n \rceil - 3) + \lceil n/8 \rceil * 20 + \lceil n/16 \rceil * 4 + \lfloor n/8 \rfloor * 2$$

Since $\lceil n/8 \rceil <= (n + 6)/8$,

$$\therefore e(G) <= n/2\lceil \log_2 n \rceil + n + 19 + n/4 + n/4$$

$$<= n/2\lceil \log_2 n \rceil + 3n/2 + 19.$$

4.8 2-ft Eight-way split for n mod 4 = 3

The construction of 2-ft mbns for n mod 4 = 3 can be done by combining the structures used in constructing 2-ft mbns for n mod 4 = 2, and for n mod 4 = 1. We can partition n vertices into 8 sets S_i and construct S_1, S_2, S_3, and S_4 in the same way as we have constructed them in the 2-ft 8-way split for n mod 4 = 2. Similarly, we can construct S_5, S_6, S_7, and S_8 in the same way as we have constructed them in the 2-ft 8-way split for n mod 4 = 1. The size of the combined graph G will have a remainder of 3 when divide by 4. Furthermore, we can use the same broadcasting scheme as in theorem 11 and give every vertex three edge disjoint calling paths.

2-ft 8-way split for n mod 4 = 3

If n mod 4 = 3, n >= 32, $n \neq 2^i - 5$, and $n \neq 2^i - 9$, a 2-ft mbn can be constructed as follows :

1. Partition n vertices into 8 sets S_i as evenly as possible such

that

a. $|S_5|$ is even.

b. $|S_8| < 2^k$.

c. $|S_6| = |S_5| = |S_7|$.

d. $|S_8| = |S_7| + 1$.

e. $|S_1|$ is even.

f. $|S_2| < 2^k$.

g. $|S_1| = |S_3|$ and $|S_2| = |S_4|$.

h. $|S_2| = |S_1| + 1$.

2. Construct a sink structure for S_1 and S_2, S_3 and S_4. Let S_2 be constructed by two 0-ft mbns P_1, P_2 such that

   $|P_1| = |P_2| + 1$.

   Let S_4 be constructed by two 0-ft mbns F_1, F_2 such that

   $|F_1| = |F_2| + 1$.

3. Construct a sink structure for S_7 and S_8. Let S_8 be constructed by two 0-ft mbns M_1, M_2 such that

   $|M_1| = |M_2| + 1$.

   Construct a sink structure for S_8 and S_6 which has only one matching between them and choose a vertex w which is different from the sink in the previous sink structure in M_2.

4. Add edges to form two edge disjoint matchings between the pair (S_5, S_6).

5. Add edges to form perfect matchings between the pairs (S_1, S_3), (S_2, S_4), $(S_6, S_8\backslash\{w\})$, and (S_5, S_7).

6. Add edges to connect the pairs : (S_1, S_5), (S_1, S_8), (S_2, S_7), (S_2, S_6), (S_3, S_6), (S_3, S_7), (S_4, S_5), and (S_4, S_8) such that every vertex in one set is connected to at least one vertex of the other set.

7. Add edges to connect the vertices in P_1 to the vertices in S_6 and S_7 which are not connected to any vertices in P_1. These edges will be used to "force" the first informed vertex in S_2 to be in P_1 if the originator is not in S_2. Similarly, add edges to connect the vertices in F_1 to the vertices in S_5 and S_8 which are not connected to any vertices in F_1, and add edges to connect the vertices in M_1 to the vertices in S_1 and S_4 which are not connected to any vertices in M_1.

Theorem 15: The graph G constructed by the 2-ft 8-way split for n mod 4=3, is a 2-ft mbn.

Proof: Using similar arguments to those used in the proof of theorem 11, G is a 2-ft mbn.

Theorem 16: The 2-ft mbn G constructed by the 2-ft 8-way split for n mod 4 = 3 has

$$e(G) <= n/2\lceil \log_2 n \rceil + 15n/8 + 18.5.$$

Proof: The number of edges in G is equal to the number of edges in a 2-ft mbn constructed by the 2-ft 8-way split method for n mod 4 = 0 plus the extra edges added.

At most $\lceil n/16 \rceil * 2$ edges are needed to connect the vertices in P_1 to vertices that are not connected to P_1 in S_6 and S_7, and at most $\lceil n/16 \rceil * 2$ edges are needed to connect the vertices in F_1 to vertices that are not connected to F_1 in S_5 and S_8. Similarly, at most $\lceil n/16 \rceil * 2$ edges are needed to connect the vertices in M_1 to vertices that are not connected to M_1 in S_1 and S_4.

At most $\lfloor n/8 \rfloor * 4$ edges are needed to connect the vertices in S_1, S_3, S_6, and S_7 to their sinks.

The total number of edges is

$$e(G) <= n/2(\lceil \log_2 n \rceil - 3) + \lceil n/8 \rceil * 20 + \lceil n/16 \rceil * 6 + \lfloor n/8 \rfloor * 4$$

Since $\lceil n/8 \rceil <= (n + 5)/8$,

$$\therefore e(G) <= n/2\lceil \log_2 n \rceil + n + 18.5 + 3n/8 + n/2$$

$$<= n/2\lceil \log_2 n \rceil + 15n/8 + 18.5.$$

4.9 Summary

Eight different methods to construct 2-ft mbns are presented. The methods give a better bound on $B_2(n)$ for almost all n than the previous methods.

For n mod 4 = 0,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + n.$$

For n mod 8 = 3 or n mod 4 = 2, $n \neq 2^i-2$, $n \neq 2^i-5$, and n >= 16,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + 3n/2.$$

For n mod 8 = 1, and 16 <= n <= 116,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + 3n/2.$$

For n mod 8 = 1, and n > 116,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + 11n/8 + 19.5.$$

For n mod 8 = 5, n >= 32 and $n \neq 2^i - 3$,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + 11n/8 + 19.5.$$

For n mod 8 = 7, n >= 32 and $n \neq 2^i - 9$,

$$B_2(n) <= n/2\lceil \log_2 n \rceil + 15n/8 + 18.5.$$

These methods do not work for $n=2^i-2$, $n=2^i-3$, $n=2^i-5$, and $n=2^i-9$. It is interesting to note that the 2-ft 8-way split for n mod 4 = 3 does work for $n = 2^i-1$ because the minimum time $T_2(n)$ for $n = 2^i-1$ is equal to $\lceil \log_2 n \rceil+3$ instead of $\lceil \log_2 n \rceil+2$. That is, we have one extra time unit for $n = 2^i-1$. Finally, it may be possible to use other splits and other approaches to get a better bound on $B_2(n)$ for $n=2^i-2$, $n=2^i-3$, $n=2^i-5$, and $n=2^i-9$. These approaches have not been investigated.

Chapter 5

Conclusion

Minimum broadcast graphs represent the cheapest possible communication networks of n members which can broadcast in minimum time regardless of originator. Mbgs may be used for message broadcasting in communication, parallel processing, and distributed computing. Unfortunately, no technique is known to generate these graphs for arbitrary n. Farley [Farley 79] suggested algorithms to construct minimal broadcast networks which are sparse graphs allowing minimum time broadcast from any originator. New methods to construct such networks are presented in chapter 2. The resulting graphs have fewer edges than Farley's graphs for three-quarters of the possible values of n. The improved graphs are estimated to have an average of 8% fewer edges than those of Farley's for $36 <= n <= 1024$. Furthermore, improvement for some of the remaining values of n may also be possible by using similar methods.

Fault-tolerant broadcasting is desirable if reliability is considered to be an important factor in a communication network. The set of k fault-tolerant minimum broadcast graphs represent the cheapest possible communication networks of n members which can complete a k fault-tolerant broadcast in minimum time regardless of originator. No technique is known to generate these graphs for arbitrary n. Algorithms to construct k fault-tolerant minimal broadcast networks have been suggested by Liestman [Liestman 81] for k=1 and k=2. In chapters 3 and 4, new methods to construct such graphs are presented. In both cases, the graphs produced by the new methods contain approximately one-half the number of edges of the previously known graphs. However, in the 2 fault-tolerant case, the new method cannot be used for $n=2^i+j$, $j=2,3,5,9$ and some small $n < 32$. Using approaches other than those used in chapter 4, it may be possible to construct improved 2 fault-tolerant mbns for these values.

Methods to construct k fault-tolerant mbns depend on the results on $T_k(n)$. It is not possible to describe such constructions without knowing the exact value of $T_k(n)$. Thus, more general results to construct k fault-tolerant mbns cannot be found without first finding more general results for $T_k(n)$. The multi-way split approach to construct 1 fault-tolerant and 2 fault-tolerant mbns does give some insight on the value of $T_k(n)$ for $k>=3$. For example, it may be possible to use an 8-way split to construct 3 fault-tolerant mbns. We can use the first $\lceil \log_2 n \rceil$ time units to create a calling path for every vertex and use three more time units to complete three more edge disjoint calling paths to each vertex. If this is possible then $T_3(n)$ is equal to $\lceil \log_2 n \rceil+3$ for n mod 8=0. Using similar arguments, we may use a 2^k way split to construct k fault-tolerant mbns. We can use the first $\lceil \log_2 n \rceil$ time units to create a calling path for every vertex and use k more time units to give k more edge disjoint calling paths to each vertex. If such a scheme exists then $T_k(n)$ is equal to $\lceil \log_2 n \rceil+k$ for n mod $2^k=0$. No further investigation has been done in this direction. However, we conjecture that $T_k(n)$ is equal to $\lceil \log_2 n \rceil+k$ for at least some values of n.

Appendix A

Table to compare the value of e(G) between Farley's algorithm and the Hybrid algorithm

n	Farley	split	Hybrid	split	difference
18	27	3	27	6	0
19	30	3	30	6	0
20	32	3	32	6	0
21	35	3	35	6	0
22	37	2	37	6	0
23	39	2	39	2	0
24	42	2	42	6	0
25	45	2	45	2	0
26	49	2	49	2	0
27	52	2	52	2	0
28	56	2	56	7	0
29	59	2	59	2	0
30	63	2	63	2	0
31	71	2	71	2	0
32	80	2	80	2	0
33	56	3	59	5	-3
34	58	3	60	6	-2
35	61	3	62	6	-1
36	63	3	63	6	0
37	67	3	66	6	1
38	70	3	69	6	1
39	74	3	72	6	2
40	77	3	75	6	2
41	81	3	78	6	3
42	84	3	81	6	3
43	88	3	86	6	2
44	91	3	90	6	1
45	95	3	94	7	1
46	101	2	97	7	4
47	104	2	100	7	4
48	108	2	103	7	5
49	111	2	107	7	4
50	115	2	111	7	4
51	119	2	116	7	3
52	124	2	121	7	3
53	127	2	126	7	1
54	131	2	131	7	0
55	135	2	135	2	0
56	140	2	140	7	0
57	143	2	143	2	0
58	147	2	147	2	0
59	151	2	151	2	0

n	Farley	split	Hybrid	split	difference
60	156	2	156	2	0
61	164	2	164	2	0
62	173	2	173	2	0
63	182	2	182	2	0
64	192	2	192	2	0
65	142	3	127	6	15
66	144	3	129	6	15
67	147	3	132	6	15
68	149	3	134	6	15
69	152	3	137	6	15
70	155	3	139	6	16
71	159	3	142	6	17
72	162	3	144	6	18
73	166	3	148	6	18
74	169	3	152	6	17
75	173	3	156	6	17
76	177	3	160	6	17
77	182	3	164	6	18
78	186	3	168	6	18
79	190	3	172	6	18
80	193	3	175	6	18
81	197	3	179	6	18
82	201	3	182	6	19
83	206	3	186	6	20
84	210	3	189	6	21
85	214	3	193	6	21
86	217	3	197	6	20
87	221	3	201	6	20
88	225	3	205	6	20
89	230	3	209	6	21
90	234	3	213	6	21
91	241	2	219	7	22
92	248	2	222	7	26
93	251	2	226	7	25
94	255	2	230	7	25
95	259	2	234	7	25
96	264	2	238	7	26
97	267	2	242	7	25
98	271	2	245	7	26
99	275	2	250	7	25
100	280	2	254	7	26
101	284	2	258	7	26
102	289	2	262	7	27
103	294	2	266	7	28
104	300	2	270	7	30
105	303	2	275	7	28
106	307	2	283	7	24
107	311	2	292	7	19
108	316	2	301	7	15
109	320	2	310	7	10
110	325	2	319	7	6
111	330	2	328	7	2
112	336	2	336	7	0
113	339	2	339	2	0
114	343	2	343	2	0
115	347	2	347	2	0
116	352	2	352	2	0

n	Farley	split	Hybrid	split	difference
117	356	2	356	2	0
118	361	2	361	2	0
119	366	2	366	2	0
120	372	2	372	2	0
121	380	2	380	2	0
122	389	2	389	2	0
123	398	2	398	2	0
124	408	2	408	2	0
125	417	2	417	2	0
126	427	2	427	2	0
127	437	2	437	2	0
128	448	2	448	2	0
129	329	3	314	6	15
130	332	3	316	6	16
131	336	3	319	6	17
132	339	3	321	6	18
133	344	3	324	6	20
134	348	3	327	6	21
135	353	3	330	6	23
136	359	3	333	6	26
137	366	3	336	6	30
138	372	3	339	6	33
139	376	3	343	6	33
140	379	3	346	6	33
141	383	3	350	6	33
142	387	3	353	6	34
143	392	3	357	6	35
144	396	3	360	6	36
145	400	3	364	6	36
146	403	3	368	6	35
147	407	3	372	6	35
148	411	3	376	6	35
149	416	3	380	6	36
150	420	3	384	6	36
151	425	3	389	6	36
152	429	3	393	6	36
153	434	3	398	6	36
154	439	3	402	6	37
155	445	3	407	6	38
156	450	3	411	6	39
157	454	3	415	6	39
158	457	3	419	6	38
159	461	3	423	6	38
160	465	3	427	6	38
161	470	3	431	6	39
162	474	3	435	6	39
163	479	3	440	6	39
164	483	3	444	6	39
165	488	3	449	6	39
166	493	3	453	6	40
167	499	3	458	6	41
168	504	3	462	6	42
169	508	3	466	6	42
170	511	3	470	6	41
171	515	3	474	6	41
172	519	3	478	6	41
173	524	3	482	6	42

n	Farley	split	Hybrid	split	difference
174	528	3	486	6	42
175	533	3	491	6	42
176	537	3	495	6	42
177	542	3	500	6	42
178	547	3	504	6	43
179	553	3	509	6	44
180	558	3	513	6	45
181	565	2	521	7	44
182	573	2	525	7	48
183	580	2	530	7	50
184	588	2	534	7	54
185	591	2	538	7	53
186	595	2	542	7	53
187	599	2	546	7	53
188	604	2	550	7	54
189	608	2	555	7	53
190	613	2	559	7	54
191	618	2	564	7	54
192	624	2	569	7	55
193	627	2	574	7	53
194	631	2	579	7	52
195	635	2	584	7	51
196	640	2	588	7	52
197	644	2	593	7	51
198	649	2	597	7	52
199	654	2	601	7	53
200	660	2	605	7	55
201	664	2	609	7	55
202	669	2	613	7	56
203	674	2	618	7	56
204	680	2	622	7	58
205	685	2	627	7	58
206	691	2	632	7	59
207	697	2	637	7	60
208	704	2	642	7	62
209	707	2	647	7	60
210	711	2	651	7	60
211	715	2	661	7	54
212	720	2	670	7	50
213	724	2	679	7	45
214	729	2	688	7	41
215	734	2	697	7	37
216	740	2	706	7	34
217	744	2	716	7	28
218	749	2	725	7	24
219	754	2	735	7	19
220	760	2	745	7	15
221	765	2	755	7	10
222	771	2	765	7	6
223	777	2	775	7	2
224	784	2	784	7	0
225	787	2	787	2	0
226	791	2	791	2	0
227	795	2	795	2	0
228	800	2	800	2	0
229	804	2	804	2	0
230	809	2	809	2	0

n	Farley	split	Hybrid	split	difference
231	814	2	814	2	0
232	820	2	820	2	0
233	824	2	824	2	0
234	829	2	829	2	0
235	834	2	834	2	0
236	840	2	840	2	0
237	845	2	845	2	0
238	851	2	851	2	0
239	857	2	857	2	0
240	864	2	864	2	0
241	872	2	872	2	0
242	881	2	881	2	0
243	890	2	890	2	0
244	900	2	900	2	0
245	909	2	909	2	0
246	919	2	919	2	0
247	929	2	929	2	0
248	940	2	940	2	0
249	949	2	949	2	0
250	959	2	959	2	0
251	969	2	969	2	0
252	980	2	980	2	0
253	990	2	990	2	0
254	1001	2	1001	2	0
255	1012	2	1012	2	0
256	1024	2	1024	2	0
257	777	3	705	6	72
258	780	3	711	6	69
259	785	3	716	6	69
260	789	3	720	6	69
261	794	3	725	6	69
262	798	3	729	6	69
263	803	3	734	6	69
264	807	3	738	6	69
265	813	3	743	6	70
266	818	3	748	6	70
267	824	3	753	6	71
268	828	3	758	6	70
269	833	3	762	7	71
270	837	3	766	7	71
271	845	3	770	7	75
272	852	3	774	7	78
273	860	3	779	6	81
274	867	3	782	6	85
275	875	3	786	6	89
276	882	3	789	6	93
277	886	3	793	6	93
278	889	3	797	6	92
279	893	3	801	6	92
280	897	3	805	6	92
281	902	3	809	6	93
282	906	3	813	6	93
283	911	3	817	6	94
284	915	3	820	6	95
285	920	3	824	6	96
286	925	3	827	6	98
287	931	3	831	6	100

n	Farley	split	Hybrid	split	difference
288	936	3	834	6	102
289	940	3	839	6	101
290	943	3	844	6	99
291	947	3	849	6	98
292	951	3	854	6	97
293	956	3	858	7	98
294	960	3	861	7	99
295	965	3	868	7	97
296	969	3	873	6	96
297	974	3	878	6	96
298	979	3	882	6	97
299	985	3	887	6	98
300	990	3	891	6	99
301	995	3	897	6	98
302	999	3	903	6	96
303	1004	3	909	6	95
304	1009	3	915	6	94
305	1015	3	921	6	94
306	1020	3	927	6	93
307	1026	3	933	6	93
308	1031	3	938	6	93
309	1037	3	944	6	93
310	1043	3	949	6	94
311	1050	3	954	7	96
312	1056	3	959	7	97
313	1060	3	964	7	96
314	1063	3	969	7	94
315	1067	3	975	7	92
316	1071	3	978	7	93
317	1076	3	982	7	94
318	1080	3	986	7	94
319	1085	3	990	7	95
320	1089	3	994	7	95
321	1094	3	998	7	96
322	1099	3	1001	7	98
323	1105	3	1006	7	99
324	1110	3	1010	7	100
325	1115	3	1014	7	101
326	1119	3	1018	7	101
327	1124	3	1022	7	102
328	1129	3	1026	7	103
329	1135	3	1031	7	104
330	1140	3	1034	7	106
331	1146	3	1038	7	108
332	1151	3	1042	7	109
333	1157	3	1046	7	111
334	1163	3	1050	7	113
335	1170	3	1054	7	116
336	1176	3	1057	7	119
337	1180	3	1063	7	117
338	1183	3	1068	7	115
339	1187	3	1073	7	114
340	1191	3	1078	7	113
341	1196	3	1083	7	113
342	1200	3	1088	7	112
343	1205	3	1094	7	111
344	1209	3	1098	7	111

n	Farley	split	Hybrid	split	difference
345	1214	3	1103	7	111
346	1219	3	1108	7	111
347	1225	3	1113	7	112
348	1230	3	1118	7	112
349	1235	3	1123	7	112
350	1239	3	1127	7	112
351	1244	3	1134	7	110
352	1249	3	1140	7	109
353	1255	3	1146	7	109
354	1260	3	1152	7	108
355	1266	3	1158	7	108
356	1271	3	1164	7	107
357	1277	3	1171	7	106
358	1283	3	1176	7	107
359	1290	3	1182	7	108
360	1296	3	1188	7	108
361	1303	2	1194	7	109
362	1311	2	1200	7	111
363	1319	2	1206	7	113
364	1328	2	1211	7	117
365	1335	2	1218	7	117
366	1343	2	1224	7	119
367	1351	2	1230	7	121
368	1360	2	1236	7	124
369	1363	2	1242	7	121
370	1367	2	1248	7	119
371	1371	2	1255	7	116
372	1376	2	1260	7	116
373	1380	2	1266	7	114
374	1385	2	1272	7	113
375	1390	2	1278	7	112
376	1396	2	1284	7	112
377	1400	2	1290	7	110
378	1405	2	1295	7	110
379	1410	2	1301	7	109
380	1416	2	1306	7	110
381	1421	2	1311	7	110
382	1427	2	1316	7	111
383	1433	2	1321	7	112
384	1440	2	1326	7	114
385	1443	2	1332	7	111
386	1447	2	1337	7	110
387	1451	2	1343	7	108
388	1456	2	1349	7	107
389	1460	2	1355	7	105
390	1465	2	1361	7	104
391	1470	2	1367	7	103
392	1476	2	1372	7	104
393	1480	2	1377	7	103
394	1485	2	1381	7	104
395	1490	2	1385	7	105
396	1496	2	1389	7	107
397	1501	2	1393	7	108
398	1507	2	1397	7	110
399	1513	2	1402	7	111
400	1520	2	1406	7	114
401	1524	2	1411	7	113

n	Farley	split	Hybrid	split	difference
402	1529	2	1416	7	113
403	1534	2	1421	7	113
404	1540	2	1426	7	114
405	1545	2	1431	7	114
406	1551	2	1435	7	116
407	1557	2	1441	7	116
408	1564	2	1446	7	118
409	1569	2	1451	7	118
410	1575	2	1456	7	119
411	1581	2	1461	7	120
412	1588	2	1466	7	122
413	1594	2	1472	7	122
414	1601	2	1477	7	124
415	1608	2	1483	7	125
416	1616	2	1489	7	127
417	1619	2	1495	7	124
418	1623	2	1501	7	122
419	1627	2	1507	7	120
420	1632	2	1512	7	120
421	1636	2	1522	7	114
422	1641	2	1531	7	110
423	1646	2	1540	7	106
424	1652	2	1549	7	103
425	1656	2	1558	7	98
426	1661	2	1567	7	94
427	1666	2	1577	7	89
428	1672	2	1586	7	86
429	1677	2	1596	7	81
430	1683	2	1606	7	77
431	1689	2	1616	7	73
432	1696	2	1626	7	70
433	1700	2	1636	7	64
434	1705	2	1645	7	60
435	1710	2	1656	7	54
436	1716	2	1666	7	50
437	1721	2	1676	7	45
438	1727	2	1686	7	41
439	1733	2	1696	7	37
440	1740	2	1706	7	34
441	1745	2	1717	7	28
442	1751	2	1727	7	24
443	1757	2	1738	7	19
444	1764	2	1749	7	15
445	1770	2	1760	7	10
446	1777	2	1771	7	6
447	1784	2	1782	7	2
448	1792	2	1792	7	0
449	1795	2	1795	2	0
450	1799	2	1799	2	0
451	1803	2	1803	2	0
452	1808	2	1808	2	0
453	1812	2	1812	2	0
454	1817	2	1817	2	0
455	1822	2	1822	2	0
456	1828	2	1828	2	0
457	1832	2	1832	2	0
458	1837	2	1837	2	0

n	Farley	split	Hybrid	split	difference
459	1842	2	1842	2	0
460	1848	2	1848	2	0
461	1853	2	1853	2	0
462	1859	2	1859	2	0
463	1865	2	1865	2	0
464	1872	2	1872	2	0
465	1876	2	1876	2	0
466	1881	2	1881	2	0
467	1886	2	1886	2	0
468	1892	2	1892	2	0
469	1897	2	1897	2	0
470	1903	2	1903	2	0
471	1909	2	1909	2	0
472	1916	2	1916	2	0
473	1921	2	1921	2	0
474	1927	2	1927	2	0
475	1933	2	1933	2	0
476	1940	2	1940	2	0
477	1946	2	1946	2	0
478	1953	2	1953	2	0
479	1960	2	1960	2	0
480	1968	2	1968	2	0
481	1976	2	1976	2	0
482	1985	2	1985	2	0
483	1994	2	1994	2	0
484	2004	2	2004	2	0
485	2013	2	2013	2	0
486	2023	2	2023	2	0
487	2033	2	2033	2	0
488	2044	2	2044	2	0
489	2053	2	2053	2	0
490	2063	2	2063	2	0
491	2073	2	2073	2	0
492	2084	2	2084	2	0
493	2094	2	2094	2	0
494	2105	2	2105	2	0
495	2116	2	2116	2	0
496	2128	2	2128	2	0
497	2137	2	2137	2	0
498	2147	2	2147	2	0
499	2157	2	2157	2	0
500	2168	2	2168	2	0
501	2178	2	2178	2	0
502	2189	2	2189	2	0
503	2200	2	2200	2	0
504	2212	2	2212	2	0
505	2222	2	2222	2	0
506	2233	2	2233	2	0
507	2244	2	2244	2	0
508	2256	2	2256	2	0
509	2267	2	2267	2	0
510	2279	2	2279	2	0
511	2291	2	2291	2	0
512	2304	2	2304	2	0

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