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by

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ABSTRACT

The well known and widely used Krylov-Bogoliubov-Mitropolskii (KBM) method in nonlinear mechanics is briefly reviewed for second and third order weakly nonlinear differential equations.

The main contribution of this thesis is the development of a technique based on the KBM asymptotic method for investigations of weakly nonlinear mechanical systems with strong damping modeled by 3-dimensional time dependent differential systems.

The motivation for this study is based on the fact that there are many oscillating processes in physics, mechanics and engineering whose consideration requires the involvment of strong damping effects.

As an illustration the KBM extended technique is applied to an elastic system with internal friction and relaxation under the action of a harmonic force. The resonance curve is sketched and the stability of the stationary regime of oscillations is examined.

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PREFACE

Various oscillating problems encountered at the present time by physicists, engineers and applied mathematicianst give difficulties due to the involvment of a significant (or strong) damping force. Since the differential equations which model physical and mechanical systems are usually nonlinear and generally can not be solved exactly, we must resort to approximate solutions that furnish adequate information about the character of the oscillations. One of the widely used methods for finding approximate solutions of nonlinear differential equations is the perturbation method of Krylov-Bogoliubov-Mitropolskii (KBM) [1,2] which involves asymptotic expansion in powers of a small parameter.

The aim of this thesis is to extend the KBM asymptotic method for the investigation of a weakly nonlinear time dependent mechanical system with strong damping modelled by a 3-dimensional differential system . As an illustration, an application to an elastic system with internal friction and relaxation under the action of a harmonic force is made.

In chapter 1, the KBM method for autonomous and nonautonomous second order differential equations is reviewed. A paper[3] by Lardner and Bojadziev which extends the method used by Osinski [4] to a third order partial differential equation in the autonomous case is outlined. Also some basic results from the Bojadziev paper [5] concerning the damped nonlinear oscillations modelled by a 3-dimensional differential system are given.

In chapter 2 a 3-dimensional weakly nonlinear nonautonomous oscillating mechanical system is studied in general.

In chapter 3, an application is made to a mechanical elastic system with internal friction,relaxation, and small external sinusoidal force.

In chapter 4 the resonance curve is investigated and the stability of the stationary regime of oscillations is examined.

1. The Asymptotic Method of Krylov-Bogoliubov-Mitropolskii

Krylov and Bogoliubov [1] introduced their method first for finding approximate solutions of the differential equation

$$\ddot{x}+\omega^2 x=\varepsilon F(x,\dot{x}) \quad . \tag{1.1}$$

In [1,2] the solution of equation (1.1) is sought in form

$$x= a\cos\psi+ \sum_{k=1}^{\infty} \varepsilon^k u_k(a,\psi) \quad , \tag{1.2}$$

where the functions $u_k(a,\psi)$, k=1,2,... are 2π periodic in and the magnitudes a and ψ are determined by the differential equations

$$\dot{a}= \sum_{k=1}^{\infty} \varepsilon^k A_k(a), \quad \dot{\psi}=\omega+ \sum_{k=1}^{\infty} \varepsilon^k B_k(a) \quad . \tag{1.3}$$

For =0 the solution (1.2) of the eq. (1,1) turns into the solution $x=a\cos(\omega t+\omega_0)$ of the generating equation $\ddot{x}+\omega^2 x=0$. We have to determine the functions

$$u_k(a,\psi), \; A_k(a), \; B_k(a), \; k=1,2,... \tag{1.4}$$

in such a manner that their substitution into (1.2) satisfies the differential equation (1.1).

The rigorous grounds of this asymptotic method is given by Bogoliubov and Mitropiskii [2].

The Taylor development of the right hand side of the eq. (1.1) is

$$\varepsilon F(x,\dot{x})=\varepsilon F(a\cos\psi,-a\omega\sin\psi)+\varepsilon^2[u_1 F'_x(a\cos\psi,-a\omega\sin\psi)$$
$$+(A_1\cos\psi-aB_1\sin\psi+\omega\frac{\partial u_1}{\partial\psi})F'_{\dot{x}}(a\cos\psi,-a\omega\sin\psi)]+\ldots \qquad (1.7)$$

Differentiating (1.2) and making use of equations (1.3), one obtains $\dot{x}$ and $\ddot{x}$. These expressions together with (1.2) can be substituted into equation (1.1) and the terms of equal orders of compared. The zero order terms cancel identically while the terms of order ε and ε^2 give the following equations:

$$\omega^2(\frac{\partial^2 u_1}{\partial\psi^2}+u_1)=F_0(a,\psi)+2\omega A_1\sin\psi+2a\omega B_1\cos\psi, \qquad (1.8)$$

$$\omega^2(\frac{\partial^2 u_2}{\partial\psi^2}+u_2)=F_1(a,\psi)+2\omega A_2\sin\psi+2a\omega B_2\cos\psi, \qquad (1.9)$$

. . . .

where

$$F_0(a,\psi)=F(a\cos\psi,-a\omega\sin\psi) \qquad (1.10)$$

$$F_1(a,\psi)=u_1 F'_x(a\cos\psi,-a\omega\sin\psi)+(A_1\cos\psi-$$

$$aB_1\sin\psi+\omega\frac{\partial u_1}{\partial\psi})F'_{\dot{x}}(a\cos\psi,-a\omega\sin\psi)+$$

$$(a B_1^2 - A_1 \frac{dA_1}{da}) \cos\psi + (2A_1 B_1 + A_1 \frac{dB_1}{da} a) \sin\psi - 2\omega A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} - 2\omega B_1 \frac{\partial^2 u_1}{\partial \psi^2} \tag{1.11}$$

. . . .

To solve the eq. (1.8) , the functions $u_1(a,\psi)$ and $F_0(a,\psi)$ are developed into Fourier series

$$u_1(a,\psi) = v_0(a) + \sum_{n=1}^{\infty} [v_n(a) \cos n\psi + w_n(a) \sin n\psi], \tag{1.12}$$

$$F_0(a,\psi) = g_0(a) + \sum_{n=1}^{\infty} [g_n(a) \cos n\psi + h_n(a) \sin n\psi], \tag{1.13}$$

where

$$g_n(a) = \frac{1}{\pi} \int_0^{2\pi} F_0(a,\psi) \cos n\psi d\psi ,$$

$$h_n(a) = \frac{1}{\pi} \int_0^{2\pi} F_0(a,\psi) \sin n\psi d\psi . \tag{1.14}$$

The coefficients $v_n(a)$ and $w_n(a)$ have to be found.

Substituting the functions (1.12) and (1.13) into the eq. (1.8), one gets

$$\omega^2 v_0(a) + \sum_{n=1}^{\infty} \omega^2 (1-n)^2 [v_n(a) \cos n\psi + w_n(a) \sin n\psi]$$

$$= g_0(a) + [g_1(a) + 2a\omega B_1] \cos\psi + [h_1(a) + 2\omega A_1] \sin\psi$$

$$+ \sum_{n=2}^{\infty} [g_n(a) \cos n\psi + h_n(a) \sin n\psi] .$$

Equating the coefficients of harmonics of the same order, i.e. the coefficients of cos $n\psi$ and sin $n\psi$ $n=1,2,\ldots,$ gives

$$g_1(a)+2aB_1=0,\quad h_1(a)+2\omega A_1=0,\quad V_0(a)=g_0(a)/\omega^2,$$

$$V_n(a)=\frac{g_n(a)}{\omega^2(1-n^2)},\quad W_n(a)=\frac{h_n(a)}{\omega^2(1-n^2)}\qquad n=2,3,\ldots \tag{1.15}$$

Then, from the equations (1.12), (1.14), and (1.15), one gets

$$A_1(a)=-\frac{1}{2\omega\pi}\int_0^{2\pi} F(a\cos\psi,-a\omega\sin\psi)\sin\psi\,d\psi,$$

$$B_1(a)=-\frac{1}{2a\omega\pi}\int_0^{2\pi} F(a\cos\psi,-a\omega\sin\psi)\cos\psi\,d\psi.$$

In a similar manner, one can find the solution of the differential equation (1.9).

The above method can be applied with some additional considerations to the nonautonomous differential equation

$$\ddot{x}+\nu^2x=\varepsilon F(\nu t,x,\dot{x}), \tag{1.16}$$

where the function $F(\nu t,x,\dot{x})$ is 2π periodic in νt and can be represented as a trigonometric polynomial

$$F(\nu t, x, \dot{x}) = \sum_{n=-N}^{N} e^{in\nu t} F_n(x, \dot{x}) \quad . \qquad (1.17)$$

Here the coefficients $F_n(x, \dot{x})$ are polynomials in x and $\dot{x}$.

The differential equation (1.16) describes vibrations of a mechanical system with one degree of freedom with unit mass and free frequency ω which is under the action of a small nonlinear perturb force with frequency ν.

If one applies the method of Krylov and Bogoliubov, for autonomous second order differential equations, starting with the solution $x = a\cos(\omega t + \omega_0)$, one will obtain terms containing $\cos(n\nu + k\omega)t$ and $\sin(n\nu + k\omega)t$ because of (1.17) where k and n are integers.

If at least one of these so called combination frequencies $n\nu + k\omega$ happens to be close enough to the free frequency ω of the system, one can expect that the amplitude will grow as in the case of linear resonance. Because of that, there are nonresonance and resonance vibrations.

Let us consider the differential equation (1.16) under the condition (1.17) but in the resonance case when

$$\omega \approx (p/q)\,\nu \qquad (1.18)$$

holds. Here p and q are integer numbers and relatively prime. According to the Krylov and Bogoliubov [1], the solution of equation (1.16) is sought in the form

$$x = a\cos\alpha + \sum_{k=1}^{\infty} \varepsilon^k u_k(a,\theta,\alpha) \quad , \qquad (1.19)$$

where a and θ are functions of t, $\theta = \nu t$, $\alpha = \frac{p}{q}\theta + \psi$ and the function $u_k(a,\theta,\alpha)$, k=1,2,3... are 2π periodic in and $\frac{p}{q}\theta + \psi$.

In the nonresonance case, as was mentioned, there is no stationary relation between the phase of the external periodic excitation and that of the oscillation. In the resonance case, on the contrary, the terms $A_K(a,\psi)$, $B_K(a,\psi)$ depend not only on a, as previously, but also on ψ, hence the amplitude a and the phase ψ are given by the equations

$$\dot{a} = \sum_{k=1}^{\infty} \varepsilon^k A_k(a,\psi), \quad \dot{\psi} = \omega - \frac{p}{q}\nu + \sum_{k=1}^{\infty} \varepsilon^k B_k(a,\psi) \quad , \qquad (1.20)$$

where the function $A_k(a,\psi)$, $B_k(a,\psi)$, k=1,2,... are 2π periodic in ψ.

For $\varepsilon=0$, from (1.5), we get

$$a = a_0 = \text{constant}, \quad \psi = \omega t - \frac{p}{q}\nu t + \psi_0 ,$$

and (1.19) turns into the generating solution $x = a\cos(\omega t + \omega_0)$.

The difference $\omega - \frac{p}{q}\nu$ generally is approximately zero in the neighborhood of the resonance, but it is possible to be not so small. Such a point of view gives us the possibility for investigation of the resonance zone starting from the nonresonance zone and approaching the resonance zone. As previously, differentiating (1.19) and making use of equations (1.20), one obtains $\dot{x}$ and $\ddot{x}$. The Taylor development of the right-hand side of the equation (1.16) is

$$\varepsilon F(\theta, x, \dot{x}) = \varepsilon F[\theta, a\cos(\frac{p}{q}\theta + \psi), -a\omega\sin(\frac{p}{q}\theta+\psi)] + \varepsilon^2 \ldots \quad . \quad (1.21)$$

After substituting the expressions of $\dot{x}$ and $\ddot{x}$ into (1.16) and equating the coefficient of ε (assume we only need to find the first approximation), one gets

$$\nu^2 \frac{\partial^2 u_1}{\partial\theta^2} + 2\nu\omega \frac{\partial^2 u_1}{\partial\theta\,\partial(\frac{p}{q}\theta+\psi)} + \omega^2 \frac{\partial^2 u_1}{\partial(\frac{p}{q}\theta+\psi)^2} + \omega^2 u_1$$

$$= F_0(a, \theta, \frac{p}{q}\theta+\psi) - [(\omega - \frac{p}{q}\nu)\frac{\partial A_1}{\partial\psi} - 2a\omega B_1]\cos(\frac{p}{q}\theta+\psi) \qquad (1.22)$$

$$+ [(\omega - \frac{p}{q}\nu)\, a\frac{\partial B_1}{\partial\psi} + 2\omega A_1]\sin(\frac{p}{q}\theta+\psi) \quad ,$$

where

$$F_0(a, \theta, \frac{p}{q}\theta+\psi) = F[\theta, a\cos(\frac{p}{q}\theta+\psi), -a\omega\sin(\frac{p}{q}\theta+\psi)] \quad . \qquad (1.23)$$

One seeks the solution of (1.22) in the form of a Fourier series

$$u_1(a,\theta,\frac{p}{q}\theta+\psi)=\sum_{n\,k} u^{(1)}_{nk}(a)\,e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}, \tag{1.24}$$

where the coefficients $u^{(1)}_{n,k}(a)$ have to be found.

The development of the function (1.23) in a sum of the same type is

$$F_0(a,\theta,\frac{p}{q}\theta+\psi)=\sum_{n\,k} F^{(0)}_{nk}(a)\,e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}, \tag{1.25}$$

where

$$F^{(0)}_{n,k}=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_0(a,\theta,\frac{p}{q}\theta+\psi)\,e^{-i[n\theta+k(\frac{p}{q}\theta+\psi)]}\,d\theta\, d(\frac{p}{q}\theta+\psi)\,. \tag{1.26}$$

Putting (1.24) and (1.25) into (1.22), one obtains

$$\sum_{n\,k}[\omega^2-(n\nu+k\omega)^2]u^{(1)}_{nk}(a)\,e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}$$

$$=\sum_{n\,k} F^{(0)}_{n,k}(a)\,e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}-[(\omega-\frac{p}{q}\nu)\frac{\partial A_1}{\partial\psi}-2a\omega B_1]\cos(\frac{p}{q}\theta+\psi)+[(\omega-\frac{p}{q}\nu)a\frac{\partial B_1}{\partial\psi}+2\omega A_1]\sin(\frac{p}{q}\theta+\psi)\,. \tag{1.27}$$

Comparing the coefficients of $e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}$ for which $\omega^2-(n\nu+k\omega)^2\neq 0$, i.e. $n\neq 0$ and $k\neq\pm 1$, we have

$$[\omega^2-(n\nu+k\omega)^2]\,u^{(1)}_{n,k}(a)=F^{(0)}_{nk}(a)$$

or

$$u^{(1)}_{n,k}(a)=\frac{1}{4\pi^2[\omega^2-(n\nu+k\omega)^2]}\,.$$

$$\int_0^{2\pi}\int_0^{2\pi}F_0(a,\theta,\frac{p}{q}\theta+\psi)\,e^{-i[n\theta+k(\frac{p}{q}\theta+\psi)]}\,d\theta\,d(\frac{p}{q}\theta+\psi)\,.$$

(1.28)

Instead of the condition $\omega^2-(n\nu+k\omega)^2\neq 0$ or $(k\mp 1)\omega+n\nu\neq 0$, we can take the nonfulfillment of the condition

$$p(k\mp 1)+nq=0\quad. \tag{1.29}$$

Therefore, the formula (1.28) gives us all coefficients of the sum (1.24) with subscripts which do not fulfill the condition (1.29) We set the remaining coefficients $u^{(1)}_{n,k}(a)$ with subscripts which satify (1.29) to be zero, i.e.

$$u^{(1)}_{n,k}(a)=0,\quad \text{for}\ (k\mp 1)\omega+n\nu=0\quad. \tag{1.30}$$

This means that there be no first harmonics in the function $u_1(a,\theta,\frac{p}{q}\theta+\psi)$ of the argument $\frac{p}{q}\theta+\psi$. Really for (1.30), we have

$$n\theta+k(\frac{p}{q}\theta+\psi)=(n+k\frac{p}{q})\theta+k\psi=\pm\frac{p}{q}\theta+k\psi$$

$$=\pm(\frac{p}{q}\theta+\psi)\mp\psi\ +k\psi=\pm(\frac{p}{q}\theta+\psi)+(k\mp 1)\psi\,.$$

Then $u^{(1)}_{n,k}(a)$ are coefficients of

$$e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]} = [\cos(\frac{p}{q}\theta+\psi) \pm i\sin(\frac{p}{q}\theta+\psi)]e^{i(k\mp1)\psi}$$

$$= e^{\pm i(\frac{p}{q}\theta+\psi)} e^{i(k\mp1)\psi} . \qquad (1.31)$$

Taking into account (1.31), we find from (1.22)

$$[(\omega-\frac{p}{q}\nu)\frac{\partial A_1}{\partial\psi}-2a\omega B_1]\cos(\frac{p}{q}\theta+\psi)-[(\omega-\frac{p}{q}\nu)a\frac{\partial B_1}{\partial\psi}+2\omega A_1]\sin(\frac{p}{q}\theta+\psi)$$

$$= \sum_{nq+p(k\ \ 1)=0} F^{(0)}_{nk}(a)\, e^{\pm i(\frac{p}{q}\theta+\psi)} e^{i(k\pm1)\psi} . \qquad (1.32)$$

One can determine A_1 and B_1 in the following manner:

(i) neglect in (1.32) the terms with the factor $\omega-\frac{p}{q}\nu$ because in the resonance case this difference is small enough;

(ii) express $\cos(\frac{p}{q}\theta+\psi)$ and $\sin(\frac{p}{q}\theta+\psi)$ by the Euler's formulas and compare the coefficients of $e^{i(\frac{p}{q}\theta+\psi)}$ or $e^{-i(\frac{p}{q}\theta+\psi)}$;

(iii) into the equality in complex form obtained, compare correspondingly the real and the imaginary parts.

Very often in the applications, it is more convenient for one to work directly or to use (1.28).

In recent years the KBM method has been used by Bojadziev and Lardner [6-8] and Osinski [4, 12] to find the nonfrequent solutions of hyperbolic partial differential equations with small nonlinearities. Now we review some results by Lardner and

Bojadziev [3] with regard to partial differential equations involving third order time derivatives.

Consider the partial differential equations.

$$u_{ttt}+k_1 u_{tt}-k_2 u_{xxt}-k_3 u_{xx}=\varepsilon F(x,u,u_x,u_t,\ldots,\varepsilon) \quad , \qquad (1.33)$$

where subscripts of x and t are used to denote the corresponding partial derivatives and the dots in the function F on the right hand side indicate that higher derivatives of u may occur in this function, k_1, k_2 and k_3 are any real constants, and ε is a positive parameter assumed to be sufficiently small so that the nonlinear right hand side of (1.36) may be treated as a perturbation.

The generating equation obtained by setting $\varepsilon=0$ in (1.33) has separable solutions of the form

$$u_i^{(0)}(x,t)=[g_i e^{-\eta_i t}+h_i e^{-\zeta_i t}\cos(\omega_i t+r_i)]\phi_i(x) \quad , \qquad (1.34)$$

where g_i, h_i, and r_i are arbitrary constants, $\phi_i(x)$ satisfies the differential equation

$$\phi_i''(x)+\mu_i\phi_i(x)=0 \quad . \qquad (1.35)$$

and $(-\eta_i, -\zeta_i+i\omega_i)$ are the three roots of the characteristic cubic equation

$$p^3+k_1 p^2+\mu_i k_2 p+\mu_i k_3=0. \qquad (1.36)$$

The coefficients of (1.36) satisfy the following identities,

$$k_1=\zeta_i+2\varsigma_i,\ \mu_i k_2=\zeta_i^2+\omega_i^2+2\zeta_i\varsigma_i,\ \mu_i k_3=\zeta_i(\varsigma_i^2+\omega_i^2) \qquad . (1.37)$$

In order to solve an initial value problem for the generating equation, we would seek the solution in the form of a sum of separable solutions:

$$u^{(0)}(x,t)=\sum_i[g_i e^{-\zeta_i t}+h_i e^{-\varsigma_i t}\cos(\omega_i t+r_i)]\,\phi_i(x) \quad . \qquad (1.38)$$

By virtue of the completeness of the set of eigenfunctions, such a solution can meet the initial condtions that u, u_t, u_{tt} are prescribed at t=0, and the orthogonality condition

$$\int_0^1 \phi_i(x)\phi_j(x)\,dx=\delta_{ij} \quad (\delta_{ij} \text{ is the Kronecker delta })$$

enables simple expressions to be extracted for the three sets of coefficients $\{g_i, h_i, r_i\}$

Consider the nonlinear equation (1.33)', and suppose initially that we wish to find the <single mode> solution $u(x,t)$ corresponding to the ith mode (1.34) of the generating equation.

We would seek u_i in the form

$$u_i(x,t) = (a_i + b_i \cos\psi_i)\phi_i(x) + \varepsilon u^{(1)}(x, a_i, b_i, \psi_i) + \varepsilon^2 \ldots, \qquad (1.39)$$

where a_i, b_i, and ψ_i, are functions of t satisfying the differential equations

$$\dot{a}_i = -\zeta_i a + \varepsilon A_i(a_i, b_i) + \varepsilon^2 \ldots,$$

$$\dot{b} = -\zeta_i b_i + \varepsilon B_i(a_i, b_i) + \varepsilon^2 \ldots, \qquad (1.40)$$

$$\dot{\psi}_i = \omega_i + \varepsilon C_i(a_i, b_i) + \varepsilon^2 \ldots .$$

Here dots denote derivatives with respect to t. Allowing $\varepsilon \to 0$ in (1.39) and (1.40), we see that the solution u_i degenerates into $u_i^{(0)}$ as required. In the case of a general initial value problem, we must seek the solution as a sum of modes of the form (1.39) in the same way as in equation (1.38) for the generating equation. It is also more convenient [8] if we change variables from a ,b to α_i, β_i defined in the form

$a_i = e^{-\zeta_i \alpha_i}$, $b_i = e^{-\zeta_i \beta_i}$. Then the solution is sought in the form

$$u(x,t) = \sum_m [e^{-\zeta_m \alpha_m} + Re(e^{-\zeta_m \beta_m + i\psi_m})]\phi_m(x) + \varepsilon u^{(1)}(x, \alpha_m, \beta_m, \psi_m) + \varepsilon^2 . \quad (1.41)$$

where

$$\dot{\alpha}_u = 1 + \varepsilon P_u + \varepsilon^2 \ldots \quad , \quad \dot{\beta}_u = 1 + \varepsilon Q_u + \varepsilon^2 \ldots ,$$
$$\dot{\psi}_u = \omega_u + \varepsilon R_u + \varepsilon^2 \ldots \qquad . \qquad (1.42)$$

In order to take fully into account the interaction between the different modes, the quantities P ,Q and R occurring in these differential equations must be allowed to depend on all the variables $\{\alpha_\gamma, \beta_\ell\}$.

Now a survey of a paper by Bojadziev [5] concerning the three dimensional autonomous differential [1]:

$$\dot{X} = AX + \varepsilon f(x) \quad , \quad \dot{X} = dX/dt \qquad (1.43)$$

is presented.

Here ε is a small positive parameter ,$X=(x_1, x_2, x_3)$ is a vector, $f(x)=(f_1(x), f_2(x), f_3(x))$ is a real vector function in a domain G with sufficient number of derivatives in G, and $f(0)=0$. It is assumed that the real 3x3 constant matrix $A=(a_{jk})$ has one real non-positive eigenvalue

$-\zeta$ ($\zeta>0$) and two complex eigenvalues $-\xi+i\omega$ with a nonpositive real part $-\xi$ ($\xi>0$).

The strong linear damping force in the system is represented by the real parts of the eigenvalues $-\zeta$ and $-\xi$

The asymptotic solution of (1.43) is sought in the form

$$x(t,\varepsilon)=\varphi a+b[\phi e^{i\psi}+\phi^{*}e^{-i\psi}]+\varepsilon u(a,b,\psi)+\varepsilon^{2}\ldots, \tag{1.44}$$

where the unknown vector functions $u=(u_1,u_2,u_3)$ is 2π periodic in ψ. The scalar variables a, b and ψ are functions of t satisfying the differential equations

$$\dot{a}=-\zeta a+\varepsilon A(a,b)+\varepsilon^{2}\ldots,$$

$$\dot{b}=-\xi b+\varepsilon B(a,b)+\varepsilon^{2}\ldots, \tag{1.45}$$

$$\dot{\psi}=\omega+\varepsilon C(a,b)+\varepsilon^{2}\ldots.$$

For $\varepsilon=0$, the solution (1.44) reduces to the solution of the linear system of (1.43) with eigenvectors $\varphi=(\varphi_1,\varphi_2,\varphi_3)$ and $\phi=(\phi_1,\phi_2,\phi_3)$ corresponding to the eigenvalues and $-\xi+i\omega$; $\phi^{*}=(\phi_1^{*},\phi_2^{*},\phi_3^{*})$ is the conjugate of ϕ. The unknown functions $u(a,b,\psi)$, $A(a,b)$, $B(a,b)$, and $C(a,b)$ in (1.44) and (1.45) are to be determined from the condition that expression (1.44) satisfies (1.43) to each order of ε.

The quantities A, B, and C are assumed to depend on both amplitudes, a and b, in order to take fully into account their mutual interaction.

The first approximate KBM solution is obtained by truncating (1.44) and (1.45) after the first two terms. Therefore, u is not included into the first approximation. However, it is necessary to keep it in the first improved approximation. Differentiating (1.44) and using (1.45), one gets x. This expression together with (1.44) is substituted into (1.43) and the terms of equal orders of are compared. The zero order terms cancel identically while the terms of order ε give the following vector equation for u :

$$-\zeta a\frac{\partial u}{\partial a}-\zeta b\frac{\partial u}{\partial b}+\omega u+\phi A+D[\phi e^{i(\psi+\delta)}+\phi^{*} e^{-i(\psi+\delta)}]$$

$$=Au+f(x^{(0)}) \quad , \qquad (1.46)$$

$$x^{(0)}=x(t,0) \quad ,$$

where

$$-\zeta+i\omega=\rho e^{i\gamma} \quad ,B(a,b) + ibC(a,b) = D(a,b)e^{i\delta} \quad . \qquad (1.47)$$

The solution of Eq. (1.47) is sought in Fourier series

$$u(a,b,\psi) = \sum_{n=-\infty}^{\infty} U_n(a,b)e^{in\psi} \quad ,$$

$$(1.48)$$

$$U_n = (U_{n1} ,U_{n2} ,U_{n3})^T \quad .$$

The known function $f(x^{(0)})$ is also expanded as a Fourier series

$$f(x^{(0)}) = \sum_{n=-\infty}^{\infty} F_n(a,b)e^{in\psi} , \qquad (1.49)$$

where

$$F_n = (F_{n1}, F_{n2}, F_{n3}) = \frac{1}{2\pi}\int_0^{2\pi} f(x^{(0)})e^{-in\psi}\, d\psi \quad .$$

Substituting expression (1.48) and (1.49) into (1.46) gives

$$\sum_{n=-\infty}^{\infty}\{-\zeta a\frac{\partial U_n}{\partial a}-\zeta b\frac{\partial U_n}{\partial b}+in\omega U_n\}e^{in\psi}+\phi A+D[\phi e^{i(\psi+\delta)}+\phi^* e^{-i(\psi+\delta)}]$$

$$= A\sum_{n=-\infty}^{\infty} U_n e^{in\psi} + \sum_{n=-\infty}^{\infty} F_n e^{in\psi} \quad . \qquad (1.50)$$

Comparing the coefficients of the terms $e^{in\psi}$ provides differential equations for U_n. In order to calculate A, B, and C, only the equations $U_0 = (U_{01}, U_{02}, U_{03})$ and $U_1 = (U_{11}, U_{12}, U_{13})$ are needed:

$$\zeta a\frac{\partial U_0}{\partial a}+\zeta b\frac{\partial U_0}{\partial b}+AU_0 = A-F_0 , \qquad (1.51)$$

$$\zeta a\frac{\partial U_1}{\partial a}+\zeta b\frac{\partial U_1}{\partial b}-i\omega U_1+AU_1=\phi De^{i\delta}-F_1 \quad . \qquad (1.52)$$

As it is costumary in the KBM method, here is assumed that the function u_1 does not contain terms proportional to $\exp(\pm i\psi)$, that is $U_{11}=0$. In addition, it is assumed that u_1 does not contain a term without the exponential factor, that is $U_{01}=0$. Using these assumptions and the substitutions $a = \exp\alpha$ and $b = \exp\beta$, equations (1.50) and (1.51) reduce to partial differential equations with constant coefficients:

$$\gamma \frac{\partial U_0}{\partial \alpha} + \zeta \frac{\partial U_0}{\partial \beta} + AU_0 = \phi A - F_0 \quad , U_{01} = 0 \quad , \tag{1.53}$$

$$\gamma \frac{\partial U_1}{\partial \alpha} + \zeta \frac{\partial U_1}{\partial \beta} - i\omega U_1 + AU_1 = De^{i\delta} - F_1 \quad , U_{11} = 0 \quad . \tag{1.54}$$

The vector equation (1.53) can be solved for U_{02}, U_{03}, and A. The vector equation (1.54) analogically gives U_{12}, U_{13} and D ; hence according to (1.47) B and C . The formulas for A, B, and C are not presented here since their practical significance is restricted. It is more convenient, for each specific problem, equations (1.52) and (1.54) to be solved directly. Only the particular solutions of the nonhomogeneous equations (1.53) and (1.54) are to be considered. Note that A,B, and C take part in equation (1.45) whose solutions give the amplitudes a and b and the phase ψ, and involve three constants of integration, say a_0, b_0, and ψ_0. The initial condition $X(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0))$ for (1.43) generate the equations for obtaining a ,b ,and ,thus the initial value problem for (1.43) can be solved. Addition constants of integration are selected to be zero. In general, equation (1.45) are to be integrated numerically.

2. Asymptotic Solutions of Time Dependent Systems

In chapter 1, we have introduced Bojadziev's results [5] with regard to a 3-dimensional weakly nonlinear autonomous oscillating mechanical system characterized by strong damping. The aim of this thesis is to extend the results of [5] to a time dependent system of the type

$$\dot{X}=AX+\varepsilon F(\theta,x), \qquad \dot{X}=dX/dt, \qquad \theta=\nu t \quad , \tag{2.1}$$

where is a small positive parameter, ν is the frequency of the external acting force, $X=(x^{(1)}, x^{(2)}, x^{(3)})$ is a vector, $F(\theta,x)=(F^{(1)}(\theta,x), F^{(2)}(\theta,x), F^{(3)}(\theta,x))^T$ is a real vector function 2π periodic in θ with sufficient number of derivatives with respect to all the arguments in a domain G, and $F(0,0)=0$. The 3x3 real constant matrix $A=(a_{jk})$ as in (1.43) is assumed to have one nonpositive eigenvalue $-\zeta$ $(\zeta\geq 0)$ and two complex eigenvalues $-\xi\pm i\omega$ with a nonpositive real part $-\xi$ $(\xi\geq 0)$. The quantities $-\zeta$ and $-\xi$ represent the strong damping force in the system (2.1). The function $F(\theta,x)$ is a finite sum of the form $\sum_n \exp(in\theta) F_n(x)$, where $F_n(x)$ are polynomials in x.

We seek the asymptotic solution of (2.1) in vector form

$$X(t,\varepsilon)=\varphi a+b[\phi e^{i\alpha}+\phi^* e^{-i\alpha}]+\varepsilon u(a,b,\theta,\alpha)+\varepsilon^2\ldots, \tag{2.2}$$

where $u=(u^{(1)}, u^{(2)}, u^{(3)})^T$ is an unknown 2π periodic function in θ and $\alpha = (p/q)\theta+\psi$, p and q are integers.
The scalar variables a, b, and are functions of t to be determined by the differential equations

$$\dot{a}=-\zeta a+\varepsilon A(a,b,\psi)+\varepsilon^2+\dots,$$
$$\dot{b}=-\xi b+\varepsilon B(a,b,\psi)+\varepsilon^2+\dots, \qquad (2.3)$$
$$\dot{\psi}=\omega-(p/q)\nu+\varepsilon C(a,b,\psi)+\varepsilon^2+\dots .$$

Actually, (2.3) are extension of formulas (1.20) and (1.45). For $\varepsilon=0$, expression (2.2) with (2.3) gives the solution of the linear system of (2.1)

$$x(t,0)=\varphi a_0 e^{-\zeta t}+b_0 e^{-\xi t}[\phi e^{i(\omega t+\psi_0)}+\phi^* e^{-i(\omega t+\psi_0)}],$$

where a_0, b_0, and ψ_0 are constants, $\varphi=(\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)})^T$, $\phi=(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})^T$, and $\phi^*=(\phi^{*(1)}, \phi^{*(2)}, \phi^{*(3)})^T$ are eigenvectors corresponding respectively to the eigenvalues $-\zeta$, $-\xi+i\omega$, and $-\xi-i\omega$; ϕ^* is the conjugate value of ϕ.

Following chapter 1, formula (1.18), we define the state of external resonance in the neighborhood of the natural frequency ν with the fulfillment of the equality

$$\omega - (p/q)\nu = O(\varepsilon), \qquad (2.4)$$

where p and q are mutually prime integers. If p=q=1, the resonance is called **main resonance** (or **prinicipal resonance**). If $O(\varepsilon)=0$, we have exact resonance. Practically, resonance is possible only for few values of p and q since the function $F(\theta, x)$ in (2.1) has a finite number of terms. If the difference $\omega - (p/q)\nu$ in equation (2.4) is not small, we have the nonresonance case. As $\omega - (p/q)\nu$ gets smaller and smaller, we have the possibility to study the phase ψ in the passing from nonresonance case to resonance case.

The functions A, B, and C are assumed to depend on both amplitudes, a and b, in order to take fully into consideration their mutual interaction. They also depend on the phase ψ to allow consideration in the resonance case.

Using and extending the KBM techinque, we will find the unknown functions u, A, B, and C involved in (2.2) and (2.3).

First from (2.2) with (2.3), we calculate X:

$$\dot{X} = \{\frac{da}{dt} + \frac{db}{dt}[\phi e^{i\alpha} + \phi^{*} e^{-i\alpha}] + b[\phi e^{i\alpha} - \phi^{*} e^{-i\alpha}]\frac{d\alpha}{dt} + [\frac{\partial u}{\partial a}\frac{da}{dt} + \frac{\partial u}{\partial b}\frac{db}{dt} + \frac{\partial u}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial u}{\partial \alpha}\frac{d\alpha}{dt}] + \varepsilon^{2} \ldots$$

$$= \{[-\zeta a + \varepsilon A + ..] + [-\xi b + \varepsilon B + ..]\}[\phi e^{i\alpha} + \phi^{*} e^{-i\alpha}] + b[\phi e^{i\alpha} + \phi^{*} e^{-i\alpha}][\omega + \varepsilon C + ..] + \varepsilon[\frac{\partial u}{\partial a}(-\zeta a + \varepsilon A + ..) + \frac{\partial u}{\partial b}(-\xi b + \varepsilon B + ..) \qquad (2.5)$$

$$+\frac{\partial u}{\partial \theta}\nu)+\frac{\partial u}{\partial \alpha}(\omega+\varepsilon C+\dots)]+\varepsilon^2\dots$$

$$= -\phi\zeta a+\rho b[\phi e^{i(\alpha+r)}+\phi^* e^{-i(\alpha+r)}]+\varepsilon\{\phi A+D[\phi e^{i(\alpha+\delta)}+$$

$$\phi^* e^{-i(\alpha+\delta)}]-\zeta a\frac{\partial u}{\partial a}-\zeta b\frac{\partial u}{\partial b}+\nu\frac{\partial u}{\partial \theta}+\omega\frac{\partial u}{\partial \alpha}\}+\varepsilon^2\dots$$

where

$$-\zeta+i\omega=\rho e^{i\gamma},$$

$$B(a,b,\psi)+ibC(a,b,\psi)=D(a,b,\psi)e^{i\delta(a,b,\psi)} \tag{2.6}$$

Substituting (2.2) and (2.5) into (2.1), we get

$$-\phi\zeta a+\rho b[\phi e^{i(\alpha+r)}+\phi^* e^{-i(\alpha+r)}]+\varepsilon\{\phi A+D[\phi e^{i(\alpha+\delta)}+\phi^{*} e^{-i(\alpha+\delta)}]$$

$$-\zeta a\frac{\partial u}{\partial a}-\zeta b\frac{\partial u}{\partial b}+\nu\frac{\partial u}{\partial \theta}+\omega\frac{\partial u}{\partial \alpha}\}+\varepsilon^2+\dots$$

$$=A\{\phi a+b[\phi e^{i\alpha}+\phi^* e^{-i\alpha}]+\varepsilon u(a,b,\theta,\alpha)+\dots\}+\varepsilon F(\theta,x).$$

Comparing the terms of same order ε gives a vector equation for u:

$$-\zeta a\frac{\partial u}{\partial a}-\zeta b\frac{\partial u}{\partial b}+\nu\frac{\partial u}{\partial \theta}+\omega\frac{\partial u}{\partial \alpha}+\phi A+D[\phi e^{i(\alpha+\delta)}+\phi^* e^{-i(\alpha+\delta)}]$$

$$=Au+F(\theta,x_0) \text{ and,} \tag{2.7}$$

$$x_0=x(t,0)=\phi a+b[\phi e^{i\alpha}+\phi^* e^{-i\alpha}].$$

Further, we expand the function $F(\theta,x_0)$ in a double Fourier sum

$$F(\theta, x_0) = \sum_n e^{in\theta} \mathcal{F}_n(x_0) = \sum_s F_{nk}(a,b)\, e^{i(n\theta+k\alpha)}, \tag{2.8}$$

where

$$F_{nk} = (F^{(1)}_{nk}, F^{(2)}_{nk}, F^{(3)}_{nk})^T, \quad F_{-n,-k} = F^*_{nk}, \text{ and}$$

$$F_{nk}(a,b) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F(\theta, x_0)\, e^{-i(n\theta+k\alpha)}\, d\theta\, d\alpha \tag{2.9}$$

are polynomials in a and b, and s is the set $\{n,k\}$.

Then we seek the solution of (2.7) in the form of a double Fourier sum

$$u(a,b,\theta,\alpha) = \sum_s U_{nk}(a,b)\, e^{i(n\theta+k\alpha)}, \tag{2.10}$$

where $U_{nk} = (U^{(1)}_{nk}, U^{(2)}_{nk}, U^{(3)}_{nk})$ and $U_{-n,-k} = U^*_{nk}$.

Substitution (2.8) and (2.10) into (2.7) gives

$$\sum_s \left\{-\zeta a \frac{\partial U_{nk}}{\partial a} - \zeta b \frac{\partial U_{nk}}{\partial b} - [A - i(n\nu + k\omega) I]\, U_{nk}\right\} e^{i(n\theta+k\alpha)}$$
$$= -\phi A - D[\phi e^{i\delta} e^{i\alpha} + \phi^* e^{-i\delta} e^{-i\alpha}] + \sum_s F_{nk} e^{i(n\theta+k\alpha)}, \tag{2.11}$$

where I is the 3x3 identity matrix.

To facilitate our study, we express the resonance condition (2.4) into the equivalent form $n\nu + k\omega = \pm\omega + O(\varepsilon)$, where n and k are integers. The pairs p, q, and n, k are connected with the relationship $qn + p(k \mp 1) = 0$. The set of integers $s = \{n,k\}$ which appears in (2.8), (2.10), and (2.11) can be presented as a sum of the two sets $S_1 = \{(n,k)\ nq + (k \mp 1)p \neq 0\}$ and $S_2 = \{(n,k)\ nq + (k \mp 1)p = 0\}$.

The comparison of the coefficients of the corresponding exponential terms $\exp[i(n\nu+k\omega)]$ in (2.11) for $(n,k)\in s_1$ gives the partial differential equations for the vectors U

$$-\zeta a\frac{\partial U_{nk}}{\partial a}-\xi b\frac{\partial U_{nk}}{\partial b}-[A-i(n\nu+k\omega)I]U_{nk}=F_{nk}, \quad n,k\neq 0 \tag{2.12}$$

and

$$-\zeta a\frac{\partial U_{00}}{\partial a}-\xi b\frac{\partial U_{00}}{\partial b}-U_{00}=-\phi A+F_{00}. \tag{2.13}$$

Making use of the equality

$$e^{i(n\theta+k\alpha)}=e^{i[n\theta+k(\frac{p}{q}\theta+\psi)]}=e^{i[(n+k\frac{p}{q})\theta+k\psi]}$$

$$=e^{i[\pm\frac{p}{q}\theta+k\psi]}=e^{i[\pm(\frac{p}{q}\theta+\psi)\mp\psi+k\psi]}$$

$$=e^{i[\pm(\frac{p}{q}\theta+\psi)+(k\mp 1)\psi]}=e^{\pm i\alpha}e^{i(k\mp 1)\psi}$$

$$(n,k)\in s_2,$$

the comparison of the coefficients of $\exp(\pm i\alpha)$ in (2.11) gives two equations combined as follow

$$\sum_{s_2^{\mp}}\{-\zeta a\frac{\partial U_{nk}}{\partial a}-\xi b\frac{\partial U_{nk}}{\partial b}-[A\mp i\omega I]U_{nk}\}e^{i(k\mp 1)\psi} \tag{2.14}$$

$$=-D\phi^{\pm}e^{\pm i\delta}+\sum_{s^{\mp}}F_{nk}e^{i(k\mp 1)\psi}, \quad \phi^{+}=\phi, \quad \phi^{-}=\phi^{*},$$

where S_2^- is the set S_2 with the upper sign, and S_2^+ with the lower sign. However, it is enough to consider the equation

(2.14) for the upper sign since the other equation gives the conjugate values of the unknown.

Equations (2.13) and (2.14) besides U_{00} and U_{nk}, $(n,k) \in s_2$, also involve the scalar functions A and D; hence, we have more variables than equations. The determination of these functions requires, as it is customary in the KBM method, the impositions of certain additional conditions on the functions U_{00} and U_{nk}, or on u, according to (2.10). Here we assume:

$$U^{(1)}_{00} = 0 \qquad (2.15)$$

which indicates that the scalar function $u^{(1)}$ does not contain a term without an exponential factor; and

$$U_{nk} = 0, \quad (n,k) \in s_2, \quad n \neq 0, \quad k \neq \pm 1; \quad U^{(1)}_{01} = U^{(1)}_{0,-1} = 0, \qquad (2.16)$$

which shows that u contains no terms $\exp(\pm i\alpha)\exp[i(k\mp 1)\psi]$, $n \neq 0$, $k \neq \pm 1$, and in addition, u_1 contains no such terms even for $n=0$, $k=\pm 1$.

We note that only the particular solutions U_{nk} of equations (2.12), (2.13) and (2.14) are to be considered. The functions A, B, and C take part in equations (2.3) whose solutions will give the amplitude a and b and the phase ψ, and will involve three constants of integration. The initial condition $X(t_0)$ for (2.1) generates the equations for obtaining these three constants which solve the initial vlaue problem for (2.1). Addition constants of integration are selected to be zero.

Since $F_{\alpha K}(a,b)$ are polynomials in a and b, we can seek the solution $U_{\alpha K}(a,b)$ of (2.12) in a polynomial form. The vector equation (2.13) is equivalent to three scalar equations involving $U^{(2)}_{oo}$, $U^{(3)}_{oo}$, and λ (according to (2.15), $U^{(1)}_{oo}=0$). The vector equation (2.14) with the assumption (2.10) gives scalar equations for $U^{(2)}_{o1}$, $U^{(3)}_{o1}$, and D. The process of solving these partial differential equations is shown in the next chapter which deals with a specific mechanical system.

3. A Mechanical Time Dependent System with Internal Friction and Relaxation

Consider an oscillating weakly nonlinear mechanical elastic system with internal friction, relaxation, and small external sinusoidal force whose motion is modelled by the following scalar equations

$$m\ddot{x}+\sigma=\varepsilon F\sin\theta \quad , \quad \theta=\nu t$$

$$\dot{\sigma}+c\sigma=dx+e\dot{x}+\varepsilon sx^3 \quad , \quad \varepsilon\ll 1 \qquad (3.1)$$

Here, x is the deformation, σ is the stress, m is the mass of the system, and c, d, e, and s are positive constants. The terms with coefficients d and s represent respectively the linear and nonlinear elasticity, the term with coefficient e corresponds to the linear viscous damping, and the term with coefficient c reflects the linear relaxation.

The influence of relaxation on the oscillating system, especially in plastic materials, sometimes may be significant. Osinski [8] has studied the system (3.1) by transforming it to a third order differential equation. He has applied the KBM method assuming the rather strong restriction that in Eqs. (2.3) there is no coupling of the amplitudes a and b , i.e., A depends on a alone. He has also considered the particular case of two purely imaginary eigenvalues of the characteristic polynomial and applied a perturbation procedure.

Using the new variables $x=x^{(1)}$, $\dot{x}=x^{(2)}$, and $\alpha=x^{(3)}$, transforms (3.1) to

$$\begin{aligned}
\dot{x}^{(1)} &= x^{(2)} \\
\dot{x}^{(2)} &= -m^{-1} x^{(3)} + m^{-1} E \sin\theta \\
\dot{x}^{(3)} &= dc^{-1} x^{(1)} + ec^{-1} x^{(2)} - c^{-1} x^{(3)} + sc^{-1} (x^{(1)})^3 .
\end{aligned} \tag{3.2}$$

with matrix which is a particular case of (2.1)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -m^{-1} \\ dc^{-1} & ec^{-1} & -c^{-1} \end{pmatrix} \tag{3.3}$$

and nonlinear part

$$F(\theta,x) = (0,0,h(x^{(1)})^3)^T + (0,m^{-1} E\sin\theta,0)^T, \quad h=sc^{-1} \tag{3.4}$$

We will consider the main resonance case $p=q=1$.

Of course the assumption made in chapter 2 with regard to the eigenvalues of A holds for (3.2). This assumption yields for a system characterized with dissipation of energy.

The characteristic polynomial of (3.2) is

$$p(\lambda) = \det|A-\lambda I| = \lambda^3 + c^{-1}\lambda^2 + e(mc)^{-1}\lambda + d(mc)^{-1} .$$

The equation $p(\lambda)=0$ has at least one real negative root $\lambda_1=-\zeta$ ($\zeta>0$) since its coefficients are positive (c, d, e, and m are positive). If $(e/c)>d$, then the other two roots are negative or have a negative real part. The condition for oscillatory solution, i.e. , $p(\lambda)=0$ has a pair of complex roots $\lambda_{2,3}=-\xi\pm i\omega$ ($\xi>0$) is more complicated:

$$\frac{1}{4c^2}\left(\frac{2}{27c^2}-\frac{e}{3cm}+\frac{d}{m}\right)^2+\frac{1}{27c^3}\left(\frac{e}{m}-\frac{1}{3c}\right)^3>0 \tag{3.5}$$

We choose the following eigenvectors of the matrix (3.3)

$$\varphi=(1,-\zeta,-m\zeta^2)^T,\quad \phi=\left(\frac{1}{2},-\frac{(\xi-i\omega)}{2},-m\frac{(\xi-i\omega)^2}{2}\right)^T,\quad {}^* \tag{3.6}$$

We find the Fourier expansion of $F(\theta,x_0)$ directly from (3.4), x_0 is given in (2.7), instead of using (2.8) and (2.9). And from (3.4), one gets $F_1(\theta,x_0)=0$,

$F_2(\theta,x_0)=-im^{-1}\frac{E}{2}(e^{i\theta}-e^{-i\theta})$, and $F_3(\theta,x_0)=h[a^3+\frac{3}{2}a^2b(e^{i\alpha}+e^{-i\alpha})$

$+3a(\frac{b}{2})^2(e^{2i\alpha}+2+e^{-2i\alpha})+(\frac{b}{2})^3(e^{3i\alpha}+3e^{i\alpha}+3e^{-i\alpha}+e^{-3i\alpha})]$.

In vector form, we have

$$F(\theta,x_0)=\begin{pmatrix}0\\0\\ha(a^2+\frac{3}{2}b^2)\end{pmatrix}-i/2\begin{pmatrix}0\\m^{-1}E\\0\end{pmatrix}(e^{i\theta}-e^{-i\theta})$$

$$
+\begin{pmatrix} 0 \\ 0 \\ \frac{3}{2}hb\left(a^2+\frac{b^2}{4}\right) \end{pmatrix}\left(e^{i\alpha}+e^{-i\alpha}\right)+\begin{pmatrix} 0 \\ 0 \\ \frac{3}{4}hab^2 \end{pmatrix}\left(e^{2i\alpha}+e^{-2i\alpha}\right)+
$$

$$
\begin{pmatrix} 0 \\ 0 \\ \frac{h}{8}b^3 \end{pmatrix}\left(e^{3i\alpha}+e^{-3i\alpha}\right) . \tag{3.7}
$$

The value A can be determined from (2.13) which with (2.15) and (3.5) in scalar form reduces to

$$
\begin{aligned}
&A=U^{(2)}_{00}\ , \quad -\zeta a\frac{\partial U^{(2)}_{00}}{\partial a}-\zeta b\frac{\partial U^{(2)}_{00}}{\partial b}+m^{-1}U^{(3)}_{00}=\zeta A\ , \\
&-\zeta a\frac{\partial U^{(3)}_{00}}{\partial a}-\zeta b\frac{\partial U^{(3)}_{00}}{\partial b}-ec^{-1}U^{(2)}_{00}+c^{-1}U^{(3)}_{00}=m\gamma^2 A+ha\left(a^2+\frac{3}{2}b^2\right) .
\end{aligned} \tag{3.8}
$$

Eliminating A from (3.8) gives two partial differential equations for $U^{(2)}_{00}$ and $U^{(3)}_{00}$

$$
\begin{aligned}
&-\zeta a\frac{\partial U^{(2)}_{00}}{\partial a}-\zeta b\frac{\partial U^{(2)}_{00}}{\partial b}+m^{-1}U^{(3)}_{00}-\zeta U^{(2)}_{00}=0\ , \\
&-\zeta a\frac{\partial U^{(3)}_{00}}{\partial a}-\zeta b\frac{\partial U^{(3)}_{00}}{\partial b}-ec^{-1}U^{(2)}_{00}+c^{-1}U^{(3)}_{00}-m\gamma^2 U^{(2)}_{00} \\
&=ha\left(a^2+\frac{3}{2}b^2\right) .
\end{aligned} \tag{3.9}
$$

We seek polynomial solutions of (3.9) in the form of

$$U^{(2)}_{00} = k_1 a^3 + k_2 ab^2 = A , \qquad U_{00} = l_1 a^3 + l_2 ab^2 . \tag{3.10}$$

Substituting it into (3.9), we obtain

$$(-4\zeta k_1 + m_1^{-1} l_1)a^3 + (-2\zeta k_2 - 2\,k_2 + m_2^{-1} l_2)ab^2 = 0$$

and

$$(-3\zeta l_1 - lc_1^{-1} k_1 - m\zeta k_1^2)a^3 + (-\zeta l_2 - 2\xi l_2 - ec_2^{-1} k_2 - m_2 k_2^2)ab^2$$

$$= h a^3 + \frac{3}{2} h ab^2 .$$

Comparing the coefficients of a^3 and ab^2 and solving the algebraic equations obtained, gives

$$k_1 = h[4mc^{-1}\zeta - 13m\zeta^2 - ec^{-1}]^{-1} ,$$

$$k_2 = \frac{3}{2} h[2m(\zeta+\xi)(c^{-1} - \zeta - 2\xi) - ec^{-1} - m^2]^{-1} , \tag{3.11}$$

$$l_1 = 4m\zeta k_1 \quad \text{and} \quad l_2 = 2m(\zeta+\xi)k_2 .$$

In order to calculate D and δ, hence B and C, we observe that equation (2.14) with the upper sign and with (2.16), (3.6), and $\exp(i\theta) = \exp(i(\alpha-\psi)) = \exp(i\alpha)\exp(-i\psi)$ in (3.7) takes the form

$$\sum_{S_2^-}\left\{-\xi a\begin{pmatrix}\frac{\partial U_{nk}^{(1)}}{\partial a}\\ \frac{\partial U_{nk}^{(2)}}{\partial a}\\ \frac{\partial U_{nk}^{(3)}}{\partial a}\end{pmatrix}-\xi b\begin{pmatrix}\frac{\partial U_{nk}^{(1)}}{\partial b}\\ \frac{\partial U_{nk}^{(2)}}{\partial b}\\ \frac{\partial U_{nk}^{(3)}}{\partial b}\end{pmatrix}-\begin{pmatrix}-i\omega & 1 & 0\\ 0 & -i\omega & -m^{-1}\\ dc^{-1} & ec^{-1} & -c^{-1}-i\omega\end{pmatrix}\begin{pmatrix}U_{nk}^{(1)}\\ U_{nk}^{(2)}\\ U_{nk}^{(3)}\end{pmatrix}\right\}e^{i(k-1)\psi}=-\frac{1}{2}\begin{pmatrix}1\\ (\xi-i\omega)\\ -m(\xi^2-\omega^2+im\xi\omega)\end{pmatrix}De^{i\delta} \tag{3.12}$$

$$-\frac{1}{2}i\begin{pmatrix}0\\ m^{-1}z\\ 0\end{pmatrix}e^{-i\psi}+\begin{pmatrix}0\\ 0\\ \frac{3}{2}hb(a^2+\frac{b^2}{4})\end{pmatrix}.$$

Here the set $S_2^- = \{(n,k) \mid n+(k-1)=0\} = \{(1,0),(0,1)\}$.

According to the assumption (2.16), we require that

$U_{10}=0$ and $U_{01}^{(1)}=0$. Then equations (3.12) in scalar form are

$$U^{(2)}_{01} = \frac{1}{2} D e^{i\delta} ,$$

$$-\zeta a \frac{\partial U^{(2)}_{01}}{\partial a} - \xi b \frac{\partial U^{(2)}_{01}}{\partial b} + i\omega U^{(2)}_{01} + m^{-1} U^{(3)}_{01} - \frac{1}{2} (\xi - i\omega) D e^{i\delta}$$

$$= -\frac{i}{2} m^{-1} E e^{-i\psi} ,$$

$$-\zeta a \frac{\partial U^{(3)}_{01}}{\partial a} - \xi b \frac{\partial U^{(3)}_{01}}{\partial b} - e c^{-1} U^{(2)}_{01} + (c^{-1} + i\omega) U^{(3)}_{01} -$$

$$\frac{m}{2} (\xi + i\omega)^2 D e^{i\delta} = \frac{3}{2} h b \left(a^2 + \frac{b^2}{4}\right) . \qquad (3.13)$$

Eliminating in Eqs. (3.13) the term $De^{i\delta}$ gives two partial differential equations for $U^{(2)}_{01}$ and $U^{(3)}_{01}$

$$-\zeta a \frac{\partial U^{(2)}_{01}}{\partial a} - \xi b \frac{\partial U^{(2)}_{01}}{\partial b} + \left(\frac{3}{2} i\omega - \frac{1}{2}\xi\right) U^{(2)}_{01} + m^{-1} U^{(3)}_{01}$$

$$= -\frac{i}{2} m^{-1} E e^{-i\psi}$$

(3.14)

$$-\zeta a \frac{\partial U^{(3)}_{01}}{\partial a} - \xi b \frac{\partial U^{(3)}_{01}}{\partial b} - [ec^{-1} + \frac{m}{2}(\xi+i\omega)^2]U^{(2)}_{01} + (c^{-1}+i\omega)U^{(3)}_{01}$$

$$= \frac{3}{2} hb(a^2+\frac{b^2}{4})$$

Assuming solutions of the form $U^{(2)}_{01} = \frac{1}{2} De^{i\delta} = \frac{b}{2}(p_1 a^2+p_2 b^2)+\mu_1 e^{-i\psi}$,

$U^{(3)}_{01} = b(q_1 a^2+q_2 b^2)+\mu_2 e^{-i\psi}$.

and substituting them into (3.14), we have

$$(-p_1\zeta - \frac{1}{2}\xi p_1 + \frac{3}{4}p_1 i\omega - \frac{1}{4}p_1 \xi + m^{-1} q_1)a^2b + (-\frac{3}{2}\xi p_2 + \frac{3}{4}i\omega p_2 -$$

$$\frac{1}{4}\xi p_2 + m^{-1} q_2)b^3 + (\frac{1}{2}i\omega\mu_1 - \frac{1}{2}\xi\mu_1 + m^{-1}\mu_2 + \frac{i}{2}m^{-1}E)e^{-i\psi} = 0 ,$$

(3.15)

$$[-2\zeta q_1 - \xi q_1 - \frac{1}{2}ec^{-1}p_1 - \frac{1}{4}mp_1(\xi+i\omega)^2 + c^{-1}q_1 + i\omega q_1 - \frac{3}{2}h]a^2b +$$

$$[-3\xi q_2 - \frac{1}{2}ec^{-1}p_2 - \frac{1}{4}mp_2(\xi+i\omega)^2 + c^{-1}q_2 + i\omega q_2 - \frac{3}{8}h]b^3 +$$

$$[-ec^{-1}\mu_1 - \frac{m}{2}(\xi+i\omega)^2\mu_1 + c^{-1}\mu_2 + i\omega\mu_2]e^{-i\psi} = 0 .$$

Comparing the coefficients of a^2b, and b^3, and, solving $e^{-i\psi}$, we obtain

$$p_1 = -3h[2m(2\zeta-c^{-1}+\xi-i\omega)(\zeta+\xi-i\omega)+m(\xi-i\omega)^2+ec^{-1}]^{-1} ,$$

$$p_2 = -\frac{3}{4}h[2m(2\xi-c^{-1}+\xi-i\omega)(2\xi-i\omega)+m(\xi-i\omega)^2+ec^{-1}]^{-1} ,$$

$$q_1 = 2m(\zeta+\xi-i\omega)p_1 ,$$

(3.16)

$$q_2 = 2m(2\xi - i\omega)P_2 \,,$$

$$\mu_1 = i\frac{E}{2m}(c^{-1}+i\omega)\{(c^{-1}+i\omega)(\xi-2i\omega) - \frac{1}{m}[m(\xi-i\omega)^2+ec^{-1}]\}^{-1} \,,$$

$$\mu_2 = i\frac{E}{2m}[m(\xi-i\omega)^2+ec^{-1}]\{(c^{-1}+i\omega)(\xi-2i\omega) - \frac{1}{m}[m(\xi-i\omega)^2+ec^{-1}]\}^{-1} \,.$$

From (2.6) and (3.15), one gets

$$B=b(P_1 a^2 + P_2 b^2) + v\cos\psi + w\sin\psi,$$

$$C=Q_1 a^2 + Q_2 b^2 + b^{-1}(w\cos\psi - v\sin\psi). \tag{3.17}$$

where

$$p_1 = P_1 + iQ_1 \,,\quad p_2 = P_2 + iQ_2 \,,\quad u_1 = v + iw \,. \tag{3.18}$$

To solve equation (2.12), we take into account that according to (3.7) only F_{0k}, $k=\pm 2$, ± 3 are different from zero. This implies that $U_{nk}=0$ for $n\neq 0$, $k\neq \pm 2, \pm 3$. For U_{02} and U_{03}, we get two vector equations:

$$-\zeta_a \frac{\partial U_{02}}{\partial a} - \zeta_b \frac{\partial U_{02}}{\partial b} - (A - 2i\omega I)\, U_{02} = \left(0, 0, \tfrac{3}{4} h a b^2\right)^T \tag{3.19}$$

and

$$-\zeta_a \frac{\partial U_{03}}{\partial a} - \zeta_b \frac{\partial U_{03}}{\partial b} - (A - 3i\omega I)\, U_{03} = \left(0, 0, \tfrac{1}{8} h b^3\right)^T \; . \tag{3.20}$$

In matrix form, (3.19) is

$$-\zeta_a \begin{pmatrix} \dfrac{\partial U^{(1)}_{02}}{\partial a} \\[2ex] \dfrac{\partial U^{(2)}_{02}}{\partial a} \\[2ex] \dfrac{\partial U^{(3)}_{02}}{\partial a} \end{pmatrix} - \zeta_b \begin{pmatrix} \dfrac{\partial U^{(1)}_{02}}{\partial b} \\[2ex] \dfrac{\partial U^{(2)}_{02}}{\partial b} \\[2ex] \dfrac{\partial U^{(3)}_{02}}{\partial b} \end{pmatrix} - \begin{pmatrix} -2i\omega & 1 & 0 \\[2ex] 0 & -2i\omega & -m^{-1} \\[2ex] dc^{-1} & ec^{-1} & -c^{-1} - 2i\omega \end{pmatrix} \begin{pmatrix} U^{(1)}_{02} \\[2ex] U^{(2)}_{02} \\[2ex] U^{(3)}_{02} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \tfrac{3}{4} h a b^2 \end{pmatrix} \; . \tag{3.21}$$

(3.21) gives three equations:

$$-\zeta_a \frac{\partial U^{(1)}_{02}}{\partial a} - \zeta_b \frac{\partial U^{(1)}_{02}}{\partial b} + 2i\omega U^{(1)}_{02} = U^{(2)}_{02} \; ,$$

$$-\zeta_a \frac{\partial U^{(2)}_{02}}{\partial a} - \zeta_b \frac{\partial U^{(2)}_{02}}{\partial b} + 2i\omega U^{(2)}_{02} = -m^{-1} U^{(3)}_{02} \; , \tag{3.22}$$

$$-\xi a \frac{\partial U^{(3)}_{02}}{\partial a} - \zeta b \frac{\partial U^{(3)}_{02}}{\partial b} + 2i\omega U^{(3)}_{02} = dc^{-1} U^{(1)}_{02} + ec^{-1} U^{(2)}_{02} - c^{-1} U^{(3)}_{02} + \frac{3}{4}hab^2 .$$

The partial derivative of $U^{(2)}_{02}$ with respect to a and b are

$$\frac{\partial U^{(2)}_{02}}{\partial a} = -\xi \frac{\partial U^{(1)}_{02}}{\partial a} - \xi a \frac{\partial^2 U^{(1)}_{02}}{\partial a^2} - \zeta b \frac{\partial^2 U^{(1)}_{02}}{\partial a \partial b} + 2i\omega \frac{\partial U^{(1)}_{02}}{\partial a} ,$$

$$\frac{\partial U^{(2)}_{02}}{\partial b} = -\xi a \frac{\partial^2 U^{(1)}_{02}}{\partial a \partial b} - \zeta \frac{\partial U^{(1)}_{02}}{\partial b} - \zeta b \frac{\partial^2 U^{(1)}_{02}}{\partial b^2} + 2i\omega \frac{\partial U^{(1)}_{02}}{\partial b} . \tag{3.23}$$

Substituting (3.23) into (3.22), we obtain

$$\xi^2 a^2 \frac{\partial^2 U^{(1)}_{02}}{\partial a^2} + 2\xi\zeta ab \frac{\partial^2 U^{(1)}_{02}}{\partial a \partial b} + \zeta^2 b^2 \frac{\partial^2 U^{(1)}_{02}}{\partial b^2} + (\xi^2 a - 4\omega\xi ai) \frac{\partial U^{(1)}_{02}}{\partial a}$$

$$+ (\zeta^2 b - 4\omega\zeta bi) \frac{\partial U^{(1)}_{02}}{\partial b} - 4\omega^2 U^{(1)}_{02} = -m^{-1} U^{(3)}_{02} \tag{3.24}$$

$$-\xi a \frac{\partial U^{(3)}_{02}}{\partial a} - \zeta b \frac{\partial U^{(3)}_{02}}{\partial b} + (c^{-1} + 2i\omega) U^{(3)}_{02} = dc^{-1} U^{(1)}_{02} +$$

$$ec^{-1}\left(-\xi a \frac{\partial U^{(1)}_{02}}{\partial a} - \zeta b \frac{\partial U^{(1)}_{02}}{\partial b} + 2i\omega U^{(1)}_{02} \right) - c^{-1} U^{(3)}_{02} + \frac{3h}{4} ab^2 .$$

We look for solutions of the form

$$U^{(1)}_{02} = Pab^2 , \quad U^{(3)}_{02} = Tab^2 . \tag{3.25}$$

Substituting (3.25) into (3.24) and comparing the coefficients of ab^2 gives

$$P(4\xi\zeta+\xi^2-4\omega\xi i-8\omega\zeta i-4\omega^2+4\zeta^2)+m^{-1}T=0 ,$$

$$P(dc^{-1}-\xi ec^{-1}-2ec^{-1}\zeta+2i\omega ec^{-1})-T(-\xi-2\zeta+2c^{-1}+2i\omega)=-\frac{3h}{4} .$$

Then, by Cramer's rule, one obtains

$$P=\frac{3h}{4R_1} , \quad T=\frac{3h}{4R_1}\{-m[\xi+2(\zeta-i\omega)]^2\} ,$$

where

$$R_1=m[\xi+2(\zeta-i\omega)]^{-1}-2cm[\xi+2(\zeta-i\omega)]^2+c^{-1}\{e[\xi+2(\zeta-i\omega)-d]\}.$$

Similarily, in matrix form, (3.20) is

$$-\xi a\begin{pmatrix}\frac{\partial U^{(1)}_{03}}{\partial a}\\ \frac{\partial U^{(2)}_{03}}{\partial a}\\ \frac{\partial U^{(3)}_{03}}{\partial a}\end{pmatrix}-\zeta b\begin{pmatrix}\frac{\partial U^{(1)}_{03}}{\partial b}\\ \frac{\partial U^{(2)}_{03}}{\partial b}\\ \frac{\partial U^{(3)}_{03}}{\partial b}\end{pmatrix}-\begin{pmatrix}-3i\omega & 1 & 0\\ 0 & -3i\omega & -m^{-1}\\ dc^{-1} & ec^{-1} & -c^{-1}-3i\omega\end{pmatrix}\begin{pmatrix}U^{(1)}_{03}\\ U^{(2)}_{03}\\ U^{(3)}_{03}\end{pmatrix}$$

$$=\begin{pmatrix}0\\ 0\\ \frac{1}{8}hb^3\end{pmatrix}. \qquad (3.26)$$

The scalar equations obtained from (3.26) are

$$-\zeta a\frac{\partial U^{(1)}_{03}}{\partial a}-\xi b\frac{\partial U^{(1)}_{03}}{\partial b}+3i\omega U^{(1)}_{03}=U^{(2)}_{03},$$

$$-\zeta a\frac{\partial U^{(2)}_{03}}{\partial a}-\xi b\frac{\partial U^{(2)}_{03}}{\partial b}+3i\omega U^{(2)}_{03}=-m^{-1}U^{(2)}_{03},\qquad (3.27)$$

$$-\zeta a\frac{\partial U^{(3)}_{03}}{\partial a}-\xi b\frac{\partial U^{(3)}_{03}}{\partial b}+3i\omega U^{(3)}_{03}=dc^{-1}U^{(1)}_{03}+ec^{-1}U^{(2)}_{03}-c^{-1}U^{(3)}_{03}+\frac{1}{8}hb^{3}$$

Taking partial derivatives of $U^{(2)}_{03}$ with respect to a and b and substituting them into (3.27), we find

$$\zeta^{2}a^{2}\frac{\partial^{2}U^{(1)}_{03}}{\partial a^{2}}+2\zeta\xi ab\frac{\partial^{2}U^{(1)}_{03}}{\partial a\partial b}+\xi^{2}b^{2}\frac{\partial^{2}U^{(1)}_{03}}{\partial b^{2}}+(\zeta^{2}a-6\omega\zeta ai)\frac{\partial U^{(1)}_{03}}{\partial a}$$

$$+(\xi^{2}b-6\omega\xi bi)\frac{\partial U^{(1)}_{03}}{\partial b}-9\omega^{2}U^{(1)}_{03}=-m^{-1}U^{(3)}_{03}$$

(3.28)

$$-\zeta a\frac{\partial U^{(3)}_{03}}{\partial a}-\xi b\frac{\partial U^{(3)}_{03}}{\partial b}+(c^{-1}+3i\omega)U^{(3)}_{03}=dc^{-1}U^{(1)}_{03}+$$

$$ec^{-1}\left(-\zeta a\frac{\partial U^{(1)}_{03}}{\partial a}-\xi b\frac{\partial U^{(1)}_{03}}{\partial b}+3i\omega U^{(1)}_{03}\right)+\frac{1}{8}hb^{3}.$$

Assume solutions are of the form

$$U^{(1)}_{03}=Ib^{3},\quad U^{(3)}_{03}=Jb^{3}.\qquad (3.29)$$

Substituting (3.29) into (3.28), and comparing the coefficients of b^3 gives

$$I(9\xi^2-9\omega^2-18\xi\omega i)+m^{-1}J=0$$

$$I(dc^{-1}-3ec^{-1}\xi+3ec^{-1}\omega i)-(c^{-1}-3\xi+3i\omega)J=-\frac{1}{8}h \quad .$$

By Cramer's rule, one gets

$$I=\frac{h}{8R_2}\,, \qquad J=\frac{h}{8R_2}\left[-9m(\xi-i\omega)^2\right],$$

where

$$R_2=9m(\xi-i\omega)^2[3(\xi-i\omega)-c^{-1}]+c^{-1}[3e(\xi-i\omega)-d].$$

From (3.22), (3.25), (3.27), and (3.29), one gets the solutions for U_{02} and U_{03}

$$U_{02}=\frac{3h}{4R_1}\begin{pmatrix}1\\ -[\gamma+2(\xi-i\omega)]\\ -m[\gamma+2(\xi-i\omega)]^2\end{pmatrix}ab^2 \tag{3.30}$$

$$U_{03}=\frac{h}{8R_2}\begin{pmatrix}1\\ -(\gamma+\xi-3i\omega)\\ -9m(\xi-i\omega)^2\end{pmatrix}b^3 \quad .$$

Finally, according to (2.2) truncated up to the order of ε and using (2.6) we can write the first improved approximate solution of (3.1) as follow:

$$x^{(1)}=a+\frac{1}{2}b(e^{i\alpha}+e^{-i\alpha})+\varepsilon(U^{(1)}_{02}e^{2i\alpha}+U^{(1)}_{0,-2}e^{-2i\alpha}+U^{(1)}_{03}e^{3i\alpha}$$

$$+U^{(1)}_{0,-3}e^{-3i\alpha}) ,$$

$$x^{(2)}=-\zeta a-\frac{1}{2}b[(\zeta-i\omega)e^{i\alpha}+(\zeta+i\omega)e^{-i\alpha}]+(U^{(2)}_{00}+U^{(2)}_{01}e^{i\alpha}+$$

$$U^{(2)}_{0,-1}e^{-i\alpha}+U^{(2)}_{02}e^{2i\alpha}+U^{(2)}_{0,-2}e^{-2i\alpha}+U^{(2)}_{03}e^{3i\alpha}+U^{(2)}_{0,-3}e^{-3i\alpha}) ,$$

$$x^{(3)}=-\blacksquare\zeta^{2}a-\frac{\blacksquare}{2}b[(\zeta-i\omega)^{2}e^{i\alpha}+(\zeta+i\omega)^{2}e^{-i\alpha}]+(U^{(3)}_{00}+ \tag{3.31}$$

$$U^{(3)}_{01}e^{i\alpha}+U^{(3)}_{0,-1}e^{-i\alpha}+U^{(3)}_{02}e^{2i\alpha}+U^{(3)}_{0,-2}e^{-2i\alpha}+U^{(3)}_{03}e^{3i\alpha}+$$

$$U^{(3)}_{0,-3}e^{-3i\alpha}) ,$$

where $U^{(2)}_{00}$ and $U^{(3)}_{00}$ are given by (3.10), $U^{(2)}_{01}$ and $U^{(3)}_{01}$ are given by (3.15), $U^{(1)}_{02}$, $U^{(2)}_{02}$, $U^{(3)}_{02}$, $U^{(1)}_{03}$, $U^{(2)}_{03}$, and $U^{(3)}_{03}$ are given by (3.19). The amplitudes a and b and the phase $\psi=\alpha-\nu t$ are to be determined from equations (2.3) truncated up to the order of with (3.9) and (3.17):

$$\dot{a}=-\zeta a+\varepsilon a(k_{1}a^{2}+k_{2}b^{2}) ,$$

$$\dot{b}=-\xi b+\varepsilon[b(P_1 a^2+P_2 b^2)+v\cos\psi+w\sin\psi] \quad , \qquad (3.32)$$

and

$$\dot{\psi}=\omega-\nu+\varepsilon[Q_1 a^2+Q_2 b^2+b^{-1}(w\cos\psi-v\sin\psi)] \, ,$$

where k_1 and k_2 are given by (3.11) and P_1, P_2, Q_1, Q_2 v, and w by (3.18) and (3.16).

In general, the system (3.32) can be solved numerically by using computer. However, valuable information can be obtained by investigating the stationary regime of oscillations in chapter 4.

4. The Resonance Curve

The stationary regime of oscillations is characterized by $\dot{a}=\dot{b}=\dot{\psi}=0$ in (3.32). We consider here only the solution $a=0$ of the first equation of (3.32) with $a=0$ which amounts to neglecting the nonoscillatory part of solution (3.31). The last two equations of (3.32) with $\dot{b}=\dot{\psi}=0$ and $a=0$ reduce to

$$B(b,\psi) = -\xi b + \varepsilon P_2 b^3 + \varepsilon(v\cos\psi + w\sin\psi) = 0,$$
$$\Psi(b,\psi) = \omega - \nu + \varepsilon Q_2 b^2 + \varepsilon b^{-1}(w\cos\psi - v\sin\psi) = 0 \quad . \qquad (4.1)$$

This implies

$$-\xi b + \varepsilon P_2 b^3 = -\varepsilon(v\cos\psi + w\sin\psi) \quad ,$$

$$(\omega - \nu) b + \varepsilon Q_2 b^3 = -\varepsilon(w\cos\psi - v\sin\psi) \quad .$$

Therefore,

$$(-\xi b + \varepsilon P_2 b^3)^2 = \varepsilon^2 (v^2\cos^2\psi + 2vw\cos\psi\sin\psi + w^2\sin^2\psi),$$
$$(4.2)$$
$$[(\omega - \nu) b + \varepsilon Q_2 b^3]^2 = \varepsilon^2 (w^2\cos^2\psi - 2vw\cos\psi\sin\psi + v^2\sin^2\psi) \quad .$$

The elimination of ψ in (4.2) gives the equation of the resonance curve

$$\nu=\omega+\varepsilon Q_2 b^2 \pm b^{-1}[\ (b^2)]^{1/2} \qquad (4.3)$$

in the plane (ν,b), where

$$\beta(b^2)=-\varepsilon^2 P_2^2 b^6+2\varepsilon\xi P_2 b^4-\xi^2 b^2+\varepsilon^2(v^2+w^2). \qquad (4.4)$$

Equation (4.3) relates the amplitude b of the oscillating term in (3.31) to the frequency ν of the external acting force. The resonance curve (4.3) consists of two branches separated by the backbone line

$$\nu=\omega+\varepsilon Q_2 b^2 \qquad (4.5)$$

To facilitate the study of the resonance curve, we consider the positive zeros of the polynomial (4.4).

Let $t=b^2$, (4.4) becomes

$$\beta(t)=-\varepsilon^2P_2^2t^3+2\varepsilon\xi P_2t^2-\xi^2t+\varepsilon^2(v^2+w^2) \quad , \qquad (4.4')$$

hence

$$\beta'(t)=-3\ {}^2P_2^2t^2+4\ P_2t-\ {}^2=0$$

has the roots

$$t_1=\frac{\xi}{\varepsilon P_2}\ ,\quad t_2=\frac{\xi}{3\varepsilon P_2}\ .$$

Case 1: $P_2 > 0$.

Since $\beta''(t) = -6\varepsilon^2 p_2^2 t + 4\varepsilon\zeta p_2$,

$$\beta''\left(\frac{\zeta}{3\varepsilon p_2}\right) = -6\varepsilon^2 p_2^2 \frac{\zeta}{3\varepsilon p_2} + 4\varepsilon\zeta p_2 = 2\varepsilon\zeta p_2 > 0 \ ,$$

$$\beta''\left(\frac{\zeta}{\varepsilon p_2}\right) = -6\varepsilon^2 p_2^2 \frac{\zeta}{\varepsilon p_2} + 4\varepsilon\zeta p_2 = 2\varepsilon\zeta p_2 < 0 \ .$$

From here we conclude that $\beta(t)$ has a local minimum at $t = \frac{\zeta}{3\varepsilon P_2}$ and a local maximum at $t = \frac{\zeta}{\varepsilon p_2}$.

Now we discuss all cases that give positive roots

We have

$$\beta_{max} = \beta\left(\frac{\zeta}{\varepsilon p_2}\right) = -\varepsilon^2 p_2^2 \left(\frac{\zeta}{\varepsilon p_2}\right)^3 + 2\varepsilon\zeta p_2 \left(\frac{\zeta}{\varepsilon p_2}\right)^2 - \zeta^2 \frac{\zeta}{\varepsilon p_2} + \varepsilon^2 (v^2 + w^2)$$

$$= \varepsilon^2 (v^2 + w^2) > 0 \ ,$$

$$\beta_{min} = \beta\left(\frac{\zeta}{3\varepsilon p_2}\right) = -\frac{4\zeta^3}{27\varepsilon p_2} + \varepsilon^2 (v^2 + w^2) \ .$$

Then

<i> if $4\zeta^3 < 27\varepsilon^3 p_2 (v^2 + w^2)$, then $\beta_{min} > 0$, (4.4') has only one positive root;

<ii> if $4\zeta^3 = 27\varepsilon^3 p_2 (v^2 + w^2)$, then $\beta_{min} = 0$, it has two double positive roots $t_1 = t_2 < t_3$;

<iii> if $4\zeta^3 > 27\varepsilon^3 p_2 (v^2 + w^2)$, then $\beta_{min} < 0$, (4.4') gives

three distinct positive roots.

Case 2: $P_2 < 0$.

Then

$$\beta''\left(\frac{\zeta}{3\varepsilon P_2}\right) = 2\varepsilon\zeta P_2 < 0 \text{ implies a relative maximum at } t = \frac{\zeta}{3\varepsilon P_2}$$

and

$$\beta''\left(\frac{\zeta}{\varepsilon P_2}\right) = -2\varepsilon\zeta P_2 > 0 \text{ means a relative minimum at } t = \frac{\zeta}{\varepsilon P_2}.$$

Similarily,

$$\beta_{max} = \beta\left(\frac{\zeta}{3\varepsilon P_2}\right) = -\frac{4\zeta^3}{27 P_2} + \varepsilon^2(v^2+w^2) > 0,$$

$$\beta_{min} = \beta\left(\frac{\zeta}{\varepsilon P_2}\right) = \varepsilon^2(v^2+w^2) > 0.$$

So (4.4') has only one positive root.

Therefore, for the zeros of (4.4), we have the following results

<I> one positive zero b_1^2 if $P_2 \le 0$ or $P_2 > 0$ and

$4\zeta^3 < 27\varepsilon^3 P_2 (v^2+w^2)$.

<II> two double positive zeros $b_1^2 = b_2^2 < b_3^2$ if

$P_2 > 0$ and $4\zeta^3 = 27\varepsilon^3 P_2 (v^2+w^2)$.

<III> three different positive zeros b_1^2, b_2^2, and b_3^2 if

$P_2 > 0$ and $4\zeta^3 > 27\ {}^3 P_2 (v^2+w^2)$.

Depending on the sign of ζ_2 , the backbone curve (4.5) sketched with broken line, and the resonance curve (4.3) are presented in Figures 1, 2 and 3. These figues show how the shape of the resonance curve changes according to the number of the positive zeros of (4.4).

<i> one positive zero b_1^2 if $P_2 \leq 0$ or $P_2 > 0$ and

$$4\zeta^3 < 27\xi^3 P_2 (v^2+w^2).$$

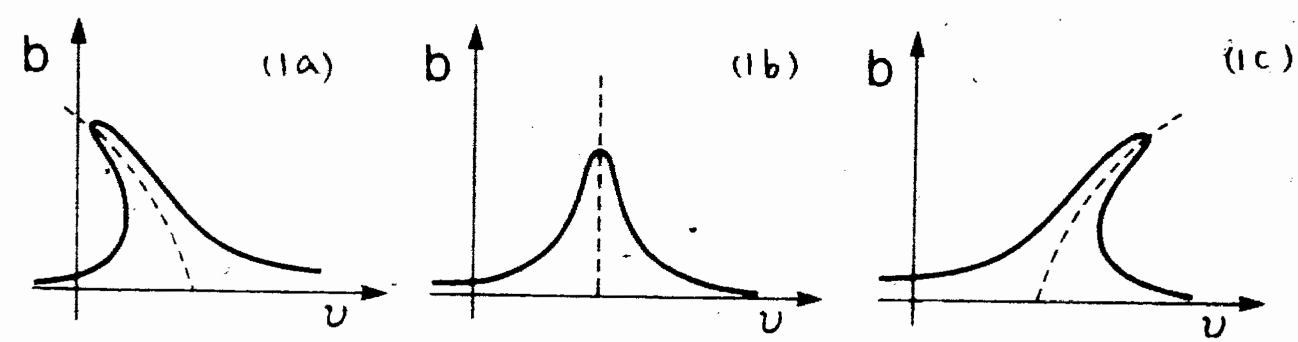

FIGURE 1

<ii>three positive zeros with two double zeros $b_1^2=b_2^2$, b_3^2:

$$P_2 > 0 \text{ and } 4\zeta^3 = 27\xi^3 P_2 (v^2+w^2).$$

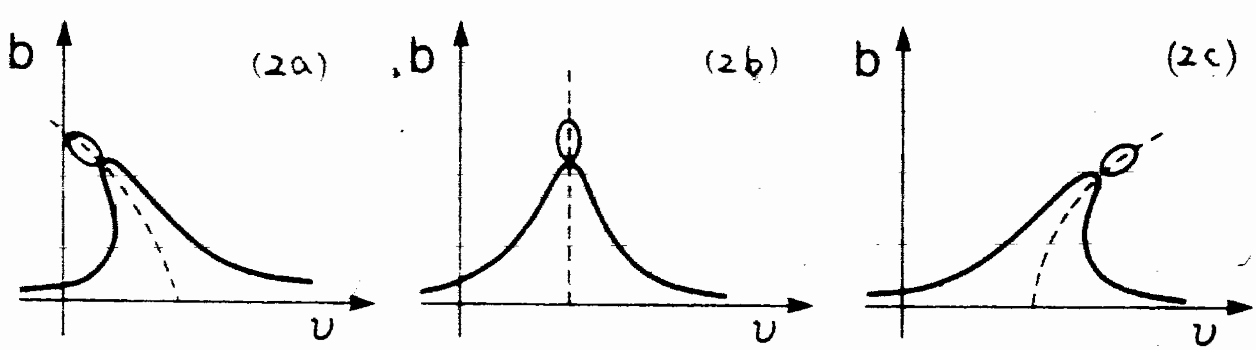

FIGURE 2

<iii>three distinct positive zeros:

$p_2>0$ and $4\zeta^3>27\varepsilon^3 p_2(v^2+w^2)$.

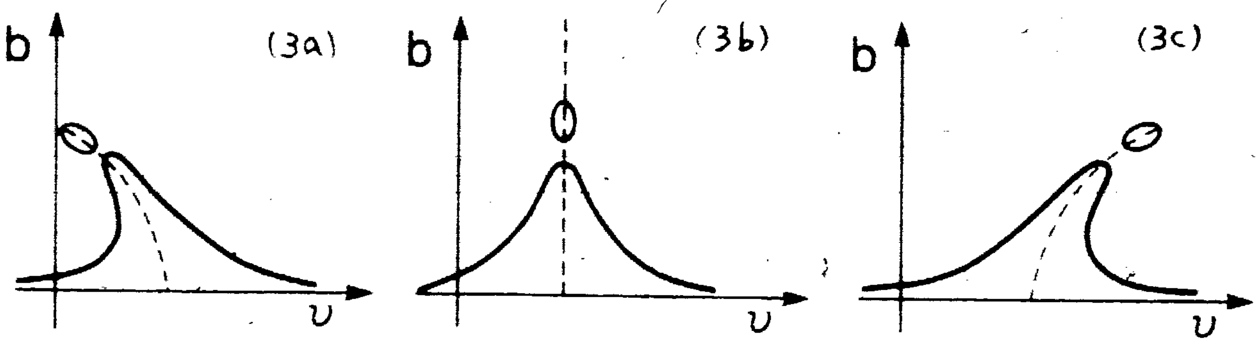

FIGURE 3

We now examine the stability of the stationary regime of oscillations in the first approximation $b\{\phi\exp[i(\psi+\nu t)+\phi^*\exp[-i(\psi+\nu t)]\}$. Here the amplitude b and the phase ψ are in the vicinity of the steady state (b_0,ψ_0) which satisfies (4.1).

The variational matrix of (4.1) is

$$V(b,\psi)=\begin{pmatrix} B_b(b,\psi) & B_\psi(b,\psi) \\ \Psi_b(b,\psi) & \Psi_\psi(b,\psi) \end{pmatrix}$$

$$=\begin{pmatrix} -\zeta+3\varepsilon p_2 b^2 & \varepsilon(-v\sin\psi+w\cos\psi) \\ 2\varepsilon\zeta_2 b-b^{-2}(w\cos\psi-v\sin\psi) & \varepsilon b^{-1}(-w\sin\psi-v\cos\psi) \end{pmatrix}$$

$$= \begin{pmatrix} -\zeta+3\varepsilon P_2 b^2 & -b(\omega-\nu)-\varepsilon Q_2 b^3 \\ (\omega-\nu)b^{-1}+3\varepsilon Q_2 b & -\zeta+\varepsilon P_2 b^2 \end{pmatrix},$$

where the subscripts of B and Ψ indicate partial differentiation with respect to the corresponding arguments. The conditions for stability of (b_0, ψ_0), using (4.1), can be expressed in the form

$$B_b(b_0,\psi_0) + \Psi_\psi(b_0,\psi_0) = -\zeta+3\varepsilon P_2 b_0^2 + (-\zeta+\varepsilon P_2 b_0^2) = 2(-\zeta+2\varepsilon P_2 b_0^2) < 0 \qquad (4.6)$$

and

$$\det \mathbf{v}(b_0,\psi_0) = b_0 [(3\varepsilon P_2 b_0^2 - \zeta)(\varepsilon P_2 b_0^2 - \zeta) + (\nu-\omega-\varepsilon Q_2 b_0^2)(\nu-\omega-3\varepsilon Q_2 b_0^2)] > 0 \ . \qquad (4.7)$$

Condition (4.6) is satified for $P_2 < 0$ and It might be satified even for some positive values of P_2, for example if ε and P_2 are much smaller than ζ. Substituting (4.1) and (4.4) into (4.3), we get

$$b^2(\nu-\omega-\varepsilon Q_2 b^2)^2 = -\varepsilon^2 P_2^2 b^6 + 2\varepsilon\zeta P_2 b^4 - \zeta^2 b^2 + \varepsilon^2(v^2+w^2). \qquad (4.8)$$

Differentiating (4.8), we obtain

$$2b\frac{db}{d\nu}(\nu-\omega-\varepsilon Q_2 b^2)^2 + 2b^2(\nu-\omega-\varepsilon Q_2 b^2)(-2\varepsilon Q_2 b\frac{db}{d\nu}+1)$$
$$= (-6\varepsilon^2 P_2^2 b^5 + 8\zeta\varepsilon P_2 b^3 - 2\zeta^2 b)\frac{db}{d\nu} \ .$$

Further, we have

$$\frac{db}{d\nu} = \frac{\omega-\nu+\varepsilon Q_2 b^2}{[(\nu-\omega-\varepsilon Q_2 b^2)(\nu-\omega-3\varepsilon Q_2 b^2)+\xi^2-4\varepsilon\xi p_2 b^2+3\varepsilon^2 p_2^2 b^4]}$$

$$= \frac{\omega-\nu+\varepsilon Q_2 b^2}{\det \underset{\sim}{v}(b,\psi)} \quad . \qquad (4.9)$$

From (4.9), we see that the condition (4.7) is satisfied if and only if db/dv and $\omega-\nu+\varepsilon Q_2 b^2_2$ have the same sign, i.e.,

$$\begin{aligned} db/d\nu &> 0 \quad \text{for} \quad \nu < \omega+\varepsilon Q_2 b^2, \\ db/d\nu &< 0 \quad \text{for} \quad \nu > \omega+\varepsilon Q_2 b^2 . \end{aligned} \qquad (4.10)$$

Provided that condition (4.6) is fulfilled, from (4.10), it follows that the stability regions of the stationary regime of oscillation correspond on the left of the backbone curve (4.5) to the arcs of the resonance curve (4.3) on which b is increasing together with ν and on the right of (4.5) to arcs of (4.3) on which b is decreasing function of ν. The other arcs of the resonance curve correspond to unstability. The passage from stability to unstability or vice versa occurs at the points of the resonance curve with vertical targent.

As an illustration below, we consider three particular cases of (3.1).

Case 1:

$m=1$, $c=0.5$, $e=1.125$, $d=0.625$, $E=5.531$, $h=100/9$, and $\varepsilon=0.1$

Then the system (3.1) becomes

$$\ddot{x}+\alpha=\varepsilon 5.531\sin\theta ,$$

$$\ddot{\alpha}+0.5\dot{\alpha}=0.625x+1.125\dot{x}+\varepsilon 5.56\,x^3 .$$

The change of varables $x=x^{(1)}$, $\dot{x}=x^{(2)}$, and $\alpha=x^{(3)}$ transforms (3.1) to (2.1). One gets

$$\begin{pmatrix}\dot{x}^{(1)}\\ \dot{x}^{(2)}\\ \dot{x}^{(3)}\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 1.25 & 2.25 & -2\end{pmatrix}\begin{pmatrix}x^{(1)}\\ x^{(2)}\\ x^{(3)}\end{pmatrix}+\varepsilon\begin{pmatrix}0\\ 5.531\sin\theta\\ 11.12(x^{(3)})^3\end{pmatrix}.$$

The characteristic polynomial of (3.2)

$$\det|A-\lambda I| = \lambda^3+2\lambda^2+2.25\lambda+1.25 = 0 \qquad (4.11)$$

has the following roots : -1 , $-0.5\pm i$.

From (3.10), (3,11), (3.17) and (3.18), we obtain

$$A = k\,a^3+k\,ab^2 = \frac{100}{9}(8-13-2.25)^{-1}a^3+\frac{50}{3}(-2.25-1)^{-1}a^2b$$
$$=-1.533a^3-5.128a^2b,$$

$$B = P_1a^2b+P_2b^3+v\cos\psi+w\sin\psi \qquad (4.12)$$

$$=-2.667a^2b+4b^3-2.115\cos\psi+0.163\sin\psi ,$$

$$C = Q_1a^2+Q_2b^2+b^{-1}(w\cos\psi-v\sin\psi)$$

$$=-5.333a^2-5.333b^2+b^{-1}(0.163\cos\psi+2.115\sin\psi),$$

where

$$p_1 = P_1 + iQ_1 = -2.667-5.333i,$$

$$p_2 = P_2 + iQ_2 = 4-5.333i,$$

and

$$u = v + iw = -2.115+0.163i \quad .$$

Substituting (4.12) into (2.3), truncating up to the order of ε, we have

$$\dot{a} = -\zeta a + \varepsilon A(a,b,\psi) = -a - \varepsilon(1.533a^3 + 5.128ab^2),$$

$$\dot{b} = -\zeta b + \varepsilon B(a,b,\psi) = -0.5b + \varepsilon(-2.667a^2+4b^3-2.115\cos\psi$$
$$+0.163\sin\psi),$$

$$\dot{\psi} = 1 - v + \varepsilon C(a,b,\psi) = 1-v+\varepsilon[-5.333a^2-5.333b^2$$
$$+b^{-1}(0.163\cos\psi + 2.115\sin\psi)] \quad .$$

Substituting P_2, Q_2, v, and w into (4,2), after elimating ψ in (4.2), we obtain the equation of the resonance curve

$$\nu = 1-0.533b^2 \pm b^{-1}(-0.16b^6+0.4b^4-0.25b^2+0.05)^{1/2}$$

which consists of two branches separated by the backbone line

$$\nu = 1-0.533b^2 \quad .$$

The polynomial (4.4) has three zeros $b_1^2=0.5$, $b_{2,3}=(2-\sqrt{1.75})/2$.
and since $Q_2 < 0$,

the resonance curve is of the type 3a. It is sketched on Fig 1. Since the stability condition (4.6) is satisfied only for

$b_0 < (5/8)^{1/2}$,

According to (4.10), only the branch (l_1) and the small arc M_1M_2 ($M_1(.77,.8)$, $M_2(.8, .79)$) of the resonance curve correspond to stability of the stationary regime of oscillations. Note that the tangent line at M of the resonance curve is parallel to b axis.

Case 2:

$m=1$, $c=0.5$, $e=1.125$, $d=0.625$, $E=5.018$, and $h=150/9$.

The system (3.1) now is

$$x + \alpha = \varepsilon 5.018\sin\theta \quad ,$$

$$\dot{\alpha} + 0.5\alpha = 0.625x + 1.125\dot{x} + \varepsilon 3.125x^3 \quad .$$

And three dimensional diferential system (2.1) becomes

$$\begin{pmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \\ \dot{x}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1.25 & 2.25 & -2 \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 5.018\sin\theta \\ 6.25(x^{(1)})^3 \end{pmatrix} \quad .$$

Similarily, one gets three roots of (4.11). They are -1 and $-0.5\pm i$. Then.

$$A = k_1 a^3 + k_2 ab^2 = -0.862a^3 - 2.885a^2 b,$$

$$B = P_1 a^2 b + P_2 b^3 + v\cos\psi + w\sin\psi$$

$$= -6.667a^2 b + 5b^3 - 1.919\cos\psi + 0.148\sin\psi ,$$

$$C = Q_1 a^2 + Q_2 b^2 + b^{-1} (w\cos\psi - v\sin\psi) \qquad (4.13)$$

$$= -13.333a^2 - 6.667b^2 + b^{-1} (0.148\cos\psi + 1.919\sin\psi),$$

where

$$p_1 = P_1 + iQ_1 = -6.667 - 13.333i,$$

$$p_2 = P_2 + iQ_2 = 5 - 6.667i, \text{and}$$

$$u = v + iw = -1.919 + 0.148i .$$

After substituting (4.13) into (2.3) , we have

$$\dot{a} = -\gamma a + \varepsilon A(a,b,\psi) = -a + \varepsilon(-1.919a^3 + 0.148ab^2),$$

$$\dot{b} = -b + \varepsilon B(a,b,\psi) = -0.5b + \varepsilon(-6.667a^2 b + 5b^3 - 1.919\cos\psi + 0.148\sin\psi),$$

$$\dot{\psi} = 1 - v + \varepsilon C(a,b,\psi) = 1 - v + \varepsilon[-13.333a^2 - 6.667b^2 + b^{-1} (0.148\cos\psi + 1.919\sin\psi)].$$

Here the equation of the resonance curve is

$$\nu = 1-0.667b^2 + b^{-1}\ (-0.25b^6+0.5b^4-0.25b^2+0.037)^{1/2} .$$

It consists of two branches separated by the backbone line

$$\nu = 1-0.667b^2.$$

The polynomial (4.4) has three positive zeros $b_1^2 = 1.15$, $b_2^2 = b_3^2 = 0.581$, two of which are equal.

Since $Q_2 < 0$ the resonance curve is of the type 2a.

Since the stability condition (4.6) is satisfied only for $b < (0.5)^{1/2}$,

According to (4.10), only the branch (1_2) of the resonance curve corresponds to stability of the stationary regime of oscillations

Case 3: m=1, c=24/49, e=1.75, d=7/12, E=10, and h=10. For the system (3.1) we have

$$\ddot{x} + \alpha = \varepsilon\, 10 \sin\theta$$

$$\alpha + \frac{24}{49}\dot{\alpha} = \frac{7}{12} x + \frac{7}{4}\dot{x} + \varepsilon \frac{240}{49} x^3$$

and the three dimensional diferential system (2.1) is

$$\begin{pmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \\ \dot{x}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \frac{343}{288} & \frac{343}{96} & \frac{-49}{24} \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 10\sin\theta \\ 10\,(x^{(1)})^3 \end{pmatrix} .$$

The characteristic equation

$$\det|A-\lambda I| = \lambda^3 + \frac{49}{24}\lambda^2 + \frac{343}{96}\lambda + \frac{343}{288} = 0. \qquad (4.15)$$

has one real and two complex roots $-0.41,\ 0.816\pm 1.496i$

From (3.10), (3.11), (3.17), and (3.18), one gets

$$A = k_1 a^3 + k_2 ab^2 = -0.565a^3 - 3.04ab^2,$$

$$B = P_1 a^2 b + P_2 b^3 + v\cos\psi + w\sin\psi$$

$$= 6.146a^2 b - 0.323\cos\psi + 0.44\sin\psi,$$

$$C = Q_1 a^2 + Q_2 b^2 + b^{-1}(w\cos\psi - v\sin\psi)$$

$$= 8.672a^2 - 0.863b^2 + b^{-1}(0.44\cos\psi + 0.323\sin\psi),$$

where

$$p_1 = P_1 + iQ_1 = 6.146 + 8.672i,$$

$$p_2 = P_2 + iQ_2 = 0.116 - 0.863i,$$

$$u = v + iw = -0.323 + 0.44i$$

$$\dot{a} = -\zeta a + \varepsilon A(a,b,\psi) = -0.41a + \varepsilon(-0.565a^3 - 3.04ab^2),$$

$\dot{b} = -\zeta b + \varepsilon B(a,b,\psi) = 0.816b + \varepsilon[6.146a^2b - 0.323\cos\psi$

$+0.44\sin\psi]$,

and

$\dot{\psi} = 1 - \nu + \varepsilon C(a,b,\psi) = 0.085 - \nu + \varepsilon[8.672a^2 - 0.863b^2$

$+b^{-1}(0.44\cos\psi + 0.323\sin\psi)]$.

The equation of the resonance curve is

$$\nu = 0.085 - 0.086b^2 \pm b^{-1}(-0.239b^2 + 0.239)^{1/2}.$$

It consists of two branches separated by the backbone line

$$\nu = 0.085 - 0.086b^2.$$

The polynomial (4.4) in this case has only one real root $b^2=1$, Since it $Q_2<0$, the stability condition (4.6) is satisified for all b, according to (4.10), only the branch (l_3) of the resonance curve is stable.

5. Conclusion

In this thesis a general case of damped oscillations modeled by a 3-dimensional time dependent differential system (2.1) has been studied , In application, the equation (3.1) governing the oscillations of an elastic system with internal friction and relaxtion under the action of a harmonic force has been discussed.

The resonance curve has been investigated and the stability of the stationary regime of oscillations in the first approximation has been examined.

Although the extended KBM method gives the impression of being complex, it can be readily applied by following four simple steps:

<1> expanding the functions $F(\theta, x)$ in Fourier series (2.8) and solving the integrals (2.9);
<2> solving the differential equations of Euler's type (2.5) using the substitution

$-\xi+i\omega=\rho e^{i\gamma}$ and $B+ibc=De^{i\delta}$;

or assuming a polynomial solution in terms of a and b, and written in expoential form harmonic terms.

<3> determining the amplitude a,b and the phase ψ by solving the equations (2.3) with separable variables.

<4> solving numerically the truncated system (2.3).

The extended KBM method essentially replaces the task of solving the nonlinear equation (2.1) by the much simpler task of solving the equations (2.3). or investigating the nonstationary regime of oscillations.

If one whishes to use a computer, then it is less costly and more accurate not to solve an equation directly (2.1),but after completing step 1, to solve equations (2.3) for a, b, and instead.

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APPENDIX

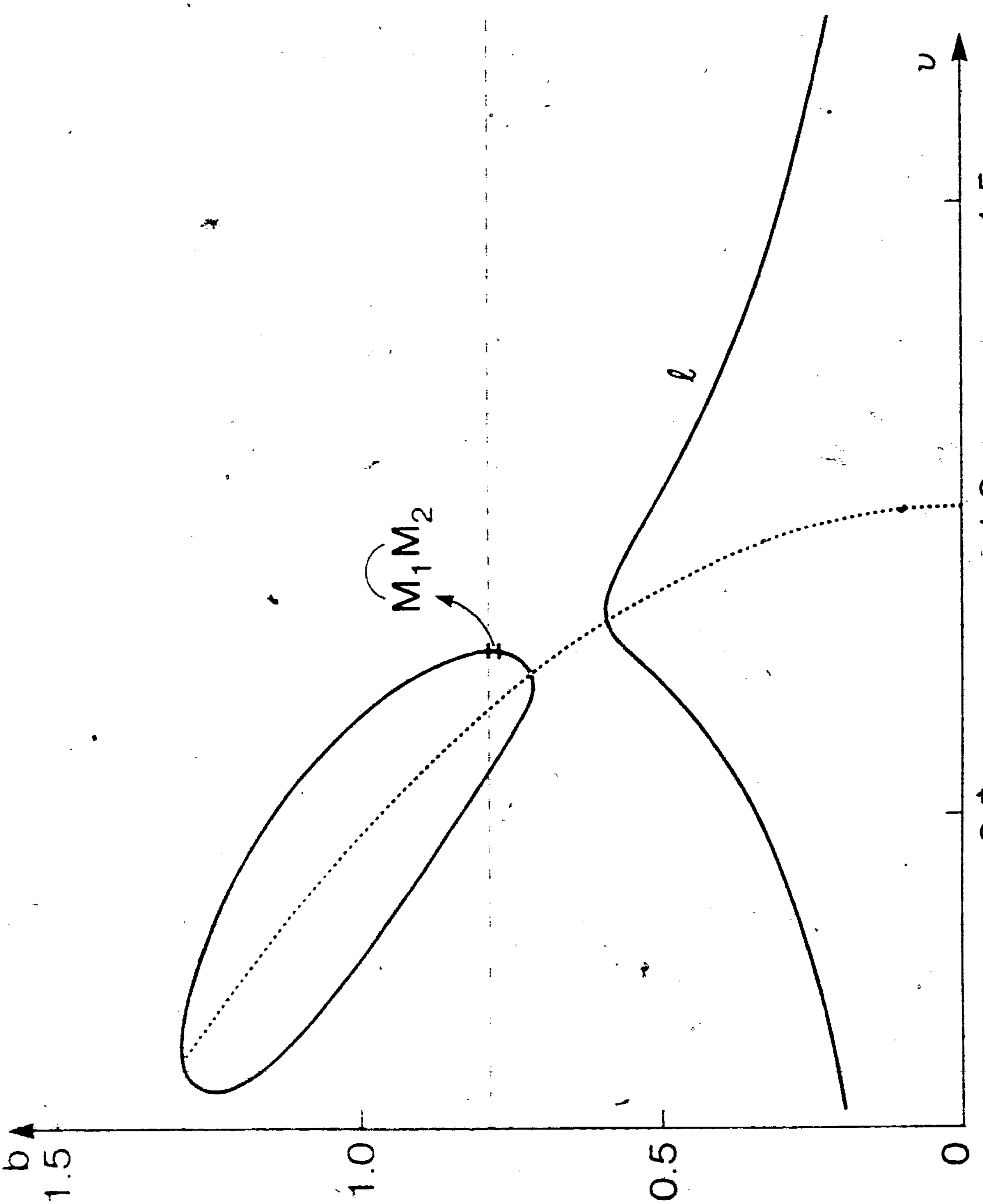

Figure 1

Figure 2