35920

NAME OF AUTHOR/*NOM DE L'AUTEUR* Marvin James Hamilton

TITLE OF THESIS/*TITRE DE LA THÈSE* On the Combined Dirac-Einstein-Maxwell Field Equations

UNIVERSITY/*UNIVERSITÉ* Simon Fraser University

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* Ph.D.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE* 1976

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DATED/*DATÉ* November 1, 1976 SIGNED/*SIGNÉ*

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ* [redacted]

ON THE COMBINED
DIRAC - EINSTEIN - MAXWELL
FIELD EQUATIONS

by

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B.S., Harvey Mudd College, 1968

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A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

October 1976

APPROVAL

Name: Marvin James Hamilton

Degree: Doctor of Philosophy

Title of Thesis: On the Combined Dirac-Einstein-Maxwell Field Equations

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Date Approved: October 7, 1976

Title of Thesis/Dissertation:

On the Combined Dirac-Einstein-Maxwell Field Equations

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November 1, 1976

(date)

ABSTRACT

This paper discusses the combined Dirac-Einstein-Maxwell equations in general relativity. A mixed tensor-spinor formalism is used. Some usual definitions of spinor structure are considered, and shown to be equivalent. The spinor calculus is then developed from an axiomatic viewpoint. The combined equations are derived from a common variational principle, and shown to satisfy the expected differential identities. A class of exact static solutions is found. These solutions are analogous to plane wave solutions in special relativity propagating along the x^3 axis which are not square integrable. Under fairly general assumptions, it is shown that there do not exist exact static solutions of the combined equations with finite total charge. It seems that the static electro-gravitational background is not compatible with localizable matter fields possessing intrinsic spin.

ACKNOWLEDGEMENT

The author wishes to thank Dr. A. Das, whose insight, inspiration, and encouragement helped to make this work possible.

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I. SPINOR STRUCTURE IN RIEMANNIAN MANIFOLDS

The purpose of this work is to investigate the combined Dirac-Einstein-Maxwell equations. Thus we are concerned with half-integer spin matter fields. While it may be possible to carry out such investigations in a purely tensorial formalism (Whittaker (36)), such a procedure seems to be awkward and unnatural. It also obscures the rather beautiful tetrad-metric relationship which exists for a manifold carrying spinor fields. Thus we choose to work with a mixed tensor-spinor formalism. In their full generality, spinors were discovered by Cartan. His book (8) is a standard reference for spinors.

The introduction of a two-component spinor theory into general relativity has a long history, beginning with the works of Weyl (34), (35), Fock (19), and Infeld and van der Waerden (17). Many authors define a spin metric, and the 'spin matrices' connecting tensors and spinors, and postulate some relations between these quantities and the metric tensor. A clear axiomatic derivation of this "algebraic" approach to spinor structure appears in Schmutzer (32). Cf. also Corson (9). In addition to its clarity, the axiomatic approach develops the complete set of spinor identities which may be consistently used, permitting the full power of the spinor technique to be brought into play.

Another approach to spinors is to relate the general relativistic spin matrices to the usual Pauli matrices via an orthonormal tetrad (OT). Algebraic identities can then be deduced from the properties of the Pauli matrices and the OT. This is basically the approach of physicists using the 'verbein' formalism.

A spin structure may also be introduced in a geometric fashion usingthe elegant constructs of modern differential geometry. This approach brings clarity and greater visualization to the subject. Penrose (27) and Geroch (15) have used this geometric definition in general relativity.

We wish to show that these three approaches to spinor structure-algebraic, tetrad, and geometric- can be made consistent with one another. We assume some familiarity with the usual spinor quantities: see Corson (9), Bade and Jehle (2), Penrose (26), or Pirani (28). Our notation and conventions: spinor indices will be denoted by capital Latin letters, with conjugate indices dashed, and take on the values 1,2. The space-time signature is taken as -2. Small Latin letters denote space-time tensor indices ranging from 1-4, and unless otherwise noted Greek letters denote spatial tensor indices ranging from 1-3. Invariant tensor or spinor indices are denoted by parentheses: thus $\chi^{(AB)}$ is a spinor invariant, while $\eta^{(mn)}$ is a tensor invariant. γ_{AB} denotes the antisymmetric spinor metric, and $\sigma^{mA'B}$

the spin matrices. Spin indices are raised and lowered using γ: $\chi^A = \gamma^{AB}\chi_B$, $\chi_A = \chi^B\gamma_{BA}$. Further, $\gamma_{A\cdot}^{\;\;B} = -\gamma_{\cdot A}^{B} = \delta_A^B$, the Kronecker delta. THe (constant) Pauli matrices are taken as:

$$\sigma^{(1)A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^{(2)A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};$$

(1)

$$\sigma^{(3)A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma^{(4)A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, $\eta^{(mn)}$ denotes the Minkowski metric (signature -2).

Let us first consider the tetrad definition of spin structure. Let M denote our space-time manifold, which we assume to be orientable and parallelizable. (If the entire manifold is not parallelizable, we take M to be a parallelizable submanifold). Let $\lambda^m_{(n)}$ be an orthonormal tetrad defined globally over M: Thus

$$(2) \qquad \lambda^m_{(n)}\lambda^{(n)k} = g^{mk},$$

or equivalently

$$(3) \qquad \lambda^m_{(n)}\lambda_{m(k)} = \eta_{(nk)}.$$

Let $\sigma^{(n)A'B}$ be the Pauli matrices. The constancy of these matrices follows from the usual coordination of proper isochronous Lorentz and unimodular spin transformations:

$$(4) \qquad \sigma^{(k)A'B} = \sigma'^{(k)A'B} = \Lambda^{(k)}_{\cdot(\ell)}U^{A'}_{\cdot C'}U^{B}_{\cdot D}\sigma^{(\ell)C'D},$$

where $U \in SL(2,C)$, and $\Lambda \in L_0$, the proper isochronous Lorentz group. Then we take the general relativistic spin matrices to be given by:

$$(5) \quad \sigma^{mA'B} = \lambda^{m}_{(n)}\, \sigma^{(n)A'B}.$$

The consistency of (4), (5) requires that a spin transformation U on $\sigma^{mA'B}$ generate the corresponding Lorentz transformation on the tetrad:

$$(6) \quad \grave{\sigma}^{mC'D} = \sigma^{mA'B}\, U^{C'}_{\cdot A'}\, U^{D}_{\cdot B} = \lambda^{m}_{(n)}\, \Lambda^{-1(n)}_{\cdot\cdot(k)}\, \sigma^{(k)C'D}.$$

Thus the transformed matrices $\grave{\sigma}^{mC'D}$ correspond to the transformed tetrad $\grave{\lambda}^{m}_{(k)} = \lambda^{m}_{(n)}\Lambda^{-1\,(n)}_{\cdot\cdot(k)}$, and the freedom of choice for the tetrad $\lambda^{m}_{(n)}$ in (5) corresponds to the freedom of a spin transformation on $\sigma^{mA'B}$.

A number of important algebraic identities involving g_{mn}, γ_{AB}, and $\sigma^{nA'B}$ follow from (5). These will be discussed after we relate this "tetrad" definition of spinor structure to an "algebraic" one.

It is also of interest to note that the Pauli matrices can be expressed in terms of a 'canonical' spin basis of C_2. Let $\overline{\xi}_A = (1,0)$ and $\overline{\tau}_A = (0,1)$. Then $(\overline{\xi},\overline{\tau})$ is a spin frame $(\overline{\xi}_A\overline{\tau}^A = 1)$, and we can define the matrices

$$(7) \quad \alpha^{(1)A'B} = -\frac{1}{\sqrt{2}}\left(\tau^{A'}\xi^{B} + \xi^{A'}\tau^{B}\right);$$

$$\alpha^{(2)A'B} = \frac{i}{\sqrt{2}}\left(\tau^{A'}\xi^{B} - \xi^{A'}\tau^{B}\right);$$

$$\alpha^{(3)A'B} = \frac{1}{\sqrt{2}}\left(\xi^{A'}\xi^{B} - \tau^{A'}\tau^{B}\right);$$

$$\alpha^{(4)A'B} = \frac{1}{\sqrt{2}}\left(\xi^{A'}\xi^{B} + \tau^{A'}\tau^{B}\right).$$

One easily checks that for $\xi = \overline{\xi}$, $\tau = \overline{\tau}$ these are the matrices defined in (1). For (ξ,τ) an arbitrary spin frame, one has $\alpha^{(n)} = U\sigma^{(n)}$, where $U(\overline{\xi},\overline{\tau}) = (\xi,\tau)$. For an arbitrary choice of spin frame the matrices (7) are not invariant under the coordinated Lorentz-spin transformations (4). One could replace the definition (5) by

$$(8) \qquad \sigma^{mA'B} = \lambda^{m}_{(n)}\alpha^{(n)A'B}$$

This merely amounts to a different association of spin frames and tetrads. (7) gives a 2-1 map of spin frames to the Hermitian matrices α.

There is still another type of spin matrices to be discussed, the invariant matrices $\sigma^{(mA'B)}$. We first quickly examine the algebra of invariant spinor components, which may be less familiar than the usual tetrad invariants. Let (ξ,τ) be a spin frame ($\xi,\tau \in C_2$, $\xi_A\tau^A = 1$). Define the spin dyad $h^{(1)}_{A} = \xi_A$, $h^{(2)}_{A} = \tau_A$. $h^{(A)}_{B}$ may be used to define invariant spinor components exactly as an OT $\lambda^{(m)}_{n}$ is used for tensor invariants. Invariant spinor indices are raised and lowered using $\gamma^{(AB)} = \gamma^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In analogy to (2), (3), one finds the identities:

$$(9) \quad h_{(A)C}\, h^{(A)}{}_{D} = \gamma_{CD} \,,$$

$$(10) \quad h^{(A)}{}_{C}\, h^{(B)C} = \gamma^{(AB)} .$$

For the mixed quantities $\sigma^{mA'B}$, consider the definition

$$(11) \quad \sigma^{(mA'B)} = \lambda^{(m)}_{n}\, h^{(A')}{}_{C'}\, h^{(B)}{}_{D}\, \sigma^{nC'D} .$$

If we choose our OT and spin frame independently of one another, this definition would have the unfortunate property that for a given OT, $\sigma^{(mA'B)}$ would depend upon the choice of spin frame. The reason for this is that a change of spin frame involves a spin transformation which should be coordinated to a Lorentz transformation. Thus OTs and spin frames should be mated and transform together. Let us suppose, then, that an OT $\overline{\lambda}^{(m)}_{n}$ has been selected, and using (5), $\sigma^{mA'B} = \overline{\lambda}^{m}_{(n)}\sigma^{(n)A'B}$. Associate to this tetrad the spin dyad $\overline{h}^{(A)}_{B}$ corresponding to the canonical spin basis $(\overline{\xi},\overline{\tau})$, where as before $\overline{\xi} = (1,0)$, $\overline{\tau} = (0,1)$. Then $\overline{h}^{(A)}_{B} = \delta^{A}_{B}$, and $\sigma^{(n)A'B} = \sigma^{(nA'B)}\big|_{(\overline{\lambda},\overline{h})}$. Given an arbitrary tetrad $\lambda^{(p)}_{n}$, there exists a Lorentz transformation Λ with $\Lambda^{(p)}{}_{(m)}\overline{\lambda}^{(m)}_{n} = \lambda^{(p)}_{n}$. Let U be the spin transformation corresponding to Λ; i.e. such that (4) holds. Define $h^{(A)}_{\ \ B} = U^{(A)}_{\ \ (C)}\bar{h}^{(C)}_{\ \ B}$. One checks that $h^{(A)}_{\ \ B}$ is a spin dyad. Then we associate the dyad h to the OT λ. Then

$$\sigma^{(mA'B)}\Big|_{(\lambda,h)} = \sigma^{nC'D}\lambda^{(m)}_{\ \ n}h^{(A')}_{\ \ C'}h^{(B)}_{\ \ D}$$

$$= \bar{\lambda}^{n}_{\ (p)}\sigma^{(p)C'D}\Lambda^{(m)}_{\ \ (q)}\bar{\lambda}^{(q)}_{\ \ n}U^{(A')}_{\ \ (F')}U^{(B)}_{\ \ (G)}\bar{h}^{(F')}_{\ \ C'}\bar{h}^{(G)}_{\ \ D}$$

$$= \sigma^{(p)C'D}\Lambda^{(m)}_{\ \ (p)}U^{(A')}_{\ \ (C')}U^{(B)}_{\ \ (D)} = \sigma^{(m)A'B}.$$

Thus with such a coupling of spin frames and OTs, we have $\sigma^{(m)A'B} = \sigma^{(mA'B)}$: the invariant spin matrices are the Pauli matrices in all frames. This invariant work provides the motivation for the definition of spinor structure which follows.

We consider the "geometric" definition of spinor structure. For greater detail, see Geroch (15) and Penrose (27). Let M be parallelizable and orientable as before, and let B denote the principal fiber of oriented orthonormal tetrads over M. Let $\tilde{B}$ denote the principal fiber bundle of spin frames over M: if $p \in M$, the fiber over p is the set $\{(p,(\xi,\tau)) : \xi,\tau \in C_2, \xi_A\tau^A = 1\}$. The group for B is taken to be the proper isochronous Lorentz group L_0, and for $\tilde{B}$ the universal covering space of L_0, SL(2,C). (We do not discuss the enlargement to the full Lorentz group here). Then a spinor structure on M is a

2-1 fiber-preserving surjection σ: $\tilde{B} \to B$ which commutes with the group operations:

Fig. (1)
$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\sigma} & B \\ U\downarrow & \circlearrowright & \downarrow\Lambda \\ \tilde{B} & \xrightarrow{\sigma} & B \end{array}$$

where U represents Λ.

Now suppose we are given a tetrad spin structure: i.e. we are given an OT $\bar{\lambda}^{m}_{(n)}$, and define $\sigma^{mA'B} = \bar{\lambda}^{m}_{(n)}\sigma^{(n)A'B}$ as in (5). Using the properties of the Pauli matrices, this may be inverted to give

$$(12)\quad \bar{\lambda}^{m}_{(n)} = \sigma^{mA'B}\sigma_{(n)A'B} .$$

Let $(\bar{\xi},\bar{\tau})$ be the canonical spin frame, and define $\sigma(p,(\bar{\xi},\bar{\tau})) = (p,\sigma^{mA'B}\sigma_{(n)A'B}) = (p,\bar{\lambda}^{m}_{(n)}) \in B$. If (ξ,τ) is an arbitrary spin frame with $(\xi,\tau) = U(\bar{\xi},\bar{\tau})$, define $(p,(\xi,\tau)) = (p,U^{C'}{}_{A'}U^{D}{}_{B}\sigma^{mA'B}\sigma_{(n)C'D})$; i.e. we use the transformed matrices $U(\sigma^{mA'B})$ to generate the map σ. Note $\sigma(p,(-\xi,-\tau)) = \sigma(p,(\xi,\tau))$, and one easily checks that σ is a 2-1 map of $\tilde{B}$ onto B. We denote the maps of fig. (1) by $\bar{U},\bar{\Lambda}$. Then

$$[\bar{\Lambda}\circ\sigma(p,(\xi,\tau))]^{m}_{(k)} = U^{A'}{}_{F'}U^{B}{}_{G}\,\sigma^{mF'G}\sigma_{(n)A'B}\,\bar{\Lambda}^{-1(n)}{}_{(k)}$$

$$= U^{A'}{}_{F'}U^{B}{}_{G}\,\sigma^{mF'G}\sigma_{(k)C'D}\,\bar{U}^{C'}{}_{A'}\bar{U}^{D}{}_{B} ,$$

since $\bar{U}$ represents $\bar{\Lambda}$. Similarly,

$$[\sigma\circ\bar{U}(p,(\xi,\tau))]^{m}_{(k)} = \sigma\circ\bar{U}(p,U(\bar{\xi},\bar{\tau}))^{m}_{(k)}$$

$$= \sigma\circ\bar{U}U(p,(\bar{\xi},\bar{\tau}))^{m}_{(k)} = \bar{U}^{C'}{}_{A'}\,\bar{U}^{D}{}_{B}\,U^{A'}{}_{F'}\,U^{B}{}_{G}\,\sigma^{mF'G}\,\sigma_{(k)C'D}.$$

Thus fig. (1) commutes, and we have shown that a "tetrad" spinor structure gives rise to a "geometric" spinor structure.

The converse result is trivial. Given a "geometric" spinor structure, let $(p,(\bar{\xi},\bar{\tau})) = (p,\bar{\lambda}^{m}_{(k)})$. Then define $\sigma^{mA'B} = \bar{\lambda}^{m}_{(k)}\sigma^{(k)A'B}$. If $(\xi,\tau) = U(\bar{\xi},\bar{\tau})$, the commutativity of fig. (1) implies $\sigma(\xi,\tau)^{m}_{(n)} = \bar{\lambda}^{m}_{(k)}\Lambda^{-1(k)}{}_{(n)}$, where U represents Λ. Then $U(\sigma^{mA'B}) = \bar{\lambda}^{m}_{(k)} \times \Lambda^{-1(k)}{}_{(n)}\sigma^{(n)A'B}$, verifying (6).

Next, we consider an "algebraic" definition of spinor structure. Suppose g_{ij}, $\sigma^{mA'B}$, γ^{AB} are chosen on M such that

$$\sigma^{mB'C}\sigma^{n}{}_{B'}{}^{D} = -\tfrac{1}{2}g^{mn}\gamma^{CD} + \tfrac{i}{2}\eta^{mnsr}\sigma_{s}{}^{B'C}\sigma_{rB'}{}^{D}, \tag{13}$$

where the $\sigma^{mA'B}$ are assumed to be Hermitian, and

$$\eta^{mnsr} = -\frac{1}{\sqrt{-g}}\delta^{mnsr}\;;\quad \eta_{mnsr} = \sqrt{-g}\,\delta_{mnsr}.$$

Here δ_{mnsr} is the generalized Kronecker delta, and η_{mnsr} is usually called the Levi-Civita tensor. Then M will be said to possess an algebraic spinor structure.

In particular, one may check that (13) is satisfied if $g^{mn} = \eta^{mn}$, the Minkowski metric, $\gamma^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\sigma^{mB'A}$ the Pauli matrices (1). In this case, the symmetric part of (13) relates the Pauli matrices to the metric, and the antisymmetric part gives the usual commutation relations. (13) may be found in Corson (9) and Schmutzer (32).

We wish to show the equivalence of an algebraic spinor structure and a tetrad spinor structure. Suppose first we are given an algebraic spinor structure. We wish to show the existence of an OT $\lambda^m_{(n)}$ satisfying (5), where $\sigma^{mA'B}$ is given by the algebraic structure. Define $\lambda^m_{(n)} = \sigma^{mA'B}\sigma_{(n)A'B}$. Then (5) follows from the properties of the Pauli matrices, and further

$$\lambda^m_{(n)}\lambda^{(n)k} = \sigma^{mA'B}\sigma_{(n)A'B}\sigma^{(n)C'D}\sigma^k{}_{C'D}$$

$$= \sigma^{mA'B}\sigma^k{}_{A'B} = g^{mk}.$$

Here we have used the identities

$$\sigma_{(n)A'B}\sigma^{(n)C'D} = \delta^{C'}_{A'}\delta^D_B \quad ,$$

$$\sigma^{mA'B}\sigma^k{}_{A'B} = g^{mk}.$$

These identities will be shown to follow from (13). Thus $\lambda^m_{(n)}$ is an OT, and existence of an algebraic spinor structure implies existence of a tetrad spinor structure.

Conversely, suppose we are given a set of spin matrices satisfying (5). The symmetric part of (13) may be written

(14) $$\sigma^{mB'C}\sigma^{n}{}_{B'D}+\sigma^{nB'C}\sigma^{m}{}_{B'D}=g^{mn}\delta^{C}_{D}.$$

To see this is satisfied, using (5) and the properties of the Pauli matrices we have

$$\sigma^{mB'C}\sigma^{n}{}_{B'D}+\sigma^{nB'C}\sigma_{mB'D}=\lambda^{m}_{(k)}\lambda^{n}_{(\ell)}\left[\sigma^{(k)B'C}\sigma^{(\ell)}{}_{B'D}+\right.$$
$$\left.+\sigma^{(\ell)B'C}\sigma^{(k)}{}_{B'D}\right]=\lambda^{m}_{(k)}\lambda^{n}_{(\ell)}\eta^{(k\ell)}\delta^{C}_{D}$$
$$=g^{mn}\delta^{C}_{D}.$$

Therefore a tetrad spinor structure satisfies the symmetric part of (13). Similarly, we have for the antisymmetric part of (13)

(15) $$\sigma^{mB'C}\sigma^{n}{}_{B'D}-\sigma^{nB'C}\sigma^{m}{}_{B'D}=i\,\eta^{mnsr}\sigma_{s}{}^{B'C}\sigma_{rB'D}.$$

Then

$$\sigma^{mB'C}\sigma^{n}{}_{B'D}-\sigma^{nB'C}\sigma_{mB'D}=\lambda^{m}_{(k)}\lambda^{n}_{(\ell)}\left[\sigma^{(k)B'C}\sigma^{(\ell)}{}_{B'D}+\right.$$
$$\left.-\sigma^{(\ell)B'C}\sigma^{k}{}_{B'D}\right]=i\,\lambda^{m}_{(k)}\lambda^{n}_{(\ell)}\eta^{(k\ell sr)}\sigma_{(s)}{}^{B'C}\sigma_{(r)B'D}$$
$$=i\,\lambda^{m}_{(k)}\lambda^{n}_{(\ell)}\lambda^{p}_{(s)}\lambda^{q}_{(r)}\eta^{(k\ell sr)}\sigma_{p}{}^{B'C}\sigma_{qB'D}$$

$$= i\eta^{mnsr}\sigma_s{}^{B'C}\sigma_{rB'D}.$$

Therefore a tetrad spinor structure is an algebraic spinor structure.

Altogether, we have shown that the existence of any of the three types of spinor structure--algebraic, tetrad, or geometric--implies the existence of the others. Besides its simplicity, this result is of practical importance. Given a metric g_{ij}, it is easiest to construct a spinor structure locally by finding an OT and using the tetrad definition. But then all the algebraic identities which follow from (13) are available in the resultant system.

This result also gives an easy proof that any set of Hermitian matrices $\sigma^{mA'B}$ satisfying (13) must be algebraically independent. Suppose $\alpha_m\sigma^{mA'B} = 0$, where α_m are complex scalar fields. By the above, there exists an OT $\lambda^m_{(n)}$ such that $\sigma^{mA'B} = \lambda^m_{(n)}\sigma^{(n)A'B}$. Then $\alpha_m\lambda^m_{(n)}\sigma^{(n)A'B} = 0$. By the algebraic independence of the Pauli matrices, this implies $\alpha_m\lambda^m_{(n)} = 0$. Then $\alpha_m\lambda^m_{(n)}\lambda^{(n)}_q = 0$, or $\alpha_q = 0$ for all q.

In this chapter we have taken $\gamma_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and have restricted our attention to unimodular spin transformations. In later work, we consider the more general form

(16) $$\gamma_{AB} = r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad r \equiv \gamma^{\frac{1}{2}}e^{i\theta},$$

and correspondingly allow arbitrary non-singular spin transformations;

i.e. the complete Gl(2,$\mathbb{C}$) group. This necessitates certain changes in our formulae. Given (16), we take for the canonical spin dyad

$$(17) \quad \overline{\xi}_A = r^{1/2}(1,0), \; \overline{\tau}_A = r^{1/2}(0,1).$$

Then $\overline{h}^{(A)}_{\;B} = r^{\frac{1}{2}}\delta^A_B$, and $\gamma^{(AB)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq \gamma^{AB}$. In particular, $\sigma^{(n)A'B} \neq \sigma^{(nA'B)}$, where $\sigma^{(nA'B)}$ are the usual Pauli matrices. Thus the matrices appearing in (1), (7) become $\sigma^{(nA'B)}$, $\alpha^{(nA'B)}$ respectively.

A spinor structure contains the entities g_{ij}, $\lambda^{(m)}_{\;n}$, $\sigma^{nA'B}$, and $h^{\;A}_{(B)}$. From an algebraic viewpoint, there is no reason for regarding some of these entities as more fundamental than the others. Bergmann (5) discusses this with reference to $\sigma^{mA'B}$ and g_{ij}. For example, given $\lambda^{(m)}_{\;n}$ and $h^{\;A}_{(B)}$, one may generate g_{ij}. Further, $h^{\;A}_{(B)}$ gives rise to $\sigma^{(nA'B)}$ (up to a spin transformation) by (7), and then $\sigma^{nA'B}$ follows from (8). Obviously other choices for the independent quantities are possible. From the variational work of Chapter III, however, we shall see that at least for the case of half-integer spin fields there is some reason for regarding the tetrad $\lambda^{(m)}_{\;n}$ as more fundamental than the metric tensor g_{ij}. Penrose (27) has considered the spinor structure of space-time as more basic than its metric structure.

In this chapter we have assumed the existence of a spinor structure. This is a natural assumption since our aim is to investigate the combined Dirac-Einstein-Maxwell equations, and therefore any solution manifolds will contain spinor Dirac fields. It is well-known that the global existence of a spinor structure implies orientability of the manifold (Penrose (27)), and Geroch (15) has shown that a non-compact space-time admits a spinor structure iff it contains a global field of orthonormal tetrads. This result in turn may be used to show that an oriented space-time ~~which~~ has a Cauchy surface, or a space-time whose Weyl tensor is everywhere of type [1,1,1,1], [2,1,1], [3,1], or [4] admits a spinor structure (Geroch, (15a)).

Because of the way relativity is usually approached, g_{ij} is often given a prescribed form (static, stationary, ...), and one wishes to obtain a consistent spinor structure. We have shown that this reduces to finding an OT for g_{ij}. In Appendix B, an explicit OT is constructed for a prescribed metric. While not always the most natural in the case of special symmetries, this explicitly defined tetrad makes the matching of a spinor structure to a metric an entirely constructive procedure. In general, of course, such a procedure is only possible locally.

II. SPINOR CALCULUS

In this chapter we develop the spinor calculus based upon the definitions of spinor structure given in Chapter I. We use the algebraic approach, so that all the usual spinor identities are seen to follow from the definitions and three axioms. These axioms may be found in Schmutzer (32). As before, γ_{AB} denotes the antisymmetric spinor metric. Spin indices are raised and lowered using γ:

$$\text{(1)} \qquad \chi^A = \gamma^{AB}\chi_B \,; \quad \chi_A = \chi^B \gamma_{BA} \,.$$

For the mixed components,

$$\text{(2)} \qquad \gamma_{A\cdot}{}^{B} = -\gamma^{B}{}_{\cdot A} = \delta^B_A \,.$$

One note of caution must be made. Since our aim is to use the spinor calculus to investigate the combined Dirac-Einstein-Maxwell equations we do not assume the covariant derivative of γ_{AB} to be zero. Our choice $\gamma_{AB};_k \neq 0$ is made in order to introduce the electromagnetic potential in as natural a way as possible. This choice is also consistent with the philosophy of minimal electromagnetic interaction (7). Bergmann (5) shows the electromagnetic field can be introduced with $\gamma_{AB};_k = 0$, but at the price of making γ a spinor density. Our choice increases the generality of the formalism over that employed by Penrose et. al., with a corresponding increase in the complexity of formulae. In particular, our expression for the spinor affinities is affected by this choice. See Bade and Jehle (2) and Penrose (26) for further discussion of this point.

The antisymmetric spin metric γ_{AB} is determined by a single complex value. Following Bade and Jehle, we write

(3) $$\gamma_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma^{\frac{1}{2}} e^{i\theta}$$

Then

(4) $$\gamma^{AB}\gamma^{CD} + \gamma^{BC}\gamma^{AD} + \gamma^{CA}\gamma^{BD} = 0,$$

and if η_{ABC} is an arbitrary 3-spinor,

(5) $$\gamma^{AB}(\eta_{ABC} + \eta_{CAB} + \eta_{BCA}) = 0.$$

If χ_A, ψ_A are spinors, then

(6) $$\chi_A \psi^A = -\psi_A \chi^A.$$

In particular, spinors of odd rank have zero "absolute value". Let η_{AN} be an arbitrary symmetric spinor. Then $\eta_A{}^A = -\gamma^{CA}\eta_{AC} = -\gamma^{CA}\eta_{CA}$ $= -\eta_C{}^C$, or

(7) $$\eta_A{}^A = 0.$$

Next, suppose $\eta_{AB} = -\eta_{BA}$. Then $\eta_A{}^A = \gamma^{AC}\eta_{AC} = (\gamma_{12})^{-1}(\eta_{12} +$

$- \eta_{21}) = 2(\gamma_{12})^{-1}\eta_{12}$, or

(8) $$\eta_{AB} = \frac{1}{2}\eta_{C\cdot}^{\ C}\gamma_{AB}.$$

If χ_{AB} is arbitrary, this implies

(9) $$\chi_{AB} - \chi_{BA} = \chi_{C\cdot}^{\ C}\gamma_{AB}.$$

Let (4) operate on the arbitrary 4-spinor η_{ABCD}, yielding

(10) $$\eta_{A\cdot C\cdot}^{\ A\ C} + \eta_{AB\cdot\cdot}^{\ \ BA} + \eta_{\cdot BC\cdot}^{C\ \ B} = 0.$$

Finally, (5) may be written

(11) $$\xi_A\eta^A\tau_C + \xi_C\eta_A\tau^A + \xi^A\eta_C\tau_A = 0,$$ or

(12) $$\eta_{A\cdot C}^{\ A} + \eta_{CA\cdot}^{\ \ A} + \eta_{\cdot CA}^{A} = 0,$$

where ξ_A, η_C, τ_A, η_{ABC} are arbitrary.

We next develop the properties of the spin matrices $\sigma^{mA'B}$. Our first axiom requires their Hermicity:

AXIOM I $$\sigma^{mB'A} = \sigma^{mAB'}.$$

Note the transpose of $\sigma^{mA'B}$ is $\sigma^{mB'A}$. Axiom II relates the basic quantities $\sigma^{mA'B}$, g_{ij}, and γ_{AB}, and may be looked upon as the generalization of the commutation relations for the Pauli matrices to general relativity. See the discussion following I-(13) for greater detail. This form of Axiom II appears in Schmutzer (32) and in Corson (9):

AXIOM II $\sigma_m{}^{B'}{}_A \sigma_{nB'C} = -\frac{1}{2} g_{mn}\gamma_{AC} + B\eta_{nmsr}\sigma^{sB'}{}_A\sigma^r{}_{B'C}$.

The Levi-Civita tensor η_{mnsr} was defined following I-(13). The constant B remains to be determined. The first and second contractions of η will be used frequently in the following:

$$\eta_{abcd}\eta^{mnsd} = \delta^m_a\delta^n_c\delta^s_b + \delta^m_c\delta^n_b\delta^s_a + \delta^m_b\delta^n_a\delta^s_c - \delta^m_a\delta^n_b\delta^s_c - \delta^m_c\delta^n_a\delta^s_b - \delta^m_b\delta^n_c\delta^s_a, \tag{13}$$

$$\eta_{abcd}\eta^{cdmn} = 2(\delta^m_b\delta^n_a - \delta^m_a\delta^n_b). \tag{14}$$

From Axiom II it follows that the spin matrices are an algebraically independent basis for the set of 2×2 complex matrices. The proof appears in Ch. I. To determine the constant B in Axiom II,

$$\sigma_m{}^{B'}{}_A\sigma_{nB'C} = -\frac{1}{2} g_{mn}\gamma_{AC} + B\eta_{mnsr}(-\frac{1}{2} g^{sr}\gamma_{AC} +$$

$$+ B\eta^{srab}\sigma_a{}^{B'}{}_A\sigma_{bB'C}\quad, \text{ or}$$

$$\sigma_m{}^{B'}{}_A\sigma_{nB'C} = -\frac{1}{2} g_{mn}\gamma_{AC} + 2B^2(\delta^a_n\delta^b_m - \delta^a_m\delta^b_n)\sigma_a{}^{B'}{}_A\sigma_{bB'C}.$$

Then

$$-\frac{1}{2} g_{mn}\gamma_{AC} + B\eta_{mnsr}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C} = -\frac{1}{2} g_{mn}\gamma_{AC} +$$

$$+ 2B^2(\sigma_n{}^{B'}{}_{A}\sigma_{mB'C} - \sigma_m{}^{B'}{}_{A}\sigma_{nB'C}) \,,$$ or

$$\eta_{mnsr}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C} = -2B^2(2)\eta_{mnsr}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C} \,,$$

and this gives

(15) $$B = \pm\frac{i}{2} \,.$$

We take $B = + \frac{i}{2}$ to agree with the results for special relativity (signature -2). In the following, we mark the most useful identities with an asterisk, (*).

From the symmetric and antisymmetric parts of Axiom II, we have

*(16) $$\sigma_m{}^{B'}{}_{A}\sigma_{nB'C} + \sigma_n{}^{B'}{}_{A}\sigma_{mB'C} = -g_{mn}\gamma_{AC} \,,$$ and

*(17) $$\sigma_m{}^{B'}{}_{A}\sigma_{nB'C} - \sigma_n{}^{B'}{}_{A}\sigma_{mB'C} = i\eta_{mnsr}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C} \,.$$

From (16),

(18) $$\sigma_m{}^{B'A}\sigma_{nB'A} + \sigma_n{}^{B'A}\sigma_{mB'C} = -g_{mn}\gamma^{A}{}_{A} \,,$$ or

*(19) $$\sigma_m{}^{B'A}\sigma_{nB'A} = g_{mn} \,.$$

Using (16) again,

$$\sigma^{mB'}{}_{A}\sigma_{mB'C} + \sigma^{mB'}{}_{A}\sigma_{mB'C} = -\delta^{m}_{m}\gamma_{AC} \quad , \qquad \text{or}$$

$$2\sigma^{mB'A}\sigma_{mB'C} = -4\gamma^{A}{}_{\cdot C} = 4\delta^{A}_{C} \quad , \qquad \text{giving the identity}$$

$$*(20) \quad \sigma^{mB'A}\sigma_{mB'C} = 2\delta^{A}{}_{\cdot C} \; .$$

Multiplying Axiom II by $\eta_{mntp}\sigma^{t}{}_{D'}{}^{C}$, we have

$$\eta_{mntp}\sigma^{t}{}_{D'}{}^{C}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C} = -\tfrac{1}{2}g^{mn}\gamma_{AC}\,\eta_{mntp}\sigma^{t}{}_{D'}{}^{C}$$

$$+\tfrac{i}{2}\eta_{mntp}\eta^{mn}{}_{\cdot\cdot sr}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C}\sigma^{t}{}_{D'}{}^{C}$$

$$= i(g_{sp}g_{rt} - g_{st}g_{rp})\sigma^{t}{}_{D'}{}^{C}\sigma^{sB'}{}_{A}\sigma^{r}{}_{B'C} \qquad \text{or}$$

$$(21) \quad \eta_{mntp}\sigma^{t}{}_{D'}{}^{C}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C} = i(\sigma_{p}{}^{B'}{}_{A}\sigma_{tB'C}\sigma^{t}{}_{D'}{}^{C}$$

$$-\sigma^{t}{}_{D'}{}^{C}\sigma_{t}{}^{B'}{}_{A}\sigma_{pB'C}).$$

Using Axiom II and (20), (21) becomes

$$\eta_{mntp}\sigma^{t}{}_{D'}{}^{C}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C} = i[-2\sigma_{p}{}^{B'}{}_{A}\gamma_{D'B'} - \sigma^{tB'}{}_{A}(-\tfrac{1}{2}g_{tp}\gamma_{D'B'}$$

$$-\tfrac{i}{2}\eta_{tpsr}\sigma^{sC}{}_{D'}\sigma^{r}{}_{CB'})]$$

$$= 2i\,\sigma_{pD'A} - \tfrac{i}{2}\sigma_{pD'A} - \tfrac{1}{2}\eta_{tpsr}\sigma^{sC}{}_{D'}\sigma^{r}{}_{CB'}\sigma^{tB'}{}_{A}\,,$$ or

$$\eta_{mntp}\sigma^{t}{}_{D'}{}^{C}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C} + \tfrac{1}{2}\eta_{tpmn}\sigma^{mC}{}_{D'}\sigma^{n}{}_{CB'}\sigma^{tB'}{}_{A}$$

$$= \tfrac{3}{2}i\,\sigma_{pD'A}\,.$$ Then

$$\tfrac{1}{2}\eta_{mntp}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C}\sigma^{t}{}_{D'}{}^{C} = \tfrac{3}{2}i\,\sigma_{pD'A}\,,$$ or

*(22) $$\sigma_{pD'A} = -\tfrac{i}{3}\eta_{mntp}\sigma^{mB'}{}_{A}\sigma^{n}{}_{B'C}\sigma^{t}{}_{D'}{}^{C}.$$

Again using Axiom II, (22) gives

(23) $$\sigma_{pD'A} = -\tfrac{2}{3}\sigma^{t}{}_{D'}{}^{C}(\sigma_{t}{}^{B'}{}_{A}\sigma_{pB'C} + \tfrac{1}{2}g_{tp}\gamma_{AC}),$$ or

$$\sigma_{pD'A} = -\tfrac{2}{3}\sigma^{t}{}_{D'}{}^{C}\sigma_{t}{}^{B'}{}_{A}\sigma_{pB'C} + \tfrac{1}{3}\sigma_{pD'A}\,.$$ Thus

(24) $$\sigma_{pD'A} = -\sigma^{t}{}_{D'}{}^{C}\sigma_{t}{}^{B'}{}_{A}\sigma_{pB'C}\,.$$

But $\sigma_{pD'A} = \sigma_{p}{}^{B'C}\gamma_{B'D'}\gamma_{CA}\,.$

Then $\sigma_{p}{}^{B'C}(\delta^{D'}_{B'}\delta^{A}_{C} + \sigma^{tD'}{}_{C}\sigma_{tB'}{}^{A}) = 0.$

Since the spin matrices form a basis for the complex 2×2 matrices, it follows that

$$*(25)\quad \sigma^{t}{}_{D'C}\,\sigma_{tB'A} = \gamma_{D'B'}\gamma_{CA}\,.$$

Next, multiplying (22) by η^{abdp} and using (13),

$$3i\eta^{abdp}\sigma_{pD'A} = \sigma^{aB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{b}{}_{D'}{}^{C} + \sigma^{dB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}$$

$$+\sigma^{bB'}{}_{A}\sigma^{a}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} - \sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} - \sigma^{bB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}$$

$$-\sigma^{dB'}{}_{A}\sigma^{a}{}_{B'C}\sigma^{b}{}_{D'}{}^{C},$$ or

$$\sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} = \sigma^{aB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{b}{}_{D'}{}^{C} + \sigma^{dB'}{}_{A}(\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}$$

$$-\sigma^{a}{}_{B'C}\sigma^{b}{}_{D'}{}^{C}) + \sigma^{bB'}{}_{A}(\sigma^{a}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} - \sigma^{d}{}_{B'C}\sigma^{a}{}_{D'}{}^{C})$$

$$-3i\eta^{abdp}\sigma_{pD'A}\,.$$ Using (16),

$$(26)\quad 2\sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} = g^{bd}\sigma^{a}{}_{D'A} - g^{ab}\sigma^{d}{}_{D'A}$$

$$+g^{ad}\sigma^{b}{}_{D'A} - 3i\eta^{abdp}\sigma_{pD'A} + 2\sigma^{dB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}$$

$$-2\sigma^{bB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}.$$

On the right side of (26), consider first $\sigma^{dB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C}$. Repeated application of (16) gives

$$\sigma^{dB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C} = -\sigma^{dB'}{}_{A}\sigma^{a}{}_{B'C}\sigma^{b}{}_{D'}{}^{C} + g^{ab}\gamma_{B'D'}\sigma^{dB'}{}_{A}$$

$$= \sigma^{aB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{b}{}_{D'}{}^{C} + g^{ab}\sigma^{d}{}_{D'A} + g^{ad}\sigma^{b}{}_{D'}{}^{C}\gamma_{AC},$$ or

$$(27)\quad \sigma^{dB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{a}{}_{D'}{}^{C} = -\sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} + g^{ab}\sigma^{d}{}_{D'A}$$

$$-g^{ad}\sigma^{b}{}_{D'A} + g^{bd}\sigma^{a}{}_{D'A}.$$ Similarly,

$$-\sigma^{bB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{a}{}_{D'}{}^{C} = \sigma^{bB'}{}_{A}\sigma^{a}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} - g^{ad}\gamma_{B'D'}\sigma^{bB'}{}_{A},$$

and a final application of (16) gives

$$(28)\quad -\sigma^{bB'}{}_{A}\sigma^{d}{}_{B'C}\sigma^{a}{}_{D'}{}^{C} = -\sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} + g^{ab}\sigma^{d}{}_{D'A}$$

$$-2g^{ad}\sigma^{b}{}_{D'A}.$$

Substituting (27), (28) into (26) yields

$$*(29)\quad \sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C} = \frac{1}{2}\left(g^{bd}\sigma^{a}{}_{D'A} - g^{ad}\sigma^{b}{}_{D'A}\right.$$

$$\left.+g^{ab}\sigma^{d}{}_{D'A}\right) - \frac{i}{2}\eta^{abdp}\sigma_{pD'A}.$$

As will be seen later, this is a particularly useful identity. It appears in Corson and Schmutzer, but is generally not explicitly mentioned in other works on spinor algebra. Multiplication of (29) with $\sigma^{fD'}{}_{A}$ yields

$$*(30)\quad \sigma^{aB'}{}_{A}\sigma^{b}{}_{B'C}\sigma^{d}{}_{D'}{}^{C}\sigma^{fD'A} = \tfrac{1}{2}\left(g^{bd}g^{af} - g^{ad}g^{bf} + g^{ab}g^{df}\right) - \tfrac{i}{2}\eta^{abdf}.$$

The identities (16), (17), (19), (20), (22), (25), (29), and (30) are thus all seen to be algebraic consequences of Axioms I, II. These identities may be looked upon as an unraveling of the generalized commutation relations given by Axiom II.

Next, we make the usual correspondence between second order Hermitian spinors $A_{K'L} = A_{LK'}$ and ordinary tensors A_m by setting

$$*(31)\quad A_{K'L} = \sigma^{m}{}_{K'L}A_m,$$

or using (19) the inverse equation

$$*(32)\quad A^m = \sigma^{mK'L}A_{K'L}.$$

Note that given the correspondence (31), the requirement that $A_{K'L}$ be Hermitian is equivalent to Axiom I. This correspondence preserves inner product structures:

$$(33)\quad g^{mn}A_m A_n = \gamma^{A'B'}\gamma^{CD}A_{B'D}A_{A'C}\,.$$

We next investigate covariant spinor differentiation. Let ψ_A denote an arbitrary 1-spinor. As usual, we introduce the spinor affinities Γ^{A}_{Bm}:

$$*(34)\quad \psi_{A;m} \equiv \psi_{A,m} - \Gamma^{B}_{Am}\psi_B \quad ;$$

$$\psi^{A}{}_{;m} \equiv \psi^{A}{}_{,m} + \Gamma^{A}_{Bm}\psi^{B}\,,$$

with similar rules for higher-order spinors. We have the usual product rule for derivatives, and if Λ^{A}_{B} represents an arbitrary spin transformation, the requirement that $\psi^{A}{}_{;m}$ and $\psi_{A;m}$ transform covariantly yields the transformation law

$$(35)\quad \Gamma'^{A}_{Dm} = \Gamma^{B}_{Cm}\Lambda^{A}_{B}\Lambda^{-1C}_{D} + \Lambda^{A}_{C}\Lambda^{-1C}_{D,m}$$

$$= \Gamma^{B}_{Cm}\Lambda^{A}_{B}\Lambda^{-1C}_{D} - \Lambda^{A}_{C,m}\Lambda^{-1C}_{D}\,.$$

For mixed quantities, we define

$$\chi^{mA}{}_{;n} = \chi^{mA}{}_{,n} + \left\{{m \atop nk}\right\}\chi^{kA} + \Gamma^{A}_{Bn}\chi^{mB}\,,$$

with the obvious extention to higher order objects. Here $\left\{{m \atop nk}\right\}$ denote the usual Christoffel symbols. We adopt as our third axiom:

AXIOM III $\sigma^{mA'B}{}_{;n} = 0.$

Axiom III is equivalent to the requirement

$$(36) \quad A^{k}{}_{;m} = \sigma^{kA'B} A_{A'B;m}.$$

Writing Axiom III explicitly,

$$(37) \quad \sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B} + \Gamma^{A'}{}_{C'k}\, \sigma^{mC'B} + \Gamma^{B}{}_{Dk}\, \sigma^{mA'D} = 0.$$

If T^m is an arbitrary tensor, and $T^m{}_{|k}$ the usual tensorial derivative,

$$*(38) \quad T^{m}{}_{|k} = T^{m}{}_{;k}.$$

To see this,

$$(\sigma^{mA'B} T_{A'B})_{;k} = \sigma^{mA'B} T_{A'B;k} = \sigma^{mA'B}(T_{A'B,k} - \Gamma^{C'}{}_{A'k} T_{C'B}$$

$$- \Gamma^{C}{}_{Bk} T_{A'C}) = (\sigma^{mA'B} T_{A'B})_{,k} - \sigma^{mA'B}{}_{,k} T_{A'B} - \sigma^{mA'B} T_{C'B} \Gamma^{C'}{}_{A'k}$$

$$- \sigma^{mA'B} T_{A'C} \Gamma^{C}{}_{Bk}.$$

Then $T^m{}_{|k} = T^m{}_{;k}$ iff

$$T^{m}{}_{,k} - \sigma^{mA'B}{}_{,k} T_{A'B} - \sigma^{mC'B} T_{A'B} \Gamma^{A'}{}_{C'k} - \sigma^{mA'C} T_{A'B} \Gamma^{B}{}_{Ck}$$

$$= T^{m}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B} T_{A'B}.$$

This is true iff

$$T_{A'B}\left(\sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B} + \Gamma^{A'}{}_{C'k}\sigma^{mC'B} + \Gamma^{B}{}_{Ck}\sigma^{mA'C}\right) = 0.$$

But this is just $T_{A'B}\sigma^{mA'B}{}_{;k}$, and the result follows from Axiom III. Using (38), we let $|$ represent either tensor or spinor derivatives. From (37),

$$\sigma_{mC'B}\,\sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B}\sigma_{mC'B} + \Gamma^{A'}{}_{D'k}\sigma^{mD'B}\sigma_{mC'B}$$

$$+ \Gamma^{B}{}_{Dk}\sigma^{mA'D}\sigma_{mC'B} = 0, \qquad \text{or}$$

$$\sigma_{mC'B}\,\sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B}\sigma_{mC'B} + 2\Gamma^{A'}{}_{C'k} + \Gamma^{B}{}_{Bk}\delta^{A'}_{C'} = 0.$$

Then

$$\text{(39)} \quad \Gamma^{A'}{}_{C'k} = -\tfrac{1}{2}\left[\sigma_{mC'B}\sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\} \sigma^{nA'B}\sigma_{mC'B} + \delta^{A'}_{C'}\Gamma^{B}{}_{Bk}\right].$$

We return to this result shortly. From (19),

$$\text{(40)} \quad g^{mn} = \sigma^{mA'B}\sigma^{n}{}_{A'B} = \sigma^{mA'B}\sigma^{nC'D}\gamma_{C'A'}\gamma_{DB}.$$

Axiom III and (40) give

$$\text{*(41)} \quad \left(\gamma_{C'A'}\gamma_{DB}\right)_{|k} = 0.$$

From the antisymmetry of γ_{AB}, this gives

$$(42)\quad (\gamma_{1'2'}\gamma_{12})_{,k} - \gamma_{1'2'}\gamma_{12}(\Gamma^{1'}_{1'k} + \Gamma^{2'}_{2'k} + \Gamma^{1}_{1k} + \Gamma^{2}_{2k}) = 0,$$

or if $\gamma = \gamma_{1'2'}\gamma_{12}$,

$$*(43)\quad \Gamma^{A}_{Ak} + \Gamma^{A'}_{A'k} = (\ln\gamma)_{,k}.$$

The 4 equations (43) and the 24 equations $\left\{{m \atop nk}\right\} = \left\{{m \atop kn}\right\}$ do not completely determine the 32 real quantities of Γ^{A}_{Bk}. Let

$$(44)\quad \Gamma'^{A}_{Bk} = \Gamma^{A}_{Bk} + i\varepsilon\phi_k\delta^{A}_{B},$$

where ϕ_k is real. Then (43) and Axiom III are still satisfied, and so we introduce the 4 additional equations

$$*(45)\quad \Gamma^{A}_{Ak} - \Gamma^{A'}_{A'k} = 4i\varepsilon\phi_k.$$

We take $\varepsilon = \sqrt{4\pi}\, e$, and as in Bade and Jehle identify ϕ_k with the electromagnetic potential. In addition to their remarks concerning this identification, we will make an observation shortly. From (43) and (45),

$$*(46)\quad \Gamma^{A}_{Ak} = 2i\varepsilon\phi_k + \ln(\gamma^{\frac{1}{2}})_{,k}.$$

Substituting in (39), we then have the expression for our spinor affinities:

$$*(47)\quad \Gamma^{A'}_{C'k} = -\frac{1}{2}\Big[\sigma_{mC'B}\,\sigma^{mA'B}{}_{,k} + \left\{ {m \atop nk} \right\}\sigma^{nA'B}\sigma_{mC'B}$$

$$+\,\delta^{A'}_{C'}\left(2i\varepsilon\phi_k + \partial_k \ln(\gamma^{\frac{1}{2}})\right)\Big].$$

While (35) shows that the spinor affinities are not themselves spinors (analogous to the properties of the usual Christoffel symbols), it is easy to show that they behave like covariant tensors with respect to general coordinate transformations.

For the spin metric, we have

$$\gamma^{12}{}_{|k} = \gamma^{12}{}_{,k} + \Gamma^{1}_{1k}\gamma^{12} + \Gamma^{2}_{2k}\gamma^{12}$$

$$= \gamma^{12}{}_{,k} + \gamma^{12}\Gamma^{A}_{Ak}.$$

Using (46),

$$\gamma^{12}{}_{|k} = \gamma^{12}{}_{,k} + \gamma^{12}\left(2i\varepsilon\phi_k + \partial_k \ln(\gamma^{\frac{1}{2}})\right).$$

By (3), we find

$$\gamma^{12}{}_{|k} = i\,\gamma^{12}\left(2\varepsilon\phi_k - \Theta_{,k}\right),$$

$$*(48)\quad \gamma^{AB}{}_{|k} = i\,\gamma^{AB}\left(2\varepsilon\phi_k - \Theta_{,k}\right).$$

Similarly,

$$*(49)\quad \gamma_{AB|k} = -i\,\gamma_{AB}\,(2\varepsilon\phi_k - \theta_{,k}).$$

By Axiom III and (41), we have

$$*(50)\quad \sigma^{k}{}_{A'B|m} = (\sigma^{kC'D}\gamma_{C'A'}\gamma_{DB})_{|m} = 0.$$

However, it does not follow that $\sigma^{kA'}{}_{B|m} = 0$.

As stated previously, (44)-(49) are not used in all formulations of the spinor calculus. We consider these equations more carefully. First, motivation for interpreting ϕ_k as the electromagnetic field: it follows from (35) that

$$(51)\quad \Gamma'^{A}{}_{Ak} = \Gamma^{A}{}_{Ak} - \partial_k \ln|\Lambda^R_S|.$$

In particular, if $\Lambda^R_S = \delta^R_S e^{i\varepsilon\sigma}$, then from (51), (46) we have

$$(52)\quad \phi'_k = \phi_k - \sigma_{,k}\,,$$

which agrees with the gauge transformation properties of the electromagnetic potential. Furthermore, suppose we make the transformation $\phi'_k = \phi_k - \sigma_{,k}$ and simultaneously make a 'phase transformation' $\psi'_A = \psi_A e^{i\sigma a}$, for some real constant a. Then

$$(53)\quad \psi'_{A|k} = (\psi_A e^{ia\sigma})_{,k} + \tfrac{1}{2}\psi_B e^{ia\sigma}\Big[\sigma_{mD'A}\,\sigma^{mD'B}{}_{,k} + \left\{ {m \atop nk} \right\}\sigma^{nD'B}\sigma_{mD'A} + \delta^B_A\big(-2i\varepsilon(\phi_k - \sigma_{,k}) + \partial_k(\ln\gamma^{\frac{1}{2}})\big)\Big]$$

$$= e^{ia\sigma}\psi_{A|k} + \psi_A\, ia\, e^{ia\sigma}\sigma_{,k} + \tfrac{1}{2}\psi_B\, e^{ia\sigma}\delta^B_A(2i\varepsilon\sigma_{,k}),$$

$$\psi'_{A|k} = e^{ia\sigma}\psi_{A|k} + i e^{ia\sigma}\psi_A\sigma_{,k}(a+\varepsilon).$$

Taking $a = -\varepsilon$, this gives

$$\psi'_{A|k} = e^{ia\sigma}\psi_{A|k}. \tag{54}$$

Thus associating to the gauge transformation $\phi'_k = \phi_k - \sigma_{,k}$ the change of phase $\psi'_A = \psi_A e^{-i\varepsilon\sigma}$ gives $\psi'_{A|k} = e^{-i\varepsilon\sigma}\psi_{A|k}$. Then a gauge transformation would leave a linear spinor equation (e.g. the Dirac equation) invariant up to a phase factor. Finally, if we define

$$\overline{\Gamma}{}^A_{Bk} = \Gamma^A_{Bk} - \delta^A_B\, i\varepsilon\phi_k$$

and the new spinor derivative

$$\psi^A{}_{||k} = \psi^A{}_{|k} - i\varepsilon\phi_k\psi^A = \psi^A{}_{,k} + \overline{\Gamma}{}^A_{Bk}\psi^B,$$

then when $\phi_k = 0$ we have $\psi^A{}_{||k} = \psi^A{}_{|k}$, and the introduction of ϕ_k corresponds to the replacement $\psi^A{}_{||k} \to \psi^A{}_{|k} = \psi^A{}_{||k} + i\varepsilon\phi_k$. This is consistent with the usual philosophy of minimal electromagnetic interaction. Altogether, then, these three observations--gauge transformation properties, invariance of linear spinor equations, minimal electromagnetic interaction--provide the motivation for our interpretation of ϕ_k as the electromagnetic potential.

Note that if we had taken $\gamma_{AB|k} = 0$ instead of (41), then $\gamma_{12,k} - \Gamma^1_{1k}\gamma_{12} - \Gamma^2_{2k}\gamma_{12} = 0$, which would imply

(55) $$\Gamma^A_{Ak} = (\ln \gamma_{12})_{,k}$$

Taking $\gamma_{12} = \gamma^{\frac{1}{2}} e^{i\theta}$, this gives

(56) $$\Gamma^A_{Ak} = (\ln \gamma^{1/2})_{,k} + i\,\theta_{,k} ,$$

and the quantities ϕ_k would not arise.

Returning to our spinor calculus, by (46) we find

*(57) $$\gamma_{AB|k} = \gamma_{AB,k} - \gamma_{AB}(2i\varepsilon\phi_k + \partial_k \ln \gamma^{1/2}).$$

Note that if $\phi_k \neq 0$, then $\gamma_{AB|k} \neq 0$ even when $\gamma_{AB,k} = 0$. The same conclusion follows from (48).

We consider more closely the roles of γ, θ. Let Λ be a general spin transformation, $\Lambda \in Gl(2,\mathbb{C})$. Then Axioms I, II are covariant under the transformation Λ, and Axiom III will be satisfied whenever the Γ^A_{BK} are given by (47). In particular, if $\Lambda^A{}_C = r^{-\frac{1}{2}}\delta^A{}_C$, $r = \alpha^{\frac{1}{2}}e^{i\beta}$, α, β real, then $\gamma'_{AB} = r\gamma_{AB}$. Under this transformation,

$$\sigma'_m{}^{B'C} = \alpha^{-1/2}\sigma_m{}^{B'C} ; \quad \sigma'_{mB'C} = \alpha^{1/2}\sigma_{mB'C} ; \quad \phi'_k = \phi_k + \frac{1}{2\varepsilon}\beta_{,k} ;$$

$$\Gamma'^A_{Bk} = \Gamma^A_{Bk} + \frac{1}{4}\delta^A_B\,\alpha^{-1}\alpha_{,k} + \frac{i}{2}\delta^A_B\,\beta_{,k} .$$

Thus we again have a spinor calculus in the primed system, with $\gamma' = \alpha\gamma$, $\theta' = \theta + \beta$. In other words, by such a "gauge transformation" γ,θ may be taken as arbitrary complex numbers, $\gamma \neq 0$. We also have the physically interesting result that given a general spin transformation $\Lambda \in Gl(2,\mathbb{C})$, Λ may be written up to an isomorphism as the product of a Lorentz transformation and a "gauge transformation".

Next, we define a spin-tensor analogous to the Riemann tensor:

$$*(58) \quad P^{A}_{\cdot Bmn} \equiv \Gamma^{A}_{Bn,m} - \Gamma^{A}_{Bm,n} + \Gamma^{C}_{Bn}\Gamma^{A}_{Cm} - \Gamma^{C}_{Bm}\Gamma^{A}_{Cn}.$$

This definition appears in Bade and Jehle. Substitution of (46) into (58) yields

$$*(59) \quad P^{A}_{\cdot Amn} = -P^{A'}_{\cdot A'mn} = 2i\varepsilon(\phi_{n,m} - \phi_{m,n}) = 2i\varepsilon F_{nm}.$$

(59) may be interpreted as 'the electromagnetic field originates in the curvature of spin space'. To find an explicit expression for $P^{A}_{\;Bmn}$, by Axiom III we have

$$0 = \sigma^{\ell A'B}{}_{|mn} - \sigma^{\ell A'B}{}_{|nm}$$

$$= \sigma^{\ell C'B} P^{A'}_{\cdot C'nm} + \sigma^{\ell A'C} P^{B}_{\cdot Cnm} + \sigma^{tA'B} R^{\ell}_{\cdot tnm}.$$

Then contracting with $\sigma_{\ell D'B}$,

$$\gamma_{D'}{}^{C'}\gamma_{B}{}^{B} P^{A'}_{\cdot C'nm} + \gamma_{D'}{}^{A'}\gamma_{B}{}^{C} P^{B}_{\cdot Cnm} = -\sigma^{tA'B}\sigma_{\ell D'B} R^{\ell}_{\cdot tnm},$$

or

$$2P^{A'}_{\cdot D'nm} + \gamma_{D'}{}^{A'} P^{B}_{\cdot Bnm} = -\sigma^{tA'B}\sigma^{\ell}_{D'B} R_{\ell tnm}.$$

Using (59), the above gives

$$*(60)\quad P^{A'}_{\cdot D'nm} = \tfrac{1}{2} R_{t\ell nm}\sigma^{tA'B}\sigma^{\ell}_{D'B} + i\varepsilon F_{nm}\delta^{A'}_{D'}.$$

One easily checks that

$$*(61)\quad \chi_{A|\ell m} - \chi_{A|m\ell} = \chi_B P^{B}_{\cdot A\ell m},$$

$$*(62)\quad \chi^{A}_{\cdot|\ell m} - \chi^{A}_{|m\ell} = \chi^{B} P^{A}_{\cdot Bm\ell},$$

in analogy to our usual tensor identities. There also exist "Bianchi identities" in the spin space:

$$(63)\quad \sigma_{mA'C} P^{C}_{\cdot B\langle n\ell|k\rangle} + \sigma_{mC'B} P^{C'}_{\cdot A'\langle n\ell|k\rangle} = 0,$$

where < > denotes cyclic permutation.

For later use, we consider the representation of infinitesimal Lorentz transformations. Consider such a transformation, given by

$$(64)\quad \Lambda^{(m)}_{\cdot(n)} = \delta^{(m)}_{(n)} + \varepsilon^{(m)}_{\cdot(n)},\quad \varepsilon^{(mn)} = -\varepsilon^{(nm)}.$$

We wish to associate to this transformation an infinitesimal spinor transformation $\Lambda^{A}_{\ B}$ such that

(65) $$\sigma'^{(m)A'B} = \sigma^{(m)A'B} = \Lambda^{(m)}{}_{(n)}\Lambda^{A'}{}_{\cdot D'}\Lambda^{B}{}_{\cdot C}\,\sigma^{(n)D'C}$$

We write $\Lambda^{A}{}_{B} = \delta^{A}{}_{B} + \eta^{A}{}_{B}$. The requirement that γ_{AB} remain invariant implies $\eta_{AB} = \eta_{BA}$. Then (65) gives

(66) $$\varepsilon^{(m)}{}_{\cdot(n)}\sigma^{(n)A'B} + \eta^{A'}{}_{\cdot D'}\sigma^{(m)D'B} + \eta^{B}{}_{\cdot C}\,\sigma^{(m)A'C} = 0.$$

To solve this, we try the natural relation

(67) $$\eta^{B}{}_{\cdot C} = C\,\varepsilon^{(mn)}\sigma_{(m)}{}^{D'B}\sigma_{(n)D'C}\,.$$

Substituting (67) into (66), we have

$$\varepsilon^{(mn)}\sigma_{(n)}{}^{A'B} + C\,\varepsilon^{(rs)}\Big[\sigma_{(r)}{}^{F'B}\sigma_{(s)F'C}\,\sigma^{(m)A'C}$$

$$+ \sigma_{(r)}{}^{A'F}\sigma_{(s)D'F}\,\sigma^{(m)D'B}\Big] = 0.$$

Using (29), we have $C = \frac{1}{2}$. Then (67) becomes

*(68) $$\eta^{B}{}_{\cdot C} = \frac{1}{2}\,\varepsilon^{(mn)}\sigma_{(m)}{}^{D'B}\sigma_{(n)D'C}\,.$$

III. THE COMBINED EQUATIONS:

VARIATIONAL DERIVATION AND DIFFERENTIAL IDENTITIES

We investigate the combined Dirac-Einstein-Maxwell equations, using the spinor calculus developed in Chapter II. Units are chosen such that $c = G = \hbar = 1$. The electromagnetic potential is denoted by ϕ_k, with $F_{kl} = \phi_{k|l} - \phi_{l|k}$. We denote the 2-component spinor wave fields by χ^A, ξ_A. The 'bare' charge and mass of our particle are denoted by e, m respectively. The following form of the Lagrangian density for the combined fields may be found in Bade and Jehle:

$$\text{(1)} \quad \mathcal{L} = \Big[R - 4\pi F^{kl}F_{kl} - 16\pi\Big\{2mi(\chi^{A'}\xi_{A'} - \chi^A\xi_A)$$

$$+ \sqrt{2}\,i\sigma^{l}{}_{B'A}(\chi^A\chi^{B'}{}_{|l} - \chi^{B'}\chi^A{}_{|l}) + \sqrt{2}\,i\sigma^{lB'A}(\xi_{B'}\xi_{A|l} - \xi_A\xi_{B'|l})\Big\}\Big]\sqrt{-g}.$$

The coupling constants have been chosen so that the combined eqtns. will reduce to the usual special relativistic eqtns. for our units. In another formalism, these equations appear in Das (10). Here $\sigma^{mB'A}$ denote the generalized spin matrices of Chapter I, which therefore satisfy Axioms I-III of our algebraic approach to spinors, and stroke represents covariant differentiation of either tensors or spinors. The form (1) is clearly invariant under general coordinate transformations, and if one uses I-(5), II-(47) to explicitly demonstrate the dependence of $\mathcal{L}$ on the choice of tetrad, it can be seen that $\mathcal{L}$ is also invariant under tetrad rotations (Lorentz transformations).

By virtue of II-(47), the electromagnetic potential appears in the covariant derivative terms of $\mathcal{L}$. One could make this dependence explicit by defining

$$\chi^{B'}{}_{\|\ell} = \chi^{B'}{}_{|\ell} + i\varepsilon\phi_\ell \chi^{B'}, \quad \varepsilon = \sqrt{4\pi}\,e, \tag{2}$$

as in Chapter II, and expressing $\mathcal{L}$ in terms of the 'double slash' covariant derivative. As mentioned before, this corresponds to the usual philosophy of minimal electromagnetic interaction in special relativity, where in the presence of an electromagnetic field, ∂_k is replaced by $\partial_k - i\varepsilon\phi_k$ (7), (30).

For the independent quantities appearing in $\mathcal{L}$, we take ϕ_k, $\lambda^{(f)}_{j}$, χ^A, ξ_A, $\chi^{A'}$, $\xi_{A'}$, and their (partial) derivatives. The choice of the tetrad components $\lambda^{(f)}_{j}$ as the fundamental geometric variables rather than the more usual metric components g_{ij} is a characteristic feature of half-integer spin fields. Using (1) and I-(5), we see our Lagrangian involves the tetrad in a manner which cannot be expressed in terms of g_{ij} alone. This will be treated in detail later.

Let us first consider the Dirac equations. The stationary principle

$$(3) \qquad \delta \int \mathcal{L}\, d^4x = 0,$$

upon variation of the fields χ^A, ξ_A gives the Euler-Lagrange eqtns.

$$(4) \qquad \frac{\partial \mathcal{L}}{\partial \chi^A} - \left(\frac{\partial \mathcal{L}}{\partial \chi^A{}_{,\ell}}\right)_{,\ell} = 0; \quad \frac{\partial \mathcal{L}}{\partial \xi_A} - \left(\frac{\partial \mathcal{L}}{\partial \xi_{A,\ell}}\right)_{,\ell} = 0.$$

For the matter fields, these equations can be written more elegantly in the form

$$(5) \qquad \frac{\partial \mathcal{L}}{\partial \chi^A} - \left(\frac{\partial \mathcal{L}}{\partial \chi^A{}_{|\ell}}\right)_{|\ell} = 0; \quad \frac{\partial \mathcal{L}}{\partial \xi_A} - \left(\frac{\partial \mathcal{L}}{\partial \xi_{A|\ell}}\right)_{|\ell} = 0.$$

See Appendix C for a proof of the equivalence of (4), (5). Substituting (1) into (5), one has for the first Dirac equations

$$(6) \qquad \sigma^{\ell}{}_{B'A}\, \chi^{B'}{}_{|\ell} = \frac{m}{\sqrt{2}}\, \xi_A,$$

and the complex conjugates. Similarly, the ξ equations are

$$(7) \qquad \sigma^{\ell B'A}\, \xi_{B'|\ell} = -\frac{m}{\sqrt{2}}\, \chi^A.$$

Turning to the Maxwell equations, we have

$$-4\pi F^{k\ell} F_{k\ell} = -4\pi g^{km} g^{\ell n} F_{mn} F_{k\ell}$$

$$= -4\pi g^{km} g^{\ell n} (\phi_{m|n} - \phi_{n|m})(\phi_{k|\ell} - \phi_{\ell|k}),$$

and this gives

$$(8)\quad \frac{\partial}{\partial\phi_{r|s}}\left(-4\pi F^{kl}F_{kl}\right) = -16\pi F^{rs}.$$

For the current terms, the relevant part of the Lagrangian is

$$(9)\quad -16\pi\sqrt{2}\,\frac{\partial}{\partial\phi_r}\Big[i\sigma^{l}{}_{B'A}\left(\chi^A\chi^{D'}\Gamma^{B'}{}_{D'l}-\chi^{B'}\chi^{C}\Gamma^{A}{}_{Cl}\right)$$

$$+i\sigma^{lB'A}\left(\xi_{D'}\xi_A\Gamma^{D'}{}_{B'l}-\xi_{B'}\xi_C\Gamma^{C}{}_{Al}\right)\Big].$$

Using I-(47), we find for (9) the result

$$(10)\quad -32\pi\sqrt{2}\,\varepsilon\sigma^{r}{}_{B'A}\left(\chi^{B'}\chi^{A}+\xi^{B'}\xi^{A}\right).$$

Combining (8), (10), we have for the Maxwell equations

$$(11)\quad F^{rs}{}_{|s} = 2\sqrt{2}\,\varepsilon\sigma^{r}{}_{B'A}\left(\chi^{B'}\chi^{A}+\xi^{B'}\xi^{A}\right),$$

from which we define the current j^k as

$$(12)\quad j^k = \varepsilon\sigma^{k}{}_{B'A}\left(\chi^{B'}\chi^{A}+\xi^{B'}\xi^{A}\right).$$

Derivation of the Einstein equations requires a considerably greater expenditure of effort. The most straightforward approach is to consider the tetrad components as the fundamental geometric variables. One then uses the relations

$$(13)\quad g_{ij} = \lambda^{(f)}_{i}\lambda^{(h)}_{j}\eta_{(fh)},\quad \sigma_{lB'A} = \lambda^{(f)}_{l}\sigma_{(f)B'A},$$

to express the geometric part of $\mathcal{L}$ entirely in terms of the tetrad.

The Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \lambda^{(f)}_j} - \left(\frac{\partial \mathcal{L}}{\partial \lambda^{(f)}_{j,k}}\right)_{,k} = 0, \tag{14}$$

then give the field equations. Because of the complicated dependence of the spinor affinities upon the tetrad, the computations in this direct approach are exceedingly long. Instead, we consider an alternative approach developed by Rosenfeld (29). Cf. also Belinfante (4).

Let us consider for the moment an arbitrary system containing gravitational, electromagnetic, and matter fields. Let Q_γ denote a maximal collection of independent gravitational variables (g_{ij}, $\lambda^{(f)}_j$, $\sigma^{mA'B}$,...). We write the principle of stationary action in the form

$$\delta \int (\mathcal{G} + \hat{\mathcal{L}}) d^4x = 0 \tag{15}$$

where $\mathcal{G}$ represents the purely geometrical part of the Lagrangian, and $\hat{\mathcal{L}}$ the terms resulting from other fields. For the gravitational variables, the Euler-Lagrange equations are

$$\frac{\delta \mathcal{G}}{\delta Q_\gamma} = -\frac{\delta \hat{\mathcal{L}}}{\delta Q_\gamma}. \tag{16}$$

For purely tensorial matter fields, one may always choose for the Q_γ the components of the metric tensor, g_{ij}. Taking $\mathcal{G} = R\sqrt{-g}$, (16) becomes

$$G_{ij} = -\tfrac{1}{2} T_{ij} \tag{17}$$

where we have used the well-known results that up to a boundary term

$$\frac{\partial R\sqrt{-g}}{\partial g^{ij}} = G_{ij}\sqrt{-g}, \quad \frac{\partial R\sqrt{-g}}{\partial g_{ij}} = -G^{ij}\sqrt{-g}$$

and the energy-momentum tensor is defined by

$$\sqrt{-g}\,T^{ij} = -\left(\frac{\delta\hat{\mathcal{L}}}{\delta g_{ij}} + \frac{\delta\hat{\mathcal{L}}}{\delta g_{ji}}\right). \tag{18}$$

In the case of spinor fields, where the tetrad cannot be replaced by g_{ij} in the Lagrangian, we take the tetrad components $\lambda^{(f)}_{j}$ as the independent variables. An interesting point arises here. For pure tensor fields, both $\mathcal{G}$ and $\hat{\mathcal{L}}$ in (15) depend upon $\lambda^{(f)}_{j}$ only through g_{ij}. For spinor fields, this is not the case for $\hat{\mathcal{L}}$, raising the possibility that it is not true for $\mathcal{G}$ as well. Thus it would be of interest to consider choices for $\mathcal{G}$ depending essentially upon the $\lambda^{(f)}_{j}$, which would reduce to $R\sqrt{-g}$ in some sense when no spinor fields were present. Such conjecture would lead us beyond the bounds of Riemannian geometry and classical relativity, and will not be considered here. Cf., however, the work of Møller (22), (23), and Kilmister (18). Thus we again set $\mathcal{G} = R\sqrt{-g}$, and therefore

$$\frac{\partial\mathcal{G}}{\partial\lambda^{(f)}_{i}} = \frac{\partial\mathcal{G}}{\partial g_{k\ell}}\frac{\partial g_{k\ell}}{\partial\lambda^{(f)}_{i}} \tag{19}$$

By (13),

$$(20)\quad \frac{\partial g_{k\ell}}{\partial \lambda^{(f)}_{i}} = \lambda_{(f)\ell}\,\delta^{i}_{k} + \lambda_{(f)k}\,\delta^{i}_{\ell}\,.$$

Combining (19), (20),

$$(21)\quad \frac{\delta \mathcal{G}}{\delta \lambda^{(f)}_{i}} = -2\,G^{i}_{(f)}\sqrt{-g}\,.$$

Thus the field equations are

$$(22)\quad G^{i}_{(f)}\sqrt{-g} = \frac{1}{2}\,\frac{\delta \hat{\mathcal{L}}}{\delta \lambda^{(f)}_{i}}\,,$$

together with the requirement

$$(23)\quad \frac{\delta \hat{\mathcal{L}}}{\delta \lambda^{(f)}_{i}}\,\lambda^{(f)j} - \frac{\delta \hat{\mathcal{L}}}{\delta \lambda^{(f)}_{j}}\,\lambda^{(f)i} = 0,$$

which follows from the symmetry of G^{ij}. Define

$$(24)\quad \sqrt{-g}\,T^{ij} = -\frac{\delta \hat{\mathcal{L}}}{\delta \lambda^{(f)}_{i}}\,\lambda^{(f)j}\,.$$

Then (22), (23) may be written

$$(25)\quad G^{ij} = -\frac{1}{2}\,T^{ij}\,,$$

$$(26)\quad T^{ij} - T^{ji} = 0,$$

respectively. For pure tensor fields, (26) is identically satisfied. For spinor fields, Rosenfeld shows it follows from the matter equations: we shall see this for our combined equations later. Using (20), one easily shows the equivalence of (24), (18) in the case that $\hat{\mathcal{L}}$ depends upon g_{ij} alone. Following Rosenfeld, we work with (24), (25). We now summarize the results of Rosenfeld we shall use.

Let Q_α denote the matter field variables for some system. Let Q_τ denote those fields contained in $\{Q_\alpha\}$ which are purely tensorial, and let Q_σ denote the remaining fields. Suppose the variation of the fields Q_σ under an infinitesimal Lorentz transformation $\Lambda^{(ab)} = \eta^{(ab)} + \varepsilon^{(ab)}$ are given by

$$(27) \qquad \delta Q_\sigma = d^{\lambda}_{\sigma(ab)}\,\varepsilon^{(ab)} ,$$

and define

$$(28) \qquad d^{\lambda\, ij}_{\sigma} = d^{\lambda}_{\sigma(ab)}\,\lambda^{(a)i}\,\lambda^{(b)j} .$$

Next, consider an infinitesimal general coordinate transformation, given by

$$(29) \qquad \delta x^i = \bar{x}^i - x^i = \xi^i(x) ,$$

and let the coefficients $c^j_{\alpha,i}$ be defined by

$$(30) \qquad \delta Q_\alpha \equiv \bar{Q}_\alpha(\bar{x}) - Q_\alpha(x) = c^j_{\alpha,i}\,\frac{\partial \xi^i}{\partial x^j} .$$

Let

(31) $$d_\alpha{}^{ji} = \frac{1}{2}\left(c_{\alpha,k}^{j}\, g^{ki} - c_{\alpha,k}^{i}\, g^{kj}\right).$$

In special relativity, for an infinitesimal Lorentz transformation ε_{ij}, one would find $\delta Q_\alpha = d_\alpha^{ij}\varepsilon_{ji}$; i.e., the d_α^{ji} simply give the variation of the matter fields. Let

(32) $$S_\tau{}^{ij} = d_\tau{}^{ij} \quad ; \quad S_\sigma{}^{ij} = d_\sigma{}^{ij} - d_\sigma{}^{\lambda ij},$$

and define

(33) $$\mathcal{D}^{kij} = \frac{\partial L}{\partial Q_{\alpha \| k}}\, S_\alpha{}^{ij} ,$$

$$R^{ijk} = \mathcal{D}^{jki} - \mathcal{D}^{ijk} + \mathcal{D}^{kij},$$

where $\sqrt{-g}\, L = \hat{\mathcal{L}}$. Here the double slash derivative is the usual covariant derivative for tensors, and is defined by (2) for spinors.

Rosenfeld proves the following important results: let $\hat{\mathcal{L}}$ represent the contribution to the Lagrangian due to an arbitrary collection of tensor and spinor matter fields. Suppose $\hat{\mathcal{L}}$ is invariant under both Lorentz and general coordinate transformations, and suppose further that $\hat{\mathcal{L}}$ depends only upon the fields and their first covariant derivatives. Define T^{ij} by (24). Then

(34) $$T^{ij} = \frac{\partial L}{\partial Q_{\alpha\| i}}\, Q_{\alpha \|}{}^{j} - L g^{ij} - R^{jki}{}_{\| k}.$$

Further, the matter field equations imply T^{ij} is symmetric and divergence free.

The generality and beauty of these results is evident. The expression (34) is of practical importance in many cases because of the savings in calculations compared to using (14) or (24). (34) also gives a relation between the 'canonical' energy-momentum tensor and the usual T^{ij} of general relativity. There are still some interesting questions regarding this relation, as we shall mention after discussing the Dirac energy-momentum.

We use (34) to calculate T^{ij} for our Lagrangian (1). First, we write

$$(35)\quad T^{ij} = E^{ij} + M^{ij},$$

where E^{ij} represents the contribution to (34) from the electromagnetic field, and M^{ij} the contribution from the Dirac spinor fields. Explicitly,

$$(36)\quad E^{ij} = \frac{\partial L}{\partial \phi_{k|i}} \phi_{k|}{}^{j} - L g^{ij} - R_{\phi}{}^{jki}{}_{|k},$$

$$(37)\quad M^{ij} = \frac{\partial L}{\partial Q_{\sigma\|i}} Q_{\sigma\|}{}^{j} - R_{\sigma}{}^{jki}{}_{\|k}.$$

For E^{ij}, by (30) and the tensor nature of ϕ_k, we have

$$(38)\quad (C^{j}_{\phi,i})_k = -\delta^j_k \phi_i .$$

Then by (31), (32),

$$(39)\quad (d_\phi{}^{ji})_k = -\tfrac{1}{2}(\delta^j_k \phi^i - \delta^i_k \phi^j) = (s_\phi{}^{ji})_k .$$

For convenience in the following, we take K = -8π. From (8), (33),

$$(40)\quad \mathcal{D}_\phi{}^{kij} = K(F^{jk}\phi^i - F^{ik}\phi^j).$$

Then

$$(41)\quad R_\phi{}^{ijk} = 2KF^{jk}\phi^i ;$$

$$(42)\quad R_\phi{}^{jki}{}_{|k} = 2K(\phi^j F^{ki})_{|k} .$$

Combining (1), (8), (36), (42), and using the Dirac equations,

$$E^{ij} = 2KF^{li}\phi_{l|}{}^j - \tfrac{1}{2}Kg^{ij}F^{ab}F_{ab} - 2K\phi^j{}_{|k}F^{ki} +$$
$$-2K\phi^j F^{ki}{}_{|k} ,$$

or

$$(43)\quad E^{ij} = 2K[F^{il}F^j{}_l - \tfrac{1}{4}g^{ij}F^{ab}F_{ab} - \phi^j F^{ki}{}_{|k}].$$

If one had used (18) directly and defined $\sqrt{-g}\,\bar{E}^{ij} = -\left(\frac{\partial}{\partial g_{ij}} + \frac{\partial}{\partial g_{ji}}\right)\times$ $\times\left(\sqrt{-g}\;F^{km}F_{km}\right)$, then $E^{ij} = \bar{E}^{ij} - 2K\phi^j F^{ki}{}_{|k}$. Using the Maxwell equations, (43) may be written

(44) $$E^{ij} = 2K[F^{ik}F^{j}{}_{k} - \tfrac{1}{4}g^{ij}F^{ab}F_{ab}] + 4\sqrt{2}K\varepsilon\,\phi^{j} \times$$

$$\times\,\sigma^{i}{}_{B'A}(\chi^{B'}\chi^{A} + \xi^{B'}\xi^{A}).$$

Next, we consider M^{ij}. In terms of our spinor derivative, we may write (37) as

(45) $$M^{ij} = \frac{\partial L}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j} - R_{\sigma}{}^{jki}{}_{|k} + \frac{\partial L}{\partial Q_{\sigma|i}}(Q_{\sigma||}{}^{j} - Q_{\sigma|}{}^{j}).$$

For our spinor variables, we have

(46) $$\frac{\partial L}{\partial \chi^{B'}{}_{|\ell}} = 2\sqrt{2}Ki\sigma^{\ell}{}_{B'A}\chi^{A};\ \frac{\partial L}{\partial \xi_{B'|\ell}} = -2\sqrt{2}Ki\sigma^{\ell B'A}\xi_{A},$$

and the conjugate equations. Further, we have

(47) $$\chi^{B'}{}_{||\ell} = \chi^{B'}{}_{|\ell} + i\varepsilon\phi_{\ell}\chi^{B'};\ \xi_{B'||\ell} = \xi_{B'|\ell} - i\varepsilon\phi_{\ell}\xi_{B'}.$$

Then

(48) $$\frac{\partial L}{\partial Q_{\sigma|i}}(Q_{\sigma||}{}^{j} - Q_{\sigma|}{}^{j}) = -4\sqrt{2}\varepsilon\,\phi^{j}\sigma^{i}{}_{B'A}(\chi^{B'}\chi^{A} + \xi^{B'}\xi^{A})K,$$

and therefore the last terms of (44) and (45) will cancel one another in the expression for T^{ij}. Then we redefine

(49) $$E^{ij} = 2K[F^{ik}F^{j}{}_{k} - \tfrac{1}{4}g^{ij}F^{ab}F_{ab}],$$

(50) $$M^{ij} = \frac{\partial L}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j} - R_{\sigma}{}^{jki}{}_{|k},$$

and (35) holds for these new definitions. To calculate M^{ij}, for an infinitesimal Lorentz transformation $\Lambda^{(mn)} = \eta^{(mn)} + \varepsilon^{(mn)}$ we have the corresponding infinitesimal spin transformation $\Lambda^{A}{}_{\dot B} = \delta^{A}{}_{B} + \eta^{A}{}_{B}$, where by II-(68)

$$\eta^{B}{}_{\cdot C} = \frac{1}{2}\varepsilon^{(mn)}\sigma_{(m)}{}^{D'B}\sigma_{(n)D'C}\,.$$

The variation of a 1-spinor χ^{A} is given by $\delta\chi^{A} = \eta^{A}{}_{C}\chi^{C}$. Then by (27)

$$(51)\quad \left(d^{\lambda}_{\chi\,(mn)}\right)^{B} = \frac{1}{4}\left(\sigma_{(m)}{}^{D'B}\sigma_{(n)D'C} - \sigma_{(n)}{}^{D'B}\sigma_{(m)D'C}\right)\chi^{C};$$

$$\left(d^{\lambda}_{\xi\,(mn)}\right)_{A} = -\frac{1}{4}\left(\sigma_{(m)}{}^{D'C}\sigma_{(n)D'A} - \sigma_{(n)}{}^{D'C}\sigma_{(m)D'A}\right)\xi_{C}\,,$$

with similar conjugate equations. Since the 1-spinors χ^{C}, ξ_{A} are invariants under coordinate transformations, we have $d^{ij}_{\chi} = d^{ij}_{\xi} = 0$. Then by (28), (32),

$$(52)\quad \left(S^{\lambda ji}_{\chi}\right)^{B} = -\frac{1}{4}\left(\sigma^{j\,D'B}\sigma^{i}{}_{D'C} - \sigma^{i\,D'B}\sigma^{j}{}_{D'C}\right)\chi^{C}$$

$$\left(S^{\lambda ji}_{\xi}\right)_{A} = \frac{1}{4}\left(\sigma^{j\,D'C}\sigma^{i}{}_{D'A} - \sigma^{i\,D'C}\sigma^{j}{}_{D'A}\right)\xi_{C}\,.$$

Combining (33), (46), (52), we have for the Dirac fields

$$(53)\quad \mathcal{D}_{\sigma}{}^{kij} = \frac{-iK}{\sqrt{2}}\Big[\sigma^{k}{}_{B'A}\chi^{A}\chi^{C'}\left(\sigma^{iB'D}\sigma^{j}{}_{C'D} - \sigma^{jB'D}\sigma^{i}{}_{C'D}\right)$$

$$+\sigma^{kB'A}\xi_{A}\xi_{C'}\left(\sigma^{iC'D}\sigma^{j}{}_{B'D} - \sigma^{jC'D}\sigma^{i}{}_{B'D}\right)\Big] + c.c.\,,$$

$$(54)\quad R_\sigma{}^{jki} = \frac{-Ki}{\sqrt{2}}\,\chi^A\chi^{C'}\Big[\sigma^k{}_{B'A}\big(\sigma^{iB'D}\sigma^j{}_{C'D} - \sigma^{jB'D}\sigma^i{}_{C'D}\big) - \sigma^j\big(\sigma^k\sigma^i - \sigma^i\sigma^k\big) + \sigma^i\big(\sigma^j\sigma^k - \sigma^k\sigma^j\big)\Big] - \frac{Ki}{\sqrt{2}}\,\xi_A\xi_{C'}\Big[\sigma^{kB'A} \times \big(\sigma^{iC'D}\sigma^j{}_{B'D} - \sigma^{jC'D}\sigma^i{}_{B'D}\big) - \sigma^j\big(\sigma^k\sigma^i - \sigma^i\sigma^k\big) + \sigma^i\big(\sigma^j\sigma^k + - \sigma^k\sigma^j\big)\Big] + \text{c.c.},$$

where omitted σ indices agree with those of the preceeding term. Using II-(16), we recombine the above terms: e.g.,

$$-\sigma^i{}_{C'D}\big(\sigma^j{}_{B'A}\sigma^{kB'D} + \sigma^k{}_{B'A}\sigma^{jB'D}\big) = -g^{jk}\sigma^i{}_{C'A}.$$

Altogether, we find

$$(55)\quad R_\sigma{}^{jki} = \frac{-Ki}{\sqrt{2}}\,\chi^A\chi^{C'}\Big[g^{ij}\sigma^k{}_{C'A} - g^{jk}\sigma^i{}_{C'A} + \sigma^j{}_{C'D} \times \big(\sigma^{iB'D}\sigma^k{}_{B'A} - \sigma^{kB'D}\sigma^i{}_{B'A}\big)\Big] + \frac{Ki}{\sqrt{2}}\,\xi_A\xi_{C'}\Big[g^{ij}\sigma^{kC'A} + - g^{jk}\sigma^{iC'A} - \sigma^{jC'D}\big(\sigma^{iB'A}\sigma^k{}_{B'D} - \sigma^{kB'A}\sigma^i{}_{B'D}\big)\Big] + \text{c.c.}$$

Canceling the real terms with the corresponding conjugate terms and taking the divergence,

$$(56)\quad R_\sigma{}^{jki}{}_{|k} = \frac{-Ki}{\sqrt{2}}\,\sigma^j{}_{C'D}\big(\sigma^{iB'D}\sigma^k{}_{B'A} - \sigma^{kB'D}\sigma^i{}_{B'A}\big)\big(\chi^A\chi^{C'}\big)_{|k} +$$

$$-\frac{Ki}{\sqrt{2}}\sigma^{jC'D}(\sigma^{iB'A}\sigma^{k}{}_{B'D}-\sigma^{kB'A}\sigma^{i}{}_{B'D})(\xi_A\xi_{C'})_{|k}+\text{c.c.}$$

In another formalism, an expression similar to this appears in Bergmann and Thomson (6). However, considerable simplification occurs by rearranging terms and using the Dirac equations. Consider first the χ terms of (56). Written out with one application of the Dirac equations (6), we have

$$(57)\quad -\frac{Kim}{2}\xi_{B'}\chi^{C'}\sigma^{iB'D}\sigma^{j}{}_{C'D}-\frac{Ki}{\sqrt{2}}\sigma^{j}{}_{C'D}(\sigma^{iB'D}\sigma^{k}{}_{B'A}\chi^A\chi^{C'}{}_{|k}$$

$$-\sigma^{kB'D}\sigma^{i}{}_{B'A}\chi^A{}_{|k}\chi^{C'}-\sigma^{kB'D}\sigma^{i}{}_{B'A}\chi^A\chi^{C'}{}_{|k})+\text{c.c.}$$

We rearrange these terms through repeated application of II-(16). For example,

$$\frac{Ki}{\sqrt{2}}\sigma^{j}{}_{C'D}\sigma^{kB'D}\sigma^{i}{}_{B'A}\chi^A\chi^{C'}{}_{|k}=-\frac{Ki}{\sqrt{2}}\sigma^{k}{}_{C'D}\chi^{C'}{}_{|k}\sigma^{jB'D}\sigma^{i}{}_{B'A}\chi^A$$

$$+\frac{Ki}{\sqrt{2}}g^{jk}\sigma^{i}{}_{B'A}\chi^A\chi^{B'}{}_{|k}=-\frac{Kim}{2}\sigma^{jB'D}\sigma^{i}{}_{B'A}\chi^A\xi_D$$

$$+\frac{Ki}{\sqrt{2}}g^{jk}\sigma^{i}{}_{B'A}\chi^A\chi^{B'}{}_{|k}.$$

Similar calculations for the other terms give for (57)

$$(58)\quad -Kim(\sigma^{iB'D}\sigma^{j}{}_{C'D}\chi^{C'}\xi_{B'}+\sigma^{jB'D}\sigma^{i}{}_{B'A}\chi^A\xi_D)+$$

$$-\frac{Ki}{\sqrt{2}}\left(g^{ik}\sigma^{j}{}_{B'A}\chi^{A}\chi^{B'}{}_{|k}-g^{ik}\sigma^{j}{}_{C'A}\chi^{C'}\chi^{A}{}_{|k}-2g^{jk}\sigma^{i}{}_{B'A}\chi^{A}\chi^{B'}{}_{|k}\right)+c.c.$$

For the ξ terms in (56), the same procedure gives

$$(59)\quad -imK\left(\sigma^{jC'D}\sigma^{i}{}_{B'D}\chi^{B'}\xi_{C'}+\sigma^{iB'A}\sigma^{j}{}_{B'D}\xi_{A}\chi^{D}\right)+$$

$$-\frac{Ki}{\sqrt{2}}\left(g^{ik}\sigma^{jC'A}\xi_{A|k}\xi_{C'}+2g^{jk}\sigma^{iC'A}\xi_{A}\xi_{C'|k}-g^{ik}\sigma^{jC'A}\xi_{A}\xi_{C'|k}\right)+c.c.$$

The conjugate of the first term in (58) is $imK\sigma^{iB'A}\sigma^{j}{}_{B'D}\chi^{D}\xi_{A}$, cancelling the second term of (59). Then combining (58) and (59), we have

$$(60)\quad R_{\sigma}{}^{jki}{}_{|k}=-Ki\sqrt{2}\left(g^{ik}\sigma^{j}{}_{B'A}-g^{jk}\sigma^{i}{}_{B'A}\right)\chi^{A}\chi^{B'}{}_{|k}+$$

$$+Ki\sqrt{2}\left(g^{ik}\sigma^{jB'A}-g^{jk}\sigma^{iB'A}\right)\xi_{A}\xi_{B'|k}+c.c.$$

Returning to (50), we next calculate the term $\frac{\partial L}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j}$. Note that if we make the usual definition of the canonical energy-momentum tensor

$$\widetilde{M}^{ij}=\frac{\partial L_M}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j}-g^{ij}L_M,\quad L_M=L+4\pi F^{kl}F_{kl},$$

we have

$$\widetilde{M}^{ij}=\frac{\partial L_M}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j}=\frac{\partial L}{\partial Q_{\sigma|i}}Q_{\sigma|}{}^{j},$$

by the Dirac equations. As is well-known, this tensor is not always symmetric even in special relativity. Using (46),

$$\text{(61)}\quad \widetilde{M}^{ij} = 2\sqrt{2}Ki(\sigma^{i}{}_{B'A}\chi^{A}\chi^{B'}{}_{|}{}^{j} - \sigma^{iB'A}\xi_{A}\xi_{B'|}{}^{j}) + \text{c.c.}$$

Using (61), (60) may be written as

$$\text{(62)}\quad R_{\sigma}{}^{jki}{}_{|k} = \tfrac{1}{2}(\widetilde{M}^{ij} - \widetilde{M}^{ji}),$$

and finally (50), (61), (62) give

$$\text{(63)}\quad M^{ij} = \tfrac{1}{2}(\widetilde{M}^{ij} + \widetilde{M}^{ji}).$$

Thus we have shown

Proposition The contribution of the Dirac fields to the energy-momentum tensor of general relativity is the symmetrized canonical energy-momentum tensor.

The result (63) is not true, e.g., for the electromagnetic field. It is interesting to speculate whether there is some physical property of a matter field which is sufficient to imply (63). This interesting and simple proposition also means that the general relativistic energy-momentum tensor for the Dirac field may be obtained from that of special relativity by the usual replacement of ordinary by covariant derivative. Mathematically, this result is something of a surprise, as it does not appear at all evident from (24) or (50).

For completeness, we summarize how the result (63) may also be obtained directly from (1) by variation of $\lambda^{(m)}_{i}$, using (24). We may write

$$\text{(64)}\quad M^{ij} = \left[-\frac{\partial L_M}{\partial \lambda^{(m)}_{i}} + \left(\frac{\partial L_M}{\partial \lambda^{(m)}_{i,\rho}}\right)_{,\rho}\right]\lambda^{(m)j},$$

where

$$\text{(65)}\quad L_M = 2\sqrt{2}Ki\left[\sqrt{2}m\chi^{A'}\xi_{A'} + \sigma^{\ell}{}_{B'A}\chi^{A}\chi^{B'}{}_{|\ell} - \sigma^{\ell B'A}\xi_{A}\xi_{B'|\ell}\right] + \text{c.c.}\,.$$

This expression depends upon $\lambda^{(m)}_{i}$ through g^{ml}, σ_m, $\chi^{B'}{}_{|1}$, and $\xi_{B'|1}$. Explicitly, define

$$\text{(66)}\quad A^{i}_{(m)} = 2\sqrt{2}Ki\frac{\partial g^{k\ell}}{\partial \lambda^{(m)}_{i}}\left(\sigma_{kB'A}\chi^{A}\chi^{B'}{}_{|\ell} - \sigma_{k}{}^{B'A}\xi_{A}\xi_{B'|\ell}\right) + \text{c.c.},$$

$$\text{(67)}\quad B^{i}_{(m)} = 2\sqrt{2}Kig^{k\ell}\left(\chi^{A}\chi^{B'}{}_{|\ell}\frac{\partial \sigma_{kB'A}}{\partial \lambda^{(m)}_{i}} - \xi_{A}\xi_{B'|\ell}\frac{\partial \sigma_{k}{}^{B'A}}{\partial \lambda^{(m)}_{i}}\right) + \text{c.c.},$$

$$\text{(68)}\quad C^{i}_{(m)} = 2\sqrt{2}Ki\left(\sigma^{\ell}{}_{B'A}\chi^{A}\chi^{D'}\frac{\partial \Gamma^{B'}_{D'\ell}}{\partial \lambda^{(m)}_{i}} + \sigma^{\ell B'A}\xi_{A}\xi_{D'}\frac{\partial \Gamma^{D'}_{B'\ell}}{\partial \lambda^{(m)}_{i}}\right) + \text{c.c.}\,.$$

Then

$$\text{(69)}\quad \frac{\partial L_M}{\partial \lambda^{(m)}_{i}} = A^{i}_{(m)} + B^{i}_{(m)} + C^{i}_{(m)}\,.$$

Further, we define

$$\text{(70)}\quad D^{i}_{(m)} = \left(\frac{\partial L_M}{\partial \lambda^{(m)}_{i,\rho}}\right)_{,\rho}\,.$$

Then

$$D^{i}_{(m)} = 2\sqrt{2}\, iK\Big(\sigma^{\ell}{}_{B'A}\chi^{A}\chi^{D'}\frac{\partial \Gamma^{B'}_{D'\ell}}{\partial \lambda^{(m)}_{i,p}} + \sigma^{\ell B'A}\xi_{A}\xi_{D'}\frac{\partial \Gamma^{D'}_{B'\ell}}{\partial \lambda^{(m)}_{i,p}}\Big)_{,p} + \text{c.c.}$$

One easily checks that

$$(71)\quad -A^{i}_{(m)}\lambda^{(m)j} = \widetilde{M}^{ij} + \widetilde{M}^{ji},$$

$$-B^{i}_{(m)}\lambda^{(m)j} = -\widetilde{M}^{ji}.$$

For the remaining terms of (69), (70), One has by I-(5), II-(47)

$$\Gamma^{B'}_{D'\ell} = -\frac{1}{2}\Big(\lambda^{k}_{(q),\ell}\lambda^{(p)}_{k}\sigma_{(p)D'C}\sigma^{(q)B'C} + \begin{Bmatrix} k \\ n\ell \end{Bmatrix}\lambda^{n}_{(q)}\lambda^{(p)}_{k}\sigma^{(q)B'C}\times$$

$$\times\,\sigma_{(p)D'C} + 2i\varepsilon\phi_{\ell}\Big),$$

or

$$(72)\quad \Gamma^{B'}_{D'\ell} = -\frac{1}{2}\Big(\sigma_{(p)D'C}\sigma^{(q)B'C}\lambda^{(p)}_{k}\lambda^{k}_{(q)|\ell} + 2i\varepsilon\phi_{\ell}\Big),$$

and after lengthy calculations one finds

$$(73)\quad \frac{\partial}{\partial \lambda^{(m)}_{i}}\big(\lambda^{(p)}_{k}\lambda^{k}_{(q)|\ell}\big) = -\lambda^{(p)i}\lambda_{(q)n|\ell}\lambda^{n}_{(m)} - \frac{1}{2}\lambda^{(p)n}\lambda^{i}_{(q)}\times$$

$$\times\big(\lambda_{(m)n|\ell} - \lambda_{(m)\ell|n}\big) + \lambda^{(p)n}\lambda^{i}_{(q)}\lambda_{(m)k}\begin{Bmatrix} k \\ n\ell \end{Bmatrix} + \lambda^{(p)n}\eta_{(qm)}\begin{Bmatrix} i \\ n\ell \end{Bmatrix} +$$

$$-\frac{1}{2}\lambda^{(p)n}\lambda_{(q)k}\delta^{i}_{\ell}\big(\lambda^{k}_{(m),n} - \lambda_{(m)n,}{}^{k}\big) - \frac{1}{2}\lambda^{(p)i}\lambda_{(q)k}\big(\lambda^{k}_{(m),\ell} - \lambda_{(m)\ell,}{}^{k}\big),$$

$$(74)\quad \frac{\partial}{\partial \lambda^{(m)}_{i,p}}\left(\lambda^{(\varphi)}_{k}\lambda^{k}_{(q)|\ell}\right) = \tfrac{1}{2}\lambda^{(\varphi)i}\eta_{(qm)}\delta^{p}_{\ell} - \tfrac{1}{2}\Big[\lambda^{(\varphi)p}\eta_{(qm)}\delta^{i}_{\ell}$$

$$-\lambda^{(\varphi)i}\lambda^{p}_{(q)}\lambda_{(m)\ell} + \lambda^{(\varphi)}_{m}\lambda^{i}_{(q)}\delta^{p}_{\ell} + \lambda^{(\varphi)p}\lambda^{i}_{(q)}\lambda_{(m)\ell} - \delta^{(\varphi)}_{(m)}\lambda^{p}_{(q)}\delta^{i}_{\ell}\Big].$$

Using (72), (73), (74), one finds after more calculation

$$(75)\quad \left(-C^{i}_{(m)} + D^{i}_{(m)}\right)\lambda^{(m)j} = \tfrac{1}{2}\left(\tilde{M}^{ji} - \tilde{M}^{ij}\right).$$

Combining (71), (75), one arrives at (63). The complete calculation of this result provides ample demonstration of the power and value of Rosenfeld's expression (34).

Before concluding our discussion of variational derivations, we note one final anomaly. For the matter field equations, we have used the result that the Euler-Lagrange equations

$$(76)\quad \frac{\partial \mathcal{L}}{\partial Q_\alpha} - \left(\frac{\partial \mathcal{L}}{\partial Q_{\alpha,k}}\right)_{,k} = 0,$$

can be replaced by the covariant form

$$(77)\quad \frac{\partial \mathcal{L}}{\partial Q_\alpha} - \left(\frac{\partial \mathcal{L}}{\partial Q_{\alpha|k}}\right)_{|k} = 0.$$

This can be proved in general for Lagrangians whose matter dependence can be expressed in terms of Q_α, $Q_{\alpha|k}$ alone. See appendix C for details. However, for the Einstein equations, the Euler-Lagrange equations (14) apparently cannot be replaced by the covariant form

$$\frac{\partial(\hat{\mathcal{L}}+\mathcal{G})}{\partial \lambda^{(m)}_{i}} - \left(\frac{\partial(\hat{\mathcal{L}}+\mathcal{G})}{\partial \lambda^{(m)}_{i|k}}\right)_{|k} = 0. \tag{78}$$

It would seem more satisfactory to be able to derive the Einstein equations from an explicitly covariant set of equations.

Summarizing the results of our variational work, we have the

<u>Proposition</u> The combined Dirac-Einstein-Maxwell equations, derived from the principle $\delta\int\mathcal{L}d^4x = 0$, where

$$\mathcal{L} = \Big[R - 4\pi F^{kl}F_{kl} - 16\pi\Big\{2mi(\chi^{A'}\xi_{A'} - \chi^{A}\xi_{A}) + \sqrt{2}\Big(i\sigma^{l}{}_{B'A} \times$$

$$\times\, \sigma^{l}{}_{B'A}(\chi^{A}\chi^{B'}{}_{|l} - \chi^{B'}\chi^{A}{}_{|l}) - i\sigma^{lB'A}(\xi_{A}\xi_{B'|l} - \xi_{B'}\xi_{A|l})\Big)\Big\}\Big]\sqrt{-g},$$

may be written

$$D^{1}_{B} \equiv \sigma^{l}{}_{A'B}\chi^{A'}{}_{|l} - \frac{m}{\sqrt{2}}\xi_{B} = 0,$$

$$D^{2B} \equiv \sigma^{lA'B}\xi_{A'|l} + \frac{m}{\sqrt{2}}\chi^{B} = 0, \tag{79}$$

$$M^{l} \equiv F^{lm}{}_{|m} - 2\sqrt{2}\, j^{l} = 0,$$

$$Q_{ij} \equiv G_{ij} + 8\pi\Big(-F_{i}{}^{l}F_{jl} + \frac{1}{4}g_{ij}F^{ab}F_{ab}$$

$$+ \frac{1}{16\pi}M_{ij}\Big) = 0,$$

where

(80) $$j^{\ell} = \varepsilon\sigma^{\ell}_{B'A}(\chi^A\chi^{B'} + \xi^A\xi^{B'}),\quad \varepsilon = \sqrt{4\pi}\,e,$$

and

(81) $$M_{ij} = -8\pi\sqrt{2}\,i\,(\sigma_{iB'A}\chi^A\chi^{B'}{}_{|j} + \sigma_{jB'A}\chi^A\chi^{B'}{}_{|i} - \sigma_i{}^{B'A}\xi_A\xi_{B'|j} - \sigma_j{}^{B'A}\xi_A\xi_{B'|i}) + c.c.\,,$$

and we also write the $Q_{ij} = 0$ equations as

(82) $$G_{ij} = -\tfrac{1}{2}T_{ij} = -\tfrac{1}{2}(M_{ij} + E_{ij}).$$

Define $m^k = \sigma^k{}_{B'A}\chi^A\chi^{B'}$, $n^k = \sigma^k{}_{B'A}\xi^A\xi^{B'}$. In the usual correspondence of 1-spinors and null vectors, m and n are the vectors associated with the spinors χ, ξ respectively. Then (80) expresses the interesting result that the current is the tensor quantity corresponding to the Dirac wave fields.

The equations (79) contain 24 unknowns: $10(g_{ij}) + 4(\chi^A) + 4(\xi_A) + 4(\phi_k) + 1(e) + 1(m)$. Because of the freedom of imposing a gauge condition on the electromagnetic field and 4 coordinate conditions, there are 27 equations: $10(Q_{ij}) + 4(M^k) + 4(D^1_B) + 4(D^2_B) + 4(\text{coord.}) + 1(\text{gauge})$. The number of independent equations is reduced by the five identities: $4(Q^{ij}{}_{|j} = 0) + 1(M^k{}_{|k} = 0)$.

Thus we have 22 independent equations for the 22 unknown functions and 2 unknown constants. This system may be made determinate by prescribing values for e,m. In addition to the 22 independent equations above, which must hold locally everywhere, there may be global restrictions upon the solutions. For example, for solutions with finite total charge, one might reasonably require that this charge be e. Such a condition appears in Das and Coffman (11). In view of the scarcity of conservation laws in general relativity, this condition seems a good candidate to replace the usual square integrability of wave functions. Such additional global requirements could affect the arbitrariness in the choice of e,m. For our static solutions, we shall see that e and m cannot be chosen independently. We do not discuss the Cauchy problem here.

We have not considered the quantities γ, θ appearing in the spin metric as unknowns. The reason for this is that the Lagrangian (1) is invariant under the two parameter "gauge transformations" $\Lambda^A{}_B = r\delta^A{}_B$, where $r = \alpha^{1/2} e^{i\beta}$, $\alpha, \beta \in R$. From Chapter II, the conserved quantity associated with β-invariance is $2\varepsilon\phi_k - \theta_{,k}$. We do not know the conservation law associated with α-invariance: the physical significance of γ is not clear.

In our variational work we also ignored the explicit dependence of the Lagrangian on γ. For the independent quantities we have selected, the χ terms of the Lagrangian involving γ may be written

$$\sqrt{2}\, i\, \sigma^{\ell}{}_{(B'A)}\, \gamma^{1/2} \Big[\chi^A \chi^{B'}{}_{,\ell} - \tfrac{1}{2} \chi^A \chi^{D'} \Big\{ \sigma_{m(D'C)}\, \gamma^{1/2} \left(\sigma^{m(B'C)} \gamma^{-1/2} \right)_{,\ell} + \left\{ {m \atop n\ell} \right\} \sigma^{n(B'C)} \sigma_{m(D'C)} + \delta^{B'}_{D'} \left(2i\varepsilon \phi_\ell + \tfrac{1}{2} \gamma^{-1} \gamma_{,\ell} \right) \Big\} \Big] + \text{c.c.} ,$$

where from Chapter I by an appropriate coordination of spin frame and tetrad we have

$$\sigma^{\ell}{}_{B'A} = \gamma^{1/2}\sigma^{\ell}{}_{(B'A)}.$$

The expression for the ξ terms is similar. Using these expressions, the Euler-Lagrange equation for γ becomes

$$\sigma^{\ell}{}_{B'A}\chi^{A}\chi^{B'}{}_{|\ell} - \sigma^{\ell B'A}\xi_{B'}\xi_{A|\ell} + \text{c.c.} = 0;$$

and this is identically satisfied from the Dirac equations.

We have already mentioned that Rosenfeld shows that the invariance of $\mathcal{L}$ under general coordinate transformations and tetrad rotations, together with the matter field equations, gives rise to the identities $Q^{ij}{}_{|j} = 0$. More precisely, one can show

$$(83)\quad \frac{1}{8\pi} Q^{k\ell}{}_{|\ell} = -\frac{\sqrt{2}}{2}(\mathcal{D}_1^k + \mathcal{D}_2^k) + F^{k}{}_{\cdot\ell} M^{\ell},$$

where $\mathcal{D}_1^k$, $\mathcal{D}_2^k$ are collections of Dirac equations. The direct proof, originally due to Infeld and van der Waerden (17), is quite long. For completeness, it is given in Appendix D.

Similarly, the invariance of $\mathcal{L}$ under gauge transformations gives rise to the identity

$$(84)\quad -\frac{1}{2\sqrt{2}\varepsilon} M^{\ell}{}_{|\ell} = D^{1}_{B'}\chi^{B'} + D^{1}_{A}\chi^{A} + D^{2}_{B'}\xi^{B'} + D^{2}_{A}\xi^{A}.$$

To see this, we first derive the contravariant ξ equation. We have

$$(85)\quad (\sigma^{\ell B'A}\xi_A)_{|\ell} + \frac{m}{\sqrt{2}}\chi^{B'} = 0.$$

Then

$$-(\sigma^{\ell B'C}\gamma_{CA}\xi^A)_{|\ell} + \frac{m}{\sqrt{2}}\chi^{B'} = 0.$$

But

$$(\sigma^{\ell B'C}\gamma_{CA})_{|\ell} = -i\sigma^{\ell B'C}\gamma_{CA}(2\varepsilon\phi_\ell - \Theta_{,\ell}),$$

by II-(48). Then (85) becomes

$$(86)\quad \sigma^\ell{}_{B'A}\xi^A{}_{|\ell} - i\sigma^\ell{}_{B'A}(2\varepsilon\phi_\ell - \Theta_{,\ell}) - \frac{m}{\sqrt{2}}\chi_{B'} = 0,$$

the contravariant Dirac ξ equation. Similarly, the covariant Dirac χ equation is

$$(87)\quad \sigma^{\ell B'A}\chi_{B'|\ell} - i\sigma^{\ell B'A}(2\varepsilon\phi_\ell - \Theta_{,\ell})\chi_{B'} + \frac{m}{\sqrt{2}}\xi^A = 0.$$

Now

$$-\frac{1}{2\sqrt{2}\varepsilon}M^\ell{}_{|\ell} = \sigma^\ell{}_{B'A}(\chi^A{}_{|\ell}\chi^{B'} + \xi^A{}_{|\ell}\xi^{B'}) + \text{c.c.}$$

$$= (\sigma^\ell{}_{B'A}\chi^A{}_{|\ell} - \frac{m}{\sqrt{2}}\xi_{B'})\chi^{B'} + [\sigma^\ell{}_{B'A}\xi^A{}_{|\ell} - i\sigma^\ell{}_{B'A}(2\varepsilon\phi_\ell - \Theta_{,\ell})\xi_A - \frac{m}{\sqrt{2}}\chi_{B'}]\xi^{B'} + \text{c.c.},$$

and (84) follows.

IV. A SPECIAL SPINOR CALCULUS

Our primary objective is to search for exact solutions of the combined Dirac-Einstein-Maxwell equations such that the electric and gravitational fields are static and the Dirac field is stationary in the wave-mechanical sense. Here we exhibit the explicit spinor calculus to be used. A static gravitational metric may always be written in the normal form

$$(1) \quad \Phi = -k(dx^1)^2 - g(dx^2)^2 - h(dx^3)^2 + f(dx^4)^2.$$

For this metric we may choose our spin matrices $\sigma^{kA'B}$ to be a conformal factor times the usual choice of special relativity:

$$(2) \begin{cases} \sigma^{1A'B} = \frac{k^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; & \sigma^{2A'B} = \frac{g^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ \sigma^{3A'B} = \frac{h^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; & \sigma^{4A'B} = \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

One easily checks that the choice (2) satisfies Axiom I of our spinor calculus. Axiom III is satisfied whenever the spinor affinities are given by II-(47): indeed, II-(47) was derived to this end. The spin matrices covariant in their spinor indices are obtained from (2) by use of γ_{AB}. Here, we take

$$(3)\quad \begin{cases} \sigma^{1}{}_{A'B} = \dfrac{K^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}; \quad \sigma^{2}{}_{A'B} = \dfrac{g^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ \sigma^{3}{}_{A'B} = \dfrac{h^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma^{4}{}_{A'B} = \dfrac{f^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

One checks that (2), (3) satisfy Axiom II of the spinor calculus, and thus we are guaranteed of a consistent spinor calculus integrally related to our space-time metric. We may therefore avail ourselves of all the identities established in Chapter II.

We have made no mention of the quantities γ, θ which appear in our expression for γ_{AB}, II-(3). γ is easily disposed of: if $\sigma^{nB'A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the rule for lowering indices gives $\sigma^{n}{}_{B'A} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\gamma$. This is consistent with (2), (3) iff

$$(4)\quad \gamma \equiv 1.$$

Further, given the matrices (2), if $\gamma \neq 1$ Axiom II will not be satisfied. This is easily checked using II-(19). Thus γ is completely determined. Using (4), II-(46) becomes

$$(5)\quad \Gamma^{A}{}_{Ak} = 2i\epsilon\phi_{k}$$

and the expression for our spinor affinities II-(47) becomes

$$(6)\quad \Gamma^{A'}{}_{C'\ell} = -\tfrac{1}{2}\left[\sigma_{mC'B}\,\sigma^{mA'B}{}_{,\ell} + \left\{ {m \atop n\ell} \right\}\sigma^{nA'B}\sigma_{mC'B} + \delta^{A'}_{C'}(2i\epsilon\phi_{\ell})\right].$$

The physical significance of θ is unclear. One easily shows that under a gauge-type spin transformation $\Lambda^A{}_B = \delta^A{}_B e^{\frac{1}{2}i\sigma}$,

(7) $\theta'_{,s} = \theta_{,s} - \frac{1}{2}\sigma_{,s}$.

~~Comparing (7) with II-(52), we see $2\epsilon\phi_s - \theta_{,s}$ is invariant with~~ respect to gauge transformations. In particular, one may choose

(8) $\theta_{,s} = 0$,

by fixing the gauge. With this choice of gauge, θ is constant, and we may take

(9) $\theta = 0$.

This gives for γ_{AB} the simple form

(10) $\gamma_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From II-(49), it then follows that

(11) $\gamma_{AB|k} = -2i\epsilon\phi_k \gamma_{AB}$.

Equations (1)-(7) are gauge-independent. (8)-(11) are true only for a special choice of gauge.

One form in which Dirac's equations are traditionally written in special relativity is

$$(12)\quad \left(i\frac{\partial}{\partial t} + i\alpha\cdot\nabla - \beta m\right)\psi = 0,$$

See e.g. Schiff (31), section 52. Here,

$$(13)\quad \begin{cases} \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix};\ \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix};\ \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}; \\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \psi^{+} = (\psi_1, \psi_2, \psi_3, \psi_4). \end{cases}$$

Writing our Dirac equations in component form for the case of special relativity and comparing with (13), we find the relations

$$(14)\quad \begin{cases} \psi_1 = -(\chi^2 + i\xi_{2'}) \ ;\ \psi_2 = \chi^{1'} + i\xi_{1'} \ ; \\ \psi_3 = -(\chi^2 - i\xi_{2'}) \ ;\ \psi_4 = \chi^{1} - i\xi_{1'} \ . \end{cases}$$

In particular, for a pure ψ_1 solution, we have

$$(15)\quad \chi^1 = \xi_1 = 0 \ ;\ \chi^2 = i\xi_{2'} \ .$$

For our static case, this corresponds to a 'spin-up' pure electron wave (no positron). See Schiff, (52.17). Finally, we note for a static solution with normal metric form (1), we may take for the electromagnetic potential

(16) $$\phi_\alpha = 0 \; ; \; \phi_4 \equiv \phi.$$

(16) implies we have a pure electrostatic field with no magnetism.

V. THE DIRAC EQUATIONS

Our aim is to calculate the explicit form of the Dirac equations in the special spinor calculus of Chapter IV, assuming the static metric form

$$(1) \quad \Phi = -k(dx^1)^2 - g(dx^2)^2 - h(dx^3)^2 + f(dx^4)^2,$$

where k, g, h, f do not depend upon x^4. From III-(79) and the definition of spinor derivatives, the Dirac equations are

$$(2) \quad \sigma^{\ell}_{A'B}\left(\chi^{A'}_{,\ell} + \Gamma^{A'}_{C'\ell}\chi^{C'}\right) - \frac{m}{\sqrt{2}}\xi_B = 0,$$

$$(3) \quad \sigma^{\ell A'B}\left(\xi_{A',\ell} - \Gamma^{C'}_{A'\ell}\xi_{C'}\right) + \frac{m}{\sqrt{2}}\chi^B = 0.$$

Writing out the summations, and using the expressions IV-(2), (3),

$$(4) \quad \sigma^1_{2'1}\chi^{2'}_{,1} + \sigma^2_{2'1}\chi^{2'}_{,2} + \sigma^3_{1'1}\chi^{1'}_{,3} + \sigma^4_{1'1}\chi^{1'}_{,4}$$

$$+ \sigma^{\ell}_{A'1}\Gamma^{A'}_{C'\ell}\chi^{C'} - \frac{m}{\sqrt{2}}\xi_1 = 0,$$

$$(5) \quad \sigma^1_{1'2}\chi^{1'}_{,1} \pm \sigma^2_{1'2}\chi^{1'}_{,2} + \sigma^3_{2'2}\chi^{2'}_{,3} + \sigma^4_{2'2}\chi^{2'}_{,4}$$

$$+ \sigma^{\ell}_{A'2}\Gamma^{A'}_{C'\ell}\chi^{C'} - \frac{m}{\sqrt{2}}\xi_2 = 0,$$

$$(6) \quad \sigma^{12'1}\xi_{2',1} + \sigma^{22'1}\xi_{2',2} + \sigma^{31'1}\xi_{1',3} + \sigma^{41'1}\xi_{1',4} +$$

$$-\sigma^{\ell A'1}\Gamma^{C'}_{A'\ell}\xi_{C'} + \frac{m}{\sqrt{2}}\chi^{1} = 0,$$

(7) $$\sigma^{11'2}\xi_{1';1} + \sigma^{21'2}\xi_{1';2} + \sigma^{32'2}\xi_{2';3} + \sigma^{42'2}\xi_{2';4}$$

$$-\sigma^{\ell A'2}\Gamma^{C'}_{A'\ell}\xi_{C'} + \frac{m}{\sqrt{2}}\chi^{2} = 0,$$

and these become

(8) $$-K^{-\frac{1}{2}}\chi^{2'}_{,1} + ig^{-\frac{1}{2}}\chi^{2'}_{,2} - h^{-\frac{1}{2}}\chi^{1'}_{,3} + f^{-\frac{1}{2}}\chi^{1'}_{,4}$$

$$+\sqrt{2}\,\sigma^{\ell}_{A'1}\Gamma^{A'}_{1'\ell}\chi^{1'} + \sqrt{2}\,\sigma^{\ell}_{A'1}\Gamma^{A'}_{2'\ell}\chi^{2'} - m\xi_1 = 0,$$

(9) $$-K^{-\frac{1}{2}}\chi^{1'}_{,1} - ig^{-\frac{1}{2}}\chi^{1'}_{,2} + h^{-\frac{1}{2}}\chi^{2'}_{,3} + f^{-\frac{1}{2}}\chi^{2'}_{,4}$$

$$+\sqrt{2}\,\sigma^{\ell}_{A'2}\Gamma^{A'}_{1'\ell}\chi^{1'} + \sqrt{2}\,\sigma^{\ell}_{A'2}\Gamma^{A'}_{2'\ell}\chi^{2'} - m\xi_2 = 0,$$

(10) $$K^{-\frac{1}{2}}\xi_{2';1} + ig^{-\frac{1}{2}}\xi_{2';2} + h^{-\frac{1}{2}}\xi_{1';3} + f^{-\frac{1}{2}}\xi_{1';4}$$

$$-\sqrt{2}\,\sigma^{\ell A'1}\Gamma^{1'}_{A'\ell}\xi_{1'} - \sqrt{2}\,\sigma^{\ell A'1}\Gamma^{2'}_{A'\ell}\xi_{2'} + m\chi^{1} = 0,$$

(11) $$K^{-\frac{1}{2}}\xi_{1';1} - ig^{-\frac{1}{2}}\xi_{1';2} - h^{-\frac{1}{2}}\xi_{2';3} + f^{-\frac{1}{2}}\xi_{2';4}$$

$$-\sqrt{2}\,\sigma^{\ell A'2}\Gamma^{1'}_{A'\ell}\xi_{1'} - \sqrt{2}\,\sigma^{\ell A'2}\Gamma^{2'}_{A'\ell}\xi_{2'} + m\chi^{2} = 0.$$

For the eight terms involving the spinor affinities, we use II-(47) and the special spinor calculus. Then e.g.

$$(12)\quad \Gamma^{1'}_{1'1} = -\frac{1}{2}\left[\sigma_{m1'B}\sigma^{m1'B}{}_{,1} + \left\{{m \atop n1}\right\}\sigma^{n1'B}\sigma_{m1'B}\right].$$

For our special spin matrices,

$$\sigma_{m1'B}\sigma^{m1'B}{}_{,1} = \sigma_{11'2}\sigma^{11'2}{}_{,1} + \sigma_{21'2}\sigma^{21'2}{}_{,1} + \sigma_{31'1}\sigma^{31'1}{}_{,1} + \sigma_{41'1}\sigma^{41'1}{}_{,1}$$

$$= \frac{1}{2}\left[K^{\frac{1}{2}}(K^{-\frac{1}{2}})_{,1} + g^{\frac{1}{2}}(g^{-\frac{1}{2}})_{,1} + h^{\frac{1}{2}}(h^{-\frac{1}{2}})_{,1} + f^{\frac{1}{2}}(f^{-\frac{1}{2}})_{,1}\right]$$

$$= -\frac{1}{4}\left[K^{-1}K_{,1} + g^{-1}g_{,1} + h^{-1}h_{,1} + f^{-1}f_{,1}\right].$$

Similarly,

$$\left\{{m \atop n1}\right\}\sigma^{n1'B}\sigma_{m1'B} = \left\{{1 \atop 11}\right\}\sigma^{11'2}\sigma_{11'2} + \left\{{2 \atop 11}\right\}\sigma^{11'2}\sigma_{21'2} + \left\{{1 \atop 21}\right\}\sigma^{21'2}\sigma_{11'2}$$

$$+ \left\{{2 \atop 21}\right\}\sigma^{21'2}\sigma_{21'2} + \left\{{3 \atop 31}\right\}\sigma^{31'1}\sigma_{31'1} + \left\{{4 \atop 41}\right\}\sigma^{41'1}\sigma_{41'1}$$

$$= -\frac{i}{2}K^{-\frac{1}{2}}g^{-\frac{1}{2}}K_{,2} + \frac{1}{4}\left[K^{-1}K_{,1} + g^{-1}g_{,1} + h^{-1}h_{,1} + f^{-1}f_{,1}\right].$$

Then we have for (12)

$$(13)\quad \Gamma^{1'}_{1'1} = \frac{i}{4}K^{-\frac{1}{2}}g^{-\frac{1}{2}}K_{,2}.$$

The remaining spinor affinities are calculated similarly; their values are given in Appendix A. Using these results,

$$\sigma^{\ell}{}_{A'1}\Gamma^{A'}{}_{1'\ell} = \sigma^{1}{}_{2'1}\Gamma^{2'}{}_{1'1} + \sigma^{2}{}_{2'1}\Gamma^{2'}{}_{1'2} + \sigma^{3}{}_{1'1}\Gamma^{1'}{}_{1'3} + \sigma^{4}{}_{1'1}\Gamma^{1'}{}_{1'4}$$

$$= \frac{1}{4\sqrt{2}}\left[-K^{-\frac{1}{2}}(K^{-\frac{1}{2}}h^{-\frac{1}{2}}K_{,3}) - g^{-\frac{1}{2}}(g^{-\frac{1}{2}}h^{-\frac{1}{2}}g_{,3}) + f^{-\frac{1}{2}}(-f^{-\frac{1}{2}}h^{-\frac{1}{2}}f_{,3} - 4i\epsilon\phi)\right]$$

$$= \frac{-h^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Kgf)_{,3} - \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\, i\epsilon\phi .$$

Altogether, similar calculations give the results

$$\sigma^{k}{}_{A'1}\Gamma^{A'}{}_{1'k} = \frac{-h^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Kgf)_{,3} - \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\, i\epsilon\phi ;$$

$$\sigma^{k}{}_{A'1}\Gamma^{A'}{}_{2'k} = \frac{-K^{-\frac{1}{2}}}{4\sqrt{2}}(\ln ghf)_{,1} + \frac{ig^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Khf)_{,2} ;$$

$$\sigma^{k}{}_{A'2}\Gamma^{A'}{}_{1'k} = \frac{-K^{-\frac{1}{2}}}{4\sqrt{2}}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Khf)_{,2} ;$$

$$\text{(14)}\quad \sigma^{k}{}_{A'2}\Gamma^{A'}{}_{2'k} = \frac{h^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Kgf)_{,3} - \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\, i\epsilon\phi ;$$

$$\sigma^{kA'1}\Gamma^{1'}{}_{A'k} = \frac{-h^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Kgf)_{,3} - \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\, i\epsilon\phi ;$$

$$\sigma^{kA'1}\Gamma^{2'}{}_{A'k} = \frac{-K^{-\frac{1}{2}}}{4\sqrt{2}}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Khf)_{,2} ;$$

$$\sigma^{kA'2}\Gamma^{1'}{}_{A'k} = \frac{-K^{-\frac{1}{2}}}{4\sqrt{2}}(\ln ghf)_{,1} + \frac{ig^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Khf)_{,2} ;$$

$$\sigma^{kA'2}\Gamma^{2'}{}_{A'k} = \frac{h^{-\frac{1}{2}}}{4\sqrt{2}}(\ln Kgf)_{,3} - \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\, i\epsilon\phi .$$

Using (8)-(11) and (14), the general Dirac equations for a static metric and electrostatic field may be written

$$(15)\quad -K^{-\frac{1}{2}}\chi^2_{,1} - ig^{-\frac{1}{2}}\chi^2_{,2} - h^{-\frac{1}{2}}\chi^1_{,3} + f^{-\frac{1}{2}}\chi^1_{,4} - \left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3} - f^{-\frac{1}{2}} i\epsilon\phi\right]\chi^1 + \left[-\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\chi^2 = m\xi_{1'},$$

$$(16)\quad -K^{-\frac{1}{2}}\chi^1_{,1} + ig^{-\frac{1}{2}}\chi^1_{,2} + h^{-\frac{1}{2}}\chi^2_{,3} + f^{-\frac{1}{2}}\chi^2_{,4} - \left[\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\chi^1 + \left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3} + f^{-\frac{1}{2}} i\epsilon\phi\right]\chi^2 = m\xi_{2'},$$

$$(17)\quad K^{-\frac{1}{2}}\xi_{2',1} + ig^{-\frac{1}{2}}\xi_{2',2} + h^{-\frac{1}{2}}\xi_{1',3} + f^{-\frac{1}{2}}\xi_{1',4} + \left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3} + f^{-\frac{1}{2}} i\epsilon\phi\right]\xi_{1'} + \left[\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1} + \frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\xi_{2'} = m\chi^1,$$

$$(18)\quad K^{-\frac{1}{2}}\xi_{1',1} - ig^{-\frac{1}{2}}\xi_{1',2} - h^{-\frac{1}{2}}\xi_{2',3} + f^{-\frac{1}{2}}\xi_{2',4} + \left[\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\xi_{1'} - \left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3} - f^{-\frac{1}{2}} i\epsilon\phi\right]\xi_{2'} = -m\chi^2.$$

For a pure ψ_1 solution, we have shown in IV-(15) that $\chi^1 = \xi_1 = 0$, and (15)-(18) reduce to

$$(19)\quad -K^{-\frac{1}{2}}\chi^2_{,1} - ig^{-\frac{1}{2}}\chi^2_{,2} + \left[-\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1} - \frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\chi^2 = 0,$$

$$(20)\quad h^{-\frac{1}{2}}\chi^2_{,3}+f^{-\frac{1}{2}}\chi^2_{,4}+\left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3}+f^{-\frac{1}{2}}i\varepsilon\phi\right]\chi^2=m\xi_{2'},$$

$$(21)\quad K^{-\frac{1}{2}}\xi_{2',1}+ig^{-\frac{1}{2}}\xi_{2',2}+\left[\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1}+\frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\xi_{2'}=0,$$

$$(22)\quad -h^{-\frac{1}{2}}\xi_{2',3}+f^{-\frac{1}{2}}\xi_{2',4}-\left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3}-f^{-\frac{1}{2}}i\varepsilon\phi\right]\xi_{2'}=-m\chi^2.$$

We have not yet used the condition $\chi^2 = i\xi_{2'}$ of IV-(15). Substituting this into (19), we have

$$(23)\quad -iK^{-\frac{1}{2}}\xi_{2',1}+g^{-\frac{1}{2}}\xi_{2',2}-i\left[\frac{K^{-\frac{1}{2}}}{4}(\ln ghf)_{,1}+\frac{ig^{-\frac{1}{2}}}{4}(\ln Khf)_{,2}\right]\xi_{2'}=0.$$

Multiplying by i, we see (23) is identical with (21). Similarly, IV-(15) gives for (20)

$$(24)\quad ih^{-\frac{1}{2}}\xi_{2',3}+if^{-\frac{1}{2}}\xi_{2',4}+i\left[\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3}+f^{-\frac{1}{2}}i\varepsilon\phi\right]\xi_{2'}=-im\chi^2.$$

Multiplying (24) by i and adding to (22), we find

$$(25)\quad h^{-\frac{1}{2}}\xi_{2',3}+\frac{h^{-\frac{1}{2}}}{4}(\ln Kgf)_{,3}\xi_{2'}=0,$$

or

$$(26)\quad (\ln\xi_{2'})_{,3}=-\frac{1}{4}(\ln Kgf)_{,3}\ ,$$

which gives

$$(27)\quad \ln\xi_{2'}=-\frac{1}{4}\ln(Kgf)+\ln X(x^1,x^2,x^4),$$

where X is arbitrary. Substitution of (25) into (24) yields

$$(\ln \xi_{2'})_{,4} = i(mf^{\frac{1}{2}} + \varepsilon\phi), \tag{28}$$

and we have shown that for a static metric and electrostatic field, the general pure ψ_1 solution of the Dirac equations is the general solution of (21), (27), (28).

As usual for stationary waves in quantum mechanics, we look for a solution satisfying the separability property

$$\psi_1 = S(\underline{x})P(t), \quad t \equiv x^4, \tag{29}$$

where S is a (possibly complex) function of x^1, x^2, x^3 only. This is equivalent to

$$\chi^2 = -\tfrac{1}{2}SP \quad ; \quad \xi_{2'} = \tfrac{i}{2}SP. \tag{30}$$

Then (28) gives

$$\frac{P'}{P} = -iC = -i(mf^{\frac{1}{2}} + \varepsilon\phi), \tag{31}$$

or

$$P = e^{-iCt}, \tag{32}$$

and

$$f^{\frac{1}{2}} = \frac{1}{m}(C - \varepsilon\phi). \tag{33}$$

From physical considerations for a particle at rest and equation (32), it seems reasonable to take C = m. Altogether, for a separable pure ψ_1 solution we have

$$(34) \quad f^{\frac{1}{2}} = 1 - \frac{\varepsilon}{m}\phi ,$$

and

$$(35) \quad \xi_{2'} = \frac{i}{2}(Kgf)^{-\frac{1}{4}} B(x^1, x^2) e^{-imt} ,$$

where B is yet to be determined, and it remains to satisfy (21). (34) is usually termed the Weyl-Majumdar relation. By (30), (35),

$$(36) \quad S = (Kgf)^{-\frac{1}{4}} B(x^1, x^2).$$

Note that if we make the further assumption

$$(37) \quad \frac{S_{,1}}{S} , \frac{S_{,2}}{S} \quad \text{real},$$

then (21) is satisfied iff

$$(38) \quad \begin{cases} \ln S = -\frac{1}{4}\ln(ghf) + \ln Y(x^2, x^3), \\ \ln S = -\frac{1}{4}\ln(Khf) + \ln Z(x^1, x^3) , \end{cases}$$

where Y, Z are arbitrary, or

$$(39)\quad S=(Kgf)^{-\frac{1}{4}}B(x^1,x^2)=(ghf)^{-\frac{1}{4}}Q(x^2,x^3)$$
$$=(Khf)^{-\frac{1}{4}}R(x^1,x^3),$$

and the Dirac equations are satisfied by (30), (32), (34), and (39), provided only B, Q, R can be chosen to make (39) an identity. However, we do not make the special assumption (37) at present.

In summary, then, we have shown the following: for a static space-time and purely electrostatic field, the Dirac equations are given by (15)-(18). If one further looks for a pure ψ_1 solution, these equations may be simplified to the equations (21), (27), (28). If the solution is to be separable, it must be given by (21), (30), (32), (34), (35). Finally, if (37) holds, the solution is given by (30), (32), (34), and (39).

VI. THE MAXWELL - EINSTEIN EQUATIONS

We investigate the Maxwell-Einstein equations, using a procedure similar to that of Das (10). For a static metric and pure ~~electrostatic field, the first three Maxwell equations are identically~~ zero. We show that when the mass and charge satisfy the balance condition

$$(1) \quad e = \pm m,$$

we have $M^4 = 0$ iff $Q_{44} = 0$. Classically, it seems reasonable that equilibrium of gravitational and electrostatic forces would only be possible provided (1) holds.

Until noted otherwise, in this section we use the normal static metric form

$$(2) \quad \Phi = -K(dx^1)^2 - g(dx^2)^2 - h(dx^3)^2 + f(dx^4)^2,$$

as in IV-(1). Barred quantities (e.g. $\bar{R}_{\alpha\beta}$) belong to the positive definite 3-space

$$(3) \quad \bar{\Phi} = K(dx^1)^2 + g(dx^2)^2 + h(dx^3)^2,$$

and a double slash $||$ represents covariant differentiation in this 3-space. The double slash spinor derivative used in Chapters II, III will not be needed here. We remind the reader that Greek indices are to take on the values 1-3, and further define

$$\text{(4)}\quad \Delta_1 W \equiv g^{ij} W_{,i} W_{,j} \; ; \; \Delta_2 W \equiv g^{ij} W_{|ij} \; ; \; d \equiv \det(g_{ij}) .$$

Thus e.g. $\bar{\Delta}_2 h = \bar{g}^{\alpha\beta} h_{||\alpha\beta}$. Finally, we assume a pure ψ_1 wave field.

We begin with the remaining Maxwell equation $M^4 = 0$. By III-(79), this equation is

$$F^{4m}{}_{|m} - 2\sqrt{2}\, j^4 = 0,$$

or

$$F^{4\alpha}{}_{|\alpha} = 2\sqrt{2}\, j^4 .$$

This gives

$$\text{(5)}\quad (-d)^{-\frac{1}{2}} \left(\sqrt{-d}\, F^{4\alpha}\right)_{,\alpha} = 2\sqrt{2}\, \varepsilon\, \sigma^4{}_{B'A} \left(\chi^A \chi^{B'} + \xi^A \xi^{B'}\right).$$

By V-(34), we have the Weyl-Majumdar condition, which we write as

$$\text{(6)}\quad f = F(\phi),$$

where $\phi = \phi_4$ is the surviving component of the electromagnetic potential. This assumption was originally due to Weyl, and for our assumptions is a necessary consequence of the Dirac equations. Now

$$\text{(7)}\quad \sqrt{-d} = F^{\frac{1}{2}} \sqrt{-\bar{d}},$$

and by (5), (7), we have

$$(\bar{d})^{-\frac{1}{2}}(\sqrt{\bar{d}}\,F^{-\frac{1}{2}}\bar{g}^{\alpha\beta}\phi_{,\beta})_{,\alpha} = -2\varepsilon(\chi^{2}\chi^{2'}+\xi_{2}\xi_{2'}).$$

Then

$$(F^{-\frac{1}{2}}\bar{g}^{\alpha\beta}\phi_{,\beta})_{\|\alpha} = -2\varepsilon(\chi^{2}\chi^{2'}+\xi_{2}\xi_{2'}).$$

This gives our remaining Maxwell equation in the form

$$(8)\quad \bar{\Delta}_{2}\phi = \tfrac{1}{2}F^{-1}F'\bar{\Delta}_{1}\phi - 2\varepsilon F^{\frac{1}{2}}(\chi^{2}\chi^{2'}+\xi_{2}\xi_{2'}).$$

We turn now to the calculation of the $Q_{44}' = 0$ field equation. First we find an expression for the curvature invariant R. We have from the dual field equations

$$(9)\quad R = \tfrac{1}{2}M$$

where $M = g^{ij}M_{ij}$, and we have used the result $g^{ij}E_{ij} = 0$. From III-(81), we have

$$(10)\quad M = -16\pi\sqrt{2}\,i\left[\sigma^{k}{}_{B'A}\chi^{A}\chi^{B'}{}_{|k} - \sigma^{kB'A}\xi_{A}\xi_{B'|k}\right] + \text{c.c.}$$

Using the Dirac equations of III-(79), we have

$$(11)\quad M = -32\pi i m(\chi^{A}\xi_{A} - \chi^{A'}\xi_{A'}).$$

Combining (9), (11),

(12) $R = 16\pi i m(\chi^{A'}\xi_{A'} - \chi^{A}\xi_{A})$.

For our static metric form (2), we may write

(13) $R_{44} = -f^{\frac{1}{2}}\overline{\Delta}_2(f^{\frac{1}{2}})$,

from which it follows that

$$R_{44} = -F^{\frac{1}{2}}\overline{g}^{\alpha\beta}(F^{\frac{1}{2}})_{\|\alpha\beta}.$$

This may be written

$$R_{44} = -F^{\frac{1}{2}}\overline{g}^{\alpha\beta}(\tfrac{1}{2}F^{-\frac{1}{2}}F'\phi_{,\alpha})_{\|\beta},$$

or

(14) $R_{44} = \frac{1}{4}F^{-1}(F')^2\overline{\Delta}_1\phi - \frac{1}{2}F''\overline{\Delta}_1\phi - \frac{1}{2}F'\overline{\Delta}_2\phi$.

The electromagnetic contribution to Q_{44} is

(15) $E_{44} = 16\pi(\frac{1}{4}g_{44}F^{ab}F_{ab} - F_{4k}F_4{}^{k})$.

In terms of the potential,

$$-F_{4k}F_4{}^{k} = -(\phi_{4|k} - \phi_{k|4})g^{ka}(\phi_{4|a} - \phi_{a|4}) = -g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} = \overline{\Delta}_1\phi,$$

$$\frac{1}{4}g_{44}F^{ab}F_{ab} = \frac{1}{4}Fg^{ac}g^{bd}F_{cd}F_{ab}$$

$$= \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} = -\frac{1}{2} \overline{\Delta}_1 \phi.$$

Altogether, the electromagnetic energy contribution is

$$(16) \quad E_{44} = -K \overline{\Delta}_1 \phi.$$

where as before, $K = -8\pi$. For the matter contribution to T_{44}, we have

$$(17) \quad M^{44} = 2\sqrt{2} K i g^{44} (\sigma^4{}_{B'A} \chi^A \chi^{B'}{}_{|4} - \sigma^4{}_{A'B} \chi^{A'} \chi^B{}_{|4}$$
$$- \sigma^{4B'A} \xi_A \xi_{B'|4} + \sigma^{4A'B} \xi_{A'} \xi_{B|4}).$$

For a pure ψ_1 solution ($\chi^1 = \xi_1 = 0$), we have

$$M^{44} = 2\sqrt{2} K i F^{-1} (\sigma^4{}_{2'2} \chi^2 \chi^{2'}{}_{|4} - \sigma^4{}_{2'2} \chi^{2'} \chi^2{}_{|4}$$
$$- \sigma^{42'2} \xi_2 \xi_{2'|4} + \sigma^{42'2} \xi_{2'} \xi_{2|4})$$
$$= 2 K i F^{-\frac{3}{2}} (\chi^2 \chi^{2'}{}_{|4} - \chi^{2'} \chi^2{}_{|4} - \xi_2 \xi_{2'|4} + \xi_{2'} \xi_{2|4}).$$

Lowering indices gives

$$(18) \quad M_{44} = 2 K i F^{\frac{1}{2}} (\chi^2 \chi^{2'}{}_{|4} - \chi^{2'} \chi^2{}_{|4} - \xi_2 \xi_{2'|4} + \xi_{2'} \xi_{2|4}).$$

Using (12), (14), (16), and (18), $Q_{44} = 0$ can be written

$$\tfrac{1}{4}F^{-1}(F')^2\bar{\Delta}_1\phi-\tfrac{1}{2}F''\bar{\Delta}_1\phi-\tfrac{1}{2}F'\bar{\Delta}_2\phi-KFim(\chi^2\xi_2-\chi^{2'}\xi_{2'})$$

$$=\tfrac{1}{2}K\bar{\Delta}_1\phi-iKF^{\frac{1}{2}}(\chi^2\chi^{2'}_{/4}-\chi^{2'}\chi^2_{/4}-\xi_2\xi_{2'/4}+\xi_{2'}\xi_{2/4}),$$

or finally

$$(19)\quad \bar{\Delta}_2\phi=\left[\tfrac{1}{2}F^{-1}F'-F''(F')^{-1}-K(F')^{-1}\right]\bar{\Delta}_1\phi\ +$$

$$-2KimF(F')^{-1}(\chi^2\xi_2-\chi^{2'}\xi_{2'})+2iKF^{\frac{1}{2}}(F')^{-1}\times$$

$$\times(\chi^2\chi^{2'}_{/4}-\chi^{2'}\chi^2_{/4}-\xi_2\xi_{2'/4}+\xi_{2'}\xi_{2/4}).$$

Comparing the $Q_{44}=0$ equation (19) with the $M^4=0$ equation (8), one sees that their consistency requires

$$(20)\quad (F''+K)(F')^{-1}\bar{\Delta}_1\phi=2\epsilon F^{\frac{1}{2}}(\chi^2\chi^{2'}+\xi_2\xi_{2'})\ +$$

$$-2KimF(F')^{-1}(\chi^2\xi_2-\chi^{2'}\xi_{2'})+2KiF^{\frac{1}{2}}(F')^{-1}\times$$

$$\times\ \chi^2\chi^{2'}_{/4}-\chi^{2'}\chi^2_{/4}-\xi_2\xi_{2'/4}+\xi_{2'}\xi_{2/4}).$$

By V-(34), $F=(1-\frac{\varepsilon}{m}\phi)^2$, which gives

$$(21)\quad F'=\frac{-2\varepsilon}{m}\left(1-\frac{\varepsilon}{m}\phi\right);\quad F''=-K\frac{e^2}{m^2}.$$

Using (21) and the 'balance' condition (1), the left side of (20) is identically zero. For the right side of (20), we calculate the spinor quantities which appear using our separable form assumed for the wave field in V-(30):

$$\chi^2 = -\tfrac{1}{2} S(x^\alpha) P(t) \; ; \; \xi_{2'} = \tfrac{i}{2} S P.$$

This assumption will be justified by the form of our final solution. First, we have

$$(22) \quad \chi^{2'}{}_{|4} = \chi^{2'}{}_{,4} + \Gamma^{2'}_{2'4} \chi^{2'},$$

and using the value of $\Gamma^{2'}_{2'4}$ given in Appendix A,

$$(23) \quad \chi^{2'}{}_{|4} = \chi^{2'}{}_{,4} + \tfrac{1}{4} f_{,3} f^{-\frac{1}{2}} h^{-\frac{1}{2}} \chi^{2'} - i\varepsilon\phi\chi^{2'}.$$

Therefore

$$(24) \quad \chi^2 \chi^{2'}{}_{|4} = \chi^2 \chi^{2'}{}_{,4} + \tfrac{1}{4} f_{,3} f^{-\frac{1}{2}} h^{-\frac{1}{2}} \chi^2 \chi^{2'} - i\varepsilon\phi\chi^2\chi^{2'}.$$

Using V-(30) and V-(32), (24) becomes

$$(25) \quad \chi^2 \chi^{2'}{}_{|4} = \frac{im}{4} |S|^2 + \tfrac{1}{4} |S|^2 \left[\tfrac{1}{4} f^{-\frac{1}{2}} h^{-\frac{1}{2}} f_{,3} - i\varepsilon\phi\right],$$

where $|S| = (S\bar{S})^{\frac{1}{2}}$. Similar calculations produce the corresponding ξ term:

$$(26) \quad \xi_{2|4} = \xi_{2,4} - \Gamma^{2}_{24}\xi_2 = \xi_{2,4} - \tfrac{1}{4} f^{-\frac{1}{2}} h^{-\frac{1}{2}} f_{,3} \xi_2 - i\varepsilon\phi\xi_2 ;$$

$$(27)\quad \xi_{2'}\xi_{2|4} = \frac{im}{4}|S|^2 + \frac{1}{4}|S|^2\left(-\frac{1}{4}f^{-\frac{1}{2}}h^{-\frac{1}{2}}f_{,3} - i\varepsilon\phi\right).$$

Then (25), (27), and V-(34) give

$$(28)\quad \chi^2\chi^{2'}_{|4} - \chi^{2'}\chi^2_{|4} - \xi_2\xi_{2'|4} + \xi_{2'}\xi_{2|4} = i|S|^2(m - \varepsilon\phi)$$

$$= imf^{\frac{1}{2}}|S|^2.$$

In terms of S, the remaining spinor terms of (20) become

$$(29)\quad \chi^2\chi^{2'} + \xi_2\xi_{2'} = \frac{1}{2}|S|^2,$$

$$(30)\quad \chi^2\xi_2 - \chi^{2'}\xi_{2'} = \frac{i}{2}|S|^2.$$

From (21) and (28)-(30), we find for the right side of (20)

$$\varepsilon f^{\frac{1}{2}}|S|^2 - 2Kim\left(\frac{-m}{2\varepsilon}f^{\frac{1}{2}}\right)\frac{i}{2}|S|^2 + 2Kif^{\frac{1}{2}}\left(\frac{-m}{2\varepsilon}\right)|S|^2 im.$$

Multiplying by $\varepsilon f^{-\frac{1}{2}}|S|^{-2}$, this gives for the r.h.s. (20)

$$(31)\quad \varepsilon^2 - 4\pi m^2,$$

and using $\varepsilon = \sqrt{4\pi}\, e$, we find the right side of (20) to be identically zero iff (1) holds. In this case, we have $M^4 = 0$ iff $Q_{44} = 0$, and the Maxwell equations will be satisfied provided the field equations are. For future use, by (18) and (28) we have for the energy density

(32) $M_{44} = -2Kmf|S|^2.$

We next turn to a consideration of the $Q_{\alpha\beta} = 0$ field equations. For the electromagnetic contribution to the energy-momentum, by III-(79), III-(82)

$$E_{\alpha\beta} = -2K[-F_{\alpha k}F_{\beta}{}^{k} + \tfrac{1}{4} g_{\alpha\beta} F_{ab}F^{ab}],$$

where

$$-F_{\alpha k}F_{\beta}{}^{k} = -g^{44}\phi_{,\alpha}\phi_{,\beta} = -f^{-1}\phi_{,\alpha}\phi_{,\beta},$$

$$\tfrac{1}{4} g_{\alpha\beta}F_{ab}F^{ab} = \tfrac{1}{2} g_{\alpha\beta} g^{\gamma\delta} g^{44} F_{\gamma 4}F_{\delta 4} = -\tfrac{1}{2} g_{\alpha\beta} f^{-1} \overline{\Delta}_1 \phi.$$

Altogether, we have

(33) $E_{\alpha\beta} = -2K(-f^{-1}\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} f^{-1}\overline{\Delta}_1\phi).$

For our static metric, we would expect from physical considerations that $M_{\alpha\beta} = 0$. By III-(61),

(34) $\widetilde{M}^{\alpha\beta} = 2\sqrt{2}Ki\, g^{\beta\gamma}[\sigma^{\alpha}{}_{B'A}\chi^{A}\chi^{B'}{}_{|\gamma} - \sigma^{\alpha B'A}\xi_{A}\xi_{B'|\gamma}] + \text{c.c.}$

It can be shown that $\widetilde{M}^{\alpha\beta} = 0$. Since $\widetilde{M}^{\alpha\beta}$ is not symmetric, all nine components must be checked independently. We show $\widetilde{M}^{22} = 0$ and $\widetilde{M}^{32} = 0$: the other calculations are similar. Using (34) and the normal form of our metric,

$$\text{(35)}\quad \widetilde{M}^{22} = 2\sqrt{2}Kig^{22}\left[\sigma^{2}{}_{B'A}\chi^{A}\chi^{B'}{}_{|2} - \sigma^{2B'A}\xi_{A}\xi_{B'|2}\right] + c.c.$$

Using our special spinor calculus,

$$\text{(36)}\quad \widetilde{M}^{22} = 2\sqrt{2}Kig^{22}\left[\sigma^{2}{}_{1'2}\chi^{2}\chi^{2'}\Gamma^{1'}_{2'2} + \sigma^{21'2}\xi_{2}\xi_{2'}\Gamma^{2'}_{1'2}\right] + c.c.$$

From Appendix A, $\Gamma^{1'}_{2'2}$ and $\Gamma^{2'}_{1'2}$ are imaginary. Then

$$\text{(37)}\quad \widetilde{M}^{22} = 2\sqrt{2}Kig^{22}\left[\chi^{2}\chi^{2'}\Gamma^{1'}_{2'2}(\sigma^{2}{}_{1'2} + \sigma^{2}{}_{2'1})\right.$$

$$\left. + \xi_{2}\xi_{2'}\Gamma^{2'}_{1'2}(\sigma^{21'2} + \sigma^{22'1})\right] = 0.$$

For $\widetilde{M}^{32}$,

$$\text{(38)}\quad \widetilde{M}^{32} = (i) + (ii),$$

where $(i) = 2\sqrt{2}Kig^{22}(\sigma^{3}{}_{2'2}\chi^{2}\chi^{2'}{}_{,2} - \sigma^{32'2}\xi_{2}\xi_{2',2}) + c.c.$, and $(ii) = 2\sqrt{2}Kig^{22}(\sigma^{3}{}_{2'2}\chi^{2'}\chi^{2} + \sigma^{32'2}\xi_{2}\xi_{2'})\Gamma^{2'}_{2'2} + c.c.$. Using the special spinor calculus,

$$(i) = 2Kg^{22}h^{-\frac{1}{2}}(\chi^{2}\chi^{2'}_{,2} + \xi_{2}\xi_{2',2}) + c.c.$$

By V-(30), $\chi^{2}\chi^{2'}{}_{,2} = \xi_{2'}\xi_{2,2} = \frac{1}{4}S\bar{S}_{,2}$. Then ~~(i) = 0~~. Next,

$$(ii) = 4K\sqrt{2}ig^{22}(\sigma^{3}{}_{2'2}\chi^{2}\chi^{2'} + \sigma^{32'2}\xi_{2}\xi_{2'})\,\mathrm{Im}\,\Gamma^{2'}_{2'2},$$

where Im Γ^A_{BC} denotes the imaginary part of Γ^A_{BC}. Using $\sigma^3{}_{2'2} = -\sigma^{32'2}$ we have

$$(ii) = 4K\sqrt{2}\, i g^{22} \sigma^3{}_{2'2}\, \mathrm{Im}\, \Gamma^{2'}_{2'2} (\chi^2\chi^{2'} - \xi_2\xi_{2'}) = 0.$$

Then by (38) we have $\widetilde{M}^{32} = 0$. Other calculations follow along these lines, giving the result

$$(39) \quad \widetilde{M}^{\alpha\beta} = 0 = M^{\alpha\beta}.$$

We turn now to the calculation of the field equations. For this purpose, it is convenient to write our metric in the form

$$(40) \quad \Phi = -e^{-\omega}(\bar{\bar{g}}_{\alpha\beta}dx^\alpha dx^\beta) + e^\omega dt^2 ,$$

where $\bar{\bar{\Phi}} = \bar{\bar{g}}_{\alpha\beta}dx^\alpha dx^\beta$ is in normal form. We have the relations

$$(41) \quad \bar{g}_{\alpha\beta} = e^{-\omega}\bar{\bar{g}}_{\alpha\beta} \quad ; \quad \bar{\Delta}_1\psi = e^\omega \bar{\bar{\Delta}}_1\psi ;$$

$$f^{\frac{1}{2}} = 1 - \frac{\varepsilon}{m}\phi = e^{\omega/2} ,$$

where Δ_1 was defined in (4). For the metric form (40), we find

$$(42) \quad \begin{cases} R_{\alpha\beta} = \bar{\bar{R}}_{\alpha\beta} + \frac{1}{2}\omega_{,\alpha}\omega_{,\beta} - \frac{1}{2}\bar{\bar{g}}_{\alpha\beta}\bar{\bar{\Delta}}_2\omega ; \\ R_{44} = -\frac{1}{2}e^{2\omega}\bar{\bar{\Delta}}_2\omega \; ; \; R = e^{-\omega}(\bar{\bar{R}} + \frac{1}{2}\bar{\bar{\Delta}}_1\omega - \bar{\bar{\Delta}}_2\omega). \end{cases}$$

As in Das (10), we will find it convenient to replace the field equations $Q_{ij} = 0$ by the dual equations

$$(43) \quad Q^*_{ij} \equiv R_{ij} + \tfrac{1}{2}\left(T_{ij} - \tfrac{1}{2} g_{ij} T\right) = 0,$$

where

$$(44) \quad T = g^{ij} T_{ij} = g^{\alpha\beta} T_{\alpha\beta} + g^{44} T_{44}.$$

To compute T, we have by (16), (32)

$$(45) \quad T_{44} = -K\bar{\Delta}_1 \phi - 2Kme^{\omega}|S|^2.$$

By (1), (41),

$$(46) \quad \phi_{,\alpha} = -\frac{m}{2\varepsilon} e^{\omega/2} \omega_{,\alpha}\,; \; \bar{\Delta}_1 \phi = -\frac{e^{\omega}}{2K} \bar{\Delta}_1 \omega = -\frac{e^{2\omega}}{2K} \bar{\bar{\Delta}}_1 \omega.$$

Using (46), (45) becomes

$$(47) \quad T_{44} = \tfrac{1}{2} e^{2\omega} \bar{\bar{\Delta}}_1 \omega - 2Kme^{\omega}|S|^2.$$

Similarly, by (33), (39),

$$(48) \quad T_{\alpha\beta} = 2K\left(e^{-\omega} \phi_{,\alpha} \phi_{,\beta} + \tfrac{1}{2} g_{\alpha\beta} e^{-\omega} \bar{\Delta}_1 \phi\right),$$

or using (46),

$$(49) \quad T_{\alpha\beta} = -\omega_{,\alpha} \omega_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} e^{\omega} \bar{\bar{\Delta}}_1 \omega.$$

Then from (47), (49),

$$T = -2Km|S|^2. \tag{50}$$

Note if S = 1 identically ($\psi_1 = e^{-imt}$), this gives the simple result $T = -2Km$. Using (43), (49), (50), we can write the $Q^*_{\alpha\beta}$ equations as

$$R_{\alpha\beta} = \frac{1}{2}\left(\omega_{,\alpha}\omega_{,\beta} - \frac{1}{2}\bar{\bar{g}}_{\alpha\beta}\bar{\bar{\Delta}}_1\omega\right) - 4\pi e^{-\omega}\bar{\bar{g}}_{\alpha\beta}\, m|S|^2. \tag{51}$$

Substituting the expression given in (42) for $R_{\alpha\beta}$, (51) gives for $Q^*_{\alpha\beta} = 0$ the equation

$$\bar{\bar{R}}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\left(-e^{\omega}\bar{\bar{\Delta}}_2\omega + \frac{1}{2}e^{\omega}\bar{\bar{\Delta}}_1\omega - Km|S|^2\right) = 0 \tag{52}$$

For the $Q^*_{44} = 0$ equation, by (43), (47), and (50) we have

$$R_{44} = -\frac{1}{4}e^{2\omega}\bar{\bar{\Delta}}_1\omega - 4\pi m e^{\omega}|S|^2. \tag{53}$$

Using (42), (53) becomes

$$-e^{\omega}\bar{\bar{\Delta}}_2\omega + \frac{1}{2}e^{\omega}\bar{\bar{\Delta}}_1\omega - Km|S|^2 = 0. \tag{54}$$

Comparing (52), (54), we find

$$\bar{\bar{R}}_{\alpha\beta} = 0, \tag{55}$$

which implies the 3-space $\bar{\bar{g}}_{\alpha\beta}dx^{\alpha}dx^{\beta}$ is flat. Then (40) may be written

(56) $$\Phi = -e^{-\omega}(dx^\alpha dx^\alpha) + e^{\omega}dt^2,$$

and for this choice the $Q^*_{44} = 0$ equation (54) becomes

(57) $$\omega_{,\alpha\alpha} - \frac{1}{2}\omega_{,\alpha}\omega_{,\alpha} = -Kme^{-\omega}|S|^2.$$

To determine S, we return to the Dirac equations. By (56), we have the relations

(58) $$Kgf = ghf = Khf = e^{-\omega}.$$

Then by V-(36) we have

(59) $$S = e^{\omega/4}B(x^1, x^2),$$

where B is to be determined. If V-(37) holds, by V-(39) the unique solution up to a constant factor is

(60) $$S = e^{\omega/4}; \quad \psi_1 = e^{\omega/4 - imt};$$

$$e^{\omega/2} = 1 - \frac{\varepsilon}{m}\phi = 1 \pm \sqrt{4\pi}\,\phi.$$

If we do not assume V-(37), then we have the Dirac equation V-(21) remaining. Using (58), this equation may be written in the simple form

$$(61)\quad S^{-1}S_{,1}-\tfrac{1}{4}\omega_{,1}+i\left(S^{-1}S_{,2}-\tfrac{1}{4}\omega_{,2}\right)=0.$$

Clearly (60) satisfies (61), but it appears there may exist still other solutions satisfying (59) and (61) (and necessarily the Weyl-Majumdar relation) for which V-(37) does not hold.

Altogether, (56) satisfies the $Q^*_{\alpha\beta} = 0$ field equations. For our assumptions, the Dirac equations are satisfied if (59), (61) hold. The Maxwell equations $M^\alpha = 0$ are identically satisfied, and $M^4 = 0$ is equivalent to $Q^*_{44} = 0$ by the 'balance' condition (1). Thus we are left to study (57), (61), and the $Q^*_{\alpha 4} = 0$ equations.

We now consider the $Q^*_{\alpha 4} = 0$ equations. For our static metric, $R_{\alpha 4} = E_{\alpha 4} = 0$, so $Q^*_{\alpha 4} = 0$ reduces to

$$(63)\quad M_{\alpha 4} = 0.$$

By III-(61),

$$(64)\quad \widetilde{M}^{\alpha 4} = 2\sqrt{2}Kig^{44}\left(\sigma^{\alpha}{}_{B'A}\chi^A\chi^{B'}{}_{/4}-\sigma^{\alpha B'A}\xi_A\xi_{B'/4}\right)+c.c.$$

For the case $\alpha = 1$, we have for our assumptions

$$(65)\quad \widetilde{M}^{14} = 2\sqrt{2}Kig^{44}\left(\sigma^{1}{}_{1'2}\Gamma^{1'}_{2'4}\chi^2\chi^{2'}+\sigma^{11'2}\Gamma^{2'}_{1'4}\xi_2\xi_{2'}\right)+c.c.$$

Then

$$\widetilde{M}^{14} = 2Ki\,K^{-\frac{1}{2}}g^{44}\chi^2\chi^{2'}\left(-\Gamma^{1'}_{2'4}+\Gamma^{2'}_{1'4}\right)+c.c.$$

Obtaining expressions for the spinor affinities from Appendix A and using V-(30),

(66) $\tilde{M}^{14} = -4\pi |S|^2 K^{-\frac{1}{2}} g^{-\frac{1}{2}} f^{-\frac{3}{2}} f_{,2}$.

Similarly, we find for the remaining components of (64)

(67) $\tilde{M}^{24} = 4\pi |S|^2 K^{-\frac{1}{2}} g^{-\frac{1}{2}} f^{-\frac{3}{2}} f_{,1}$;

(68) $\tilde{M}^{34} = 0$.

Again using III-(61),

(69) $\tilde{M}^{4\alpha} = 2\sqrt{2}\, K i g^{\alpha\beta} (\sigma^{4}{}_{B'A} \chi^A \chi^{B'}{}_{|\beta} - \sigma^{4B'A} \xi_A \xi_{B'|\beta}) + c.c.$

For our special assumptions,

$$\tilde{M}^{4\alpha} = 2K i g^{\alpha\beta} f^{-\frac{1}{2}} (\chi^2 \chi^{2'}{}_{,\beta} - \xi_2 \xi_{2',\beta}) +$$

$$+ K i g^{\alpha\beta} f^{-\frac{1}{2}} |S|^2 \Gamma^{2'}_{2'\beta} + c.c.$$

If $\alpha = 1$, we find

$$2K i g^{\alpha\beta} f^{-\frac{1}{2}} (\chi^2 \chi^{2'}{}_{,\beta} - \xi_2 \xi_{2',\beta}) + c.c. = -K i K^{-1} f^{-\frac{1}{2}} (S \bar{S}_{,1} - \bar{S} S_{,1}),$$

and

$$K i g^{11} f^{-\frac{1}{2}} |S|^2 \Gamma^{2'}_{2'1} + c.c. = 4\pi K^{-\frac{3}{2}} g^{-\frac{1}{2}} f^{-\frac{1}{2}} |S|^2 K_{,2}.$$

Altogether,

$$(70)\quad \widetilde{M}^{41} = -K i K^{-1} f^{-\frac{1}{2}}(S\bar{S}_{,1} - \bar{S}S_{,1}) + 4\pi |S|^2 K^{-\frac{3}{2}} g^{-\frac{1}{2}} f^{-\frac{1}{2}} K_{,2}\,.$$

Similarly,

$$(71)\quad \widetilde{M}^{42} = -K i g^{-1} f^{-\frac{1}{2}}(S\bar{S}_{,2} - \bar{S}S_{,2}) - 4\pi |S|^2 g^{-\frac{3}{2}} K^{-\frac{1}{2}} f^{-\frac{1}{2}} g_{,1}\,,$$

$$(72)\quad \widetilde{M}^{43} = -K i h^{-1} f^{-\frac{1}{2}}(S\bar{S}_{,3} - \bar{S}S_{,3})\,.$$

Then the equations (63) may be written

$$(73)\quad \begin{cases} |S|^2 g^{-\frac{1}{2}}(K^{-1}K_{,2} - f^{-1}f_{,2}) + 2iK^{-\frac{1}{2}}(S\bar{S}_{,1} - \bar{S}S_{,1}) = 0, \\ |S|^2 K^{-\frac{1}{2}}(f^{-1}f_{,1} - g^{-1}g_{,1}) + 2ig^{-\frac{1}{2}}(S\bar{S}_{,2} - \bar{S}S_{,2}) = 0, \\ S\bar{S}_{,3} - \bar{S}S_{,3} = 0. \end{cases}$$

The last equation of (73) is equivalent to the condition that $\frac{S_{,3}}{S}$ be real, and is automatically satisfied by (59) and the reality of ω. Using the special metric (56), the remaining equations of (73) may be written

$$(74)\quad \begin{cases} -|S|^2 \omega_{,2} + i(S\bar{S}_{,1} - \bar{S}S_{,1}) = 0, \\ |S|^2 \omega_{,1} + i(S\bar{S}_{,2} - \bar{S}S_{,2}) = 0. \end{cases}$$

Note that if we assume V-(37), then (74) implies $\omega = \omega(x^3)$ only. The remaining Dirac equation may be written

$$(75) \quad S^{-1}S_{,1} = \frac{1}{4}(\omega_{,1} + i\omega_{,2}) - i S^{-1}S_{,2} .$$

Substituting (75) into the first of (74), we find

$$(76) \quad \frac{1}{2}\omega_{,2} = -(\bar{S}^{-1}\bar{S}_{,2} + S^{-1}S_{,2}) = -2(\ln|S|)_{,2} \quad ;$$

and this gives

$$(77) \quad |S| = e^{-\frac{\omega}{4}}|P(x^1, x^3)|, \ P \text{ arbitrary}.$$

Similarly, the other equation of (74) gives

$$(78) \quad |S| = e^{-\omega/4}|Q(x^2, x^3)|, \ Q \text{ arbitrary}.$$

Now (77), (78) are consistent iff $|Q| = |P| = |P|(x^3)$. Since ω depends only upon $|P|$, we take $P = P(x^3)$. Then (59), (77) give

$$(79) \quad e^{\omega/2} = \left|\frac{P(x^3)}{B(x^1, x^2)}\right| .$$

Therefore (59) becomes

$$(80) \quad S = \pm \left|\frac{P(x^3)}{B(x^1, x^2)}\right|^{1/2} B(x^1, x^2).$$

Then

$$S^{-1}S_{,1} = \frac{3}{4}B^{-1}B_{,1} - \frac{1}{4}\bar{B}^{-1}\bar{B}_{,1} \ ; \quad \omega_{,1} = -(\ln B\bar{B})_{,1} \quad ,$$

and similarly for $S^{-1}S_{,2}$, $\omega_{,2}$. Then (75) becomes

$$B^{-1}B_{,1} + i\,B^{-1}B_{,2} = 0, \tag{81}$$

and therefore B is analytic. The remaining $M^{\alpha 4} = 0$ equations (74) are identically satisfied by (81). Thus with our assumptions we have a solution of the combined equations with metric (56) and spatial part of the wave functions given by (80), provided (81) and the $Q^*_{44} = 0$ equation (57) can be satisfied.

The following observation is of interest: if $M_{\alpha 4}$ had been zero, then the $Q^*_{\alpha 4} = 0$ equations would have been identically satisfied. In this case, (56), (60) would provide a solution of the combined equations, provided only that ω satisfied the $Q^*_{44} = 0$ equation (57). One can show that there would then exist spherically symmetric solutions with finite total charge. The fact that $M_{\alpha 4} \neq 0$ may be interpreted as showing that the intrinsic spin of the particle is affecting the geometry in the same fashion as orbital angular momentum would. In the light of a full spinor calculus, we see that the true physical situation does not appear to be static. This may explain the lack of physically reasonable solutions in the following.

Turning to the $Q^*_{44} = 0$ equation, by (79)

$$\omega_{,3} = P^{-1}P_{,3} + \bar{P}^{-1}\bar{P}_{,3} .$$

Then the equation (57) is of the form

$$(82) \quad M(x^1, x^2) + N(x^3) = 8\pi m \frac{|B|^3}{|P|},$$

where $M = \omega_{,11} + \omega_{,22} - \tfrac{1}{2}((\omega_{,1})^2 + (\omega_{,2})^2)$; $N = \omega_{,33} - \tfrac{1}{2}(\omega_{,3})^2$. This implies $|B|$ = constant or $|P|$ = constant. If $|P|$ = constant, we set $|P| = 1$, and by (79)

$$(83) \quad \omega = -\ln(B\bar{B}).$$

Using the analytic property (81), we find

$$(84) \quad \nabla^2\omega = 0.$$

Then the $Q^*_{44} = 0$ equation (57) becomes

$$(85) \quad -\tfrac{1}{2}\left[(\omega_{,1})^2 + (\omega_{,2})^2\right] = 8\pi m e^{-\omega}|S|^2.$$

Since the right side of (85) is positive definite and ω is real, this equation has no solutions.

If $|B|$ = constant = 1, then $S = e^{\frac{1}{4}\omega}$ as in (60), $|P| = e^{\frac{1}{2}\omega}$ by (79), and $\omega = \omega(x^3)$ only. Then the remaining $Q^*_{44} = 0$ equation is

$$(86) \quad \omega'' - \tfrac{1}{2}(\omega')^2 = 8\pi m e^{-\omega/2}.$$

Letting $V = e^{-\frac{1}{2}\omega}$, this becomes

$$(87) \quad V'' = -4\pi m V^2,$$

which may also be written

$$(88)\quad (V')^2 = \frac{-2}{3} a V^3 + C_1 , \quad a = 4\pi m .$$

The general solution of (88) is seen to be

$$(89)\quad V = \frac{-3}{2\pi m} \wp(x^3 + C_2 | g_2 = 0, g_3 = C_1),$$

where p represents the Weierstrauss p-function, and C_1, C_2 are arbitrary real constants. See MacRobert (19) for details. Cf. also (1).

The solutions given by (89) are not flat: by (42),

$$R_{44} = -\frac{1}{2} e^{2\omega} \omega''(x^3) \lambda,$$

and by (86)

$$\omega'' = 8\pi m e^{-\omega/2} + \frac{1}{2} (\omega')^2,$$

which may vanish only when $V = 0$.

Any solutions of this nature are physically uninteresting, since the total charge, given by

$$(90)\quad Q = \int_{V_3} j^4 \eta_4 \sqrt{g}\, d_3 x ,$$

would diverge. Here η_4 represents the unit normal to the hypersurface V_3. For our situation, a necessary condition is that ω be a function of all three spatial coordinates. Thus a solution with finite total charge is not possible under the assumptions we have made. These assumptions are: (i) static space-time; (ii) spinor calculus related to the geometry using Axioms I-III of chapter II, with the electromagnetic field introduced via the minimal electromagnetic interaction; (iii) purely electrostatic e.m. field; (iv) pure ψ_1 solution, space-time separable; (v) $e = \pm m$.

Further, we can show that (v) is unnecessary. Suppose $e = Cm$, $C^2 \neq 1$. From (20), the consistency of $Q_{44} = 0$ and $M^4 = 0$ requires

$$(91)\quad (F'' - 8\pi)(F')^{-1}\bar{\Delta}_1\phi = 2\varepsilon F^{\frac{1}{2}}(\chi^2\chi^{2'} + \xi_2\xi_{2'}) + 16\pi i m \times$$

$$\times F(F')^{-1}(\chi^2\xi_2 - \chi^{2'}\xi_{2'}) - 16\pi i F^{\frac{1}{2}}(F')^{-1}(\chi^2\chi^{2'}_{\ |4} - \chi^{2'}\chi^2_{\ |4} - \xi_2\xi_{2'|4} + \xi_{2'}\xi_{2|4}).$$

From (21),

$$(92)\quad F' = \frac{-2\varepsilon}{m}\left(1 - \frac{\varepsilon}{m}\phi\right);\quad F'' = 8\pi\frac{e^2}{m^2} = 8\pi C^2.$$

Using V-(34),

$$(93)\quad F' = -2\sqrt{4\pi}\,C f^{\frac{1}{2}}.$$

Then for the left side of (91), we have

$$(94)\quad -\sqrt{4\pi}\,\frac{C^2 - 1}{C} f^{-\frac{1}{2}}\bar{\Delta}_1\phi.$$

Using the expression following (30), the right side of (91) becomes

$$(95)\quad \sqrt{4\pi}\, f^{\frac{1}{2}} |S|^2 \frac{m}{C} (C^2-1).$$

Then if $C^2 \neq 1$, the consistency of $Q_{44} = 0$ and $M^4 = 0$ requires

$$(96)\quad \overline{\Delta}_1 \phi = -m|S|^2 f.$$

Since $\overline{\Delta}_1$ is positive definite, (96) cannot be satisfied. We may summarize these results in the

Proposition There does not exist a solution of the combined Dirac-Einstein-Maxwell equations with finite total charge satisfying:

(i) Static space-time;

(ii) Algebraic spinor structure of Ch. II, with minimal electromagnetic interaction;

(iii) Purely electrostatic Maxwell field;

(iv) Space-time separable, pure ψ_1 Dirac field.

As stated previously, there seems some hope that this situation can be remedied by searching for stationary rather than static solutions. Assumption (iii) ignores the magnetic moment of the electron and therefore should be dispensed with for a realistic solution. For assumption (iv), we found no solutions with wave field $\psi_1 = e^{-imt}$: our wave fields have a non-trivial x^3 dependence. From the corresponding situation in special relativity, one would then expect that

$\psi_3 \neq 0$. Thus it seems unlikely that there exist physically realistic solutions in general relativity with a pure ψ_1 Dirac field.

APPENDIX A

For the static metric form

$$\Phi = -K(dx^1)^2 - g(dx^2)^2 - h(dx^3)^2 + f(dx^4)^2$$

where k, g, h, f are functions of x^1, x^2, x^3 only, the non-vanishing Christoffel symbols are given by:

$$\begin{Bmatrix}1\\11\end{Bmatrix} = \tfrac{1}{2}K^{-1}K_{,1}\,; \quad \begin{Bmatrix}2\\22\end{Bmatrix} = \tfrac{1}{2}g^{-1}g_{,2}\,; \quad \begin{Bmatrix}3\\33\end{Bmatrix} = \tfrac{1}{2}h^{-1}h_{,3}\,;$$

$$\begin{Bmatrix}1\\22\end{Bmatrix} = -\tfrac{1}{2}K^{-1}g_{,1}\,; \quad \begin{Bmatrix}1\\33\end{Bmatrix} = -\tfrac{1}{2}K^{-1}h_{,1}\,; \quad \begin{Bmatrix}2\\11\end{Bmatrix} = -\tfrac{1}{2}g^{-1}K_{,2}\,;$$

$$\begin{Bmatrix}2\\33\end{Bmatrix} = -\tfrac{1}{2}g^{-1}h_{,2}\,; \quad \begin{Bmatrix}3\\11\end{Bmatrix} = -\tfrac{1}{2}h^{-1}K_{,3}\,; \quad \begin{Bmatrix}3\\22\end{Bmatrix} = -\tfrac{1}{2}h^{-1}g_{,3}\,;$$

$$\begin{Bmatrix}1\\12\end{Bmatrix} = \tfrac{1}{2}K^{-1}K_{,2}\,; \quad \begin{Bmatrix}1\\13\end{Bmatrix} = \tfrac{1}{2}K^{-1}K_{,3}\,; \quad \begin{Bmatrix}2\\21\end{Bmatrix} = \tfrac{1}{2}g^{-1}g_{,1}\,;$$

$$\begin{Bmatrix}2\\23\end{Bmatrix} = \tfrac{1}{2}g^{-1}g_{,3}\,; \quad \begin{Bmatrix}3\\31\end{Bmatrix} = \tfrac{1}{2}h^{-1}h_{,1}\,; \quad \begin{Bmatrix}3\\32\end{Bmatrix} = \tfrac{1}{2}h^{-1}h_{,2}\,;$$

$$\begin{Bmatrix}1\\44\end{Bmatrix} = \tfrac{1}{2}K^{-1}f_{,1}\,; \quad \begin{Bmatrix}2\\44\end{Bmatrix} = \tfrac{1}{2}g^{-1}f_{,2}\,; \quad \begin{Bmatrix}3\\44\end{Bmatrix} = \tfrac{1}{2}h^{-1}f_{,3}\,;$$

$$\begin{Bmatrix}4\\41\end{Bmatrix} = \tfrac{1}{2}f^{-1}f_{,1}\,; \quad \begin{Bmatrix}4\\42\end{Bmatrix} = \tfrac{1}{2}f^{-1}f_{,2}\,; \quad \begin{Bmatrix}4\\43\end{Bmatrix} = \tfrac{1}{2}f^{-1}f_{,3}\,.$$

The spin matrices for this metric may be written

$$\sigma^{1A'B} = \frac{K^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \; ; \; \sigma^{2A'B} = \frac{g^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} ;$$

$$\sigma^{3A'B} = \frac{h^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \; ; \; \sigma^{4A'B} = \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} ;$$

$$\sigma^{1}{}_{A'B} = \frac{K^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \; ; \; \sigma^{2}{}_{A'B} = \frac{g^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} ;$$

$$\sigma^{3}{}_{A'B} = \frac{h^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \; ; \; \sigma^{4}{}_{A'B} = \frac{f^{-\frac{1}{2}}}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} .$$

Using these expressions, the non-vanishing spinor affinities are given by:

$$\Gamma^1_{11} = \frac{-i}{4} K^{-\frac{1}{2}} g^{-\frac{1}{2}} K_{,2} + i\varepsilon\phi_1 \; ; \; \Gamma^2_{11} = \frac{1}{4} K^{-\frac{1}{2}} h^{-\frac{1}{2}} K_{,3} \; ;$$

$$\Gamma^1_{21} = -\frac{1}{4} h^{-\frac{1}{2}} K^{-\frac{1}{2}} K_{,3} \; ; \; \Gamma^2_{21} = \frac{i}{4} g^{-\frac{1}{2}} K^{-\frac{1}{2}} K_{,2} + i\varepsilon\phi_1 \; ;$$

$$\Gamma^1_{12} = \frac{i}{4} K^{-\frac{1}{2}} g^{-\frac{1}{2}} g_{,1} + i\varepsilon\phi_2 \; ; \; \Gamma^2_{12} = -\frac{i}{4} g^{-\frac{1}{2}} h^{-\frac{1}{2}} g_{,3} \; ;$$

$$\Gamma^1_{22} = -\frac{i}{4} g^{-\frac{1}{2}} h^{-\frac{1}{2}} g_{,3} \; ; \; \Gamma^2_{22} = -\frac{i}{4} K^{-\frac{1}{2}} g^{-\frac{1}{2}} g_{,1} + i\varepsilon\phi_2 \; ;$$

$$\Gamma^1_{13} = i\varepsilon\phi_3 \; ; \; \Gamma^2_{13} = -\frac{1}{4}\left(h^{-\frac{1}{2}} K^{-\frac{1}{2}} h_{,1} - i h^{-\frac{1}{2}} g^{-\frac{1}{2}} h_{,2}\right) ;$$

$$\Gamma^1_{23} = \frac{1}{4}\left(h^{-\frac{1}{2}} K^{-\frac{1}{2}} h_{,1} + i h^{-\frac{1}{2}} g^{-\frac{1}{2}} h_{,2}\right) ; \; \Gamma^2_{23} = i\varepsilon\phi_3 \; ;$$

$$\Gamma^1_{14} = -\tfrac{1}{4} f^{-\frac{1}{2}} h^{-\frac{1}{2}} f_{,3} + i\varepsilon\phi_4 \; ; \; \Gamma^2_{14} = -\tfrac{1}{4}\left(f^{-\frac{1}{2}} k^{-\frac{1}{2}} f_{,1} - i f^{-\frac{1}{2}} g^{-\frac{1}{2}} f_{,2}\right);$$

$$\Gamma^1_{24} = -\tfrac{1}{4}\left(k^{-\frac{1}{2}} f^{-\frac{1}{2}} f_{,1} + i g^{-\frac{1}{2}} f^{-\frac{1}{2}} f_{,2}\right); \; \Gamma^2_{24} = \tfrac{1}{4} f^{-\frac{1}{2}} h^{-\frac{1}{2}} f_{,3} + i\varepsilon\phi_4 .$$

APPENDIX B

Let g_{ij} be a given space-time metric form. In Chapter I, we have shown that the introduction of a spinor structure reduces to the problem of finding an orthonormal tetrad for g_{ij}. In general, the tetrad should be chosen to make fullest use of any space-time symmetries. However, for the sake of mathematical completeness, we show that a tetrad may always be chosen by an entirely constructive procedure. This construction gives the usual OT for metrics in normal form, but e.g. does not give the simplest tetrad for a stationary space. The construction is due to S. Kloster (private communication).

Consider the forms

$$\lambda^{(1)}_i \sim (a_1, 0, 0, 0) \; ; \; \lambda^{(2)}_i \sim (a_2, b_2, 0, 0) \; ;$$

$$\lambda^{(3)}_i \sim (a_3, b_3, c_3, 0) \; ; \; \lambda^{(4)}_i \sim (a_4, b_4, c_4, d_4),$$

and define

$$B \equiv (g_{43})^2 - g_{33} g_{44}$$

Then it follows that

$$\lambda^{(4)}_i = \frac{g_{4i}}{\sqrt{g_{44}}} \; ; \; \lambda^{(3)}_i = \left(\frac{g_{44}}{B}\right)^{1/2} \left(g_{43}g_{41} - g_{31}g_{44}, g_{43}g_{42} - g_{32}g_{44}, \frac{B}{g_{44}}, 0\right);$$

$$\lambda^{(2)}_{2} = (g_{44})^{-\frac{1}{2}}\left[g_{42}^{2} - g_{22}g_{44} - \frac{1}{B}(g_{43}g_{42} - g_{32}g_{44})^{2}\right]^{1/2};$$

$$\lambda^{(2)}_{1} = (\lambda^{(2)}_{2}g_{44})^{-1}\left[g_{42}g_{41} - g_{21}g_{44} - \frac{1}{B}(g_{43}g_{42} - g_{32}g_{44})(g_{43}g_{41} - g_{31}g_{44})\right];$$

$$\lambda^{(1)}_{1} = (g_{44})^{-\frac{1}{2}}\left[(g_{41})^{2} - g_{11}g_{44} - \frac{1}{B}(g_{43}g_{41} - g_{31}g_{44})^{2} - (\lambda^{(2)}_{1})^{2}\right]^{1/2}.$$

Further,

$$\sqrt{-g} = \det\left[\lambda^{(A)}_{i}\right] = \lambda^{(1)}_{1}\lambda^{(2)}_{2}\lambda^{(3)}_{3}\lambda^{(4)}_{4},$$

and the components $\lambda^{n}_{(m)}$ take on the form

$$\lambda^{i}_{(4)} = (0,0,0,(g_{44})^{-\frac{1}{2}});\ \lambda^{i}_{(3)} = B^{-\frac{1}{2}}\left(0,0,\sqrt{g_{44}},\ \frac{-g_{33}}{\sqrt{g_{44}}}\right);$$

$$\lambda^{i}_{(2)} = (\lambda^{(2)}_{2}B)^{-1}(0,1,g_{44}g_{32} - g_{43}g_{42},\ g_{42}g_{33} - g_{43}g_{32});\ \lambda^{i}_{(1)} = -\lambda^{(1)}_{1}g^{i1}.$$

For the spin matrices, let

$$a = -\lambda^{(1)}_{1}g^{13} - i(\lambda^{(2)}_{2}B)^{-1}(g_{44}g_{32} - g_{43}g_{42});\ b = -\lambda^{(1)}_{1}g^{14} - i(\lambda^{(2)}_{2}B)^{-1}(g_{42}g_{33} - g_{43}g_{32});$$

$$c = \frac{g_{42}}{\sqrt{g_{44}}} + \left(\frac{g_{44}}{B}\right)^{\frac{1}{2}}(g_{43}g_{42} - g_{32}g_{44});\ d = \frac{g_{41}}{\sqrt{g_{44}}} - \left(\frac{g_{44}}{B}\right)^{1/2}(g_{43}g_{41} - g_{31}g_{44});$$

Then

$$\sigma^{1A'B} = \frac{-\lambda^{(1)}_{1}g^{11}}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \sigma^{2A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix}0, & -\lambda^{(1)}_{1}g^{12} - i(\lambda^{(2)}_{2}B)^{-1}\\ -\lambda^{(1)}_{1}g^{12} + i(\lambda^{(2)}_{2}B)^{-1}, & 0\end{pmatrix};$$

$$\sigma^{3A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} B^{-\frac{1}{2}}\sqrt{g_{44}} , & a \\ \bar{a} , & -B^{-\frac{1}{2}}\sqrt{g_{44}} \end{pmatrix};$$

$$\sigma^{4A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} (g_{44})^{-\frac{1}{2}}(1-g_{33}B^{-\frac{1}{2}}), & b \\ \bar{b} , & (g_{44})^{-\frac{1}{2}}(1+g_{33}B^{-\frac{1}{2}}) \end{pmatrix};$$

$$\sigma_1{}^{A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} d , & -\lambda_1^{(1)} - i\lambda_1^{(2)} \\ -\lambda_1^{(1)} + i\lambda_1^{(2)}, & -d + 2(g_{44})^{-\frac{1}{2}} g_{41} \end{pmatrix};$$

$$\sigma_2{}^{A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} c , & -i\lambda_2^{(2)} \\ i\lambda_2^{(2)} , & -c + 2(g_{44})^{-\frac{1}{2}} g_{42} \end{pmatrix};$$

$$\sigma_3{}^{A'B} = \frac{1}{\sqrt{2}}\begin{pmatrix} (g_{44})^{-\frac{1}{2}}(g_{43}-B^{\frac{1}{2}}), & 0 \\ 0 , & (g_{44})^{-\frac{1}{2}}(g_{43}+B^{\frac{1}{2}}) \end{pmatrix};$$

$$\sigma_4{}^{A'B} = \frac{\sqrt{g_{44}}}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For particular coordinate patches, this construction may fail, e.g. if $B = 0$, but a similar construction is always possible locally.

APPENDIX C

Here we discuss various choices for the independent variables of a general relativistic Lagrangian, and show that the Euler-Lagrange equations for matter fields in general relativity may be written in explicitly covariant form. Consider a Lagrangian density $\mathcal{L} = L\sqrt{-g}$, where L is a scalar dependent upon g_{ij}, the matter fields, and their first covariant derivatives. The matter fields may be tensor fields, spinor fields, or mixed. We consider three ways of obtaining the Euler-Lagrange equations:

Case I: Let $\phi^{\cdots}$ represent a matter field appearing in L. We treat $\phi^{\cdots}$ and the partial derivatives $\phi^{\cdots}{}_{,d}$ as independent variables, and arrive at the usual equations

$$(1) \quad \left(\frac{\partial\sqrt{-g}\,L}{\partial\phi^{\cdots}{}_{,d}}\right)_{,d} - \frac{\partial\sqrt{-g}\,L}{\partial\phi^{\cdots}} = 0.$$

Case II: We take $\phi^{\cdots}$ and $\phi^{\cdots}{}_{|d}$ as independent, where slash is our covariant derivative of Chapter II, and write for the Euler-Lagrange equations

$$(2) \quad \left(\frac{\partial L}{\partial\phi^{\cdots}{}_{|d}}\right)_{|d} - \frac{\partial L}{\partial\phi^{\cdots}} = 0.$$

Case III: We formally treat $\phi^{\cdots}$ and $\phi^{\cdots}{}_{\|d}$ as independent, where the double slash covariant derivative is the usual covariant derivative for tensors, and is defined by III-(2) for spinor indices. We then write the Euler-Lagrange equations as

$$(3)\quad \left(\frac{\partial L}{\partial \phi^{\cdots}{}_{\|d}}\right)_{\|d} - \frac{\partial L}{\partial \phi^{\cdots}} = 0$$

Let us first consider (1), (2). To save bookkeeping, we suppress all but a representative pair of indices for $\phi^{\cdots}$, writing $\phi^{\cdots} = \phi^{aA\cdots}$. Let

$$\frac{\partial L}{\partial \phi^{aA\cdots}{}_{|d}} = \psi^{d}{}_{aA\cdots}$$

Then

$$\left(\frac{\partial L}{\partial \phi^{aA\cdots}{}_{|d}}\right)_{|d} = \left(\psi^{d}{}_{aA\cdots}\right)_{,d} + \left\{{d \atop dm}\right\}\psi^{m}{}_{aA\cdots} - \left\{{m \atop ad}\right\}\psi^{d}{}_{mA\cdots} - \Gamma^{F}{}_{Ad}\,\psi^{d}{}_{aF\cdots} + \cdots ,$$

or

$$(4)\quad \left(\frac{\partial L}{\partial \phi^{aA\cdots}{}_{|d}}\right)_{|d} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\psi^{d}{}_{aA\cdots}\right)_{,d} - \left\{{m \atop ad}\right\}\psi^{d}{}_{mA\cdots} - \Gamma^{F}{}_{Ad}\,\psi^{d}{}_{aF\cdots} + \cdots .$$

For case II, where ϕ^{aA}, $\phi^{aA}{}_{|d}$ are independent, we take

$$(5)\quad \frac{\partial L}{\partial \phi^{aA\cdots}} = \Sigma_{aA\cdots}.$$

Using (4), (5), we have for (2)

$$(6)\quad \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\psi^{d}{}_{aA\cdots}\right)_{,d} - \left\{{m \atop ad}\right\}\psi^{d}{}_{mA\cdots} - \Gamma^{F}{}_{Ad}\,\psi^{d}{}_{aF\cdots} + \cdots - \Sigma_{aA\cdots} = 0.$$

Turning to case I,

$$(7)\quad \left(\frac{\partial\sqrt{-g}\,L}{\partial\phi^{aA\cdots}{}_{,d}}\right)_{;d} = \left(\sqrt{-g}\,\psi^{d}_{aA}\right)_{;d}\,.$$

Since we are now taking ϕ^{aA}, $\phi^{aA}{}_{;d}$ as independent,

$$(8)\quad \frac{\partial\sqrt{-g}\,L}{\partial\phi^{aA}{}_{\cdots}} = \sqrt{-g}\left[\Sigma_{aA\cdots} + \frac{\partial L}{\partial\phi^{cB\cdots}{}_{|d}}\,\frac{\partial\phi^{cB\cdots}{}_{|d}}{\partial\phi^{aA\cdots}}\right].$$

Now

$$\phi^{cB\cdots}{}_{|d} = \phi^{cB\cdots}{}_{;d} + \left\{{c \atop md}\right\}\phi^{mB\cdots} + \Gamma^{B}_{Fd}\,\phi^{cF\cdots} + \cdots$$

By II-(47), Γ^{A}_{Fd} is independent of all matter fields except the electromagnetic field, which is a pure tensor field. Then

$$\frac{\partial\phi^{cB\cdots}{}_{|d}}{\partial\phi^{aA\cdots}} = \left\{{c \atop ad}\right\}\delta^{B}_{A} + \Gamma^{B}_{ad}\,\delta^{c}_{a} + \cdots$$

and this gives

$$(9)\quad \frac{\partial\sqrt{-g}\,L}{\partial\phi^{aA\cdots}} = \sqrt{-g}\left[\Sigma_{aA\cdots} + \left\{{c \atop ad}\right\}\psi^{d}_{cA} + \Gamma^{B}_{Ad}\,\psi^{d}_{aB} + \cdots\right]$$

Combining (7), (9), we find for (1) the equations

$$(10)\quad \left(\sqrt{-g}\,\psi^{d}_{aA\cdots}\right)_{;d} - \left[\Sigma_{aA\cdots} - \left\{{m \atop ad}\right\}\psi^{d}_{mA\cdots} - \Gamma^{F}_{Ad}\,\psi^{d}_{aF\cdots} + \cdots\right]\sqrt{-g} = 0.$$

By direct comparison, (6) $\equiv$ (10). Turning next to case III, we suppose $\phi^{aA\cdots}$, $\phi^{aA\cdots}{}_{\|d}$ are independent. Then

$$\frac{\partial L}{\partial\phi^{aA\cdots}} = \Sigma_{aA\cdots} + \frac{\partial L}{\partial\phi^{cB\cdots}{}_{|d}}\,\frac{\partial\phi^{cB\cdots}{}_{|d}}{\partial\phi^{aA\cdots}}$$

But

$$\phi^{cB\cdots}{}_{|d} = \phi^{cB\cdots}{}_{\|d} + i\varepsilon\phi_d\phi^{cB\cdots},$$

and this gives

$$(11)\quad \frac{\partial L}{\partial\phi^{aA\cdots}} = \Sigma_{aA\cdots} + i\varepsilon\phi_d\psi^d_{aA\cdots}.$$

Now

$$\frac{\partial L}{\partial\phi^{aA\cdots}{}_{\|d}} = \psi^c_{aA\cdots}\frac{\partial\phi^{aA\cdots}{}_{|c}}{\partial\phi^{aA}{}_{\|d}} = \psi^d_{aA\cdots},$$

$$\psi^d_{aA\|d} = \psi^d_{aA|d} + i\varepsilon\phi_d\psi^d_{aA\cdots}.$$

Combining these results with (11), we have for (3)

$$\psi^d_{aA\cdots|d} - \Sigma_{aA\cdots} = 0,$$

which is identical with (2). Thus all three approaches lead to the same equations.

Cases II, III do not imply that one can treat e.g. $\phi^{aA\cdots}{}_{|d}$, ξ^{bB} as independent, where ϕ, ξ are different fields. Since the spinor covariant derivative depends in general upon the electromagnetic field, one cannot treat $\phi^{aA\cdots}{}_{|d}$ and the electromagnetic field ϕ_1 as independent in deriving the Maxwell equations. This is similar to the treatment of $\phi^{aA}{}_{|d}$ and g_{ij} in deriving the Einstein equations.

APPENDIX D

In III-(83), we claimed that the combined equations III-(79) satisfy the identities

$$(1)\quad \frac{1}{8\pi} Q^{kl}{}_{|l} = -\frac{\sqrt{2}}{2}(\mathcal{D}_1^k + \mathcal{D}_2^k) + F^k{}_{\cdot l} M^l .$$

Here we give a direct proof of (1). A similar proof was first given by Infeld and Van der waerden (17). Now

$$(2)\quad \frac{1}{8\pi} Q^{kl}{}_{|l} = \left(-F^{km}F^l{}_{\cdot m} + \frac{1}{4} g^{kl} F^{ab} F_{ab}\right)_{|l} + \frac{1}{16\pi} M^{kl}{}_{|l} ,$$

where

$$M^{kl} = \frac{1}{2}(\tilde{M}^{kl} + \tilde{M}^{lk}),$$

$$(3)\quad \tilde{M}^{kl} = -16\pi\sqrt{2}\, i\left(\sigma^k{}_{A'B}\chi^B \chi^{A'}{}_{|\cdot}{}^l - \sigma^{kA'B}\xi_B \xi_{A'|\cdot}{}^l\right) + \text{c.c.} .$$

Let us first consider $\tilde{M}^{lk}{}_{|l}$. In the first term, we have

$$(4)\quad i g^{km} \sigma^l{}_{A'B}(\chi^{A'}{}_{|m}\chi^B)_{|l} = i g^{mk} \sigma^l{}_{A'B}(\chi^{A'}{}_{|ml}\chi^B + \chi^{A'}{}_{|m}\chi^B{}_{|l})$$

$$= i g^{mk} \sigma^l{}_{A'B}(\chi^{A'}{}_{|m}\chi^B{}_{|l} + \chi^B \chi^{A'}{}_{|lm} + \chi^B \chi^{C'} P^{A'}{}_{\cdot C'lm}),$$

where we have used II-(62). To express our result in a special form, we add the following factors to the four terms of $\tilde{M}^{lk}{}_{|l}$ resulting from differentiation of (3):

First term: $-\frac{im}{\sqrt{2}} g^{jk}(\chi^{A'}{}_{|j}\xi_{A'} + \chi^{B}\xi_{B|j})$;

Second term: $\frac{im}{\sqrt{2}} g^{jk}(\chi^{A}{}_{|j}\xi_{A} + \chi^{B'}\xi_{B'|j})$;

with similar expressions for the conjugate terms. Note these terms cancel one another and therefore affect only the form of the final result. Adding the first factor above to (4), we have

$$i g^{mk}\chi^{A'}{}_{|m}(\sigma^{\ell}{}_{A'B}\chi^{B}{}_{|\ell} - \frac{m}{\sqrt{2}}\xi_{A'}) + i g^{mk}\chi^{B}(\sigma^{\ell}{}_{A'B}\chi^{A'}{}_{|\ell} + -\frac{m}{\sqrt{2}}\xi_{B})_{|m} + i g^{mk}\chi^{B}\chi^{C'}\sigma^{\ell}{}_{A'B} P^{A'}{}_{C'\ell m} .$$

Using II-(60), this becomes

$$i g^{mk}\chi^{A'}{}_{|m}(\sigma^{\ell}{}_{A'B}\chi^{B}{}_{|\ell} - \frac{m}{\sqrt{2}}\xi_{A'}) + i g^{mk}\chi^{B}(\sigma^{\ell}{}_{A'B}\chi^{A'}{}_{|\ell} - \frac{m}{\sqrt{2}}\xi_{B})_{|m} + + i g^{mk}\chi^{B}\chi^{C'}\sigma^{\ell}{}_{A'B}(\tfrac{1}{2}R_{tr\ell m}\sigma^{tA'D}\sigma^{r}{}_{C'D} + i\varepsilon F_{\ell m}\delta^{A'}_{C'}),$$

or

$$(5) \quad i g^{mk}\chi^{A'}{}_{|m}D^{1}_{A'} + i g^{mk}\chi^{B}D^{1}_{B|m} - \varepsilon F_{\ell m} g^{mk}(\sigma^{\ell}{}_{A'B}\chi^{A'}\chi^{B} + \tfrac{i}{2} g^{mk}\chi^{B}\chi^{C'}R_{tr\ell m}\sigma^{tA'D}\sigma^{r}{}_{C'D}\sigma^{\ell}{}_{A'B}.$$

For the Riemann term, by II-(29)

$$(6)\quad \frac{i}{2} g^{mk} \chi^{B} \chi^{C'} R_{trlm} \sigma^{r}{}_{C'D} \sigma^{tA'D} \sigma^{l}{}_{A'B}$$

$$= \frac{i}{4} g^{mk} \chi^{B} \chi^{C'} R_{trlm} \big(g^{tl} \sigma^{r}{}_{C'B} - g^{rl} \sigma^{t}{}_{C'B} + g^{rt} \sigma^{l}{}_{C'B}$$

$$- \frac{i}{2} \eta^{ltrp} \sigma_{pC'B} \big).$$

Now

$$\eta^{rtlp} R_{trlm} = \eta^{rtlp} R_{mlrt} = \frac{1}{3} \big(R_{mlrt} \eta^{rtlp}$$

$$+ R_{mtrl} \eta^{lrtp} + R_{mrtl} \eta^{tlrp} \big)$$

$$= \frac{1}{3} \eta^{rtlp} \big(R_{mlrt} + R_{mtlr} + R_{mrtl} \big) = 0.$$

Then (6) becomes

$$(7)\quad \frac{i}{4} g^{mk} \chi^{B} \chi^{C'} \sigma^{r}{}_{C'B} \big(g^{tl} R_{trlm} - g^{tl} R_{rtlm} + g^{tl} R_{tlrm} \big),$$

which reduces to

$$(8)\quad -\frac{i}{2} g^{mk} \chi^{B} \chi^{C'} \sigma^{r}{}_{C'B} R_{rm}.$$

Then for (5) we have

$$(9)\quad i g^{mk} \chi^{A'}{}_{|m} D^{1}_{A'} + i g^{mk} \chi^{B} D^{1}_{B|m} - \varepsilon g^{mk} F_{lm} \sigma^{l}{}_{A'B} \chi^{A'} \chi^{B} +$$

$$-\frac{i}{2} g^{mk}_{j} \chi^B \chi^{C'} \sigma^r{}_{C'B} R_{rm} .$$

Up to a multiplicative factor, this is the first term of $\tilde{M}^{lk}{}_{|l}$. Altogether we obtain by similar calculations

$$(10)\quad -\frac{1}{16\pi\sqrt{2}} \tilde{M}^{lk}{}_{|l} = i g^{mk} \chi^{A'}{}_{|m} D^1_{A'} + i g^{mk} \chi^B (D^1_B)_{|m} +$$

$$+ i g^{mk} \xi_{A|m} D^{2A} + i g^{mk} \xi_{B'} (D^{2B'})_{|m} - g^{mk} F_{lm} j^l + \text{c.c.}$$

(10) may be written symbolically as

$$(11)\quad -\frac{1}{16\pi\sqrt{2}} \tilde{M}^{lk}{}_{|l} = \mathcal{D}^k_1 + 2 F^k{}_{\cdot m} j^m ,$$

where $\mathcal{D}^k_1$ is composed of Dirac equations and their derivatives.

Before calculating $\tilde{M}^{kl}{}_{|l}$, we develop the second-order Dirac equations. From III-(79) we obtain

$$(12)\quad g^{kj} \chi^A{}_{|kj} = -m^2 \chi^A - \sigma^{kB'A} \sigma^j{}_{B'C} \chi^D P^C{}_{\cdot Dkj} .$$

Using II-(60), this gives

$$(13)\quad g^{kj} \chi^A{}_{|kj} = -m^2 \chi^A - \frac{1}{2} \chi^D R_{tlkj} \sigma^{kB'A} \sigma^j{}_{B'C} \sigma^{tF'C} \sigma^l{}_{F'D} +$$

$$+ i\varepsilon F_{kj} \chi^D \sigma^{kB'A} \sigma^j{}_{B'D} .$$

By II-(30),

$$(14)\quad R_{t\ell kj}\sigma^{kB'A}\sigma^{j}{}_{B'C}\sigma^{tF'C}\sigma^{\ell}{}_{F'D} = R_{t\ell kj}\Big[\tfrac{1}{4}\gamma_{D}{}^{A}(g^{jt}g^{k\ell} - g^{kt}g^{j\ell} + g^{kj}g^{t\ell}) - \tfrac{i}{4}\sigma_{m}{}^{H'A}\sigma_{nH'D}(\eta^{\ell kmn}g^{jt} - g^{kt}\eta^{\ell jmn} + g^{kj}\eta^{\ell tmn}) + \tfrac{i}{2}\eta^{kjtp}\sigma_{pF'}{}^{A}\sigma^{\ell F'}{}_{D}\Big].$$

A calculation similar to that preceeding (7) shows $\eta^{kjtp}R_{tlkj} = 0$. The other imaginary terms also vanish, and (14) implies

$$(15)\quad R_{t\ell kj}\sigma^{kB'A}\sigma^{j}{}_{B'C}\sigma^{\ell}{}_{F'D} = \tfrac{1}{4}\delta^{A}_{D}R_{t\ell kj}(g^{tj}g^{k\ell} - g^{kt}g^{j\ell}) = \tfrac{1}{2}R\delta^{A}_{D}.$$

Substituting (15) into (13), we have

$$(16)\quad g^{kj}\chi^{A}{}_{|kj} = -m^{2}\chi^{A} - \tfrac{1}{4}R\chi^{A} + i\varepsilon F_{kj}\sigma^{kB'A}\sigma^{j}{}_{B'D}\chi^{D}.$$

This is the second-order Dirac χ equation. A similar calculation gives for the second-order Dirac ξ equation

$$(17)\quad g^{\ell m}\xi_{C'|\ell m} = -m^{2}\xi_{C'} - \tfrac{1}{4}R\xi_{C'} - i\varepsilon F_{\ell m}\sigma^{\ell}{}_{C'B}\sigma^{mA'B}\xi_{A'}.$$

We now proceed to calculate $\tilde{M}^{kl}{}_{|1}$. Using (3), the χ terms of $\tilde{M}^{kl}{}_{|1}$ up to a constant factor are

$$(18)\quad ig^{m\ell}\sigma^{k}{}_{B'A}\chi^{A}{}_{|\ell}\chi^{B'}{}_{|m} + ig^{m\ell}\sigma^{k}{}_{B'A}\chi^{A}\chi^{B'}{}_{|m\ell} + \text{c.c.}\,.$$

Now the first term of (18) and its conjugate give

$$i g^{mj} \sigma^{k}{}_{B'A} (\chi^{A}{}_{|j} \chi^{B'}{}_{|m} - \chi^{A}{}_{|j} \chi^{B'}{}_{|m}) = 0.$$

The second term of (18) and its conjugate may be written

$$(19) \quad i\sigma^{k}{}_{B'A}\chi^{A} \big(g^{ml}\chi^{B'}{}_{|ml} + m^{2}\chi^{B'} + \tfrac{1}{4} R\chi^{B'} + i\varepsilon F_{ml}\sigma^{mB'C}\sigma^{l}{}_{D'C}\chi^{D'}\big) + \varepsilon F_{ml}\chi^{A}\chi^{D'}\sigma^{k}{}_{B'A}\sigma^{l}{}_{D'C}\sigma^{mB'C} + c.c.$$

By (16), the first term of (19) and its conjugate are Dirac equations. For the second term of (19), by II-(29)

$$(20) \quad \varepsilon F_{ml}\chi^{A}\chi^{D'}\sigma^{k}{}_{B'A}\sigma^{mB'C}\sigma^{l}{}_{D'C} = \tfrac{1}{2}\varepsilon F_{ml}\chi^{A}\chi^{D'}\big(g^{ml}\sigma^{k}{}_{D'A} - g^{kl}\sigma^{m}{}_{D'A} + g^{km}\sigma^{l}{}_{D'A} - i\eta^{kmlp}\sigma_{pD'A}\big).$$

The imaginary term of (20) will cancel with the conjugate term of (19). Simplifying (20), we arrive at

$$(21) \quad \varepsilon F_{ml} g^{km}\sigma^{l}{}_{D'A}\chi^{A}\chi^{D'}.$$

Altogether, we find for (18) the result

$$(22) \quad \varepsilon F^{k}{}_{l}\sigma^{l}{}_{D'A}\chi^{D'}\chi^{A} + i\sigma^{k}{}_{B'A}\chi^{A}\big(g^{ml}\chi^{B'}{}_{|ml} + m^{2}\chi^{B'} + \tfrac{1}{4}R\chi^{B'} + i\varepsilon F_{ml}\sigma^{mB'C}\sigma^{l}{}_{D'C}\chi^{D'}\big) + c.c.$$

A similar calculation with the ξ terms of $\tilde{M}^{kl}{}_{|1}$ leads finally to the result

$$(23)\quad \frac{-1}{16\pi\sqrt{2}}\tilde{M}^{kl}{}_{|l} = i\sigma^{k}{}_{B'A}\chi^{A}\left(g^{ml}\chi^{B'}{}_{|ml} + m^2\chi^{B'} + \tfrac{1}{4}R\chi^{B'}\right.$$

$$\left. + i\varepsilon F_{ml}\sigma^{mB'C}\sigma^{l}{}_{D'C}\chi^{D'}\right) - i\sigma^{kB'A}\xi_{A}\left(g^{ml}\xi_{B'|ml} + m^2\xi_{B'}\right.$$

$$\left. + \tfrac{1}{4}R\xi_{B'} + i\varepsilon F_{ml}\sigma^{m}{}_{B'C}\sigma^{lD'C}\xi_{D'}\right) + F^{k}{}_{\cdot m}j^{m} + \text{c.c.},$$

or symbolically we may write

$$(24)\quad \frac{-1}{16\pi\sqrt{2}}\tilde{M}^{kl}{}_{|l} = \mathcal{D}_2^{k} + 2F^{k}{}_{\cdot m}j^{m},$$

where $\mathcal{D}_2^k$ is composed of Dirac equations. Finally, (24), (11) give the result

$$(25)\quad M^{kl}{}_{|l} = -8\pi\sqrt{2}\,(\mathcal{D}_1^{k} + \mathcal{D}_2^{k}) - 32\pi\sqrt{2}\,.$$

For the electromagnetic contribution, we find

$$(26)\quad E^{kj}{}_{|j} = 16\pi F^{km}F_{m\cdot}{}^{j}{}_{|j}\,.$$

Altogether, we have for the Einstein equations the divergence (1).

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