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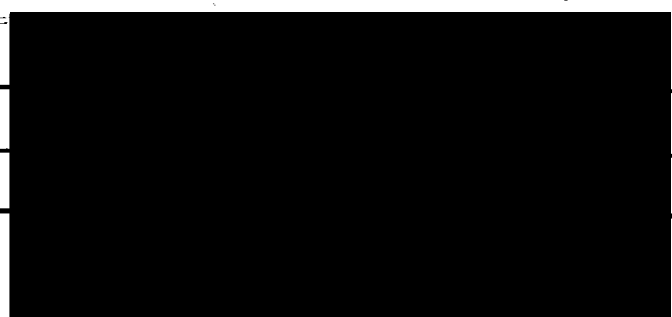
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## ERRATUM


Due to a typing error the text begins on page 3.

ON THE STATIONARY GRAVITATIONAL FIELDS

by

Lewis Garrett Zenk

B. Sc., Simon Fraser University, 1973

A THESIS SUBMITTTED IN PARTIAL FULFILLMENT OF

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ABSTRACT

In this paper the use of "invariant" methods in General
Relativity is emphasized. The theory of connections on principal
fiber bundles serves as a vehicle for the introduction of the
Cartan approach to affine connections, which is naturally adapted
to the invariant calculus. The theory also provides a unified
framework from which to view many of the "formalisms" which can be
introduced in General Relativity. Chief among these is the Spinor
Calculus; an account is given of how it arises from the bundle
viewpoint. In stationary spaces a natural "spinor structure" can
be defined in an obvious manner on the associated $V_3$ of the space.

   Corresponding to the conditions for a symmetric connection
to be semi-Riemannian, new necessary conditions for a metric
connection to be symmetric (and hence semi-Riemannian) are obtained.
Some of these conditions are purely algebraic. An account is given
of the "geometrical optics" of congruences in a positive definite
$V_3$, similar to the well-known version for null congruence in a
$V_4$.

   The field equations in invariant form are derived by the
method of differential forms (the Cartan method). The elegance of
this method compared to the once exclusive tensor calculus is
pointed out. Proceeding either from spinorial considerations, or
from the geometrical optics, a complex version of the field equa-
tions is formulated, which is well adapted to finding solutions.

   As an example, the class of stationary vacua with
eigenvalues of the Ricci subtensor equal is dealt with. This class

includes the stationary subcase of plane-fronted gravitational waves (with parallel rays); the remainder of the problem is reduced to a pair of second-order partial differential equations, one of which is independent. This latter subclass belongs to the Petrov type III and has rays with nonvanishing divergence and twist (complex dilatation) in the general case.

## ACKNOWLEDGEMENT

Errata. Underlined portions to be added.


The Thesis starts on page 3 due to an error in pagination.

Ch. 1, p. 12, line 6: The tensor field must also be nonsingular.

Ch. 1, p. 23, 7th line from bottom: are not in general geodesics.

Ch. 2, p. 43, l. 8: null bivectors

Ch. 4, p. 61, l. 11: (Bade & Jehle [33] ) in invariant form.

Ch. 5, p. 73, eqn. (5.5): The terms shear-free and complex-

dilatation-free for $\widetilde{\beta} = 0$, $\widetilde{\gamma} = 0$ should really only be used in

case the congruences of $k^{\alpha}$ are geodesics.

Ch. 6, p. 84, l. 3: closed form solutions with an arbitrary

functional dependence.

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Introduction

In writing this thesis an implicit commitment has been made for bringing the concepts of modern differential geometry to bear in a more intimate way upon the theory of gravitation. We include among the rudiments of these "concepts" the Cartan approach [1] to affine connections and the natural global setting for that approach, connections in principal fiber bundles. The latter was first defined by Ehresman [2].

The Cartan approach, also called the "moving frames method" or the "method of differential forms," while not recent, has at least not been traditionally used by relativists and other specialists in tensor analysis. Among its users we may mention Lichnerowicz [3], Israel [4], and Jordan, Ehlers, and Kundt [5]. We will demonstrate the utility of this method when used in conjunction with orthonormal frames. It is also the key to our obtaining some interesting conditions for a metric connection to be symmetric.

The theory of fiber bundles has recently received attention in connection with the theory of relativity in some expository articles by Trautman [6] and Lichnerowicz [7]. It has been used to clarify the relationship of the spinor calculus to the ordinary tensor analysis or Ricci-calculus (Geroch [8], [9]; Ehlers [10]; Lichnerowicz [7]). We will touch upon this aspect of

its applications in Chapter 4 when discussing the spinor structure. The principal fiber bundles are the natural arena for the discussion of connections. We develop most of the necessary tools of Riemannian geometry from these foundations.

On the physical side (or perhaps just another mathematical side), we will be concerned with stationary vacuum space-times. For definiteness we set ourselves the problem of eigenvalues of the Ricci subtensor equal. The result proves to be interesting in relation to algebraically special space-times. In the solution it is expedient to use a set of equations derived either from spinorial considerations, or, equivalently, from a geometrical standpoint of "optical" properties of a congruence of curves in $V_3$.

The use of spinor calculus in general relativity was initiated by Witten [11] and Penrose [12], and culminated in the Newman-Penrose [13] formalism, which is particularly suited to the study of algebraically special space-times. A contracted version of the spinor calculus, adapted to stationary spaces, was used by Perjes [14] and by Kóta and Perjés [15]. A similar formulation in the static case was attained by Das [16] from geometric considerations. The equations in this paper are the same as those in Perjes, except for being restricted to the vacuum case; but we have brought out the structure of those equations in a clearer way. Our concrete example turns out to be a subcase of one considered (but not solved) by Perjes [14].

1. Fiber bundles and the Cartan approach to connections.

We begin with the definition of a differentiable manifold (Hicks [17]), in order to get into the spirit of things, and help establish some notation. Let $M$ be a set. An n-coordinate pair $(\phi, U)$ on $M$ is a subset $U$ of $M$ together with a one-to-one map $\phi: U \to R^n$ such that $\phi(U)$ is open. A $C^\infty$ n-subatlas on $M$ is a collection of n-coordinate pairs $\{(\phi_h, U_h)\}$ which are differentiably related in the sense that $\phi_h \circ \phi_k^{-1}$ is $C^\infty$ wherever it is defined, for all $h$ and $k$, and where the $U_n$ cover $M$. A maximal $C^\infty$ n-subatlas is called a $C^\infty$ n-atlas. A differentiable manifold $M$ is a set together with a $C^\infty$ n-atlas; $M$ is said to be provided with a differentiable structure. Since all the manifolds we deal with will be $C^\infty$, we often drop that adjective. "Differentiable" will mean the $C^\infty$ version. We work with Hausdorff and paracompact manifolds exclusively.

We assume the standard concepts in differential geometry pertaining to the differentiable structure of a manifold. For the relevant definitions and much of the notation we refer to Hicks [17]. We recall a few of these for convenience:

If $M$ is a $C^\infty$ manifold and $m \in M$, then the tangent space at $m$ is denoted by $M_m$. If $X$ is a vectorfield on $M$, we use either $X_m$ or $X(m)$ to denote the value of the vectorfield at $m \in M$, whichever is more convenient. If $N$ is another $C^\infty$ manifold and $f: M \to N$ is $C^\infty$, the Jacobian or differential $f_*$

of  f  is a linear map  $f_*: M_m \to N_{f(m)}$  for each  $m \in M$.  If the cotangent space is denoted by  $M_m^*$ , we denote by  $f^*$  the mapping which takes forms in the opposite direction,  $f^*: N_{f(m)}^* \to M_m^*$ .  For this chapter, a <u>curve</u> in  M  is a  $C^\infty$-mapping from a connected subset of  R  into  M;  but later we will also mean simply the range of such a mapping.  If  X  and  Y  are two  $C^\infty$  vector fields, their <u>bracket</u>  [X,Y]  is the vector field defined by  [X,Y]f = X(Yf) - Y(Xf),  where  f  is any  $C^\infty$  real-valued function.  .We have the Jacobi identity:  [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.

A <u>Lie group</u> G  is a group which is a differentiable manifold such that the group operations are differentiable  $(C^\infty)$; it is said to be a <u>Lie transformation group</u> <u>on</u> <u>a</u> <u>differentiable</u> <u>manifold</u>  M  provided:

(a)  To each  $a \in G$  there corresponds a differentiable transformation of  M,  denoted by  $m \mapsto ma, m \in M$.

(b)  If  $a,b \in G$,  then  (ma)b = m(ab)  for every  $m \in M$.

(c)  The mapping  $(m,a) \in M \times G \mapsto ma \in M$  is differentiable.

The group is said to act <u>freely</u> on  M  if  ma = m  for some  $m \in M \Rightarrow$ m = e,  where  e  is the identity.  The mapping in (a) may be characterized as a right translation by  a,  and we often use the symbol  $R_a: R_a(m) = ma$.  A vector field  X  on  G  is <u>right-invariant</u> if  $(R_a)_* X_b = X_{ba}$ ,  for all  $a,b \in G$

Some final remarks on notation.  A comma will mean the

partial or invariant derivative with respect to a variable or indexed vector: $B^{\cdots}{}_{,i} = (\partial/\partial x^i)(B^{\cdots})$, $B^{\cdots}{}_{,A} = \lambda_A(B^{\cdots})$, where $B^{\cdots}$ are the components of any tensor. The summation convention is assumed to hold for repeated indices, unless indicated otherwise. Square brackets surrounding a set of indices means, as usual, full antisymmetrization: $B_{[i_1 i_2 \cdots i_n]} = (1/n!)\left( \varepsilon^{i_{k_1} i_{k_2} \cdots i_{k_n}} B_{i_{k_1} i_{k_2} \cdots i_{k_n}} \right)$ where $\varepsilon^{i_{k_1} i_{k_2} \cdots i_{k_n}}$ is the Levi-Civita permutation symbol.

The basis of our work will be the Cartan approach to affine connections. (Cartan [1]; also Israel [4], Flanders [18].) But the original "moving frames" method of Cartan, though admirably suited to computations of a certain kind (as we shall see!), relies on the existence of a differentiable base field for the region in which its structural equations are to be formulated. A more abstract setting is needed for the global formulation, which can be properly related to the usual approach to affine connections (such as that in Hicks). This setting is furnished by the theory of fiber bundles; so we begin our study with a sketch of the relevant parts of that theory. The main references for the material on this subject will be Bishop and Crittenden [19], Nomizu [20], and Hicks [17]. For a concrete (down-to-earth) treatment of some of these topics we refer the reader to Flanders [18], especially Chapter 8.

Let $M$ be a differentiable manifold and $G$ a Lie group. A differentiable manifold $P$ is a _principal_ _fiber_ _bundle_ over the _base_ _space_ $M$ with _structure_ _group_ $G$, denoted $P(M,G)$, if:

(1.1)    (a)  G acts on P on the right differentiably and freely.

(b)  M is the quotient space of P by the equivalence
     relation induced by G, and the projection $\pi:P \to M$
     is $C^\infty$.

(c)  P is locally trivial: that is, every $m \in M$ has a
     neighbourhood U such that $\pi^{-1}(U)$ is diffeomorphic
     to $U \times G$, by a mapping $p \in \pi^{-1}(U) \longmapsto (\pi(p),\phi(p)) \in$
     $U \times G$ with $\phi(pa) = \phi(p)a$ for every $a \in G$.

According to condition (a), no element other than $e \in G$ has a
fixed point in P. Condition (b) means that $M = P/\{pG:p \in P\}$ with
the quotient topology, which is the strongest topology making $\pi$
continuous. Palais [21] shows that the quotient topology is
uniquely characterized by the conditions that with respect to it $\pi$
is both continuous and open. The quotient differentiable structure
is fixed by $\pi$ being $C^\infty$. Condition (c) may be paraphrased by
saying that P is locally a product bundle, with the right action
of G naturally defined. For each $m \in M$, $\pi^{-1}(m)$ is a closed
submanifold of P, called the _fiber_ over m, and diffeomorphic to
G.

To put principal fiber bundles into something of a
"categorical" setting we should define the morphisms between them.
Let P(M,G) and $P^\nabla(M^\nabla,G^\nabla)$ be two principal bundles. A _bundle
map_ $f:P(M,G) \to P^\nabla(M^\nabla,G^\nabla)$ is a set of $C^\infty$ maps $(f_P,f_M,f_G)$ between

the obvious pairs, such that $f_G$ is a homomorphism, and we have

(1.2)    (i)  $f_M \circ \pi = \pi' \circ f_P$

(ii)  $f_P \circ R_a = R_{f_G(a)} \circ f_P$  for every  $a \in G$.

For a more complete picture we may mention some more general concepts. A <u>bundle</u> over a manifold M is just a $C^\infty$ manifold P and a $C^\infty$ onto map $\pi: P \to M$. A <u>fiber bundle</u> is one which is locally a product bundle in the sense of (c) above, but without the group aspect. As an example, one may define a fiber bundle <u>associated</u> to a principal bundle $P(M,G)$. This is essentially a fiber bundle over M whose fiber is a differentiable manifold on which G acts to the left. However, the principal bundles form the proper setting for the introduction of a connection.

We now give an important example, which will help us to fix notation [17]. Let M be a $C^\infty$ n-manifold and let B(M) = $\{(m; \lambda_{(1)}, \ldots, \lambda_{(n)}) : m \in M$ and $\lambda_{(1)}, \ldots, \lambda_{(n)}$ an ordered basis of $M_m\}$. The natural projection $\pi$ acts by $\pi(m; \lambda_{(1)}, \ldots, \lambda_{(n)}) = m$. We give B(M) a differentiable structure as follows. If $(\phi, U)$ is a coordinate pair on M with $x_i = u_i \circ \phi$ ($u_i$ is the $i^{th}$ slot function), let $(\bar{\phi}, \bar{U})$ be a coordinate pair on B(M) with $\bar{U} = \pi^{-1}(U)$ and $\bar{\phi}: \bar{U} \to R^{n+n^2}$ defined by $\bar{\phi}(m; \lambda_{(1)}, \ldots, \lambda_{(n)}) = (x_1, \ldots, x_n, \lambda_{(1)}^1, \lambda_{(1)}^2, \ldots, \lambda_{(n)}^n)$,

where $x_i = u_i \circ \phi(m)$ and $\lambda_{(k)} = \lambda_{(k)}^i \partial/\partial.x_i$. The $\overline{U}$'s obviously

cover $B(M)$, and on any overlap the coordinates are $C^\infty$-related.

Thus the set of all $(\overline{\phi},\overline{U})$ generates a $C^\infty(n + n^2)$-atlas. The

group $Gl(n,R)$ acts on $B(M)$ on the right: if $A_{k\ell} \in Gl(n,R)$,

then $(m;\lambda_{(1)}, \ldots, \lambda_{(n)}) \mapsto (m;\lambda_{(k)}A_{k1}, \ldots, \lambda_{(k)}A_{kn})$. It is

easy to show that $B(M)$ with this structure is a principal bundle;

we call it the bundle of bases over $M$.

In a similar way one may define the bundle of orthonormal

frames in any Riemannian space. The group in this case is $0(n,R)$

where $n$ = dimension of the space. Other examples of principal

fiber bundles figure heavily in the theory of relativity; we

believe that the theory of fiber bundles allows the treatment of

many different formalisms from a unified point of view. The most

important among these is the bundle of oriented orthonormal tetrads

with the proper, isochronous Lorentz group $L^+_+$, which we denote

by $OT(M)$, where $M$ = space-time. The bundle of oriented null

tetrads is isomorphic to $OT(M)$. This bundle is closely associated

with the spinor structure, a principal bundle dealt with in Chapter

4. If our space-time is spatially and temporally oriented and

parallelizable, and $(K,L)$ is a fixed pair of future oriented null

direction fields, then the collection of null tetrads $\zeta(M) =$

$\{(m;k,\ell,t):m \in M, k \in K, \ell \in L, t$ is complex, $k \cdot \ell = - t \cdot \overline{t} =$

$- 1, k \cdot t = \ell \cdot t = 0\}$ becomes a principal fiber bundle with the

structure group $C$ (multiplicative group of complex numbers), if

we define a right action of $C$ by $(m;k,\ell,t)z =$ $(m;|z|^2 k,|z|^{-2}\ell,z/\overline{z}t)$, where $z \in C$. This example was pointed out by Ehlers [10]. One can regard a stationary space-time itself as a principal bundle $M(W_3,R^1)$ with the world-lines of the stationary observers as the equivalence classes $mR^1$, $m \in M$. The condition that the killing motion have no fixed points accords with the requirement in the definition that the group act freely. The fiber bundle concept can also be applied to more general space-times; see Lichnerowicz [7].

Many of the examples above can be obtained from the bundle of bases via a general process of reduction of the structural group, which is a special case of a bundle map. Let $P(M,G)$ be a principal bundle and $H$ a subgroup of $G$. Then $G$ is reducible to $H$ iff there exists a principal bundle $P^\nabla(M,H)$ and a $C^\infty$, one-to-one mapping $f:P^\nabla \to P$ such that:

(1.3)    (i)  $\pi^\nabla = \pi \circ f$

         (ii)  $f \circ R_h = R_{j(h)} \circ f$ where $h \in H$ and $j$ is the
                            inclusion map, $j(h) \in G$.

It should be remarked that this concept has no a priori relationship to any given connection on $M$. The reductions considered in the above examples were very special.

In regard to the existence of such reductions, we may cite

the following theorem, due to Steenrod (Bishop and Crittenden [19],
p. 50): If H is a maximal compact subgroup of G, then G can
be reduced to H. Now O(n,R) is a maximal compact subgroup of
Gl(n,R). In the case of B(M), any particular reduction of
Gl(n,R) to O(n,R) is equivalent to a Riemannian structure on
M (that is, a $C^\infty$ symmetric 2-covariant tensor field on M).
For an explicit construction of such a structure, see Hawking and
Ellis [22], p. 38. In the case of most interest to us, however,
the proper isochronous Lorentz group $L^+_+$ is not compact. Thus
it turns out that the ability to reduce B(M) to OT(M) where M
is a space-time leads to a certain restriction on the manifold M.
This restriction is that M admit a global line-element field [22].
This is not much of a restriction, however, since any non-compact
manifold satisfies it. These considerations are of more consequence
in the case of a spinor structure (Chapter 4).

We are now prepared to define a connection in a princi-
pal fiber bundle, and demonstrate how this definition gives us the
usual parallel translation. Let P(M,G) be a principal bundle
over the n-dimensional base manifold M with structure group G.
At each point p of P, let $V_p$ be the subspace of $P_p$ tangent
to the fiber through p. Then $\pi_*(V_p) = 0$.

A connection H on P is an n-dimensional distribution
on P; i.e., an assignment of a subspace $H_p$ of $P_p$ (called
horizontal) to each p $\in$ P, satisfying

(1.4)     (a)   $P_p = V_p \oplus H_p$   (direct sum).

          (b)   If  $a \in G$  and  $p \in P$  then  $H_{pa} = (R_a)_* H_p$ .

          (c)   $H_p$  depends differentiably on  p.


Note that the distribution is not necessarily involutive; if it is,

the connection turns out to be flat--i.e., have vanishing curvature

tensor (to be defined). According to properties (a) and (c), if  X

is a  $C^\infty$  vector field on  P,  at each  $p \in P$  we have a unique

decomposition of the form  $X_p = Y_p + Z_p$ ,  where  $Y_p \in V_p$ ,

$Z_p \in H_p$ ,  and the vector fields Y  and  Z  are  $C^\infty$.  The vector

field  Y  is called the <u>vertical component</u>, and  Z  the <u>horizontal</u>

<u>component</u> of  X.  For an intuitive picture of this definition, we

specialize to the frame bundle  B(M)  (or  B  for short).  Each

vector in  $B_p$  corresponds in  M,  roughly speaking, to  (1) a

framed point (information provided by  p),  (2) a direction, and

(3) associated "rates of turn" of the frame.  If the vector lies in

$H_p$ ,  these "rates of turn" are those associated with a parallel

translation in the direction (2).  Property (b) then says that this

infinitesimal parallel translation commutes with the action of the

group on the frame.

          For a more exact, but still simple, elucidation of the

meaning of this definition we indicate how it is equivalent to the

specification of a parallel translation in $M$; as is well known, this is equivalent in the frame bundle to the determination of a classical or infinitesimal affine connection on $M$ ([23] or [19], p. 77). Let $\gamma$ be a broken $C^\infty$ curve in $M$, $\gamma : [0,1] \to M$. A **horizontal** **lift** of $\gamma$ is a broken $C^\infty$ curve $\tilde{\gamma}$ in $P$ such that (i) $\tilde{\gamma}$ is horizontal, that is $\tilde{\gamma}_*$ is horizontal, and (ii) $\pi \circ \tilde{\gamma} = \gamma$. It can be shown that, given $p \in \pi^{-1}(\gamma(0))$, there exists a unique lift $\tilde{\gamma}$ of $\gamma$ such that $\tilde{\gamma}(0) = p$ ([19], p. 77). Then $T_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ is a diffeomorphism satisfying $T_\gamma \circ R_a = R_a \circ T_\gamma$, and $T_{\gamma\sigma} = T_\sigma \circ T_\gamma$ for any other broken $C^\infty$ curve $\sigma$ in $P$ with $\sigma : [0,1] \to P$ and $\sigma(0) = \gamma(1)$.

The proper setting for the definition of an **affine** connection (in which torsion has meaning) is the bundle of bases. So we specialize once more to $B(M)$, and draw on Hicks. Let $p \in B(M)$. There is a mapping $f_p : Gl(n,R) \to B(M)$ defined by $f_p(a) = pa$. Let $\{\lambda_{(i)}\}$ be the basis at $M_{\pi(p)}$ corresponding to $p$. Any other basis at $M_{\pi(p)}$, corresponding to $p^\nabla$ in $B(M)$ $(\pi(p^\nabla) = \pi(p))$, say, may be given by its components $A_{ki} \in Gl(n,R)$ with respect to $\lambda_{(i)}$: $\lambda_{(i)}^\nabla = \lambda_{(k)} A_{ki}$. If we make the $A_{ki}$ double as coordinate functions then we may define unique right-invariant vector fields $X_{ki}$ which take the values $X_{ki}(e) = \partial/\partial A_{ki}(e)$ at the identity. Then $(f_p)_* X_{ki}(e) = \varepsilon_{ki}(p)$ define vector fields in $B(M)$ (the vector fields of "rate of twist" of the frame). These vector fields are **vertical**: $\pi_* \varepsilon_{ki} = 0$ (Appendix A).

They are globally defined, $C^\infty$, and intrinsic to the bundle of bases--i.e., a part of the differentiable and group structure. We do not yet have a basis for $B_p$ ; and we cannot find one in any well-determined fashion without adding more structure. We must settle for $n$ naturally defined one-forms: if $X \in B_p$ , $p = (m; \lambda_{(1)}, \ldots, \lambda_{(n)})$, define the $n$ one-forms $\bar{\omega}_i$ by $\pi_* X = \bar{\omega}_i(x) \lambda_{(i)}$ (these are also intrinsic to the bundle of bases). We can now add the necessary assumption: Let $\bar{\omega}_{ij}$ be any one-forms dual to $\varepsilon_{ij}$ , subject to a forthcoming restriction, then we assert that the dual base to $\{\bar{\omega}_i, \bar{\omega}_{ij}\}$, namely $\{\varepsilon_i, \varepsilon_{ij}\}$ provides us with the breakdown into vertical and horizontal vectors. Hence, if $X \in B_p$ , define the vertical part of $X$ by $X_V = \bar{\omega}_{ij}(X) \varepsilon_{ij}$ , from which we get $X_H = X - X_V$ for the horizontal component. This specification of $X_H$ is equivalent to giving the $H_p$ of the above definition. For an explicit construction of these basic vectors and forms see Appendix A. Formally, the <u>connection one-forms</u> $\bar{\omega}_{ij}$ satisfy the following defining properties:

(1.5)    (a)$^\nabla$  $\bar{\omega}_{ij}|_{V_p}$ form a dual base to $\varepsilon_{ij}$ at all $p \in B$.

(b)$^\nabla$  $\bar{\omega}_{ij}((R_a)_* X) = A_{ir}^{-1} \bar{\omega}_{rs}(X) A_{sj}$ ,   all $X \in B_p$ .

(c)$^\nabla$  $\bar{\omega}_{ij}$ are $C^\infty$.

By taking exterior derivatives of the one-forms $\{\bar{\omega}_i, \bar{\omega}_{ij}\}$

we obtain the Cartan structural equations

(1.6) $\qquad d\bar{\omega}_i = -\bar{\omega}_{ij} \wedge \bar{\omega}_j + \bar{\tau}_i$ ,

$\qquad\qquad d\bar{\omega}_{ij} = -\bar{\omega}_{ik} \wedge \bar{\omega}_{kj} + \bar{\Omega}_{ij}$ ,

which may be considered as a definition of the <u>torsion forms</u> $\bar{\tau}_i$
and <u>curvature forms</u> $\bar{\Omega}_{ij}$. More elegant definitions are,
$\bar{\Omega}_{ij}(X,Y) = (d\bar{\omega}_{ij})(X_H, Y_H)$ and $\bar{\tau}_i(X,Y) = (d\bar{\omega}_i)(X_H, Y_H)$; but they
are less suited to our purposes. If $\bar{\tau}_i = 0$ the connection is
said to be <u>symmetric</u>.

Under the process of reduction of the structural group,
a connection $\bar{\omega}_{ij}$ on $B(M)$ gives rise to a natural connection on
the reduced bundle $B^{\nabla}(M,H)$. If $f:B^{\nabla} \to B$ is the mapping which
gives the reduction, then the induced connection is just $f^*\bar{\omega}_{ij}$.
If a connection is given on $B^{\nabla}$ instead, we can define one on $B$
by a dual process. If $H^{\nabla}$ is the connection (first definition)
on $B^{\nabla}$, let $f_* H^{\nabla}_p = H_{f(p)}$ and extend $H$ to a distribution on
all of $B$ by right translation. If the structural group of the
reduced bundle is $0(n,R)$, then the induced connection $f^*\bar{\omega}_{ij}$ is
called a <u>metric connection</u>. We use the same term even if the
reduced group is $L^+_+$; thus any connection on $OT(M)$, where $M$
is a space-time, is a metric connection. In such a connection,
parallel translation always preserves inner products.

We are now ready to derive the more conventional, local

version of the structural equations by "descending" to M. Let us not think, however, that we are abandoning fiber bundles: What follows can equally well be thought of as a reduction of Gl(n,R) to the trivial {e}, which is possible on any sufficiently small open subset U of M by the local triviality of B(M); the reduction is defined by some $f: B^\nabla(U, \{e\}) \to B(U)$.

Now let $\lambda_{(1)}, \ldots, \lambda_{(n)}$ be a base field on the open set U of M. Define a $C^\infty$ map $f: U \to B(M)$ by $f(m) = (m; \lambda_{(1)}(m), \ldots, \lambda_{(n)}(m))$, where $m \in U$. Since $\pi \circ f$ is the identity on U we call f a section over U. Define the "connection one-forms" $\omega_{ij}$ on U by $\omega_{ij} = f^* \bar\omega_{ij}$. By allowing $f^*$ to operate on the structural equations (1.6), they are brought down to a corresponding set of equations on M. Defining $\omega_i = f^* \bar\omega_i$ and taking account of $f^* d\sigma = df^* \sigma$ for any form $\sigma$, we have

(1.7)     $d\omega_i = -\omega_{ij} \wedge \omega_j + \tau_i$

$d\omega_{ij} = -\omega_{ik} \wedge \omega_{kj} + \Omega_{ij}$

where the one-forms $\omega_i$ are the dual base to $\lambda(i)$. By descending to U, of course, we have introduced the additional complication of the consistency of these equations for different sections over U (that is, their covariance under changes of frame). But the way we have proceeded guarantees this covariance. In a sense, we have taken "moving frames" and the (still more dizzying) transformations

between them, with their somewhat kinematical flavour, and frozen
them all in the bundle of bases; much as the theory of relativity
turns kinematical relations of physics into "static" geometrical
relations of space-time.

Let us investigate how a change of section affects the
basic and connection one-forms. Let $f:U \to B(U)$ and $\hat{f}:U \to B(U)$
be two sections over $U$ as before. We wish to pull down the
forms $\overline{\omega}_i$ and $\overline{\omega}_{ij}$ to $M$ via the two mappings $f*$ and $\hat{f}*$. If
the points of $f(U)$ are of the form $p = (m; \lambda_{(1)}, \ldots, \lambda_{(n)})$
then we may write $\hat{f}(U) = \{(m: \lambda_k A_{k1}, \ldots, \lambda_k A_{kn}): m \in U, \text{ etc.}\}$,
where $A_{ki}$ is pointwise a member of $Gl(n,R)$, but over $U$ will
be a function of $m$. Thus $\hat{f} = R_A \circ f$, where we have abused our
notation by using $R_A$ for a nonconstant $A_{ki}$. Then we calculate
$\hat{\omega}_i = \hat{f}*\overline{\omega}_i$ and $\hat{\omega}_{ij} = \hat{f}*\overline{\omega}_{ij}$ in terms of $\omega_i = f*\overline{\omega}_i$ and $\omega_{ij} = f*\overline{\omega}_{ij}$. The results are (Appendix C)

(1.8) $\quad \hat{\omega}_i = A^{-1}{}_{ij}\omega_j$

$$\hat{\omega}_{ij} = A^{-1}{}_{ik}\omega_{k\ell}A_{\ell j} + A^{-1}{}_{i\ell}dA_{\ell j}.$$

In the calculation we have used the intermediary of a coordinate
section; with more advanced notation this would not have been
necessary.

From now on we restrict our attention to such a

parallelizable submanifold U of M, and we change our notation

somewhat. We use $\lambda^N$ or $\lambda^i$ instead of $\omega_i$ , and reserve the

use of lower case indices (Latin or Greek) for coordinate frames,

with but a minor exception in the case of bivector indices in the

Petrov classification. Thus let $\lambda^N$, N = 1, . . . ,n be a general

base for the one-forms on U. In local coordinates we write

$\lambda^N = \lambda^N{}_i dx^i$. The dual vector fields are written as $\lambda_N = \lambda^i{}_N \partial/\partial x^i$,

where $(\lambda^i{}_N)$ is the matrix inverse to $(\lambda^N{}_i)$. In general, from this

point on we will be careful about the ups and downs in the place-

ment of indices. The structural equations (1.7) now take the form [1]


(1.9) $\qquad d\lambda^N = \omega^N{}_{\cdot M} \wedge \lambda^M + \tau^N$

$\qquad\qquad \Omega^N{}_{\cdot M} = - d\omega^N{}_{\cdot M} + \omega^N{}_{\cdot P} \wedge \omega^P{}_{\cdot M} \quad .$


The generalized Ricci rotation coefficients are defined as the

coefficients in an expansion of $\omega^N{}_{\cdot M}$ :


$$\omega^N{}_{\cdot M} = \gamma^N{}_{\cdot MP} \lambda^P.$$


For a coordinate frame these are the (negative of the) christoffel

symbols. In that case, $\lambda^i = dx^i$ and the first equations of

structure (1.9) yield $\Gamma^i{}_{[kj]} dx^j \wedge dx^k = \tau^i$. Hence the name,

symmetric, for a connection in which $\tau^i = 0$.

---

[1] Although it may not be natural from a fundamental point of
view, we have here changed the definition of $\omega^N{}_{\cdot}M$ by a sign. This
accords with widespread usage in tensor calculus.

With this notation the components $\gamma^N_{MP}$ transform as follows under a change of frame (change of section) $\hat{\lambda}_\Sigma = A^N_\Sigma \lambda_N$ :

$$A^N_\Sigma \hat{\gamma}^\Sigma_{\Omega X} = \gamma^N_{MP} A^M_\Omega A^P_X - A^N_{\Omega,M} A^M_X .$$

In particular if $\{\hat{\lambda}_\Sigma\} = \{\partial/\partial X^i\}$ is a coordinate frame we obtain $\gamma^N_{MP} = \lambda^N_{i|j} \lambda^i_M \lambda^j_P$ , where we have written $\lambda^N_{i|j} \equiv \lambda^N_{i,j} - \lambda^N_k \Gamma^k_{ij}$ . We have thus recovered the ordinary definitions of the covariant differentiation (in essence) and Ricci rotation coefficients. One has in general $\gamma^N_{MP} = - \gamma^N_{M\ P}$ , where $\gamma^N_{M\ P} = \lambda^i_{M|j} \lambda^N_i \lambda^j_P$ . For an arbitrary change of frame we may write, by analogy,

$$A^N_{\Omega \| X} = A^N_{\Omega,X} + A^N_\Sigma \hat{\gamma}^\Sigma_{\Omega X} .$$

From whence we get $\gamma^N_{MP} = A^N_{\Sigma \| X} A^\Sigma_M A^X_P$ . The frame-covariant differentiation so defined is nothing but covariant differentiation with respect to "anholonomic coordinates" in the classical version (Schouten [24], p. 169). In our work we will ordinarily use two types of frames: a coordinate frame and an orthonormal frame. We prefer to distinguish the symbols for covariant differentiation with respect to each, reserving "|" for coordinates and "||" for the orthonormal frame.

In the classical language we would say (Eisenhart [25])

that an ennuple of vectors $\{\lambda^i{}_N\}$ and any set of $\gamma^N{}_{MP}$ determine

a connection. It makes sense to ask what kinds of geometry may be

specified by various choices for the Ricci rotation coefficients

and associated frames. A trivial example is that a coordinate

frame and any set of $\Gamma^i{}_{jk}$ , where $\Gamma^i{}_{[jk]} = 0$, determine a sym-

metric connection. More to the point for us are the Riemannian

(metric and symmetric) connections. But the conditions on the

frame and rotation coefficients which will guarantee that the

connection so determined is Riemannian are not nearly so handy.

If the connection is symmetric, we have to consider conditions of

integrability for the existence of a metric $g_{ij}$ which are of the

form [25]:

$$g_{ih}R^h{}_{jk\ell} + g_{jh}R^h{}_{ik\ell} = 0$$

$$g_{ih}R^h{}_{jk\ell|m} + g_{jh}R^h{}_{ik\ell|m} = 0$$

$$\cdots$$

$$g_{ih}R^h{}_{jk\ell|m_1\ldots m_r} + g_{jh}R^h{}_{ik\ell|m_1\ldots m_r} = 0 \ .$$

More recently the fiber bundle outlook and algebraic topology has

been used for insight into global aspects [26], but the local tools

have remained the same (even though couched in different language).

The conditions for a metric connection to be symmetric do not seem

to have attracted much attention.

Necessary conditions for a connection to be metric are readily found, if the frame is assumed to be orthonormal. If $\lambda_N \cdot \lambda_M = \eta_{NM}$, where $\eta_{NM} = \rho_N \delta_{NM} (\rho_N = \pm 1,\ N$ not summed), then the Ricci rotation coefficients satisfy $\eta_{NP} \gamma^P_{MR} + \eta_{MP} \gamma^P_{NR} = 0$. But these conditions are not in a very useful form, since the frame was pre-selected.

To make a start towards the analysis of this problem we pose two questions: First, what kind of geometry is specified by any $\lambda_N$'s and $\gamma^N_{MP}$'s with the restriction that $\eta_{NR} \gamma^R_{MP} + \eta_{MR} \gamma^R_{NP} = 0$ $(\eta_{NM} = \rho_N \delta_{NM},\ N$ not summed); and second, what are the conditions we can put on the $\gamma^N_{MP}$ in order to guarantee that there exist $\lambda_N$ for which the connection is symmetric—i.e., for which $d\lambda^N = -\gamma^N_{MP} \lambda^M \wedge \lambda^P$?

First, let $\{\lambda^i_N\}$ be an ennuple of vectors and let $\gamma^N_{MP}$ be given functions obeying the restriction $\eta_{NR} \gamma^R_{MP} + \eta_{MR} \gamma^R_{NP} = 0$. Then $\eta_{NM}$ is covariantly constant:

$$(\eta_{NM})_{\|P} = \eta_{NM,P} + \eta_{NR} \gamma^R_{MP} + \eta_{RM} \gamma^R_{NP}$$

$$= 0 .$$

If we define an inner product by $\lambda_N \cdot \lambda_M = \eta_{NM}$, from this it is easy to conclude that the connection is metric (i.e., it is pulled

down from a connection on the bundle of orthonormal frames). We digress to consider the classical formulation. If the coordinate form of the metric is denoted by $g_{ij}$, where $\eta_{NM} = g_{ij}\lambda^i{}_N\lambda^j{}_M$, then it follows that $g_{ij|k} = 0$ as well. Then we may write ([24], p. 132)

$$\Gamma^i{}_{hj} = \{^i_{hj}\} + \Omega_{hj}{}^i{}_\cdot - \Omega_{j}{}^i{}_{\cdot h} + \Omega^i{}_{\cdot hj}$$

where $\{^i_{hj}\}$ are the christoffel symbols formed with respect to $g_{ij}$, $\Gamma^i{}_{[hj]} = \Omega_{hj}{}^i{}_\cdot$ is the classical torsion tensor, and the indices on $\Omega_{hj}{}^i{}_\cdot$ are raised and lowered by $g_{ij}$ and $g^{ij}$. For the symmetric part of the connection we may write

$$\Gamma^i{}_{(hj)} = \{^i_{hj}\} + \Omega^i{}_{\cdot(hj)} \ .$$

It follows that the curves of extremal arc-length, as measured by $ds^2 = g_{ij}dx^i dx^j$, are not geodesics. This is the form in which metric connections have been recently considered by Hehl [27] as a candidate for a "Unified" field theory.

Our second question reduces to determining the conditions of integrability for the equations $d\lambda^N = -\gamma^N{}_{MP}\lambda^M \wedge \lambda^P$ where the $\gamma^N{}_{MP}$ are given functions. A complete set of first-order conditions of integrability can easily be obtained by exterior differentiation.

With the abbreviations $\lambda^{PNM} = \lambda^N \wedge \lambda^M$, $\lambda^{NMP} = \lambda^{NM} \wedge \lambda^P$, we have for the first set of conditions

$$(1.10) \qquad 0 = d(\gamma^N_{\ MP}) \wedge \lambda^{MP} - \gamma^N_{\ AQ}\gamma^A_{\ MP}\lambda^{MPQ} + \gamma^N_{\ MA}\gamma^A_{\ PQ}\lambda^{MPQ}$$

which may be conveniently rewritten $R^N_{\ [MPQ]} = 0$; i.e., it is just the cyclic identities for the curvature tensor (see Chapter 2). Now this is 16 independent equations in the 16 unknown quantities $\lambda^N_{\ i}$ ($\lambda^N = \lambda^N_{\ i}dx^i$); generally speaking, a solution will exist. Thus given $\gamma^N_{\ MP}$, we can generally find a frame for which the first conditions of integrability (1.10) are satisfied. The second set of conditions, obtained from (1.10) by exterior differentiation, have the remarkable property that they are of an algebraic form and involve $\gamma^N_{\ MP}$ <u>alone</u>:

$$(1.11) \qquad 0 = \gamma^N_{\ [AB]}\gamma^A_{\ [MP}\gamma^B_{\ \cdot QR]} \ .$$

They may be rewritten in the form

$$0 = [\lambda_A, \lambda_B]\gamma^A_{\ \cdot [NM}\gamma^B_{\ \cdot PQ]} \ .$$

The geometric significance of these conditions is at present obscure, but should be interesting. These conditions are as far as

we can go in 4 dimensions, since there, further exterior

differentiation will make all terms vanish identically.   Therefore

we shall not pursue further conditions here.   It is not clear at

the present time how much these conditions of integrability, com-

plete as they are, guarantee the existence of a solution to $d\lambda^N =$

$- \gamma^N_{MP} \lambda^{MP}$;   nevertheless they may prove useful.

## 2. Riemannian geometry.

As we have intimated in Chapter 1, a _Riemannian connection_ is one which is both metric and symmetric. That is, the torsion is zero and there exists a nonsingular, symmetric, and covariantly constant 2-covariant tensor $g$, called the metric of course. If $g = g_{NM} \lambda^N \otimes \lambda^M$ on $U \subseteq M$, where $g_{NM}$ is symmetric, the latter condition means $g_{NM\|P} = 0$. We use $g_{NM}$ to raise and lower indices, that is provide a connection between forms and vectors which commutes with covariant differentiation. We have then $dg_{NM} = \omega_{NM} + \omega_{MN}$, where $\omega_{NM} = g_{NP} \omega^P{}_M$. A basis in which $g_{NM}$ takes the simple form $g_{NM} = \eta_{NM} \equiv e_N \delta_{NM}$ ($e_N = \pm 1$, $N$ not summed) is, of course, _orthonormal_. In that case the Ricci rotation coefficients obey the rule $\eta_{NP} \gamma^P{}_{MR} + \eta_{MP} \gamma^P{}_{NR} = 0$. We assume from now on that the frame is orthonormal.

The structural equations (1.9) for Riemannian geometry may now be written as

$$(2.1) \qquad \text{(a)} \quad d\lambda^N = \omega^N{}_{\bullet M} \wedge \lambda^M$$

$$\text{(b)} \quad \mathcal{Q}^N{}_{\bullet M} = - d\omega^N{}_{\bullet M} + \omega^N{}_{\bullet P} \wedge \omega^P{}_{\bullet M}$$

$$\text{(c)} \quad 0 = \omega_{NM} + \omega_{MN} .$$

The Riemann tensor is defined by

$$(2.2) \qquad \frac{1}{2} R^N_{\bullet MPQ} \lambda^P \wedge \lambda^Q = \Omega^N_{\bullet M} \, ,$$

and the Ricci tensor is $R_{MP} = R^N_{MPN}$ . These definitions trans-
late into coordinate form as:

$$\phi_{i|jk} - \phi_{i|kj} = \phi_\ell R^\ell_{\bullet ijk}$$

$$R_{ij} = R^k_{ijk}$$

$$\gamma^N_{MP} = \lambda^N_{i|j} \lambda^i_M \lambda^j_P$$

to accord with the sign convention of most authors.

If $T^{NM\cdots}_{\qquad PQ\cdots}$ is any "invariant," its frame-covariant
derivative $T^{NM\cdots}_{\qquad PQ\cdots\bullet\|R}$ is easily defined by using the product
rule on a decomposition of $T$ into a sum of products of vectors
and forms, and the rules

$$S^N_{\ \|M} = S^N_{\ ,M} + S^P \gamma^N_{P\ M}$$

$$S_{N\|M} = S_{N,M} + S_P \gamma^P_{NM} \, .$$

We note that if $S_N = S_{i \cdot} \lambda^i_N$ , then

$$S_{i|j} \lambda^i_N \lambda^j_M = S_{N\|M}$$

and similarly for $S^N$ . So the frame covariance is manifest, as well as the coordinate _invariance_. This relation makes it easy to go between tensorial and invariant formulae. Since the operation of raising and lowering invariant indices is pretty nearly trivial (because of the special form for the metric tensor), we essentially simplify formulae without losing _anything_ in return.

Allowing the first equations of structure to act on the orthonormal frame, $(d\lambda^N - \omega^N_M \wedge \lambda^M)(\lambda_P, \lambda_Q) = 0$, results in

(2.3)  (M)  $\lambda^i_{N,M} - \lambda^i_{M,N} = \lambda^i_P (\gamma^P_{\cdot MN} - \gamma^P_{\cdot NM})$ .

These will be called the metric equations; a misnomer, since they really determine the symmetric nature of the connection.

The second equations of structure amount to the definition of the Riemann tensor in terms of the Ricci rotation coefficients:

(2.4)  $\frac{1}{2} R^N_{\cdot MPQ} = \gamma^N_{\cdot M[P,Q]} + \gamma^N_{\cdot MR} \gamma^R_{\cdot [PQ]} + \gamma^N_{\cdot R[P} \gamma^R_{\cdot |M|Q]}$ .

Imposing the vacuum conditions on these equations in 4 dimensions, $R_{MP} = 0$, results in the system of first-order equations which we denote by (F):

(F) $\qquad \gamma^N_{\cdot M[P,N]} + \gamma^N_{\cdot MR} \gamma^R_{\cdot [PN]} + \gamma^N_{\cdot R[P} \gamma^R_{\cdot |M|N]} = 0$ .

It is clear from the derivation that the system of equations (M) and (F) in a $V_4$ of signature $+2$ are a necessary and sufficient set of first-order equations for the determination of a gravitational universe. But certain additional conditions of integrability can be useful in the solution of the problem. These are most easily obtained by exterior differentiation of the structural equations (2.1). From (2.1a) we obtain the cyclic identities

$$\Omega^N_{\phantom{N}M} \wedge \lambda^M = 0$$

which are equivalent to $R^N_{[MPQ]} = 0$. From (2.1b) exterior differentiation gives us the Bianchi identities

$$d\Omega^N_{\phantom{N}M} = \omega^N_{\cdot P} \wedge \Omega^P_{\cdot M} - \Omega^N_{\cdot P} \wedge \omega^P_{\cdot M}$$

which may be expressed as $R^N_{\phantom{N}M[PQ\|R]} = 0$. Further application of

the exterior derivative cannot result in any relations independent

of these. Contracting the Bianchi identities over  N  and  R,

then over  M  and  P,  we have the useful identities

$$(2.5) \qquad R^{N}_{M\|N} = \frac{1}{2} R_{,M}$$

'which we shall call the "potential equations" in the context of the

associated  $V_3$  of stationary spaces.

The Jacobi identities for the frame  $\{\lambda_N\}$  arise from the

cyclic identity  $R_{N[MPQ]} = 0$.  To see this we take partial

coordinate components of the Riemann tensor:

$$\lambda^{N}_{\ell} R_{NMPQ} = - \lambda_{M\ell \,|\, [jk]} \lambda^{j}_{P} \lambda^{k}_{Q} \ .$$

Cyclically permuting  M, P, Q  and adding, we eventually arrive at

the Jacobi identities

$$[\lambda_N, [\lambda_M, \lambda_P]] + [\lambda_M, [\lambda_P, \lambda_N]] + [\lambda_P, [\lambda_N, \lambda_M]] = 0 \ .$$

If one now replaces each bracket in this expression by its value

from (M), he will arrive back at the cyclic identity. The Jacobi

identities find an application as differential criteria for certain

special choices of the frame, but the details will not be pursued here. We only cite an example. In the $V_3$, the cyclic identity is equivalent to the identity $R_{NMPQ} = R_{PQNM}$ . If $\lambda_{(1)}$ , say, is normal to surfaces, then the differential criteria for that specialization are supplied by substituting (2.4) into $R_{1213} - R_{1312} = 0$. Oddly enough, the same differential criteria arise if $\lambda_{(1)}$ is to be along a killing motion.

The Riemann tensor is easily seen to satisfy the following identities in a Riemannian space:

(2.6)     (i)   $R_{NMPQ} = - R_{MNPQ}$

          (ii)  $R_{NMPQ} = - R_{NMQP}$

          (iii) $R_{N[MPQ]} = 0$

          (iv)  $R_{NMPQ} = R_{PQNM}$

of these, (i), (ii) and (iii) may be considered to be independent, while (iv) is a consequence of the others. Any other tensor $W_{NMPQ}$ which satisfies these symmetries and, in addition, the "vacuum" condition

          (v)  $W^N{}_{MPN} = 0$

may be termed a Weyl tensor. An important example is the conformal curvature tensor

$$(2.7) \qquad C^N_{MPQ} = R^N_{MPQ} + \frac{2}{n-2}(\delta^N_{[P}R_{Q]M} + \eta_{M[Q}R^N_{\cdot P]})$$

$$+ \frac{2R}{(n-1)(n-2)}(\delta^N_{[Q}\eta_{P]M})$$

which forms the basis of the Petrov classification in $V_4$. In any $V_3$ we have $C^N_{MPQ} = 0$, the proof being particularly simple in invariants.

Finally the commutation relations for invariant differentiation, which merely express $d^2f = 0$, and in a symmetric connection are equivalent to $f_{\|NM} - f_{\|MN} = 0$, are

$$(2.8) \qquad f_{,NM} - f_{,MN} = f_{,P}(\gamma^P_{MN} - \gamma^P_{NM}) \; .$$

We now develop the criteria for various geometric specializations of the frame in terms of the Ricci rotation coefficients. If the congruences of $\lambda_N$ are _normal to non-null hypersurfaces_ $f(X^1, \ldots, X^n) = $ const., then $\lambda_A$, $A \neq N$, are tangent to $f = $ constant. Choose coordinates in which $f = X^n$; then $\lambda^n_A = 0$ ($A \neq N$) and the metric equations yield $0 = \lambda^n_N \cdot (\gamma^N_{AB} - \gamma^N_{BA})$ ($N$ not summed), from whence $\gamma_{NAB} = \gamma_{NBA}$, $A,B \neq N$.

The congruences of $\lambda_N$ are _geodesics_ provided that $\lambda^i_{N|j}\lambda^j_N = \mu\lambda^i_N$ . In invariants we have simply $\delta^A_{N\|B}\delta^B_N = \mu\delta^A_N$ , which yields $\gamma^A_{NN} = 0$, $A \neq N$, $N$ not summed. $\lambda_N$ is along the trajectories of a _killing motion_ in case $\pounds_{\mu\lambda_N}g_{ij} = 0$ or $(\mu\lambda_{Ni})|_j + (\mu\lambda_{Nj})|_i = 0$ for some function $\mu$. In invariants, $(\mu\eta_{NA})_{\|B} + (\mu\eta_{NB})_{\|A} = 0$ which gives

$$(2.9) \qquad \gamma_{NAB} + \gamma_{NBA} = 0,$$

$$\gamma_{NBN} = -\eta_{NN}(\ell n\mu),_B \qquad (A,B \neq N) .$$

The criteria for the congruences of $\lambda_N$ in a $V_n$ to be shear-free or divergenceless will be needed for null congruences in a $V_4$ and for congruences of a $V_3$. The results in $V_4$ are well-known, and will be cited at the appropriate time. We outline here a derivation for $V_3$.

Let $\lambda_{(1)}$ be tangent to the congruences in question, and forming along with $\lambda_{(2)}$ and $\lambda_{(3)}$ an orthonormal frame. Fix a trajectory T of $\lambda_{(1)}$. We could characterize the behaviour of neighbouring trajectories of $\lambda_{(1)}$ by a pair of vectors $\{\hat{\xi}_1,\hat{\xi}_2\}$ undergoing Lie-transport along the trajectories: $\pounds_{(1)}\hat{\xi}_A = [\lambda_{(1)},\hat{\xi}_A] = 0$, $A = 1,2$ . However, we find it more convenient to use a pair of vectors $\{\xi_A\}$ normal to the trajectories, since we

can then describe the behaviour of the trajectories in terms of familiar transformations of the plane spanned by $\{\xi_A\}$. The resulting description will be independent of the initial choice of $\{\xi_A\}$. Thus, we allow the normals $\xi_A$ to "slide" along the trajectories, that is $\mathcal{L}_{\varphi\nabla_A(1)}\xi_A = 0$ for some function $\varphi^\nabla_A$, $A = 1,2$. We can rewrite this as $\mathcal{L}_{(1)}\xi_A = [\lambda_{(1)},\xi_A] = \varphi_A\lambda_{(1)}$.

Then we say that the congruences of $\lambda_{(1)}$ are <u>shear-free</u>, if all it takes is an expansion and rotation of $\lambda_{(2)}$ and $\lambda_{(3)}$ to bring them into coincidence with $\xi_2$ and $\xi_3$ along T:

$$\mathcal{L}_{(1)}(\rho(\lambda_{(2)} + i\lambda_{(3)})) = \varphi\lambda_{(1)} ,$$

where $\rho$ and $\varphi$ are complex. As a consequence of this we get (Appendix D)

$$(2.10) \quad 2\beta = \gamma_{122} - \gamma_{133} + i(\gamma_{123} + \gamma_{132}) = 0.$$

Thus we may call $\beta$ the <u>complex shear</u>, and $|\beta|$ the <u>shearing</u>. It may be noted that $|\beta|$ is an invariant property of the trajectories of $\lambda_{(1)}$ and does not depend on the choice of $\lambda_{(2)}$ and $\lambda_{(3)}$.

For the next case we fix the frame by requiring $\gamma_{231} = 0$, i.e., the vectors $\lambda_{(2)}$ and $\lambda_{(3)}$ do not "rotate" along

$\lambda_{(1)}$ ([28], p. 99). The congruences of $\lambda_{(1)}$ will be said to be <u>dilatation-free</u>, if $\lambda_{(2)}$ and $\lambda_{(3)}$ are such that, infinitesimally, a shearing motion is all that is needed to achieve a $\xi_A$ :

$$\mathcal{L}_{(1)}(e^{i\theta}\lambda_{(2)} + ie^{-i\theta}\lambda_{(3)})\Big|_{\theta=0} = \varphi\lambda_{(1)} .$$

As a consequence of this we find (Appendix D)

$$(2.11) \qquad 2\gamma = \gamma_{122} + \gamma_{133} + i(\gamma_{123} - \gamma_{132}) = 0.$$

Notice that this implies $\lambda_{(1)}$ was normal to surfaces, since $\gamma_{123} = \gamma_{132}$. We call $\gamma_{123} - \gamma_{132}$ the <u>twist</u>, and $\gamma$ the <u>complex dilatation</u>. The twist and complex dilatation are separately independent of the choice of frame.

If $\gamma_{122} + \gamma_{133} = 0$, but possibly $\gamma_{123} - \gamma_{132} \neq 0$, the congruences of $\lambda_{(1)}$ are said to be <u>divergenceless</u>. This term can be justified by an area argument. At a given point $t \in T$, pick $\xi_2$ and $\xi_3$ so that $\xi_2(t) = \lambda_{(2)}(t)$, and $\xi_3(t) = \lambda_{(3)}(t)$ (on nearby points, of course, this equality does not hold). The rate of change along $T$ of $\xi_2{}^2\xi_3{}^2$, evaluated at $t$, equals the rate of change of the (area)$^2$ of the parallelogram spanned by $\xi_2$ and $\xi_3$ (by an order of magnitudes argument). Now

$$\pounds_{(1)}(\xi_2{}^2\xi_3{}^2)\Big|_t = \pounds_{(1)}(\xi_2{}^2) + \pounds_{(1)}(\xi_3{}^2), \quad \text{since} \quad \xi_A{}^2\Big|_t = 1$$

$$= \xi_2{}^i\xi_2{}^j\pounds_{(1)}g_{ij} + \xi_3{}^i\xi_3{}^j\pounds_{(1)}g_{ij}$$

$$+ 2\xi_2{}^ig_{ij}\pounds_{(1)}\xi_2{}^j + 2\xi_3{}^ig_{ij}\pounds_{(1)}\xi_3{}^j$$

$$= \lambda_{(2)}{}^i\lambda_{(2)}{}^j\lambda_{(1)(i|j)} + \lambda_{(3)}{}^i\lambda_{(3)}{}^j\lambda_{(1)(i|j)}$$

$$= \gamma_{122} + \gamma_{133} \ .$$

Thus $\gamma_{122} + \gamma_{133} = 0$ implies $\pounds_{(1)}(\xi_2{}^2\xi_3{}^2) = 0$, so that the area of the parallelogram spanned by $\xi_2$ and $\xi_3$ is fixed.

Finally, for completeness we will write down the invariant form of the Gauss and Codacci equations ([22], p. 46), though we will not need them. The formulation in invariants is particularly appealing. If $\lambda_N$ is normal to the hypersurfaces in question, then the invariant form of the second fundamental form is just (and we may take this as a definition)

$$\chi_{AB} = \gamma_{NAB} , \quad A,B \neq N.$$

Gauss' equation, which relates the curvature tensor of the $V_n$ to that of the hypersurface $V^\nabla_{n-1}$, is just

$$R^{\nabla A}_{\phantom{\nabla A}BCD} = R^A_{\phantom{A}BCD} \pm \gamma^A_{N\phantom{A}C} \gamma_{NBD} \mp \gamma^A_{N\phantom{A}D} \gamma_{NBC}$$

and Codacci's equation is

$$\gamma^A_{N\phantom{A}B\|A} - \gamma^A_{N\phantom{A}A\|B} = R_{NB}$$

where $A,B,C,D \neq N$, $\|$ refers to frame-covariant differentiation in the $V_{n-1}$, and the sign depends on the indicator of $\lambda_N$: $\lambda_N \cdot \lambda_N = \pm 1$.

The treatment of duality in invariants is especially simple. We restrict ourselves in this discussion to four dimensions. The invariant components of the permutation <u>tensor</u> turn out to be just the Levi-Civita permutation <u>symbol</u>:

$$\eta_{ijk\ell} \lambda^i_{\phantom{i}N} \lambda^j_{\phantom{j}M} \lambda^k_{\phantom{k}P} \lambda^\ell_{\phantom{\ell}Q} = \varepsilon_{NMPQ}$$

where $\eta_{ijk\ell} = \sqrt{|g|}\, \varepsilon_{ijk\ell}$. The <u>dual</u> of any bivector $F_{AB}$ is defined as

$$F^*_{AB} = \frac{1}{2} \varepsilon_{ABCD} F^{CD} .$$

Taking the dual twice affects as a factor of $-1$:

$$F^{*}_{AB} = - F_{AB} \; .$$

With any such bivector we can form the complex <u>self-dual</u> combination [29]

$$\mathcal{F}_{AB} = \tfrac{1}{2}(F_{AB} + iF^{\star}_{AB}) \; .^{1}$$

It is self-dual in the sense that

$$\mathcal{F}^{*}_{AB} = - i\mathcal{F}^{\star}_{AB} \; .$$

Any Weyl tensor $W_{NMPQ}$ behaves as a bivector with respect to the index pairs $NM$ and $PQ$, so we may form two duals:

$$W^{\star}_{NMPQ} = \frac{1}{2} \, \varepsilon_{NMAB} W^{AB}_{PQ}$$

$$W_{NMPQ}{}^{\star} = \frac{1}{2} \, \varepsilon_{PQAB} W_{NM}{}^{AB} \; .$$

---

[1] An alternative definition is [4]: $\mathcal{F}_{AB} = F_{AB} - iF^{\star}_{AB} \; .$
With this one has, perhaps more naturally, $\mathcal{F}^{*}_{AB} = i\mathcal{F}_{AB} \; .$

It may be shown that (by a calculation in Israel [4], or by inspection)

$$W^*{}_{NM}{}^*{}_{PQ} = - W_{NMPQ}$$

From this it follows that the two duals are equal:

$$W^*{}_{NMPQ} = W_{NMPQ}{}^*$$

It should be noted that this is a consequence of the "vacuum" conditions satisfied by W.

Petrov [30] has introduced bivector indices as follows:

| NM | 14 | 24 | 34 | 23 | 31 | 12 |
|----|----|----|----|----|----|----|
| a  | 1  | 2  | 3  | 4  | 5  | 6  |

Thus $F_{NM} \leftrightarrow F_a$, where $(NM) \leftrightarrow a$, and for a Weyl tensor we have $W_{NMPQ} \leftrightarrow W_{ab}$, with $W_{ab} = W_{ba}$. For reasons which will become apparent, we restrict the range of $a, b, c, \ldots$ to 1,2,3, from now on.

Duality in this formulation behaves as follows:

$$W^*_{ab} = W_{(a+3)b} = W_{a(b+3)}$$

$$W^*_{(a+3)b} = - W_{ab} \quad , \quad a,b = 1,2,3 \ .$$

The self-dual combinations in bivector space are,

$$\widetilde{W}_{ab} = \frac{1}{2}(W_{ab} + iW^*_{ab})$$

and we have $\widetilde{W}^*_{ab} = - i\widetilde{W}_{ab}$ . Working with these self-dual combinations enables us to dispense entirely with the indices 4, 5, and 6.

It may be noticed that the bivectors form a six-dimensional vector space $b_6$ over the reals, and the self-dual bivectors a three-dimensional complex vector space, which we denote by $b_3$. If $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an orthonormal tetrad with $\lambda_4$ timelike (the signature + 2), a basis for the space $b_6$ is defined by the choice

$$e_1{}^{\alpha\beta} = \lambda^\alpha_{[1}\lambda^\beta_{4]}, \ e_2 = \lambda_{[2}\lambda_{4]}, \ e_3 = \lambda_{[3}\lambda_{4]}$$

$$e_4 = \lambda_{[2}\lambda_{3]}, \ e_5 = \lambda_{[3}\lambda_{1]}, \ e_6 = \lambda_{[1}\lambda_{2]} \ .$$

We have $e_1{}^* = e_4$ , $e_2{}^* = e_5$ , and $e_3{}^* = e_6$ , so the following

combinations are self-dual:

(2.14) $\quad \tilde{e}_1 = e_1 + ie_1{}^* = e_1 + ie_4$

$\tilde{e}_2 = e_2 + ie_5$

$\tilde{e}_3 = e_3 + ie_6$

and they form a basis for the space $b_3$. An inner product on bivectors is defined by $UV = U^{\alpha\beta}V_{\alpha\beta}$ . Using this one finds the orthonormality relations $\tilde{e}_a\tilde{e}_b = -\delta_{ab}$ . It follows that any proper orthochronous Lorentz transformation in $M$ corresponds to an $SO(3,C)$ transformation in $b_3$. Indeed, we have the isomorphism $L^+_+ \cong SO(3,C)$ (Israel [4], for example).

Petrov has based a classification of Einstein spaces in $V_4$ on the algebraic properties of the Riemann tensor. In vacuum space-times $R_{ABCD} = C_{ABCD}$ which is a Weyl tensor, so we may use all the machinery above. Petrov's classification may then be carried out on the self-dual Riemann tensor $\tilde{R}_{ab}$ . According to his result (translated into our terms) transformations from $SO(3,C)$ suffice to put the matrix $\tilde{R}_{ab}$ into one of the following three forms (Petrov [30], p. 110):

$$(2.15) \quad \text{I.} \quad 2\widetilde{R}_{ab} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \quad \text{where} \quad \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\text{II.} \quad 2\widetilde{R}_{ab} = \begin{bmatrix} -2\alpha & 0 & 0 \\ 0 & \alpha+1 & i \\ 0 & i & \alpha-1 \end{bmatrix}$$

$$\text{III.} \quad 2\widetilde{R}_{ab} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

Further classification is based upon the various possible specializations of the complex quantities $\alpha$, $\alpha_1$, etc. (for details see [5]). In particular we single out the types

$$(2.16) \quad \text{D or } I_d : \quad 2\widetilde{R}_{ab} = \begin{bmatrix} -\alpha/2 & 0 & 0 \\ 0 & -\alpha/2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

$$\text{N or } II_d : \quad 2\widetilde{R}_{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{bmatrix}$$

$$0 : \quad 2\widetilde{R}_{ab} = 0$$

and reserve the symbols  I, II  for the nondegenerate cases.

We could expand  $\widetilde{R}_{ab}$  in our basis of self-dual bivectors $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$.  However it is more convenient to use a modified basis

(2.17)    $\widetilde{e}_0 = 1/\sqrt{2}\,(\widetilde{e}_2 + i\widetilde{e}_3)$

$\widetilde{e}_0^{\bullet} = 1/\sqrt{2}\,(\widetilde{e}_2 - i\widetilde{e}_3)$

$\widetilde{e}_1 = \widetilde{e}_1$ .

$\widetilde{e}_0$  and  $\widetilde{e}_0^{\bullet}$  are <u>null</u>:  $\widetilde{e}_0^{\,2} = \widetilde{e}_0^{\bullet\,2} = 0$.  They thus correspond in the real bivector space to null vectors ([4], p. 39).  Indeed, we find $(\lambda_{(1)\alpha} + \lambda_{(4)\alpha})\widetilde{e}_0^{\,\alpha\beta} = 0$  so that  $\lambda_{(1)} + \lambda_{(4)}$  is tangent to the null direction defined by  $\widetilde{e}_0$ .  For brevity we may sometimes write  $U = \widetilde{e}_0$ ,  $V = \widetilde{e}_0^{\bullet}$ ,  $M = \widetilde{e}_1$ .  Making slight variations in the expansion given by R. Sachs [31], we have

(2.18)    $2\widetilde{R}_{ab} = R^{(1)} V_a V_b + R^{(2)} (V_a M_b + M_a V_b)$

$+ R^{(3)} (M_a M_b - \frac{1}{2} U_a V_b - \frac{1}{2} V_a U_b) + R^{(4)} (U_a M_b + M_a U_b)$

$+ R^{(5)} U_a U_b$ ,

where  $V_a = \widetilde{e}_a^{\,\alpha\beta} V_{\alpha\beta}$ , etc., and  $R^{(i)}$  are scalars, $i = 1, \ldots , 5$.

The scalars $R^{(i)}$ are readily computed:

$$(2.19) \quad R^{(1)} = 2\widetilde{R}_{00} = \widetilde{R}_{22} - \widetilde{R}_{33} + 2i\widetilde{R}_{23}$$

$$R^{(2)} = 2\widetilde{R}_{10} = 2/\sqrt{2}\,(\widetilde{R}_{12} + i\widetilde{R}_{13})$$

$$R^{(3)} = 2\widetilde{R}_{11}$$

$$= -4\widetilde{R}_{0\dot{0}} = -2\widetilde{R}_{22} - 2\widetilde{R}_{33}$$

$$R^{(4)} = 2\widetilde{R}_{1\dot{0}} = 2\sqrt{2}\,(\widetilde{R}_{12} - i\widetilde{R}_{13})$$

$$R^{(5)} = 2\widetilde{R}_{\dot{0}\dot{0}} = \widetilde{R}_{22} - \widetilde{R}_{33} - 2i\widetilde{R}_{23} \ .$$

A basis $\{\lambda_1, \ldots, \lambda_4\}$ in which the matrix $\widetilde{R}_{ab}$ takes one of the forms (2.15)-(2.16) is called a $\underline{principal}$ $\underline{tetrad}$. If a basis is chosen in which $R^{(1)} = 0$, that is in which $\widetilde{R}_{ab}U^a U^b = 0$, the null direction defined by $U$ is called a $\underline{principal}$ $\underline{null}$ $\underline{direction}$. There is at least one and at most four such fields of directions [31].

An alternative characterization of the Petrov classification may be given by the various possible coincidences of these principal null directions [32]:

| type | description of principal null directions | symbol |
|------|------------------------------------------|--------|
| I    | four distinct                            | [1111] |
| D    | two repeated                             | [22]   |
| O    | indeterminate                            | [-]    |
| II   | one repeated, two single                 | [211]  |
| N    | one repeated                             | [4]    |
| III  | one repeated, one single                 | [31]   |

Finally, we may remark that the bivector space also admits of a convenient formulation in fiber bundles. In this formulation certain <u>connecting quantities</u> $B_b{}^{\alpha\beta}$ defined by Petrov ([30], p. 90), play a role analogous to that of the $\sigma_i{}^{a\dot{b}}$ in the spinor calculus (see Chapter 4).

3. Stationary spaces and derivation of the field equations.

We now turn to stationary space-times which satisfy the vacuum field equations. To be precise, a space-time is a pair $(M, \Phi)$ where $M$ is the manifold and $\Phi$ is a Lorentz metric on $M$. The associated Riemannian connection is assumed to satisfy the field equations $R_{NM} = 0$. Finally, the space-time is stationary means there exists a one-parameter group of motions on $M$ which leaves $\Phi$ invariant, whose trajectories are timelike curves, and which has no fixed points. We could be much more abstract and reformulate everything in fiber bundle language, but such generality at this stage would be beside the point. It might be appropriate in the study of global topological questions, but we are concerned only with the local geometry, where the topology is quite trivial.

In any system of coordinates adapted to the motion, that is in which the killing vector field $\xi$ has components $\xi^i = \delta_4{}^i$, say, we may write the metric form as

$$(3.1) \qquad \Phi = \gamma_{ij} dx^i dx^j = e^{-\omega}(g_{\alpha\beta} dx^\alpha dx^\beta) - e^{\omega}(a_\alpha dx^\alpha + dt)^2$$

where $i, j = 1, \ldots, 4$; $\alpha, \beta = 1, 2, 3$; $g_{\alpha\beta}$, $a_\alpha$, and $\omega$ are functions of $x^1, x^2, x^3$; and we have written $t = x^4$. We have chosen $-e^{\omega}$ as the coefficient of $dt^2$ because of the timelikeness of the trajectories and because the killing vector field is everywhere

nonzero. A decomposition of $\Phi$ into squares shows that in order for a signature of $+2$ to obtain we must have the metric form

$$(3.2) \qquad ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

positive definite ([3], p. 111). It defines a Riemannian $V_3$ which we call the <u>associated</u> $V_3$ . It is also known as the space-quotient; and if we were being abstract we would identify it as the base-space of the principal fiber bundle $M(V_3, R)$.

We still have the following freedom in the choice of coordinates:

$$(3.3) \qquad \hat{\chi}^\alpha = f^\alpha(\chi^\beta)$$

$$\hat{t} = t + \lambda(\chi^\beta), \quad \alpha, \beta = 1,2,3.$$

Under such a transformation, $\hat{\xi}^i = \xi^i = \delta^i_4$, so the coordinates remain adapted to the motion.

In the following, indices $i,j,k, \ldots$ and $N,M,P, \ldots$ will take the range $1,2,3,4$; while $\alpha,\beta,\gamma, \ldots$ and $A,B,C, \ldots$ will take $1,2,3$.

Let $\Lambda^{(4)} = e^{\omega/2}(a_\alpha dx^\alpha + dx^4)$, so that $\Lambda^{(4)}{}_\alpha = e^{\omega/2} a_\alpha$ and $\Lambda^{(4)}{}_4 = e^{\omega/2}$. If $\lambda_A$ is an orthonormal frame with respect to the metric of the associated $V_3$ , i.e.,

$$\lambda_A{}^\alpha \lambda_B{}^\beta g_{\alpha\beta} = \delta_{AB} \ , \quad \lambda_A{}^\alpha \lambda^A{}_\beta = \delta^\alpha{}_\beta \ ,$$

then define forms $\Lambda^A$ in $M$ by their components

$$\Lambda_\alpha{}^A = e^{-\omega/2}\lambda_\alpha{}^A, \quad \Lambda_4{}^A = 0.$$

We may then write

$$(\Lambda_i{}^N) = \begin{bmatrix} e^{-\omega/2}\lambda_i{}^N & 0 \\ \hline e^{\omega/2}a_i & e^{\omega/2} \end{bmatrix}$$

The matrix inverse to $(\Lambda_i{}^N)$, which we denote by $(\Lambda_N{}^i)$ expresses the components of the vector frame dual to $\{\Lambda^N\}$; it takes the form

$$(\Lambda_N{}^i) = \begin{bmatrix} e^{\omega/2}\lambda_N{}^i & 0 \\ \hline -e^{\omega/2}a_N & e^{-\omega/2} \end{bmatrix}$$

The frame $\{\Lambda_N\}$ forms an orthonormal frame, since we have the decomposition

$$\Phi = \Lambda^{(1)}\Lambda^{(1)} + \Lambda^{(2)}\Lambda^{(2)} + \Lambda^{(3)}\Lambda^{(3)} - \Lambda^{(4)}\Lambda^{(4)}$$

i.e., we have $\Phi(\Lambda_N, \Lambda_M) = \eta_{NM} = \text{diag}\{1,1,1,-1\}$.

Explicitly,

$$\begin{cases} \Lambda_{(4)}{}^i = e^{-\omega/2}\delta_4{}^i \\[2em] \Lambda_A{}^i = e^{\omega/2}\lambda_A{}^i - e^{\omega/2}a_A\delta_4{}^i \quad (\text{where } \lambda_A{}^4 = 0). \end{cases}$$

Note that $\Lambda_{(4)}$ is thus codirectional with the killing motion, as might have been expected.

In the following, except for the places where $\Lambda^N$, $\Lambda_N$ explicitly occur, we will hold to the convention that all tensors or connection coefficients whose components are taken on the frame $\Lambda_N$ be proceeded by a superscript (4); all other quantities shall have components taken on the frame $\lambda_A$. Thus $^{(4)}\omega^A{}_{\bullet B}$ refer to the connection one-forms in space-time for the frame $\{\Lambda_A, \Lambda_4\}$ which may be expanded in that basis as $^{(4)}\omega^A{}_{\bullet B} = {}^{(4)}\gamma^A{}_{\bullet BN}\Lambda^N$; while $\omega^A{}_{\bullet B}$ are the corresponding quantities in $V_3$, with $\omega^A{}_{\bullet B} = \gamma^A{}_{\bullet BC}\lambda^C$. Moreover if $f = f(\chi^1, \chi^2, \chi^3)$ is any function (independent of time)

then $df = f, \alpha \, dx^\alpha = e^{\omega/2} f_{,A} \Lambda^A$ where by the above convention $f_{,A} = f_{,\alpha} \lambda_A{}^\alpha$. Similar remarks hold for higher-order tensors. We define $f_{\alpha\beta} = a_{\alpha,\beta} - a_{\beta,\alpha}$.

To find the relationship between ${}^{(4)}\omega_{\bullet M}^{N}$ and $\omega_{\bullet B}^{A}$, we compute the exterior derivatives of the basic one-forms:

$$(3.4) \qquad d\Lambda^4 = \frac{1}{2} e^{3\omega/2} f_{BA} \Lambda^A \wedge \Lambda^B + \frac{1}{2} e^{\omega/2} \omega_B \Lambda^B \wedge \Lambda^4$$

$$d\Lambda^A = (\frac{1}{2} e^{\omega/2} \omega_B \Lambda^A + \omega_{\bullet B}^{A}) \wedge \Lambda^B.$$

The first equations of structure (2.1a) for the $V_4$ may be written

$$d\Lambda^4 = {}^{(4)}\omega_{\bullet B}^{4} \wedge \Lambda^B$$

$$d\Lambda^A = {}^{(4)}\omega_{\bullet B}^{A} \wedge \Lambda^B + {}^{(4)}\omega_{\bullet 4}^{A} \wedge \Lambda^4.$$

Equating coefficients of $\Lambda^B$, $\Lambda_4$ in the only way possible we have

$$(3.5) \qquad {}^{(4)}\omega_{\bullet B}^{4} = \frac{1}{2} e^{3\omega/2} f_{BA} \Lambda^A - \frac{1}{2} e^{\omega/2} \omega_B \Lambda^4$$

$${}^{(4)}\omega_{\bullet B}^{A} = \frac{1}{2} e^{\omega/2} (\omega_B \Lambda^A - \omega_A \Lambda^B) + \frac{1}{2} e^{3\omega/2} f_{AB} \Lambda^4 + \omega_{\bullet B}^{A}.$$

These equations give the relationship between the Ricci rotation coefficients of the $V_3$ and $V_4$. We record here, for future reference, the explicit formulae. Let $A, B, C \neq$. Then

(3.6)    (a)  $^{(4)}\gamma_{ABB} = e^{\omega/2}\gamma_{ABB} - \frac{1}{2}e^{\omega/2}\omega_{,A}$   (B not summed),

(b)  $^{(4)}\gamma_{ABC} = e^{\omega/2}\gamma_{ABC}$ ,

(c)  $^{(4)}\gamma_{4B4} = -\,^{(4)}\gamma_{\cdot B4}^{4} = \frac{1}{2}e^{\omega/2}\omega_{,B}$ ,

(d)  $^{(4)}\gamma_{AB4} = \frac{1}{2}e^{3\omega/2}f_{AB} = \frac{1}{2}e^{-\omega/2}\phi_{,C}$ ,

(e)  $^{(4)}\gamma_{4BC} = -\,^{(4)}\gamma_{\cdot\,BC}^{4} = -\frac{1}{2}e^{3\omega/2}f_{BC} = -\frac{1}{2}e^{-\omega/2}\phi_{,A}$ .

where
$\varepsilon_{ABC} = +1$

For the sake of convenience we have included, in the last equalities of (3.6d) and (3.6e), equivalent expressions which are a consequence of the field equations. Note how the equations (3.6c) and (3.6e) realize the criteria for $\Lambda_{(4)}$ to be in the direction of a killing motion (2.9).

Towards the computation of $^{(4)}R^{N}_{MPQ}$ from the second equations of structure, we have the following:

(3.7)    (a)   $^{(4)}\Omega^4_{\bullet B} = -d\,^{(4)}\omega^4_{\bullet B} + \,^{(4)}\omega^4_{\bullet A} \wedge \omega^{(4)\ A}_{\ \ \bullet B}$

$$= \frac{1}{2}\,e^{2\omega}\left[f_{BA,C} - f_{BN}\gamma^N_{\bullet CA} - f_{NC}\gamma^N_{\bullet BA} + \omega_C f_{BA}\right.$$

$$\left. - \omega_B f_{AC} + \frac{1}{2}\,\omega_N f_{NC}\delta_A^{\ B}\right]\Lambda^A \wedge \Lambda^C$$

$$+ \frac{1}{2}\,e^{\omega}\left[\omega_{B,C} + \omega_N\gamma^N_{\bullet BC} + \frac{3}{2}\,\omega_B\omega_C - \frac{1}{2}\,\omega_N\omega^N\delta_C^{\ B}\right.$$

$$\left. + \frac{1}{2}\,e^{2\omega}f^N_{\bullet C}f_{NB}\right]\Lambda^C \wedge \Lambda^4.$$

(b)   $^{(4)}\Omega^A_{\bullet B} = -d\,^{(4)}\omega^A_{\bullet B} + \,^{(4)}\omega^A_{\bullet C} \wedge \,^{(4)}\omega^C_{\bullet B} + \,^{(4)}\omega^A_{\bullet 4} \wedge \,^{(4)}\omega^4_{\bullet B}$

$$= \Omega^A_{\bullet B} + \frac{1}{4}\,e^{\omega}\left[8e^{-\omega/2}(e^{\omega/2})_{\|D[B}\delta^{A]}_{\ \ C} - \omega^N\omega_N\delta_C^{\ A}\delta_D^{\ B}\right.$$

$$\left. + e^{2\omega}(f_{AB}f_{CD} + f_{AC}f_{BD})\right]\Lambda^C \wedge \Lambda^D$$

$$- \frac{1}{2}\,e^{2\omega}\left[f_{AB\|C} + 2\omega_C f_{AB} - 2\omega_{[A}f_{B]C}\right.$$

$$\left. - \omega_N f_{N[B}\delta^{A]}_{\ \ C}\right]\Lambda^C \wedge \Lambda^4.$$

From these we read off,

(3.8)
$$^{(4)}R^4_{\cdot BC4} = \frac{1}{2} e^\omega [\omega_{\|BC} - \frac{1}{2} \Delta_1 \omega \delta^B_C + \frac{3}{2} \omega_B \omega_C + \frac{1}{2} e^{2\omega} f_{NC} f_{NB}] \ .$$

(3.9)
$$^{(4)}R^4_{\cdot BAC} = e^{2\omega} [f_{B[A\|C]} + f_{B[A}\omega_{C]} - \omega_B f_{AC} + \frac{1}{2} \omega^N f_{N[C} \delta^B_{A]}] \ .$$

(3.10)
$$^{(4)}R^A_{\cdot BC4} = - \frac{1}{2} e^{2\omega} [f_{AB\|C} + 2\omega_C f_{AB} - 2\omega_{[A} f_{B]C} - \omega^N f_{N[B} \delta^A_{C]}] \ .$$

(3.11)
$$^{(4)}R^A_{\cdot BCD} = e^\omega R^A_{\cdot BCD} + 2e^{\omega/2} [(e^{\omega/2})_{\|A[C} \delta^B_{D]} - (e^{\omega/2})_{\|B[C} \delta^A_{D]}]$$

$$- \frac{1}{2} e^\omega \Delta_1 \omega \delta^A_{[C} \delta^B_{D]} + \frac{1}{2} e^{3\omega} [f_{AB} f_{CD} - f_{A[C} f_{D]B}] \ .$$

Equations (3.9) and (3.10) come, respectively, from (3.6) and (3.7),
but are consistent. At this point it is worthwhile to note the
saving in writing of the right-hand sides which we have achieved by
going to the invariant formulation. Compared with the formulae
(2.3-2.6) of Kloster, Som, and Das [35], we see that our (3.8)
performs the function of both (2.3) and (2.4); and in the rest of
the formulae the number of terms are more than halved.

The field equations, $^{(4)}R_{NM} = 0$ are now a simple matter
to write down. From (3.8), $^{(4)}R_{44} = 0$ yields

(3.12)    (a)    $\omega_{\|AA} + \frac{1}{2} e^{2\omega} f_{AB} f_{AB} = 0$

for which the coordinate version is

(3.12)    (b)    $\Delta_2 \omega + \frac{1}{2} e^{2\omega} f^{\alpha\beta} f_{\alpha\beta} = 0$ .

From (3.9),   $^{(4)}R_{4B} = 0$   yields

(3.13)    (a)    $(e^{2\omega} f_{BA})_{\|A} = 0$

          (b)    $(e^{2\omega} f^\mu{}_\nu)_{|\mu} = 0$ .

Suppose we write   $\phi_A = \frac{1}{2} \varepsilon_{ABC} e^{2\omega} f^{BC}$ ,   or simply   $\phi_A = e^{2\omega} f_{BC}$   where $\varepsilon_{ABC} = +1$;   then this becomes   $\phi_{C\|D} - \phi_{D\|C} = 0$   for   $C, D = 1, 2, 3$; these are the conditions of integrability for the equations   $\phi_{,C} = \phi_C$ .   Thus the <u>twist potential</u> $\phi$   is defined by

(3.14)    (a)    $\phi_{,A} = \frac{1}{2} \varepsilon_{ABC} e^{2\omega} f^{BC}$

which, in coordinates, becomes

(3.14)    (b)    $\phi_{,\alpha} = \frac{1}{2} e^{2\omega} \eta_{\alpha\beta\gamma} f^{\beta\gamma}$ .

At this point we note that the cyclic identities $^{(4)}R^4{}_{[BAC]} = 0$
applied to (3.9) yield $f_{[AB\|C]} = 0$. This may be rewritten in
terms of the potential $\phi$ as

(3.15)     $\Delta_2\phi - 2\Delta_1(\phi,\omega) = 0$ .

Finally, combining the vacuum conditions from (3.8) and (3.11) and
denoting $G_{BC} = R_{BC} - \frac{1}{2} R\delta_{BC}$ , we have the two equivalent sets of
equations

(3.16)     $0 = G_{BC} + \frac{1}{2}(\omega_B\omega_C - \frac{1}{2}\Delta_1\omega\delta^B{}_C) + \frac{1}{2}e^{2\omega}(-f_{AB}f_{AC} + \frac{1}{4}f^{AD}f_{AD}\delta^B{}_C)$ .

(3.17)     $0 = R_{BC} + \frac{1}{2}\omega_B\omega_C + \frac{1}{2}e^{-2\omega}\phi_B\phi_C$ .

It may be shown that the identities $(R^B{}_C)_{\|B} = \frac{1}{2}R_{,C}$   when applied
to (3.17), yield the previously obtained field equations, that is
to say (3.12), (3.13), (3.15). Hence equations (3.17) are
sufficient field equations for the stationary vacuum; the problem
has been reduced to finding positive definite $V_3$'s satisfying
(3.17) for some $\omega$'s and $\phi$'s.

From the structure equations for $V_3$ , we write down the
complete set of first-order equations: letting A,B,C $\neq$, A,B  not
summed,

(3.18)    (a)   $R_{ABAB} = \frac{1}{4}(\omega_A{}^2 + \omega_B{}^2 - \omega_C{}^2) + \frac{1}{4} e^{-2\omega}(\phi_A{}^2 + \phi_B{}^2 - \phi_C{}^2)$ .


(b)   $R_{ABAC} = \frac{1}{2}(\omega_B\omega_C) + \frac{1}{2} e^{-2\omega}(\phi_B\phi_C)$ .


(3.19)    $\lambda^\alpha{}_{A,B} - \lambda^\alpha{}_{B,A} = \lambda^\alpha{}_C(\gamma^C{}_{BA} - \gamma^C{}_{AB})$, $\alpha = 1,2,3$.


For convenience we also use the "potential equations" (3.12) and (3.15):


(3.20)    (a)   $\omega_{,CC} + \omega_{,C}\gamma^C{}_{DD} + e^{-2\omega}(\phi^C\phi_C) = 0$ .


(b)   $\phi_{,CC} + \phi_{,C}\gamma^C{}_{DD} - 2\omega^C\phi_C = 0$ .


Finally, we will find it convenient to use the Riemann tensor of the $V_4$ expressed in terms of the twist potential $\phi$. A separate consideration of subcases is needed: let (ABC) ≠ in the following, A, B, C not summed.


(3.21)    $^{(4)}R^4{}_{BC4} = \frac{1}{2} e^\omega(\omega_{\|BC} + \frac{3}{2} \omega_B\omega_C - \frac{1}{2} e^{-2\omega}\phi_B\phi_C)$ .

$$^{(4)}R^4{}_{BB4} = \frac{1}{2} e^\omega (\omega_{\|BB} - \frac{1}{2}(\omega_A{}^2 + \omega_C{}^2) + \omega_B{}^2 + \frac{1}{2} e^{-2\omega}(\phi_A{}^2 + \phi_C{}^2)),$$

(B not summed).

$$^{(4)}R^A{}_{BC4} = - \frac{1}{2}[\phi_{\|CC} - \omega_A \phi_A - \omega_B \phi_B],$$

$(\varepsilon_{ABC} = + 1)$.

$$^{(4)}R^A{}_{BA4} = \mp \frac{1}{2}[\phi_{\|CA} + \frac{1}{2} \omega_A \phi_C + \frac{1}{2} \omega_C \phi_A],$$

$(\varepsilon_{ABC} = \pm 1)$.

$$^{(4)}R_{ABAB} = \frac{1}{2} e^\omega [- \omega_{\|CC} + \frac{1}{2}(\omega_A{}^2 + \omega_B{}^2) - \omega_C{}^2 - \frac{1}{2} e^{-2\omega}(\phi_A{}^2 + \phi_B{}^2)],$$

$(\varepsilon_{ABC} = + 1)$.

$$^{(4)}R_{ABAC} = \frac{1}{2} e^\omega [\omega_{\|BC} + \frac{3}{2} \omega_B \omega_C - \frac{1}{2} e^{-2\omega} \phi_B \phi_C].$$

The symmetries of the Vacuum Riemanh tensor demand that the following identities be satisfied:

$$^{(4)}R_{ABAB} = - {}^{(4)}R_{C4C4}$$
$$^{(4)}R_{ABAC} = {}^{(4)}R_{B4C4}$$
$$^{(4)}R_{ABA4} = - {}^{(4)}R_{CBC4}$$
$$^{(4)}R_{N4N4} = 0 \quad (N \text{ summed})$$

a glance tells us that these are, indeed, satisfied by (3.21).

4. Spinors and complexification of the field equations.

The spinors arise in the theory of relativity because the group Sl(2,C) is a double-valued representation of the proper, isochronous Lorentz group $L^+_+$. To be precise, considering Sl(2,C) (or more exactly something isomorphic to it) as a subgroup of Gl(4,R), it is the universal covering space of $L^+_+$. To take advantage of this fact one can introduce, locally at least, a <u>spinor structure</u> on the space-time M (Geroch [8] and [9]; Lichnerowicz [7]). This can be introduced in two ways. In both we assume M is time- and space-oriented, and the bases concerned are likewise oriented.

For the first approach, recall that, given a principal fiber bundle P(M,G), if H is a maximal compact subgroup of G then G may be reduced to H. Here, Sl(2,C) is not compact; it turns out that the reduction of Gl(4,R), the structural group of the bundle bases, to Sl(2,C) is only possible under certain conditions on M. These conditions are mentioned in connection with the second approach. If such a reduction is possible, we may identify the reduced principal fiber bundle S(M) as the spinor structure. This approach, in its abstractness, may be said to slight the representational advantages of spinors.

The second approach was that actually used by Geroch; he started from OT(M) and went "up" to S(M). The situation is

slightly different in that OT(M) cannot be obtained from S(M) by a reduction of the group, strictly speaking. Now the definition: A _spinor structure_ on M is a principal fiber bundle S(M ,Sl(2,C)) or S(M) with group Sl(2,C) over M, along with a 2-1 mapping $\sigma:S(M) \to OT(M)$, such that

> (i) $\sigma$ maps each fiber of S(M) onto a single fiber of OT(M),
>
> (ii) $\sigma$ commutes with the group operations: for $U \in Sl(2,C)$, $\sigma \cdot U = \Lambda(U) \cdot \sigma$ where $\Lambda:Sl(2,C) \to L^+_+$ is the covering mapping of the restricted Lorentz group.

A _spinor_ at $m \in M$ is a mapping from $\pi^{-1}(m)$ (where $\pi:S(M) \to M$ is the natural projection) into arrays of complex numbers $\xi^{A\cdots\dot{B}\cdots}_{C\cdots\dot{D}\cdots}$ such that, if $v,w \in \pi^{-1}(m)$ are related by $U^A_B \in Sl(2,C)$, then

$$\xi^{A\cdots\dot{B}\cdots}_{C\cdots\dot{D}\cdots}(w) = U^A_E\cdots\bar{U}^{\dot{B}}_{\dot{F}}\cdots(U^{-1})^G_C\cdots(\bar{U}^{-1})^{\dot{H}}_{\dot{D}}\cdots\xi^{E\cdots\dot{F}\cdots}_{G\cdots\dot{H}\cdots}(v) .$$

One can now define a _spinor field_ to be a $C^\infty$ piecing together of these spinors; alternatively it is a section of some associated tensor bundle to S(M). If OT(M) has a semi-Riemannian connection then S(M) may be defined so the induced connection $\sigma^*\bar{\omega}_{ij}$ is semi-Riemannian. Geroch [9] has shown that M admits a spinor

structure iff M is parallelizable. We will never have to worry about this because, in solving the field equations, we always work on a parallelizable submanifold of M.

Notice that $\sigma$ is a bundle map, but not quite a reduction of the structural group, since it fails to be one-to-one. In the notation of Chapter 1, our bundle map

$$f: S(M, Sl(2,C)) \to OT(M, L^+_+)$$

is given by

$$f_S = \sigma$$

$$f_M = \text{identity}$$

$$f_G = \Lambda, \quad \text{the covering map of } Sl(2,C) \to L^+_+ ,$$

the following conditions being satisfied:

$$f_M \circ \pi = \pi^\nabla \circ \sigma \quad (\text{where } \pi^\nabla : OT(M) \to M)$$

$$\sigma \circ R_U = R_{\Lambda(U)} \circ \sigma \quad \text{for } U \in Sl(2,C).$$

We can obtain a more concrete representation of $\sigma$. Fix

a section $\{(m;\zeta_1,\zeta_2)\} \subseteq S(U)$ for some open set $U \subseteq M$, and let

the image under $\sigma$ of this section be $\{(m;\lambda_{(1)}, \ldots, \lambda_{(4)}\} \subseteq$

$\sigma T(U)$. We denote these sections by $(\zeta_a)$ and $(\lambda_H)$,

respectively. Then the action of $\sigma$ on any other section over

$U$, $(\zeta_a^\nabla) \subseteq S(U)$, where $\zeta_a^\nabla = U_a{}^b \zeta_b$ and $U_a{}^b \in Sl(2,C)$ is given

by

$$\sigma(\zeta_a^\nabla) = R_{\Lambda(U)} \bullet \sigma(\zeta_a) \; .$$

Upon converting this into numerical relations between the

components of $\zeta_a^\nabla$ and $\sigma(\zeta_a^\nabla) = (\lambda_H^\nabla)$ relative to the bases

$(\zeta_a)$ and $(\lambda_H)$, respectively, we obtain the well-known

connecting quantities of Van der Waerden (Bade and Jehle [33]).

In lieu of a proof, we only cite the well-known correspondence

between $L^+_+$ and $Sl(2,C)$ using these connecting quantities [33]:

$$(4.1) \qquad \sigma_H{}^{c\dot{d}} \Lambda_K{}^H = \sigma_K{}^{ab} U_a{}^c \bar{U}_b{}^{\dot{d}} \; .$$

Here, $\Lambda_K{}^H \in L^+_+$ represents the components of $\sigma(\zeta_a^\nabla)$ relative to

$(\lambda_K)$ and $U_a{}^C \in Sl(2,C)$ gives the components of $(\zeta_a^\nabla)$ relative

to $(\zeta_a)$. This provides us _ipso_ _facto_ with a mapping $\sigma$ with all

the right properties. The $\sigma_H{}^{ab}$ are four constant hermitian

matrices for $H = 1,2,3,4$. We may note that these are the analogues

in "invariants" of the $\sigma_i{}^{ab}$ of Newman and Penrose [13], not of

their $\sigma_i^{A\dot{B}}$.

We do not need the full spinor structure in our analysis of stationary space-times. The stationarity condition picks out a timelike killing vector field, which if we regard as fixed induces a contraction of the usual Lorentz group to the rotation group of $R^3$. In the spin space this corresponds to a contraction $Sl(2,C) \rightarrow SU(2)$, the group of unimodular, unitary $2 \times 2$ complex matrices [33]. In order to parallel the work of Chapter IV in spinorial terms we would have to invent a bundle map $\tau$ from $S(M,Sl(2,C))$ to a new bundle $S^\nabla(V_3,SU(2))$ with structure group $SU(2)$ and base space the associated $V_3$, paying due attention to the respective connections. This is essentially what Perjes [14] has done, although a more elegant formulation than his could perhaps be provided by the spinor forms of Bichteler [34].

However, we have already done the work in real terms; all that is necessary is to define a new <u>spinor</u> <u>structure</u> on the $V_3$ in a way analogous to that for the $V_4$. That is, we work "up" from $OT(V_3)$ (the bundle of oriented orthonormal <u>triads</u>) to $S^\nabla(V_3,SU(2))$. Now the group $SU(2)$ is the simply connected covering group of $SO(3,R)$; we have

$$SU(2)/Z_2 \cong SO(3,R)$$

so the mapping $\sigma: S^\nabla(V_3) \rightarrow OT(V_3)$ which defines the spinor structure

will again be a 2-1 mapping. Proceeding in a way analogous to the above, we arrive at <u>connecting quantities</u> $\sigma_H{}^{\dot{a}\dot{b}}$ satisfying

$$(4.2) \qquad \sigma_H{}^{\dot{c}\dot{d}}\Omega_K{}^H = \sigma_K{}^{\dot{a}\dot{b}}V_a{}^{\dot{c}}\bar{V}_b{}^{\dot{d}}$$

where $\Omega_K{}^H \in SO(3,R)$ and $V_a{}^c \in SU(2)$. $\sigma_H{}^{\dot{a}\dot{b}}$ are the <u>Pauli matrices</u>. Since $V \in SU(2)$ must be unitary, $\bar{V}_b{}^{\dot{d}} = V_d{}^b$ and by defining a new set of $\sigma$'s we can rewrite the above equation in the form

$$\sigma_H{}^{cd}\Omega_K{}^H = -\sigma_K{}^{ab}V_a{}^c V_b^d{}_b$$

where $\sigma_H{}^{ab}$ are certain <u>symmetric</u> connecting quantities first defined in their coordinate form by Perjes [14]. However, we will stick with the previous formulation in terms of Pauli matrices.

Now the whole complex vectorial formalism follows naturally. First of all $\sigma$ may also be considered as a mapping from hermitian spinors into vectors (linear, in this case). We may identify the preimage of $\sigma$ in this interpretation with complex combinations of vectors by the recipe

$$(4.3) \qquad \lambda_H \sigma^H_{a\dot{b}} = \lambda_{a\dot{b}} = \begin{cases} \lambda_{(1)} - i\lambda_{(2)} \\ \\ \lambda_{(1)} + i\lambda_{(2)} \\ \\ \lambda_{(3)} \end{cases}$$

Having placed the complex methods which we will use in the larger context of the spinor calculus, and having shown their essential simplicity, we now proceed with the "complexification" of the field equations.

We choose the frame

$$\begin{cases} \lambda_{(1)} \\ \\ \Lambda_{(0)} = 1/\sqrt{2}(\lambda_{(2)} + i\lambda_{(3)}), \ \Lambda_{(\dot{0})} = 1/\sqrt{2}(\lambda_{(2)} - i\lambda_{(3)}) \ . \end{cases}$$

Complex Ricci rotation coefficients may be defined in the obvious way. We denote them as follows:

$$(4.4) \qquad \alpha = \Gamma_{101}, \ \beta = \Gamma_{100}, \ \gamma = \Gamma_{1\dot{0}0}$$

$$\delta = \Gamma_{0\dot{0}1}, \ \varepsilon = \Gamma_{0\dot{0}0} \ .$$

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Here, $\alpha$, $\beta$, and $\gamma$ have the same meaning as in Chapter 2 when written in terms of $\gamma_{ABC}$'s.

We may employ the frame-covariance of $R_{ABCD}$ to complexify the right-hand sides of equations (3.18); for example,

$$2R_{1010} = R_{1212} - R_{1313} + i(R_{1213} + R_{1312}) \ .$$

For the left-hand sides we use the complex version of the definition of the Riemann tensor,

(4.5) $$\frac{1}{2} R_{ABCD} = \Gamma_{AB[C,D]} + \eta^{NM}\Gamma_{ABM}\Gamma_{N[CD]} + \eta^{NM}\Gamma_{AM[C}\Gamma_{|NB|D]}$$

where

$$\eta^{NM} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which may also be obtained from abstract spinorial considerations.

The commutation relations (2.8) for invariant differentiation, in the complex version, are

(4.6) $$f_{\|AB} - f_{\|BA} = 0, \quad A,B = 0,\dot{0},1$$

where

$$f_{\|AB} = f_{,AB} + \eta^{NM} f_{,N} \Gamma_{MAB} \, .$$

Defining $F = \frac{1}{2}(e^{\omega} + i\phi)$ (differing by a factor of $1/2$ from Kloster, Som and Das [35]) the result is

($\mathcal{F}$)

(1) $\quad \alpha_{,(0)} - \beta_{,(1)} - \alpha(\alpha + \varepsilon) - \beta(\gamma + \overline{\gamma} - 2\delta) = 2e^{-2\omega} F_{,(0)} \overline{F}_{,(0)}$

(2) $\quad \overline{\gamma}_{,(1)} - \alpha_{,(\dot{0})} + |\alpha|^2 + |\beta|^2 + \overline{\gamma}^2 - \alpha\overline{\varepsilon} = - e^{-2\omega} F_{(1)} \overline{F}_{(1)}$

(3) $\quad \beta_{,(\dot{0})} - \overline{\gamma}_{,(0)} + \alpha(\overline{\gamma} - \gamma) + 2\beta\overline{\varepsilon} = e^{-2\omega}(F_{(1)}\overline{F}_{(0)} + \overline{F}_{(1)}F_{(0)})$

(4) $\quad \varepsilon_{,(\dot{0})} + \overline{\varepsilon}_{,(0)} + \delta(\overline{\gamma} - \gamma) + 2|\varepsilon|^2 - |\beta|^2 + |\gamma|^2$

$$= e^{-2\omega}(F_{(1)}\overline{F}_{(1)} - F_{(0)}\overline{F}_{(\dot{0})} - \overline{F}_{(0)}F_{(\dot{0})})$$

(5) $\quad \varepsilon_{,(1)} - \delta_{,(0)} + \alpha(\delta + \gamma) + \varepsilon(\gamma - \delta) - \beta(\overline{\alpha} + \overline{\varepsilon})$

$$= -e^{-2\omega}(F_{(1)}\overline{F}_{(0)} + \overline{F}_{(1)}F_{(0)})$$

(6) $\quad F_{,(1)(1)} + 2F_{,(0)(\dot{0})} + 2\overline{\gamma}F_{,(1)} + (2\overline{\varepsilon} - \overline{\alpha})F_{(0)} - \alpha F_{(\dot{0})}$

$$= 2e^{-\omega}(F_{(1)}{}^2 + 2F_{(0)}F_{(\dot{0})})$$

(𝔐)   1. $f_{,(1)(0)} - f_{,(0)(1)} = \alpha f_{(1)} + (\gamma - \delta) f_{,(0)} + \beta f_{,(\dot{0})}$

2. $f_{,(0)(\dot{0})} - f_{,(\dot{0})(0)} = (\gamma - \bar{\gamma}) f_{,(1)} - \bar{\varepsilon} f_{,(0)} + \varepsilon f_{,(\dot{0})}$ .

The first three equations (intentionally grouped) it may

be noticed, involve derivatives of $\alpha$, $\beta$, and $\gamma$ only, on the

left. This is related to the fact that these quantities are of

geometric significance for the congruences of $\lambda_{(1)}$. We recall

that $|\alpha|$ = first curvature, $|\beta|$ = shearing, and $\gamma$ = dilatation

of the congruences of $\lambda_{(1)}$. The question arises as to whether the

equations (4) and (5) are of similar import for the "congruences"

of $\Lambda_{(0)}$, in an appropriately generalized sense.

We may mention that we need not stop at the level of the

orthonormal frame in our complexification. We can introduce com-

plex variables, in which the invariant derivatives take the form

$$\lambda_{(1)}{}^{\alpha} = U^{-1} \delta_1{}^{\alpha} + \eta \delta_z{}^{\alpha} + \bar{\eta} \delta_{\dot{z}}{}^{\alpha}$$

$$\Lambda_{(0)}{}^{\alpha} = \phi \delta_1{}^{\alpha} + \rho \delta_z{}^{\alpha} + \sigma \delta_{\dot{z}}{}^{\alpha} .$$

In order to investigate general questions in the

classification of stationary space-time, we construct the complex

self-dual combinations of the Riemann Tensor. They are, very simply,

$$(4.8) \qquad 2\tilde{R}_{ab} = \overline{F}_{\|AB} + \frac{1}{2}(\text{ReF})^{-1}\overline{F}_{,A}\overline{F}_{,B} - \frac{1}{2}(\text{ReF})^{-1}\Delta_1\overline{F}\delta_{AB}$$

where A,B take 1,2,3. The potential equations are (N summed over 1,2,3)

$$(4.9) \qquad \overline{F}_{\|NN} - (\text{ReF})^{-1}\overline{F}_N\overline{F}_N = 0. \quad [1]$$

We can easily write down another version using the complex frame $\{\lambda_{(1)}, \Lambda_0, \Lambda_{\dot{0}}\}$.

$$(4.10) \qquad 2\tilde{R}_{1\dot{1}} = \overline{F}_{\|(1)(1)} - \frac{1}{2}(\text{ReF})^{-1}\overline{F}_0\overline{F}_{\dot{0}}$$

$$2\tilde{R}_{10} = \overline{F}_{\|10} + \frac{1}{2}(\text{ReF})^{-1}\overline{F}_1\overline{F}_0$$

$$2\tilde{R}_{00} = \overline{F}_{\|00} + \frac{1}{2}(\text{ReF})^{-1}\overline{F}_0\overline{F}_0$$

$$2\tilde{R}_{0\dot{0}} = -2\tilde{R}_{1\dot{1}}$$

For $\tilde{R}_{1\dot{0}}, \tilde{R}_{\dot{0}\dot{0}}$ just replace $0$ by $\dot{0}$ in $\tilde{R}_{10}, \tilde{R}_{00}$ respectively.

---

[1] With the alternative definition of the self-dual combinations, mentioned in Chapter 2, $\overline{F}$ is replaced by F in equations (5.8) and (5.9).

5.  Consequences of the field equations.

In this chapter we list some simple observations and
techniques dealing with the bare equations derived in the previous
sections, which will allow us to classify the example of the next
chapter with a minimum of complications.

In general we are particularly interested in spaces of
nondegenerate algebraic types  I, II and III; type I for physical
reasons, because it is the most general type of vacuum space-time,
and all three types because they determine their principal tetrads
uniquely.  We may use this latter property to develop an invariant
approach to asymptotic flatness and the presence of other motions.
We can go about this in two ways.

The most obvious route is to transform to the principal
tetrad.  The corresponding Ricci rotation coefficients then supply
us with an abundance of invariant functions.  These must all go to
zero at infinity in order for the space to be asymptotically flat.
A more convenient, necessary condition for space-times outside the
P.E. class is to use coordinates  $\omega = \chi^1$, $\phi = \chi^2$; i.e., the
gravitational potential and twist potential, and an appropriate
invariant for  $\chi^3$; then all invariants and their derivatives must
vanish at the origin for asymptotic flatness to obtain.  (The con-
dition is only necessary because the space may instead become flat
at some finite point.)  For Petrov types II and III it may be more

appropriate for the invariants to approach the values for the

corresponding spaces of maximum mobility (Petrov [30]). Any group

of motions must manifest itself as the invariance group for this

set of functions, making its determination easier than would be

the case with the classical theorem [30].

The second route, which has a chance of giving a complete

answer only for spaces of type I, is to investigate the orthogonal

invariants of the matrix of the self-dual Riemann tensor $\widetilde{R}_{ab}$.

These are defined as follows: denoting $(\widetilde{R}) = (\widetilde{R}_{ab})$,

$$(5.1) \qquad I_{\widetilde{R}} = \operatorname{tr} \widetilde{R}$$

$$II_{\widetilde{R}} = \frac{1}{2}(\operatorname{tr}(\widetilde{R}^2) - (\operatorname{tr} \widetilde{R})^2)$$

$$III_{\widetilde{R}} = \det \widetilde{R} .$$

they will be invariant under changes of basis defined by transforma-

tions in $SO(3,C)$, which of course suffice to bring us to the

principal tetrad. In vacuum space-time $I_{\widetilde{R}} = 0$ always, so only

$II_{\widetilde{R}}$ and $III_{\widetilde{R}}$ remain; four real functions in the most general

cases. In spaces of type O, N, and III, all the invariants

vanish; in spaces of type II and D, there are two real functions,

and in type I, four. These easily obtained invariants can yield

important necessary conditions for asymptotic flatness and the

existence of motions.  It may be noted that they are algebraically

related to the well-known second-order invariants formed from the

metric tensor and curvature tensor [11], of which are at most four

in empty space-time.

It will be noticed that these orthogonal invariants

enable us to distinguish between spaces of type  D  and  III,

which have the same number of principal null directions.  The

algebraic classification can then be completed by finding the total

number of principal null directions defined by the space.  This is

most easily done as follows.  Beginning from a tetrad

$\{\Lambda_{(1)}, \ldots, \Lambda_{(4)}\}$  adapted to the timelike killing vector field,

with  $\lambda_{(4)}$  tangent to the trajectories, we define the null tetrad

$$k = 1/\sqrt{2}(\Lambda_{(1)} + \Lambda_{(4)}), \quad \ell = 1/\sqrt{2}(-\Lambda_{(1)} + \Lambda_{(4)})$$

$$t = 1/\sqrt{2}(\Lambda_{(2)} + i\Lambda_{(3)}).$$

The "null rotation" of Sachs [31], defined by

(5.2)      $\hat{k} = k - \alpha t - \bar{\alpha}\bar{t} + \alpha\bar{\alpha}\ell$

$\hat{\ell} = \ell \qquad \hat{t} = t - \bar{\alpha}\ell$

for  $\alpha$  complex, maps null tetrads into null tetrads.  By applying

such null rotations we can map $k$ into a principal null direction $\hat{k}$. The condition for $\hat{k}$ to be a principal null direction is that $\hat{R}^{(1)} = 0$; i.e., $\tilde{R}_{ab}\hat{e}_0{}^a\hat{e}_0{}^b = 0$, which translates into a quartic equation for the complex parameter $\alpha$:

$$(5.3) \qquad 0 = R^{(1)} + 2\sqrt{2}\bar{\alpha}R^{(2)} + 3\bar{\alpha}R^{(3)} - 2\sqrt{2}\bar{\alpha}^3 R^{(4)} + \bar{\alpha}^4 R^{(5)}.$$

The number of roots of this equation is equal to the number of principal null directions. This equation also offers a method for the construction of the principal null directions, by the specification (5.2).

Another way of classifying stationary spaces is by the geometry of the associated $V_3$. There is an inherent difficulty in this type of classification, in that if there is more than one killing vector field there are in fact an infinite number, with different associated $V_3$'s. This difficulty aside, we find it useful to distinguish three special congruences of curves in the associated $V_3$: The lines of force are the normals to the equipotential surfaces $\omega$ = constant; the lines of twist are the normals to surfaces of constant twist potential, $\phi$ = constant; and the eigenrays, first studied by Perjes [14], are defined as the trajectories of $\lambda_{(1)}$ in an orthonormal triad $\{\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}\} \leftrightarrow \{\lambda_{(1)}, \Lambda_{(0)}\}$ satisfying

$$\overline{F}_{,(0)} = 0$$

or, equivalently,

$$(5.4) \qquad \begin{cases} \omega_{,(2)} = -e^{-\omega}\phi_{,(3)} \\[2ex] \omega_{,(3)} = e^{-\omega}\phi_{,(2)} \ . \end{cases}$$

To every curve in $V_3$ may be associated a null curve of $V_4$ by the mapping of unit tangents $\lambda_{(1)} \to \lambda_{(1)} + \Lambda_{(4)}$, where $\Lambda_{(4)}$ is along the timelike killing vector field. It thus becomes of significance for the algebraic classification to express the geometric properties of null trajectories (in $V_4$) in terms of the properties of corresponding curves of $V_3$. To this end we may use the formulae (3.6) for the Ricci rotation coefficients of $V_3$ and $V_4$. Let $\{k, \ell, t\}$ be a null triad as above, and $\{\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}\}$ an orthonormal triad in $V_3$ codirectional with $\Lambda_{(1)}, \Lambda_{(2)}, \Lambda_{(3)}$. Then, referring to Israel [4], the null curves with tangent $k^\alpha$ are:

$$(5.5) \qquad \text{geodesics iff } \tilde{\alpha} = k_{\alpha|\beta} t^\alpha k^\beta = 0$$

$$\text{shear-free iff } \tilde{\beta} = k_{\alpha|\beta} t^\alpha t^\beta = 0$$

$$\text{complex-dilatation-free iff } \tilde{\gamma} = k_{\alpha|\beta} \overline{t}^\alpha t^\beta = 0 \ .$$

If the null curves are geodesics, the real and imaginary parts of $\widetilde{\gamma}$ have the significance of divergence and twist, just as is the case for $\gamma$ in $V_3$. Thus if $\widetilde{\gamma}$ is real for geodesic null curves, the curves are normal to (hence tangent to) null hypersurfaces [37]. We find:

(5.6)    (a)  $\widetilde{\alpha} = \frac{1}{2} e^{\omega/2}\alpha + e^{-\omega/2}\overline{F}_0$

(b)  $\widetilde{\beta} = 1/\sqrt{2} \ e^{\omega/2}\beta$

(c)  $\widetilde{\gamma} = 1/\sqrt{2} \ e^{\omega/2}\gamma - 1/\sqrt{2} \ e^{-\omega/2}F_{,(1)}$ .

We take note of a number of relationships:

(a)  $\Lambda_{(1)} + \Lambda_{(4)}$ is shear-free iff $\lambda_{(1)}$ is in the $V_3$.

(b)  If $\lambda_{(1)}$ is chosen along the eigenrays, $\Lambda_{(1)} + \Lambda_{(4)}$ is geodesic iff $\lambda_{(1)}$ is.

(c)  If the eigenrays are geodesic and shear-free, the space is algebraically special.

(d)  By a fortunate choice of terminology, the amount of twisting in $\lambda_{(1)}$'s direction, in the sense of $\phi_{,\alpha}\lambda^{\alpha}{}_{(1)}$ , supplies the disparity between the "twist" of $\lambda_{(1)}$ in $V_3$ and $\Lambda_{(1)} + \Lambda_{(4)}$ in $V_4$. Similarly the gravitational potential $\omega$ is related

to the respective divergences.

Property (d) may admit of a convenient physical interpretation in terms of stationary observers and signals travelling with the fundamental velocity.

The eigenrays have an even closer relationship to the principal null directions. We have indicated that the congruences of $\Lambda_{(1)} + \Lambda_{(4)}$ are along a principal null direction iff $\tilde{R}_{00} = 0$, that is iff

$$\overline{F}_{\|00} + \frac{1}{2}(\mathrm{Re}F)^{-1}\overline{F}_0\overline{F}_0 = 0.$$

If $\lambda_{(1)}$ is chosen along the eigenrays, and the eigenrays are shear-free, this equation is identically satisified. Thus shear-free eigenrays correspond to principal null directions.

It may be noted that in the static and Papapetrou-Ehlers class ($\omega$ and $\phi$ functionally related) spaces, the eigenrays coincide with the lines of force. Also, one could define a similar set of "eigenrays" for static electrovac, and generalizations in the stationary Einstein-Maxwell fields. It would be interesting to see the corresponding geometrical relationships in these cases.

6. Case of the eigenvalues of the Ricci subtensor equal.

The class of stationary gravitational universes which we deal with in this chapter can be solved up to a pair of simple partial differential equations, which hold out the promise of further analysis. The problem is a "natural" one, in view of the significance of other specializations of the Ricci subtensor: The case of only one nonzero eigenvalue corresponds either to the static space-time, with vanishing twist potential, or Papapetrou-Ehlers class, in which the lines of twist coincide with the lines of force.

The field equations in invariant form are

$$R_{AB} = -\frac{1}{4}(\mathrm{ReF})^{-2}(F_A \overline{F}_B + \overline{F}_A F_B) .$$

The determinental equation is (Ericksen [36])

$$0 = \det(\lambda\delta_{AB} - R_{AB}) = \lambda^3 + I(-R_{AB})\lambda^2 + II(-R_{AB})\lambda + III(-R_{AB})$$

where I, II, III denote the orthogonal invariants defined in Chapter 6. As roots we find

$$\lambda_1 = 0$$

$$\lambda_2 = -\frac{1}{4}(\mathrm{ReF})^{-2}(\Delta_1(F,\overline{F}) + |\Delta_1 F|)$$

$$\lambda_3 = -\frac{1}{4}(\mathrm{ReF})^{-2}(\Delta_1(F,\overline{F}) - |\Delta_1 F|)$$

Thus $\lambda_2 = \lambda_3$ iff $\Delta_1 F = 0$ iff $\Delta_2 F = 0$, the latter a result of the potential equation (4.9). This results in

$$(6.1) \qquad\qquad \Delta_1(\omega, \phi) = 0$$

$$e^{2\omega} \Delta_1 \omega = \Delta_1 \phi \ .$$

Hence the lines of twist are normal to the lines of force.

Choose $\lambda_{(2)}$ along the lines of force and $\lambda_{(3)}$ along the lines of twist. Then it follows from (6.1) that

$$e^{2\omega} \omega_{,(2)}^2 = \phi_{,(3)}^2 \ .$$

Choose the orientation so that $e^{\omega} \omega_{,(2)} = - \phi_{,(3)}$. Since $e^{\omega} \omega_{(3)} = \phi_{,(2)}$ automatically, it follows that

$$\bar{F}_{,(0)} = 0 \quad \text{where} \quad \Lambda_0 = 1/\sqrt{2}(\lambda_{(2)} + i\lambda_{(3)})$$

so $\lambda_{(1)}$ is along the eigenrays. We have, as well, $F_{,(1)} = 0$.

Putting $F$ and $\bar{F}$ into the metric equations $\mathcal{M}1$ and $\mathcal{M}2$, and using $\mathcal{F}6$, we get:

from $\mathcal{M}2$:  $F_{,0\dot{0}} = -\overline{\epsilon}F_0$ .  Putting this in

$\mathcal{F}6$:  $-2\overline{\epsilon}F_0 + (2\overline{\epsilon} - \overline{\alpha})F_0 = 0 \Leftrightarrow \overline{\alpha}F_0 = 0$ .

$\mathcal{M}1$:  $\beta\overline{F}_{\dot{0}} = 0$ .

Since  $F_0 \neq 0$,  $\alpha = \beta = 0$,  hence the eigenrays are geodesic and shear-free, and the space is algebraically special.

From the fact that  $\lambda_{(2)}$  and  $\lambda_{(3)}$  are normal to surfaces, we have

$$\delta = \frac{1}{2}(\gamma - \overline{\gamma}) .$$

From  $\mathcal{F}3$,  $\overline{\gamma}_0 = 0$  so  $\delta_0 = \frac{1}{2}\gamma_0$.  Since  $0 = (F - \overline{F})_0 + (F - \overline{F})_{\dot{0}} = F_0 - \overline{F}_{\dot{0}}$ ,  $F_0$  is real and we may put  $F_0 = Q$.  Since  $\omega$  and  $\phi$  are functionally independent we can choose complex coordinates  $\{x^1, z, \overline{z}\}$  with  $F = z$,  leaving  $x^1$  free.  These results imply that the frame takes the form

$$\lambda_{(1)}{}^\alpha = P\delta_1{}^\alpha, \ \Lambda_0{}^\alpha = \eta\delta_1{}^\alpha + Q\delta_z{}^\alpha , \quad P \text{ and } Q \text{ real.}$$

And the reduced field equations become

$\mathcal{I}^{\nabla}$)   (a)   $\gamma_{,(1)} + \gamma^2 = 0$

(b)   $\gamma_{,\dot{0}} = 0$

(c)   $\gamma_{,0} - 2\varepsilon_{,(1)} = \varepsilon(\gamma + \bar{\gamma})$

(d)   $\varepsilon_{,\dot{0}} + \bar{\varepsilon}_{,0} + 2|\varepsilon|^2 + 2|\gamma|^2 - \frac{1}{2}(\gamma^2 + \bar{\gamma}^2) + \frac{Q^2}{(z + \bar{z})^2} = 0$

$\mathcal{M}^{\nabla}$)   (e)   $P_{,0} - \eta_{,(1)} = \frac{1}{2}(\gamma + \bar{\gamma})\eta$

(f)   $Q_{,(1)} + \frac{1}{2}(\gamma + \bar{\gamma})Q = 0$

(g)   $\eta_{,\dot{0}} - \bar{\eta}_{,0} = (\gamma - \bar{\gamma})P - \eta\bar{\varepsilon} + \bar{\eta}\varepsilon$

(h)   $Q_{,\dot{0}} = -\bar{\varepsilon}Q$ .

With our choice of frame the metric is

$$ds^2 = \frac{1}{P^2}\left(\frac{\eta dz}{Q} + \frac{\bar{\eta}d\bar{z}}{Q} - dx^1\right)^2 + \frac{2}{Q^2}dzd\bar{z} .$$

We may note that the case of $\gamma$ imaginary is ruled out by $\mathcal{I}^{\nabla}$(a), but $\gamma$ real is not. This is an example of the more general theorem in any space-time $M$ to the effect that, if $K^i$ is tangent to a geodesic and shear-free null congruence with

$^{(4)}R_{ij}K^iK^j = 0$, then $\text{Re}\widetilde{\gamma} = 0 \Rightarrow \text{Im}\widetilde{\gamma} = 0$ [37]. If $\gamma$ is complex, then the eigenrays have twist; in view of $F_{,(1)} = 0$, so do the corresponding repeated principal null directions. At this point we divide into two subcases, according to whether or not $\gamma$ is zero.

Case I. $\gamma = 0$.

We note that this has as an immediate consequence that $\lambda_{(1)}$ is along the trajectories of a killing motion in the $V_3$.

By a transformation $x^1 \to \hat{x}^1$, $\hat{z} = z$, we may put $P = 1$. Then the field equations $(\mathcal{f}^\nabla)$, $(\mathcal{m}^\nabla)$ become

$(c^\nabla)$  $\varepsilon_{,1} = 0$

$(d^\nabla)$  $\varepsilon_{,0} + \overline{\varepsilon}_{,0} + 2|\varepsilon|^2 + \dfrac{Q^2}{(z + \overline{z})^2} = 0$

$(e^\nabla)$  $\eta_{,1} = 0$

$(f^\nabla)$  $Q_{,1} = 0$

$(g^\nabla)$  $\eta_{,0} - \overline{\eta}_{,0} = \overline{\eta}\varepsilon - \eta\overline{\varepsilon}$

$(h^\nabla)$  $Q_{,0} \rightleftharpoons \overline{\varepsilon}Q$ .

Equation $(h^\nabla)$ yields $\varepsilon = -Q_z$; substituting this into $(d^\nabla)$ gives an equation in $Q$ alone, which is transformable to

$$(\ell nQ)_{,z\bar{z}} = \frac{1/2}{(z + \bar{z})^2}$$

with the solution $Q = \dfrac{h(z)\bar{h}(\bar{z})}{(z + \bar{z})^{1/2}}$ , $h$ arbitrary. Equation $(g^\nabla)$ is transformable into

$$\left(\frac{\eta}{Q}\right)_{,\bar{z}} = \left(\frac{\bar{\eta}}{Q}\right)_{,z}$$

which implies $\dfrac{\eta}{Q} = f_{,z}$ for real $f$. We still have some freedom in $x^1$, namely the change of "gauge" $\hat{x}^1 = x^1 + \lambda(z,\bar{z})$. Put $\hat{x}^1 = x^1 - f$, $\hat{z} = z$, then

$$\hat{\eta} = \Lambda_0{}^\alpha \hat{x}^1{}_{,\alpha} = \eta - Qf_{,z} = 0 .$$

Thus, dropping hats, $\eta = 0$, $P = 1$, and the metric is

$$ds^2 = dx^1{}^2 + \frac{2(z + \bar{z})dzd\bar{z}}{h^2(z)\bar{h}^2(\bar{z})}$$

$$= (dx^1)^2 + (z + \bar{z})H\bar{H}dzd\bar{z} \quad \text{where} \quad H(z) = \frac{\sqrt{2}}{h^2(z)} .$$

The metric of the $V_4$ is transformable to

$$(6.2) \qquad \Phi = dx^2 + dy^2 + 2d\theta dt - \omega(x,y)dt^2 \quad \text{where}$$

$$\omega_{xx} + \omega_{yy} = 0 .$$

This is a special case of the well-known plane-fronted gravitational waves (with parallel rays), first found by Brinkman [38] and described in detail by Ehlers and Kundt [37]. The general case can be recovered by an application of a result of Pechlaner and Das [39] (their theorem 3): the resulting metric form is

$$\Phi = dx^2 + dy^2 + 2d\theta dt - \omega(x,y,t)dt^2$$

where $\omega_{xx} + \omega_{yy} = 0$. Of course, this metric no longer falls into the subclass of stationary spaces with which we are concerned.

Case II. $\gamma \neq 0$.

We again choose $x^1$ to make $P = 1$: then, integrating $\mathcal{F}^\nabla(a)$, $\gamma^{-1} = x^1 + \psi(z,\bar{z})$ where $\psi$ is arbitrary. We are still allowed a change of gauge $\hat{x}^1 = x^1 + \lambda(z,\bar{z})$; we choose $\lambda = \frac{1}{2}(\psi + \bar{\psi})$. Recalling that $\gamma$ is an invariant, we then have $\gamma^{-1} = x^1 + \tau(z,\bar{z})$ where $\tau$ is imaginary.

We have the following consequences:

From $\mathcal{F}^\nabla$(b), $\eta = Q\tau_{,z}$ .

From $\mathcal{M}^\nabla$(f), $Q = h|\gamma|$ where $h = h(z,\bar{z})$ is a real function.

From the preceding relations

$$\eta = h\tau_{,z}|\gamma|$$

which satisfies $\mathcal{M}^\nabla$(e) identically.

From $\mathcal{M}^\nabla$(h), $\varepsilon = \eta\gamma - h_{,z}|\gamma| = h|\gamma|(\tau_z\gamma - (\ell nh)_{,z})$ .

From $\mathcal{M}^\nabla$(g), after tedious calculations, we obtain the differential equation

(6.3)   $\tau_{,z\bar{z}} = -\dfrac{\tau}{h^2}$ .

Equation $\mathcal{F}^\nabla$(c) is identically satisfied by the above relations. Finally, after quite lengthy calculations, we get from $\mathcal{F}^\nabla$(d), with the help of (6.3), the "potential equation"

(6.4)   $\dfrac{h_{z\bar{z}}}{h} = \dfrac{h_z h_{\bar{z}}}{h^2} + \dfrac{1}{2h^2} + \dfrac{1}{2(z + \bar{z})^2}$ .

Making the substitution $V = -2\ell nh$, we can rewrite the system (6.3), (6.4) in the form

(6.5)   (a)  $V_{,z\bar{z}} = -e^V - \dfrac{1}{(z + \bar{z})^2}$

(b)  $\tau_{,z\bar{z}} = -\tau e^V$

which represents the reduced set of field equations.

Although at first glance our equations appear innocent enough, they may not admit of any closed-form solutions. However, from general considerations, there should be a functional dependence upon two arbitrary harmonic functions. A similar, soluble equation to (6.5) has appeared in static fields (Das [16]), but the $1/(z + \bar{z})^2$ term was missing. (We might say that the additional term in our equation represents "stationarity.") A generalization of (6.5a) has appeared in null Einstein-Maxwell fields (Trollope [40]); the substitution $W = V - \ln(z + \bar{z})$ brings us to a special case of equation (4.1) of that paper. These two points of contact with other types of space-time naturally suggest ways of generalizing the present work, but we cannot pursue that here.

However, we can make some progress in a special case. If $V$ is assumed to be a function of $z + \bar{z}$ alone, then (6.5a) turns into an ordinary differential equation. If we assume $\tau = \tau(z + \bar{z})$ as well, then we can explicitly solve for a metric depending on two arbitrary parameters (though the solutions may still not admit of a closed expression). A slightly more difficult (or tedious) problem results if we leave $\tau$ general.

In any case we may write down the metric form for the $V_3$ as it presently stands:

$$\Phi = (dx^1 + \tau_{\bar{z}}d\bar{z} - \tau_z dz)^2 + 2((x^1)^2 - \tau^2)e^V dz d\bar{z} .$$

## Petrov Classification.

We now carry out the algebraic classification for both cases. We have $\alpha = \beta = 0$, $F_0 = Q$, $F_{\dot{0}} = F_{(1)} = 0$. Substituting these into the formulae for the self-dual combinations of the Riemann Tensor (4.10), we have

$$2\widetilde{R}_{11} = 0, \quad \widetilde{R}_{10} = 0, \quad \widetilde{R}_{00} = 0$$

$$2\widetilde{R}_{1\dot{0}} = - Q\bar{\gamma}$$

$$2\widetilde{R}_{\dot{0}\dot{0}} = - 2Q\bar{\varepsilon} + \frac{Q^2}{z + \bar{z}}$$

The counterparts of these in the basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ are

$$\widetilde{R}_{22} - \widetilde{R}_{33} + 2i\widetilde{R}_{23} = 0$$

$$\widetilde{R}_{22} - \widetilde{R}_{33} - 2i\widetilde{R}_{23} = - Q\bar{\varepsilon} + \frac{Q^2}{2(z + \bar{z})}$$

$$\widetilde{R}_{22} + \widetilde{R}_{33} = 0$$

$$\widetilde{R}_{12} + i\widetilde{R}_{13} = 0$$

$$\widetilde{R}_{12} - i\widetilde{R}_{13} = - \frac{1}{2} Q\overline{\gamma}.$$

From whence the matrix $\widetilde{R}_{ab}$ takes the form

$$(\widetilde{R}_{ab}) = \begin{bmatrix} 0 & \Gamma & i\Gamma \\ \Gamma & \Lambda & i\Lambda \\ i\Gamma & i\Lambda & -\Lambda \end{bmatrix}$$

where for brevity we have defined

$$\Gamma = - \frac{Q\overline{\gamma}}{4}, \quad \Lambda = - \frac{Q\overline{\epsilon}}{4} + \frac{1}{8} \frac{Q^2}{z + \overline{z}}.$$

Since all the principal invariants of such a matrix vanish, the space-time in both cases must be of types $N$ or $III$.

A glance at the quartic equation (5.3) will tell us which type. We have $R^{(1)} = R^{(2)} = R^{(3)} = 0$ when $\gamma \neq 0$ and $R^{(1)} = \ldots = R^{(4)} = 0$ when $\gamma = 0$. Hence in case I ($\gamma = 0$), the space is type $N$, and in case II ($\gamma \neq 0$), the space is type $III$.

If $\nu$ is complex, in addition, that is $\tau \neq 0$, the rays have twist. According to Kinnersley [41] such a solution has not been found previously.

Discussion.

We have developed a portion of the theory of connections on principal fiber bundles and pointed out some of its applications to the theory of gravitation. It seems to offer the best way of understanding and generalizing such formalisms as the spinor calculus; in fact, any irreducible representation of the Lorentz group will give rise to an associated vector bundle with a natural concept of covariant differentiation. Also, it may be noted that the bundle of bases itself is essentially a phase space; this is shown in a concrete way by the calculations of Appendix A. It is natural to ask if we could profitably raise Einstein's equations to global ones on B(M) or to some associated tensor bundle.

The method of differential forms offers the best algorithm for computing the connection coefficients and curvature tensor in any frame. We have used the method in conjunction with orthonormal frames, but the same advantages are apparent in a context of spinor forms [34].

As to the physical significance of the subcase of stationary spaces which we have considered, we will have to wait for further (numerical or qualitative) analysis of our reduced equations. In the meantime there is the suggestion that it could be a physically meaningful generalization of the plane-wave solutions.

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Appendix A.  Some calculations.

1.  The vector fields $\varepsilon_{ki}$ are vertical.

Using the definition,

$$\pi_* \varepsilon_{ki} = \pi_* \circ (f_p)_* X_{ki}(e)$$

$$= (\pi \circ f_p)_* X_{ki}(e)$$

$$= 0 \ .$$

Since $\pi \circ f_p$ is the constant function on $Gl(n,R)$.

2.  Explicit constructions of $\varepsilon_i, \varepsilon_{ki}, \overline{\omega}_i, \overline{\omega}_{ki}$ .

Let $x_i = \overline{u}_i \circ \overline{\phi}$ and $x_{k\ell} = \overline{u}_{k\ell} \circ \overline{\phi} = \lambda_{(\ell)}{}^k$ be the coordinate functions on $\overline{U} \subseteq B(M)$, and $p \in \overline{U}$. Then

$$\varepsilon_{ij,k\ell}(p) \equiv [\varepsilon_{ij}(p)](x_{k\ell})$$

$$= [(f_p)_* (\partial/\partial A_{ij})(e)](x_{k\ell})$$

$$= [\partial/\partial A_{ij}(e)](x_{k\ell} \circ f_p)$$

$$= \frac{\partial}{\partial A_{ij}}(\lambda_{(m)}{}^k A_{m\ell})\Big|_e$$

$$= \lambda_{(i)}{}^k \delta_{j\ell}$$

Thus

$$(A.1) \qquad \varepsilon_{ij}(p) = \lambda_{(i)}{}^k \frac{\partial}{\partial \lambda_{(j)}{}^k} \quad .$$

For the components of $\bar{\omega}_i$ , let $X = b^i \partial/\partial x^i + b_i{}^k \partial/\partial \lambda_{(i)}{}^k$. Then $\pi_* X = b^i \partial/\partial x^i = b^k \lambda_k{}^{(i)} \lambda_{(i)}$ . Hence $\bar{\omega}_i(x) = b^k \lambda_k{}^{(i)}$ , and

$$(A.2) \qquad \bar{\omega}_i = \lambda_k{}^{(i)} dx^k = \lambda^{(i)} \quad \text{considered as a form.}$$

The connection one-forms $\bar{\omega}_{ij}$ must be dual to $\varepsilon_{ij}$ , so we may write

$$\bar{\omega}_{ij} = a_{ijk} dx^k + \lambda_k{}^{(i)} d\lambda_{(j)}{}^k \quad .$$

So far $a_{ijk}$ can be any functions of $x^i$ and $\lambda_{(i)}{}^k$, which in B(M) are independent variables. But property (1.5) (b)$^\nabla$ restricts us: From $R_A{}^* \bar{\omega}_{ij} = A^{-1}{}_{ir} A_{sj} \bar{\omega}_{rs}$ it follows that $a_{ijk}$ are "invariant" components in the first two indices--that is

$$a_{ijk} = \lambda_\ell{}^{(i)} \lambda_{(j)}{}^m \Gamma_m{}^\ell{}_k \quad \text{where} \quad \Gamma_m{}^\ell{}_k = \Gamma_m{}^\ell{}_k(x^i) \quad \text{are}$$
$$\text{arbitrary functions.}$$

So we have

$$(A.3) \qquad \overline{\omega}_{ij} = \lambda_{\ell}{}^{(i)}\lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{k}dx^{k} + \lambda_{k}{}^{(i)}d\lambda^{k}{}_{(j)} \; .$$

Finally, to calculate $\varepsilon_{k}$, let $\varepsilon_{k} = b_{k}{}^{n}\,\partial/\partial x^{n} + b_{\ell k}{}^{n}\,\partial/\partial\lambda_{(\ell)}{}^{n}$. Then $\{\varepsilon_{k}, \varepsilon_{ki}\}$ is the dual frame to $\{\overline{\omega}_{i}, \overline{\omega}_{ij}\}$ implies

$$0 = \overline{\omega}_{ij}(\varepsilon_{k}) = \lambda_{\ell}{}^{(i)}\lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{n}b_{k}{}^{n} + b_{jk}{}^{n}\lambda_{n}{}^{(i)}$$

$$\Leftrightarrow 0 = b_{jk}{}^{\ell} + \lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{n}b_{k}{}^{n}$$

and

$$\delta_{ij} = \overline{\omega}_{i}(\varepsilon_{j}) = b_{j}{}^{k}\lambda_{k}{}^{(j)}$$

$$\Leftrightarrow b_{j}{}^{k} = \lambda_{(j)}{}^{k} \; .$$

Therefore $b_{jk}{}^{\ell} + \lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{n}\lambda_{(k)}{}^{n} = 0$ and

$$(A.4) \qquad \varepsilon_{k} = \lambda_{(k)}{}^{n}\,\partial/\partial x^{n} - \lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{n}\lambda_{(k)}{}^{n}\,\partial/\partial\lambda_{(j)}{}^{\ell} \; .$$

A coordinate basis of $H_{p}$ is, since $\lambda_{(k)}{}^{n}$ is nonsingular

$$(A.5) \qquad \widetilde{\varepsilon}_{n} = \partial/\partial x^{n} - \lambda_{(j)}{}^{m}\Gamma_{m}{}^{\ell}{}_{n}\,\partial/\partial\lambda_{(j)}{}^{\ell} \; .$$

Appendix B.  Brief account of covariant differentiation in an
associated vector bundle.

The associated bundles were mentioned in passing in
Chapter 1.  But the important topic of covariant differentiation in
associated vector bundles was not touched upon.  We will avoid the
explicit definition of these fiber bundles, but remark that they
are characterized by having a vector space as typical fiber, and
cite as examples the well-known tangent bundle and the various tensor
bundles.

If  $\hat{B}(M)$  has a connection  $H$,  then any associated vector
bundle  $W$  has a naturally defined induced connection  $H^{\nabla}$.  (In the
case of the tangent bundle this makes possible the parallel trans-
port of vectors.)  Let  $X : U \to W$  be a section over  $U \subseteq M$,  and let
$t \in M_m$ ,  $m \in U$.  Then we define the <u>covariant derivative of  $X$  with
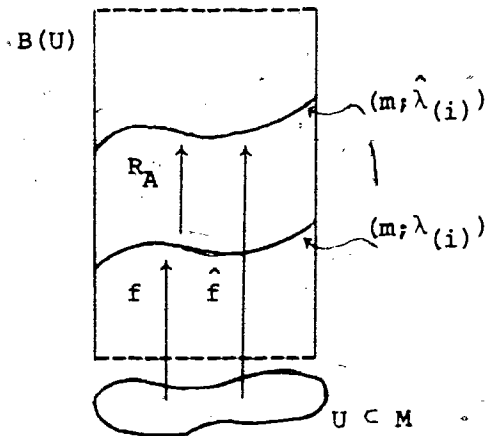respect to  $t$</u>  to be

$$D_t X = X_*(t) - H^{\nabla}(X_*(t)).$$
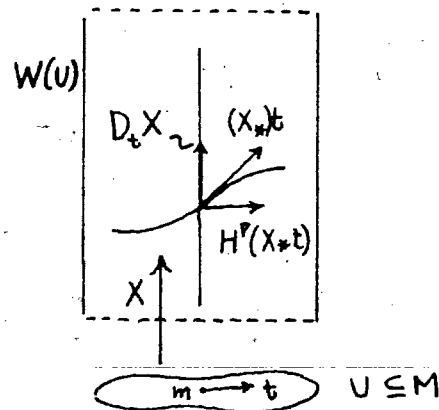


i.e., it is just the <u>vertical part</u> of  $X_*(t)$.  Since  $D_t X$  is

tangent to the fiber over  m  and this fiber is a vector space, it
may be identified with a point in  W.  Hence  $D_t X$  is a quantity
of the same "type" as  X.

Appendix C.  How a change of section affects the basic and connection

one-forms on M.



let $\hat{\lambda}_{(k)} = \lambda_{(i)} A_{ik}$ ,

$A_{ik}(m) \in Gl(n,R)$  for each  $m \in U$.

I.  Let  $X_p \in B_p$ ,  where  $p = (m; \lambda_{(i)}) \in f(U)$.

Then  $\pi_{\bullet} X_p = \bar{\omega}_i(X_p) \lambda_{(i)}$    (definition)

$$= A^{-1}_{k\ell} \bar{\omega}_\ell(X_p) \lambda_{(i)} A_{ik}$$

(c.1)  $$= A^{-1}_{k\ell} \bar{\omega}_\ell(X_p) \hat{\lambda}_{(k)} .$$

But also,  $\pi_* X_p = \pi_* (R_{A*} X_p)$,

where  $R_{A*} X_p \in B_{pA}$   with  $pA = (m; \lambda_{(i)} A_{ik})$.

Then (c.1) together with the definition of  $\bar{\omega}_i$  gives

$$\bar{\omega}_k (R_{A*} X_p) = A^{-1}_{k\ell} \bar{\omega}_\ell(X_p).$$

Hence

(c.2)  $$\hat{\omega}_k = \hat{f}^* \bar{\omega}_k = f^* \bullet R_A^* \circ \bar{\omega}_k = f^* \circ (A^{-1}_{k\ell} \bar{\omega}_\ell) = A^{-1}_{k\ell} \bar{\omega}_\ell .$$

II. $R_A{}^{\star}\overline{\omega}_{ij} = R_A{}^{\star}(\hat{a}_{ijk}dx^k + \hat{\lambda}_k{}^{(i)}d\hat{\lambda}_{(j)}{}^k)$, from (A.3)

$$= A^{-1}{}_{im}A_{nj}\,a_{ijk}dx^k + \lambda_k{}^{(m)}A^{-1}{}_{im}d(\lambda_{(n)}{}^k A_{nj})$$

$$= A^{-1}{}_{im}A_{nj}(a_{ijk}dx^k + \lambda_k{}^{(m)}d\lambda_{(n)}{}^k) + \lambda_k{}^{(m)}A^{-1}{}_{im}\lambda_{(n)}{}^k dA_{nj}$$

$$= A^{-1}{}_{im}A_{nj}\overline{\omega}_{mn} + A^{-1}{}_{i\ell}dA_{\ell j}.$$

Hence

$$\hat{\omega}_{ij} = \hat{f}^{\star}\overline{\omega}_{ij} = f^{\star}R_A{}^{\star}\overline{\omega}_{ij}$$

$$= f^{\star}(A^{-1}{}_{im}A_{nj}\overline{\omega}_{mn} + A^{-1}{}_{i\ell}dA_{\ell j})$$

$$= A^{-1}{}_{im}A_{nj}\omega_{ij} + A^{-1}{}_{i\ell}dA_{\ell j}.$$

Appendix D.   Geometrical optics in $V_3$:   Proofs.

I.   Conditions for congruences of $\lambda_{(1)}$ to be shear-free.

$$\xi_{(2)} + i\xi_{(3)} = \rho(\lambda_{(2)} + i\lambda_{(3)})$$



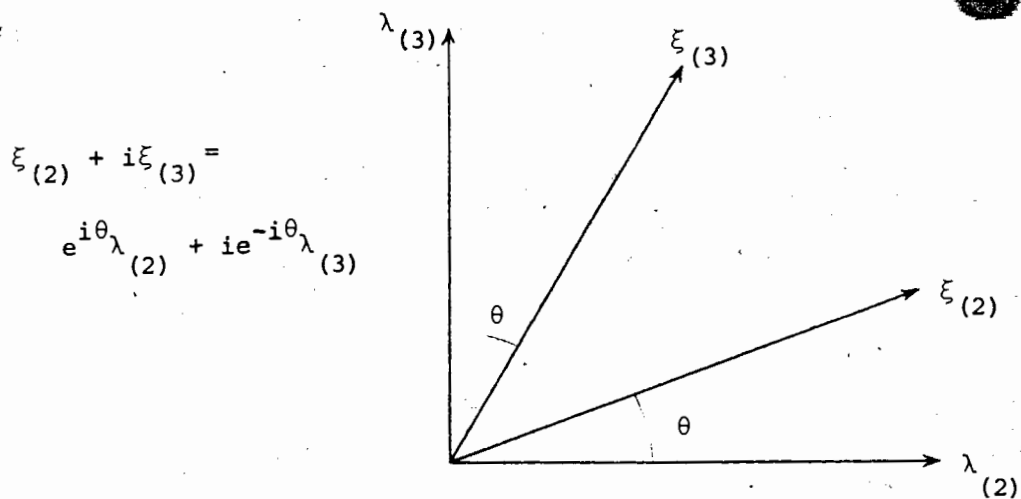$$\mathcal{L}_{(1)}(\rho(\lambda_{(2)} + i\lambda_{(3)})) = \varphi\lambda_{(1)}$$

$$\Rightarrow \rho,_{(1)}(\lambda_{(2)} + i\lambda_{(3)}) + \rho[\lambda_{(1)},\lambda_{(2)}] + i\rho[\lambda_{(1)},\lambda_{(3)}] = \varphi\lambda_{(1)}.$$

Using the metric equations (2.3) and choosing $\varphi$ to take care of the $\lambda_{(1)}$-part, we get

$$(\ell n\rho),_{(1)}(\lambda_{(2)} + i\lambda_{(3)}) + \lambda_{(2)}\gamma_{212} + \lambda_{(3)}(\gamma_{312} - \gamma_{321})$$

$$+ i\lambda_{(2)}(\gamma_{213} - \gamma_{231}) + i\lambda_{(3)}\gamma_{313} = 0$$

$$\Rightarrow [(\ell n\rho),_{(1)} + \tfrac{1}{2}(\gamma_{212} + \gamma_{313} + i\gamma_{213} - i\gamma_{312})](\lambda_{(2)} + i\lambda_{(3)})$$

$$+ \tfrac{1}{2}(\gamma_{212} - \gamma_{313} + i(\gamma_{312} + \gamma_{213}))(\lambda_{(2)} - i\lambda_{(3)})$$

$$+ i\gamma_{321}(\lambda_{(2)} + i\lambda_{(3)}) = 0$$

$$\Rightarrow \Upsilon_{212} - \Upsilon_{313} + i(\Upsilon_{312} + \Upsilon_{213}) = 0.$$

II. Conditions for congruences of $\lambda_{(1)}$ to be dilatation-free.

$$\xi_{(2)} + i\xi_{(3)} =$$

$$e^{i\theta}\lambda_{(2)} + ie^{-i\theta}\lambda_{(3)}$$



$$\pounds_{(1)}(e^{i\theta}\lambda_{(2)} + ie^{-i\theta}\lambda_{(3)}) = \varphi\lambda_{(1)}$$

$$\Rightarrow ie^{i\theta}\theta_{,(1)}\lambda_{(2)} + e^{-i\theta}\theta_{,(1)}\lambda_{(3)} + e^{i\theta}[\lambda_{(1)},\lambda_{(2)}]$$

$$+ ie^{-i\theta}[\lambda_{(1)},\lambda_{(3)}] = \varphi\lambda_{(1)} \quad .$$

Evaluating this at $\theta = 0$, and using $\varphi$ to take care of the $\lambda_{(1)}$-dependence, we have

$$\theta_{,(1)}(i\lambda_{(2)} + \lambda_{(3)}) + \lambda_{(2)}\gamma_{212} + \lambda_{(3)}(\gamma_{312} - \gamma_{321})$$

$$+ i\lambda_{(2)}(\gamma_{213} - \gamma_{231}) + i\lambda_{(3)}\gamma_{313} = 0$$

$$\Rightarrow [i\theta_{,(1)} + \frac{1}{2}(\gamma_{212} + \gamma_{313} + i(\gamma_{213} + \gamma_{312}))](\lambda_{(2)} - i\lambda_{(3)})$$

$$+ \frac{1}{2}(\gamma_{212} + \gamma_{313} + i(\gamma_{213} - \gamma_{312} + 2\gamma_{321}))(\lambda_{(2)} + i\lambda_{(3)}) = 0$$

As a consequence of our choosing $\gamma_{321} = -\gamma_{231} = 0$, we have

$$\gamma_{212} + \gamma_{313} + i(\gamma_{213} - \gamma_{312}) = 0.$$

Appendix E.   Some Observations, Made Too Late For Inclusion in the
Main Body of the Thesis.

Ch. 1, p. 25.   In fact, no further conditions of integrability can
be obtained from these in any case.   This is evident from the form of
(1.11) as finite relations.

Ch. 5, eqn. (5.5).   It should be noted that the actual magnitudes of the
quantities $\tilde{\beta}$ and $\tilde{\gamma}$ have the indicated interpretation only when the
gauge is such that $k^{\alpha}$ is of the form $k^{\alpha}= dx^{\alpha}/dv$, where $v$ is an affine
parameter.   However, the vanishing of $\tilde{\beta}$, $\mathrm{Re}\tilde{\gamma}$, and $\mathrm{Im}\tilde{\gamma}$, does not
depend on the gauge.

Ch. 6, p. 86, bottom.   A. Held (Lett. Nuovo Cim. 11: 545, 1974)
using the Newman-Penrose formalism has found solutions falling
within our case (II):   they correspond to the case of equation (6.5)
where the unknown V is a certain function of $z + \tilde{z}$ alone, and $\tau$
is chosen to be exponential in $z - \bar{z}$.   However it is necessary to
extend $e^{V}$ to negative values.   I. Hauser (Phys. Rev. Lett. 33: 1112,
1974) has found solutions of type N with twisting rays.   But they
will not coincide with any of our solutions.   These recent articles
were called to my attention by Steve Kloster.