NAME OF AUTHOR/*NOM DE L'AUTEUR* ___WOODROW, Robert Edward___

TITLE OF THESIS/*TITRE DE LA THÈSE*___Theories with a finite number of countable models and a___

___small language___

_____

UNIVERSITY/*UNIVERSITÉ*___SIMON FRASER UNIVERSITY___

DEGREE FOR WHICH THESIS WAS PRESENTED/
GRADE POUR LEQUEL CETTE THESE FUT PRÉSENTÉE___Ph.D.___

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE*___1976___

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE*___Dr. A. Lachlan___

DATED/*DATÉ*___December 1, 1976___ SIGNED/*SIGNÉ*___

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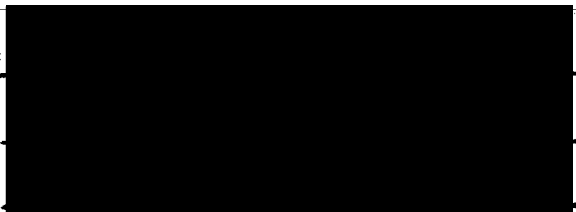
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THEORIES WITH A FINITE NUMBER OF COUNTABLE

MODELS AND A SMALL LANGUAGE

by

Robert E. Woodrow

B.Sc.(Hons), The University of Calgary

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

APPROVAL

Name: Robert E. Woodrow

Degree: Doctor of Philosophy (Mathematics)

Title of Thesis: Theories with a finite number of countable

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language

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December 1, 1976

(date)

ABSTRACT

This work treats countable complete theories having a finite number of countable models and satisfying restrictions on complexity of formulae. Most of the emphasis is on theories with more than one countable model, but some related results on the nature of complete theories which admit elimination of quantifiers and have one binary relation symbol are given. It is shown that there cannot be two incomparable definable equivalence relations if such a theory has infinite models and one 1-type. Tournaments whose theories admit elimination of quantifiers are considered, and it is shown that there is one finite example and there are two countable examples where the set of successors of a member are linearly ordered. It is shown that there are only four countable undirected graphs which omit the complete graph on three vertices and whose theories admit elimination of quantifiers.

It is shown that if a countable complete theory has three countable models and admits elimination of quantifiers in a language with one binary relation and constant symbols then the theory is "essentially" the Ehrenfeucht example $- < Q , < , n >_{n \in \omega}$.

Some conjectures regarding the complexity of theories satisfying restrictions on language and number of countable models are formulated and discussed. A theory T_0 is constructed which has nine countable models and a nonprincipal 1-type containing infinitely many 2-types. A theory T_1 is constructed which has four countable models and an inessential extension T_2 having infinitely many countable models.

ACKNOWLEDGMENTS

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Introduction

This work is concerned with those countable complete theories which have a finite number of countable models. An elegant characterization of those theories which have one countable model is the theorem of Engeler [3], Ryll-Nardzewski [9] and Svenonius [11]: a theory is ω_0-categorical just in case for each n there are a finite number of n-types. The work of Vaught [12, p.320] shows that no countable complete theory can have exactly two countable models. Work of Baldwin and Lachlan [2] and Lachlan [6] shows that a countable complete theory with a finite number of countable models but more than one cannot be superstable. In a letter to Professor Lachlan, Shelah conjectured that no such theory could be stable. To our knowledge the conjecture remains open. Further progress on the general problem appears to be very difficult. We shall restrict ourselves to theories which are simple in complexity of language. The main emphasis will be on theories with more than one countable model, but we also present some results which are strictly concerned with ω_0-categorical theories. We shall first give a brief account of the history of theories with more than one model.

For fourteen years the only widely known examples of theories with a finite number of countable models but more than one were those due to Ehrenfeucht [12, §6]. The archetype of the example is $< Q , < , n , D_m >_{n \in \omega, m < K}$ where $< Q , < >$ is the rational numbers under the usual order, and the D_m are disjoint dense subsets of Q whose union is Q. The theory of this structure will have $(K-1) + 3$ countable models. In a like manner certain other countable

order types can be distinguished by constants in a model of dense

linear order to give a structure whose theory will have finitely many

countable models.

Certain techniques can then be applied to known examples to

yield others. For example the constants in Ehrenfeucht's example may

be replaced by unary predicates which determine initial segments of

the order. At the expense of elimination of quantifiers an example

with a finite language can be given [12, §6]. For this example one

takes an equivalence relation E which is a congruence relation with

respect to a binary relation R . The equivalence classes under E

are densely ordered by R , and for each n there is exactly one

class with $(n+1)$ members. For $n < m$ the equivalence class with

$(n+1)$-members precedes that with $(m+1)$ in the ordering induced by

R . Another way of forming a new theory is to construct a disjoint

union. Given two theories U_0 , U_1 which have n , m countable

models, respectively we can form the disjoint union W having $n \cdot m$

countable models in the following manner. Let P_0 , P_1 be new unary

predicate symbols and c a new constant symbol. We may assume that

U_0 and U_1 have no nonlogical symbols in common. Then the language

of W is the union of the languages of U_0 and U_1 with

$\{P_0 , P_1 , c\}$. The nonlogical axioms of W are the following:

    1) $(P_0 v_0 \vee P_1 v_0 \vee v_0 = c) \wedge \neg P_0 c \wedge \neg P_1 c$

    2) $R v_0 , \ldots , v_n \to \bigwedge_{j \leq n} P_i v_j$ for R an $(n+1)$-ary

        predicate symbol of U_i , $i = 0,1$

3) $\neg P_i v_j \to f v_0, \ldots, v_n = c$ for f an $(n+1)$-ary

function symbol of U_i and $j \leq n$, $i = 0,1$

4) The relativization to P_i of each nonlogical

axiom of U_i for $i = 0,1$.

With "linking" of the copies in the disjoint union some further control

on the number of countable models may be exerted. A link between an

m-type p and an n-type q is an $(m+n)$-type r such that $r \supseteq p$

and for every formula φ

$$\varphi \in q \to \varphi_{v_0, \ldots, v_{n-1}} [v_m, \ldots, v_{m+n-1}] \in r . \quad [1].$$

Linking then refers to the addition of suitable links. Yet another

way to proceed is to form the product $U_0 \cdot U_1$ of two theories U_0, U_1

which have a finite number of countable models, say m,n respectively.

Essentially its models are obtained from models M,N of U_0 and U_1

respectively by replacing each member of M by a copy of N .

Formally we assume that U_0, U_1 have no nonlogical symbols in common.

A new constant c , two unary predicates P_0, P_1 and two unary func-

tion symbols p_0, p_1 are added. The nonlogical axioms of $U_0 \cdot U_1$ are:

1) The relativization to P_i of the nonlogical

axioms of U_i for $i = 0,1$

2) Axioms assuring that outside P_i the nonlogical

symbols of U_i have trivial interpretation.

a) $R v_0, \ldots, v_n \to \bigwedge_{j \leq n} P_i v_j$ for R an n-ary relation

symbol of U_i and $i = 0,1$

b) $\bigvee_{j \leq n} \neg P_i v_j \rightarrow f v_0, \ldots, v_n = c$ for f an

$(n+1)$-ary function symbol of U_i, and $i = 0, 1$

3) $\neg \exists v_0 (P_0 v_0 \wedge P_1 v_0) \wedge \neg P_0 c \wedge \neg P_1 c$

4) p_0, p_1 are coordinate projections onto P_0, P_1

a) $\forall v_0 (P_0 v_0 \vee P_1 v_0 \vee v_0 = c \rightarrow p_i v_0 = c)$ for $i = 0, 1$

b) $\forall v_0 (v_0 \neq c \wedge \neg P_0 v_0 \wedge \neg P_1 v_0 \rightarrow P_i p_i v_0)$ for $i = 0, 1$

c) $\forall v_0 \forall v_1 (P_0 v_0 \wedge P_1 v_1 \rightarrow \exists! v_2 (P_0 v_2 = v_0 \wedge P_1 v_2 = v_1)$.

It is not difficult to see that $U_0 \cdot U_1$ will have $m \cdot n$ countable

models; and if $m > 1$ or $n > 1$, $U_0 \cdot U_1$ will have infinitely many non-

principal k-types for some k . As far as we know no more sophisti-

cated techniques than those discussed above have been mentioned in the

literature.

The restricted nature of the Ehrenfeucht examples and others

closely related to them led to the following conjectures.

(C1): If p is a nonprincipal 1-type in a complete theory with

a finite number of countable models, and M the countable saturated

model then the relation on $p(M)$: "a is prime over b" is a total

partial order.

(C2): Let T be a countable complete theory with a finite

number of countable models and let p be a nonprincipal 1-type in T .

Let M be the countable saturated model of T and $p(M)$ the set of

individuals of M which realize p . Then there is a definable linear

order on an infinite subset of $p(M)$. The subset need not be

definable.

(C3): A nonprincipal 1-type in a countable complete theory with a finite number of countable models can contain at most a finite number of 2-types.

(C4): If T is a complete theory with a finite number of countable models then every inessential extension of T has a finite number of countable models.

(C5): Every model of a complete recursive theory with at most four countable models is totally recursive.

(C1), (C2), (C3) arose when we were first attempting to characterize theories with three countable models in 1974. As far as we know (C2) remains open to this day. The conjecture is given some support by Lemma 2.1 Chapter 2 where it is shown that a nonprincipal 1-type in a complete theory with three countable models contains a 2-type q such that if (a,b) realizes q in a model then b is prime over a while a is not prime over b. (For precise definitions refer to Chapter 1). C3 would confirm C2 for theories with three countable models by an application of Ramsey's theorem. C4 derives from work by Benda [1], and C5 is due to M. Morley (unpublished). The conjecture C1 never seemed very plausible but does focus attention on an interesting class of theories.

The state of knowledge described above was drastically changed when we learned by letter of the example of Peretyiatkin [8, §5]. He introduced a dense binary tree. With thanks to A.H. Lachlan we reproduce the following model of the theory in question, $M = < M , \wedge >$ where $\wedge$ is a meet operation on M. $< Q , < >$ is the rational numbers under the usual order.

$$M = \{f : \exists \alpha \in \overline{Q} \ f : \{\beta : \beta \leq \alpha\} \rightarrow \{0,1\} \text{ and } \exists N \exists \beta_0, \ldots, \beta_N \text{ such that}$$

$$\beta_N = \alpha \text{ and } \forall \beta \leq \beta_0 (f(\beta) = 0) \text{ and } \forall i < N (\beta_i < \beta_{i+1} \ \& \ f(\beta_i) \neq f(\beta_{i+1}) \ \&$$

$$\forall \beta (\beta_i < \beta \leq \beta_{i+1} \rightarrow f(\beta) = f(\beta_{i+1})))\}$$

For $f, g \in M$ let α be the largest rational such that α belongs
to the domains of f and of q and

$$\forall \beta \leq \alpha \ (f(\beta) = g(\beta)) \ .$$

Define $f \wedge g = f \upharpoonright \{\beta : \beta \leq \alpha\}$.

It is then not difficult to show that the class of finite substructures
of M is just the class of finite binary trees. Also by a back and
forth argument one can show that any isomorphism of two finite sub-
structures of M can be extended to an automorphism of M . Using
the techniques of Chapter 3 we then know that the theory of M ,
$Th(M)$, admits elimination of quantifiers and also that any countable
binary tree can be embedded in M . These facts ensure that $Th(M)$
i's equivalent to the theory T_0 of §2 of [8]. If a binary tree which
is a subtree of the complete binary tree $^{<\omega}2$ and which has only one
infinite branch is distinguished in M by constants the theory of the
structure will have three countable models. Since there is a
recursively enumerable tree with one infinite branch and that branch
is nonrecursive Peretyiatkin was able to give a counterexample to C5.
That C5 could not be strengthened by replacing four by six was known
because of an example of Lachlan (unpublished) where the order type
$\omega^* + \omega$ was distinguished by constants in a model of dense linear
order.

Peretyiatkin's example also shows that Cl is false. As non-principal 1-type we take the type of an upper bound for the members of the infinite branch. For the 2-type q we take the 2-type in p which contains $(v_0 \wedge v_1 \neq v_0) \wedge (v_0 \wedge v_1 \neq v_1)$ i.e. the 2-type of two incomparable elements realizing p. Those theories for which Cl is true will be considered in Chapter 6.

A comparison of Ehrenfeucht's example with that of Peretyiatkin raises the question of whether the presence of a function symbol in the latter is really essential. We may formulate the question precisely as follows. Call the language of a theory (or structure) *small* if the language contains a finite number of nonlogical symbols other than unary predicate symbols or constant symbols and the theory (of the structure) admits elimination of quantifiers. The question can then be phrased:

Q1: Let T be a countable complete theory in a small language which has no n-ary function symbols for $n > 0$. Does T satisfy Cl through C4?

A second question which naturally arises is:

Q2: Which of C2 through C3 hold for theories with small languages?

In Chapter 2 we give a partial answer to Q1. There we define precisely what we mean when we say that a structure is like the Ehrenfeucht example. Roughly it means that the model can be viewed as the union of a finite number of "dense orders" and an ω_0-categorical set. Each "order" is actually the induced relation with respect to a definable equivalence relation and it densely

orders the equivalence classes. In each "order" there are infinitely
many principal 1-types which are "almost" arranged in an ω-sequence.
We conjecture that:

C6: Any complete theory with exactly three countable models in
a small language without function symbols other than constants is like
the Ehrenfeucht example.

In Chapter 2 it is shown that any complete theory which has only
constant symbols, which admits elimination of quantifiers and which
has exactly three countable models is like the Ehrenfeucht example.

The examples of Ehrenfeucht and Peretyiatkin show that it is
fruitful to examine ω_0-categorical theories in small language in order
to build examples of theories with other finite numbers of countable
models. Of particular interest are those having one binary relation
symbol. In Chapter 3, methods of Jonsson [5] and Morley and Vaught [7]
are redeveloped for the purpose of constructing and characterizing
examples of theories with small languages. Given a class Σ of
finite structures for a finite language such that Σ has the
amalgamation property and has a prime structure, a denumerable
Σ-homogeneous Σ-universal structure M is constructed such that
each finite subset of M is contained in a member of Σ. When Σ
contains arbitrarily large finite structures M is countable and
when Σ is closed under substructure and there are uniform bounds on
generated substructures, $Th(M)$ admits elimination of quantifiers.
This result is given a partial converse. Let M be a countable
structure in a finite language such that the theory of M admits

elimination of quantifiers and every finite subset of M is contained in a finite substructure. Let Σ be the class of finite substructures of M. Then Σ has the amalgamation property and M is the denumerable Σ-homogeneous Σ-universal structure.

Using the results of Chapter 3 we turn our attention to those countable complete theories which admit elimination of quantifiers and have one binary relation symbol. There are two approaches which may be used to classify this class of theories. From the work of Chapter 3 we see that each such theory is determined by the class of finite graphs which can be embedded in its countable model. A natural question is then:

Q3: Is there a proper subclass of the class of finite graphs which serves to determine every countable graph whose theory admits elimination of quantifiers?

Another approach uses the notion of neighbourhood. Let $H = (F,G)$ be such that F and G are finite classes of finite graphs. Then $N(H)$ is the set of isomorphism types of those countable graphs whose theories admits elimination of quantifiers, in which each graph in F can be embedded, and in which no graph in G can be embedded. $N(H)$ is a *neighbourhood* of K just in case the isomorphism type of K is in $N(H)$. The examples of Peretyiatkin [8, §4] show that not all graphs have even a countable neighbourhood and also show that the first approach is little help in classification. He builds 2^{ω_0} graphs whose theories admit elimination of quantifiers as follows. For each subset A of ω he shows that the class $\Sigma(A)$ of finite graphs in

which the undirected (n+3)-cycle C_n cannot be embedded for $n \in A$

has the amalgamation property. By the results of Chapter 3 there

is a graph $G(A)$ whose theory admits elimination of quantifiers and

such that $\Sigma(A)$ is the class of finite graphs which can be embedded

in $G(A)$. Thus there are many neighbourhoods which contain uncount-

ably many members. Several natural questions are immediate.

Q4: Are there countably infinite neighbourhoods? Are there

finite neighbourhoods? Are there nontrivial empty

neighbourhoods?

Q5: Does every neighbourhood have a finite subneighbourhood?

An examination of the examples of Peretyiatkin reveals that all possi-

ble 2-types occur. Let σ be the graph with one element a and

relation $\{(a,a)\}$. Let $\rightarrow$ be the graph with universe $\{a,b\}$ and

relation $\{(a,b)\}$. Let $\circ \ \circ$ be the graph with universe $\{a,b\}$ and

empty relation and let $\circ\!\!-\!\!\circ$ be the graph with universe $\{a,b\}$ and

relation $\{(a,b), (b,a)\}$. We conjecture

C7: Each of $H(\phi,\{\sigma,\rightarrow\})$, $H(\phi,\{\sigma,\circ\ \circ\})$ and $H(\phi,\{\sigma,\circ\!\!-\!\!\circ\})$

are countable.

In Chapter 4 we give examples which show that the answers to Q4 are

"yes" and to Q5 "no". Also we provide some arguments which lend

support to C7. In this chapter three basic situations are considered,

and the theories are assumed to have only one 1-type, and indeed it

is assumed that their models are irreflexive. In the second section

we show that such a theory cannot have two incomparable definable

equivalence relations if it has an infinite model. This provides us

with a nontrivial empty neighbourhood. In the third section we give
partial results for those theories whose models are tournaments, i.e.
which omit σ , $\circ\!\!-\!\!\circ$, and $\circ \ \circ$. It is shown that there is only one
finite tournament whose theory admits elimination of quantifiers, and
that there are only two such countable tournaments which have the
property that the successors of an element are linearly ordered by
the relation. There is also the countable tournament homogeneous and
universal for the class of finite tournaments. We do not know what
other examples exist. In the fourth section we turn to undirected
graphs whose theories admit elimination of quantifiers. Work of
A. Gardiner [4] neatly classifies the finite examples. We show that
for each n the class Σ_n of finite undirected graphs which omit the
complete graph with n+2 vertices K_{n+2}, has the amalgamation property
and thus obtain infinitely many countable graphs. The conjecture that
$H(\phi,\{\sigma,\rightarrow\})$ is countable is supported by the main result of this
section where it is shown that there are only four countable graphs
which omit the triangle and whose theories admit elimination of
quantifiers - i.e. $H(\phi,\{\sigma,\rightarrow,K_3\})$ has four members.

In Chapter 5 we use the results of Chapter 3 to furnish a counter-
example to C2. The example is related to that of Peretyiatkin. We
produce a dense tree with infinite branching - the subtrees above a
node being indexed by the rationals. Using this a complete theory T_0
with nine countable models is produced. The nonlogical symbols of T_0
are two binary function symbols, a unary predicate symbol, a binary
predicate symbol and countable many constant symbols. T_0 admits

elimination of quantifiers and has a nonprincipal 1-type which contains infinitely many 2-types. However we conjecture:

(C2'): No nonprincipal 1-type in a theory with three countable models can contain infinitely many 2-types.

Another way to focus the attack on theories with a finite number of countable models is to consider the implications of C1 and C4. In Chapter 6 we investigate these conjectures at greater depth. It is shown that if every complete inessential extension of a theory has a finite number of countable models then the theory has a universal model which is prime over a finite set, thus improving a result of Benda [1, Thm 2]. In the second section we consider a complete theory T with a nonprincipal 1-type p with the property that in the countable saturated model M the relation "a is prime over b" is a linear order of $p(M)$. We show that in fact it is a dense order defined by a formula, thus echoing the result of Chapter 2. Finally the results of Chapter 3 are applied to produce a complete theory T_1 with a binary relation symbol, a 3-ary function symbol and infinitely many unary predicate symbols. T_1 admits elimination of quantifiers and has four countable models. In the saturated model of T_1 the relation "a is prime over b" is a linear order of the set of elements realizing each nonprincipal 1-type. T_1 has a complete inessential extension with infinitely many countable models. An interesting unresolved question is whether an example like T_1 can be found with three countable models.

Chapter 1

Notation and Preliminaries

In this chapter we set down the basic notation, definitions and preliminary results which we shall employ throughout. Generally we shall follow the conventions of Shoenfield [10] but in certain important respects our notation will differ from his.

Let L be a first order language. We use upper case Latin letters to denote structures for L. If M is a structure for L its underlying set will be denoted $|M|$ if confusion is possible and otherwise more simply by M. We shall write $a \in M$ for $a \in |M|$ and $\bar{a} \in M$ for $\bar{a} \in |M|^n$ when $\bar{a}$ is an n-tuple of elements of M. $L(A)$ will denote the language L augmented by constant symbols for names of elements of A. In general we shall abuse the notation by using an element as its own name. If $\underline{u}$ is a symbol of L and M is a structure for L then $\underline{u}_M$ is the interpretation of $\underline{u}$ in M. For ease in reading we shall also write $\underline{u}(M)$ or simply $\underline{u}$ when it is clear from context that $\underline{u}_M$ is meant. If $A \subseteq M$ (M,A) is the expansion of M to the theory with names for individuals $a \in A$ with $a_M = a$. $(M,\bar{a})$ is $(M, \text{Rng} \bar{a})$.

We assume a fixed list of variables $\{v_n \cdot n \in \omega\}$. We shall let x, x_n, y, y_n, z, z_n range among variables and constants of the language. For a formula φ, and a sequence $\vec{x} = \langle x_0, \ldots, x_{n-1} \rangle$ we write $\varphi(\vec{x})$ for the formula $\varphi_{v_0, \ldots, v_{n-1}}[x_0, \ldots, x_{n-1}]$. It is assumed that no clash of quantifiers results, and unless the context implies otherwise that the free variables of $\varphi(\vec{x})$ occur among $x_0, \ldots, x_{n-1}$.

If $\vec{x}$, $\vec{y}$ are two finite sequences then $\varphi(\vec{x},\vec{y})$ is $\varphi(\vec{x} \cap \vec{y})$ where $\cap$ is the operation of concatenation of finite sequences. We write x for $\langle x \rangle$. $\text{lh}(\vec{x})$ is the length of $\vec{x}$.

If L is a first order language and $n \in \omega$ then L_n is the set of formulae whose free variables occur among $v_0, \ldots, v_{n-1}$. L_0 is thus the set of sentences for L. $L_n(A)$ is $(L(A))_n$.

We shall use lower case Greek letters χ, ϕ, ψ, θ, λ to range among formulae and σ, ρ, ξ among mappings. For a formula ψ, ψ^1 denotes $\neg\psi$ and ψ^0 denotes ψ.

Let T be a complete theory with language L and let $\vec{x}$ be a sequence of n new constant symbols. A set Γ of formulae in L_n is an *n-type in* T just in case $T[\Gamma] = T \cup \{\varphi(\vec{x}) : \varphi \in \Gamma\}$ is a consistent extension of T. Γ is a *complete n-type* just in case $T[\Gamma]$ is complete. We reserve the letter p to denote a complete 1-type in T. Upper case Greek Λ, Γ, Φ will normally denote complete n-types. Unless otherwise specified n-type will mean complete n-type.

If M is a structure for L and $\varphi \in L(M)$ then $M \models \varphi$ means that φ is valid in M. If $\Gamma \subseteq L$ then $M \models \Gamma$ means $M \models \varphi$ for all $\varphi \in \Gamma$. $M \models T$ has the usual meaning. If $\bar{a}$ is an n-tuple in M then $\text{tp}(\bar{a}) = \{\varphi \in L_n : M \models \varphi(\bar{a})\}$, $\text{tp}(\bar{a})$ is an n-type in $\text{Th}(M)$. $\text{tp}(\bar{a},\bar{b}) = \text{tp}(\bar{a} \cap \bar{b})$. We say that $\bar{a}$ *realizes* q if $q \subseteq \text{tp}(\bar{a})$. If $\varphi \in L_n(M)$ then $\varphi(M) = \{\bar{a} \in M : M \models \varphi(\bar{a})\}$.

$I(T,\kappa)$ is the cardinality of the set of isomorphism types of models of T of power κ. If S is a set, $|S|$ denotes its cardinality. Countable means infinite and countable.

If T is a theory then the language of T is $L(T)$. When no confusion is likely we shall write L for $L(T)$.

T' is an *inessential extension* of the complete theory T if T' is complete and an extension of T in a language which differs from L by finitely many constant symbols.

Let T be a complete theory. We shall say that an $(m+n)$ - type Λ is *principal over an* m-*type* Δ just in case $\Delta \subseteq \Lambda$ and there is some $\theta \in \Lambda$ such that whenever $M \models T$ $\bar{d} \in M$ and $\bar{d}$ realizes Δ we have for each $\chi \in \Lambda$, $M \models (\theta \to \chi)$ $(\bar{d}^{\cap} <v_m, \ldots, v_{m+n-1}>)$. In this case θ is said to *generate* Λ *over* Δ . If $M \models T$ and $\bar{a}$, $\bar{b} \in M$ then $\bar{b}$ is *prime over* $\bar{a}$ just in case $tp(\bar{a}, \bar{b})$ is principal over $tp(\bar{a})$.

If $m, n < \omega$ are given let the transposition operator $T_{m,n}$ be defined for each θ by

$$T_{m,n}(\theta) = \theta'_{v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m-1}} [v_m, \ldots, v_{m+n-1}, v_0, \ldots, v_{m-1}]$$

where θ' is a variant of θ chosen so that no v_i for $i < m+n$ occurs bound in θ' . We extend $T_{m,n}$ to $(m+n)$-types in the obvious way. Note that if $\bar{a}$ is an n-tuple and $\bar{b}$ is an m-tuple then $tp(\bar{b}, \bar{a}) = T_{m,n}(tp(\bar{a}, \bar{b}))$.

We shall make frequent use of the following well-known result.

Lemma 1.1   Let M be a model of the complete theory T and let $\bar{a}, \bar{b}, \bar{c} \in M$ be such that $\bar{b}^{\cap} \bar{c}$ is prime over $\bar{a}$ and $\bar{a}$ is prime over $\bar{b}$. Then $\bar{b}^{\cap} \bar{c}$ is prime over $\bar{b}$.

Proof.   Let $lh(\bar{a}) = m$ $lh(\bar{b}) = n$ $lh(\bar{c}) = k$. Let θ generate the type of $\bar{a}^{\cap} \bar{b}^{\cap} \bar{c}$ over $tp(\bar{a})$ and let ψ generate

$tp(\bar{b} \cap \bar{a})$ over $tp(\bar{b})$. Then

$$T_{m,n+k}(\exists v_0, \ldots, \exists v_{m-1}(\theta \wedge T_{n,m}(\psi)))$$

will generate the type of $\bar{b} \cap \bar{c}$ over $tp(\bar{b})$.

If Γ is a 2-type, p,q 1-types with $p \subset \Gamma$ and $q \subset T_{1,1}(\Gamma)$ then Γ is said to be *in* $p \times q$.

A formula ψ with one free variable v_0 is said to be ω_0-categorical just in case for each $m \in \omega$ there are a finite number of m-types Γ which contain $\psi(v_i)$ for $i < m$.

A graph is a structure for the language with one binary relation symbol R . If G is a graph $|G|$ is the *vertex set* of G and members are called vertices while R_G is the *edge set* and members are called edges.

Chapter 2

Ehrenfeucht-like theories

1. Definitions

In this chapter we give some results which indicate a distinction between two types of theories in a small language - those with only relation symbols and constant symbols and those which allow function symbols. These differences are related to conjectures C1, C2, and C3. The main theorem uses severe restrictions on the number of countable models and the language. Before introducing these however we shall present some more general results, and the notion of being "like" an Ehrenfeucht structure.

Vaught's argument of [12; p.320] that no countable complete theory has exactly two isomorphism types of countable models may be modified using his Theorem 3.5 [12; p.311] on the existence of prime models to give the following observation.

If $I(T,\omega) = 3$ then T has countable models M_0, M_1, and M where M_0 is prime, M_1 is saturated, and for each nonprincipal n-type Λ there is a sequence $\bar{a} \in M$ such that $tp(\bar{a}) = \Lambda$ and M is prime over $\bar{a}$. The modification is to note that T and any complete inessential extension of T have prime and saturated countable models. Thus given a nonprincipal n-type Λ there is a countable model N of T and $\bar{b} \in N$ such that N is prime over $\bar{b}$ and $\bar{b}$ realizes Λ. But then N cannot be saturated and it cannot be prime. Thus if M_0 is a prime model of T, M_1 is a countable saturated model of T and M is the third countable model of T M must be isomorphic to N. We call M *the middle model* of T.

Another result which was discovered independently by Benda [1] and the author is:

Lemma 2.1   Let U be a countable complete theory with $I(U,\omega) = 3$. Then if Λ is any nonprincipal m-type there is a 2m-type Γ with $\Lambda \subseteq \Gamma$ and $\Lambda \subseteq T_{m,m}(\Gamma)$ such that Γ is principal over Λ while $T_{m,m}(\Gamma)$ is not.

Proof. Let M be prime over $\bar{a}$ realizing Λ . There are infinitely many m-types since Λ is nonprincipal and hence infinitely many m-types over $\bar{a}$, i.e. in $Th(M,\bar{a})$. Thus there is a 2n-type $\Gamma' \supseteq \Lambda$ which is not principal over Λ . Choose $\bar{a}'$ realizing Λ and $\bar{b}$ such that $\bar{a}' \cap \bar{b}$ realizes Γ' and M is prime over $\bar{a}' \cap \bar{b}$. Consider the type $\Gamma = tp(\bar{a} \cap \bar{a}')$. Γ is principal over Λ since M is prime over $\bar{a}$ but $T_{m,m}(\Gamma)$ cannot be principal over Λ for otherwise Γ' would be principal over Λ by Lemma 1.1 . This establishes the lemma.

Let $\dot{T}$ be a countable complete theory which has a binary relation symbol R . For $\varphi \in L_1$ define

$$\lambda^{\varphi}xy = ((Rxy \wedge Ryx) \vee (\neg Rxy \wedge \neg Ryx)) \wedge \varphi(x) \wedge \varphi(y) .$$

We say that *property E holds* of φ if the following seven conditions are satisfied:

(i) $\vdash_T \lambda^{\varphi}xy \wedge \lambda^{\varphi}yz \to \lambda^{\varphi}xz$

i.e. λ^{φ} is an equivalence relation on φ .

(ii) $\vdash_T \lambda^{\varphi}xx_1 \wedge \lambda^{\varphi}yy_1 \wedge Rxy \wedge \neg Ryx \to Rx_1y_1 \wedge \neg Ry_1x_1$

i.e. λ^{φ} is a congruence relation with respect to $Rv_0v_1 \wedge \neg Rv_1v_0$ on φ .

(iii) $\vdash_T \varphi(x) \wedge \varphi(y) \wedge \varphi(z) \wedge (Rxy \wedge \neg Ryx) \wedge (Ryz \wedge \neg Rzy) . \rightarrow .$

$Rxz \wedge \neg Rzx$

i.e. $Rv_0 v_1 \wedge \neg Rv_1 v_0$ is transitive on φ .

(iv) $\vdash_T \varphi(x) \wedge \varphi(y) \wedge (Rxy \wedge \neg Ryx) . \rightarrow . \exists z(\varphi \wedge (Rxz \wedge \neg Rzx) \wedge$

$(Rzy \wedge \neg Ryz))$ where z is a new variable

i.e. $(Rv_0 v_1 \wedge \neg Rv_1 v_0)$ is dense on φ/λ^φ .

(v) $\vdash_T \varphi(x) . \rightarrow . \exists z(\varphi(z) \wedge (Rxz \wedge \neg Rzx))$ where z is a new

variable.

(vi) $\vdash_T \varphi(x) . \rightarrow . \exists z(\varphi(z) \wedge (Rzx \wedge \neg Rxz))$ where z is a new

variable.

i.e. $Rv_0 v_1 \wedge \neg Rv_1 v_0$ is "without endpoints" on φ/λ^φ .

and:

(vii) *Either* for each principal 1-type p containing φ there

are at most finitely many 1-types q containing φ such

that there is a 2-type Λ in pxq containing

$((Rv_1 v_0 \wedge \neg Rv_0 v_1) \vee \lambda^\varphi v_0 v_1)$,

or for each principal 1-type p containing φ there

are at most finitely many 1-types q containing φ such

that there is a 2-type Λ in pxq containing

$((\neg Rv_1 v_0 \wedge Rv_0 v_1) \vee \lambda^\varphi v_0 v_1)$.

Roughly speaking property E holds of φ if in each model N

of T the structure obtained by restriction to φ resembles the

Ehrenfeucht example: R_N is a dense ordering of equivalence classes,

and principal types are almost arranged in a sequence.

A model M of the complete theory T whose language includes the binary relation symbol R is said to be E-*like* if there are a finite number of formulae $\varphi_0, \ldots, \varphi_n \in L_1$ such that:

(i)   property E holds of φ_i for $i \leq n$

(ii) $\vdash_T \neg \exists v_0 (\varphi_i \wedge \varphi_j)$ for $i \neq j$

(iii) $\neg \bigvee_{i \leq n} \varphi_i$ is ω_0-categorical .

In this case we also say that T is E-*like*.

2.   The Theorem

For the remainder of this chapter we assume that T is a complete theory in the language with one binary relation symbol R and constant symbols $\{a_i : i \in \omega\}$. We also assume that T admits elimination of quantifiers and that it has three countable models.

Our main result is the following:

Theorem 2.1   There is $n \in \omega$ and there are formulae $\varphi_0, \ldots, \varphi_n \in L_1$ such that $\neg \bigvee_{i \leq n} \varphi_i$ is ω_0-categorical and for each $i \leq n$ property E holds of φ_i .

The proof of this theorem rests on the following two lemmas whose proofs are deferred to §3 and §4.

Lemma 2.2   Let $\varphi' \in L_1$ be contained in a nonprincipal 1-type p . Then there is a formula $\varphi \in p$ such that $\vdash_T \varphi \rightarrow \varphi'$ and property E holds of φ .

Lemma 2.3   There are only finitely many nonprincipal 1-types in T .

The following is immediate:

Lemma 2.4 If $\bar{b}$ is an n-tuple in a model N of T then $tp(\bar{b})$ is the unique n-type Λ such that $T_{n-j,j}(\Lambda) \supset tp(b_j)$, $v_i = v_j \in \Lambda$ iff $b_i = b_j$ and $Rv_i v_j \in \Lambda$ iff $N \models Rb_i b_j$ for $i , j < n$.

Proof of the theorem from the lemmas. By Lemma 2.4 and Ryll-Nandzewski's theorem there must be a nonprincipal 1-type in T . By Lemma 2.3 there are finitely many nonprincipal 1-types, say $p_1, \ldots, p_n$. Choose ψ_i such that p_i is the only nonprincipal 1-type that contains ψ_i for $i \leq n$. By Lemma 2.2 we may choose $\varphi_i \in p_i$ such that $\vdash_T \varphi_i \to \psi_i$ and property E holds of φ_i for $i \leq n$. There can be no nonprincipal 1-type which contains $\neg \bigvee_{i \leq n} \varphi_i$. From Lemma 2.4 it is easy to see that $\neg \bigvee_{i \leq n} \varphi_i$ is ω_0-categorical. This completes the proof.

3. Proof of Lemma 2.2

Recall that we must show that if T satisfies the conditions of this chapter, p is a nonprincipal 1-type of T and $\varphi' \in p$ then there is a formula $\varphi \in p$ such that $\vdash_T \varphi \to \varphi'$ and φ has property E.

Let p be a nonprincipal 1-type of T . Let M be the middle model of T .

Consider now those 2-types Δ in $p \times p$ which contain $v_0 \neq v_1$. By Lemma 2.4 we see that each such Δ is specified by the pair $(i,j) \in \{0,1\}^2$ such that $(Rv_0 v_1)^i \wedge (Rv_1 v_0)^j \in \Delta$. By Lemma 2.1 there is some type Γ in $p \times p$ which is principal over p while $T_{1,1}(\Gamma)$ is not. Let θ generate Γ with respect to p . By elimination of quantifiers without loss of generality we may take θ to be

$$\varphi_0 \wedge \varphi_0(v_1) \wedge Rv_0 v_1 \wedge \neg Rv_1 v_0 \wedge v_0 \neq v_1 \text{ where } \varphi_0 \in p .$$

Note that θ cannot be symmetric in v_0, v_1 since $T_{1,1}(\Gamma)$ is not principal over p.

Claim. $T_{1,1}(\Gamma)$ is the only 2-type in $p \times p$ which is nonprincipal over p.

For proof by contradiction assume that Δ is a 2-type in $p \times p$ which is nonprincipal over p and different from $T_{1,1}(\Gamma)$. Let a be a member of the middle model M which realizes p and over which M is prime. Let M be prime over (c,d) realizing Δ. Now (a,c) and (a,d) realize types principal over p and from Lemma 1.1 neither (c,a) nor (d,a) realizes a type principal over p. Therefore $\Gamma = tp((a,c)) = tp((a,d))$ by Lemma 2.4. The type Λ of (a,c,d) is principal over p and $T_{1,2}(\Lambda)$ is principal over Δ. Since d realizes p there is a pair $(e,f) \in M$ such that (d,e,f) realizes Λ. We shall show that (a,e,f) also realizes Λ. Now (e,f) realizes Δ and as before by Lemmas 1.1 and 2.4 $\Gamma = tp(a,e) = tp(a,f)$ since M is prime over a but not over e or f. For M prime over e or f would imply by Lemma 1.1 that M was prime over d, a contradiction. But then by Lemma 2.4 the type of (a,e,f) is Λ. By homogeneity of prime models (see Vaught [12, p.310]) there is an automorphism of M fixing a and taking (c,d) onto (e,f). Thus M is also prime over (e,f). But (d,e,f) realizes a type principal over p. Hence by Lemma 1.1 M is prime over d whence (d,a) realizes a type principal over p, a contradiction. This verifies the claim.

From the definition of Γ, $\Gamma \neq T_{1,1}(\Gamma)$. Thus from the claim we may choose $\varphi \in p$ such that for i, $j < 2$

either $\vdash_T \neg \exists v_0 \exists v_1 (\varphi \wedge \varphi(v_1) \wedge (Rv_0v_1)^i \wedge (Rv_1v_0)^j \wedge v_0 \neq v_1)$

or there is a unique 2-type $\Delta_{i,j}$ in $p \times p$ containing

$$\lambda_{i,j}^\varphi = \varphi \wedge \varphi(v_1) \wedge (Rv_0v_1)^i \wedge (Rv_1v_0)^j \wedge v_0 \neq v_1$$

and $\lambda_{i,j}^\varphi$ generates $\Delta_{i,j}$ over p or $T_{1,1}(\lambda_{i,j}^\varphi)$

generates $T_{1,1}(\Delta_{i,j})$ over p .

We call this the *basic restriction* on φ , and assume that φ satisfies the basic restriction in the remainder of this section.

We may also suppose without loss that

$$\vdash_T \forall v_0 (\varphi \to Rv_0v_0) \vee \forall v_0 (\varphi \to \neg Rv_0v_0) .$$

For if $\psi \in p$ and $\vdash_T \psi \to \varphi$ and φ satisfies the basic restriction then ψ does as well.

The proof is now reduced to a number of claims which permit one to choose a formula $\varphi \in p$ such that $\vdash_T \varphi \to \varphi'$ and φ is E-like.

Claim 2.1 Given a formula $\varphi \in p$ there is a formula ψ in p such that if φ^* is any formula in p implying $\varphi \wedge \psi$ then φ^* satisfies $E(i)$.

Let ψ be the formula

$$\psi = df \; \forall v_1 \forall v_2 (\lambda^\varphi v_0 v_1 \wedge \lambda^\varphi v_1 v_2 : \to : \lambda^\varphi v_0 v_2) .$$

We shall show first that $\psi \in p$. Let M be prime over a realizing p and let b, c be elements such that

$$M \vdash \lambda^\varphi ab \wedge \lambda^\varphi bc .$$

From the definition of λ^φ either $\lambda^\varphi_{0,0}$ or $\lambda^\varphi_{1,1}$ belongs to the

type realized by (a,b). Suppose that $M \models \lambda^\varphi_{0,0}(a,b)$ without loss

of generality. There is a unique 2-type $\Delta_{0,0}$ in $p \times p$ containing

$\lambda^\varphi_{0,0}$ from Lemma 2.4. Since $\lambda^\varphi_{0,0}(v_1,v_0) = \lambda^\varphi_{0,0}(v_0,v_1)$, $\Delta_{0,0} =$

$\tau_{1,1}(\Delta_{0,0})$ whence $\Delta_{0,0}$ is generated over p by $\lambda^\varphi_{0,0}$. Thus

(b,a) realizes $\Delta_{0,0}$ and M is prime over b by Lemma 1.1.

Similarly, M is prime over c . It follows that the types realized

by (a,c) and (c,a) are both in $p \times p$ and principal over p .

Neither of these types can contain $\lambda^\varphi_{0,1}$ whence $M \models \lambda^\varphi(a,c)$ as

required.

If φ^* is any formula in p implying $\varphi \wedge \psi$ it is clear that

$$\vdash_T \lambda^{\varphi^*}xy \wedge \lambda^{\varphi^*}yz \rightarrow \lambda^{\varphi^*}xz$$

since by inspection $\lambda^{\varphi^*}v_0v_1$ is equivalent to $\varphi^* \wedge \varphi^*(v_1) \wedge \lambda^\varphi v_0 v_1$.

Thus any φ^* in p which implies $\varphi \wedge \psi$ will serve to satisfy $E(i)$.

Thus claim 2.1 is verified.

Let θ^χ be the formula $\chi \wedge \chi(v_1) \wedge Rv_0v_1 \wedge \neg Rv_1v_0$

where χ has at most v_0 occurring free.

Claim 2.2 Let $\varphi \in p$. There is a formula $\psi \in p$ such that

any $\varphi^* \in p$ which implies ψ will satisfy $E(ii)$.

Let ψ be the formula

$$\forall v_1 \forall v_2 \forall v_3 (\lambda^\varphi v_0 v_1 \wedge \theta^\varphi(v_0,v_2) \wedge \lambda^\varphi v_2 v_3 . \rightarrow . \theta^\varphi(v_1,v_3)) \ .$$

We aregu that $\psi \in p$. Let a,b,c,d be elements of M such that M

is prime over a , a realizes p and $M \models \lambda^\varphi ab \wedge \theta^\varphi(a,c) \wedge \lambda^\varphi cd$.

It is easy to see that b,c and d realize p and that (a,c)

realizes Γ because of the basic restriction on φ . Also if

(b,d) does not realize Γ then M is prime over d and thus also over c contradicting the fact that (a,c) realizes Γ . Arguing as for Claim 2.1 we see that any φ^* in p which implies ψ will satisfy $E(ii)$.

Claim 2.3 Let $\varphi \in p$ satisfy $E(ii)$. Then there is $\psi \in p$ such that if $\varphi^* \in p$ and $\vdash_T \varphi^* \to \varphi \wedge \psi$ then φ^* satisfies $E(iii)$.

Let ψ be the formula

$$\forall v_1 \forall v_2 (\theta^\varphi(v_0,v_1) \wedge \theta^\varphi(v_1,v_2) \to \theta^\varphi(v_0,v_2)) .$$

We show that $\psi \in p$. If M is prime over a realizing p and b,c are such that $M \models \theta^\varphi(a,b) \wedge \theta^\varphi(b,c)$, then b,c realize p from the basic restriction on φ . The type of (a,c) is determined by the unique pair $(i,j) \in \{0,1\}^2$ such that $M \models \lambda^\varphi_{i,j}(a,c)$. If (i,j) is $(0,0)$ or $(1,1)$ then $M \models \lambda^\varphi ac$ which gives $M \models \theta^\varphi(c,b)$ since φ satisfies $E(ii)$. This contradicts $M \models \theta^\varphi(b,c)$. If (i,j) is $(1,0)$ then (c,a) realizes Γ , which means that (a,c) realizes $T_{1,1}(\Gamma)$ contradicting the fact that M is prime over a . Thus (i,j) is $(0,1)$ which yields $M \models \theta^\varphi(a,c)$ as required. Arguing as for Claim 2.1 any φ^* in p which implies $\varphi \wedge \psi$ will serve to satisfy $E(iii)$.

Claim 2.4 Let p be the only nonprincipal 1-type which contains φ . Then if $\varphi^* \in p$ and $\vdash_T \varphi^* \to \varphi$ we have that φ^* satisfies $E(vii)$.

Let $\varphi \in p$ be such that p is the unique nonprincipal 1-type containing φ . Let q be a principal 1-type which contains φ .

Notice that

$$\{q' : \text{ there exists } \Delta \text{ in } q' \times q \text{ such that } \lambda^{\varphi} v_0 v_1 \vee \theta^{\varphi} \in \Delta\}$$

is finite. Otherwise, by the compactness theorem there exists non-principal q' and Δ in $q' \times q$ such that $\lambda^{\varphi} v_0 v_1 \vee \theta^{\varphi} \in \Delta$. Since p is the only nonprincipal 1-type containing φ , we have $q' = p$. But $(\lambda^{\varphi} v_0 v_1 \vee \theta^{\varphi}) \in \Delta$, $\Delta \supset p$ and $T_{1,1}(\Delta) \supset q$ imply that $q = p$ by the basic restriction on φ . Thus q is nonprincipal, a contradiction. This shows that φ satisfies $E(\text{vii})$ and the claim follows easily.

Claim 2.5 If φ satisfies $E(\text{i})$, $E(\text{ii})$, $E(\text{iii})$ and p is the unique nonprincipal 1-type containing φ then there is a formula X in p such that $\vdash_T X \to \varphi$ and X satisfies $E(\text{iv})$.

Let ψ be

$$\forall v_1 \exists v_2 (\theta^{\varphi}(v_0, v_1) \ .\to. \ \theta^{\varphi}(v_0, v_2) \wedge \theta^{\varphi}(v_2, v_1)) \ .$$

We have $\psi \in p$ since φ satisfies $E(\text{iii})$ and θ^{φ} generates Γ over p. For if M is prime over a realizing p there are then b,c such that $M \models \theta^{\varphi}(a,b) \wedge \theta^{\varphi}(b,c)$. By $E(\text{iii})$ $M \models \theta^{\varphi}(a,c)$ whence $\exists v_2 (\theta^{\varphi}(v_0, v_2) \wedge \theta^{\varphi}(v_2, v_1))$ belongs to Γ . By choice of φ , $\varphi \wedge \neg \psi$ is ω_0-categorical. Let $p_0, \ldots, p_m$ enumerate those 1-types containing $\varphi \wedge \neg \psi$. We claim that there is a finite list of 1-types $q_0, \ldots, q_n$ extending $p_0, \ldots, p_m$ such that for each $i \le n$ $\varphi \in q_i$ and

$(*) \forall q (q \text{ a 1-type} \wedge \exists \Delta (\Delta \text{ a 2-type } \Delta \supset q \wedge T_{1,1}(\Delta) \supseteq q_i \wedge (\lambda^{\varphi} v_0 v_1 \vee \theta^{\varphi})$

$\in \Delta) \ :\Rightarrow: \ \exists j \le n \ (q = q_j)) \ .$

$(*)$ says that $\{q_0, q_1, \ldots, q_n\}$ is closed under "predecessors" with respect to $(\theta^{\varphi} \vee \lambda^{\varphi})$. The list $\langle q_i : i \le n \rangle$ exists since φ already satisfies $E(\text{i})$, $E(\text{ii})$, $E(\text{iii})$ and $E(\text{vii})$: one need only extend the list $p_0, \ldots, p_m$ by adding those q for which

$(\exists_i \leq m) \, \exists \Delta (\Delta$ in $q \times p_i$ and $\lambda^\varphi v_0 v_1 \vee \theta^\varphi \in \Delta)$.

Let $X = df \ \varphi \wedge \bigwedge_{i \leq n} \neg X_i$. Now X satisfies $E(i)$, $E(ii)$, $E(iii)$ and

and (vii). Also since the list $q_0, \ldots, q_n$ extends $p_0, \ldots, p_m$ we have

$$\vdash_T \forall v_0 \forall v_1 \exists v_2 (\theta^X \, . \to . \ \theta^\varphi(v_0, v_2) \wedge \theta^\varphi(v_2, v_1)) \ .$$

From (*) we can deduce

$$\vdash_T \forall v_0 \forall v_1 (\theta^\varphi \wedge X \, . \to . \ X(v_1)) .$$

Thus X satisfies $E(iv)$.

Claim 2.6 If $\varphi \in p$ and p is the unique nonprincipal 1-type

containing φ then $\vdash_T \forall v_0 \exists v_1 (\varphi \to \theta^\varphi)$ and φ satisfies $E(v)$. For

let M be prime over a realizing p and let $b \in M$ be such that

$M \vDash \varphi(b)$. If b realizes a nonprincipal 1-type then that type

is p and since θ^φ generates Γ over p $M \vDash \exists v_1 \ \theta^\varphi(b, v_1)$.

Otherwise b realizes a principal type and again because of the basic

restriction on φ , $M \vDash \theta^\varphi(b, a)$. Thus φ satisfies $E(v)$.

Now we shall apply Claims 2.1, 2.2, 2.3 . Choose $\varphi_0 \in p$ such

that φ satisfies the basic restriction, $\vdash_T \varphi_0 \to \varphi'$, p is the unique

nonprincipal 1-type containing φ_0 and if $\varphi^* \in p$ and $\vdash_T \varphi^* \to \varphi_0$

then φ^* satisfies $E(i)$, $E(ii)$ and $E(iii)$. By Claim 2.5 we may assume

that φ_0 satisfies $E(iv)$. Now let φ be $\exists v_1 (\theta^{\varphi_0}(v_1, v_0))$. Clearly

$\varphi \in p$. Also φ satisfies $E(i)$, $E(ii)$ and $E(iii)$ and by Claim 2.4

$E(vii)$ and by Claim 2.6 $E(v)$. φ satisfies $E(iv)$ and $E(vi)$ because

φ_0 satisfies $E(iv)$.

Thus φ has property E and the lemma is proved.

4. Proof of Lemma 2.3

Under the conditions of the theorem we prove that there are only finitely many nonprincipal 1-types.

Assume there are infinitely many. Then there is a formula φ which is contained in infinitely many nonprincipal 1-types and is "minimal" among such formulae: i.e. for no formula ψ is it the case that $\varphi \wedge \psi$ and $\varphi \wedge \neg \psi$ are each contained in infinitely many nonprincipal 1-types. Otherwise there would be 2^{ω_0} 1-types and thus 2^{ω_0} countable models. [12, §5.1]

It is easy to see that there are exactly ω_0 nonprincipal 1-types p' containing φ such that for some ψ p' is the unique nonprincipal 1-type containing ψ. Such 1-types we call *isolated nonprincipal 1-types*. Let $\langle p_i : i \in \omega \rangle$ be an enumeration without repetition of all isolated nonprincipal 1-types containing φ and let φ_i' be chosen such that p_i is the unique nonprincipal type containing φ_i'.

Let φ_0 be $\varphi_0' \wedge \varphi$ and for each $i > 0$ let φ_i be
$$\varphi_i' \wedge (\neg \bigvee_{j<i} \varphi_i) \wedge \varphi .$$

Then the sequence $\langle \varphi_i : i \in \omega \rangle$ satisfies for each $i, j < \omega$

(i) $\vdash_T \varphi_i \to \varphi$.

(ii) φ_i belongs to exactly one nonprincipal 1-type namely p_i

(iii) $\vdash_T \neg \exists v_0 (\varphi_i \wedge \varphi_j)$ if $i \neq j$

and

(iv) if φ belongs to an isolated nonprincipal 1-type q

there is $i \in \omega$ with $\varphi_i \in q$.

By the compactness theorem there is some nonprincipal 1-type $p \supseteq \{\varphi\} \cup \{\neg \varphi_i : i \in \omega\}$. By the choice of φ and $\langle \varphi_i : i \in \omega\rangle$ there is only one such p .

By Lemma 2.1 let Γ_i be a 2-type in $p_i \times p_i$ which is principal over p_i but such that $T_{1,1}(\Gamma_i)$ is not principal over p_i and let Γ be a type in $p \times p$ with the same property. Let M be prime over c_i realizing p_i for each $i \in \omega$ and also prime over c realizing p .

Let c_i' , $i \in \omega$, and c' be elements of M such that (c_i, c_i') realize Γ_i and (c, c') realizes Γ . The contradiction will be obtained by showing that there are five distinct 2-types in $p_0 \times p$. From Lemma 2.4 there can be at most four.

We claim that there is a strictly increasing sequence of positive integers $\langle i_j : j \in \omega\rangle$ and there are pairs $\langle k_0, k_1\rangle$, $\langle \ell_0, \ell_1\rangle$ and $\langle m_0, m_1\rangle$ in $\{0,1\}$ such that for each $j \in \omega$.

(i) $(Rv_0 v_1)^{k_0} \wedge (Rv_1 v_0)^{k_1} \in \mathrm{tp}(c_0, c_{i_j})$

(ii) $(Rv_0 v_1)^{\ell_0} \wedge (Rv_1 v_0)^{\ell_1} \in \mathrm{tp}(c_0, c_{i_j}')$

(iii) $(Rv_0 v_1)^{m_0} \wedge (Rv_1 v_0)^{m_1} \in \mathrm{tp}(c_0', c_{i_j})$.

To find $\langle k_0, k_1\rangle$ and a subsequence $\langle i_j' : j \in \omega\rangle$ which satisfies (i) we need only observe that some one of the four possible pairs, say (k_0, k_1) must be such that for infinitely many $i \in \omega$

$$(Rv_0 v_1)^{k_0} \wedge (Rv_1 v_0)^{k_1} \in \mathrm{tp}(c_0, c_i) .$$

By thinning $\langle i_j' : j \in \omega\rangle$ twice we can find $\langle i_j : j \in \omega\rangle$, $\langle \ell_0, \ell_1\rangle$ and $\langle m_0, m_1\rangle$ such that (i), (ii) and (iii) are all satisfied.

From Lemma 1.1 $tp(c_0, c_i)$ is principal over p_0 and $T_{1,1}(tp(c_0, c_i))$ is principal over p_i. Also $tp(c_0, c_i')$ is principal over p_0 and $T_{1,1}(tp(c_0, c_i))$ is not principal over p_i; while $tp(c_0', c_i)$ is not principal over p_0 and $T_{1,1}(tp(c_0', c_i))$ is principal over p_i. Thus the pairs $<k_0, k_1>$, $<\ell_0, \ell_1>$ and $<m_0, m_1>$ are distinct by Lemma 2.4. By the compactness theorem there are distinct 2-types Λ_0, Λ_1, Λ_2 in $p_0 \times p$ such that

$$(Rv_0v_1)^{k_0} \wedge (Rv_1v_0)^{k_1} \in \Lambda_0$$

$$(Rv_0v_1)^{\ell_0} \wedge (Rv_1v_0)^{\ell_1} \in \Lambda_1$$

$$(Rv_0v_1)^{m_0} \wedge (Rv_1v_0)^{m_1} \in \Lambda_2 .$$

Notice that none of Λ_0, Λ_1, Λ_2 is principal over p_0. Suppose, for example, that Λ_0 is generated by θ over p_0. By Lemma 2.4 and elimination of quantifiers we may take θ to be

$$(Rv_0v_1)^{k_0} \wedge (Rv_1v_0)^{k_1} \wedge \psi(v_1) \text{ where } \psi \in p . \text{ Every}$$

formula in p is contained in all but finitely many of the types p_i. Thus there is some $j \in \omega$ such that $M \models \psi(c_{i_j})$. But then (c_0, c_{i_j}) realizes Λ_0, an impossibility. Thus we have three 2-types in $p_0 \times p$ none of which is principal over p_0. At the same time there are at least two 2-types in $p_0 \times p$, namely those realized by (c_0, c) and (c_0, c') which are principal over p_0. This is the desired contradiction.

Thus there are only a finite number of nonprincipal 1-types in T.

5. Summary

Our analysis shows that under considerable restrictions $I(T,\omega) = 3$ implies that any model of T is like the Ehrenfeucht example. Benda has pointed out that we may not have used the full force of the assumption $I(t,\omega) = 3$ since we only use the existence of the middle model M. This remark suggests the following question.

Q6: If a countable complete theory has a middle model, does

it have three countable models?

An analysis of the proofs of Lemmas 2.2 and 2.3 shows that T satisfies (C1), (C2) and (C3) of the introduction.

Chapter 3

Σ-generic structures

In this chapter we redevelop methods of Jonsson [5] and Morley and Vaught [7] for the purpose of constructing and characterizing theories in a small language.

The most important concept involved is that of a class of structures having the amalgamation property. Let Σ be a class of structures for a language L . We say that Σ has the *amalgamation property* (AP) provided that whenever G, H_0, H_1, e_0, e_1 are such that $G, H_0, H_1 \in \Sigma$, $e_0 : G \to H_0$ and $e_1 : G \to H_1$ are embeddings then there is $H \in \Sigma$ and there are embeddings f_0, f_1 where $f_0 : H_0 \to H$, $f_1 : H_1 \to H$ and $f_0 \circ e_0 = f_1 \circ e_1$.

For the remainder of this chapter let L be a fixed finite language. The reader should bear in mind that L may contain function symbols.

Let Σ be a class of finite structures for L and let M be a denumerable structure for L .

M is said to be *Σ-universal* provided that every structure in Σ can be embedded in M .

M is said to be *Σ-homogeneous* provided that whenever G, H are substructures of M which are isomorphic to members of Σ and when $f : G \to H$ is an isomorphism then there is an automorphism g of M which extends f (i.e. $f \subset g$).

M is said to be *Σ-generic* provided that M is Σ-homogeneous, Σ-universal and every finite subset of M is contained in a substructure of M isomorphic to a member of Σ .

The following lemma provides a convenient test to determine whether a model M is Σ-homogeneous.

Lemma 3.1 Assume that Σ is closed under isomorphism and that every finite subset of M is contained in a substructure of M belonging to Σ. Then M is Σ-homogeneous just in case whenever H_0, K_0, H are substructures of M belonging to Σ, $H_0 \subseteq H$ and $g : H_0 \to K_0$ is an isomorphism there is a substructure K of M with $K_0 \subseteq K$ and an isomorphism $\hat{g} : H \to K$ such that $g \subseteq \hat{g}$.

Proof. Necessity is trivial. The proof of sufficiency uses a form of back and forth argument. Let H_0, K_0 be substructures of M and let $g_0 : H_0 \to K_0$ be an isomorphism. Let $\{a_k : k < \omega\}$ enumerate M. We extend g_0 to an automorphism as follows. Construct by recursion sequences $\langle H_n : n \in \omega \rangle$, $\langle K_n : n \in \omega \rangle$ and $\langle g_n : n \in \omega \rangle$ where H_n, K_n are substructures of M, and $g_n : H_n \to K_n$ is an isomorphism for $n \in \omega$, such that $M = \bigcup_{n \in \omega} H_n = \bigcup_{n \in \omega} K_n$ and for $n \in \omega$ $H_n \subseteq H_{n+1}$, $K_n \subseteq K_{n+1}$ and $g_n \subseteq g_{n+1}$. For n even we may choose k minimal such that $a_k \notin H_n$ and choose $H_{n+1} \supseteq H_n \cup \{a_k\}$ such that $H_{n+1} \in \Sigma$ and $H_{n+1} \subseteq M$. Then we may take $K_{n+1} \supseteq K_n$ and $g_{n+1} \supseteq g_n$ so that $g_{n+1} : H_{n+1} \to K_{n+1}$ is an isomorphism. For n odd we may choose k minimal such that $a_k \notin K_n$, $K_{n+1} \supseteq K_n \cup \{a_k\}$ with $K_{n+1} \in \Sigma$ and $K_{n+1} \subseteq M$. We may now apply the condition to K_n, H_n, K_{n+1} and g_n^{-1} to obtain H_{n+1} and g_{n+1}.

In the next lemma we provide a sufficient condition for the existence of a Σ-generic structure.

Lemma 3.2   Let Σ be closed under isomorphism and assume that Σ has AP and that there is a structure S in Σ which can be embedded in every member of Σ . Then there is a Σ-generic structure M .

Proof.   If there is a bound on the cardinality of members of Σ let M be a member of Σ of largest cardinality. Without loss of generality we may take $S \subset M$. Let $G \in \Sigma$. There is no loss in assuming that $S \subseteq G$ and $G \cap M = S$. Since Σ has AP there is N in Σ and there are embeddings $f_0 : M \to N$ and $f_1 : G \to N$. But then f_0 is the identity and f_1 is an embedding of G in M . Thus M is Σ-universal. Now assume that H,K are substructures of M which belong to Σ and let $f : H \to K$ be an isomorphism. If H = M we are done. For an application of AP consider the identity mapping $1_H : H \to M$ and $f : H \to M$. There are therefore $N \in \Sigma$, $g_1 : M \to N$, $g_2 : M \to N$ such that g_1, g_2 are embeddings and $g_1 \circ I_H = g_2 \circ f$. But then g_1, g_2 are isomorphisms and there is no loss of generality in assuming g_1 is the identity mapping on M . The required automorphism of M is then g_2^{-1} . Thus M is Σ-generic.

If there is no bound on the cardinality of members of Σ we proceed as follows. The collection of isomorphism types of structures in Σ is countable so we may choose a sequence $\langle S_n , n \in \omega \rangle$ of members of Σ such that $S_0 = S$ and if $G \in \Sigma$ then $G \cong S_n$ for some n . Now construct an increasing sequence $\langle M_n : n \in \omega \rangle$ of members of Σ such that for all n $M_n \in \Sigma$, $M_n \subseteq M_{n+1}$ and also

(i) $M_0 = S_0$

(ii) $\forall n \forall H \forall k \forall f \exists m \exists g (H \subset M_n$ & $f : H \to S_k$ is an embedding

$\Rightarrow g : S_k \to M_m$ is an embedding such that for all

$h \in H$ $h = g \circ f (h))$.

At stage $m+1$ choose $n \leq m$, $H \subset M_n$, k and an embedding $f : H \to S_k$ such that there is no embedding $g : S_k \to M_m$ making $g \circ f = i_H$. Further, make these choices so as to minimize $n+k$. By an application of AP we may choose M_{m+1} such that $M_m \subset M_{m+1}$ and there is an embedding $g : S_k \to M_{m+1}$ making $g \circ f = 1_H$.

To see that the sequence $\{M_n : n \in \omega\}$ satisfies (ii) assume that $n+k$ is minimal such that there are $H \subset M_n$ and $f : H \to S_k$ with no embedding $g : S_k \to M_m$ with $g \circ f = 1_H$ for any m . There are a finite number, say ℓ, of pairs (H',f') where $H' \subset M_{n'}$ and $f' : H' \to S_{k'}$ is an embedding and $n'+k' \leq n+k$. Thus we may choose $m \leq n$ such that for each such (H',f') with $n'+k' < n+k$ there is an embedding $g' : S_{k'} \to M_m$ with $g' \circ f' = 1_H$. The construction ensures that there must be an embedding $g : S_k \to M_{m+\ell}$ making $g \circ f = 1_H$, a contradiction.

Let $M = \bigcup_{n \in \omega} M_n$. Since S_0 may be embedded in each member of Σ (ii) ensures that M is Σ-universal. It is easy to see from (ii) and Lemma 3.1 that M is Σ-homogeneous.

Thus M is Σ-generic and countable and the lemma is proved.

A back and forth argument which closely resembles the proof of Lemma 3.1 provides us with a uniqueness result for Σ-generic structures.

Lemma 3.3 Any two Σ-generic structures are isomorphic.

Proof. Let M,N be two Σ-generic structures. It suffices to show that if H_0, K_0 $H \in \Sigma$, $H_0 \subseteq H \subseteq M$, $K_0 \subseteq N$, and if $f : H_0 \to K_0$ is an isomorphism then there is an embedding $g : H \to N$ such that $g \upharpoonright H_0 = f$. But there is an embedding h of H into N . Also K_0 and the image of H_0 under h are isomorphic by $h \circ f^{-1}$. Since N is Σ-homogeneous there is an automorphism k of N which extends $h \circ f^{-1}$. We may take $g = (k)^{-1} \circ h$.

The next lemma establishes that construction of a class of struc-tures Σ satisfying the conditions of Lemma 3.2 and a condition on substructures is a way of generating ω_0-categorical theories.

Lemma 3.4 If M is Σ-generic and if there is a function $f : \omega \to \omega$ such that whenever A is a subset of a structure in Σ there is a member of Σ containing A whose cardinality is at most $f(|A|)$ then $Th(M)$ is ω_0-categorical. Moreover if Σ is closed under substructure then $Th(M)$ admits elimination of quantifiers.

Proof. Let M be Σ-generic and let $f : \omega \to \omega$ be such that whenever $A \subseteq H \in \Sigma$ $\exists K \in \Sigma$ $A \subseteq K$ and $|K| \leq f(|A|)$. If M is finite there is nothing to prove. Otherwise by Lemma 3.3 it suffices to show that any countable model of $Th(M)$ is Σ-generic. Since L is finite the isomorphism type of any member of Σ can be described by a formula and thus every model of $Th(M)$ is Σ-universal. We claim that any finite subset of a model of $Th(M)$ is contained in a substructure which belongs to Σ . Let n be fixed. There are a finite number of

isomorphism types of structures which belong to Σ and have cardinality at most $f(n)$. For each $\bar{a}$ a sequence of n distinct elements from a structure H in Σ there is a formula $\varphi \in L_n$ which fixes the isomorphism type of H over $\vec{a}$; that is we have $K \models \varphi(\vec{b})$ where K is an L-structure and $\vec{b} \in K$ iff there is an embedding of H into K which takes $\bar{a}$ onto $\bar{b}$. It is easy to see that a finite disjunction of such formulas is valid in M and the claim follows. From Lemma 3.1 any countable model of Th(M) is also Σ-homogeneous. The extension property in the criterion of Lemma 3.1 for each H_0, K_0, H and σ can be encoded in a formula which is valid in M.

Thus Th(M) is ω_0-categorical.

Now if Σ is closed under substructure we may employ the following well-known proposition which we do not prove, to see that Th(M) admits elimination of quantifiers.

Proposition 3.1 Let T be a countable complete theory. Then T admits elimination of quantifiers just in case for each denumerable model N of T and for each pair of sequences $\vec{a}, \vec{b}$ of the same length, n say, such that for each open formula $\psi \in L_n$ $N \models \psi(\vec{a}) \leftrightarrow \psi(\vec{b})$, there is a denumerable elementary extension N' of N with an automorphism carrying $\vec{a}$ to $\vec{b}$.

When Σ is closed under substructure Σ-homogeneity ensures that the conditions of Proposition 3.1 are fulfilled and therefore Th(M) admits elimination of quantifiers.

This completes the proof of Lemma 3.4 .

The following simple example shows that the uniform bound on the cardinality of the smallest member of Σ containing a subset A is really needed in Lemma 3.4 .

Example 3.1 Let L be the language with one unary function symbol P and let Σ be the class of structures for L isomorphic to one of $\langle n, P_n \rangle$ for $n \geq 1$ where $n = \{m : m < n\}$ and $P_n(m) = m-1$ if $m > 0$ and $P_n(0) = 0$. The Σ-generic model is then $\langle \omega, P\omega \rangle = \bigcup_{n < \omega} \langle n, P_n \rangle$.

The following lemma provides a partial converse to Lemma 3.4 and strengthens the connection between classes of structures with AP and theories which admit elimination of quantifiers.

Lemma 3.5 Let M be a denumerable structure such that $\mathrm{Th}(M)$ admits elimination of quantifiers, and such that every finite subset of M is contained in a finite substructure of M . Let Σ be the class of finite structures which can be embedded in M . Then Σ has the amalgamation property, and M is Σ-generic.

Proof. Let $A, B_0, B_1 \in \Sigma$ and let $f_i : A \to B_i$ be embeddings for $i = 0,1$. There is no loss of generality in assuming that A, B_0, B_1 are substructures of M and that f_0 is the inclusion of A in B_0 . Since A and B_1 are finite and L is finite there are open formulas which fix their isomorphism types as substructures of M . But $A, f_1(A)$ satisfy the same open formula and so f_1 is in fact elementary since $\mathrm{Th}(M)$ admits elimination of quantifiers. But then f_1^{-1} can be extended to an isomorphism of B_1 and an extension B_1' of

A . Now $(B_0 \cup B_1')$ generates a member of Σ . This argument also shows that M satisfies the condtions of Lemma 3.1 and is Σ-homogeneous. Therefore Σ has AP and M is Σ-generic.

Chapter 4

Quantifier eliminable graphs

1. Examples

In this chapter we apply the results of Chapter 3 to graphs whose theories admit elimination of quantifier.   Theories are assumed to be complete theories with one binary relation symbol R , and to admit elimination of quantifiers. We also stipulate that each model be irreflexive, i.e. model $\forall v_0 \neg R v_0 v_0$. This ensures that there is only 1-type.

In this section we shall present several simple examples and some basic definitions.

We first introduce abbreviations for some basic formulas. These abbreviations are $I, \Gamma, \Delta_0, \Delta_1, \Delta_2, U$ defined as follows:

$$Ixy = df \quad x = y$$

$$\Gamma xy = df \quad Rxy \wedge \neg Ryx$$

$$\Delta_0 xy = df \quad (Rxy \wedge Ryx) \vee x = y$$

$$\Delta_1 xy = df \quad (\neg Rxy \wedge \neg Ryx) \vee x = y$$

$$\Delta_2 xy = df \quad (\Gamma xy \vee \Gamma yx) \vee x = y$$

$$Uxy = df \quad x = x$$

Several well-known examples are the following:

DO - The theory of $<Q,<>$ the rationals under the usual ordering.

E_k^n - The theory of an equivalence relation with n equivalence classes of power k , where $1 \leq n$, $k \leq \omega$.

G_A - The $\Sigma(A)$ generic structure for the class $\Sigma(A)$ of finite graphs in which the (n+3) cycle cannot be embedded for $n \in A \subseteq \omega$.

DO^n - The theory of the direct product of a model of DO
and the complete graph on n points where $n \leq \omega$.

Given T we may form its *dual* $\bar{T}$ from the denumerable model
G of T. $\bar{G}$ is the graph with universe G and relation
$R_{\bar{G}} = (G \times G) \setminus (R_G \cup \{(g,g) : g \in G\})$. $\bar{T}$ is the theory of $\bar{G}$.
Clearly $\bar{T}$ also admits elimination of quantifiers. DO is self dual,
while the duals of the other theories are not included in the list.

The following proposition gives a way of constructing new
examples from known ones.

Proposition 4.1 Let i,j,k be $0,1,2$ in some order and let
T_0, T_1 be such that $\vdash_{T_0} (\Delta_i v_0 v_1 \vee \Delta_j v_0 v_1)$ and $\vdash_{T_1} (\Delta_k v_0 v_1)$.
Let A, B be denumerable models of T_0, T_1 respectively. Define the
structure C by $|C| = |A| \times |B|$ and $R_C = \{((a_0, b_0), (a_1, b_1)) :$
$(a_0 = a_1 \wedge (b_0, b_1) \in R_B) \vee ((a_0, a_1) \in R_A)\}$. Define the structure D
by $|D| = |C|$ and $R_D = \{((a_0, b_0), (a_1, b_1)) : ((b_0, b_1) \in R_B) \vee$
$((a_0, a_1) \in R_A \wedge b_0 = b_1))\}$. Then $Th(C)$ and $Th(D)$ admit
elimination of quantifiers.

Proof: We present the proof that $Th(C)$ admits elimination of quanti-
fiers. The proof for D is similar. The main idea is that from C
we can retrieve A and B in order to construct enough automorphisms
in the Wreath product of the automorphism groups. It suffices to show
that if $\vec{c}, \vec{d}$ are two sequences of n-elements of C such that

$$C \models c_\ell = c_m \leftrightarrow d_\ell = d_m ,$$

$$C \models \Delta_\mu \; c_\ell c_m \leftrightarrow \Delta_\mu \; d_\ell d_m ,$$

and $\qquad C \models \Gamma \; c_\ell c_m \leftrightarrow \Gamma \; d_\ell d_m$

for ℓ, $m < n$ and $\mu < 2$ then there is an automorphism h of C which carries $\vec{c}$ onto $\vec{d}$. The conditions ensure that there are distinct $a_0, \ldots, a_{\ell-1} \in A$ and distinct $a'_1, \ldots, a'_{\ell-1} \in A$ such that $\{m : a_p \text{ is the first coordinate of } c_m\} = \{m : a_{p'} \text{ is the first coordinate of } d_m\}$ for $p < \ell$. and $\vec{a}$ realizes the same type as $\vec{a}'$ in T_0. We may choose an automorphism f of A which carries $\vec{a}$ onto $\vec{a}'$. Let $p < \ell$ and consider now $I_p = \{m : a_p \text{ is the first coordinate of } c_m\}$. Let $I_p = \{m_0, \ldots, m_{r-1}\}$ and consider the r-tuples $\vec{b}$ and $\vec{b}'$ from B with $\vec{b}_s$ the second coordinate of a_{m_s} and $\vec{b}'_s$ the second coordinate of d_{m_s}. Then $\vec{b}$ and $\vec{b}'$ realize the same type in T_1 and there is an automorphism g_{a_p} of B which sends $\vec{b}$ onto $\vec{b}'$. For $a \not\in \{a_0, \ldots, a_{\ell-1}\}$ let σ_a be the identity on B. Consider now the automorphism h of C defined by

$$h((a,b)) = (f(a), g_a(b))$$

which carries $\vec{c}$ onto $\vec{d}$.

Note that in the construction above $\{(c,d) : C \models \Delta_k cd\}$ is an equivalence relation on C.

2.   Definable equivalence relations

In this section we examine the structure of the lattice of definable equivalence relations of a model of a fixed theory T . The main result is the following:

Theorem 4.1   If T has an infinite model then no model has two incomparable definable equivalence relations.

The main theorem will follow easily from the lemmas of this section.

Fix a theory T , which has an infinite model. Note first that if Λ is a formula in L_2 which defines an equivalence relation in T then Λ is equivalent to a disjunction of some of $\Delta_0 v_0 v_1, \Delta_1 v_0 v_1,$ $\Delta_2 v_0 v_1$ and $I v_0 v_1$.

The following lemma deals with the most difficult case in the proof of Theorem 4.1 .

Lemma 4.1   Assume that $T \vdash \Delta_i v_0 v_1 \wedge \Delta_i v_1 v_2 \rightarrow \Delta_i v_0 v_2$ for $i = 0,1$. Then either $\Delta_0 v_0 v_1$ or $\Delta_1 v_0 v_1$ is equivalent to $v_0 = v_1$ in T .

Proof.   Let $A \models T$ and let $A \models \Delta_0 ab \wedge a \neq b$. Note that

$$A \models \forall v_0 (\Delta_1 b v_0 \wedge v_0 \neq b \rightarrow (\Gamma a v_0 \vee \Gamma v_0 a)) \ .$$

Claim. $A \models \exists v_0 (\Delta_1 b v_0 \wedge \Gamma a v_0)$

Otherwise we have

$$A \models \forall v_0 (\Delta_1 b v_0 \wedge b \neq v_0 \rightarrow \Gamma v_0 a) \qquad\qquad I$$

Let $A \models \Delta_0 ab \wedge \Delta_1 bc \wedge \Gamma ca \wedge \Delta_1 ad \wedge a \neq d$.

By I we see that $A \models \neg \Delta_0 cd$. Otherwise (c,d) realizes the same type as (a,b) yielding $A \models \Gamma ac$, contradiction. Also it is trivial that $A \models \neg \Delta_1 cd \wedge c \neq d$ since $\Delta_1 v_0 v_1(A)$ is an equivalence relation. Thus $A \models (\Gamma cd \vee \Gamma dc)$. Suppose $A \models \Gamma cd$, then (c,d) and (c,a) realize the same type and we have $e \in A$ with $A \models \Delta_1 ce \wedge \Delta_0 de$. It is easy to see that e is none of a,b,c,d . Since $\Delta_1 v_0 v_1(A)$ is an equivalence relation and I holds $A \models \Gamma ea$. Comparing the triples (a,b,c) and (e,d,a) we see that $A \models \Gamma ae$ since I must hold with e,d for a,b respectively. This contradiction leaves only the possibility that $A \models \Gamma dc$. Now (c,a) and (d,c) realize the same type whence there exists f such that $A \models (\Delta_1 fd \wedge \Delta_0 fc)$. Comparing the triples (a,b,c) and (c,f,a) we get $A \models \Gamma ac$, a contradiction. This establishes the claim.

Clearly the above argument can be modified by exchanging the roles of Δ_0 and Δ_1 to establish that

$$A \models \exists v_0 (\Delta_0 ab \wedge \Delta_1 b \, v_0 \wedge \Gamma v_0 a) \ .$$

Thus

$$\exists v_0 \exists v_1 \exists v_2 (\Delta_0 v_0 v_1 \wedge \Delta_1 v_1 v_2 \wedge \Gamma v_0 v_2) \tag{1}$$

and

$$\exists v_0 \exists v_1 \exists v_2 (\Delta_0 v_0 v_1 \wedge \Delta_1 v_1 v_2 \wedge \Gamma v_2 v_0) \tag{2}$$

are both theorems of T .

Fix $a,b,c \in A$ such that $A \models \Delta_0 ab \wedge \Delta_1 bc \wedge \Gamma ac$ for the remainder of the proof. By (2) and elimination of quantifiers there exists d such that $A \models \Delta_1 ad \wedge \Delta_0 dc$. Note that since $\Delta_0 v_0 v_1(A)$ and $\Delta_1 v_0 v_1(A)$ are both equivalence relations

$$A \models \exists! v_0 (\Delta_1 a \ v_0 \wedge \Delta_0 v_0 c) \tag{3}$$

Now it will be established that

$$T \models \exists v_0 \exists v_1 \exists v_2 \exists v_3 (\Delta_0 v_0 v_2 \wedge \Delta_1 v_0 v_1 \wedge \Delta_1 v_2 v_3 \wedge \Delta_0 v_1 v_3 \wedge \Gamma v_0 v_3 \wedge \Gamma v_1 v_2). \tag{4}$$

We require the following lemma which will be proved later.

Lemma 4.2   Suppose that the hypothesis of Lemma 4.1 holds and that the conclusion fails, then for each $n \leq \omega$

$$T \models \forall v_0 \ \exists^{\geq n} v_1 (\Delta_0 v_0 v_1) \wedge \forall v_0 \ \exists^{\geq n} v_1 (\Delta_1 v_0 v_1) \ .$$

If (4) fails, then

$$A \models \forall v_0 (\Delta_1 a \ v_0 \wedge \Delta_0 c \ v_0 \rightarrow \Gamma b \ v_0) \ . \tag{5}$$

But there is a unique d such that $A \models \Delta_1 ad \wedge \Delta_0 cd$. The idea is to exploit the existence of this unique point and the condition (5) to define a partition of $\{e : A \models \Delta_1 be\}$ with parameters a,b which is independent of a in the sense that if $A \models \Delta_1 a'a$ and a' is substituted for a the same partition will result. The existence of many points in the $\Delta_1 v_0 v_1$ (A) equivalence class of b will then furnish a contradiction. Let $\varphi(z)$ be

$$\Delta_1 bz \wedge \exists v_1 (\Delta_1 a \ v_1 \wedge \Delta_0 v_1 z \wedge \Gamma b \ v_1) \ .$$

From Lemma 4.2 $S = \{e : A \models \Delta_1 be\}$ is infinite and without loss of generality $S' = \{e : A \models \varphi(e)\}$ is also infinite. This is because otherwise we may apply a similar argument to $\{e : A \models \neg \varphi(e) \wedge e \neq b \wedge \Delta_1 eb\}$. This corresponds to replacing $\Gamma b \ v_1$ by $\Gamma v_1 b$ in the definition of $\varphi(z)$.

Let e_0 be such that $A \models \varphi(e_0)$ and let f_0 be such that $A \models \Delta_1 a \, f_0 \wedge \Delta_0 e_0 f_0 \wedge \Gamma b \, f_0$. Then $A \models \Gamma a \, e_0$ since (5) holds. Let φ' result from φ by replacing a by f_0 . Then

$$A \models \forall v_0 (\varphi v_0 \leftrightarrow \varphi' v_0) \ .$$

But T admits elimination of quantifiers and there is an open formula θ with parameters b , f_0 such that

$$A \models \varphi' v_0 \leftrightarrow \theta \ .$$

Hence either

$$\forall e (e \in S' \ \& \ e \neq e_0 \leftrightarrow (A \models \Gamma \, f_0 e)) \tag{6}$$

$or \qquad \forall e (e \in S' \ \& \ e \neq e_0 \leftrightarrow (A \models \Gamma e \, f_0)) \ . \tag{7}$

This is because there is a member of S' other than e_0 , and there is an element other than b which satisfies $\Delta_1 b \, v_0 \wedge \neg \varphi(v_0)$.

Clearly there are e_1, e_2 in S' such that for the corresponding f_1, f_2 either (6) holds when the subscript 0 is replaced by either 1 or 2, or (7) holds whichever of 1 and 2 replaces 0. In the first case $A \models \Gamma f_1 e_2 \wedge \Gamma f_2 e_1$ and in the second $A \models \Gamma e_1 f_2 \wedge \Gamma e_2 f_1$. Thus (4) holds in either case.

This argument does not use the full strength of Lemma 4.2. One needs only that $\vdash_T \exists^{\geq 4} v_1 (\Delta_0 v_0 v_1) \wedge \exists^{\geq 4} v_1 (\Delta_1 v_0 v_1)$. The argument used for (4) when applied to $\bar{T}$ gives

$$T \vdash \exists v_0 \exists v_1 \exists v_2 \exists v_3 (\Delta_0 v_0 v_2 \wedge \Delta_0 v_1 v_3 \wedge \Delta_1 v_0 v_1 \wedge \Delta_1 v_2 v_3 \wedge \Gamma v_0 v_3 \wedge \Gamma v_2 v_1) \tag{8}$$

From (4) and (8) however we easily derive a contradiction for we must have

$$A \models \exists v_1 (\Delta_0 v_1 c \wedge \Delta_1 a \, v_1 \wedge \Gamma b \, v_1) \wedge \exists v_1 (\Delta_0 v_1 c \wedge \Delta_1 a \, v_1 \wedge \Gamma v_1 b) \ .$$

This contradicts uniqueness in (3), and completes the proof of Lemma 4.1.

We now turn to the proof of Lemma 4.2 .

Assume the hypothesis of Lemma 4.1 holds and that the conclusion fails. Let $A \models T$ and let $a \in A$. It will suffice to show that $S_0 = \{a' : A \models \Delta_0 aa'\}$ is infinite, for the same argument applied to $\bar{T}$ shows that $S_1 = \{b : A \models \Delta_1 ab\}$ is also infinite. Assume for contradiction that A is finite and let $A = \{a = a_0, a_1, \ldots, a_m\}$ $m \geq 1$. One may now establish as follows that S_1 is infinite. It is easy to see that at least one of $\{c : A \models \Gamma ac\}$ and $\{c : A \models \Gamma ca\}$ must be infinite. Assume that $\{c : A \models \Gamma ac\}$ is infinite, the argument given below can easily be adapted for the other eventuality. In the proof of Lemma 4.1 we showed that (1) holds, i.e.

$$T \vdash \exists v_0 \exists v_1 \exists v_2 (\Delta_0 v_0 v_1 \wedge \Delta_1 v_1 v_2 \wedge \Gamma v_0 v_2) .$$

Because of elimination of quantifiers this ensures that for each c with $A \models \Gamma ac$ there is $j \leq m$ such that $A \models \Delta_1 a_j c$. But by the pigeon-hole-principle there is some $j \leq m$ with $A \models \Delta_1 a_j c$ for infinitely many c . But then S_1 is infinite since a_0 and a_j realize the same 1-type in T .

Consider now the equivalence relation Θ defined on S_1 by the formula $\Theta v_0 v_1$

$$\Delta_1 a v_0 \wedge \Delta_1 a v_1 \wedge \forall v_2 (\Delta_0 a v_2 :\rightarrow: (R v_0 v_2 \leftrightarrow R v_1 v_2) \wedge (R v_2 v_0 \leftrightarrow R v_2 v_1)) .$$

Given $j \leq m$ we can define an equivalence relation Θ_j on S_1 where two elements of S_1 are equivalent iff they realize the same 1-type over a_j . Θ is then the intersection of the Θ_j .

Consider now the partition of S_1 induced by Θ. Clearly $\{a\}$ is one equivalence class. Since T admits elimination of quantifiers the classes are either $\{a\}$ and $S_1 \setminus \{a\}$ *or* all singletons $\{e\}$ $e \in S_1$. The latter is impossible because it would make Θ the identity on B yet from the form of Θ the number of classes in the partition is $\leq 2^m + 1$.

Consider the equivalence relation Λ defined on S_0 by $\lambda v_0 v_1$:

$$\Delta_0 a v_0 \wedge \Delta_0 a v_1 \wedge \forall v_2 (\Delta_1 a v_2 :\rightarrow: (Rv_0 v_2 \leftrightarrow Rv_1 v_2) \wedge (Rv_2 v_0 \leftrightarrow Rv_2 v_1))$$

Again $\{a\}$ is an equivalence class under Λ. Also the equivalence classes are either $\{a\}$ and $\{a_1, \ldots, a_m\}$ *or* $\{a_0\}, \ldots, \{a_m\}$. The latter can only occur if $m+1 \leq 2^1 + 1 = 3$. Thus either

$$\Delta_0 a v_0 \wedge \Delta_1 a v_1 \rightarrow (\Gamma v_0 v_1 \vee a = v_0)$$
or $$\Delta_0 a v_0 \wedge \Delta_1 a v_1 \rightarrow (\Gamma v_1 v_0 \vee a = v_0)$$

is valid in A, *or* $m = 2$ and without loss of generality

$$A \models \forall v_1 (\Delta_1 a v_1 \rightarrow (\Gamma v_1 a_1 \wedge \Gamma a_2 v_1) \vee a = v_1).$$

The first two cases occur only when the Λ equivalence classes are $\{a\}$ and $\{a_1, \ldots, a_m\}$ and are impossible from (1) and (2) which were established in the proof of Lemma 4.1. The third case is also impossible because a_1 would be definable from a. This completes the proof of Lemma 4.2.

Thus we see that if T has an infinite model then $\Delta_0 v_0 v_1$ and $\Delta_1 v_0 v_1$ cannot both be nontrivial equivalence relations. In the next lemma we show that neither both of $\Delta_0 v_0 v_1$ and $\Delta_2 v_0 v_1$ nor both of $\Delta_1 v_0 v_1$ and $\Delta_2 v_0 v_1$ can be nontrivial equivalence relations.

Lemma 4.3 Let $i \in \{0,1\}$. If $\Delta_i v_0 v_1$ and $\Delta_2 v_0 v_1$ both define equivalence relations in T and $\vdash_T \exists v_0 \exists v_1 (v_0 \neq v_1 \wedge \Delta_i v_0 v_1)$ then $\Delta_2 v_0 v_1$ is equivalent to $v_0 = v_1$.

Proof. Assume for contradiction that $T \vdash \exists v_0 \exists v_1 (v_0 \neq v_1 \wedge \Delta_2 v_0 v_1)$.
Let $A \models T$ and let a,b,c be distinct such that

$$A \models \Delta_i ab \wedge \Delta_2 bc .$$

Then $A \models \Delta_{1-i} ac$ follows from the hypothesis of the lemma. Without loss of generality suppose that $A \models \Gamma bc$. There exists c' such that $A \models \Gamma c'b$. From the hypothesis $A \models \Delta_1 ac'$. Since (a,c) and (a,c') realize the same type and $A \models \exists v_0 (\Gamma c' v_0 \wedge \Delta_i a v_0)$ there exists b' such that $A \models \Gamma cb' \wedge \Delta_i ab')$. Then we have $A \models \Delta_2 bb'$ and $A \models \Delta_i ab \wedge \Delta_i ab' \wedge b \neq b'$, contradiction. This proves Lemma 4.3 .

From Lemmas 4.1 and 4.3 no two of $\Delta_0 v_0 v_1$, $\Delta_1 v_0 v_1$ and $\Delta_2 v_0 v_1$ can be nontrivial equivalence relations in T . We use this observation to establish the theorem, that is to show that the definable equivalence relations of T form a chain. For this it suffices to show that the following two statements are true:

(I) Let i,j,k be 0,1,2 in some order and that

$(\Delta_i v_0 v_1 \vee \Delta_j v_0 v_1)$ and $(\Delta_k v_0 v_1)$ define equivalence

relations in T . Then one of these formulas is

equivalent to $v_0 = v_1$.

(II) Let i,j,k be 0,1,2 in some order and that

$T \vdash \exists v_0 \exists v_1 (v_0 \neq v_1 \wedge \Delta_\ell v_0 v_1)$ for $\ell = 0,1,2$. Then

not both of $(\Delta_i v_0 v_1 \vee \Delta_j v_0 v_1)$ and $(\Delta_j v_0 v_1 \vee \Delta_k v_0 v_1)$

can define an equivalence relation in T .

To see I let $A \models T$ and a,b,c be distinct such that $A \models \Delta_k ab$ and $A \models \Delta_i ac \vee \Delta_j ac$ then either $A \models \Delta_k bc$ or $A \models (\Delta_i \vee \Delta_j)bc$ a contradiction.

For II let a,b,c be distinct members of $A \models T$ such that $A \models \Delta_i ac \wedge \Delta_k ab$. Then $A \models (\Delta_i bc \vee \Delta_j bc)$ and $A \models (\Delta_j bc \vee \Delta_k bc)$ contradiction.

But this establishes Theorem 4.1. Lemmas 4.1 and 4.2 give interesting examples of nontrivial empty neighbourhoods, namely $N(\{E_0, E_1, E_2\}, \{\emptyset, D_{i,j}, D_{i,k}, D_{j,i} D_{j,k}\})$ where i,j,k are distinct in $\{0,1,2\}$ and for ℓ,m distinct members of $\{0,1,2\}$ $D_{\ell,m}$ is the structure with universe $\{a,b,c\}$ which models $\Delta_\ell ab \wedge \Delta_\ell bc \wedge \Delta_m ac$, and where E_i is the structure with universe $\{a,b\}$ which models $\Delta_i ab$.

3. Tournaments

In this section we turn our attention to those T for which $\forall v_0 \forall v_1 (\Delta_2 v_0 v_1)$ is a theorem. A model of T with two or more members is a tournament. In the first part of this section we characterize those T which have a finite model. In so doing we present a lemma which will be useful in Section 5 where we make a beginning on the problem of characterizing all directed graphs without loops whose theories admit elimination of quantifiers.

Lemma 4.4 Let $\theta \in L_{n+1}$ generate an $(n+1)$-type in T such that $T \vdash \theta \to v_0 \neq v_i$ for $1 \leq i \leq n$. Let $G \models T$ and let $\vec{a}$ be an n-tuple from G for which $G \models \exists v_0 \theta(v_0, \vec{a})$. Let H be the substructure of G determined by $\theta(v_0, \vec{a})(G)$. Then Th(H) also

admits elimination of quantifiers.

Proof. It suffices to show that if H_0 and H_1 are finite substructures of H and if $f : H_0 \to H_1$ is an isomorphism then there is an automorphism of H which extends f. But consider the substructures of G determined by $H_0 \cup \text{Rng } \vec{a}$ and $H_1 \cup \text{Rng } \vec{a}$ and the extension of f which leaves $\vec{a}$ fixed. This is an isomorphism of two finite substructures of G and can thus be extended to an automorphism of G. This automorphism fixes $\vec{a}$ and therefore the restriction of it to H is an automorphism of H which extends f. This completes the proof.

For the remainder of this section we assume that $T \vdash \Delta_2 v_0 v_1 \wedge \exists v_0 \exists v_1 (v_0 \neq v_1)$. Let $A \models T$ and a_0, a_1, a_2 be three members of A. We say that $\{a_0, a_1, a_2\}$ is a *3-cycle in* A just in case $A \models R a_0 a_1 \wedge R a_1 a_2 \wedge R a_2 a_0$. We say that $\{a_0, a_1, a_2\}$ is a *transitive triple* just in case $A \restriction \{a_0, a_1, a_2\} \models \forall v_0 \forall v_1 \forall v_2$ $(R v_0 v_1 \wedge R v_1 v_2 \to R v_0 v_2)$. The statement "A is a 3-cycle" has the obvious meaning.

We may now state the first of the two major results of this section.

Theorem 4.2 If T has a finite model then the only possibility for T is the theory of a 3-cycle.

Proof. Let $G \models T$ and let $a \in G$. By Lemma 4.4 we see that $O_a = \{b : G \models R\,ab\}$ and $I_a = \{b : G \models R\,ba\}$ determine substructures of G which either have one member or are finite tournaments whose theories admit elimination of quantifiers. Since the cardinalities

of O_a and I_a are independent of the choice of a and since $\sum_{a \in G} |O_a| = \sum_{a \in G} |I_a|$, we see that $|O_a| = |I_a|$. But then $|G| = 2m+1$ where $m = 1$ or m is the cardinality of a finite tournament whose theory admits elimination of quantifiers. But then the cardinality of G must be (2^k-1) for some $k \geq 2$, and for all k' where $2 \leq k' \leq k$ there must be a tournament of power $(2^{k'}-1)$ whose theory admits elimination of quantifiers. It will thus suffice to show that there is no such tournament of power seven.

For contradiction assume that G is a tournament of power seven whose theory admits elimination of quantifiers. Let the universe of G be $\{a_i : i < 7\}$ where $O_{a_0} = \{a_1, a_2, a_3\}$ and $I_{a_0} = \{a_4, a_5, a_6\}$. By the above there is no loss in assuming that $G \models Ra_1a_2 \wedge Ra_2a_3 \wedge Ra_3a_1$ and that $G \models Ra_4a_5 \wedge Ra_5a_6 \wedge Ra_6a_4$. Now there is no loss of generality in assuming that $G \models Ra_1a_4 \wedge Ra_1a_5$ since by inspection there must be two elements in $I_{a_0} \cap O_{a_1}$. Then $G \models Ra_6a_1$. Now consider the edge (a_0, a_1). To each j with $2 \leq j \leq 6$ we may associate an ordered pair $p_j = (k, \ell)$ from $\{0,1\}$ determined by

$$G \models (Ra_0a_j)^k \wedge (Ra_1a_j)^\ell.$$

Then these pairs are $(0,0)$, $(0,1)$, $(1,0)$, $(1,0)$, and $(1,1)$ in order. Now it is obvious that any automorphism of G which fixes (a_0, a_1) must also fix a_2, a_3 and a_6 since they are determined by the pairs p_2, p_3 and p_6 respectively. Also there must be an automorphism which fixes (a_0, a_1) and carries a_4 to a_5. This must interchange them since it fixes a_0, a_1, a_2, a_3 and a_6 ..

But then $G \models Ra_4a_5 \leftrightarrow Ra_5a_4$ a contradiction.

This completes the proof of Theorem 4.2. We are grateful to A.H. Lachlan for helping us simplify the above proof.

It is natural to ask what examples exist which have infinite models. There are, of course, D0 and the model generic for the class of all finite tournaments. There is also another countable model in which the successors of a point are linearly ordered by R. The countable model C may be described in the following way. First choose a countable dense subset $|C|$ of the unit circle in the plane with the property that if $\alpha \in |C|$ then the opposite end of the diameter through α does not belong to $|C|$. Now let $R_C = \{(\alpha,\beta) \in C \times C :$ signed angle subtended by the arc $\overset{\frown}{\beta\alpha}$ lies in $(0,\pi)\}$, that is β lies on the semicircle clockwise from α and is not α itself. Let $C = <|C|,R_C>$. We claim that Th(C) admits elimination of quantifiers and that the set of successors of a point is linearly ordered by R_C.

The second part of the claim is easy to see for if (α,β_1), (α,β_2), $(\alpha,\beta_3) \in R_C$ then whether $(\beta_i,\beta_j) \in R_C$ depends on the magnitude of $\overset{\frown}{\beta_i\alpha}$ and $\overset{\frown}{\beta_j\alpha}$. To see that Th(C) admits elimination of quantifiers we shall apply the results of Chapter 3. By Lemma 3.4 it suffices to show that C is Σ- homogeneous where Σ is the class of finite structures which can be embedded in C. By Lemma 3.1 to establish this it is sufficient to show that if $\bar{\alpha} = <\alpha_0,\ldots,\alpha_{n-1}>$ and $\bar{\beta} = <\beta_0,\ldots,\beta_{n-1}>$ are two sequences from C such that the mapping g taking α_i to β_i for $i<n$ is an

isomorphism, and if $\alpha \in C$ then there is $\beta \in C$ such that the exten-sion to $\bar{\alpha} \cap <\alpha>$ of g which takes α to β is an isomorphism.
If $n = 0$ or $\alpha \in \{\alpha_0, \ldots, \alpha_{n-1}\}$ this is trivial. Otherwise the atomic formulae satisfied by $<\alpha_i, \alpha>$ permit one to translate the problem into a question about an intersection of open intervals in the rationals, when it can be seen to be the case that β must exist as above.

It turns out that the rationals and C are the only two infinite models in which the successors of a member are linearly ordered.

Theorem 4.3 If T satisfies the following three conditions then C is a model of T.

(i) T has an infinite model.

(ii)   There is a three cycle in T.

(iii)   The successors of a point are linearly ordered.

Proof. Let G be countable and let $G \models T$. We shall show that $G \cong C$. First however we establish the following two claims.

Claim 1.   The predecessors of a point are linearly ordered, that is

$$T \vdash [\bigwedge_{1 \leq i \leq 3} Rv_i v_0) \rightarrow Rv_1 v_2 \rightarrow Rv_2 v_3 \rightarrow Rv_1 v_3].$$

For a contradiction assume that

$$G \models Rb_0 b_1 \wedge Rb_1 b_2 \wedge Rb_2 b_0 \wedge \bigwedge_{i<3} Rb_i a.$$

Now since T admits elimination of quantifiers there is $c \in G$ such that $G \models Rac \wedge Rcb_0$. It is easy to see that c is distinct from a, b_0, b_1, b_2 since $G \models Rac$. Consider $\{b_2, c\}$. Since $G \models Rb_2 c$ would mean that the three cycle $\{a, c, b_0\}$ was in the set of successors

of b_2 we must have that $G \models Rcb_2$. Also, since $G \models Rb_1c$ would mean that the three cycle $\{b_2,a,c\}$ lies in the outset of b_1 we have $G \models Rcb_1$. But then b_0,b_1,b_2 lie in the outset of c, contradiction.

Claim 2. Let $a \in G$ and let $B_0 = \{b : G \models Rab\}$ and let $B_1 = \{b : G \models Rba\}$. Then $\langle B_i, R_G \upharpoonright B_i \rangle \models DO$ for $i = 0,1$.

This claim is an easy consequence of elimination of quantifiers. To give the idea we show that B_i is infinite for $i = 0,1$. Clearly one of B_0, B_1 is infinite, and each is linearly ordered by R_G. Suppose B_0 is infinite. Then there are b_0,b_1,b_2 such that

$$G \models \bigwedge_{i<3} Rab_i \wedge Rb_0b_1 \wedge Rb_1b_2 \wedge Rb_0b_2 .$$

Now a,b_2 realize the same 1-type and it easily follows that R_G is a dense order on B_0 and B_1 and they are infinite.

It is clear that G is the generic structure for the class of finite structures which can be embedded in G. By Lemma 3.3 to show $G \cong C$ it will suffice to show that if H is a finite structure then H can be embedded in G just in case it can be embedded in C. We show the condition holds by induction on the cardinality n of H. When $n \leq 3$ the condition holds trivially. Let H be given and assume that whenever $|H| \geq 4$ and whenever $|H_0| < |H|$ then H_0 can be embedded in G just in case it can be embedded in C.

Case 1. H is a linear order. Since B_0, B_1 are infinite this case is trivial.

Case 2. Otherwise there is a 3-cycle $\{h_0, h_1, h_2\}$ in H, where $H \models \Gamma h_0 h_1 \wedge \Gamma h_1 h_2 \wedge \Gamma h_2 h_0$. We shall assume that $H \subseteq G$ and show that H can be embedded in C. The other implication is proved similarly. There is no loss of generality in assuming that $h \in H \setminus \{h_0, h_1, h_2\}$ and $H \models (\Gamma h h_0 \leftrightarrow \Gamma h h_1)$. Let H_0 be the sub-structure determined by $H \setminus \{h\}$ and let H_1 be that determined by $H \setminus \{h_2\}$. By the induction hypothesis H_0 and H_1 can be embedded in C. Since C is generic for the class of structures which can be embedded in it we may assume the embeddings agree on $H_0 \cap H_1$. Let these embeddings be e_0 and e_1 respectively. Now $C \models (\Gamma e_0(h) e_0(h_0) \leftrightarrow \Gamma e_0(h) e_0(h_1))$. Thus $e_0(h) \neq e_1(h_2)$. But also it entails that $G \models \Gamma h h_2$ iff $C \models \Gamma e_0(h) e_1(h_2)$. But then $e_0 \cup e_1$ is an embedding of H into C.

This establishes Theorem 4.3. An open question (Q7) is just what other countable tournaments have theories which admit elimination of quantifiers.
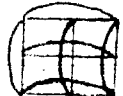
4. Undirected graphs

In this section we turn to another case in which combinatorial arguments have characterized those finite structures whose theories admit elimination of quantifiers - symmetric irreflexive graphs. In this section we assume that the following formula is a theorem of the theory T

$$\forall v_0 \forall v_1 (R v_0 v_1 \leftrightarrow R v_1 v_0).$$

The models of such theories are just *undirected graphs*.

A. Gardiner [4; Th. 7, Th. 12] proved the following:

Theorem 4.4 Let $G \models T$ and assume that G is finite. Then either

G or $\bar{G}$ is isomorphic to one of the following:

(i) K_k^n - a model of E_k^n for some $n, k < \omega$,

(ii) P , the pentagon ,

or (iii) $L(K_{3,3})$,

Here the pentagon P is the structure with universe $\{i : i < 5\}$

and with $R_p = \{(i,j) : i = j+1 \lor j = i+1 \lor (i = 4 \land j = 0) \lor$

$(i = 0 \land j = 4)\}$; and $L(K_{3,3})$ is the line graph on $K_{3,3} = \overline{K_3^2}$:

the vertices of $L(K_{3,3})$ are the unordered pairs $\{a,b\}$ such that

$(a,b) \in R_{K_{3,3}}$ and the edges of $L(K_{3,3})$ are the pairs (e,f) such

that $e \cap f \neq \phi$. $L(K_{3,3})$ is easily seen to be isomorphic to the

product of two copies of the complete graph on three vertices. Note

that P and $L(K_{3,3})$ are both self-dual.

Following Gardiner we call a graph G *ultrahomogeneous* just in

case it is Σ-homogeneous where Σ is the class of finite structures

which may be embedded in G .

In the remainder of this section we assume that T has an

infinite model. An interesting family of ultrahomogeneous graphs is

provided by the methods of Chapter 3. Let K_n be the complete graph

with n vertices. For $2 \leq n$ let Σ_n be the class of undirected

graphs which omit K_n . Then Σ_n has arbitrarily large members,

for example $\bar{K}_m$ for $m \in \omega$. We may see that Σ_n has the amalgamation

property as follows.

If G, H_0, $H_1 \in \Sigma_n$ and $|H_0| \cap |H_1| = |G|$ then let H be the structure with $|H| = |H_0| \cup |H_1|$ and $R_H = R_{H_0} \cup R_{H_1}$. Then $H \in \Sigma_n$ trivially. Now let G_n be a Σ_n-generic structure by Lemma 3.2 . G_n is ultrahomogeneous.

The main result of this section is the following theorem which lends some further support to C7 .

Theorem 4.5 There are only four countable ultrahomogeneous undirected graphs which omit the triangle, K_3 . The proof of the theorem is broken down into a sequence of lemmas. For the remainder of this section graph will mean undirected graph. Also fix G a countable ultrahomogeneous graph which omits K_3 . Let Σ be the class of finite graphs which can be embedded in G . Let S_2 be the graph $\diagdown\!\!\diagup$ where $|S_2| = \{a_0, a_1, a_2\}$ and $R_2 = \{(a_i, a_j) :$ $i \neq j \wedge i \cdot j = 0\}$. Let M^* be the graph $\cdots\big|$, i.e. $<\{b_0, b_1, b_2\}$, $\{(b_1, b_2)$, $(b_2, b_1)\}>$.

Let K_k^n be a model of E_k^n , that is an equivalence relation with n classes each containing k members. Recall the abbreviations given in §1.

Lemma 4.5 If $M^* \notin \Sigma$ or $S_2 \notin \Sigma$ then G is isomorphic to one of the following:

$\qquad$ (i) K_1^ω .

$\qquad$ (ii) K_2^ω

$\qquad$ (iii) $\overline{K_\omega^2}$

Proof. If $M^* \notin \Sigma$ then $\Delta_1 v_0 v_1(G)$ is an equivalence relation and the equivalence classes have the same cardinality. Since $K_3 \notin \Sigma$ there are at most two classes for if g_1, g_2, g_3 all belong to different classes we must have $(g_i, g_j) \in R(G)$ for $i \neq j$ which is impossible. Thus G is isomorphic to K_1^ω or $\overline{K_\omega^2}$.

If $S_2 \notin \Sigma$ then $\Delta_0 v_0 v_1(G)$ is an equivalence relation all of whose equivalence class have the same cardinality. The fact that $K_3 \notin \Sigma$ ensures that each class contains at most two elements. Thus G is either isomorphic to K_1^ω or K_2^ω, and the proof of Lemma 4.5 is completed.

For the remainder of this section we assume that M^* and S_2 belong to Σ. From Lemma 3.5 Σ has the amalgamation property and we rely very heavily on this fact below. We shall show that $\Sigma = \Sigma_3$, whence G will be isomorphic to G_3. To establish that $\Sigma = \Sigma_3$ some definitions are required.

For $n \geq 2$ define the *n-star* $S_n = \langle \{a_i : i \leq n\}, R_n \rangle$ where

$R_n = \{(a_i, a_j) : i \neq j \wedge i \cdot j = 0\}$, i.e. $S_n = a_0 \begin{array}{c} \bullet a_1 \\ \vdots \\ \bullet a_n \end{array}$.

Let $I = K_1^1$, i.e. $I = \bullet\bullet$ and let $M = K_2^1$, i.e. $M = \big|$.

Let $\{H_\alpha ; \alpha \in A\}$ be a nonempty family of graphs where $H_\alpha = \langle |H_\alpha|, R_\alpha \rangle$. Define the *disjoint union* $\langle H, E \rangle = \biguplus \{H_\alpha : \alpha \in A\}$ as follows:

$$|H| = \{(h, \alpha) : h \in H_\alpha \wedge \alpha \in A\}$$

and

$$E = \{((h, \alpha), (g, \beta)) : \alpha = \beta \wedge (h, g) \in R_\alpha\} .$$

If κ is a cardinal and H is a graph $(\kappa \cdot H)$ is the disjoint union of κ copies of H . There is no loss in assuming that $\uplus$ is commutative and associative. If H and J are graphs then $(0 \cdot H) \uplus J$ is just J . Note that $M^* \cong I \uplus M$.

We shall complete the proof of Theorem 4.5 by proving the following four lemmas:

Lemma 4.6   For all n $(n \cdot I) \uplus S_2 \in \Sigma$

Lemma 4.7 $(n \cdot S_2) \in \Sigma$ for all $n \geq 1$

Lemma 4.8 $(n \cdot S_m) \in \Sigma$ for all $n \geq 1$ and $m \geq 2$

Lemma 4.9 $\Sigma = \Sigma_3$

Of course the theorem is immediate from Lemma 4.9 for which the other lemmas are a necessary preliminary.

Proof of Lemma 4.6   We prove by induction that $(n \cdot I) \uplus S_2 \in \Sigma$ and $((n+1) \cdot I) \uplus M \in \Sigma$. With $n=0$ this is simply that S_2 and M^* belong to Σ . The induction step follows from Propositions 1, 2, and 3 below.

Proposition 1.   If $(n \cdot I) \uplus S_2 \in \Sigma$ then $((n+1) \cdot I) \uplus S_2 \in \Sigma$ or $(n \cdot I) \uplus S_3 \in \Sigma$.

Proof.   For an application of AP we may assume that $(n \cdot I) \cup S_2 \cong B_0$ where $|B_0| = N \cup \{a_0, a_1, a_2\}$ and $R(B_0) = \{(a_i, a_j) : i \neq j \ \& \ i \cdot j = 0\}$ Let A be the substructure of B_0 generated by $N \cup \{a_1, a_2\}$. Let b be new and define B_1 such that $|B_1| = N \cup \{a_1, a_2, b\}$, and $B_1 \cong (n+3) \cdot I$. Now $A, B_0 \in \Sigma$. Also $B_1 \in \Sigma$ since from Ramsey's theorem we easily get that $\omega \cdot I$ can be embedded in G .

Thus let $C \in \Sigma$ and let $f_0 : B_0 \to C$ and $f_1 : B_1 \to C$ be embeddings such that $f_0 \upharpoonright A = f_1 \upharpoonright A$. Then $f_0(a_0) \neq f_1(b)$ since $(a_0, a_1) \in R(B_0)$ while $(b, a_1) \notin R(B_1)$. The disjunction depends on whether $(f_0(a_0), f_1(b)) \in R(C)$.

The reader may find the proofs less difficult to follow from consideration of the figures which we shall present. The following figure corresponds to Proposition 1.
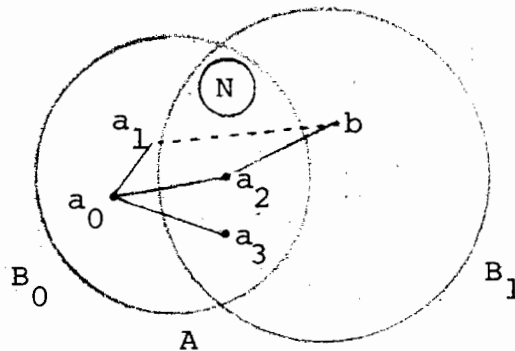
Figure 1.
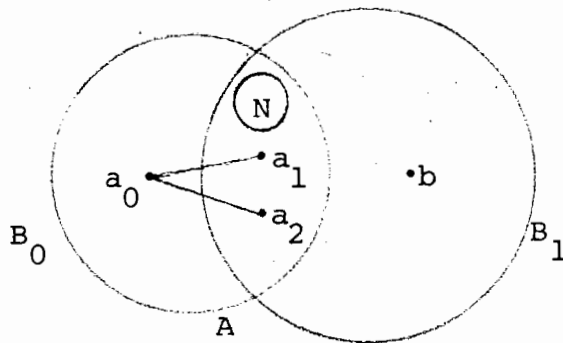
Proposition 2. If $(n \cdot I) \uplus S_3 \in \Sigma$ and $(((n+1) \cdot I) \uplus M) \in \Sigma$ then $((n+1) \cdot I) \uplus S_2 \in \Sigma$.

Proof. The figure for this proposition is figure 2.

Figure 2.

For an application of AP we may choose $B_0 \cong (n \cdot I) \uplus S_3$ where $|B_0| = N \cup \{a_0, a_1, a_2, a_3\}$ and $R(B_0) = \{(a_i, a_j) : i \neq j$ and $i \cdot j = 0\}$.

Let $A = N \cup \{a_2, a_3\}$ and let b be new and let $B_1 \cong ((n+1) \cdot I) \uplus M$ where $|B_1| = A \cup \{b\}$ and $R(B_1) = \{(a_2, b), (b, a_2)\}$.

Now let $C \in \Sigma$ and let $f_0 : B_0 \to C$ and $f_1 : B_1 \to C$ be embeddings such that $f_0 \restriction A = f_1 \restriction A$. Then $f_1(b) \neq f_0(a_0)$ since $(a_0, a_3) \in R(B_0)$ while $(b, a_3) \notin R(B_1)$ and similarly $f_1(b) \neq f_0(a_1)$ because of a_2. There is no loss of generality in assuming that f_0, f_1 are inclusions. Now $(a_0, b) \notin R(C)$ since $K_3 \notin \Sigma$. There are now two cases:

Case 1. $(a_1, b) \notin R(C)$

and Case 2. $(a_1, b) \in R(C)$.

In Case 1 consider $N \cup \{a_0, a_1, a_3, b\}$. This subgraph is isomorphic to $((n+1) \cdot I) \uplus S_2$.

In Case 2 consider $N \cup \{b, a_1, a_2, a_3\}$ which is isomorphic to $((n+1) \cdot I) \uplus S_2$.

Proposition 3. If $((n+1) \cdot I) \uplus S_2 \in \Sigma$ then $((n+2) \cdot I) \uplus M \in \Sigma$.

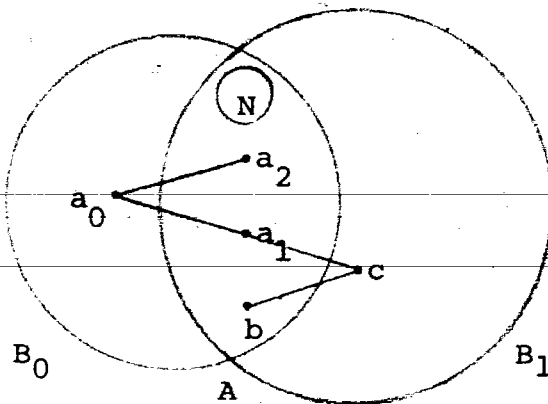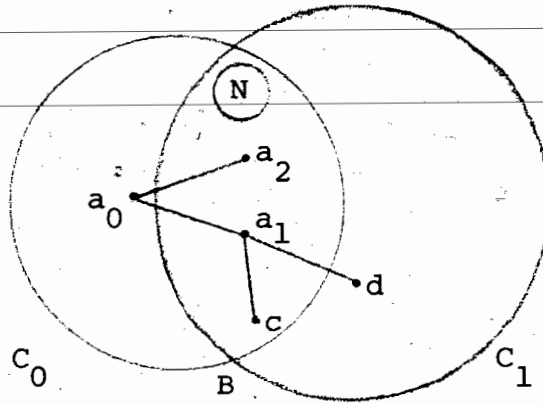Proof. For this proposition consider figures 3 and 4.

Figure 3.

Figure 4.

For a first application of AP let $B_0 \cong ((n+1) \cdot I) \uplus S_2$ where $|B_0| = N \cup \{b, a_0, a_1, a_2\}$ and $R(B_0) = \{(a_i, a_j) : i \neq j$ and $i \cdot j = 0\}$. Let $A = N \cup \{b, a_1, a_2\}$ and letting c be new define $B_1 \cong B_0$ by $|B_1| = A \cup \{c\}$ and $R(B_1) = \{(c, b), (b, c), (c, a_1), (a_1, c)\}$. Now let $C \in \Sigma$ and $f_0 : B_0 \to C$ and $f_1 : B_1 \to C$ be embeddings such that $f_0 \upharpoonright A = f_1 \upharpoonright A$. Then $f_0(a_0) \neq f_1(c)$ and without loss of generality f_0, f_1 are inclusions. Since $K_3 \not\in \Sigma$ we have $(a_0, c) \not\in R(C)$.

But then the substructure C_0 determined by $N \cup \{a_2, a_0, a_1, c\}$ belongs to Σ. For a second application of AP let $B = N \cup \{a_2, a_1, c\}$. Let d be a new individual and let $C_1 \cong ((n+1) \cdot I) \uplus S_2$ be such that $|C_1| = B \cup \{d\}$ and $R(C_1) = \{(a_1, c), (c, a_1), (a_1, d), (d, a_1)\}$. Now let $D \in \Sigma$ and $g_0 : C_0 \to D$ and $g_1 : C_1 \to D$ be embeddings with $g_0 \upharpoonright B = g_1 \upharpoonright B$. Now $g_0(a_0) \neq g_1(d)$ since $(a_0, a_2) \in R(C_0)$ while $(d, a_2) \not\in R(C_1)$. Thus we may assume that g_0, g_1 are inclusions. Since $K_3 \not\in \Sigma, (a_0, d) \not\in R(D)$. Now the substructure determined by $N \cup \{c, d, a_0, a_2\}$ is isomorphic to $((n+2) \cdot I) \uplus M$.

This completes the proof of Lemma 4.6.

We now turn to the proof of Lemma 4.7: If $(n \cdot I) \uplus S_2 \in \Sigma$ for all n then for all $n \geq 1$ $(n \cdot S_2) \in \Sigma$.

Proof. We prove by induction on $n \geq 1$ that $((m \cdot I) \uplus (n \cdot S_2)) \in \Sigma$

for all m. With $n=1$ this is simply the conclusion of Lemma 4.6.

The induction step involves three applications of AP and is greatly

simplified by consideration of figures 5, 6, and 7.
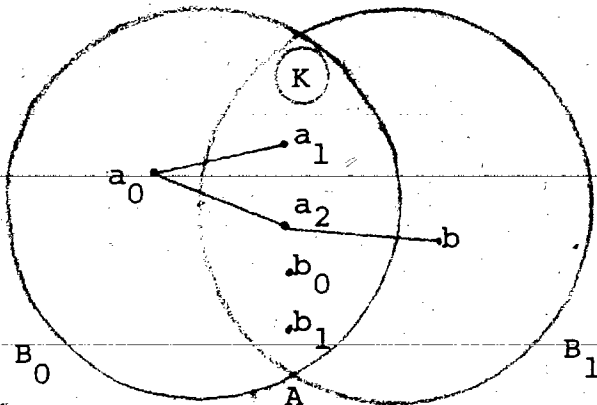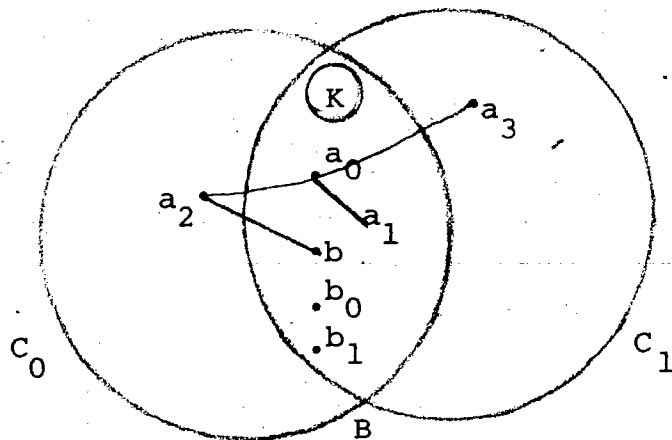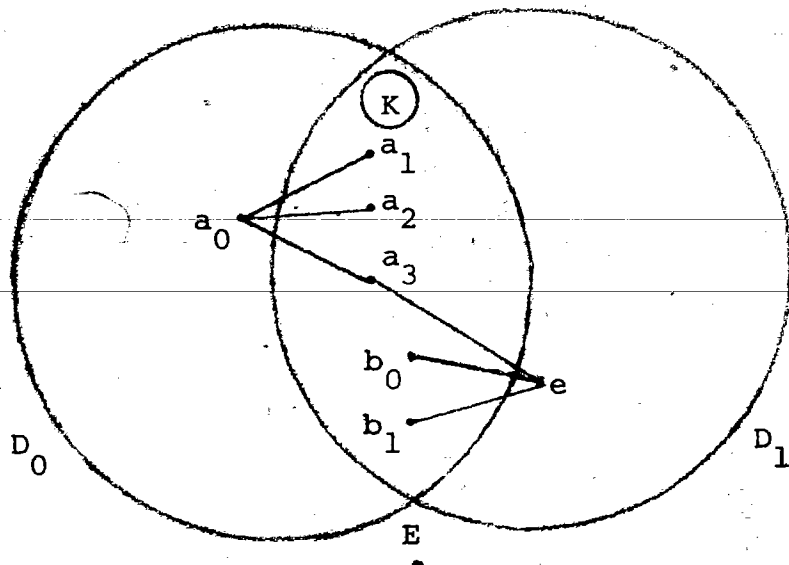
Figure 5.

Figure 6.

Figure 7.

In the first two amalgamations a label is attached to an outer vertex of S_2 and with this a 3-star is created. Then the 3-star is used to create two 2-stars. For the reader who can follow the diagrams this should suffice. However to avoid any possible obscurity let $m \geq 1$ be fixed and fix $K \cong ((m \cdot I) \uplus ((n-1) \cdot S_2))$.

Claim. $K \uplus (2 \cdot I) \uplus S_3 \in \Sigma$.

Fix new entities $a_0, a_1 a_2, b_0, b_1$, and b. For an application of AP let B_0 be determined by $|B_0| = |K| \cup \{a_0, a_1', a_2, b_0, b_1\}$ and $R(B_0) = R(K) \cup \{(a_i, a_j) : i \neq j$ and $i \cdot j = 0\}$ and let $A = K \cup \{a_1, a_2, b_0, b_1\}$. $B_0 \cong (((m+2) \cdot I) \uplus (n \cdot S_2))$ and so belongs to Σ. Let B_1 be such that $|B_1| = |A| \cup \{b\}$ and $R(B_1) = R(K) \cup \{(a_2, b), (b, a_2)\}$. Now $B_1 \in \Sigma$.

Let $C \in \Sigma$ and let $e_0 : B_0 \to C$ and $e_1 : B_1 \to C$ be embeddings with $e_0 \upharpoonright A = e_1 \upharpoonright A$. Now $e_0(a_0) \neq e_1(b)$ since $(a_0, a_1) \in R(B_0)$ while $(b, a_1) \notin R(B_1)$. Thus without loss e_0, e_1 are inclusions. $(a_0, b) \notin R(C)$ since $K_3 \notin \Sigma$.

For another application of AP let C_0 be the subgraph determined by $K \cup \{a_0, a_1, a_2, b_0, b_1, b\}$. Let $B = K \cup \{a_0, a_1, b, b_0, b_1\}$. Let a_3 be new and let C_1 be such that $|C_1| = B \cup \{a_3\}$ and $R(C_1) = R(K) \cup \{(a_3, a_0), (a_0, a_3), (a_0, a_1), (a_1, a_0)\}$. Then $C_1 \cong ((m+3) \cdot I) \uplus (n \cdot S_2) \in \Sigma$.

Now let $D \in \Sigma$ and let $f_0 : C_0 \to D$ and $f_1 : C_1 \to D$ be embeddings with $f_0 \upharpoonright B = f_1 \upharpoonright B$. By inspection $f_0(a_2) \neq f_1(a_3)$ and without loss f_0, f_1 are inclusions. $(a_3, a_2) \notin R(D)$ since $K_3 \notin \Sigma$. But then the subgraph determined by $K \cup \{a_0, a_1, a_2, a_3, b_0, b_1\}$ is isomorphic to $K \uplus (2 \cdot I) \uplus S_3 \in K$, establishing the claim.

With another application of AP we may see that $K \uplus (2 \cdot S_2) \in \Sigma$.

Let D_0 be the subgraph determined by $K \cup \{a_0 a_1, a_2, a_3, b_0, b_1\}$ in the above. Let e be new and let D_1 be determined by $|D_1| = K \cup \{a_1, a_2, a_3, b_0, b_1, e\}$ and $R(D_1) = R(K) \cup \{(a_3, e), (e, a_3),$ $(b_0, e), (e, b_0), (b_1, e), (e, b_1)\}$. Let $E = D_0 \cap D_1$. Then $D_0, D_1 \in \Sigma$.

Let $F \in \Sigma$ and let $g_0 : D_0 \to F$ and $g_1 : D_1 \to F$ be embeddings with $g_0 \upharpoonright E = g_1 \upharpoonright E$. Then $g_0(a_0) \neq g_1(e)$ and we may take g_0, g_1 to be inclusions. The substructure of F determined by $K \cup \{a_0, a_1, a_2, e, b_0, b_1\}$ is isomorphic to $(m \cdot I) \uplus ((n+1) \cdot S_2)$. This completes the proof of Lemma 4.7.

Now we turn to the proof of Lemma 4.8, that $(n \cdot S_m) \in \Sigma$ for $n \geq 1$ and $m \geq 2$.

Proof. The proof of this lemma is very similar to the first part of the proof of Lemma 4.7. Using the amalgamation property we tag one of the outer vertices of an m-star and then by another amalgamation produce an (m+1)-star. This is shown in figures 8 and 9.
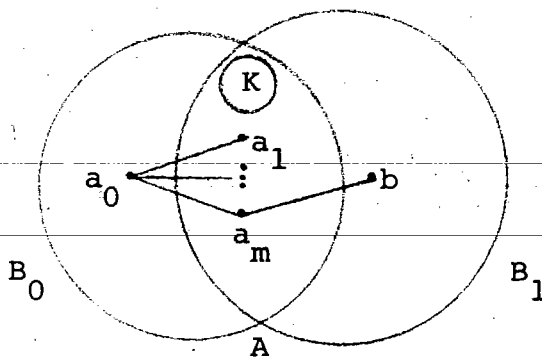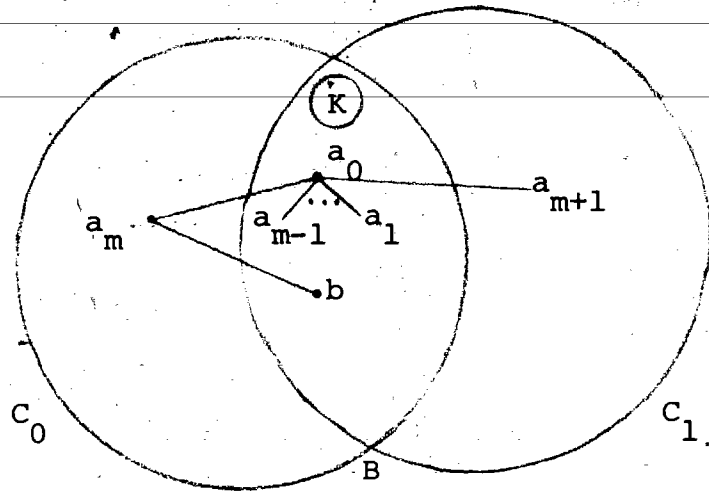
Figure 8.

Figure 9.

Formally we prove by induction on $m \geq 2$ that for all $n \geq 1$ $(n \cdot S_m) \in \Sigma$.

With $m = 2$ this is just the conclusion of Lemma 4.7. Assume that

$(k \cdot S_m) \in \Sigma$ for all $k \geq 1$. Now we prove by induction on $n \geq 0$ that

for all $k \geq 1$ $((k \cdot S_m) \uplus (n \cdot S_{m+1})) \in \Sigma$. Assume that $((k \cdot S_m) \uplus (n \cdot S_{m+1}))$

$\in \Sigma$ for all $k \geq 1$. Let $K \cong ((k \cdot S_m) \uplus (n \cdot S_{m+1}))$ and let

$a_0, \ldots, a_m$ be distinct new individuals. Define $B_0 \cong (K \uplus S_m)$ by

$|B_0| = |K| \cup \{a_i : i \leq m\}$ and $R(B_0) = R(K) \cup \{(a_i, a_j) : i \neq j \text{ and } i \cdot j = 0\}$

Let $A = K \cup \{a_1, \ldots, a_m\}$. Let b be new and define B_1 by

$|B_1| = A \cup \{b\}$ and $R(B_1) = R(K) \cup \{(b, a_m), (a_m, b)\}$. Then $B_1 \in \Sigma$.

By AP let $C_0 \in \Sigma$ and let $f_0 : B_0 \to C_0$ and $f_1 : B_1 \to C_0$ be

embeddings for which $f_0 \restriction A = f_1 \restriction A$. Then $f_0(a_0) \neq f_1(b)$ since

$(a_0, a_1) \in R(B_0)$ while $(b, a_1) \notin R(B_1)$. Thus we may assume that

f_0 and f_1 are inclusions and then $(a_0, b) \notin R(C_0)$ since $K_3 \notin \Sigma$.

For another application of AP let $B = K \cup \{a_0, \ldots, a_{m-1}, b\}$ and

let a_{m+1} be new and let C_1 be such that $|C_1| = B \cup \{a_{m+1}\}$ and

$R(C_1) = R(B) \cup \{(a_{m+1}, a_0), (a_0, a_{m+1})\}$. Then $C_1 \in \Sigma$.

Let $D \in \Sigma$ and let g_0, g_1 be embeddings of C_0 and C_1 into D respectively such that $g_0 \upharpoonright B = g_1 \upharpoonright B$. Now $g_0(a_m) \neq g_1(a_{m+1})$ since $(a_m, b) \in R(C_0)$ while $(a_{m+1}, b) \notin R(C_1)$. Without loss g_0, g_1 are inclusions. Now $(a_m, a_{m+1}) \notin R(D)$ and $K \cup \{a_0, \ldots, a_{m+1}\}$ is isomorphic to $(k \cdot S_m) \uplus ((n+1) \cdot S_{m+1})$.

This complete proof of Lemma 4.8.

We now turn to the proof of Lemma 4.9, that is $\Sigma = \Sigma_3$.

Proof. It is clear that $\Sigma \subset \Sigma_3$. We prove by induction on the lexicographic order on pairs $(\|B\|, |R(B)|)$ that if $B \in \Sigma_3$ then $B \in \Sigma$.

Thus let $H \in \Sigma_3$ and assume that if $B \in \Sigma_3$ and B has fewer vertices than H or if $\|B\| = \|H\|$ but B has fewer edges than H then $B \in \Sigma$. We now consider three cases.

Case 1. There are elements $h_0, h_1, h_2 \in H$ such that (h_0, h_1), $(h_0, h_2) \in R(H)$ and such that the mapping $f : (H \setminus \{h_1\}) \to (H \setminus \{h_2\})$ which has $f(h_2) = h_1$, and $f(h) = h$ for $h \in H \setminus \{h_1, h_2\}$ is not an isomorphism.

By the induction hypothesis $H \setminus \{h_1\}$ and $H \setminus \{h_2\} \in \Sigma$ and we may choose $K \in \Sigma$ and embeddings $f_0 : (H \setminus \{h_1\}) \to K$ and $f_1 : (H \setminus \{h_2\}) \to K$ such that f_0 and f_1 agree on $H \setminus \{h_1, h_2\}$. Then $f_0(h_2) \neq f_1(h_1)$ since the mapping f is not an isomorphism. Thus there is no loss in taking f_0, f_1 to be inclusions. We have $(h_1, h_2) \notin R(K)$ since $K_3 \notin \Sigma$. But then $H \subset K \subset G$ and $H \in \Sigma$.

Case 2. There are $h_0, h_1, h_2 \in H$ such that $(h_0, h_1), (h_0, h_2) \in R(H)$ and such that the mapping $f : H \backslash \{h_1\} \to H \backslash \{h_2\}$ defined in Case 1 is an isomorphism but such that $|R(H \backslash \{h_1\})| + 2 < |R(H)|$.

Then let k be new and let K_2 be the graph for which $|K_2| = (H \backslash \{h_1\}) \cup \{k\}$ and $R(K_2) = R(H \backslash \{h_1\}) \cup \{(h_2, k), (k, h_2)\}$. Now $K_2 \in \Sigma_3$ and by the induction hypothesis $K_2 \in \Sigma$ since $|R(K_2)| = |R(H \backslash \{h_1\})| + 2$.

Let K_1 be the graph such that $|K_1| = (H \backslash \{h_2\}) \cup \{k\}$ and $R(K_1) = R(H \backslash \{h_2\})$. $K_1 \in \Sigma$ and $K_1 \cap K_2 = (H \backslash \{h_1, h_2\}) \cup \{k\}$. By the amalgamation property let $K \in \Sigma$ and let $f_1 : K_1 \to K$ $f_2 : K_2 \to K$ be embeddings such that $f_1 \upharpoonright (K_1 \cap K_2) = f_2 \upharpoonright (K_1 \cap K_2)$.

Then $f_1(h_1) \neq f_2(h_2)$ since $(h_1, k) \notin R(K_1)$ while $(h_2, k) \in R(K_2)$. Thus without loss f_1, f_2 are inclusions. Note that $(h_1, h_2) \notin R(K)$. But then $H \subseteq K$ and so $H \in \Sigma$.

Case 3. Otherwise for all $h_0, h_1, h_2 \in \Sigma$ if $(h_0, h_2), (h_0, h_1) \in R(H)$ then $(h, h_1) \in R(H)$ iff $(h, h_2) \in R(H)$ for all $h \in H$ since Case 1 fails; and indeed if $(h, h_1) \in R(H)$ then $h = h_0$ since Case 2 fails. But then it is easy to see that there are m, n such that H is isomorphic to a subgraph of $n \cdot S_m$ and so $H \in \Sigma$.

This completes the proof of Lemma 4.9 and hence the proof of Theorem 4.5.

We conjecture

(C8) For each n there are a finite number of countable undirected graphs whose theories admit elimination of quantifiers and which omit K_n .

The extension of the techniques employed in this paper is not obvious even for the case $n=4$.

5. Finite ultrahomogeneous graphs

In this section we examine briefly the problem of describing all finite graphs which model $\neg Rv_0v_0$ and whose theories admit elimination of quantifiers. This project is far from complete, but continues from the work of Theorem 4.2 and the work of Gardiner [4]. In this section G is assumed to be a finite ultrahomogeneous graph which models $(\neg Rv_0v_0)$.

Lemma 4.4 holds out the hope of obtaining some inductive characterization of such G : . If $a \in G$ then each of the substructures determined by $\{b : G \models \Gamma ba\}$, $\{b : G \models \Gamma ab\}$, $\{b : G \models \Delta_0 ab \wedge a \neq b\}$ and $\{b : G \models \Delta_1 ab \wedge a \neq b\}$ is a graph whose theory admits elimination of quantifiers.

If $G \models \Delta_0 v_0 v_1$ or if $G \models \Delta_1 v_0 v_1$ then G is K_n or $\overline{K}_n$ for some n. If $G \models \Delta_2 v_0 v_1$ then G is K_1 or a 3-cycle. The work of Gardiner describes those G in which $(\Delta_0 v_0 v_1 \vee \Delta_1 v_0 v_1)$ is valid. In this section we give some results on those G which model $(\Delta_1 v_0 v_1 \vee \Delta_2 v_0 v_1)$. For the remainder of this section assume that $G \models \Delta_1 v_0 v_1 \vee \Delta_2 v_0 v_1$.

Say that G is *connected* just in case there are no non-empty $G_0, G_1 \subseteq G$ with $G_0 \cap G_1 = \phi$, $G_0 \cup G_1 = G$ and $(R_G \restriction G_0) \cup (R_G \restriction G_1) = R_G$. *Component* has the usual meaning.

The first easy observation is the following:

Lemma 4.10. If G is not connected then G is isomorphic to

$\bar{K}_n$ for some n or G is a disjoint union of 3-cycles.

Proof. Since G is finite the equivalence relation " a lies in

the same component as b " is a definable equivalence relation. From

this it follows that any automorphism of G must permute its components.

The subgraph determined by a component C is then itself ultrahomo-

geneous since the restriction to C of the appropriate automorphism

of G will furnish an automorphism of C extending an isomorphism of

subgraphs of C .

If a,b lie in different components of G then $G \models \Delta_1 ab$.

Hence each component models $\Delta_2 v_0 v_1$ by elimination of quantifiers.

Therefore each component is either a 3-cycle or each component is a

singleton.

In the sequel we assume that G is connected. Then we conjecture

(C9) G has a Hamiltonian circuit.

This is supported by

Lemma 4.11 G has a spanning circuit, that is there is some sequence

$\langle a_n : n \leq m \rangle$ such that $G = \{a_n : n \leq m\}$, $a_0 = a_m$ and for all $n < m$

$G \models R a_n a_{n+1}$.

Proof. If $|G| = 1$ there is nothing to prove. Thus assume that

$G \models \exists v_0 \exists v_1 (\Gamma v_0 v_1)$. Now since G is finite the relation Sab ,

defined on G by "there is a sequence $\langle a_n : n \leq m \rangle$ such that $a_0 = a$,

$a_m = b$ and $m > 0$ and $G \models R a_n a_{n+1}$ for all $n < m$ " , is definable.

Note that S is transitive. Since Th(G) admits elimination of quantifiers |{b : Sab holds}| is independent of the choice of a .
Fix $a \in G$ and let a_0 be such that $G \models Ra_0 a$. Since $Sa_0 a$ holds we have {b : Sab holds} $\subset$ {b : $Sa_0 b$ holds} whence equality follows.
Clearly Saa holds, i.e. S is reflexive. If Sab holds, then Sba holds since Saa holds and {c : Sac holds} = {c : Sbc holds} .
But then S is a definable equivalence relation on G . If $G \models Rab$ then a,b are in the same equivalence class under S .
Since G is connected any $a,b \in G$ are S-equivalent. The desired conclusion now follows easily.

Chapter 5

The Theory T_0

In this chapter we construct an example of a complete theory T_0 in a small language which has a nonprincipal 1-type p containing infinitely many 2-types q and such that T_0 has a finite number of isomorphism types of countable models. The idea is to generalize the construction used by Peretyiatkin [8]. We produce a dense tree with infinite branching and indexing of subtrees above a node by the rationals.

Let L be the language with one unary function symbol U , a binary predicate symbol $<$, a binary function symbol $\wedge$, a binary function symbol I and a constant symbol c .

Let T be the theory with language L and nonlogical axioms:

Group 1. $<U, \wedge \upharpoonright U>$ is a tree, otherwise $\wedge$ is trivial. Thus letting $x \alpha y \equiv_{D_f} Ux \wedge Uy \wedge (x \wedge y = x) \wedge x \neq y$ the axioms are

$$v_0 \wedge v_1 = v_1 \wedge v_0 \ , \ (v_0 \wedge v_1) \wedge v_2 = v_0 \wedge (v_1 \wedge v_2), \ v_0 \wedge v_0 = v_0 \ ,$$

$$v_0 \alpha v_2 \wedge v_1 \alpha v_2 \ . \rightarrow . \ (v_0 \alpha v_1) \vee (v_1 \alpha v_0) \vee (v_0 = v_1) \ ,$$

$$\neg U v_0 \rightarrow (v_0 \wedge v_1 = c) \ , \ \neg U c \ .$$

Group 2. $<\neg U \backslash \{c\}, <>$ is a linear order, otherwise $<$ is trivial:

$$v_0 < v_1 \rightarrow \neg U v_0 \wedge \neg U v_1 \wedge v_0 \neq c \wedge v_1 \neq c$$

$$(v_0 < v_1) \wedge (v_1 < v_2) \rightarrow v_0 < v_2 \ , \ \neg v_0 < v_0$$

$$\neg U v_0 \wedge \neg U v_1 \wedge v_0 \neq c \wedge v_1 \neq c \wedge v_0 \neq v_1 \rightarrow v_0 < v_1 \vee v_1 < v_0$$

Group 3. $\check{I}$ indexes the subtrees above a node by points of the linear order:

$$\neg \, v_0 \, \alpha \, v_1 \;\rightarrow\; I(v_0, v_1) = c$$

$$\dot{v}_0 \, \alpha \, v_1 \,\wedge\, v_1 \, \alpha \, v_2 \;\rightarrow\; I(v_0, v_1) = I(v_0, v_2)$$

$$v_0 \, \alpha \, v_1 \;\rightarrow\; \neg \, U\,(\bar{I}(v_0, v_1)) \,\wedge\, (I(v_0, v_1) \neq c) \;.$$

Let Σ be the class of finite models of T .

Lemma 5.1 Σ has the amalgamation property.

Proof. It will suffice to show that if $A, B_0, B_1 \in \Sigma$, and $A \subset B_0$, $A \subset B_1$ and $|B_0| \cap |B_1| = |A|$ then there is $C \in \Sigma$ with $B_i \subset C$ for $i = 0, 1$. Let A, B_0, B_1 be as above. We prove by induction that the operations of B_0, B_1 may be extended to $|B_0| \cup |B_1| = |C|$ to construct the required C .

Let $U_C = U_{B_0} \cup U_{B_1}$ and let $c_C = c_{B_0}$.

It is not difficult to extend $<_{B_0} \cup <_{B_1}$ to a linear order $<_C$ on $|C| \setminus (U_C \cup \{c_A\})$. First define $\vartriangleleft$ by $c \vartriangleleft d$ iff there is a sequence $\langle z_i : i \leq n \rangle$ where $n > 0$ and $c = z_0$, $d = z_n$ and for all $i < n$ $(z_i, z_{i+1}) \in (<_{B_0} \cup <_{B_1})$. An induction on the minimun length of the sequence $\langle z_i \rangle$ shows that if $c \vartriangleleft d$ and $c, d \in B_i$ then $(c, d) \in <_{B_i}$. Indeed $c = z_0$ and $d = z_{n+1}$ are both in B_i . If $z_n \in B_i$ we are done for $(z_0, z_n) \in <_{B_i}$. Otherwise $z_n \in B_{1-i} \setminus A$ whence both z_{n-1} and $z_{n+1} \in B_{1-i}$ assuming $n \geq 1$. Now $z_{n-1} <_{B_{1-i}} z_{n+1}$ so the sequence $\langle z_i : i \leq n+1 \rangle$ can be shortened.

Thus in each case the shortest sequence has length 2 if $c, d \notin B_i$.

That $\triangleleft$ can be extended to a linear order is clear.

We now show by induction on $|U_C|$ that there exist functions $\wedge_C$, I_C on $|C|$ extending $\wedge_{B_i}$, I_{B_i} respectively for $i = 0, 1$ such that $\langle |C|, U_C, <_C, \wedge_C, I_C, c \rangle$ is a model of T. Let $\wedge_i$, I_i, denote $\wedge_{B_i}$, I_{B_i} respectively. Let α_i be the partial order corresponding to $\wedge_i$ as in the definition in Group 1.

There are now two cases.

Case 1. $\exists i < 2 \; \exists x \in B_i \setminus A \; \forall z \in B_i (\neg x \, \alpha_i \, z)$;

Choose such an i and x. By the induction hypothesis there are extensions $\wedge'$ I' of $\wedge_{1-i}$, $\wedge_i \upharpoonright (B_i \setminus \{x\})$ and I_{1-i}, $I_i \upharpoonright (B_i \setminus \{x\})$ to $C \setminus \{x\}$ such that the resulting structure is a model of T. Now let x_0 be the maximum of $\{y : y \, \alpha_i \, x\}$ and for $z \in C$ define

$$z \wedge x = x \wedge z = \begin{cases} x & \text{if } z = x \\ z \wedge' x_0 & \text{otherwise} \end{cases}$$

and

$$I(x, z) = c_A$$

$$I(z, x) = \begin{cases} I'(z, x_0) & \text{if } z \notin \{x_0, x\} \\ I_i(x_0, x) & \text{if } z = x_0 \end{cases}.$$

Case 2. For every $x \in B_i$ there is $a \in A$ such that $x = a \lor x \, \alpha_i \, a$.

Choose i from $0, 1$ and $x \in B_i \setminus A$ such that

$$\forall y \forall z (x \, \alpha_i \, y \to y \in A \, .\wedge. \, x \, \alpha_i \, y \wedge y \, \alpha_{1-i} \, z \to z \in A)$$

Let $A_0 = \{a : x \, \alpha_i \, a\}$.

As in Case 1 let $\wedge'$, I' extend $\wedge_{1-i}$, $\wedge_i \restriction (B_i \setminus \{x\})$ and I_{1-i}, $I_i \restriction (B_i \setminus \{x\})$ respectively to $C \setminus \{x\}$ such that one obtains a model of T.

Extend $\wedge'$ to C by letting x_0 be the least element of A which is greater than x and for all $z \in C$

$$z \wedge x = x \wedge z = \begin{cases} x & \text{if } z \in A_0 \cup \{x\} \\ x_0 \wedge' z & \text{otherwise} \end{cases}$$

Let

$$I(x,z) = \begin{cases} I_i(x,z) & \text{if } z \in A_0 \\ c_A & \text{otherwise} \end{cases}$$

and

$$I(z,x) = \{I'(z,x_0) \quad \text{if } z \neq x .$$

It is an easy calculation to check that C is a model of T. This completes the proof of Lemma 5.1.

Σ clearly has a member which can be embedded in each other member - a structure with one element which is a model of T. Also Σ has arbitrarily large finite members and is closed under substructure. Finally it is easy to see that there is a bounding function for the cardinality of generated substructures and so we may apply Lemmas 3.2 and 3.4. Let M be a Σ-generic structure. Then $Th(M)$ admits elimination of quantifiers.

Let $c_n : n \in \omega$ and $d_n : n \in \omega$ be new constant symbols and let L_0 be the language $L \cup \{c_n : n \in \omega\} \cup \{d_n : n \in \omega\}$. Let M_0 be an expansion of M to the language L_0 to model:

$\neg U c_m \wedge c_m \neq c$ for all m, $c_m < c_n$ for $m < n$,

$U d_m$ for all m, $d_m \wedge d_n = d_m$ for $m < n$,

and $I(d_m, d_{m+1}) = c_0$ for all m.

Let $T_0 = Th(M_0)$. We shall show that T_0 has 9 countable models and that there is a nonprincipal 1-type which contains infinitely many 2-types.

First note that T_0 admits elimination of quantifiers since $Th(M)$ does.

The 9 models are characterized by the upper bounds for $\{c_n : n \in \omega\}$ and $\{d_n : n \in \omega\}$. Let the models H, K agree on the following four infinitary sentences:

$$\exists x \left(\bigwedge_{i < \omega} c_i < x \right)$$

$$\exists x \left(\bigwedge_{i < \omega} d_i \; \alpha \; x \right)$$

$$\exists x \forall y \left(\bigwedge_{i < \omega} c_i < x \; . \&. \; \left(\bigwedge_{i < \omega} c_i < y \right) \to x \leq y \right)$$

$$\exists x \forall y \left(\bigwedge_{i < \omega} d_i \; \alpha \; x \; . \&. \; \left(\bigwedge_{i < \omega} d_i \; \alpha \; y \right) \to x \; \underline{\alpha} \; y \right)$$

The first two sentences assert that $\{c_i\}$, $\{d_i\}$ respectively have upper bounds, the third and fourth sentences that $\{c_i\}$, $\{d_i\}$ respectively have least upper bounds. When these least upper bounds exist we shall denote them by $\sup\{c_i\}$, $\sup\{d_i\}$ respectively.

We shall show that $H \cong K$.

Write $\neg U (A)$ for $(\neg U v_0)(A)$.

We may apply the argument for the Ehrenfeucht structures discussed in [12, §6] to obtain an isomorphism

$$\sigma : [\neg U(H)]_H \to [\neg U(K)]_K$$

where $[S]_A$ denotes the substructure of A generated by S.

We now apply a back and forth argument on U_H and U_K to extend σ to an isomorphism of H and K.

Let $\{e_n : n < \omega\}$ enumerate U_H and let $\{f_n : n < \omega\}$ enumerate U_K. We construct enumerations $\{h_n : n < \omega\}$ of U_H and $\{k_n : n < \omega\}$ of U_K such that for each n there is an isomorphism

$$\rho_n : [\neg U(H) \cup \{h_i : i < n\}]_H \to [\neg U(K) \cup \{k_i : i < n\}]_K$$

and such that the following conditions are satisfied for all n

(i) ρ_n extends σ

(ii) $\rho_n(h_i) = k_i$ for $i < n$

(iii) $h_0 = (\sup\{d_i\})_H$ and $k_0 = (\sup\{d_i\})_K$ if these exist

(iv) h_n is an upperbound of $\{d_i\}$ in H iff k_n is an upper bound of $\{d_i\}$ in K.

There is no difficulty in choosing h_0 and k_0 to be $\sup\{d_i\}$ in H, K respectively when these exist since for any upper bound h of $\{d_i\}$ in H and any upper bound k of $\{d_i\}$ in K

$$H \models I(d_m, h) = c_0 \text{ and } K \models I(d_m, k) = c_0$$

for all m.

Thus we may assume that $\{h_n : n < N\}$ and $\{k_n : n < N\}$ have been chosen to satisfy the conditions and $N > 0$ if $\sup\{d_i\}$ exists. We present the argument for even N, with N odd we extend the range of ρ_N in a symmetric way.

Let n be minimal such that $e_n \notin \{h_n : n < N\}$ and set $h_N = e_n$.

Let $H' = H \upharpoonright L$ and $K' = K \upharpoonright L$. There are now two cases:

Case 1. $H \models d_m \alpha h_N$ for all m .

Let A be the finite substructure of H' generated by $\{h_n : n < N+1\}$.

Let a be the least of the upper bounds of $\{d_i\}$ which belongs to A .

If $a \in [\{h_n : n < N\}]_H$ then let B be the substructure of H' generated

by $\{h_n : n < N\} \cup (A \cap \neg U (H))$. Now $\rho_N \upharpoonright B$ is an embedding and K'

is Σ-universal and Σ-homogeneous. Thus $\rho_N \upharpoonright B$ can be extended to an

embedding of A in K' . Let k_N be the image of h_N in this

embedding. This determines ρ_{N+1} . We do not verify all the details

that ρ_{N+1} is an L_0-isomorphism but check the relevant clause to

see that ρ_{N+1} respects I . If $k \in \rho_{N+1} ([\{h_1, \ldots, h_N\} \cup \neg U (H)]_H)$

and $k \alpha k_N$ then either $k \alpha \rho_N(a)$ in which case

$I_K(k, k_N) = I_K(k, \rho_n(a))$, or $\rho_N(a) \underline{\alpha} k$ in which case the extension of

$\rho_N \upharpoonright B$ to A ensures that ρ_{N+1} is an isomorphism.

Now suppose that $a \notin [\{h_n : n < N\}]_H$. Then $\{d_i\}_H$ has no least

upper bound, but there are upper bounds. If there are no upper bounds

for $\{d_i\}_H$ which belong to $[\{h_n : n < N\}]_H$ then we may choose for k_N

any upper bound of $\{d_i\}_K$. Otherwise let b be the minimum of those

upper bounds for $\{d_i\}_H$ which belong to $[\{h_n : n < N\}]_H$. Choose an

upper bound k for $\{d_i\}_K$ such that $k \alpha \rho_N(b)$. Let $j \in \neg U(H)$ be such

that $\sigma(j) = I_K(k, \rho_N(b))$. The idea from here is to see that we can

use k in the way we used $\rho_N(a)$ in the first part of this case.

We construct a member of Σ in order to use the Σ-generic nature

of H' to find a suitable preimage for k , the analogue of a

in the first part.

Let S be the substructure of H' generated by $\{h_i : i \leq N\} \cup \{j\}$
and let h be a new individual. Consider the structure C for L
whose universe is $|S| \cup \{h\}$ and which satisfies in addition to the
condition that $C \upharpoonright |S| = S$ the conditions

   (i) $h \in U_C$

  (ii) $x \wedge_C h = h = h \wedge_C x$ for $x \in S$ satisfying $a \underset{\sim}{\alpha}_S x$ or

        if $x = h$

 (iii) $x \wedge_C h = h \wedge_C x = \begin{cases} c_H & \text{if } x \notin U_C \\[2ex] x \wedge_S a & \text{if } x \in U_S \text{ but } a \underset{\sim}{\not\alpha}_S x \ . \end{cases}$

  (iv) $I_C(h,x) = \begin{cases} j & \text{if } x \in S \text{ and } a \underset{\sim}{\alpha}_S x \\[2ex] c_H & \text{otherwise} \end{cases}$

   (v) $I_C(x,h) = \begin{cases} I_S(x,a) & \text{for } x \ \alpha_C \ h \ . \end{cases}$

Then $C \in \Sigma$ and since H' is Σ-generic there is no loss of generality
in assuming that $h \in H$.

   Now observe that the substructure of H'
$[\{h_i : i < N\} \cup \{h\} \cup (\neg U (C))]_{H'}$ and the substructure of K'
$[\{k_i : i < N\} \cup \{k\} \cup (\sigma(\neg U(C))]_{K'}$ are isomorphic by the unique
mapping σ_0 compatible with ρ_N which takes h to k . Using the
genericness of K' we can find k_N so that σ_0 extends to an
isomorphism $\sigma_1 : [\{h_i : i < N+1\} \cup \{h\} \cup (\neg U(C))]_{H'} \rightarrow$
$[\{k_i : i < N+1\} \cup \{k\} \cup \sigma(\neg U(C))]_{K'}$ where $\sigma_1(h_N) = k_N$.

   Since $k \ \alpha \ k_N$ k_N is an upper bound of $\{d_i\}_K$ and for ρ_{N+1}
we can take a restriction of the unique common extension of σ_1 and
ρ_N . It is easy to see that ρ_{N+1} is an L_0-isomorphism. This
completes the first case.

Case 2. Otherwise, i.e. $\exists m \ [(d_m)_H \not\alpha \ h_N]$.

Let m be minimal such that $(d_m)_H \not\alpha \ h_N$. Let A be the sub-structure of H' generated by $\{h_i : i < N+1\} \cup \{d_i)_H : i < m+1\}$. Let B be the substructure of A generated by $\{h_i : i < N\} \cup \{d_i)_H : i < m+1\} \cup \neg \ U(A)$. By hypothesis $\rho_N \restriction B$ is an L-isomorphism from B to the substructure of K' generated by $\{k_i : i < N\} \cup \{d_i)_K : i < m+1\} \cup \sigma(\neg U(A))$. Since K' is Σ-generic we may find k_N a suitable image in K for h_N and an L-embedding σ_1 of A in K' which extends $\rho_N \restriction B$ such that $\sigma_1(h_N) = k_N$. As in the first case we let ρ_{N+1} be the unique common extension of ρ_N and σ_1. Again it is easy to see that ρ_{N+1} is an L_0-isomorphism.

Thus $H \cong K$ as was to be proved.

We have shown that the isomorphism type of a countable model of T_0 is determined by the truth values of four infinitary sentences. These sentences are not independent because existence of a least upper bound implies existence of an upper bound. The reader will easily see that there are in fact 9 different possibilities for the quadruple of truth-values. From this it is easy to see that T_0 has 9 countable models.

Let p be the type determined by $\{Ux \wedge d_i \ \alpha \ x : i < \omega\}$. p is nonprincipal. Also there are ω 2-types q in pxp . There are at most ω since $I(T_0, \omega) < \omega$ and $\{q_n : n \in \omega\}$ where q_n is the 2-type in pxp determined by $\{x \ \alpha \ y \wedge I(x,y) = c_n\}$ is a family of distinct 2-types in pxp .

We summarize these results in the following theorem.

Theorem 5.1   There exists a complete theory with a small language which has 9 countable models and a nonprincipal 1-type which contains infinitely many 2-types.

In the next chapter we shall see how to construct a complete theory T with a small language with countably many unary predicate symbols such that T has 4 countable models and a nonprincipal 1-type containing infinitely many 2-types.   The following is open.

Q9:   If T has finitely many nonlogical symbols except

for constant symbols, if T admits elimination of

quantifiers and $I(T,\omega) < 9$ can T have a non-

principal 1-type containing infinitely many 2-types?

Another question is the following:

Q10:   If $I(T,\omega) = 3$ and T has a small language can T

have a nonprincipal 1-type containing infinitely

many 2-types?

Chapter 6

C1, C4 and the Theory T_1

1. C1 and C4

In this chapter we shall consider further complete theories with a small language which have a finite number of countable models. In this section we describe some properties of theories which satisfy C1 or C4. In the next section we shall construct the theory T_1, which satisfies C1 but for which C4 fails.

In this section we assume that T is a complete theory with a countable language and that T has a finite number of countable models and more than one. Let M be a countable saturated model of T, and let p be a nonprincipal 1-type of T.

In the next lemma we explore the consequence of assuming that T satisfy a strengthening of C1.

Lemma 6.1 Assume that the relation R on $p(M) = \{a \in M : a$ realizes $p\}$ defined by

aRb iff b is prime over a

is a linear order. Then T satisfies the following conditions:

(i) there are a finite number of 2-types in p

(ii) R is definable (in the sense that there is a formula whose restriction to p defines R)

(iii) R is a dense order.

Proof. Let $\{\theta_n : n < \alpha \leq \omega\}$ be a family of formulas in $(L)_2$ such that if a,b realize p in M and aRb then for some n θ_n generates $tp(a,b)$ over p, and for $n \neq m$ $p \cup \{\exists v_1 (\theta_n \wedge \theta_m)\}$ is contradictory.

To see (i) assume for contradiction that there are infinitely many 2-types in p. Then $\alpha = \omega$. But then by the compactness theorem there is a 2-type q in $p \times p$ containing

$$\{\neg \theta_n : n \in \omega\} \cup \{T_{1,1}(\neg \theta_n) : n \in \omega\} \cup \{v_0 \neq v_1\} .$$

But then if a',b' realize q in M we have neither $a'Rb'$ nor $b'Ra'$ a contradiction. Thus (i) holds.

We now easily obtain (ii) from (i), for $\alpha < \omega$ and $\bigvee_{n < \alpha} \theta_n$ will define R on $p(M)$.

Since p is nonprincipal it is clear that $p(M)$ is infinite and that R is without endpoints. Now let $n < \alpha$ and observe that for b realizing p in M $\{a \in M : M \models \theta_n(a,b)$ & a realizes $p\}$ must be infinite since R is linear. For the type q_n generated by θ_n over p is not principal in its second coordinate, i.e. $T_{1,1}(q_n)$ is not principal over p. But now if a_1, a_2, b realize p in M and

$$M \models \theta_n (a_1,b) \wedge \theta_n (a_2,b) \wedge a_1 \neq b$$

we must have that $a_1 R a_2$ holds or $a_2 R a_1$ holds. Thus $\exists v_2 (\bigvee_{k < \alpha} \theta_k(v_0,v_2) \wedge \bigvee_{k < \alpha} \theta_k(v_2,v_1))$ belongs to q_n. Since this is true for each $n < \alpha$ it easily follows that R is dense on $p(M)$. This completes the proof of the lemma.

In the next lemma we present a strengthening of a result of Benda [1, Thm. 2]. He showed that if every inessential extension of T has

a finite number of countable models then T has a universal model which is not saturated.

Lemma 6.2 If every inessential extension of T has a finite number of countable models then T has a universal model prime over a finite set.

Proof. Recall first that if a theory has a finite number of countable models then the theory has a powerful n-type for some n. An n-type is powerful just in case every model of the theory realizing the type realizes every m-type in the theory [1, p. 111].

Assume that every inessential extension of T has a finite number of countable models. Let $\{a_n : n \in \omega\}$ be an enumeration of M. Let $\bar{a}_n$ be the n-tuple $\{a_j : j < n\}$. Now $\mathrm{Th}(M, \bar{a}_n)$ has a finite number of countable models. Let Π_n be a powerful k_n-type in $\mathrm{Th}(M, \bar{a}_n)$ and let $\bar{b}_n$ realize Π_n in $(M, \bar{a}_n)$. Let Σ_n be the $(n+k_n)$-type of $\bar{a}_n \cap \bar{b}_n$. Note that Σ_0 is a powerful k_0-type in T. We shall show that the model N prime over $\bar{b}_0'$ realizing Σ_0' is universal.

We choose by recursion sequences $\{a_n' : n \in \omega\}$ and $\{\bar{b}_n' : n \in \omega\}$ in N such that $\bar{a}_n' \cap \bar{b}_n'$ realizes Σ_n where $\bar{a}_n'$ is the sequence $\{a_j' : j < n\}$. Then M is isomorphic to an elementary substructure of N so that N is universal.

Assume that a'_{m-1} and $\bar{b}_m'$ have been chosen for $m < n$ such that $\bar{a}_m' \cap \bar{b}_m'$ realizes Σ_m for $m < n$, and $n > 0$. Now $(N, \bar{a}_{n-1}')$ is a model of $\mathrm{Th}(M, \bar{a}_{n-1})$ and $\bar{b}_{n-1}'$ realizes Π_{n-1}, a powerful type in $\mathrm{Th}(M, \bar{a}_{n-1})$. But then it is easy to see that we may choose

a_{n-1}' and $\bar{b}_n'$ in N such that $\bar{a}_n' \cap \bar{b}_n'$ realize Σ_n since the type

of $<a_n> \cap \bar{b}_n$ in $Th(M,\bar{a}_{n-1})$ must be realized in $(N,\bar{a}_{n-1}')$.

This completes the proof of Lemma 6.2.

2.   The Theory T_1

In this section we construct a theory T_1 which has 4 countable

models but which has an inessential extension T_2 having infinitely

many countable models. T_1 is in a small language with countably

many unary predicate symbols. T_1 satisfies C1, indeed it has a

nonprincipal 1-type p on which the relation "a is prime over b"

is linear.

The construction of T_1 uses the tools of Chapter 3 and resembles

the development in Chapter 5. First an ω_0-categorical theory is

constructed, the models of which are linear orders with two sorts of

elements: "rationals", and "irrationals" indexed by increasing triples

of rationals. This theory is obtained by amalgamation techniques.

Consider the language L with nonlogical symbols: a unary predi-

cate symbol U, a binary relation symbol <, a 3-ary function symbol I,

and a constant symbol c . Let T be the theory with language L and

the following axioms.

Group 1. < is a linear order with least elements · c :

$\quad\quad$ $(x<y \wedge y<z .\rightarrow. x<z)$, $\neg x<x$,

$\quad\quad$ $(x<y \vee x=y \vee y<x)$, $\quad c\leq x$.

Group 2. I is a mapping from increasing triples of U to

$\neg U \setminus \{c\}'$ and is otherwise trivial:

$$Iv_0v_1v_2 \neq c \;.\leftrightarrow\; v_0 < v_1 \wedge v_1 < v_2 \wedge Uv_0 \wedge Uv_1 \wedge Uv_2 \;,$$

$$Iv_0v_1v_2 \leq v_0, \neg U(Iv_0v_1v_2) \;.$$

Group 3. Distinct increasing tuples of U index distinct irrationals:

$$Iv_0v_1v_2 = Iv_3v_4v_5 \wedge Iv_0v_1v_2 \neq c \;.\rightarrow. \; v_0 = v_3 \wedge v_1 = v_4 \wedge v_2 = v_5$$

Let Σ be the class of finite models of T.

Lemma 6.3 Σ has the amalgamation property.

Proof. To see this it suffices to consider $A_0, A_1, A_2 \in \Sigma$ with $A_0 \subset A_1$, $A_0 \subset A_2$ and $|A_0| = |A_1| \cap |A_2|$. There is no loss of generality in assuming that $U_{A_1} \cup U_{A_2} \cup \{c_{A_0}\}$ contains no triple and

$$(\neg Uv_0 \wedge v_0 \neq c)(A_i) = \{(u,v,w) : A_i \models u < v \wedge v < w \wedge Uu \wedge Uv \wedge Uw\}$$

for $i = 0,1,2$.

First let $<'$ be an extension of $<_{A_1} \cup <_{A_2}$ to a linear order of $|A_1| \cup |A_2|$. In the proof of Lemma 5.1 we have already seen how to amalgamate two linear orders.

Now define A_3 as follows:

$$U_{A_3} = U_{A_1} \cup U_{A_2} \;,$$

$$|A_3| = |A_1| \cup |A_2| \cup \{c_{A_0}\} \cup \{(u,v,w) : u,v,w \in U_{A_3} \text{ and } u <' v, v <' w\} \;,$$

$$c_{A_3} = c_{A_0} \;,$$

$$I_{A_3}(u,v,w) = \begin{cases} (u,v,w) & \text{if } u,v,w \in U_{A_3} \text{ and } u <' v, v <' w \\ c_{A_0} & \text{otherwise}, \end{cases}$$

$<_{A_3} \upharpoonright \{(u,v,w):(u,v,w) \in |A_3| \setminus (|A_1| \cup |A_2|)\}$ is the

lexicographic order induced by $<'$.

$c_{A_3} < (u,v,w) < x$ for $x \in (|(A_1| \cup |A_2|) \setminus \{c_{A_1}\}$ and $(u,v,w) \in |A_3| \setminus$

$(|A_1| \cup |A_2|)$ and $<_{A_3}$ is a linear order which extends $<'$ to $|A_3|$.

Then clearly $A_1 \subset A_3$, $A_2 \subset A_3$ and $A_3 \in \Sigma$.

This completes the proof that Σ has AP and Lemma 6.3 is established.

Clearly there is a one element structure in Σ which can be

embedded in every structure in Σ , Σ has arbitrarily large finite

members, and Σ is closed under substructure. Also there is a

bounding function for the cardinality of generated substructure so we

may apply Lemmas 3.2 and 3.4.

Let M be a countable Σ-generic structure. Th(M) admits elimina-

tion of quantifiers. It is easy to see that $<|M|, <_M>$ is a dense

order with first element in which U_M and $(|M| \setminus U_M)$ are dense sub-

sets. Also $M \models \neg U v_0 \rightarrow \exists v_1 \exists v_2 \exists v_3 (I v_1 v_2 v_3 = v_0)$.

Now let L_1 be the first order language obtained from L by

adding countably many unary predicate symbols $\{U_i : i < \omega\}$. Let M

be expanded to a structure M_1 for L_1 to model for each $i < \omega$:

$$U_i v_0 \wedge v_0 < v_1 \rightarrow U_i v_1 ,$$

$$U_{i+1} v_0 \rightarrow U_i v_0 ,$$

$$U_i v_0 \rightarrow \exists v_1 (v_1 < v_0 \wedge U_i v_1) ,$$

$$\neg U_i v_0 \rightarrow \exists v_1 (v_0 < v_1 \wedge \neg U_i v_1) ,$$

$$\neg U_i c ,$$

and $$\exists v_0 (\neg U_{i+1} v_0 \wedge U_i v_0) .$$

That is we expand M by distinguishing an increasing sequence of irrational cuts in $<_M$.

Now let $T_1 = Th(M_1)$. We shall show that T_1 has four countable models. Consider the following four conditions on a model A of T :

Condition I. $\quad \bigcap_{i<\omega} U_i(A) = \phi$.

Condition II. There is $a \in U_A$ such that $a \in \bigcap_{i<\omega} U_i(A)$ and for all $b<a$ $b \notin \bigcap_{i<\omega} U_i(A)$.

Condition III. There is $a \in A \setminus U_A$ such that $a \in \bigcap_{i<\omega} U_i(A)$ and for all $b<a$ $b \notin \bigcap_{i<\omega} U_i(A)$.

Condition IV. None of Conditions I, II, III, that is for all

$a \in A$ there is $b \in A$ such that $a \in \bigcap_{i<\omega} U_i(A) \to (b<a \wedge b \in \bigcap_{i<\omega} U_i(A))$.

If Condition II or III holds we say a is $\sup\{U_i\}$. It is clear that the conditions are exhaustive and mutually exclusive. We shall show that if A and B are two countable models of T_1 which satisfy the same condition then $A \cong B$. To do this we first require some further properties of M_1 .

For $j<\omega$ let $L(j)$ be $L \cup \{U_i : i<j\}$ and let $\Sigma(j)$ be the class of finite structures for $L(j)$ which are expansions of members of Σ which model for each $i<j$:

$$U_i v_0 \wedge v_0 < v_1 \to U_i v_1$$

$$U_{i+1} v_0 \to U_i v_0 \quad \text{when} \quad i+1 < j$$

$$\neg U_i c \ .$$

We then have $\Sigma(0) = \Sigma$ and $M_1 \upharpoonright L(0) = M$ and the following lemma.

Lemma 6.4. $M_1 \upharpoonright L(j)$ is $\Sigma(j)$-generic for each $j < \omega$.

Proof. The proof is by induction on j . With $j=0$ this is just the choice of M . Assume that $M_1 \upharpoonright L(j)$ is $\Sigma(j)$-homogeneous and $\Sigma(j)$-universal. We shall let a symbol refer to the interpretation in M_1 unless otherwise specified.

We first show that $M_1 \upharpoonright L(j+1)$ is $\Sigma(j+1)$-homogeneous. Assume that H,K are finite substructures of $M_1 \upharpoonright L(j+1)$ and that σ is an isomorphism of H onto K . In order to see that we may extend σ to an automorphism of $M_1 \upharpoonright L(j+1)$ by a back and forth argument it will suffice to see that if H_0 is a finite substructure of $M_1 \upharpoonright L(j+1)$ with $H \subseteq H_0$ then σ may be extended to H_0 .

The idea is to reduce the problem to $\Sigma(j)$-genericness by introducing "brackets" for U_j . Given a structure $N \in \Sigma(j+1)$ define the *canonical extension* $N' \in \Sigma(j)$ of N as follows. Let ℓ_N, r_N be new individuals and let N' satisfy

1) $|N'| = |N| \cup \{\ell_N, r_N\} \cup D$ and $N \upharpoonright L(j) \subseteq N'$

   $U_{N'} = U_N \cup \{\ell_N, r_N\}$ $(U_i)_{N'} = (U_i)_N \cup \{\ell_N, r_N\}$ for $i < j$,

2) $<_{N'}$ is a linear order of $|N'|$ such that $\ell_N < r_N$ and

   (a) if $a \in |N| \setminus (U_j)_N$ then $a <_{N'} \ell_N$

   (b) if $a \in (U_j)_N$ then $r_N <_{N'} a$,

3) $D = \{(u,v,w) :$ at least one of u,v,w is ℓ_N or r_N and $u,v,w \in U_{N'}$ and $N' \models (u < v \land v < w)\}$,

4) $<_{N'} \restriction D$ is the lexicographic order induced by

$<_{N'} \restriction U_{N'}$,

5) $N' \models c<d \wedge d<b$ for $d \in D$ and $b \in |N'| \setminus (D \cup \{c_N\})$,

6) $N' \models \neg U_i d$ for $d \in D$ and $i<j$,

and 7) $N' \models I u v w = (u,v,w)$ for $(u,v,w) \in D$.

We say that ℓ_N, r_N bracket $(U_j)_N$.

We first show that we may take the canonical extension of K to be a substructure of $M_1 \restriction L(j)$ with $\ell_K \notin U_j$ and $r_K \in U_j$. We first choose first approximations ℓ_0, r_0 , that is we choose $\ell_0 \in U$ such that

(a) $\ell_0 \in U_i$ for $i<j$

(b) $\ell_0 \notin U_j$

and (c) $k \notin U_j \to k < \ell_0$ for $k \in K$; and choose $r_0 \in U$ such that

(a) $r_0 \in U_i$ for $i \leq j$

and (b) $k \in U_j \to r_0 < k$ for $k \in K$.

Let K^1 be the substructure of $M_1 \restriction L(j)$ generated by $|K| \cup \{\ell_0, r_0\}$. We may assume that ℓ_K, r_K are new. We now construct a common extension K^2 of K^1 and K' such that $K^2 \in \Sigma(j)$. Let K^2 satisfy clauses 1), 3), 4), 5), 6) and 7) of the definition of canonical extension with K^1 in place of N, K^2 in place of N' , ℓ_0 for ℓ_N , and r_0 for r_N ; and let K^2 satisfy

2') $<_{K^2}$ is a linear order of $|K^2|$ such that ℓ_0 is the immediate successor of ℓ_K and r_0 is the immediate predecessor of r_K .

By the induction hypothesis there is no loss in assuming that $K^2 \subset M_1 \upharpoonright L(j)$. But then $K' \subset M_1 \upharpoonright L(j)$ and since $\ell_0 \notin U_j$ $\ell_K \notin U_j$ and since $r_0 \in U_j$ we have $r_K \in U_j$.

The next step is to introduce brackets for H_0 . It is clear that H_0' can be taken to be a substructure of $M \upharpoonright L(j)$ and that the substructure of H_0' generated by $|H| \cup \{\ell_{H_0}, r_{H_0}\}$ is isomorphic to H' and so we may identify it and H' . Now σ may be immediately extended to an isomorphism σ_1 of H' and K' with $\sigma_1(\ell_{H_0}) = \ell_K$ and $\sigma_1(r_{H_0}) = r_K$. But then since $M_1 \upharpoonright L(j)$ is $\Sigma(j)$-generic we must have that σ_1 can be extended to an isomorphism σ_2 of H_0' and a substructure of $M_1 \upharpoonright L(j)$. Let K_0 be the substructure of $M_1 \upharpoonright L(j+1)$ generated by the image of $|H_0|$ under σ_2 and let ρ be the restriction of σ_2 to $|H_0|$. Then ρ is an isomorphism of H_0 and K_0 since ℓ_K and r_K bracket U_j , for

$$h \in (U_j)_{H_0} \rightarrow r_{H_0} < h \rightarrow r_K < \rho(h) \rightarrow \rho(h) \in (U_j)_{K_0}$$

and

$$h \notin (U_j)_{H_0} \rightarrow h < \ell_{H_0} \rightarrow \rho(h) < \ell_K \rightarrow \rho(h) \notin (U_j)_{K_0} .$$

This completes the proof that $M_1 \upharpoonright L(j+1)$ is $\Sigma(j+1)$-universal.

Next we show that $M_1 \upharpoonright L(j+1)$ is $\Sigma(j+1)$-homogeneous. Let $H \in \Sigma(j+1)$. Consider H' . Without loss of generality we may assume that $\ell_H, r_H \in M_1$ and $\ell_H \in \bigcap_{i<j} U_i \setminus U_j$ and $r_H \in U_j$. Now the substructure of H' generated by $\{\ell_H, r_H\}$ is a substructure of $M_1 \upharpoonright L(j)$ and so by the $\Sigma(j)$-genericness of $M_1 \upharpoonright L(j)$ we may assume that $H' \subset M_1 \upharpoonright L(j)$. But then it is easy to see that

$H \subset M_1 \upharpoonright L(j+1)$ since $(h \notin U_j(H) \to h < \ell_H)$ and $(h \in U_j(H) \to r_H < h)$.

This completes the proof that $M_1 \upharpoonright L(j+1)$ is $\Sigma(j+1)$-generic and the lemma follows.

With lemma 6.4 we can prove the following theorem.

Theorem 6.1 T_1 has four countable models.

Proof. Assume that A and B are two countable models of T_1 which satisfy the same one of Conditions I, II, III or IV. We shall prove that A and B are isomorphic by a back and forth argument. Let $\{e_n : n \in \omega\}$ and $\{f_n : n \in \omega\}$ be enumerations of U_A and U_B respectively. We construct enumerations $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ of U_A and U_B such that for each n :

1) if A satisfies Condition II then $a_0 = \sup \{U_i\}$ in A

   and $b_0 = \sup \{U_i\}$ in B .

2) if A satisfies Condition III then $I(a_0, a_1, a_2)$ is

   $\sup \{U_i\}$ in A and $I(b_0, b_1, b_2)$ is $\sup \{U_i\}$ in B .

3) There is an isomorphism ρ_n of $[\{a_j : j < n\}]_A$ and

   $[\{b_j : j < n\}]_B$ which sends a_j to b_j for $j < n$.

Here $[S]_N$ is the substructure of N generated by S . When S is finite and $N \models T_1$ then $[S]_N$ is finite also. It is easy to satisfy 1). Also 2) is straightforward for if $a_0, a_1, a_2 \in A$ and $b_0, b_1, b_2 \in B$ are such that

$$A \models a_0 < a_1 \land a_1 < a_2 \land U_i \ Ia_0a_1a_2 \quad \text{for } i < \omega$$

and $B \models b_0 < b_1 \land b_1 < b_2 \land U_i \ Ib_0b_1b_2 \quad \text{for } i < \omega$

then $[\{a_0, a_1, a_2\}]_A$ and $[\{b_0, b_1, b_2\}]_B$ are isomorphic since there

are no other increasing tuples in U . We may assume $n > 0$ in case

Condition II holds and $n > 2$ in case Condition III holds.

We present only the argument for even n . When n is odd we

extend $\{b_i : i < n\}$ symmetrically.

Let m be minimal such that $e_m \notin \{a_i : i < n\}$. Let $a_n = e_m$.

Let $A_n = [\{a_i : i < n\}]_A$, $B_n = [\{b_i : i < n\}]_B$ and $A_{n+1} = [\{a_i : i < n+1\}]_A$.

Case 1. $a' = \min\{a \in A_n : a \in \bigcap_{i < \omega} U_i(A)\} = \min\{a \in A_{n+1} : a \in \bigcap_{i < \omega} U_i(A)\}$

or $\{a \in A_{n+1} : a \in \bigcap_{i < \omega} U_i(A)\}$ is empty.

In this case let $j > 0$ be sufficiently large that for $a \in A_{n+1}$

if $a \in U_{j-1}(A)$ then $a \in \bigcap_{i \in \omega} U_i(A)$. Now $B \upharpoonright L(j)$ is $\Sigma(j)$-generic

from Lemma 6.4 and A_n and B_n are isomorphic under ρ_n . Also

$A_{n+1} \upharpoonright L(j) \in \Sigma(j)$. Thus there is an extension σ of ρ_n which

embeds $A_{n+1} \upharpoonright L(j)$ in $B \upharpoonright L(j)$. Let $b_n = \rho(a_n)$. Then it is not

difficult to see that we may take $\rho_{n+1} = \sigma$ to obtain an L_1-isomorphism

of A_{n+1} and $[\{b_i : i < n+1\}]_B$. The only detail to check is that

$a \in U_k(A) \leftrightarrow \rho_{n+1}(a) \in U_k(B)$, for $a \in A_{n+1}$ and $k \geq j$.

If $k \geq j$ and $a \in U_k(A_{n+1})$ then $a \in \bigcap_{i < \omega} U_i(A)$ and so

$\rho_n(a') \leq \rho_{n+1}(a)$ so $\rho_{n+1}(a) \in \bigcap_{i < \omega} U_i(B)$. Otherwise $a \notin U_i(A)$ for

some $i < j$ and the result follows.

Case 2. Otherwise, that is there is $a \in A_{n+1}$ with $a \in \bigcap_{i < \omega} U_i(A)$

and for all $a' \in A_n$ with $a' \in \bigcap U_i(A)$ $a < a'$.

In this case A and B must satisfy Condition IV. Before applying

the techniques of Case 1 we must first establish an upper bound in B

to play the rôle of $\rho_n(a')$, suited to the introduction of a preimage

to play the rôle of a' .

Let $b'' \in \cap_{i<\omega} U_i(B)$ be such that $b'' \in U(B)$ and $b'' < b$ for

each $b \in \cap_{i<\omega} U_i(B_n)$. Since $B \upharpoonright L(1)$ is $\Sigma(1)$-generic we may choose

$b' \in U(B)$ such that

1) $b'' \leq b'$

2) if $b \in \cap_{i<\omega} U_i(B_n)$ then $b' < b$

3) $B \models \neg U_0(I(u \vee w))$ whenever $u,v,w \in |B_n| \cup \{b'\}$ and

at least one is b'

4) $B \models I(u \vee w) < b$ when $b \in B_n \setminus \{c_B\}$, $u,v,w \in |B_n| \cup \{b'\}$

and at least one of u,v,w is b'

5) the ordering on $\{I(u,v,w) : u,v,w \in U(B_n) \cup \{b'\}$,

one of u,v,w is b' , and $B \models u < v \wedge v < w\}$ is the

lexicographic order.

We leave the explicit application of the techniques of the proof of

Lemma 6.4 to the reader.

Now choose $j > 0$ such that for all $a \in A_{n+1}$ if $a \in U_{j-1}(A)$

then $a \in \cap_{i<\omega} U_i(A)$.

From Lemma 6.4 $A \upharpoonright L(j)$ is $\Sigma(j)$-generic. Thus we can choose

$a' \in U(A)$ such that

1) $a' \in U_{j-1}(A)$

2) if $a \in A_{n+1}$ and $a \in U_{j-1}$ then $a' < a$

3) $A \models \neg U_0(I(u,v,w))$ whenever at least one of u,v,w is a'

and $u,v,w \in |A_{n+1}| \cup \{a'\}$

4) $A \models I(u \vee w) < a$ when $a \in A_{n+1} \setminus \{c_A\}$ at least

one of u,v,w is a' and $u,v,w \in |A_{n+1}| \cup \{a'\}$

5) the ordering on $\{I(u,v,w) :$ one of u,v,w is a',

$u,v,w \in U(A_{n+1}) \cup \{a'\}$, and $A \models u<v \wedge v<w\}$ is the

lexicographic order.

Now $[|A_n| \cup \{a'\}]_A \upharpoonright L(j)$ and $[|B_n| \cup \{b'\}]_B \upharpoonright L(j)$ are

isomorphic by the extension of ρ_n which takes a' to b'.

Since B is $\Sigma(j)$-generic this may be extended to an embedding ρ

of $[|A_{n+1}| \cup \{a'\}]_A \upharpoonright L(j)$ into $B \upharpoonright L(j)$. Let $b_n = \rho(a_n)$ and

let $\rho_{n+1} = \rho \upharpoonright A_{n+1}$. The choice of a', b' and ρ is sufficient

to show that ρ_{n+1} is an isomorphism by the argument from case 1.

Thus $A \cong B$, and T_1 has at most 4 countable models. Since

T_1 is easily seen to have at least 4 countable models the proof of

the theorem is complete.

Let p be the 1-type of a member of $\bigcap\limits_{i<\omega} U_i$, i.e. the 1-type gen-

erated by $\{U_i v_0 \wedge U v_0 : i<\omega\}$. Then there are only a finite number of

2-types in $p \times p$ - each principal over at least one coordinate: the

types q_0, q_1, q_2 where $q_0 \supset \{v_0 = v_1\}$, $q_1 \supset \{v_0 < v_1\}$ and

$q_2 \supset \{v_1 < v_0\}$. Indeed one may show that if A is the countable

saturated model of T_1 then given a,b in A a is prime over b

or b is prime over a. One need only modify the first steps in

the proof of the theorem to construct the desired automorphism of A.

There are, however, infinitely many 3-types in p. For each

$j > 0$ there is the 3-type r_j in p such that

$r_j \supset \{v_0 < v_1 \wedge v_1 < v_2 \wedge U_{j-1}(I v_0 v_1 v_2) \wedge \neg U_j(I v_0 v_1 v_2)\}$.

Clearly $r_j \neq r_k$ if $j \neq k$. There are also the 3-types r_0 and r_ω in p where

$$r_0 \supset \{v_0 < v_1 \wedge v_1 < v_2 \wedge \neg U_0 (I\, v_0 v_1 v_2)\}$$

and $r_\omega \supset \{v_0 < v_1 \wedge v_1 < v_2\} \cup \{U_i (I\, v_0 v_1 v_2) : i < \omega\}$.

T_1 has an inessential extension T_2 which has infinitely many countable models. Let c_0, c_1 be new constant symbols and let T_2 be a completion of $T \cup \{U_j c_0 : j < \omega\} \cup \{c_0 < c_1\}$. There are now infinitely many models satisfying Condition II. For each $j < \omega$ there is a model M_j with an element $a = \sup\{U_i\}$ such that $(a, (c_0)_{M_j}, (c_1)_{M_j})$ realizes r_j . Thus the M_j are distinct countable models of T_2 .

The construction of T_1 uses an infinite number of unary predicate symbols. We have not succeeded in finding a construction that gives an example in a small language that is finite except for constant symbols. Also T_1 has 4 countable models. This raises the following questions:

Q11: If T has a finite number of countable models and if T is in a small language which has only a finite number of symbols other than constant symbols, does every inessential extension of T have a finite number of countable models? , and

Q12: If T has three countable models does every inessential extension of T have a finite number of countable models?

Conclusion

It is to be hoped that the techniques employed in this work can be used to develop more examples of theories with a finite number of countable models. The reader will have noticed that all of the examples considered have an underlying dense, albeit partial, order and thus satisfy C2. Some further examples which cast light on this conjecture would be welcome.

The properties of theories with a small language are of independent interest as shown by Q11 and C6. The class of such theories with function symbols appears to be much richer than the class of theories in a small language which do not have function symbols.

However, the results of Chapter 4 indicate that considerable complexity is possible even for those theories with one binary relation symbol. The problem of classifying all finite ultrahomogeneous graphs is an illustration.

Finally it appears that theories with three countable models may have very special properties since they have middle models. A separate investigation of such theories would be worthwhile. An open question is whether the existence of a middle model is equivalent to the property of having three countable models.

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