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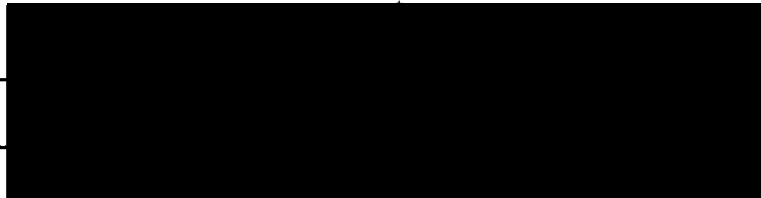NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE___ DR. R.W. LARDNER

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FURTHER RESULTS RELATING TO THE WEAKLY NONLINEAR

WAVE EQUATION OF VAN DER POL TYPE

by

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B.Sc., Simon Fraser University, 1975

M.Sc., Simon Fraser University, 1978

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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ABSTRACT

The two-time method is applied to the wave equation with small nonlinearity to obtain the general form for the lowest order approximation to the solution. This is then specialized to the case where the nonlinearity is of Van der Pol type and the integro-differential equation governing the lowest order approximation is explicitly derived. Viewed as an ordinary differential equation, it turns out that it belongs to a class of equations which ordinarily cannot be explicitly solved. The stationary solutions are found and a stability analysis provides a complete classification of the possible asymptotic solutions that can be generated. These solutions represent waves having sawtooth profiles, one wave moving in each direction. An application to a model for wind-induced oscillations of overhead power lines is given.

Two non-stationary solutions for the integro-differential equations are also given and in each case the asymptotic solutions attained are consistent with those found from the stability analysis. Several theorems on general time dependent behaviour of the solution reduce the problem of predicting which asymptotic solution will be generated from given initial values. Numerical solutions of the equation for a variety of initial values show asymptotic solutions of the same type as those predicted by the stability analysis.

DEDICATION

To my FATHER in Heaven, the true Author of all things.

ACKNOWLEDGEMENTS

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CHAPTER 1

1. Introduction.

Much recent research has necessitated finding the solutions of certain nonlinear differential equations and while general solutions are normally not available, a considerable amount of success has been achieved, particularly in problems related to nonlinear vibrations, by using certain asymptotic methods. The principle techniques include the KBM method [1], [2], the method of averaging [2] and the two-time [3] or multiple scale [4] methods. They were pioneered for use in ordinary differential equations but more recently they have been successfully applied to certain partial differential equations. In this regard we can mention the work of Bojadziev and Lardner [5], [6], [7] and Myerscough [8] using the KBM method and the closely related work of Davy and Ames [9], Keller and Kogelman [10] and Lardner [11] using the method of averaging. Two-time expansions have been used by Chikwendu and Kevorkian [12], Nayfeh [13] and Lardner [14] and more recently Lardner [15] has used a three-time expansion to obtain the lowest order approximation valid for time $t = 0(1/\epsilon^2)$ for the equation $u_{tt} - u_{xx} = \epsilon(2u_x u_{xx} - 2\lambda u_t)$.

In this thesis we shall be considering the application of the two-time method to the weakly nonlinear wave equation. For certain classes of ordinary differential equations it has been shown by Morrison [16] that the two-time method and the method of

averaging yield identical results. Then, due to a theorem of Bogoliubov [2] justifying the method of averaging, we have indirect justification for the two-time method for these equations. For certain classes of partial differential equations, it has been shown by Lardner [14] that the method of averaging and the two-time method lead to the same lowest order approximation. A justification of the two-time method for a certain class of nonlinear wave equations has been given by Ekhaus [19].

In the next chapter we apply the two-time method to obtain an integro-differential equation which governs the lowest order approximation to the solution of the weakly nonlinear wave equation

$$u_{tt} - u_{xx} = \epsilon\, E(x,t,u,u_x,u_t,\epsilon) \ .$$

Here ϵ is a small parameter and E a general, suitably differentiable function of its arguments. The solution is obtained in the form of a uniform expansion

$$u(x,t) = u_0(x,t,\tau) + \epsilon\, u_1(x,t,\tau) + 0(\epsilon^2)$$

where $\tau = \epsilon t$. This solution remains valid for times $t = 0(1/\epsilon)$. In §2.2 we show explicitly that when $E = -2u_t - \epsilon u$, the difference between the exact solution and the approximate solution given by the two-time method is $o(1)$. For fixed end conditions $u(0,t) = u(\ell,t) = 0$, $t \geq 0$ and initial conditions of the type $u(x,0) = \chi(x)$ and

$u_t(x,0) = \psi(x)$, it is shown that the lowest order solution becomes

$$u(x,t) = u_0(x,t,\tau) = G(\alpha,\tau) - G(\beta,\tau)$$

where $\alpha = t + x$, $\beta = t - x$. The integro-differential equation and initial values for the function $g(\theta,\tau) = G_\theta(\theta,\tau)$ (θ representing α or β as required) are then obtained. These are then applied to the case when the nonlinearity is of Van der Pol type (i.e., $E = u_t - \frac{1}{3} u_t^3$). We obtain the equation

$$2g_\tau = (1-\overline{g^2})g - \frac{1}{3} g^3 + \frac{1}{3} \overline{g^3}$$

where   (1.1)

$$\overline{g^n}(\tau) = \frac{1}{2\ell} \int_{-\ell}^{\ell} g^n(\theta,\tau)d\theta \quad \text{with} \quad \overline{g^1} = \overline{g} \equiv 0 .$$

This equation, which will be the primary topic of consideration in this work, was derived in this form by Lardner [14]. A special case which turns out to be valid only in the case of impulsive initial conditions $(u(x,0) = 0)$ was obtained by Myerscough [17]. Chikwendu and Kevorkian [12] have investigated an application to progressive waves $(u(x,0) = \chi(x)$, $u_t(x,0) = -\chi'(x))$. A direct derivation of this same equation using the method of averaging is also given at the end of Chapter 2.

If we view eqn. (1.1) as an ordinary differential equation (θ appearing only parametrically), then we see that it belongs to a class of equations which cannot, in general, be explicitly solved. It is possible however to obtain its stationary solutions and this we do in Chapter 3. (Much of the material in this chapter has recently appeared in the literature [18]). Due to the cubic nature of the equation obtained by setting $g_\tau = 0$, there are three stationary values g_1, g_2, g_3, each of which represents a possible solution along some portion I_1, I_2, I_3 respectively of the interval $-\ell \leq \theta \leq \ell$. The length of each of these intervals is $2\ell\lambda_p$ ($p = 1,2,3$) where $\lambda_1 + \lambda_2 + \lambda_3 = 1$. A stability analysis (only these solutions which are stable are obtainable as possible asymptotic solutions from time dependent behaviour) shows that in no case is one of these three root cases stable. The only possible stable configurations arise when one of the intervals has length zero. That is, one $\lambda_p = 0$ and the corresponding root g_p does not appear in the stationary solution. For example, if we take without loss of generality $\lambda_3 = 0$, then the stationary solution is given by $g_1 = \pm \sqrt{3\lambda_2}$ on I_1 and $g_2 = \mp \sqrt{3\lambda_1}$ on I_2. Moreover, we must have $\frac{1}{3} < \lambda_1 < \frac{2}{3}$ for stability. A maximum asymptotic displacement of $2\sqrt{3\lambda_1(1-\lambda_1)}\ell$ is obtained. This value is maximum when $\lambda_1 (=\lambda_2) = \frac{1}{2}$ and corresponds to the special case given by Myerscough [8]. He considered the equation

$$y_{tt} - c^2 y_{xx} = \alpha y_t - \beta y_t^3$$

as a model for wind induced oscillations of overhead transmission

lines. In §3.4 we relate our results to his for various values of

the parameters c, α and β . Finally, in §3.5 we show that

eqn. (1.1) can be transformed into a coupled, infinite system of

differential equations. A form for the stationary solutions of these

is derived, but due to the complexity of the system no stability

analysis is attempted.

In Chapter 4 we consider various non-stationary solutions

and general theorems concerning the time dependent behaviour of the

solution of eqn. (1.1). For odd initial conditions

$g(\theta,0) = -g(-\theta,0)$ we have $g(\theta,\tau) = -g(-\theta,\tau)$ for all τ and hence

$\overline{g}^3(\tau) \equiv 0$. Then eqn. (1.1) can be solved [14]. Asymptotically, it

is shown that $g \to \pm\sqrt{3}/2$, the sign depending on whether the initial

value was positive or negative. This is consistent with the type of

behaviour found by Myerscough [17]. If the initial conditions are piece-

wise constant (essentially independent of θ) subject to the constraint

that $\overline{g} \equiv 0$, a solution to the full equation (1.1) can be found. It

is shown that the solution retains the same distribution of positive

and negative values as it originally had and attains an asymptotic

solution of a type similar to those found in chapter 3. A third

non-stationary solution is sought using a perturbation on the odd

solution. Observing that equation (1.1) can be solved in the case

of odd initial conditions, the question arises as to the feasibility

of obtaining a solution when the initial conditions are slightly

different from those in the odd case. Unfortunately, it turns out

that the perturbation grows exponentially in the neighbourhoods of the zeroes of the odd function and hence the attempted perturbation solution is non-uniform.

Although it does not seem possible to solve the general problem $(\overline{g^3}(\tau) \neq 0)$, it would be desirable to be able to pick which of the asymptotic solutions a given set of initial values would produce. This would involve determining the value $\lambda = \lambda_1$ (hence $\lambda_2 = 1-\lambda_1 = 1-\lambda$) and the location of the intervals I_1' and I_2. It turns out that this problem can be reduced to simply finding the value of λ. In §4.4 we show that if $\bar{g}(\theta_1,0) > g(\theta_2,0)$ for $\theta_1, \theta_2 \in [-\ell,\ell]$ then $g(\theta_1,\tau) > g(\theta_2,\tau)$ for all finite τ. Knowing λ, this result allows us to predict the location of the intervals from the initial values. Furthermore, we show that if two sets of initial values are related by $g_1(\theta,0)/g_2(\theta,0) = $ constant, then the same asymptotic solution is attained. From this it is speculated that λ may be found in terms of certain scale invariant ratios. Lastly we prove that the migration of the zeroes of the solution function is directly related to the presence of the $\overline{g^3}$ term.

In the last chapter various numerical solutions for eqn. (1.1) are presented. In each case the form of the asymptotic solution attained is found to be consistent with the predicted stable asymptotic solutions.

CHAPTER 2

THE TWO TIME METHOD FOR THE WAVE EQUATION WITH SMALL NONLINEARITY

In this chapter we consider the equation

$$u_{tt} - u_{xx} = \epsilon E(x,t,u,u_x,u_t,\epsilon) \tag{2.1}$$

where ϵ is a small parameter and E a general, suitably differentiable function of its arguments. The two time method [3] is employed to obtain the general integro-differential equation which governs the lowest order solution of eqn. (2.1). Two time expansions have been used previously by Chikwendu and Kevorkian [12] and Lardner [14] to investigate wave equations with small nonlinearities.

For the case when E is of Van der Pol type, that is $E = u_t - \frac{1}{3} u_t^3$, the integro-differential equation describing the lowest order approximation is explicitly derived. This has been given previously by Lardner [14]. An alternative derivation of this equation using the method of averaging [2] is given. Myerscough [8], [17] has also investigated this equation but it turns out that his conclusions are valid only for the case of impulsive initial conditions (i.e., $u(x,0) = 0$). Keller and Kogelman [10] have applied the method of averaging to the Klein-Gordon equation with small nonlinearity and have discussed in detail the case where the nonlinear term is of Van der Pol type. The wave equation presents a more difficult problem than the Klein-Gordon equation because infinitely many

internal resonances are brought into operation by the Van der Pol nonlinearity.

2.1 The Two Time Method.

According to the two time method we seek a solution to eqn. (2.1) in the form

$$u(x,t) = u_0(x,t,\tau) + \epsilon\, u_1(x,t,\tau) + 0(\epsilon^2) \qquad (2.2)$$

where $\tau = \epsilon t$. We suppose that ϵu_1 is a small correction to the u_0 term and that the expansion is uniform in the sense that the ratio $\epsilon u_1/u_0$ is $o(1)$ for all t in the interval of interest. Since the solution, eqn. (2.2), is to be valid for times as large as $0(1/\epsilon)$, this condition requires that we cannot allow terms in u_1 which grow linearly with respect to t. That is, we must have

$$\lim_{t \to \infty} \frac{1}{t}\, u_1(x,t,\tau) = 0 . \qquad (2.3)$$

Substituting eqn. (2.2) into eqn. (2.1) and comparing terms of orders ϵ^0 and ϵ^1, we obtain

$0(1)$
$$u_{0tt} - u_{0xx} = 0 \qquad (2.4)$$

$0(\epsilon)$
$$u_{1tt} - u_{1xx} = -2u_{0t\tau} + E(x,t,\{u_0(x,t,\tau)\},0) \qquad (2.5)$$

where $E(x,t,\{u_0(x,t,\tau)\},0) = E(x,t,\, u_0(x,t,\tau), u_{0x}(x,t,\tau), u_{0t}(x,t,\tau),0).$

The solution of eqn. (2.4) is simply given by

$$u_0(x,t,\tau) = G(t+x,\tau) - H(t-x,\tau) \qquad (2.6)$$

for two suitably differentiable functions G and H. To this order of approximation τ appears only parametrically. Setting $\alpha = t+x$ and $\beta = t-x$ it follows that eqn. (2.5) can be written as

$$4u_{1\alpha\beta} = -2(G_{\alpha\tau} - H_{\beta\tau}) + E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - H(\beta,\tau)\},0) . \qquad (2.7)$$

Integration of eqn. (2.7) gives

$$u_1 = -\frac{\beta}{2} G_\tau + \frac{\alpha}{2} H_\tau + \frac{1}{4} \iint E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - H(\beta,\tau)\},0)d\alpha d\beta$$

$$\qquad (2.8)$$

$$+ G_1(\alpha,\tau) - H_1(\beta,\tau)$$

where the last two terms represent the complementary function. To eliminate any secular terms we impose condition (2.3) which in this case takes the form

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} u_1 = 0 \quad , \quad \lim_{\beta \to \infty} \frac{1}{\beta} u_1 = 0 .$$

Hence we obtain the following integro-differential equations for the determination of G and H

$$G_\tau = \lim_{\beta \to \infty} \frac{1}{2\beta} \iint E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - H(\beta,\tau)\},0)\,d\alpha d\beta$$

$$\hspace{12cm} (2.9).$$

$$H_\tau = \lim_{\alpha \to \infty} -\frac{1}{2\alpha} \iint E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - H(\beta,\tau)\},0)\,d\alpha d\beta .$$

2.2   A Simple Example.

In this section we shall compare a known exact solution to a wave equation with small nonlinearity to its approximation solution obtained by the two time method. Consider the equation.

$$u_{tt} - u_{xx} = -2\epsilon u_t - \epsilon^2 u .$$

It is straightforward to show that the general solution of this equation is

$$u(x,t) = e^{-\epsilon t}[f(t+x) - g(t-x)]$$

where f and g are suitably differentiable functions of their arguments.

To apply the results from the previous section, we consider the equation

$$u_{tt} - u_{xx} = -2\epsilon u_t - \epsilon^2 u = \epsilon[-2u_t - \epsilon u]$$

where we note that E is now a function of ϵ . The lowest order solution is given by eqn. (2.6)

$$u_0(x,t,\tau) = G(\alpha,\tau) - H(\beta,\tau)$$

where $\alpha = t+x$ and $\beta = t-x$. The τ-dependence of G and H is to be determined from eqns. (2.9). It then follows that

$$u_t = u_\alpha + u_\beta = G_\alpha(\alpha, \tau) - H_\beta(\beta, \tau) \ .$$

Setting $E(x,t,\{u_0(x,t,\tau)\},0) = -2u_t$ in eqns. (2.9), we obtain

$$G_\tau(\alpha, \tau) = \lim_{\beta \to \infty} \frac{1}{2\beta} \iint -2\{G_\alpha(\alpha, \tau) - H_\beta(\beta, \tau)\} \, d\alpha d\beta$$

$$= -\lim_{\beta \to \infty} \frac{1}{\beta} [\beta G(\alpha, \tau) - \alpha H(\beta, \tau)]$$

$$= -G(\alpha, \tau) \ .$$

Similarly $H_\tau(\beta, \tau) = -H(\beta, \tau)$. These equations have solutions

$$G(\alpha, \tau) = f_1(\alpha) e^{-\tau} \ , \qquad H(\beta, \tau) = g_1(\beta) e^{-\tau}$$

where f_1 and g_1 are arbitrary differentiable functions of their arguments. Thus the lowest order solution becomes

$$u_0(x,t,\tau) = e^{-\tau}\left[f_1(\alpha) - g_1(\beta)\right]$$

where $\tau = \epsilon t$. This solution is valid for $t = 0(\frac{1}{\epsilon})$. If we impose arbitrary initial conditions $u(x,0) = \chi(x)$ and $u_t(x,0) = \psi(x)$ on

the solution, we shall find that $f = f_1$ and $g = g_1$. Clearly

then for the time interval in which the approximate solution is valid,

we would have

$$|u - u_0| = o(1)$$

· as $\epsilon \to 0$.

2.3 Application to Fixed End Conditions.

Here we consider the problem

$$u_{tt} - u_{xx} = \epsilon E(x,t,u,u_x,u_t,\epsilon)$$

$$u(0,t) = u(\ell,t) = 0 \qquad t \geq 0 \qquad (2.10)$$

$$u(x,0) = \chi(x) \quad , \quad u_t(x,0) = \psi(x) \qquad 0 \leq x \leq \ell$$

To the order of approximation being sought $u(x,t) = u_0(x,t,\tau) + O(\epsilon)$,

we have from eqn. (2.6) that

$$u(0,t) = G(t,\tau) - H(t,\tau) = 0$$

and

$$u(\ell,t) = G(t+\ell,\tau) - H(t-\ell,\tau) = 0 .$$

These conditions are satisfied if we take

$$G(\theta,\tau) = H(\theta,\tau) \ , \qquad G(\theta+2\ell,\tau) = G(\theta,\tau) \tag{2.11}$$

for all θ,τ . Hence in this approximation, the functions G and H are equal and 2ℓ-periodic in their first arguments.

Since the functions G and H are to be equal, it follows that eqns. (2.9) should reduce to a single equation. If we extend the definition of the function E by

$$E(-x,t,-u,u_x,-u_t,\epsilon) = - E(x,t,u,u_x,u_t,\epsilon) \ ,$$

and

$$E(x+2\ell,t,u,u_x,u_t,\epsilon) = E(x,t,u,u_x,u_t,\epsilon) \ ,$$

then eqn. (2.9) can be replaced by the single equation

$$G_\tau = \lim_{\beta \to \infty} \frac{1}{2\beta} \iint E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - G(\beta,\tau)\}, 0) \, d\alpha d\beta \ .$$

Differentiating with respect to α and setting $g = G_\alpha$, we obtain the usual form for this equation

$$g_\tau = \lim_{\beta \to \infty} \frac{1}{2\beta} \int E(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}, \{G(\alpha,\tau) - G(\beta,\tau)\}, 0) \, d\beta \ . \tag{2.12}$$

The lowest order solution (2.6) becomes

$$u(x,t) = G(\alpha,\tau) - G(\beta,\tau) + 0(\epsilon) \tag{2.13}$$

with corresponding derivatives

$$u_t(x,t) = g(\alpha,\tau) - g(\beta,\tau) + 0(\epsilon)$$

$$(2.14)$$

$$u_x(x,t) = g(\alpha,\tau) + g(\beta,\tau) + 0(\epsilon) \ .$$

From the initial conditions we have

$$g(x,0) - g(-x,0) = \psi(x)$$

$$g(x,0) + g(-x,0) = \chi'(x) \qquad 0 \le x \le \ell \ .$$

These can be inverted to give

$$g(\theta,0) = \frac{1}{2}[\psi(\theta) + \chi'(\theta)]$$

$$(2.15)$$

$$g(-\theta,0) = \frac{1}{2}[-\psi(\theta) + \chi'(\theta)] \qquad 0 \le \theta \le \ell \ .$$

We see that the initial values of g are known on the interval $-\ell \le \theta \le \ell$ and so by periodicity can be extended to all values of θ.

The problem of the first approximation for fixed end conditions consists of solving the integro-differential equation (2.12) subject to the initial condition (2.15).

2.4   Two Time Method for Wave Equation with Van der Pol Type

Nonlinearity.

Here we specialize the results of the preceding sections

to the case where

$$E(x,t,u,u_x,\bar{u}_t,\epsilon) = u_t - \frac{1}{3} u_t^3 .$$

The lowest order solution for fixed end conditions is given by (2.13)

while from (2.14) it is clear that eqn. (2.12) becomes

$$g_\tau = \lim_{\beta \to \infty} \frac{1}{2\beta} \int \left[\{ g(\alpha,\tau) - g(\beta,\tau) \} - \frac{1}{3} \{ g(\alpha,\tau) - g(\beta,\tau) \}^3 \right] d\beta .$$

Performing the integrations, this can be written as

$$g_\tau(\alpha,\tau) = \frac{1}{2} g(\alpha,\tau) - \lim_{\beta \to \infty} \frac{1}{2\beta} \int g(\beta,\tau) d\beta - \frac{1}{6} g^3(\alpha,\tau)$$

$$+ \frac{1}{2} g^2(\alpha,\tau) \lim_{\beta \to \infty} \int g(\beta,\tau) d\beta - \frac{1}{2} g(\alpha,\tau) \lim_{\beta \to \infty} \int g^2(\beta,\tau) d\beta$$

$$+ \frac{1}{6} \int g^3(\beta,\tau) d\beta .$$

Since $g(\alpha,\tau)$ is 2ℓ-periodic in α, it follows that the long time

averages can be replaced by the average over one period. Setting

$$\overline{g^n}(\tau) = \lim_{\beta \to \infty} \frac{1}{\beta} \int g^n(\beta,\tau) d\beta = \frac{1}{2\ell} \int_{-\ell}^{\ell} g^n(\beta,\tau) d\beta$$

we obtain

$$2g_\tau = (1 - \overline{g^2})g - \frac{1}{3} g^3 + \frac{1}{3} \overline{g^3} . \tag{2.16}$$

Since G is a 2ℓ-periodic function in its first argument, the first moment of its derivative g ,

$$\overline{g^1}(\tau) = \frac{1}{2\ell} \int_{-\ell}^{\ell} g(\beta,\tau) d\beta$$

is identically zero. Eqn. (2.16) is to be solved together with the initial conditions (2.15) to obtain the lowest order solution for the case of the Van der Pol nonlinearity.

2.5 The Method of Averaging for the Wave Equation with Van der Pol Type Nonlinearity.

In the previous section we derived the integro-differential equation which governs the lowest approximation to the problem

$$u_{tt} - u_{xx} = \epsilon(u_t - \frac{1}{3} u_t^3)$$

$$u(0,t) = u(\ell,t) = 0 \qquad\qquad t \geq 0$$

$$u(x,0) = \chi(x) , \quad u_t(x,0) = \psi(x) \qquad 0 \leq x \leq \ell .$$

In this section, we shall use the method of averaging to give an alternate derivation of eqn. (2.16). Lardner [14] has previously shown that for a large class equations, the two time method and method

of averaging lead to the same lowest order approximation. Here we shall demonstrate explicitly that this is the case for the above problem.

To apply the method of averaging we first obtain the solution for the case $\epsilon = 0$. This is given by

$$u(x,t) = \sqrt{\frac{2}{\ell}} \sum_{n=1}^{\infty} \frac{1}{n} \left[z_n e^{i\lambda_n t} + \bar{z}_n e^{-i\lambda_n t} \right] \sin \lambda_n x$$

where $\lambda_n = n\pi/\ell$ and $z_n, \bar{z}_n$ are constants determined from the initial conditions. For $\epsilon \neq 0$ we assume that a solution can be obtained in the form

$$u(x,t) = \sqrt{\frac{2}{\ell}} \sum_{n=1}^{\infty} \frac{1}{n} \left[z_n(\tau) e^{i\lambda_n t} + \bar{z}_n(\tau) e^{-i\lambda_n t} \right] \sin \lambda_n x \qquad (2.17)$$

where z_n and $\bar{z}_n$ are now considered to be functions of the slow time $\tau = \epsilon t$. Also, we take

$$u_t(x,t) = \sqrt{\frac{2}{\ell}} \sum_{n=1}^{\infty} i \frac{\pi}{\ell} \left[z_n(\tau) e^{i\lambda_n t} - \bar{z}_n(\tau) e^{-i\lambda_n t} \right] \sin \lambda_n x \qquad (2.18)$$

which requires that

$$z_n'(\tau) e^{i\lambda_n t} + \bar{z}_n'(\tau) e^{-i\lambda_n t} = 0 \qquad (2.19)$$

for each n. Substituting (2.17) and (2.18) into the differential equation, we obtain

$$\sum_{n=1}^{\infty} \left[z_n' e^{i\lambda_n t} - \overline{z_n'} e^{-\lambda_n t} \right] \sin \lambda_n x = \sum_{n=1}^{\infty} P_n \sin \lambda_n x$$

$$+ \frac{2\pi^2}{3\ell^3} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} P_n P_p P_q \sin(\lambda_n x) \sin(\lambda_p x) \sin(\lambda_q x)$$

where $P_n = z_n e^{i\lambda_n t} - \overline{z_n} e^{-i\lambda_n t}$. Multiplying by $\frac{2}{\ell} \sin \lambda_m x$ and

integrating over $[0,\ell]$ gives

$$z_m' e^{i\lambda_m t} - \overline{z_m'} e^{-i\lambda_m t} = P_m + \frac{1}{3} \sum_{n,p,q=1}^{\infty} P_n P_p P_q J_{mnpq}$$

where

$$J_{mnpq} = \frac{4\pi^2}{\ell^4} \int_0^\ell \sin(\lambda_m x) \sin(\lambda_n x) \sin(\lambda_p x) \sin(\lambda_q x) \, dx \ .$$

Making use of eqn. (2.19), we obtain the exact system of equations

$$2z_m' = P_m e^{-i\lambda_m t} + \frac{1}{3} \sum_{n,p,q=1}^{\infty} P_n P_p P_q e^{-i\lambda_m t} J_{mnpq} \ .$$

This system is now replaced by the following averaged system

$$2z_m' = \lim_{T \to \infty} \frac{1}{T} \int_0^T P_m e^{-i\lambda_m t} dt$$

$$+ \frac{1}{3} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \sum_{n,p,q=1}^{\infty} P_n P_p P_q e^{-i\lambda_m t} J_{mnpq} \right\} dt \ .$$

Bogoliubov [2] has shown that if τ is restricted to a finite interval $0 \leq \tau \leq L$, where L is any constant, then the difference between the solutions $z_n(\tau)$ of the original system and the averaged system is $O(1)$ as $\epsilon \to 0$. The initial values $z_n(0)$ for each system are presumed to be the same.

Performing the integrations and taking the limits, we obtain the autonomous system

$$2z_m' = z_m + \frac{1}{3} \sum_{n,p,q=1}^{\infty} I_{npq} J_{mnpq} \qquad (2.20)$$

where

$$I_{npq} = \lim_{T \to \infty} \frac{1}{T} \int_0^T (z_n e^{i\lambda_n t} - \bar{z}_n e^{-i\lambda_n t})(z_p e^{i\lambda_p t} - \bar{z}_p e^{-i\lambda_p t})$$

$$(z_q e^{i\lambda_q t} - \bar{z}_q e^{-i\lambda_q t})e^{-i\lambda_m t} dt .$$

$$\begin{aligned}
&= z_n z_p z_q &&\text{if} && m = n+p+q \\
&-z_n z_p \bar{z}_q &&\text{if} && m = n+p-q \\
&-z_n \bar{z}_p z_q &&\text{if} && m = n-p+q \\
&z_n \bar{z}_p \bar{z}_q &&\text{if} && m = n-p-q \\
&-\bar{z}_n z_p z_q &&\text{if} && m = -n+p+q \\
&\bar{z}_n \bar{z}_p z_q &&\text{if} && m = -n-p+q \\
&\bar{z}_n z_p \bar{z}_q &&\text{if} && m = -n+p-q \\
&-\bar{z}_n \bar{z}_p \bar{z}_q &&\text{if} && m = -n-p-q .
\end{aligned}$$

We also have

$$J_{mnpq} = \frac{4\pi^2}{\ell^4} \int_0^\ell \sin \lambda_m x \, \sin \lambda_n x \, \sin \lambda_p x \, \sin \lambda_q x \, dx$$

$$= \frac{\pi^2}{2\ell^4} \int_0^\ell \{\cos(\lambda_m - \lambda_n + \lambda_p - \lambda_q)x + \cos(\lambda_m - \lambda_n - \lambda_p + \lambda_q)x - \cos(\lambda_m - \lambda_n + \lambda_p + \lambda_q)x$$

$$- \cos(\lambda_m - \lambda_n - \lambda_p - \lambda_q)x - \cos(\lambda_m + \lambda_n + \lambda_p - \lambda_q)x - \cos(\lambda_m + \lambda_n - \lambda_p + \lambda_q)x$$

$$+ \cos(\lambda_m + \lambda_n + \lambda_p + \lambda_q)x + \cos(\lambda_m + \lambda_n - \lambda_p - \lambda_q)x\}dx \ .$$

Noting that a contribution to this integral is obtained only when one
of the arguments is equal to zero, we obtain that

$$J_{mnpq} = \frac{\pi^2}{2\ell^3} \sum_{\pm\text{'s}} \pm \, \delta(m, \pm n \pm p \pm q)$$

where $\delta(i,j)$ is the Kronecker delta. This summation consists of the
sum of eight distinct delta terms. In summing, the + sign is chosen
when the number of internal minus signs is odd and the − sign is
chosen when the number of internal minus signs is even.

Introducing these expressions for I_{npq} and J_{mnpq} into the
sum in eqn. (2.20), we obtain that

$$\sum_{n,p,q=1}^{\infty} I_{npq} \, J_{mnpq} = \frac{\pi^2}{2\ell^3} \sum_{n,p,q=1}^{\infty} [z_n z_p z_q \, \delta(m,n+p+q) - z_n z_p \bar{z}_q \, \delta(m,n+p-q)$$

$$- z_n \bar{z}_p z_q \, \delta(m,n-p+q) + z_n \bar{z}_p \bar{z}_q \, \delta(m,n-p-q) - \bar{z}_n z_p z_q \, \delta(m,-n+p+q)$$

$$+ \bar{z}_n \bar{z}_p z_q \, \delta(m,-n-p+q) + \bar{z}_n z_p \bar{z}_q \, \delta(m,-n+p-q) - \bar{z}_n \bar{z}_p \bar{z}_q \, \delta(m,-n-p-q)] \cdot$$

$$[-\delta(m,n+p+q) + \delta(m,-n+p+q) + \delta(m,n-p+q) + \delta(m,n+p-q)$$

$$- \delta(m,-n-p+q) - \delta(m,-n+p-q) - \delta(m,n-p-q) + \delta(m,-n-p-q)] \ .$$

Performing the multiplications, disregarding terms which give no contribution and making use of the symmetry in n,p and q leads to

$$\sum_{n,p,q=1}^{\infty} I_{npq} J_{mnpq} = -\frac{\pi^2}{2\ell^3} \sum_{n,p,q=1}^{\infty} z_n z_p z_q \, \delta(m,n+p+q) - \frac{3\pi^2}{\ell^3} z_m \sum_{n=1}^{\infty} z_n \bar{z}_n$$

$$-\frac{3\pi^2}{2\ell^3} \sum_{n,p,q=1}^{\infty} z_n z_p \bar{z}_q \, \delta(m,n+p-q) - \frac{3\pi^2}{2\ell^3} \sum_{n,p,q=1}^{\infty} z_n \bar{z}_p \bar{z}_q \, \delta(m,n-p-q) \ .$$

Using the above results, eqn. (2.20) becomes

$$2z'_m = z_m - \frac{\pi^2}{6\ell^3} \sum_{m=n+p+q} z_n z_p z_q - \frac{\pi^2}{\ell^3} z_m \sum_{n=1}^{\infty} z_n \bar{z}_n$$

$$- \frac{\pi^2}{2\ell^3} \sum_{m=n+p-q} z_n z_p \bar{z}_q - \frac{\pi^2}{2\ell^3} \sum_{m=n-p-q} z_n \bar{z}_p \bar{z}_q \qquad (2.21)$$

Taking the complex conjugate gives the corresponding conjugate equation

$$2\bar{z}'_m = \bar{z}_m - \frac{\pi^2}{6\ell^3} \sum_{m=n+p+q} \bar{z}_n \bar{z}_p \bar{z}_q - \frac{\pi^2}{\ell^3} \bar{z}_m \sum_{n=1}^{\infty} z_n \bar{z}_n$$

$$- \frac{\pi^2}{2\ell^3} \sum_{m=n+p-q} \bar{z}_n \bar{z}_p z_q - \frac{\pi^2}{2\ell^3} \sum_{m=n-p-q} \bar{z}_n z_p z_q \quad . \tag{2.22}$$

In each case the index m and the summation indices are assumed positive. The above eqns. can be simplified by the following technique. Define $\bar{z}_m = z_{-m}$ and $z_0 = 0$ and consider the sum

$$\sum_{m=n+p+q} z_n z_p z_q \quad \circ$$

Here we again assume that $m \geq 1$, but that n, p and q are allowed to take on both positive and negative values. A little consideration then shows that we can write

$$\sum_{m=n+p+q} z_n z_p z_q = \sum_{\substack{n,p,q>0 \\ m=n+p+q}} z_n z_p z_q + 3 \sum_{\substack{n,p,q>0 \\ m=n+p-q}} z_n z_p \bar{z}_q + 3 \sum_{\substack{n,p,q>0 \\ m=n-p-q}} z_n \bar{z}_p \bar{z}_q \quad .$$

Using this result, it follows that eqn. (2.21) can be written as

$$2z'_m = z_m - \frac{\pi^2}{6\ell^3} \left[\sum_{m=n+p+q} z_n z_p z_q + 3 z_m \sum_{n=-\infty}^{\infty} z_n z_{-n} \right] \quad . \tag{2.23}$$

Taking the complex conjugate of this equation, we obtain

$$2\bar{z}'_m = \bar{z}_m - \frac{\pi^2}{6\ell^3} \left[\sum_{m=n+p+q} \bar{z}_n \bar{z}_p \bar{z}_q + 3 \bar{z}_m \sum_{n=-\infty}^{\infty} z_n z_{-n} \right] \quad .$$

Letting $\bar{z}_k = z_{-k}$ and $k' = -k$ for $k = m,n,p,q$ gives

$$2z'_{m'} = z'_{m'} - \frac{\pi^2}{6\ell^3}\left[\sum_{m'=n'+p'+q'} z_{n'}z_{p'}z_{q'} + 3z_{m'}\sum_{n=-\infty}^{\infty} z_n z_{-n}\right].$$

Here $m' \leq -1$. Hence eqn. (2.23) is valid for all $m \neq 0$.

Define

$$z(\theta,\tau) = \sum_{m=-\infty}^{\infty} z_m(\tau)e^{i\lambda_m\theta}. \tag{2.24}$$

Multiplying (2.23) by $e^{i\lambda_m\theta}$ and summing over all $m \neq 0$, we obtain

$$2\sum_{m\neq 0} z'_m e^{i\lambda_m\theta} = \sum_{m\neq 0} z_m e^{i\lambda_m\theta} - \frac{\pi^2}{6\ell^3}\sum_{m\neq 0} e^{i\lambda_m\theta}(\sum_{n+p+q=m} z_n z_p z_q).$$

$$-\frac{\pi^2}{2\ell^3}\sum_{m\neq 0} z_m e^{i\lambda_m\theta}(\sum_{n=-\infty}^{\infty} z_n z_{-n})$$

$$= \sum_{m\neq 0} z_m e^{i\lambda_m\theta} - \frac{\pi^2}{6\ell^3}\sum_{m=-\infty}^{\infty} e^{i\lambda_m\theta}(\sum_{n+p+q=m} z_n z_p z_q)$$

$$+ \frac{\pi^2}{6\ell^3}\sum_{n+p+q=0} z_n z_p z_q - \frac{\pi^2}{2\ell^3}\sum_{m=-\infty}^{\infty} z_m e^{i\lambda_m\theta}(\sum_{n=-\infty}^{\infty} z_n z_{-n}). \tag{2.25}$$

From (2.24) we also have that

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} z^2(\theta,\tau)d\theta = \frac{1}{2\ell}\int_{-\ell}^{\ell} \sum_{n,p=-\infty}^{\infty} z_n z_p e^{i(\lambda_m+\lambda_p)\theta}d\theta$$

$$= \sum_{n+p=0} z_n z_p = \sum_{n=-\infty}^{\infty} z_n z_{-n}$$

and

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} z^3(\theta,\tau)\, d\theta = \sum_{n+p+q=0} z_n z_p z_q \ .$$

Finally, when use is made of (2.24), eqn. (2.25) can be expressed as

$$2\frac{\partial z}{\partial \tau} = z - \frac{\pi^2}{6\ell^3} z^3 + \frac{\pi^2}{12\ell^4} \int_{-\ell}^{\ell} z^3(\theta,\tau)\, d\theta - \frac{\pi^2}{4\ell^4} z \int_{-\ell}^{\ell} z^2(\theta,\tau)\, d\theta \ .$$

Rescaling z by

$$z(\theta,\tau) = \frac{\sqrt{2\ell^3}}{\pi} g(\theta,\tau) \tag{2.26}$$

and writing

$$\overline{g^n}(\tau) = \frac{1}{2\ell} \int_{-\ell}^{\ell} g^n(\theta,\tau)\, d\theta \ ,$$

we finally obtain

$$2g_\tau = (1-\overline{g^2}) g - \frac{1}{3} g^3 + \frac{1}{3} \overline{g^3} \ . \tag{2.27}$$

We have now derived using the two-time method and the method of averaging the integro-differential equation which governs the

lowest order approximation to the problem defined by eqn. (2.10).

We can now show that the functions g defined by eqns. (2.16) and (2.27) are the same to this order of approximation.

From eqn. (2.14) we have that

$$u_t(x,t) = g(\alpha,\tau) - g(\beta,\tau)$$

$$= g(t+x,\tau) - g(t-x,\tau) .$$

The corresponding derivative u_t for the method of averaging is given by eqn. (2.18). This can be rewritten as

$$u_t(x,t) = \frac{\pi}{\sqrt{2\ell}^3} \sum_{n=1}^{\infty} \left[z_n(\tau) e^{i\lambda_n(t+x)} + \bar{z}_n(\tau) e^{-i\lambda_n(t+x)} \right]$$

$$- \frac{\pi}{\sqrt{2\ell}^3} \sum_{n=1}^{\infty} \left[z_n(\tau) e^{i\lambda_n(t-x)} + \bar{z}_n(\tau) e^{-i\lambda_n(t-x)} \right] .$$

Making use of eqns (2.24) and (2.26) we obtain

$$u_t(x,t) = \frac{\pi}{\sqrt{2\ell}^3} \sum_{n=-\infty}^{\infty} z_n(\tau) e^{i\lambda_n(t+x)} - \frac{\pi}{\sqrt{2\ell}^3} \sum_{n=-\infty}^{\infty} z_n(\tau) e^{i\lambda_n(t-x)}$$

$$= \frac{\pi}{\sqrt{2\ell}^3} z(t+x,\tau) - \frac{\pi}{\sqrt{2\ell}^3} z(t-x,\tau)$$

$$= g(t+x,\tau) - g(t-x,\tau) .$$

Hence we have identical expressions for the derivative u_t. In a

similar fashion it can be shown that the derivative u_x is obtainable

as

$$u_x(x,t) = g(t+x,\tau) + g(t-x,\tau)$$

using either method. Thus to the order of approximation being

sought here, $u(x,t) = u_0(x,t,\tau) + O(\epsilon)$, we have exact agreement between

the two methods.

CHAPTER 3

THE STATIONARY SOLUTIONS OF THE g-EQUATION

Here we investigate the stationary solutions of the equation

$$2g_\tau = (1-\overline{g^2})g - \frac{1}{3}g^3 + \frac{1}{3}\overline{g^3} \qquad (3.1)$$

and determine which of these stationary solutions are stable. For only the stable solutions are asymptotically attainable from time dependent behaviour.

3.1 The Stationary Solutions.

Setting $g_\tau = 0$ in eqn. (3.1), we obtain

$$g^3 + 3(\overline{g^2}-1)g - \overline{g^3} = 0 \qquad (3.2)$$

where $\overline{g^n}$ are now all constants. This equation is cubic in g. Denoting the three possible roots by g_1, g_2 and g_3, we can replace (3.2) by the three conditions

$$g_1 + g_2 + g_3 = 0$$

$$g_1 g_2 + g_2 g_3 + g_3 g_1 = 3(\overline{g^2}-1)$$

$$g_1 g_2 g_3 = \overline{g^3} \quad .$$

To express $\overline{g^n}$ in terms of these three values, suppose that $g(\theta) = g_p$ for a fraction λ_p of the interval $-\ell < \theta < \ell$. $(p = 1,2,3)$. Then

$$\overline{g^n} = \frac{1}{2\ell} \int_{-\ell}^{\ell} g^n(\theta)\,d\theta = \lambda_1 g_1^n + \lambda_2 g_2^n + \lambda_3 g_3^n$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Therefore, we obtain the following set of conditions to be satisfied by $\{\lambda_p, g_p\}$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \tag{3.3}$$

$$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 = 0 \tag{3.4}$$

$$g_1 + g_2 + g_3 = 0 \tag{3.5}$$

$$g_1 g_2 + g_2 g_3 + g_3 g_1 = 3(\lambda_1 g_1^2 + \lambda_2 g_2^2 + \lambda_3 g_3^2 - 1) \tag{3.6}$$

$$g_1 g_2 g_3 = \lambda_1 g_1^3 + \lambda_2 g_2^3 + \lambda_3 g_3^3 \tag{3.7}$$

The second of these conditions comes from the fact that we require $\overline{g} = 0$. The last of these equations (3.7) can in fact be derived from the first four and hence is redundant.

To obtain the solutions to this set of equations, we first assume that at least two of the λ_p are unequal. Without any loss of generality we can assume that $\lambda_1 \neq \lambda_2$. If we now set $g_3 = (\lambda_1 - \lambda_2)\Delta$ and substitute into (3.4) and (3.5), we obtain that $g_1 = (\lambda_2 - \lambda_3)\Delta$ and $g_2 = (\lambda_3 - \lambda_1)\Delta$. To obtain the expression for Δ, substitute these forms for the g_p into eqn. (3.6). After some manipulation

we find that this equation reduces to

$$\Delta^2(1-27\lambda_1\lambda_2\lambda_3) = 3$$

when use is made of eqn. (3.3). Hence the general solution for the case when at least two of the λ_p are different is obtained as

$$g_1 = (\lambda_2-\lambda_3)\Delta, \quad g_2 = (\lambda_3-\lambda_1)\Delta, \quad g_3 = (\lambda_1-\lambda_2)\Delta \tag{3.8}$$

where

$$\Delta = \pm\sqrt{\frac{3}{1-27\lambda_1\lambda_2\lambda_3}} \quad . \tag{3.9}$$

For the case when $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ this solution becomes indeterminate. To obtain the stationary solution in this case, we set $\lambda_p = \frac{1}{3}$ (p = 1,2,3) in equations (3.4)-(3.7). We easily find that they reduce to

$$g_1+g_2+g_3 = 0$$

$$g_1g_2+g_2g_3+g_3g_1 = g_1^2+g_2^2+g_3^2-3$$

$$3g_1g_2g_3 = g_1^3+g_2^3+g_3^3 \quad .$$

Again the last of these equations is redundant, and the solution of the other two can be written in the form

$$g_2 = \frac{1}{2}\left[-g_1 \pm \sqrt{4-3g_1^2}\right] , \qquad g_3 = \frac{1}{2}\left[-g_1 \mp \sqrt{4-3g_1^2}\right] \qquad (3.10)$$

Here we require that $g_1^2 \leq 4/3$. Hence for each p we have that $g_p^2 \leq 4/3$. Also, from eqn. (3.10), when $g_1 = 1/\sqrt{3}$ we have either g_2 or $g_3 = -2/\sqrt{3}$. Thus we must also have $g_p^2 \geq 1/3$.

3.2 Stability Analysis - General Case.

Having obtained convenient expressions for the stationary solutions, we must now investigate their stability. Only those which are stable are obtainable as asymptotic solutions of a general time dependent solution $g(\theta,\tau)$ of eqn. (3.1).

Let $g_0(\theta)$ be a stationary solution of the type discussed in the previous section. We suppose that

$$g_0(\theta) = \begin{cases} g_1 & \text{for } \theta \in I_1 \\[2mm] g_2 & \text{for } \theta \in I_2 \\[2mm] g_3 & \text{for } \theta \in I_3 \end{cases}$$

where $I_1 \cup I_2 \cup I_3 = [-\ell, \ell]$ and the length of the set I_p is $2\ell\lambda_p$.

To investigate the stability of this proposed solution, we consider a perturbed solution

$$g(\theta,\tau) = g_0(\theta) + v(\theta,\tau) . \qquad (3.11)$$

Substituting this into eqn. (3.1), using the fact that $g_0(\theta)$ is a solution of this equation, and retaining only linear terms in v, we obtain the following first-order perturbation equation

$$2v_\tau = v - g_0^2 v - \overline{g_0^2 v} - 2\overline{g_0 v g_0} + \overline{g_0^2 v} \tag{3.12}$$

where, once again, the bar denotes an average over θ.

We can observe that since $\overline{g} = \overline{g_0} = 0$ it must follow that $\overline{v} = 0$. Taking the average of eqn. (3.12) over θ, we obtain

$$2\overline{v}_\tau = (1 - \overline{g_0^2})\,\overline{v}$$

which is consistent with the condition $\overline{v} = 0$.

Suppose that

$$v(\theta,\tau) = \begin{cases} v_1(\theta,\tau) & \text{for } \theta \in I_1 \\ v_2(\theta,\tau) & \text{for } \theta \in I_2 \\ v_3(\theta,\tau) & \text{for } \theta \in I_3 \end{cases} \tag{3.13}$$

and

$$k_p(\tau) = \frac{1}{2\ell} \int_{I_p} v_p(\theta,\tau)\,d\theta \ .$$

It then follows that we can write

$$\overline{g_0^n v} = g_1^n k_1(\tau) + g_2^n k_2(\tau) + g_3^n k_3(\tau) \, .$$

Choosing θ to be respectively in I_1, I_2 and I_3, we see that equation (3.12) is equivalent to the following system of equations for v_1, v_2 and v_3

$$\left. \begin{aligned} 2v_{1\tau} &= (1 - \overline{g_0^2} - g_1^2) v_1 - 2(g_1 k_1 + g_2 k_2 + g_3 k_3) g_1 + (g_1^2 k_1 + g_2^2 k_2 + g_3^2 k_3) \\ 2v_{2\tau} &= (1 - \overline{g_0^2} - g_2^2) v_2 - 2(g_1 k_1 + g_2 k_2 + g_3 k_3) g_2 + (g_1^2 k_1 + g_2^2 k_2 + g_3^2 k_3) \\ 2v_{3\tau} &= (1 - \overline{g_0^2} - g_3^2) v_3 - 2(g_1 k_1 + g_2 k_2 + g_3 k_3) g_3 + (g_1^2 k_1 + g_2^2 k_2 + g_3^2 k_3) \end{aligned} \right\} \quad (3.14)$$

We can obtain differential equations for the averages $k_p(\tau)$ by integrating these equations respectively over the sets I_1, I_2 and I_3. We obtain

$$\left. \begin{aligned} 2k_{1\tau} &= \left[1 - (1 + 2\lambda_1) g_1^2 - \lambda_2 g_2^2 - \lambda_3 g_3^2 \right] k_1 + \lambda_1 g_2 (g_2 - 2g_1) k_2 + \lambda_1 g_3 (g_3 - 2g_1) k_3 \\ 2k_{2\tau} &= \lambda_2 g_1 (g_1 - 2g_2) k_1 + \left[1 - (1 + 2\lambda_2) g_2^2 - \lambda_1 g_1^2 - \lambda_3 g_3^2 \right] k_2 + \lambda_2 g_3 (g_3 - 2g_2) k_3 \\ 2k_{3\tau} &= \lambda_3 g_1 (g_1 - 2g_3) k_1 + \lambda_3 g_2 (g_2 - 2g_3) k_2 + \left[1 - (1 + 2\lambda_3) g_3^2 - \lambda_1 g_1^2 - \lambda_2 g_2^2 \right] k_3 \end{aligned} \right\} \quad (3.15)$$

This system of equations with constant coefficients can readily be solved for k_1, k_2 and k_3 and the solutions substituted into eqns. (3.14). This latter system can then be immediately integrated to give v_1, v_2 and v_3. (If we sum together the eqns. of (3.15) we note that we obtain

$$2(k_1+k_2+k_3)_\tau = (1-\lambda_1 g_1^2 - \lambda_2 g_2^2 - \lambda_3 g_3^2)(k_1+k_2+k_3)$$

$$= (1-\overline{g_0^2})(k_1+k_2+k_3) \ .$$

This is consistent with the requirement that $\overline{v} = k_1+k_2+k_3 = 0$.)

It turns out, however, that we can eliminate most of the general cases by simply considering eqns. (3.14). It is evident from the form of these eqns. that if the coefficient of v_p is positive in any of the equations, then the solution will have exponentially growing terms and will therefore be unstable. That is, we have instability if $(1-\overline{g_0^2}-g_p^2) > 0$ for any p.

Substituting from eqns. (3.8) and (3.9) we obtain for the first of these coefficients

$$1-\overline{g_0^2}-g_1^2 = 1-(\lambda_1 g_1^2 + \lambda_2 g_2^2 + \lambda_3 g_3^2)-g_1^2$$

$$= \frac{1}{3}\Delta^2(1-3\lambda_2)(1-3\lambda_3) \ ,$$

with similar expressions for the other two coefficients. Now, the three quantities $(1-3\lambda_1)$, $(1-3\lambda_2)$ and $(1-3\lambda_3)$ sum up to zero. Hence, provided that none of them is equal to zero, two of them must be of the same sign, and the product of these two must be positive. The corresponding coefficient $(1-\overline{g_0^2}-g_p^2)$ is then also positive. Consequently, when each $\lambda_p \neq \frac{1}{3}$, the stationary state $g_0(\theta)$ is always unstable.

Thus, in searching for stable stationary solutions, we are immediately restricted to considering certain special cases. These are:

(a) One $\lambda_p = \frac{1}{3}$, for example $\lambda_3 = \frac{1}{3}$, $\lambda_1, \lambda_2 \neq \frac{1}{3}$;

(b) $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$;

(c) One $\lambda_p = 0$, for example $\lambda_3 = 0$, $\lambda_1 + \lambda_2 = 1$.

The reason why the last case must be considered separately is that I_3 is empty, and so the third of eqns. (3.14) and (3.15) must be dropped. It turns out that case (c) is the only one which is capable of producing stable stationary solutions.

3.3   Stability Analysis - Special Cases.

We have seen in the previous section that the search for stable stationary solutions has been narrowed down to several special cases. We consider the stability analysis for case (c) first since it is the only one which produces stable solutions.

·(a) The special case $\lambda_3 = 0$.

Here we investigate the stability of the stationary solutions (3.8) when one of the λ_p's is equal to zero. Without any loss of generality we can take $\lambda_3 = 0$. Then the set I_3 is empty and there is no function $v_3(\theta, \tau)$ in eqn. (3.13) to consider. The

third of eqns. (3.14) is absent and $k_3 = 0$, so we have

$$2v_{1\tau} = (1-\overline{g_0^2}-g_1^2) v_1 - 2(g_1 k_1 + g_2 k_2) g_1 + (g_1^2 k_1 + g_2^2 k_2) \left.\vphantom{\begin{array}{c}1\\2\end{array}}\right\}$$

$$2v_{2\tau} = (1-\overline{g_0^2}-g_2^2) v_2 - 2(g_1 k_1 + g_2 k_2) g_2 + (g_1^2 k_1 + g_2^2 k_2) . \qquad (3.16)$$

When $\lambda_3 = 0$, it follows from eqns. (3.8) and (3.9) that

$$g_1 = \pm \sqrt{3} \lambda_2 , g_2 = \mp \sqrt{3} \lambda_1$$

where $\lambda_1 + \lambda_2 = 1$. Substituting these values in eqns. (3.16), we find that these equations reduce to

$$2v_{1\tau} = (1-3\lambda_2) v_1 - 3\lambda_2^2 k_1 + 3\lambda_1 (1+\lambda_2) k_2 \left.\vphantom{\begin{array}{c}1\\2\end{array}}\right\}$$

$$2v_{2\tau} = (1-3\lambda_1) v_2 + 3\lambda_2 (1+\lambda_1) k_1 - 3\lambda_1^2 k_2 \qquad (3.17)$$

Furthermore, the first two of eqns. (3.15) take the form

$$2k_{1\tau} = (1-3\lambda_2-3\lambda_1\lambda_2^2) k_1 + 3\lambda_1^2 (1+\lambda_2) k_2 \left.\vphantom{\begin{array}{c}1\\2\end{array}}\right\}$$

$$2k_{2\tau} = 3\lambda_2^2 (1+\lambda_1) k_1 + (1-3\lambda_1-3\lambda_1^2\lambda_2) k_2 . \qquad (3.18)$$

Bearing in mind the constraint $\overline{v} = k_1 + k_2 = 0$, eqns. (3.18) reduce to

$$k_{1\tau} = -k_1 \;, \quad k_{2\tau} = -k_2$$

whose solution is of the form

$$k_1 = Ae^{-\tau} \;, \quad k_2 = -Ae^{-\tau}$$

where A is a constant. Substituting these into eqns. (3.17) we obtain

$$2v_{1\tau} = (1-3\lambda_2)v_1 - 3Ae^{-\tau}$$

$$2v_{2\tau} = (1-3\lambda_1)v_2 + 3Ae^{-\tau} \;.$$

(3.18)

These equations will have solutions which tend exponentially to zero
as $\tau \to \infty$ provided that $(1-3\lambda_1)$ and $(1-3\lambda_2)$ are both negative.
That is, provided that $1/3 < \lambda_1 < 2/3$. In these cases therefore,
the corresponding stationary solution is stable.

We conclude that the stationary solution of eqn. (3.1)
given by

$$g_0(\theta) = \begin{cases} \pm \sqrt{3}(1-\lambda_1) & \text{for } \theta \in I_1 \\[2ex] \mp \sqrt{3}\,\lambda_1 & \text{for } \theta \notin I_1 \end{cases}$$

(3.20)

where the length of I_1 is $2\ell\lambda_1$, is stable if and only if
$1/3 < \lambda_1 < 2/3$. There are no other stable stationary solutions.

From eqns. (3.17) we can also calculate the rate at which the stable solution is approached. The solution of these equations is given by

$$v_1(\theta,\tau) = f_1(\theta) e^{-\frac{1}{2}(2-3\lambda_1)\tau} + \frac{3}{2} Ae^{-\tau}$$

$$v_2(\theta,\tau) = f_2(\theta) e^{-\frac{1}{2}(3\lambda_1-1)\tau} - \frac{3}{2} Ae^{-\tau}$$

where f_1 and f_2 are any two functions which satisfy

$$\int_{I_1} f_1(\theta)\,d\theta = \int_{I_2} f_2(\theta)\,d\theta = 0 .$$

The slowest of the three exponential terms in the above equations determines the rate at which v_1 and v_2 decay to zero. It is easily seen that v_1 and/or v_2 decay in proportion to $e^{-q\tau}$ where

$$q = \text{Min}\{\tfrac{1}{2}(3\lambda_1-1), \tfrac{1}{2}(2-3\lambda_1)\} \qquad (3.21)$$

and so the decay time will be q^{-1}.

In the next chapter we develop an explicit solution for eqn. (3.1) for the case of odd initial conditions. That is $g(-\theta,0) = -g(\theta,0)$. It turns out that asymptotically, the solution tends to one of the values $\pm\sqrt{3}/2$, the sign depending on whether the initial value is positive or negative. It can also be easily shown that the rate of approach to these asymptotic solutions is $e^{-\tau/4}$. This

case corresponds to taking $\lambda_1 = \lambda_2 = \frac{1}{2}$ in the above analysis.

Then the stationary solution given by eqn. (3.20) becomes

$$
g_0(\theta) = \begin{cases} \pm \sqrt{3}/2 & \text{for } \theta \in I_1 \\ \\ \mp \sqrt{3}/2 & \text{for } \theta \notin I_1 \end{cases}
$$

and eqn. (3.21) becomes simply $q = \frac{1}{4}$. Thus we have consistency between the results from the stability analysis and the analytic solution.

A typical solution of the type (3.20) is shown in Figure 1 . The upper graph shows $g(\theta,\tau)$ and the lower shows $G(\theta,\tau)$ (recall that $g = G_\theta$). We have illustrated a case where $\lambda_1 = 0.4$ and I_1 and I_2 each consist of two disjoint intervals. The solution $u(x,t)$ in lowest order consists of the difference between two such sawtooth waves, one moving in each direction.

The difference between the maximum and minimum values of G is greatest, for given value λ_1 , when the wave has a simple triangular profile. That is, the sets I_1 and I_2 consist of single intervals (see Fig. 2). This difference is then equal to $2\sqrt{3}\lambda_1(1-\lambda_1)\ell$. From eqn. (2.13), the maximum value of $u(x,t)$ is equal to the maximum value of G minus its minimum value. Hence for given λ_1 , the maximum possible amplitude of $u(x,t)$ is $2\sqrt{3}\lambda_1(1-\lambda_1)\ell$. This maximum is greatest when $\lambda_1 = \lambda_2 = \frac{1}{2}$ and is then equal to $\sqrt{3}\ell/2$.
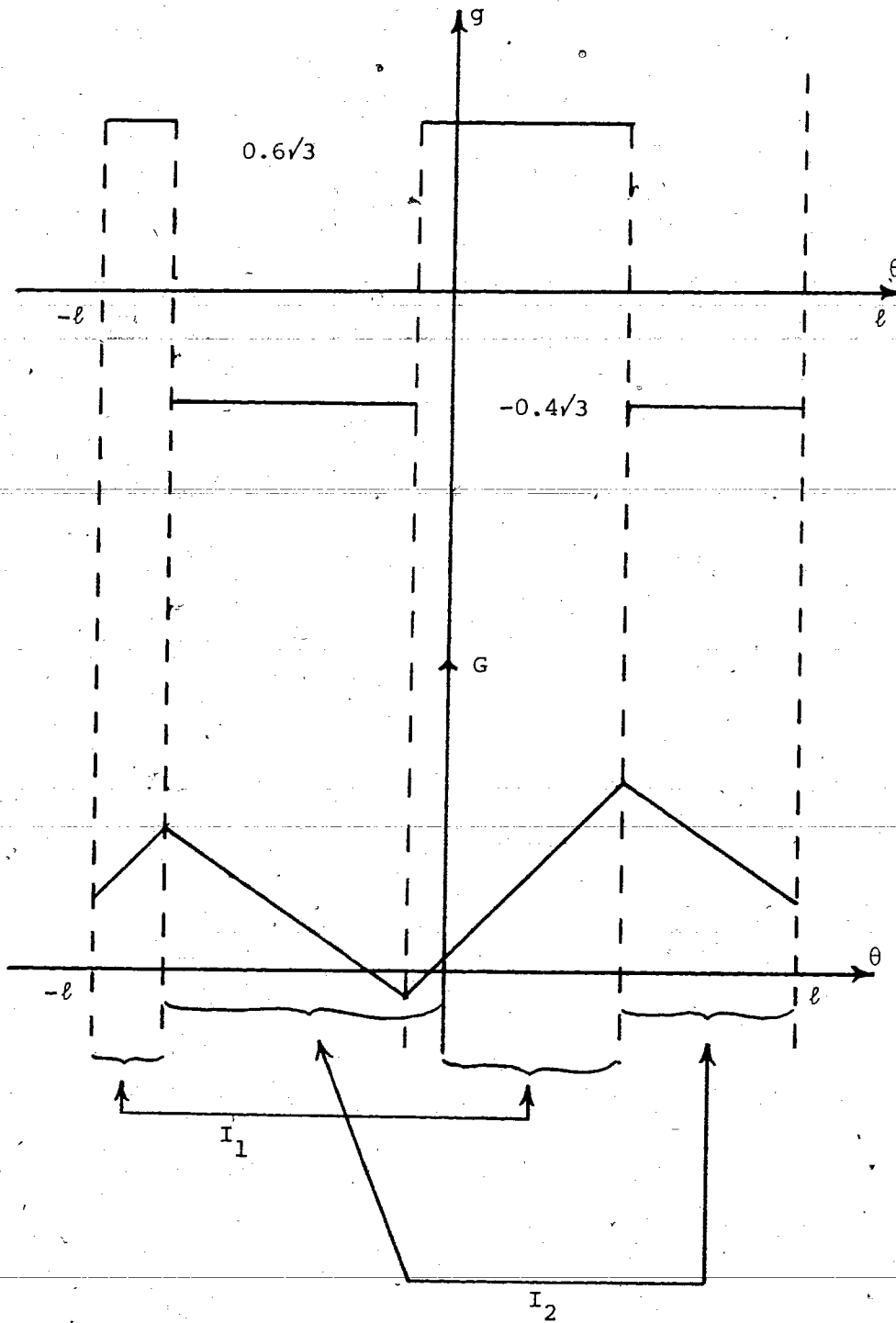
Fig. 1
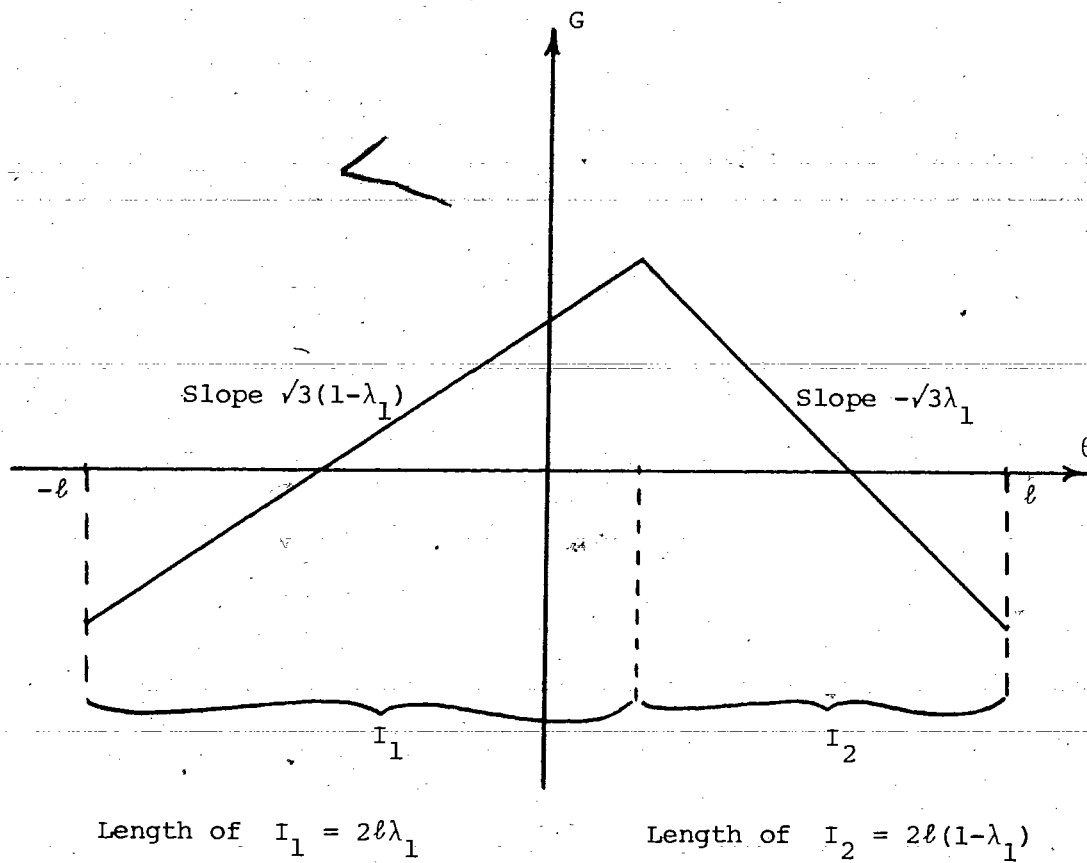
Fig. 2

(b) The special case $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$.

In this case, g_2 and g_3 are given in terms of g_1 by eqns. (3.10). It follows that

$$\overline{g_0^2} = \frac{1}{3}(g_1^2 + g_2^2 + g_3^2) = 2/3 \ .$$

Here we can proceed as in the general case. If we examine the leading coefficients in equations (3.14), we find that

$$1 - \overline{g_0^2} - g_1^2 = \frac{1}{3} - g_1^2.$$

$$1 - \overline{g_0^2} - g_2^2 = \frac{1}{2}(4 - 3g_1^2)\left[\frac{g_1}{(4 - 3g_1^2)^{\frac{1}{2}}} - \frac{1}{3} \right]$$

$$1 - \overline{g_0^2} - g_3^2 = \frac{1}{2}(4 - 3g_1^2)\left[-\frac{g_1}{(4 - 3g_1^2)^{\frac{1}{2}}} - \frac{1}{3} \right] \ .$$

The first of these is negative provided $g_1 > 1/\sqrt{3}$ or $g_1 < -1/\sqrt{3}$. If $g_1 > 1/\sqrt{3}$ it is not difficult to see that the second coefficient is positive except when $g_1^2 = 4/3$ (Recall that we must have $4 - 3g_1^2 \geq 0$). If $g_1 < -1/\sqrt{3}$ the third coefficient is positive provided $g_1^2 \neq 4/3$.

Thus among this class of stationary solutions, one of the three coefficients is always positive, and hence we have instability, except in the cases $g_1^2 = 1/3$ and $g_1^2 = 4/3$. In each of these cases one of the leading coefficients is negative and the other two are zero. This leads us then to the final special case to be considered.

(c) The special case $\lambda_3 = \frac{1}{3}$.

When $\lambda_3 = \frac{1}{3}$, $\lambda_1 + \lambda_2 = 2/3$, it is readily seen from eqns. (3.8) and (3.9) that the stationary solution reduces to

$$g_1 = g_2 = \pm 1/\sqrt{3}, \quad g_3 = -2g_1 \qquad (3.22)$$

(The remaining solution from subsection (b) belongs to this class.)

Then eqns. (3.14) become

$$2v_{1\tau} = -\frac{1}{3}(k_1 + k_2) + \frac{8}{3} k_3 \quad \left. \begin{array}{c} \\ \\ \\ \end{array} \right.$$

$$2v_{2\tau} = -\frac{1}{3}(k_1 + k_2) + \frac{8}{3} k_3 \qquad (3.23)$$

$$2v_{3\tau} = -v_3 + \frac{5}{3}(k_1 + k_2) - \frac{4}{3} k_3 \quad \left. \begin{array}{c} \\ \\ \\ \end{array} \right\}$$

and eqns. (3.15) become

$$2k_{1\tau} = -\frac{\lambda_1}{3}(k_1 + k_2) + \frac{8\lambda_1}{3} k_3$$

$$2k_{2\tau} = -\frac{\lambda_2}{3}(k_1 + k_2) + \frac{8\lambda_2}{3} k_3 \qquad (3.24)$$

$$2k_{3\tau} = \frac{5}{9}(k_1 + k_2) - \frac{13}{9} k_3 .$$

Bearing in mind the condition $k_1 + k_2 + k_3 = 0$, we can easily solve equations (3.24). The solutions can be written in the form

$$k_1 = C - 3\lambda_1 A e^{-\tau}, \quad k_2 = -C - 3\lambda_2 A e^{-\tau}, \quad k_3 = 2A e^{-\tau}$$

where A and C are arbitrary constants. Solving equations (3.23)
therefore, we obtain

$$v_1(\theta,\tau) = -3Ae^{-\tau} + f_1(\theta)$$

$$v_2(\theta,\tau) = -3Ae^{-\tau} + f_2(\theta)$$

$$v_3(\theta,\tau) = 6Ae^{-\tau} + f_3(\theta)e^{-\tau/2}$$

where $f_p(\theta)$ (p = 1,2,3) are arbitrary functions of θ satisfying
the conditions

$$\int_{I_1} f_1(\theta)\,d\theta = -\int_{I_2} f_2(\theta)\,d\theta = 2\ell C, \quad \int_{I_3} f_3(\theta)\,d\theta = 0 \ .$$

From this solution it appears that the quantities $v_p(\theta,\tau)$
do not grow exponentially with time, but as $\tau \to \infty$ they approach the
values

$$v_1(\theta,\tau) \to f_1(\theta), \ v_2(\theta,\tau) \to f_2(\theta), \ v_3(\theta,\tau) \to 0 \ .$$

Thus we appear to have a form of neutral stability.

However, this outcome is solely a result of our having considered
only a first order stability analysis. Otherwise the solution $g(\theta,\tau)$
given by eqn. (3.11) would approach asymptotically the limiting values

$$g(\theta,\tau) \to \begin{cases} g_1 + f_1(\theta) & \text{for } \theta \in I_1 \\[2mm] g_2 + f_2(\theta) & \text{for } \theta \in I_2 \\[2mm] g_3 & \text{for } \theta \in I_3 \end{cases} \qquad (3.25)$$

and this solution would be a stable stationary solution of eqn. (3.1). However such solutions are not among the possible stationary solutions.

It is clear therefore that a solution of the type (3.25) cannot be a stationary solution of the full nonlinear equation (3.1). Its apparent stability disappears in a higher order analysis. In substituting eqn. (3.11) into eqn. (3.1) let us retain terms which are of second order in v. Then the perturbation equation (3.12) is replaced by the following equation

$$2v_\tau = (1-\overline{g_0^2}-g_0^2)v - 2\overline{g_0 v}g_0 + \overline{g_0^2}v - g_0 v^2 - 2\overline{g_0 v v} - g_0\overline{v^2} + \overline{g_0 v^2} .$$

Introducing $v_p(\theta,\tau)$ as before via eqn. (3.13) and

$$k_p(\tau) = \frac{1}{2\ell}\int_{I_p} v_p(\theta,\tau)\,d\theta , \quad h_p(\tau) = \frac{1}{2\ell}\int_{I_p} v_p^2(\theta,\tau)\,d\theta ,$$

we see that this equation is equivalent to the system of equations

$$2v_{p\tau} = (1-\overline{g_0^2}-g_p^2)v_p - 2(g_1 k_1 + g_2 k_2 + g_3 k_3)(v_p + g_p) + (g_1^2 k_1 + g_2^2 k_2 + g_3^2 k_3)$$

$$- g_p v_p^2 - g_p(h_1 + h_2 + h_3) + (g_1 h_1 + g_2 h_2 + g_3 h_3) \quad (p = 1,2,3) .$$

In the case $\lambda_3 = \frac{1}{3}$, the stationary solution is given by eqn. (3.22) and these second order perturbation equations take the form

$$2v_{1\tau} = -\frac{1}{3}(k_1+k_2) + \frac{8}{3}k_3 - gv_1^2 - 2gv_1(k_1+k_2-2k_3) - 3gh_3$$

$$2v_{2\tau} = -\frac{1}{3}(k_1+k_2) + \frac{8}{3}k_3 - gv_2^2 - 2gv_2(k_1+k_2-2k_3) - 3gh_3$$

$$2v_{3\tau} = -v_3 + \frac{5}{3}(k_1+k_2) - \frac{4}{3}k_3 + 2gv_3^2 - 2gv_3(k_1+k_2-2k_3) + 3g(h_1+h_2)$$

where $g \equiv g_1 = \pm 1/\sqrt{3}$. Since $k_1+k_2+k_3 = 0$, these can be simplified to

$$2v_{1\tau} = 3k_3 - gv_1^2 + 6gk_3v_1 - 3gh_3$$

$$2v_{2\tau} = 3k_3 - gv_2^2 + 6gk_3v_2 - 3gh_3 \tag{3.26}$$

$$2v_{3\tau} = -v_3 - 3k_3 + 2gv_3^2 + 6gk_3v_3 + 3g(h_1+h_2) \ .$$

Since we simply wish to demonstrate instability we need only find a perturbation v_p which grows with time. Therefore, for simplicity let us suppose that each v_p is independent of θ. Then $k_p = \lambda_p v_p$ and $h_p = \lambda_p v_p^2$. The condition $k_1+k_2+k_3 = 0$ then implies that $v_3 = -3(\lambda_1 v_1+\lambda_2 v_2)$ and the third of equations (3.26) can be dropped. Setting $u_1 = gv_1$ and $u_2 = gv_2$, the first two of these equations become

$$2u_{p\tau} = -3(\lambda_1 u_1 + \lambda_2 u_2) - [u_p + 3(\lambda_1 u_1 + \lambda_2 u_2)]^2 \quad (p = 1,2) \; .$$

Introducing $r = u_1 - u_2$ and $s = \lambda_1 u_1 + \lambda_2 u_2$, these are transformed to

$$4r_\tau = r[3(\lambda_1 - \lambda_2)r - 18s]$$

$$\text{(3.27)}$$

$$4s_\tau = -[3\lambda_1\lambda_2 r^2 + 27s^2 + 4s] \; .$$

Let us suppose without loss of generality that $\lambda_1 > \lambda_2$ and let

$$D = \{ (r,s) \,|\, r > 0, \; s < 0, \; 3\lambda_1\lambda_2 r^2 + 27s^2 + 4s > 0 \} \; .$$
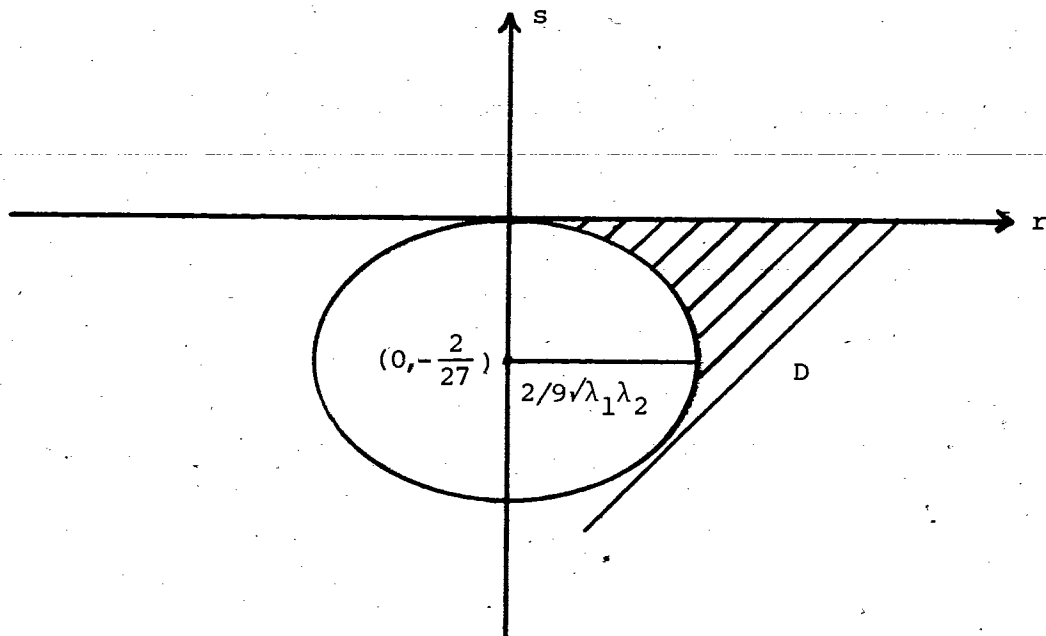
Fig. 3

This set of points in the rs-plane is illustrated in Figure 3. Then we can show that any trajectory of eqns. (3.27) which starts in D moves away from the origin. And furthermore such a trajectory remains in D and therefore continues to move away from the origin. This will establish the required instability of the point $r = s = 0$ (or $u_1 = u_2 = 0$).

First of all, it is easily seen from eqns. (3.27) that if (r,s) lies in D, then $r_\tau \geq 0$ and $s_\tau < 0$. Thus the trajectory moves further away from $(0,0)$ and in particular does not move out of D across the boundaries $r = 0$ and $s = 0$.

Setting $W(\tau) = 3\lambda_1\lambda_2 r^2 + 27s^2 + s$ it follows from eqns. (3.27) that

$$\frac{dW}{d\tau} + W = 6\lambda_1\lambda_2 rr_\tau + 54ss_\tau > 0 .$$

Therefore, the function $W(\tau)e^\tau$ is an increasing function of τ. Since it is positive in D, it remains positive, and so a trajectory in D cannot move out of D across the elliptical boundary on which $W = 0$. This completes the proof of instability.

3.4 Application to the Powerline Model.

In his investigation of overhead transmission lines, Myerscough [8] considered the equation

$$y_{tt} - c^2 y_{xx} = \alpha y_t - \beta y_t^3 .$$

For typical values (see Table 1) of α, β and ℓ as given by him, we can estimate the value of the maximum amplitude $\sqrt{3}\ell/2$ which was obtained from the analysis of case (a) of the preceding section.

Table 1

Typical values for α and β.

α	β
0.0228	0.0151
0.0694	0.00514

If we set $t' = ct$ and $y = \frac{1}{c} \sqrt{\frac{\alpha}{3\beta}} u$, we shall transform the above equation into

$$u_{t't'} - u_{xx} = \epsilon \left[u_{t'} - \frac{1}{3} u_{t'}^3 \right]$$

where $\epsilon = \alpha/c$. Also, for a stretched cable, we can relate the wave speed c to the distance between the pylons ℓ according to the approximate relation

$$c^2 = \frac{1}{2} g\ell \left[\frac{\ell}{6(d-\ell)} \right]^{\frac{1}{2}}$$

where d is the actual length of the cable between the pylons. For typical values $c = 133\text{m/sec}$ and $\ell = 400\text{m}$ given by Myerscough, we find that $d-\ell \approx 0.8\text{m}$. If we fix the value of ℓ to be 400m and use the quantity $d-\ell$ as a small parameter, we obtain the following values for the maximum amplitude.

Table 2

Typical values for the maximum amplitude $\sqrt{\frac{3}{2}}\,\ell$.

$d-\ell$ m.	c m/sec	α – sec^{-1} β – m^2/sec	$u = \frac{\sqrt{3}}{2}\ell$ m.	$y = \frac{1}{c}\sqrt{\frac{\alpha}{3\beta}}u$ m.	characteristic decay time $= \frac{4}{\alpha}$ – sec.
0.4	159	$\alpha_1 = 0.0228$ $\beta_1 = 0.0151$	346	1.55	175
0.4	159	$\alpha_2 = 0.0694$ $\beta_2 = 0.00514$	346	4.61	58
0.6	144	$\alpha_1 = 0.0228$ $\beta_1 = 0.0151$	346	1.71	175
0.6	144	$\alpha_2 = 0.0694$ $\beta_2 = 0.00514$	346	5.08	58
0.8	134	$\alpha_1 = 0.0228$ $\beta_1 = 0.0151$	346	1.83	175
0.8	134	$\alpha_2 = 0.0694$ $\beta_2 = 0.00514$	346	5.46	58
1.0	127	$\alpha_1 = 0.0228$ $\beta_1 = 0.0151$	346	1.93	175
1.0	127	$\alpha_2 = 0.0694$ $\beta_2 = 0.00514$	346	5.76	58

For the case of overhead transmission lines which are separated by a vertical distance of about 10 metres, no displacement greater than 4.5 m. is allowable. (It is undesirable to have the conductors closer than 1 m. apart.) To the extent which this model is applicable to the actual situation, we have, for the range of values of $d-\ell$ considered, acceptable displacement values for the parameters α_1 and β_1 .

3.5 Stationary Values for the Moments.

Thus far our discussion has been concerned with finding the stationary solutions of eqn. (3.1) and determining which of them are stable. There is however, another formulation of the problem which is available to us. If we multiply eqn. (3.1) by $1/2\ell \; g^{n-1}(\theta,\tau)$ and integrate with respect to θ over $[-\ell,\ell]$, we obtain

$$\frac{2}{n} \frac{dI_n}{d\tau} = (1-I_2)I_n - \frac{1}{3} I_{n+2} + \frac{1}{3} I_3 I_{n-1} \tag{3.28}$$

where we have now set $\overline{g^n}(\tau) \equiv I_n(\tau)$. Setting $I_0(\tau) \equiv 1$, we can then define this system for values of n such that $n \geq 1$. (Note that $I_1(\tau) \equiv 0$ then implies that $\frac{dI_1}{d\tau} = 0$.) This is an infinite system of coupled, first order equations. It clearly indicates the dependence of the moments I_2 and I_3 which occur in eqn. (3.1) on each of the higher moments. From eqns. (3.28) one can gain a certain appreciation of the complexities involved in trying to solve eqn. (3.1). Attempted numerical integration of these equations has met, thus far, with little success.

If we set $\dfrac{dI_n}{d\tau} = 0$ in eqn. (3.28), we obtain the following non-linear recurrence relation

$$I_{n+2}^0 - 3(1-I_2^0)I_n^0 - I_3^0 I_{n-1}^0 = 0 \ . \tag{3.29}$$

Setting $I_n^0 = p^n$ we obtain the characteristic equation

$$p^3 - 3(1-I_2^0)p - I_3^0 = 0$$

which is essentially the same equation (cf. eqn. (3.2)) as that considered for the stationary solutions for g . If we denote the three roots of the characteristic equation by p_1, p_2 and p_3 , we obtain

$$I_n^0 = \mu_1 p_1^n + \mu_2 p_2^n + \mu_3 p_3^n$$

where μ_1, μ_2 and μ_3 are arbitrary parameters.

We have previously seen (§3.1) that this is precisely the form derived for the I_n^0's using an essentially different approach. Moreover, we have also seen that the above definition can be extended to I_1^0, I_2^0 and I_3^0 as well.

It would of course be desirable to investigate the stability of these stationary solutions. However, due to the complexity of the infinite system given by eqn. (3.28), this approach seems to be at least considerably more difficult than the approach adopted earlier in this chapter.

Letting $\{\mu_i\}$ correspond to $\{\lambda_i\}$ and $\{p_i\}$ correspond to $\{g_i\}$ with $\mu_3 = 0$, we have

$$I_n^0 = \lambda_1 g_1^n + \lambda_2 g_2^n$$

$$= \sqrt{3}\lambda(1-\lambda)\left[(\sqrt{3}\lambda)^{n-1} + (-1)^n (\sqrt{3} - \sqrt{3}\lambda)^{n-1}\right]. \qquad (3.30)$$

If $\lambda > 1/\sqrt{3} \approx .577$ or $\lambda < 1 - 1/\sqrt{3} \approx .423$ (recall $\frac{1}{3} < \lambda < \frac{2}{3}$), then we have that $I_n^0 \to \infty$ as $n \to \infty$. This of course means that we are not able to approximate eqns. (3.28) by a suitably truncated system, but that we must keep the entire system.

If we were able to determine any one of I_n's in terms of the initial values $\{I_n(0)\}$, then we could obtain the appropriate value for λ from eqn. (3.30). In the next chapter we show that this value of λ is all we really need to construct the asymptotic solution $q(\theta, \infty)$.

CHAPTER 4

NON-STATIONARY SOLUTIONS AND GENERAL THEOREMS

In this Chapter we investigate various non-stationary
solutions of eqn. (3.1) along with several results regarding general
behaviour of its solution. The integrability of eqn. (3.1), considered
as a differential equation, is seen to be directly related to the
presence of the $\overline{g^3}$ term. If this term is absent, then eqn. (3.1)
belongs to a class of differential equations which can be integrated
in a straightforward manner.

4.1   Impulsive Initial Conditions.

If the initial conditions for the problem defined by eqn.
(2.10) are of impulsive type (i.e., $u(x,0) = 0$) then it follows
from eqn. (2.15) that the initial values to be satisfied by the
solution of eqn. (3.1) are odd. That is $g(\theta,0) = -g(-\theta,0)$. It can
then be shown as follows that if eqn. (3.1) is assumed to have a
unique solution, this solution remains an odd function of θ for
all time.

Setting $g(\theta,\tau) = -h(-\theta,\tau)$ in eqn. (3.1), we obtain

$$2h_\tau = (1-\overline{h^2})h - \frac{1}{3}h^3 + \frac{1}{3}\overline{h^3} .$$

Furthermore, if $g(\theta,0) = -g(-\theta,0)$, it follows that $h(\theta,0) = g(\theta,0)$.

Hence g and h satisfy the same equation and have the same initial

values. Assuming a unique solution exists, we have $g(\theta,\tau) = h(\theta,\tau) =$

$-g(-\theta,\tau)$ for all τ.

As a consequence, for the case of odd initial conditions,

eqn. (3.1) reduces to

$$2g_\tau = (1-\overline{g^2})g - \frac{1}{3} g^3 \qquad\qquad (4.1)$$

since $\overline{g^3}(\tau) \equiv 0$. This can be regarded as a Bernoulli or Ricatti

type equation which can be directly integrated in terms of the

unknown function $\overline{g^2}$. Introducing an integrating factor, we can

rewrite eqn. (4.1) as

$$\frac{\partial}{\partial\tau}\left\{\exp\left(-\frac{1}{2}\int_0^\tau [1-\overline{g^2}(\tau')]d\tau'\right)g\right\} = -\frac{1}{6}\exp\left(-\frac{1}{2}\int_0^\tau [1-\overline{g^2}(\tau')]d\tau'\right)g^3 .$$

Setting $w(\tau) = \exp\left\{\frac{1}{2}\int_0^\tau [1-\overline{g^2}(\tau')]d\tau'\right\}$ and $g(\theta,\tau) = w(\tau) f(\theta,\tau)$ we

obtain

$$f_\tau(\theta,\tau) = -\frac{1}{6} w^2(\tau) f^3(\theta,\tau) .$$

This may be integrated directly and the solution written in the form

$$f(\theta,\tau) = \frac{f(\theta,0)}{\left[1 + \frac{1}{3}\phi(\tau)f^2(\theta,0)\right]^{\frac{1}{2}}} \Big)$$

where

$$\phi(\tau) = \int_0^\tau w^2(\tau')d\tau' = \int_0^\tau \exp\left\{\int_0^{\tau'}\left[1-\overline{g^2}(\tau'')\right]d\tau''\right\}d\tau' \quad \text{(4.2)}$$

Since $w(0) = 1$ and $\phi'(\tau) = w^2(\tau)$ it follows that we can write

$$g(\theta,\tau) = g(\theta,0)\left[\frac{\phi'(\tau)}{1+\frac{1}{3}\phi(\tau)g^2(\theta,0)}\right]^{\frac{1}{2}}. \quad \text{(4.3)}$$

We have thus obtained a formal solution to eqn. (4.1) in terms of the unknown function $\phi(\tau)$. If we substitute this solution back into eqn. (4.2) we obtain an equation for $\phi(\tau)$ which takes the form

$$\tau - \ell n\phi'(\tau) = \frac{3}{2\ell}\int_0^{2\ell}\ell n\left[1 + \frac{1}{3}\phi(\tau)g^2(\theta,0)\right]d\theta. \quad \text{(4.4)}$$

Having solved this equation for $\phi(\tau)$, the solution function $g(\theta,\tau)$ is then found from eqn. (4.3). It is immediately evident from the form (4.3) of the solution that the sign of $g(\theta,\tau)$ is the same as the sign of $g(\theta,0)$. In particular, the zeros of the solution function are simply the zeroes of the initial value function.

It is of interest to examine the form of the solution for large times and to compare this solution with the previously determined stationary solutions. In order for eqn. (4.4) to be satisfied as $\tau \to \infty$, it follows that $\phi(\tau)$ and $\phi'(\tau) \to \infty$ as well. Then we can replace eqn. (4.4) by the approximate equation

$$\tau - \ell n \phi'(\tau) = \frac{3}{2\ell} \int_0^{2\ell} \ell n \left[\frac{1}{3} \phi(\tau) g^2(\theta, 0) \right] d\theta$$

$$= \frac{3}{2\ell} \int_0^{2\ell} \ell n \left[\frac{1}{3} g^2(\theta, 0) \right] d\theta + 3\ell n \phi(\tau) .$$

Hence we find that

$$\phi(\tau) \sim c e^{\tau/4} \quad \text{as} \quad \tau \to \infty$$

where $c^4 = 4 \exp \left\{ -\frac{3}{2\ell} \int_0^{2\ell} \ell n \left[\frac{1}{3} g^2(\theta, 0) \right] d\theta \right\}$. Substituting this form for ϕ in eqn. (4.3) we obtain

$$g(\theta, \tau) \sim \frac{\sqrt{3}}{2} \, \text{sgn}[g(\theta, 0)] \quad \text{as} \quad \tau \to \infty .$$

Thus $g(\theta, \tau)$ tends asymptotically to one or the other of the two constants $\pm \frac{\sqrt{3}}{2}$. This result can be seen as follows to be in complete agreement with the previously obtained forms for the stationary solutions.

In deriving the solution (4.3) we have seen that if the initial value is an odd function of θ, then the solution function remains odd for all time. Since the solution always retains the same sign as it had originally, it follows that the asymptotic distribution of positive and negative values must be exactly the same as the initial distribution. But for an odd function this means that one half the interval $[-\ell,\ell]$ is comprised of positive values for $g(\theta,0)$ (assuming $g(\theta,0) \neq 0$) and the other half negative. That is, asymptotically, $\lambda_1 = \lambda_2 = \frac{1}{2}$. For these values, we obtain from eqn. (3.20) that g tends asymptotically to $\pm \frac{\sqrt{3}}{2}$, as explicitly derived.

It is of interest to note that the solution given by eqn. (4.3) can be extended to include any initial values $g(\theta,0)$ which have a point of antisymmetry. If this is the case, then a suitable coordinate translation will transform $g(\theta,0)$ into an odd function with respect to the new coordinate system. For example, $g(\theta,0) = \cos \pi\theta/\ell$ is such a function. A numerical solution for this initial value is given in the next chapter and this type of asymptotic behaviour with $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $g \sim \pm \frac{\sqrt{3}}{2}$ is clearly indicated.

4.2 Piecewise Constant Solution.

Although the full equation (3.1) cannot be solved in general, it turns out that there is one simple case when an exact solution can be obtained. This occurs when g is taken to have a piecewise constant profile.

Suppose that g has the form

$$g(\theta,\tau) = \begin{cases} g_+(\tau) & \text{for} \quad \theta \in I \\ \\ g_-(\tau) & \text{for} \quad \theta \notin I . \end{cases}$$

Here we assume that g_+ is positive and g_- negative. Let the interval I have length $(1-\lambda)(2\ell)$. We must set $g_+ = \alpha\lambda$ and $g_- = -\alpha(1-\lambda)$ in order to have $\bar{g} = 0$. We then find

$$\bar{g}(\tau) = (1-\lambda)g_+ + \lambda g_- = \alpha\lambda(1-\lambda) - \alpha\lambda(1-\lambda) = 0$$

$$\overline{g^2}(\tau) = (1-\lambda)g_+^2 + \lambda g_-^2 = \alpha^2\lambda(1-\lambda) \tag{4.5}$$

$$\overline{g^3}(\tau) = (1-\lambda)g_+^3 + \lambda g_-^3 = -\alpha^3\lambda(1-\lambda)(1-2\lambda) .$$

From eqn. (3.1) we obtain the following pair of equations

$$2g_{+\tau} = (1-\overline{g^2})g_+ - \frac{1}{3}g_+^3 + \frac{1}{3}\overline{g^3}$$

$$2g_{-\tau} = (1-\overline{g^2})g_- - \frac{1}{3}g_-^3 + \frac{1}{3}\overline{g^3}$$

for g_+ and g_-. Substitution of the assumed forms for g_+ and g_- along with eqns. (4.5) into these equations then gives a system of equations for α and λ

$$2 \frac{d}{d\tau} (\alpha\lambda) = [1-\alpha^2\lambda(1-\lambda)]\alpha\lambda - \frac{1}{3}\alpha^3\lambda^3 - \frac{1}{3}\alpha^3\lambda(1-\lambda)(1-2\lambda)$$

$$2 \frac{d}{d\tau} \{-\alpha(1-\lambda)\} = - [1-\alpha^2\lambda(1-\lambda)]\alpha(1-\lambda) + \frac{1}{3}\alpha^3(1-\lambda)^3 - \frac{1}{3}\alpha^3\lambda(1-\lambda)(1-2\lambda).$$

After some algebraic manipulations, these reduce simply to

$$\frac{d\alpha}{d\tau} = \frac{1}{2}\alpha\left(1 - \frac{1}{3}\alpha^2\right) , \qquad \frac{d\lambda}{d\tau} = 0 .$$

The solutions of these equations are given by

$$\alpha(\tau) = \frac{\alpha(0)\sqrt{3}}{\left[\alpha^2(0) + [3-\alpha^2(0)]e^{-\tau}\right]^{\frac{1}{2}}} \qquad \lambda(\tau) = \text{constant} = \lambda(0) .$$

We may easily extract the asymptotic behaviour from these solutions.
As $\tau \to \infty$ we see that $\alpha \to \sqrt{3}$ and λ remains constant for all time.
These results are consistent with the asymptotic solutions obtained
in the last chapter namely $g_+ \sim \sqrt{3}\lambda$, $g_- \sim - \sqrt{3}(1-\lambda)$, except here
we note that there is no restriction placed upon λ . It may have
any value between zero and one. (If $\lambda < \frac{1}{3}$ or $\lambda > \frac{2}{3}$ the
asymptotic solution is unstable under certain non-constant
perturbations.)

4.3 A Perturbation Approach.

In Section 4.1, we investigated the case where the initial
conditions satisfy $g(\theta,0) = -g(-\theta,0)$ for all θ . Since this implies

that the solution always remains odd, we were able to develop an explicit solution. The question then arises as to what happens when the initial conditions differ slightly from those in the odd case.

To investigate this situation set

$$g(\theta,\tau) = g_0(\theta,\tau) + \eta g_e(\theta,\tau) \tag{4.6}$$

where g_0 and ηg_e are the odd and even parts of the solution function g and η is a small parameter. Substituting this form for g into the full equation (3.1) and comparing powers of η we obtain

$$\eta^0 \;:\; 2g_{0\tau} = (1-\overline{g_0^2})g_0 - \frac{1}{3}g_0^3$$

$$\tag{4.7}$$

$$\eta^1 \;:\; 2g_{e\tau} = (1-\overline{g_0^2}-g_0^2)g_e + \overline{g_0^2}g_e \;.$$

Here we have made use of the even and odd properties of g_e and g_0 respectively to set $\overline{g_0 g_e} = \overline{g_0^3} = 0$.

The first equation of the system (4.7) is the same as eqn. (4.1) and hence its solution is given by eqn. (4.3). The second equation can, in principle, be solved and in the following we briefly outline the procedure. Write it as

$$g_{e\tau} - \frac{1}{2}(1-\overline{g_0^2}-g_0^2)g_e = \frac{1}{2}\overline{g_0^2 g_e}$$

and note from the first equation that

$$\frac{1}{2}(1-\overline{g_0^2}-g_0^2) = \frac{g_{0\tau}}{g_0} - \frac{1}{3}g_0^2 .$$

Making use of eqn. (4.3), it follows that the integrating factor for

the above equation can be expressed as

$$\exp\left[-\frac{1}{2}\int_0^\tau \{1-\overline{g_0^2}(\sigma) - g_0^2(\theta,\sigma)\}d\sigma\right] = \frac{g_0^3(\theta,0)\,\phi'(\tau)}{g_0^3(\theta,\tau)} .$$

Then the solution in terms of the unknown function $\overline{g_0^2 g_e}$ is

$$g_e(\theta,\tau) = \frac{g_0^3(\theta,\tau)}{g_0^3(\theta,0)\phi'(\tau)} \cdot \left[g_e(\theta,0) + \frac{1}{2}\int_0^\tau \frac{g_0^3(\theta,0)\phi'(\sigma)}{g_0^3(\theta,\sigma)}\,\overline{g_0^2 g_e}\,d\sigma\right]. \quad (4.8)$$

Finally, an expression for $\overline{g_0^2 g_e}(\tau)$ can be obtained from this equation.

Multiplying by $\frac{1}{2\ell}g_0^2(\theta,\tau)$ and integrating over $[-\ell,\ell]$, we obtain

the following linear Volterra integral equation for $f(\tau) = \overline{g_0^2 g_e}(\tau)$

$$f(\tau) = h(\tau) + \int_0^\tau f(\sigma)\,K(\tau,\sigma)d\sigma . \quad (4.9)$$

Here we have set

$$h(\tau) = \frac{1}{2\ell}\int_{-\ell}^\ell \frac{g_0^5(\theta,\tau)g_e(\theta,0)}{g_0^3(\theta,0)\phi'(\tau)}\,d\theta$$

and

$$K(\tau,\sigma) = \frac{1}{4\ell} \int_{-\ell}^{\ell} \frac{g_0^{5}(\theta,\tau)\phi'(\sigma)}{\phi'(\tau)g_0^3(\theta,\sigma)}\, d\theta$$

These are known functions in terms of $g_0(\theta,\tau)$. Then solving eqn. (4.9) for f and substituting in eqn. (4.8), we obtain the complete solution to the second equation of (4.7). However, while this procedure is always possible, it is rather more instructive to consider the asymptotic forms of eqns. (4.7).

In Section 4.1, we saw that $g(\theta,\tau) \sim \pm \frac{\sqrt{3}}{2}$ as $\tau \to \infty$. Thus g_0^2 and $\overline{g_0^2} \sim \frac{3}{4}$ and $\overline{g_0^2 g_e} \sim \frac{3}{4}\overline{g_e} = 0$. (If g defined by eqn. (4.6) is a solution, then to be consistent with $\overline{g} = 0$ we must have $\overline{g_e} \equiv 0$.) Using these values in the equation for the even perturbation, we obtain

$$g_{e\tau} + \frac{1}{4}\, g_e = 0 .$$

asymptotically. This result seems to indicate that the perturbation ultimately dies out. It is based on the assumption that the perturbation remains uniform for all θ. Unfortunately, for this approach this is not the case.

We can see directly as follows from eqn. (4.8) that the solution g_e must grow exponentially in the neighbourhood of a zero $\theta = \theta_i$ of the odd function. From eqn. (4.3) we have that

$$g_0(\theta,\tau) \sim g_0(\theta,0)\, [\phi'(\tau)]^{\frac{1}{2}} \quad \text{for } \theta \text{ close to } \theta_i . \quad \text{The solution}$$

g_e then becomes

$$g_e(\theta, \tau) \sim [\phi'(\tau)]^{\frac{1}{2}} \left\{ g_e(\theta, 0) + \frac{1}{2} \int_0^\tau \frac{1}{[\phi'(\sigma)]^{\frac{1}{2}}} \overline{g_0^2 g_e} \, d\sigma \right\}$$

near θ_i. Since $\phi(\tau) \sim ce^{\tau/4}$ and $\overline{g_0^2 g_e} \to 0$ as $\tau \to \infty$, the

integral is convergent and hence

$$g_e(\theta, \tau) \sim \text{const} \cdot [\phi'(\tau)]^{\frac{1}{2}} \sim \text{const} \cdot e^{\tau/8} \quad \text{as} \quad \tau \to \infty .$$

To see how this relates to the system (4.7) let us

suppose for simplicity that our odd function is such that

$g_0(\theta, 0) < 0$ for $\theta < 0$ and $g_0(\theta, 0) > 0$ for $\theta > 0$ where

$\theta \in [-\ell, \ell]$. Then the asymptotic solution for this function will

simply be $\pm \frac{\sqrt{3}}{2}$, the sign at each θ depending on whether the

initial value there was positive or negative. Now let us consider

an asymptotic solution (Fig. 4a) which differs slightly from this odd

asymptotic solution and decompose it (Fig. 4b) into its even and odd

parts. It is clear from this simple consideration that in some

neighbourhood of the zero of the odd function at $\theta = 0$, the odd

solution itself must die out and the asymptotic solution in that region

is totally dominated by the even solution. This is apparently what

happens in the case of the even perturbation.

If we denote by Δ the interval over which g_0 tends to zero,

then the long time behaviour of g_e is more appropriately described by

the equations (from (4.7b) : $\overline{g_0^2} \approx \frac{3}{4}$, $g_0 = 0$ in Δ, $\pm \frac{\sqrt{3}}{2}$ outside Δ)
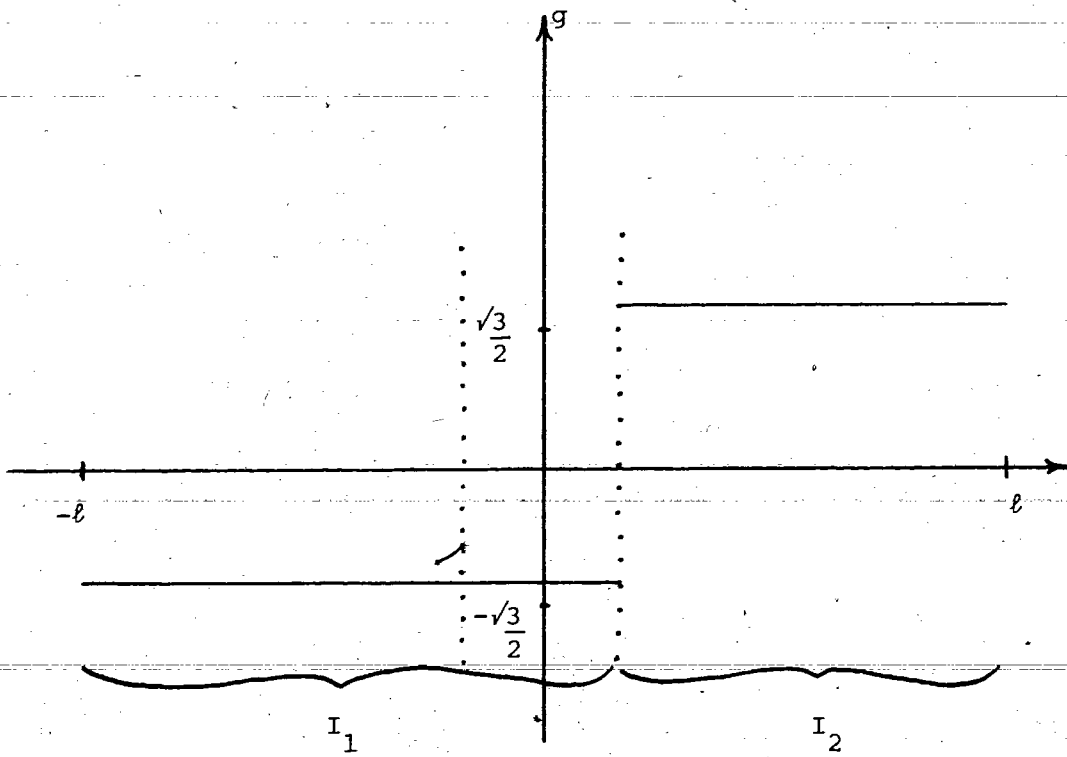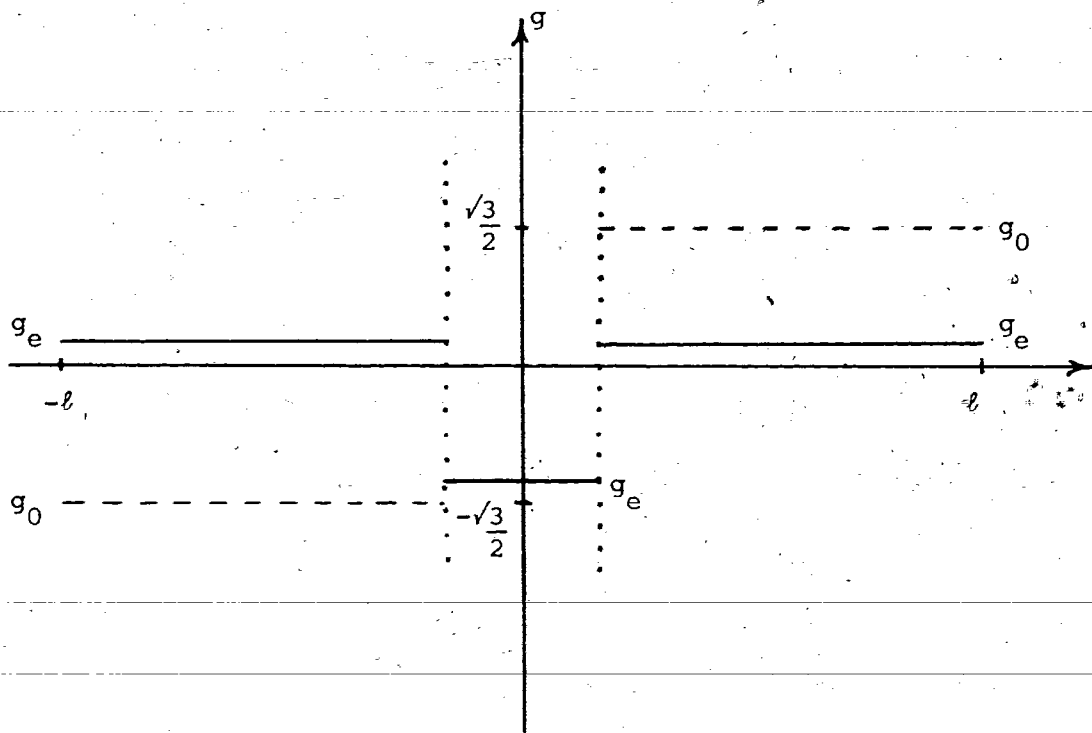
Fig. 4(a)

Fig. 4(b)

$$g_{e\tau} - \frac{1}{8} g_e = \frac{1}{2} \overline{g_0^2 g_e} \qquad \text{for } \theta \in \Delta$$

and

$$g_{e\tau} + \frac{1}{4} g_e = \frac{1}{2} \overline{g_0^2 g_e} \qquad \text{for } \theta \notin \Delta .$$

Thus, for sufficiently large times, the even perturbation must grow exponentially for those values of $\theta \in \Delta$ and the resulting expansion (4.6) is no longer uniformly valid.

It is possible to carry out an asymptotic analysis of eqn. (4.9). The procedure is long and somewhat involved, and ultimately leads to conclusions which are consistent with those reached above. That is, the even perturbation grows exponentially in the neighbourhoods of the zeroes of the odd function.

4.4   General Results.

Although it has not yet been possible to develop a complete solution to eqn. (3.1), the task of determining the appropriate asymptotic solution of the type found in Chapter 3 may be further reduced by applying the results of several general theorems. It is shown in the following that the problem is really that of finding the value of λ and from that constructing the asymptotic solution in terms of the initial values.

Theorem 1.

Let θ_1 and θ_2 be two values of θ such that $g(\theta_1, 0) > g(\theta_2, 0)$. Then $g(\theta_1, \tau) > g(\theta_2, \tau)$ for all finite τ .

Proof.

Substituting $g(\theta_1, \tau)$ and $g(\theta_2, \tau)$ into eqn. (3.1) and subtracting the two resulting equations, we obtain

$$\frac{d}{d\tau}(\Delta g) = A(\tau) \; \Delta g$$

where we have set $\Delta g(\tau) = g(\theta_1, \tau) - g(\theta_2, \tau)$ and $A(\tau) = \frac{1}{2}(1 - \overline{g^2})$
$-\frac{1}{6}[g^2(\theta_1, \tau) + g(\theta_1, \tau)g(\theta_2, \tau) + g^2(\theta_2, \tau)]$. Since $A(\tau)$ is a bounded function of τ , it follows that

$$\Delta g(\tau) = \Delta g(0)\{\exp \int_0^\tau A(\tau')d\tau'\}$$

cannot be zero for any finite τ . Hence $\Delta g(\tau) > 0$ and the result follows.

This result allows us to calculate the final form of the asymptotic solution if the value of λ is known. Suppose λ is known, then for $\theta \in I_1$, $g(\theta, \tau) \to -\sqrt{3}(1-\lambda)$ as $\tau \to \infty$ and for $\theta \in I_2$, $g(\theta, \tau) \to \sqrt{3}\,\lambda$ as $\tau \to \infty$. The lengths of the intervals I_1 and I_2 respectively are $2\ell\lambda$ and $2\ell(1-\lambda)$. Then in order to determine the location of these intervals we simply must find a value g_0 such that the length of the set $\{\theta \mid g(\theta, 0) < g_0\}$ is equal to $2\ell\lambda$. This set is then I_1 and its complement with respect to $[-\ell, \ell]$ is I_2 .

Thus the problem is reduced to determining the value of λ from $g(\theta, 0)$. We next show that the value of λ does not depend on the scale of the initial data.

Theorem 2.

If $g_1(\theta,0)$ and $g_2(\theta,0)$ are any two initial values related by $g_2(\theta,0) = k\, g_1(\theta,0)$ where k = constant > 0, then the corresponding solutions $g_1(\theta,\tau)$ and $g_2(\theta,\tau)$ tend to the same asymptotic solution as $\tau \to \infty$.

Proof.

Suppose $g(\theta,\tau)$ is a solution of eqn. (3.1). Then, introducing a time dependent scale $s = s(\tau)$ and new time $\sigma = \sigma(\tau)$, we can easily show that $g_1(\theta,\tau) = s(\tau)g(\theta,\sigma)$ is also a solution of eqn. (3.1) provided that

and

$$\begin{cases} s(\tau) = [1 + Ae^{-\tau}]^{-\frac{1}{2}} \\[2em] \sigma(\tau) = \ln\{\dfrac{e^{\tau}+A}{1+A}\} \end{cases}$$

(4.10)

where A is an arbitrary constant. Substituting for $g(\theta,\sigma) = g_1(\theta,\tau)/s(\tau)$ in eqn. (3.1) and rearranging, we obtain

$$2\frac{d\tau}{d\sigma}[s^2 g_{1\tau} - ss'g_1] = (s^2 - \overline{g_1^2})g_1 - \frac{1}{3}g_1^3 + \frac{1}{3}\overline{g_1^3}.$$

This equation can be cast in the form of eqn. (3.1) if we require that

$$\frac{d\tau}{d\sigma} = \frac{1}{s^2} \quad \text{and} \quad \frac{2}{s}\cdot\frac{ds}{d\tau} + s^2 = 1.$$

These equations can be solved in a straightforward manner and the solutions are given by eqns. (4.10). We can now use these results to establish the theorem.

For $\tau = 0$ we have $s = (1+A)^{-\frac{1}{2}}$ and $\sigma = 0$, so that

$$g_1(\theta,0) = (1+A)^{-\frac{1}{2}} g(\theta,0) .$$

Also, as $\tau \to \infty$, $\sigma(\tau) \sim \tau$ and $s(\tau) \sim 1$ so $g_1(\theta,\tau) \sim g(\theta,\tau)$. Since the value of A is arbitrary, it follows that if $g_1(\theta,0)$ and $g_2(\theta,0)$ are related by $g_2(\theta,0)/g_1(\theta,0) = k = $ constant, then the corresponding solutions $g_1(\theta,\tau)$ and $g_2(\theta,\tau)$ approach the same asymptotic solution. That is, the value of λ is not dependent on the scale of the initial data.

In view of this result, it can be speculated that a convenient expression for λ may be found in terms of scale invariant ratios of the type

$$\frac{I_3^2}{I_2 I_4} , \quad \frac{I_5^2}{I_2 I_8} , \quad \frac{I_5^2}{I_4 I_6} , \quad \ldots \text{ etc.}$$

However no such expression has yet been found.

In the two cases (§4.1 and 4.2) where we have been able to develop explicit solutions for eqn. (3.1), the significant aspect of the initial value and solution functions is that the positions of the zeroes of these functions remains fixed. But for arbitrary initial

conditions this is not generally the case, and it may be shown that the local migration of the zeroes is directly related to the presence of the $\overline{g^3}(\tau)$ term.

Suppose that $\theta = \theta_i$ is the location of a simple zero of the initial value function $g(\theta, 0)$. (i.e., $g(\theta_i, 0) = 0$). Then for fixed θ near θ_i, it follows that we can write

$$g(\theta, \tau) \sim k_1(\tau)[\theta - \theta_i(\tau)] + \frac{1}{2}k_2(\tau)[\theta - \theta_i(\tau)]^2 + \ldots$$

where $\theta_i(\tau)$ represents the new location of the zero which was originally at θ_i. In a neighbourhood of $\theta_i(\tau)$, we then have from eqn. (3.1) that

$$\frac{\partial g}{\partial \tau} \approx k_1'(\tau)\{[\theta - \theta_i(\tau)] - \frac{k_1(\tau)}{k_1'(\tau)}\theta_i'(\tau)\} - k_2(\tau)\theta_i'(\tau)[\theta - \theta_i(\tau)]$$

$$\approx \frac{1}{2}[1 - \overline{g^2}]g + \frac{1}{6}\overline{g^3}$$

$$= \frac{1}{2}[1 - \overline{g^2}]k_1(\tau)[\theta - \theta_i(\tau)] + \frac{1}{6}\overline{g^3}.$$

Since it is assumed that $\theta - \theta_i(\tau)$ remains small, we have on comparison of terms

$$k_1'(\tau) = \frac{1}{2}[1 - \overline{g^2}]k_1(\tau) + k_2(\tau)\theta_i'(\tau)$$

and

$$k_1(\tau)\theta_i'(\tau) = -\frac{1}{6}\overline{g^3}.$$

Integration of the second of these leads to

$$\theta_i(\tau) = \theta_i(0) - \frac{1}{6} \int_0^\tau \frac{\overline{g^3}(\tau')}{k_1(\tau')} d\tau'$$

where $k_1(\tau)$ is ultimately determinable from an infinite system of differential equations for the coefficients $k_n(\tau)$ of $[\theta - \theta_i(\tau)]^n$. Thus the location of the zeroes of the solution is seen to be directly related to the presence of the $\overline{g^3}$ term. For the odd solution this term is identically zero and the zeroes remain fixed for all time.

CHAPTER 5

NUMERICAL SOLUTIONS

In this chapter we present various numerical solutions of eqn. (3.1). For arbitrary initial conditions, it was found that the asymptotic solution which was attained was entirely consistent with the type of solution found in Chapter 3. However, the question of how this asymptotic solution relates to the given initial values remains unanswered.

5.1 The Numerical Method.

For the purposes of numerical integration eqn. (3.1) was treated as a system of ordinary differential equations. The value $\ell = 1$ was chosen and the resulting θ interval $[-1,1]$ was then uniformly partitioned. The averages $\overline{g^2}$ and $\overline{g^3}$ were replaced by either trapezoidal or Simpson's rule type approximations and the resulting system of equations for the mesh values,

$$g_p(\tau) = g(\theta_p,\tau) \qquad p = 1,\ldots,N$$

was then integrated with respect to τ using a Runge-Kutta method. Many different runs using a variety of initial values were made to compare the trapezoidal and Simpson's approximations. On comparing the final asymptotic solution obtained from each approximation for a

given initial value, no significant difference was found. The asymptotic solution was found in every case to be very well established for values of $\tau \geq 35$.

Certain problems arose in the course of obtaining these numerical results. Particularly for the case of odd initial conditions, it was found that convergence to the final solution was greatly assisted by choosing a partition for which none of the mesh points coincided with the location of a zero of the initial value function. (For example, for an odd function of θ, the point $\theta = 0$ should be excluded as a mesh point.) This is because the zeroes of the odd solution remain fixed for all time.

A second, more significant problem arose due to the requirement that $\overline{g} = 0$ for all time. In earlier runs, when this restriction was not applied, it was found that the first moment was growing in accordance to the relation

$$\overline{g}_\tau = \frac{1}{2}(1-\overline{g^2})\overline{g} \sim \frac{1}{2}(1-3\lambda+3\lambda^2)\overline{g} \quad \text{as} \quad \tau \to \infty.$$

In order to counteract this growth, the calculated value of $\overline{g}$ was periodically subtracted from the coordinate values. This ensured that $\overline{g}$ remained identically zero.

To test the possibility that the final solution might be directly related to the positioning of the zeros of the initial value function, a large number of runs was made using a trial function of the form

$$g(\theta,0) = \frac{\beta\theta+1}{(\theta+\beta)^3} \qquad \beta > 1 \ . \qquad\qquad (5.1)$$

This function has the following two properties:

(a) for any value of $\beta > 1$

$$\int_{-1}^{1} \frac{(\beta\theta+1)\,d\theta}{(\theta+\beta)^3} = 0$$

and   (b) it is possible by appropriately choosing β to place a zero of the function anywhere in the interval $(-1,0)$.

After working with this and other trial functions, it could be concluded that for the general case $(\overline{g^3} \neq 0)$, there is no apparent simple relationship between the location of the zeros and the asymptotic solution. It seems more likely that the final solution is directly related to the moments $\overline{g^n}$ (cf. eqn. (3.28)) of the initial value function.

5.2   Examples of Numerical Solutions.

In this section we present several examples of numerical solutions for eqn. (3.1) obtained by the method of the previous section. In each case the solution has been graphed for various times which clearly show the evolution of the asymptotic solution. As noted previously, the final solutions are of the same type as those predicted in Chapter 3.

We have presented the following:

  (a) Figs. (5) and (6) show the evolution of the final solution from odd initial conditions,

  (b) Fig. (7) for the initial value $g(\theta, 0) = \cos \pi \theta$ shows that the solution to eqn. (3.1) given by the eqn. (4.3) can be extended to certain non-odd functions,

  (c) Figs. (8), (9) and (10) are examples of the trial function given by eqn. (5.1),

and (d) Figs. (9) and (10) also demonstrate the scaling invariance result proved in Chapter 4. In this case we have $g_1(\theta, 0) / g_2(\theta, 0) = 8$.

  The values of λ_1 and λ_2 shown with these solutions were obtained by estimating $\overline{g^2}$ numerically and equating

$$\overline{g^2} = \lambda_1 g_1^2 + \lambda_2 g_2^2 \sim 3\lambda_1 \lambda_2 \ .$$
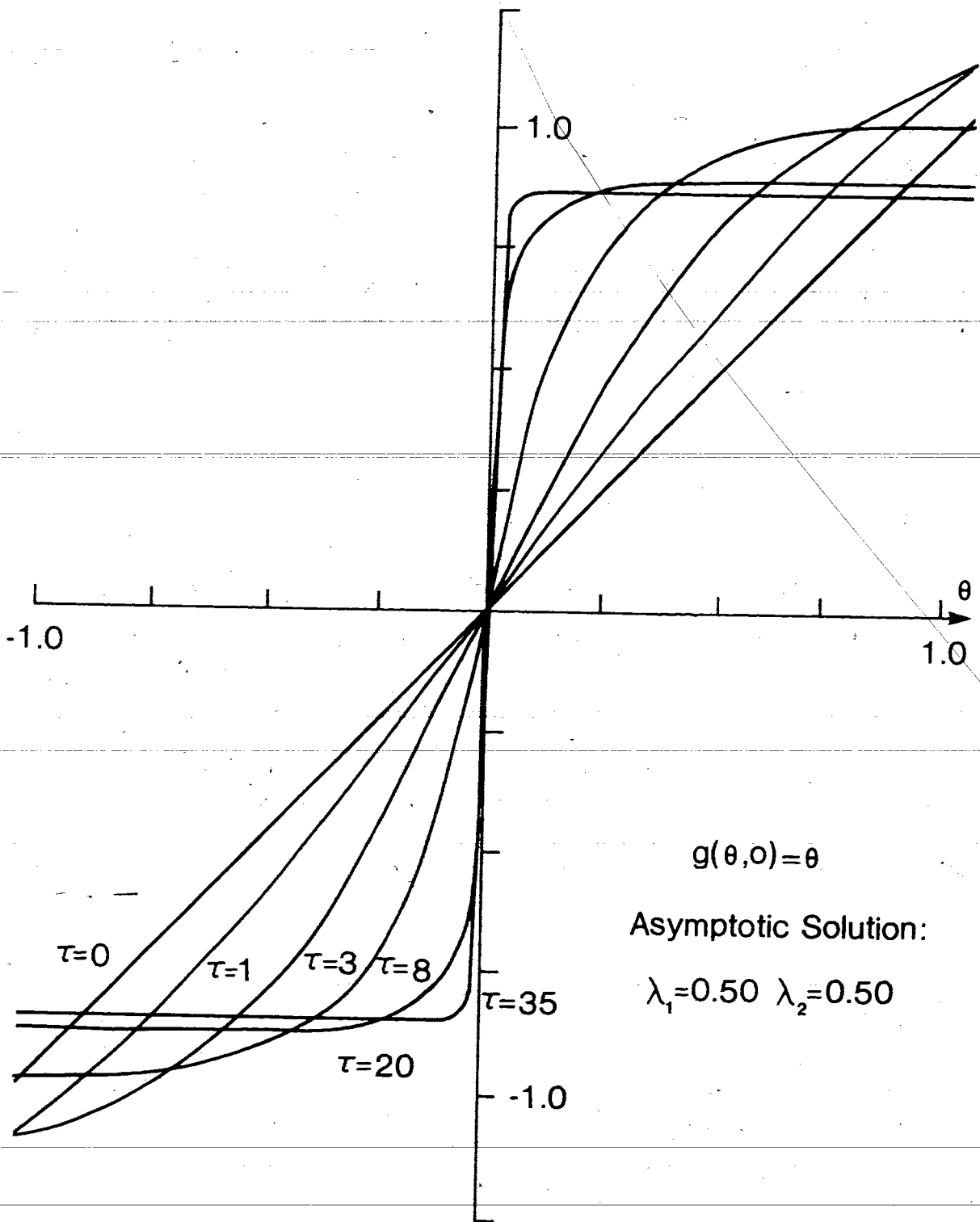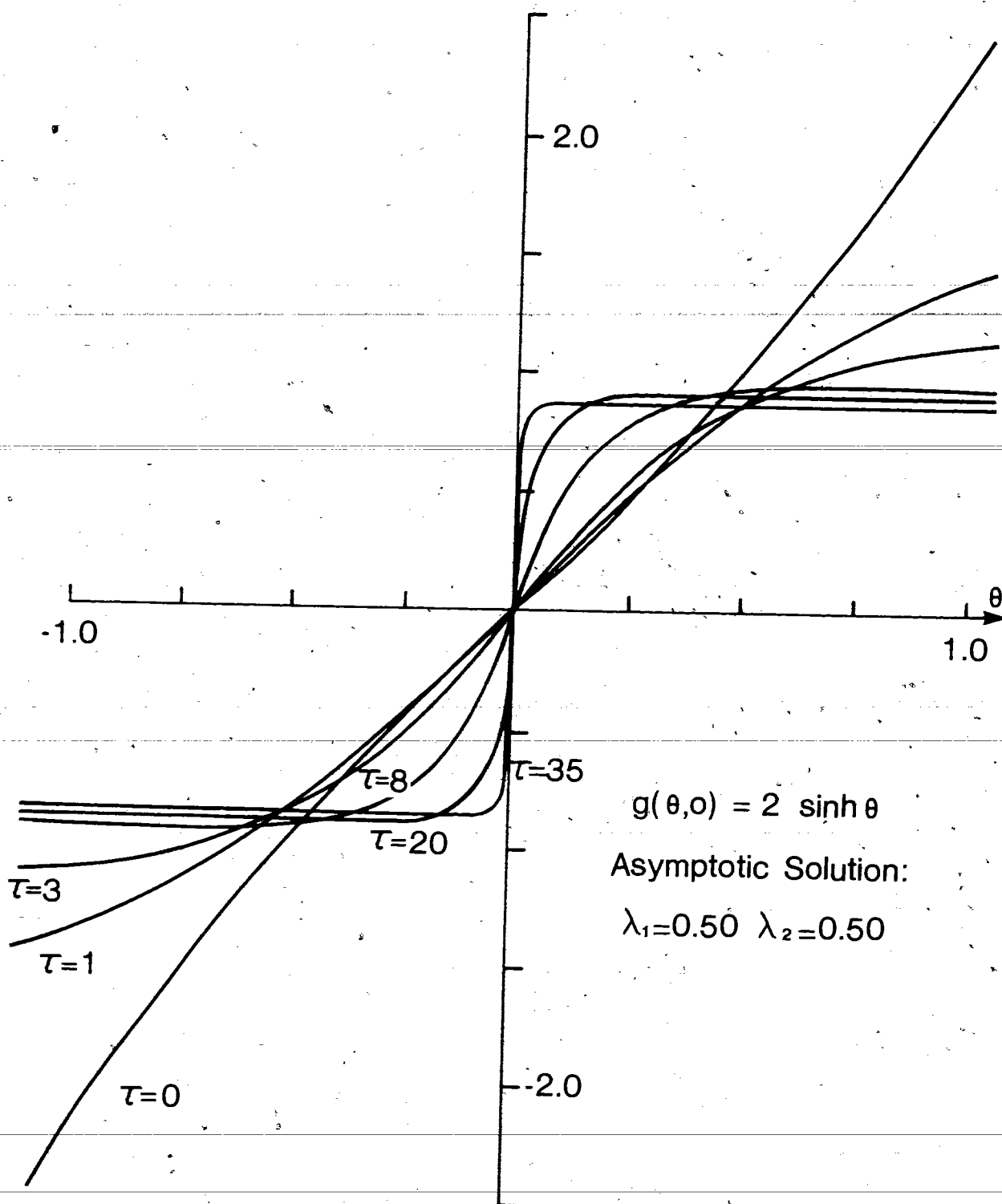
$g(\theta,0)=\theta$

Asymptotic Solution:

$\lambda_1=0.50 \quad \lambda_2=0.50$

$\tau=0$

$\tau=1$

$\tau=3$ $\tau=8$

$\tau=35$

$\tau=20$

Fig. 5

$$g(\theta, o) = 2 \text{ sinh } \theta$$

Asymptotic Solution:

$$\lambda_1 = 0.50 \quad \lambda_2 = 0.50$$

Fig. 6

$g(\theta,o)=\cos \pi \theta$

Asymptotic Solution:

$\lambda_1 = 0.50 \quad \lambda_2 = 0.50$

Fig. 7

$$g(\theta,0)= \frac{5\,(\beta\theta+1)}{(\theta+\hat{\beta})^3} \quad \beta=3.0$$

Asymptotic Solution:

$\lambda_1=0.56 \quad \lambda_2=0.44$

Fig. 8

$$g(\theta,0) = \frac{10\,(\beta\theta+1)}{(\theta+\beta)^3} \quad \beta = 5.0$$

Asymptotic Solution:

$$\lambda_1 = 0.53 \quad \lambda_2 = 0.47$$

Fig. 9

$$g(\theta,o) = \frac{80(\beta\theta+1)}{(\theta+\beta)^3} \qquad \beta = 5.0$$

Asymptotic Solution:

$\lambda_1 = 0.53$, $\lambda_2 = 0.47$
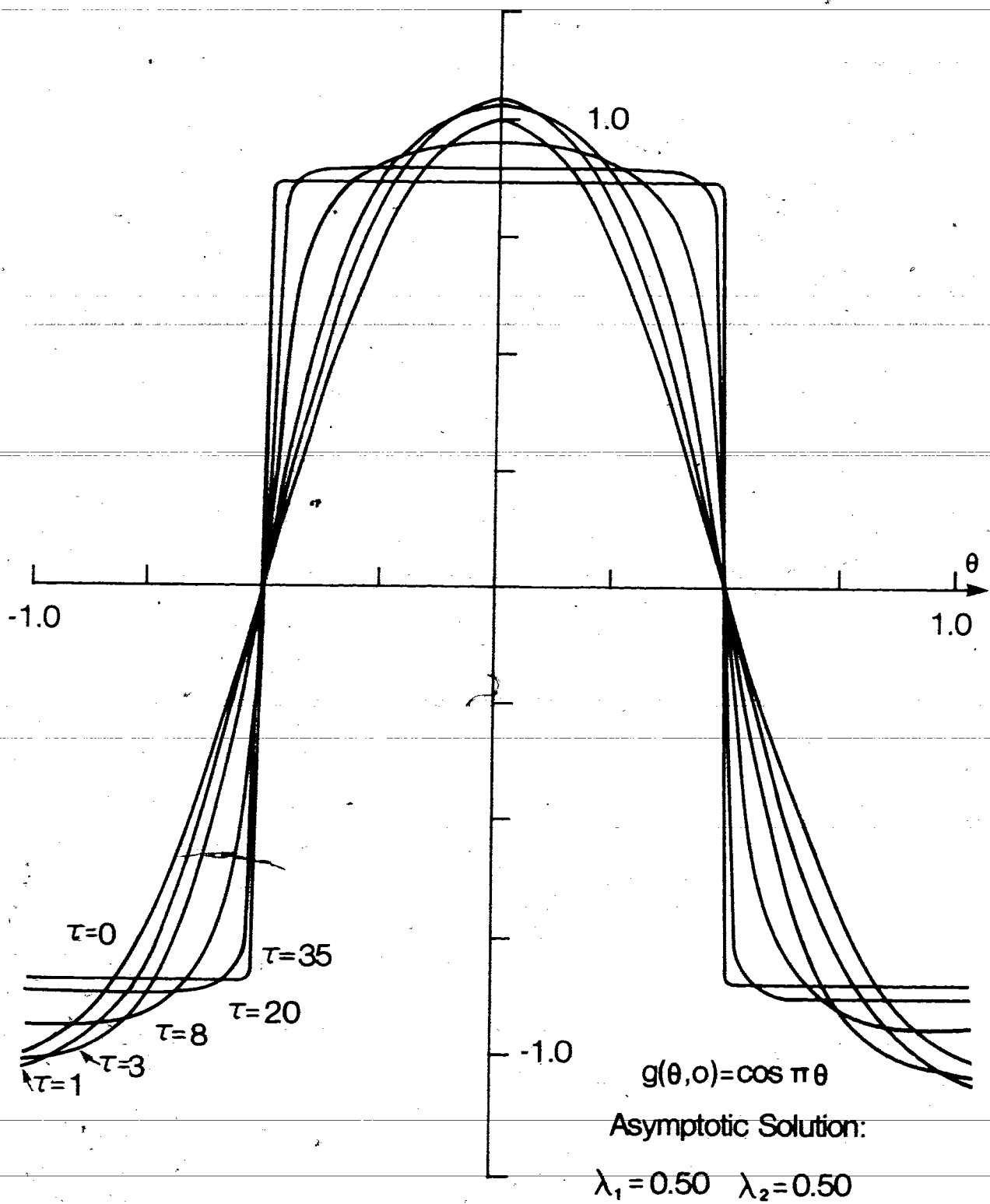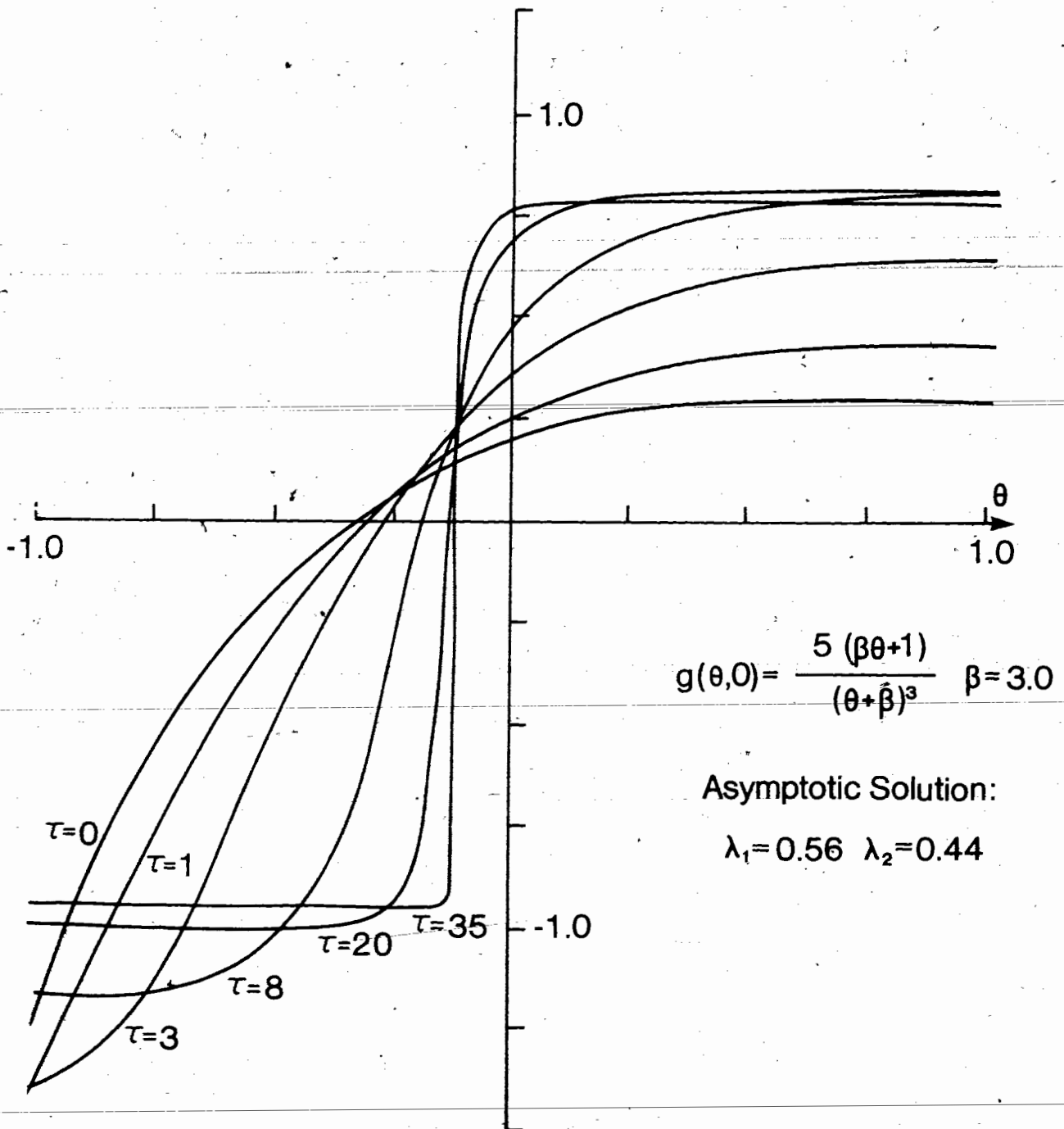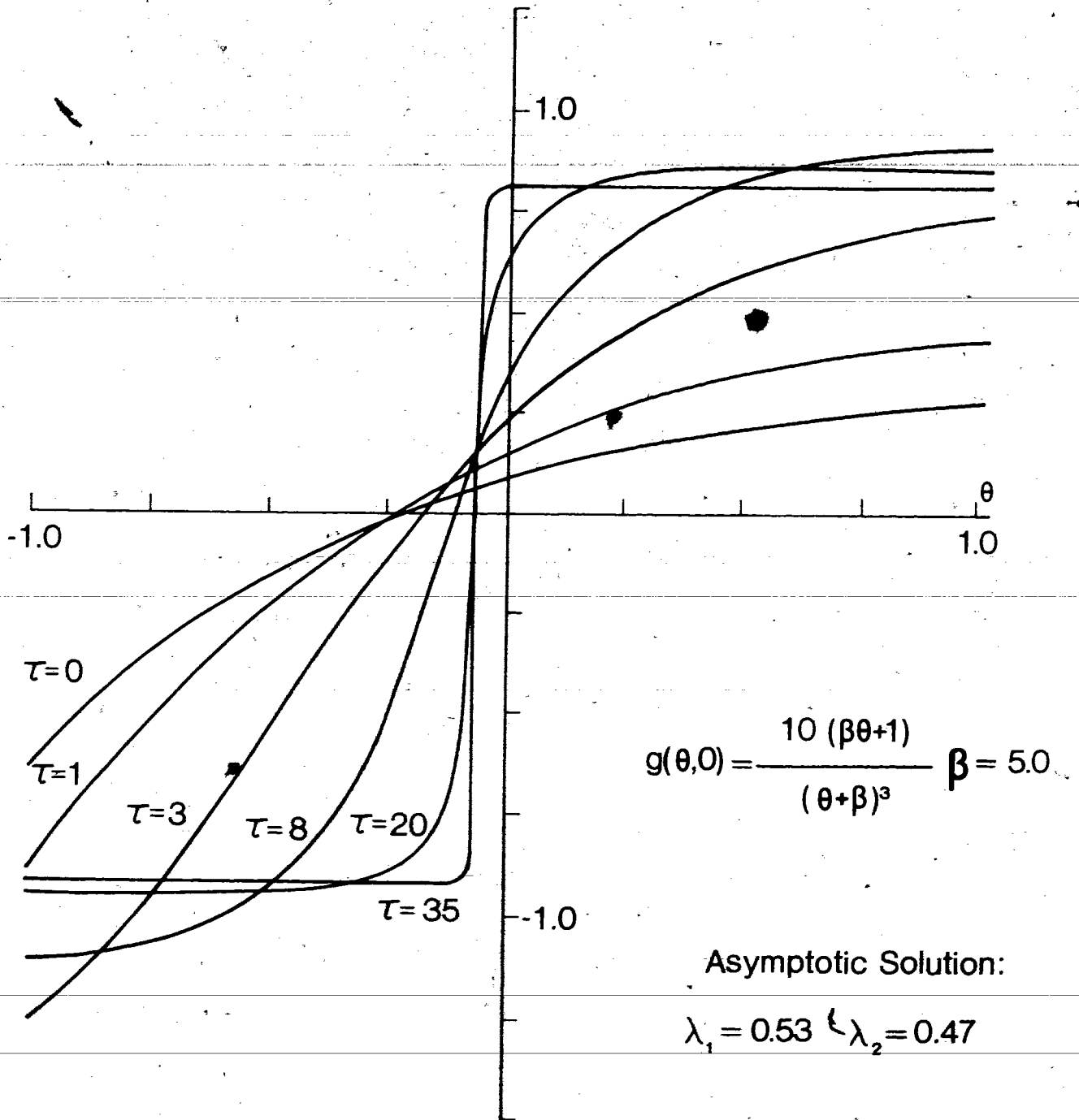
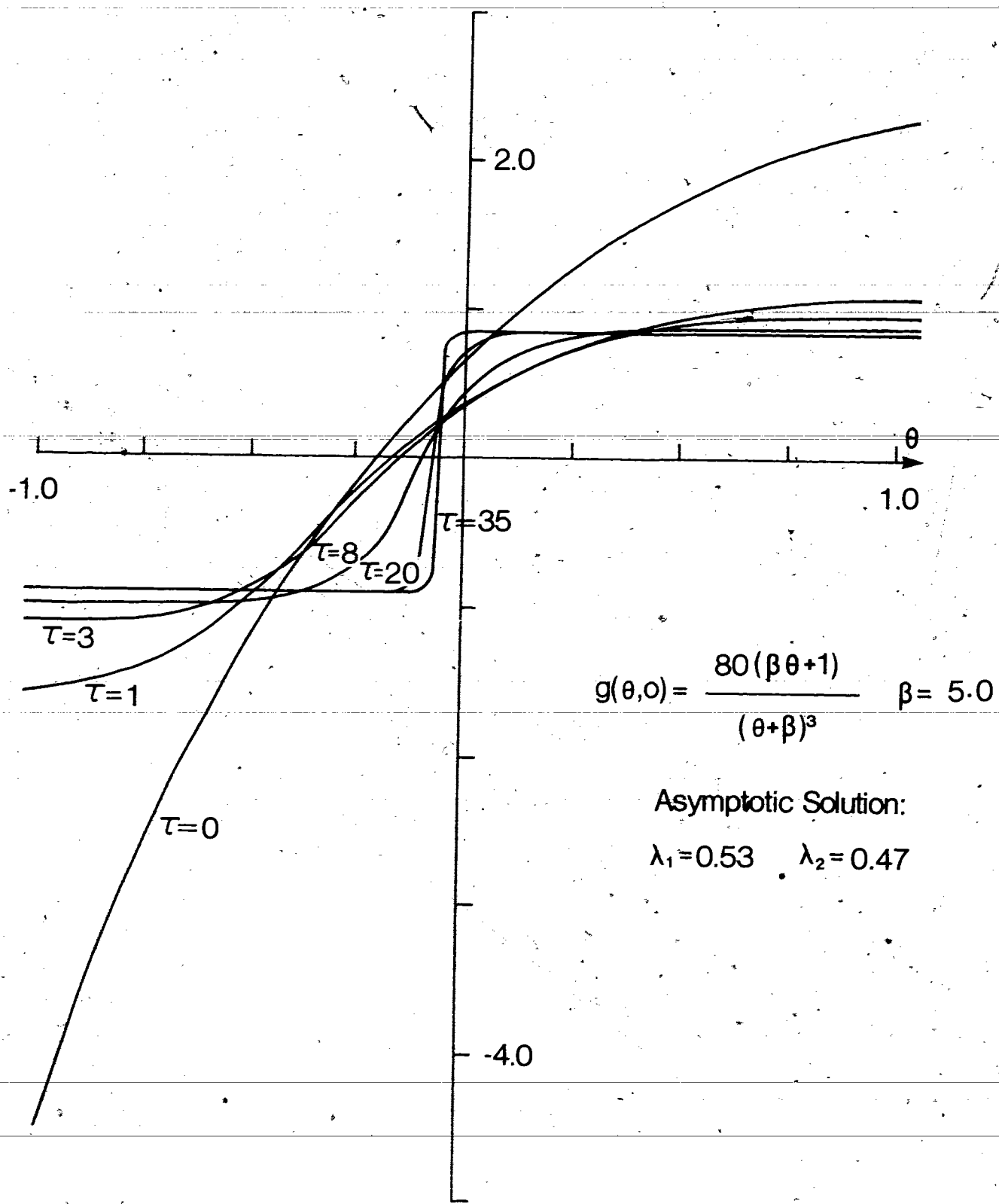Fig. 10

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