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by

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ABSTRACT

This thesis consists of a survey of the properties of a class of hamiltonian graphs called hamiltonian-connected graphs. Chapter 1 consists of a brief survey of the current status of studies on necessary and sufficient conditions for a graph to be hamiltonian. Chapter 2 is devoted to a survey of necessary and sufficient conditions for a graph to be hamiltonian-connected. In chapter 3, a characterization of Cayley graphs on an abelian group which are hamiltonian-connected is given. In chapter 4, some necessary and sufficient conditions for the existence of two special classes of hamiltonian-connected graphs, called, respectively, PLD-maximal graphs and panconnected graphs, are investigated. Various examples and counterexamples concerning some open questions and a few conjectures on the path length distribution (PLD) are presented at the end of Chapter 4.

to my mother

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TABLE OF CONTENTS

CHAPTER 1

AN INTRODUCTION TO SOME CURRENT RESULTS ON HAMILTONIAN GRAPHS

Section 1.1 Introduction

Studies on the existence and properties of paths and cycles on a specified number of vertices in a graph have been of considerable interest to both pure and applied mathematicians as well as researchers in other disciplines. Important practical applications of such studies can be found in fields related to operations research, electrical engineering, computer algorithm analysis as well as many other areas of scientific research. This thesis primarily concerns itself with the study of graphs which have a path connecting any two distinct vertices such that all other vertices in the graph are contained in the path.

Section 1.2 Recent results on hamiltonian graphs

The definition of a graph and all the notations employed primarily follows that of Bondy and Murty [5]. A graph G=(V(G),E(G)) as defined in [5] is assumed to be loopless and without multiple edges. Otherwise it is called a pseudograph.

Definition 1.2.1 Let G=(V(G),E(G)) be a graph. If a cycle C in

G satisfies $|V(C)| = |V(G)|$, then it is said to be a hamiltonian cycle. A graph is said to be a hamiltonian graph if it has a hamiltonian cycle.

Definition 1.2.2 A graph $G=(V(G),E(G))$ is said to be a hamiltonian-connected graph if for every pair of distinct vertices $u,v \in V(G)$, there exists a hamiltonian u,v-path in G.

Since the four-color theorem has recently been proven with the aid of a computer, the oldest and the most famous unsolved problem in the theory of graphs is undoubtedly that of constructing an elegant and practical characterization of hamiltonian graphs. Indeed, these two problems are not entirely unrelated. It is known that [46] every hamiltonian plane map is 4-colorable. The problem of recognizing a graph to be hamiltonian is notoriously difficult. In fact Karp, Lawler, and Tarjan [32] proved that it is an NP-complete problem. Combined with a theorem of S.A. Cook [16], the existence of a good characterization of nonhamiltonian graph seems unlikely. Consequently, major efforts have been devoted to constructing certain varieties of sufficient conditions for different classes of graphs and a few necessary conditions for a graph to be hamiltonian.

There are two main objectives in this thesis. The first objective is to provide a thorough up to date survey on the existence and the properties of a special class of hamiltonian graphs called hamiltonian-connected graphs; in particular, the panconnected graphs and PLD-maximal graphs. Chapter 2 of this thesis is concerned directly with the necessary and sufficient conditions for a graph to be hamiltonian-connected and generalizations of such conditions to graphs which are n-hamiltonian-connected. In Chapter 4, the concept of hamiltonian-connectedness will be generalized to consider even a smaller class of hamiltonian-connected graphs called panconnected graphs and PLD-maximal graphs. A graph $G=(V(G),E(G))$ is said to be panconnected if for each pair of distinct vertices $u,v \in V(G)$ and for each ℓ satisfying $d_G(u,v) \leq \ell \leq |V(G)|-1$, where $d_G(u,v)$ denotes the distance between u and v, there exists a uv-path of length ℓ. Clearly, each PLD-maximal graph is a panconnected graph.

The second objective is to provide a complete characterization of the hamiltonian-connectedness of a Cayley graph on an abelian group. This is given in Chapter 3.

Finally the present chapter is intended to provide a brief survey on the relevant general necessary and sufficient conditions for a graph to be hamiltonian. Since it is not the objective of this thesis to pursue the details of such conditions, the proof of the theorems presented in this chapter will not be provided. Instead, numerous examples will be presented to illuminate the strength and the sharpness of these theorems. The following categories of sufficient conditions for a graph to be hamiltonian will be discussed: degree conditions, edge conditions, the condition of being the power of a graph and conditions which involve some topological parameters like connectedness and independence number.

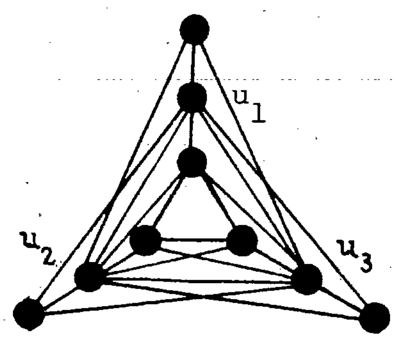
We now begin with a few simple but important necessary conditions for a graph to be hamiltonian.

<u>Theorem 1.2.3 [5]</u> Let $G=(V(G),E(G))$ be a hamiltonian graph. Let S be any non-empty proper subset of the vertex set $V(G)$ and let $c(G-S)$ be the number of components of the subgraph $\langle V(G)-S\rangle$. Then $c(G-S)\leq |S|$.

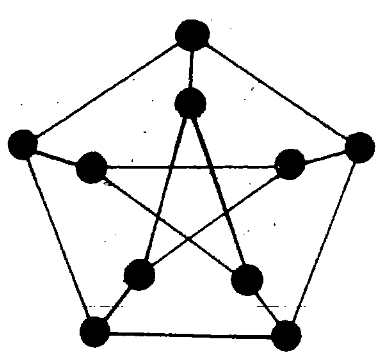
As a consequence of this theorem we have

<u>Corollary 1.2.4</u> Every hamiltonian graph is necessarily 2-connected.

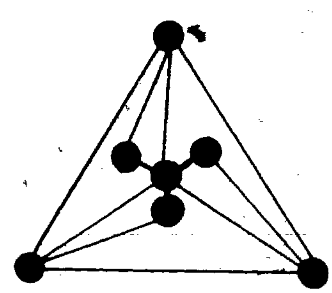
Necessary conditions are very useful for identifying nonhamiltonian graphs. For example, the graph in Figure 1.1 is non-hamiltonian since the removal of the vertices labelled u_1, u_2, u_3 results in a disconnected graph with 4 components. However, the converse of Theorem 1.2.3 is false as illustrated by the graphs in Figure 1.2 and Figure 1.3. In fact the graph in Figure 1.3 is the smallest graph which



<u>figure 1.1</u>



<u>figure 1.2</u>



<u>figure 1.3</u>

is a counterexample to the converse of Theorem 1.2.3. The converse of

Corollary 1.2.4 is also false as the complete bipartite graph $K_{m,n}$, where $m<n$, serves as a counterexample.

Chvatal [14] introduced the definition of a graph to be 1-tough if $c(G-S)\leq|S|$ is satisfied for every nonempty subset S in V(G). It follows from Theorem 1.2.3 that every hamiltonian graph is necessarily 1-tough. Chvatal has further extended this idea by defining the toughness t(G) of a graph G by $t(G)=\min\left\{\frac{S}{c(G-S)}\middle| S \text{ is a cut set in } G\right\}$. A subset S of V(G) is a cut set in G if $c(G-S)>1$. Intuitively, the toughness t(G) of a graph is the measure of the ability of a graph to hold together when subsets of vertices in V(G) are removed. G being 1-tough implies that $t(G)\geq 1$. A lower bound and an upper bound for the toughness of a graph may be established from the connectivity κ and independence number β of a graph G.

<u>Theorem 1.2.5 [14]</u> Let G be a graph, not isomorphic to a connected graph with connectivity κ and independence number β. Then,

$\frac{\kappa}{\beta}\leq t(G)\leq\frac{\kappa}{2}$.

The lower bound of Theorem 1.2.5 follows directly from the fact that for any cut set S in G, $|S|\geq\kappa$ and $c(G-S)\leq\beta$. The upper bound is obvious. Note that $\frac{\kappa}{\beta}$ is exactly the bound for the complete bipartite graph $K_{m,n}$. The concept of toughness provides a necessary condition for a graph to be r-hamiltonian.

<u>Definition 1.2.6 [36]</u> A graph G is r-hamiltonian if the removal of any k vertices from G, $0\leq k\leq r$ results in a hamiltonian graph. Note that a r-hamiltonian graph is (r+2)-connected.

John Molluzzo [36] proved the following necessary condition.

Theorem 1.2.7 [36] If G is n-hamiltonian, the $t(G) \geq 1 + \frac{n}{\beta}$.

He also proposed the following conjectures.

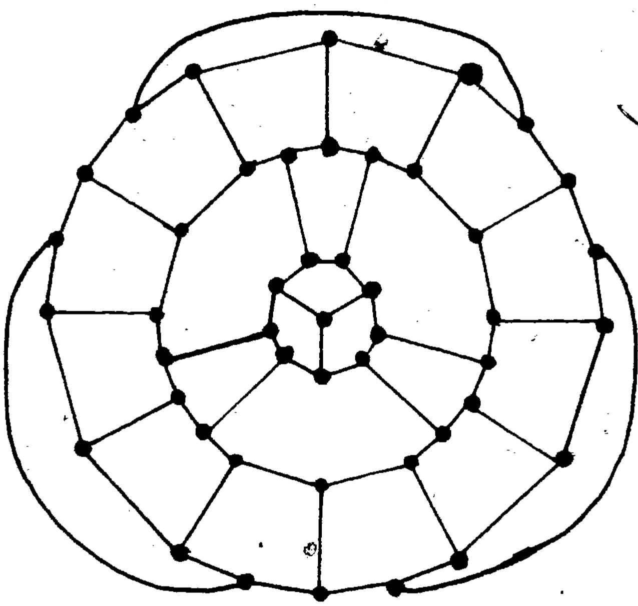
Conjecture 1.2.8 [36] If $t(G) > 2$, then G is hamiltonian-connected.

Conjecture 1.2.9 [36] If $t(G) > 2 + \frac{r}{\beta}$, then G is r-hamiltonian.

The next well-known and useful necessary condition to be discussed is the equation of Grinberg.

Theorem 1.2.10 [4] Let G be a planar graph with a hamiltonian cycle C. Then $\sum_{i=1}^{|V(G)|} (i-2)(\phi'_i - \phi''_i) = 0$, where ϕ'_i and ϕ''_i are the numbers of faces of degree i contained in Int C and Ext C, respectively.

With the aid of Theorem 1.2.10, one can easily show that the graphs in Figure 1.4 and Figure 1.5 are nonhamiltonian.



Grinsberg's graph

figure 1.4

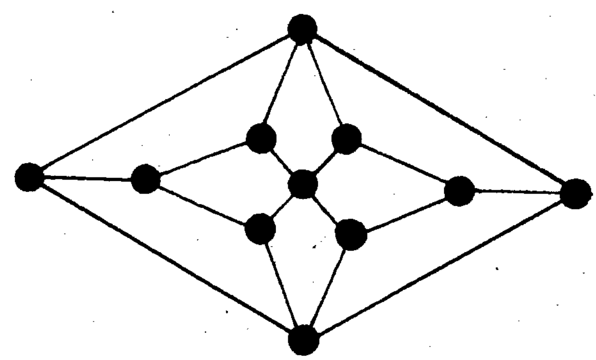


figure 1.5

The graph in Figure 1.4 is the Grinberg graph. Each number in a region of the graph represents the degree of the region (that is, the number of edges which constitutes the regions). To show that it is nonhamiltonian we assume the contrary. Then the fact that the Grinberg graph has faces of degrees 5, 8 and 9, the Grinberg equation yields $3(\phi_5'-\phi_5'')+(\phi_8'-\phi_8'')+7(\phi'-\phi_9'')=0$ which implies that $7(\phi_9'-\phi_9'')=0 \pmod 3$. This, however, is impossible since the value of the left-handed side of the last equation is -7 or 7 depending on whether the face of degree 9 is in Ext C or Int C.

A similar argument shows that the graph in Figure 1.2 is nonhamiltonian. Assume the contrary. If a hamiltonian cycle C can be found, then the resulting Grinberg's equation is $2(\phi_4'-\phi_4'')=0$. This, however, is impossible since there are an odd number of regions of degree 4 in the graph.

The equation of Grinberg will be employed again in Chapter 4 of this thesis to help construct an important set of counterexamples to a well known conjecture by R.J. Faudree and R.H. Schelp concerning hamiltonian-connected graphs [23,24]. (See Theorem 4.4.8).

There is a well known sufficient condition [15] for a graph to be hamiltonian which is expressed in terms of the topological parameters connectedness κ and independence number β.

Theorem 1.2.11 [15] Let $G=(V(G),E(G))$ be a κ-connected graph with independence number β. If $\kappa \geq \beta$, then G is hamiltonian.

Two more sufficient conditions for a graph to be hamiltonian expressed in terms of connectedness and locally connectedness can be found in Section 2.2. A considerable amount of effort has been devoted to the development of sufficient conditions for a graph to be hamiltonian expressed in terms of the degrees of the vertices. Chronologically, Dirac [17], Ore [39], Posa [41], Bondy [6] and Chvatal [13] have determined such conditions, with each successive result strengthening those preceeding it. Let G=V(G),E(G)) be a graph on n vertices. Without loss of generality we assume that the set of vertices $V(G)=\{v_1,...,v_n\}$ satisfies the degree sequence $d_1 \le d_2 \le ... \le d_n$, where d_i is the degree of v_i in G.

Suppose that $n=|V(G)| \geq 3$. Then Theorems 1.2.12 through Theorems 1.2.16 hold.

Theorem 1.2.12 (Dirac [17]) If $\delta(G)=\mathrm{Min}\{d_i | 1 \le k \le n\} \geq \frac{n}{2}$, then G is hamiltonian.

Theorem 1.2.13 (Ore [39]) Suppose that for each pair of nonadjacent vertices $u,v \in V(G)$ $\deg(u)+\deg(v) \geq n$. Then G is hamiltonian.

Theorem 1.2.14 (Posa [41]) Suppose that for each k, $1 \le k \le \frac{n}{2}$, $d_k > k$. Then, G is hamiltonian.

Theorem 1.2.15 (Bondy [6]) Suppose that for each j,k satisfying $d_k \le k$ and $d_j \le j (j \ne k)$ implies that $d_j + d_k \geq n$. Then, G is hamiltonian.

Theorem 1.2.16 (Chvatal [13]) Suppose that for each k, $d_k \le k < \frac{n}{2}$, $d_{n-k} \geq n-k$. Then G is hamiltonian.

A sequence of nonnegative integers $d_1 \le d_2 \le ... \le d_n$ is said to be graphic if it is the degree sequence of a graph on n vertices. Let $S: d_1 \le d_2 \ ... \le d_n$ and $S': d'_1 \le d'_2 \le ... \le d'_n$ be two graphic sequences such that for each i, $1 \leq i \leq n$, $d_i \geq d'_i$.

The sequence S' is said to be degree-majorized by the sequence S. In particular, every graphic sequence $d_1 \leq \ldots \leq d_n$ which fails to satisfy Theorem 1.2.16 is degree majorized by the degree sequence $\underset{\sim}{S}$: $\underset{\sim}{d}_1 \leq \ldots \leq \underset{\sim}{d}_n$, where $\underset{\sim}{d}_i = k$ for each i, $1 \leq i \leq k$, $\underset{\sim}{d}_i = n-k-1$, for each i, $k+1 \leq i \leq n-k$, and $\underset{\sim}{d}_i = n-1$ for $n-k+1 \leq i \leq n$. $\underset{\sim}{S}$ is precisely the degree sequence of the graph $K_k + (\bar{K}_k \cup K_{n-2k})$ in Figure 1.9a. The graph of Figure 1.9a is not hamiltonian because of Theorem 1.2.3.

$K_1 + (\bar{K}_1 \cup K_{n-1})$

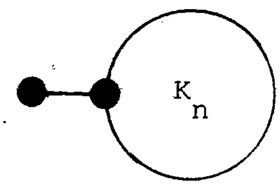


figure 1.9b

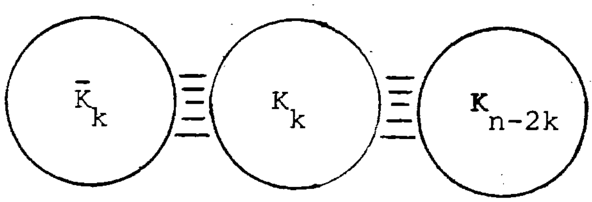


figure 1.9a

$K_2 + (\bar{K}_2 \cup K_1)$

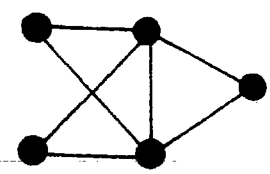
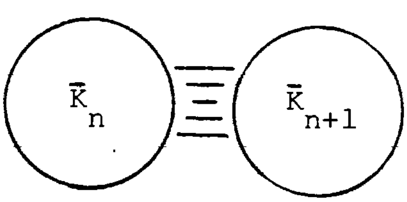


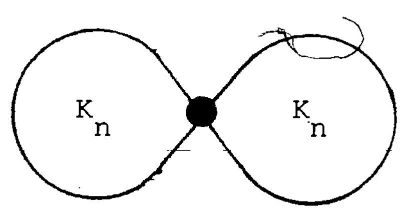
figure 1.9c

It is in this sense that Chvatal's theorem (1.2.16) is sharp. Note that the lower bound in Dirac's theorem (1.2.12) cannot be further reduced as the examples in Figure 1.10 illustrate.



$K_{n,n+1} = \bar{K}_n + \bar{K}_{n+1}$

figure 1.10a



$K_1 + (K_{n-1} \cup K_{n-1})$

figure 1.10b

By Theorem 1.2.3, it is clear that the graphs in Figure 1.9 and 1.10 are nonhamiltonian.

Clearly, if a graphic sequence S which satisfies the hypothesis of any one of Theorem 1.2.12 to Theorem 1.2.16, so does any graphic sequence which majorizes S. Although in the sense discussed above that Chvatal's condition (Theorem 1.2.16) is the strongest possible, it has been generalized by Las Vergnas [33] and by Chvatal and Bondy [7] as follows.

Theorem 1.2.17 Let $G=(V(G),E(G))$ be a graph on $n\geq 3$ vertices with degree sequence $d_1\leq \ldots \leq d_n$. Let $V(G)=\{v_1,\ldots,v_n\}$ Suppose that for each j,k such that $j<k$, $k\geq n-j$, $v_jv_k\notin E(G)$, $d_j\leq j$, and $d_k\leq k-1$ we have $d_j+d_k\geq n$. Then G is hamiltonian.

Using Chvatal's condition, one can easily deduce a sufficient condition for a graph to be hamiltonian expressed in terms of the number of edges.

Theorem 1.2.18 [5] Let $G=(V(G),E(G))$ be a graph on $n=|V(G)|\geq 3$ vertices and $|E(G)|>\binom{n-1}{2}+1$. Then, G is hamiltonian. Furthermore, the only nonhamiltonian graph with n vertices and $\binom{n-1}{2}+1$ edges is precisely $K_1+(\bar{K}_1\cup K_{n-1})$, and additionally, for $n=5$, $K_2+(\bar{K}_2\cup K_1)$ as shown in Figure 1.9b and 1.9c respectively.

Bondy [5] defined the colsure Cl(G) of a graph G to be the graph obtained from G by recursively joining pairs of nonadjacent vertices with degree sum at least $|V(G)|$. It is easily shown that the construction of Cl(G) depends only on G and not on the order in which the new edges are added to constitute Cl(G). The following closure theorem is easily verified.

Theorem 1.2.19 G is hamiltonian if and only if Cl(G) is hamiltonian.

An immediate corollary is as follows.

Corollary 1.2.20 Let G be graph on $n \geq 3$ vertices. If $Cl(G)=K_n$, then G is hamiltonian.

Corollary 1.2.20 can be a result employed to deduce Chvatal's condition in Theorem 1.2.16.

The sufficient conditions of Dirac, Ore and Posa (Theorem 1.2.12 through Theorem 1.2.14, respectively) have been generalized to r-hamiltonian graphs [11]. Theorem 1.2.21 below is the generalization of Dirac's condition in Theorem 1.2.12.

Theorem 1.2.21 [11] Let G be a graph on n 3 vertices and let $0 \leq n \leq r-3$. If every vertex of G has degree at least $\frac{n+r}{2}$, then G is r-hamiltonian.

The generalizations of Theorem 1.2.15 and Theorem 1.2.16 are as presented in Theorem 1.2.22 and Theorem 1.2.23, respectively, in the following.

Theorem 1.2.22 [11] Let G be a graph on $n \geq 3$ vertices and let $0 \leq r \leq n-3$. If for every pair of nonadjacent vertices u and v of G, $\deg_G(u)+\deg_G(v) \geq n+r$, then G is r-hamiltonian.

Theorem 1.2.23 [11] Let G be a graph on $n \geq 3$ vertices, and let $0 \leq r \leq n-3$. If

(1) for each j, $r+1 \leq j < \frac{n+r-1}{2}$, the number of vertices of degree not exceeding j is less than j-r

and (2) the number of vertices of degree not exceeding $\frac{n+r-1}{2}$

does not exceed $\frac{n-r-1}{2}$, then G is r-hamiltonian.

The bounds in Theorems 1.2.21, 1.2.22 and 1.2.23 are indeed the best possible as we will now show. Denote by K_{m_1,m_2,m_3} the complete tripartite graph defined by having its set of vertices $V(K_{m_1,m_2,m_3})$ partitioned into three non-empty independent subsets V_1, V_2, V_3 where $|V(V_i)|=m_i, i=1,2,3$, and $u,v \in V(K_{m_1,m_2,m_3})$ are adjacent if and only if $u \in V_j$, $v \in V_k$, $j \neq k$.

For each pair of nonadjacent vertices u,v in $K_{r,r,r+1}$ $\deg(u)+\deg(v) \geq n+r-1$ and every vertex has degree at least $\frac{n+r-1}{2}$. But $K_{r,r,r+1}$ is not r-hamiltonian since the removal of a partition of size r leaves a bipartite graph isomorphic to $K_{r,r+1}$ which is non-hamiltonian. Theorems 1.2.21 and 1.2.22 are in this sense sharp.

For r=0, Theorem 1.2.21, 1.2.22 and 1.2.25 are reduced to Theorems 1.2.12, 1.2.13 and 1.2.14 respectively. In particular, Theorem 1.2.23 is reduced to the condition of Posa. To show that Posa's theorem is sharp, we consider the graph G isomorphic to $K_1+(K_k \cup K_{n-k-1})$ for $1 \leq k < \frac{n-1}{2}$. G is nonhamiltonian since it has a cut vertex and it has exactly k vertices of degree k.

Suppose now that n is odd and $k=\frac{n-1}{2}$. A graph $G=(V(G),E(G))$ is defined on 2k+1 vertices as follows. Let $V(G)=\{v_1,\ldots,v_{2k+1}\}$. $E(G)=\{x_i x_j \mid 1 \leq i \leq k < j \leq 2k+1\}$. G is nonhamiltonian and it has exactly $\frac{n+1}{2}$ vertices of degree $\frac{n-1}{2}$.

It remains to consider the case $r>0$. The graph $K_{r,r,r+1}$ clearly satisfies Condition (1) in Theorem 1.2.23 and it is not r-hamiltonian. If to the graph $K_{r,r,r+2}$, a new edge is added to V_3, where $|V_3|=r+2$. The resulting graph satisfies Condition (2) of Theorem 1.2.23 and Condition

(1) except for the value $j=\frac{n+r-2}{2}$ and G is not r-hamiltonian.

The necessary condition in Theorem 1.2.18 has also been generalized in [11] as follows.

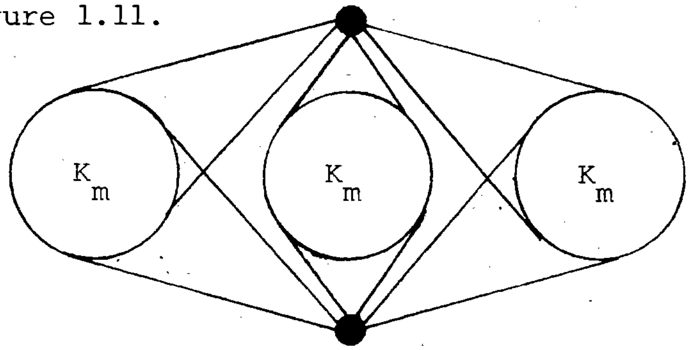
Theorem 1.2.24 [11] Let G=(V(G),E(G)) be a graph on $n\geq 3$ vertices. If $|E(G)| > \binom{n-1}{2}+r+1$, then G is r-hamiltonian.

We now have several necessary and sufficient conditions for a graph to be r-hamiltonian. Despite the apparent difficulty in giving a practical characterization for a graph to be r-hamiltonian, for some large values of r, however, there are conditions which are both necessary and sufficient. Clearly, for any graph G on $n\geq 3$ vertices, G is (n-3)-hamiltonian if and only if it is complete, and a graph G on $n\geq 4$ vertices is (n-4)-hamiltonian if and only if G is a complete graph from which a collection of mutually nonadjacent edges has been removed. One can observe that for $n\geq 3$, there is exactly one (n-3)-hamiltonian graph up to isomorphism. For $n\geq 4$, there are exactly $1-\left[\frac{n}{2}\right]$ nonisomorphic (n-4)-hamiltonian graph on n vertices.

It has been shown previously that the lower bound on $\delta(G)$ in Dirac's theorem (Theorem 1.2.12)cannot be further reduced. Thus, if we wish to generalize such a condition, additional constraints must be imposed. Perhaps the first condition which may come to anyone's mind would be that of 2-connectedness since every hamiltonian graph is necessarily 2-connected. Nash-Williams [48] has indeed established the following basic Lemma.

Lemma 1.2.25 [48] Let G be a 2-connected graph on n vertices with $\delta(G)\geq\frac{n+2}{3}$, and let C be a longest cycle in G. Then, V(G)-V(C) is an independent set in G.

The sharpness of Lemma 1.2.25 is demonstrated by the graph in Figure 1.11.



$\delta(G) = \frac{|V(G)| + 1}{3}$

figure 1.11

As a consequence of Lemma 1.2.25, we have,

Theorem 1.2.26 [49] If G is 2-connected graph on n vertices, with $\delta(G) \geq \beta$ and $\delta(G) \geq \frac{1}{3}(n+2)$, then G is hamiltonian.

With the help of the Hopping Lemma [52], Woodall established a theorem on the lower bound of $\delta(G)$ as follows.

Theorem 1.2.27 [52] If G is a 2-connected graph on n vertices, with $\delta(G) \geq \frac{1}{3}(n+2)$ and $|N(S)| \geq \frac{1}{3}(n+|S|-1)$ for all $S \subseteq V(G)$, then G is hamiltonian.

Regular graphs possess additional structure that gives them interesting and often stronger properties. As we shall see in the following that the lower bound of $\delta(G)$ in Dirac's condition can be further reduced for regular graphs.

Nash-Williams has shown the following.

Theorem 1.2.28 [50] Every k-regular graph on 2k+1 vertices is hamiltonian. This bound was further reduced by Erdos and Hobbs [20,21] by imposing 2-connected as an additional constraint.

Theorem 1.2.29 [22] For $k \geq 4$, every 2-connected k-regular graph on 2k+4 vertices is hamiltonian.

Theorem 1.2.30 [21] Let G be a 2-connected, k-regular graph on n-vertices, where $k \geq \frac{1}{2}(n - c\sqrt{n})$ and $C = \begin{cases} \sqrt{2} \text{ if n is even.} \\ 1 \text{ if n is odd.} \end{cases}$ Then G is hamiltonian.

Bollobas and Hobbs [4] obtained the following stronger result.

Theorem 1.2.31 Every 2-connected, k-regular graph on at most $\left\lceil \frac{9k}{4} \right\rceil$ vertices is hamiltonian.

Finally, a much stronger result on graph is that of the generalization of Theorem 1.2.31 by Bill Jackson [30] in the following.

Theorem 1.2.32 [30] Every 2-connected, k-regular graph on at most 3k vertices is hamiltonian.

Furthermore, Theorem 1.2.32 is sharp since the Petersen graph is a nonhamiltonian, 2-connected, 3-regular graph on 10 vertices. For $k \geq 4$, there exists nonhamiltonian, 2-connected, k-regular graphs on 3k+4 vertices for even k, and on 3k+5 vertices for all k, respectively, as illustrated in [19,28].

Definition 1.2.33 Let G=(V(G),E(G)) be a graph. For each integer $m \geq 1$, let the m^{th}-power of G, $G^m = (V(G^m), E(G^m))$, be the graph with $V(G^m) = V(G)$ and for each $x, y \in V(G^m)$, $xy \in E(G^m)$ if and only if $d_G(x,y) \leq m$.

Being the m^{th}-power of a graph, $m \geq 2$, is a strong condition for a graph to be hamiltonian. In a paper by J.J. Karaganis [31], it has been shown by induction on the number of vertices that G^3 is always hamiltonian-connected for any connected graph G.

Y. Alavi and E. Williamson [3] have generalized this result to the effect that G^3, G a connected graph, is in fact panconnected as will be shown in Chapter 4 of this thesis. Following are two well known results of Herbert Fleischner.

Theorem 1.2.34 [26] The square G^2 of every 2-connected graph G is hamiltonian.

Theorem 1.2.35 [27] The square G^2 of a connected bridgeless DT- graph G (that is, every edge of G is incident to a vertex of degree 2) is hamiltonian-connected.

Furthermore, R.J. Faudree and R.H. Schelp [23] have generalized both Theorems 1.2.34 and 1.2.35 by showing that the square of bridgeless connected DT-graphs and 2-connected graphs are, in fact, panconnected. These results undoubtedly might lead one to speculate whether or not there exists a good characterization of graphs with hamiltonian squares. P. Underground [46], however, has shown that such a characterization is extremely unlikely by constructing the following example. Given a graph G with the set of vertices $V(G)=\{u_1,\ldots,u_n\}$, a new graph H is defined with the vertex set $V(H)=\{u_1,v_1,w_1,\ldots,u_n,v_n,w_n\}$ and two vertices u_iu_j in H are adjacent if and only if u_iu_j are adjacent in G. For each i, v_i is adjacent to both u_i and w_i. It can be easily verified that G is hamiltonian if and only if H^2 is hamiltonian. This implies that the problem of recognizing graph with a hamiltonian square is NP-complete and the hope for a good characterization of such graphs is further diminished.

Due to the vast amount of research taking place on the hamiltonian properties of graphs, many important topics have been omitted in this chapter. Among these are the decomposition of graphs into hamiltonian cycles, hypohamiltonian graphs, hamiltonian line graphs, hamiltonian directed graphs (with the exception of Cayley color digraphs), the existence of paths or cycles in graphs random graphs and m-partite graphs for $m \geq 2$.

For classifications of the notations employed in this thesis, the reader is referred to the Appendix.

CHAPTER 2

ON HAMILTONIAN-CONNECTED GRAPHS

Section 2.1 Degree and Edge Conditions

As far as results on necessary and sufficient conditions for a graph to be hamiltonian-connected, it is perhaps most appropriate to begin with the classic work of Oystein Ore on edge and degree conditions [40]. It is clear that a complete graph $K_n=(V(K_n), E(K_n))$ on $n \geq 2$ vertices has $|E(K_n)| = \frac{n(n-1)}{2}$ edges and is always hamiltonian-connected. One natural question one would like to ask is as follows:- "What is the smallest b(n) such that given any graph G=(V(G),E(G)) on n vertices G is necessarily hamiltonian-connected if $E(G) \geq b(n)$ is satisfied?" For small values of n, in particular, n=1,2,3,4, it is clear that G on n vertices is hamiltonian-connected if and only if it is isomorphic to the complete graph K_n. For n=5, the graph obtained by the deletion of a single edge from K_5 is hamiltonian-connected. The deletion of 2 non-adjacent edges (two edges are nonadjacent if and only if they do not share a common vertex) from K_5 results in a graph which is again hamiltonian-connected. However, the deletion of two adjacent edges results in a graph on 5 vertices which is not hamiltonian-connected since there exists no hamiltonian path connecting the two vertices adjacent to the vertex of degree 2. Thus, we can conclude that:

$b(1)=0$, $b(2)=1$, $b(3)=3$, $b(4)=6$, $b(5)=9$.

A sharp lower bound on b(n) can easily be obtained by generalizing the above observation. Let $G=(V(G), E(G))$ be a graph on $n \geq 4$ vertices with a vertex $v \in V(G)$ satisfying $\deg_G(v)=2$. Then, there exists no hamiltonian path connecting the two vertices adjacent to v. Therefore, if n-3 edges incident with a single vertex x in a complete graph K_n on $n \geq 4$ vertices are removed, then the resulting graph is not hamiltonian-connected.

This implies that for $n \geq 4$,

$$b(n) \geq \tfrac{1}{2}n(n-1)-(n-3)+1=\tfrac{1}{2}(n-1)(n-2)+3.$$

The following theorem which is parallel to Theorem 1.2.13, which can be found in many standard text books on Graph Theory (See Behzad and Chartrand [8]), allows an upper bound for b(n) to be determined.

Theorem 2.1.1[8] Let $G=(V(G),E(G))$ be a graph on n vertices. Suppose that for each pair of nonadjacent vertices $\{u,v\}$, $\deg_G(u)+\deg_G(v) \geq n+1$ is satisfied. Then, G is hamiltonian-connected.

Theorem 2.1.1 is sharp in the sense that the lower bound n+1 of the inequality cannot be replaced by n as indicated by the example mentioned in the previous paragraph. Namely, for each pair of nonadjacent vertices $\{x,w\}$ satisfies $\deg_G(x)+\deg_G(w)=2+(n-2)=n$, and the resulting graph is not hamiltonian-connected as we observed earlier. Note that the resulting graph is still hamiltonian according to Theorem 1.2.13. Theorem 2.1.2 below follows readily from Theorem 2.1.1.

Theorem 2.1.2 For $n \geq 4$, $b(n)=\tfrac{1}{2}(n-1)(n-2)+3$.

Proof: From the previous discussion, what remains to be shown is that any graph $G=(V(G),E(G))$ on $n \geq 4$ vertices which is not

isomorphic to K_n and with $|E(G)| \geq \frac{1}{2}(n-1)(n-2)+3$ is necessarily a hamiltonian-connected graph. Let G be a graph on n vertices. Let $x,y \in V(G)$ be an arbitrary pair of nonadjacent vertices in G and let G_{n-2} be the induced subgraph $\langle V(G)-\{x,y\}\rangle$ on n-2 vertices. Clearly, $|E(G)| = deg_G(x)+deg_G(y)+|E(G_{n-2})|$. Since $|E(G_{n-2})| \leq \frac{(n-2)(n-3)}{2}$, $deg_G(x)+deg_G(y) \geq |E(G)|-|E(G_{n-2})| \geq n+1$.
By Theorem 2.1.1, G is hamiltonian-connected and the theorem follows. ■

Following Theorem 2.1.2, one might naturally be prompted to investigate which graphs G on $n \geq 4$ vertices satisfying $|E(G)| = \frac{1}{2}(n-1)(n-4)+2$ remain hamiltonian-connected. It will be shown that all such graphs with only a few exceptions are still hamiltonian-connected as Theorem 2.1.3 below indicates. Let $G_6^0 = (V(G_6^0)), E(G_6^0))$ be a graph on 6 vertices defined as follows:- $V(G_6^0) = \{u_1,u_2,u_3,v_1,v_2,v_3\}$, $E(G_6^0) = \{u_i u_j | 1 \leq i < j \leq 3\} \cup \{v_i u_j | i, j=1,2,3\}$. The graph G_6^0 is shown in Figure 2.1. Given

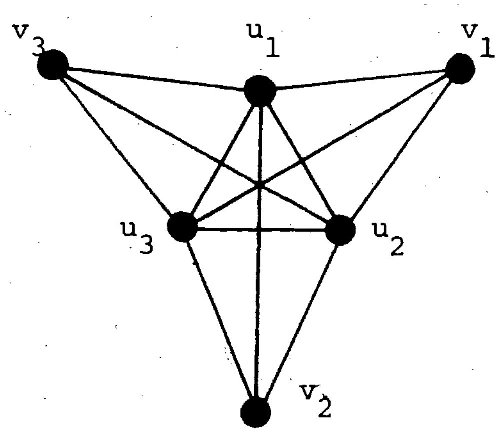


figure 2.1

any $u_i,u_j \in V(G_6^0)$, $i \neq j$, it is easy to show that there exists no hamiltonian $u_i u_j$-path in G_6^0.

<u>Theorem 2.1.3</u>[40] Let $G=(V(G),E(G))$ be a graph on $n\geq 4$ vertices with $\deg(v)\geq 3$ for each $v\in V(G)$. If $|E(G)|=\frac{(n-1)(n-2)}{2}+2$, then G is hamiltonian-connected except when it is isomorphic to G_6^0.

<u>Proof</u>: Let G be a graph satisfying the hypotheses of the theorem and let $x,y\in V(G)$ be a pair of nonadjacent vertices. As in Theorem 2.1.2, let $G_{n-2}=\langle V(G)-\{x,y\}\rangle$. It is then clear that $\deg_G(x)+\deg_G(y)$ $=|E(G)|-|E(G_{n-2})|\geq n$ which still allows Theorem 1.2.13 to guarantee G to be a hamiltonian graph.

By Theorem 2.1.1, G can fail to be a hamiltonian-connected graph only in the case that there exists a pair of nonadjacent vertices $x,y\in V(G)$ such that $\deg_G(x)+\deg_G(y)=n$. This, however, implies that $|E(G_{n-2})|=|E(G)|-(\deg(x)+\deg(y))=\frac{(n-1)(n-2)}{2}+2-n=\frac{(n-2)(n-3)}{2}$. It follows that the subgraph G_{n-2} is isomorphic to the complete graph on n-2 vertices. Hence, G in this case consists of a complete subgraph G_{n-2} on n-2 vertices with the property that n edges from two vertices x,y not in $V(G_{n-2})$ are incident. When either of x or y has degree less than or equal to 2, the graph G is clearly not hamiltonian-connected as indicated in a previous discussion. It remains to show that, with the single exception of G_6^0, the case with $3\leq\deg_G(x)\leq n-3$ and $3\leq\deg_b(y)\leq n-3$ always results in a hamiltonian-connected graph. This implies that $n=\deg_G(x)+\deg_G(y)\geq 6$. Since a total of n edges are incident with n-2 vertices in G_{n-2} and x,y, there are at least two vertices $w_1,w_2\in V(G_{n-2})$ which are adjacent to both x and y.

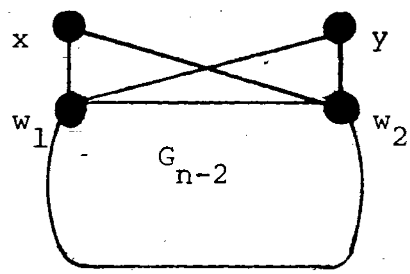


figure 2.2

We proceed to construct a hamiltonian path between any two vertices in G if G is not isomorphic to G_6^0 . There are three cases to be considered.

Case 1 A hamiltonian x,y-path P(x,y):

Let $Q(w_1,w_2)$ be a hamiltonian w_1,w_2-path in G_{n-2}. A hamiltonian x,y-path can be P(x,y): $xw_1Q(w_1,w_2)w_2y$.

Case 2 A hamiltonian x,u-path (or x,y-path) P(x,u) (or P(y,u) for any $u \in V(G_{n-2})$:

Without loss of generality, $W_1 \neq u$ is assumed. Let $w_i \in V(G_{n-2})$ such that $w_i \neq u$ and $w_iy \in E(G)$. Let $Q(w_i,u)$ be a hamiltonian path in the induced subgraph $\langle V(G)-\{x,y,w_1\}\rangle$. Then, a hamiltonian x,u-path can be consturcted by the concatenation

$$P(x,u): xw_1yw_iQ(w_i,u).$$

P(u,u) is constructed similarly.

Case 3 Given any $u,v \in V(G_{n-2})$, a hamiltonian u,v-path in G is constructed as follows provided that $n>6$.

Case 3.1 We first assume that $\{u,v\} \cap (N(x)\ N(y)) = \emptyset$. Under this assumption, a hamiltonian u,v-path can readily be constructed by the concatanation.

$$P(u,v): uw_1xw_2yw_jQ(w_j,v),$$

where $w_j \in V(G)-\{x,y,w_1,w_2\}$ and $yw_j \in E(G)$ and $Q(w_j,v)$ is a hamiltonian w_j,v-path in the induced subgraph $\langle V(G)-\{x,y,w_1,w_2\}\rangle$. We next assume that $\{u,v\}\cap(N(x)\cup N(y))\neq\phi$. Without loss of generality, $u=w_i$ and $w_i\neq w_1$ for some $w_i \in N(X)$. If there exists a vertex $w_j \in V(G_{n-2})$ different from u,v and w_1, one can form a hamiltonian u,v-path with the concatenation

$$P(u,v):u=w_i y w_j Q(W_j,v)$$

where $Q(w_j,v)$ is a hamiltonian w_j,v-path in the induced subgraph $\langle V(G)-\{x,y,w_1,w_i\}\rangle$.

It remains to consider the case where all the edges from y are incident with the vertices w_1,u and v, hence, deg(y)=3. Due to the symmetry of x and y, a hamiltonian u,v-path can also be similarly constructed in G if x does not satisfy these conditions. Finally, we are left with the exceptional case where n=6 and $\deg_G(x)=\deg_G(y)=3$ and N(x)=N(y). This describes precisely the graph isomorphic to G_6^0 which has been shown previously to be not hamiltonian-connected. This completes the proof of the theorem. ■

For a graph G on n vertices, it is now known that $b(n)=\frac{1}{2}(n-1)(n-2)+3$ is the least number of edges sufficient to guarantee that G is hamiltonian-connected. It is therefore most natural for one to determine the least number of edges e(n) a hamiltonian-connected graph necessarily has (that is, e(n) is the greatest integer such that for any graph G=(V(G),E(G) on n vertices, $|E(G)|<e(n)$ implies that G is not hamiltonian-connected).

Theorem 2.1.4 [J.W. Moon,37] The minimum number of edges e(n) a hamiltonian-connected graph on $n\geq 4$ vertices can have is

$[\frac{1}{2}(3n+1)]$.

Proof: Let $G=(V(G),E(G))$ be a graph on n vertices, having less than $[\frac{1}{2}(3n+1)]$ edges. Then, there exists a vertex of degree at most 2. Therefore, G is not hamiltonian-connected. To complete the proof, it remains to construct a hamiltonian-connected graph on $n\geq 4$ vertices with exactly $[\frac{1}{2}(3n+1)]$ edges. We first consider the case where n is odd. Let $n=2m+1$ for $m\geq 2$. Let F_n denote the graph containing two disjoint paths of length m $P_m:p_1p_2\ldots p_m$, and $Q:q_1q_2\ldots q_m$ together with the edges p_iq_i, $i=1,\ldots,m$. An additional vertex x is adjacent the vertices p_1,q_1,p_m,q_m. The graph F_n is shown in Figure 2.3. It can be easily verified that F_n is hamiltonian-connected.

F_n

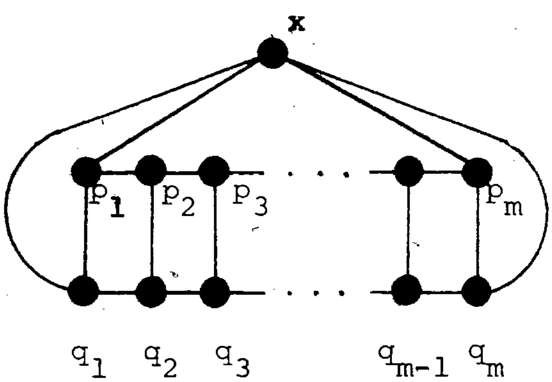


figure 2.3

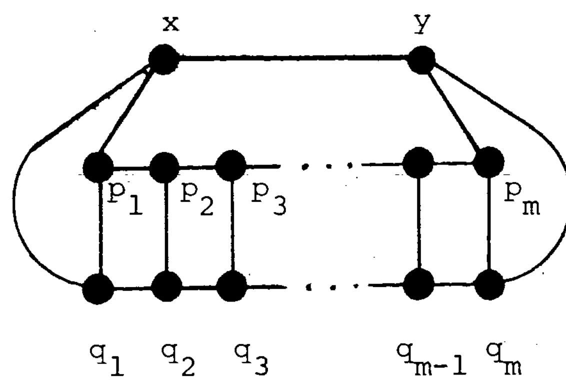


figure 2.4

It remains to consider even values of n. If $n=4$, the only graph on $\left[\frac{(3.4+1)}{2}\right]=6$ edges is the complete graph K_4. If $n=2m+2$, $m\geq 2$, let F_n differ from F_{n-1} in that p_m and q_m are not adjacent to x but to an additional vertex y, where x and y are adjacent (see Figure 2.4). It can be easily verified that F_n in this case is also hamiltonian-connected and this completes the proof. ∎

A sufficient condition for a graph to be hamiltonian-connected which is parallel to Theorem 1.2.15 has been established by D.R. Lick [34] as follows.

Theorem 2.1.5 [34] Let $G=(V(G),E(G))$ be a graph on n vertices with degree sequence $d_1 \le d_2 \le \ldots \le d_n$. Suppose that for each $k \ne \ell$, $d_k \le k+1$ and $d_\ell \le \ell+1$ imply that $d_k + d_\ell \ge n+1$. Then, G is hamiltonian-connected.

Proof: Let $x,y \in V(G)$ be any two distinct vertices. A new graph H is formed from G by introducing an additional vertex z and two new edges zx, zy. Since the degree sequence of (G satisfies the hypotheses of the theorem, the degree sequence of) H clearly satisfies the hypothesis of Theorem 1.2.15. Hence, there exists a hamiltonian cycle C in H containing the edges xz,zy. The segment of C obtained by the deletion of the edges xz,zy from C results in a hamiltonian x,y-path in G. This shows that G is hamiltonian-connected.

Section 2 Topological Conditions

Chvatal and Erdos [15] have established a sufficient condition for hamiltonian-connectedness parallel to Theorem 1.2.11 as follows.

Theorem 2.2.1 [15] Let G be a κ-connected graph with independence number β. If $\kappa-1 \ge \beta$, then G is hamiltonian-connected. The bound for this theorem is sharp.

Proof: Suppose to the contrary that there exists a graph $G=(V(G),E(G))$ which satisfies the hypothesis of the theorem and is not hamiltonian-connected.

Then there exist vertices $u,v,w \in V(G)$ where $w \notin V(P)$ for a longest u,v-path P in G. Since G is κ-connected, there are κ paths starting from w and terminating in P which are pairwise vertex-disjoint apart from the vertex w and share with P their terminal vertices $p_1, p_2, \ldots p_\kappa$ [See theorem 1,18]. Without loss of generality, we may assume that $w_i \neq v$ for $i<\kappa$. Denote the successor (in the direction from u to v) of each p_i $(i<\kappa)$ by u_i. Since G has no κ independent vertices, there is an edge wu_i or $u_i u_j$. In both cases, a path connecting u to v longer that P can be constructed which contradicts the choice of P. This completes the proof. The complete bipartite graph $K_{\kappa,\kappa}$ shows that the bound for β is indeed sharp. ∎

It is interesting to observe how the relationship between κ and β governs the existence of a hamiltonian path or cycle as Theorems 1.2.11, 2.2.1 and 2.2.2 below show.

<u>Theorem 2.2.2</u> [15] Let G be a κ-connected graph with independence number β. If $\kappa+1 \geq \beta$, then G is traceable (that is, there exists a hamiltonian path in G). The bound for β in this case is sharp.

<u>Proof</u>: Let G be a κ-connected graph which satisfies $\kappa+1 \geq \beta$. A new graph H is formed from G by introducing a new vertex x and joining to it all vertices of G. Then, the graph H clearly satisfies the hypothesis of Theorem 1.2.11 with $\kappa+1$ in place of κ. Therefore, there exists a hamiltonian cycle in H. This implies that G is traceable. The complete bipartite graph $K_{\kappa+2,\kappa}$ shows that the bound for β is sharp. ∎

In what follows, it will be shown how locally m-connectedness contributes to a sufficient condition for a graph to be hamiltonian-connected.

<u>Definition 2.2.3</u> For $m \geq 1$, a graph $G=(V(G),E(G))$ is said to be locally m-connected if the induced subgraph $\langle N(u) \rangle$ is m-connected

for each $u \in V(G)$.

There are various articles on the hamiltonian properties of graphs which concern connectedness and local connectedness.

Theorem 2.2.4 [12] If G is a connected, locally connected (locally 1-connected) graph with maximum degree $\Delta(G) \leq 4$, then G is either a hamiltonian graph or isomorphic to the tripartite graph $K_{1,1,3}$.

Theorem 2.2.5 [38] If G is a connected, locally-connected graph on at least 3 vertices which contains no induced subgraph isomorphic to the bipartite graph $K_{1,3}$, then G is hamiltonian.

G. Chartrand, R.J. Gould and A.D. Polimeni [9] have established a sufficient condition for a graph to be hamiltonian-connected by employing hypotheses similar to that of Theorem 2.2.5. A preliminary lemma is required to establish this sufficient condition.

Lemma 2.2.6 [9] If G is a connected, locally m-connected graph, $m \geq 1$, then G is $(m+1)$-connected.

Proof: We proceed by induction on m. Suppose that there exists a graph G which is both connected and locally connected and is not 2-connected. Then, there exists a cut vertex $u \in V(G)$ such that $\langle V(G)-\{u\} \rangle$ is disconnected. Let $C_1, C_2, \ldots C_t$, $t \geq 2$, be the components of $\langle V(G)-\{u\} \rangle$. Since for each i, $1 \leq i \leq t$, $V(C_i) \cap N(u) \neq \phi$, $\langle N(u) \rangle$ is necessarily disconnected which contradicts the hypothesis on G. Hence, G is 2-connected.

We next assume that for some $k \geq 1$, it has been established for each ℓ satisfying $1 \leq \ell \leq k$, that every graph which is connected and locally ℓ-connected is necessarily $(\ell+1)$-connected. Let G be a graph which is connected and locally $(k+1)$-connected.

For any vertex $v \in V(G)$, $\langle V(G)-\{v\} \rangle$ is clearly connected and locally k-connected. Since v has been arbitrarily chosen, $\langle V(G)-\{v\} \rangle$ is by the induction hypothesis, k+1-connected and the result follows. ■

We are now in the position to investigate the sufficient condition mentioned above.

<u>Theorem 2.2.7</u> [9] Let G be a connected, locally 3-connected graph which contains no induced subgraph isomorphic to the bipartite graph $K_{1,3}$. Then, G is hamiltonian-connected.

<u>Proof</u>: Suppose to the contrary that there exists a graph $G=(V(G),E(G))$ which satisfies the hypotheses of the theorem and is not hamiltonian-connected. By Lemma 2.2.6, G is 4-connected. This implies that G is ℓ-connected for $1 \leq \ell \leq 4$. Let $u,v \in V(G)$ be such that no hamiltonian u,v-path exists. Since G is 2-connected, there exists two internally disjoint u,v-paths in G (see Theorem 3.2,[5]). This implies the existence of a u,v-path of length at least 2 in G. Let $P: u=u_0 u_1 \ldots u_m = v$, $m \geq 2$, be a u,v-path of maximum length in G. Since G is a connected graph, there is a vertex $x \in (V(G)-V(P))$ which is adjacent to some vertex u_i on P, $0 \leq i \leq m$. One can in fact assume without loss of generality that $0<i<m$; for otherwise, if each vertex $y \in V(G)-V(P)$ is adjacent to only the terminal vertices $u_0=u$ and $u_m=v$ on P, then $\langle V(G)-\{u,v\} \rangle$ is a disconnected graph and this would contradict the assumption that G is 3-connected. Since G is locally 3-connected, the induced subgraph $\langle N(u_i) \rangle$ contains an x,u_{i+1}-path Q which contains neither u nor v as internal vertices. Furthermore, the path Q either does not contain the vertex u_{i-1} or

it contains an x, u_{i-1} subpath of Q not containing u_{i+1}. By symmetry, there is no loss in generality to assume that Q does not contain the vertex u_{i-1}. Clearly, x can be adjacent to neither u_{i-1} nor u_{i+1}; for otherwise, a u,v-path of length m+1 can be constructed in G which contradicts the choice of P. Since the induced subgraph $\langle\{x, u_{i-1}, u_i, u_{i+1}\}\rangle$ cannot be isomorphic to the bipartite graph $K_{1,3}$, $u_{i-1}u_{i+1} \in E(G)$. In addition, if $V(P) \cap V(Q) = \{u_{i+1}\}$, then a u,v-path of length greater than m can be constructed in G. Hence, $V(P) \cap V(Q) \neq \{u_{i+1}\}$ is assumed.

A vertex $w \in (V(P) \cap V(Q)) - \{u_{i+1}\}$ is said to be a singular vertex if no vertex in $N(w) \cap V(P)$ is adjacent to u_i. Since $V(Q) \subseteq N(u_i)$, for any singular vertex w and its neighbouring vertices w_1, w_2 on P, $\{w_1, w_2\} \cap V(Q) = \phi$. Also, $w_1w_2 \in E(G)$ since u_i, w, w_1, w_2 cannot be isomorphic to $K_{1,3}$. It remains to consider the following two cases.

Case 1 Each vertex in $(V(P) \cap V(Q)) - \{u_{i+1}\}$ is a singular vertex.

In such a case, for each $w \in (V(P) \cap V(Q)) - \{u_{i+1}\}$ and the two neighbouring vertices w_1, w_2 of w on P. $w_1w_2 \in E(G)$ is satisfied. We proceed to construct a u,v-path of length greater than m accordingly. Starting with the initial vertex u, traverse the path P where for each vertex $u_j \in V(P) \cap V(Q)$, $j < i$, we bypass u_j using the edge $u_{j-1}u_{j+1}$. This process is continued until u_i is reached. From u_i proceed to the vertex x along the edge u_ix and then subsequently along the path Q t. u_{i+1}. Then, from u_{i+1}, proceed along P and for each u_k $V(P) \cap V(Q)$, $k > i+2$, and bypass the vertex u_k using the edge $u_{k-1} u_{k+1}$ until v is reached. The resulting u,v-path contains all vertices u_i, $0 \leq i \leq n$, and the vertex x. This contradicts the choice of P.

Case 2 $(V(P)\cap V(Q))-\{u_{i+1}\}$ contains nonsingular vertices. Let u_k be the first nonsingular vertex incountered if Q is traversed in the direction from x to u_{i+1}. Either $u_{k-1}u_i \in E(G)$ or $u_{k+1}u_i \in E(G)$ must be satisfied. Without loss of generality, $u_{k-1}u_i \in E(G)$ is assumed. The path P can now be replaced by a different u,v-path P* of length m as follows.

If $k<i$, then set

$$P^*: u_0u_1\dots u_{k-1}u_iu_ku_{k+1}\dots u_{i-1}u_{i+1}u_{i+2}\dots u_m.$$

If $k>i$, then set

$$P^*: u_0u_1\dots u_{i-1}u_{i+1}\dots u_{k-1}u_iu_ku_{k+1}\dots u_m.$$

In either case consider the x,u_k subpath Q* of the path Q. By the choice of u_k, $(V(P^*)\cap V(Q^*))-\{u_k\}$ does not contain nonsingular vertices. A construction similar to that in Case 1 allows a u,v-path of length greater than m to be produced. This contradiction completes the proof of the theorem. ■

At this point, it is important to observe that there are graphs which satisfy the hypotheses of Theorem 2.2.5 and are not hamiltonian-connected. For example, for each $n\geq 3$, the graphs $K_2+(K_{n-2}\cup K_2)$ satisfy these properties.

However, it still is unknown whether or not Theorem 2.2.7 is the best possible in the sense that it might be possible to replace the condition "locally 3-connected" by "locally 2-connected."

We next investigate a necessary condition for a graph to be hamiltonian-connected expressed in terms of toughness [36]. As indicated in Chapter 1, every hamiltonian graph is 1-tough. It is natural to attempt to determine a similar condition for hamiltonian-connected graphs.

Theorem 2.2.8 [36] If G is a hamiltonian-connected graph, then the toughness of G satisfies $t(G)>1$. This is best possible in the sense that there exists a sequence of hamiltonian-connected graphs $\{G_n\}_{n=1}^{\infty}$ with the property that $\lim_{n\to\infty} t(G_n)=1$.

Proof: Let G be a hamiltonian-connected graph. Hence, G is hamiltonian and $t(G)=\min_S \frac{S}{c(\langle V(G)-S\rangle)}\geq 1$, where the minimum is taken over the cut sets of G. To show that $t(G)\neq 1$, suppose the contrary and let $S\subset V(G)$ be a cut set with $|S|=c(\langle V(G)-S\rangle)=k>1$. Let $S=\{x_1,x_2,\ldots,x_k\}$ and let $C_1,C_2,\ldots,C_k$ be the k components of $\langle V(G)-S\rangle$. Suppose that P is a hamiltonian x_1,x_k-path in G. Without the loss of generality, let the labelling of the vertices in S be so arranged that $x_1,x_2,x_3,\ldots,x_k$ are precisely in that order as the hamiltonian x_1,x_k-path P is traversed. For $i\neq j$, there exists no edge between C_i and C_j. Hence, for each i, $1\leq i\leq k-1$, every vertex between x_i and x_{i+1} on P belongs to a single component C_ℓ, $1\leq \ell\leq k$. This implies that P can contain vertices of at most k-1 of the components $C_1,C_2,\ldots,C_k$. This contradicts the choice of P and this completes the first part of the proof.

To show that this result is best possible, for each $n\geq 3$ let G_n be the graph on 2n+1 vertices as defined by J.W. Moon in Theorem 2.1.4 (See Figure 2.3 and 2.5).

It can be easily shown that $t(G_n)=\frac{n+1}{n}$ and this completes the proof. ■

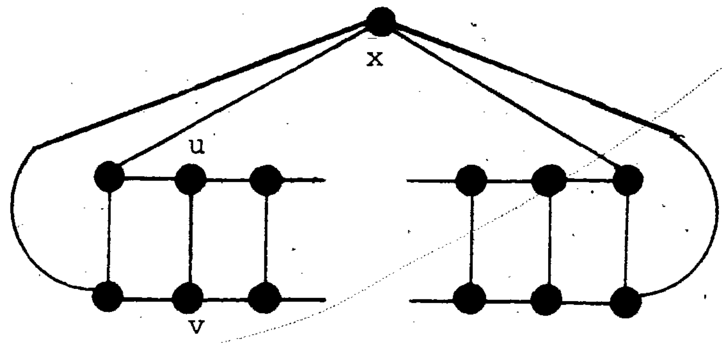


figure 2.5

Note that the converse of Theorem 2.2.8 is false as the following example shows. For each $n \geq 3$, let H_n be the graph obtained from G_n defined above by removing the vertices u,v. Then for each $n \geq 3$, H_n is not hamiltonian-connected and $t(H_n) = \frac{n+1}{n}$.

The combination of Theorems 1.2.5, 2.2.1 and 2.2.8 indeed have shed some light on the relationship between the connectivity and the hamiltonian-connectedness of a graph.

Let G be a graph with connecting κ, independence number β and toughness t(G). There are three possibilities for the graph G:

1. $1 < \frac{k}{\beta} \leq t(G)$
2. $\frac{k}{\beta} < 1 \leq t(G)$
3. $\frac{k}{\beta} \leq t(G) < 1$

One can at least conclude the following.

By Theorem 2.2.8, G is not hamiltonian-connected if 3. is satisfied and by Theorem 2.2.1, G is a hamiltonian-connected graph if 1. is satisfied.

Section 3 r-hamiltonian-connected graphs

A natural extension of the concept of a hamiltonian-connected graph is that of an r-hamiltonian-connected graph, $r \geq 0$.

Definition 2.3.1 A graph $G=(V(G),E(G))$ is said to be r-hamiltonian-connected if for each $S \subseteq V(G)$, with $|S| \leq r$, the induced subgraph $\langle V(G)-S \rangle$ is hamiltonian-connected. It is clear that the 0-hamiltonian-connected graphs are simply the hamiltonian-connected graphs.

An immediate observation is that for any graph $G=(V(G),E(G))$ with a cut set of 2 vertices $\{u,v\}$ in $V(G)$, there can be no hamiltonian u,v-path in G. This implies that a hamiltonian-connected graph is at least 3-connected. This for each $r \geq 0$, an r-hamiltonian-connected graph on $n \geq 4$ vertices is necessarily $(r+3)$-connected. If κ is the connectivity of G, then the minimum degree $\delta(G)$ of G satisfies $\delta(G) \geq \kappa \geq r+3$. It follows that the minimum number of edges that G can have satisfies $|E(G)| \geq \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq \frac{1}{2} n(\delta(G)) \geq \frac{n(r+3)}{2}$.

It is interesting to observe that many of the degree and edge sufficiency conditions for a graph to be hamiltonian-connected can be generalized in a similar manner with amazingly minor modifica--tions to cover r-hamiltonian graphs as D.R. Lick [35] has shown. A theorem on degree sufficiency conditions for a graph to be r-hamiltonian connected parallel to Theorems 2.1.1 and 1.2.13 is as follows.

Theorem 2.3.1 [35] Let $G=(V(G),E(G))$ be a graph on $n \geq 4$ vertices such that for every nonadjacent pair of vertices **u** and **v** in $V(G)$, $\deg_G(u)+\deg_G(v) \geq n+r+1$, with $0 \leq r \leq n-4$. Then, G is r-hamiltonian-connected.

Proof: The case where $r=0$ is simply Theorem 2.1.1.

It remains to investigate the cases where $1 \leq r \leq n-4$. Let $\{v_1,\ldots,v_k\} \subset V(G)$ be an arbitrary set of k vertices in G, $0 \leq k \leq r$. Define $G^* = \langle V(G)-\{v_1,\ldots,v_k\}\rangle$ and for each vertex $u \in V(G^*)$, denote the degree of U in G and G* by $deg_G(u)$ and $deg_{G^*}(u)$, respectively. Let u,v be a pair of nonadjacent vertices in G*. Clearly, $uv \notin E(G)$. By the hypothesis of the theorem, $deg_G(u)+deg_G(v) \geq n+r+1$. Therefore,

$$\begin{aligned} deg_{G^*}(u)+deg_{G^*}(v) &\geq n+r+1-2k=(n-k)+(r-k)+1 \\ &= |V(G^*)|+r-k+1 \\ &\geq |V(G^*)|+1 . \end{aligned}$$

Hence, by Theorem 2.1.1, G* is hamiltonian-connected and this shows that G is r-hamiltonian-connected. ■

The sharpness of the bound in Theorem 2.3.1 is given by the following example. Let $K_{3,3,3}$ be the complete tripartite graph on 3n vertices, $n \geq 2$. Let $V(K_{3,3,3})=V_1 \cup V_2 \cup V_3$ be the partitions of $K_{3,3,3}$. (That is, $x \in V_i$ and $y \in V_j$ imply that $xy \in E(K_{3,3,3})$ if and only if $i \neq j$). Note that for each pair of vertices $u,v \in V(K_{3,3,3})$, $deg(u)+deg(v)=4n=3n+n$. It is clear that $K_{3,3,3}$ is not n-hamiltonian-connected since for each $i=1,2,3$, $\langle V(K_{3,3,3})-V_i\rangle$ is easily shown not to be hamiltonian-connected.

A moment of reflection allows Theorem 1.2.14 to be related in the following manner.

Theorem 1.2.14a Let $G=(V(G),E(G))$ be a graph on $n \geq 3$ vertices such that for each j, $i \leq j \leq \frac{n}{2}$, the number of vertices with degree not exceeeding j is less than or equal to j-2, that is,

$|\{v \in V(G) \mid deg(v) \leq j\}| \leq j-1$. Then, G is hamiltonian.

A sufficient condition for a graph to be hamiltonian-connected very much in the flavor of Theorem 1.2.14a is now discussed.

Lemma 2.3.2 [35] Let $G=(V(G),E(G))$ be a graph on $n \geq 4$ vertices such that for each j, $2 \leq j \leq \frac{n}{2}$, $|\{v \in V(G) \mid deg(v) \leq j\}| \leq j-2$. Then, G is hamiltonian-connected.

Proof: Let G be a graph that satisfies the hypotheses of the theorem. Note that G is hamiltonian since it also satisfies the hypotheses of Theorem 1.2.14.a. Let $u,v \in V(G)$ be any two distinct vertices in G. We proceed to construct a hamiltonian u,v-path. There are two cases to be considered.

Case 1 $uv \in E(G)$.

Let ℓ be the maximum value of the lengths of paths in G which contains the edge uv. Let $P: u_1u_2 \ldots u_{\ell+1}$ be a path of length ℓ containing the edge uv, where $u_j = u$ and $u_{j+1} = v$ with $1 \leq j \leq \ell$, such that the sum of the degrees of the initial vertex u_1 and the terminal vertex $u_{\ell+1}$ is maximum. Let $S=\{u_i \in V(P) \mid u_1u_{i+1} \in E(G)\}$. Since no path in G containing the edge uv can have length greater than ℓ, $N(u_1) \subseteq V(P)$. Also, for each $u_i \in S$, $P^*: u_iu_{i-1} \ldots u_1u_{i+1}u_{i+2} \ldots u_{\ell+1}$ is a path of length ℓ which contains the edge uv whenever $i \neq j$. By the choice of P, it is clear that for each $i \neq j$ and $u_i \in S$, $deg_G(u_i) \leq deg_G(u_1)$. Thus, there are at least $(deg_G(u_1)-1)$ vertices in S with degrees not exceeding $deg_G(u_1)$. By the hypotheses of the theorem, $deg_G(u_1) > \frac{n}{2}$. A similar argument shows that $deg_G(u_{\ell+1}) > \frac{n}{2}$. Therefore, $deg_G(u_1)+(u_{\ell+1}) > n$ and this implies that $|N(u_{\ell+1}) \cap S| \geq 2$, in which case there exists at least one vertex $u_i \in N(u_{l+1}) \cap S$ with $i \neq j$. Hence, $C: u_i u_{i-1}, \ldots u_1 u_{i+1} u_{i+2}, \ldots u_{\ell+1} u_i$ is a cycle of length $\ell+1$.

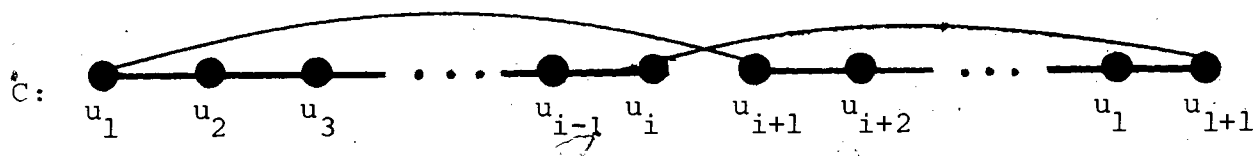


figure 2.6

The proof of Case 1 is complete if n= +1. It remains to consider the case where $\ell+1<n$. Let the vertices of the cycle C be relabelled as $C: v_1 v_2 \ldots v_{\ell} v_{\ell+1} v_1$. Since G is a hamiltonian graph, there is a vertex w V(G) adjacent to some vertex $v_h \in V(C)$. Then, it is clear that at least one of the following two paths P_1, P_2

$$P_1: w v_h v_{h+1} \ldots v_{\ell} v_{\ell+1} v_1 v_2 \ldots v_{h-1}$$

$$P_2: w v_h v_{h-1} \ldots v_1 v_{\ell+1} v\ v_{\ell-1} \ldots v_{h+1}$$

is a path of length $\ell+1$ which contains the edge uv. This contradicts the choice of P.

Case 2 $uv \notin E(G)$.

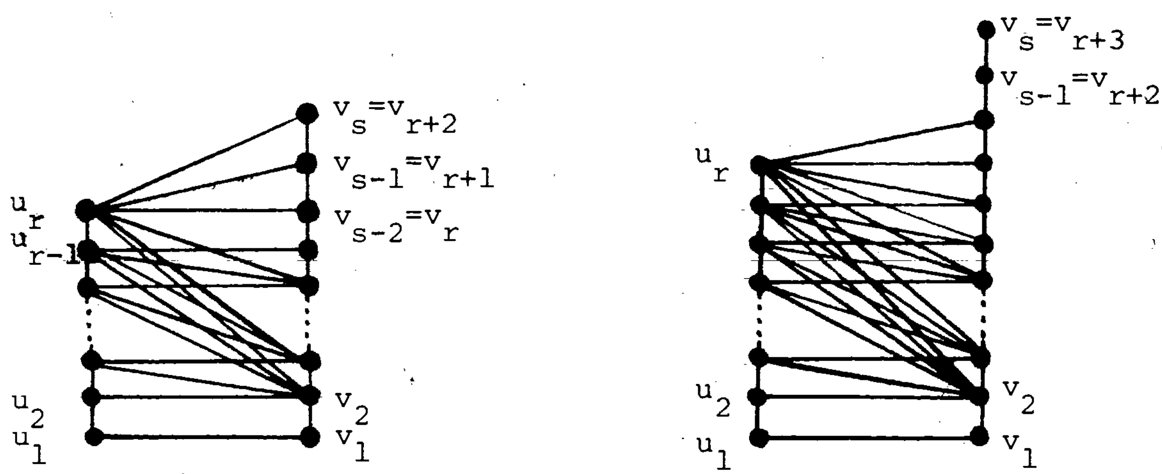
Define a new graph $G^*=G+uv$ (that is, $V(G)=V(G^*)$ and $E(G^*)=E(G) \cup \{uv\}$). Clearly, G^* satisfies the hypothesis of the theorem and the argument in Case 1 asserts that the edge uv in G^* is contained in a hamiltonian cycle. Hence, there exists a hamiltonian u,v-path in G and this completes the proof. ■

It is worthwhile to mention that a graph is said to be edge-hamiltonian if every edge of the graph is contained in a hamiltonian cycle. Thus, a graph which satisfies the hypotheses of Lemma 2.3.2 is necessarily edge-hamiltonian.

Lemma 2.3.2 above is needed to prove the following result for r-hamiltonian connected graphs.

<u>Theorem 2.3.3</u> [34] Let $G=(V(G),E(G))$ be a graph on $n\geq 4$ vertices such that for each j, $r+2\leq j\leq \frac{n+r}{2}$, $|\{v\in V(G)\,|\,\deg(v)\leq j\}|\leq j-2-r$. Then, G is r-hamiltonian-connected.

<u>Proof</u>: Let G be a graph which satisfies the hypotheses of the theorem. Lemma 2.3.2 yields the desired result for the case where $r=0$. It remains to investigate the case where $1\leq r\leq n-4$. Let $0\leq k\leq r$ and let $S=\{v_1,\ldots,v_k\}\subset V(G)$ be a set of k arbitrary vertices in G. Consider the induced subgraph $G^*=\langle V(G)-S\rangle$ on $n^*=n-k$ vertices. To verify that G^* is a hamiltonian-connected graph, let j be such that $2\leq j\leq \frac{n^*}{2}$ and $m=|\{v\in V(G^*)\,|\,\deg_{G^*}(v)\leq j\}|$. Clearly for each $y\in\{v\in V(G)\,|\,\deg_{G^*}(v)\leq j\}$, $\deg_G(y)\leq j+k$. Observe that $2+k\leq j+k\leq \frac{n^*}{2}+k=\frac{n+k}{2}\leq\frac{n+r}{2}$. If $j+k<2+r$, then $\{v\in V(G^*)\,|\,\deg_{G^*}(v)\leq j\}\subseteq\{v\in V(G)\,|\,\deg_G(v)\leq j+k\}\subseteq\{v\in V(G)\,|\,\deg_G(v)\leq r+2\}$. By the hypotheses of the theorem, $|\{v\in V(G)\,|\,\deg_G(v)\leq r+2\}|\leq(r+2)-(r+2)=0$. This implies that $m=0$. Otherwise, for $2+r\leq j+k\leq\frac{n+r}{2}$ $m<j+k-r-1\leq j-1$. Hence, G^* is hamiltonian-connected. It follows that G is r-hamiltonian-connected. ■



<u>figure 2.7</u>

The bounds in both Lemma 2.3.2 and Theorem 2.3.3 are sharp as Figure 2.7 shows. The graph which demonstrates the sharpness of the bound in Lemma 2.3.2 is schematically illustrated in Figure 2.7.

For each $n>7$, define a graph G_n on n vertices as follows. Let P_r be a path on r vertices, where $r=\left[\frac{n}{2}\right]-1$, and K_s be a complete graph on s vertices with $s=\frac{n+3}{2}$. Let $V(P_r)=\{u_1,\ldots,u_r\}$ and $V(K_s)=\{v_1,\ldots,v_s\}$, respectively. The graph G_n is constituted accordingly by defining $V(G_n)=V(P_r)\cup V(K_s)$ and $E(G_n)=E(P_r)\cup E(K_s)\cup\{u_1v_1, u_rv_{r+1}\}\cup\{u_iv_j \mid 2\leq j\leq i\leq r\}$ if n is odd. If n is even, then the edge u_rv_{r-1} in the previous set is replaced by u_rv_{r+2}. It can be easily shown that for each j, $2\leq j\leq\frac{n}{2}$, the number of vertices of degree not exceeding j in G_n is exactly j-1. However, G_n is not hamiltonian-connected since $\deg_{G_n}(u_1)=2$ even though it is a hamiltonian graph. This verifies the sharpness of the bound in Lemma 2.3.2.

In order to construct a graph which demonstrates the sharpness of Theorem 2.3.3, a preliminary lemma is required. We first review the definition of the join G_1+G_2 of two graphs G_1 and G_2 as defined in [8, 47].

<u>Definition 2.3.4</u> Let $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$ be two graphs. The <u>join</u> $G_1+G_2=(V(G_1+G_2),E(G_1+G_2))$ is defined to be the graph where

$V(G_1+G_2)=V(G_1)\cup V(G_2)$

$E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{xy \mid x\in V(G_1) \text{ and } y\in V(G_2)\}$.

In particular, if one of the graphs G_1 and G_2, is isomorphic to K_1, then G_1+G_2 is denoted by G_1+v.

<u>Lemma 2.3.5</u> If G is an (r-1)-hamiltonian-connected graph, then G+v is r-hamiltonian-connected.

Proof: Let G be an (r-1)-hamiltonian-connected graph on n vertices and $S=\{x_1,\ldots,x_{r-1}\}$ be an arbitrary subset of r-1 vertices in G.

Clearly, $G'=\langle V(G)-S\rangle$ is hamiltonian-connected and G' is isomorphic to $G''=\langle V(G+v)-\{v,x_1,\ldots,x_{r-1}\}\rangle$. Hence, G" is hamiltonian-Connected. Let y_1,y_2,y_3 be any three vertices in G" and $P:y_1=v_1v_2\ldots v_{n-r+1}=y_3$ be a hamiltonian y_1,y_3-path in G", where $y_2=v_k$. Then, $Q:y_1=v_1v_2\ldots v_{k-1}vv_{k+1}v_{k+2}\ldots u_{n-r+1}=y_3$ is a hamiltonian y_1,y_3-path, in $G'''=\langle V(G+v)-\{y_2,x_1,x_2,\ldots x_{r-1}\}\rangle$. This implies that that the graph G+v is r-hamiltonian-connected. ■

Note that for any hamiltonian graph G and an additional vertex v, the join G+v contains a spanning subgraph isomorphic to the wheel graph on $|V(G+v)|$ vertices. Hence, G+v is PLD-maximal. In particular, it is hamiltonian-connected.

We are now in a position to construct the following example to demonstrate the sharpness of the bound in the inequality of Theorem 2.3.3.

Let G_n (see Figure 2.7) be the graph defined in the example preceding.

Definition 2.3.4. For a fixed integer r, let $\{x_1,\ldots,x_r\}$ be r additional vertices. By lemma 2.3.5 and the preceding discussion it is clear that the graph $G=(\ldots(((G_n+x_1)+x_2)+x_3)+\ldots+x_{r-1})+x_r$ is (r-1)-hamiltonian-connected. However, G is not r-hamiltonian-connected since it has a vertex of degree r+2. Also, it can be easily shown that for each j, $r+2\leq j\leq \frac{n+r}{2}$, there are exactly j-r-1 vertices of degree not exceeding j. This example consolidates the sharpness of Theorem 2.3.3.

The sufficiency conditions for a graph to r-hamiltonian-connected expressed in terms of the number of edges in the graph is the next most natural generalization of Theorem 2.1.2.

We first characterize the (n-4)-hamiltonian-connected graphs on $n\geq 4$ vertices.

Theorem 2.3.6 [35] Let G be a graph on $n\geq 4$ vertices. Then G is (n-4)-hamiltonian-connected if and only if G is isomorphic to the graph K_n.

Proof: Let G be a graph on $n\geq 4$ vertices. If $G\cong K_n$, then the removal of k vertices from G, $0\leq k\leq n-4$, results in a graph isomorphic to the complete g aph on n-k vertices K_{n-k}. Hence, G is (n-4)-hamiltonian-connected. Conversely, suppose that G is not complete. There exists a vertex $v\in V(G)$ such that $\deg_G(v)\leq n-2$. The removal of n-4 vertices adjacent to v results in a graph G' with $\deg_G(v)\leq 2$. Hence, G' is not hamiltonian-connected and G is not (n-4)-hamiltonian-connected. ■

The generalization of Theorem 2.1.2 can now be established.

Theorem 2.3.7 [35] Let $G=(V(G),E(G))$ be a graph on $n\geq 4$ vertices such that $|E(G)|\geq \frac{(n-1)(n-2)}{2}+3+r$, for $0\leq r\leq n-4$. Then G is r-hamiltonian-connected.

Proof: For r=0, Theorem 2.1.2 gives the desired result. If r-n-4, then $|E(G)|=\frac{(n-1)(n-2)}{2}+3+(n-4)=\frac{n(n-1)}{2}$, in which case G is isomorphic to K_n which is consistent with Theorem 2.3.6. It remains to consider the case $0<r<n-4$. Let $0\leq k\leq r$ and let $S=\{v_1,v_2,\ldots,v_k\}$ be an arbitrary set of vertices in G. Let $G^*=\langle V(G)-S\rangle$ on $n^*=n-k$ vertices. Since at most $\frac{k(k-1)}{2}+k(n-k)$ edges have been removed from

G to form G*. $|E(G^*)| \geq \frac{(n-k-1)(n-k-2)}{2}+(r-k)+3 \geq \frac{(n^*-1)(n^*-2)}{2}+3$.

Hence G* satisfies the hypotheses of Theorem 2.1.2 and is, therefore, hamiltonian-connected. It now follows that G is r-hamiltonian-connected. ■

Indeed, the bound in Theorem 2.3.7 is sharp as the graph in the following example shows. For each $n \geq 5$ and $0 \leq r \leq n-4$, a graph G on n vertices and $\frac{(n-1)(n-2)}{2}+r+2$ edges is constituted by a complete graph K_{n-1} and an additional vertex v adjacent to (r+2) vertices $\{v_1, v_2, \ldots, v_{r+2}\} \subseteq V(K_{n-1})$. It is then clear that G has $\frac{(n-1)(n-2)}{2}+r+2$ edges and the graph $G' = \langle V(G) - \{v_1, \ldots, v_{r-1}, v_r\} \rangle$ has $n-r \geq 4$ vertices including the vertex v of degree 2 in G'. Hence, G' is not hamiltonian-connected.

As Theorem 2.3.6 suggests, for sufficiently large values of r, graphs which are r-hamiltonian-connected can be characterised without a great deal of difficulty. So it is perhaps most appropriate to close this chapter by presenting a characterization of graphs on $n \geq 5$ vertices which are (n-5)-hamiltonian-connected.

Theorem 2.3.8 [35] Let G=(V(G),E(G)) be a graph on $n \geq 5$ vertices. Then, G is (n-5)-hamiltonian-connected if and only if G is a complete graph from which a set of mutually nonadjacent edges has been deleted.

Proof: Let G be a complete graph on $n \geq 5$ vertices with two adjacent edges xu, yu removed. Then, $\deg_G(u) \leq n-3 = (n-5)+2$. Hence, the removal of n-5 vertices adjacent to u in G results in a graph G' such that $\deg_{G'}(u) \leq 2$. Therefore G' connot be hamiltonian-connected and it follows that G is not (n-5)-hamiltonian-connected.

Conversely, if G is a complete graph from which a set of mutually nonadjacent edges has been removed, then $\delta(G) \geq n-2$. In particular, for each pair of nonadjacent vertices $u,v \in V(G)$, $\deg(u) + \deg(v) \geq (n-2)+(n-2)=n+(n-5)+1$. By Theorem 2.3.1, G is (n-5)-hamiltonian-connected and the theorem now follows. ∎

It is now clear that for each $n \geq 4$, there is uniquely one (n-4)-hamiltonian-connected graph up to isomorphism; and for each $n \geq 5$, there are precisely $1+\left[\frac{n}{2}\right]$ non-isomorphic (n-5)-hamiltonian-connected graphs up to isomorphism.

CHAPTER 3

ON THE HAMILTONIAN-CONNECTEDNESS OF CAYLEY GRAPHS OF A FINITE ABELIAN GROUP

The study of the hamiltonian properties of Cayley graphs is important; especially because a Cayley graph is a vertex-transitive graph. L. Lovasz in 1968 conjectured that every connected vertex-transitive graph contains a hamiltonian path. This conjecture has been verified for vertex-transitive graphs on a prime number of vertices by J. Turner. The validity of this conjecture in general, however, remains far from being settled. Furthermore, it has been conjectured that every connected Cayley graph is hamiltonian. It is hoped that the studies of the hamiltonian properties of Cayley graph will provide some helpful insight into constructing at least certain partial solutions to the conjecture of L. Lovasz and related problems. In this chapter, a characterization of the hamiltonian-connectedness of a connected Cayley graph of an abelian group will be presented based on the work of C.C. Chen and N.F. Quimpo [42].

Section 3.1 Cayley graphs

Definition 3.1.1 Let U be a group with the identity element e. A subset $S \subseteq U$ is said to be a symbol for U if the following two conditions are satisfied:

(1) $e \notin S$.

(2) For each $s \in U$ satisfying $s^2 \neq e$, $s \in S$ implies $s^{-1} \notin S$.

Let $S^{-1}=\{s^{-1} | s \in S\}$. It is clear that the subset consisting of the set of all finite products of elements in $S \cup S^{-1}$, denoted by $\langle S \rangle$, is a subgroup in U.

Definition 3.1.2 Let S be a symbol for a group U. S is said to generate U is and only if $U=\langle S \rangle$.

Definition 3.1.3 Let S be a symbol for a group U. An element $s \in S$ is said to be redundant if and only if $\langle S-\{s\} \rangle = \langle S \rangle$. S is called a minimal symbol if it contains no redundant element.

Based on the concepts introduced in the last few definitions, the definition of a Cayley graph of a group can now be given.

Definition 3.1.4 Let U be a group and $S \subseteq U$ be a symbol for U. A Cayley graph of U with respect to S, denoted by C(S,U), is the undirected graph whose vertices are the elements of U and whose edge-set is described below. Without ambiguity, one can write $V(C(S,U))=U$. Any two distinct vertices $x,y \in V(C(S,U))$ are joined by an edge if and only if $s \in S$ and either $s=x^{-1}y$ or $s=y^{-1}x$ (that is, $x^{-1}y \in S \cup S^{-1}$).

All the Cayley graphs studied in this chapter are assumed to be finite. In particular, an abelian group will be denoted by A. It is instructive to consider an example of a Cayley graph of an abelian group.

Example 3.1.5 Consider the abelian group $A=Z_2 x Z_2 x Z_3$ and a minimal symbol $S=\{(1,0,0),(0,1,0),(0,0,1)\}$.

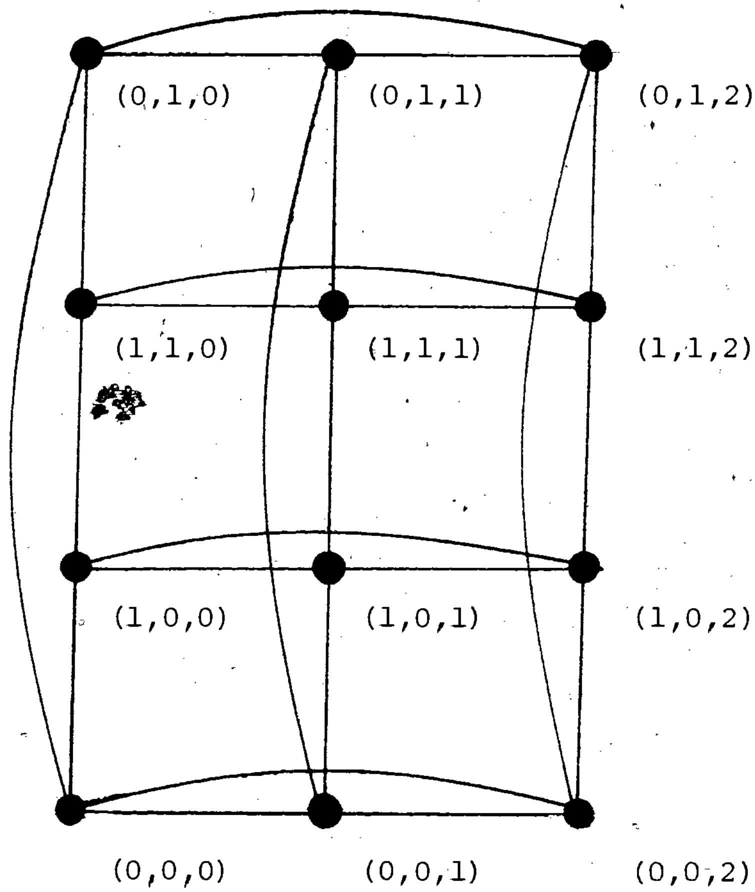


figure 3.1

Note that the Cayley graph C(S,A) in this above example is connected and A=⟨S⟩. In general, it is clear that for any group U and a symbol $S \subseteq U$, the associated Cayley graph C(S,U) is connected if and only if U=⟨S⟩.

Definition 3.1.5 (Bondy and Murty [5])

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. The graph G_1xG_2 consists of the vertex set $V(G_1xG_2)=V_1xV_2$ such that for all (x_1,x_2), $(y_1,y_2) \in V(G_1xG_2)$, (x_1,x_2) is adjacent to (y_1,y_2) if and only if $x_1=y_1$ and $x_2y_2 \in E_2$ or $y_1=y_2$ and $x_1x_2 \in E_1$.

For any integers $n \geq 1$ and $m \geq 2$, let L_n be a path on n vertices and C_m be a cycle on m vertices, respectively. The Cayley graph as shown in Figure 3.1 contains a spanning subgraph isomorphic to C_3xL_4. This is suggestive of the possibility that a similar spanning subgraph H isomorphic to $C_m x L_n$, satisfying $n \geq 1$ and $m \geq 2$, is in general contained in a connected Cayley graph of an abelian group of order mn. In general, this is due to the fact that a finite abelian group A has a decomposition $A = Z_{m_1} x \ldots x Z_{m_n}$, for some collection of positive integers $\{m_1, \ldots, m_n\}$ satisfying $|A| = m_1 \ldots m_n$. It will be shown in the next section that the hamiltonian-connectedness of a connected Cayley graph of an abelian group is due to the existence of appropriate hamiltonian paths in such spanning subgraphs. It is clear that $L_m x L_n$ is a spanning subgraph of $C_m x L_n$ and $C_m x L_n$ is a spanning subgraph of $C_m x C_n$. We now proceed to study the hamiltonian properties of $L_m x L_n$, $C_m x L_n$ and $C_n x C_m$ and their applications to the characterization of the hamiltonian-connectedness of a Cayley graph of an abelian group of order mn.

Section 3.2 Some hamiltonian properties of $L_m x L_n$, $C_m x L_n$ and $C_m x C_n$.

Without loss of generality the vertices of either L_n or C_n can be conveniently labelled by $\{0,1,\ldots,n-1\}$, for some n satisfying $n \geq 1$. Let $G=(V(G),E(G))$ be a graph isomorphic to $L_m x L_n$ or $C_m x L_n$ or $C_m x C_n$, for some positive integers n and m.

For the analysis in the remainder of this chapter, it will be assumed that the vertices of G will be partitioned into two sets by the following coloring scheme. A vertex $(i,j) \in V(G)$ is colored black if and only if $i+j$ is even. Otherwise, it is colored white. An edge which joins two black vertices will be called a black edge and an edge which joins two white vertices will be called a white edge. An edge which is incident with a black vertex on one end and is incident with a white vertex on the other end is conveniently called a grey edge. For each i and j satisfying $0 \le i \le m-1$ and $0 \le j \le n-1$, the vertices (i,0), (i,n-1), (0,j) and (m-1,j) are called side vertices. In particular, the four vertices (0,0), (m-1,0), (m-1,n-1) and (0,n-1) are called corner vertices. The remaining vertices in V(G) are called interior vertices.

Lemma 3.2.1 Let G=(V(G),E(G)) be isomorphic to $L_m \times L_n$ such that mn is an even integer and $m,n \ge 2$. If x is a corner vertex in V(G), then for any vertex y in V(G) colored differently from u, there exists a hamiltonian x,y-path in G.

Proof: From the symmetry of the problem, there is no loss of generality in assuming x is the black vertex (0,0) and m is an even integer. Let m=2k such that $k \ge 1$. We proceed by induction on n and k. We first consider the case where k=1 and induct on n. Let n=2. Then, y can either be (0,1) or (1,0). The hamiltonian paths (0,0) (1,0) (1,1) (0,1) and (0,0) (0,1) (1,1) (1,0) will suffice, respectively. Suppose that the lemma has been proven for k=1 and for each integer $n=\ell, \ell-1 \ldots, 1$, for some $\ell \ge 2$. Consider the graph $G=L_2 \times L_n$ such that $n=\ell+1$. If y=(1,0), then the path (0,0) (0,1)...(0,n-1)(1,n-1)(1,n-2)...(1,0) is a hamiltonian x,y-path in G. Let y be a white vertex such that $y \ne (1,0)$. In this case y can either be in the form (0,j), for some j satisfying $2 \le j \le n-1$ or (1,j), for $1 \le j \le n-1$. We partition V(G) as follows.

$H_1 = \{(i,r) \mid i=0 \text{ or } 1 \text{ and } 0 \le r < j\}$ and

$H_2 = V(G) - H_1$

If $y=(1,j)$, then by the induction hypothesis, there exists a hamiltonian $(0,0),(0,j-1)$-path P_1 in $\langle H_1 \rangle$ and there exists a hamiltonian $(0,j),(1,j)$-path P_2 in $\langle H_2 \rangle$. A hamiltonian x,y-path in G can now be constructed by concatenating P_1 and P_2 using the edge $(0,j-1)(0,j)$. If $y=(0,j)$, then by the induction hypothesis, there exists a hamiltonian $(0,0),(1,j-1)$-path P_1' in $\langle H_1 \rangle$ and a hamiltonian $(1,j),(0,j)$-path P_2' in $\langle H_2 \rangle$. Similarly, the concatenation of the paths P_1' and P_2' using the edge $(1,j-1)(1,j)$ constitutes a hamiltonian x,y-path in G. This establishes the validity of the lemma for all $n \geq 2$ and for $k=1$ (that is, $m=2$). Suppose that the lemma has now been established for each $n \geq 2$ and for each $k=\ell, \ell-1, \ldots 1$, for some $\ell \geq 1$. Let $k=\ell+1$ and consider the graphs $G=L_{2k} \times L_n$ for some arbitrary n satisfying $n \geq 2$. We partition the set V(G) as follows:

$H_3=\{(s,j) \mid 0 \leq s \leq m-3,\ 0 \leq j \leq n-1\}$, and

$H_4=V(G)-H_3$. (See Figure 3.2)

Let $y=(a,b)$ be any white vertex in H_4. Observe that $(m-2,0)$ is a black vertex in H_4 and $(m-3,0)$ is a white vertex in H_3. By the induction hypothesis, there exists a hamiltonian $(0,0),(m-3,0)$-path P_3 in $\langle H_3 \rangle$ and a hamiltonian $(m-3,0)(a,b)$-path P_4 in $\langle H_4 \rangle$. The concatenation of P_3 and P_4 using the edge $(m-3,0)(m-2,0)$ constitutes a hamiltonian x,y-path in G. Suppose that y is any white vertex in H_3 such that $y \neq (m-3,0)$. By the induction hypothesis, there exists a hamiltonian x,y-path P_3' in $\langle H_3 \rangle$. Notice that the vertex (m-3,0) is a white corner vertex of degree 2 in $\langle H_3 \rangle$. The edge (m-3,0)(m-3,1) is necessarily on the path P_3'. By the induction hypothesis, there exists a hamiltonian (m-2,0),(m-2,1)-path P_4' in $\langle H_4 \rangle$. The hamiltonian x,y-path P_3' in $\langle H_3 \rangle$ can now be extended to become a hamiltonian x,y-path in G by replacing the edge (m-3,0)(m-3,1) on the path P_3' by the path (m-3,0)(m-2,0) P_4' (m-2,1)(m-3,1). Finally, let y=(m-3,0). By the induction hypothesis, there exists a hamiltonian x,y-path P_3'' in $\langle H_3 \rangle$. If the edge (m-3,0)(m-3,1) is on the path P_3'', then P_3'' can be extended to become a hamiltonian x,y-path in G in a manner similar to the last construction. Suppose that the edge (m-3,0)(m-3,1) is not on the path P_3''. Then, it is clear that $n \geq 3$. Since the vertex (m-3,n-1) is a corner vertex in H_3, it is clear that the edge (m-3,n-1)(m-3,n-2) must be on the path P_3''. By the induction hypothesis, there exists a hamiltonian (m-2,n-1),(m-2,n-2)-path P_4'' in $\langle H_4 \rangle$. Then, the hamiltonian x,y-path P_3'' in $\langle H_3 \rangle$ can be extended to become a hamiltonian x,y-path in G by replacing the edge (m-3,n-1)(m-3,n-2) on P_3'' by the path (m-3,n-1)(m-2,n-1) P_4'' (m-2,n-2)(m-3,n-2). This completes the proof of the lemma. ■

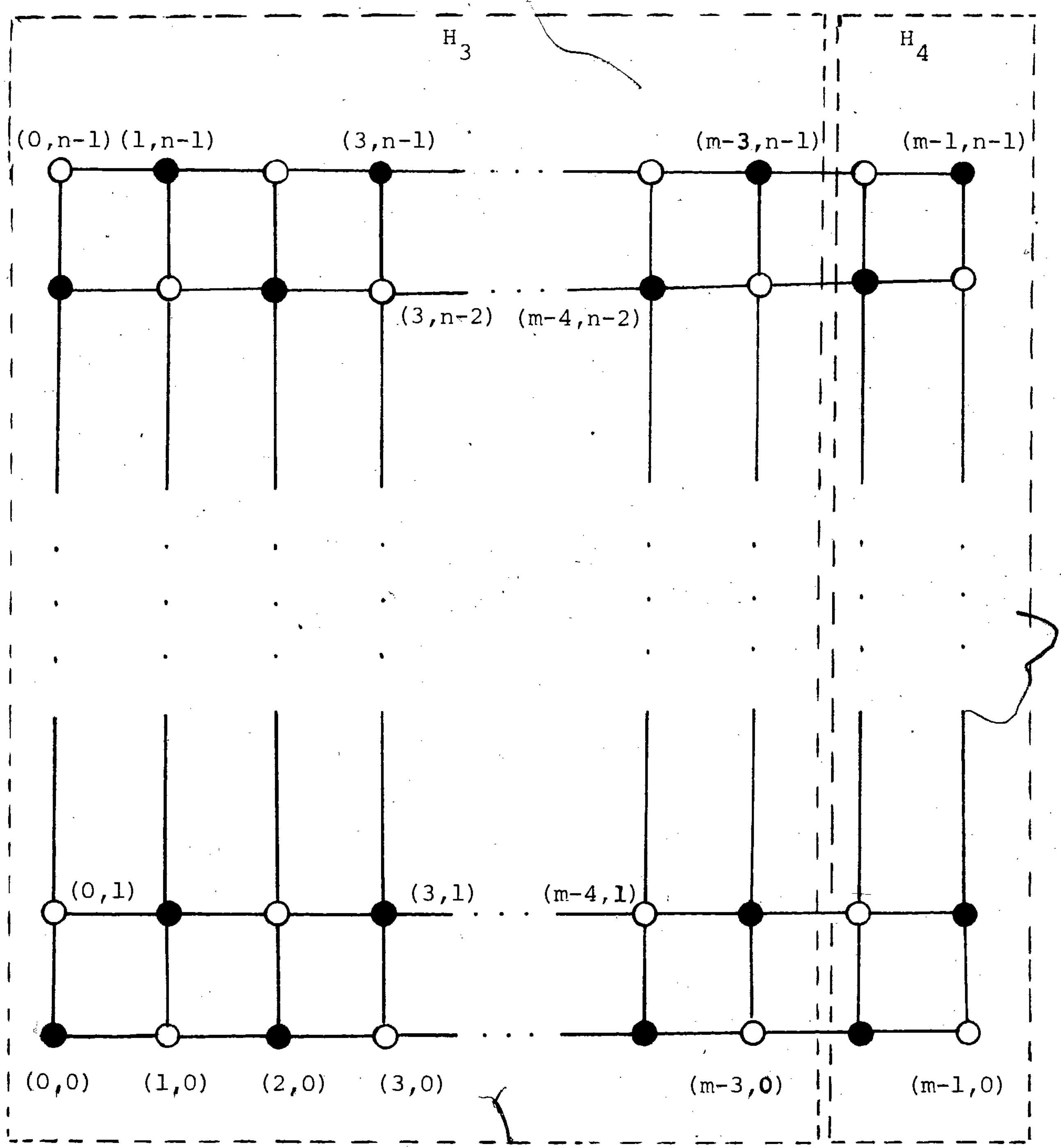


figure 3.2

Lemma 3.2.2 Let $G=L_m x L_n$ such that both m and n are positive odd integers greater than or equal to 3. Then there exists a hamiltonian path connecting any black corner vertex and a distinct black vertex in G.

Proof: From the symmetry of the problem, there is no loss of generality in assuming that the corner vertex $x=(0,0)$. Let $y=(a,b)$ be any other black vertex in G. If $a\geq 1$, then we partition the vertex set of G as follows. Let $H_1=\{(0,j)\mid 0\leq j\leq n-1\}$ and $H_2=V(G)-H_1$. Since $|H_2|$ is even, by Lemma 3.2.1 there exists a hamiltonian $(1,n-1),(a,b)$-path P_2 in $\langle H_2\rangle$. Let $P_1:(0,0)(0,1)\ldots(0,n-1)$. It is clear that the concatenation of P_1 and P_2 using the edge $(0,n-1)(1,n-1)$ constitutes a hamiltonian x,y-path in G. Suppose that $a=0$. Then, we partition V(G) as follows. Let $H_3=\{(i,0)\mid 0\leq i\leq m-1\}$ and $H_4=V(G)-H_3$. As before, by Lemma 3.2.1, there exists a hamiltonian $(m-1,1),(a,b)$-path P_4 in $\langle H_4\rangle$. Let $P_3:(0,0)(1,0)\ldots(m-1,0)$ be a hamiltonian path in $\langle H_3\rangle$. It is clear that concatenating P_3 and P_4 using the edge $(m-1,0)(m-1,1)$ results in a hamiltonian x,y-path in G. This completes the proof of the lemma. ■

Let m and n be positive integers satisfying $m,n\geq 2$. A very useful construction of a hamiltonian path between any two side vertices in a graph is isomorphic to $L_m x L_n$ is summarised in the following corollary.

Corollary 3.2.3 Let m and n be two positive integers greater than or equal to 2 such that mn is an even integer. Then there exists a hamiltonian path connecting any two adjacent side vertices in a graph $G=L_m x L_n$.

Proof: Without loss of generality, we assume that m is an even integer. Let $x,y\in V(G)$ be two arbitrary adjacent side vertices. If either x or y is a corner vertex in G, then by Lemma 3.2.1 there exists a hamiltonian x,y-path in G. Hence, it is assumed for the remainder of the proof that neither x nor y is a corner vertex. Suppose that n is also an even integer. By the symmetry of the problem, it suffices to consider the case where $x=(i,0)$ and $y=(i+1,0)$, for some i satisfying $1 \le i \le m-3$. We partition the set V(G) as follows. Let $H_1=\{(r,s) \mid 0 \le r \le i, 0 \le s \le n-1\}$ and $H_2=V(G)-H_1$. Then, by Lemma 3.2.1 there exists a hamiltonian (i,0),(i,1)-path P_1 in $\langle H_1 \rangle$ and a hamiltonian (i+1,1),(i+1,0)-path P_2 in $\langle H_2 \rangle$. A hamiltonian x,y-path in G can be constructed by concatenating P_1 and P_2 using the edge (i,1)(i+1,1). Suppose that n is an odd integer and let $x=(i,0)$ and $y=(i+1,0)$, for some i satisfying $1 \le i \le m-3$. We partition V(G) as above. If both $|H_1|$ and $|H_2|$ are even, then the construction used in the last argument will provide a hamiltonian x,y-path in G.

If both $|H_1|$ and $|H_2|$ are odd, then by Lemma 3.2.2 there exists a hamiltonian (i,0),(i,n-1)-path P_1' in $\langle H_1 \rangle$ and a hamiltonian (i+1,0),(i+1,n-1)-path P_2' in $\langle H_2 \rangle$. The concatenation of P_1' and P_2' using the edge (i,n-1)(i+1,n-1) provides a hamiltonian x,y-path in G. By the symmetry of the problem, the only case which still is required to be examined is when $x=(0,j)$ and $y=(0,j+1)$, for some j satisfying $1 \le j \le n-3$, assuming that n is an odd integer. Let V(G) be partitioned as follows.
Let $H_3=\{(r,s) \mid 0 \le r \le m-1, s \le j\}$ and $H_4=V(G)-H_3$. Since m is an even integer, an argument similar to the first construction employed in the proof of this corollary will provide a hamiltonian x,y-path in G. ∎

We now apply Lemma 3.2.1, Lemma 3.2.2 and Corollary 3.2.3 to produce more generalized hamiltonian properties in $L_m x L_n$ and $C_m x L_n$. The technique of extending a path used at the end of the proof of Lemma 3.2.1 will be useful in many constructions in the remainder of this chapter. It can be summarized in the following definition.

Definition 3.2.4 Let $G=(V(G),E(G))$ be a graph and H_1 and H_2 be nonempty disjoint subsets of $V(G)$. Suppose that there exist four distinct vertices $x,y \in H_1$ and $u,v \in H_2$ such that there exists a hamiltonian x,y-path P_1 in $\langle H_1 \rangle$ and a hamiltonian u,v-path P_2 in $\langle H_2 \rangle$. Furthermore, suppose there is an edge $u'v'$ on the path P_1 such that $uu', vv' \in E(G)$. Then, a hamiltonian x,y-path in $\langle H_1 \cup H_2 \rangle$ can be constructed by replacing the edge $u'v'$ on P_1 by the path $u'uP_2vv'$. Such an extension scheme is called a $[P_1,P_2]$-extension.

Notice that a $[P_1,P_2]$-extension has a different meaning from a $[P_2,P_1]$-extension.

Lemma 3.2.5 Let m and n be two positive integers greater than or equal to 4 such that the product mn is even. Let $G=L_m x L_n$. Then there exists a hamiltonian path connecting any two vertices in G which are colored differently.

Proof: Let G be a graph which satisfies the hypotheses of the lemma and m be an even integer. By the symmetry of the problem, it suffices to consider two vertices $x=(i,j)$ and $y=(h,k)$ which are colored black and white, respectively, such that $j \le k$. Furthermore, if either x or y is a corner vertex, then by Lemma 3.2.1, there exists a hamiltonian x,y-path in G. Hence, we assume that neither x nor y is a corner vertex and consider the following cases.

Case 1 $j < k$. Suppose that $j > 0$ and $k < n-1$. We partition $V(G)$ as follows. Let $H_1 = \{(r,s) | 0 \le r \le m-1 \text{ and } s \le j\}$ and $H_2 = V(G) - H_1$. Since m is even, it is clear that either $(0,j)$ or $(m-1,j)$ is a white vertex. If $(0,j)$ is the white vertex, then $(0,j+1)$ is a black vertex. By Lemma 3.2.1, there exists a hamiltonian $(i,j),(0,j)$-path P_1 in $\langle H_1 \rangle$ and a hamiltonian $(0,j+1),(h,k)$-path P_2 in $\langle H_2 \rangle$. The concatenation of the paths P_1 and P_2 using the edge $(0,j)(0,j+1)$ will result in a hamiltonian x,y-path in G. An argument similar to this last construction will provide a hamiltonian x,y-path in G if $(m-1,j)$ instead of $(0,j)$ is the white vertex.

For the remainder of Case 1, it will be assumed that either $k=n-1$ or $j=0$. We first investigate those situations where $k=n-1$. If $k > j+1$, then let $H_1 = \{(r,s) | 0 \le r \le m-1 \text{ and } s \le n-3\}$ and $H_2 = V(G) - H_1$. An argument similar to the last construction above will provide a hamiltonian x,y-path in G. If $k=j+1=n-1$ and n is an even integer, then it suffices to consider only the case where $i \le h$. We introduce the following partitioning on $V(G)$. Let $H_1 = \{(r,s) | r \ge i \text{ and } s \ge j\}$, $H_2 = \{(r,s) | r \ge i \text{ and } s < j\}$ and $H_3 = V(G) - (H_1 \cup H_2)$. It is clear that $|H_1|$, $|H_2|$ and $|H_3|$ are even integers. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1 \rangle$. For some integer ℓ satisfying $i \le \ell \le m-2$, there exists an edge of the form $(\ell,j)(\ell+1,j)$ on the path P_1. By Corollary 3.2.3, there exists a hamiltonian $(\ell,j-1)(\ell+1,j-1)$-path P_2 in $\langle H_2 \rangle$. A $[P_1,P_2]$-extension will constitute a hamiltonian x,y-path P_1' in $\langle H_1 \cup H_2 \rangle$. If $H_3 = \emptyset$, then P_1' is a hamiltonian x,y-path in G. Otherwise, there exists an edge of the form $(i,t)(i,t+1)$ on the path P_1', for some t satisfying $0 \leq t \leq n-2$. By Corollary 3.2.3, there exists a hamiltonian $(i-1,t),(i-1,t+1)$-path P_3 in $\langle H_3 \rangle$. A hamiltonian x,y-path can now be constructed by $[P_1',P_3]$-extension. We next suppose that $k=j+1=n-1$ and n is an odd integer. If $i<h$, then the set V(G) is partitioned as follows. Let $H_1=\{(r,s) \mid r \leq i \text{ and } s \geq j\}$, $H_2=\{(r,s) \mid r \leq i \text{ and } s<j\}$ and $H_3=V(G)-(H_1 \cup H_2)$. Notice that $|H_1|$, $|H_2|$ and $|H_3|$ are all even integers. For some ℓ such that $\ell<i$, let (ℓ,j) be a white vertex in H_1. By Lemma 3.2.1, there exists a hamiltonian $(i,j),(\ell,j)$-path P_1 in $\langle H_1 \rangle$. Since $(\ell,j-1)$ and $(i,0)$ are black white vertices in H_2, respectively, by Lemma 3.2.1, there exists a hamiltonian $(\ell,j-1),(i,0)$-path P_2 in $\langle H_2 \rangle$. Since $(i+1,0)$ is a black vertex in H_3, by Lemma 3.2.1, there exists a hamiltonian $(i+1,0)$, (h,k)-path P_3 in $\langle H_3 \rangle$. It is clear that the concatenation of the paths P_1,P_2 and P_3 using the edges $(\ell,j)(\ell,j-1)$ and $(i,0)$ $(i+1,0)$ constitutes a hamiltonian x,y-path in G. We now consider the case where $i \geq h$ and we use the same partition on V(G) as in this last construction. Note that $x,y \in H_1$ under such an assumption. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1' in $\langle H_1 \rangle$. Let ℓ be an integer such that $\ell<i$ and $(\ell,j)(\ell+1,j)$ is an edge on the path P_1'. Since $|H_2|$ is an even integer, by Corollary 3.2.3, there exists a hamiltonian $(\ell,j-1),(\ell+1,j-1)$-path P_2' in $\langle H_2 \rangle$. Let P_1'' be a hamiltonian x,y-path in $\langle H_1 \cup H_2 \rangle$ constructed by $[P_1',P_2']$-extension. Since $n \geq 5$, there exists an edge of the form $(i,t)(i,t+1)$ on the path P_1'' for some t

satisfying $0 \le t \le n-2$. By Corollary 3.2.3, there exists a hamiltonian $(i+1,t),(i+1,t+1)$-path P_3' in $\langle H_3 \rangle$. A hamiltonian x,y-path can be constructed by $[P_1'',P_3']$-extension.

We next investigate the situations where $j=0$. Under this assumption, i is necessarily even. If $k > j+1$, then let $H_1 = \{(r,s) \mid 0 \le r \le m-2, s \le 1\}$ and $H_2 = V(G) - H_1$. It is clear that the vertices $(0,1) \in H_1$ and $(0,2) \in H_2$ are colored white and black, respectively. By Lemma 3.2.1, there exists a hamiltonian $(i,0),(0,1)$-path P_1 in $\langle H_1 \rangle$ and a hamiltonian $(0,2),(h,k)$-path P_2 in $\langle H_2 \rangle$. A hamiltonian x,y-path can be constructed by concatenating P_1 and P_2 using the edge $(0,1)(0,2)$. For the remainder of the investigation of case 1, we assume that $y=(h,1)$. If $i=0$, then Lemma 3.2.1 guarantees the existence of a hamiltonian x,y-path in G. Hence, $i \ge 2$ is assumed. Suppose that $i \le h$. Let $H_1 = \{(r,s) \mid r \ge i \text{ and } 0 \le s \le n-1\}$ and $H_2 = V(G) - H_1$. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1' in $\langle H_1 \rangle$. Let t be an integer satisfying $0 \le t \le n-1$ such that the edge $(i,t)(i,t+1)$ is on the path P_1'. By Corollary 3.2.3, there exists a hamiltonian $(i-1,t),(i-1,t+1)$-path P_2' in $\langle H_1 \rangle$. A hamiltonian x,y-path can be constructed by $[P_1',P_2']$-extension. If n is an odd integer, then the additional case where $i > h$ must be considered. We partition V(G) into the following three subsets. Let $H_1 = \{(r,s) \mid r \ge i \text{ and } s \le 1\}$, $H_2 = \{(r,s) \mid r < i \text{ and } s \le 1\}$ and $H_3 = V(G) - (H_1 \cup H_2)$. Since the vertices $(i,1)$ and $(i-1,1)$ are colored white and black, respectively, by Lemma 3.2.1, there exists a hamiltonian $(i,0),(i,1)$-path P_1 in $\langle H_1 \rangle$ and a hamiltonian

$(i-1,1),(h,1)$-path P_2 in $\langle H_2\rangle$. Let $P_{1,2}$ be the hamiltonian x,y-path in $\langle H_1 \cup H_2\rangle$. Since $|H_3|$ is an even integer, by Corollary 3.2.3, there exists a hamiltonian $(i-1,2)$, $(i,2)$-path P_3 in $\langle H_3\rangle$. A hamiltonian x,y-path can now be constructed by $[P_{1,2},P_3]$-extension. This completes the constructions for Case 1.

Case 2 $j=k$ and $i<h$. Suppose that i is an even integer and $j\neq n-1$. We partition $V(G)$ in the following manner. Let $H_1=\{(r,s)\mid r\geq i$ and $s\geq j\}$, $H_2=\{(r,s)\mid 0\leq r\leq m-1$ and $s<j\}$ and $H_3=V(G)-(H_1\cup H_2)$. Notice that $|H_1|$, $|H_2|$ and $|H_3|$ are all even integers. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. Let t be an integer satisfying $i\leq t\leq n-2$ such that the edge $(i,t)(i,t+1)$ is an edge on the path P_1. If $|H_3|\neq 0$, then by Corollary 3.2.3 there exists a hamiltonian $(i-1,t),(i-1,t+1)$-path P_3 in $\langle H_3\rangle$. Let $P_{1,3}$ be the hamiltonian x,y-path in $\langle H_1\cup H_3\rangle$ constructed by $[P_1,P_3]$-extension. If $|H_2|\neq 0$, then let q be an integer satisfying $0\leq q\leq m-1$ such that $(1,j)(q+1,j)$ is an edge on the path $P_{1,3}$. By Corollary 3.2.3, there exists a hamiltonian $(q,j-1),(q+1,j-1)$-path P_2 in $\langle H_2\rangle$. A hamiltonian x,y-path can now be constructed by $[P_{1,3},P_3]$-extension. Suppose that i is an even integer and $j=n-1$. This implies that n is necessarily an odd integer and we consider the following partitioning on $V(G)$. Let $H_1=\{(r,s)\mid r\geq i$ and $0\leq s\leq n-1\}$ and $H_2=V(G)-H_1$. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. Similarly, if $|H_2|\neq 0$, there exists an integer t' satisfying $0\leq t'\leq n-2$ such that the edge $(i,t')(i,t'+1)$ is on the path P_1. By Corollary 3.2.3, there exists a hamiltonian $(i-1,t'),(i-1,t'+1)$-path P_2 in $\langle H_2\rangle$.

A hamiltonian x,y-path in G can be constructed by $[P_1, P_2]$-extension. For the remaining constructions in Case 2, we assume that i is an odd integer and the set V(G) is partitioned as follows. Let $H_1=\{(r,s) \mid r \leq i \text{ and } 0 \leq s \leq n-1\}$ and $H_2=V(G)-H_1$. It is clear that the vertices $(i,0) \in H_1$ and $(i+1,0) \in H_2$ are colored white and black, respectively. By Lemma 3.2.1, there exists a hamiltonian (i,j),(i,0)-path P_1 in $\langle H_1 \rangle$ and a hamiltonian (i+1,0),(h,k)-path P_2 in $\langle H_2 \rangle$. The concatenation of P_1 and P_2 using the edge (i,0)(i+1,0) results in a hamiltonian x,y-path in G. This completes the constructions required for Case 2.

Case 3 j=k and i>h. From the symmetry of the problem this case is required to be considered only if n is an odd integer. Furthermore, it suffices to assume that $j=k \leq \left[\frac{n}{2}\right]+1$. Suppose that i is an even integer. We consider the following partitioning on V(G). Let $H_1=\{(r,s) \mid r \geq i \text{ and } s \leq j\}$, $H_2=\{(r,s) \mid r<i \text{ and } s \leq j\}$ and $H_3=V(G)-(H_1 \cup H_2)$. Observe that $(m-1,j) \in H_1$ and $(0,k) \in H_2$ are colored white and black, respectively. By Lemma 3.2.1, there exists a hamiltonian (i,j),(m-1,j)-path P_1 in $\langle H_1 \rangle$ and a hamiltonian (0,j),(h,k)-path P_2 in $\langle H_2 \rangle$. Since $|H_3|$ is an even integer, Lemma 3.2.1 also guarantees the existence of a hamiltonian (m-1,j+1)(0,j+1)-path P_3 in $\langle H_3 \rangle$. The concatenation of P_1, P_2 and P_3 using the edges (m-1,j)(m-1,j+1) and (0,j)(0,j+1) results in a hamiltonian x,y-path in G.

Finally, we assume that i is an odd integer and consider the following partition on V(G). Let $H_1=\{(r,s) \mid r \leq i \text{ and } s \leq j\}$,

$H_2=\{(r,s)|\ r\leq i$ and $s>j\}$ and $H_3=V(G)-(H_1\cup H_2)$. This partitioning yields that $|H_1|$, $|H_2|$ and $|H_3|$ are even integers. By Lemma 3.2.1 there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. Suppose that $|H_3|\neq 0$. There exists an integer q' satisfying $0\leq q'\leq j-1$ such that the edge $(i,q')(i,q'+1)$ is on the path P_1. By Corollary 3.2.3, there exists a hamiltonian $(i+1,q')$, $(i+1,q'+1)$-path P_3 in $\langle H_3\rangle$. Let $P_{1,3}$ be the hamiltonian x,y-path obtained by $[P_1,P_3]$-extension. Since $j=k\leq\left\lceil\frac{n}{2}\right\rceil+1$, there exists an integer p satisfying $j+1\leq p\leq n-1$ such that the edge $(i+1,p)(i+1,p+1)$ is on the path $P_{1,3}$. By Corollary 3.2.3, there exists a hamiltonian $(i,p),(i,p+1)$-path P_2 in $\langle H_2\rangle$. A hamiltonian x,y-path can now be obtained by $[P_{1,3},P_2]$-extension. If $|H_3|=0$, then $x=(m-1,j)$ and there exists an integer ℓ satisfying $0\leq\ell\leq m-2$ such that the edge $(\ell,j)(\ell+1,j)$ is on the path P_1. By Corollary 3.2.3, there exists a hamiltonian $(\ell,j+1)$, $(\ell+1,j+1)$-path P_2' in $\langle H_2\rangle$. A hamiltonian x,y-path in G can now be obtained by $[P_1,P_2']$-extension. This completes the proof of the lemma. ■

It is important to realize that the lower bounds on m and n in Lemma 3.2.5 are sharp.

<u>Lemma 3.2.6</u> Let m and n be two odd positive integers greater than or equal to 3. Let $G=L_m\times L_n$. Then there exists a hamiltonian path connecting any two black vertices in G.

<u>Proof</u>: Let G be a graph which satisfies the hypotheses of the lemma. Consider two arbitrary

distinct black vertices $x=(i,j)$ and $y=(h,k)$ in G. From the symmetry of the problem, there is no loss of generality in assuming that $i \leq h$ and $j \leq k$. There are three cases to be considered.

Case 1 i is an odd integer and $i<h$. Consider the following partitioning on V(G). Let $H_1 = \{(r,s) \mid r \leq i, 0 \leq s \leq n-1\}$ and $H_2 = V(G) - H_1$. It is clear that the vertices $(i,0) \in H_1$ and $(i+1,0) \in H_2$ are colored white and black, respectively. Observe that $|H_1|$ is an even integer and $|H_2|$ is an odd integer. By Lemma 3.2.1, there exists a hamiltonian $(i,j),(i,0)$-path P_1 in $\langle H_1 \rangle$. By Lemma 3.2.2, there exists a hamiltonian $(i+1,0),(h,k)$-path in $\langle H_2 \rangle$. The concatenation of the paths P_1 and P_2 using the edge $(i,0)(i+1,0)$ results in a hamiltonian x,y-path in G.

Case 2 i is an odd integer and $i=h$. Under the assumptions of the case, $j<k$ must be satisfied. Let $H_1 = \{(r,s) \mid 0 \leq r \leq m-1 \text{ and } s \leq j\}$ and $H_2 = V(G) - H_1$. It is clear that the vertices $(0,j) \in H_1$ and $(0,j+1)$ are colored white and black, respectively. Furthermore, $|H_1|$ and $|H_2|$ are even and odd integers, respectively. By Lemma 3.2.1, there exists a hamiltonian $(i,j),(0,j)$-path P_1' in $\langle H_1 \rangle$. By Lemma 3.2.2, there exists a hamiltonian $(0,j-1),(h,k)$-path P_2' in $\langle H_2 \rangle$. A hamiltonian x,y-path in G can be constructed by concatenating P_1' and P_2' using the edge $(0,j)\ (0,j+1)$.

<u>Case 3</u> i is an even integer. If either x or y is a corner vertex, then Lemma 3.2.3 can be applied directly to construct a hamiltonian x,y-path in G. For the remainder of the proof, it will be assumed that neither x nor y is a corner vertex. We first assume that $i \neq m-1$ and consider the following partitioning on V(G). Let $H_1 = \{(r,s) \mid r \geq i \text{ and } s \geq j\}$, $H_2 = \{(r,s) \mid r \geq i \text{ and } s < j\}$ and $H_3 = V(G) - (H_1 \cup H_2)$. Then, $|H_1|$ is an odd integer while both $|H_2|$ and $|H_3|$ are even integers. By Lemma 3.2.2, there exists a hamiltonain x,y-path P''_1 in $\langle H_1 \rangle$. Suppose that $|H_2| \neq 0$. Let ℓ be an integer satisfying $i \leq \ell \leq m-2$ such that the edge $(\ell,j)(\ell+1,j)$ is on the path P''_1. By Corollary 3.2.3, there exists a hamiltonian $(\ell,j-1),(\ell+1,j-1)$-path P''_2 in $\langle H_2 \rangle$. Let $P''_{1,2}$ be the hamiltonian x,y-path obtained by $[P''_1,P''_2]$-extension. Suppose that $|H_3| \neq 0$. It is clear that there exists an integer t satisfying $0 \leq t \leq n-1$ such that the edge $(i,t)(i,t+1)$ is on the path $P''_{1,2}$. By Corollary 3.2.3, there exists a hamiltonian $(i-1,t),(i-1,t+1)$-path P''_3 in $\langle H_3 \rangle$. A hamiltonian x,y-path in G can now be obtained by $[P''_{1,2},P''_3]$-extension. The cases where either $|H_2|=0$ or $|H_3|=0$ can be treated similarly. It remains to consider the case where $i=m-1$. This, however, implies that $h=m-1$ and $j<k$. By the symmetry of the problem, this is equivalent to constructing a

hamiltonian (0,j),(0,k)-path in G which has already been accounted for in the last construction.

This completes the proof of the lemma. ∎

It will be shown in the following how Lemmas 3.2.1 through 3.2.6 can be used to establish the hamiltonian-connectedness of a graph isomorphic to $C_m x L_n$, for any integer $n \geq 2$ and any odd integer $m \geq 3$.

<u>Lemma 3.2.7</u> Let G be a graph isomorphic to $C_m x L_n$ for any integer $n \geq 2$ and any odd integer $m \geq 3$. Then G is hamiltonian-connected.

<u>Proof</u>: Let G be a graph which satisfies the hypothesis of the lemma. Let $x=(i,j)$ be a black vertex in G and $y=(h,k)$ be any vertex in G distinct from x. From the symmetry of the problem, it is clear that there is no loss of generality in assuming that $0 \leq i \leq 1$, $i \leq h$ and $j \leq k$. In order to apply Lemmas 3.2.5 and 3.2.6, we first investigate the hamiltonian-connectedness of G where the lower bounds $n \geq 4$ and $m \geq 5$ are satisfied. The remaining cases will be examined separately after the following four cases have been considered.

<u>Case 1</u> n is an odd integer and $h > 1$. If y is a black vertex, then by Lemma 3.2.6 there exists a hamiltonian x,y-path in G. Hence, y is assumed to be a white vertex in G for the remaining constructions. Consider the following partition on V(G). Let

$H_1=\{(r,s) \mid 0 \le r \le 1$ and $0 \le s \le n-1\}$ and $H_2=V(G)-H_1$. The vertices (1,0) and (2,0) are of colors white and black, respectively. By Lemma 3.2.1 there exists a hamiltonian (i,j),(1,0)-path P_1 in $\langle H_1 \rangle$, and a hamiltonian (2,0),(h,k)-path P_2 in $\langle H_2 \rangle$. The concatenation of P_1 and P_2 using the edge (1,0)(2,0) results in a hamiltonian x,y-path in G.

Case 2 n is an odd integer and $h \le 1$. For the reason mentioned in Case 1, it suffices to assume that y is a white vertex. If x=(1,j), for some j satisfying $1 \le j \le n-2$, then we partition V(G) as follows. Let $H_1=\{(r,s) \mid 0 \le r \le 1,\ j \le s \le n-1\}$, $H_2=\{(r,s) \mid 2 \le r \le m-1,\ 0 \le s \le n-1\}$ and $H_3=V(G)-(H_1 \cup H_2)$. Observe that $|H_2|$ is an odd integer while both $|H_1|$ and $|H_3|$ are even integers. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1 \rangle$. By Lemma 3.2.6, there exists a hamiltonian (2,n-1),(m-1,n-1)-path P_2 in $\langle H_2 \rangle$. Since y=(1,k), for some k satisfying $j+1 \le k \le n-1$, it is clear that the edge (0,n-1)(1,n-1) is on P_1. Furthermore, (0,n-1) is adjacent to (m-1,n-1) and (1,n-1) is adjacent to (2,n-1). Let $P_{1,2}$ be the hamiltonian x,y-path in $\langle H_1 \cup H_2 \rangle$ which is obtained by $[P_1,P_2]$-extension. It is clear that the edge (0,j)(1,j) is on the P_1 portion of the path $P_{1,2}$. By Lemma 3.2.1, there exists a hamiltonian (0,j-1),(1,j-1)-path P_3 in $\langle H_3 \rangle$. Then, a hamiltonian x,y-path in G can be obtained by

$[P_{1,2}, P_3]$-extension. We next assume that x is of the form $(0,j)$ for some j satisfying $0 \le j \le n-1$. As long as $j < k$, the last construction can still be used to provide a hamiltonian x,y-path in G. If $j=k$, then consider the following partition on $V(G)$. Let $H_1 = \{(r,s) \mid 0 \le r \le 1,\ j \le s \le n-1\}$, $H_2 = \{(r,s) \mid 2 \le t \le m-1,\ j \le s \le n-1\}$ and $H_3 = V(G) - (H_1 \cup H_2)$. It is clear that $|H_2|$ is an odd integer and $|H_1|$ and $|H_3|$ are even integers. Suppose that $j \le n-1$. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1' in $\langle H_1 \rangle$ and by Lemma 3.2.6, there exists a hamiltonian $(2,n-1),(m-1,n-1)$-path P_2' in $\langle H_2 \rangle$. Since the edge $(0,n-1)(1,n-1)$ is on the path P_1', a hamiltonian x,y-path $P_{1,2}'$ in $\langle H_1 \cup H_2 \rangle$ can be obtained by $[P_1', P_2']$-extension. Let ℓ be an integer satisfying $2 \le \ell \le m-1$ such that the edge $(\ell,j)(\ell+1,j)$ is on the path $P_{1,2}'$. Suppose that $|H_3| > 0$. By Corollary 3.2.3, there exists a hamiltonian $(\ell,j-1),(\ell+1,j-1)$-path P_3' in $\langle H_3 \rangle$. A hamiltonian x,y-path in G can be obtained by $[P_{1,2}', P_3']$-extension. The cases where $j=n-1$ or $j=0$ can be treated similarly.

<u>Case 3</u> n is even integer and $h > 1$. If y is a white vertex, then Lemma 3.2.5 guarantees the existence of a hamiltonian x,y-path in G. Hence we assume that y is black. Clearly, $(0,n-1)$ and $(m-1,n-1)$ are white vertices in G and consider the following partition on $V(G)$. Let $H_1 = \{(r,s) \mid 0 \le r \le 1,\ 0 \le s \le n-1\}$ and $H_2 = V(G) - H_1$.

By Lemma 3.2.1, there exists a hamiltonian (i,j), (0,n-1)-path P_1 in $\langle H_1\rangle$ and a hamiltonian (m-1,n-1),(h,k)-path P_2 in $\langle H_2\rangle$. The concatenation of P_1 and P_2 using the edge (0,n-1)(m-1,n-1) results in a hamiltonian x,y-path in G.

<u>Case 4</u> n is an even integer and $h\le 1$. As in case 3, it suffices to assume that y is a white vertex. Suppose that $y\neq(1,n-1)$. Then, $k\le n-2$ and we partition V(G) as follows. Let $H_1=\{(r,s)\mid 0\le r\le m-1,\ 0\le s\le n-2\}$ and $H_2=V(G)-H_1$. By Lemma 3.2.6, there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. Let ℓ be an integer satisfying $0\le\ell\le m-2$ such that $(\ell,n-2)(\ell+1,n-2)$ is an edge on the path P_1. Let P_2 be the hamiltonian $(\ell,n-1)(\ell+1,n-1)$-path in $\langle H_2\rangle$ defined by $(\ell,n-1)(\ell-1,n-1)\ldots(0,n-1)$ $(m-1,n-1)(m-2,n-1)\ldots(\ell+1,n-1)$. A hamiltonian x,y-path in G can be obtained by $[P_1,P_2]$-extension. It remains to assume that $y=(1,n-1)$. If $x=(0,n-2)$, then by Lemma 3.2.6 there exists a hamiltonian (0,n-2),(2,n-2)-path P_1' in $\langle H_1\rangle$. It is clear that the path P_2': $(2,n-1)(3,n-1)\ldots(m-1,n-1)(0,n-1)(1,n-1)$ is a hamiltonian (2,n-1),(1,n-1)-path in $\langle H_2\rangle$. A hamiltonian x,y-path in G can be constructed by concatenating P_1' and P_2' using the edge (1,n-2)(1,n-1). If $x\neq(0,n-2)$, then by Lemma 3.2.6 there exists a hamiltonian (i,j), (0,n-2)-path P_1'' in $\langle H_1\rangle$. Furthermore,

P_2'': (0,n-1)(m-1,n-1)(m-2,n-1)...(2,n-1)(1,n-1) is a hamiltonian (0,n-1),(1,n-1)-path in $\langle H_2 \rangle$. The concatenation of P_1'' and P_2'' using the edge (0,n-2)(0, n-1) will result in a hamiltonian x,y-path in G.

For the remainder of the proof, we assume that either $n \le 3$ or $m=3$.

We first consider the situation where m=3 and n is an integer greater than or equal to 2 and examine the following two cases.

Case 1 n is an odd integer and m=3. It is clear that it suffices to consider only the case where y is a white vertex. Suppose the x=(0,j) for some j satisfying $0 \le j \le n-1$. If j=n-1, then we let $H_1 = \{(r,s) \mid 0 \le r \le 1,\ 1 \le s \le n-1\}$ and $H_2 = V(G) - H_1$. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1 \rangle$. Let ℓ be an integer satisfying $0 \le \ell \le 1$ such that the edge $(\ell,1)(\ell+1,1)$ is on the path P_1. Let P_2 be a hamiltonian $(\ell,0),(\ell+1,0)$-path in $\langle H_2 \rangle$ which contains the edge (0,0)(0,2). A hamiltonian x,y-path in G can be obtained by $[P_1,P_2]$-extension. If $j < n-1$, then the following partition on V(G) applies. Let $H_1 = \{(r,s) \mid 0 \le r \le 2, 1 \le s \le n-1\}$, $H_2 = \{(r,s) \mid 0 \le r \le 2,\ s=n-1\}$ and $H_3 = V(G) - (H_1 \cup H_2)$. If $y \ne (1,n-1)$, then $y \in H_1$ and by Lemma 3.2.1 there exists a hamiltonian x,y-path P_1' in $\langle H_1 \rangle$. Let t be an integer satisfying $0 \le t \le 1$ such that the edge $(t,n-2)(t+1,n-2)$ is on the path P_1'. There exists a hamiltonian $(t,n-1),(t+1,n-1)$-path P_2' in $\langle H_2\rangle$ which contains the edge $(0,n-1)(2,n-1)$. Let $P_{1,2}'$ be the hamiltonian x,y-path in $\langle H_1 \cup H_2\rangle$ obtained by $[P_1',P_2']$-extension. Observe that $|H_3|$ is even. If $|H_3|>0$, then $P_{1,2}'$ can be extended in a similar way to a hamiltonian x,y-path in G by applying Corollary 3.2.3 on $\langle H_3\rangle$. Otherwise, $G=\langle H_1 \cup H_2\rangle$ and $P_{1,2}'$ itself is a hamiltonian path in G. Suppose that $y=(1,n-1)$. Since $(2,n-2)$ is a white vertex in H_1, by Lemma 3.2.1, there exists a hamiltonian $(0,j)(2,n-2)$-path P_1'' in $\langle H_1\rangle$. Let P_2'': $(2,n-1)(0,n-1)(1,n-1)$ be the hamiltonian $(2,n-1),(1,n-1)$-path in $\langle H_2\rangle$. Let P" be the hamiltonian x,y-path in $\langle H_1 \cup H_2\rangle$ obtained by concatenating P_1'' and P_2'' using the edge $(2,n-2)(2,n-1)$. If $|H_3|=0$, then P" is a hamiltonian x,y-path in G. Otherwise, P" can be extended to a hamiltonian x,y-path in G by applying Corollary 3.2.3 on $\langle H_3\rangle$ in a way similar to the last construction. Finally, we assume that $x=(1,j)$ for some j satisfying $1 \le j \le n-2$, and we partition V(G) in the following manner. Let $H_1=\{(r,s) \mid 0 \le r \le 2,\ 0 \le s \le j\}$, $H_2=\{(r,s) \mid 0 \le r \le 2, j+1 \le s \le n-2\}$ and $H_3=\{(r,s) \mid 0 \le r \le 2, s=n-1\}$. If $y \in H_1$, then $y=(2,j)$. By Lemma 3.2.1, there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. Without loss of generality, P_1 can be chose in such a way that the edge (0,j)(1,j) is on the path P_1. If $|H_2|>0$, then by applying Corollary 3.2.3 there exists a hamiltonian (0,j+1),(1,j+1)-path P_2 in $\langle H_2\rangle$. Let $P_{1,2}$ be the hamiltonian x,y-path in G obtained by $[P_1,P_2]$-extension. In a similar manner $P_{1,2}$ can be extended to a hamiltonian x,y-path in G. If $y\in H_2$, then by Lemma 3.2.1, there exists a hamiltonian (2,j+1),(h,k)-path P_2' in $\langle H_2\rangle$ and a hamiltonian (1,j),(2,j)-path P_1' in $\langle H_1\rangle$. Let P' be a hamiltonian x,y-path in $\langle H_1\cup H_2\rangle$ obtained by concatenating P_1' and P_2' using the edge (2,j)(2,j+1). P' can again be extended to a hamiltonian x,y-path in G. If $y\in H_3$, then by Lemma 3.2.1, there exists a hamiltonian (1,j),(2,n-2)-path $P_{1,2}''$ in $\langle H_1\cup H_2\rangle$. Let P_3'': (2,n-1)(0,n-1)(1,n-2). Then the concatenation of $P_{1,2}''$ and P_3'' using the edge (2,n-2)(2,n-1) results in a hamiltonian x,y-path in G.

Case 2 n is an even integer and m=3. We first assume that y is a white vertex and x is a black vertex of the form (0,j) for some j satisfying $0\le j\le n-2$. We partition

$V(G)$ into two subsets $H_1=\{(r,s) \mid 0\le r\le 2,\ j\le s\le n-1\}$ and $H_2=V(G)-H_1$. Since $|H_1|$ is even and x is a corner vertex in H_1, by Lemma 3.2.1 there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$. By Corollary 3.2.3, there exists a hamiltonian $(1,j-1),(\ell+1,j-1)$-path P_2 in $\langle H_2\rangle$ for some satisfying $0\le\ell\le 1$ which can be used to extend P_1 to a hamiltonian x,y-path in G by $[P_1,P_2]$-extension. Next let $x=(1,j)$ for some j satisfying $1\le j\le n-1$ and partition $V(G)$ into the two subsets $H_1=\{(r,s) \mid 0\le r\le 1,\ 0\le s\le j\}$ and $H_2=V(G)-H_1$. If $y=(2,j)$, then by Lemma 3.2.1 there exists a hamiltonian x,y-path P_1' in $\langle H_1\rangle$ which contains the edge $(0,j)(1,j)$. By Corollary 3.2.3, there exists a hamiltonian $(0,j+1),(1,j+1)$-path P_2' in $\langle H_2\rangle$ which is used to construct a hamiltonian x,y-path in G using $[P_1',P_2']$-extension. It remains to consider the case where y is also a black vertex. If $y\ne(1,n-1)$, then let $V(G)$ be partitioned into two subsets $H_1=\{(r,s) \mid 0\le r\le 2,\ 0\le s\le n-2\}$ and $H_2=V(G)-H_1$. By Lemma 3.2.6, there exists a hamiltonian x,y-path P_1 in $\langle H_1\rangle$ which contains the edge $(1,n-2)(\ell+1,n-2)$ for some satisfying $0\le\ell\le 1$. If $\ell=1$, then the path $P_2:(1,n-1)(0,n-1)(2,n-1)$ can be used to construct a hamiltonian x,y-path in G by $[P_1,P_2]$-extension. The case where $\ell=0$ can be treated similarly. If $y=(1,n-1)$, then $y \in H_2$ and let $P_2' = (1,n-1)(0,n-1)(2,n-1)$. By Lemma 3.2.6, there exists a hamiltonian $(i,j),(2,n-2)$-path P_2' in $\langle H_1 \rangle$. The concatenation of P_1' and P_2' using the edge $(2,n-2)(2,n-1)$ is a hamiltonian x,y-path in G.

We proceed to complete the proof of the lemma by considering the case where n=2 or 3 and m is an odd integer greater than or equal to 5. Suppose that n=2 and x=(0,0). If y is a white vertex in G, then by Lemma 3.2.1, x is connected to y by a hamiltonian path. If y is a black vertex, then partition V(G) into two subsets $H_1 = \{(0,0),(0,1)\}$ and $H_2 = V(G) - H_1$. Since $(m-1,1)$ is a white vertex in H_2, by Lemma 3.2.1, there exists a hamiltonian $(m-1,1),(h,k)$-path P_2 in $\langle H_2 \rangle$. A hamiltonian x,y-path can be constructed from the concatenation of the path P_2 with the path $P:(0,0)(0,1)(m-1,1)$. If x=(1,1), then consider the partition on V(G) as follows. Let $H_1 = \{(r,s) \mid 0 \le r \le 1, 0 \le s \le 1\}$ and $H_2 = V(G) - H_1$. By Lemma 3.2.1, there exists a hamiltonian $(1,1),(0,1)$-path P_1' in $\langle H_1 \rangle$ and a hamiltonian $(m-1,1)(h,k)$-path P_2' in $\langle H_2 \rangle$. The concatenation of P_1' and P_2' results in a hamiltonian x,y-path in G. Finally, we assume that n=3. Lemma 3.2.6 accounts for the case when both x and y are black vertices and it remains to assume that y is a white vertex. By the symmetry of G, it suffices to assume that x=(0,0) or (0,2). Suppose that x=(0,0). If y=(h,k) such that

$h \geq 1$, then we partition $V(G)$ into two subsets $H_1 = \{(r,s) \mid r=0, 0 \leq s \leq 2\}$ and $H_2 = V(G) - H_1$. Since $|H_2|$ is even and $(m-1,2)$ is a black vertex, there exists a hamiltonian $(m-1,2),(h,k)$-path P_2 in $\langle H_2 \rangle$. A hamiltonian x,y-path can be constructed by concatenating P_2 with the path $P_1:(0,0)(0,1)(0,2)(m-1,2)$. If $y=(0,1)$, then we partition $V(G)$ into the two subsets $H_1 = \{(r,s) \mid 0 \leq r \leq m-1, \ 0 \leq s \leq 1\}$ and $H_2 = V(G) - H_1$. From the construction used in the case where $n=2$, it is clear that $\langle H_1 \rangle$ is hamiltonian-connected. P_1' be a hamiltonian $(0,0),(0,1)$-path in $\langle H_1 \rangle$. There exists an integer t satisfying $0 \leq t \leq m-1$ such that $(t,1)(t+1,1)$ is an edge on the path P_1'. Let $P_2':(t,2)(t-1,2)\ldots(0,2)(m-1,2)(m-2,2)\ldots(t+1,2)$ be a hamiltonian $(t,2),(t+1,2)$-path in $\langle H_2 \rangle$. A hamiltonian x,y-path can be obtained from $[P_1',P_2']$-extension. Suppose that $x=(1,1)$. We consider the same partition on $V(G)$ as in the last argument. If $y \in H_1$, then construct a hamiltonian x,y-path in $\langle H_1 \rangle$ and extend it to a hamiltonian x,y-path in G as in the last construction. If $y=(1,2)$, then let P_1'' be a hamiltonian $(1,1),(0,1)$-path in $\langle H_1 \rangle$. Let $P_2''=(0,2)(m-1,2)(m-2,2)\ldots(1,2)$ and the concatenation of P_1'' and P_2'' constitutes a hamiltonian x,y-path in G. If $y=(h,2)$ for some $h \geq 3$, then $V(G)$ is partitioned into $H_1=\{(r,s) \mid 0 \le r \le m-1,\ 1 \le s \le 2\}$ and $H_2=V(G)-H_1$. Since $\langle H_1 \rangle$ is hamiltonian-connected, a hamiltonian x,y-path in $\langle H_1 \rangle$ is obtained which can be extended to become a hamiltonian x,y-path in G as in a previous argument. This completes the proof of the lemma. ■

Lemma 3.2.8 Let n and m be two integers greater than or equal to 2 and G be a graph isomorphic to $L_m x L_n$. If G contains a white edge, then there exists a hamiltonian (0,0),(m-1,n-1)-path in G.

Proof: Let x=(0,0) and y=(m-1,n-1). If both m and n are odd integers, then by Lemma 3.2.6, there exists a hamiltonian x,y-path in G. If one of the integers m and n is even while the other one is odd, then y is a white vertex and by Lemma 3.2.1 there exists a hamiltonian x,y-path in G. If both m and n are even integers, then a straight forward double induction on m and n in a manner similar to the proof in Lemma 3.2.1 will establish the existence of a hamiltonian x,y-path in G. This completes the proof of the lemma. ■

Before we apply the results obtained so far to establish a characterization of the hamiltonian-connectedness of a graph isomorphic to $C_m x C_n$ for some integers $n \ge 2$ and $m \ge 3$, two useful comments are in order. It is important to observe that if n=2 or 3 and m is an even positive integer, then the graph $C_m x L_n$ is not hamiltonian-connected. However, any two vertices in the graph which are colored differently by a hamiltonian path.

The second comment concerns with simplifying of notations. Let $G=(V(G),E(G))$ be a graph and H_1 and H_2 be disjoint subsets of $V(G)$ such that for some $x,y \in H_1$ and $u,v \in H_2$, there exists a hamiltonian x,y-path P_1 in $\langle H_1 \rangle$ and a hamiltonian u,v-path P_2 in $\langle H_2 \rangle$. We shall simply say that P_1 is extended to a hamiltonian path in $\langle H_1 \cup H_2 \rangle$ by including H_2 to denote a $[P_1,P_2]$-extension, without specifying explicitly the forms of the vertices u,v and the vertices $u',v' \in V(P_1)$ such that $uu',vv' \in E(G)$.

Theorem 3.2.9 Let n and m be positive integers greater than or equal to 2. Let G be a graph isomorphic to $C_m x C_n$. Then G is hamiltonian-connected if and only if G contains a white edge and a black edge.

Proof: Let $G=C_m x C_n$ for some positive integers m and n greater than or equal to 2. Suppose that one of the integers n or m is odd. Without loss of generality, let m be an odd integer. Then, G contains a spanning subgraph isomorphic to $C_m x L_n$ which satisfies the hypotheses of Lemma 3.2.7. This implies that G is hamiltonian-connected. Furthermore, G contains at least one black edge and one white edge. Hence, for the remainder of the proof, both m and n are assumed to be even integers greater than or equal to 4.

We first establish the necessity and assume that G is hamiltonian-connected. Observe that $V(G)$ is an even integer greater than or equal to 16. Let u and v be two arbitrary white vertices and P be a hamiltonian u,v-path in G. It is clear that the number of black vertices and the number of white vertices on P are equal. Let u' and v' be two vertices on P such that the edges uu' and vv' are on the path P. Then the number of black vertices on the u',v'-path P', obtained from P by removing the initial and terminal vertices u and v, exceed the number of white vertices on P' by 2. It is clear that there exists at least one black edge on P'. If u and v were chosen to be black vertices, then a similar argument will show that there exists at least one white edge on P'. This established the necessity of the theorem.

It remains to establish the sufficiency of the theorem. Let x=(i,j) and y=(h,k) be two artitrary vertices in G. If x and y are of different colors, then by Lemma 3.2.5 and the comment preceeding the theorem, there exists a hamiltonian x,y-path in G. It remains to assume that both x and y are of the same color. Due to the symmetry of the problem, it suffices to assume that both x and y are colored black and x=(0,j) for some $j \in \{0,2,...,n-2\}$. Let y=(h,k) be an arbitrary black vertex in G distinct from x. We proceed by induction on $|j-k|$ to construct a hamiltonian x,y-path in the underlying spanning subgraph of G which is isomorphic to $C_m x L_n$. We first consider the case where $|j-k|=0$. Under this assumption, $j=k$ and $h \in \{2,4,...,m-2\}$. We partition V(G) into four subsets as follows (see Figure 3.3). Let

$H_1 = \{(r,s) | 0 \le r \le h-1,\ j \le s \le n-1\}$, $H_2 = \{(r,s) | h \le r \le m-1, j \le s \le n-1\}$

$H_3 = \{(r,s) | 0 \le r \le h-1,\ 0 \le s \le j-1\}$ and $H_4 = \{(r,s) | h \le r \le m-1, j \le s \le n-1\}$.

Observe that $|H_1|$, $|H_2|$, $|H_3|$ and $|H_4|$ are even integers.

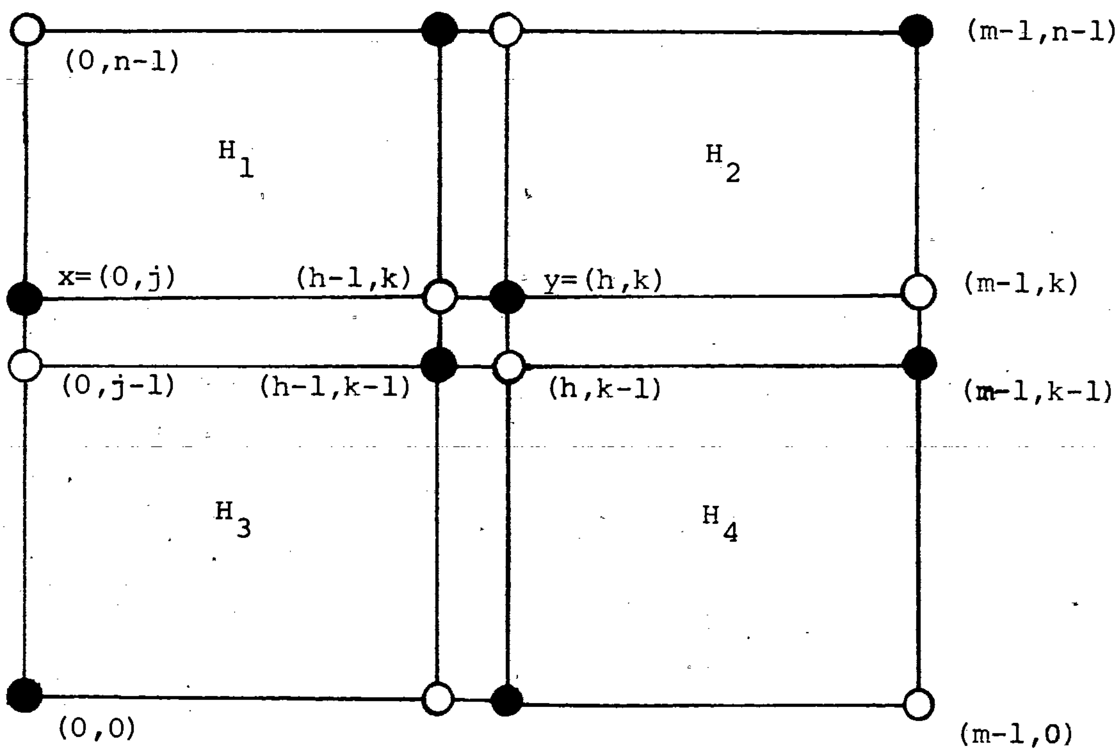


figure 3.3

By the hypotheses of the theorem, there exist two white vertices w_1 and w_2 in V(G) such that $w_1 w_2 \in E(G)$. There are four cases to consider.

Case 1 $w_1 w_2 \in H_i$, for each i=1,2,3,4. Suppose $w_1 w_2 \in H_2$.

By Lemma 3.2.8, there exists a hamiltonian $(0,j),(h-1,n-1)$-path P_1 in $\langle H_1 \rangle$. By Lemma 3.2.1, there exists a hamiltonian $(h,n-1),(h,k)$-path P_2 in $\langle H_2 \rangle$. Let P be the hamiltonian x,y-path in $\langle H_1 \cup H_2 \rangle$ constructed from concatenating P_1 and P_2 using the edge $(h-1,n-1)(h,n-1)$. The path P can be extended to a hamiltonian path in G by including $H_3 \cup H_4$. If $w_1, w_2 \in H_2$ then by Lemma 3.2.8 there exists a hamiltonian $(m-1,n-1),(h,k)$-path P_2' in $\langle H_2 \rangle$. Let P_1' be a hamiltonian $(0,j),(0,n-1)$-path in $\langle H_1 \rangle$. The concatenation of P_1' and P_2' using the edge $(0,n-1)(m-1,n-1)$ constitutes a hamiltonian x,y-path P' in $\langle H_1 \cup H_2 \rangle$. P' can now be extended to a hamiltonian x,y-path G by including $H_3 \cup H_4$. If $w_1, w_2 \in H_3$, then by Lemma 3.2.8, there exists a hamiltonian $(h-1,k-1),(0,0)$-path P_3'' in $\langle H_3 \rangle$. By Lemma 3.2.1, there exists a hamiltonian $(0,j),(h-1,k)$-path P_1'' in $\langle H_1 \rangle$, a hamiltonian $(m-1,0)(m-1,k-1)$-path P_4'' in $\langle H_4 \rangle$ and a hamiltonian $(m-1,k),(h,k)$-path P_2'' in $\langle H_2 \rangle$. The concatenation of the paths P_1'', P_2'', P_3'' and P_4'' using the edges $(h-1,k)(h-1,k-1)$, $(0,0)(m-1,0)$ and $(m-1,k-1)(m-1,k)$ constitutes a hamiltonian x,y-path in G. If $w_1, w_2 \in H_4$, then there exists a hamiltonian $(h,0),(m-1,k-1)$-path P_4''' in $\langle H_4 \rangle$. By Lemma 3.2.1, there exists a hamiltonian $(0,j),(h-1,k)$-path P_1''' in $\langle H_1 \rangle$, a hamiltonian $(h-1,k-1),(h-1,0)$-path P_3''' in $\langle H_3 \rangle$ and a hamiltonian $(m-1,k),(h,k)$-path P_2''' in

H_2 . The concatenation of P_1''', P_2''', P_3''' and P_4''' using the edges (h-1,k)(h-1,0), (h-1,0)(h,0) and (m-1,k-1)(m-1,k) results in a hamiltonian x,y-path in G.

Case 2 ($w_1 \in H_1$ and $w_2 \in H_3$) or ($w_1 \in H_2$ and $w_2 \in H_4$) or ($w_1 \in H_3$ and $w_2 \in H_4$). Suppose that $w_1 \in H_1$ and $w_2 \in H_3$. By Lemma 3.2.1, there exists a hamiltonian x,w_1-path P_1 in $\langle H_1 \rangle$, a hamiltonian w_2,(h-1,k-1)-path P_3 in $\langle H_3 \rangle$, a hamiltonian (h,k-1),(m-1,k-1)-path P_4 in $\langle H_4 \rangle$ and a hamiltonian (m-1,k),(h,k)-path P_2 in $\langle H_2 \rangle$. The concatenation of P_1, P_2, P_3 and P_4 using the edges w_1w_2, (h-1,k-1)(h-1,k) and (m-1,k-1)(m-1,k) results in a hamiltonian x,y-path in G. The other two cases where $w_1 \in H_2$ and $w_2 \in H_4$ or $w_1 \in H_3$ and $w_2 \in H_4$ can be treated similarly.

Case 3 ($w_1 \in H_1$ and $w_2 \in H_4$) or ($w_1 \in H_2$ and $w_2 \in H_3$). Suppose that $w_1 \in H_1$ and $w_2 \in H_4$. By Lemma 3.2.1, there exists a hamiltonian x,w_1-path P_1 in $\langle H_1 \rangle$, a hamiltonian w_2,(m-1,k-1)-path P_4 in $\langle H_4 \rangle$ and a hamiltonian (m-1,k),(h,k)-path P_2 in $\langle H_2 \rangle$. By concatenating P_1, P_2 and P_4 using the edges w_1w_2 and (m-1,k-1)(m-1,k), we obtain a hamiltonian x,y-path P in $\langle H_1 \cup H_2 \cup H_4 \rangle$. The path P can now be extended to a hamiltonian x,y-path in G by including H_3. Using a similar argument, a hamiltonian x,y-path in G can be constructed if $w_1 \in H_2$ and $w_2 \in H_3$.

Case 4 $w_1 \in H_1$ and $w_2 \in H_2$. We first suppose that either

$m \geq 6$ or the white edge w_1w_2 is not in the form $(h-1,k)(m-1,k)$. By Lemma 3.2.1 there exists a hamiltonian x,w_1-path P_1 in $\langle H_1 \rangle$ and a hamiltonian w_2,y-path P_2 in $\langle H_2 \rangle$. Let P be the hamiltonian x,y-path in $\langle H_1 \cup H_2 \rangle$ produced by concatenating P_1 and P_2 using the white edge w_1w_2. If $|H_3 \cup H_4| > 0$, then P can be extended to a hamiltonian x,y-path in G by including $H_3 \cup H_4$. If $|H_3| = |H_4| = 0$, then P itself is a hamiltonian x,y-path in G. It remains to consider the case where m=4 and the white vertices w_1 and w_2 are in the forms (1,j) and (3,j), respectively. Consider the x,y-path P':(0,j)(1,j)(3,j)(2,j). If $|H_3 \cup H_4| > 0$, then by Corollary 3.2.3, there exists a hamiltonian (0,j-1),1,j-1)-path P_{34} in $\langle H_3 \cup H_4 \rangle$. Let P^* be the x,y-path in $\langle H_3 \cup H_3 \cup V(P') \rangle$ constructed by $[P', P_{34}]$-extension. It is clear that there exists a hamiltonian (2,j+1),(3,j+1)-path $P_{1,2}$ in $\langle (H_1 \cup H_2) - V(P') \rangle$. P^* can now be extended to a hamiltonian x,y-path in G by $[P^*, P_{1,2}]$-extension.

This establishes the hamiltonian-connectedness of G when j=k. Suppose that for some nonnegative integer $\ell^* \geq 0$, the theorem has been esatblished for each value of $|j-k|$ satisfying $0 \leq |j-k| \leq \ell^*$. Let x=(0,j) and y=(h,k) such that $j-k = \ell^*+1$. The case where $k>j$ will be considered separately from the case where $k<j$. We first investigate the former case.

Case a $k=j+(\ell^*+1)>j$.

We first assume that $j \geq 2$ and $k \leq n-2$ and partition $V(G)$ into two subsets $H_1=\{(r,s) \mid 0 \leq r \leq m-1,\ 0 \leq s \leq j\}$ and $H_2=V(G)-H_1$. Suppose that the white edge w_1w_2 is in $\langle H_1 \rangle$. By the induction hypothesis, there exists a hamiltonian $(0,j),(m-2,j)$-path P_1 in $\langle H_1 \rangle$. Since $(m-2,j)$ is a black vertex in H_1, $(m-2,j+1)$ is a white vertex in H_2. By a comment preceeding Theorem 3.2.9 or by Lemma 3.2.5, there exists a hamiltonian $(m-2,j+1),(h,k)$-path P_2 in $\langle H_2 \rangle$. The concatenation of P_1 and P_2 using the edge $(m-2,j)(m-2,j+1)$ provides a hamiltonian x,y-path in G. Suppose that the white edge w_1w_2 is in $\langle H_2 \rangle$. Let q be a integer satisfying $1 \leq q \leq m-1$ such that $(q,j+1)$ is a black vertex distinct from y. By the induction hypothesis, there exists a hamiltonian $(0,j+1),(h,k)$-path P_2' in $\langle H_2 \rangle$. By Lemma 3.2.1, there exists a hamiltonian $(0,j),(q,j)$-path P_1' in $\langle H_1 \rangle$. A hamiltonian x,y-path in G can be constructed from the concatenation of P_1' and P_2' using the edge $(q,j)(q,j+1)$. Next, we assume that $w_1 \in H_1$ and $w_2 \in H_2$. By Theorem 3.2.5 or by the comment preceeding Theorem 3.2.9, there exists a hamiltonian $(0,j),w_1$-path P_1'' in $\langle H_1 \rangle$ and a hamiltonian $w_2,(h,k)$-path P_2'' in $\langle H_2 \rangle$. A hamiltonian x,y-path can be constructed by concatenating P_1'' and P_2'' using the white edge w_1w_2. It remains to assume that either $j=0$

or k=n-1. If G contains $C_m x L_2$ as a spanning subgraph, then it can be easily shown that G is hamiltonian-connected. Hence, we let $n \geq 4$ and consider the case where $x=(0,j)$ and $y=(\ell,n-1)$, for some odd integer ℓ satisfying $1 \leq \ell \leq m-1$. If $j \leq n-4$, then a construction similar to the previous argument will account for the existence of a hamiltonian x,y-path in G. If $x=(0,n-2)$ then $y=(h,n-1)$ and we let V(G) be partitioned into two subsets $H_3=\{(r,s) \mid 0 \leq r \leq m-1,\ n-2 \leq s \leq n-1\}$ and $H_4=V(G)-H_3$. If the white edge w_1w_2 is in $\langle H_3 \rangle$, then it is clear that there exists a hamiltonian x,y-path in $\langle H_3 \rangle$ which can be extended to a hamiltonian x,y-path in G by including H_4. If the white edge w_1w_2 is in $\langle H_4 \rangle$, then by the induction hypothesis there exists a hamiltonian path P_4 connecting any two black vertices in H_4 of the form $(p,n-3)$ and $(q,n-3)$, respectively. In particular, we choose p=1 and q to be the greatest odd integer such that $q \leq h$ and let H_3 be partitioned into two subsets $H_{31}=\{(r,s) \mid 0 \leq r < q, n-2 \leq s \leq n-1\}$ and $H_{32}=H_3-H_{31}=\{(v,s) \mid q \leq r \leq m-1, n-2 \leq s \leq n-1\}$. It is clear that $x \in H_{31}$ and $y \in H_{32}$. It is clear that there exists a hamiltonian $(0,n-2)$, $(1,n-2)$-path P_{31} in $\langle H_{31} \rangle$ and a hamiltonian $(q,n-2),(h,n-1)$-path P_{32} in $\langle H_{32} \rangle$. A hamiltonian x,y-path can be constructed by concatenating P_4, P_{31} and P_{32} using edges $(1,n-1)(1,n-2)$ and $(q,n-1)(1,n-2)$. The construction is valid as long as $y \neq (1,n-1)$. If $y=(1,n-1)$, then by the induction hypothesis, there exists a hamiltonian $(0,n-2),(2,n-2)$-path P' in $\langle H_4 \cup \{(r,s) \mid 0 \le r \le m-1, s=n-2\} \rangle$. Let $P'': (1,n-1)(0,n-1)(m-1,n-1)(m-2,n-1)\ldots(2,n-2)$ be a hamiltonian $(1,n-1),(2,n-2)$-path in $\langle H_3 - \{(r,s)\ 0 \le r \le m-1, s=n-2\} \rangle$. The concatenation of P' and P" using the edge $(2,n-2)(2,n-1)$ constitutes a hamiltonian x,y-path in G. It remains to consider the case where $w_1 \in H_3$ and $w_2 \in H_4$. In particular, let $w_1=(q,\ell)$. We first assume that $y \ne (1,n-1)$ and use the same partition on H_3 as the above. If $q \ge h$, then there exists a hamiltonian y,w_1-path P'_{32} in $\langle H_{32} \rangle$ and a hamiltonian $(0,n-2),(1,n-2)$-path P'_{31} in $\langle H_{31} \rangle$. Furthermore, either by Lemma 3.2.5 or by the comment preceeding Theorem 3.2.9, there exists a hamiltonian $(1,n-2),w_2$-path P_4 in $\langle H_4 \rangle$. The concatenation of the paths P'_{31} P'_4 and P'_{32} using the edges $(1,n-2)(1,n-3)$ and w_1w_2 constitutes a hamiltonian x,y-path in G. If $q < h$, then $w_1 \in H_{31}$ and by Lemma 3.2.1, there exists a hamiltonian x,w_1-path P''_{31} in $\langle H_{31} \rangle$. Observe that $(m-1,n-3)$ is a black vertex in H_4. By Lemma 3.2.1, there exists a hamiltonian $w_2,(m-1,n-3)$-path P''_4 in $\langle H_4 \rangle$ and a hamiltonian $(m-1,n-2),(h,n-1)$-path P''_{32} in $\langle H_{32} \rangle$. A hamiltonian x,y-path can be constructed by concatenating P''_{31}, P''_{32} and P''_4 using the edges w_1w_2 and $(m-1,n-3)(m-1,n-2)$. Now, suppose that $y=(1,n-1)$. If $w_1 \ne (0,n-1)$, then we let H_3 be partitioned into two subsets $H_{33}=\{(0,n-2),(0,n-1),(m-1,n-1)(m-1,n-2)\}$ and and $H_{34}=\{(r,s) \mid 1 \le r \le m-2, n-2 \le s \le n-1\}$. It is clear that $y,w_1 \in H_{34}$ and by the comment preceeding Theorem 3.2.9, there exists a hamiltonian w_1,y-path P''_{33} in $\langle H_{33} \rangle$. Let

P_{34}''': (0,n-2)(0,n-1)(m-1,n-1)(m-1,n-2) be the hamiltonian path in $\langle H_{34}\rangle$. Since (m-1,n-2) is a black vertex, by Lemma 3.2.1 there exists a hamiltonian (m-1,n-3),w_2-path P_4''' in $\langle H_4\rangle$. The concatenation of P_{33}''', P_{34}''' and P_4''' using the edges w_1w_2 and (m-1,n-3)(m-1,n-2) constitutes a hamiltonian x,y-path in G. However, if w_1=(0,n-1), then we partition V(G) into two subsets $H_5=\{(r,s)\mid 0\le r\le m-1,\ 0\le s\le n-2\}$ and $H_6=V(G)-H_5$. By Lemma 3.2.1, there exists a hamiltonian x,w_2-path P_5 in $\langle H_5\rangle$. Consider the hamiltonian path in $\langle H_6\rangle$ defined as P_6: (0,n-1)(m-1,n-1)(m-2,n-1)... .(1,n-1). A hamiltonian x,y-path can now be constructed by concatenating the paths P_5 and P_6 using the edge w_1w_2. This completes the constructions of all the hamiltonian x,y-paths for the case where x=(0,n-1) and y=(h,n-1). It remains to assume that x=(0,0) and y=(h,k) with $n\ge 4$. Let the end vertices of the white edge be in the form $w_1=(a_1,b_1)$ and $w_2=(a_2,b_2)$. Suppose that $0\le b_1\le 1$ and $0\le b_2\le 1$. Let V(G) be partitioned into two subsets $H_1=\{(r,s)\mid 0\le r\le m-1, 0\le s\le 1\}$ and $H_2=V(G)-H_1$. It is clear in this case that the white edge w_1w_2 is in $\langle H_1\rangle$. If y=(h,k) $\in H_2$, then by Lemma 3.2.1 there exists a hamiltonian (h,k),(m-1,2)-path P_2 in $\langle H_2\rangle$. By Lemma 3.2.8, there exists a hamiltonian (0,0),(m-1,1)-path P_1 in $\langle H_1\rangle$. The concatenation of P_1 and P_2 using the edge (m-1,1)(m-1,2) constitutes a hamiltonian x,y-path in G. Suppose that $y\in H_1$. Since $\langle H_1\rangle$ contains $C_m x L_2$ as a spanning subgraph, it is clear from a previous comment that $\langle H_1\rangle$ is hamiltonian-connected. Then, there exists a hamiltonian x,y-path in $\langle H_1\rangle$ which can be extended to become a hamiltonian x,y-path in G by including H_2. We now consider the situation when $w_1 \in H_1$ and $w_2 \in H_2$ (that is, $0 \le b_1 \le 1$ and $b_2 \ge 2$). If $y \in H_2$, then by Lemma 3.2.5 or by the comment preceeding Theorem 3.2.9, there exists a hamiltonian x,w_1-path P_1' in $\langle H_1 \rangle$ and a hamiltonian w_2,y-path P_2' in $\langle H_2 \rangle$. A hamiltonian x,y-path in G can be produced by concatenating P_1' and P_2' using the white edge w_1w_2. If $y \in H_1$, then we further partition H_1 into two subsets $H_{11} = \{(0,0),(0,1)\}$ and $H_{12} = H_1 - H_{11} = \{(r,s) \mid 1 \le r \le m-1, 0 \le s \le 1\}$. Suppose that $w_1 = (0,1)$. It is clear there exists a hamiltonian $w_2,(m-2,2)$-path P_2'' in $\langle H_2 \rangle$ and a hamiltonian $(m-2,1),y$-path P_{12}'' in $\langle H_{12} \rangle$. Let $P_{11}'' = (0,0)(0,1)$ which is a hamiltonain path of length 1 in H_{11}. A hamiltonian x,y-path in G can be constructed by concatenating the paths P_{11}'', P_{12}'' and P_2'' using the edges w_1w_2 and $(m-2,2)(m-2,1)$. Suppose that $w_1 \in H_{12}$. By Lemma 3.2.1, there exists a hamiltonian $(0,2),w_2$-path P_2''' in $\langle H_2 \rangle$ and by the comment preceeding Theorem 3.2.9, there exists a hamiltonian w_1,y-path P_{12}''' in $\langle H_{12} \rangle$. Let P_{11}'' be the same path of length 1 as the above. We construct a hamiltonian x,y-path in G by concatenating P_{11}'', P_{12}''' and P_2''' using the edges $(0,1)(0,2)$ and w_1w_2. It remains to assume that $w_1,w_2 \in H_2$ (that is, $b_1 \ge 2$ and $b_2 \ge 2$). Let q be an even integer such that $0 \le q \le m-2$ and such the vertex $(q,2)$ is colored black and is distinct from $y=(h,k)$. By the induction hypothesis, there exists a hamiltonian $(h,k),(q,2)$-path P_2 in $\langle H_2 \rangle$. By Lemma 3.2.1, there exists a hamiltonian $(0,0),(q,1)$-path P_1 in $\langle H_1 \rangle$. The concatenation of the paths P_1 and P_2 using the edge $(q,2)(q,1)$ constitutes a hamiltonian x,y-path in G.

This completes the constructions of all the hamiltonian

x,y-paths in G in case 1.a. It remains to investigate the case where $j>k$.

Case b: $j=k+(\ell^*+1)>k$. Let V(G) be partitioned into two subsets $H_1=\{(r,s) \mid 0 \le r \le m-1, j \le s \le n-1\}$ and $H_2=\{(r,s) \mid 0 \le r \le m-1, 0 \le s \le j-1\}$. It is clear that $2 \le j \le n-2$, $x \in H_1$ and $y \in H_2$. Suppose that $w_1, w_2 \in H_1$ and let q be an even integer such that $q>0$ and (q,j) is a black vertex in H_1. By the induction hypothesis, there exists a hamiltonian $(0,j),(q,j)$-path P_1 in $\langle H_1 \rangle$. It is clear that $(q,j-1)$ is a white vertex in H_2 and by either the comment preceeding Theorem 3.2.9 or by Lemma 3.2.5, there exists a hamiltonian $(q,j-1),(h,k)$-path P_1 in $\langle H_1 \rangle$. The concatenation of the paths P_1 and P_2 using the edge $(q,j-1)(q,j)$ results in a hamiltonian x,y-path in G. A similar construction will account for the existence of a hamiltonian x,y-path in G in the case where $w_1, w_2 \in H_2$. It remains to assume that $w_1 \in H_1$ and $w_2 \in H_2$. By Lemma 3.2.5 or by the comment preceeding Theorem 3.2.9, there exists a hamiltonian x,w_1-path P_1' in $\langle H_1 \rangle$ and a hamiltonian w_2,y-path P_2' in $\langle H_2 \rangle$. The concatenation of P_1' and P_2' using the white edge w_1w_2 constitutes a hamiltonian x,y-path in G.

This completes the proof of the theorem. ∎

In the following section, the results obtained so far on the hamiltonian properties of graphs isomorphic to $L_m \times L_n$, $C_m \times L_n$ or $C_m \times C_n$ will be applied to give a characterization of the hamiltonian-connectedness of a Cayley graph of an abelian group. For consistency of notations, a graph isomorphic to $C_m \times L_2$ will be written as $C_m \times C_2$ without loss of generality.

Section 3.3 The hamiltonian-connected of a Cayley graph of an abelian group.

Theorem 3.3.1 Let A be an abelian group of order $|A| \geq 3$ and S be a symbol for A such that $\langle S \rangle = A$. Then the Cayley graph C(S,A) contains an underlying spanning subgraph isomorphic to either $C_{|A|}$ or $C_m \times C_n$ for some integers n and m satisfying $|A| = nm, n \geq 2$ and $m \geq 2$. Furthermore, C(S,A) is hamiltonian.

Proof: Let A be an abelian group and S be a generating set for A which satisfies the hypothesis of the theorem. Without loss of generality, there exists a minimal generating set S* for A such that $S^* \subsetneq S$. We proceed by induction on $|S^*|$.

If $|S^*| = 1$, then $C(S^*,A)$ is an underlying subgraph of C(S,A) isomorphic to $C_{|A|}$. Clearly, C(S,A) is hamiltonian. Suppose that for some positive integer $k \geq 1$, the theorem has been proven for each value for $|S^*|$ satisfying $1 \leq |S^*| \leq k$. We now consider the case where $|S^*| = k+1$. Let h be an arbitrary generator in S* of order O(h). It is clear that the subgroup $F = \langle S^* - \{h\} \rangle$ is a proper subgroup in A. Furthermore, the Cayley graph $C(S^* - \{h\}, A)$ consists of O(h) many disjoint copies of connected subgraphs $\{B_0, B_1, \ldots B_{O(h)-1}\}$, such that for each $j \in \{0,1,\ldots,O(h)-1\}$, the coset $h^j F$ uniquely represents the set of all vertices of the copy B_j. It is clear that for each $j \in \{0,1,\ldots,O(h)-1\}$, the mapping $f: h^j F \rightarrow h^{j+1} F$ defined by $f(v) = hv$ is a graph isomorphism from B_j to B_{j+1}, where j+1 is obtained by addition module O(h). Let $n = |F|$ and $m = O(h)$. By the induction hypothesis, there exists a hamiltonian cycle of length n in B. Without loss of generality, let the vertices of this hamiltonian cycle be labelled $\{V_0, V_1, \ldots, V_{n-1}\}$. Then for each $j \in \{0,1,\ldots,m-1\}$, a corresponding hamiltonian cycle in B_j will be in the form $f^j(V_0), f^j(V_1), \ldots, f^j(V_{n-1})$ such that $f^j(V_i)$ and $f^j(V_{i+1})$ are adjacent on the cycle for each i satisfying $0 \leq i \leq n-1$. Since A is an abelian group, $f^{j+1}(V_i) = h^{j+1}V_i = (h^j V_i)h = (f^j(V_i))h$. This implies that the vertex $f^j(V_i)$ on the hamiltonian cycle in B_j is adjacent to the vertex $f^{j+1}(V_i)$ on the hamiltonian cycle in B_{j+1}. This constitutes an underlying subgraph G in $C(S^*,A)$ isomorphic to $C_m \times C_n$ such that $n \geq 2$ and $m \geq 2$ (see Figure 3.4). If either of n or m is an odd integer, then by Lemma 3.2.7, G is hamiltonian-connected which implied G is hamiltonian. If both n and m are even integers, then G contains a hamiltonian cycle in the form

$$(V_0)(V_1)\ldots(V_{n-1})(hV_{n-1})(hV_{n-2})\ldots(hV_1)(h^2V_1)(h^2V_2)\ldots(h^2V_{n-1})$$
$$(h^3V_{n-1})(h^3V_{n-2})\ldots(h^3V_1)(h^4V_1)(h^4V_2)\ldots(h^{m-2}V_{n-2})(h^{m-2}V_{n-1})$$
$$(h^{m-1}V_{n-1})(h^{m-1}V_{n-2})\ldots(h^{m-1}V_{n-2})\ldots(h^{m-1}V_0)(h^{m-2}V_0)\ldots(h^1V_0)(V_0).$$

(See Figure 3.5). This completes the proof of the theorem. ■

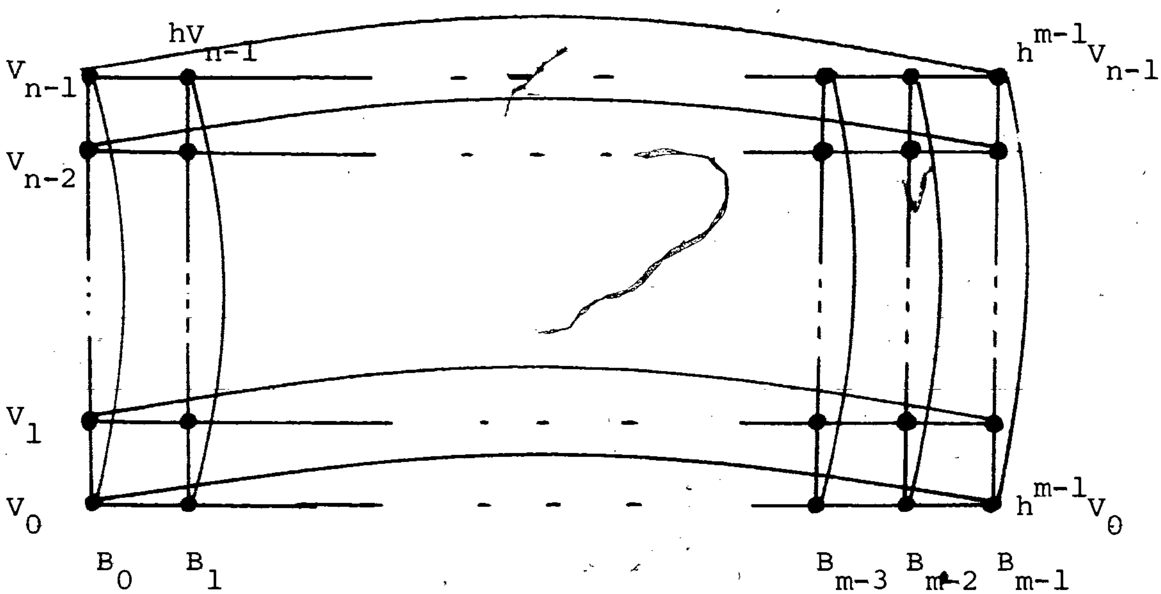


figure 3.4

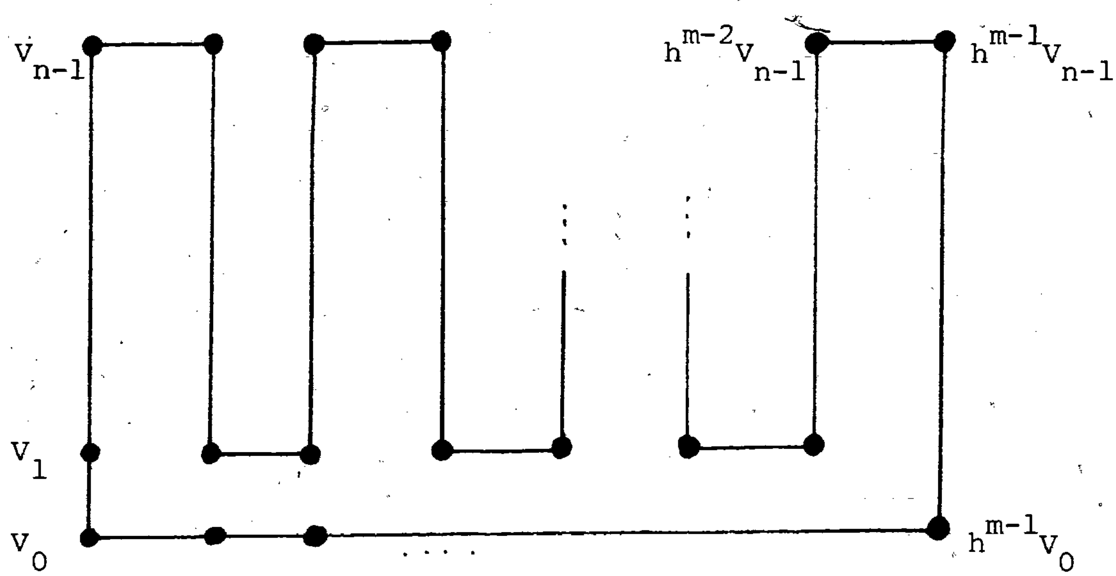


figure 3.5

Theorem 3.3.2 Let A be an abelian group of order $|A|\geq 3$ and S be a symbol for A. Suppose that $\langle S\rangle = A$ and that there exists a generator $h\in S$ such that $\langle\{h\}\rangle$ is a proper subgroup in A. Then C(S,A) is hamiltonian-connected if and only if C(S,A) is not bipartite.

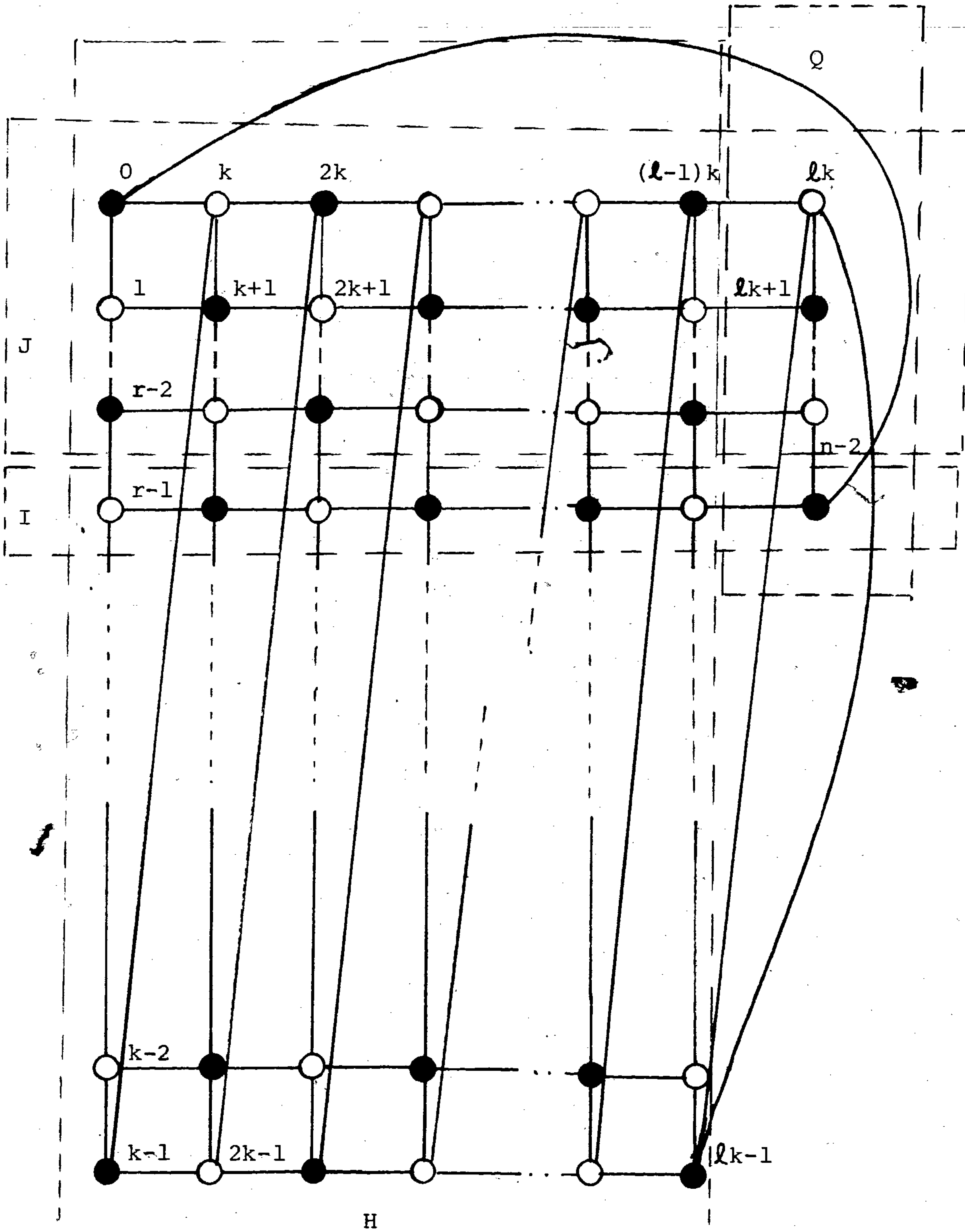
Proof: If C(S,A) is bipartite, then it is clear that C(S,A) is not hamiltonian-connected. To establish the converse, we assume that C(S,A) is not bipartite. Let m be the order of the subgroup $\langle\{h\}\rangle$. Since $\langle\{h\}\rangle$ is a proper subgroup in A, $m<|A|$. Let n be the integer which satisfies $|A|=mn\geq 4$. It is clear that C(S,A) contains an underlying spanning subgraph G isomorphic to $C_m \times C_n$. If either of m or n is an odd integer, then $|A|=mn\geq 6$ and G contains a spanning subgraph which, by Lemma 3.2.7, is hamiltonian-connected. It ramains to assume that both m and n are even integers for the remainder of the proof.

Let the vertices of $G=C_m \times C_n$ be colored in the same way as in the previous sections. Let B and W represent the set of all black vertices and the set of all white vertices in G, respectively. It is clear that $|B|=|W|$. Since C(S,A) is not bipartite, there exists either a black edge or a white edge in C(S,A). Suppose that C(S,A) contains a black edge b_1b_2. Let $k\in S$ such that $b_1k=b_2$ and define a bijection $f:A\to A$ by $f(x)=xk$ for each x in A. Since $f(b_1)=b_2$, $f(B)\nsubseteq W$ and this implies that $f(B)\nsubseteq W$. This implies that there exist two white vertices $w_1,w_2\in W$ such that $f(w_1)=w_1k=w_2$. Hence, w_1w_2 is a white edge in C(S,A). Therefore, C(S,A) contains at least one black edge and one white edge. By Theorem 3.2.9, C(S,A) is hamiltonian-connected. ■

<u>Theorem 3.3.3</u> Let A be an abelian group and S be a symbol for A such that for each $h\in S$, $\langle\{h\}\rangle=A$ and that there exist two distinct generators r and s in S such that $r\neq s^{-1}$. Then C(S,A) is hamiltonian-connected if and only if $|A|$ is odd.

<u>Proof</u>: Let A be an abelian group and S be a symbol for A which satisfies the hypotheses of the theorem. Since A is cyclic, there is no loss of generality in assuming that $A=Z_n$, where $n=|A|$, and Z_n is the addition cyclic group whose elements are members of the set $\{0,1,\ldots,n-1\}$. Furthermore, we can assume that $1\in S$ and each generator in S is an odd integer in Z_n. If n is even, then it is clear that by coloring each vertex x of C(S,A) black or white depending on whether x is even or odd, respectively, no two vertices of the same color are adjacent. Namely, C(S,A) is bipartite and by Theorem 3.3.2 C(S,A) is not hamiltonian-connected.

To establish the converse, we assume that n is an odd integer.



This figure assumes k and ℓ to be odd as an example.

figure 3.6

Since $|S|>1$, there exists an integer $k\in\{0,1,\ldots,\left[\frac{n}{2}\right]\}$ such that $k\neq 1$ and $\gcd(k,n)=1$. Let ℓ and r be the unique integers such that $n=\ell k+r$ with $0<r<k$. It is clear that $C(S,A)$ contains a spanning subgraph in the form as illustrated in Figure 3.6. Since $C(S,A)$ is vertex-transitive, in order to show that $C(S,A)$ is hamiltonian-connected it suffices to construct a hamiltonian $(n-1,x)$-path in $C(S,A)$ for each $x\in\{0,1,\ldots,n-2\}$. Let $V(C(S,A))$ be labelled by the following subsets (see figure 3.6):

$H=\{0,1,\ldots,k-1\}$,

$I=\{r-1,k+(r-1),2k+(r-1),\ldots,n-1\}$,

$J=\{ik+j\mid i=0,1,\ldots,\ell, j=0,1,\ldots,r-2\}$, and

$Q=\{\ell k,\ell k+1,\ldots,n-1\}=V(S,A)-H$.

Since H is isomorphic to $L_\ell \times L_k$, let the vertices be colored the usual way by colors black and white such that the vertex 0 is always colored black. For each vertex in Q, $\ell k+j$ is colored black if and only if $(\ell-1)k+j$ is colored white, for each $j\in\{0,1,\ldots,r-1\}$, and vice-versa, (see Figure 3.6).

There are three cases to be considered.

<u>Case 1</u> k is even. Since k is even, the vertices $(\ell-1)k$ and $\ell k-1$ are colored differently. Let x be an arbitrary vertex in H and y be either $(\ell-1)k$ or $\ell k-1$ such that y is colored differently from x. By Lemma 3.2.1, there exists a hamiltonian x,y-path P in $\langle H\rangle$. Let $P'=(n-1)(n-2)\ldots(\ell k+1)(\ell k)$, which is a hamiltonian $n-1,\ell k$-path in $\langle Q\rangle$. Since k is adjacent to both $(\ell-1)k$ and $\ell k-1$, ℓk is adjacent to y. The concatenation of P and P' using the edge $(\ell k)y$ is a hamiltonian $n-1,x$-path in $C(S,A)$. Observe

that this construction accounts for each hamiltonian (n-1,x)-path, for each $x \in \{0,1,\ldots,\left[\frac{n}{2}\right]\}$. Let x be an arbitrary integer in the set $\{\left[\frac{n}{2}\right]+1, \left[\frac{n}{2}\right]+2,\ldots,n-2\}$. Consider the graph automorphism $f:A \to A$ defined by $f(z)=(n-2)-z$. It is clear that $f(n-1)=n-1$. For each $x \in \{0,1,\ldots,\left[\frac{n}{2}\right]\}$, let the vertices of a hamiltonian n-1,x-path in C(S,A) be labelled as $P^*:V_{n-1}V_{n-2}\ldots V_0$ such that $V_{n-1}=n-1$ and $V_0=x$. It is clear that $f(V_{n-1})=n-1$ and $f(V_0) \in \{\left[\frac{n}{2}\right]+1,\ldots,n-2\}$. Then the hamiltonian path $f(P^*):f(V_{n-1})f(V_{n-2})\ldots f(V_0)$ is necessarily a path which has n-1 as the initial vertex and a terminal vertex in the set $\{\left[\frac{n}{2}\right]+1,\ldots,n-2\}$. Mapping each of the (n-1,x)-paths obtained in the previous construction with f will account for all the hamiltonian (n-1,x)-path in C(S,A), for each $z \in \{\left[\frac{n}{2}\right]+1, \left[\frac{n}{2}\right]+2,\ldots,n-2\}$.

Case 2 Both k and ℓ are odd integers. Under this assumption, the vertices $0, k-1, (\ell-1)k, \ell k-1$ and $n-1$ are all of color black and the vertex ℓk is colored white. Let x be an arbitrary black vertex in H. Let y be either $(\ell-1)k$ or $\ell k-1$ such that $y \neq x$. By Lemma 3.2.2, there exists a hamiltonian y,x-path P in $\langle H \rangle$. Let $P'=(n-1)(n-2)\ldots(\ell k+1)(\ell k)$. A hamiltonian (n-1,x)-path can be constructed by concatenating P and P' using the edge $(\ell k)y$. Suppose that x is a black vertex in Q. Since ℓk is a white vertex, by Lemma 3.2.1, there exists a hamiltonian ℓk,x-path P" in $\langle J \rangle$ and a hamiltonian $(\ell k-1),(n-1-k)$-path P''' in $\langle H-J \rangle$. The concatenation of P",P''' and the edge $(n-1)(n-1-k)$ using the edge $(\ell k-1)(\ell k)$ constitutes a hamiltonian n-1,x-path in C(S,A). This accounts for all the hamiltonian n-1,x-paths for all black terminal vertices x in C(S,A). Let f be the same graph automorphism defined in case 1. Since n is an odd integer, for each x which is colored black in C(S,A), f(x) is a white vertex in C(S,A). For each hamiltonian n-1,x-path P* in C(S,A), f(P*) defined in a manner as illustrated in case 1 is a hamiltonian n-1,f(x)-path in C(S,A). This accounts for all the hamiltonian paths which have n-1 as the initial vertex and a white vertex as a terminal vertex.

Case 3 k is odd and ℓ is even. Under this assumption, the vertices 0,k-1, ℓk and n-1 are all colored black while (ℓ-1)k and ℓk-1 are colored white. Let x be an arbitrary black vertex in H. Then, by Lemma 3.2.1 there exists a hamiltonian (ℓ-1)k,x-path P' in $\langle H \rangle$. Let P'=(n-1)(n-2)...(ℓk+1)(ℓk). The concatenation of P and P' using the edge ((ℓ-1)k)-(ℓk) constitutes a hamiltonian n-1,x-path in C(S,A). If x is a black vertex in Q then by Lemma 3.2.2 there exists a hamiltonian n-1,x-path P" in $\langle I \cup J \rangle$. Since $|A-(I \cup J)|$ is even, P" can be extended to a hamiltonian n-1,x-path in C(S,A) by including $\langle A-(I \cup J) \rangle$. This accounts for all the hamiltonian n-1,x-path where x is a black vertex in C(S,A) distinct from n-1. For any white vertex x in C(S,A), a hamiltonian n-1,x-path can be obtained by using the graph automorphism f in the manner discribed in Case 2.

This completes the proof of the theorem. ■

Using all the results obtained so far, a characterization on the hamiltonian-connected of a cayley graph of an abelian group can now be given.

Theorem 3.3.4 Let A be an abelian group and S be a symbol for A such that $\langle S\rangle = A$. Then the Cayley graph C(S,A) is hamiltonian-connected if and only if C(S,A) is neither a cycle of length $|A|$ nor bipartite.

Proof: The necessity of the theorem is obvious. To establish the sufficiency, we assume that C(S,A) is neither isomorphic to $C_{|A|}$ nor is bipartite. It is clear that there exist at least two distinct generators $h_1, h_2 \in S$ such that $h_1 \neq h_2^{-1}$. Suppose that for all $h \in S$, $\langle\{h\}\rangle = A$. Then it is clear that $|A|$ is odd and by Theorem 3.3.3, the Cayley graph C(S,A) is hamiltonian-connected. Otherwise, if there exists $h' \in S$ such that the $\langle\{h'\}\rangle$ is a proper subgraph in A, then by Theorem 3.3.2, C(S,A) is hamiltonian-connected. This completes the proof of the theorem. ■

Since a graph is bipartite if and only if all its cycles are even, Theorem 3.3.4 can be rephrased as follows.

Theorem 3.3.4a Let A be an abelian group and S be a symbol for A such that $\langle S\rangle = A$. Then the Cayley graph C(S,A) is hamiltonian-connected if and only if C(S,A) is not isomorphic to $C_{|A|}$ and contains an odd cycle.

Theorem 3.3.4b Let A be an abelian group and S be a symbol for A such that $\langle S\rangle = A$. Then, the Cayley graph C(S,A) is hamiltonian-connected if and only if C(S,A) is not isomorphic to $C_{|A|}$ and there exists a sequence of integers $\{P_1, P_2, \ldots, P_n\}$ and a sequence of generators $\{h_1, h_2, \ldots, h_n\}$ in S such that $h_1^{P_1} h_2^{P_2} \ldots h_m^{P_n} = e$ and the sum $P_1 + P_2 + \ldots + P_m$ is odd.

CHAPTER 4

ON SOME SPECIAL CLASSES OF HAMILTONIAN-CONNECTED GRAPHS

In Chapter 2, several necessary and sufficient conditions for a graph to be hamiltonian-connected have been examined. In this chapter, it will be shown that the implications of some of those conditions go beyond that of assuring a graph to be hamiltonian-connected. Sometimes with additional constraints, some of those sufficient conditions for a graph to be hamiltonian-connected also guarantee a graph to be PLD-maximal, among other interesting properties. The intention of the chapter is to study these extended implications.

Section 4.1 A generalization of a result of Oystein Ore

In the beginning of chapter 2, a condition related to the sums of the degrees of pairs of nonadjacent vertices has been shown to be sufficient for a graph to be hamiltonian-connected. This sufficient condition has been shown by Faudree and Schelp [22] to be much stronger as Theorem 4.1.6 below indicates.

Before presenting Theorem 4.1.6, several preliminary definitions are needed. All graphs considered in this chapter are undirected and have no multiple edges or loops.

Definition 4.1.1 Let $G=(V(G),E(G))$ be a graph. Suppose that the number of vertices and the number of edges of G are respectively $n=|V(G)|$ and $m=|E(G)|$. Then, G is called an (n,m)-graph. In particular, n is called the order of G and m is called the size of G.

Definition 4.1.2 Let $G=(V(G),E(G))$ be an (n,m) graph. Let $\sigma(G)=\min\{\deg_G(u)+\deg_G(v) \mid \text{For each } u,v\in V(G) \text{ such that } u\neq v\}$, and $\bar{\sigma}(G)=\min\{\deg_G(u)+\deg_G(v) \mid \text{For each } u,v\in V(G) \text{ such that } uv\notin E(G)\}$.

Definition 4.1.3 Let $G=(V(G),E(G))$ be an (n,m)-graph. Let i be a positive integer which satisfies $1\leq i\leq n-1$. P_i is said to hold in G if and only if for any arbitrary pair of distinct vertices $u,v\in V(G)$, $P_i(u,v)$ holds in G.

In light of Definitions 4.1.2 through 4.1.4, Theorem 2.1.1 can be restated as follows in Theorem 2.1.1a.

Theorem 2.1.1a [40] Let $G=(V(G),E(G))$ be an (n,m)-graph. If If $\bar{\sigma}(G)\geq n+1$, then P_{n-1} holds in G.

Definition 4.1.5 Let G be an (n,m)-graph. G is said to satisfy Ore's condition (OC) if and only if $\bar{\sigma}(G)\geq n+1$. R.J. Faudree and R.H. Schelp in [22] have shown, as in Theorem 4.1.6, that the implication of a graph satisfying OC is much stronger than Theorem 2.1.1a.

Theorem 4.1.6 [22] Let $G=(V(G),E(G))$ be an (n,m)-graph such that $n\geq 4$. If G satisfies OC, then P_i holds in G for each i satisfying $4\leq i\leq n-1$.

Due to the complexity of the proof of Theorem 4.1.6, the proof will be postponed until after several important lemmas have been introduced.

Lemma 4.1.7 Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies OC. Then for each pair of nonadjacent vertices $u,v\in V(G)$, there exist $v_1,v_2,v_3\in V(G)$ such that for each i, $1\le i\le 3$, uv_i, $vv_i\in E(G)$.

Proof: Let $x,y\in V(G)$ be a pair of nonadjacent vertices in G. Since $\bar{\sigma}(G)\ge n+1$, $n\ge 5$. Let $N(u)=\{x\in V(G)|\ xu\in E(G)\}$ and $N(v)$ be, respectively, the neighbourhoods of u and v. It is clear that $N(u)\cup N(v)\subseteq V(G)-\{u,v\}$ and $|N(u)|+|N(v)|=\bar{\sigma}(G)\ge n+1$. By the principle of inclusion-exclusion, $|N(u)\cap N(v)|=|N(u)|+|N(v)|-|N(u)\cup N(v)|\ge 3$ and the result follows. ■

For the remainder of the chapter, $N(u)$ will always be used to denote the neighbourhood of a vertex $u\in V(G)$ as defined in Lemma 4.1.7.

Lemma 4.1.8 Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies OC and let $u,v\in V(G)$ be two arbitrary vertices in G. If $uv\notin E(G)$ or $\deg_G(u)+\deg_G(v)\ge n+2$, then $P_2(u,v)$ and $P_3(u,v)$ hold in G.

Proof: Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies OC and the hypotheses of the lemma. It is clear that either $G\cong K_4$ or $n\ge 5$. If $G\cong K_4$, then the result follows trivially. It remains to consider the case where $n\ge 5$.

Let $u,v\in V(G)$ be two arbitrary vertices in G. By Lemma 4.1.10, $uv\notin E(G)$ implies that $|N(u)\cap N(v)|\ge 3$. If $uv\in E(G)$, then $\deg_G(u)+\deg_G(v)\ge n+2$ which implies that $|N(u)\cap N(v)|\ge 2$. Hence, $P_2(u,v)$ holds in G. In order to show that $P_3(u,v)$ holds in G, there are two cases to be considered.

Case 1 $N(u)-\{v\}=N(v)-\{u\}$.

According to the hypotheses of the lemma, it is clear that $|N(u)-\{v\}|=|N(v)-\{u\}|\ge \frac{n}{2}>2$. In such a case, let $N=N(u)-\{v\}=N(v)-\{u\}$.

If there exists a pair of vertices $x,y \in N$ such that $xy \in E(G)$, then obviously $P_3(u,v)$ holds in G. Suppose that no adjacent pair of vertices exists in N. Then, for each $x \in N$, $deg_G(x) \leq n-1-(|N|-1) \leq \frac{n}{2}$. Since $|N| > 2$, there exist $y_1, y_2 \in N$ such that $deg_G(y_1)+deg_G(y_2) \leq n$. This contradicts the fact that G satisfies OC.

Case 2 $N(u)-\{v\} \neq N(v)-\{u\}$.

Without loss of generality, let $x \in N(u)-\{v\}$ such that $x \notin N(v)-\{u\}$. By Lemma 4.1.7, there exists $w \neq u$ and $w \in N(x) \cap N(v)$. It follows that uxwv is a u,v-path of length 3. In both cases $P_3(u,v)$ holds and this completes the proof. ■

The following theorem brings us one step closer to completing the proof of Theorem 4.1.6 which is our major result in this section.

Theorem 4.1.9 [22] Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies OC. Then, for each i satisfying $\frac{n}{2} < i \leq n-1$, P_i holds in G.

Proof: Let G be an (n,m)-graph which satisfies the hypotheses of the theorem. For each $n \leq 4$, $\bar{\sigma}(G) \geq n+1$ implies that $G = K_n$ and the result follows trivially. It remains to consider the cases where $n \geq 5$.

Suppose the contrary and assume that there exists $u,v \in V(G)$ and an integer i satisfying $\frac{n}{2} < i \leq n-1$ such that $P_i(u,v)$ fails to hold. Since G satisfies OC, by Theorem 2.1.1, $P_{n-1}(u,v)$ holds. Let the vertices of G be labelled in such a way that

$p: u=x_1x_2x_3 \ldots x_{n-i+1} \ldots x_i \ldots x_{n-1}x_n=v$ is a hamiltonian u,v-path in G. Let $x=x_{n-i+1}$ and $y=x_i$. If $ux \in E(G)$, then $ux_{n-i+1}x_{n-i+2} \ldots x_n=v$ is a u,v-path of length i which is impossible. Similarly, if $yv \notin E(G)$, then $u=x_1x_2x_3 \ldots x_iv$ is a u,v-path of length i which is impossible.

Hence, $ux \notin E(G)$ and $uv \notin E(G)$. Since G satisfies OC,

$deg_G(u)+deg_G(x) \geq \bar{\sigma}(G) \geq n+1$ and $deg_G(y)+deg_G(v) \geq \bar{\sigma}(G) \geq n+1$

Let j be an integer which satisfies $n-i-1 \leq j \leq n-1$. If $ux_j \in E(G)$ and $xx_{j+1} \in E(G)$, then $ux_j x_{j-1} \cdots x\ x_{j+1} x_{j+2} \cdots x_n = v$ is a u,v-path of length i which is impossible. This implies that the pairs of vertices $\{x, x_{j+1}\}$ and $\{u, x_j\}$ cannot be simultaneously adjacent in G. In a similar fashion, suppose that k is a positive integer which satisfies $2 \leq k \leq i$. If $vx_k \in E(G)$ and $yx_{k-1} \in E(G)$, then $u = x_1 x_2 \cdots x_{k-1} y x_{i-1} x_{i-2} \cdots x_k v = x_n$ is a u,v-path of length i which again is a contradiction. Hence, the pair of vertices $\{v, x_k\}$ and $\{y, x_{k-1}\}$ cannot be simultaneously adjacent.

Let r and s be the numbers of vertices in $\{x_1, x_2, \ldots, x_{n-i+1}\}$ to which u and x are respectively nonadjacent. Similarly, let r' and s' be the number of vertices in $\{x_i, x_{i+1}, x_{i+2}, \ldots, x_n\}$ to which v and y are respectively nonadjacent. According to the discussion in the previous paragraph, it follows that for each vertex in $\{x_{n-i+1}, x_{n-i+2}, \ldots, x_{n-1}\}$ to which u is adjacent, there exists a vertex in $\{x_{n-i+2}, x_{n-i+3}, \ldots, x_n\}$ to which x is not adjacent. Suppose that there exist ℓ vertices in $\{x_{n-i+1}, x_{n-i+2}, \ldots, x_{n-1}\}$ to which u is adjacent, it is clear that $deg_G(u) \leq r+\ell+1$ and $deg_G(x) \leq s+((i-1)-\ell)$. Hence, $n+1 \leq \bar{\sigma}(G) \leq deg_G(u)+deg_G(v) \leq r+s+1$. Similarly, $n-i+1\ \ r'+s'$.

Since $i > \frac{n}{2}$, $i \geq n-i+1$. Suppose that there exists j satisfying $2 \leq j \leq n-i+1$ such that $ux_j \in E(G)$ and $yx_{n+2-j} \in E(G)$. Then $ux_{j+1} x_{j+2} \cdots y x_{n+2-j} x_{n+3-j} \cdots v$ is a u,v-path of length i which is impossible. Hence, for each j, $2 \leq j \leq n-i+1$, $ux_j \in E(G)$ implies that

$yx_{n+2-j} \notin E(G)$. Similarly, if there exists k satisfying $i \leq k \leq n-1$ such that $ux_k \in V(G)$ and $xx_{n-k} \in E(G)$, then $ux_2x_3 \ldots x_{n-k}xx_{n-i+2} \ldots x_kv$ is a u,v-path of length i. It follows that for each k satisfying $i \leq k \leq n-1$, $ux_k \in E(G)$ implies that $xx_{n-k} \notin E(G)$. Hence, it is clear that $r' \leq ((n-i+1)-1)-s$, $s' \leq ((n-i+1)-1)-r$ and $r'+s' \leq 2(n-i+1)-2-(r+s) \leq 2(n-i+1)-2-(n-i+1)=n-i-1$. This contradicts the inequality $r'+s' \geq n-i+1$ introduced in the previous paragraph. This implies that the assumption that there exists an i satisfying $\frac{n}{2} < i \leq n-1$ for which $P_i(u,v)$ fails for some $u,v \in V(G)$ is a false assumption. This completes the proof. ∎

In order to prove Theorem 4.1.9, one more Lemma is required.

Lemma 4.1.10 Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies OC. If there exists a path $P: u=x_1x_2x_3 \ldots x_ix_{i+1}=v$, then at least one of the following four conditions holds.

(i) $P_{i+1}(u,v)$ holds in G,

(ii) the subgraph $G'=\langle V(G)-V(P) \rangle$ satisfies OC, where $V(P)=\{x_1,x_2,\ldots,x_i,x_{i+1}\}$,

(iii) there exist $w_1,w_2 \in V(G)$ distinct from u,v such that the subgraph $G''=\langle V(G)-\{w_1w_2\} \rangle$ satisfies OC, or

(iv) $i=2$ and $P_4(u,v)$ holds in G.

Proof: As in the proof of Theorem 4.1.12, the cases where $n \leq 4$ will result in $G=K_n$ and the theorem follows trivially. Hence, we consider only the cases where $n \geq 5$.

If $i > \frac{n}{2}$, then $P_{i+1}(u,v)$ holds according to Theorem 4.1.12. We assume that $i \leq \frac{n}{2}$. If there exists $y \in V(G)-V(P)$ such that for some

j satisfying $1 \le j \le i$, $yx_j \in E(G)$ and $yx_{j+1} \in E(G)$, then $u=x_1x_2\ldots x_jyx_{j+1}x_{j+2}\ldots x_{i+1}=v$ is a u,v-path of length $i+1$. So $P_{i+1}(u,v)$ holds and condition (i) is satisfied. We assume that no such a vertex y exists in G for the remainder of the proof.

If the subgraph $G'=\langle V(G)-V(P)\rangle=K_{n-(i+1)}$, then it is clear that $P_{i+1}(u,v)$ holds since $i \le \frac{n}{2}$ and G is 2-connected. Let $w_1, w_2 \in V(G')$ be nonadjacent vertices. Since G satisfies OC, $\deg_G(w_1)+\deg_G(w_2) \ge n+1$. Let $\deg_{G'}(w_i)$ denote the degrees of w_i in G', for $i=1,2$. Hence, $\deg_{G'}(w_1)+\deg_{G'}(w_2) \ge n+1-i-1 = |V(G')|+1$, unless it is the case that i is even and $w_1x_j, w_2x_j \in E(G)$, for each $j=1,3,5,\ldots,i-1,i+1$.

If this latter case is not encountered, then $\bar{\sigma}(G') \ge |V(G')|+1$. This implies that G' satisfies OC and condition (ii) of this lemma is satisfied. Therefore, it remains to investigate the case that i is even and for each $j=1,3,5,\ldots,i+1, w_1x_j, w_2x_j \in E(G)$ distinct from u and v such that $zw_1, zw_2 \in E(G)$. In such a case, uw_1zw_2v is a u,v-path of length 4 and $P_4(u,v)$ holds in G. This implies that condition (iv) of the lemma is satisfied. Thus, we assume that $i \ge 4$. Since $\deg_{G'}(w_1)+\deg_{G'}(w_2) \ge n+1-(i+2)=n-(i+1)=|V(G')|$, there exists $w \in V(G')-\{w_1,w_2\}$ such that $ww_1, ww_2 \in E(G)$. Let $G''=\langle V(G)-\{w,w_1\}\rangle$. If w and w_1 are adjacent to no common vertex in G, then for each pair of nonadjacent vertices $y,z \in V(G'')$, $\deg_{G''}(y)+\deg_{G''}(z) \ge \deg_G(y)+\deg_G(z)-2 \ge n+1-2=(n-2)+1=|V(G'')|+1$. It remains to consider the case where there exists $x \in V(G)$ such that $xw_1, xw \in E(G)$. If $x \in V(G')$, then $uw_1xww_2x_5x_6\ldots v$ is a u,v-path of length $i+1$. This means (i) is satisfied. If $x \in V(P)$, then $x=x_j$ for some odd value of j since no two adjacent pair of vertices can be simultaneously adjacent to w. Without loss of generality, $j \geq 3$ can be assumed. In such a case, $ux_2 \ldots x_{j-2}w_1ww_2x_jx_{j+1} \ldots x_i$ is a u,v-path of length i+1. This again means $P_i(u,v)$ holds G and this completes the proof of the lemma. ■

Equipped with the information introduced in this chapter so far, one can now proceed to construct the proof of the main theorem of this section.

Proof of Theorem 4.1.6 By Theorem 4.1.9, the result holds for all $n \leq 7$. It remains only to consider the cases where $n \geq 8$. Suppose the contrary and let $G=(V(G),E(G))$ be an (n,m)-graph where n is the least number of vertices on which a graph G fails to satisfy the theorem. There exist $u,v \in V(G)$ and an integer j which satisfies $4 \leq j \leq n-1$ such that $P_j(u,v)$ fails to hold in G. By Lemma 4.1.8, either $P_1(u,v)$ or $P_2(u,v)$ holds in G. If $P_4(u,v)$ holds in G, then let i be the greatest integer, such that for each k satisfying $4 \leq k \leq i$, $P_k(u,v)$ holds in G and $P_{k+1}(u,v)$ fails to hold in G. Otherwise, let i be the greatest integer satisfying $i < 4$ such that $p_i(u,v)$ holds in G. It is clear that $i < \left\lceil \frac{n}{2} \right\rceil$ in both cases. Choose a u,v-path of length i $P: u=x_1x_2 \ldots x_{i-1}x_ix_{i+1}=v$ and apply Lemma 4.1.10 accordingly. It is clear by the choice of i that conditions (i) and (iv) cannot hold in G. Suppose that condition (iii) of Lemma 4.1.10 holds in G and let $w_1,w_2 \in V(G)$ be distinct from u and v. By the choice of G, in the subgraph $G''=\langle V(G)-\{w_1,w_2\}\rangle$, $P_k(u,v)$ is satisfied for each k satisfying $4 \leq k \leq n-3$. Note that $n \geq 8$ and $\frac{n}{2} < n-3$ as a result. This, however, implies that

$P_k(u,v)$ holds in G, for each k satisfying $4 \le k \le n-1$, which contradicts the choice of G.

We may assume that only condition (ii) of Lemma 4.1.10 holds in G for the remainder of the proof.

Let $G' = \langle V(G) - V(P) \rangle$. By Lemma 4.1.10, G' satisfies OC. Define $N'(u) = \{x \in V(G') \mid xu \in E(G)\}$ and $N'(v) = \{y \in V(G') \mid yv \in E(G)\}$.

There are three cases to be examined.

Case 1 There exists $x \in N'(u)$ and $y \in N'(v)$ such that $x \ne y$. Note that $|V(G')| = n-(i+1) \ge 4$. If $|V(G')| = 4$, then G' satisfying OC implies that $G' = K_4$. Hence, $|V(G')| \ge 5$ is assumed for the remainder of the proof. By the choice of G, $|V(G')| < |V(G)|$ and this implies that for each k satisfying $4 \le k \le |V(G')| - 1$, $P_k(x,y)$ holds in G'. Together with the edges ux, vy in G, $P_\ell(u,v)$ holds in G for each ℓ satisfying $6 \le \ell \le |V(G')| + 1$. Since $|V(G')| + 1 > \frac{n}{2}$, $P_\ell(u,v)$ holds in G for each satisfying $6 \le \ell \le n-1$. Suppose that either $xy \notin E(G)$ or one of $\deg_{G'}(x) = (n-1)-(i+1)$ and $\deg_{G'}(y) = (n-1)-(i+1)$ holds. Since $\deg_{G'}(x) \ge 3$ and $\deg_{G'}(y) \ge 3$, by Theorem 4.1.8, $P_2(x,y)$ and $P_3(x,y)$ hold in G'. This implies that $P_4(u,v)$ and $P_5(u,v)$ hold in G and this implies that P_ℓ holds in G for each ℓ satisfying $4 \le \ell \le n-1$ and which contradicts the choice of G. Hence, it remains to assume that no vertex in $N'(u) \cup N'(v)$ has degree $(n-1)-(i+1) = n-i-2$ and each vertex in $N'(u)$ is adjacent to every vertex in $N'(v)$.

Let $w \in V(G')$ such that $wx \notin E(G')$. By Lemma 4.1.7, there exists $w' \in V(G')$ such that $xw' \in E(G')$ and $ww' \in E(G')$. Since $w \in N'(v)$, again by Lemma 4.1.7, there exists a vertex $y' \in V(G)$ such that $vy' \in E(G)$, $wy' \in E(G)$ and $y \notin \{u,x,w'\}$. Then, $uxw'wy'v$ is a u,v-path of length 5 and this implies $P_5(u,v)$ holds in G.

If there exists a vertex $w\in V(G)$ such that $uw\in E(G)$ and $vw\in E(G)$, then by Lemma 4.1.10, there exist two distinct vertices $a,b\in V(G)$ such that $ua,wa,vb,wb\in E(G)$. Then, uawbv constitutes a u,v-path of length 4 in G which is impossible. This implies that $N'(u)\cup N'(v)=V(G')$. Since none of x and y has degree $(n-1)-(i+1)$, neither $N'(u)\subseteq N'(v)$ not $N'(v)\subseteq N'(u)$ holds. Hence, $x\notin N'(v)$ and $y\notin N'(u)$. If there exists a vertex $z\in N'(u)$ such that $zx\in E(G')$, then uxzyv is a u,v-path of length 4 and $P_4(u,v)$ holds in G which is impossible. This implies that no two vertices in $N'(u)$ are adjacent. Similarly, no two vertices in $N'(v)$ are adjacent. Hence, $N'(u)\cap N'(v)=\phi$. It follows that $\deg_{G'}(u)=|N'(v)|$ and $\deg_{G'}(v)=|N'(u)|$. Without loss of generality, we assume that $|N'(u)|\leq|N'(v)|$. If $|N'(v)|\geq 2$, then for each pair of distinct vertices $y_1,y_2\in N'(v)$,
$\deg_{G'}(y)+\deg_{G'}(y_2)=2|N'(u)|\leq|N'(u)|+|N'(v)|\leq|V(G')|$ which is a contradiction. If $N'(v)=\{y\}$, then $\deg_{G'}(y)=n-i-2$ which again is a contradiction.

What has been proven so far is that if there exist distinct vertices $x\in N'(u)$ and $y\in N'(v)$, then $P_\ell(u,v)$ holds in G for each ℓ satisfying $4\leq\ell\leq n-1$. This contradicts the choice of G. Hence, Case 1 cannot occur in G.

<u>Case 2</u> $N'(u)=N'(v)=\{x\}$or $N'(u)\neq\phi$ and $N(v)=\phi$.

Let $x\in N'(u)$ such that x is adjacent to every other vertex in $V(G')$, in which case $\deg_{G'}(x)=(n-1)-(i+1)=n-i-2$. Let $y\in V(G')$ be an arbitrary vertex in G' distinct from x. Since G' satisfies

OC, $\deg_{G'}(y) \geq 3$. By Lemma 4.1.8, $P_2(x,y)$ and $P_3(x,y)$ hold in G'. If $\deg_{G'}(x) < n-i-2$, then there exists a vertex $y \in V(G')$ distinct from x such that $xy \notin E(G')$. Lemma 4.1.8 again implies that $P_2(x,y)$ and $P_3(x,y)$ hold in G'. Since $|V(G')| < |V(G)|$, $P_k(x,y)$ holds in G' for each k satisfying $2 \leq k \leq |V(G')|-1$. Since $N'(v) = \phi$ or $N'(v) = N'(u) = \phi$, $vy \notin E(G)$. By Lemma 4.1.7, there exists a vertex $w \in V(G)-(V(G') \cup \{u\})$ such that $vw \in E(G)$ and $yw \in E(G)$. It follows that for each k satisfying $5 \leq k \leq |V(G')|+2$, $P_k(u,v)$ holds in G. Note that $|V(G')|+2 > \frac{n}{2}$. By choosing y such that $xy \in E(G')$, uxywv is a u,v-path of length 4. Hence, $P_4(u,v)$ holds in G. If G satisfies the hypothesis of this second case, then $P_k(u,v)$ holds for each k satisfying $4 \leq k \leq n-1$. This contradicts the hypothesis that $P_{i+1}(u,v)$ fails to hold. Hence, Case 2 cannot occur in G.

<u>Case 3</u> $N'(u) = N'(v) = \phi$.

Let $x,y \in V(G')$. If $xy \notin E(G)$, then by Lemma 4.1.8, $P_2(x,y)$ and $P_3(x,y)$ hold in G'. Since $|V(G')| < |V(G)|$, $P_k(u,v)$ holds in G for each k satisfying $4 \leq k \leq n-1$. Since $ux \notin E(G)$ and $vy \notin E(G)$, by Lemma 4.1.7, there exist two distinct vertices $a,b \in V(G)-(V(G')-\{u,v\})$ such that $ua, xa, vb, yb \in E(G)$. It is clear that $P_k(u,v)$ holds in G for each k satisfying $6 \leq k \leq |V(G')|+3$. Note that $\frac{n}{2} < |V(G')|+3$

If x,y are chosen in such a way that $xy \in E(G)$, then uaxybv is a u,v-path of length 5. Choosing x=y allows $P_4(u,v)$ to hold in G. Once again $P_k(u,v)$ holds in G for each k satisfying $4 \leq k \leq n-1$. In each of the three cases above, we have shown that $G' = \langle V(G)-V(P) \rangle$ satisfies OC and $P_\ell(u,v)$ holds in G for each ℓ satisfying $4 \leq \ell \leq n-1$.

Therefore, the original assumption of $P_{i+1}(u,v)$ having failed to hold in G must be invalid and this completes the proof. ■

Section 4.2 Path length distribution (PLD) and PLD-maximal graphs

In Section 4.1, we have examined how OC goes much beyond being a sufficient condition for a graph to be hamiltonian-connected. In this second section, the concept of path length distribution (PLD) of an (n,m)-graph will be introduced. Several necessary and sufficient conditions for an (n,m)-graph to be PLD-maximal will be examined.

Let $G=(V(G),E(G))$ be an (n,m)-graph. Let $[V(G)]^2$ denote the set of all two-element subsets of V(G). For each k satisfying $1 \leq k \leq n-1$, define

$$S_k = \{\{u,v\} \in [V(G)]^2 \mid P_k(u,v) \text{ holds in } G\}$$

$$\text{and } S_0 = [V(G)]^2 - \bigcup_{k=1}^{n-1} S_k.$$

Definition 4.2.1 Let $G=(V(G),E(G))$ be an (n,m)-graph. The path length distribution (PLD) of G is the sequence of non-negative integers $(|S_0|, |S_1|, |S_2|, \ldots, |S_{n-2}|, |S_{n-1}|)$.

It is clear that $|S_0|=0$ if and only if G is connected; $m=|E(G)|=|S_1|$ so that $|S_1|=\frac{n(n-1)}{2}$ if and only if $G=K_n$. Also, for all k, $|S_k| \leq \frac{n(n-1)}{2}$.

For the sake of completeness, the definition of a PLD-meximal graph will be introduced again in light of Definition 4.2.1.

Definition 4.2.2 Let $G=(V(G),E(G))$ be an (n,m)-graph. Then, G is said to be PLD-maximal if and only if the elements in the PLD of G satisfy $|S_0|=0$ and $|S_\ell|=\frac{n(n-1)}{2}$ for each ℓ,

$2 \leq \ell \leq n-1$.

The first sufficient condition for a graph to be PLD-maximal is related to the topological parameter 'connectivity'.

Theorem 4.2.3 [22] Let $G=(V(G),E(G))$ be an (n,m)-graph which is $\left[\frac{n}{2}+1\right]$-connected. Then, G is PLD-maximal. Furthermore, the connectivity condition in this theorem connot be reduced.

Proof: Since G is $\left[\frac{n+1}{2}\right]$-connected, $\delta(G) \geq \left[\frac{n+1}{2}\right]$. Therefore, $\bar{\sigma}(G) \geq \sigma(G) \geq 2\left[\frac{n+1}{2}\right] \geq n+1$ which implies that G satisfies OC. By theorem 4.1.6, P_ℓ holds in G for each ℓ satisfying $4 \leq \ell \leq n-1$. For any two arbitrary distinct vertices $u,v \in V(G)$, $deg_G(u)+deg_G(v) \geq \sigma(G) \geq n+1$. Hence, there exists $x \in V(G)$ such that $xu \in E(G)$ and $xv \in E(G)$ which implies that P_2 holds in G. It remains to show that P_3 holds in G.

Suppose the contrary and let $u,v \in V(G)$ such that $P_3(u,v)$ fails in G. Define

$$A=\{x \in V(G) \mid xu \in E(G) \text{ and } xv \in E(G)\}.$$

$$B=\{x \in V(G) \mid xu \in E(G) \text{ and } xv \notin E(G)\}.$$

$$C=\{x \in V(G) \mid xu \notin E(G) \text{ and } xv \in E(G)\}.$$

$$D=\{x \in V(G) \mid xu \notin E(G) \text{ and } xv \notin E(G)\}.$$

In order for $P_3(u,v)$ not to hold in G, the following two conditions must hold.

(1) For any $a_1,a_2 \in A$, $a_1a_2 \notin E(G)$

(2) For each $a \in A$ and $x \in B \cup C$, $ax \notin E(G)$.

If $A=\phi$, then the subgraphs $\langle V(G)-(B \cup \{v\})\rangle$ and $\langle V(G)-(C \cup \{u\})\rangle$ must be disconnected. Since G is $\left[\frac{n+1}{2}\right]$-connected,

$|B|+1\geq\left[\frac{n+1}{2}\right]$ and $|C|+1\geq\left[\frac{n+1}{2}\right]$. Hence, $|B|\geq\left[\frac{n}{2}\right]$ and $|C|\geq\left[\frac{n}{2}\right]$. This implies that $n\geq|B|+|C|+2\geq 2\left[\frac{n}{2}\right]+2$ which is impossible. Therefore, $|A|\geq 1$. Since Conditions (1) and (2) above hold, it follows that the subgraph $\langle V(G)-(D\cup\{u,v\})\rangle$ is disconnected. This implies that $|D|\geq\left[\frac{n}{2}\right]-1$ and $n\geq|A|+|D|+2\geq|A|+(\left[\frac{n}{2}\right]-1)+2$. Hence $\left[\frac{n}{2}\right]\geq|A|$. Similarly, the subgraphs $\langle V(G)-(A\cup B\cup\{v\})\rangle$ and $\langle V(G)-(A\cup C\cup\{u\})\rangle$ are disconnected and it follows that $|B|\geq\left[\frac{n}{2}\right]-|A|$ and $|C|\geq\left[\frac{n}{2}\right]-|A|$.

Combining the inequalities developed so far yields

$$n=|V(G)|=|A|+|B|+|C|+|D|+2\geq|A|+2\left(\left[\frac{n}{2}\right]-|A|\right)+\left[\frac{n}{2}\right]-1+2=3\left[\frac{n}{2}\right]-|A|+1$$

which implies $|A|\geq\left[\frac{n}{2}\right]$. This together with $|A|\leq\left[\frac{n}{2}\right]$, gives $|A|=\left[\frac{n}{2}\right]$. Hence, $n=|A|+|B|+|C|+|D|+2\geq\left[\frac{n}{2}\right]+|B|+|C|+\left[\frac{n}{2}\right]-1+2$. This implies that $|B|=|C|=0$ and it follows that G is at most $\left[\frac{n}{2}\right]$-connected, which contradicts the hypotheses on G. Therefore, the original assumption that P_3 fails is false and this completes the proof. ■

The sharpness of the theorem will be discussed following Corollary 4.2.9.

In Theorem 4.2.4 and Theorem 4.2.5 below, two sufficient conditions for a graph to be PLD-maximal will be introduced. These two theorems are in similar spirit to Theorem 2.1.1.

<u>Theorem 4.2.4</u> [22] Let G be an (n,m)-graph with $n\geq 2$ and $\bar{\sigma}(G)\geq\left[\frac{3n-1}{2}\right]$. Then, G is PLD-maximal. Furthermore, the degree condition in this theorem cannot be reduced.

<u>Proof</u>: For $n\leq 4$, G satisfying the condition of the theorem implies that $G\cong K_n$ and the result follows trivially. It remains to consider the cases where $n\geq 5$.

For $n\geq 5$, G satisfies OC and by Theorem 4.1.6 again, P_i holds in G for each i satisfying $4 \leq i \leq n-1$. Furthermore, by Lemma 4.1.8, for any pair of nonadjacent vertices $u,v \in V(G)$, $P_2(u,v)$ and $P_3(u,v)$ holds in G. To complete the proof of the theorem, it suffices to show that for any pair of adjacent vertices $u,v \in V(G)$, $P_2(u,v)$ and $P_3(u,v)$ hold in G.

Suppose the contrary and let $u,v \in V(G)$ be two vertices such that $uv \in E(G)$ and $P_2(u,v)$ fails in G. Define

$$A = \{x \in V(G) \mid xu \in E(G) \text{ and } x \neq v\}.$$

$$B = \{x \in V(G) \mid xv \in E(G) \text{ and } x \neq u\}.$$

$$C = \{x \in V(G) \mid xu \notin E(G) \text{ and } xv \notin E(G)\}.$$

Since $P_2(u,v)$ fails to hold in G, the sets A,B and C are mutually disjoint. Since G satisfies OC, $\delta(G) \geq 3$. Hence, $|A| \geq 1$ and $|B| \geq 1$. Let $a \in A$ and $b \in B$. Clearly,

$$\deg_G(a) + \deg_G(v) \leq (n-2) + |B| + 1 + (n-1) + |B| \text{ and}$$

$$\deg_G(b) + \deg_G(u) \leq (n-2) + |A| + 1 = (n-1) + |A|.$$

Since $av \notin E(G)$ and $bu \notin E(G)$

$$\left[\frac{3n-1}{2}\right] \leq \bar{\sigma}(G) \leq \deg_G(a) + \deg_G(v) \leq (n-1) + |B| \text{ and}$$

$$\left[\frac{3n-1}{2}\right] \leq \bar{\sigma}(G) \leq \deg_G(b) + \deg_G(u) \leq (n-1) + |A|.$$

Combining these last two inequalities, we obtain $3n-3 \leq 2\left[\frac{3n-1}{2}\right] \leq 2n-2+|A|+|B|$.

Hence, $n \leq 1+|A|+|B| < 2+|A|+|B| \leq |V(G)| = n$ which is a contradiction. Therefore, the assumption that $P_2(u,v)$ fails in G is false and this implies that P_2 holds in G.

We now suppose that P_3 fails in G and let $u,v \in V(G)$ be two vertices such that $uv \in E(G)$ and $P_3(u,v)$ fails in G. Let

$A=\{x\in V(G) \mid xu\in E(G) \text{ and } xv\notin E(G) \text{ and } x\neq v\}$.

$B=\{x\in V(G) \mid xu\notin E(G) \text{ and } xv\in E(G) \text{ and } x\neq u\}$.

$C=\{x\in V(G) \mid xu\in E(G) \text{ and } xu\in E(G)\}$.

$D=\{x\in V(G) \mid xu\notin E(G) \text{ and } xv\notin E(G)\}$.

It is clear that the sets A,B,C and D are mutually disjoint which implies that $n=|V(G)|=|A|+|B|+|C|+|D|+2$.

In order that $P_3(u,v)$ fails to hold in G, the following three conditions must hold.

(1) For each $a\in A$ and for each $x\in B\cup C$, $ax\notin E(G)$.

(2) For each $b\in B$ and for each $y\in A\cup C$, $by\notin E(G)$.

(3) For each $c_1,c_2\in C$, $c_1c_2\notin E(G)$.

Clearly, $\deg_G(u)\leq |C|+|A|+1$ and

$\deg_G(v)\leq |C|+|B|+1$.

For any $a\in A$ and $b\in B$ $\deg_G(a)\leq |A|+|D|$ and

$\deg_G(b)\leq |B|+|D|$.

Note that $av\notin E(G)$ and $bu\notin E(G)$. If either $A\neq\phi$ or $B\neq\phi$, then $n+1\leq\bar{\sigma}(G)\leq |A|+|B|+|C|+|D|+1=n-1$ which is a contradiction. Hence, $A=B=\phi$ and it follows that $n=|C|+|D|+2$. Therefore, $2n-2|C|=2|D|+4$. Since $\deg_G(u)\geq 3$ and $\deg_G(v)\geq 3$, $|C|\geq 2$. Let $c_1,c_2\in C$. It is clear that $c_1c_2\notin E(G)$ and $\deg_G(c_1)\leq |D|+2$ and $\deg_G(c_2)\leq |D|+2$. Hence, $\left[\frac{3n-1}{2}\right]\leq\bar{\sigma}(G)\leq\deg_G(c_1)+\deg_G(c_2)\leq 2|D|+4=2n-2|C|$. For any $x\in D$, $\deg_G(x)\leq n-3$ and $xv\notin E(G)$. This implies that

$\left[\frac{3n-1}{2}\right]\leq\bar{\sigma}(G)\leq\deg_G(v)+\deg_G(x)\leq |C|+1+(n-3)=n+|C|-2$. Combining the last two inequalities in the appropriate manner, we have

$9n-9\leq 6\left[\frac{3n}{2}-1\right]\leq 4(n+|C|-2)+2(2n-2|C|)=8n-8$, which implies that $n\leq 1$

and this contradicts thecchoice of $n=|V(G)|\geq 5$. Thus, the original assumption of $P_3(u,v)$ having failed to hold in G is false and it follows that P_3 holds in G. Hence, G is PLD-maximal.

The example which shows that the conditions in this theorem are sharp will be illustrated following Corollary 4.2.9.

Theorem 4.2.5 [25] Let $G=(V(G),E(G))$ be an (n,m)-graph such that

(1) $\sigma(G)\geq n+1$ and when n is odd, in addition to (1)

(2) $\bar{\sigma}(G)\geq n+2$ is satisfied. Then, G is PLD-maximal.

Furthermore, conditions (1) and (2) are sharp. Three preliminary lemmas are required before the proof of Theorem 4.2.5 is to be presented.

Lemma 4.2.6 Let $G=(V(G),E(G))$ be an (n,m)-graph with $n\geq 5$. Let $v_1,v_2\in V(G)$ be two nonadjacent vertices in G such that for any $x,y\in V(G)-\{v_1,v_2\}$, $\deg_G(x)+\deg_G(y)\geq n+2$. Then, $G'=\langle V(G)-\{v_1,v_2\}\rangle$ is hamiltonian.

Proof: It is clear for each pair of vertices $x,y\in V(G')$ such that $xy\notin E(G')$, $\deg_{G'}(x)+\deg_{G'}(y)\geq n+2-4=|V(G')|$. Hence, $\bar{\sigma}(G')\geq |V(G')|$ which implies that G' is hamiltonian. ■

Lemma 4.2.7 Let $G=(V(G),E(G))$ be an (n,m)-graph with a cycle $C:x_0x_1x_2\ldots x_{s-1}x_0$ of length s, $3\leq s\leq n-2$. Let $v_1,v_2\in V(G)-V(C)$ and let $\deg_C(v_i)=|\{x\in V(C)\,|\,xv_i\in E(G)\}|$ for i=1 and 2. If $\deg_C(v_1)+\deg_C(v_2)\leq s+1$, then for each ℓ satisfying $2\leq\ell\leq s+1$, $P_\ell(v_1,v_2)$ holds in G.

Proof: It is sufficient to show that for each ℓ satisfying $2\leq\ell\leq s+1$, there exists $x_i,x_j\in V(C)$ with $i-j=\ell-2 \pmod s$ such that $v_1x_i,v_2x_j\in E(G)$. Suppose the contrary. For each $i\in\{0,1,\ldots,s-1\}$

such that $x_i \in C$ and $x_i v_1 \in E(G)$, there exists a unique $j \in \{0,1,\ldots,s-1\}$ satisfying $i-j \equiv \ell-2 \pmod{s}$ and $v_2 x_j \notin E(G)$. This implies that $\deg_C(v_1) \leq s-\deg_C(v_2)$ which is a contradiction. Hence, there exist $x_i, x_j \in V(C)$ such that $i-j \equiv \ell-2 \pmod{s}$ and $v_1x_i, v_2x_j \in E(G)$. Without loss of generality, let $i<j$ in which case $v_1 x_i x_{i+1} \ldots x_j v_2$ is a v_1,v_2-path of length ℓ and this completes the proof. ■

Lemma 4.2.8 Let $G=(V(G),E(G))$ be a $(2k,m)$-graph with $k \geq 2$ such that there exists a vertex $u \in V(G)$ with $\deg_G(x) \geq k$ for all $x \in V(G)-\{u\}$. Then, either G is hamiltonian or $\langle V(G)-\{u\}\rangle$ is hamiltonian.

Proof: Let $G=(V(G),E(G))$ be a $(2k,m)$-graph with $k \geq 2$ which satisfies the conditions of the theorem. If G is hamiltonian, then the proof is complete. Thus we assume that G is not hamiltonian and establish that $G'=\langle V(G)-\{u\}\rangle$ is hamiltonian. Let $A_1: x_1x_2\ldots x_{2k}x_1$ be a cyclic arrangement of the vertices in G such that the number of pairs of vertices of the form x_i, x_{i+1}, with $1 \leq i \leq 2k-1$, which satisfy $x_ix_{i+1} \in E(G)$ is maximum. We next establish that for any i, $1 \leq i \leq 2k-1$, $x_ix_{i+1} \notin E(G)$ implies that either $x_i=u$ or $x_{i+1}=u$. Suppose the contrary and let $x_ix_{i+1} \notin E(G)$ for some i, $1 \leq i \leq 2k-1$, such that neither $x_i=u$ nor $x_{i+1}=u$. Without loss of generality, $i=1$ is assumed. Then, for each j satisfying $3 \leq j \leq 2k-1$, if $x_1x_j \in E(G)$, then $x_2x_{j+1} \notin E(G)$; for otherwise, $A_2: x_1x_jx_{j-1}x_{j-2}\ldots x_2x_{j+1}x_{j+2}\ldots x_{2k-1}x_1$ is a cyclic arrangement of the vertices of G for which the number of pairs of contiguous pairs of vertices which are adjacent is greater than that of the original arrangement A_1. This, however implies that $\deg_G(x_1) \le 2k-1-\deg_G(x_2)$, which contradicts the assumption that neither $x_1=u$ nor $x_2=u$. Hence, there are at most two pairs of contigueous vertices of the form $\{x_i, x_{i+1}\}$ which are nonadjacent, with each nonadjacency involving the vertex u. Without loss of generality, it can now be assumed that $A_1: x_1x_2x_3x_4\ldots x_{2k-1}x_1$ is a cyclic arrangement of the vertices of G with at most $\{x_1,u\}$ and $\{u,x_2\}$ being the nonadjacent pairs, where $x_2=u$. If $x_1x_3 \in E(G)$, then $x_1x_3x_4\ldots x_{2k-2}x_{2k-1}x_1$ is a hamiltonian cycle in $G'=\langle V(G)-\{u\}\rangle$ and the proof is complete. Thus, $x_1x_3 \notin E(G)$ is assumed. Note that $A': x_1x_3x_4\ldots x_{2k-1}x_1$ is also an arrangement of vertices in G' which has the maximum number of adjacent contigueous pairs. Without loss of generality, let $ux_1 \notin E(G)$. Similarly for each j satisfying $4 \le j \le 2k-1$, $x_1x_j \in E(G)$ implies that $x_3x_{j+1} \notin E(G)$. Then, either $\deg_G(x_1) \le 2k-1-\deg_G(x_3)$ or $\deg_G(x_1) \le 2k-2-\deg_G(x_3)$ depending on whether $ux_3 \in E(G)$ or $ux_3 \notin E(G)$ respectively. In either case, one of x_1, x_3 has degree strictly less than k. This contradicts the choice of $u \ne x$, and $u \ne x_3$. Thus, $x_1x_3 \in E(G)$ and $C: x_1x_3x_4\ldots x_{2k-1}x_1$ is a hamiltonian cycle in $G'=\langle V(G)-\{u\}\rangle$ which completes the proof. ∎

We are now ready to prove Theorem 4.2.5.

<u>Proof of Theorem 4.2.5</u> Let $G=(V(G),E(G))$ be an (n,m)-graph which satisfies the hypotheses of Theorem 4.2.5. The result is obvious for the case where $n \le 4$. We assume that $n \ge 5$.

Suppose that both conditions (1) and (2) are satisfied. Let $v_1, v_2 \in V(G)$ be two arbitrary vertices in G. Then, for any pair of nonadjacent vertices $x,y \in V(G)-\{v_1,v_2\}$, $\deg_G(x)+\deg_G(y) \geq \bar{\sigma}(G) \geq n+2$. By Lemma 4.2.6, $G'=\langle V(G)-\{v_1,v_2\}\rangle$ is hamiltonian. Let $C:x_1x_2\ldots x_{n-2}x_1$ be a hamiltonian cycle in G'. Clearly, $\deg_C(v_1)+\deg_C(v_2) \geq \sigma(G)-2 \geq n+1-2=(n-2)+1=|V(C)|+1$. By Lemma 4.2.7, for each ℓ satisfying $2 \leq \ell \leq n-1$, $P_\ell(v_1,v_2)$ holds in G. Hence, G is PLD-maximal.

It remains to show that for the case where $n=2k$, $k \geq 3$, condition (1) alone is sufficient to ensure that G is PLD-maximal. Note that if for each $u \in V(G)$, $\deg_G(u) \geq k+1$, then $\bar{\sigma}(G) \geq \sigma(G) \geq n+2$. The argument in the first part of the proof then shows that G is PLD-maximal. Thus, it remains to consider the case where there exists a vertex $u \in V(G)$ such that $\deg_G(u) \leq k$. There are two subcases to be examined.

Case 1 Suppose there exists a vertex $u \in V(G)$ such that $\deg_G(u)=k$. Note that $\bar{\sigma}(G) \geq \sigma(G) \geq n+1$. By Theorem 2.1.1, P_{n-1} holds in G. Let $v \in V(G)-\{u\}$ be a vertex such that $\deg_G(v)+\deg_G(u) \geq \sigma(G) \geq n+1=2k+1$. Hence, $\deg_G(v) \geq k+1$. Therefore, for any $x,y \in V(G)-\{u,v\}$ and for any fixed $v \in V(G)-\{u\}$, $\deg_G(x)+\deg_G(y) \geq 2k+2=n+2$. By Lemma 4.2.6, the induced subgraph $G'=\langle V(G)-\{u,v\}\rangle$ contains a hamiltonian cycle $C:x_1x_2\ldots x_{2k-2}x_1$. Clearly, $\deg_C(u)+\deg_C(v) \geq \sigma(G)-2 \geq n+1-2=|V(C)|+1$. It follows from Lemma 4.2.7 that for all ℓ satisfying $2 \leq \ell \leq n-1$, $P_\ell(u,v)$ holds in G.

Now consider an arbitrary pair of distinct vertices $x,y \in V-\{u\}$ and let $G''=\langle V(G)-\{x,y\}\rangle$. Note that for any vertex $z \in V(G'')-\{u\}$, $\deg_G(z)+\deg_G(u) \geq \sigma(G) \geq 2k+1$. Hence, $\deg_G(z) \geq k+1$ and this implies that $\deg_G(z) \geq k-1=\frac{|V(G'')|}{2}$. By Lemma 4.2.8, either the subgraph G"

G" has a hamiltonian cycle or G" has a cycle of length $|V(G")|-1$ passing through every vertex in V(G") except the vertex u. If V(G") has a hamiltonian cycle $C':uy_2\dots y_{n-2}u$, then $\deg_{C'}(x)+\deg_{C'}(y)\geq\sigma(G)-2\geq 2k+1-2=|V(C')|+1$. By Lemma 4.2.8, $P_\ell(x,y)$ hold for each ℓ satisfying $2\leq\ell\leq 2k-1$. Next we consider the case where there exists a cycle $C":y_1y_2\dots y_{2k-3}y_1$ in $V(G')-\{u\}$. Since $\deg_G(x)+\deg_G(u)\geq\sigma(G)\geq 2k+1$, it follows that $\deg_G(x)\geq k+1$. Similarly, $\deg_G(y)\geq k+1$. Therefore, $\deg_G(x)+\deg_G(y)\geq 2k+2$ and it follows that $\deg_{C"}(x)+\deg_{C"}(y)\geq 2k+2-4=(2k-3)+1=|V(G")|+1$. By Lemma 4.2.7, for each ℓ satisfying $2\leq\ell\leq 2k-2$, $P_\ell(x,y)$ holds in G. In either of the cases above, it has been proven that $P_\ell(x,y)$ holds for each ℓ, $2\leq\ell\leq 2k-2$. Together with the fact that P_{n-1} holds in G, it follows that G is PLD-maximal.

Case 2 Suppose there exists $u\in V(G)$ such that $\deg_G(u)\leq k-1$. Again, $\bar{\sigma}(G)\geq\sigma(G)\geq n+2$ which implies that P_{n-1} hold in G. Let $G'=\langle V(G)-\{u\}\rangle$. For any two vertices $x,y\in V(G')$, $\deg_G(x)+\deg_G(u)\geq\sigma(G)\geq 2k+1$. Hence, $\deg_G(x)\geq k+2$. Similarly, $\deg_G(y)\geq k+2$. Therefore, $\deg_{G'}(x)+\deg_{G'}(y)\geq\deg_G(x)+\deg_G(y)-2\geq 2k+2=(2k-1)+3=|V(G')|+3$. This implies that $\bar{\sigma}(G')\geq\sigma(G')\geq|V(G')|+3>|V(G')|+2$. Hence, for each ℓ satisfying $2\leq\ell\leq 2k-2$, $P_\ell(x,y)$ holds in G. Now consider any vertex $z\in V(G')$. Since $\deg_G(z)+\deg_G(u)\geq\sigma(G)\geq n+1$, there exists $w\in V(G')$ such that zwu is a z,u-path of length 2 there exists a z,w-path $Q_\ell:z\dots w$ of length ℓ in G'. A z,u-path in G of length $\ell+1$ can then be obtained by concatenating the path Q_ℓ with the edge uw. Thus for each ℓ satisfying $2\leq\ell\leq 2k-2$, $P_{\ell}(z,u)$ holds in G. Together with the fact that P_{-1} holds in G, it follows that G is PLD-naximal and this completes the proof. ■

The example that shows the sharpness of the conditions in Theorem 4.2.5 will be given after the following obvious corollary to Theorem 4.2.5.

Corollary 4.2.9 Let $G=(V(G),E(G))$ be an (n,m)graph such that $\delta(G)\geq\frac{n+2}{2}$. Then, G is PLD-maximal. Moreover, the condition of this corollary is sharp.

Proof: Clearly, for any $x,y\in V(G)$, $deg_G(x)+deg_G(x)+deg_G(y)\geq 2\delta(G)\geq n+2$. Hence, $\bar{\sigma}(G)\geq\sigma(G)\geq n+2$. By Theorem 4.2.5, G is PLD-maximal. The following examples will show that the condition given in this corollary is sharp. ■

Examples to show the sharpness of the conditions in Theorems 4.2.3 and 4.2.5 and Corollary 4.2.9, respectively, are as follows.

Definition 1.2.10 Let $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$ be any two (n_1,m_1)-graph and (n_2,m_2)-graphs, respectively. Now $G_1\cup G_2$ is the (n_1+n_2, m_1+m_2)-graph with the set of vertices $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and the set of edges $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$ and G_1+G_2 is the (n_1+n_2,q)-graph with set of vertices $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and the set of edges $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{xy\mid \text{for all } x\in V(G_1) \text{ and for all } y\in V(G_2)\}$. Clearly, $q=m_1+m_2+n_1n_2$.

Definition 4.2.11 Let m,n,k be any three positive integers. An (m,n,k)-graph $G_{m,n,k}$ is defined by $G_{m,n,k}=(K_m\cup K_k)+K_n$.

Then,

$$\deg(x)=m+n-1 \text{ for each } x\in V(K_m),$$

$$\deg(x)=m+k \text{ for each } x\in V(\bar{K}_n),$$

and $$\deg(x)=k+n-1 \text{ for each } x\in V(K_k).$$

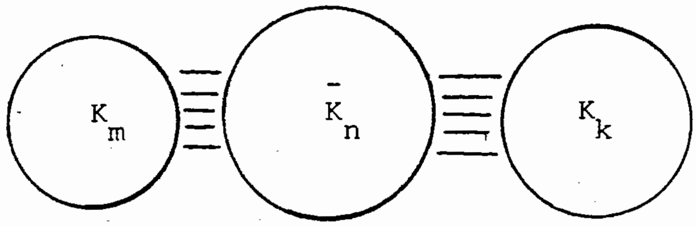
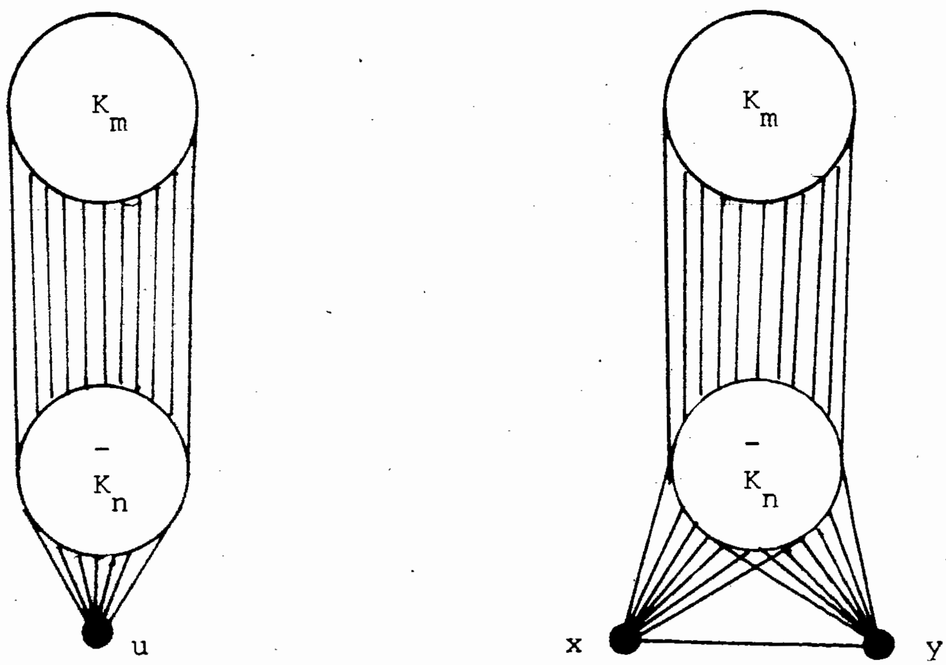


figure 4.1

It is clear that $G_{m,n,k}$ satisfies OC as long as $m+k\geq n+1$ and $n\geq 3$. In particular, we consider the graphs $G_{m,n,1}$ and $G_{m,n,2}$ shown in Figure 4.2 below.



$G_{m,n,1}$ $G_{m,n,2}$

figure 4.2

For any $v\in V(\bar{K}_n)$ and $u\in K_1$ of $G_{m,n,2}$, it is clear that $P_2(u,v)$ fails to hold. Let $x,y\in V(K_2)$ of $G_{m,n,2}$. It is clear that $P_3(x,y)$ fails to hold in $G_{m,n,2}$. Thus, for any $m,n \geq 1$, neither $G_{m,n,1}$ nor $G_{m,n,2}$ is PLD-maximal. However, if we choose m and n such that $m \geq n \geq 3$ and $m+1 \geq n \geq 3$ for $G_{m,n,1}$ and $G_{m,n,2}$, respectively, then both of them satisfy OC and are not PLD-maximal. This shows that the lower bound in Theorem 4.1.6 cannot be further reduced.

To show that the condition in theorem 4.2.3 is sharp, simply consider $G_{n-2-\left[\frac{n}{2}\right],\left[\frac{n}{2}\right],2}$ for any $n \geq 4$ and it can be easily shown that the graph is $\left[\frac{n}{2}\right]$-connected while P_3 fails to hold.

Next, let $G_1=G_{N-1,N-1,2}$, for each $N \geq 3$,

$G_2=G_{N-1,N,2}$, for each $N \geq 3$,

and $G_3=G_{N+1,N-1,1}$, for each $N \geq 6$, respectively.

G_1 has $n=2N$ vertices with $\sigma(G_1)=n=2N$ and is not PLD-maximal. This establishes the sharpness of the condition in Theorem 4.2.5 for the case where the number of vertices n is even. For the case where n is odd, we examine G_2 and G_3 in a similar manner. It is clear that $|V(G_2)|=2N+1$ and $\bar{\sigma}(G_2)=\sigma(G_2)=|V(G_2)|+1$. Thus G_2 satisfies Condition (1) and fails to satisfy Condition (2) of Theorem 4.2.5 and P_3 fails to hold in G_2. We next observe that $|V(G_3)|=2N+1$ with $\sigma(G_3)=|V(G_3)|$ and $\bar{\sigma}(G_3)=|V(G_3)|+3$. Hence, G_3 satisfies Condition (2) and fails to satisfy Condition (1) in Theorem 4.2.5 while P_2 fails in G_3. The sharpness of the conditions in Theorem 4.2.5 is now established both for the cases when the size of the set of vertices is odd or even.

Moreover, observe that $\delta(G_1)=\frac{1}{2}|V(G_1)|$ and $(G_2)=\frac{1}{2}(|V(G_2)|+1)$. This establishes the sharpness of the condition in Corollary 4.2.9.

Finally, to establish the sharpness of the conditions in

Theorem 4.2.4, we consider the graph $G=K_{\left[\frac{n-2}{2}\right]}\cup\{v\}+K_{n-\left[\frac{n-2}{2}\right]-2}\cup\{u\}$,

for each $n\geq 4$.

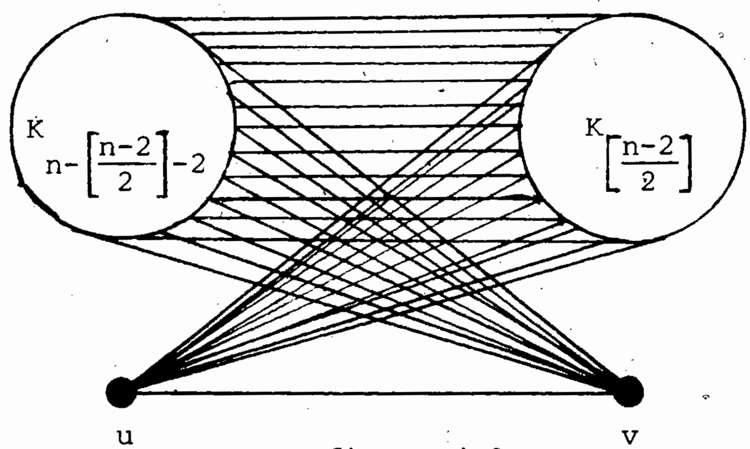


figure 4.3

It can easily be shown that $\bar{\sigma}(G)=\left[\frac{3n-2}{2}\right]$ and $P_2(u,v)$ fails to hold in G. Hence, the condition in Theorem 4.2.4 is best possible.

With the examples illustrated thus far in this section, it is clear that in addition to having a graph G satisfy OC, additional constraints must be imposed before the graph G can become PLD-maximal. This is entirely consistent with the results in Theorems 4.2.3 and 4.2.4.

As another corollary to Theorem 4.2.5, we establish a sufficient condition for a graph to be PLD-maximal that is related to the number of edges in a graph.

Corollary 4.2.12 Let G be an (n,m)-graph such that $m\geq \frac{1}{2}(n)(n-1)-(n-4)$, then G is PLD-maximal. Furthermore, the condition in the theorem is sharp.

Proof: Let G be an (n,m)-graph constructed by removing at most q edges from the complete graph on n vertices K_n, $q\leq p-4$. Let $u,v\in V(G)$ be arbitrary vertices in G. Then, it is clear that

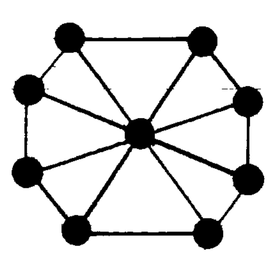
$\deg_G(u)+\deg_G(v) \geq (n-1)+(n-1)-((n-4)+1)=n+1$. As a matter of fact, $\deg_G(u)+\deg_G(v)=n+1$ only if $uv \in E(G)$ and all the edges removed are incident with either u or v or uv itself is removed. Otherwise, $\deg_G(u)+\deg_G(v) \geq n+2$. Therfore, $\sigma(G) \geq n+1$. A similar argument clearly shows that for any nonadjacent vertices $u,v \in V(G)$, $\deg_G(u)+\deg_G(v) \geq n+2$. Hence, $\sigma(G) \geq n+1$ and $\bar{\sigma}(G) \geq n+2$. By Theorem 4.2.5, G is PLD-maximal.

To show that the condition of the theorem is best possible, consider the graph G consisting of the set of vertices $V(G)=\{u,v_1,v_2,\ldots,v_{n-1}\}$ and the set of edges $E(G)=\{uv_1,uv_{n-1}\}\cup\{v_iv_j \mid 1\leq i<j\leq n-1\}$. Since $\deg_G(u)=2$, G defined in such a manner is not hamiltonian-connected and $|E(G)| = \frac{1}{2}n(n-1)-(n-3)$ Hence, G is not PLD-maximal and this completes the proof. ■

A counting argument analogous to the proof of Theorem 4.2.5 will produce the following necessary conditions for a graph to be PLD-maximal.

Theorem 4.2.13 [25] Let $G=(V(G),E(G))$ be an (n,m)-graph with $n \geq 4$. If both P_2 and P_{n-1} hold in G, then $m \geq 2(n-1)$. Furthermore, $m=2(n-1)$ if and only if G is isomorphic to the wheel graph W_n on n vertices.

Proof: The details of this proof can be found in [25]



W_9

figure 4.4

As a corollary to Theorem 4.2.13, we have established a necessary condition for an (n,m)-graph to be PLD-maximal in terms of a lower bound for m.

Corollary 4.1.14 [25] Let $G=(V(G),E(G))$ be an (n,m)-graph. If G is PLD-maximal, then $m=|E(G)|\geq 2(n-1)$ and $m=2(n-1)$ if and only if $G\cong W_n$.

More on path length distribution will be discussed in Section 4.4.

Section 4.3 Panconnected graphs

The objective of this section is to present a survey of results that relate the powers of a connected graph, Path Length Distribution (PLD) of a graph and panconnectedness. In the attempt to make this section self-contained, the concept of panconnectedness will be introduced again. Some of the results related to the power of a graph briefly mentioned in Theorems 1.2.34 and 1.2.35 will be introduced once more in a logical order in this section with respect to the concept of the Path Length Distribution (PLD) of a graph. Proofs which are representative of the types of arguments used in analysing these types of problems will be presented in detail for two theorems by Y. Alavi and J.E. Williamson.

Recall that in Definition 1.2.33, the m^{th} power of a graph $G^m=(V(G^m),E(G^m))$ has been formally defined. As mentioned in the discussion of Theorem 1.2.34, Herbert Fleischner [26] earlier in 1971 produced a significant result relating the power of a graph to its hamiltonianicity. He proved that the square of a 2-connected graph is always hamiltonian.

<u>Definition 4.3.1</u> Let $G=(V(G),E(G))$ be an (n,m)-graph. If for each vertex $v \in V(G)$ (edge $e \in E(G)$), there exists a cycle C_ℓ of length ℓ, for each ℓ satisfying $3 \leq \ell \leq n$, such that $v \in V(C_\ell)$ ($e \in E(C_\ell)$), then the graph G is said to be <u>vertex-pancyclic (edge-pancyclic)</u>.

In 1971, J.A. Bondy asked if the square G^2 of a graph G being hamiltonian implies that vertex pancyclic.

Arthur M. Hobbs [29] gave an affirmative answer to Bondy's question for the classes of 2-connected graphs and connected bridgeless DT-graph if and only iff each edge is incident to a vertex of degree 2). Bondy also introduced the concept of edge-pancyclicity as defined in Definition 4.3.1. The concept of edge-pancyclicty is closely related to the concept of panconnectedness as we now indicate. As in definition 1.2.33, for any (n,m)-graph $G=(V(G),E(G))$ and for any $x,y \in V(G)$, $d_G(x,y)$ denotes the distance between the vertices x and y in G.

<u>Definition 4.3.2</u> Let $G=(V(G),E(G))$ be an (n,m)-graph. Suppose that for each $x,y \in V(G)$, $P_i(x,y)$ holds in G for all i satisfying $d_G(x,y) \leq i \leq n-1$. Then G is said to be a panconnected graph.

It is clear that if a graph is panconnected, then it is necessarily edge pancyclic. These two concepts, however, are not equivalent as the graph in Figure 4.5 indicates. By inspection, it can be easily shown that the graph is edge-pancyclic. However, there exist no x,y-paths in of length 5,6, or $7=|V(G)|-1$, when $d_G(x,y)=2$. Hence, G is not panconnected.

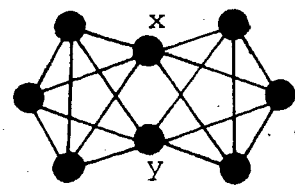


figure 4.5

Later in 1971, G. Chartrand, A. H½bbs, H. Jung, S. Kapour and Nash-Williams gave the following generalization of the result of Fleischner [26] as stated in Theorem 1.2.34.

Theorem 4.3.3 [10] Let G=(V(G),E(G)) be a 2-connected (n,m)-graph. Then, P_{n-1} holds in G^2, that is, G^2 is hamiltonian-connected.

Earlier in the same year. Fleischner [27] established another result relating the power of a graph to a hamiltonian property.

Theorem 4.3.4 [27] Let G be a connected bridgeless DT-graph. Then, G^2 is hamiltonian-connected.

Furthermore, Karaganis [30] and Sekania [41] earlier had established the following result.

Theorem 4.3.5 [31,43] If G=(V(G),E(G)) is a connected (n,m)-graph, then G^3 is hamiltonian-connected.

Being the power of a graph, however, is much stronger a condition than merely being sufficient to ensure that a graph is hamiltonian-connected. In a paper by Y. Alavi and J.E. Williamson [3], the following two results have been established.

Theorem 4.3.6 If G is a connected graph, then G^3 is panconnected.

Theorem 4.3.7 If G is hamiltonian graph, then G^2 is panconnected.

Clearly, the result of Karaganis [31] and Sekanina [43] in Theorem 4.3.5 is merely a corollary to Theorem 4.3.6.

In 1973, R.J. Faudree and R.H. Schelp [23] established the following much stronger result to which Theorems 4.3.3, 4.3.4 and 4.3.7 are all corollaries.

Theorem 4.3.8 [23] If G is either a bridgeless DT-graph or a 2-connected graph, then G^2 is a panconnected graph.

Later in 1975, Fleischner [28] established the fact that in the square of a connected graph G, panconnectedness and hamiltonian-connectedness are equivalent concepts as stated formally in the following theorem.

Theorem 4.3.9 [28] Let G be a connected graph. Then, G^2 is hamiltonian-connected if and only if G^2 is panconnected.

To complete the section, the proofs of Theorems 4.3.6 and 4.3.7 will be given to illustrate the types of arguments used in developing results of a similar nature. The proof of Theorem 4.3.7 given at the end of this section will not employ directly the fact that Theorem 4.3.7 is a corollary to Theorem 4.3.8. It is an independent proof.

Proof of Theorem 4.3.6 Let G=(V(G),E(G)) be a connected (n,m)-graph. We proceed by induction on n. Note that for $n \leq 4$, $G^3 = K_n$ and the result follows immediately. Now, suppose that the theorem has been established for each value n-1,n-2,...,4,3,2,1, for some $n \geq 5$. Let $u,v \in V(G)$ be any two distinct vertices in G. Let T=(V(T),E(T)) be a spanning tree in G such that $d_G(u,v) = d_T(u,v)$. The choice of such a tree is always possible. Note that $d_{G^3}(u,v) = d_{T^3}(u,v)$. Hence, it is sufficient to justify that for each ℓ satisfying $d_{G^3}(u,v)=d_{T^3}(u,v)\leq\ell\leq n-1$, there exists a u,v-path of length ℓ in T^3.

There are two cases to be examined according to whether $uv\in E(T)$ or $uv\notin E(T)$.

Case 1 Suppose that $uv\in E(T)$. Let $T_1=(V(T_1),E(T_1))$ and $T_2=(V(T_2),E(T_2))$ be the two components of the subgraph $T-uv=\langle E(T)-\{uv\}\rangle$ such that $u\in V(T_1)$ and $v\in V(T_2)$. Furthermore, let $|V(T_1)|=n_1$ and $|V(T_2)|=n_2$. First suppose that either $n_1=1$ or $n_2=1$. Without loss of generality, let $n_1=1$ and $n_2=n-1$. Let $v'\in V(T_2)$ be such that $vv'\in E(T_2)$. By the induction hypothesis, there exists in T_2^3 a v,v'-path Q_ℓ of length ℓ, for each ℓ satisfying $1\leq\ell\leq n_2-1=n-2$. Since $uv'\in E(T^3)$, for each ℓ, concatenate the u,v'-path Q_ℓ with the edge uv' to obtain a u,v-path of length $\ell'=\ell+1$ in T^3 with $2\leq\ell'\leq n-1$. Clearly, the edge uv is the u,v-path of length $1=d_{T^3}(u,v)$ and this accounts for the case where $n_1=1$ and $n_2=n-1$.

Next, suppose that $n_1>1$ and $n_2>1$. Let $u'\in V(T_1)$ and $v'\in V(T_2)$ be such that $uu'\in E(T_1)$ and $vv'\in E(T_2)$, respectively. Since $d_T(u',v')=3$. $u'v'\in E(T^3)$. By the induction hypothesis, T_1^3, T_2^3 are panconnected. It follows that for each ℓ_1,ℓ_2 satisfying $1\leq\ell_1\leq n_1-1$ and $1\leq\ell_2\leq n_2-1$, there exists a u,u'-path Q_{ℓ_1} in T_1^3 of length ℓ_1 and a v,v'-path Q_{ℓ_2} of length ℓ_2. For each pair of such integers ℓ_1,ℓ_2, concatenate the edge $u'v'$ and the paths Q_{ℓ_1} and Q_{ℓ_2} to obtain a u,v-path in T^3 of length $\ell=\ell_1+\ell_2+1$.

This accounts for each u,v-path of length ℓ, for each ℓ satisfying $3 \leq \ell \leq n-1$. Since $uv' \in E(T^3)$, $uv'v$ is a u,v-path of length 2. Clearly, The edge uv is a u,v-path of length ℓ and the result follows.

Case 2 Now suppose that $uv \in E(T)$. Let U be the unique u,v-path in T and let $w \in V(U)$ be such that $uw \in E(T)$. Similar to Case 1, let $T_1=(V(T_1),E(T_1))$ and $T_2=(V(T_2),E(T_2))$ be the two components of the subgraph $T-uw=\langle E(T)-\{uw\}\rangle$, such that $u \in V(T_1)$ and $w \in V(T_2)$. Again, let $n_1=|V(T_1)|$ and $n_2=|V(T_2)|$. Suppose that either $n_1=1$ or $n_2=1$. Without loss of generality, $n_1=1$ and $n_2=n-1$ is assumed. By the induction hypothesis, T_2^3 is panconnected. Thus for each ℓ_2 satisfying $d_{T^3}(w,v)=d_{T_2^3}(w,v) \leq \ell_2 \leq n_2-1=n-2$, there exists a w,v-path Q_{ℓ_2} in T_2^3 of length ℓ_2. For each ℓ_2, concatenate the edge uw to the w,v-path Q_{ℓ_2} to obtain a u,v-path of length ℓ with $1=d_{T^3}(w,v) \leq \ell \leq n-1$. Let $|U(w,v)|$ denote the length of the subpath of the u,v-path U connecting the vertices w and v. If $|U(w,v)| \equiv 0 \pmod 3$, then $d_{T^3}(u,v)=1+d_{T^3}(w,v)$. It follows in this case that every u,v-path in T^3 of length ℓ for each ℓ satisfying $d_{T^3}(u,v) \leq \ell \leq n-1$ has been accounted for. If $|U(w,v)| \equiv 0 \pmod 3$, then $d_{T^3}(u,v)=d_{T^3}(w,v)$. However, by the definition of $d_{T^3}(u,v)$, there exists a u,v-path in T^3 of length $d_{T^3}(u,v)$. Every u,v-path in T^3 of the required length has now been accounted for.

Assume that $n_1 > 1$ and $n_2 > 1$. Let $u' \in V(T_1)$ such that $uu' \in E(T_1)$. By the induction hypothesis, both T_1^3 and T_2^3 are panconnected. It follows that for each ℓ_1 satisfying $1 \leq \ell_1 \leq n_1-1$, there exists in T_1^3 a u,u'-path Q_{ℓ_1} of length ℓ_1. Similarly, for each ℓ_2

satisfying $d_{T_2^3}(w,v)=d_{T^3}(w,v)\leq \ell_2 \leq n_2-1$, there exists a w,v-path Q_{ℓ_2} of length ℓ_2 in T_2^3. Since $d_T(u',w)=2$, $wu'\in E(T^3)$. For each ℓ_1 and ℓ_2, a u,v-path of length $\ell=\ell_1+\ell_2+1$ can be obtained by concatenating the paths Q_{ℓ_1} and Q_{ℓ_2} with the edge wu'. This accounts for every u,v-path of length ℓ such that $2+d_{T^3}(w,v)\leq\ell\leq n-1$. Again, if $|U(w,v)|\equiv 0 \pmod 3$, then $d_{T^3}(u,v)=1+d_{T^3}(w,v)$. This inequality allows all u,v-paths in T^3 of length ℓ for every ℓ satisfying $1+d_{T^3}(u,v)\leq\ell\leq n-1$ to be accounted for. Since there always exists a u,v-path in T^3 of length $d_{T^3}(u,v)$, every u,v-path of desired length has now been accounted for. Next, assume that $|U(w,v)|\not\equiv 0 \pmod 3$ which implies that $d_{T^3}(u,v)=d_{T^3}(w,v)$. Under this condition, a u,v-path of length $1+d_{T^3}(u,v)$ can be obtained by concatenating a w,v-path of length $d_{T^3}(w,v)$ in T_2^3 with the edge uw. All the u,v-paths of desired length have now been accounted for and this completes the proof of the theorem. ∎

In chapter 1, an example constructed by P. Underground [46] clearly suggests that there exist infinitely many graphs whose square is not hamiltonian. This is consistent with the fact that only when additional constraints are imposed on a graph G can G^2 have properties like hamiltonian-connectedness or panconnectedness as demonstrated by many of the theorems stated earlier in this section. In what follows, a proof of Theorem 4.3.7 will be given in detail to illustrate a constructive argument used to show that the square of a hamiltonian graph is panconnected.

Proof of Theorem 4.3.7 Let $G=(V(G),E(G))$ be a hamiltonian (n,m)-graph and let $u,v\in V(G)$ be two arbitrary vertices in $V(G)$. Since G is hamiltonian, there exists a hamiltonian cycle $C: x_1 x_2 \ldots x_n x_1$. Without loss of generality, let $x_1=u$ and $x_k=v$ for some k satisfying $2\le k\le n$. Choose a shortest u,v-path Q in G. By definition, the length of the path Q is $d_G(u,v)$. Observe that in the graph G^2, the subgraph $\langle V(Q) \rangle$ induced by the vertices of Q contains a u,v-path Q' which has p edges in which p is the least positive integer satisfying $2p\ge d_G(u,v)$ and with at most one exception, all the edtes which constitute the path Q' are in $E(G^2)-E(G)$. In fact no edge in Q' is an edge in G if and only if $d_G(u,v)\equiv 0 \pmod 2$. A path Q' chosen in such a manner is necessarily a shortest u,v-path in G^2. It follows that $p=d_{G^2}(u,v)$.

To complete the proof of the theorem, we proceed to construct a u,v-path Q_ℓ of length ℓ in G^2 for each ℓ satisfying $d_{G^2}(u,2)\le \ell \le n-1$. There are three cases to be examined in the following.

Case 1 Construction of Q_ℓ for each ℓ satisfying $d_{G^2}(u,v)\le \ell \le d_G(u,v)$. Let $B=(V(B),E(B))$ be the subgraph in G2 defined by $V(B)=V(Q)$ and $E(B)=E(Q)\cup E(Q')$. Since both Q,Q' are subgraphs of B, it is clear that there exists a u,v-path of length $d_{G^2}(u,v)$ and a u,v-path of length $d_G(u,v)$ in B. Next, consider any u,v-path Q_ℓ of length ℓ in B such that $d_{G^2}(u,v)\le \ell < d_G(u,v)$. By the choice of Q , there exists an edge $xy\in E(Q_\ell)$ such that $sy\in E(Q')-E(Q)$. Then, there exists $w\in V(B)$ such that $xw,wy\in E(Q)$ and $w\notin\{u,v\}$. Then, a u,v-path $Q_{\ell+1}$ of length $\ell+1$ in B can be obtained by simply replacing the edge xy in Q_ℓ by the x,y-path xwy. The process can be repeated until a u,v-path of length ℓ has been obtained for each ℓ satisfying $d_{G^2}(u,v) \le \ell \le d_G(u,v)$. This accounts for all the u,v-paths of lengths required for case 1.

Recall that $u=x_1$ and $v=x_2$. We proceed to the second case.

<u>Case 2</u> Construction of Q_ℓ for each ℓ satisfying $d_G(u,v) \le \ell \le k-1$. Note that the portion $C': u=x_1x_2x_3\ldots x_k=v$ of the cycle C is a u,v-path of length k-1. If $d_G(u,v)=k-1$, then this second case is complete. Hence, we assume that $d_G(u,v)<k-1$.

As the u,v-path Q is traversed from $u=x_1$ to $v=x_k$, let x_i be the first vertex encountered on Q such that x_j is the next vertex on Q and $x_j \ne x_{i+1}$. Note that $1 \le i \le k$. Let m be the least positive integer such that $x_m \in V(Q)$ and $m \ge i+1$. Since $v=x_k$ is the terminal vertex of Q, $m \le k$, by the choice of Q being a shortest u,v-path in G, it is clear that $m \ge i+2$. Now, observe that $x_ix_j \in E(G)$ and $x_ix_{i+1} \in E(G)$. It follows that $x_jx_{i+1} \in E(G^2)$. Therefore, a u,v-path Q_L of length $L=1+d_G(u,v)$ can be obtained by replacing the edge x_ix_j in $E(Q)$ by the x_i,x_j-path $x_ix_{i+1}x_j$ of length 2.

Observe that for each index t satisfying $i+1 \le t \le m-2$, $x_{t-1}x_t \in E(G)$ and $x_tx_{t+1} \in E(G)$. Therefore, $x_{t-1}x_{t+1} \in E(G^2)$. A u,v-path Q_{L+1} of length $L+1=2+d_G(u,v)$ can now be obtained by replacing the edge x_ix_{i+1} by the x_i,x_{i+1}-path $x_ix_{i+2}x_{i+1}$. In a similar manner, a u,v-path Q_{L+2} of length $L+2=3+d_G(u,v)$ is obtained by replacing the edge $x_{i+1}x_{i+2}$ in Q_{L+1} by the x_{i+2},x_{i+1}-path $x_{i+2}x_{i+3}x_{i+1}$. This process can be repeated recursively to construct a sequence of u,v-paths such that each member of the sequence is of length one greater than the preceeding member until all the vertices on C between x_i and x_m have been used. The last member of this sequence of paths, $Q_{L+(m-i-1)}$, must be of the form $Q_{L+(m-i-1)}: P_i, x_{i+2}x_{i+4}\ldots x_{i+2s}x_{i+2s+1}x_{i+2s-1}$ $x_{i+2s-3}\ldots x_{i+1}, P_j$, when m-i-1 is odd and $s=\frac{m-i}{2}-1$, and $Q_{L+(m-i-1)}$: $P_i, x_{i+2}x_{i+4}\ldots x_{i+2s}x_{i+2s-1}x_{i+2s-3}x_{i+2s-5}\ldots x_{i+1}, P_j$, when m-i-1 is even and $s=\frac{m-i-1}{2}$, where P_i and P_j are x_1,x_i-subpaths of Q and x_j,x_k-subpaths of Q, respectively.

Now, for each successive vertex x_q of Q such that $1\leq q\leq k$, and the next vertex encountered on Q is not x_{q+1}, the same process discussed so far in this case can be repeated to obtain a sequence of u,v-paths of consecutive lengths, until every vertex between x_i and x_k on C has been exhausted. This accounts for all the u,v-paths of the required lengths for Case 2.

Case 3 Construction of Q_ℓ for each ℓ satisfying $k\leq\ell\leq n-1$. Note that $D_{k-1}: x_1x_2x_3\ldots x_{k-1}x_k$, the portion of C between $u=x_1$ and $v=x_k$, is a u,v-path of length k-1. Since $x_{k-1}x_k\in E(G)$ and $x_kx_{k+1}\in E(G)$, it follows that $x_{k-1}x_{k+1}\in E(G^2)$. A u,v-path D_k of length k can now be obtained by replacing the edge $x_{k-1}x_k$ by the x_{k-1},x_k-path $x_{k-1}x_{k+1}x_k$. Similarly, a u,v-path D_{k+1} of length k+1 can be obtained from L_k by simply replacing the edge x_kx_{k+1} in D_k by the x_{k+1},x_k-path $x_{k+1}x_{k+2}x_k$. This process is repeated recursively until a sequence of u,v-paths $D_k,D_{k+1},D_{k+2},\ldots D_{n-1}$, is obtained by exhausting all the vertices in C between x_k and x_n.

The accounts for all the u,v-paths of the desired length and this completes the proof of the theorem. ■

Section 4.4 Some useful examples on path length distributions

In Section 4.2, several sufficient conditions for a graph to be PLD-maxinal have been examined in detail. However, many questions concerning the path length distribution of a graph and PLD-maximal graphs remained unexplored. In light of Theorems 4.2.12 and 4.2.13, a question one would naturally ask would be the following.

Question 4.4.1 Does the path length distribution of a graph determine the graph up to isomorphism?

The answer to this question is 'no', as the following two theorems in [25] indicate.

Theorem 4.4.2 [25] For each pair of positive integers $\{n,m\}$ such that $2n-1 \le m \le \frac{1}{2}n(n-1)-2$, there exists at least two non-isomorphic PLD-maximal (n,m)-graphs.

Theorem 4.4.2 implies that for the same restriction on a pair of integers $\{n,m\}$, there exists at least two non-isomorphic (n,m)-graph with the same path length distribution.

Theorem 4.4.2 [25] For all $n \ge 9$, there exist non-isomorphic **trees** with the same PLD. Moreover, for each $N \ge 1$, there exist N mutually non-isomorphic trees with the same PLD.

Since every path in a tree is unique, if $(x_0, x_1, \ldots, x_{n-1})$ is a sequence of positive integers which represents the path length distribution of a tree on n vertices, then $x_0=0$, $x_2=n-1$ and

$\sum_{i=2}^{n} x_i = \frac{1}{2}n(n-1)$. It is easy to show by direct calculations that there exist no non-isomorphic trees on $n \leq 8$ vertices with the same path length distribution.

Clearly, some graphs are determined uniquely by their PLD. For example, the complete graph K_n on n vertices. The complete graph K_n-e with an edge deleted and the wheel graph W_n are the only graphs on n vertices which have their second terms x_1 in their corresponding PLD equal to $\frac{1}{2}n(n-1)$, $\frac{1}{2}n(n-1)$ -1 and $2(n-1)$, respectively. Furthermore, a path Q_n on n vertices is the only tree which has diameter n-1 and satisfies $x_0=0$, $x_1=n-1$ and $x_{n-1}>0$ in its PLD. Trivially, one can verify that for $n \leq 5$, a graph on n vertices is determined uniquely up to isomorphism by its PLD. The discussions up to this point inevitably leave the following difficult question open.

Question 4.4.4 Characterize the class of all connected graphs which are determined uniquely up to isomorphism by their PLD's.

For a full answer to be given to this question, great insight into the problem would be required. A more specific question which is equally fundamental is the following.

Question 4.4.5 Give a characterization of the class of all PLD-maximal graphs and panconnected graphs.

An even more basic question is the following.

Question 4.4.6 Which sequences of nonnegative integers $(0,x_1,x_2,\ldots,x_{n-1})$ represent the path length distribution of a connected graph on n vertices?

In light of Theorems 4.1.6,4.3.3, 4.3.4 and especially Theorems

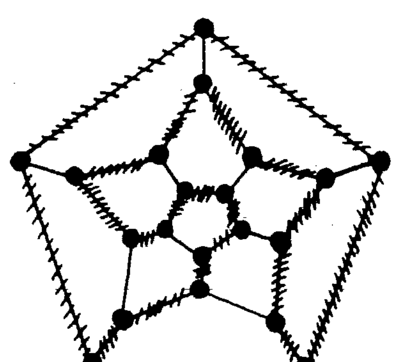
4.3.8 and 4.3.9, R.J. Faudree and R.H. Schelp were lead to propose the following conjecture.

Conjecture 4.4.7 [23] Let G be a hamiltonian-connected (n,m)-graph. Then for each i satisfying $\frac{n}{2} \leq i \leq n-1$, P_i holds in G.

A counter-example to this conjecture was found in a paper published in 1978 by Carsten Thomassen [43] in which he has established that there exist infinitely many exceptions to this conjecture. The details of the construction of these sophisticated counter-examples will be presented in the remaining part of this section.

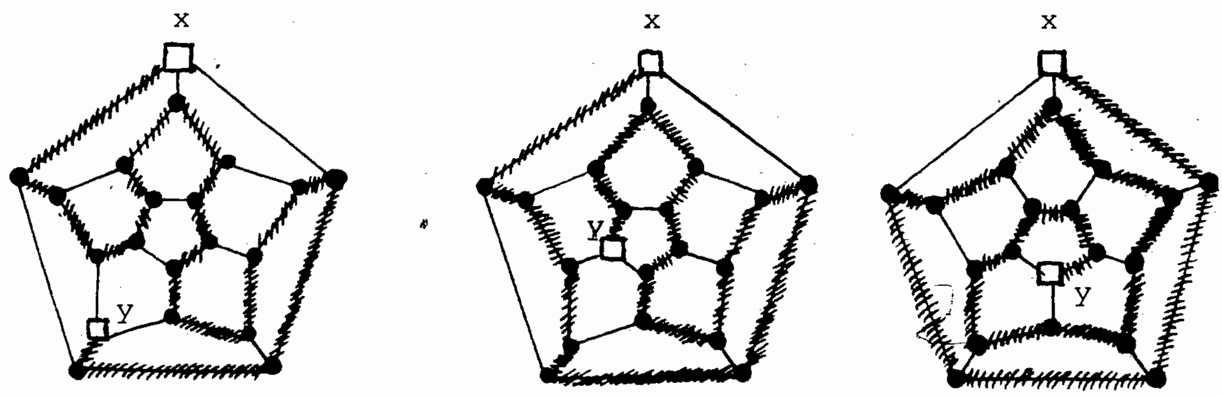
Counterexamples to Conjecture 4.4.7 A series of theorems relevant to the construction of these counter-examples are required due to the complexity of the construction. We begin with a careful investigation of the dodecahedron graph.

Dodecahedron Graph For the remainder of this section, D will denote the dodecahedron graph. As shown in Figure 4.6, D is hamiltonian. It is also well known that D is edge-transitive and distance transitive (that is, for any pairs $\{x,y\}$, $\{x',y'\}$ of vertices in V(D), there exists an automorphism $\pi \in Aut(D)$ such that $\pi(x)=x'$ and



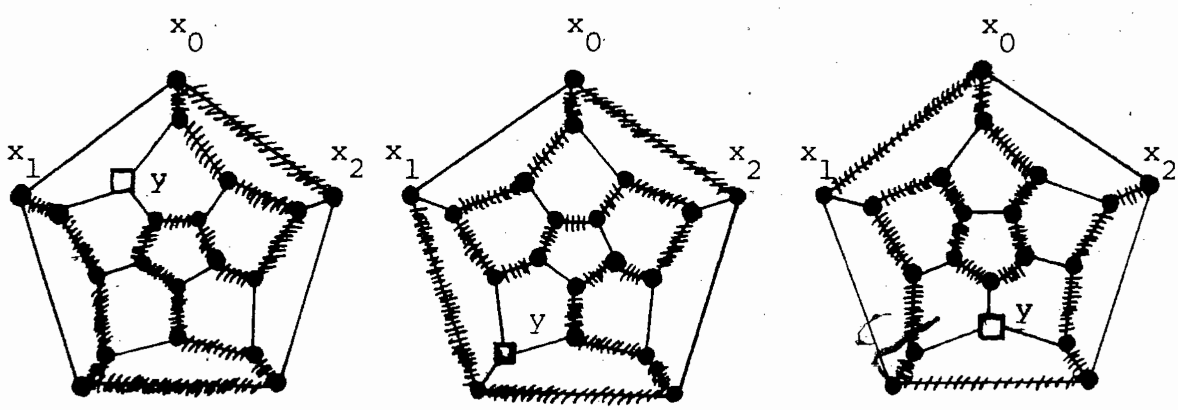
A hamiltonian cycle in D

figure 4.6



$d_D(x,y)=3$ $d_D(x,y)=4$ $d_D(x,y)=5$

figure 4.7



$d_D(x_0,y)=2$ $d_D(x_0,y)=3$ $d_D(x_0,y)=4$

figure 4.8

$\pi(y)=y'$ if and only if $d_D(x,y)=d_D(x',y')$. Moreover, D is a union of its hamiltonian cycles. D, however, has no cycle of length

$|V(D)|-1=19$ and D is not hamiltonian-connected. This latter fact is a consequence of the following theorem.

Theorem 4.4.8 [44] Let $G=(V(G),E(G))$ be a 2-connected planar (n,m)-graph such that each cycle which constitutes the boundary of a region has a number of edges congruent to 2 modulo 3. Then, the following conditions are satisfied.

(1) G has no cycle of length $n-1$.

(2) If uvw is a u,w-path of length 2 that is on the boundary of a region, then $P_{n-1}(u,w)$ fails to hold in G.

Proof Let $G=(V(G),E(G))$ be a 2-connected planar (n,m)-graph which satisfies the boundary length condition of the theorem and let $v \in V(G)$ be an arbitrary vertex in G. Then, the subgraph $G'=\langle V(G)-\{v\}\rangle$ contains exactly one region R which has a number of edges on the boundary of R congruent to 0 modulo 3. Using Grinberg's equation (see Theorem 1.2.10), it is clear that $\sum_{i=1}^{n}(i-2)(\phi_i'-\phi_i'')\neq 0$. Hence, G' is not hamiltonian. Since v is an arbitrary vertex in G, it follows that there exists no cycle of length $n-1$ in G.

Let uvw be a u,w-path of length 2 that is on the boundary of a region. Then, the graph $G''=G\cup\{u\,w\}$, defined by $V(G'')=V(G)$ and $E(G'')=E(G)\cup\{u\,w\}$, has exactly one region R_0 bounded by a number of edges congruent to 0 modulo 3 and one region R_1 bounded by a number of edge congruent to 1 modulo 3.

By Grinberg's equation, $\sum_{i=1}^{n}(i-2)(\phi_i'-\phi_i'')=0$ can be satisfied only if both regions R_0 and R_1 lie on the same side of every hamiltonian cycle C in G". This implies that there exists no hamiltonian cycle in G" which contains the edge uw. Therefore, $P_{n-1}(u,w)$ fails to hold in G and this completes the proof. ■

Other relevant properties concerning the graph D are reflected in Figures 4.7 and 4.8. We first give the following theorem.

Theorem 4.4.9 [44] Let $\{x,y\}$ be two arbitrary vertices in D. Then, there exists a hamiltonian x,y-path in D if and only if $d_D(x,y)\neq 2$. If $d_D(x,y)=1$, then there is a hamiltonian x,y-path.

Proof Shown in Figure 4.7 are three hamiltonian x,y-paths for some vertices $x,y\in V(D)$ with $d_D(x,y)=3,4$ and 5, respectively. Since D is distance transitive, it follows that for any two vertices $x',y'\in V(D)$ such that $3\leq d_D(x',y')\leq 5$, there exists a hamiltonian x',y'-path in D. Also, there is a hamiltonian x,y-path if $d_D(x,y)=1$. Conversely, if $x,y\in V(D)$ and $d_D(x,y)=2$, then by Theorem 4.4.8, there exists no hamiltonian x,y-path in D and the result follows. ■

Theorem 4.4.10 [44] Let $x_0\in V(D)$ be an arbitrary vertex in D. For any vertex $y\in V(D)$ such that $2\leq d_D(x_0,y)\leq 4$, $D'=\langle V(D)-\{y\}\rangle$ has a hamiltonian path connecting two neighbours x_1,x_2 of x_0 such that neither x_1 nor x_2 is a neighbour of y.

Proof Shown in Figure 4.8 are three hamiltonian x_1,x_2-paths in $D'=\langle V(D)-\{y\}\rangle$, where $d_D(x_0,y)=2,3$ and 4, respectively. Now x_1,x_2 are two neighbours of x_0 such that neither x_1 nor x_2 is a neighbour of y in D. Since D is distance transitive, the result follows immediately. ■

Using theorems 4.4.9 and 4.4.10, the following useful result can now be established.

Theorem 4.4.11 Let $x_0 \in V(D)$ be any vertex in D and let $N(x_0)=\{x_1,x_2,x_3\}$ be the set of neighbours of x_0 in D. Let $u,v \in V(D)$ be any two vertices in D such that at least one of u,v does not belong to $N(x_0)$. Then, by adding at most one edge between two vertices in $N(x_0)$, a hamiltonian u,v-path can be obtained.

Proof: Let $u,v \in V(D)$ be two vertices in V(D) which satisfy the hypotheses of the theorem. If $d_G(u,v)>2$, then by Theorem 4.4.9, there exists a hamiltonian u,v-path in D and the result follows immediately. Thus, we assume that $d(u,v)=2$. Without loss of generality, $2 \leq d_D(u,x_0) \leq 4$ is assumed. By Theorem 4.4.10, a cycle C which contains all vertices in D except u can be obtained by adding at most one edge x_1x_2 between two vertices in $N(x_0)$.

Let w be a common neighbour of u and v. Since each vertex in D has degree 3, the edge wv is necessarily on C. A hamiltonian u,v-path which contains the edge x_1x_2 can be obtained by replacing the edge wv by the edge wu. The theorem now follows. ■

Next, we proceed to define a class of graphs called generalized dodecahedron graphs which will be used to construct our counter-examples to Conjecture 4.4.7.

GENERALIZED DODECAHEDRON GRAPHS

For each positive integer $k \geq 3$ construct two concentric cycles of length 3k:

$C_1: x_1y_1z_1x_2y_2z_2 \ldots x_ky_kz_kx_1$ and $C_2: u_1v_1w_1u_2v_2w_2 \ldots u_kv_kw_ku_1$. For each i, $1 \le i \le k$, the edges y_iu_i and z_iw_i are then connected between C_1 and C_2. Two new vertices x_0, v_0 are introduced such that for each i, $1 \le i \le k$, x_0 is joined to x_i and v_0 is joined to v_i. The resulting graph is the generalized dodecahedron graph D_k. It is clear that P_3 is isomorphic to the dodechedron graph D itself. Shown in Figure 4.9 is the generalized dodechahedron graph D_4.

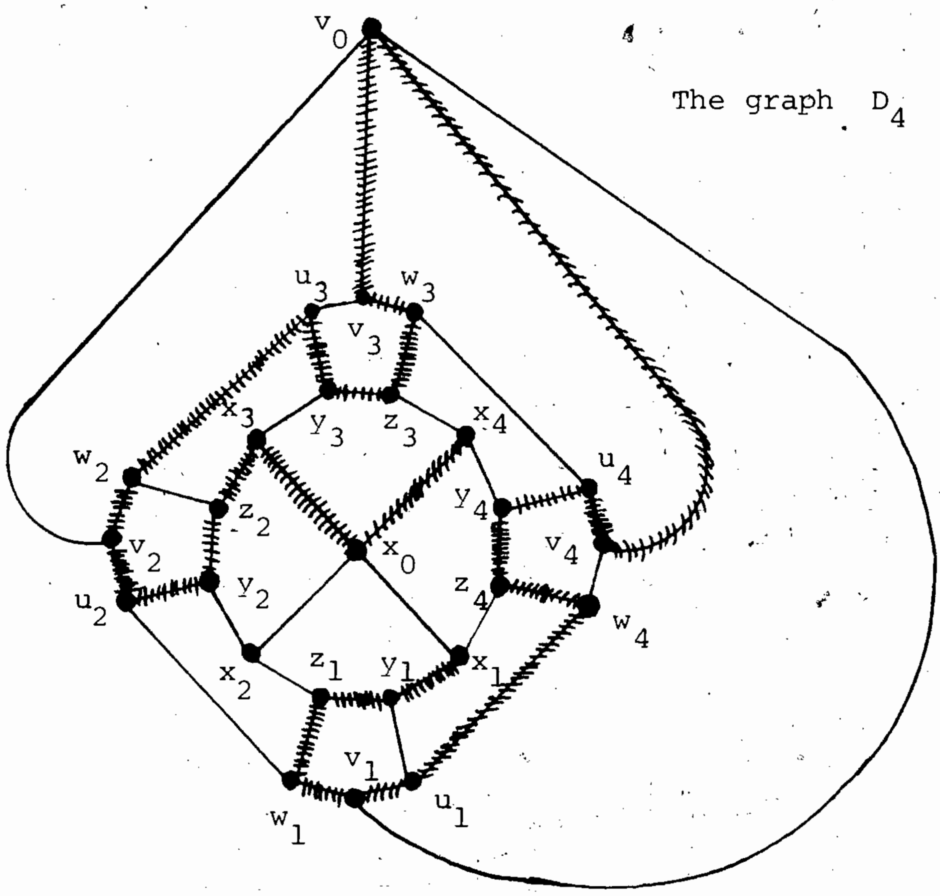


figure 4.9

Furthermore, the graph D_4 as shown in Figure 4.9 contains an x_1,x_4-path Q which contains all vertices of D_4 except x_1 and x_2. By adding the edge x_2x_4 to the path Q, a hamiltonian x_1,x_2-path is obtained. Similarly, adding the edge x_1x_2 to the path Q will result in a hamiltonian

x_2,x_4-path. By the symmetry of D_4, for any two neighbours x,y of x_0, a hamiltonian x,y-path can be obtained by joining two neighbours of x_0,x_i,x_j such that at least one of x_i and x_j is not in $\{x,y\}$. These facts can now be used to produce the following more general result.

Theorem 4.4.12 [44] Let $k\geq 4$. If x and y are any two vertices of D_k, then by adding at most one edge joining two neighbours of x_o, a hamiltonian x,y-path in D_k is obtained.

Proof: Suppose the contrary and let $k, k\geq 4$, be the smallest positive integer such that D_k contains two neighbours $\{x,y\}$ of x_0 which fail to satisfy the conclusion of the theorem. Clearly, there exists i, with $1\leq i\leq k$, such that neither x nor y is in the set $\{x_i,y_i,z_i,u_i,v_i,w_i,x_{i+1}\}$ unless k=4 and $\{x,y\}\subset\{x_1,x_2,x_3,x_4\}$. This latter case, however, has previously been accounted for in a discussion associated with Figure 4.9. Without loss of generality, it is assumed for the remainder of the proof that for $k>4$, neither x nor y is in the set $\{x_k,y_k,z_k,u_k,v_k,w_k,x_1\}$ and and for k=4, at least one of x and y is not in $\{x_1,x_2,x_3,x_4\}$. Form a new graph by deleting from D_k the vertices x_k,y_k,z_k,u_k,v_k,w_k and adding the edges $z_{k-1}x_1$ and $w_{k-1}u_1$. The resulting graph is necessarily isomorphic to the graph D_{k-1} and will be referred to as such. By the choice of k or by Theorem 4.4.11 if k=4, a hamiltonian x,y-path P can be obtained in D_{k-1} by adding an edge x_ix_j ($i<j$) between two neighbours x_i and x_j of x_0. We now proceed to transform the path P into a hamiltonian x,y-path in D_k by adding an edge between two neighbours of x_0 and thereby obtaining the

desired contradiction. There are several cases to be considered.

Case 1 Suppose that the path P contains the edges $z_{k-1}x_1$ and $w_{k-1}u_1$. A hamiltonian x,y-path in D_k can be obtained simply by replacing the edges $z_{k-1}x_1$ and $w_{k-1}u_1$ by the paths $z_{k-1}x_k y_k z_k x_1$ and $w_{k-1}u_k v_k w_k u_1$, respectively.

Case 2 Suppose that the path P contains the edge $z_{k-1}x_1$ but not the edge $w_{k-1}u_1$. A hamiltonian x,y-path in D_k in this case can be obtained by replacing the edge $z_{k-1}x_1$ by the path $z_{k-1}x_k y_k u_k v_k w_k z_k x_1$.

Case 3 Suppose that the path P contains the edge x_0x_1 but not the edge $z_{k-1}x_1$. If the edge $w_{k-1}u_1$ is contained in P, then replace the edges x_0x_1 and $w_{k-1}u_1$ by the paths $x_0x_k y_k z_k x_1$ and $w_{k-1}u_k v_k w_k u_1$, respectively, and obtain an x,y-path in D_k accordingly. Otherwise, if the edge $w_{k-1}u_1$ is not in P, then replace the edge x_0x_1 by the path $x_0x_k y_k u_k v_k w_k z_k x_1$ to obtain a hamiltonian x,y-path in D_k.

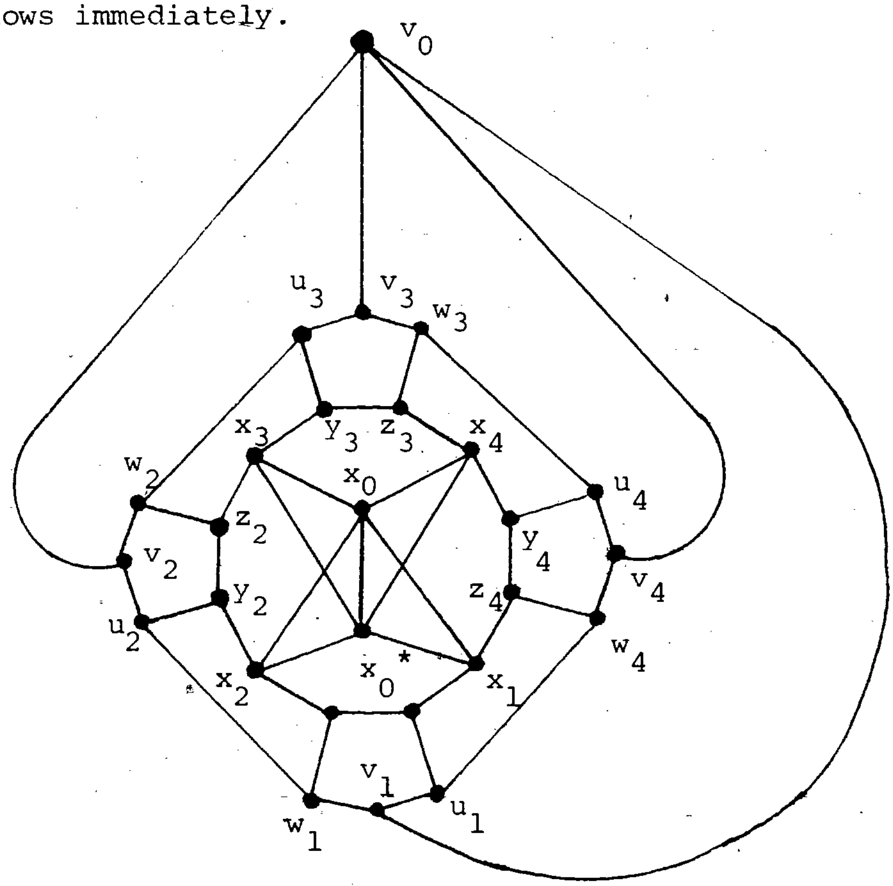
Case 4 Finally, suppose that neither $z_{k-1}x_1$ nor x_0x_1 is in P. Since x_1 cannot be an end vertex of the x,y-path P in D_{k-1}, i=1. The edges x_ix_j and $w_{k-1}u_1$ can now be replaced by the paths $x_1z_k y_k x_k x_j$ and $w_{k-1}u_k v_k w_k u_1$, respectively, if $w_{k-1}u_1$ is in P, to obtain a hamiltonian x,y-path in the graph $D_k \cup \{x_kx_j\}$. If the edge $w_{k-1}u_1$ is not in P, then replace the edge x_1x_j by the path $x_1z_k w_k v_k u_k y_k x_k x_j$ to obtain a hamiltonian x,y-path in $D_k \cup \{x_kx_j\}$. The result now follows from this contradiction. ■

We are now equipped to construct a class of counter examples to conjecture 4.4.7 as indicated in Theorem 4.4.13 below.

For each $k \geq 4$, let D_k^* denote the graph obtained from D_k by introducing one new vertex x_0^* such that x_0^* is adjacent to every neighbour of x_0 and to x_0 itself. Figure 4.10 illustrates the graph D_4^*.

Theorem 4.4.13 [44] For each $k \geq 4$, D_k^* is a hamiltonian-connected graph which has no x_0, x_0^*-path of length $|V(D_k^*)|-2$.

Proof: The fact that D_k^* is hamiltonian-connected is an immediate consequence of Theorem 4.4.12. Observe that for each x_0, x_0^*-path of length $\ell \geq 2$ in D_k^*, a cycle of length ℓ in D_k can be obtained by contracting the edge $x_0x_0^*$. However, by Theorem 4.4.8, there exists no cycle of length $|V(D_k)|-1=|V(D_k^*)|-2$ and the result follows immediately. ■



The graph D_4^*

figure 4.10

Theorem 4.4.13 has demonstrated that there exists an infinite number of hamiltonian-connected (n,m)-graphs each of which contains an edge that is not contained in any cycle of length n-1. This constitutes an infinite set of counter-examples to Conjecture 4.4.7. We conclude this section by mentioning a few open questions concerning the path length distribution of a graph.

<u>Question 4.4.14</u> Does there exist a hamiltonian-connected (n,m)-graph which fails to satisfy P_ℓ for each ℓ satisfying $3 \le \ell \le n-2$?

<u>Question 4.4.15</u> To be more general, one can ask for which integers ℓ, $3 \le \ell \le n-2$, there exist a hamiltonian connected graph G on n vertices such that P_ℓ fails to hold in G?

<u>Question 4.4.16</u> Does there exist a hamiltonian-connected (n,m)-graph G such that no cycle of length n-1 is in G?

APPENDIX

NOTATION

C(S,U)	The Cayley Graph of a group U with respect to symbol S.
$\|B\|$	The order of a group, graph or set.
$\langle S \rangle$	If S is a subset of a group, then this notation denotes the subgroup generated by S. If S is a subset of the vertex set of a graph, then this notation represents the subgraph induced by S.
Z_n	The addition cyclic group modulo n.
κ	Connectivity.
β	Independence Number.
δ	Minimum degree
t(G)	The toughness of a graph G.

BIBLIOGRAPHY

1. Alspach, B. "Hamiltonian cycles in Vertex-Transitive Graphs of order 2p". Proc. Tenth S-E Conf. Combinatorics, Graph Theory and Computing (1981) pp. 131 - 139.

2. Alspach, B. " On Point-Symmetric Tournaments". Canadian Mathematics Bulletin Volume 13 (1970) pp. 317 - 323.

3. Alavi, Y. and Williamson, J.E. "Panconnected Graphs". Studia Scientiarum Mathicarum Hungarica Volume 10 (1975) pp. 19 - 22.

4. Bollobas, B. and Hobbs, A.M. "Hamiltonian Cycles in Regular Graphs". Advances in Graph Theory. North Holland, Amsterdam (1978).

5. Bondy, J.A. and Murty, U.S.R. "Graph Theory with Applications". Elsevier North Holland, Inc. (1976).

6. Bondy, J.A. "Properties of Graphs with constraints on Degrees". Studia Scientiarum Mathicarum Hungarica Volume 4 (1969) pp. 473 - 475.

7. Bondy, J.A. and Chvatal, V. "A Method in Graph Theory". Discrete Mathematics Volume 15 (1976) pp. 111 - 135.

8. Behzad, M. and Chartrand, G. "Introduction to the Theory of Graphs". Boston, Allyn and Bacon (1971).

9. Chartrand, G., Gould, R.J. and Polimeni, A.D. "A Note on Locally-Connected and Hamiltonian-Connected Graphs". Israel Journal of Mathematics Volume 33, Number 1 (1979) pp. 5 - 8.

10. Chartrand, G., Hobbs, A.M., Jung, H.A., Kapoor, S.F. and Nash-Williams, St.J.A. "Note- The Square of a Block is Hamiltonian-Connected". Journal of Combintorial Theory (B) Volume 16 (1974) pp. 290 - 292.

11. Chartrand, G., Kapoor, S.F. and Lick, D.R. "n-Hamiltonian Graphs". Journal of Combinatorial Theory Volume 9 (1970) pp.308 - 312.

12. Chartrand, G. and Pippert, R.E. "Locally-Connected Graphs". Casopis Pest Mat. Volume 99 (1974) pp.158 - 163.

13. Chvatal, V. "On Hamiltonian Ideals". Journal of Combinatorial Theory(B) Volume 12 (1972) pp. 163 - 168.

14. Chvatal, V. "Tough Graphs and Hamiltonian Circuits". Discrete Mathematics Volume 5 (1973) pp. 215 - 228.

15. Chvatal, V. and Erdos, P. "A Note on Hamiltonian Circuits". Discrete Mathematics Volume 2 (1972) pp. 111 - 113.

16. Cook, S.A. "The Complexity of Theorem Proving Procedures". Proc. Third Ann. ACM Symposium on Theory of Computing (1970) pp. 151 - 158.

17. Dirac, G.A. "Some Theorems On Abstract Graphs". Proc. London Math. Soc. Volume 2 (1952) pp. 69 - 81.

18. Dirac, G.A. "Generalisation du Theoreme de Menger". C. R. Acad.Sci. Paris 250 Volume 26 (1960) pp. 4252 - 4253.

19. Elpas, B. and Turner, J. "Graphs with Circulant Adjacency Matrices". Journal of Combinatorial Theory Volume 9 (1970) pp. 297 - 307.

20. Erdos, P. and Hobbs, A.M. "Hamiltonian Cycles in Regular Graphs of Moderate Degree". Journal of Combinatorial Theory(B) Volume 23 (1977) pp. 139 - 142.

21. Erdos, P. and Hobbs, A.M. "A Class of Hamiltonian Regular Graphs". Journal of Graph Theory Volume 2 (1978) pp. 129 - 137.

22. Faudree, R.J. and Schelp, R.H. "Path Connected Graphs". Acta Mathematica Academiae Scientiarum Hungaricae Tomus 25 (3-4) (1974) pp. 313 - 319.

23. Faudree, R.J. and Schelp, R.H. "The Square of a Block is Strongly Path Connected". Journal of Combinatorial Theory(B) Volume 20 (1976) pp. 47 - 61.

24. Faudree, R.J. and Schelp, R.H. "Various Length Paths in Graphs". Theory and Applications of Graphs (Edited by Alavi, Y. and Lick, D.R.) Springer Verlag, Berlin (1978) pp. 160 - 173.

25. Faudree, R.J., Rouseau, C.C. and Schelp, R.H. "Theory of Path Length Distribution I". Discrete Mathematics Volume 6 (1973) pp. 35 - 52.

26. Fleischner, H. "The Square of Every Two-Connected Graph is Hamiltonian". Journal of Combinatorial Theory(B) Volume 16 (1974) pp. 29 - 34.

27. Fleischner, H. "On Spanning Subgraphs of a Connected Bridgeless Graphs and their applications to DT-Graphs". Journal of Combinatorial Theory(B) Volume 16 (1974) pp. 17 - 28.

28. Fleischner, H. "In the Square of Graphs, Hamiltonicity and Pancyclicity, Hamiltonian-Connectedness and Panconnectedness are equivalent concepts". Monatschefte fur Mathematik Volume 82 (1976) pp. 125 - 149.

29. Hobbs, A.M. "The Square of a Block is Vertex Pancyclic". Journal of Combinatorial THeory(B) Volume 20 (1976) pp.1 - 4.

30. Jackson, B. "Hamiltonian Cycles in Regular 2-Connected Graphs". Journal of Combinatorial Theory(B) Volume 29 Number 1 (1980) pp. 27 - 46.

31. Karaganis, J.J. "On the Cube of a Graph". Canadian Mathematical Bulletin Volume 11 (1969) pp. 295 - 296.

32. Karp, R.M. "Reducibility among Combinatorial Problems". Miller, R.E. and Thatcher, J.W. Eds., Complexity of Computer Computations, Plenum Press, New York (1972) pp. 85 - 104.

33. Las Vergnas, M. Ph.D. Thesis, University of Paris (1972).

34. Lick, D.R. "A sufficient Condition on Hamiltonian-Connectedness". Journal of Combinatorial Theory Volume 8 (1970) pp. 444 - 445.

35. Lick, D.R. "n-Hamiltonian-Connected Graphs". Duke Mathematical Journal Volume 37 Number 2 (1970) pp. 387 - 392.

36. Molluzzo, J.C. "Toughness, Hamiltonian-Connected Graphs and n-Hamiltonicity". Annals New York Academy of Science (1979) pp. 402 - 404.

37. Moon, J.W. "On a Problem of Ore". Math. Gaz. Volume 49 (1965) pp. 40 - 41.

38. Oberly, D.J. and Sumner, D.P. "Every Connected, Locally-Connected Nontrivial Graph with no induced Claw is Hamiltonian". Journal of Graph Theory Volume 3 (1979) pp. 351 - 356.

39. Ore, O. "A Note on Hamiltonian Circuits". Amer. Math. Monthly Volume 67 (1960) p. 55.

40. Ore, O. "Hamilton Connected Graphs". J. Math. Pures. Appl. Volume 42 (1963) pp. 21 - 27.

41. Posa, L. "A Theorem Conerning Hamilton Lines". Magyar Tud. Akad. Mat. Kutato Int. Kozl Volume 7 (1962) pp. 225 - 226.

42. Quimpo, N.F. and Chen, C.C. " On Strongly Hamiltonian Abelian Group Graphs". Proc. Eighth Austral. Comb. Conf., Lecture Notes in Mathematics, Springer-Verlag, Berlin (1981) to appear.

43. Sekanina, M. "On an Ordering of the Set of Vertices of a Connected Graph". Publ. Fac. Sci. Univ. Brno Volume 412 (1960) pp. 137 - 142.

44. Thomassen, C. "Counterexamples to Faudree and Schelp's Conjecture on Hamiltonian-Connected Graphs". Journal of Graph Theory Volume 2 (1978) pp. 341 - 347.

45. Turner, J. "Point Symmetric Graphs with a Prime Number of Points". Journal of Combinatorial Theory Volume 3 (1967) pp. 136 - 145.

46. Underground, P. "On Graphs with Hamiltonian Squares". Discrete Mathematics Volume 21 (1978) p. 323.

47. White A.T. "Graphs, Groups and Surfaces". Mathematical Studies 8, North Holland Publishing Company, Amsterdam (1973).

48. Whitney, H. "A Theorem on Graphs". Ann. Math. Volume 32 (1931) pp. 378 - 390.

49. Williams,Nash C.St.J.A. "Edge Disjoint Hamiltonian Circuits in Graphs with Vertices of Large Valency". Studies in Pure Mathematics, Academic Press, London (1971) pp. 157 - 183.

50. Williams,Nash C.St.J.A. "Valency Sequences which forces Graphs to be Hamiltonian Circuits". Interim Report, University of Waterloo Research Report (1969).

51. Woodall, D.R. "The Binding Number of a Graph and Its Anderson Number". Journal of Combinatorial Theory (B) Volume 15 (1973 pp. 225 - 255.

52. Woodall, D.R. "A Sufficient Condition For Hamiltonian Circuits". Journal of Combinatorial Theory (B) Volume 25 (1978) pp. 184 - 186.