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THE LATTICES OF SUBGROUPS AND VARIETIES OF LATTICE ORDERED GROUPS

by

Mary Elizabeth Huss

B.Sc., University of Nottingham, 1975

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THE LATTICES OF SUBGROUPS AND VARIETIES OF LATTICE ORDERED GROUPS

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ABSTRACT

A lattice ordered group is a group $(G,+)$ with a lattice order $\leq$ that is compatible with the group operation. In Chapter 1 we develop the basic properties of lattice ordered groups and include a discussion of ℓ-subgroups (subgroups which are also sublattices) and of ordered permutation groups.

The second chapter is devoted to lattices of subgroups of lattice ordered groups. In particular we answer a question posed by Conrad, showing that the lattice of ℓ-subgroups of a lattice ordered group, G, is distributive if and only if G is isomorphic to a subgroup of the additive rationals.

In Chapter 3 we consider several examples of varieties of lattice ordered groups and see how they are related in the lattice of varieties of lattice ordered groups. We also describe a generalization of the wreath product, the twisted wreath product, showing that the twisted wreath product of a lattice ordered group by a totally ordered group may be lattice ordered. We conclude by looking at a specific example of a twisted wreath product and see how this is related to a standard wreath product.

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INTRODUCTION

The fundamental results of the theory of lattice ordered groups were first presented by Birkhoff [2]. He described many of the basic properties of the elements of lattice ordered groups, showing, for example, that any lattice ordered group is a distributive lattice.

In Chapter 1 we begin by outlining the elementary properties of lattice ordered groups, many of which are due to Birkhoff. We also consider ℓ-subgroups, that is, those subgroups which are sublattices. Finally ordered permutation groups are described. An important example of a lattice ordered permutation group is the group, A(X), of all order preserving permutations of a totally ordered set, X . The last result of this chapter, Holland's representation theorem, is an important tool in the study of lattice ordered groups, and is an analogue of Cayley's theorem in group theory. Holland's representation theorem establishes that every lattice ordered group is isomorphic to an ℓ-subgroup of the group of all order preserving permutations of a totally ordered set.

In the second chapter we are concerned with lattices of subgroups. It is well known that the lattice of convex ℓ-subgroups of a lattice ordered group is distributive and we consider those lattice ordered groups for which the lattice of all ℓ-subgroups is distributive. The main result of this section answers a question posed by Conrad [4]: we show that the lattice of ℓ-subgroups of a lattice ordered group G is distributive if and only if G is isomorphic to a subgroup of the additive rationals. Finally we pursue the question with an important generalization of lattice ordered groups, Riesz groups. These are groups which, although not lattice ordered, satisfy an interpolation property, and we show that the lattice of convex directed subgroups of a Riesz group is distributive.

For any type of abstract algebra, a variety is an equationally defined class of such algebras. The extensive work on varieties of groups, much of which is described by H. Neumann [14], prompted an interest in the study of lattice ordered group varieties. The early works in this area are mainly concerned with specific lattice ordered group varieties. For example, Weinberg [19] showed that the abelian variety, A , is the smallest proper variety of lattice ordered groups. Wolfenstein [20] showed that the normal valued lattice ordered groups form a variety, N , which was later found by Holland [10], to be the largest proper lattice ordered group variety. Martinez [13] undertook a more comprehensive study of varieties of lattice ordered groups, describing an associative multiplication of lattice ordered group varieties, and determined that the set, L , of all lattice ordered group varieties forms a lattice ordered semigroup under this multiplication, the partial order being set inclusion. More recently, Glass, Holland and McCleary [7] have extended this work. One of their main results shows that the powers of the abelian variety, $A^{\cdot}$, generate the normal valued variety, N .

In Chapter 3 we consider several examples of lattice ordered group varieties and see how they are placed in the lattice of all lattice ordered group varieties. As in group theory, wreath products provide a useful tool in the study of varieties and these are described. However lattice ordered group theory considers only wreath products of lattice ordered permutation groups. We also consider a generalization of the wreath product, the twisted wreath product and show that the twisted wreath product of a lattice ordered group by a totally ordered group may be lattice ordered. Finally we look at our example of a twisted wreath product which is isomorphic to $G(mn)$ an ℓ-subgroup of Z Wr Z first described by Martinez [12], and later by Scrimger [17] who used lattice ordered groups of this kind to provide an infinite number of covers of the abelian variety, A .

CHAPTER 1

ℓ-groups

In this chapter we introduce and outline some of the basic properties of partially ordered groups, and more particularly, lattice ordered groups. Many results are quoted without proof. Along with a more detailed discussion, these proofs may be found in Conrad [4], Fuchs [5] and Bigard, Keimel and Wolfenstein [1].

In group theory it is conventional to use multiplicative notation for the binary operation of non-commutative groups and additive notation for commutative groups. The development of ℓ-group theory has tended to use additive notation more extensively, however, the close ties to group theory make the use of multiplicative notation desireable in some situations. In this thesis both notations are used for ℓ-groups and it should be noted that additive notation does not imply commutativity.

Some knowledge of group theory and lattice theory is assumed. Information on these topics may be found in Schenkman [16] and Birkhoff [3] respectively.

Section 1. Basic Results.

We begin with the definition of a partially ordered group and in particular a lattice ordered group. From our definition we are then able to develop some of the basic properties of lattice ordered groups which will be used in later sections.

A <u>partially ordered group</u> (po-group) is a group $(G,+)$ with a partial order $\leq$ that is compatible with the group operation; that is, for all $a, b, x, y \in G$

$$a \leq b \text{ implies } x + a + y \leq x + b + y .$$

If the partial order is a lattice order, then G is called a <u>lattice ordered group</u> (ℓ-group). If the partial order is a total order then G is called a <u>totally ordered group</u> (o-group).

We shall denote the <u>positive cone</u> of G by G^+ :

$$G^+ = \{g \in G \mid g \geq 0\} .$$

<u>Proposition 1.1.1</u>. If G is a po-group, then G^+ is a normal subsemigroup that contains 0 but no other element and its inverse. Conversely if P is such a subsemigroup and if we define $a \leq b$, provided $b - a \in P$, then $\leq$ is a partial order for G and $P = G^+$.

A po-group G is <u>directed</u> if, for all $a, b \in G$ there exists $x \in G$ such that

$$x \geq a \quad \text{and} \quad x \geq b .$$

<u>Proposition 1.1.2</u>. For a po-group G, the following are equivalent:

(1) G is directed.

(2) For each $a \in G$, $G = \{y-z \mid y \geq a \text{ and } z \geq a\}$.

(3) G^+ generates G.

(4) For each $g \in G$ there is an upper bound for g and 0.

We now establish some elementary properties of ℓ-groups.

For $a,b \in G$ an ℓ-group we denote by $a \vee b$ $(a \wedge b)$ the least upper bound (greatest lower bound) of a and b.

Proposition 1.1.3. For $a, b, x, y \in G$, an ℓ-group

$$x + (a \vee b) + y = (x + a + y) \vee (x + b + y)$$

and

$$x + (a \wedge b) + y = (x + a + y) \wedge (x + b + y) .$$

Proof. Since $a \vee b \geq a,b$ we have

$$x + (a \vee b) + y \geq x + a + y , \quad x + b + y$$

and thus

$$x + (a \vee b) + y \geq (x + a + y) \vee (x + b + y) .$$

Assume that for some $z \in G$

$$z \geq (x + a + y) \vee (x + b + y)$$

Then $\quad z \geq x + a + y$ and $z \geq x + b + y$

hence $\quad -x + z - y \geq a,b$

and so $\quad -x + z - y \geq a \vee b$

whence $\quad z \geq x + (a \vee b) + y$.

Thus we have

$$x + (a \vee b) + y = (x + a + y) \vee (x + b + y) .$$

The dual may be proved similarly.

<u>Proposition 1.1.4.</u> If G is an ℓ-group and $a,b \in G$, then

$$-(a \vee b) = -a \wedge -b$$

and dually.

<u>Proof.</u> Since $a \vee b \geq a,b$, we have $-(a \vee b) \leq -a,-b$.

Thus $\quad -(a \vee b) \leq -a \wedge -b$.

If $z \in G$ and $z \leq -a \wedge -b$, then $-z \geq a,b$.

Thus $\quad -z \geq a \vee b$

and $z \leq -(a \vee b)$.

Hence $-(a \vee b)$ is the greatest lower bound of $-a$ and $-b$. The dual result may be proved similarly.

Proposition 1.1.5. A po-group G is an ℓ-group if and only if, for all $g \in G$, $g \vee 0$ exists.

Proof. If G is an ℓ-group, then it is clear that $g \vee 0 \in G$ for all $g \in G$.

Conversely, if $g \vee 0$ exists in G , for all $g \in G$, than for all $a,b \in G$

$$[(a-b) \vee 0] + b = a \vee b$$

and

$$a \wedge b = -(-a \vee -b) .$$

Proposition 1.1.6. Let G be an ℓ-group. Then, for each positive integer n , $na \geq 0$ implies $a \geq 0$.

Proof. We first use induction to show that for each positive integer n ,

$$n(a \wedge 0) = na \wedge (n-1)a \wedge \ldots \wedge a \wedge 0 .$$

The result holds for $n = 1$. Assume the result is true for $n-1$. Then

$$n(a \wedge 0) = (n-1)(a \wedge 0) + (a \wedge 0)$$

$$= [(n-1)(a \wedge 0) + a] \wedge [(n-1)(a \wedge 0) + 0]$$

(by Proposition 1.1.3)

$$= [((n-1)a \wedge (n-2)a \wedge \ldots \wedge a \wedge 0) + a] \wedge [(n-1)a \wedge (n-2)a \wedge \ldots \wedge a \wedge 0]$$

$$= [na \wedge (n-1)a \wedge \ldots \wedge 2a \wedge a] \wedge [(n-1)a \wedge (n-2a) \wedge \ldots \wedge a \wedge 0]$$

$$= na \wedge (n-1)a \wedge \ldots \wedge a \wedge 0 \, .$$

Now, if $na \geq 0$, then $na \wedge 0 = 0$ and we have

$$n(a \wedge 0) = na \wedge (n-1)a \wedge \ldots \wedge a \wedge 0$$

$$= (n-1)a \wedge \ldots \wedge a \wedge 0$$

$$= (n-1)(a \wedge 0) \, .$$

Thus $a \wedge 0 = 0$ and hence $a \geq 0$.

<u>Corollary 1.1.7.</u> An ℓ-group is torsion free.

<u>Proof</u>. Let $g \in G$ and $n \in N$. If $ng = 0$, then $ng \geq 0$ and so $g \geq 0$. However if $g > 0$, then $g + g > g > 0$ and by induction $ng > 0$. Thus $ng = 0$ implies $g = 0$.

<u>Proposition 1.1.8.</u> For an ℓ-group G, and $a, b, c \in G$ if $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$, then $a = b$. Consequently G is a distributive lattice.

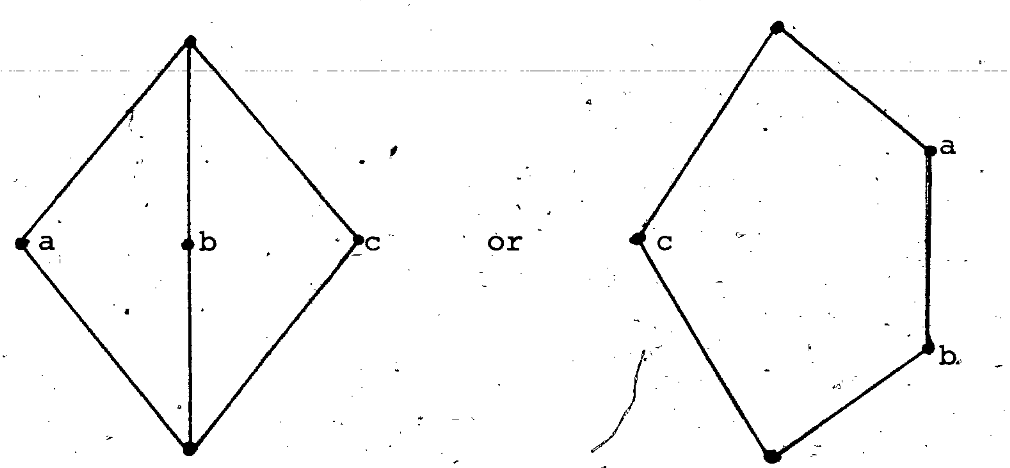
<u>Proof.</u> Since $a \vee c = a - (a \wedge c) + c$, we have

$$a = (a \vee c) - c + (a \wedge c)$$

$$= (b \vee c) - c + (b \wedge c)$$

$$= b .$$

Thus G does not contain a sublattice of the form



and hence G is a distributive lattice.

For elements a and b belonging to an ℓ-group G, if $a \wedge b = 0$ we say that a and b are <u>disjoint</u> or <u>orthogonal</u>. We note that disjoint elements commute, for if $a \wedge b = 0$, we have

$$a + b = a - (a \wedge b) + b$$

$$= a + (-a \vee -b) + b$$

$$= b \vee a$$

$$= a \vee b$$

$$= b + (-b \vee -a) + a$$

$$= b - (b \wedge a) + a$$

$$= b + a .$$

We define the <u>positive part of a</u>, a^+, to be $a \vee 0$, and the <u>negative part of a</u>, a^-, to be $-a \vee 0$.

<u>Proposition 1.1.9.</u> For an ℓ-group G and an element a belonging to G, $a^+ \wedge a^- = 0$, $a = a^+ - a^-$ and this the unique representation of a as the difference of disjoint elements.

<u>Proof.</u> For an element a of an ℓ-group G we have

$$a + a^- = a + (-a \vee 0) = 0 \vee a = a^+$$

and hence $$a = a^+ - a^- .$$

Also

$$a^+ \wedge a^- = (a+a^-) \wedge a^- = (a\wedge 0) + a^- = (a\wedge 0) - (a\wedge 0) = 0 .$$

Suppose $a = x - y$ where $x \wedge y = 0$.

$$a^- = -a \vee 0$$

$$= -(a \wedge 0)$$

$$= -((x-y) \wedge 0)$$

$$= -((x \wedge y) - y)$$

$$= y .$$

$$x = a + y = a + a^- = a^+ .$$

<u>Proposition 1.1.10</u>. An ℓ-group G is an o-group if and only if, $a > 0$ and $b > 0$ implies $a \wedge b > 0$.

<u>Proof</u>. If G is an o-group, then $a \wedge b = \min\{a,b\} > 0$. Conversely, suppose the condition holds and consider $g \in G$, $g \neq 0$. By Proposition 1.1.9,

$$g = g^+ - g^- \text{ and } g^+ \wedge g^- = 0$$

and hence either $g^+ = 0$ or $g^- = 0$. Thus either $g = -g^- < 0$ or $g = g^+ > 0$. Therefore G is an o-group.

The absolute value of an element $a \in G$, denoted by $|a|$ is defined to be $a \vee -a$.

<u>Proposition 1.1.11</u>. For G an ℓ-group and $a,b \in G$, we have

(1) $|a| \geq 0$

(2) $|a+b| \le |a| + |b| + |a|$.

Proof.

(1) $|a| = a \vee -a$, and thus $|a| \ge a, -a$.

So $\quad 2|a| \ge a - a = 0$,

hence by Proposition 1.1.6 $\quad |a| \ge 0$.

(2) $a + b \le |a| + |b| + |a|$.

Also, $\quad -|a| - |b| - |a| \le -|a| - |b| \le a+b$

and so $\quad -(a+b) \le |a| + |b| + |a|$.

Therefore, $\quad |a+b| = (a+b) \vee -(a+b) \le |a| + |b| + |a|$.

Section 2. Homomorphisms, Isomorphisms and Subgroups.

Space does not allow the inclusion of all proofs. Having described some of the basic concepts of ℓ-groups in detail, we will now discuss, without proof, some important results from the next level of development of the theory. However the inclusion of proofs will resume when closer to the focus of the thesis. Details of the following results, which are concerned with homomorphisms, isomorphisms and subgroups of ℓ-groups, may be found in Conrad [4] and Bigard et al [1].

A subgroup S of an ℓ-group G is said to be an <u>ℓ-subgroup</u> of G if S is a sublattice of G . From Proposition 1.1.5 it follows that S is an ℓ-subgroup of G if and only if, $s \vee 0 \in S$ for each $s \in S$. We note that S may be an ℓ-group with respect to the induced partial order, but not an ℓ-subgroup of G.

<u>Proposition 1.2.1.</u> Let π be an isomorphism from an ℓ-group A into an ℓ-group B. If π preserves $\wedge$ or $\vee$, then $A\pi$ is an ℓ-subgroup of B and both π and π^{-1} preserve $\leq$, $\wedge$ and $\vee$. Such a map is called an <u>ℓ-isomorphism</u>.

A homomorphism π of an ℓ-group A into an ℓ-group B is called an <u>ℓ-homomorphism</u> if π preserves $\wedge$ and $\vee$.

<u>Proposition 1.2.2.</u> For an isomorphism π of an ℓ-group A into an ℓ-group B, the following are equivalent

(1) π is an ℓ-isomorphism

(2) $x \wedge y = 0$ implies $x\pi \wedge y\pi = 0$ for all $x,y \in A$

(3) $(a \vee 0)\pi = a\pi \vee 0$ for all $a \in A$.

We turn our attention to subgroups. A subgroup S of a po-group G is said to be convex if, for $a,b \in S$, $g \in G$ $a \leq g \leq b$ implies $g \in S$.

<u>Proposition 1.2.3.</u> For an ℓ-subgroup S of an ℓ-group G, the following are equivalent.

(1) S is a convex ℓ-subgroup.

(2) The set of right cosets of S forms a distributive lattice under the order given by: $S + x \leq S + y$ if there exists $s \in S$ such that $s + x \leq y$ and, for this order, $(S+x) \wedge (S+y) = S + (x \wedge y)$.

(3) If $g \in G$, $s \in S$ and $|g| \le |s|$, then $g \in S$.

A normal convex ℓ-subgroup is called an <u>ℓ-ideal</u>.

<u>Corollary 1.2.4.</u> If H is an ℓ-ideal of G, then the right cosets of H form an ℓ-group.

<u>Proposition 1.2.5.</u> Let π be an ℓ-homomorphism from an ℓ-group G into an ℓ-group H. Then $\ker\pi$ is an ℓ-ideal of G and $G/\ker\pi$ is ℓ-isomorphic to $G\pi$.

A convex ℓ-subgroup M of an ℓ-group G is called <u>regular</u> if M is maximal with respect to not containing some $g \in G$ and in this case M is called a <u>value</u> of g.

A convex ℓ-subgroup P of an ℓ-group G is called <u>prime</u> if, for, A,B convex ℓ-subgroups of G, $P \supseteq A \cap B$ implies either $P \supseteq A$ or $P \supseteq B$. The concept of a prime subgroup is important in the various ways used to represent ℓ-groups.

<u>Proposition 1.2.6.</u> For a convex ℓ-subgroup M of an ℓ-group G, the following are equivalent

(1) M is regular.

(2) $M \subsetneq M^* = \bigcap\{C \mid M \subsetneq C,\ C$ convex ℓ-subgroup of $G\}$.

(3) M is meet irreducible in the collection of convex ℓ-subgroups of G.

If M is normal, each of the above is equivalent to

(4) G/M is an O-group with a convex subgroup that covers the identity M in G/M.

<u>Proposition 1.2.7.</u> For an ℓ-group G and a convex ℓ-subgroup P of G, the following are equivalent

(1) P is prime.

(2) If A, B are convex ℓ-subgroups of G and $P \subsetneq A$, $P \subsetneq B$ then $P \subsetneq A \cap B$.

(3) If $a,b \in G \backslash P$ then $a \wedge b \in G \backslash P$.

(4) The lattice of right cosets of P is totally ordered.

If P is normal each of the above is equivalent to

(5) G/P is an o-group.

It follows from Proposition 1.2.6 and 1.2.7 that each regular subgroup is prime.

Section 3. Ordered Permutation Groups.

In this section we introduce ordered permutation groups and will discuss Holland's Representation Theorem, one of the fundamental tools used in examining ℓ-groups. The importance of ordered permutation groups will be seen again in Chapter 3, where we consider ordered wreath products of ordered permutation groups and their role in the study of varieties of ℓ-groups. An extensive treatment of ordered permutation groups may be found in Glass [6].

As we are dealing with permutation groups, we change from additive to multiplicative notation.

An <u>ordered permutation group</u> (G,Ω) is a permutation group G acting on a totally ordered set Ω such that

(1) for all $\alpha,\beta \in \Omega$ $\alpha < \beta$ if and only if $\alpha g < \beta g$ for all $g \in G$,

and (2) $\{g \in G \mid \alpha g = \alpha$ for all $\alpha \in \Omega\} = \{1\}$, where 1 is the identity element of G.

The group $(G,\leq)$ is a po-group, where the partial order $\leq$ on G is given by:

for $g,h \in G$, $g \leq h$ if and only if $\alpha g \leq \alpha h$ for all $\alpha \in \Omega$.

If this partial order on G is a lattice order, (G,Ω) is called a <u>lattice ordered permutation group</u> (ℓ-permutation group). In this case, for all $g,h \in G$ and all $\alpha \in \Omega$

$$\alpha(g \vee h) = \alpha g \vee \alpha h \quad \text{and dually.}$$

An example of an ℓ-permutation group is the group $A(\Omega)$ of all order preserving permutations of a totally ordered set Ω.

The last main result of this chapter, Holland's representation theorem, is an analogue of Cayley's theorem in group theory. Holland [9] developed his result by means of several lemmas and we will outline the main ideas used.

For each $1 \neq g \in G$, there exists a regular (and therefore prime) convex ℓ-subgroup, $P(g)$ of G which is a value of g. Then $G/P(g)$, the set of right cosets of $P(g)$ in G, is totally ordered in the partial order given by:

$P(g)x \leq P(g)y$ if there exists $z \in P(g)$ such that $zx \leq y$.

We denote the group of order preserving automorphisms of $G/P(g)$ by $A(G/P(g))$.

Now the mapping $\alpha(g) : G \to A(G/P(g))$ defined by

$$x\,\alpha(g) = \beta(x,P(g)) ,$$

where $\beta(x,P(g)) \in A(G/P(g))$ is given by

$$(P(g)y)\ \beta(x,P(g)) = P(g)yx ,$$

is an ℓ-homomorphism of G onto a transitive ℓ-subgroup $B(g)$ of $A(G/P(g))$.

Holland's main embedding theorem states that G is ℓ-isomorphic to a subdirect sum of the ℓ-groups $\{B(g) \mid g \in G\}$. Further than this, we may totally order the set $\bigcup_{g \in G} G/P(g)$. We first order the collection $\{G/P(g) \mid g \in G\}$ in any way. Then for $x,y \in \bigcup_{g\in G} G/P(g)$ let $x < y$ if $x,y \in G/P(g)$ and $x < y$ as elements of $G/P(g)$, or if $x \in G/P(g)$ and $y \in G/P(h)$ where $G/P(g) < G/P(h)$. If H is the direct sum of the ℓ-groups $\{A(G/P(g)) \mid g \in G\}$, then each $\phi \in H$ induces an automorphism of $\bigcup_{g\in G} G/P(g)$ as follows

$$x \longrightarrow x\,\phi_g$$

where $x \in G/P(g)$ and ϕ_g is the gth component of ϕ.

From this we have the following proposition.

<u>Proposition 1.3.1.</u> If G is an ℓ-group, G is ℓ-isomorphic to an ℓ-subgroup of the ℓ-group of order preserving automorphisms of a totally ordered set.

CHAPTER 2

Certain Lattices of Subgroups.

In this chapter we will be concerned with lattices of subgroups. We begin by considering the lattice of convex ℓ-subgroups of an ℓ-group and the lattice of ℓ-subgroups of an ℓ-group. Later, in the second section, we will drop the requirement that our group is lattice ordered and will look at circumstances under which the directed convex subgroups of a partially ordered group form a distributive lattice.

Section 1. Lattices of L-Subgroups.

In this section we shall show that the lattice of convex ℓ-subgroups of an ℓ-group is distributive and shall determine those ℓ-groups for which the lattice of all ℓ-subgroups is distributive.

The next result, the Riesz decomposition property, is important in the consideration of the lattice of convex ℓ-subgroups of an ℓ-group.

<u>Proposition 2.1.1</u>. (Conrad [4]). Let G be an ℓ-group and $0 \le a, b_1, \ldots, b_n \in G$ such that $a \le b_1 + \ldots + b_n$, then there exist $c_1, \ldots, c_n \in G$ such that $a = c_1 + \ldots + c_n$ where $0 \le c_i \le b_i$ for $i = 1, \ldots, n$.

<u>Proof.</u> We use induction on n .

If $0 \le a \le b_1 + b_2$, let $c_1 = a \wedge b_1 \ge 0$ and $c_2 = -c_1 + a \ge 0$.

Then,
$$0 \le a = c_1 + c_2 .$$

and
$$\begin{aligned} 0 \le c_2 &= -c_1 + a \\ &= -(a \wedge b_1) + a \\ &= (-a \vee -b_1) + a \\ &= 0 \vee (-b_1 + a) \\ &\le b_2 . \end{aligned}$$

Thus the result is true when $n = 2$. Assume the result holds for all positive integers less than n .

Let $a, b_1, \ldots, b_n \ge 0$ and $a \le b_1 + \ldots + b_n$.

Then
$$a \le (b_1 + \ldots + b_{n-1}) + b_n$$

and hence, by the induction hypothesis, there exist $d, c_n \in G$ such that

$$a = d + c_n$$

and
$$0 \le d \le b_1 + \ldots + b_{n+1} ,\quad 0 \le c_n \le b_n .$$

Again, by the induction hypothesis, $0 \le d \le b_1 + \ldots + b_{n-1}$ implies that there exist $c_1, \ldots, c_{n-1} \in G$ such that

$$d = c_1 + \ldots + c_{n-1}$$

and $\quad 0 \le c_i \le b_i \quad$ for $\quad i = 1,\ldots,n-1$.

Thus we have

$$a = c_1 + \ldots + c_n$$

where $\quad 0 \le c_i \le b_i \quad$ for $\quad i = 1,\ldots,n$.

We shall denote the collection of all convex ℓ-subgroups of an ℓ-group G , by $\mathcal{C}(G)$.

Proposition 2.1.2. (Conrad [4]). Let G be an ℓ-group, then $\mathcal{C}(G)$ is a complete distributive sublattice of the lattice of all subgroups of G , and for $A, B_\lambda \in \mathcal{C}(G)$, $(\lambda \in \Lambda)$,

$$A \wedge (\vee B_\lambda) = \vee(A \wedge B_\lambda) .$$

Proof. We first show that if $\{B_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{C}(G)$, then $[\cup B_\lambda] \in \mathcal{C}(G)$. By Proposition 1.2.3 it is sufficient to show that for $g \in G$, $b_1,\ldots,b_n \in \cup B_\lambda$, $|g| \le |b_1 + \ldots + b_n|$ implies $g \in [\cup B_\lambda]$. Now, $g^+ \le |g| \le |b_1 + \ldots + b_n|$

$$\le |b_1| + \ldots + |b_{n-1}| + |b_n| + |b_{n-1}| + \ldots + |b_1|$$

(by Proposition 1.1.11).

By the Riesz decomposition property, there exist $g_1, \ldots, g_{2n-1} \in [\cup B_\lambda]$ such that

$$g^+ = g_1 + \ldots + g_{2n-1}$$

where
$$0 \leq g_1 \leq |b_1| \;,$$
$$0 \leq g_2 \leq |b_2'|, \;\ldots\ldots$$
$$0 \leq g_n \leq |b_n|,$$
$$0 \leq g_{n+1} \; |b_{n-1}|, \;\ldots\ldots$$
$$0 \leq g_{2n-1} \leq |b_1| \;.$$

Since each $B_\lambda \in C(G)$, $g_i \in \cup B_\lambda$ for $i = 1,\ldots,2n-1$, and hence $g^+ \in [\cup B_\lambda]$.

Similarly $g^- \in [\cup B_\lambda]$ and so $g = g^+ - g^- \in [\cup B_\lambda]$.

Thus we have that $[\cup B_\lambda] \in C(G)$.

Clearly $\cap B_\lambda \in C(G)$ and therefore $C(G)$ is a complete sublattice of the lattice of all subgroups of G.

We must now show that

$$A \wedge (\vee B_\lambda) = \vee(A \wedge B_\lambda) \;.$$

Clearly
$$A \wedge (\vee B_\lambda) \supseteq \vee(A \wedge B_\lambda) \;.$$

Let $a \in A \wedge (\vee B_\lambda)$, $a \geq 0$.

Then $a = b_1 + \ldots + b_n$ where $b_i \in \cup B_\lambda$ for $i = 1,\ldots,n$.

As above,

$$a = c_1 + \ldots + c_{2n-1} \quad \text{where} \quad 0 \le c_i \in \cup B_\lambda .$$

Now, $c_i = -c_{i-1} - \ldots - c_1 + a - c_{2n-1} - \ldots - c_{i+1} \le a$,

and since $0 \le c_i \le a$, and $A \in \mathcal{C}(G)$, we have $c_i \in A$.

Thus $a \in [\cup(A \cap B_\lambda)] = \vee(A \wedge B_\lambda)$ and so $A \wedge (\vee B_\lambda) \subseteq \vee(A \wedge B_\lambda)$.

Conrad [4] has posed the question: when is the lattice of ℓ-subgroups of an ℓ-group distributive? In answering this question, we will use the following results from group theory.

We first note that a group is said to be <u>locally cyclic</u> if each of its finitely generated subgroups is cyclic.

<u>Proposition 2.1.3</u>. (Hall [8], Theorem 19.2.1). The lattice of subgroups of a group G is distributive if and only if G is locally cyclic.

<u>Proposition 2.1.4</u>. (Schenkman [16], Theorem II.2.k). A group G is locally cyclic if and only if it is isomorphic to a subgroup of a homomorphic image of the additive rationals, $(Q,+)$. Moreover, if G is torsion free, then G is isomorphic to a subgroup of the additive rationals.

If A and B are ℓ-groups, then $A + B$ will denote the <u>cardinal sum</u> of A and B, that is, the direct sum of A and B with order given by

$$(A + B)^+ = \{(a,b) \mid a \ge 0 \text{ and } b \ge 0\} .$$

Lemma 2.1.5. The lattice of ℓ-subgroups of $Z + Z$ is not distributive.

Proof. Consider the following subgroups of $Z + Z$.

$$Z_2 = \{(a,2a) \mid a \in Z\},$$

$$Z_3 = \{(a,3a) \mid a \in Z\},$$

$$Z_5 = \{(a,5a) \mid a \in Z\}.$$

Z_2 is an ℓ-subgroup of $Z + Z$ since

$$(a,2a) \wedge (0,0) = \begin{cases} (a,2a) & a < 0 \\ (0,0) & a \geq 0 \end{cases}$$

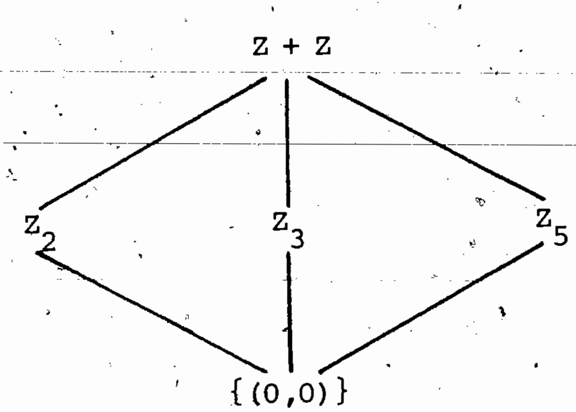
belongs to Z_2.

Similarly Z_3 and Z_5 are ℓ-subgroups of $Z + Z$. In fact these subgroups are totally ordered.

Now, $Z_2 \cap Z_3 = Z_3 \cap Z_5 = Z_5 \cap Z_2 = \{(0,0)\}$

and, $[Z_2 \cup Z_3] = [Z_3 \cup Z_5] = [Z_5 \cup Z_2] = Z + Z$.

Thus we have the following sublattice of the lattice of all ℓ-subgroups of $Z + Z$:



and we see that the lattice of ℓ-subgroups of $Z + Z$ is not distributive.

<u>Proposition 2.1.6</u>. The lattice of ℓ-subgroups of an ℓ-group, G, is distributive if and only if G is isomorphic to a subgroup of the additive rationals.

<u>Proof</u>. We consider two cases.

(i) G is totally ordered.

Then every subgroup of G is an ℓ-subgroup, and thus the lattice of ℓ-subgroups of G is distributive if and only if G is locally cyclic, that is, if and only if G is isomorphic to a subgroup of the additive rationals (by Propositions 2.1.3 and 2.1.4).

(ii) G is not totally ordered.

There exist $a,b \in G^+$ with $a \wedge b = 0$ (by Propositional 1.1.10)

$\Rightarrow$ $a + b = b + a$

$\Rightarrow$ the subgroup generated by the elements a and b,

$$[a,b] = [a] + [b] \cong Z + Z .$$

Further, $[a,b]$ is an ℓ-subgroup of G. Thus the lattice of ℓ-subgroups of G contains a sublattice which is not distributive (Lemma 2.1.5). Hence the lattice of ℓ-subgroups of G is not distributive.

Section 2. Lattices of Convex Directed Subgroups.

We now proceed in a different direction by relaxing the assumption that our groups are lattice ordered. We shall consider the circumstances under which the directed convex subgroups of a partially ordered group form a distributive lattice.

A partially ordered group $(G,\leq)$ is a <u>Riesz group</u> if, for any elements $a, b, c, d \in G$ with $a,b \leq c,d$, there exists an element $x \in G$ such that $a,b \leq x \leq c,d$.

Clearly ℓ-groups are Riesz groups.

<u>Proposition 2.2.1.</u> (Birkhoff [2]). A partially ordered group is a Riesz group if and only if it has the Riesz decomposition property.

<u>Proof</u>. Let G be a Riesz group and let $a, b, x \in G^+$ such that $0 \leq x \leq a+b$.

Then $0,x-b \leq x,a$ and since G is a Riesz group, there exists $s \in G$ such that $0,x-b \leq s \leq x,a$.

Let $t = -s+x$. Then $t \geq 0$ and $s+t = x \leq s+b$ whence $0 \leq t \leq b$.

Thus given $a, b, x \in G^+$ with $0 \leq x \leq a+b$ there exist s and t in G such that $x = s+t$ and $0 \leq s \leq a$, $0 \leq t \leq b$.

An easy induction argument then yields the full Riesz decomposition property (see Proposition 2.1.1.).

Conversely let G be a po-groups with the Riesz decomposition property and let $0,x \leq y,z$.

Then $$0 \leq x + (-x+z) \leq y + (-x+z)$$

$$0 \leq z \leq y + (-x+z)$$

and by the Riesz decomposition property, there exist s and t in G such that $0 \leq s \leq y$, $0 \leq t \leq -x+z$ and $z = s+t$.

Then $x \leq x+t \leq z = s+t$ and hence $x \leq s$, and also $s \leq z$.

Thus we have $0,x \leq s \leq y,z$ and therefore G is a Riesz group.

<u>Proposition 2.2.2.</u> Let $(G,\leq)$ be a Riesz group and $C(G)$ be the set of all convex directed subgroups of G . Then $C(G)$ is a distributive sublattice of the lattice of all subgroups of G .

<u>Proof</u>. Let H and $K \in C(G)$.

Clearly $H \cap K$ is convex.

Let $a,b \in H \cap K$. Since H and K are directed, there exist $h \in H$, $k \in K$ such that $h \geq a,b$, $k \geq a,b$.

Now, G is a Riesz group and so there exists $x \in G$ such that $a,b \leq x \leq h,k$.

However $x \in H \cap K$ since both H and K are convex.

Therefore $H \cap K$ is directed.

Thus $H \cap K \in C(G)$.

We now show that $[H \cup K] \in C(G)$.

Let $h \in [H \cup K]$, then

$$h = h_1 + \ldots + h_n \text{ where } h_i \in H \text{ or } h_i \in K \text{ for } i=1,\ldots,n .$$

Now for $h_i \in H \cap K$, $(i = 1,\ldots,n)$, let k_i be an upper bound for h_i and 0 in $H \cup K$; such an upper bound exists as both H and K are directed.

Then, $\quad k_i \geq h_i, 0 \quad$ for $\quad i = 1,\ldots,n$

$$\Rightarrow \quad k_1 + \ldots + k_n \geq h_1 + \ldots + h_n \, , \, 0$$

$$\Rightarrow \quad k_1 + \ldots + k_n \in [H \cup K] \text{ is an upper bound for } h \text{ and } 0$$

$$\Rightarrow \quad [H \cup K] \text{ is directed.}$$

Now let $g \in G$, $h \in [H \cup K]$ such that $0 \leq g \leq h$.

Let $h = h_1 + \ldots + h_n$ for some $h_i \in H \cup K$ $\quad i = 1,\ldots,n$.

Again let k_i be an upper bound in $H \cup K$ for h_i and 0.

Thus we have

$$0 \leq g \leq h = h_1 + \ldots + h_n \leq k_1 + \ldots + k_n \, .$$

By the Riesz decomposition property there exist $c_i \in G$, $i = 1,\ldots,n$, such that

$$g = c_1 + \ldots + c_n \quad \text{and} \quad 0 \leq c_i \leq k_i \quad \text{for} \quad i = 1,\ldots,n \, .$$

Now H,K convex implies $c_i \in H \cup K$ for $i = 1,\ldots,n$, and thus $g \in [H \cup K]$, and $[H \cup K]$ is convex.

Therefore $[H \cup K] \in \mathcal{C}(G)$.

We now have that $\mathcal{C}(G)$ is a sublattice of the lattice of all subgroups of G. It remains to show that the distributive laws hold in $\mathcal{C}(G)$, that is, for $A, B, C \in \mathcal{C}(G)$

$$A \wedge (B \vee C) = (A \wedge b) \vee (A \wedge C) .$$

Clearly $A \cap [B \cup C] \supseteq [(A \cap B) \cup (A \cap C)]$.

Let $a \in A \cap [B \cup C]$, $a > 0$.

Then $a = b_1 + \ldots + b_n$ for some $b_i \in B \cup C$, $i = 1,\ldots,n$.

Now, let $i = 1,\ldots,n$, let $c_i \in B \cup C$ be an upper bound for b_i and 0 .

Thus $a = b_1 + \ldots + b_n \leq c_1 + \ldots + c_n$.

Then by the Riesz decomposition property,

$$a = d_1 + \ldots + d_n$$

where $0 \leq d_i \leq c_i$ for $i = 1,\ldots,n$, and by convexity $d_i \in B \cup C$.

Also, we have

$$0 \leq d_i = -d_{i-1} - \ldots - d_1 + a - d_n - \ldots - d_{i+1} \leq a$$

and so $d_i \in A$ since A is convex .

Hence $d_i \in (A \cap B) \cup (A \cap C)$

and thus $a \in [(A \cap B) \cup (A \cap C)]$.

Therefore $A \cap [B \cup C] \subseteq [(A \cap B) \cup (A \cap C)]$

and we have

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) \ .$$

Finally we consider compatible tight Riesz groups. Such groups are Riesz groups and thus the convex directed subgroups will form a distributive lattice. However we are able to obtain a stronger result linking this lattice to a sublattice of the lattice of convex ℓ-subgroups of an associated ℓ-group.

A partially ordered group, $(G,\leq)$, is a tight Riesz group if for any elements $a, b, c \in G$ with $a < c$ and $b < c$, there exists an element $d \in G$ such that

$$a < d < b \quad \text{and} \quad a < d < c \ .$$

The order, $\leq$, is then called a tight Riesz order.

An element $g \in (G,\leq)$ is said to be pseudopositive if $g \not\geq 0$ but $a > 0$ implies $a + g > 0$.

An element $g \in (G,\leq)$ is said to be pseudozero if both g and $-g$ are pseudopositive.

If $(G,\leq)$ has no pseudozeros we will write $g \succ 0$ to mean $g > 0$ or g is pseudopositive. $(G,\preccurlyeq)$ is a partially ordered group and we say $\preccurlyeq$ is the associated order.

Given an ℓ-group $(G,\preccurlyeq)$ and a tight Riesz group $(G,\leq)$ without pseudozeros, if $\preccurlyeq$ is the associated order for $\leq$, then $\leq$ is said to be a <u>compatible tight Riesz order</u> for $(G,\preccurlyeq)$, and $(G,\leq)$ is called a <u>compatible tight Riesz group</u>.

<u>Lemma 2.2.3.</u> Let $(G,\preccurlyeq)$ be an ℓ-group with compatible tight Riesz order $\leq$. Let $H \neq \{0\}$ be a convex directed subgroup of $(G,\leq)$, then H is a convex ℓ-subgroup of $(G,\preccurlyeq)$.

<u>Proof.</u> Let $a,b \in H$, $x \in G$ with $a \preccurlyeq x \preccurlyeq b$.

If $a = b$, then $x = a \in H$.

If $a \neq b$, since $(H,\leq)$ is directed, we may assume $a < b$. Then,

$$a - (b-a) < a \preccurlyeq x \preccurlyeq b < b + (b-a)$$

or

$$2a-b < x < 2b-a .$$

Since H is convex with respect to $\leq$, $x \in H$ and so H is convex with respect to .

Now let y be an upper bound in $(H,\leq)$ for a and b.

$$y \geqslant a,b$$

$$\Rightarrow \quad y \succcurlyeq a,b$$

$$\Rightarrow \quad y \succcurlyeq a \vee b \succcurlyeq a,b$$

Thus $a \vee b \in H$, since $(H,\preccurlyeq)$ is convex.

Therefore, $(H,\preccurlyeq)$ is a convex ℓ-subgroup of $(G,)$.

We shall denote by $\mathcal{C}(G,\preccurlyeq)$ the lattice of convex ℓ-subgroups of $(G,\preccurlyeq)$, and by $\mathcal{C}(G,\leq)$ the set of convex directed subgroups of $(G,\leq)$.

Proposition 2.2.4. $\mathcal{C}(G,\leq)$ is a sublattice of $\mathcal{C}(G,\preccurlyeq)$ and so is distributive.

Proof. By lemma 2.2.2, $\mathcal{C}(G,\leq) \subseteq \mathcal{C}(G,\preccurlyeq)$.

Let $H,K \in \mathcal{C}(G,\leq)$.

It is clear that $H \cap K$ is convex.

Let $a,b \in H \cap K$, then since H and K are directed, there exist $r \in H$, $s \in K$ such that

$$r > a,b \qquad s > a,b \ .$$

Since $(G,\leq)$ is a tight Riesz group, there exists $x \in G$ such that

$$a,b < x < r,s \ .$$

Now, $a < x < r$ and $(H,\leq)$ convex implies $x \in H$.

Similarly, $b < x < s$, and $(K,\leq)$ convex implies $x \in K$.

Thus we have $x \in H \cap K$ with $x > a,b$ and hence $H \cap K$ is directed and $H \cap K \in \mathcal{C}(G,\leqslant)$.

We must now show that the join of H and K in the lattice $\mathcal{C}(G,\leq)$ is equal to the join of H and K in the lattice $\mathcal{C}(G,\preccurlyeq)$.

Let L_1 be the smallest convex directed subgroup of $(G,\leq)$ containing H and K.

Let L_2 be the smallest convex ℓ-subgroup of $(G,\preccurlyeq)$ containing H and K.

We wish to show $L_1 = L_2$. Clearly, $L_1 \supseteq L_2$.

Let $T = \{g \in G \mid g > 0\}$.

We claim $H \cap T \neq \emptyset$.

Let $a \in H$, $a \neq 0$. If $a > 0$, then $a \in H \cap T$. If $a < 0$, then $-a \in H \cap T$. If a and 0 are incomparable, then, since H is directed, there exists $b \in H$ such that $b \geq a,0$. Since $a \not\leq 0$, $b > 0$ and thus $b \in H \cap T$. In all cases $H \cap T \neq \emptyset$.

Now, $L_2 \supseteq H$ and so $L_2 \cap T \neq \emptyset$.

Let $t \in L_2 \cap T$ and let $a,b \in L_2$. Then,

$$t + (a \vee b) > a \vee b \succcurlyeq a,b$$

that is $t + (a \vee b) > a,b$ and thus L_2 is directed with respect to $\geq$.

L_2 is convex with respect to $\preccurlyeq$, and since $\preccurlyeq$ is a refinement of $\leq$, L_2 must also be convex with respect to $\leq$.

Thus $L_2 \in C(G,\leq)$ and since $L_2 \supseteq H,K$, we must have $L_2 \supseteq L_1$.

Thus $L_1 = L_2$ as required.

CHAPTER 3

Varieties of L-Groups and Wreath Products

In this Chapter we present several examples of varieties of lattice ordered groups and discuss where they are placed in the lattice of ℓ-group varieties. Wreath products are important in the study of varieties of ℓ-groups and these are also described, together with a generalisation, the twisted wreath product.

Section 1. Varieties of ℓ-groups.

A variety of ℓ-groups or ℓ-variety is a class of ℓ-groups closed under taking ℓ-subgroups, ℓ-homomorphic images and cardinal products. Equivalently an ℓ-variety is the class of all ℓ-groups for which a given set (~~possibly~~ infinite) of equations, which may involve both group and lattice operations, are laws.

Some examples of well known ℓ-group varieties are as follows:

Example 3.1.1. The trivial variety, E , consists of all those ℓ-groups with one element. The law defining this variety is: $x = y$.
Clearly this variety contains one element, the trivial ℓ-group $\{1\}$.

Example 3.1.2. At the other extreme, we have the variety L consisting of all ℓ-groups. This variety has as its defining law: $x = x$.

Example 3.1.3. The abelian variety $\mathcal{A}$ is the variety consisting of all abelian ℓ-groups. The law defining this variety is:
$xy = yx$.

Example 3.1.4. An ℓ-group is said to be representable if it is a subdirect product of totally ordered groups. The collection of all representable ℓ-groups is a variety, denoted by $\mathcal{R}$ with defining law given by: $(x \wedge (y^{-1}x^{-1}y)) \vee 1 = 1$.

This result may be found in Conrad [4]. Theorem 1.8. We note that this is the first of our examples for which the lattice operations appear in the defining law.

Example 3.1.5. Let G be an ℓ-group. A convex ℓ-subgroup M of G is called a value of $g \in G$ if M is maximal with respect to not containing g. Further, M is called a normal value if

$$M \lhd M^{*} = \cap\{C \in \mathcal{C}(G) \mid M \underset{\neq}{\subset} C\} .$$

An ℓ-group G is called a normal valued ℓ-group if each value M in G is a normal value. Wolfenstein [18] has shown that the class of all normal valued ℓ-groups forms a variety $\mathcal{N}$ which has as its defining law: $(x \vee 1)(y \vee 1) \leq (y \vee 1)^2(x \vee 1)^2$.

Example 3.1.6. Let n be a positive integer. Then $\mathcal{L}(n)$ denotes the variety for which the defining law is: $x^n y^n = y^n x^n$.

We use L to denote the set of all ℓ-group varieties. We may then consider L as a partially ordered set, the partial order being inclusion, that is

$$\mathcal{U} \leq \mathcal{V} \quad \text{if and only if} \quad \mathcal{U} \subseteq \mathcal{V} .$$

This partial order becomes a lattice order if we define, for $\{\mathcal{V}_i \mid i \in I\} \subseteq L$,

$$\bigwedge_{i \in I} \mathcal{V}_i = \bigcap_{i \in I} \mathcal{V}_i$$

and

$$\bigvee_{i \in I} \mathcal{V}_i = \bigcap_{i \in I} \{\mathcal{U} \in L \mid \mathcal{U} \geq \mathcal{V}_i \text{ for all } i \in I\} .$$

These definitions make L a complete lattice since L contains both a largest element ($\mathcal{L}$) and a smallest element ($\mathcal{E}$) .

In a similar manner for a collection of ℓ-groups, $\{G_i \mid i \in I\}$, we define the <u>ℓ-variety generated by $\{G_i \mid i \in I\}$</u> by

$$\ell\text{-var}\,\{G_i \mid i \in I\} = \bigcap \{\mathcal{U} \in L \mid G_i \in \mathcal{U} \text{ for all } i \in I\} .$$

We may define a multiplication of ℓ-group varieties as follows: for $\mathcal{U}$ and $\mathcal{V} \in L$, an ℓ-group G belongs to $\mathcal{U}\mathcal{V}$ if and only if G contains an ℓ-ideal H such that $H \in \mathcal{U}$ and $G/H \in \mathcal{V}$.

It can be shown that L is closed under this multiplication and that the multiplication is associative. Thus we have that L is a semigroup. Further than this, L is a lattice ordered semigroup with an identity, namely $\mathcal{E}$.

It is of interest to see where our examples of ℓ-group varieties are placed in the lattice L .

Proposition 3.1.7. (Martinez [12]). The normal valued variety, N , is idempotent.

Proposition 3.1.8. (Weinberg [17]). The abelian variety, A , is the smallest proper ℓ-variety.

Proposition 3.1.9. (Holland [9]). The normal valued variety, N , is the unique largest proper ℓ-variety.

The last two results mark one of the differences between the lattice of varieties of ℓ-groups and that of groups. A connection between the smallest proper ℓ-variety, A , and the largest proper ℓ-variety, N , is given by the following theorem.

Proposition 3.1.10. (Glass, Holland and McCleary [6]).

$$N = \bigvee_{n=1}^{\infty} A^n .$$

This result has the following two corollaries.

Corollary 3.1.11. If V is any proper ℓ-variety, then the powers of V generate N .

Corollary 3.1.12. The only idempotent ℓ-varieties are E , N and L .

Martinez [11] and Scrimger [15] have proved the following properties concerning the ℓ-varieties $L(n)$.

Proposition 3.1.13. (i) $L(n) \subseteq L(m)$ if and only if n divides m .

(ii) For each positive integer n, $\mathcal{L}(n) \cap \mathcal{R} = \mathcal{A}$.

(iii) If m and n are relatively prime positive integers, then $\mathcal{L}(n) \cap \mathcal{L}(m) = \mathcal{A}$.

Scrimger also introduced a new class of ℓ-varieties $\{\mathcal{S}(n) \mid n \in N\}$ such that for each $n \in N$, $\mathcal{S}(n) \subset \mathcal{L}(n)$. These ℓ-varieties will be discussed in a later section.

Section 2. Standard Wreath Product.

In ℓ-group theory, as in group theory, the wreath product is of much use in the study of varieties. We will describe the construction of the standard wreath product of ordered permutation groups. For a more general construction of wreath products of ordered permutation groups see Holland and McCleary [10].

Let (G,Γ) and (H,Λ) be order preserving permutation groups.

Let $\Omega = \Gamma \times \Lambda$; then Ω is totally ordered with respect to the order given by, for all (α_1,β_1), $(\alpha_2,\beta_2) \in \Omega$,

$$(\alpha_1,\beta_1) \geq (\alpha_2,\beta_2) \text{ if and only if}$$

$$\beta_1 > \beta_2 \text{ or } \beta_1 = \beta_2 \text{ and } \alpha_1 \geq \alpha_2 .$$

Now let $W = \{(\hat{g},h) \mid h \in H, \hat{g} : \Lambda \to G\}$.

Then (W,Ω) is an order preserving permutation group, the action of W on Ω being given by, for all $(\alpha,\beta) \in \Omega$ and all $(\hat{g},h) \in W$

$$(\alpha,\beta)(\hat{g},h) = (\alpha\hat{g}(\beta),\beta h),$$

and multiplication being given by, for all $(\hat{g},h)$, $(\hat{f},k) \in W$

$$(\hat{g},h)(\hat{f},k) = (\hat{c},hk)$$

where $\hat{c} : \Lambda \to G$ is given by

$$\hat{c}(\beta) = \hat{g}(\beta)\hat{f}(\beta h) .$$

(W,Ω) is then an ordered permutation group with respect to the usual ordering of order preserving permutations. Further if (G,Γ) and (H,Λ) are lattice ordered, then so is (W,Ω).

(W,Ω) is called the <u>standard wreath product of (G,Γ) and (H,Λ)</u> and is denoted by $(G,\Gamma)\mathrm{Wr}(H,\Lambda)$; the subgroup of (W,Ω) consisting of those $(\hat{g},h)$ such that $\hat{g}(\beta) \neq 1$ for only finitely many $\beta \in \Lambda$ is called the <u>restricted wreath product of (G,Γ) and (H,Λ)</u> and is denoted by $(G,\Gamma)\,\mathrm{wr}(H,\Lambda)$.

We note that the wreath product $(G,\Gamma)\mathrm{Wr}(H,\Lambda)$ is independent of the totally ordered set Γ on which G acts.

As previously mentioned the standard wreath product is important in the study of ℓ-varieties. In particular we have the following results, which have group theoretic analogues and which may be found in Glass, Holland and McCleary [6].

Lemma 3.2.1. Let (G,Γ) and (H,Λ) be ℓ-permutation groups. Then $\ell\text{-var}\{(G,\Gamma)\mathrm{Wr}(H,\Lambda)\} = \ell\text{-var}\{(G,\Gamma)\mathrm{wr}(H,\Lambda)\}$.

Lemma 3.2.2. Let (G,Γ) be an ℓ-permutation group in the ℓ-variety $\mathcal{U}$ and let (H,Λ) be a transitive ℓ-permutation group in the ℓ-variety $\mathcal{V}$. Then $(G,\Gamma)\mathrm{Wr}(H,\Lambda)$ belongs to the ℓ-variety $\mathcal{U}\mathcal{V}$.

We may define $\mathrm{Wr}^n(G)$ inductively by

$$\mathrm{Wr}^n(G) = (\mathrm{Wr}^{n-1}(G))\mathrm{Wr}G .$$

Now let (Z,Z) be the regular representation of the integers and (R,R) the regular representation of the reals. An easy induction argument yields

Lemma 3.2.3. $\mathrm{Wr}^n(Z) \in \mathcal{A}^n$ and $\mathrm{Wr}^n(R) \in \mathcal{A}^n$ for all positive integers n.

We say that a collection $\{(G_i,\Omega_i) \mid i \in I\}$ of ℓ-permutation groups mimics an ℓ-variety $\mathcal{V}$ if the following two conditions are satisfied:

(i) $G_i \in \mathcal{V}$ for all $i \in I$;

(ii) for any transitive ℓ-permutation group (H,Λ) with $H \in \mathcal{V}$, for any $\lambda \in \Lambda$, any finite set of words $\{wp(x)\}$ and any substitution $x \to h$ in (H,Λ), there exist elements $i \in I$, $\alpha \in \Omega_i$ and a substitution $x \to g$ in G_i such that $\lambda wp(h) < \lambda wq(h)$ if and only if $\alpha wp(g) < \alpha wq(g)$.

With this definition Glass, Holland and McCleary [6] proved the following results on product varieties.

Proposition 3.2.4. If $\mathcal{U} = \ell\text{-var}\{(U_i,\Gamma_i) \mid i \in I\}$ and $\{(G_j,\Omega_j) \mid j \in J\}$ mimics $\mathcal{V}$, then $\ell\text{-var}\{(U_i,\Gamma_i)\mathrm{wr}(G_j,\Omega_j) \mid i\in I, j\in J\} = \mathcal{U}\mathcal{V}$.

Proposition 3.2.5. $\ell\text{-var}(\mathrm{wr}^n Z) = A^n$ for each positive integer n.

Section 3. More ℓ-varieties.

We will now consider a class of ℓ-group varieties generated by certain ℓ-subgroups of $Z \text{ Wr } Z$.

For each positive integer n, let

$$G(n) = \{(F,k) \mid K \in Z,\ F\colon Z \to Z,\ F(i) = F(j) \text{ if } i \equiv j \pmod n\}$$
$$\subseteq Z \text{ Wr } Z .$$

$G(n)$ is an ℓ-subgroup of $Z \text{ Wr } Z$. This may easily be verified if we note that the binary operation in $Z \text{ Wr } Z$ is given by, for $(F,k), (G,\ell) \in Z \text{ Wr } Z$,

$$(F,k) + (G,\ell) = (F + G^k, k + \ell)$$

where $G^k(z) = G(k + z)$ for all $z \in Z$.

The inverse of (F,k) is $(-F^{-k},-k)$ and the identity element of $Z \text{ Wr } Z$ is $(\overline{0},0)$, where $\overline{0}(z) = 0$ for all $z \in Z$.

It is clear that if $(F,k), (G,\ell) \in G(n)$, then

$$(F,k) - (G,\ell) = (F,k) + (-G^{-\ell},-\ell)$$
$$= (F-G^{-\ell+k},k-\ell) .$$

Now suppose $i \equiv j \pmod n$, we have

$$(F-G^{-\ell+k})(i) = F(i) - G(i-\ell+k)$$

$$= F(j) - G(j-\ell+k)$$

$$= (F-G^{-\ell+k})(j) .$$

Thus $(F,k) - (G,\ell) \in G(n)$ and hence $G(n)$ is a subgroup of $Z \text{ Wr } Z$.

To check that $G(n)$ is an ℓ-subgroup of $Z \text{ Wr } Z$, we observe that for $(F,k) \in G(n)$, we have in $Z \text{ Wr } Z$

$$(F,k) \vee (\overline{0},0) = (H,h) \quad \text{where}$$

$$(H,h) = \begin{cases} (\overline{0},0) & k < 0 \\ (F,k) & k > 0 \\ (K,0) & k = 0 \end{cases}$$

where $K(z) = F(z) \vee 0$ for all $z \in Z$;

and we see that $(H,h) \in G(n)$.

For each positive integer n, we define the <u>Scrimger variety</u>, $\mathcal{S}(n)$, to be the ℓ-variety generated by $G(n)$. The ℓ-varieties $\mathcal{S}(n)$, $n \in N$, play an important role in the lattice of varieties of ℓ-groups. We give some basic properties of these varieties which may be found in Scrimger [15] and Smith [16].

<u>Proposition 3.3.1.</u> For each positive integer n, $\mathcal{S}(n) \subseteq \mathcal{L}(n)$, and if n is not prime the containment is proper.

<u>Proposition 3.3.2.</u> If m and n are relatively prime, then $\mathcal{S}(n) \cap \mathcal{S}(m) = \mathcal{L}(n) \cap \mathcal{L}(m) = \mathcal{A}$.

<u>Proposition 3.3.3.</u> For any prime p, $\mathcal{S}(p)$ covers $\mathcal{A}$ in the lattice of ℓ-varieties; that is, no ℓ-variety lies strictly between $\mathcal{S}(p)$ and $\mathcal{A}$.

Section 4. Twisted Wreath Products of Groups.

In this section we will consider the twisted wreath product, a generalisation of the wreath product, due to B.H. Neumann [13].

The construction will use a group B , a subgroup S of B, and transversal T of S in B , a second group A and a homomorphism α of S into the group of automorphisms of A .

Given B, S and T as above, every $b \in B$ may be uniquely factorised in the form

$$b = st \ , \quad s \in S \ , \quad t \in T \tag{1}$$

Denote by τ the mapping of B onto T that maps each element b of B to its coset representative $t \in T$.

Thus $b^{\tau} = t$ where t and b are as in (1) .

Further, denote by σ the mapping of B onto S that maps each $b \in B$ to its representative $s \in S$.

Thus $b^{\sigma} = s$ where s and b are as in (1).

We have the following identities where $x,y \in B$

$$\left.\begin{aligned}(x^{\tau}y)^{\tau} &= (xy)^{\tau} \\ (xy)^{\sigma} &= x^{\sigma}(x^{\tau}y)^{\sigma}\end{aligned}\right\} \quad 3.4.1\ .$$

Right multiplication by elements of B permutes the right cosets of S transitively, thus we can consider B acting as a transitive permutation group on the transversal T by putting for $b \in B$, $t \in T$

$$t^{b} = (tb)^{\tau}.$$

Then for all $b,b' \in B$ and all $t \in T$,

$$t^{bb'} = (t^{b})^{b'}$$

and

$$t^{1} = t \qquad \text{where } 1 \text{ is the unit element of } B\ .$$

Given a group A and a homomorphism $\alpha : S \to \text{Aut } A$, by a^{s} we will mean the image of $a \in A$ under the automorphism $\alpha(s)$, where $s \in S$. We then have, for all $a,a' \in A$ and all $s,s' \in S$

$$\begin{aligned}(aa')^{s} &= a^{s}a'^{s} \\ a^{(ss')} &= (a^{s})^{s'} \\ a^{1} &= a\ .\end{aligned}$$

We now proceed with the construction of the twisted wreath product.

We form the cartesian power $F = A^T$; this consists of all functions f on T to A , with componentwise multiplication. Thus $f_1 f_2 = f_3$ means $f_1(t) f_2(t) = f_3(t)$ for all $t \in T$.
Those functions f^* whose support, $\{t \in T \mid f^*(t) \neq 1\}$, is finite, form the direct power F^* contained in F .

We will next define an antihomomorphism β of B into the group of automorphisms of F . Again the notation will be simplified by denoting the image of $f \in F$ under the automorphism $\beta(b)$ by f^b .
For all $f \in F$, all $b \in B$ we define $f^b \in F$ by

$$f^b(t) = f(t^b)^{s(b,t)} \quad \text{for all } t \in T$$

where
$$s(b,t) = ((tb)\sigma)^{-1} .$$

We must now verify that

(1) the mapping $f \to f^b$ is an automorphism of F and

(2) β is an antihomomorphism,

that is, for all $b,b' \in B$

$$\beta(bb') = \beta(b')\beta(b) .$$

(1).

(i) the mapping $\beta(b)$ which maps f to f^b is $1 - 1$:

let $f^b = g^b$

then for all $t \in T$,

$$f^b(t) = g^b(t)$$

$$\Rightarrow \; f(t^b)^{s(b,t)} = g(t^b)^{s(b,t)}$$

$$\Rightarrow \; f(t^b) = g(t^b) \quad \text{since} \quad \alpha(s(b,t)) \quad \text{is an automorphism.}$$

Thus, since $\{t^b \mid t \in T\} = T$, we have for all $t \in T$,

$$f(t) = g(t)$$

and so $f = g$ and $\beta(b)$ is $1-1$.

(ii) $\beta(b)$ is onto :

Given $f \in F$, let $g \in F$ be given by

$$g(t) = f(t^{b^{-1}})^{s(b,t^{b^{-1}})^{-1}} \quad \text{for all} \quad t \in T .$$

Then $g^b(t) = g(t^b)^{s(b,t)}$

$$= \left[f((t^b)^{b^{-1}})^{s(b,t^{-b^{-1}})^{-1}} \right]^{s(b,t)}$$

$$= f(t) \qquad \text{for all } t \in T .$$

Thus $g^b = f$ and so the mapping $\beta(b)$ is onto.

(iii) $\beta(b)$ is a homomorphism:

Let $f, f' \in F$, then for all $t \in T$

$$(ff')^b(t) = (ff'(t^b))^{s(b,t)}$$

$$= f(t^b)^{s(b,t)} f'(t^b)^{s(b,t)}$$

$$= f^b(t)f'^b(t)$$

$$= f^b f'^b(t) \ .$$

Thus $\beta(b)$ is an automorphism and (1) is proved.

(2) The map β is an antihomomorphism:

For all $t \in T$ we have

$$f^{bb'}(t) = f(t^{bb'})s(bb',t)$$

and $$(f^{b'})^b(t) = f^{b'}(t^b)s(b,t)$$

$$= (f(t^{bb'})s(b',t^b))s(b,t)$$

$$= f(t^{bb'})(s(b',t^b)s(b,t)) \ .$$

However, recalling identities 3.4.1, we have

$$s(bb',t) = [(tbb')^\sigma]^{-1}$$

$$= [(tb)^\sigma((tb)^\tau b')^\sigma]^{-1}$$

$$= [((tb)^\tau b')^\sigma]^{-1}[(tb)^\sigma]^{-1}$$

$$= s(b',t^b)s(b,t) \ .$$

Thus for all $t \in T$,

$$f^{bb'}(t) = (f^{b'})^b(t)$$

and so $$f^{bb'} = (f^{b'})^b \ .$$

This holds for each $f \in F$ and hence

$$\beta(bb') = \beta(b')\beta(b)$$

and we see that β is an antihomomorphism.

In this way we have B acting as a group of automorphisms of F and can now form the group P whose elements are pairs (f,b) where $f \in F$ and $b \in B$. Multiplication in P is defined by

$$(f_1,b_1)(f_2,b_2) = (f_1f_2^{b_1},b_1b_2) .$$

We must now verify that this gives rise to a group.

(i) It is clear that P is closed under multiplication.

(ii) Multiplication is associative:

Let $(f_1,b_1), (f_2,b_2), (f_3,b_3) \in P$. Then

$$[(f_1,b_1)(f_2,b_2)](f_3,b_3) = (f_1f_2^{b_1},b_1b_2)(f_3,b_3)$$

$$= (f_1f_2^{b_1}f_3^{b_1b_2},b_1b_2,b_3)$$

$$= (f_1f_2^{b_1}(f_3^{b_2})^{b_1},b_1b_2b_3)$$

$$= (f_1(f_2f_3^{b_2})^{b_1},b_1b_2b_3)$$

$$= (f_1,b_1)(f_2f_3^{b_2},b_2b_3)$$

$$= (f_1,b_1)[(f_2,b_2)(f_3,b_3)] .$$

(iii) P has identity element $(\bar{1},1)$ where $\bar{1}$ is the function defined by $\bar{1}(t) = 1$ for all $t \in T$.

Let $(f,b) \in P$, then

$$\begin{aligned}(\bar{1},1)(f,b) &= (\bar{1}f^{1},1b)\\ &= (f,b)\\ &= (f\bar{1}^{b},b1)\\ &= (f,b)(\bar{1},1) .\end{aligned}$$

(iv) Each element (f,b) of P has an inverse, namely $((f^{-1})^{b^{-1}},b^{-1})$:

$$\begin{aligned}((f^{-1})^{b^{-1}},b^{-1})(f,b) &= ((f^{-1})^{b^{-1}}f^{b^{-1}},b^{-1}b)\\ &= ((f^{-1}f)^{b^{-1}},1)\\ &= (\bar{1}^{b^{-1}},1)\\ &= (\bar{1},1) ,\end{aligned}$$

and similarly

$$(f,b)((f^{-1})^{b^{-1}},b^{-1}) = (\bar{1},1) .$$

The group P is called the <u>unrestricted twisted wreath product</u> of A by B and will be denoted by $TWr(A,B,S,\alpha)$. Our construction depended upon the choice of a transversal T of S in B, but this dependence is only apparent.

The <u>restricted twisted wreath product</u>, P^*, is the subgroup of P consisting of those (f,b) such that $f \in F^*$ and is denoted

$twr(A,B,S,\alpha)$. We can see that $TWr(A,B,S,\alpha)$ and $twr(A,B,S,\alpha)$ are the same if the index of S in B is finite or if A is the trivial group.

The standard wreath product of A by B (unrestricted and restricted) is obtained as a special case of the twisted wreath product of A by B when S is taken to be the trivial group.

Section 5. Ordering the Twisted Wreath Product.

In this section we return our attention to the theory of ℓ-groups and show that the twisted wreath product of an ℓ-group by an O-group may be lattice ordered.

<u>Proposition 3.5.1</u>. Let A be a lattice ordered group, B a totally ordered group, S a subgroup of B and α a homomorphism from S into the group of ℓ-automorphisms of A. $TWr(A,B,S,\alpha)$ can be lattice ordered.

<u>Proof</u>. Let T be a transversal of S in B.

Define an ordering on $TWr(A,B,S,\alpha)$ by

$(F,k) \geq (G,\ell) \Leftrightarrow k > \ell$ or $k = \ell$ and $F(t) \geq G(t)$ for all $t \in T$.

It is clear that $\geq$ is a partial order.

Let $(F,k),(G,\ell) \in TWr(A,B,S,\alpha)$ with $(F,k) \geq (G,\ell)$.

Let (X,x), $(Y,y) \in TWr(A,B,S,\alpha)$.

We claim $(X,x)(F,k)(Y,y) \geq (X,x)(G,\ell)(Y,y)$.

From this it will follow that $(\mathrm{TWr}(A,B,S,\alpha), \geq)$ is a partially ordered group.

We note that

$$(X,x)(F,k)(Y,y) = (XF^{x}Y^{xk}, xky) \quad \text{and}$$

$$(X,x)(G,\ell)(Y,y) = (XG^{x}Y^{x\ell}, x\ell y).$$

Now $(F,k) \geq (G,\ell)$ implies either

(i) $k > \ell$ or

(ii) $k = \ell$ and $F(t) \geq G(t)$ for all $t \in T$.

(i) If $k > \ell$, then B is totally ordered implies

$xk\ell > yk\ell$

and so $\quad (X,x)(F,k)(Y,y) \geq (X,x)(G,\ell)(Y,y)$.

(ii) If $k = \ell$ and $F(t) \geq G(t)$ for all $t \in T$ then

$$xky = x\ell y$$

and.
$$\begin{aligned}
XF^{x}Y^{xk}(t) &= XF^{x}Y^{x\ell}(t) \\
&= X(t)F^{x}(t)Y^{x\ell}(t) \\
&= X(t)F((tx)^{\tau})\alpha_{s(x,t)}Y^{x\ell}(t) \\
&\geq X(t)G((tx)^{\tau})\alpha_{s(x,t)}Y^{x\ell}(t)
\end{aligned}$$

(since A is an ℓ-group and $\alpha_{s(x,t)}$ is an ℓ-automorphism of A)

$$= (X(t)G^{x}(t)Y^{x\ell}(t)$$

$$= XG^{x}Y^{x\ell}(t) .$$

Thus in this case we also have

$$(X,x)(F,k)(Y,y) \geq (X,x)(G,\ell)(Y,y)$$

as required.

We have that $(TWr(A,B,S,\alpha), \geq)$ is a po-group. Further than this it is an ℓ-group for we can see that for all $(F,k) \in TWr(A,B,S,\alpha)$

$$(F,k) \vee (\bar{1},1) = \begin{cases} (\bar{1},1) & k < 1 \\ (F,k) & k > 1 \\ (F \vee \bar{1}, 1) & k = 1 \end{cases} .$$

Section 6. An Example.

We have seen in sections 2 and 3 that the standard wreath product has an important place in the study of ℓ-group varieties. Bearing this in mind it is of interest to consider specific examples of twisted wreath products. We shall illustrate the theory of sections 4 and 5 by describing a twisted wreath product of Z^n by Z, and will show that our example is ℓ-isomorphic to a subgroup $G(mn)$ of $Z \text{ Wr } Z$ which generates the Scrimger variety $S(mn)$.

<u>Example 3.6.1.</u> We will consider $TWr(Z^n,Z,mZ,\alpha)$ where $\alpha : mZ \to Aut(Z^n)$ is defined by, for all $r \in Z$

$$(mr)\alpha = \alpha_r$$

where $\alpha_r \in Aut(Z^n)$ is given by, for all $(a_1,\ldots,a_n) \in Z^n$

$$(a_1,\ldots,a_n)\alpha_r = (a_{\overline{1-r}},\ldots,a_{\overline{n-r}})$$

where $\quad 1-r \equiv \overline{i-r} \pmod n$

and $\quad 1 \leq \overline{i-r} \leq n$.

Thus the homomorphism α maps an element mr of mZ to an automorphism α_r of Z^n where α_r rotates the coordinates of elements of Z^n r positions to the right.

We will take $\{0,\ldots,m-1\}$ to be the transversal, T, of mZ in Z.

The elements of $TWr(Z^n,Z,mZ,\alpha)$ are pairs (f,b) where $f \in (Z^n)^T$, $b \in Z$.

Using additive notation, addition in $TWr(Z^n,Z,mZ,\alpha)$ is given by

$$(f_1,b_1) + (f_2,b_2) = (f_1+f_2^{\ b_1},b_1+b_2)$$

and we can easily verify that if $b = mr + s$, where $s \in \{0,\ldots,m-1\}$, then for all $t \in \{0,\ldots,m-1\}$

$$f^b(t) = \begin{cases} f[(t+s)^\tau]\alpha_{-r} & t < m-s \\ f[(t+s)^\tau]\alpha_{-(r+1)} & t \geq m-s \ . \end{cases}$$

We note that

$$(t+s)^\tau = \begin{cases} t+s & t+s \leq m-1 \\ t+s-m & t+s > m-1 \ ; \end{cases}$$

also if $b_1 \equiv b_2 \mod mn$ then $f^{b_1} = f^{b_2}$.

It may be clearer to use an alternative representation for the elements of $TWr(Z^n, Z, mZ, \alpha)$. We can write the elements in the following form

$$\begin{array}{c} \quad\quad\quad b \\ \left[(a_{0,1},\ldots,a_{0,n}),\ (a_{1,1},\ldots,a_{1,n}),\ \ldots\ ,\ (a_{m-1,1},\ldots,a_{m-1,n}) \right] \end{array}$$

where $b \in Z$, and for $i \in \{0,\ldots,m-1\}$, $(a_{i,1},\ldots,a_{i,n}) \in Z^n$.

With this notation addition in $TWr(Z^n, Z, mZ, \alpha)$ is given by

$$\begin{array}{c} b \quad\quad\quad\quad\quad\quad\quad\quad \\ \left[(a_{0,1},\ldots,a_{0,n}),\ldots,(a_{m-1,1},\ldots,a_{m-1,n}) \right] \end{array} + \begin{array}{c} \quad\quad\quad c \\ \left[(d_{0,1},\ldots,d_{0,n}),\ldots,(d_{m-1,1},\ldots,d_{m-1,n}) \right] \end{array}$$

$$= \left[\begin{array}{l} (a_{0,1}+d_{s,1+r},\ldots,a_{0,n}+d_{s,n+r}),\ldots,(\overset{b+c}{a}_{m-s-1,1}+d_{m-1,1+r},\ldots, \\ a_{m-s-1,n}+d_{m-1,n+r}),(a_{m-s,1}+d_{m,1+r+1},\ldots,a_{m-s,n}+d_{m,n+r+1}),\ldots, \\ (a_{m-1,1}+d_{m-1+s,1+r+1},\ldots,a_{m-1,n}+d_{m-1+s,n+r+1}) \end{array} \right]$$

where $b = mr + s$ with $s \in \{0,\ldots,m-1\}$ and the arithmetic of the subscripts is done modulo m for the first subscript and modulo n for the second.

With this notation the 'twisting' affect of the homomorphism α in $TWr(Z^n,Z,mZ,\alpha)$ can readily be seen.

<u>Proposition 3.6.2.</u> $TWr(Z^n,Z,mZ,\alpha)$ is ℓ-isomorphic to $G(mn)$.

<u>Proof.</u> Define a mapping $\gamma : G(mn) \to TWr(Z^n,Z,mZ,\alpha)$ by, for all $(F,k) \in G(mn)$

$$(F,k)\gamma = (F_\gamma,k)$$

where, if $F = (x_1,\ldots,x_{mn})$

then $F_\gamma = ((y_1,\ldots,y_n),(y_{n+1},\ldots,y_{2n}),\ldots,(y_{(n-1)n+1},\ldots,y_{mn}))$

where, for $0 \le i \le m-1$ and $0 \le j \le n-1$,

$$y_{in+j+1} = x_{jm+i+1} .$$

Clearly γ is one to one and onto.

We now show that γ is a homomorphism.

Let (F,k), $(G,\ell) \in G(mn)$.

Then
$$\begin{aligned}[(F,k) + (G,\ell)]\gamma &= (F+G^k, k+\ell)\gamma \\ &= ((F+G^k)_\gamma, k+\ell) \\ &= (F_\gamma + (G^k)_\gamma, k+\ell),\end{aligned}$$

and
$$\begin{aligned}(F,k)\gamma + (G,\ell)\gamma &= (F_\gamma, k) + (G_\gamma, \ell) \\ &= (F_\gamma + (G_\gamma)^k, k+\ell) .\end{aligned}$$

It is sufficient to show that $(G^k)_\gamma = (G_\gamma)^k$.

Let
$$\begin{aligned} G &= (x_1, \ldots \ldots, x_i, \ldots \ldots, x_{mn}) \\ G^k &= (y_1, \ldots \ldots, y_i, \ldots \ldots, y_{mn}) \\ (G^k)_\gamma &= ((z_1, \ldots \ldots, z_i, \ldots \ldots, z_{mn})) \\ G_\gamma &= ((a_1, \ldots \ldots, a_i, \ldots \ldots, a_{mn})) \\ (G_\gamma)^k &= ((b_1, \ldots \ldots, b_i, \ldots \ldots, b_{mn})) .\end{aligned}$$

We wish to show that for $1 \leq i \leq mn$, $z_i = b_i$.

Without loss of generality we may assume $1 \leq k \leq mn$.

Let $$k = pm + q \qquad 0 \leq q \leq m-1 .$$

Let $i = rn + s + 1$ where $0 \leq r \leq m-1$, $0 \leq s \leq n-1$.

Then we have

$$
\begin{aligned}
z_i &= z_{rn+s+1} \\
&= y_{sm+r+1} \\
&= x_{sm+r+1+k} \\
&= x_{sm+r+1+pm+q}
\end{aligned}
$$

where arithmetic on the subscripts is done modulo mn .

Now $b_i = b_{rn+s+1}$ and hence is in the rth n-tuple of $(G_\gamma)^k$. But this is just $G_\gamma[(r+k)^\tau][(r+k)^\sigma]^{-1}$.

We have
$$
\begin{aligned}
(r+k)^\tau &= (pm+q+r)^\tau \\
&= (q+r)^\tau \\
&= \begin{cases} q+r & q+r < m \\ q+r-m & q+r \geq m . \end{cases}
\end{aligned}
$$

Also

$$
[(r+k)^\sigma]^{-1} = \begin{cases} -pm & q+r < m \\ -(p+1)m & q+r \geq m . \end{cases}
$$

Thus the rth n-tuple of $(G_\gamma)^k$ is given by

$$G_\gamma(q+r)\alpha_{-p} \qquad q+r < m$$

$$G_\gamma(q+r-m)\alpha_{-(p+1)} \qquad q+r \geq m$$

$$= (a_{(q+r)n+1},\ldots,a_{(q+r)n+n})\alpha_{-p} \qquad q+r < m$$

$$(a_{(q+r-m)n+1},\ldots,a_{(q+r-m)n+n})\alpha_{-(p+1)} \qquad q+r \geq m$$

$$= (a_{(q+r)n+\overline{1+p}},\ldots,a_{(q+r)n+\overline{n+p}}) \qquad q+r < m$$

$$(a_{(q+r-m)n+\overline{1+p}},\ldots,a_{(q+r-m)n+\overline{n+p}}) \qquad q+r \geq m$$

where $1 \leq \overline{p+i} \leq n$ and $\overline{p+i} \equiv p+i \pmod{n}$.

From this, and recalling that $0 \leq p \leq n$, $0 \leq q \leq m-1$ and $0 \leq s \leq n-1$, we can see that

$$b_i = b_{rn+s+1}$$

$$= a_{(q+r)n+\overline{s+1+p}} \qquad q+r < m$$

$$a_{(q+r-m)n+\overline{s+1+p+1}} \qquad q+r \geq m$$

$$
\begin{array}{ll}
= a_{(q+r)n + (s+p)+1} & q+r < m \,, \quad s+p < n \\[1em]
 a_{(q+r)n + (s+p-n)+1} & q+r < m \,, \quad s+p \quad n \\[1em]
 a_{(q+r-m)n + (s+p+1)+1} & q+r \geq m \,, \quad s+p+1 < n \\[1em]
 a_{(q+r-m)n + (s+p+1-n)+1} & q+r \geq m \,, \quad s+p+1 \geq n
\end{array}
$$

$$
\begin{array}{ll}
= x_{(s+p)m + q+r+1} & q+r < m \,, \quad s+p < n \\[1em]
 x_{(s+p-n)m + q+r+1} & q+r < m \,, \quad s+p \geq n \\[1em]
 x_{(s+p+1)m + q+r-m+1} & q+r \geq m \,, \quad s+p+1 < n \\[1em]
 x_{(s+p+1-n)m + q+r-m+1} & q+r \geq m \,, \quad s+p+1 \geq n \,.
\end{array}
$$

$$= x_{(s+p)m + q+r+1}$$

$$= z_i \qquad \text{as required.}$$

We now have that γ is an isomorphism, it remains to show that γ is an ℓ-isomorphism. We must show that for all (F,k), $(G,\ell) \in G(mn)$

$$[(F,k) \wedge (G,\ell)]\gamma = [(F,k)\gamma] \wedge [(G,\ell)\gamma] \,.$$

Now

$$(F,k) \wedge (G,\ell) = \begin{cases} (G,\ell) & \ell < k \\ (F,k) & \ell > k \\ (F \wedge G,k) & \ell = k\,, \end{cases}$$

hence

$$[(F,k) \wedge (G,\ell)]\gamma = \begin{cases} (G,\ell)\gamma & \ell < k \\ (F,k)\gamma & \ell > k \\ (F \wedge G,k)\gamma & \ell = k \end{cases}$$

$$= \begin{cases} (G_\gamma,\ell) & \ell < k \\ (F_\gamma,k) & \ell < k \\ ((F \wedge G)_\gamma,k) & \ell = k\,. \end{cases}$$

Also $[(F,k)\gamma] \wedge [(G,\ell)\gamma] = (F_\gamma,k) \wedge (G_\gamma,\ell)$

$$\begin{aligned} &= (G_\gamma,\ell) && \ell < k \\ &\ (F_\gamma,k) && \ell > k \\ &\ (F_\gamma \wedge G_\gamma),k) && \ell = k\,. \end{aligned}$$

Thus it is sufficient to show

$$(F \wedge G)_\gamma = F_\gamma \wedge G_\gamma .$$

Let $F = (f_1,\ldots,f_{mn})$ and $G = (g_1,\ldots,g_{mn})$

Then $(F \wedge G)_\gamma = ((h_1,\ldots,h_n),\ldots,(h_{(m-1)n+1},\ldots,h_{mn}))$

where , for $0 \le i \le m-1$ and $0 \le j \le n-1$,

$$h_{in+j+1} = \min\{f_{jm+i+1}, g_{jm+i+1}\} .$$

Also $F_\gamma = ((x_1,\ldots,x_n),\ldots,(x_{(m-1)n+1},\ldots,x_{mn}))$

and $G_\gamma = ((y_1,\ldots,y_n),\ldots,(y_{(m-1)n+1},\ldots,y_{mn}))$

where, for $0 \le i \le m-1$ and $0 \le j \le n-1$

$$x_{in+j+1} = f_{jm+i+1} \quad \text{and} \quad y_{in+j+1} = g_{jm+i+1} .$$

Clearly $F_\gamma \wedge G_\gamma = ((z_1,\ldots,z_n),\ldots,(z_{(m-1)n+1},\ldots,z_{mn}))$

where for $1 \le i \le mn$, $z_i = \min\{x_i, y_i\}$.

Thus for $0 \le i \le m-1$ and $0 \le j \le n-1$,

$$z_{in+j+1} = \min\{x_{in+j+1}, y_{in+j+1}\}$$

$$= \min\{f_{jm+i+1}, g_{jm+i+1}\}$$

$$= h_{in+j+1}$$

and so $(F \wedge G)_\gamma = F_\gamma \wedge G_\gamma$, as required .

BIBLIOGRAPHY

[1] A. Bigard, K. Keimel and S. Wolfenstein, Groupes et anneaux réticulés, Lecture Notes in Mathematics 608, Springer-Verlag, 1977.

[2] G. Birkhoff, Lattice-ordered groups, Ann. of Math. 43 (1942), 298-331.

[3] G. Birkhoff, Lattice theory, 3rd edition, Amer. Math. Soc. Coll. Publ. 25, New York (1968).

[4] P.E. Conrad, Lattice-ordered groups, Lecture Notes, Tulane University (1970).

[5] L. Fuchs, Partially ordered algebraic systems, Addison-Wesley, (1963).

[6] A.M.W. Glass, Ordered permutation groups, Bowling Green State University (1976).

[7] A.M.W. Glass, W.C. Holland, and S.H. McCleary, The structure of ℓ-group varieties, Algebra Universalis 10, (1980), 1-20.

[8] M. Hall, The theory of groups, MacMillan (1959).

[9] W.C. Holland, The lattice ordered group of automorphisms of an ordered set, Michigan Math. J. 10 (1963), 399-408.

[10] W.C. Holland, The largest proper variety of lattice ordered groups, Proc. Amer. Math. Soc. 57 (1976), 25-28.

[11] W.C. Holland and S.H. McCleary, Wreath products of ordered permutation groups, Pacific Math. J. 31 (1969), 703-716.

[12] J. Martinez, Free products in varieties of lattice ordered groups, Czech. Math. J. 22 (1972), 535-553.

[13] J. Martinez, Varieties of lattice-ordered groups, Math. Zeitchr., 137 (1974), 265-284.

[14] B.H. Neuman, Twisted wreath products of groups, Arch. Math. 14, (1963), 1-6.

[15] H. Neuman, Varieties of groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 37, Springer-Verlag, New York, 1967.

[16] E. Schenkman, Group theory, D. Van Nostrand Co., Princeton, 1965.

[17] E.B. Scrimger, A large class of small varieties of lattice ordered groups, Proc. Amer. Math. Soc., 51 (1975), 301-306.

[18] J.E. Smith, The lattice of ℓ-group varieties, Ph.D. thesis, Bowling Green State University, 1976.

[19] E.C. Weinberg, Free lattice ordered abelian groups II, Math. Ann. 159 (1965), 217-222.

[20] S. Wolfenstein, Valeurs normales dans un groupe réticulé, Atti Accad. Naz. Lincei, 44 (1968), 337-342.