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RAMSEY'S THEOREM FOR SPACES

by

Patricia Brearley

B.Sc., Simon Fraser University, 1978

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

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of

Mathematics

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Patricia Brearley, 1980

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ABSTRACT

A brief history of Ramsey's Theorem and related problems is given. A Ramsey Theorem is proved to the effect that any r -colouring of the t -spaces of any sufficiently large vector space must result in some k -space with all of its t -spaces coloured the same. Spaces can be affine or vector subspaces of a vector space. The theorem is also shown to be true where vector space is replaced with projective space. The proof uses the Hales-Jewett Theorem which is proved for the case $t = 3$; the proof of the general case follows the same lines. Finally, parameter systems are defined and the above Ramsey Theorem is proved to be true where parameter system replaces vector space.

DEDICATION

Dedicated to the memory of my parents, Frank James Collett
(1912-1979) and Mary Edith Collett (née Bridle) (1916-1955).

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INTRODUCTION

The purpose of this thesis is to give a clear detailed proof of Ramsey's Theorem for spaces. Our main source is J. Spencer's "Ramsey's Theorem for Spaces" [7].

We begin with a history of the problem. This is followed in Chapter 2 by a proof of the Hales-Jewett Theorem for the case $r = 3$. Chapters 3 and 4 consist of definitions, lemmas and corollaries which are used in the main proof. In the final chapter, Chapter 6, we define parameter systems and modify the proof given in Chapter 5 to show that parameter systems are Ramsey.

CHAPTER 1

HISTORY OF THE PROBLEM

In 1927, B.L. van der Waerden proved the following theorem for arithmetic progressions.

Theorem 1.1. Given integers t and r , there exists an integer $N(t,r)$ such that if $n \geq N(t,r)$ and the non-negative integers $< n$, i.e. $0,1,2,\dots,n-1$ are arbitrarily r -coloured, then there must exist a monochromatic arithmetic progression of length t . (By the length of an arithmetic progression is meant simply the number of terms in the progression.)

In its essence, van der Waerden's theorem turns out to be a special case of a result dealing not with integers but rather with finite sequences formed from a finite set. This was first discovered by Hales and Jewett. Before stating their basic result, we will define the concept of a (combinatorial) line. A fuller definition will be found at the beginning of Chapter 2 (page 8).

Let $A = \{a_1, a_2, \dots, a_t\}$. A line in A^n consists of t n -tuples,

$$\begin{array}{c}
 (x_{11}, x_{12}, \dots, x_{1n}) \\
 \\
 (x_{21}, x_{22}, \dots, x_{2n}) \\
 \\
 \vdots \quad \vdots \quad \vdots \\
 \\
 (x_{t1}, x_{t2}, \dots, x_{tn})
 \end{array}$$

where each column $\begin{matrix} x_{1i} \\ \vdots \\ x_{ti} \end{matrix}$, $1 \leq i \leq n$ is a constant or is $\begin{matrix} a_1 \\ \vdots \\ a_t \end{matrix}$.

Hales and Jewett's basic result which was proved in 1963 now follows [4].

Theorem 1.2 (Hales and Jewett). For all finite sets A and positive integers r , there exists $N(A,r)$ such that for $n \geq N(A,r)$, in any r -colouring of A^n there is a monochromatic line.

Theorem 1.2 is a special case of Ramsey's Theorem for vector spaces, (see Corollary 3.1). Ramsey's Theorem (Theorem 1.3 below) was proved by F.R. Ramsey in 1930. In the early 1960's, Rota conjectured that Ramsey's Theorem would still hold if set was replaced with vector space. This is Theorem 1.4 (below).

Theorem 1.3. Let t, k, r be positive integers. Then there is a number $N = N(t, k, r)$ depending only on t, k and r with the following property: If S is a set with at least N elements and if the t element subsets of S are divided into r classes in any way, then there is some subset of S consisting of k elements with all of its t element subsets belonging to a single class.

Theorem 1.4 (Ramsey's Theorem for vector subspaces). Let k, t, r be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, k, t)$ depending only on q, r, k and t with the following property: If V is a vector space over F of dimension at least N , and if all the t -dimensional vector subspaces of V are divided into r classes in any way, then there is

some k -dimensional vector subspace with all of its t -dimensional vector subspaces in a single class.

If we replace the notion of vector subspace with that of affine subspace we obtain Ramsey's Theorem for affine subspaces which we will call Theorem 1.4(a). By an affine subspace we mean a translate of a vector subspace. A more formal definition of an affine subspace will be given at the beginning of Chapter 3. Theorem 1.4(a) will be proved in detail in this paper and Theorem 1.4 is an immediate corollary of Theorem 1.4(a). It is also possible to replace "vector" by "projective" in Theorem 1.4 giving Ramsey's Theorem for projective spaces which we will call Theorem 1.4(b). We will refer to Theorems 1.4, 1.4(a) and 1.4(b).

Now we are going to describe the history of progress on Theorem 1.4. In 1967, B. Rothschild showed the equivalence between the affine and projective versions of Ramsey's Theorem and used this equivalence to prove the case for $t = 1$ and for $q = 2, 3$ and 4 [2]. Before this, Kleitman proved the special case of Theorem 1.4(a) for $t = 1$ and $q = 2$ [6].

In 1968, R. Graham used the Hales-Jewett Theorem (Theorem 1.2) to prove Ramsey's Theorem for spaces for the case $k = 1$ and for all q [6]. Again, in 1971, R. Graham and B. Rothschild used n -parameters sets to prove the case $t = 2$ for all q [3]. A year later, in 1972, Graham, Leeb and Rothschild used category theory to prove the theorem for all t [2]. In 1979, J. Spencer published an outline of a more stream-lined proof of Ramsey's Theorem for

spaces [7]. In Chapters 4 and 5 we will give this proof in detail. Simultaneously, with Spencer's proof B. Voigt published another short proof of the theorem [6].

Conjecture on possible density versions of Rota's conjecture.

We return to van der Waerden's Theorem. A simple infinite version of van der Waerden's Theorem states that if $\omega = A \cup B$, where $\omega = \{0, 1, \dots\}$, then either A or B contains an infinite arithmetic progression. Szemmeredi proved a density version of the above theorem (Theorem 1.5 below) which can be used to determine whether the arithmetic progression lies in A or B.

Theorem 1.5. Given k and $\epsilon > 0$, there exists n such that if $B \subset \{1, 2, \dots, n\}$ with $|B| > \epsilon n$, then B contains an arithmetic progression of length k .

This was proved for $k = 3$ by Roth in 1950 [1] and for $k = 4$ by Szemmeredi in 1972 [1]. Finally, it was proved for all k by Szemmeredi in 1975 [1].

Since the Hales-Jewett Theorem implies van der Waerden's Theorem, it is natural to ask if the density form of the Hales-Jewett Theorem holds. The density version of the Hales-Jewett Theorem is the following conjecture:-

Given $\epsilon > 0$ and t , there exists n such that if $B \subseteq A^n$ (where $|A| = t$) and $|B| > \epsilon |A^n|$ then B contains a combinatorial line.

This has been observed to be true for $t = 2$ but is still unknown for values of $t > 2$ [1]. We will state an easier result for $t = 3$ but first we will define the concept of an affine line.

Let F be a finite field and $|F| = q$. Let V be a vector space over F and let $\bar{x} = (x_1, x_2, \dots, x_n)$, $\bar{y} = (y_1, y_2, \dots, y_n) \in V$. Then $L = \bar{y} + \{\alpha\bar{x} : \alpha \in F\}$ is an affine line. We observe that combinatorial lines are affine lines but that not all affine lines are combinatorial lines. We now offer the following theorem for vector spaces. This has just been proved by T.C. Brown and J.P. Buhler [1].

Theorem 1.6. Let $F = \{0, 1, 2\}$, i.e. F is the 3 element field. Then given $\epsilon > 0$, there exists n such that if $B \subseteq F^n$, $|B| > \epsilon 3^n$ then B contains an affine line.

This is a special case of a general density version conjecture which naturally arises from Ramsey's Theorem for spaces. First, we will restate Ramsey's Theorem for vector subspaces.

For $k \geq 0$, we define

$$\begin{bmatrix} V \\ k \end{bmatrix} = \{K \subseteq V : \dim(K) = k\} \text{ where } \dim \text{ is the dimension.}$$

Using this definition, Ramsey's Theorem for vector subspaces is: -

Let k, t, r be non-negative integers and F the field of q elements. Then there is a number $N = N(q, r, k, t)$ depending only on q, r, k and t with the following property: If V is a vector space over F of dimension at least N and $\begin{bmatrix} V \\ k \end{bmatrix}$ is divided into r classes in any way, then there is a vector subspace $W \in \begin{bmatrix} V \\ k \end{bmatrix}$ such that $\begin{bmatrix} W \\ t \end{bmatrix}$ is contained in a single class.

The density version of Ramsey's Theorem for vector subspaces would be:-

Let k and t be non-negative integers and F the field of q elements. Let $\epsilon > 0$ be given and let B be a collection of t vector subspaces of V such that $|B| > \epsilon \binom{|V|}{t}$. Then the union of the elements of B , denoted $\cup B$, is such that $\cup B \cap \binom{V}{k} \neq \phi$.

CHAPTER 2

THE HALES-JEWETT THEOREM

The Hales-Jewett Theorem for any number of colours r was stated in Chapter 1 (Theorem 1.2, p. 3). Two corollaries of this Theorem (Corollaries 3.1 and 3.2) are used in the proof of Ramsey's Theorem for spaces. Before we prove the theorem, we will discuss the concept of a combinatorial line.

For a fixed finite set, $A = \{a_1, a_2, \dots, a_t\}$, we would like to form some kind of a structure from the n -tuples A^n analogous to n -dimensional vector spaces over a finite field. Of course, A is not assumed to be endowed with any special algebraic structure so that the linear equations in the coordinates we can write for determining analogs of vector subspaces are very limited. In fact they are just:-

$$(1) \quad x_i = x_j$$

and

$$(2) \quad x_i = \text{constant} .$$

We use these equations to define a combinatorial line in A^n . In future, we shall use the term "line" for combinatorial line. $[n] = \{1, 2, \dots, n\}$

and $I \subseteq [n]$ is a non-empty set of coordinates. Let $L = \{(x_1, x_2, \dots, x_n)\};$
 $x_i = x_{i'}$, for all $i, i' \in I$ and $x_j = b_j \in A$ for $j \notin I$. Then L
 is a combinatorial line in A^n . Since $I \neq \emptyset$, L consists of t
 distinct n -tuples and these will be of the form:

$$\begin{array}{c}
 \left. \begin{array}{cccccc}
 a & x_1 & x_1 & b & \dots & x_1 \\
 \vdots & \vdots & \vdots & \vdots & & \vdots \\
 a & x_i & x_i & b & & x_i \\
 \vdots & \vdots & \vdots & \vdots & & \vdots \\
 a & x_t & x_t & b & & x_t
 \end{array} \right\} t
 \end{array}$$

(Note that if A does happen to be a finite field, then what has just been defined is an ordinary affine line. However, an affine line need not be a combinatorial line.)

We will now illustrate this concept with a simple example. Let $A = \{1,2,3,4,5\} = [5]$ and $n = 2$. We are going to describe "lines" in A^2 . It can easily be seen that a line in A^2 will consist of 5 2-tuples. $I \subseteq [2]$ so $|I| = 1$ or 2 . For $|I| = 1$, we have the lines $\{(1,a), (2,a), (3,a), (4,a), (5,a)\}$ (which can be displayed

conveniently as $\begin{array}{c} 1,a \\ 2,a \\ \vdots \\ 5,a \end{array}$) where a is any fixed value between 1 and 5.

Thus, the five different values for a create five lines. We can

create another five lines of the form $a,1$
 $a,2$
 \vdots
 $a,5$, where a again takes on

any fixed value between 1 and 5.

If $|I| = 2$, we have the single line $1,1$
 $2,2$
 $3,3$
 $4,4$
 $5,5$.

These 11 lines are illustrated in Figure 2.1.

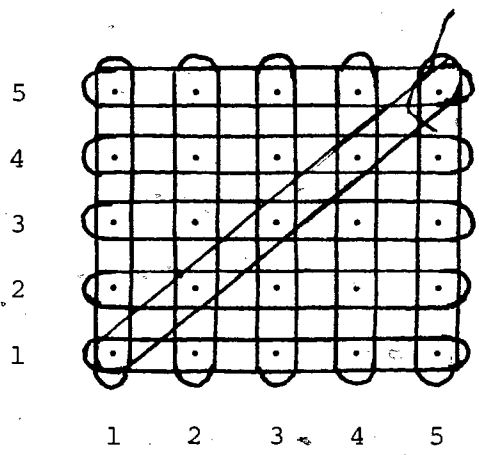


Figure 2.1

Now suppose $A = \{1,2,3\}$ and n is 7. Then abbreviating the line

1 2 1 3 1 1 1 in A^7 by 1 2 * 3 * * 1, it is clear that the
 1 2 2 3 2 2 1
 1 2 3 3 3 3 1

lines in A^7 correspond to 7-tuples on the symbols $\{1, 2, 3, *\}$ in

which at least one $*$ appears, so the total number of lines is $4^7 - 3^7$.

Thus, in general if $|A| = k$, there are exactly $(k+1)^n - k^n$ lines in A^n .

Using this terminology, we state the Hales-Jewett Theorem.

Theorem 2.1 (Hales-Jewett). For all finite sets A and positive integers r , there exists $N(A,r)$ such that for $n \geq N(A,r)$, in any r -colouring of A^n , there is a monochromatic line.

The notational difficulties in the proof of the Hales-Jewett Theorem are more difficult than the proof itself. Therefore in the induction step of the proof of this theorem, we will use $r = 3$ rather than the general case.

The Hales-Jewett number, $HJ(t,r)$, is the least positive integer, N such that if $n \geq N$ and the set $A^n = \{a_1, a_2, \dots, a_t\}^n$ is r coloured then A^n contains a monochromatic line. For example, let $A = \{a_1, a_2\}$ then $HJ(2,3)$ is the least positive integer such that if $n \geq HJ(2,3)$ and A^n is coloured with 3 colours, then A^n contains a monochromatic line. The Hales-Jewett Theorem states that such numbers exist.

Proof of the Hales-Jewett Theorem. Let

$A = \{a_1, a_2, \dots, a_t\}$ then $|A| = t$. We prove the Theorem by induction on $|A|$.

For $t = 1$, the result is immediate since A^n then consists of simply 1 point and this point is a line in A^n (from the definition of combinatorial line). Then it follows that any r -colouring of A^n gives a monochromatic line.

We assume that the theorem holds for all values of $|A| < t$ where $t \geq 2$. This means that for $k < t$ and for all values of r , we are assuming the existence of $HJ(k,r)$. We now prove for this t and all r but only illustrate it with $r = 3$.

$$\text{Let } N_3 = HJ(t-1, 3)$$

$$N_2 = HJ(t-1, 3^{N_3})$$

$$N_1 = HJ(t-1, 3^{N_3+N_2})$$

$$\text{and } n = N_3 + N_2 + N_1 \quad ..$$

Let $\chi : A^n \rightarrow [3]$ be a colouring of A^n with 3 colours. We can think of A^n as $A^{N_3+N_2} \times A^{N_1}$ and $a \in A^n$ as $(\bar{x} \times \bar{y})$ where

$$\bar{x} = (x_1, x_2, \dots, x_{N_3+N_2}) \text{ and } \bar{y} = (y_1, y_2, \dots, y_{N_1}).$$
 We can use the

colouring χ of A^n to induce a colouring $\chi^{(1)}$ of A^{N_1} by

defining $\chi^{(1)}$ as follows:-

$$\chi^{(1)}(\bar{y}) = \chi(\bar{x} \times \bar{y}_1)$$

iff

$$[\chi(\bar{x} \times \bar{y}) = \chi(\bar{x} \times \bar{y}_1) \text{ for all } \bar{x} \in A^{N_3+N_2}].$$

This defines the colouring implicitly by defining the partition the colouring induces.

We can think of each \bar{y} as the "tail" of an n -tuple. Attached to this "tail" are many different "heads", \bar{x} . (See diagram, Figure 2.2).

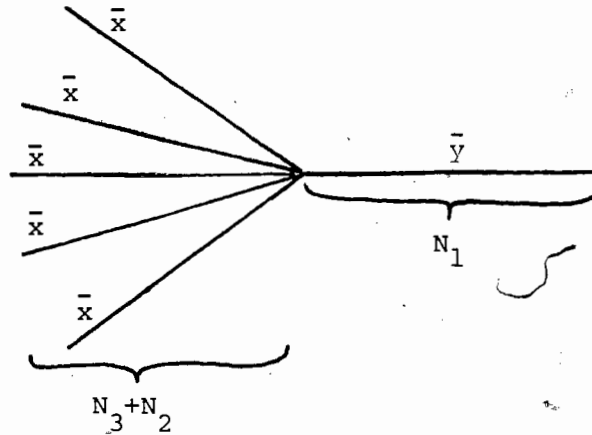


Figure 2.2

Each "head" is a $N_3 + N_2$ -tuple in $A^{N_3 + N_2}$ where the underlying set A

has order t . Then there are $t^{N_3 + N_2}$ different "heads", \bar{x} in

$A^{N_3 + N_2}$. χ is a colouring which uses three colours and the number

of different ways of colouring the $t^{N_3 + N_2}$ "heads" with three colours

is $3^{t^{N_3 + N_2}}$. Thus, $\chi^{(1)}$ is a colouring of A^{N_1} which requires

$3^{t^{N_3 + N_2}}$ colours. Each of the colours used in $\chi^{(1)}$ can be viewed

as a function $f: A^{N_3 + N_2} \rightarrow [3]$.

Let $B = \{a_1, a_2, \dots, a_{t-1}\}$.

Then $\chi^{(1)}$ is also a colouring of B^{N_1} . From the induction hypothesis and the definition of N_1 , there is a $\chi^{(1)}$ monochromatic line L_1 in B^{N_1} . We will call this line $X_1(i_1)$, $1 \leq i_1 \leq t-1$.

Figure 2.3 shows this line in diagrammatic form.

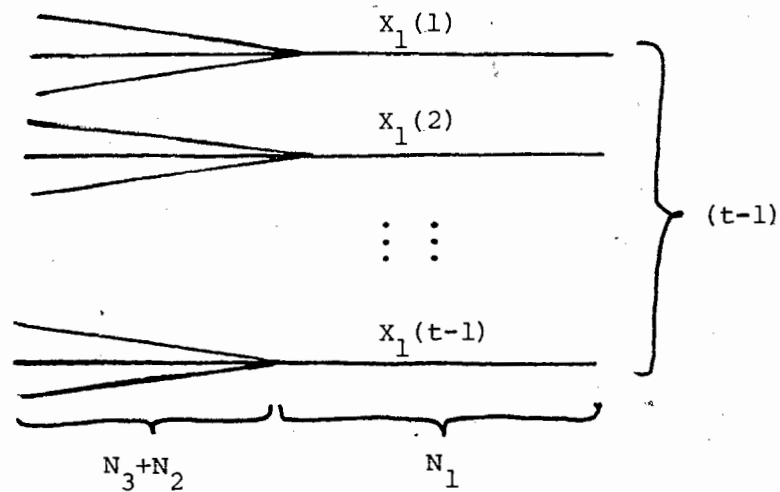


Figure 2.3

If to each of these $(t-1)$ N_1 -tuples we attach the same head \bar{x} , then the resulting $(t-1)$ N_1 -tuples have the same χ -colour.

Next we focus on the "heads", $\bar{a} \in A^{N_3+N_2}$. We can think of

$A^{N_3+N_2}$ as $A^{N_3} \times A^{N_2}$ and $\bar{a} \in A^{N_3+N_2}$ as $(\bar{x} \times \bar{y})$ where

$\bar{x} = (x_1, \dots, x_{N_3})$ and $\bar{y} = (y_1, \dots, y_{N_2})$. Again using the colouring χ

of A^n we can induce a colouring $\chi^{(2)}$ of A^{N_2} by defining

$$\chi^{(2)}(\bar{y}) = \chi^{(2)}(\bar{y}_1)$$

iff

$$[\chi(\bar{x} \times \bar{y} \times X_1(1)) = \chi(\bar{x} \times \bar{y}_1 \times X_1(1)) \text{ for all } \bar{x} \text{ in } A^{N_3}]$$

$\chi^{(2)}$ is a colouring of A^{N_2} which uses 3^{N_3} colours; each colour corresponds to a function $f : A^{N_3} \rightarrow [3]$. Note that since

$L_1(X_1(i_1), 1 \leq i_1 \leq t-1)$ is monochromatic, we could have used $X_1(i_1)$ for any $i_1 \in [t-1]$ in place of $X_1(1)$ in the definition of $\chi^{(2)}$.

We can restrict $\chi^{(2)}$ to B^{N_2} and find a $\chi^{(2)}$ monochromatic line

L_2 in B^{N_2} . We call this line $X_2(i_2), 1 \leq i_2 \leq t-1$. Figure 2.4 illustrates L_2 .

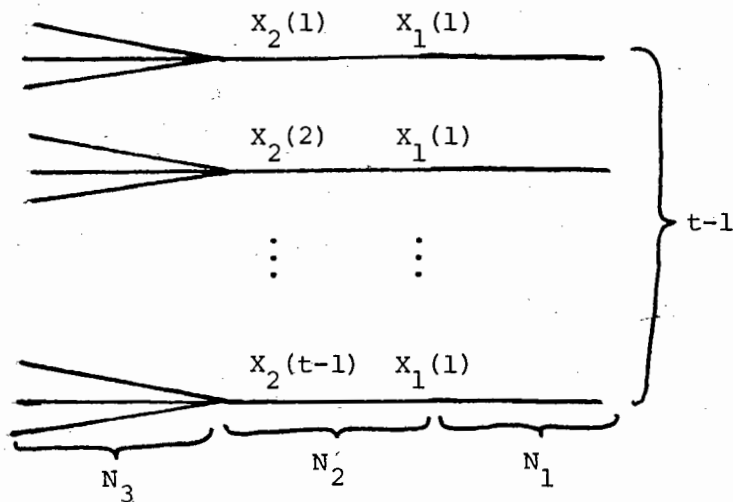


Figure 2.4

If to each of the $(t-1) N_2 + N_1$ -tuples $X_2(i_2) \times X_1(i_1)$, $1 \leq i_1, i_2 \leq t-1$ we attach the same "head" $\bar{x} \in A^{N_3}$, then the

resulting $(t-1)$ n -tuples have the same colour. Note that by construction the χ -colour of any of the $(t-1)^2$ points $(x_1, \dots, x_{N_3}) \times X_2(i_2) \times X_1(i_1)$, $1 \leq i_1, i_2 \leq t-1$ depends only on the choice of (x_1, \dots, x_{N_3}) and not on the particular values of i_1 and i_2 . Furthermore, suppose $X_2(i_2) = (\dots, c, \dots, a_{i_2}, \dots)$ where c is a constant and consider $X_2(t) = (\dots, c, \dots, a_t, \dots)$. $X_2(t)$ has the same "constant components" c as $X_2(i_2)$, $1 \leq i_2 \leq t-1$, but has a_t for its "variable components". The colour assigned to

$(x_1, \dots, x_{N_3}) \times X_2(t) \times X_1(i_1)$ by χ does not depend on the choice of $i_1 \in [t-1]$. This follows from the fact that the $X_1(i_1)$, $1 \leq i_1 \leq t-1$, form a $\chi^{(1)}$ monochromatic line in B^{N_1} and from the definition of $\chi^{(1)}$. If two N_1 -tuples say $X_1(j)$ and $X_1(k)$ in A^{N_1} have been assigned the same colour by $\chi^{(1)}$, then if $\bar{x} \in A^{N_3+N_2}$, $\bar{x} \times X_1(j)$ and $\bar{x} \times X_1(k)$ must have been assigned the same colour by χ .

Finally, we use χ to induce a $\chi^{(3)}$ colouring of A^{N_3} by $\chi^{(3)}(\bar{x}) = \chi(\bar{x} \times X_2(1) \times X_1(1))$. From the induction hypothesis we can find a monochromatic $\chi^{(3)}$ line L_3 in B^{N_3} which we call $X_3(i_3)$, $1 \leq i_3 \leq t-1$.

We now examine the set of n -tuples defined by

$$X_3(i_3) \times X_2(i_2) \times X_1(i_1), \quad 1 \leq i_1, i_2, i_3 \leq t-1.$$

From the definitions of $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$ and the lines,

L_1 , L_2 , and L_3 we have:-

$$\begin{aligned} \chi(X_3(i_3) \times X_2(i_2) \times X_1(i_1)) &= \chi(X_3(i_3) \times X_2(i_2) \times X_1(1)) \\ &= \chi(X_3(i_3) \times X_2(1) \times X_1(1)) = \chi(X_3(1) \times X_2(1) \times X_1(1)), \end{aligned}$$

for $1 \leq i_1, i_2, i_3 \leq t-1$. The first equality is from the definition of $\chi^{(1)}$ and the fact that L_1 is a monochromatic line with respect to $\chi^{(1)}$. The second and third equalities follow from the definitions of $\chi^{(2)}$ and L_2 , $\chi^{(3)}$ and L_3 respectively. This gives us $(t-1)^3$

n -tuples $X_3(i_3) \times X_2(i_2) \times X_1(i_1)$, $1 \leq i_1, i_2, i_3 \leq t-1$, all assigned

the same colour by χ . Further since L_1 is a $\chi^{(1)}$ monochromatic

line in A^1 then $\chi(X_3(t) \times X_2(t) \times X_1(i_1)) = \chi(X_3(t) \times X_2(t) \times X_1(1))$

for $1 \leq i_1 \leq t-1$. Also $\chi(X_3(t) \times X_2(1) \times X_1(1)) =$

$\chi(X_3(t) \times X_2(1) \times X_1(i_1)) = \chi(X_3(t) \times X_2(i_2) \times X_1(i_1))$, $1 \leq i_1, i_2 \leq t-1$.

Again, this follows from the definition of $\chi^{(1)}$, L_1 , $\chi^{(2)}$ and L_2 .

We now consider the following 6 lines L_i^* , $1 \leq i \leq 6$, in A^n .

$$\begin{aligned} L_1^* &= \begin{matrix} X_3(1) \times X_2(1) \times X_1(1) \\ \vdots \\ X_3(t) \times X_2(1) \times X_1(1) \end{matrix} & L_2^* &= \begin{matrix} X_3(1) \times X_2(1) \times X_1(1) \\ \vdots \\ X_3(t) \times X_2(t) \times X_1(1) \end{matrix} \end{aligned}$$

$$\begin{array}{l}
 L_3^* = \begin{array}{ccc}
 x_3(1) \times x_2(1) \times x_1(1) & & \\
 \vdots & \vdots & \vdots \\
 x_3(t) \times x_2(t) \times x_1(t) & &
 \end{array}
 \end{array}
 \qquad
 \begin{array}{l}
 L_4^* = \begin{array}{ccc}
 x_3(1) \times x_2(1) \times x_1(1) & & \\
 \vdots & \vdots & \vdots \\
 x_3(t) \times x_2(t) \times x_1(1) & &
 \end{array}
 \end{array}$$

$$\begin{array}{l}
 L_5^* = \begin{array}{ccc}
 x_3(t) \times x_2(1) \times x_1(1) & & \\
 \vdots & \vdots & \vdots \\
 x_3(t) \times x_2(t) \times x_1(t) & &
 \end{array}
 \end{array}
 \qquad
 \begin{array}{l}
 L_6^* = \begin{array}{ccc}
 x_3(t) \times x_2(t) \times x_1(1) & & \\
 \vdots & \vdots & \vdots \\
 x_3(t) \times x_2(t) \times x_1(t) & &
 \end{array}
 \end{array}$$

For each L_i^* , $1 \leq i \leq 6$, the previous discussion established that its first $(t-1)$ n -tuples are χ -monochromatic. All we must now show is that for at least one L_i^* , the last n -tuple is the same colour as the previous $(t-1)$ n -tuples. To establish this, we now consider the following four n -tuples:

$$x_3(1) \times x_2(1) \times x_1(1)$$

$$x_3(t) \times x_2(1) \times x_1(1)$$

$$x_3(t) \times x_2(t) \times x_1(1)$$

$$x_3(t) \times x_2(t) \times x_1(t)$$

Since the χ -colouring uses only three colours, at least two of the above n -tuples have the same colour; whichever two they are, they are exactly the first and last n -tuples in one of the six lines,

say the line L_k^* . Then L_k^* is a χ -monochromatic line as desired and this establishes that $HJ(t,3) \leq N_3 + N_2 + N_1$.

From the proof we can derive that $HJ(2,3) \leq 3$ (since $HJ(1,r) = 1$ for all values of r). On the other hand if $A = \{0,1\}$ then $A^2 = \{00,01,10,11\}$ and the 3-colouring given by $00 \rightarrow 1, 11 \rightarrow 2, 01 \rightarrow 3, 10 \rightarrow 3$ does not give a monochromatic line. (This is illustrated in Figure 2.5). Then $HJ(2,3) > 2$ giving $HJ(2,3) = 3$.

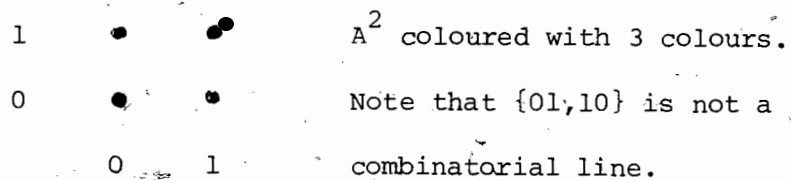


Figure 2.5

The upper bound for $HJ(3,3)$ is $3 + 3^3 + 3^3 + 3^3$ which is considerably larger than $HJ(2,3)$! This bound is obtained in the following way. We use the fact that $HJ(1,r) = 1$ for all values of r . Then $HJ(3,3) \leq N_3 + N_2 + N_1$

$$\text{where } N_3 = HJ(2,3) = 3$$

$$N_2 = HJ(2,3^3)$$

$$N_1 = HJ(2,3^3 + N_2)$$

Now for all r , $HJ(2,r) \leq N_r + N_{r-1} + \dots + N_1$

$$\text{where } N_r = \text{HJ}(1, r) = 1$$

$$N_{r-1} = \text{HJ}(1, r^{2^r}) = 1$$

$$\vdots$$

$$N_1 = \text{HJ}(1, r^{2^{N_r + N_{r-1} + \dots + N_2}}) = 1.$$

Hence $\text{HJ}(2, r) \leq r$ which gives the following values for

$$N_3, N_2 \text{ and } N_1. \quad N_3 = 3, \quad N_2 = 3^{3^3} \text{ and } N_1 = 3^{3^3} + 3^{3^3}.$$

$$\text{Hence, } \text{HJ}(3, 3) \leq 3 + 3^{3^3} + 3^{3^3} + 3^{3^3}.$$

CHAPTER 3

SOME COROLLARIES OF THE HALES-JEWETT THEOREM

The first corollary is a special case of Ramsey's Theorem for affine subspaces (Theorem 1.4(a), p.3). First, we will give some definitions and then state Ramsey's Theorem for affine subspaces.

F is an arbitrary, but fixed finite field. Functions defined in this section are dependent on F in addition to the written variables. Next we define an affine subspace. If V is a vector space, $\bar{v} \in V$ and W is a vector subspace of V then the set $W + \bar{v}$ is an affine subspace of V whose dimension is the dimension of W . For example, let F be a finite field, $|F| = q$ and V an n -dimensional vector space over F . If A_2 is any 2-dimensional affine subspace of V then $A_2 = W + \bar{v}$ where W is a 2-dimensional vector subspace of V and $\bar{v} \in V$. We can write $W = \{\alpha\bar{y} + \beta\bar{z}; \alpha, \beta \in F, \bar{y}, \bar{z} \in V, \bar{y} \text{ and } \bar{z} \text{ are linearly independent}\}$. Then $A_2 = \{\alpha\bar{y} + \beta\bar{z} + \bar{v}; \alpha, \beta \in F\}$. Addition and scalar multiplication are defined as usual. We note that an affine subspace is not closed under addition and scalar multiplication. Throughout this chapter and following chapters, an affine subspace will frequently be referred to simply as a subspace or space.

$\dim(T)$ is the dimension of a space T . If $T = W + \bar{v}$ where W is a vector subspace, then $\dim(T)$ is defined to be $\dim(W)$. T is called a t -space if $\dim(T) = t$.

Let V be an n -space; V may be a vector subspace or an affine subspace. We define for $t \geq 0$

$$\left[\begin{array}{c} V \\ t \end{array} \right] = \{T \subset V; \dim(T) = t\},$$

i.e. $\left[\begin{array}{c} V \\ t \end{array} \right]$ is all the t -spaces in V .

The points of V are its 0-spaces.

We now state Ramsey's Theorem for affine subspaces which was referred to in Chapter 1 as Theorem 1.4(a).

Theorem 1.4(a). (Ramsey's Theorem for affine subspaces). Let k, t, r be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, k, t)$ depending only on q, r, k and t with the following property: If V is a vector space over F of dimension at least N and if all the t -dimensional affine subspaces of V are divided into r classes, then there is some k -dimensional affine subspace with all of its t -dimensional affine subspaces in a single class.

Dividing the t -spaces of V into r classes is equivalent to colouring them using r colours. We restate Theorem 1.4(a) in these terms.

Theorem 1.4(a)¹. Let k, t, r be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, k, t)$ depending only on q, r, k and t with the following property: If V is a vector space over F of dimension at least N

and χ is an r -colouring of the t -dimensional affine subspaces of V , then there exists a k -dimensional affine subspace $W \subset V$ such that all the t -dimensional affine subspaces $T \subset W$ are the same colour.

Corollary 3.1 is the case $t = 0$ and we shall use the Hales-Jewett Theorem to prove Corollary 3.1. First, we shall repeat the definition of the Hales-Jewett number and restate the Hales-Jewett Theorem in terms of the Hales-Jewett number.

The Hales-Jewett number, $n = \text{HJ}(k, r)$ is the least number, n such that if A is a set, $|A| = k$ and χ is any r -colouring of A^n then A^n contains a monochromatic line.

Hales-Jewett Theorem. For all positive integers, k, r , there exists $n = \text{HJ}(k, r)$ with the following property. Let $|A| = k$ and χ be an r -colouring of A^n . Then there exists a monochromatic line $L \subset A^n$.

Corollary 3.1. For positive integers, r, k , and F a field of q elements there exists a number $n = N(q, r, k)$ with the following property. Let V be an n -dimensional vector space over F and let χ be an r -colouring of the points of V . Then there exists a k -dimensional affine subspace, W such that all the points of W are the same colour.

Proof. Let $n = km$ where $m = \text{HJ}(|F|^k, r)$.

We identify V with F^n .

Let $\psi : F^n \rightarrow (F^k)^m$ denote the natural bijection given by grouping the coordinates of $\bar{x} \in F^n$ into disjoint sets of k coordinates each.

Then, for $\bar{x} \in F^n$, $\bar{x} = (x_1, x_2, \dots, x_n)$

$$\begin{aligned} \psi(\bar{x}) &= \psi(x_1, x_2, \dots, x_n) = ((x_1, x_2, \dots, x_k), \\ &\quad (x_{k+1}, \dots, x_{2k}), \dots, \\ &\quad (x_{(m-1)k+1}, \dots, x_{mk})). \end{aligned}$$

Let $\chi : F^n \rightarrow [r]$ be any r -colouring of F^n . Then χ induces the r -colouring $\chi\psi^{-1} : (F^k)^m \rightarrow [r]$.

By the definition of m , (i.e., the Hales-Jewett Theorem), there is a $\chi\psi^{-1}$ monochromatic line $L \subset (F^k)^m$. Then the set $\psi^{-1}(L) \subset F^n$ is a χ monochromatic set in F^n . We claim that $\psi^{-1}(L)$ is an affine k -space in V . The proof is conceptionally easy but notationally difficult so we illustrate by an example.

Let $k = 2$ and $m = 3$.

In $(F^2)^3$, the set

$$L = \{(x, y), (2, 5), (x, y); (x, y) \in F^2\}$$

is a combinatorial line which corresponds to the affine plane

$$\psi^{-1}(L) = \{(x, y, 2, 5, x, y); x, y \in F\} \subset F^6.$$

Corollary 3.2. Let $m = \text{HJ}(|F|^{u+1}, r)$ and let χ be an r -colouring of the ordered $(u+1)$ -tuples $(\bar{x}_0, \dots, \bar{x}_u)$, $\bar{x}_i \in F^m$. Then there exist parallel combinatorial lines $L_0, \dots, L_u \subset F^m$ so that $\{(\bar{x}_0, \dots, \bar{x}_u); \bar{x}_i \in L_i\}$ is monochromatic.

Proof. Consider the natural bijection

$$\phi : (F^m)^{u+1} \rightarrow (F^{u+1})^m$$

that sends $(\bar{x}_0, \dots, \bar{x}_u), \bar{x}_i \in F^m$, into $(\bar{y}_1, \dots, \bar{y}_m), \bar{y}_j \in F^{u+1}$

where $\bar{x}_i = (x_{i1}, \dots, x_{im}), 0 \leq i \leq u$ and $\bar{y}_j = (x_{0j}, \dots, x_{uj}),$

$1 \leq j \leq m.$

ϕ in fact regroups and reorders a $(u+1)$ - m -tuple in the following way. Let $\bar{x} \in (F^m)^{u+1}$, then

$$\bar{x} = \underbrace{(x_{01}, x_{02}, \dots, x_{0m}), (x_{11}, \dots, x_{1m}), \dots, (x_{u1}, \dots, x_{um})}_{(u+1)\text{-}m\text{-tuple}}$$

We can write \bar{x} as follows:-

$$\begin{array}{c} \left. \begin{array}{c} \overbrace{(x_{01}, \dots, x_{0m})}^m \\ (x_{11}, \dots, x_{1m}) \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ (x_{u1}, \dots, x_{um}) \end{array} \right\} u+1 \\ \\ \text{then } \phi(\bar{x}) = \left. \begin{array}{c} (x_{01}, x_{11}, \dots, x_{u1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ (x_{0m}, x_{1m}, \dots, x_{um}) \end{array} \right\} m \\ \underbrace{\hspace{10em}}_{u+1} \end{array}$$

We have actually written \bar{x} as a $(u+1) \times m$ matrix and $\phi(\bar{x})$ is the transpose of \bar{x} formed by interchanging the rows and columns of \bar{x} .

Let $\chi : (F^m)^{u+1} \rightarrow [r]$ be an r -colouring of $(F^m)^{u+1}$.

Then χ induces an r -colouring $\chi\phi^{-1}$ of $(F^{u+1})^m$.

From the definition of m , (i.e., the Hales-Jewett Theorem), there is a combinatorial line $L \subset (F^{u+1})^m$ which is monochromatic. Then $\phi^{-1}(L)$ is also monochromatic.

Now we must show that

$$\phi^{-1}(L) = L_0 \times \dots \times L_u$$

where the $L_i \subset F^m$ are parallel combinatorial lines.

Once again, as in Corollary 3.1, because of difficult notation we illustrate with an example.

Let $u = 1, m = 3$ and

$$L = \{(x,y), (2,5), (x,y)\}; (x,y) \in F^2\}.$$

Then $\phi^{-1}(L) = \{(x,2,x), (y,5,y)\}; x,y \in F\}$.

So $L_0 = \{(x,2,x); x \in F\}$ and $L_1 = \{(y,5,y); y \in F\}$ and L_0 is parallel to L_1 .

$\therefore \phi^{-1}(L) = L_0 \times L_1$ where the L_i are parallel combinatorial lines.

Corollary 3.3. (Gallai's Theorem). Let k and r be positive integers and let χ be any r -colouring of $[\omega]^2$. Then $[\omega]^2$ contains a monochromatic affine copy of $[k]^2$, that is a monochromatic set $\{a[k]^2 + \bar{b}\}$.

Proof. Let $A = \{00, 01, \dots, 0k-1, 10, 11, \dots, 1k-1, \dots, k-1 k-1\} = [k]^2$.

Let $\phi : [\omega]^2 \rightarrow A \cup A^2 \cup A^3 \cup A^4 \cup \dots \cup A^n \cup \dots$ be defined as follows:-

for $\bar{x} \in [\omega]^2$, $x = (a_1 a_2 \dots a_n, b_1 b_2 \dots b_n)$ where $a_1 a_2 \dots a_n$ and $b_1 b_2 \dots b_n$ are base k representations possibly with $a_1 = 0$, etc.)

$$\phi(x) = (a_1 b_1, a_2 b_2, \dots, a_n b_n) \quad a_i b_i \in A.$$

Then any r -colouring, χ of $[\omega]^2$ induces an r -colouring, $\chi\phi^{-1}$ of $A \cup A^2 \cup \dots \cup A^n \dots$.

For $y \in A^m$, $y = (a_1 b_1, \dots, a_m b_m)$

and $\chi\phi^{-1}(y) = \chi(a_1 a_2 \dots a_n, b_1 b_2 \dots b_n)$.

Now let $m = HJ(k^2, r)$, then from the Hales-Jewett Theorem, A^m contains a $\chi\phi^{-1}$ monochromatic line, L ;

$$L = \begin{matrix} \alpha_1\alpha_2, \beta_1\beta_2, 00, & \dots, & \gamma_1\gamma_2, 00 \\ \cdot & \cdot & 01, & \dots, & \cdot & 01 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_1\alpha_2, \beta_1\beta_2, k-1 k-1, & \dots, & \gamma_1\gamma_2, k-1 k-1 \end{matrix}$$

and $\phi^{-1}(L)$ is a χ monochromatic set in $[\omega]^2$.

$$\begin{aligned} \phi^{-1}(L) = & \begin{matrix} \alpha_1\alpha_2^0 \dots \gamma_1^0, & \alpha_2\beta_2^0 & \dots & \gamma_2^0 \\ \alpha_1\beta_1^0 \dots \gamma_1^1, & \alpha_2\beta_2^1 & \dots & \gamma_2^0 \\ \vdots & \vdots & & \vdots \\ \alpha_1\beta_1^{k-1} \dots \gamma_1^{k-1}, & \alpha_2\beta_2^{k-1} & \dots & \gamma_2^{k-1} \end{matrix} \\ & = \{(\alpha_1\beta_1^x \dots \gamma_1^x, \alpha_2\beta_2^y \dots \gamma_2^y); 0 \leq x, y \leq k-1\}. \end{aligned}$$

Setting $\bar{b} = (\alpha_1\beta_2^0 \dots \gamma_1^0, \alpha_2\beta_2^0 \dots \gamma_2^0)$ and $a = 001 \dots 01$ in base k representation then $\phi^{-1}(L) = \{a[k]^2 + \bar{b}\}$ is a monochromatic set of the required form.

For clarity, we illustrate this concept with an example.

Let $k = 3$, $A = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}$ and for $n = 13$, consider the line L given as follows:

12 12 12 01 00 00 02 01 00 00 22 21 00
 12 12 12 01 01 01 02 01 01 00 22 21 01
 12 12 12 01 02 02 02 01 02 00 22 21 02
 12 12 12 01 10 10 02 01 10 00 22 21 10
 L = 12 12 12 01 11 11 02 01 11 00 22 21 11
 12 12 12 01 12 12 02 01 12 00 22 21 12
 12 12 12 01 20 20 02 01 20 00 22 21 20
 12 12 12 01 21 21 02 01 21 00 22 21 21
 12 12 12 01 22 22 02 01 22 00 22 21 22

= $\{(12,12,12,01,xy,xy,02,01,xy,00,22,21,xy) : 0 \leq x,y \leq 2\}$.

If L is $\chi\phi^{-1}$ monochromatic, then the set, $S = \{(1110xx00x022x, 2221yy21y021y) ; 0 \leq x,y \leq 2\}$ is χ monochromatic.

Let $\bar{b} = (111000000220, 2221002100210)$, $a = (3^8 + 3^7 + 3^4 + 1)$

and for $n = 13$, $a = 0000110010001$ in base 3. Then $a(1,1) = (0000110010001, 0000110010001)$ and $S = (a[2]^2 + \bar{b})$.

This corollary generalizes to:

Let k, r and m be positive integers and let χ be any r -colouring of $[\omega]^m$. Then $[\omega]^m$ contains a χ monochromatic set, $\{a[k]^m + \bar{b}\}$.

Corollary 3.4. Let S be a finite commutative semigroup and define

$$nS = \underbrace{\{s + s + s + \dots + s; s \in S\}}_n.$$

Then there exists a positive integer, n and $x \in S$ such that $nS + x$ is a singleton.

Proof. Let $S = \{s_1, s_2, s_3, \dots, s_r\}$, then $|S| = r$ and let $m = HJ(r, r)$.

Let $\chi : S^m \rightarrow S$ be defined by $\chi(x_1, \dots, x_m) = x_1 + x_2 + \dots + x_m$.

Then since $|S| = r$, χ is an r -colouring of S^m

and so from the definition of m , there exists a χ monochromatic line, $L \subset S^m$. L consists of r m -tuples, the columns of which are

either constant or of the form $\begin{matrix} s_1 \\ \vdots \\ s_r \end{matrix}$. For example,

$\begin{matrix} s_1 & s_1 & s_1 & s_r & s_1 & \dots & s_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ s_1 & s_r & s_r & s_r & s_r & \dots & s_3 \end{matrix}$ is a line in S^m . If L has n

"moving columns" and $m-n$ "constant columns" and has colour $s \in S$,

let x be the sum of the $m-n$ constants appearing in the constant columns. Then $nS + x = \{s\}$.

CHAPTER 4

DEFINITIONS AND LEMMAS REQUIRED IN THE PROOF

OF RAMSEY'S THEOREM FOR AFFINE SUBSPACES

In Chapter 5, we shall prove Ramsey's Theorem for affine subspaces (Theorem 1.4(a)). Ramsey's Theorem for vector subspaces (Theorem 1.4) and Ramsey's Theorem for projective spaces (Theorem 1.4(b)) will be immediate corollaries of Ramsey's Theorem for affine subspaces. In this chapter, we will give some definitions and explanations of notation used as well as some lemmas which are necessary to the proof of Theorem 1.4(a).

At the beginning of Chapter 3, we gave the definition of an affine subspace and noted that an affine subspace is not closed under addition and scalar multiplication. We emphasize this point. To the reader, the lemmas in this chapter may appear to be basic facts of linear algebra but they are basic facts for vector subspaces not affine subspaces.

Also at the beginning of Chapter 3, we gave definitions of $\dim(T)$ and $\begin{bmatrix} v \\ t \end{bmatrix}$. We are now going to define a projection.

In order to do this, we will first define a direct sum. If U and V are vector spaces (over the same field, F), their direct sum is the vector space W (denoted by $U \oplus V$) whose elements are all the

ordered pairs (\bar{x}, \bar{y}) with \bar{x} in U and \bar{y} in V with linear operations defined by:-

$$\alpha_1(\bar{x}_1, \bar{y}_1) + \alpha_2(\bar{x}_2, \bar{y}_2) = (\alpha_1\bar{x}_1 + \alpha_2\bar{x}_2, \alpha_1\bar{y}_1 + \alpha_2\bar{y}_2)$$

$$\alpha_1, \alpha_2 \in F, \bar{x}_1, \bar{x}_2 \in U \text{ and } \bar{y}_1, \bar{y}_2 \in V.$$

Let V be a direct sum of M and N , $V = M \oplus N$. We identify $M \oplus \{0\} \subset V$ with M and $\{0\} \oplus N \subset V$ with N . Then every \bar{z} in V may be written uniquely in the form $\bar{z} = \bar{x} + \bar{y}$ with \bar{x} in M and \bar{y} in N . The projection on M along N is the transformation, E defined by $E\bar{z} = \bar{x}$ where $\bar{z} = \bar{x} + \bar{y}$, $\bar{x} \in M$, $\bar{y} \in N$. We illustrate with an example. Let V be a $(u+1)$ -dimensional vector space and let $\bar{x} \in V$, $\bar{x} = (x_1, \dots, x_{u+1})$, then $(x_1, \dots, x_{u+1}) = (x_1, \dots, x_u, 0) + (0, \dots, 0, x_{u+1})$ where $(x_1, \dots, x_u, 0) \in M$, a u -dimensional vector subspace of V and $(0, \dots, 0, x_{u+1}) \in N$, a 1-dimensional vector subspace of V . Let p be the projection on M along N , then $p(x_1, \dots, x_{u+1}) = (x_1, \dots, x_u, 0)$.

The next concept is important in understanding the proof of Ramsey's Theorem for affine subspaces. Let V be an n -dimensional vector space and $\{\bar{y}_i\}_0^u$ be a set of $(u+1)$ linearly independent vectors in V with $(u+1) < n$. Let X be the smallest affine subspace containing $\{\bar{y}_i\}_0^u$. Then we say that $\{\bar{y}_i\}_0^u$ generate X . Lemma 4.1 shows that $\dim(X) = u$ and that $X = W + \bar{y}_0$ where W is the vector subspace generated by $\{\bar{y}_i - \bar{y}_0\}_1^u$. In Corollary 4.1(b), we prove

that X also consists of all linear combinations of the set $\{\bar{y}_i\}_0^u$ where the sum of the coefficients is 1 .

Next we prove three very useful lemmas.

Lemma 4.1. Let V be an n -dimensional vector space and let $\{\bar{y}_i\}_0^u$ be a set of $(u+1)$ linearly independent vectors in V with $(u+1) < n$. Then the smallest affine subspace, X , containing $\{\bar{y}_i\}_0^u$ has dimension u . We say that $\{\bar{y}_i\}_0^u$ generate X .

Proof. Let Y be any affine subspace which contains $\{\bar{y}_i\}_0^u$, say $Y = W + \bar{c}$ where W is a u -dimensional vector subspace of V . Let X be the intersection of all such Y . We show that $X = W_0 + \bar{y}_0$ where W_0 is the vector subspace generated by $\{\bar{y}_i - \bar{y}_0\}_1^u$ and thus that $\dim(X) = u$.

Set $X' = (W + \bar{c}) - \bar{y}_0$. Then X' is an affine subspace of V which contains $\bar{0}$ and is therefore a vector subspace. Further X' contains $\{\bar{y}_i - \bar{y}_0\}_1^u$.

Let W_0 be the vector subspace generated by $\{\bar{y}_i - \bar{y}_0\}_1^u$. Then $W_0 \subset X'$, and $W_0 + \bar{y}_0 \subset X' + \bar{y}_0 = Y$. Then $W_0 + \bar{y}_0$ is in every affine subspace, Y which contains $\{\bar{y}_i\}_0^u$. Thus $W_0 + \bar{y}_0 \subset X$, the intersection of all such Y . Since $\{\bar{y}_i\}_0^u \subset W_0 + \bar{y}_0$, then $X \subset W_0 + \bar{y}_0$ and thus $X = W_0 + \bar{y}_0$.

Corollary 4.1(a). Let V be an n -dimensional vector space and $\{\bar{y}_i\}_0^u$ a set of $(u+1)$ linearly independent vectors in V (with

$(u+1) < n$). If X is the affine subspace generated by $\{\bar{y}_i\}_0^u$ and $X = W + \bar{c}$ where W is a vector subspace then W is unique.

Proof. Suppose that $X = W + \bar{c}$ and $X = U + \bar{b}$, where W and U are vector subspaces in V . We show that $W = U$. Let $\bar{w} \in W$, then $\bar{w} + \bar{c} \in X = U + \bar{b}$, thus $\bar{w} + \bar{c} = \bar{u} + \bar{b}$ for some $\bar{u} \in U$.

Then, we have $\bar{w} = \bar{u} + \bar{b} - \bar{c}$.

Since W is a vector subspace, $\bar{0} \in W$ and $\bar{0} + \bar{c} \in W + \bar{c} = U + \bar{b}$. Therefore, $\bar{c} \in U + \bar{b}$ so $\bar{c} = \bar{u}^* + \bar{b}$ for some $\bar{u}^* \in U$.

It follows that $\bar{w} = \bar{u} + \bar{b} - (\bar{u}^* + \bar{b})$

$$= (\bar{u} - \bar{u}^*) \in U.$$

Therefore, $W \subset U$.

Similarly, we can show that $U \subset W$ and thus $W = U$.

Corollary 4.1(b). Let V be an n -dimensional vector space and X a u -dimensional affine subspace in V . Let $\{\bar{y}_i\}_0^u$ be a set of $(u+1)$ linearly independent vectors in X . Then $\{\bar{y}_i\}_0^u$ generate X and a vector \bar{v} is in X if and only if $\bar{v} = \sum_0^u c_i \bar{y}_i$ where $c_i \in F$ (the underlying field), $0 \leq i \leq u$ and $\sum_0^u c_i = 1$.

Proof. Set $W = X - \bar{y}_0$. Then W is an affine subspace of V which contains $\bar{0}$ and is therefore a vector subspace. Since $\dim(X) = u$, $\dim(W) = u$ and it follows that $\{\bar{y}_i - \bar{y}_0\}_1^u$ generate W .

Thus X is generated by $\{\bar{y}_i\}_0^u$.

Let $\bar{v} \in X$, then $\bar{v} = \bar{w} + \bar{y}_0$ where $\bar{w} \in W$ and since

$\{\bar{y}_i - \bar{y}_0\}_1^u$ form a basis for W , we can write

$$\bar{v} = \sum_1^u d_i (\bar{y}_i - \bar{y}_0) + \bar{y}_0 = \sum_1^u d_i \bar{y}_i + (1 - \sum_1^u d_i) \bar{y}_0.$$

Therefore $\bar{v} = \sum_0^u c_i \bar{y}_i$ where $c_i = d_i$, $1 \leq i \leq u$ and

$$c_0 = (1 - \sum_1^u d_i) \text{ then } \sum_0^u c_i = \sum_1^u d_i + (1 - \sum_1^u d_i) = 1.$$

Let $\bar{x} = \sum_0^u c_i \bar{y}_i$ and $\sum_0^u c_i = 1$. Then we can write

$$\bar{x} = \sum_1^u c_i (\bar{y}_i - \bar{y}_0) + (\sum_1^u c_i) \bar{y}_0 + c_0 \bar{y}_0 = \sum_1^u c_i (\bar{y}_i - \bar{y}_0) + (\sum_0^u c_i) \bar{y}_0.$$

Since $\sum_0^u c_i = 1$,

$$\text{then } \bar{x} = \sum_1^u c_i (\bar{y}_i - \bar{y}_0) + \bar{y}_0$$

so $\bar{x} \in W + \bar{y}_0 = X$.

Lemma 4.2. Let V be an $(u+m)$ -dimensional vector space over a finite field F . Let L_0, \dots, L_u be parallel lines in V , with $L_i \subset A_i$ where $A_i = \bar{e}_i F^m$ $0 \leq i \leq u$, $\bar{e}_0 = \bar{0}$, and $\bar{e}_i = (0, \dots, 1, \dots, 0)$ where the '1' is in the i th position. Then $\{L_i\}_0^u$ generate a $(u+1)$ -affine subspace, B , in V .

Proof. We identify V with F^{u+m} .

Consider $L_0, L_0 \subset A_0 = \underbrace{00\dots 0}_u \times F^m$. So if $L_0 = \bar{a}_0 + L^*$

where L^* is a line through the origin, then $L^* \subset \underbrace{00\dots 0}_u \times F^m$

and $\bar{a}_0 \in A_0$. Also, since $\{L_i\}_0^u$ are parallel lines we have

$$L_i = \bar{a}_i + L^*, \quad 0 \leq i \leq u, \quad L^* \subset A_0 \quad \text{and} \quad \bar{a}_i \in A_i, \quad 0 \leq i \leq u.$$

Set $L'_i = L_i - \bar{a}_0$, $0 \leq i \leq u$.

Since $\bar{a}_0 \in A_0$, it follows that $\bar{a}_i - \bar{a}_0 \in A_i$, $0 \leq i \leq u$.

Writing $\bar{a}'_i = \bar{a}_i - \bar{a}_0$, we have

$$L'_i = L_i - \bar{a}_0 = \bar{a}_i + L^* - \bar{a}_0 = \bar{a}'_i + L^*, \quad 0 \leq i \leq u,$$

where $\bar{a}'_i \in A_i$, $0 \leq i \leq u$ and $\bar{a}'_0 = \bar{0}$.

Let B be the affine subspace generated by $\{L_i\}_0^u$ i.e., B is the smallest subspace containing the $\{L_i\}_0^u$ and let V be the vector subspace generated by $\{L'_i\}_0^u$. Then $B = V + \bar{a}_0$. Now

$L'_0 = \bar{a}_0 + L^*$ and $\bar{a}'_0 = \bar{0}$, thus $L'_0 = L^*$, a line through the origin and $L^* \subset A_0$. Let $\bar{\ell} \neq \bar{0}$, $\bar{\ell} \in L^*$, $\bar{\ell} = \underbrace{00\dots 0}_u \bar{f}$ where $\bar{f} \in F^m$ and

$\bar{f} \neq \bar{0}$. Then V is spanned by $\bar{\ell} + \bar{a}'_0$, $\bar{\ell} + \bar{a}'_1, \dots, \bar{\ell} + \bar{a}'_u$ and

$\{\bar{\ell} + \bar{a}'_i\}_0^u$ is a linearly independent set. For if $\sum_0^u c_i (\bar{\ell} + \bar{a}'_i) = \bar{0}$

then looking at the first u coordinates we get $c_1 = c_2 = \dots = c_u = 0$

and this implies that $c_0 = 0$.

Thus $\dim(V) = u+1$

and $\dim(B) = \dim(V) = u+1$.

Lemma 4.3. Let V be an n -dimensional vector space, L a combinatorial line in V and \bar{x} a vector not in L . Let X be the affine subspace generated by L and \bar{x} . Then $\dim(X) = 2$. Also, suppose that the first u coordinates of \bar{x} and each vector in L are the same, say (x_1, \dots, x_u) then the first u coordinates of each vector in X are (x_1, \dots, x_u) .

Proof. Let $X = W + \bar{c}$, W a vector subspace.

Then W contains a translate of L , i.e., a line through the origin and a translate of \bar{x} which is not on the translate of L . Thus W contains two linearly independent vectors and has dimension ≥ 2 .

Then $\dim(X) = \dim(W) = 2$.

Let $L = L^* + \bar{v}$ where $L^* \subset W$ and is a line through the origin. Then $\bar{0} \in L^*$ and $\bar{v} \in L$. Since each vector in L begins with (x_1, \dots, x_u) ,

the first u coordinates of \bar{v} are (x_1, \dots, x_u) . Then the first u coordinates of each vector in L^* are $\underbrace{(0, 0, \dots, 0)}_u$. Further since

\bar{x} has (x_1, x_2, \dots, x_u) for its first u coordinates, the translate of \bar{x} in W must have $\underbrace{(0, 0, \dots, 0)}_u$ for its first u coordinates.

Thus every vector in W has $\underbrace{(0, 0, \dots, 0)}_u$ for its first u coordinates.

It then follows that every vector in X has (x_1, x_2, \dots, x_u) for its first u coordinates.

CHAPTER 5

RAMSEY'S THEOREM FOR SPACES

In this Chapter, we are going to prove Ramsey's Theorem for affine subspaces (Theorem 1.4(a)). Ramsey's Theorem for vector subspaces and Ramsey's Theorem for projective spaces are immediate corollaries of Theorem 1.4(a). We again state Ramsey's Theorem for affine subspaces in the form which we called Theorem 1.4(a)¹. We will now call it Theorem 5.1.

Theorem 5.1. (Ramsey's Theorem for affine subspaces). Let k, r, t be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, k, t)$ depending only on q, r, k and t with the following property: If V is a vector space over F of dimension at least N and χ is an r -colouring of the t -dimensional affine subspaces of V , then there exists a k -dimensional affine subspace $W \subset V$ such that all the t -dimensional affine subspaces $T \subset W$ are the same colour.

We restate Theorem 5.1 in formal terms. Let $\dim(V) = n$ and $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \rightarrow [r]$. Then there exists $W \in \begin{bmatrix} V \\ k \end{bmatrix}$ such that χ is constant on $\begin{bmatrix} W \\ t \end{bmatrix}$.

The proof of Theorem 5.1 uses the Hales-Jewett Theorem which was proved for the special case of $r = 3$ in Chapter 2. As was stated at the end of Chapter 2, the proof for the general case follows the same lines as the case $r = 3$. The definitions and lemmas of

Chapter 4 are also necessary to the proof. Before proving Theorem 5.1 we give some further definitions and prove a vital lemma.

Let χ be a colouring of $\begin{bmatrix} V \\ t \end{bmatrix}$; χ is thus a colouring of the t dimensional affine subspaces of V . Let $B \in \begin{bmatrix} V \\ u+1 \end{bmatrix}$. Let $p : B \rightarrow F^u$ be a surjective projection. If $T \in \begin{bmatrix} B \\ t \end{bmatrix}$ then $p|_T : T \rightarrow F^u$ is either bijective or it is not. In the former case, $p(T) \in \begin{bmatrix} F^u \\ t \end{bmatrix}$ and T is called transversal. In the latter case, $p(T) \in \begin{bmatrix} F^u \\ t-1 \end{bmatrix}$, $T = p^{-1}(p(T))$ and T is called vertical. (Intuitively, p defines a vertical direction).

We now define the term "special". This definition is essential to the entire proof.

A space $B \in \begin{bmatrix} V \\ u+1 \end{bmatrix}$ is special (with respect to a colouring, χ and a projection, p) if whenever $T_1, T_2 \in \begin{bmatrix} B \\ t \end{bmatrix}$, are transversals and $p(T_1) = p(T_2)$ then $\chi(T_1) = \chi(T_2)$. That is, B is special if the colour of a transversal t -space in B is determined by its projection.

We now prove the following lemma which is central to the proof of Theorem 5.1.

Lemma 5.1. Let t, u, r be non-negative integers and F a field of q elements. Then there exists a number $m = M(q, t, u, r)$ such that for any r -colouring of the t -spaces of F^{u+m} there exists a special $(u+1)$ -space, B .

Let $p : F^{u+m} \rightarrow F^u$ be the natural projection given by taking the first u coordinates. Then there exists B which is special with respect to p (technically : $p|_B$).

Proof. Let $v = v(t, u)$ be the number of t -spaces of a u -space and $c = r^v$. Let $m = \text{HJ}(|F|^{u+1}, c)$; then m is such that if S is a set of cardinal $|F|^{u+1}$ and S^m is coloured with c colours then S^m contains a monochromatic line.

Let χ be an r -colouring of $\begin{bmatrix} F^{u+m} \\ t \end{bmatrix}$,

$$\chi : \begin{bmatrix} F^{u+m} \\ t \end{bmatrix} \rightarrow [r].$$

Let $p : F^{u+m} \rightarrow F^u$ be the natural projection of F^{u+m} onto F^u given by taking the first u coordinates.

Define $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_u \in F^u$ by $\bar{e}_0 = \bar{0}$ and

$\bar{e}_i = (0, \dots, 1, \dots, 0)$ where the '1' is in the i^{th} position,

$1 \leq i \leq u$. Then the \bar{e}_i , $1 \leq i \leq u$, are the basis vectors of F^u

and \bar{e}_0 is the zero vector. Set $A_i = p^{-1}(\bar{e}_i) \subset F^{u+m}$, $0 \leq i \leq u$. In

order to see more clearly what A_i looks like, consider $\bar{m} \in F^m$ then

$\bar{e}_i \bar{m} \in A_i$. Thus A_i is just a copy of F^m and consists of all vectors

"starting" with \bar{e}_i .

Let $(\bar{x}_0, \dots, \bar{x}_u)$ be a $(u+1)$ -tuple, $\bar{x}_i \in F^m$, $0 \leq i \leq u$.

Set $\bar{y}_i = \bar{e}_i \bar{x}_i \in A_i$, $0 \leq i \leq u$, (i.e., we place \bar{e}_i to the left of

\bar{x}_i forming a $u+m$ -tuple \bar{y}_i which is in A_i .) Technically, \bar{y}_i is

defined by $\bar{y}_i \in A_i$ and $p_1(\bar{y}_i) = \bar{x}_i$, $0 \leq i \leq u$, where p_1 is the

natural projection of F^{u+m} onto F^m given by taking the last m coordinates.

The $\{\bar{y}_i\}_0^u$ are a set of $(u+1)$ linearly independent vectors or a set of u linearly independent vectors and the zero vector. To

see this, let $\sum_0^u c_i \bar{y}_i = \bar{0}$; then $(c_1, \dots, c_u, \bar{x}) = \bar{0}$. Thus $c_i = 0$, $\bar{x} \in F^m$

$1 \leq i \leq u$ which implies that $c_0 = 0$ or $\bar{y}_0 = \bar{0}$. In the latter case, the $\{\bar{y}_i\}_0^u$ generate a vector subspace, $X \subset F^{u+m}$, of dimension u .

We note that a vector subspace is simply an affine space which contains the zero vector. In the former case, from Lemma 4.1, the $\{\bar{y}_i\}_0^u$ generate

a unique u -space $X \subset F^{u+m}$. Let $X = W + \bar{y}_0$ where W is a u -dimensional vector subspace generated by $\{\bar{y}_i - \bar{y}_0\}_1^u$. Then

$p(X) = p(W) + p(\bar{y}_0) = p(W)$. Since $p(\bar{y}_i - \bar{y}_0) = \bar{e}_i$, $1 \leq i \leq u$, then

$p(W)$ contains a basis for F^u and it follows that $p|_X$ is onto.

Since $|X| = |F^u|$, $p|_X$ is one to one. Thus $p|_X : X \rightarrow F^u$ is bijective (i.e., X "transverses" F^u).

Let T_1, \dots, T_v denote the t -spaces of F^u in some pre-assigned order. Let S_i be the unique t -space in X such that $p(S_i) = T_i$, $1 \leq i \leq v$. We will use the expression X is generated by a $(u+1)$ -tuple $(\bar{x}_0, \dots, \bar{x}_u)$ where $\bar{x}_i \in F^m$, $0 \leq i \leq u$.

In fact, X is generated by $(\bar{e}_0 \bar{x}_0, \dots, \bar{e}_u \bar{x}_u)$ where

$\bar{e}_i \bar{x}_i = \bar{y}_i \in A_i \subset F^{u+m}$, $\bar{e}_0 = \bar{0}$ and \bar{e}_i , $1 \leq i \leq u$, are the basis

vectors for F^u . We are going to use the r -colouring χ of the

t -spaces of F^{u+m} to induce a c -colouring $\chi^{(1)}$ of the $(u+1)$ -tuples

of $(F^m)^{u+1}$. Let $\bar{x}, \bar{x}' \in (F^m)^{u+1}$, $\bar{x} = (\bar{x}_0, \dots, \bar{x}_u)$ and $\bar{x}' = (\bar{x}'_0, \dots, \bar{x}'_u)$ and let X and X' be the unique u -spaces generated by \bar{x} and \bar{x}' respectively. Thus X is generated by $\{\bar{e}_i \bar{x}_i\}_0^u$ and X' is generated by $\{\bar{e}_i \bar{x}'_i\}_0^u$. $\{S_i\}_1^v$ and $\{S'_i\}_1^v$ are the t -spaces of X and X' respectively. For each i , $1 \leq i \leq v$, we have $p(S_i) = T_i = p(S'_i)$. We define the c -colouring $\chi^{(1)}$ as follows:

$$\chi^{(1)}[(\bar{x}_0, \dots, \bar{x}_u)] = \chi^{(1)}[(\bar{x}'_0, \dots, \bar{x}'_u)]$$

if and only if,

$$[\chi(S_1), \chi(S_2), \dots, \chi(S_v)] = [\chi(S'_1), \chi(S'_2), \dots, \chi(S'_v)].$$

That is, two $(u+1)$ -tuples in $(F^m)^{u+1}$ are coloured the same by $\chi^{(1)}$ if and only if the u -spaces generated by them are coloured identically by χ (identifying under the projection, p).

We are now going to apply Corollary 3.2 of the Hales-Jewett Theorem [p. 24] to F^m . Since $m = \text{HJ}(|F|^{u+1}, c)$ and $\chi^{(1)}$ is a c -colouring of the ordered $(u+1)$ -tuples $(\bar{x}_0, \dots, \bar{x}_u)$, $\bar{x}_i \in F^m$, $0 \leq i \leq u$, then there exist parallel combinatorial lines $L'_0, \dots, L'_u \subset F^m$ such that $\{(\bar{x}_0, \dots, \bar{x}_u); \bar{x}_i \in L'_i\}$ is $\chi^{(1)}$ monochromatic. A full description of

these lines is given in Appendix A at the end of Chapter 6. Since $|F| = q$, then each line L_i^1 , $0 \leq i \leq u$ consists of q m -tuples and is of the form:

$$L_i^1 = \begin{cases} a_i \dots x_1 \dots b_i \dots x_1 \\ a_i \dots x_j \dots b_i \dots x_j \\ a_i \dots x_q \dots b_i \dots x_q \end{cases}$$

where a_i, b_i are fixed elements in F and x_j , $1 \leq j \leq q$, runs through all the elements of F .

To each m -tuple in L_i^1 , $0 \leq i \leq u$, we are going to attach a "head", $\bar{e}_i \in F^u$, $0 \leq i \leq u$ forming parallel combinatorial lines, L_i , $0 \leq i \leq u$ in F^{u+m} .

$$L_i = \begin{cases} \bar{e}_i a_i \dots x_1 \dots b_i \dots x_1 \\ \bar{e}_i a_i \dots x_j \dots b_i \dots x_j \\ \bar{e}_i a_i \dots x_q \dots b_i \dots x_q \end{cases}$$

$\underbrace{\hspace{10em}}_{\substack{\text{u-tuple} \quad \text{m-tuple} \in F^m \\ \in F^u}} \\ \text{(u+m)-tuple} \in F^{u+m}$

From Lemma 4.2, the $\{L_i\}_0^u$ generate a $(u+1)$ -space, B which we will now show is special with respect to the r -colouring, χ and the projection $p|_B$ which we shall denote by p .

In Appendix B, we illustrate the preceding discussion with a simple example.

Let $T \in \begin{bmatrix} B \\ t \end{bmatrix}$ be a transverse space. Then for some j , $1 \leq j \leq v$, $p(T) = T_j$ where T_j is a t -space in F^u . We extend T to a transverse u -space, $X \subset B$ in the following way. Choose $\bar{x} \in F^u - T_j (= p(T))$ and \bar{x} is linearly independent from T_j . Choose $\bar{y} \in p^{-1}(\bar{x})$ and add to T forming $T^* = (T + \bar{y})$, a transverse $(t+1)$ -affine space. Continue in this way until we have a transverse u -space X .

We now want to show that X is generated by $\{\bar{e}_i \bar{x}_i\}_0^u$ where $\bar{e}_i \bar{x}_i \in L_i$, $0 \leq i \leq u$. To accomplish this it suffices to show that for each i , $0 \leq i \leq u$, $X \cap L_i = \bar{e}_i \bar{x}_i$. For then X is a u -space containing the set of $(u+1)$ linearly independent vectors, and from Corollary 4.1(b) X is generated by $\{\bar{e}_i \bar{x}_i\}_0^u$. Since X is a transverse space, $p|_X : X \rightarrow F^u$ is bijective, thus for each i , $0 \leq i \leq u$, X contains one and only one element of the form $\bar{e}_i \bar{x}_i$, $\bar{x}_i \in F^m$. Now $p : B \rightarrow F^u$ is surjective and since $\dim(B) = u+1$, the dimension of the null space of $p = 1$. It follows then that for a t -space $T \subset B$, $p(T)$ is either a t -space in F^u or a $(t-1)$ -space in F^u . Now each L_i , $0 \leq i \leq u$ is a 1-space in B and $p(L_i) = \bar{e}_i$, a 0-space in F^u . Suppose for some i , $0 \leq i \leq u$, that there is another element in X and therefore in B of the form $\bar{e}_i \bar{z}$, $\bar{z} \in F^u$, which does not belong to L_i . Then from Lemma 4.3 $[L_i + \bar{e}_i \bar{z}]$ is a 2-space and all the vectors in $[L_i + \bar{e}_i \bar{z}]$

begin with \bar{e}_i . Thus $p[L_i + \bar{e}_i \bar{z}_i] = \bar{e}_i$ a 0-space in F^u but we have already shown that if $\dim(T) = 2$, then $\dim(p(T)) = 2$ or 1 . Therefore, it follows that any element in X of the form $\bar{e}_i \bar{x}_i$ also belongs to L_i , $0 \leq i \leq u$ and $(X \cap L_i) = \bar{e}_i \bar{x}_i$. Then X is generated $\{\bar{e}_i \bar{x}_i\}_0^u$. It follows that T is contained in the u -space X generated by $\{\bar{y}_i\}_0^u$ (where $\bar{y}_i = \bar{e}_i \bar{x}_i$, $0 \leq i \leq u$). Hence $\chi(T)$ is the j^{th} coordinate of $\chi^{(1)}[(\bar{x}_0, \dots, \bar{x}_u)]$. Since $\chi^{(1)}$ is constant on these $(\bar{x}_0, \dots, \bar{x}_u)$, $\chi(T)$ does depend only on j as required and thus B is special with respect to the r -colouring χ and the projection, p .

Before proceeding with the proof of the main theorem, we prove the following short lemma which is used in the proof.

Lemma 5.2. Let V be an n -dimensional vector space ($n = u+m$) and $p : V \rightarrow F^u$ be the natural projection given by taking the first u coordinates. Let B be a $(u+1)$ affine subspace of V such that $p|_B : B \rightarrow F^u$ is a surjective projection. Then for each $(t-1)$ -space, $T \subset F^u$, $p|_B^{-1}(T)$ is a t -space and hence is the unique vertical t -space in B which projects onto T .

Proof. Let N be the null space of p . Set $N_B = N \cap B$. N_B consists of all vectors $\bar{x} \in B$ with $p(\bar{x}) = \underbrace{(0, \dots, 0)}_n \in F^u$.

Since $p : B \rightarrow F^u$ is onto, it follows that $N_B \neq \emptyset$. Let

$\bar{c} \in N_B$, then $\bar{c} = \underbrace{(0, \dots, 0)}_{\in F^u} \underbrace{\bar{c}^*}_{\in F^m}$. Since B is a $(u+1)$ affine space,

we can write $B = W + \bar{c}$ where W is a $(u+1)$ vector subspace.

We now show that $\dim(N_B) = 1$. \bar{e}_i is a $(u+m)$ vector with '1' in the i^{th} position and '0's everywhere else. Then $\{\bar{e}_i\}_1^{u+m}$ are the basis vectors for V . Since p projects W onto F^u , there is a basis of W of the form $\{\{\bar{e}_i\}_1^u\}$ and $\bar{f} = (\underbrace{0, \dots, 0}_u, \bar{y}) \in F^m$. Then $N \cap W = F\bar{f}$, a subspace of dimension 1. Then $N_B = (N \cap W) + \bar{c}$ and thus $\dim(N_B) = 1$. We can write $N_B = \{\bar{x}; \bar{x} = (\underbrace{0, 0, \dots, 0}_u, \bar{x}_i), \bar{x}_i \in F^m, 1 \leq i \leq q\}$. (We note that $\{\bar{x}_i\}_1^q$ is a 1-dimensional affine space in F^m .)

Let $T \subset F^u$ and $\dim(T) = t-1$. Let $(\beta_1, \dots, \beta_u) = \bar{\beta} \in T$, then $(\beta_1, \dots, \beta_u, \bar{x}_i) \in B$, $1 \leq i \leq q$. Thus each point in T is $\in F^m$ a 1-space in B . It follows that $\dim(p|_B^{-1}(T)) = k$.

Hence, $p^{-1}(T)$ is the unique vertical t -space in B which projects onto T .

We are now ready to prove Ramsey's Theorem for affine subspaces, Theorem 5.1. In order to do so, we first restate it in an equivalent form more suitable for an inductive proof.

Theorem 5.1¹. Let t, r, k_1, \dots, k_r be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, t, k_1, \dots, k_r)$ depending only on q, r, t, k_1, \dots, k_r with the following property: If V is a vector space over F of dimension at least N and $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \rightarrow [r]$, then for some $i, 1 \leq i \leq r$, there exists $W \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ such that for all $T \in \begin{bmatrix} W \\ t \end{bmatrix}$, $\chi(T) = i$. (That is, there is a k_i -space with all of its t -spaces coloured i).

Before proving Theorem 5.1¹, we establish the equivalence between Theorems 5.1 and 5.1¹.

First we assume that Theorem 5.1¹ is true and set $k_1 = k_2 = \dots = k_r = k$ then Theorem 5.1¹ states:-

Let t, r, k be non-negative integers and F a field of q elements. Then there is a number $N = N(q, r, t, k)$ depending only on q, r, t and k with the following property: If V is a vector space over F of dimension at least N and $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \rightarrow [r]$, then there exists a k -space $W \subset V$ such that all the t -spaces T of W are the same colour.

Thus Theorem 5.1¹ implies Theorem 5.1.

Now we assume that Theorem 5.1 is true. Let t, r, k_1, \dots, k_r be given and set $k = \max\{k_i\}$. Then from Theorem 5.1, there is a

k -space W such that all the t -spaces $T \subset W$ are the same colour, say colour i where $1 \leq i \leq r$. Then we can write that for all $T \in \begin{bmatrix} W \\ t \end{bmatrix}$, $\chi(T) = i$. Since $k = \max\{k_i\}$ we can choose any k_i -space W_i inside W and must still have that for all $T \in \begin{bmatrix} W_i \\ t \end{bmatrix}$, $\chi(T) = i$. Thus Theorem 5.1 implies Theorem 5.1¹ and we have established the equivalence between Theorem 5.1 and 5.1¹. We now proceed with the proof of Theorem 5.1¹.

Proof of Theorem 5.1¹. The proof is by double induction - first on t (for all k_1, \dots, k_r) and then on (k_1, \dots, k_r) . For all t and $k_i = t$, this is trivially true.

For $t = 0$, Theorem 5.1¹ is simply Corollary 3.1 of the Hales-Jewett Theorem and was proved in Chapter 3. We now state it in the same form as Theorem 5.1¹:

Let r, k_1, \dots, k_r , be positive integers and F a field of q elements. Then there is a number $N = N(q, r, k_1, \dots, k_r)$ depending only on q, r, k_1, \dots, k_r with the following property:

If V is a vector space over F of dimension at least N

and $\chi : \begin{bmatrix} V \\ 0 \end{bmatrix} \rightarrow [r]$ then for some $i, 1 \leq i \leq r$ there exists $W \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ such that for all $T \in \begin{bmatrix} W \\ 0 \end{bmatrix}$, $\chi(T) = i$.

(We note that in this case of the theorem k and r are positive integers whereas in the general case t, k and r were non-negative integers. This arises from the fact that $r = 0$ or $k = 0$ give rise to trivial situations.)

We now assume the existence of N for $t' < t$ (all k_1, k_2, \dots, k_r) and for t and all $(k_1^t, \dots, k_r^t) < (k_1, \dots, k_r)$. For (k_1^t, \dots, k_r^t) to be less than (k_1, \dots, k_r) , at least one k_i^t is less than the corresponding k_i , $1 \leq i \leq r$, and for $j \neq i$, $k_j^t \leq k_j$.

Let $s = \max\{N(q, r, t, k_1, \dots, k_{i-1}, \dots, k_r); 1 \leq i \leq r\}$ and $u = N(q, r, t-1, s)$. u is such that if V is a u -dimensional vector space over F (with $|F| = q$) and χ an r -colouring of the $(t-1)$ -spaces of V , then there exists an s -space W such that all the $(t-1)$ -spaces $T \subset W$ are the same colour. Let $m = M(q, t, u, r)$ and let $n = u+m$. We shall now show that n has the desired property. We identify the n -dimensional vector space V with F^{u+m} . Let $\chi : \begin{bmatrix} F^{u+m} \\ t \end{bmatrix} \rightarrow [r]$ be arbitrary and let $p : F^{u+m} \rightarrow F^u$ be the natural projection given by taking the first u coordinates. By the definition of m , there exists a $(u+1)$ -space B which is special with respect to the colouring χ and the projection $p|_B$. Henceforth, p refers to the projection with domain B .

We use χ to induce an r -colouring $\chi^{(1)}$ of the $(t-1)$ -spaces of F^u . Let $T \in \begin{bmatrix} F^u \\ t-1 \end{bmatrix}$, then from Lemma 5.2, $p^{-1}(T) \subset B$ is the unique vertical t -space which projects onto T . Then we can define $\chi^{(1)}$ as follows:-

$$\chi^{(1)}(T) = \chi(p^{-1}(T)) .$$

(That is a $(t-1)$ -space, $T \subset F^u$ is coloured by the colour of the vertical t -space, $p^{-1}(T) \subset B$ which projects onto it.)

By the definition of u (i.e., induction on t), there exists $X \in \begin{bmatrix} F^u \\ s \end{bmatrix}$ which is monochromatic under $\chi^{(1)}$, that is all the $(t-1)$ -spaces of the s -space X are of the same colour. We will call this colour 1. Then from Lemma 5.2, $p^{-1}(X)$ is an $(s+1)$ -space in B .

Since B is special and $p^{-1}(X) \subset B$, then $p^{-1}(X)$ is also special. Further, since all the $(t-1)$ -spaces of X are coloured 1 by $\chi^{(1)}$ then from the definition of $\chi^{(1)}$, all the vertical t -spaces of $p^{-1}(X)$ are coloured 1 by χ .

We now use the r -colouring χ to induce an r -colouring $\chi^{(2)}$ of the t -spaces of X . $\chi^{(2)}$ is defined as follows:

$$\chi^{(2)}(T) = \chi(T_1) \text{ where } T \in \begin{bmatrix} X \\ t \end{bmatrix} \text{ and } T_1 \in \begin{bmatrix} p^{-1}(X) \\ t \end{bmatrix} \text{ with } p(T_1) = T.$$

As already shown, $p^{-1}(X)$ is an $(s+1)$ -space while X is an s -space

so that $p|_{p^{-1}(X)} : p^{-1}(X) \rightarrow X$ is not a one to one projection. So

there are many T_1 with $p(T_1) = T$; however, all such T_1 have the same colour since $p^{-1}(X)$ is special. (That is, the colouring $\chi^{(2)}$ is produced by projecting the colour of the transverse t -spaces $T_1 \subset p^{-1}(X)$ down onto the t -spaces of X .)

Now s was chosen such that $s \geq N(q, r, t, k_1, \dots, k_r)$ and from the induction hypothesis on (k_1, \dots, k_r) , there exists $W_1 \subset X$ so that either

(i) $\dim(W_1) = k_1 - 1$ and $\begin{bmatrix} W_1 \\ t \end{bmatrix}$ is coloured 1 under $\chi^{(2)}$

or

(ii) $2 \leq j \leq r$, $\dim(W_1) = k_j$ and $\begin{bmatrix} W_1 \\ t \end{bmatrix}$ is coloured j under $\chi^{(2)}$.

In case (ii) suppose that W_1 is generated by $\{\bar{w}_i\}_0^{k_j}$ where $\bar{w}_i = (w_{i1}, \dots, w_{iu})$, $0 \leq i \leq k_j$. Set $\bar{w}_i^* = (w_{i1}, \dots, w_{iu}, 0)$, $0 \leq i \leq k_j$. Then $\{\bar{w}_i^*\}_0^{k_j}$ generate a transverse k_j -space $W \subset p^{-1}(X)$ so that $p(W) = W_1$. (That is we can lift W_1 to W .) Then $\begin{bmatrix} W \\ t \end{bmatrix}$ is coloured j under χ . (This comes automatically from the definition of $\chi^{(2)}$.)

Case (i) is the most important case since it is the moment of induction. We set $W = p^{-1}(W_1)$. From Lemma 5.2, $p^{-1}(W_1)$ is a $(k_1 - 1) + 1 = k_1$ -space in B . Let T be a t -space of W . Then T is either a transverse space or a vertical space. If T is a transverse space, $\chi(T) = \chi^{(2)}(p(T)) = 1$ as $p(T) \in \begin{bmatrix} W_1 \\ t \end{bmatrix}$. If T is a vertical space, it is a vertical t -space of $p^{-1}(X)$ and all the vertical t -spaces of $p^{-1}(X)$ are coloured 1. (This comes from the definition of $\chi^{(1)}$ and the induction on t .) Hence $\chi(T) = 1$.

This completes the proof of Theorem 5.1. To review: inside an arbitrary n -dimensional vector space, we find a special $(u+1)$ -space, B . Inside B , we find a $(s+1)$ -space $p^{-1}(X)$ which is special and such that all of its vertical t -spaces are coloured 1. Inside $p^{-1}(X)$, we find (in case (i) a $((k_1 - 1) + 1) = k_1$ -space, W all of whose transverse t -spaces are coloured 1. Then $\begin{bmatrix} W \\ t \end{bmatrix}$ is coloured 1. We will now outline the preceding proof in diagrammatic form. In each

diagram, $r = 3$, (i.e., we use these colours) and represent transverse spaces by horizontal lines and vertical spaces by vertical lines. With each diagram, we give a brief written description.

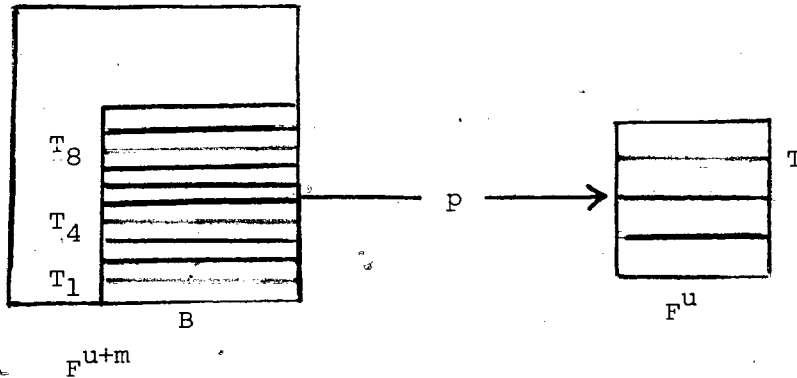


Figure 5.1

In Figure 5.1, χ is a 3-colouring of the t-spaces of F^{u+m} and B is special with respect to χ and the natural projection $p : F^{u+m} \rightarrow F^u$. This means, for example, that $p(T_1) = p(T_4) = p(T_8) = T$. Note that although the t-spaces of F^u are coloured in Figure 1, they are not actually coloured by χ . We have coloured them in order to illustrate the concept "special".

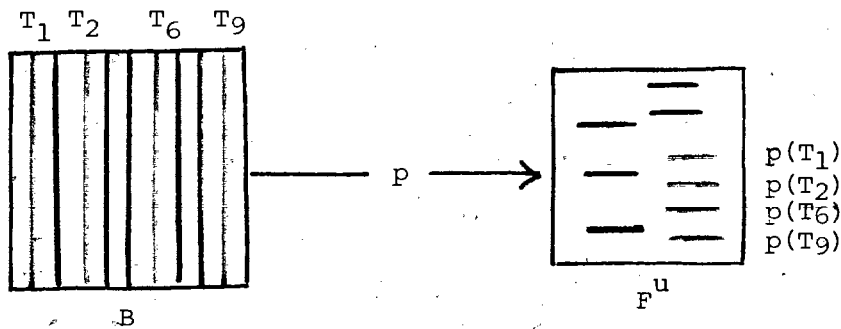


Figure 5.2

Figure 5.2 illustrates how the 3-colouring $\chi^{(1)}$ is formed from χ . $\chi^{(1)}$ assigns to each $(t-1)$ -space, $T \subset F^u$, the χ -colour of the vertical t -space, T_1 which projects onto it.

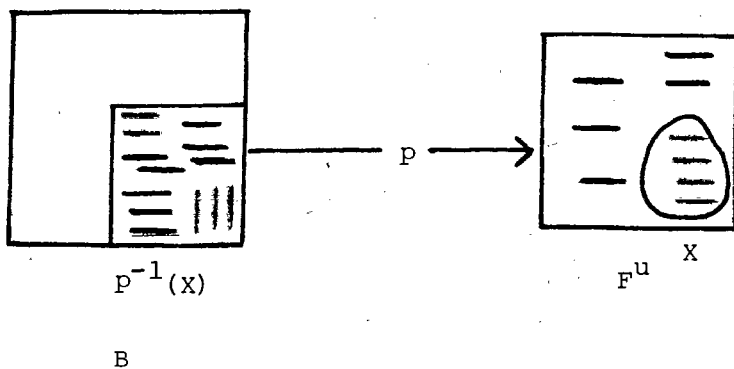


Figure 5.3

Figure 5.3 shows the monochromatic s -space $X \subset F^u$. From the definition of u , F^u contains a s -space X which has all of its $(t-1)$ -spaces coloured the same colour (i.e., orange) by $\chi^{(1)}$. From the definition of $\chi^{(1)}$, it follows that $p^{-1}(X)$ is a $(s+1)$ -space in B which has all of its vertical t -spaces coloured orange by χ .

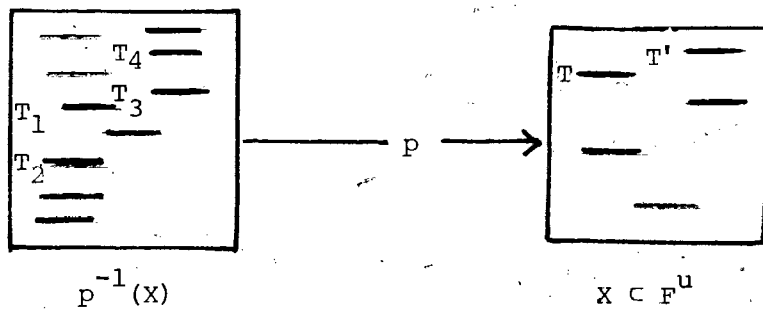


Figure 5.4

Figure 5.4 illustrates the 3-colouring $\chi^{(2)}$. Since $p^{-1}(X) \subset B$ and B is special with respect to χ and p , then $p^{-1}(X)$ is also special. This means that the χ -colour of transverse t -spaces $T_1, T_2 \subset X$ is the same if $p(T_1) = p(T_2)$. We can then use χ to define the 3-colouring $\chi^{(2)}$ of the t -spaces of X . For example, $p(T_1) = p(T_3) = T \subset X$ is coloured green and $p(T_2) = p(T_4) = T'$ is coloured blue by $\chi^{(2)}$.

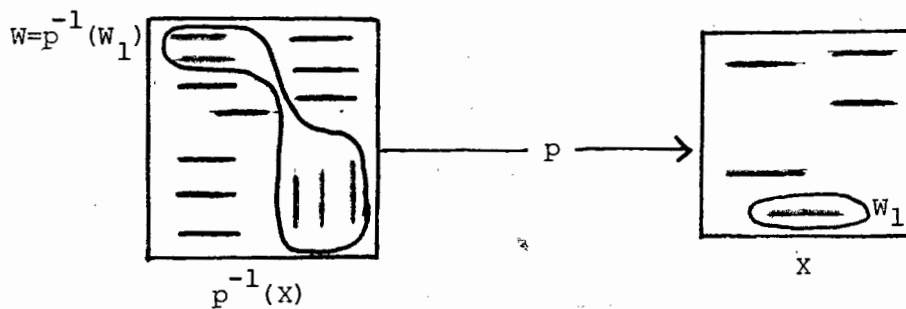


Figure 5.5(i)

Figure 5.5(i) illustrates case (i). W_1 is a $(k_1 - 1)$ -space in X and all the t -spaces of W_1 are coloured orange by $\chi^{(2)}$. We now consider $p^{-1}(W_1) = W$ which is a vertical k_1 -space in B . From the definition of $\chi^{(2)}$, all the transverse t -spaces of W are coloured orange. If T is any vertical t -spaces in W_1 then T is also a vertical t -space in $p^{-1}(X)$ and all the vertical t -spaces in $p^{-1}(X)$ are coloured orange.

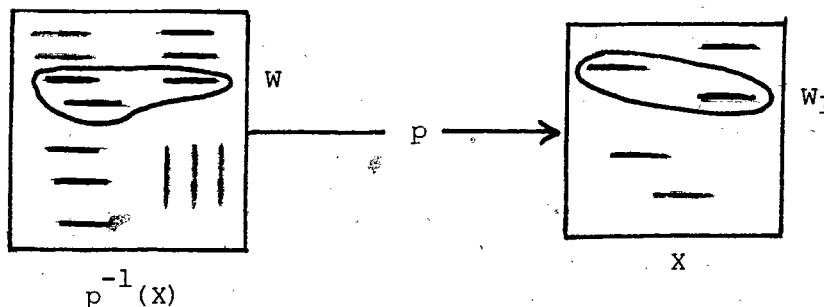


Figure 5.5(ii)

Figure 5.5(ii) illustrates case (ii). W_1 is a k_2 -space in X with all of its t -spaces coloured green (colour 2). Then as already explained we can find a transverse k_2 -space, W in $p^{-1}(X)$ and thus in B such that $p(W) = W_1$. Then if T is a t -space in W , $p(T) \subset W_1$ and $p(T)$ is coloured green by $\chi^{(2)}$. From the definition of $\chi^{(2)}$, T must also be coloured green. Thus W is a k_2 -space in B such that $\begin{bmatrix} W \\ t \end{bmatrix}$ is monochromatic.

Corollary 5.1. (Ramsey's Theorem for projective spaces).

Theorem 5.1 holds where "space" refers to projective space.

Corollary 5.2. (Ramsey's Theorem for vector subspaces).

Theorem 5.1 holds where "space" refers to vector subspace.

First, we explain the canonical association between vector $(t+1)$ -space and projective t -spaces. Because of this canonical association, the above corollaries are equivalent. Then we will prove Corollary 5.2. The proof is very short.

Let V be an $(n+1)$ -dimensional vector space over a field F of q elements. We will denote the points (vectors) of V by the $(n+1)$ -tuples $(x_1, x_2, \dots, x_{n+1})$, $x_i \in F$, $1 \leq i \leq n+1$. For a point of the projective space, P , we take the set consisting of a non-zero point \bar{v} in V and all nonzero scalar multiples of \bar{v} . This construction gives

$$\frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + \dots + q + 1 \text{ points.}$$

For a line, L , in P let $\bar{v}_1 = (x_1, x_2, \dots, x_{n+1})$ and $\bar{v}_2 = (y_1, y_2, \dots, y_{n+1})$, where \bar{v}_1, \bar{v}_2 are linearly independent non-zero points in V . Then $L = \{(a\bar{v}_1 + b\bar{v}_2) = (ax_1 + by_1, ax_2 + by_2, \dots, ax_{n+1} + by_{n+1})\}$; $a, b \in F$, a and b are not both equal to zero.)

There are $q^2 - 1$ possible choices for the part (a, b) ; but since we identify scalar multiples, there are $(q^2 - 1)/(q - 1) = q + 1$ points on a line and hence by the principle of duality for projective spaces, each point has $q + 1$ lines intersecting at it. Briefly, a point in P is a line in V and a line in P is a plane in V . For higher dimensional subspaces a u -space in P is a $u+1$ -space in V . Thus an n -dimensional projective space P is equivalent to an $(n+1)$ -dimensional vector space V , and hence Corollaries 5.1 and 5.2 are equivalent.

Proof of Corollary 5.2. Let χ be a r -colouring of the t -dimensional vector subspaces of V , $\chi : \begin{bmatrix} V \\ t \end{bmatrix} \rightarrow [r]$. We induce

an r -colouring $\chi^{(1)}$ of the affine t -spaces of V as follows. Let A be an affine t -space in V , then for some unique vector t -space T and vector \bar{c} in V , $A = T + \bar{c}$. Then colour A with the same colour as χ colours T .

$$\chi^{(1)}(A) = \chi(T) .$$

From Theorem 5.1, there exists an affine k -space B such that all of the affine t -spaces in B are coloured red by $\chi^{(1)}$. Now $B = X + \bar{d}$ where X is vector k -space in V and \bar{d} a vector in V . If T is any vector t -space in X , then $T + \bar{d}$ is an affine t -space in B and is coloured red by $\chi^{(1)}$. Then from the definition of $\chi^{(1)}$, T is coloured red by χ . Thus all the vector t -spaces in X are coloured red by χ and Corollary 5.2 is proved.

CHAPTER 6

PARAMETER SYSTEMS AND THE RAMSEY THEOREMS

Since Ramsey's Theorem (Theorem 1.3) appeared, there has been interest in finding generalizations, applications and analogues. It is this interest which motivated the work of this chapter. In Chapter 5, we proved Ramsey's Theorem for spaces using the Hales-Jewett Theorem. We can use the ideas in this proof to prove another analogue, Ramsey's Theorem for parameter systems. Before we define parameter systems, we will give an informal description of the n -parameter sets of Graham-Rothschild [3]. These n -parameter sets are generalized by parameter systems. In [3], R.L. Graham and B.L. Rothschild proved Ramsey's Theorem for n -parameter sets. Among the immediate corollaries of this theorem are Ramsey's Theorem for spaces for the special cases $t = 0$ and 1 and Ramsey's Theorem itself. Other corollaries are listed in Appendix C.

It will be seen that in general the affine subspaces of a vector space do not correspond to parameter subsets whereas affine subspaces are an example of a parameter system. **The advantage, then, of parameter systems is that they include both n -parameter sets and the affine spaces of a vector space.**

n -parameter sets. Basically, just as an n -dimensional vector space, V as a set consists of all q^n n -tuples of elements from a field F where $|F| = q$, so an n -parameter set essentially consists of all t^n n -tuples

of elements of a set A with t -elements, $A = \{a_1, \dots, a_t\}$. Any affine line, L (i.e., a 1-dimensional affine space) of V is a set of the form, $L = \bar{y} + \{f\bar{x}, f \in F\}$ where $\bar{x}, \bar{y} \in V$ and $\bar{x} \neq 0$. L consists of q n -tuples which can be written in a column as

$$\begin{pmatrix} x_{11}, \dots, x_{1n} \\ x_{21}, \dots, x_{2n} \\ \vdots \\ x_{q1}, \dots, x_{qn} \end{pmatrix}$$

where for each i , $1 \leq i \leq n$, either $x_{1i} = x_{2i} = \dots = x_{qi}$, or else x_{1i}, \dots, x_{qi} is a permutation of the elements f_1, \dots, f_q constituting F . The permutations obtainable in this way constitute a subset K of all the $q!$ possible permutations of f_1, \dots, f_q . K will consist of the $q-1$ permutations of the form $\{\alpha f_i, \alpha \in F\}$ for each non-zero f_i in F . For example, if $|F| = 5$ then $K = \{(0,1,2,3,4), (0,2,4,1,3), (0,3,1,4,2), (0,4,3,2,1)\}$.

In a similar way, then, we define a 1-parameter subset of A^n (the n -tuples of A) as any set of t n -tuples which can be listed

$$\begin{pmatrix} a_{11}, \dots, a_{1n} \\ \vdots \\ a_{t1}, \dots, a_{tn} \end{pmatrix}$$

such that for each i , $1 \leq i \leq n$, either $a_{1i} = a_{2i} = \dots = a_{ti} \in B \subseteq A$, or else a_{1i}, \dots, a_{ti} is one of a certain set K_H of permutations of a_1, \dots, a_t (the set of permutations considered is defined by a permutation group H acting on A .)

The general idea for k -parameter subsets can be illustrated by considering the case $k = 2$. For $k \geq 2$, only certain special k -dimensional affine subspaces correspond to k -parameter subsets. If B_2 is any 2-dimensional affine subspace of V then

$$B_2 = \{\bar{x} + \alpha\bar{y} + \beta\bar{z}; \alpha, \beta \in F\} \text{ where } \bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_n)$$

and $\bar{z} = (z_1, \dots, z_n)$ are in V and \bar{y}, \bar{z} are linearly independent.

Addition and scalar multiplication are defined as usual. We will now examine a specific example. This will enable us to describe the special affine subspaces in which we are interested. Let $F = \{0, 1, 2\}$ and $n = 4$. Consider the following 2-dimensional affine subspaces in the 4-dimensional vector space V .

$$B_2^{(1)} = \{(1, 1, 1, 1) + \alpha(1, 0, 0, 0) + \beta(0, 1, 0, 0); \alpha, \beta \in F\}$$

$$\text{and } B_2^{(2)} = \{(1, 1, 1, 1) + \alpha(0, 0, 1, 1) + \beta(1, 2, 2, 0); \alpha, \beta \in F\}.$$

We observe that in $B_2^{(1)}$, $\bar{y}^{(1)} = (1, 0, 0, 0)$, $\bar{z}^{(1)} = (0, 1, 0, 0)$

and $y_i^{(1)} z_i^{(1)} = 0$ for $1 \leq i \leq 4$ whereas in $B_2^{(2)}$, we have

$\bar{y}^{(2)} = (0, 0, 1, 1)$, $\bar{z}^{(2)} = (1, 2, 2, 0)$ and $y_3^{(2)} z_3^{(2)} \neq 0$. Further,

although we have several choices for $\bar{y}^{(2)}$ and $\bar{z}^{(2)}$, we always find that for all choices there is a value of i , $1 \leq i \leq 4$ for which $y_i^{(2)} z_i^{(2)} \neq 0$. We are interested in affine subspaces which have the property exhibited by $B_2^{(1)}$; i.e., affine subspaces

$B_2 = \{\bar{x} + \alpha\bar{y} + \beta\bar{z} ; \alpha, \beta \in F\}$ and $y_i z_i = 0, 1 \leq i \leq n$. In such cases,

we can partition the n coordinates into three disjoint sets: the coordinates i where $z_i = 0$ but $y_i \neq 0$, those where $z_i \neq 0$ but $y_i = 0$ and those where $z_i = y_i = 0$. We will call these sets,

S_1, S_2 and S_0 respectively and let $S_1 = \{i_1, \dots, i_{n_1}\}$,

$S_2 = \{j_1, \dots, j_{n_2}\}$, $S_0 = \{k_1, \dots, k_{n_0}\}$. If $\bar{v} = (v_1, \dots, v_2) \in B_2$,

then there are only q possibilities for $(v_{i_1}, \dots, v_{i_{n_1}})$, q

possibilities for $(v_{j_1}, \dots, v_{j_{n_2}})$ and one possibility for $(v_{k_1}, \dots, v_{k_{n_0}})$

Hence B_2 can be formed precisely by listing the q values for each of

S_1 and S_2 and the one value from S_0 , q times and then selecting one

value from each of the lists in all q^2 possible ways:

$$\begin{array}{ccc}
 S_0 & & S_1 & & S_2 \\
 (x_{k_1}, \dots, x_{k_{n_0}}) & & (y_{i_1}, \dots, y_{i_{n_1}}) & & (z_{j_1}, \dots, z_{j_{n_2}}) \\
 \vdots & & \vdots & & \vdots \\
 (x_{k_1}, \dots, x_{k_{n_0}}) & & (y_{q_1}, \dots, y_{q_{n_1}}) & & (z_{q_1}, \dots, z_{q_{n_2}})
 \end{array}$$

The possible columns $(y_{1i_\ell}, \dots, y_{q_i i_\ell})$, $1 \leq \ell \leq n_1$ and $(z_{1j_m}, \dots, z_{q_j j_m})$, $1 \leq m \leq n_2$ are just the same as the set K of permutations in the 1-dimensional case which was discussed previously.

Returning to our example, $B_2^{(1)} = \{(1,1,1,1) + \alpha(1,0,0,0) + \beta(0,1,0,0); \alpha, \beta \in \mathbb{F}\}$, we have $S_0 = \{3,4\}$, $S_1 = \{1\}$ and $S_2 = \{2\}$.

Our three lists are: -

S_0	S_1	S_2
(1,1)	(0)	(0)
(1,1)	(1)	(1)
(1,1)	(2)	(2)

From this we can find the 3^2 elements of $B_2^{(1)}$ which are $\{0011, 0111, 1011, 1100, 0211, 2011, 1211, 2111, 2211\}$.

2-parameter sets are described in a similar way. For a set A and a subset K_H of the permutations of A , we form a 2-parameter subset A^n as follows: First partition the set $\{1, \dots, n\}$ into three disjoint subsets S_0, S_1, S_2 with S_1 and S_2 non-empty. Then write three lists

S_0	S_1	S_2
(a, \dots, b)	(x, \dots, x')	(z, \dots, z')
\vdots	\vdots	\vdots
(a, \dots, b)	(y, \dots, y')	(w, \dots, w')

such that the columns under S_1 and S_2 are in K_H . All t^2 elements of the 2-parameter subset are obtained by taking one row from each list.

To obtain k -parameter subsets, we do the same thing with partitions into $k+1$ subsets S_0, \dots, S_k . For $k \geq 2$, we have seen that these correspond to special affine subspaces of an n -dimensional vector space over a field F .

Now we are going to define parameter systems. These generalize the k -parameter sets of Graham-Rothschild which have just been described. All the affine subspaces of a vector space also form a parameter system whereas only certain special affine subspaces correspond to k -parameter sets.

Let A be a finite set and $F = \bigcup_{i=1}^n F_i$, where F_i is a family of functions $f: A^i \rightarrow A$. If A is a finite field, then F_i is the family of affine linear functions f such that

$$f(x_1, \dots, x_i) = c + \sum_{j=1}^i a_j x_j \text{ where } c, a_j \in A, 1 \leq j \leq i. \text{ We fix}$$

A and F throughout. A subset $S \subset A^n$ is called a t -space if there exists $J = \{j_1, \dots, j_t\} \subset [n]$ and for $i \notin J$, functions $f_i \in F_t$ so that

$$S = \{(x_1, \dots, x_n); x_i = f_i(x_{j_1}, \dots, x_{j_t}), i \notin J\}.$$

To see clearly what S looks like we will examine the two 2-dimensional affine subspaces $B_2^{(1)}$ and $B_2^{(2)}$ already described.

$$B_2^{(1)} = \{(1,1,1,1) + \alpha(1,0,0,0) + \beta(0,1,0,0); \alpha, \beta \in F\}$$

$$B_2^{(2)} = \{(1,1,1,1) + \alpha(0,0,1,1) + \beta(1,2,2,0); \alpha, \beta \in F\}$$

and $F = \{0,1,2\}$.

$$\text{For } B_2^{(1)}, J = \{1,2\}, f_3 = 1 + \sum_{j=1}^2 0x_j, f_4 = 1 + \sum_{j=1}^2 0x_j \text{ and}$$

we can write $B_2^{(1)}$ as $\{(x_1, x_2, x_3, x_4); x_1, x_2 \in F, x_3 = 1 + \sum_{j=1}^2 0x_j,$

$$x_4 = 1 + \sum_{j=1}^2 0x_j\}.$$

$$\text{For } B_2^{(2)}, J = \{3,4\}, f_1(x_3, x_4) = 1 + 2x_3 + x_4, f_2(x_3, x_4)$$

$$= 1 + x_3 + 2x_4. B_2^{(2)} = \{(x_1, x_3, x_3, x_4); x_3, x_4 \in F, x_1 = 1 + 2x_3 + x_4,$$

$$x_2 = 1 + x_3 + 2x_4\}.$$

We call J a basis for S . J is not unique with respect to S . In the above example, $J = \{2,3\}$ is also a basis for $B_2^{(2)}$

with $x_1 = f_1(x_2, x_3) = 2 + 2x_2$ and $x_4 = f_4(x_2, x_3) = 1 + 2x_2 + x_3$.

However, if J_1 and J_2 are both bases for S then $|J_1| = |J_2| = t$

and we call t the dimension of S and write $\dim(S) = t$. The singleton subsets of A^n are called 0-spaces.

For any distinct $j_1, \dots, j_t \in [n]$, we define

$$p_{j_1 j_2 \dots j_t} : A^n \rightarrow A^t \text{ by } p_{j_1 j_2 \dots j_t}(x_1, \dots, x_n) = (x_{j_1}, \dots, x_{j_t}).$$

As order will be unimportant, we write p_J for $p_{j_1 j_2 \dots j_t}$,

$J = \{j_1, \dots, j_t\}$. We call $p : A^n \rightarrow A^t$ a projection if $p = p_J$ for

some J . If $S \subset A^n$, we write $p_J|_S : S \rightarrow A^{|J|}$ for the restriction

$p_J|_S$.

We say that (A, \mathcal{F}) is Ramsey if for all $t \geq 0$, r, k there exists $n = n(t, r, k)$ with the following property. If V is an n -space and the t -spaces of V are r -coloured there exists a k -space W all of whose t -spaces are the same colour.

We call (A, \mathcal{F}) a parameter system if it satisfies the following six axioms.

(A₁) Constants: For all $a \in A$ and for all m , the constant function $f(x_1, \dots, x_m) = a$ is in \mathcal{F}_m . (A generalization (A'₁) is given at the end of the chapter.)

(A₂) Identity: \mathcal{F}_1 contains the identity function $f(x) = x$.

(A₃) Extension: If $f \in \mathcal{F}_u$ and $p : A^n \rightarrow A^u$ is a projection then $f' = f_p \in \mathcal{F}_n$. (E.g., If $f \in \mathcal{F}_2$ then $f' \in \mathcal{F}_3$ where $f'(x, y, z) = f(x, z)$.)

(A₄) Composition: If $f_1, \dots, f_s \in F_u$, $f \in F_s$ then $f' \in F_u$ where
 $f'(x_1, \dots, x_u) = f(f_1(x_1, \dots, x_u) \dots f_s(x_1, \dots, x_u))$.

(A₅) Basis: If $S \subset A^n$, $\dim(S) = t$, $J \subset [n]$ and $p_J : S \rightarrow A^{|J|}$ is
 bijective then J is a basis for S .

(A₆) Projection: If $S \subset A^n$ is a subspace and $J \subset [n]$ then $p_J(S)$
 is a subspace of $A^{|J|}$.

Theorem 6.1. Parameter systems are Ramsey.

Of course axioms (A₁) - (A₆) have been chosen so that
 Theorem 6.1 will hold. Before proving Theorem 6.1, we will derive
 some elementary corollaries of (A₁) - (A₆). Let S be a t -space
 in A^n and $I \subset [n]$. We say that I is a spanning set for S if
 the projection $p_I : S \rightarrow A^{|I|}$ is injective and that I is independent
 (for S) if the projection $p_I : S \rightarrow A^{|I|}$ is surjective. From (A₅),
 I is a basis iff it is independent and a spanning set.

(A₇) If I is a spanning set for S there exists a basis $J \subset I$.

Proof. By (A₆), $p_I(S)$ is a subspace of $A^{|I|}$. Since I is a
 spanning set, then $p_I : S \rightarrow A^{|I|}$ is an injective projection and $p_I(S)$
 must be a t -space. Then there is a bijective projection $p : p_I(S) \rightarrow A^t$.
 From this it follows that $pp_I : S \rightarrow A^t$ is bijective and $pp_I = p_J$
 for some $J \subset I$. By (A₅), J is a basis for S .

(A₈) If I is independent for S , there exists a basis $J \subseteq I$.

Proof. Let J be a maximal independent set, $J \supseteq I$. By this we mean that $p_J : S \rightarrow A^{|J|}$ is surjective whereas for any $u, u \notin J$ and $J \cup \{u\} \subseteq [n]$, $p_{J \cup \{u\}} : S \rightarrow A^{|J|+1}$ is not surjective. If $p_J : S \rightarrow A^{|J|}$ is not bijective then it is not injective and we can find $x, y \in S$ with $p_j(x) = p_j(y)$ for $j \in J$ but $p_u(x) \neq p_u(y)$ for some $u \notin J$. Let $J^* = J \cup \{u\}$ then $p_{J^*} : S \rightarrow A^{|J|+1}$ has an image with more than $|A^{|J|}|$ elements. From (A_6) , the image $p_{J^*}(S)$, is a space, thus it must be all of $A^{|J|+1}$, contradicting the maximality of J . Thus $p_J : S \rightarrow A^{|J|}$ is bijective and J is a basis for S .

(A_9) . If I is a basis for S , the map p_I is a bijection such that it and its inverse preserve subspaces and their dimensions.

Proof. p_I is bijective by definition and sends spaces into spaces by (A_6) . If U is a subspace of $A^{|I|}$ we may by (A_4) , express $p_I^{-1}(U)$ as a subspace of S . In both cases dimension is preserved as cardinality is.

(A_9) says, essentially, that all n -spaces (A, F) fixed are isomorphic.

We have established that n -parameter systems have properties which parallel those of the affine subspaces of a vector space.

To prove Theorem 6.1, we follow the same lines as that of Theorem 5.1. Suppose $\dim(B) = u+1$ and $p : B \rightarrow A^u$ is a surjective

projection. Let $T \subset B$, $\dim(T) = t$. Then $p : T \rightarrow A^u$ is either bijective or it is not. If it is, we call T transverse (relative to p) and T has a basis $I \subset [u]$. If it is not, we call T vertical (relative to p). In the latter case, as $p(T)$ is a space and the inverse image of $x \in A^u$ under $p|_T$ (i.e., $p^{-1}(x)$) has at most $|A|$ points (since $p^{-1}(x) \in B$ and $\dim(B) = u+1$) then $p(T)$ must be $(t-1)$ -space, $p|_T$ is an $|A|$ -to-1 function and $T = p^{-1}(p(T))$. Now let χ be a colouring of the t -spaces of B . We call B special (relative to χ, p) if wherever $T_1, T_2 \subset B$, both transverse t -spaces with $p(T_1) = p(T_2)$ then $\chi(T_1) = \chi(T_2)$.

Lemma 6.1. Let t, u, r be non-negative integers and (A, F) a parameter system. Then there exists a number $m = M(t, u, r)$ (dependent on (A, F) of course) such that for any r -colouring χ of the t -spaces of A^{u+m} there exists a $(u+1)$ -space B special relative to χ and the projection $p : A^{u+m} \rightarrow A^u$ onto the first u coordinates.

Proof. Let v be the number of t -spaces in a u -space. We prove Lemma 6.1 for $m = HJ(|F_u|, r^v)$. Let $T \subset A^u$, $\dim(T) = t$,

$f_1, \dots, f_m \in F^u$. We define $(T, f_1, \dots, f_m) \subset A^{u+m}$ by

$$(T, f_1, \dots, f_m) = \{(x_1, \dots, x_u, y_1, \dots, y_m); (x_1, \dots, x_u) \in T, y_i = f_i(x_1, \dots, x_u)\}.$$

Let J be a basis for T , then $T = \{(x_1, \dots, x_u);$

$x_k = f_k(x_j, \dots, x_{j_m}), k \in J\}$ where $f_k \in F_t$. It follows from (A_4)

that there are functions $f_i \in F_t$ such that $(T, f_1, \dots, f_m) =$

$\{(x_1, \dots, x_u, x_{u+1}, \dots, x_{u+m}); x_i = f_i(x_{j_1}, \dots, x_{j_t}), i \in J\}$ and thus J

is also a basis for (T, f_1, \dots, f_m) so it is a transverse t -space in

A^{u+m} . Conversely, let T' be a transverse t -space in A^{u+m} . From

the definition of a transverse t -space, the projection $p : T' \rightarrow A^u$

is bijective and thus it follows from the definition of a spanning set

that $[u]$ is a spanning set for T' . Then by (A_7) , T' has a basis

$J \subset [u]$. Let $\bar{x} \in T'$, $\bar{x} = (x_1, \dots, x_u, y_1, \dots, y_m)$. Since T' has a

basis $J \subset [u]$, all y_i may be written as functions f_i of the basis

variables, i.e., $y_i = f_i(x_{j_1}, \dots, x_{j_t})$. By (A_3) , we can extend f_i

to f_i' such that $y_i = f_i'(x_1, \dots, x_u)$ so $T' = (T, f_1', \dots, f_m')$. (In

general, T' will not have a unique expression in this form.)

Now, the critical step, we induce a colouring

$$\chi^* : (F_u)^m \rightarrow [r^v]$$

setting $\chi^*(f_1, \dots, f_m) = \chi^*(f_1', \dots, f_m')$ iff for all $T \subset A^u$,

$\dim(T) = t$, $\chi((T, f_1, \dots, f_m)) = \chi((T, f_1', \dots, f_m'))$. To see clearly

what is happening in this induced colouring we will consider the

following simple case. Let $u = 3$, $m = 2$, $t = 0$, $|A| = 2$, $r = 2$. Then

v , the number of points in A^3 is 8, χ is a 2-colouring of the points of A^{3+2} and we use it to induce a 2^8 -colouring, χ^* of $(F_3)^2$.

For all $x \in A^3$, $x = (x_1, x_2, x_3)$ and $f \in F_3$, $f(x) \in A$. Let

$f_1, f_2, f_3, f_4 \in F_3$, then $(x_1, x_2, x_3, f_1(x), f_2(x))$ and

$(x_1, x_2, x_3, f_2(x), f_3(x)) \in A^{3+2}$. Then χ^* colours (f_1, f_2) the same

as (f_3, f_4) in $(F_3)^2$ iff $\chi(x_1, x_2, x_3, f_1(x), f_2(x)) =$

$\chi(x_1, x_2, x_3, f_3(x), f_4(x))$ for all x in A^3 . Since there are 8

points in A^3 and χ is a 2-colouring of A^5 , then χ^* requires

at the most 2^8 colours. (In general, we may consider (f_1, \dots, f_m) as

representing a lifting from A^u to A^{u+m} , inverse to the projection p .

Two liftings are coloured the same iff the images of A^u are coloured identically - identifying the images under p .)

From the definition of m (i.e., the Hales-Jewett Theorem)

there is a monochromatic "line" in $(F_u)^m$. This line is of the form:-

$$f_{11}, f_{12}, \dots, f_{1m}$$

$$\vdots$$

$$f_{s1}, f_{s2}, \dots, f_{sm}$$

where each column f_{li} , $i = 1, m$ is a constant or is f_1 , $s = |F_u|$

$$\vdots$$

$$f_{si}$$

$$\vdots$$

$$f_s$$

where $|F_u| = |A|^{|A^u|}$. For convenience, we will renumber so that this monochromatic "line" in $(F_u)^m$ varies in the first $r(>0)$ coordinates with constants f_i , $r < i \leq m$. That is, for all $T \subset A^u$, $\dim(T) = t$,

$$\chi((T, f, f, \dots, f, f_{r+1}, \dots, f_m))$$

is independent of $f \in F_u$. Now set

$$B = \{(x_1, \dots, x_u, y_1, \dots, y_m); y_i = y_1, 2 \leq i \leq r, y_i = f_i(x_1, \dots, x_u), \\ r \leq i \leq m\}.$$

B is the desired space. Its basis is the first $u+1$ coordinates.

Let T' be a transverse t -space of B , with $p(T') = T$. Then we may express $T' = (T, g_1, \dots, g_m)$ for some $g_i \in F_u$. Since $T' \subset B$, we can also express $T' = (T, g_1, \dots, g_1, f_{r+1}, \dots, f_m)$ so $\chi(T')$ depends only on T .

From this point on the proof of Theorem 6.1 follows practically word for word (changing F to A) that of Theorem 5.1. (We note that in Corollary 3.1 the set $\psi^{-1}(L)$ is a k -space in A^n .)

In case (ii), the lifting of W_1 to W is accomplished by setting $W = (W_1, f)$ for arbitrary f . We omit the details.

The following result given in [3], somewhat strengthens Theorem 6.1. Let $\phi \neq c \subset A$ and replace (A_1) by

(A₁¹) Constants: The constant function $f(x_1, \dots, x_m) = c$ is in F_m iff $c \in C$.

We call (A, F, c) satisfying $(A_1^1, A_2 - A_6)$ a parameter system with restricted coefficients. The 0-spaces of A^n are the points $x \in C^n$.

Theorem 6.2. Parameter systems with restricted coefficients are Ramsey.

Outline of Proof. Let an element $0 \in C$ be specified. In A^n , we call spaces in (A, F) "spaces" and spaces in (A, F, c) "restricted spaces". If S is a t -space we define $\text{Rest}(S)$, the restriction of S to be that restricted t -space given by changing all coordinates which are equal to a $\neq C$ to the constant 0 . Now we assume that $t \geq 0$, r, k, n satisfy the Ramsey property for (A, F) and consider an r -colouring of the restricted t -spaces of A^n . We induce a colouring of all t -spaces, giving S the colour of $\text{Rest}(S)$. By Theorem 6.1, there exists a k -space W all of whose t -spaces have the same induced colour. $W^1 = \text{Rest}(W)$ is a restricted k -space all of whose restricted t -spaces have the same colour.

When A is a finite field, F the set of all linear functions (without constant term) and $c = \{0\}$, Theorem 6.2 gives Ramsey's Theorem for vector spaces.

Theorem 6.3. n -parameter sets are Ramsey.

Proof. This follows automatically from Theorem 6.2 since an n -parameter set is parameter system, (A, F) where $F = \bigcup_{i=1}^{\infty} F_i$ where F_i is a family of constant functions $f : A^i \rightarrow A$, i.e., $f \in F_i, f(x_1, \dots, x_i) = a \in A$. Appendix C lists several corollaries of Theorem 6.3.

Having produced two examples of parameter systems, it is natural to wonder if there are any more. This may prove a fruitful area of research for further analogues of Ramsey's Theorem although it should be noted that axioms (A_1-A_6) require (A, F) to have quite a lot of structure.

Appendix A

In this appendix, we take a closer look at the parallel combinatorial lines L_0', L_1', \dots, L_u' in F^m . Let $|F| = q$, the a combinatorial line in F^m consists of q m -tuples and will be of the form:-

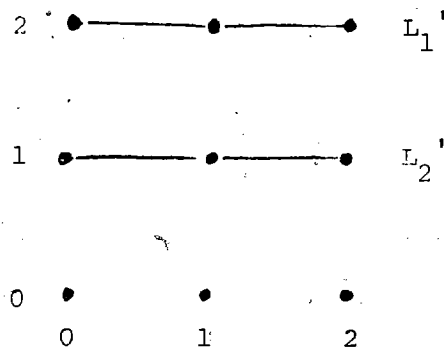
$$L_i' = \left\{ \begin{array}{cccc} a_i & \dots & x_1 & \dots & b_i & \dots & x_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_i & \dots & x_j & \dots & b_i & \dots & x_j \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_i & \dots & x_q & \dots & b_i & \dots & x_q \end{array} \right.$$

where a_i, b_i, \dots are fixed elements in F and $x_j, j = 1, q$ runs through the elements of F . L_i' has n ($1 \leq n \leq m$) "moving" columns and $(m-n)$ "constant" columns. As shown above, the "moving" columns all move in unison through the elements of F .

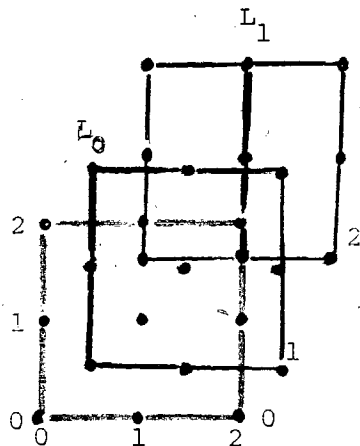
If L_k' is parallel to L_i' , then L_k' has the same n "moving" columns as L_i' and L_k' has $(m-n)$ "constant" columns, at least one of which will differ from the corresponding "constant" column of L_i' . For example, suppose L_i' is given as above, then:-

Appendix B

Let $F = Z_3 = \{0,1,2\}$, $u = 1$, $m = 2$. Let $L_0' = \{(x,1); x \in F\}$ and $L_1' = \{(y,2); y \in F\}$. Figure B.1 illustrates these lines.



$F^u = F^1 = \{0,1,2\}$ and so $\bar{e}_0 = (0)$ and $\bar{e}_1 = (1)$. Then we have $L_0 = \{(0,x,1); x \in F\}$ and $L_1 = \{(1,y,2); y \in F\}$. We observe that L_0' and L_1' are lines in 2-dimensional space whereas L_0 and L_1 are lines in 3-dimensional space. Figure B.2 shows the lines L_0 and L_1 in F^3 . It can be seen that L_0 and L_1 are vertical lines in F^3 .



To obtain B , the 2-space generated by L_0 and L_1 we find L_* , the line through the origin of which L_0 and L_1 are translates. $L_* = \{(0,0,0), (0,1,0), (0,2,0)\}$ and $L_0 = L_* + (0,0,1)$, $L_1 = L_* + (1,0,2)$. Next we choose any two vectors $\bar{y}_i, \bar{y}_i \in L_i, i = 0,1$ say $(0,0,1)$ and $(1,0,2)$ and use them to generate the 1-space $Y = \{(0,0,1), (1,0,2), (2,0,0)\} = V_0 + (0,0,1)$ or $V_0 + (1,0,2)$ where $V_0 = \{(0,0,0), (1,0,1), (2,0,2)\}$. Then $B = L_* + Y = \{(0,0,1), (0,1,1), (0,2,1), (1,0,2), (1,1,2), (1,2,2), (2,0,0), (2,1,2), (2,2,2)\}$. $B = V_2 + (0,0,1)$ where $V_2 = \{(0,0,0), (0,1,0), (0,2,0), (1,0,1), (1,1,1), (1,2,1), (2,0,2), (2,1,2), (2,2,2)\}$, a 2-dimensional vector space. Figure B.3 illustrates B . We observe that in the Figure, B appears to be 3-dimensional; however by definition $\dim(B) = \dim(V_2) = 2$.

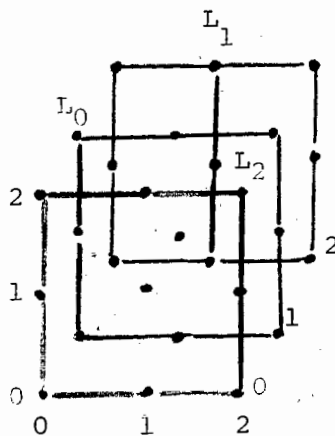


Figure B.3

B consists of the lines L_0, L_1 and L_2 . We observe that $L_i, i = 0, 2$ are vertical spaces of B and that any transverse 1-space T in B would contain a vector from each $L_i, i = 0, 2$. Since T is a 1-space, i.e., a 1-dimensional affine space it is generated by two vectors. In fact, T can be generated by any two of the three vectors of T. Thus, it follows that T is generated by $\{\bar{y}_i\}_0^1, \bar{y}_i \in L_i$.

Appendix CCorollaries of Theorem 6.3.

Corollary 6.1. Given integers ℓ and r , there exists an integer $N(\ell, r)$ such that if A is a finite set with $|A| \geq N(\ell, r)$ and the subsets of A are r -coloured, then there exist ℓ disjoint non-empty subsets A_1, \dots, A_ℓ of A such that all $2^\ell - 1$ unions

$\bigcup_{j \in J} A_j$, $\emptyset \neq J \subseteq \{1, 2, \dots, \ell\}$ have one colour.

Corollary 6.2. Given positive integers ℓ and r , there exists an integer $N(\ell, r)$ such that if $n \geq N(\ell, r)$ and the positive integers $\leq n$ are r -coloured then there exist ℓ integers a_1, \dots, a_ℓ such that all the sums $\{\sum_{i=1}^{\ell} \varepsilon_i a_i; \varepsilon_i = 0 \text{ or } 1, \text{ not all } \varepsilon_i = 0\}$ have one colour.

Corollary 6.3. Given integers ℓ, r , there exists an integer $N(\ell, r)$ such that if G is any group with $|G| \geq N(\ell, r)$ and if the elements of G are r -coloured, then there exist ℓ elements a_1, \dots, a_ℓ in G such that all the products a_{i_1}, \dots, a_{i_j} have one colour for all $j \geq 1$ and all choices of distinct i_1, \dots, i_j in $\{1, 2, \dots, \ell\}$.

Corollary 6.4. Let $L = L_i(x_1, \dots, x_m)$, $1 \leq i \leq h$, be a system of homogeneous linear equations with real coefficients with the property that for each j , $1 \leq j \leq m$, there exists a solution $(\varepsilon_1, \dots, \varepsilon_m)$ to the system L with $\varepsilon_i = 0$ or 1 and $\varepsilon_j = 1$. Then given an integer

r there exists an integer $N(r)$ such that if $n \geq N(r)$ and the positive integers $< n$ are r -coloured, then L can be solved with integers having one colour.

Let $C_n = \{(x_1, \dots, x_n); x_i = 0 \text{ or } 1\}$ be the set of 2^n vertices of a unit n -cube in R^n . Let us call a subset $Q_k \subseteq C_n$ a k -subspace of C_n if $|Q_k| = 2^k$ and Q_k is contained in some k -dimensional euclidean subspace of R^n .

Corollary 6.5. Given integers k, ℓ, r , there exists an integer $N(k, \ell, r)$ such that if $n \geq N(k, \ell, r)$ and the k -subspaces of C_n are r -coloured, then there exists an ℓ -subspace of C_n all of whose k -subspaces have one colour.

van der Waerden's Theorem (Theorem 1.1), and Ramsey's Theorem (Theorem 1.3) can also be proved as corollaries of Theorem 6.3.

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