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TITLE OF THESIS/*TITRE DE LA THÈSE* Classical SU(3) Yang-Mills Gauge Theory with External Static Sources

UNIVERSITY/*UNIVERSITÉ* Simon Fraser University

DEGREE FOR WHICH THESIS WAS PRESENTED/*GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* M.Sc.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1980

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DATED/*DATÉ* AUG 20th 1980 SIGNED/*SIGNÉ* [redacted]

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CLASSICAL SU(3) YANG-MILLS GAUGE THEORY WITH EXTERNAL STATIC SOURCES

by

Dubravko Horvat

B.Sc., University of Zagreb, Yugoslavia, 1976

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Physics

SIMON FRASER UNIVERSITY

August 1980

APPROVAL

Name: Dubravko Horvat

Degree: Master of Science

Title of Thesis: Classical SU(3) Yang-Mills Gauge Theory with External Static Sources

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Classical SU(3) Yang-Mills Gauge Theory with External Static Sources

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ABSTRACT

A classical SU(3) Yang-Mills gauge theory with prescribed external static sources is studied for finite energy non-Coulombic solutions. Starting from an Ansatz consistent with SU(3) and rotational symmetry, the gauge field equations are reduced to a set of nonlinear, coupled, ordinary differential equations. These equations are solved numerically by collocation method using the code COLSYS.

Finite energy solutions require that (a) the source be extended in space and (b) two different sets of boundary conditions be imposed on the solutions.

The first type of solutions are non-trivial generalizations of non-Coulombic solutions studied previously in the literature. These solutions and their energies are studied in detail in this thesis.

An exhaustive study is made of the type-two solutions. These solutions belong to the second type of boundary conditions mentioned above: namely they tend to a pure gauge at infinity. These solutions have remarkable properties: i) existence of critical source strengths below which solutions (of finite energy) do not exist; ii) bifurcation of solutions above the critical source strength, i.e., there exist two distinct branches of solutions above the critical point.

In our study we have found three pairs of bifurcating

solutions. The first pair represents SU(3) generalizations of the SU(2) solutions known in the literature. The other two groups are unique to SU(3) gauge fields and have no SU(2) counterparts. They were discovered during the course of this work.

The stability of these bifurcating solutions is analysed and it is shown that the upper branch is unstable relative to the lower branch. For the lower branch solutions it is shown that no normalizable zero energy oscillations exist, therefore the lower branch solutions are absolutely stable.

ACKNOWLEDGEMENTS

I wish to thank my supervisor Dr.K.S.Viswanathan for suggesting this topic of research and for his guidance and support of my work.

I am grateful to Dr.U.Ascher, Mr.J.Christiansen and Dr.R.D.Russell for allowing me to use their collocation program.

Special thanks goes to Mr. Jan Christiansen for helping me understand and apply the numerical techniques used in this thesis.

I also want to thank Mr.A.Kurn for his help and for allowing me to use his plotting program.

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I. INTRODUCTION

Gauge theories offer the greatest promise for describing the elementary forces in nature. The Weinberg-Salam (W-S) model (Weinberg 1967, Salam 1968) (or generalizations of it) and Quantum Cromodynamics (QCD) (see for example Marciano and Pagels 1978) are the two existing non-Abelian gauge theories of real phenomenological importance.

The W-S model which combines the weak and electromagnetic interaction through the use of the Higgs mechanism (Abers and Lee 1973) was proposed about thirteen years ago. QCD, the renormalizable quantum gauge theory of the unbroken local symmetry group SU(3) of colour, is widely believed to be the correct theory of the strong interactions.

Non-Abelian gauge theories were introduced by Yang and Mills (Yang and Mills 1954) over twenty-five years ago. For most of this period it was not known whether any of the interactions observed in nature could be described by a non-Abelian gauge theory. Quantization and renormalizability were the central topic of research.

In 1971 't Hooft ('t Hooft 1971a, 1971b) proved renormalizability of the Yang-Mills (Y-M) theory, and after that one had confidence in Feynman diagram calculational techniques.

The door to the quantitative Y-M phenomenology was finally open.

The W-S model and QCD can be formulated in terms of Feynman path integrals (Feynman and Hibbs 1965, Popov 1977), i.e. functional integrals over all classical field configurations weighted by a factor exp(-action). If one knew everything about classical field configurations then in principle all questions concerning the quantum theory could be answered.

Partial information about classical fields might yield at least some insight into the quantum theory. This is the basic hope which motivates present research activity in the classical Y-M theory.

In this work we will study the solutions to the classical Y-M field equations in the presence of static external (non-dynamical) sources. The physical, quantum-mechanical significance of such solutions has not thus far been as profound as that of solitons, e.g. monopoles with Higgs-field sources (where sources are dynamical)('t Hooft 1974, Polyakov 1974) nor as that of instantons, (where sources are absent but the equations are continued to imaginary time (Sciuto 1977)).

In the Abelian case, the Coulomb field is the unique and absolutely stable solution for an arbitrary number of charged particles of given position. This establishes the dominant role played by the classical Coulomb interaction in the low-energy regime of both classical and quantum electrodynamics.

Mandula (1976) showed how the Abelian solution could be trivially translated to the Y-M case (Sikivie and Weiss 1978a, 1978b). Classical sources are described by a charged vector in a gauge group space, and if all the sources are oriented in commuting directions in the Y-M space (for example, along the λ_3 and λ_8 directions for the SU(3) gauge group), it is consistent to make an Ansatz that the only nonvanishing components of the Y-M fields are along these directions. This Ansatz immediately linearizes the field equations and reduces them to a set of decoupled copies of Maxwell's equations in the presence of external fixed charges.

Since that time there have appeared in the literature several papers that generalize Mandula's solution and consider a stability problem in the Y-M system with external sources (Mandula 1977, Magg 1978a, 1978c, 1979) as well as a total (colour or Y-M) charge screening (Magg 1978b, Cahill 1978). A particular strong coupling gauge theory, the coloured quark-gluon model (QCD) is widely favoured as a theory of strong interactions, because of its successful interpretations of scattering data and of hadron spectra. Therefore the questions of stability and charge screening are important because classical stability is necessary if the classical solution is to resemble the quantum situation (but it by no means ensures such resemblance) and in the colour gauge theory the confinement of quark quantum numbers requires that physical states be colour

singlets, that is, that all colour charges be totally screened.

Jackiw, Jacobs and Rebbi (1979) give another example of a solution for the sources of arbitrary strength[1] which they call the non-Abelian Coulomb solution and which differs from the Abelian one by the property that it does not vanish with the vanishing of the source;[2] rather it becomes a pure gauge. They solve the problem perturbatively in the source strength and find that the energy of a non-Abelian Coulomb solution is always lower then a corresponding Abelian one. When the source strength increases, the previous solution continues to be present but in addition, solutions which require a critical, minimal source strength appear. One example of such a solution is given by Sikivie and Weiss (1978a, 1978b). It is their "magnetic dipole solution" which is gotten by aligning an external source within the Cartan subalgebra of a gauge group. The extended charge distribution then yields a type of solution which has the long-range behaviour of a magnetic dipole field.

The other example of a minimum-strength solution is given by Jackiw, Jacobs and Rebbi (1979). They consider the gauge group SU(2) and use an Ansatz for the Y-M fields where the explicit radial symmetry is absent, but the rotational

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[1] Solutions naturally fall into two classes: those that exist for arbitrary source strengths and those that require a critical, finite source strength.

[2] In the absence of sources, finite energy solutions are necessarily trivial (Deser 1976)

non-invariance could be compensated by a gauge transformation.[3] Since their Ansatz contains A^i and A^o different from zero (one obtains the Coulomb solution by setting $A^i=0$), the full nonlinearity of the equations comes into play and there are no analytic methods available. For two different sets of boundary conditions they get two types of solutions, where the first type is in fact the non-Abelian Coulomb solution encountered before. The second type comes in two branches once the source strength exceeds a critical magnitude.

It has been a challenge for us to generalize results and ideas to a much more complicated case with SU(3) gauge symmetry, the symmetry which is used to describe hadrons and their interactions in terms of quark fields coupled to gluon gauge fields. Once the spherically symmetric Ansatz had been chosen (Corrigan et al. 1976, Yates 1977) and the source had been characterized in a gauge invariant manner, a whole diversity of gauge-field configurations was found. Within the Ansatz, energy, the gauge invariant quantity, displayed interesting bahaviour. The approach we use is based upon the decomposition of gauge fields and external sources into symmetric and anti-symmetric parts (Ragiadakos and Viswanáthan 1979) and a specific choice of SU(3) generators (Marshak et al.

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[3] This is the so called non-Abelian fashion of the radial symmetry. The Abelian fashion is realized when everything points in a fixed direction in a gauge group space and is radially invariant (Sikivie and Weiss 1978b).

1969, Li Ling-Fong 1974), in order to achieve a more general monopole plus quadrupole-like gauge-field response to a corresponding external source configuration.

In chapter 2 we review the general structure of classical field theories based on the principle of minimum action and explain some basic notions of symmetry transformations and gauge invariance with emphasis on the non-Abelian gauge invariance. In chapter 3 we briefly present the Y-M theory in the presence of external, static sources and give the Ansatz we will be working with.

In chapter 4 a survey of numerical methods as applied to our particular problem is made including details of the techniques we use in order to prepare the problem for the collocation method. A brief description of collocation is given.

Chapter 5 is devoted to a review of the results. The complete numerical analysis is presented for different external source configurations, their strengths and positions.

In chapter 6 the stability theory is briefly reviewed and stability properties of the Y-M fields with respect to radial oscillations are determined.

Chapter 7 contains the conclusion and suggestions for the further research.

II. THE FORMALISM Of CLASSICAL FIELD THEORY; SYMMETRIES AND GAUGE INVARIANCE

The dynamical behaviour of a physical system is usually described by means of integral, differential or integro-differential equations, where the unknowns are arbitrary functions (in classical physics) or operators (in quantum theory). In most of the important practical cases it is possible to derive these equations from a variational principle. This means that the equations in question may be considered to be the necessary condition for the extremum of a functional, the so called action functional S, which we write as

$$S = \int d^4x \, \mathcal{L}\left(\phi(x), \partial^\mu \phi(x), \partial^\mu \partial^\nu \phi(x), \ldots\right) \qquad (2.1)$$

where $\phi(x)$ are in general complex fields, taken at the point x.

We assume in the following that the integrand $\mathcal{L}$ which is called the Lagrange density, together with all derivatives which occur in the course of the calculations, are continuous functions of all their arguments. We will keep only the first partial derivatives of the fields, and the corresponding differential equations will contain, at most, second order derivatives of the fields. Introduction of higher

order derivatives into $\mathcal{L}$ would result in the appearance of higher order derivatives in the corresponding field equations, but equations of this type, discussed in the literature, contain some physically unreasonable features such as infinite energy for the dynamical system.

A physical principle from which we can infer the dynamical behaviour of the system of fields under consideration is provided by the principle of stationary action. The necessary and sufficient conditions for S to be stationary are the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^{\mu} \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi)} = 0 \qquad (2.2)$$

In order that the field equations should be covariant, the action must be invariant under the Lorentz transformations and, consequently, the Lagrangian density must be a Lorentz scalar. Additional requirements on $\mathcal{L}$ come from an assumed invariance of the theory (e.g. internal symmetries, discrete symmetries etc.).

It is important to observe that the choice of the Lagrangian density in the variational functional (2.1) which leads to the actual equations of motion, is not unique. The simplest type of change which can be made is the multiplication of $\mathcal{L}$ by a quantity which does not depend on the variables or state functions of the system; that is, the multiplication of $\mathcal{L}$ by a constant. In addition we observe that the addition of a

four-divergence to the Lagrangian density - i.e., the substitution

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial^{\mu} K^{\mu}(\phi(x)) \tag{2.3}$$

does not modify the equations of motion.

The conservation laws usually express the fact that certain physical quantities, such as momentum, energy, angular momentum, charge, etc., are constant in time. These laws follow as a consequence either from the field equations (2.2) or directly from the action functional (2.1). The latter point of view is intimately connected with the structure of this functional, in particular with its invariance properties with respect to a certain set of transformations.

Noether's theorem assures the existence of conserved quantities when the action functional is left invariant by a set of transformations of the coordinates and the fields.

The combination of Noether's theorem and the principle of stationary action clarifies the relationship between conservation laws and symmetry groups.[1] We will consider those groups which can be generated by infinitesimal transformations in the neighbourhood of the identity transformation, so-called Lie groups. The following theorem is thus obtained

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[1] We call a transformation a symmetry transformation when it has the property of leaving the equations of motion invariant in form (form-invariant).

If the Lagrangian density function is invariant under a p-parameter Lie group, the equations of motion ensure p conservation laws.

The Lie groups of special interest in classical mechanics are the four-dimensional translation group and the three-dimensional rotation group (forming the Euclidean group), the Galilean group and the Lorentz group. The Euclidean group corresponds to the following seven conservation laws: conservation of energy (time-translation invariance), conservation of linear momentum (space-translation invariance) and conservation of angular momentum (rotation invariance). The additional three parameters of the Galilean and the Lorentz group yield the center-of-mass theorem nonrelativistically and relativistically, respectivly.[2]

Now we shall extend these ideas to non-classical symmetries, called internal symmetries, i.e., transformations which do not involve space-time coordinates.

The discovery of internal symmetry principles or invariances under certain gauge or phase transformations leads to conservation laws that greatly reduce the multiplicity of processes and states of a theory.

One group of internal symmetry transformations that plays an important role in particle physics and seems to be unrelated to the space-time symmetries is the gauge

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[2] This theorem states that the position of the center of mass of a system is a linear function of time (the velocity of the center of mass is conserved).

transformation of the first kind. Simplest gauge transformations of the first kind are transformations of the unitary group U(1), which is a one-parameter Abelian Lie group possessing only one dimensional represenatations. Under these transformations the fields change according to

$$\phi(x) \longrightarrow \phi'(x) = e^{i\alpha}\phi(x) \tag{2.4}$$

with α real. When the Lagrangian density function is invariant under the gauge transformation of the first kind we refer to it as global gauge invariance.

Gauge invariance of the first kind gives rise to the conservation of a "charge". Besides the universally known conservation of the electric charge, experimental evidence exists for the conservation of the baryon charge or baryon number, electron-lepton number, muon-lepton number, etc.

A theory may be invariant under a more complicated group of transformations than U(1). We can define a gauge transformation by

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\Theta_a L_a}\phi(x) \equiv U(\Theta)\phi(x) \tag{2.5}$$

where $\phi(x)$ is a column vector and L_a is a matrix representation of the generators of an internal symmetry group G. The number of parameters needed to specify $U(\Theta) \in G$ is the dimensionality of G.

A generalized gauge transformation can depend on the space-time point. Let us take again the U(1) group

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\Theta(x)}\phi(x) \tag{2.6}$$

This is the so-called gauge transformation of the second kind, or local gauge transformation. Since the Lagrangian density function contains the derivatives of fields $\phi(x)$, it cannot in general be invariant under these transformations. The invariance is restored by introducing a gauge field $A^{\mu}(x)$

$$\partial^{\mu} \longrightarrow \partial^{\mu} - ieA^{\mu}(x) \tag{2.7}$$

which transforms in a special way so that the combination (2.7) transforms like in (2.6). e is a coupling constant which characterizes strength of the interaction.

Yang and Mills generalized the local gauge transformation in (2.6) to non-Abelian groups.

Let T_a (a=1,...,N) be the generators of a compact Lie group with structure constants C_{abc} (see Appendix C). Let a collection of scalar fields transform according to

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\Theta_a(x)L_a}\phi(x) \equiv U(\Theta)\phi(x) \tag{2.8}$$

where $\phi(x)$ is a column vector and L_a is a matrix representation of T_a. The Lagrangian density is assumed to be invariant under a transformation with constant Θ_a. The problem of how to construct a theory which is invariant under a local

gauge transformation $\Theta_a(x)$ as well, is solved by introducing a covariant derivative $\mathcal{D}^\mu$ which transforms like

$$\mathcal{D}^\mu \phi(x) \longrightarrow U(\Theta)\, \mathcal{D}^\mu \phi(x) \tag{2.9}$$

Then, if $\partial^\mu \phi(x)$ appears in $\mathcal{L}$ only as a part of $\mathcal{D}^\mu \phi(x)$, $\mathcal{L}$ will be invariant under local gauge transformations.

The covariant derivative is constructed by introducing a vector field $A_a^\mu(x)$ for each dimension of the Lie algebra, hence

$$\mathcal{D}^\mu \equiv \partial^\mu + i g T_a A_a^\mu(x) \tag{2.10}$$

In (2.10) g is the gauge coupling constant.

In this way we have introduced an interaction between $\phi(x)$ and the gauge fields A_a^μ. This implies that only fields in interaction can be invariant under gauge transformations of the second kind.

Now we must add to $\mathcal{L}$ the part in which the $A_a^\mu(x)$ and their derivatives occur only in combinations

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + i g c_{abc} A_b^\mu A_c^\nu \tag{2.11}$$

The full Lagrangian density, which is invariant under the gauge group G is

$$\mathcal{L} = \mathcal{L}(\phi, \mathcal{D}^\mu \phi) + \mathcal{L}_{YM} = \mathcal{L}(\phi, \mathcal{D}^\mu \phi) - \tfrac{1}{4} F_a^{\mu\nu} F_a^{\mu\nu} \tag{2.12}$$

and $\mathcal{L}_{YM}$, the Y-M Lagrangian, is invariant since the transformation rule for the gauge fields $A^{\mu}_{a}(x)$ is defined to be

$$A^{\mu}(x) \longrightarrow U(\Theta) A^{\mu} U^{-1}(\Theta) - \frac{i}{g}[\partial^{\mu} U(\Theta)]\, U^{-1}(\Theta) \tag{2.13}$$

It is the last term of the right-hand side of this equation which reflects the local character of the transformation. is defined in (2.16).

The gauge fields or the Y-M fields $A^{\mu}_{a}(x)$ are Lorentz vectors which carry integral spin angular momentum. Because of these reasons they are often called "photons" or gauge bosons. The gauge field mass term would have the form

$$-\frac{1}{2} m^2 A^{\mu}_{a} A^{\mu}_{a} \tag{2.14}$$

which obviously violates local gauge invariance. The conclusion is that local gauge invariance is impossible unless the gauge boson is massless.

It is convenient at this stage to introduce matrix notation. In the adjoint representation (see Appendix B) the generators T_a have the form

$$(T_a)_{bc} = - C_{abc} \tag{2.15}$$

Let

$$A^{\mu}(x) \equiv T_a A^{\mu}_{a}(x) \tag{2.16}$$

Namely, $A^{\mu}(x)$ takes value in the Lie algebra of G. Similarly

$$F^{\mu\nu} \equiv T_a F^{\mu\nu}_a \tag{2.17}$$

Then we can write

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} + ig [A^{\mu}, A^{\nu}] \tag{2.18}$$

Now the $\mathcal{L}_{YM}$ of the total $\mathcal{L}$ can be written in the form

$$\mathcal{L}_{YM} = -\frac{1}{4N} Tr (F^{\mu\nu} F^{\mu\nu}) \tag{2.19}$$

where the normalization for the generators T_a is defined to be

$$Tr (T_a T_b) = N \delta_{ab} \tag{2.20}$$

and since $F^{\mu\nu}$ transforms under the local gauge transformation according to

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = U(\Theta) F^{\mu\nu} U^{-1}(\Theta) \tag{2.21}$$

the $\mathcal{L}_{YM}$ is obviously invariant due to the cyclic property of trace.

III. THE CLASSICAL YANG-MILLS THEORY WITH EXTERNAL SOURCES

3.1 INTRODUCTION

A theory of free fields alone has no physical content. The nature of the physical world is revealed to us only through interactions. In constructing an interaction Lagrangian between Y-M fields and an external, prescribed source we shall be motivated by analogy with the classical electromagnetic field. This analogy can be easily seen just by looking at the Lagrangian density (2.12).

There is no general recipe for finding the Lagrangian function which will provide a given set of equations. Only certain structural features can be assured from the outset. Its exact determination is, in the last analysis, educated guesswork. Physically, it adds nothing that is not contained in the basis equations.

The Lagrangian density for an electromagnetic field interacting with an external electric current is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F^{\mu\nu} - j^{\mu} A^{\mu} \tag{3.1}$$

where

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{3.2a}$$

The action principle yields the Maxwell equations

$$\partial^{\mu} F^{\mu\nu} = j^{\nu} \tag{3.2b}$$

The other two Maxwell equations are automatically satisfied because of the definition of $F^{\mu\nu}$.

The differential law of charge conservation

$$\partial^{\mu} j^{\mu} = 0 \tag{3.3}$$

which can be obtained from (3.2b) using the antisymmetry of $F^{\mu\nu}$ has an equivalent integral form

$$Q = -\int j^{\mu} \hat{n}^{\mu} d^4x \tag{3.4a}$$

thus

$$\partial_t Q = \int (\partial^{\mu} j^{\mu}) d\sigma = 0 \tag{3.4b}$$

We conclude this short exposition by stressing that under the condition

$$\Box \chi = 0 \tag{3.5}$$

where χ is defined by the gauge transformation

$$A^{\mu} \longrightarrow A^{\mu} + \partial^{\mu} \chi \tag{3.6}$$

the gauge invariance of the action functional is equivalent to charge conservation (Rohrlich 1965).

We now consider the generalization of the classical electric and magnetic fields in interaction with a given charge and current distribution j^ν to the non-Abelian case. The last chapter enables us to write the Lagrangian density which yields the equations of motion for the fields in interaction with prescribed "colour" charge and current density [1] in the form

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_{ab} F^{\mu\nu}_{ab} - j^{\mu}_{ab} A^{\mu}_{ab} \quad (a,b = 1,2,3) \tag{3.7}$$

(see p.116 for the notation). We choose, for reasons mentioned before, the semisimple compact Lie group SU(3) (see Appendix C) as the underlying gauge symmetry group. The field equations satisfied by the field strengths written in the matrix notation

$$F^{\mu\nu} \equiv F^{\mu\nu}_{ab} U_{ab} \tag{3.8}$$

so that $F^{\mu\nu}$ can be written in the form (2.18), are

$$\mathcal{D}^{\mu} F^{\mu\nu} = j^{\nu} \tag{3.9}$$

where

$$\mathcal{D}^{\mu} \equiv \partial^{\mu} + ig[A^{\mu}, \;]; \quad j^{\nu} = j^{\nu}_{ab} U_{ab} \tag{3.10}$$

Here $[\ ,\ ]$ has meaning of the ordinary commutator (see footnote

---

[1] The terms "colour" imply only that the fields transform according to some representation of a non-Abelian gauge group G in a "colour space" (gauge-group space), and their non-comutativity is purely classical, as a consequence of non-comutativity of matrices but not the quantum mechanical operator non-comutativity.

1) so (3.9) can be written

$$\partial^\mu F^{\mu\nu} + ig A^\mu F^{\mu\nu} - ig F^{\mu\nu} A^\mu = j^\nu \tag{3.11}$$

The matrices U_{ab} are the matrix representation of generators of SU(3) (explained in details in Appendix B).

Since the gauge transformation properties of the gauge fields A^μ_{ab} and $F^{\mu\nu}_{ab}$ are given by (2.13) and (2.21), the equations of motion (3.9) are covariant provided j^ν also transforms covariantly

$$j^\nu \longrightarrow j'^\nu = U(\Theta) j^\nu U^{-1}(\Theta) \tag{3.12}$$

Hence, if A^μ is a solution of the equations (3.9) with the current j^ν, then A'^μ, the gauge transform of A^μ, given by (2.13), is the solution of the equation (3.9) with the current j'^ν defined by (3.12).

The gauge covariant current j^ν, although not conserved, satisfies

$$\partial^\nu \{ j^\nu - ig[A^\mu, F^{\mu\nu}] \} = 0 \tag{3.13}$$

Hence, the current J^ν

$$J^\nu = \partial^\mu F^{\mu\nu} \tag{3.14}$$

is conserved due to the antisymmetry of $F^{\mu\nu}$. Thus $J^\nu(x)$ is a conserved but not gauge covariant current, whereas $j^\nu(x)$ is gauge covariant but not conserved. However, the current $j^\nu(x)$

satisfies

$$\mathcal{D}^{\nu} j^{\nu} = 0 \tag{3.15}$$

In order to prove this relation we start with

$$[\mathcal{D}^{\mu}, \mathcal{D}^{\nu}]\phi = ig[F^{\mu\nu}, \phi] \tag{3.16}$$

Indeed

$$\{[\partial^{\mu} + ig[A^{\mu},], \partial^{\nu} + ig[A^{\nu},]\}\phi = \partial^{\mu}\partial^{\nu} +$$
$$+ ig[\partial^{\mu}A^{\nu}, \phi] + ig[A^{\nu}, \partial^{\mu}\phi] + ig[A^{\mu}, \partial^{\nu}\phi] + \tag{3.17a}$$
$$+ ig[A^{\mu}, A^{\nu}\phi] - (ig)^2[A^{\mu}, \phi A^{\nu}] -$$
$$- \left\{ \begin{array}{l} \nu \longrightarrow \mu \\ \mu \longrightarrow \nu \end{array} \right\}$$

so

$$[\mathcal{D}^{\mu}, \mathcal{D}^{\nu}]\phi = ig[\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \phi] + (ig)^2\{[A^{\mu}, A^{\nu}]\phi +$$
$$+ A^{\nu}[A^{\mu}, \phi] - [A^{\mu}, \phi]A^{\nu} - \phi[A^{\mu}, A^{\nu}] - \tag{3.17b}$$
$$- [A^{\nu}, A^{\mu}]\phi - A^{\mu}[A^{\nu}, \phi] + [A^{\nu}, \phi]A^{\mu} + \phi[A^{\nu}, A^{\mu}]\}$$

and after simple manipulations the relation (3.16) is obtained.

Now

$$\mathcal{D}^{\mu} j^{\mu} = \partial^{\mu} j^{\mu} + ig[A^{\mu}, j^{\mu}] = \tag{3.18}$$
$$= \mathcal{D}^{\mu}\mathcal{D}^{\nu} F^{\mu\nu} = \frac{1}{2}[\mathcal{D}^{\mu}, \mathcal{D}^{\nu}] F^{\mu\nu} = 0$$

We shall be concerned with static external sources. We call a current $j^{\mu}(x)$ static if

$$j^{i}(x) = 0$$
$$j^{\circ}(x) = g\,\rho(\underset{\sim}{x}) \tag{3.19a}$$

and

$$\partial_t \rho(\underset{\sim}{x}) = 0 \tag{3.19b}$$

From (3.15) we can obtain a useful relation between A° and ρ

$$[A^{\circ}, \rho] = 0 \tag{3.20}$$

which means that in a solution the time component of the field A^{μ} is always parallel to the source. This remark is important because it will in a way determine the form of an Ansatz for A° once ρ is given.

Using (3.7) and (3.19) we can write the total energy of the system

$$H = \frac{1}{2}\int d\tau\,[(F^{oi}_{ab})^2 + \frac{1}{2}(F^{ij}_{ab})^2] \tag{3.21}$$

The colour electric and magnetic fields are

$$E^{i}_{ab} = F^{oi}_{ab}$$
$$B^{i}_{ab} = -\frac{1}{2}\,\varepsilon^{ijk}\,F^{jk}_{ab} \tag{3.22}$$

so the energy (3.21) can be written

$$H = \frac{1}{2} \int d\tau \, (\underset{\sim}{E}^2_{ab} + \underset{\sim}{B}^2_{ab}) \tag{3.23}$$

By identifying E^i_{ab} as momenta conjugate to the canonical coordinate A^i_{ab}, it is easy to derive the static Y-M equations in the presence of an externally prescribed time independent source. However, the variations of H, defined by (3.23), with respect to A^i_{ab} and E^i_{ab} which are needed for obtaining the dynamical equations are not unrestricted: they are constrained by the three relations that comprise Gauss' law. In other words, the equations (3.9) for $\mu=1,2,3$ are equations of motion, but for $\mu=0$, the equation is the restriction on what configurations are admissible. Gauss' law requires the divergence of the colour electric field E^i_{ab} to equal the total colour density J^o (3.14).

The constraint may be implemented with Lagrange multipliers which we call A^o_{ab} (Magg 1979), and unrestricted variations are performed on the effective Hamiltonian

$$\bar{H} = H - \int d\tau \, A^o_{ba} \{\partial^i E^i_{ab} + ig[A^i, E^i]_{ab} - g\rho_{ab}\} \tag{3.24}$$

In this way the full Y-M equations are regained

$$\frac{\delta \bar{H}}{\delta A^o_{ba}} = 0 \tag{3.25}$$

$$\Longrightarrow \partial^i E^i_{ab} + ig[A^i, E^i]_{ab} - g\rho_{ab} = 0$$

(Gauss' law constraint - time component of Y-M equations)

$$\frac{\delta \bar{H}}{\delta A^i_{ba}} = \partial_t E^i_{ab} =$$

$$= (\underset{\sim}{\partial} \times \underset{\sim}{B}_{ab})^i - ig\,\varepsilon^{ijk}[A^k, B^j]_{ab} - ig[A^\circ, E^i]_{ab} \tag{3.26}$$

(Ampere's law - spatial components of Y-M equations)

$$\frac{\delta \bar{H}}{\delta A^i_{ba}} = \partial_t E^i_{ab} =$$

$$= E^i_{ab} + \partial^i A^\circ_{ab} + ig[A^i, A^\circ]_{ab} \tag{3.27}$$

(definiton of the canonical momentum E^i_{ab}).

3.2 THE STATIC YANG-MILLS EQUATIONS

In simple dynamical systems it follows from Hamilton's equations of motion that the energy is stationary for time independent solutions.

Therefore, from the equations of motion for the Y-M fields (3.26) and (3.27) and from the constraint (3.25) we get the static Y-M equations

$$\partial^i E^i_{ab} + ig[A^i, E^i]_{ab} = g\,\rho_{ab}$$

$$(\underset{\sim}{\partial} \times \underset{\sim}{B}_{ab})^i = ig\,\varepsilon^{ijk}[A^k, B^j]_{ab} + ig[A^\circ, E^i]_{ab} \qquad (3.28)$$

$$E^i_{ab} = -\partial^i A^\circ_{ab} - ig[A^i, A^\circ]_{ab}$$

The most obvious solution to the equations (3.28) is the static Coulomb one, where it is given by

$$\underline{A} = 0 \qquad (3.29a)$$

and

$$A^\circ = \phi \quad ; \quad \Delta\phi = -g\rho \qquad (3.29b)$$

but our task will be to solve the equations (3.28) retaining all the components of the vector potential.

3.3 ANSÄTZE FOR SOURCES AND GAUGE POTENTIALS

In order to solve the set of highly non-linear differential equations, we have to choose an Ansatz which will simplify the system but which will still exhibit all interesting features incorporated in the Y-M theory. Proceeding in this direction, we make the following decomposition of the colour charge density ρ_{ab}

$$\rho_{ab} = \rho_{\{ab\}} + \rho_{[ab]} \tag{3.30}$$

where $\rho_{\{ab\}}$ is symmetric and $\rho_{[ab]}$ is antisymmetric under the interchange of a and b (Ragiadakos and Viswanathan 1979). In the same way we decompose the potential

$$A^{\mu}_{ab} = A^{\mu}_{\{ab\}} + A^{\mu}_{[ab]} \tag{3.31}$$

For an external source we propose a spherically symmetric Ansatz which transforms according to the adjoint representation of SU(3). Recall that spherical symmetry in the non-Abelian gauge theory can be realized in the following way: if explicit spherical symmetry is absent, any rotational non-invariance can be compensated by a gauge transformation (see remarks on the page 19 about gauge transformed quantities).

Hence for an external source being spherically symmetric we have

$$[\underset{\sim}{J} + \underset{\sim}{T}, \rho] = 0 \tag{3.32}$$

where $\underset{\sim}{J}$ generates spatial rotations and $\underset{\sim}{T}$ is a triplet of generators of SU(3), that generate an SU(2) (or SO(3)) subgroup.

There are essentially two different ways of identifying $\underset{\sim}{T}$ among the SU(3) generators:

$$\left\{\tfrac{1}{2}\lambda_1, \tfrac{1}{2}\lambda_2, \tfrac{1}{2}\lambda_3\right\} \quad \text{or} \quad \{\lambda_2, -\lambda_5, \lambda_7\} \tag{3.33}$$

These generate SU(2) and SO(3) groups respectively, which are subgroups of SU(3).[2] We take T's to be generators of the SO(3) subgroup, which have the same commutation properties as SU(2). This is so-called SO(3) embedding of SU(2) in SU(3) or sometimes called "nuclear physics" embedding.[3]

An invariant under the combined rotations (3.32) which is antisymmetric in the colour indices is of the type

$$\varepsilon_{abk}\,\frac{x^k}{r} \tag{3.34}$$

therefore we can write for $\rho_{[ab]}$

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[2] SO(3) is isomorphic to $SU(2)/Z_2$ where Z_2 is center of the group. We can say also that any element of SO(3) can be represented by a matrix SU(2) defined up to a sign, or that matrices of SU(2) form a two-valued representation of the group SO(3).

[3] The other embedding is SU(2) or "U-spin embedding" (see Appendix B).

$$\rho_{[ab]} = i\,\varepsilon_{abk}\frac{x^k}{r}q_1(r)\frac{1}{g^2} \tag{3.35}$$

and $q_1(r)$ is a spherically symmetric colour charge distribution. We shall be concerned with gauge-fields configurations of finite total energy - a requirement which imposes regularity constraints on the charge density. Consequently, we do not discuss point sources.

The form of the symmetric part of ρ_{ab} will depend upon transformation properties of the coset of SU(3) generators

$$\{\lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8\} \tag{3.36}$$

under SO(3) rotations generated by $\underset{\sim}{T}$. Since the coset transforms as a quadrupole moment tensor (Corrigan et al. 1976), it is reasonable to construct an invariant by using the quadrupole moment tensor in the spatial coordinates so the symmetric Ansatz becomes

$$\rho_{\{ab\}} = \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\frac{q_2(r)}{g^2} \tag{3.37}$$

and the total colour source tensor can be written as

$$\rho_{ab} = i\,\varepsilon_{abk}\frac{x^k}{r}\frac{q_1(r)}{g^2} + \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\frac{q_2(r)}{g^2} \tag{3.38}$$

The condition (3.20) enables us immediately to write the Ansatz for the zero-component of the vector potential

$$A^{\circ}_{ab} = i\,\varepsilon_{abk}\frac{x^k}{r}\frac{f_1(r)}{gr} + \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\frac{f_2(r)}{gr} \tag{3.39}$$

The spatial part of the vector potential transforms as a vector

under combined rotations (3.32). So its symmetric part can be immediately written as an invariant (quadrupole moment tensor) "times" a three vector, i.e.,[4]

$$A^{i}_{\{ab\}} = \left[\partial^{i}\left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\right]\frac{H(r)}{gr} \tag{3.40}$$

Since the commutation relations among the SO(3) generators are the same as those for SU(2), the antisymmetric spatial part of the Ansatz for the gauge potential has the form

$$A^{i}_{[ab]} = i\,\varepsilon_{abc}\,A^{i}_{c} \tag{3.41}$$

and with

$$A^{i}_{c} = \varepsilon_{ick}\,\frac{x^k}{r}\,\frac{G(r)-1}{gr} \tag{3.42}$$

it is simple to obtain the form for the Ansatz of the spatial components of the gauge potential which transforms as a vector under the combined rotations in Euclidean and gauge-group space

$$A^{i}_{ab} = \frac{H}{gr}\,\partial^{i}\left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right) - i\left(\delta_{ai}\frac{x_b}{r} - \delta_{bi}\frac{x_a}{r}\right)\frac{G-1}{gr} \tag{3.43}$$

By imposing the Coulomb condition on the field $\underset{\sim}{A}$ (see for instance Bogoliubov and Shirkov 1976)

$$\partial^{i} A^{i}_{ab} = 0 \tag{3.44}$$

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[4] As is known from the literature (Corrigan et al. 1976, Yates 1977), from the rotational invariants x^i/r, δ^{ij} and ε^{ijk} one can construct essentially six traceless, Hermitian, 3×3 matrices.

the antisymmetric part is identically zero

$$\partial^i A^i_{[ab]} = 0 \tag{3.45}$$

and the only solution of (3.44) is H(r)=0. Therefore, for the spatial part of the vector potential A^i we can write the Ansatz in the form

$$A^i_{ab} = i\left(\delta_{bi}\frac{x_a}{r} - \delta_{ai}\frac{x_b}{r}\right)\frac{G(r)-1}{gr} \tag{3.46}$$

It should be stessed here that the Ansätze (3.38), (3.39) and (3.46) can not be gauge rotated to lie entirely in the Cartan subalgebra of SU(3). Namely, these solutions are not gauge equivalent to Coulomb solutions. This will be seen from the equations of motion which are nonlinear in G(r) (Chakrabarti 1980) and from the existence of bifurcating solutions which are not certainly characteristic of Coulomb solutions.

The differential equations for $f_1(r)$, $f_2(r)$ and G(r) emerging from (3.28) are

$$-f_1''(r) + \frac{2}{r^2}G^2(r)f_1(r) = 2rq_{f_1}(r) \tag{3.47a}$$

$$-f_2''(r) + \frac{6}{r^2}G^2(r)f_2(r) = 2rq_{f_2}(r) \tag{3.47b}$$

$$-G''(r) + \frac{1}{r^2}\left[G^2(r)-1\right]G(r) - \frac{1}{r^2}\left[f_1^2(r) + f_2^2(r)\right]G(r) = 0 \tag{3.47c}$$

while the energy is

$$\mathcal{E} = \frac{4\pi}{g^2} \int_0^\infty dr \left[\frac{1}{2}(f_1')^2 + \frac{1}{6}(f_2')^2 + (G')^2 + \right.$$

$$\left. + \frac{1}{r^2}(f_1^2 + f_2^2)G^2 + \frac{1}{2r^2}(G^2-1)^2 \right] \tag{3.48}$$

The equations (3.47) follow from varying the effective Hamiltonian (3.24) which now can be written as

$$\bar{H} = \frac{4\pi}{g^2} \int dr \left[-\frac{1}{2}(f_1')^2 - \frac{1}{6}(f_2')^2 + (G')^2 + \frac{1}{2r^2}(G^2-1)^2 - \right.$$

$$\left. - \frac{1}{r^2}(f_1^2 + f_2^2)G^2 + 2r(f_1 q_1 + \frac{1}{3} f_2 q_2) \right] \tag{3.49}$$

which coincides with (3.48) when Gauss' law, (3.47a) and (3.47b), is satisfied.

3.4 BOUNDARY CONDITIONS

Finiteness of the energy (3.48) requires that G goes to 1 and that f_1 and f_2 vanish at the origin. At large distances f_1 and f_2 vanish and G(r) becomes +1 or -1. Therefore, we are led to two types of solutions to (3.47), which differ in their asymptotic behaviour at large distances:

type I

$$\begin{aligned} f_1(0) &= 0 \qquad & f_1(\infty) &= 0 \\ f_2(0) &= 0 \qquad & f_2(\infty) &= 0 \\ G(0) &= 1 \qquad & G(\infty) &= 1 \end{aligned} \tag{3.50}$$

type II

$$\begin{aligned} f_1(0) &= 0 \qquad & f_1(\infty) &= 0 \\ f_2(0) &= 0 \qquad & f_2(\infty) &= 0 \\ G(0) &= 1 \qquad & G(\infty) &= -1 \end{aligned} \tag{3.51}$$

In other words, for type I solutions we have vanishing scalar and vector potential at the origin and at infinity whereas for type II solutions, the scalar potential has the same behaviour as the type I and the spatial part vanishes at the origin but it

becomes a non-trivial pure-gauge as r tends to infinity

$$A^{i}_{ab} \xrightarrow[r\to\infty]{} \frac{2i}{gr}\left(\delta_{ai}\frac{x_b}{r} - \delta_{bi}\frac{x_b}{r}\right) \tag{3.52}$$

The results which we will present have all been derived for a delta-function (more precisely, delta-shell) source, that is to say

$$q_{f_1}(r) = Q\,\delta(r-r_1) \tag{3.53}$$

and

$$q_{f_2}(r) = R\,\delta(r-r_2) \tag{3.54}$$

An advantage of the choice (3.53) and (3.54) is that Y-M equations (3.47) reduce to free equations (i.e., with $q_i(r)=0$) in the intervals $0<r<r_1$, $r_1<r<r_2$ and $r_2<r<+\infty$, together with requirement that f_1, f_2, G, and G′ be continuos at r_1 and r_2 whereas f_1' must have a discontinuity of magnitude -Q at r_1 and f_2' a discontinuity of magnitude -R at r_2.

The alternate formula for energy of the static Y-M theory can be written as

$$\mathcal{E} = \frac{8\pi}{g^2}\int dr\, r^2\left[\left(f_1' - \frac{f_1}{r}\right)Q\,\delta(r-r_1) + \frac{1}{3}\left(f_2' - \frac{f_2}{r}\right)R\,\delta(r-r_2)\right] \tag{3.55}$$

Thus the whole class of solutions depends on two parameters, the magnitudes (source strengths) Q and R.

Since $\mathcal{G}_{ab}$ is the gauge covariant quantity, a gauge transformation will mix the "monopole" (antisymmetric) and

"quadrupole" (symmetric) components. This fact forces us to find a gauge invariant characterization of the source.[5]

The gauge invariant characterization can be accomplished by calculating two invariant operators for the gauge group SU(3) - two Casimir operators (see Appendix B) which are nonlinear combinations of the group generators which commute with all the generators of the group. The first one is the quadratic operator (B-14)

$$C_2(r) = \mathrm{Tr}\,\rho^2 = \rho_{ab}\,\rho_{ba} \tag{3.56}$$

and since

$$\rho_{ac}^2 = (\delta_{ac} - \frac{x_a x_b}{r^2}) q_{f_1}^2 - \frac{i}{3}\,\varepsilon_{ack}\,\frac{x^k}{r}\,q_{f_1} q_{f_2} + \tag{3.57}$$

$$+ \frac{1}{3}\left(\frac{x_a x_c}{r^2} + \frac{1}{3}\,\delta_{ac}\right) q_{f_2}^2$$

the first invariant is

$$C_2(r) = 2\left[q_{f_1}^2(r) + \frac{1}{3}\,q_{f_2}^2(r)\right] \tag{3.58}$$

and with the specific choice of the colour charge distributions (3.53) and (3.54) upon integrating over r we get

$$C_2 = 2Q^2 + \frac{2}{3}R^2 \tag{3.59}$$

Integration of (3.58) is justified by the specific form of the

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[5]That the source should be described in a gauge invariant manner was pointed out to us by Prof.H. Pagels (Pagels 1980).

energy (3.55) as a consequence of the choices (3.53) and (3.54).

The second invariant is calculated from

$$\rho^3_{ab} = i\,\varepsilon_{abk}\,\frac{x^k}{r}\,q_1^3 + \frac{2}{3}\left(\frac{x_a x_b}{r^2} - \delta_{ab}\right) q_1^2\, q_2 + \tag{3.60}$$

$$+ \frac{1}{3}\left(\frac{x_a x_b}{r^2} - \frac{1}{9}\,\delta_{ab}\right) q_2^3$$

so

$$C_3(r) = \mathrm{Tr}\,\rho^3(r) = \rho_{ab}\,\rho_{bc}\,\rho_{ca} \tag{3.61}$$

and

$$C_3 = \frac{2}{9}\,R^3 \tag{3.62}$$

The energy, a gauge invariant quantity, still depends on the two source strengths, but only those combinations of them are admissible which simultaneously satisfy the equations (3.59) and (3.62), for a given pair of C_1 and C_2 .

The two expressions for the invariants can be brought to a more simple form by substitutions

$$r = \sqrt{\frac{2}{3}}\,R \quad ; \quad q = \sqrt{2}\,Q \tag{3.63a}$$

and

$$\tilde{C}_3 = \left(\frac{4}{3}\,C_3\right)^{1/3} \tag{3.63b}$$

so

$$C_2 = q^2 + r^2 \tag{3.64}$$

and

$$\tilde{C}_3 = r \tag{3.65}$$

Hence we can write

$$C_2 = \tilde{C}_3^2 + q^2 \tag{3.64}$$

The character of the solutions to the Y-M equations will give us restrictions on a choice of C_1 and C_2. In other words, the numerical calculations will show that there exist critical values of the strength parameters, so not all combinations of C_1 and C_2 are admissible but only those which yield the strength parameters above their critical values. The numerical calculation will give very accurate values of these critical strengths, so it will be simple to determine the suitable combinations of C_1 and C_2.

IV. NUMERICAL TECHNIQUES; COLSYS

In this chapter we will discuss some of the aspects of the numerical calculation as applied to the system of coupled, non-linear, ordinary differential equations developed in Chapter 3.

4.1 NUMERICAL APPROACH

The boundary value problem, i.e., the differential equations (3.47) and the boundary conditions (3.50), is defined on the semi-infinite interval $[0,\infty)$, but the program which we use requires a finite interval. Therefore we map the semi-infinite to the finite interval $[0,1]$ by means of the transformation

$$x = \frac{r}{1+r} \; ; \; r \in [0, \infty] \; ; \; x \in [0, 1] \tag{4.1}$$

The differential equations (3.47) now have the form

$$-f_1''(x) = \frac{2}{1-x} f_1'(x) - \frac{2}{x^2(1-x)^2} G^2(x) f_1(x) - \frac{2x}{(1-x)^5} q_1(x) \tag{4.2a}$$

$$-f_2''(x) = \frac{2}{1-x} f_2'(x) - \frac{6}{x^2(1-x)^2} G^2(x) f_2(x) - \frac{2x}{(1-x)^5} q_2(x) \tag{4.2b}$$

$$G''(x) = \frac{2}{1-x} G'(x) - \frac{1}{x^2(1-x)^2} \left[G^2(x) - 1 - f_1^2(x) - f_2^2(x) \right] G(x) \qquad (4.2c)$$

Having two sources of the delta-function type in the finite interval, it is convenient to take advantage of symmetry. Thus we choose radii of delta-shells in the semi-infinite interval in such way that they are mapped in $x_1 = 1/3$ and $x_2 = 2/3$. From the property of the delta-function

$$\delta^{(\alpha)}[g(x)] = \sum_n \frac{1}{|g'(x_n)|} \left[\frac{1}{g'(x)} \cdot \frac{d}{dx} \right]^{\alpha} \delta(x - x_n) \qquad (4.3)$$

(Krein 1972) it is easy to calculate that $r_1 = 1/2$ and $r_2 = 2$ will be mapped in the appropriate way.

From equations (4.2) it is evident that since f" is proportional to a delta-function, f' is discontinuous. Upon integrating the first equation around x_1 we get

$$-\int_{x_1-\varepsilon}^{x_1+\varepsilon} f_1''(x)\, dx = 2 \int_{x_1-\varepsilon}^{x_1+\varepsilon} \frac{x}{(1-x)^5} Q\, \delta\left(\frac{x}{1-x} - \frac{1}{2} \right) dx \qquad (4.4)$$

so the discontinuity is given by the following expression

$$f_1'\left(\tfrac{1}{3} - \varepsilon\right) - f_1'\left(\tfrac{1}{3} + \varepsilon\right) = \frac{9}{4} Q \qquad (4.5)$$

and repeating the same calculation with the other equation we obtain

$$f_2'\left(\tfrac{2}{3}-\varepsilon\right) - f_2'\left(\tfrac{2}{3}+\varepsilon\right) = 36\,R \tag{4.6}$$

As pointed out in Chapter 3, the advantage of the choice of the so-called forcing terms in the form of delta-shells is in the fact that the equations become free in corresponding intervals. But some difficulties remain when the problem must be formulated in a way that the numerical calculation can be applied. Thus we define for f_1 and f_2 four functions in the following way

$$\begin{aligned} f_1(x) &= \varphi_1(x) \qquad 0 \le x < \tfrac{1}{3} \\ f_1(x) &= \varphi_2(x) \qquad \tfrac{1}{3} \le x \le 1 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} f_2(x) &= \varphi_3(x) \qquad 0 \le x < \tfrac{2}{3} \\ f_2(x) &= \varphi_4(x) \qquad \tfrac{2}{3} \le x \le 1 \end{aligned} \tag{4.8}$$

Therefore we have enlarged the system of differential equations, since the functions φ_i satisfy

$$\varphi_1''(x) = 0 \quad \tfrac{1}{3} \le x \;;\; \varphi_2''(x) = 0 \quad x < \tfrac{1}{3} \;; \tag{4.9a}$$

and

$$\varphi_3''(x) = 0 \quad \tfrac{2}{3} \le x \;;\; \varphi_4''(x) = 0 \quad x < \tfrac{2}{3}\,. \tag{4.9b}$$

Therefore, the new solution vector now has the components

$$\underset{\sim}{u}(x) = \left[\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x), G(x)\right] \tag{4.10}$$

The functions must be matched at x=1/3 and x=2/3 so the boundary conditions become

$$\varphi_1(0)=0\ ;\ \varphi_3(0)=0\ ;\ G(0)=1\ ;\ \varphi_1\left(\tfrac{1}{3}\right)=\varphi_2\left(\tfrac{1}{3}\right);\ \varphi_2(1)=0 \tag{4.11a}$$

and

$$\varphi_1'\left(\tfrac{1}{3}\right)-\varphi_2'\left(\tfrac{1}{3}\right)=\frac{9}{4}Q\ ;\ \varphi_3\left(\tfrac{2}{3}\right)=\varphi_4\left(\tfrac{2}{3}\right)\ ;$$
$$\varphi_3'\left(\tfrac{2}{3}\right)-\varphi_4'\left(\tfrac{2}{3}\right)=36R\ ;\ \varphi_4(1)=0;\ G(1)=\pm 1 \tag{4.11b}$$

The expression for the energy (3.48) now has the form

$$\mathcal{E}=\frac{4\pi}{g^2}\int_0^1 dx\left\{(1-x)^2\left[(f_1')^2+\frac{1}{6}(f_2')^2+(G')^2\right]+\right. \tag{4.12}$$
$$\left.+\frac{1}{x^2}(f_1^2+f_2^2)G^2+\frac{1}{2x^2}(G^2-1)^2\right\}$$

4.2 COLSYS

In this section we will present a very simplified description of the general purpose code COLSYS (COLlocation for SYStems) for solving boundary value problems for mixed order systems of ordinary differential equations. More detailed explanations are given in Ascher, Christiansen and Russell 1979a and 1979b. [1]

Consider a mixed order system of d nonlinear differential equations of orders $1 \le m_1 \le m_2 \cdots \le m_d < 4$

$$u_n^{(m_n)}(x) = F_n(x; \underset{\sim}{Z}(\underset{\sim}{u})) \; ; \quad a < x < b \; ; \quad n = 1, \ldots, d \tag{4.13a}$$

where

$$\underset{\sim}{u}(x) = [u_1(x), u_2(x), \ldots, u_d(x)] \tag{4.13b}$$

is a solution vector and

$$\underset{\sim}{Z}(\underset{\sim}{u}) = (u_1, u_1', \ldots, u_1^{(m_1-1)}, \ldots, u_d^{(m_d-1)}) \tag{4.13c}$$

is the vector of unknowns that would result from converting (4.13a) to a first order system. The system is subject to

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[1] It is our pleasure to point out that one of the authors, Mr. Jan Christiansen with his unselfish help made this work possible.

$$m^* = \sum_{n}^{d} m_n \qquad (4.14)$$

nonliner multipoint separated boundary conditions

$$g_j(\xi_j ; \underset{\sim}{Z}(\underset{\sim}{u})) = 0 \; ; \quad j = 1,\ldots,m^*, \qquad (4.15)$$

where ξ_j is the location of the j-th boundary (or side) condition, and

$$a \le \xi_1 \le \xi_2 \le \ldots \le \xi_{m^*} \le b. \qquad (4.16)$$

In our case, for the system (4.2) enlarged by the definitions (4.7) and (4.8) we have: d=5, m_i=2, (i=1,..,5), a=0, b=1, F is everything except the second derivative in (4.2). In addition

$$\underset{\sim}{Z}(\underset{\sim}{u}) = (\varphi_1, \varphi_1', \varphi_2, \varphi_2', \varphi_3, \varphi_3', \varphi_4, \varphi_4', G, G') \; ; \quad m^* = 10$$

$$\xi_1 = 0 \; ; \; \xi_2 = 0 \; ; \; \xi_3 = 0 \; ; \; \xi_4 = \tfrac{1}{3} \; ; \; \xi_5 = \tfrac{1}{3} \; ; \; \xi_6 = \tfrac{2}{3} \; ; \; \xi_7 = \tfrac{2}{3}$$

$$\xi_8 = 1 \; ; \; \xi_9 = 1 \; ; \; \xi_{10} = 1 \; ; \; g_1 = Z_1 \; ; \; g_2 = Z_5 ; \qquad (4.17)$$

$$g_3 = Z_9 - 1 \; ; \; g_4 = Z_1 - Z_3 \; ; \; g_5 = Z_2 - Z_4 - \tfrac{9}{4} Q \; ; \; g_6 = Z_5 - Z_7$$

$$g_7 = Z_6 - Z_8 - 36R \; ; \; g_8 = Z_3 \; ; \; g_9 = Z_7 \; ; \; g_{10} = Z_9 \mp 1.$$

To solve (4.13) and (4.15) numerically, collocation at Gaussian points (zeroes of the appropriate Legendre polynomials) is applied, using piecewise polynomial functions. If π is a partition of [a,b]

$$\pi : \; a = x_1 < x_2 < \ldots < x_N < x_{N+1} = b \qquad (4.18a)$$

where $\{x_i\}$ points form a mesh

$$I_i = (x_i, x_{i+1}) ;\ h_i = x_{i+1} - x_i ;\ i = 1, \ldots, N \tag{4.18b}$$

and

$$h = \max_{1 \le i \le N} h_i \tag{4.18c}$$

and

$$P_{k,\pi} = \{ v \mid v \text{ is a polynomial of order } k , \quad (\deg < k) \text{ on } I_i , \quad i = 1, \ldots, N \} \tag{4.18d}$$

then an appropriate solution is

$$\underset{\sim}{v} = (v_1 , v_2 , \ldots , v_d) \tag{4.19}$$

such that

$$v_n \in P_{k+m,\pi} \cap C^{(m_n - 1)}_{[a,b]} ;\ n = 1, \ldots, d \tag{4.20}$$

where $C_{[a,b]}$ is the class of continuous functions on the interval [a,b] and by the definition

$$f(x) \in C^{(\ell)}_{[a,b]} \iff f^{(\ell)}(x) \in C_{[a,b]} ;\ x \in [a,b] \tag{4.21}$$

The requirement is that k>m where k is the number of collocation points per subinterval.

It is convenient to define the collocation points in the following way

$$x_{ij} = \frac{x_i + x_{i+1}}{2} + \frac{1}{2} h_i \rho_j \qquad (4.22)$$

where $\{\rho_j\}_{j=1}^{k}$ are roots of the Legendre polynomials mapped in each interval.

The collocation equations which v has to satisfy are

$$v_n^{(m_n)}(x_{ij}) = F_n(x_{ij}, \underset{\sim}{Z}(\underset{\sim}{v})) \; ; \qquad (4.23)$$

$$j = 1, \ldots, k \; ; \; i = 1, \ldots, N \; ; \; n = 1, \ldots, d$$

and (4.15). Therefore the solution is determined by requiring that it satisfy the differential equations at the images of the k zeros of the appropriate Legendre polynomials in each element. The nonlinear problems are solved using the Newton iteration.

Error estimates are

$$\| u_n^{(\ell)} - v_n^{(\ell)} \| = \Theta(h^{k+m_n-\ell}) \; ; \; \ell = 0, \ldots, m_n \; ; \; n = 1, \ldots, d \, , \qquad (4.24)$$

and at mesh points

$$|(u_n^{(\ell)} - v_n^{(\ell)})(x_{ij})| = \Theta(h^{2k}) \; ; \; i = 1, \ldots, N \; ; \qquad (4.25)$$

$$j = 1, \ldots, k \; ; \; \ell = 0, \ldots, m_n - 1 \; ; \; n = 1, \ldots, d.$$

To use COLSYS the user must specify a set of tolerances. COLSYS allows the user to specify different tolerances for different components.

Mesh points are selected automatically, although an initial mesh can be defined by the user. The mesh selection

procedure is free to change every interior mesh point during refinement. This provides for flexible adjustment of meshes to incorporate extreme characteristics of the solution with relatively small meshes.

For instance, for the solution G(x) for the type II family, third group (see Fig.5.21) with a sharp peak at 2/3, and rather irregular shape, the mesh was selected on 38 subintervals, with error tolerances equal 10^{-5} .

COLSYS allows continuation, i.e., it is possible to use a former solution as an initial approximation for the current nonlinear iteration with the possible choice of mesh: either first mesh for the current problem is constructed from every second point of the final mesh for the former problem, or the meshes are identical.

An important feature of COLSYS is that the appropriate solution values at any points, and hence solution plots are readily available. Plots with high resolution as those in Chapter 5 can be easily generated by using mesh points plus a sufficient number of equally spaced points (500 in our case).

V. RESULTS

The finite energy SU(3), static, spherically symmetric solutions corresponding to our Ansatze (3.39)and (3.46) for the gauge fields in the presence of sources of the form (3.38) satisfy the coupled non-linear ordinary differential equations (3.47a, 47b and 47c), in the radial coordinate r. Their energy is given by (3.48) or equivalently, by (3.49).

The differential equations are supplemented by the boundary conditions (3.50) and (3.51).

We have not succeeded in obtaining analytic solutions to these equations for any reasonable choice of the source strength characterized by the functions $q_1(r)$ and $q_2(r)$. However we have obtained very interesting results by numerical techniques. These results form the subject matter of this chapter.

All results below are for extended SU(3) static source of the delta-shell type. These are taken to be

$$q_1(x) = \frac{9}{4} Q \, \delta(x - \tfrac{1}{3}) \tag{3.53}$$

and

$$q_2(x) = 36 R \, \delta(x - \tfrac{2}{3}) \tag{3.54}$$

We remark that the static non-Abelian source is characterized by two parameters Q and R and these do not represent two different sources as one will obtain for Abelian sources. The solutions are presented as functions of Q and R. Let us recall that there are two distinct types of boundary conditions imposed on the spatial components A^i_{ab} of the gauge potential. For type-I (given by (3.50))

$$A^i_{ab} \xrightarrow[r\to\infty]{} 0 \tag{5.1}$$

while for type-II solutions

$$A^i_{ab} \xrightarrow[r\to\infty]{} \frac{2i}{gr}\left(\delta_{ai}\frac{x_b}{r} - \delta_{bi}\frac{x_a}{r}\right) \tag{5.2}$$

which is a non-trivial pure gauge, i.e., it is not homotopically equivalent to zero gauge potential.

This chapter is organized as follows: in section 5.1 type-I solutions are presented for various combinations of Q and R and their energy content is plotted.

In section 5.2 the more interesting type-II solutions and their properties are discussed.

5.1 TYPE-I SOLUTIONS

As explained in Chapter 4, the semi-infinite interval $[0,\infty)$ in r is mapped on to unit circle [0,1] in the variable x. The radii r_1 and r_2 of the two delta-shells of strengths Q and R respectively are taken to be $x_1=1/3$ for Q and $x_2=2/3$ for R.

Figures 5.1 and 5.2 show the behaviour of G(x) (which is the radial part of A^i_{ab}) for different values of the strength combination (Q,R).

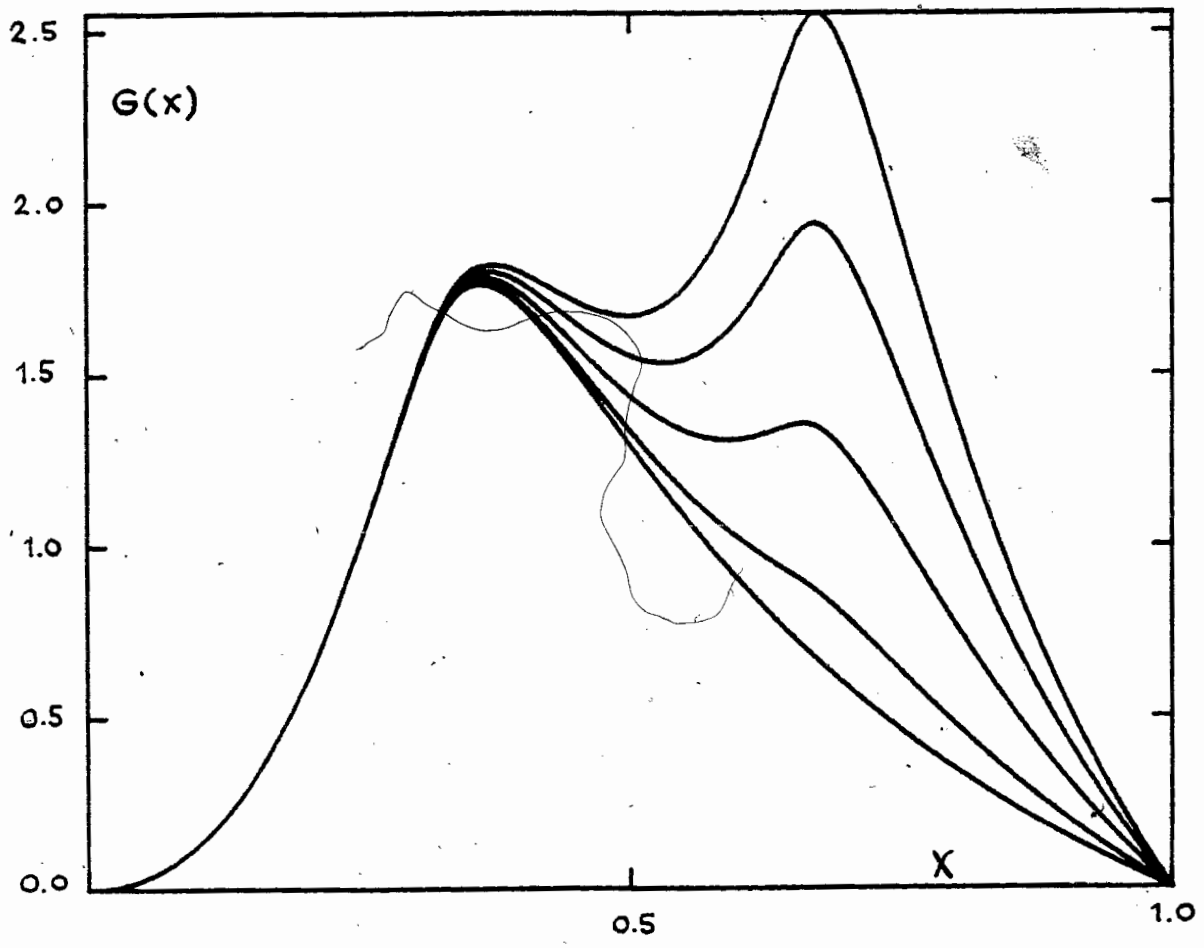

Fig.5.1 Solutions for G(x) of type-I family (Q fixed at 12.0)

In Fig.5.1 G(x) is displayed for the following strength combinations, starting from the lowest curve: (12.0,0.0), (12.0,0.5), (12.0,1.0), (12.0,1.5) and (12.0,2.0).

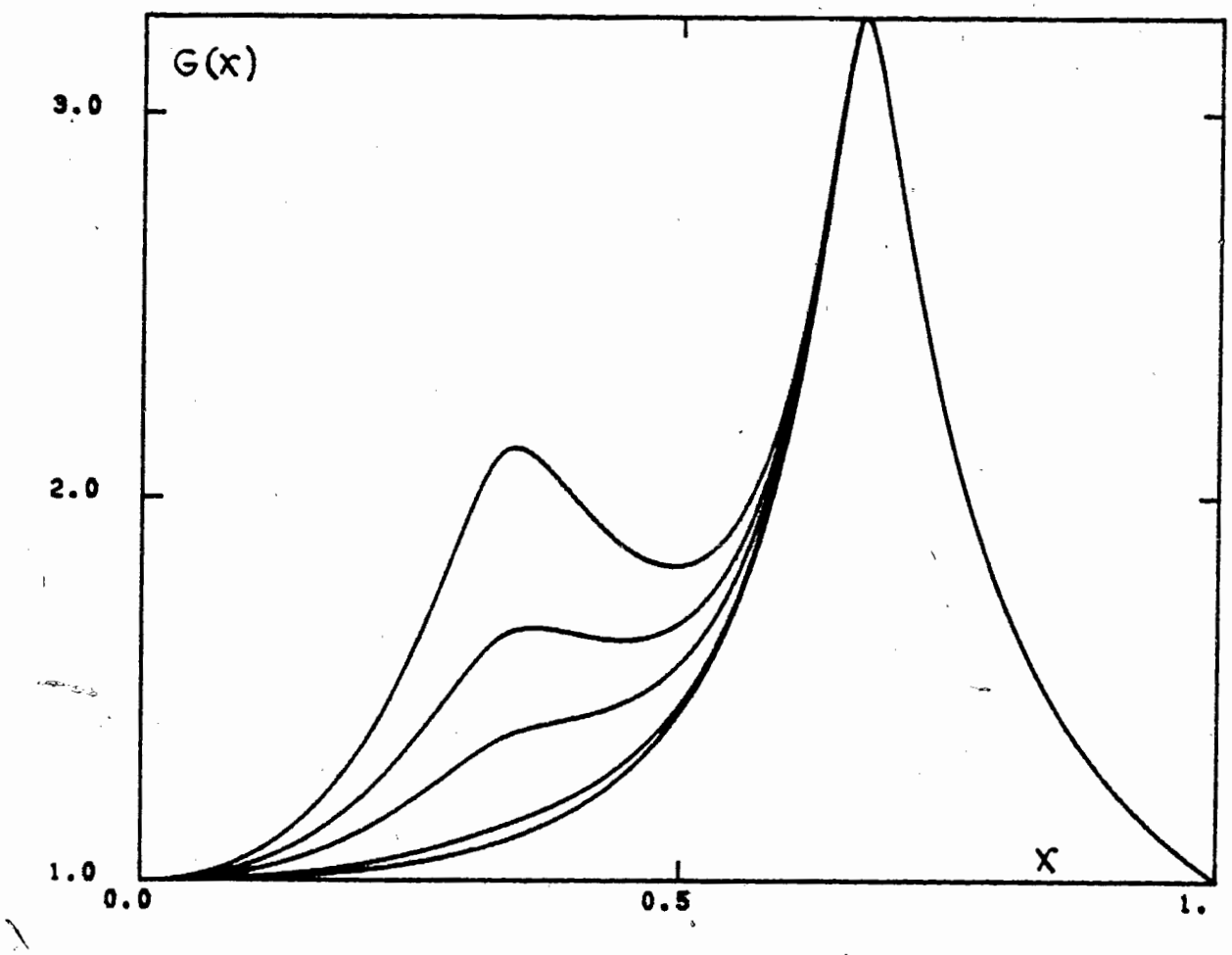

Fig.5.2 Solutions for G(x) of type-I family (R fixed at 12.0)

For Fig.5.2 data are: (0.0,12.0), (4.0,12.0), (12.0,12.0), (20.0,12.0) and (36.0,12.0), starting from the lowest curve.

The behaviour of $f_1(x)$ and $f_2(x)$ which characterize the SO(3) and SU(3)/SO(3) parts of the scalar part of A^{μ}_{ab} of the gauge potential are shown in Figs.5.3 and 5.4, again for different (Q,R) combinations.

We note that $f_2(x)$ is identically zero for R=0 and Q arbitrary. This can be easily inferred from the equations of motion (3.47). This of course corresponds to the SO(3) embedding in SU(3) and this is identical to the solutions for SU(2) previously found by Jackiw, Jacobs and Rebbi (1979).

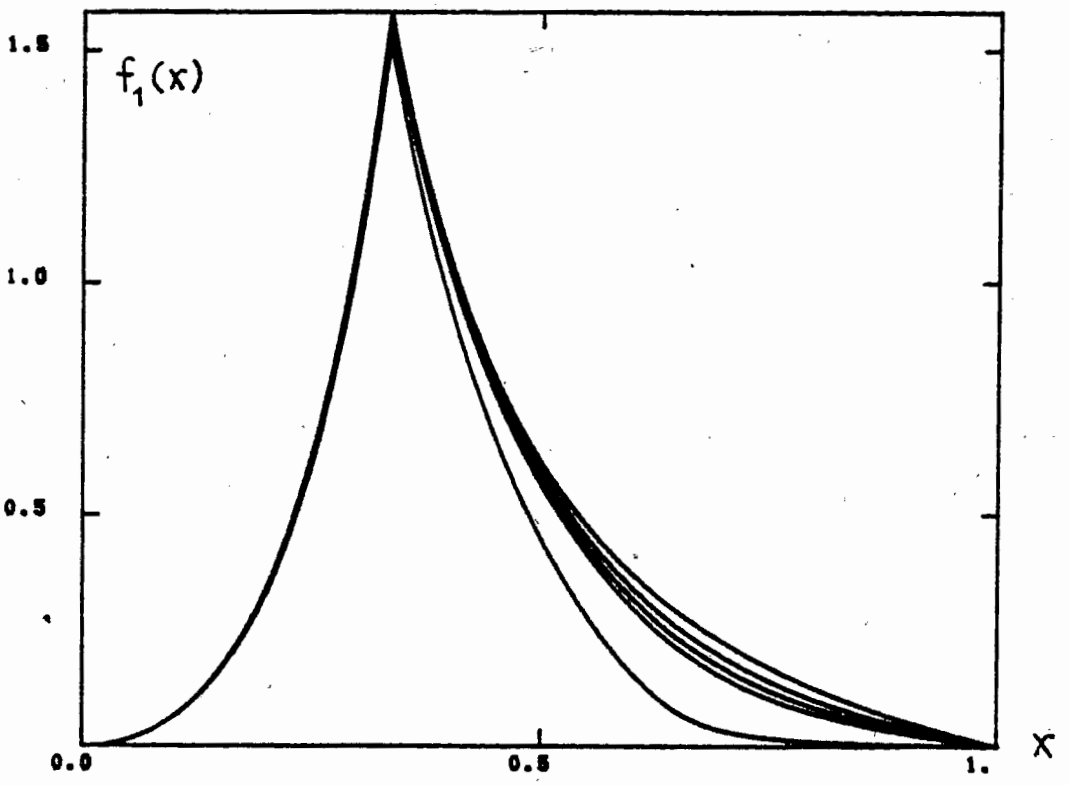

Fig.5.3 $f_1(x)$ for fixed Q=12.0

The five solutions correspond to the following combinations: (12.,0), (12.,1.0), (12.,1.5) (12.,2.0) and (12.,12.), starting from the highest curve.

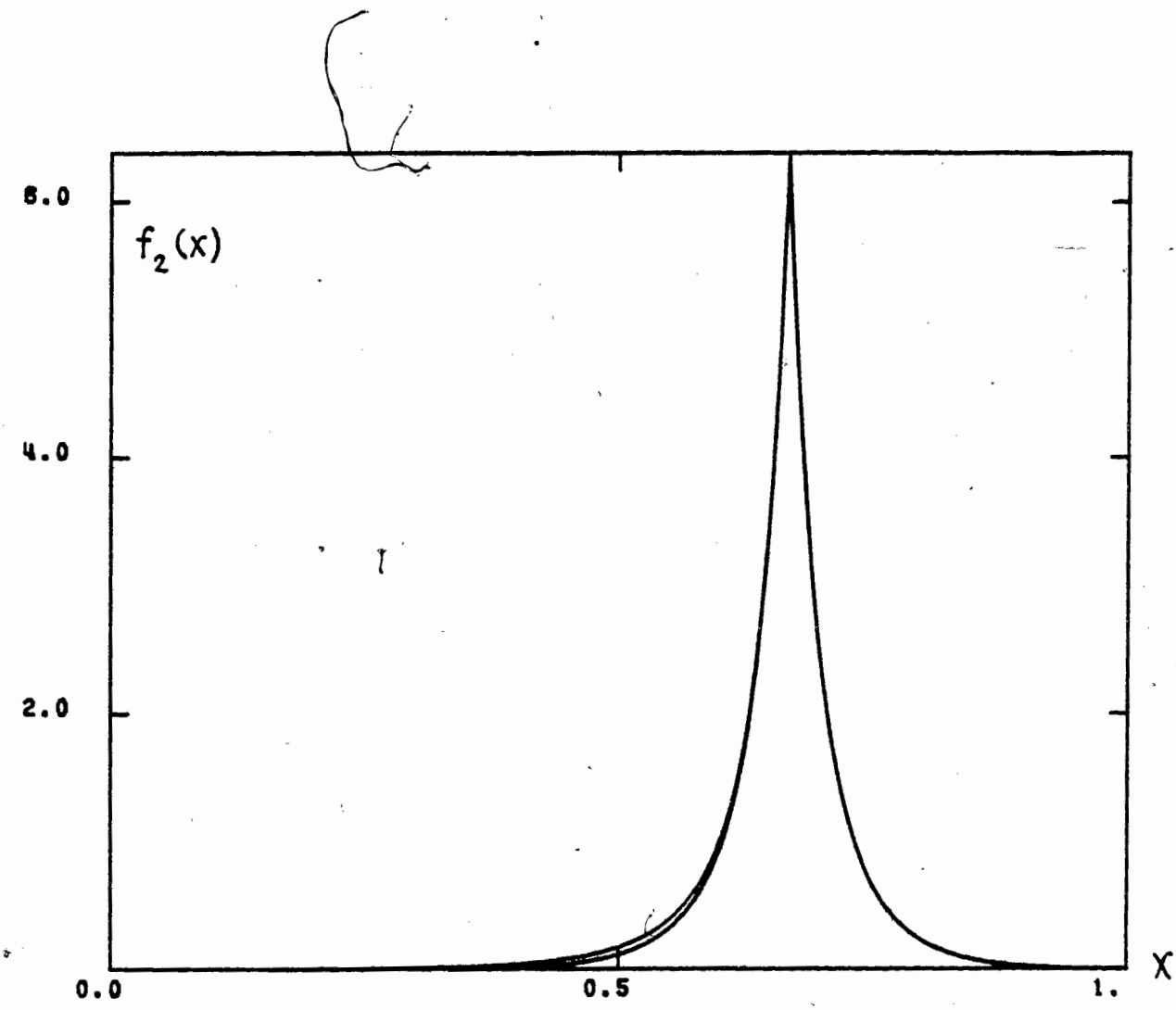

Fig.5.4 $f_2(x)$ for fixed R=12.0

There cotrespond two curves to Q=12 (lower) and Q=36 with fixed R=12.

It is worth remarking that the dependence of the solutions on Q and R is somewhat asymmetric. This arises for the following reason: when one transforms from the variable r to x, there arise extra factors in front of the delta shell sources as

explained in Chapter 4. The quadrupole-like source described by R gets multiplied by a larger factor (due to its being placed at x=2/3), than the monopole term.

Energies of these solutions are plotted in Fig.5.5.

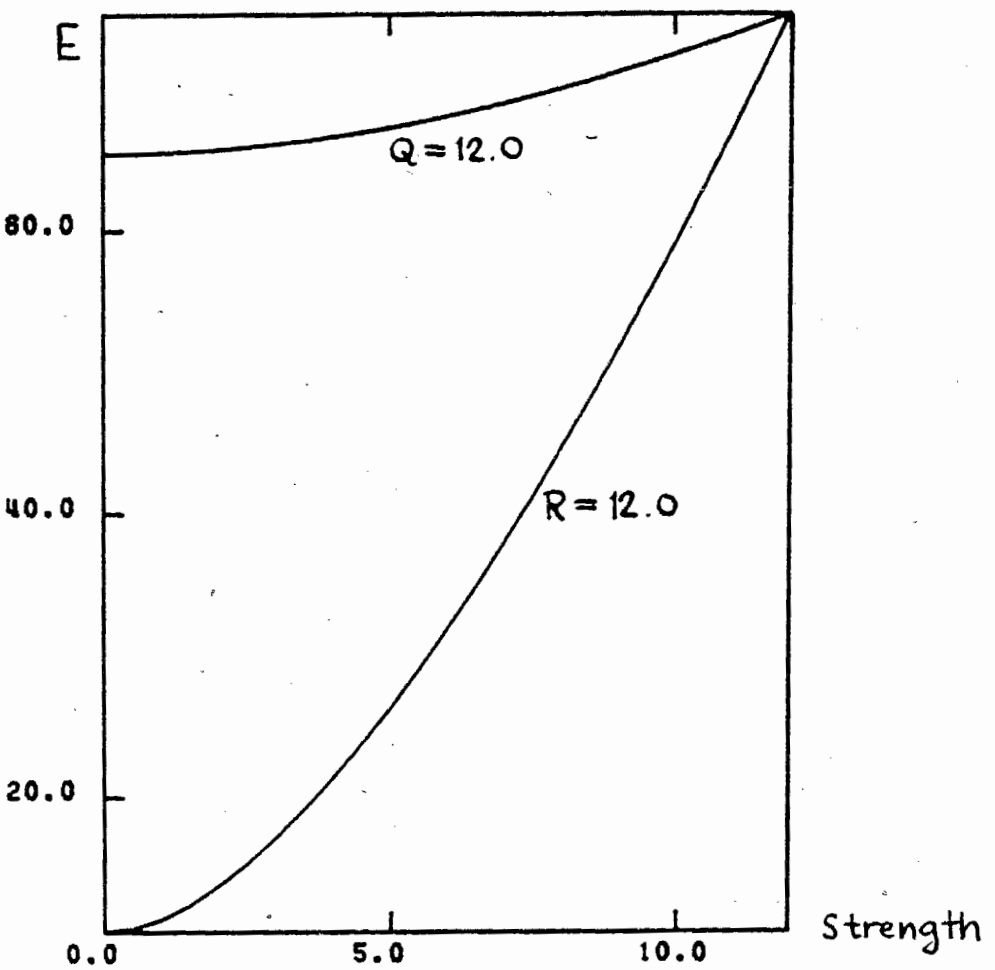

Fig.5.5 Energy vs. strength of type-I family

These are computed from formula (3.48) in units of $4\pi/g^2$.

Energy is a gauge invariant quantity. It is plotted here as a function of Q and R. Even though they are by themselves not a priori gauge invariant, the discussion in Chapter 3 on the Casimir invariants of SU(3) shows that since C_2 is proportional to R hence R is gauge invariant. Hence Q can also be used as gauge invariant object, instead of C_1 .

5.2 TYPE-II SOLUTIONS

Solutions of this type which differ form the ones in the previous section in their asymptotic behaviour, showed suprisingly remarkable properties. Numerical calculations yielded six different sets of solutions for a given combination of Q and R. All these solutions exist only for sufficiently strong sources. Below a critical value of Q or R or both Q and R, solutions cease to exist. For Q and R greater than the critical values, solutions bifurcate, i.e., there exist two branches of solutions which have different energies for a given combination of Q and R. In the case of SU(2) we reproduce the results of Jackiw, Jacobs and Rebbi (1979) to better precision. In this case R=0 and there exists only one critical source strength Q_c. However when both Q and R are different from zero, we observe many more bifurcating solutions. We try to systematize these solutions below.

Type-II solutions divide into three groups.

a. The first group contains two solutions for a given value of Q and R. The solutions reduce to SU(2) solutions mentioned above as R tends to zero. These solutions are characterized by the existence of a critical value of Q which changes as R is varied.

b. The second group again contains two solutions for given (Q,R), but they are different from the group-one solutions by not only having different energy content: they are characterized by a critical value in R, which exists for every Q. Hence these solutions have no SU(2) counterparts. In fact these are genuine SU(3) solutions which we have discovered, as far as we know, for the first time. Again for $R>R_c$ the solutions bifurcate.

c. In this group solutions cease to exist for $(Q,R)<(Q_c,R_c)$, contrary to previous cases where there exists only one critical strength either in Q or in R. For every $R>R_c$ there are two separate branches of solutions for $Q>Q_c$. R_c's are different for the two branches, and solutions simply terminate at $R=R_c$. For $R>R_c$, as Q approaches Q_c the solutions coincide. Therefore, this is another pair of genuine SU(3) solutions which we have discovered, with no lower symmetry counterpart.

We elaborate on these solutions now.

5.2.1 Type-II First group solutions

In all the calculations the radii of delta shells are taken as for type-I solutions to be 1/3 and 2/3 for Q and R respectively. We take an appropriate value for Q and R and change one of these, say Q, while the other is kept fixed. This way we found that there exists a Q_c. For every value of the pair ($Q>Q_c$,R) we found two distinct sets of solutions which merge into one as Q tends to Q_c. When R is continuously varied we find that in the limit of $R \to 0$ we reproduce the solution of Jackiw and collaborators.

In Fig.5.6 we have plotted G(x) for different values of Q (for a fixed R=12.0), as a function of x.

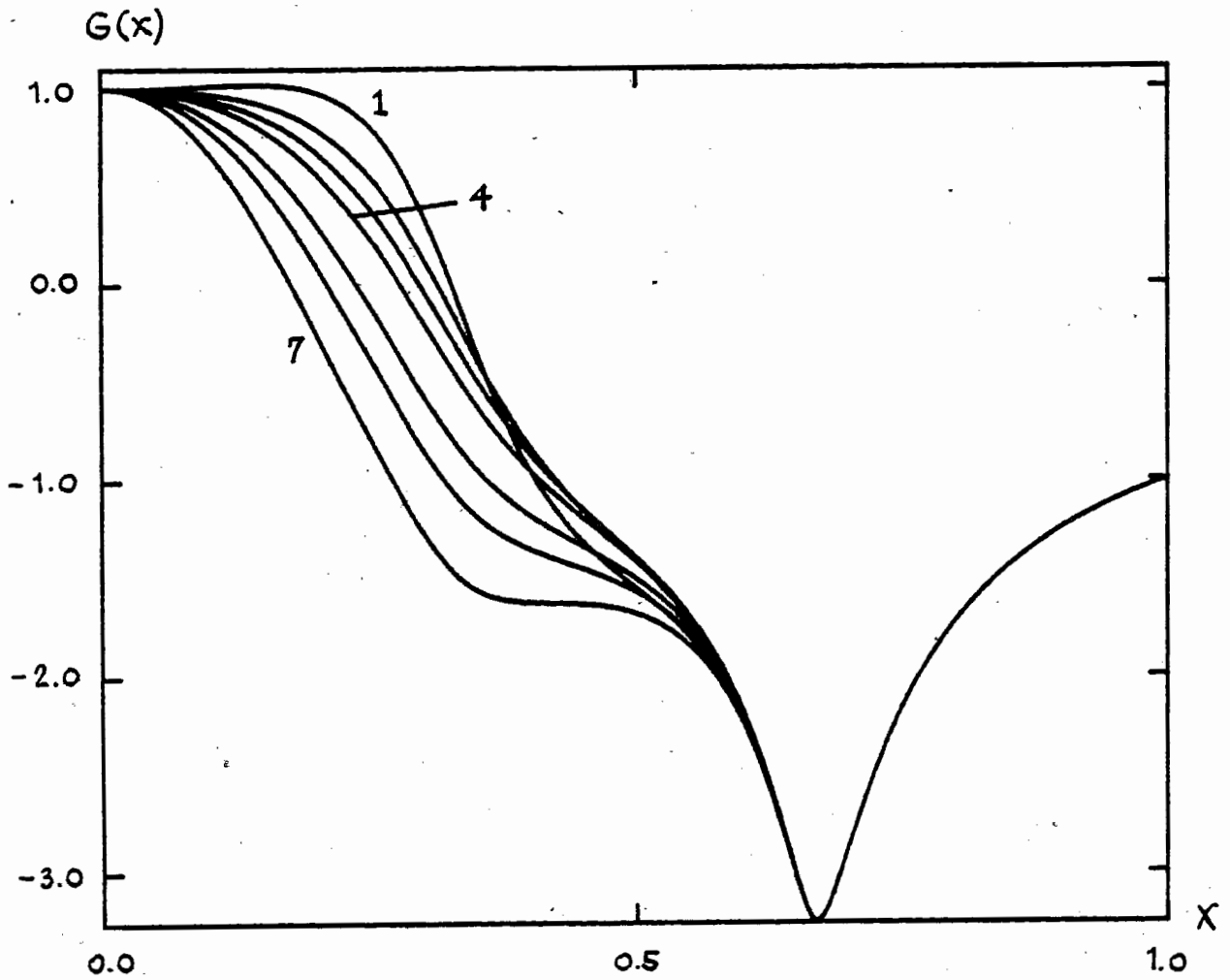

Fig.5.6 G(x) for the group one of type-II family (R fixed at 12.0)

The critical value Q for this case is 11.236. To that combination corresponds the fourth curve from the above in the figure. The curves in Fig.5.6 correspond to the configuration-strength combinations with the corresponding energies:

Q=20., R=12., E=116.774
Q=14., R=12., E=95.377

Q=12., R=12., E=89.603
$Q=Q_c$=11.3, R=12., E=87.794
Q=12., R=12., E=89.417
Q=14., R=12., E=94.047
Q=20., R=12., E=108.803

We can exploit the behaviour of the function G(x) to establish a method for finding a second branch solution once the first has been found.

Suppose we have found a solution which is suspected to have a mate since there is a critical strength below which the solution does not exist. Suppose that this is the solution which corresponds to curves 1,2,3 and 4 in Fig.5.6. In the region about $x_o \cong 0.2$ we can see that $G(x_o)$ decreases as Q approaches Q_c, and that the region below the curve corresponding to the critical strength is "forbidden". We now enlarge the system of differential equations by putting

$$Q' = 0 \tag{5.3}$$

and impose an additional boundary condition by requiring that G(x) at the point x_o has a value in the forbidden region (for instance, $G(x_o)$=0.0 in the case from Fig.5.6). In other words, we force a supposed solution to pass through the point which is not allowed for the solutions we already have. Hence, we let Colsys determine an additional "function" Q, plus the new solution vector, "similar" to the existing one but on the other

branch. If an initial approximation is good enough (it is convenient to take the solution vector for the initial approximation not far from the critical point) and the additional boundary condition is carefully chosen (not too far from the critical value) the method turns out to be feasible.

$f_1(x)$ and $f_2(x)$ are plotted in Fig.5.7 as functions of x for the same source strengths as used to plot G(x).

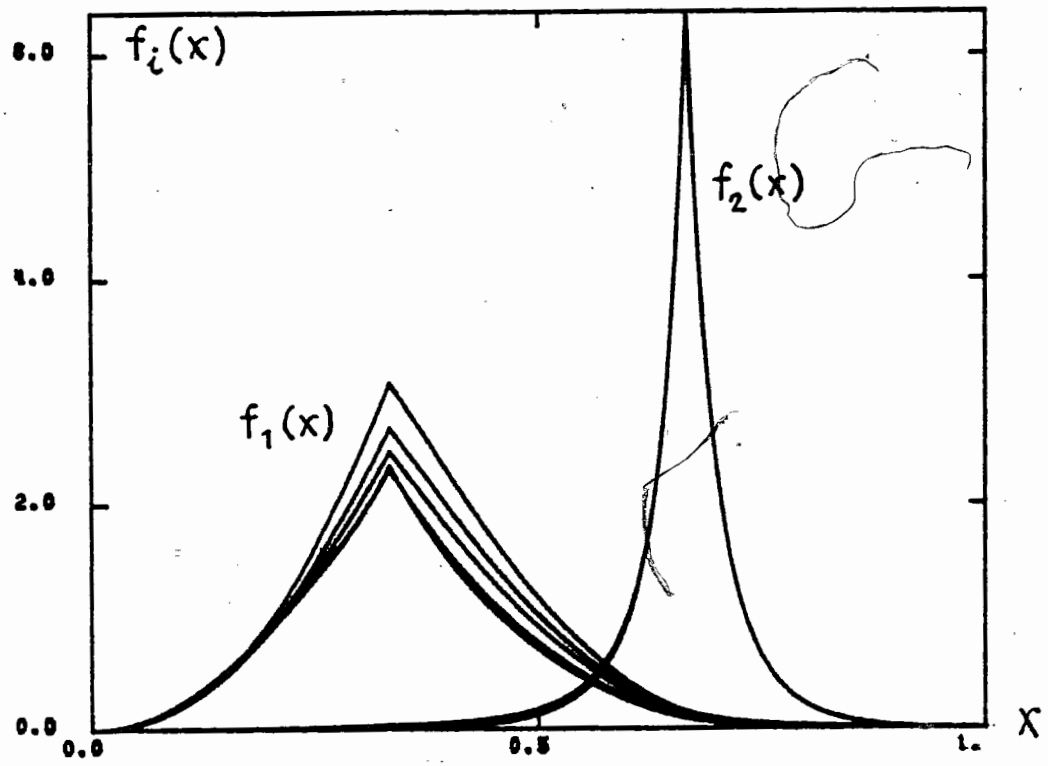

Fig.5.7 $f_1(x)$ and $f_2(x)$ solutions for the type II family (R fixed at 12.0)

In Fig.5.7 there are 5 solutions for $f_1(x)$ shown for the following values of Q: 14.0 (Energy=94.047), 12.0 (E=89.603), 11.3 - critical strength (E=87.794), 12.0 (E=89.417) and 14.0

(E=95.377). $f_2(x)$ is displayed for Q=20.0 with the upper branch energy being 116.774 and the lower branch energy being 108.803.

The solutions shown in Figs.5.6 and 5.7 are calculated for R=12.0. In the following two figures Q strength is fixed, and solutions are plotted as functions of x for different Q's.

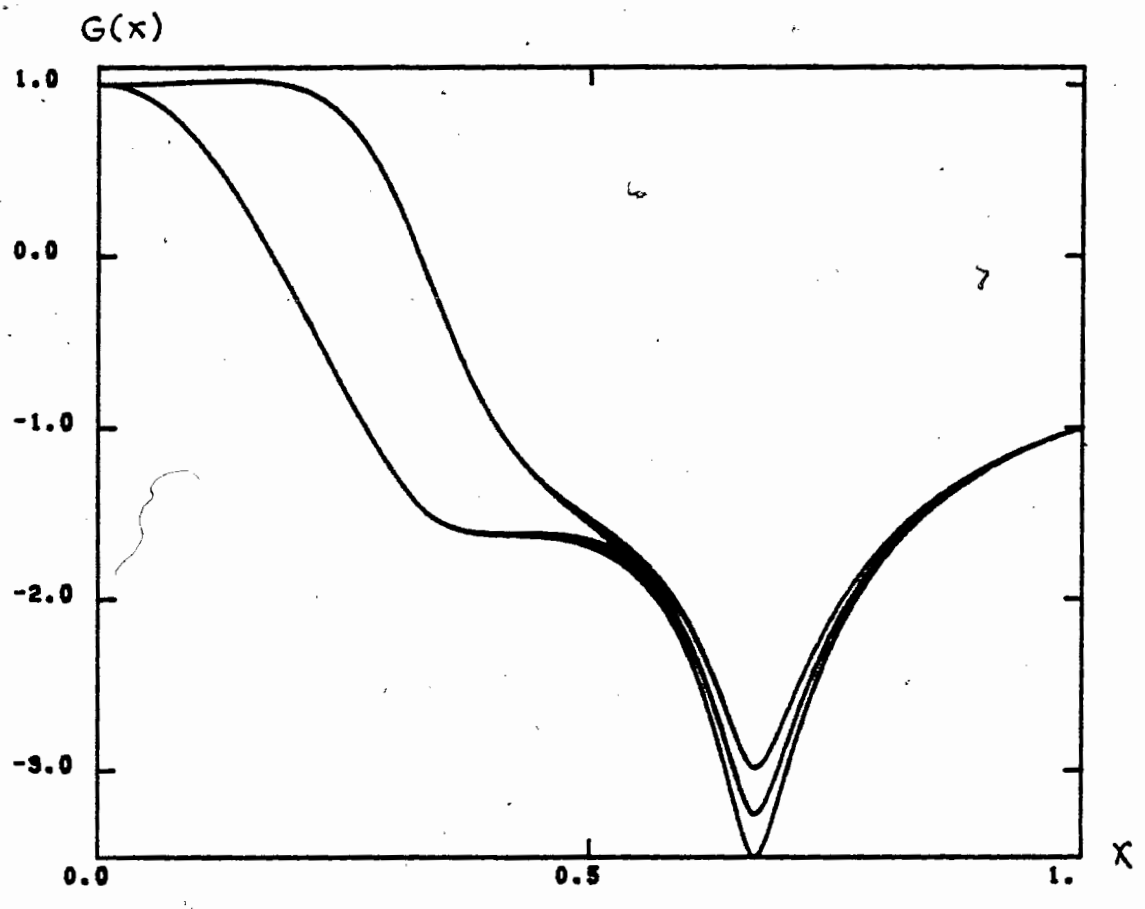

Fig.5.8 G(x) solutions for the group one, type II family (Q fixed at 12.0)

A carefull examination will show that there are 6 solutions displayed in Fig.5.8 which are divided in pairs in the region x<0.5 corresponding to two different branch solutions. In the

region x>0.5 they are grouped in a triplet according to three R strengths, 10,12 and 14.

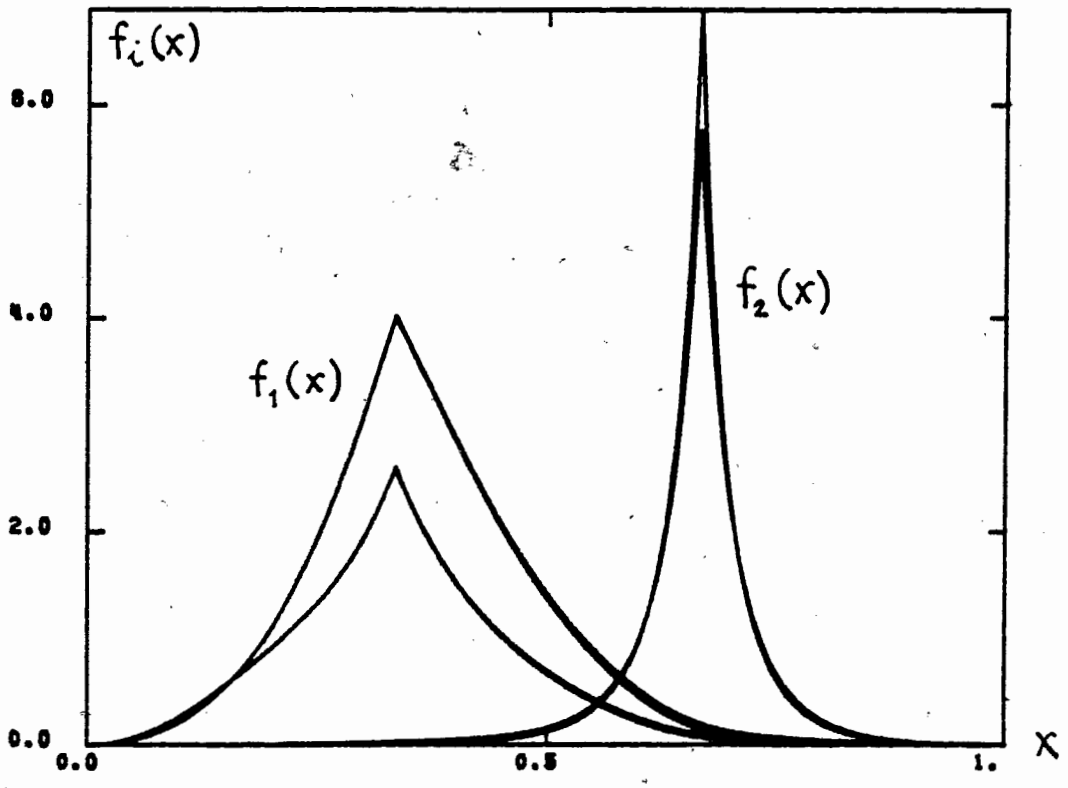

Fig.5.9 $f_1(x)$ and $f_2(x)$ of type-II family, first group (Q fixed at 12.0)

Two pairs of $f_1(x)$ are shown for (Q,R)=(10.0,12.0) and (20.0,12.0) respectively. $f_2(x)$ is plotted for R=12.0 and 20.0.

The typical energy dependence on Q and R is shown in Fig.5.10 and 5.11.

In Fig.5.10 E is plotted as a function of Q. R is fixed at 12.0. The energy bifurcates at Q_c=11.236. For comparison we have plotted the energy of type-I solutions in the

same graph. We note that type-II solutions have higher energy for all values of Q.

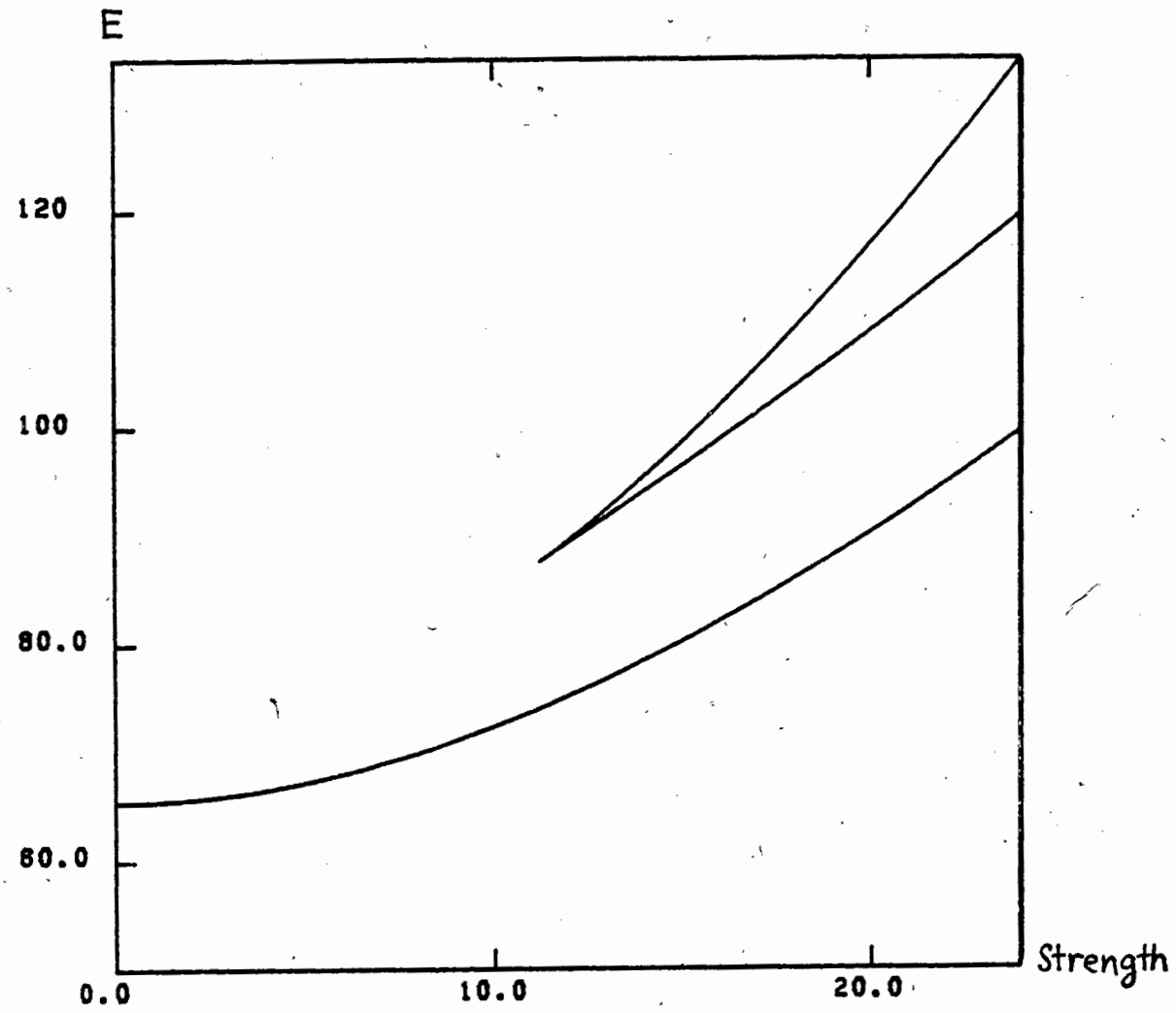

Fig.5.10 Energy vs. Q of type-I and type-II, group one solutions (R is fixed at 12.0)

In Fig.5.11 we exhibit the behaviour of E as a function of R when Q is fixed at 12.0. There are three parabolic shaped curves, and cusp-like behaviour of E from Fig.5.10.

Evidently, the energy is defined for all values of R. For the fixed value of Q=12.0 there are two different parabolic-like curves which are almost identical (at least in the figure) since Q =11.23632812 for R=12.0 and the energy difference between the upper and the lower branch is very small:

$$\Delta \mathcal{E} = 0.063 \tag{5.4}$$

These two curves are presented by the lowest parabola. Two other curves describe E vs. R for Q=20.0.

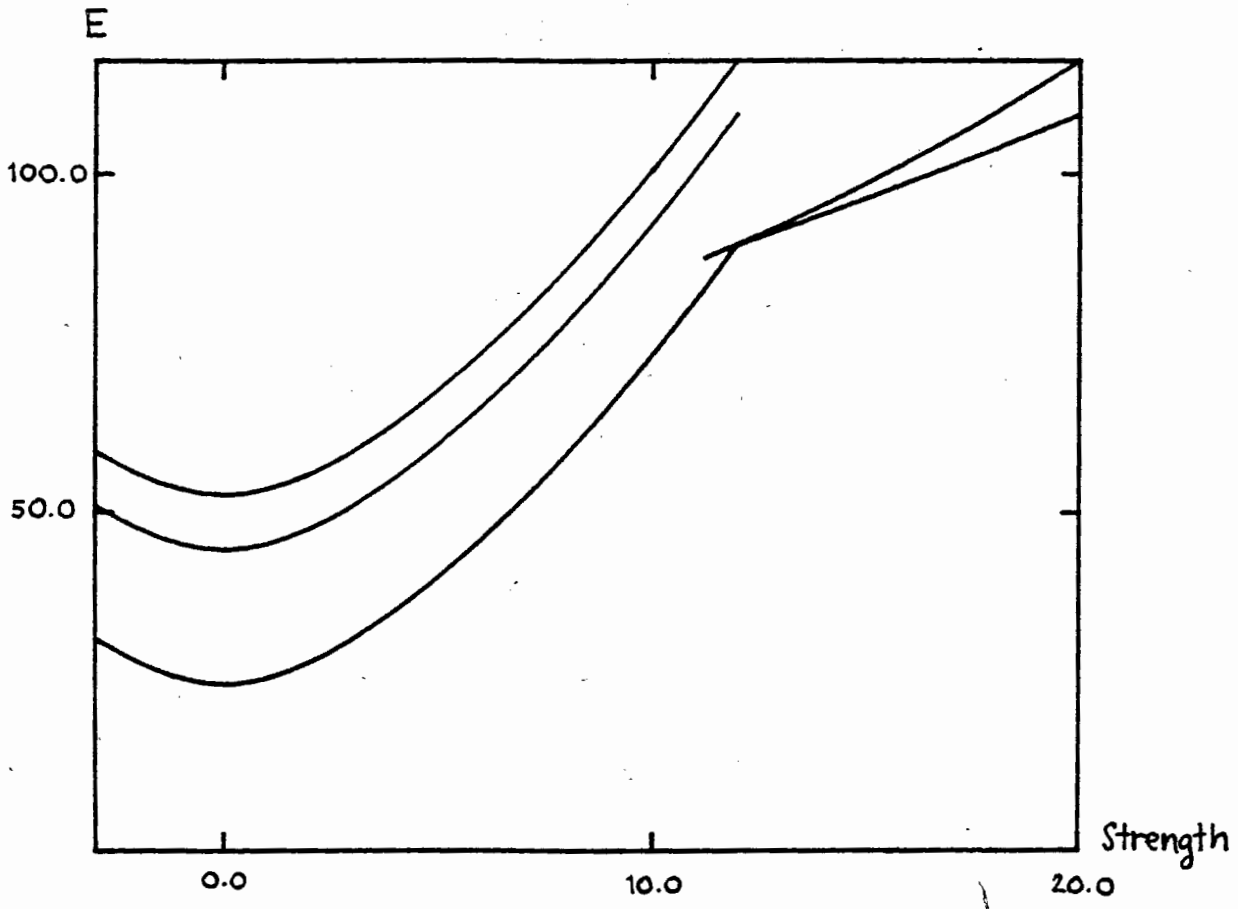

Fig.5.11 Energy vs. strength for the group one of type-II family

By moving the delta-shells along the x axis, we can bring both Q and R on the same delta-shell. Due to the symmetry of the unit interval, we choose the radius of the delta shell as 0.5. Solutions are presented in Figures 5.12, 5.13 and 5.14.

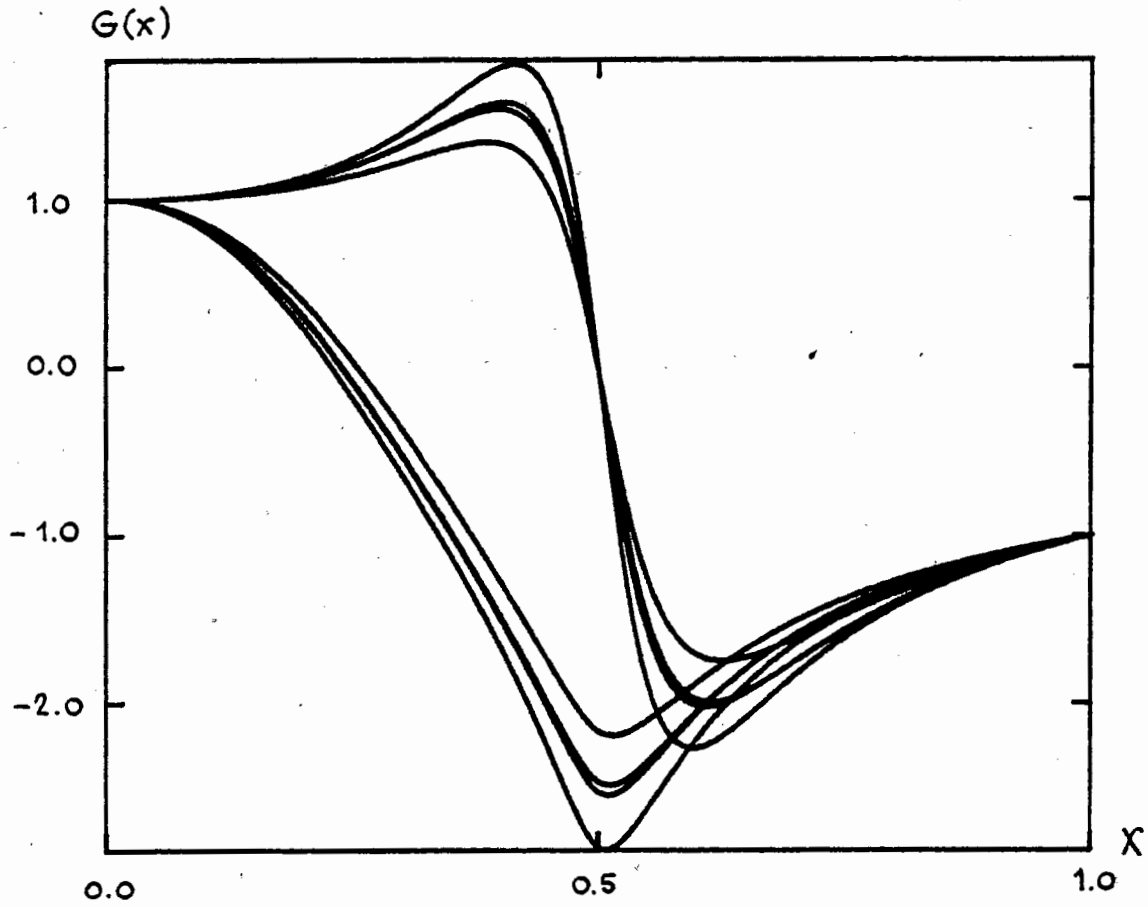

Fig.5.12 G(x) for single delta-shell distribution; group one, type-II solutions

We have again two sets of solutions for the same combination of Q and R as before.

The curves displayed in the figures are results for the (Q,R) combinations: (16.0,12.0), (12.0,12.0), (5.0,12.0) and

(12.0,5.0). From data is evident that Q_c is not any more 11.236: critical value goes down when the radius of a delta-shell over which a corresponding charge is distributed increases.

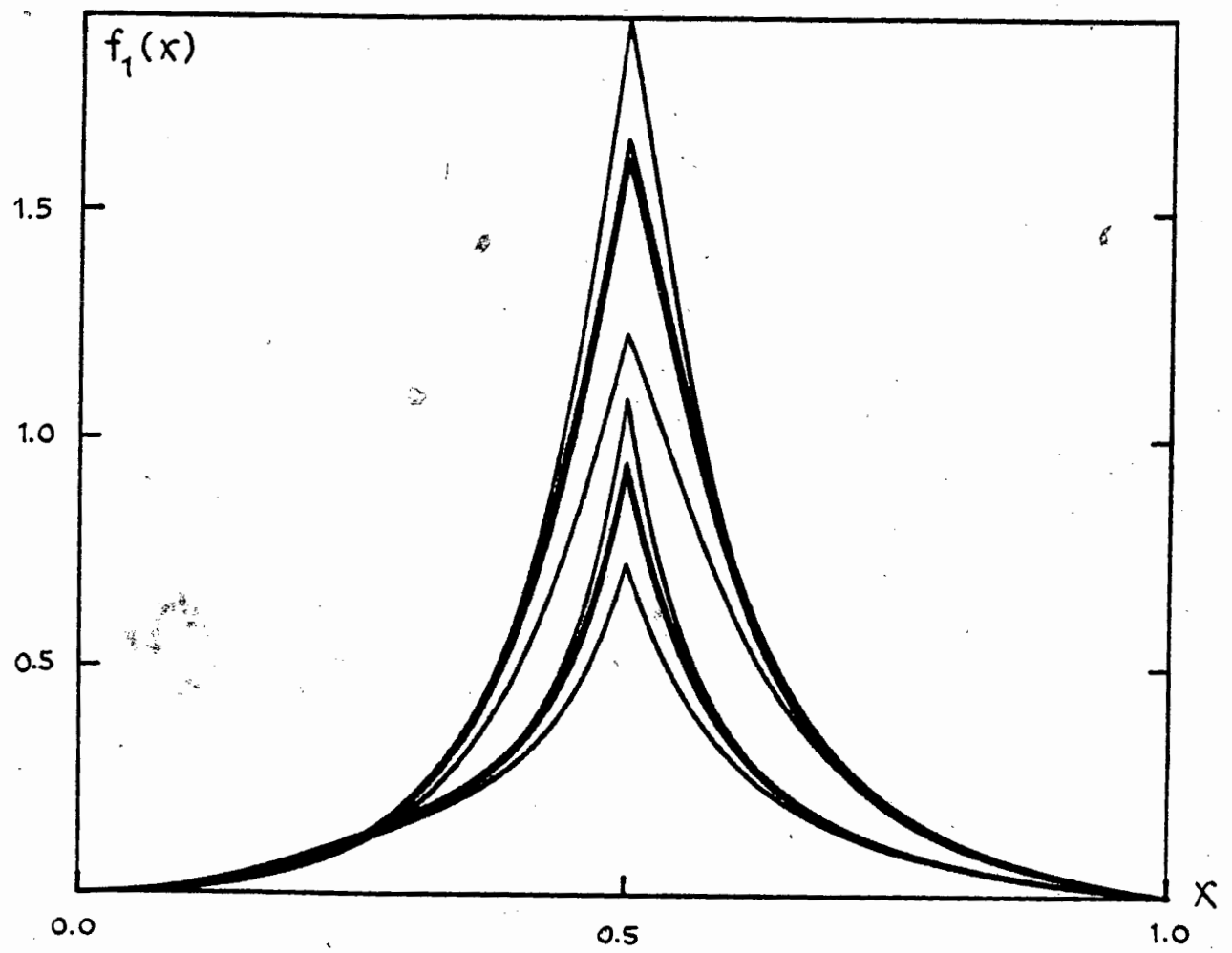

Fig.5.13 $f_1(x)$ for single delta-shell distribution; group one, type-II solutions

In their analysis for SU(2) gauge group, Jackiw, Jacobs and Rebbi computed their solutions with the radius of Q as x=0.5.

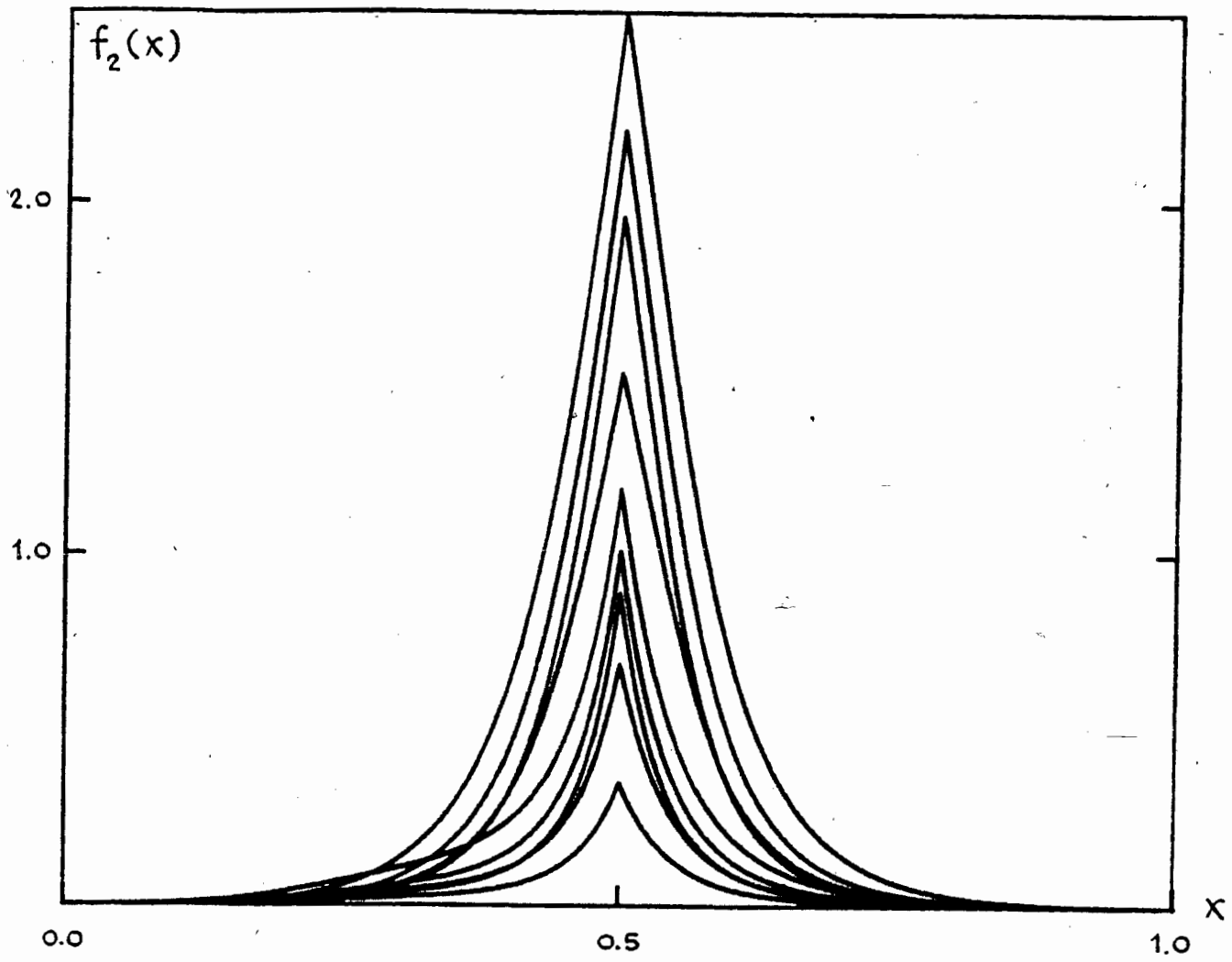

Fig.5.14 $f_2(x)$ for single delta-shell distribution; group one, type-II solutions

As a check, we found that for R=0, our results are in excellent agreement with theirs. This provides a nice check on the numerical accuracy.

In order to investigate how the energy depends on the distance between delta-shells, we fix a position of the monopole

charge contribution and move the quadrupole along the x axis. The strength parameters must be chosen carefully because, as we pointed out before, the critical strength depends on the radius of the shell.

For (Q,R) equal (12,12) and for x_1=1/3 the energy dependence is crudely drawn in the following figure.

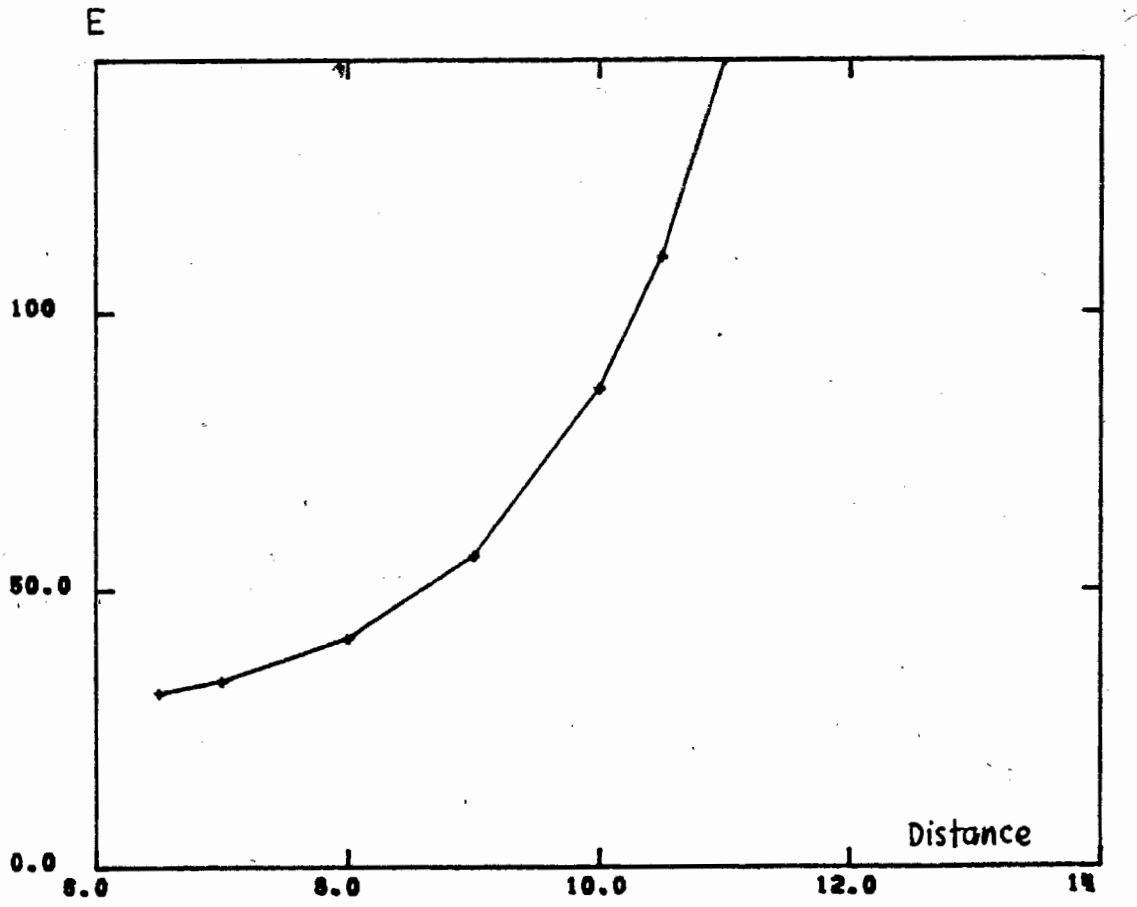

Fig.5.15 Energy vs. distance between delta-shells

This is the typical energy behaviour and this calculation will

not be repeated for the other groups since it does not give, there, any new feature.

The total energy of the solutions increases with relative distance. This is an indication of the fact that the solutions have larger spatial extent as one of the sources is moved further from the origin.

5.2.2 Type II - second group

The problem setup is as before: radii are 1/3 and 2/3, boundary conditions are as for group-one and Q=12, R=12. But there is a difference which gives the solutions of this group quite a different character: instead of having a Q_c, the solutions exist for every Q. But there is a R_c, basically with the same consequences: solutions split and the energy bifurcates for $R>R_c$.

Therefore there is no possibility of obtaining the lower symmetry i.e., SU(2) situation as was the case for the group one solutions. However, by continuing Q to zero we get a "coset-symmetry", the case when only the quadrupole contribution remains.

The critical strength R_c depends upon Q. This dependence is weak, as can be seen from Table 5.1.

Q	0.0	2.5	4.5	5.0	12.0
R_c	1.1360	$\gtrsim$ 1.2	$\gtrsim$ 1.3	1.357	1.812

Table 5.1 R_c dependence upon Q

The most significant difference between solutions for the type I and type II family can be seen by looking at G(x). In Fig. 5.16 and 5.17 G(x) is shown for different sequence of strengths.

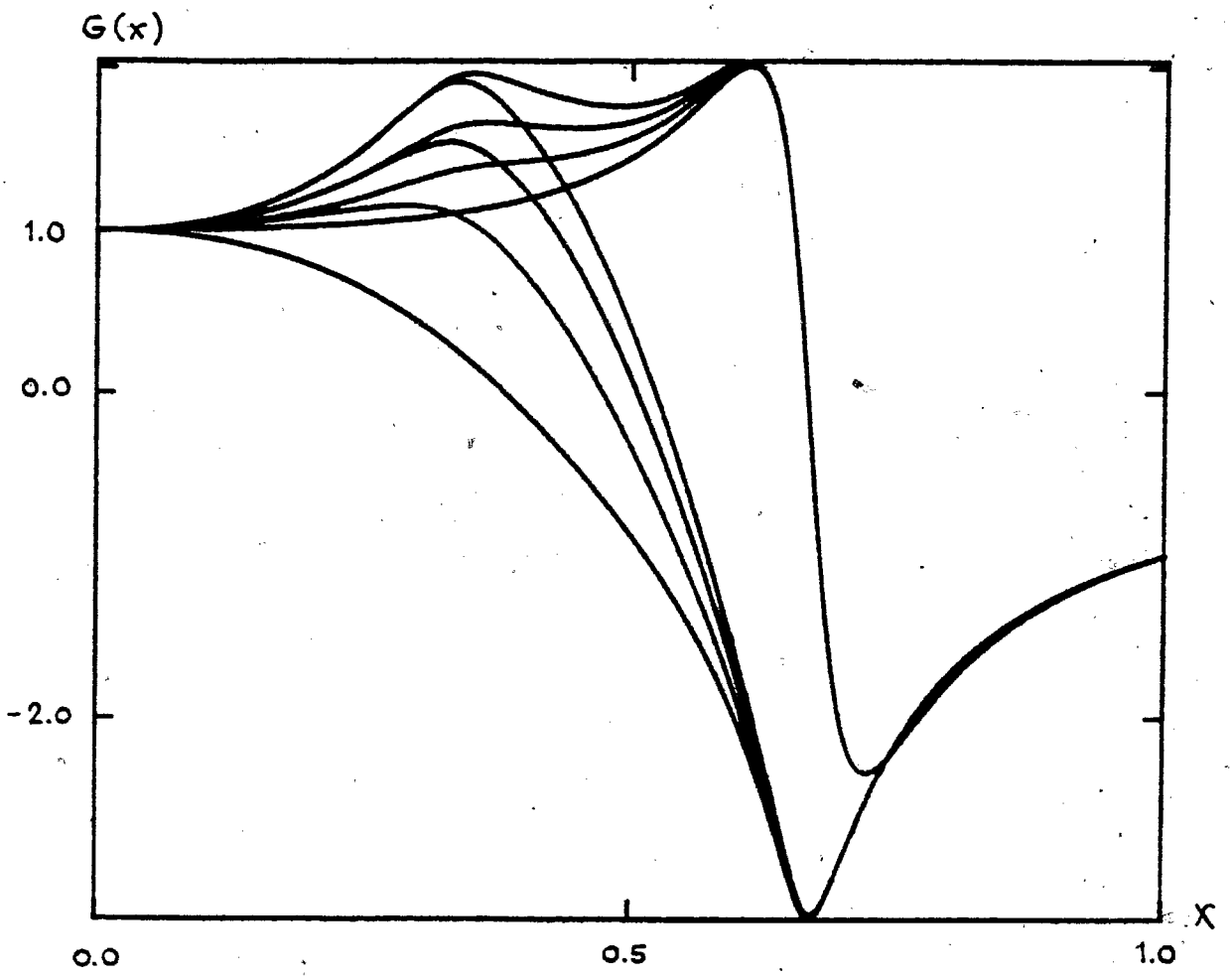

Fig.5.16 G(x) of type-II family, group two (R fixed at 12.0)

In Fig.5.16 solutions are displayed for Q equals: 0.0, 12.0, 20.0 and 30.0 (starting from the lowest curve in region $x \lesssim 0.5$). It is easy to recognize pairs of solutions which give upper and

lower energies: at the point $x \simeq 0.25$ mates stem from the same line (except the solutions for Q=0). All these solutions almost coincide in the region where the quadrupole contribution is dominant.

This is not the case when R changes, as is shown in Fig.5.17.

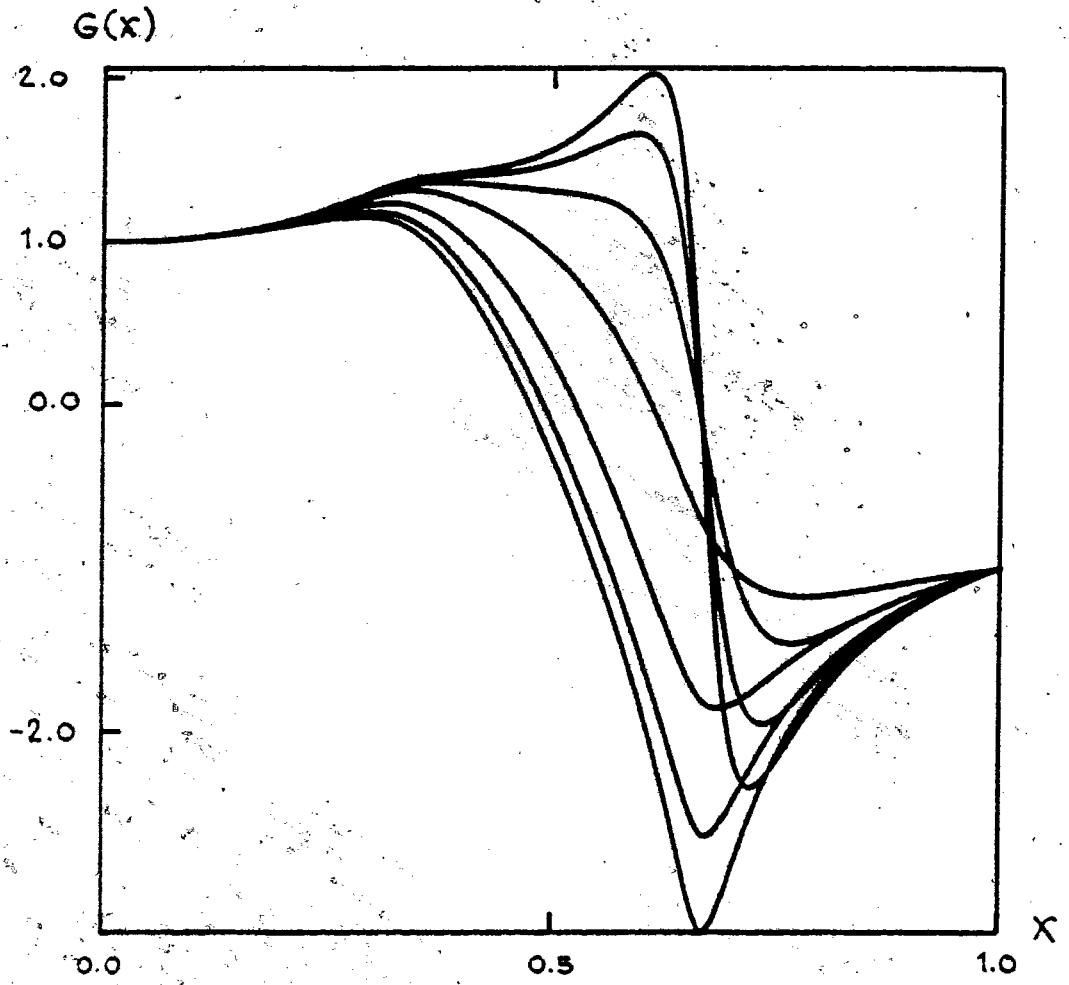

Fig.5.17 G(x) of type-II family (Q fixed at 12.0)

Here we have seven solutions for Q fixed at 12.0. For this value

of Q, R_c=1.8120 and the curve in the middle in Fig.5.17 is the solution for this combination. At $x \tilde{=} 0.75$ one can recognize which pair belongs to a given (Q,R). R values depicted in the figure are 1.82, 4.0, 8.0 and 12.0.

In the next two figures $f_1(x)$ and $f_2(x)$ are shown.

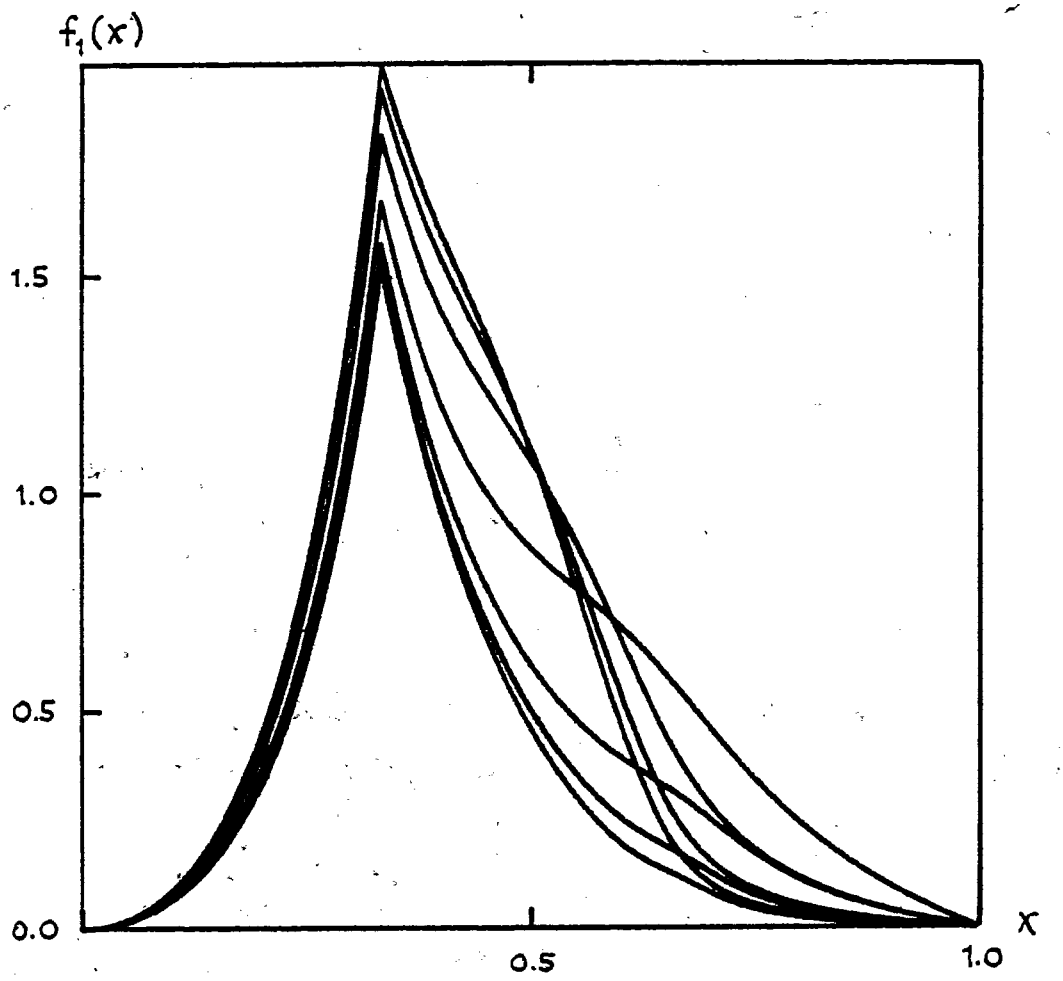

Fig.5.18 $f_1(x)$ of type-II family, group two (Q fixed at 12.0)

Their shapes are similar to the solutions of the first group. Their pairing is evident again, expecially for $f_1(x)$ in the

region around $x \tilde{=} 0.8$. The curves displayed in the figures are solutions for the strength combinations: (Q=12,R=12), (12,8), (12,4) and (12,1.82).

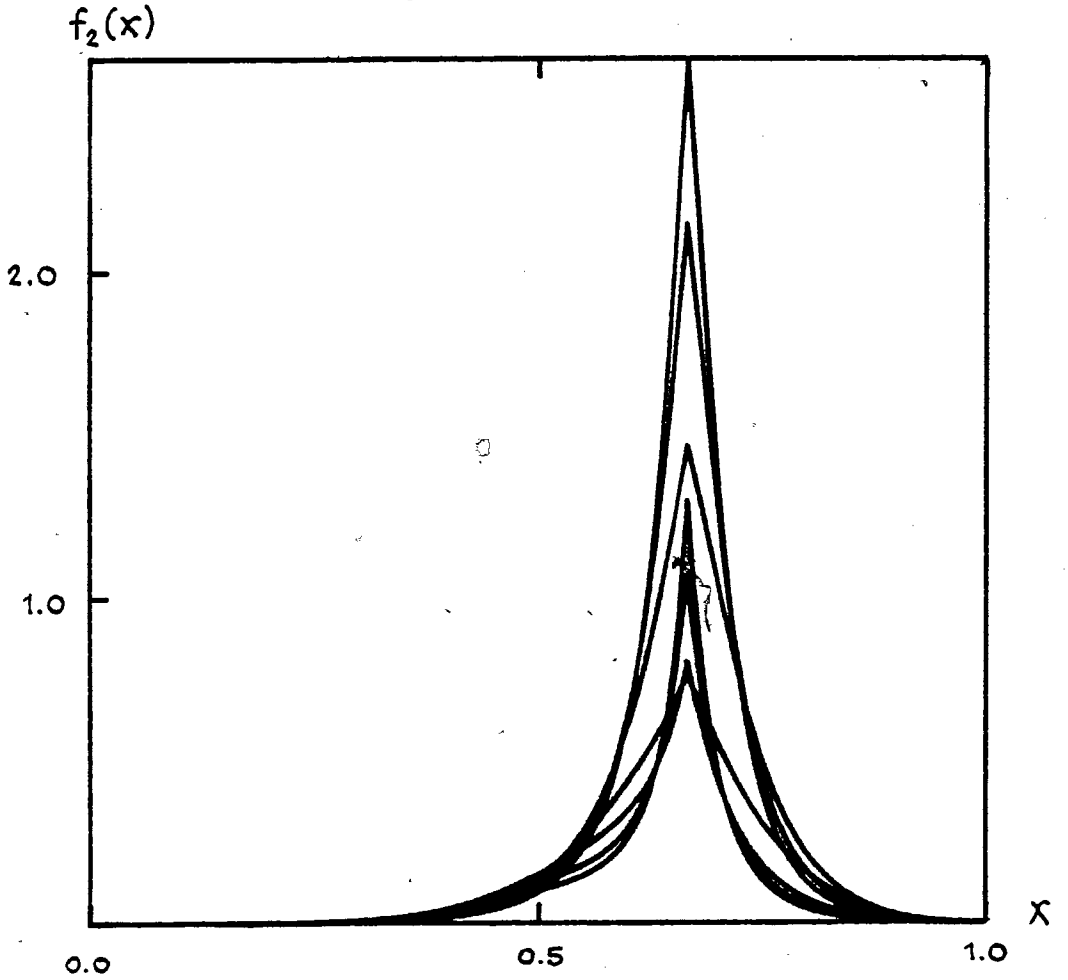

Fig.5.19 $f_2(x)$ of type-II family (Q fixed at 12.0)

The complete energy dependence on R is shown in Fig.5.20. The figure requires a careful description: going along the energy axis, the curve belonging to type-I family (Fig.5.5) is met first: Q is fixed at 12.0 and R varies from zero to 20.0.

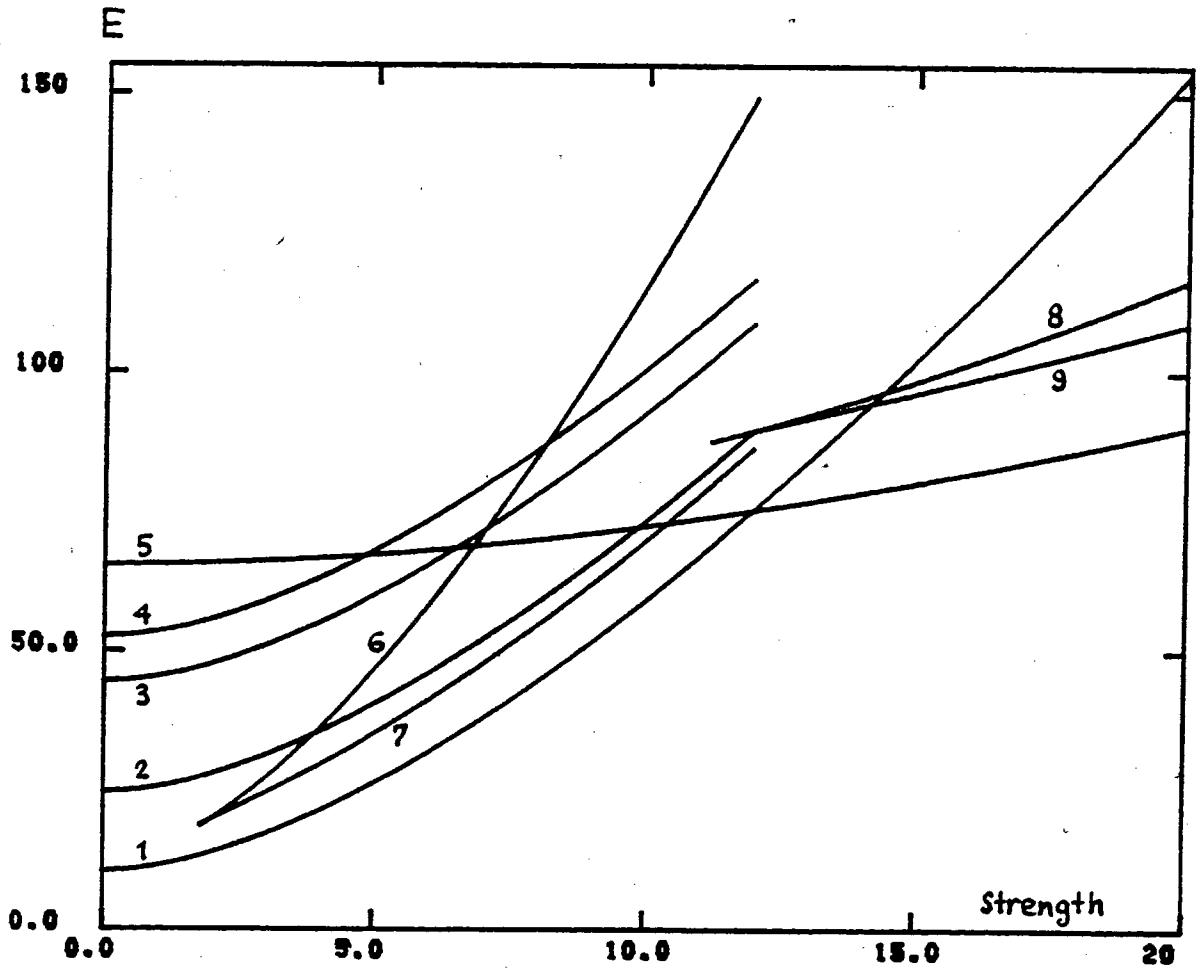

Fig.5.20 Energy vs. strength

The next curve (2) encountered on the energy axis is actually the double curve explained on page 63 and shown in Fig.5.11. Next two curves (3 and 4) are also known from Fig.5.11: they represent the lower and the upper branch for Q=20.0 as R varies from 0.0 to 12.0 (therefore, this is the group one energy, with Q_c). The fifth curve on the energy axis describes type-I energy

and it intersects the first type I energy curve at the point where Q=12.0. Here Q varies from 0 to 20.

The bifurcating curves on the left (6 and 7) represent energy vs. R for fixed Q=12.0. R_c is 1.8120 (Table 5.1). Two other bifurcating energy curves are the known group-one curves, which emerge at a critical Q=11.236.

5.2.3 Type II - third group

The third group of the type II family contains again two solutions. But the fact that there are two critical strengths makes this group essentially different from the other two.

For a given pair of appropriately chosen parameters there exist two solutions. As Q approaches a critical value, the two solutions coincide. But when R decreases an upper-branch solution terminates at R_c^{upper} and a lower at R_c^{lower}. Hence, there is no finite-energy solution below $R_c^{up,low}$. We found that generally $R_c^{upper} \neq R_c^{lower}$. At R_c therefore the solutions are distinct as well as energies calculated from them.

The types of the critical points encountered here are known in the literature. Many problems arising in applications can be described by nonlinear differential equations which depend on real parameters. The structure of solutions may change dramatically at certain critical points of a parameter, called branch points or as we call them here, bifurcation points. It is common to classify the branch points according to the number of emerging branches, and the two examples investigated here, in the case of group one or two and the group three fall into this categorization.

In the case of group-one and two, the main difference, apart from energy behaviour, was in shapes of the function G(x). The same happens again: the shape of the function G(x) is rather unusual in comparison with the other solutions.

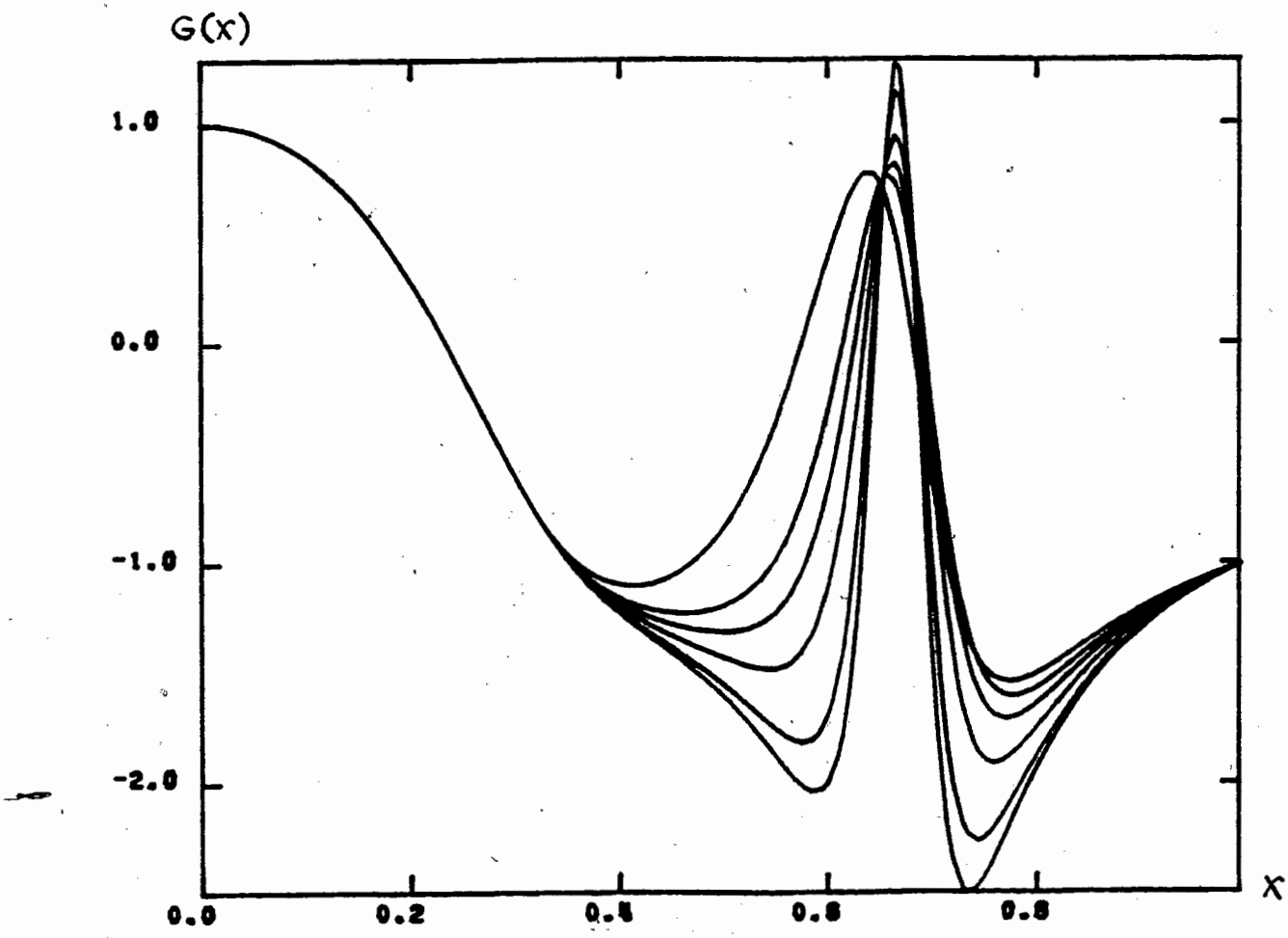

Fig.5.21 G(x) lower branch solutions of type-II, third group (Q fixed at 12.0)

The shape of G(x) for different strength parameters is shown in Fig.5.21. The parameters (Q,R) have the following values: (12.0,15.0), (12.0,12.0), (12.0,8.0), (12.0,6.0) and (12.0,R_c=4.2), starting from the lowest curve at the point x=0.5. The highest curve at x=0.5 is the boundary curve for this solution which yields the lower branch energy when the energy is plotted against Q with some fixed R value.

It is not hard to imagine how the other branch solution will behave in the same situation.

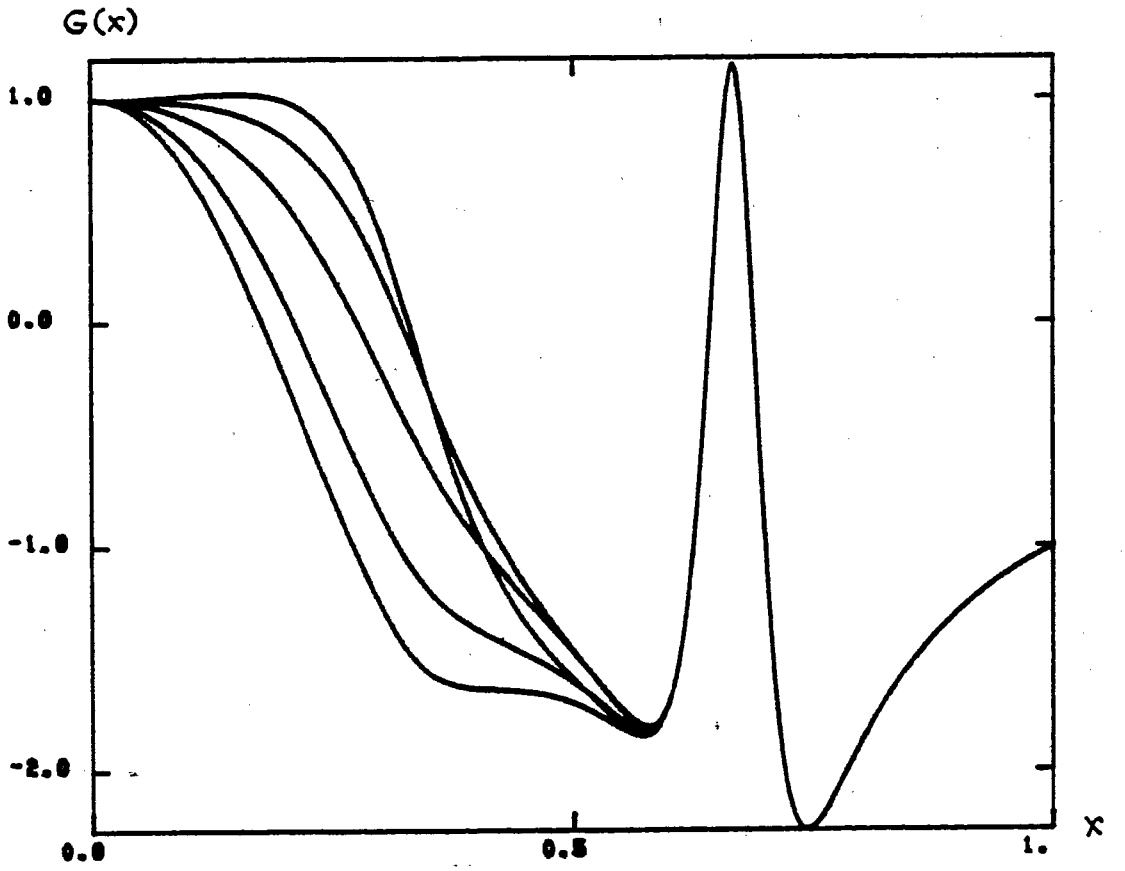

Fig.5.22 G(x) for the third group of type-II family (R fixed at 12.0)

The bifurcating solution is shown in Fig.5.22. The boundary curve for this solution will have an identical shape in the region of the quadrupole dominance, and the difference between upper and lower solutions will be evident only in the region $x \lesssim 0.5$.

In Fig.5.23 all six G(x) solutions of type-II family are shown, for the same combination of strengths, (12.0,12.0). It is easy to identify corresponding mates and groups.

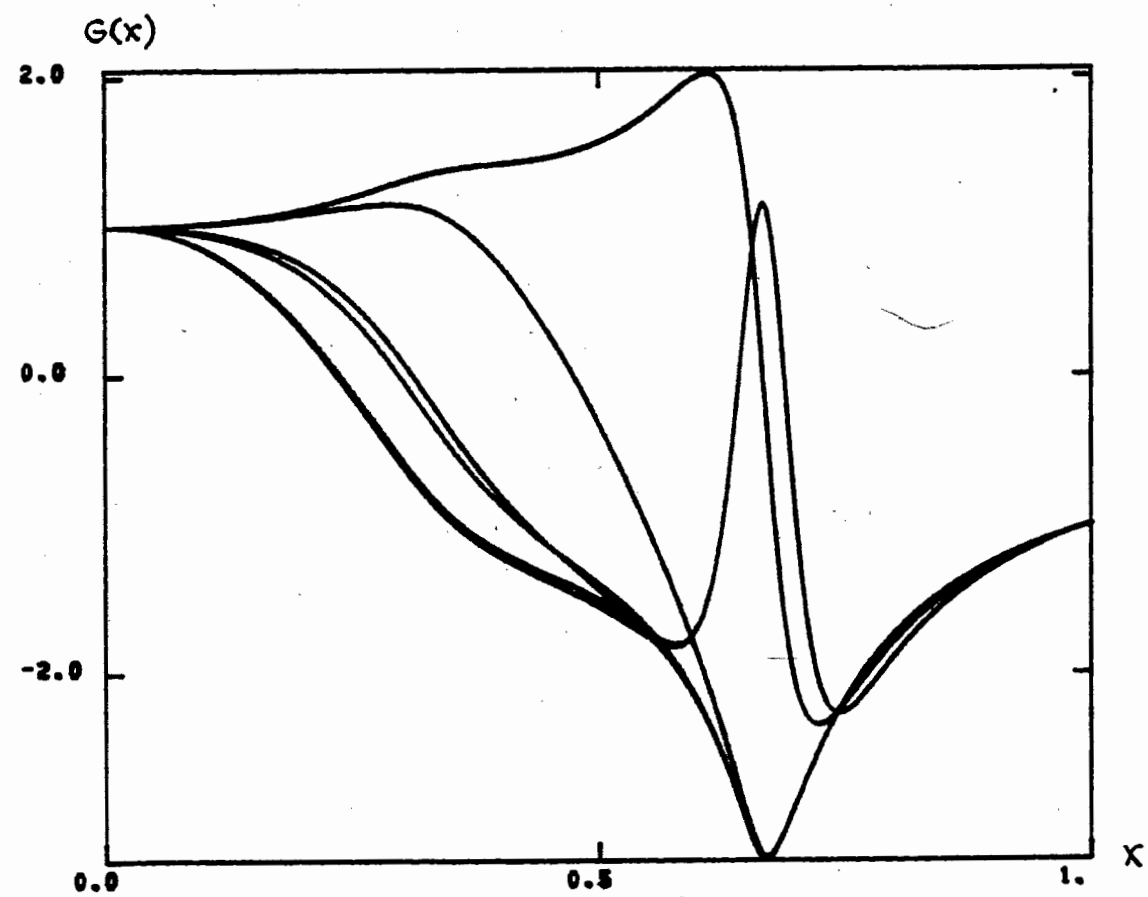

Fig.5.23 All six G(x) solutions of type-II family

The third group solutions yield the highest energy found in the calculation. In Fig.5.24 all bifurcating enery solutions are shown.

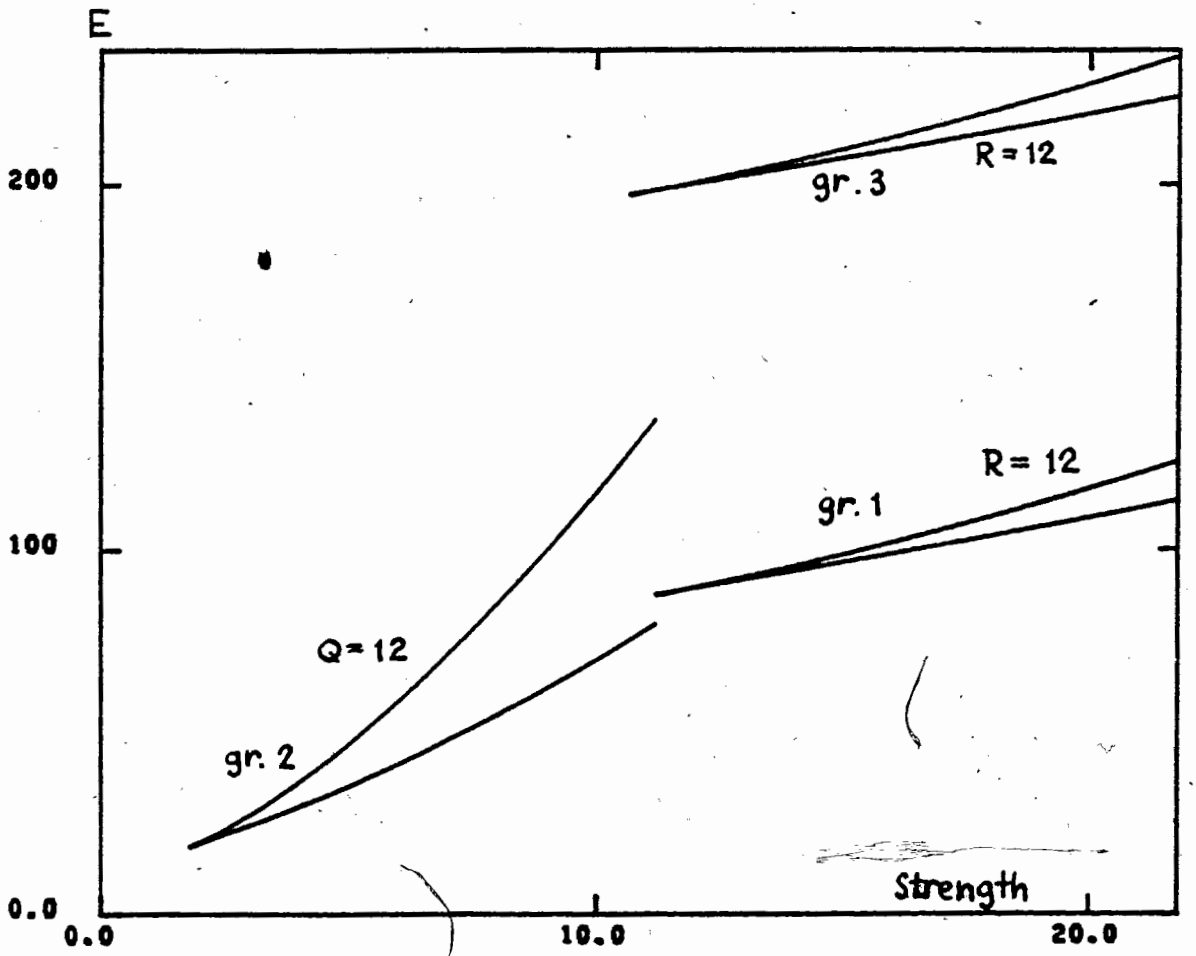

Fig.5.24 Bifurcating energies of type-II family

For a fixed Q, a uniform curve which represents energy vs. R is simply bounded below. A strength at which this happens is called primary branching point. To clarify this situation we plot in Fig.5.25 E vs. R in the same graph where E vs. Q is

plotted. The cusp-like behaviour of the energy with Q_c=10.692 is easy to recognize. The lowest curve is in fact a double curve which consists of an upper an a lower branch for Q=12.0, therefore not far enough from the critical point to give a good resolution in the graph. But for Q=20.0, two upper parabola-like curves clearly show the critical or the primary branching point R_c^{up} and R_c^{low}.

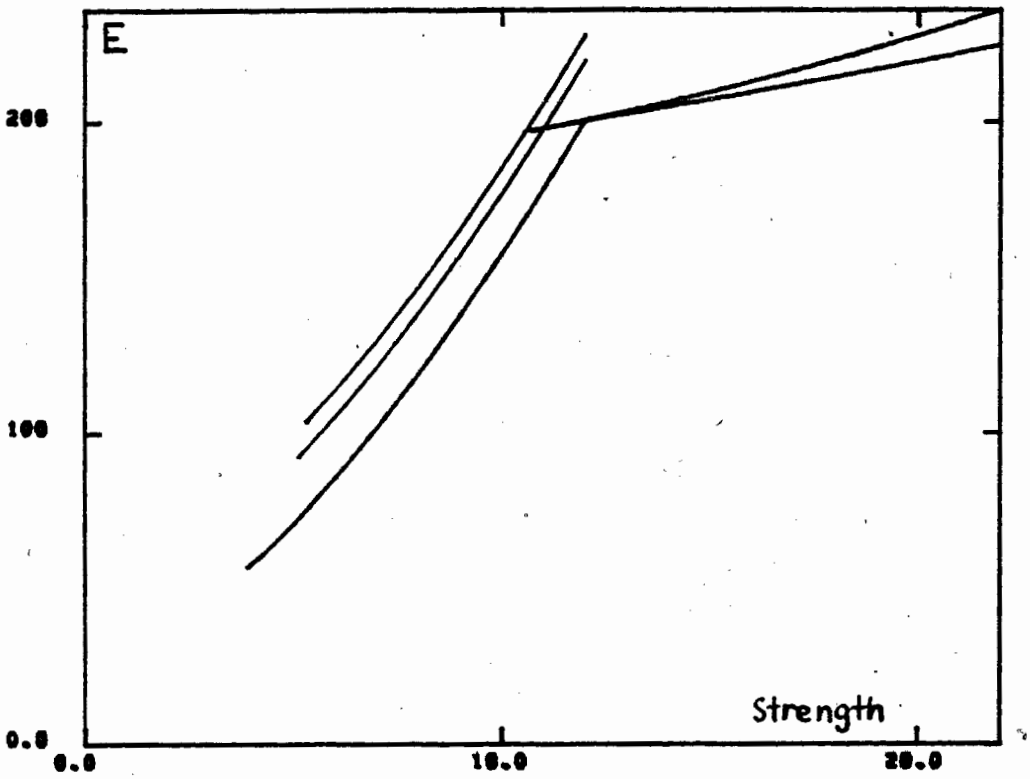

Fig.5.25 Energy vs. strength of type-II family, third group

5.3 SUMMARY

In this section we try to systematize the results presented in sections 5.1 and 5.2.

There are two essentially different families of solutions: type I and type II.

The type-I solutions described in Section 5.1 exhibit regular properties with respect to strengths: they are defined for all possible combinations of R and Q, and energy plot is similar to the Abelian Coulomb parabola.

Type II solutions can be classified according their properties with respect to strengths parameters: the first pair of solutions possess a critical Q strength whereas R can be continuously changed to zero. These solutions are generalizations of SU(2) solutions investigated by Jackiw, Jacobs and Rebbi (1979). The second pair of solutions are characterized by an R_c, whereas Q is free to approache zero. These are genuine SU(3) solutions, as are the third pair of solutions.

In Fig.5.26 a sketch of type-II energies for the first and the second group is made.

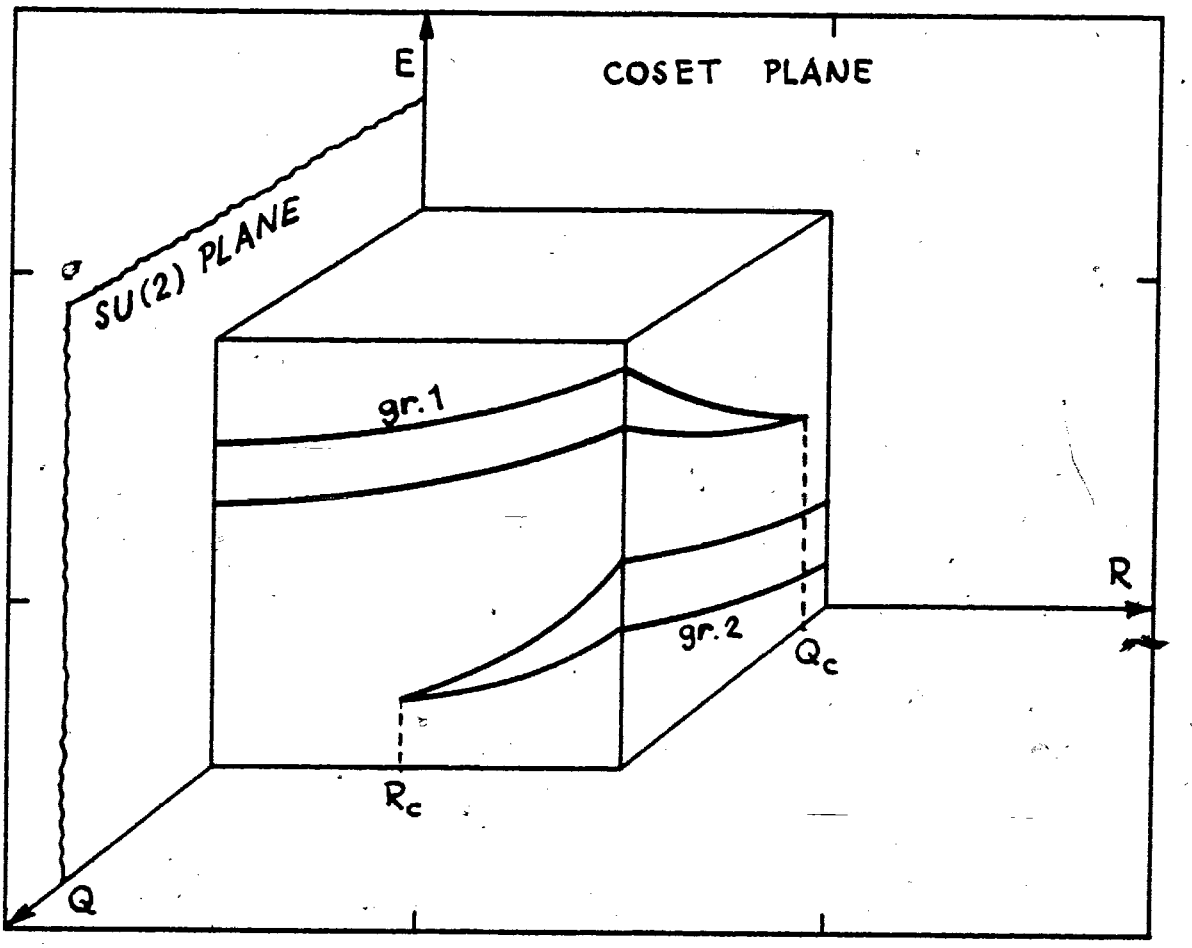

Fig.5.26 Three-dimensional sketch of energy of type-II family, first and second group

The third-pair solutions possess a remarkable property: they are defined only for a certain combination of $(Q,R)>(Q_c,R_c)$. Q_c is so-called secondary branching point, or as it is often called bifurcation point: solutions which are

identical at $Q=Q_c$ bifurcate for $Q>Q_c$. R_c is a primary branching point: a solution (upper or lower) does not exist for $R<R_c$, but it does not coincide with its mate (lower or upper) at R_c. Moreover, upper-branch solutions and lower-branch solutions have different R_c. These energy features are shown in Fig.5.27.

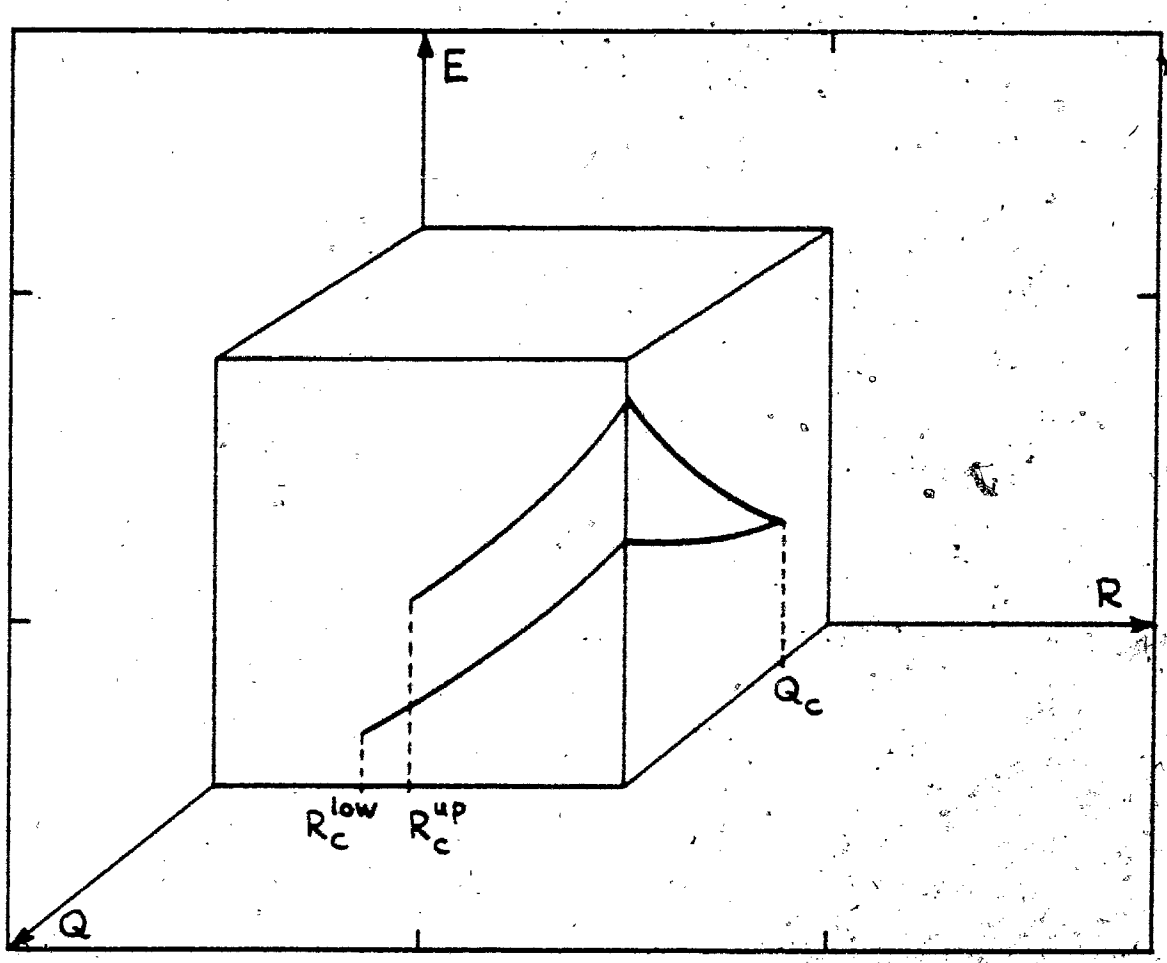

Fig.5.27 Three-dimensional sketch of energy of type-II family, third group

The question of whether or not a primary branching point is in fact secondary, has a simple answer: if there was an other solution emerging from R_c this would imply that this solution has a bifurcation point Q_c from which another solution must emerge, which possess R_c, etc. Therefore, the assumption that R_c is a secondary branching point leads to an infinite number of solutions, which certainly does not exist.

VI. STABILITY IN THE YANG-MILLS THEORY

The question of stability in the classical Y-M theory is important because it addresses the issue of whether or not the classical solutions are relevant to the quantum-mechanical situation. Classical instability is a sure sign that the quantum situation will be quite unlike the classical one, but classical stability by no means ensures a resemblance between the classical and quantum situation. Therefore, classical stability is necessary if the classical solution is to resemble the quantum situation.

In this chapter we consider stability properties of the solutions to the Y-M equations discussed in Chapter 5. We concentrate on bifurcating solutions for which we discuss their relative stability: due to the fact that for an appropriately chosen combination of Q and R we have two bifurcating solutions, we will be able to develop a formalism which enables us to determine the relative stability of solutions, without extremely complicated numerical calculations.

We start the discussion by defining stability in Hamiltonian systems. From the definition we get an eigenvalue problem which can be generalized to the Y-M case to obtain relations between eigen-frequencies and increments in energy and

gauge fields. From simple algebraic considerations, important conclusions about relative stability are drawn.

Small oscilation equations for increments of the gauge potentials are developed. Numerical computations based on the theoretical considerations will justify the arguments.

The chapter is based on a general discussion given by Jackiw and Rossi (1980).

6.1 THE STABILITY CONCEPT IN MECHANICS

Consider a system with finite degrees of freedom described by a Hamiltonian $H(P,Q)$, which is also the conserved energy. Equations of motion are

$$\dot{P}_n = -\frac{\partial H}{\partial Q_n} \tag{6.1a}$$

and

$$\dot{Q}_n = \frac{\partial H}{\partial P_n} \tag{6.1b}$$

where $n=1,\ldots,N$.

A static solution is one for which P and Q vanish. A natural question is whether or not a static solution $\{\hat{P},\hat{Q}\}$ is stable. We shall take stability as the requirement that the equations of motion for δP and δQ, when linearized about the static solution (δP and δQ are "small" quantities), do not yield exponential growth in time. In other words, for stable motion the quantities δP, δQ fluctuate harmonically in time with real frequency, while complex frequencies signal instability. To derive the condition for this, we expand the equations of motion around their static solutions. The Hamiltonian has the form

$$H(P,Q) = H(\hat{P},\hat{Q}) + \tfrac{1}{2}\,\delta P_n T_{nm}\,\delta P_m + \delta P_n\, G_{nm}\,\delta Q_m +$$

$$+ \tfrac{1}{2}\,\delta Q_n V_{nm}\,\delta Q_m \equiv \tag{6.2}$$

$$\equiv H(\hat{P},\hat{Q}) + \tfrac{1}{2}\,\tilde{X}\,\mathcal{H}\,X$$

we find

$$\mathcal{H}\,x = \omega\,\eta\,x \tag{6.3}$$

where

$$\mathcal{H} = \begin{bmatrix} T & G \\ \tilde{G} & V \end{bmatrix} \; ; \quad \eta = \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix} \tag{6.4}$$

and

$$X = \begin{bmatrix} \delta P \\ \delta Q \end{bmatrix} \; ; \quad X = e^{-i\omega t}\,x \; ; \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.5}$$

We recognize the (constant) x as simplectic eigenvectors of $\mathcal{H}$ with simplectic eigenvalue ω (Abraham and Marsden 1967, Siegel and Moser 1971). Therefore, the stability condition becomes

$$\det(\mathcal{H} - \omega\eta) = 0 \Longrightarrow \quad \omega \text{ real} \tag{6.6}$$

The off-diagonal matrix G arises from mixed terms in the Hamiltonian which are frequently called gyroscopic or Coriolis terms. The kinetic energy matrix T with an appropriate definition of coordinates can be choosen to be the identity. Furthermore, the off diagonal matrix G is always antisymmetric:

$$\tilde{G} = -G \tag{6.7}$$

so we are led to the simpler form for $\mathcal{H}$

$$\mathcal{H} = \begin{bmatrix} I & G \\ -G & V \end{bmatrix} \tag{6.8}$$

The simplectic eigenvalue problem becomes

$$[(i\omega + G)(i\omega + G) + V]\,\delta Q = 0 \tag{6.9}$$

and the stability condition (6.6) reduces to

$$\det(V + G^2 + 2i\omega G - \omega^2 I) = 0 \Rightarrow \omega \text{ real.} \tag{6.10}$$

6.2 STABILITY ANALYSIS FOR THE YANG-MILLS THEORY

We turn now to the stability analysis of static solutions to the Y-M equations. It should be pointed out that the Y-M field theory differs from the simple Hamiltonian: rather than 2N degrees of freedom, there is an infinite number. This causes matrices to be replaced by differential operators, summations by integration, etc. Furthermore, the Hamiltonian formulation now has the Gauss law constraints encountered before and implemented with Lagrange multipliers A°_{ab}.

To determine the quadratic Hamiltonian (6.2) we set

$$\underset{\sim}{A}_{ab} \longrightarrow \hat{\underset{\sim}{A}}_{ab} + \delta\underset{\sim}{A}_{ab} \tag{6.11}$$

and

$$\underset{\sim}{E}_{ab} \longrightarrow \hat{\underset{\sim}{E}}_{ab} + \delta\underset{\sim}{E}_{ab} \tag{6.12}$$

and with the help of the static equations (3.28) satisfied by $\hat{\underset{\sim}{A}}_{ab}$ and $\hat{\underset{\sim}{E}}_{ab}$, we find

$$H = \hat{H} + \frac{1}{2}\int d\mathcal{T}\left\{(\delta B^{i}_{ab})^2 + (\delta E^{i}_{ab})^2 + 2E^{i}_{ab}\,\delta E^{i}_{ab} + \right. \tag{6.13}$$

$$\left. + ig\,B^{i}_{ab}\,\epsilon^{ijk}[\delta A^{j}, \delta A^{k}]_{ba} + 2ig[A^{\circ}, E^{i}]_{ab}\,\delta A^{i}_{ba}\right\}$$

where

$$\delta B^i_{ab} = -(\underset{\sim}{\partial} \times \delta \underset{\sim}{A}_{ab})^i + ig\,\varepsilon^{ijk}[A^k, \delta A^j]_{ab} \tag{6.14}$$

We have suppressed the label on the static background field.

The linear terms do not disapper, because as yet the constraint has not been taken into account.

Expanding Gauss' law around the static background field we get

$$\partial^i \delta E^i_{ab} + ig[A^i, \delta E^i]_{ab} + ig[\delta A^i, E^i]_{ab} + \tag{6.15}$$
$$+ ig[\delta A^i, \delta E^i]_{ab} = 0$$

so

$$H = \hat{H} + \tfrac{1}{2}\int d\tau \{(\delta B^i_{ab})^2 + (\delta E^i_{ab})^2 + \tag{6.16}$$
$$+ ig\, B^i_{ab}\, \varepsilon^{ijk}[\delta A^j, \delta A^k]_{ba} + 2ig[\delta A^i, \delta E^i]_{ab} A^\circ_{ba}\}$$

We can identify the elements of the quadratic Hamiltonian matrix (6.8) with

$$G^{ij}_{cbad} = \delta^{ij} ig\,\delta(\underset{\sim}{r}-\underset{\sim}{r}')[A^\circ_{ba}\,\delta_{cd} - \tag{6.17a}$$
$$- \delta_{ca}\,\delta_{be}\,A^\circ_{ba}\,\delta_{ae}\,\delta_{db}] \text{ not summed over the repeated indices.}$$

Indeed

$$\delta E^i_{cb}(\underset{\sim}{r})\, G^{ij}_{cbad}\, \delta A^j_{ad}(\underset{\sim}{r}') = ig(A^\circ_{ba}\,\delta E^i_{db}\,\delta A^i_{ad} - \tag{6.17b}$$
$$- \delta E^i_{ad}\,A^\circ_{ba}\,\delta A^i_{db}) = ig[\delta A^i, \delta E^i]_{ab}\,A^\circ_{ba}$$

The next term in the Hamiltonian (6.16) becomes

$$ig B^i_{ab} \varepsilon^{ijk} \delta A^j_{bc} \delta A^k_{ca} - ig B^i_{ab} \varepsilon^{ijk} \delta A^k_{bc} \delta A^j_{ca} = \tag{6.18}$$

$$= \delta A^k_{ca} \, 2ig B^i_{ab} \varepsilon^{kij} \delta A^j_{bc}$$

so the corresponding matrix is

$$V^{kj}_{(1)\,cabd} = 2ig \, \delta(\underset{\sim}{r} - \underset{\sim}{r}') \, \delta_{cd} \, \varepsilon^{kij} B^i_{ab} \tag{6.19}$$

The expression for δB^i_{ab} (eq.(6.14)) can be written in the form

$$\delta B^i_{ab} = - (\underset{\sim}{\mathcal{D}} \times \delta \underset{\sim}{A})^i_{ab} \tag{6.20}$$

so the first term in the Hamiltonian (6.16) becomes

$$(\delta B^i_{ab})^2 = (\underset{\sim}{\mathcal{D}} \times \delta \underset{\sim}{A})^i_{ab} \, (\underset{\sim}{\mathcal{D}} \times \delta \underset{\sim}{A})^i_{ba} \tag{6.21}$$

or

$$\delta B^i_{ab} \, \delta B^i_{ba} = - \delta A^k_{cb} \, \varepsilon^{kji} \mathcal{D}^j_{ac} \, \mathcal{D}^l_{bd} \, \varepsilon^{ilm} \delta A^m_{da} \tag{6.22}$$

The corresponding matrix is

$$V^{km}_{(2)\,cbda} = - \delta(\underset{\sim}{r} - \underset{\sim}{r}') \, \varepsilon^{kji} \mathcal{D}^j_{ac} \, \mathcal{D}^l_{bd} \, \varepsilon^{ilm} \tag{6.23}$$

so the total matrix V in the quadratic Hamiltonian is the sum of (6.19) and (6.23)

$$V^{ij}_{abcd} = \delta(\underset{\sim}{r} - \underset{\sim}{r}') (2ig \varepsilon^{ilj} B^l_{bc} \, \delta_{ad} - \varepsilon^{ilm} \mathcal{D}^l_{da} \mathcal{D}^k_{bc} \, \varepsilon^{mkj}) \tag{6.24}$$

The total effective quadratic Hamiltonian is

$$\bar{\mathcal{H}}^{(2)} = \frac{1}{2}\int d\tau \{(\delta B^i_{ab})^2 + (\delta E^i_{ab})^2 + 2ig[\delta A^i, \delta E^i]_{ab} A^\circ_{ba} +$$

$$+ ig B^i_{ab} \epsilon^{ijk} [\delta A^j, \delta A^k]_{ba} - \delta A^\circ_{ba}(\partial^i \delta E^i_{ab} + \qquad (6.25)$$

$$+ ig[A^i, \delta E^i]_{ab} - ig[\delta A^i, E^i]_{ab})\}$$

and from it we can derive the fluctation equations. We get

$$\frac{\delta \bar{\mathcal{H}}^{(2)}}{\delta(\delta A^\circ_{ba})} = 0 = \partial^i \delta E^i_{ab} + ig[A^i, \delta E^i]_{ab} - ig[\delta A^i, E^i]_{ab} \qquad (6.26a)$$

and

$$\frac{\delta \bar{\mathcal{H}}^{(2)}}{\delta(\delta A^i_{ba})} = \partial_t \delta E^i_{ab} = (\underset{\sim}{\partial} \times \underset{\sim}{\delta B}_{ab})^i - ig\epsilon^{ijk}[\delta A^k, B^j]_{ab} - \qquad (6.26b)$$

$$- ig\epsilon^{ijk}[A^k, \delta B^j]_{ab} - ig[\delta A^\circ, E^i]_{ab} - ig[A^\circ, \delta E^i]_{ab}$$

and

$$\frac{\delta \bar{\mathcal{H}}^{(2)}}{\delta(\delta E^i_{ab})} = -\partial_t \delta A^i_{ab} = \delta E^i_{ab} - \partial^i \delta A^\circ_{ab} + \qquad (6.26c)$$

$$+ ig[\delta A^i, A^\circ]_{ab} + ig[A^i, \delta A^\circ]_{ab}$$

It can be recognized that these equations are the linearization of the Y-M equations around the static background fields.

When a monochromatic Ansatz is made for

$$X \equiv \begin{bmatrix} \delta E^i_{ab} \\ \delta A^i_{ab} \end{bmatrix} \qquad (6.27)$$

the above equations take the form of a simplectic eigenvalue

problem (6.3).

6.3 THE STABILITY ANALYSIS OF THE BIFURCATING SOLUTIONS

The bifurcating solution described in detail in Chapter 5, section 2, exhibit properties which enable us to say a considerable amount about their stability without explicit computation.

We will be using alternatively the matrix and the integral notation bearing in mind the differences between the Y-M and the simple Hamiltonian theory. So the bracket notation of the type

$$\langle \psi_1 | M | \psi_2 \rangle$$

involves taking matrix elements over whatever degrees of freedom are appropriate.

At the bifurcation point the Y-M equations with the static solution vector

$$X_c = \begin{bmatrix} \hat{E}^i_{ab} \\ \hat{A}^i_{ab} \end{bmatrix} \tag{6.28}$$

are solved. Next, imagine changing the source slightly

$$\rho_c \longrightarrow \rho_c + \delta\rho \tag{6.29}$$

where ρ_c supports a unique static solution X_c. The increment in the Y-M fields defined in (6.11) and (6.12) will satify the fluctuation equations (6.26) except that $\delta\rho$ occurs on the

left-hand side of Gauss' law constraint, and

$$\partial_t \delta E^i_{ab} = 0 \; ; \; \partial_t \delta A^i_{ab} = 0 \tag{6.30}$$

However, if we are at the bifurcation point, it must be impossible to solve these equations. This happens when the homogenous system has a nontrivial solution. Therefore, the stability condition (6.10) implies that at the bifurcation point the stability equations have zero-eigenvalue mode. [1]

By introducing a parameter β which is chosen to be positive and which systemizes the study of the theory around the bifurcation point we write

$$\varrho_c \longrightarrow \varrho_c + \beta \, \delta\varrho \tag{6.31}$$

It is now appropriate to expand the static solution about the bifurcation point in the form

$$A^\mu_{ab} = \hat{A}^\mu_{ab} + \beta^{1/2} c \, \bar{A}^\mu_{ab} \tag{6.32a}$$

or

$$|\varphi\rangle = |\varphi_c\rangle + \beta^{1/2} c \, |\chi\rangle \tag{6.32b}$$

where $|\varphi_c\rangle$ is the static solution at the bifurcation point, and $|\chi\rangle$ is the normalized zero-eigenvalue mode. A constant c is a real number to be determined from the fluctuation

---

[1] Whether these zero-frequency modes are normalizable or not will be addressed later (see page 102).

equations. Therefore $|\psi\rangle$ solves the eigenvalue problem (6.9)

$$(G_c^2 + V_c)|\psi\rangle \equiv W_c\,|\psi\rangle = 0 \tag{6.33}$$

and

$$W_c\,|\varphi_c\rangle = -\,\rho_c \tag{6.34}$$

where W_c is evaluated at the bifurcation point.

In general, an oscillatory mode is defined by

$$|\psi(r,t)\rangle \equiv |\psi(r)\rangle + e^{-i\omega_n t}\,|\psi_n\rangle \tag{6.35}$$

so slightly above the bifurcation point we make an expansion of the oscillating solution with a small eigen-frequency ω_o

$$|\psi_o\rangle = |\psi\rangle + |\delta\psi_o\rangle \tag{6.36}$$

The eigen-value equation (6.9) now has the form

$$(W_c + 2i\omega_o G_c)\,|\psi_o\rangle = \omega_o^2\,|\psi_o\rangle \tag{6.37}$$

and with the definition (6.33) we can write as a consequence of (6.31)

$$W = W_c + \beta^{1/2} c\,\Delta W \tag{6.38}$$

and since ω_o is a small quantity the eigen-value equation becomes

$$(W_c + \beta^{1/2} c\,\Delta W + 2i\omega_o G_c)(|\psi\rangle + |\delta\psi_o\rangle) = \omega_o^2\,|\psi\rangle \tag{6.39}$$

From this equation, bearing in mind that (6.33) holds and that G is an antisymmetric matrix, we get

$$\omega_o^2 = \frac{c\beta^{1/2}\langle 4|\Delta W|4\rangle}{1+4\langle G4|W^{-1}|G4\rangle} \tag{6.40}$$

Here $|G4\rangle$ is the zero eigenvalue solution transformed by the gyroscopic term and $WxW^{-1}=1$. Therefore, since $\beta^{1/2}$ is defined to be a positive number, ω_o^2 depends upon values c can take. ΔW is defined to be a positive quantity.

The total effective Hamiltonian (6.25) can be formally written as

$$H = \int d\tau \; \mathcal{H}(\varphi_{ab}) \tag{6.41}$$

where φ_{ab} denotes the collection of fields.

Above the bifurcation point taking into account (6.31) and the corresponding response of the fields (6.32a) and (6.32b), the quadratic Hamiltonian is seen to be a function of β. When (6.41) is differentiated with respect to β we find

$$\frac{\partial H}{\partial \beta} = \int d\tau \left(\frac{\partial \mathcal{H}}{\partial \varphi_{ab}} \frac{\partial \varphi_{ab}}{\partial \beta} - A^{\circ}_{ba} \frac{\partial \rho_{ab}}{\partial \beta} \right) \tag{6.42}$$

But since for the static solution the first term on the right-hand side vanishes, we get

$$\frac{\partial H}{\partial \beta} = -\int d\tau \, \hat{A}^{\circ}_{ba} \delta\rho_{ab} - \beta^{1/2} c \int d\tau \, \bar{A}^{\circ}_{ba} \delta\rho_{ab} \tag{6.43}$$

or

$$H = H_c + \beta\, Tr \int d\sigma\, \hat{A}^\circ \delta\varrho - \tfrac{2}{3}\beta^{3/2} c\, Tr \int d\sigma\, \bar{A}^\circ \delta\varrho \tag{6.44}$$

H_c in the above expression is the energy at the bifurcation point.

In order to determine the constant c we use (6.31) in Eq. (6.34) and expand the static solution according (6.32a). We find

$$\int d\sigma\, (c\beta^{1/2} \Delta W_{ab})(c\beta^{1/2} \bar{A}^\circ_{ba}) = -\int d\sigma\, \beta\, \delta\varrho_{ab}\, \bar{A}^\circ_{ba} \tag{6.45}$$

where ΔW is the energy difference defined in (6.38). Hence

$$c^2 = \frac{-\,Tr \int d\sigma\, \bar{A}^\circ d\varrho}{Tr \int d\sigma\, \Delta W\, \bar{A}^\circ} \tag{6.46}$$

Therefore we can see that with an appropriate choice of sign of $\delta\varrho$ we get

$$c = \pm c_0 \equiv \pm \left| \frac{Tr \int d\sigma\, \bar{A}^\circ d\varrho}{Tr \int d\sigma\, \Delta W\, \bar{A}^\circ} \right| \tag{6.47}$$

and consequently, the expression (6.44) shows that the energy bifurcates with an energy difference rising as $\beta^{3/2}$.

Since c can have either sign, we see from (6.40) that for one branch ω_0^2 is negative, hence there is an instability.

Comparison with (6.46) and (6.44) shows that c is $(-c_o)$ for the upper branch, so ω_o^2 has the opposite sign from the energy difference. Therefore, the upper branch of the bifurcating solutions is unstable with respect to radial oscillations, while the lower branch shares the stability properties of the unique solution at the bifurcation point.

In order to determine stability properties of the lower branch solutions, we calculate the small oscillation equations (6.26). These equations are in fact linearized small radial oscillations equations (3.47) when f_i is replaced by $f_i + \delta f_i$ and G by $G + \delta G$. The equations are

$$-\delta f_1'' + \frac{2}{r^2}\left(2G f_1\, \delta G + G^2 \delta f_1\right) = 0$$

$$-\delta f_2'' + \frac{6}{r^2}\left(2G f_2\, \delta G + G^2 \delta f_2\right) = 0 \qquad (6.48a)$$

and

$$-\delta G'' + \frac{1}{r^2}\left(3G^2 - 1 - f_1^2 - f_2^2\right)\delta G -$$

$$-\frac{2}{r^2}\left(f_1 \delta f_1 + f_2 \delta f_2\right) G = 0 \qquad (6.48b)$$

These equations arise by varying $\mathcal{H}$ (4.25) here given by

$$\bar{\mathcal{H}} = \frac{4\pi}{g^2}\int_0^\infty dr\left\{\frac{1}{r^2}(3G^2-1)(\delta G)^2 + (\delta G')^2 - \frac{1}{2}(\delta f_1')^2 - \right.$$

$$\left. -\frac{1}{r^2} G^2 (\delta f_1)^2 - \frac{1}{r^2} f_1^2 (\delta G)^2 - \frac{4}{r^2} G f_1\, \delta f_1\, \delta G -\right.$$

$$- 12\,\frac{1}{r^2}\,G f_2\,\delta f_2\,\delta G - \frac{1}{2}(\delta f_2')^2 - \frac{1}{r^2}\,G^2(\delta f_1)^2 \Big\} \qquad (6.49)$$

Analytic solutions to equations (6.48) are of course out of the question since we do not have closed analytic form for f_1, f_2, and G. Clearly from the known behaviour of these functions as $r \to 0$ and as $r \to \infty$, we can conclude that

$$\left.\begin{array}{l} \delta f_1 \longrightarrow 0 \\ \delta f_2 \longrightarrow 0 \\ \delta G \longrightarrow 0 \end{array}\right\} \quad \begin{array}{l} \text{as } r \longrightarrow 0 \\ \text{and} \\ r \longrightarrow \infty. \end{array} \qquad (6.50)$$

If we insist on small fluctuations with finite energy content, we have to look for normalizable (square integrable) solutions to (6.48). Numerical calculations have shown that there are no normalizable zero-frequency fluctuation modes. This indicates that the lower branch of bifurcating solutions is absolutely stable. In this respect the solutions we have found share the properties of fluctuations of Prasad-Sommerfield monopole solutions in flat space for SU(2) gauge fields (Adler 1979, Akhoury et al. 1980) and in curved space-time (Chakrabarti and Viswanathan 1980). On the other hand, one knows that for a class of instanton solutions there exist non-trivial zero energy fluctuation modes that are normalizable.

If non-trivial normalizable (finite energy) solutions to the small oscillation equations had been found this would have implied that non-zero frequency modes must be investigated. Since all higher frequency modes include the zero frequency

mode, it was enough to prove that this mode is not normalizable.

6.4 THE NUMERICAL CALCULATION OF STABILITY OF THE BIFURCATING SOLUTIONS

Using the numerical results for the bifurcating solutions (Chap. 5, sections 5.2.1, 5.2.2 and 5.2.3) we verify correctness of the formalism developed in the previous sections.

For the group one of the bifurcating solutions (section 5.2.1) we found the critical value of Q and the corresponding energy as

$$Q_c\,(R=12) = 11.24 \;;\; \mathcal{E}_c = (Q_c, 12) = 87.64 \qquad (6.51)$$

From the following table we can see that the ratio of the energy difference between the upper and lower branch and $(Q-Q_c)^{3/2}$ is constant as predicted by (6.44) and equals 0.3. So the parameter β is taken to be a measure how far from the bifurcation point the solutions and the energies are calculated.

Q	$(Q - Q_c)^{3/2}$	$\Delta \mathcal{E}$	RATIO :
12.0	0.66	0.18	0.3
14.0	4.59	1.33	0.3
16.0	10.39	3.09	0.3
18.0	17.58	5.32	0.3
20.0	25.93	7.97	0.3
22.0	35.30	10.97	0.3
24.0	45.58	14.29	0.3

Table 6.1 - Results of stability analysis for the group-one bifurcating solutions

For the group two of the bifurcating solutions (section 5.3) the same calculation gives results presented in Table 6.2. The critical value of R for fixed Q=12.00 is 1.81 and the corresponding energy is 18.74 (in appropriate units - see Chapter 5).

R	$(R-R_c)^{3/2}$	$\Delta\mathcal{E}$	RATIO :
2.0	0.08	0.17	0.5
3.2	1.64	3.34	0.5
4.0	3.24	6.58	0.5
5.2	6.24	12.54	0.5
6.0	8.58	17.14	0.5
8.0	15.40	30.39	0.5
10.0	23.44	45.78	0.5
12.0	32.53	63.00	0.5

Table 6.2 - Results of the stability calculation for the group two of the bifurcating solutions

Recall that these solutions are genuine SU(3) solutions with energies above the energies of group-one solutions (Fig.5.24).

VII. CONCLUSION

In this work we have investigated solutions to classical SU(3) Yang-Mills theory with static external sources. The solutions are non-Coulombic and non-Abelian in nature.

From the Ansatze for the sources and the gauge potentials, a system of nonlinear differential equations is derived. The system is solved using COLSYS, a code for solving boundary value problems for mixed order systems of ordinary differential equations. The presented results are surprisingly new and rich.

Investigations known in literature are concentrated more or less on SU(2) gauge symmetry, or on the SU(3) gauge symmetry but with point sources, when all the sources are oriented in commuting directions in the gauge space. The main difficulty in the Yang-Mills theory is the high nonlinearity of equations governing behaviour of gauge fields, so there are no analytic methods available in this case.

We found basically two different types of solutions for the SU(3) Yang-Mills theory, which differ from one another in their asymptotic behaviour.

The first type of solutions are nontrivial generalizations of solutions found by Jackiw and his

collaborators, to the higher symmetry group SU(3), and with an appropriate choice of parameters we regained their results. The generalization gave a class of solutions that tend to SU(2) solutions continuously, as one part of the SU(3) source is turned off.

The other type of solutions exist only for sufficiently strong sources, and they differ from the first type in their spatial asymptotic behaviour: they become a non-trivial pure gauge as r tends to infinity.

Surprisingly, many solutions of the second type exist. Apart from the generalizations of bifurcating solutions of Jackiw et al. with SU(2) gauge symmetry, there are four new bifurcating solutions with the genuine SU(3) symmetry properties.

It is possible to divide the four genuine SU(3) solutions into two groups, since there is an essential difference between the two pairs of solutions: while the first group of solutions exhibit a bifurcation in one (quadrupole) strength parameter and they are defined for every value of the monopole strength, the second set of solutions exhibit a bifurcation in the monopole strength, but for the quadrupole strength they are simply bounded: below some critical value, neither an upper nor a lower branch solution exists.

A natural question in every dynamical system is stability of the solutions under small fluctuations. In

classical non-Abelian gauge theory the question is extremely important since the classical instability is a sure sign that the quantum situation will be quite unlike the classical one.

By taking advantage of the fact that the solutions bifurcate, we were able to prove that all upper branch solutions are unstable. Although this result can be said to be expected it is not obvious since there exist physical systems which are stable even though their energy is not locally minimal.

We then proceed to show that the lower branches of the bifurcating solutions have absolute stability. It is found that no normalizable zero energy oscillations exist.

Some questions remain to be answered: how to classify the static SU(3) Yang-Mills solutions, is a topological classification applicable as in other branches of Yang-Mills theory (e.g., instantons, merons etc.)? Are there other static solutions and what is their energy in comparison to the ones we have found?

And some further computations obviously suggest themselves, expecially concerning stability properties of bifurcating solutions under non-radial oscillations. But all these questions and problems are immersed in the most pressing question at the present time: what is the relevance of these mathematical investigations to the quantum physics of Yang-Mills theory.

APPENDIX A - NOTATION CONVENTIONS

Throughout the text we will be using natural units defined by

$$\hbar = c = 1 \tag{A-1}$$

The dimensions of the fundamental quantities become:

$$[E] = [M] = [L^{-1}] = [T^{-1}]$$

and $[Q] = 1$. The fine structure constant takes the form:

$$\alpha = \frac{e^2}{4\pi} \cong \frac{1}{137} \tag{A-2}$$

We will be working in the Minkowski space so the metric tensor $\eta^{\mu\nu}$ will be taken to be

$$\eta^{\mu\nu} = diag\,(1,-1,-1,-1) \tag{A-3}$$

Any repeated index will be summed over its range unless stated otherwise. A point in the space will be represented by x^μ:

$$X^\mu = (t, \underset{\sim}{x}) \equiv (x^\circ, \underset{\sim}{x}) \tag{A-4}$$

In that case we define:

$$\underset{\sim}{\partial} \equiv \vec{\nabla}\,; \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}\,; \quad \partial^i \equiv \frac{\partial}{\partial x^i}\,; \quad \partial_t = \partial_\circ \equiv \frac{\partial}{\partial t} \tag{A-5}$$

As a general rule, Greek indices $\alpha, \beta, \gamma, \ldots$ will be Lorentz indices, while the Latin indices i,j,k,l,m... will run from one to three. The Latin indices a,b,c,d,e, will be reserved to label colour-space quantities.

A product of 2 contravariant 4-vectors is therefore defined as

$$a^{\mu} \cdot b^{\mu} = a^{\circ} \cdot b^{\circ} - \underset{\sim}{a} \cdot \underset{\sim}{b} \tag{A-6}$$

We will consistently keep the Lorentz indices up and the coloured indices down.

The three dimensional integration will be performed over the volume element

$$\int d\sigma = \int dx^1 dx^2 dx^3 \tag{A-7}$$

whereas a four dimensional integral will have the form

$$\int d^4x \tag{A-8}$$

A unit vector pointing out of the 4-dimensional volume is $\hat{n}^{\mu}$. It is perpendicular on the differential 4-surface $d\sigma^{\mu}$.

The D'Alambertian operator is given in the form

$$\Box \equiv -\Delta + \partial_{\circ}^{2} \tag{A-9}$$

APPENDIX B - SU(3) GROUP

The group SU(3) is the set of complex unitary unimodular 3 3 matrices. The elements of this group are characterized by 18 real parameters. The unitarity and unimodularity conditions give 10 real relations among these parameters, so that we remain with 8 real parameters.

The generic element of the SU(3) group can be written in the form:

$$U = \exp\left(i \frac{\lambda_a}{2} \theta_a\right) \tag{B-1}$$

with θ_a real parameters and λ_a eight 3×3 linearly independent Hermitian matrices. Explicitly the matrices are given by (Gell-Mann 1962, Carruthers 1966):

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}; \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}; \quad \lambda_8 = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} \tag{B-2}$$

The generators T_a of the infinitesimal transformations are defined by means of the relation:

$$T_a = \frac{1}{2} \lambda_a \tag{B-3}$$

According to the fundamental theorem proved by Lie and Engels, the structure of the group is completely specified by the commutation relations among the generators T_a of the infinitesimal transformations:

$$[T_a , T_b] = i f_{abc} T_c \tag{B-4}$$

or

$$[\lambda_a , \lambda_b] = 2i f_{abc} \lambda_c \tag{B-5}$$

The structure constants are real, as it must be, and are given by

$$C^c_{ab} = - f_{abc} \tag{B-6}$$

and they satisfy the conditions

$$C^c_{ab} = - C^c_{ba} \qquad \text{(antisymmetry)}$$

$$C^c_{ab} C^e_{cd} + C^c_{bd} C^e_{ca} + C^c_{da} C^e_{cb} = 0$$

(Jacoby identity)

The nonvanishing f_{abc} are:

$$f_{123} = 1$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \qquad (B-7)$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

Using (B-6) we can show that SU(3) is a semisimple[1] and compact Lie group (Racah 1964, Speiser 1964).

Gell-Mann (1962) has introduced some other coefficients defined by the relation:[2]

$$\{\lambda_a , \lambda_b\} = \frac{4}{3}\delta_{ab} I + 2 d_{abc} \lambda_c \qquad (B-8)$$

where I is the unit matrix and d_{abc} are fully symmetric. The nonvanishing d_{abc} coefficients are:

$$d_{118} = d_{228} = d_{338} = d_{888} = \frac{1}{\sqrt{3}}$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = \qquad (B-9)$$

$$= d_{355} = -d_{366} = -d_{377} = \frac{1}{2}$$

$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}$$

From the relation

$$\lambda_a \lambda_b = \frac{2}{3}\delta_{ab} I + (d_{abc} + i f_{abc}) \lambda_c \qquad (B-10)$$

we can obtain the properties:

---

[1] A group is semisimple if it has no Abelian invariant subgroup.

[2] In the case of SU(2) the generators σ_a (Pauli matrices) satisfy the simple relations $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$

$$\mathrm{Tr}\,\lambda_a\lambda_b = 2\delta_{ab} \tag{B-11}$$

$$\mathrm{Tr}\,\lambda_a\lambda_b\lambda_c = 2(d_{abc} + i f_{abc})$$

Using the identity

$$[\{A,B\},C] = \{[A,C],B\} + \{A,[B,C]\}$$

we get

$$f_{abe}\,d_{cde} + f_{cbe}\,d_{dae} + f_{dbe}\,d_{ace} = 0 \tag{B-12}$$

The group SU(3) has rank 2. This follows from the fact that there are pairs of commuting elements[3] as shown by (B-5) and (B-7). It is then possible to build up two Casimir operators for SU(3), i.e. non-linear invariant operators which commute with all T_a's

$$[C_i, T_a] = 0 \;;\; (a = 1,\ldots,8)(i = 2,3) \tag{B-13}$$

The first one, apart from a multiplicative factor, is the quadratic operator

$$C_2 = 3\,T_a\,T_b \tag{B-14}$$

The second one turns out to be a cubic operator

$$C_3 = d_{abc}\,T_a\,T_b\,T_c \tag{B-15}$$

---

[3] To be contrasted with commuting operators of the group, e.g., Casimir operators which are non-linear in T_a.

For pratical purposes (see Chapter 3) it is convenient to make a change in the parametrization of the group by introducing the operators U_{ab} (Marshak et al. 1969, Werle 1966) with properties:

$$[U_{ab}, U_{cd}] = \delta_{bc} U_{ad} - \delta_{ad} U_{cb}$$

$$(U_{ab})^{\dagger} = U_{ba} \; ; \; U_{cc} = 0 \qquad \text{(B-16)}$$

The relationship between U_{ab} and T_a is:

$$U_{12} = T_1 + i T_2 \; ; \; U_{21} = T_1 - i T_2 \; ; \; \tfrac{1}{2}(U_{11} - U_{22}) = T_3 \; ;$$

$$U_{13} = T_4 + i T_5 \; ; \; U_{31} = T_4 - i T_5 \; ; \qquad \text{(B-17)}$$

$$U_{23} = T_6 + i T_7 \; ; \; U_{32} = T_6 - i T_7 \; ; \; U_{33} = -\frac{2}{\sqrt{3}} T_8$$

In compact form this relationship can be written as:

$$U_{ab} = (\lambda_c)_{ba} T_c \qquad \text{(B-18)}$$

or

$$T_a = \tfrac{1}{2} (\lambda_a)_{cd} U_{cd} \qquad \text{(B-19)}$$

The generators of SU(3) that are conveniently taken to define isospin and hypercharge are

$$I_4 = U_{12} \; ; \; I_- = U_{21} \; ; \; I_3 = \tfrac{1}{2}(U_{11} - U_{22})$$

$$Y = U_{11} + U_{22} = -U_{33} \qquad \text{(B-20)}$$

We can employ other subsets of SU(3) generators to describe "U-spin" and "V-spin" generators

$$U_+ = U_{23};\ U_- = U_{32};\ U_3 = \tfrac{1}{2}(U_{22} - U_{33})$$

$$V_+ = U_{31};\ V_- = U_{13};\ V_3 = \tfrac{1}{2}(U_{33} - U_{11}) \qquad \text{(B-21)}$$

Note that I_+, I_-, I_3 constitute a subalgebra of SU(3) that is isomorphic to the algebra of SU(2). This corresponds to the obvious fact that SU(2) is a subgroup of SU(3) (but not an invariant one[4]). Similarly the operators (U_1, U_2, U_3) and (V_1, V_2, V_3) define other subgroups of SU(3) where

$$U_1 = \tfrac{1}{2}(U_+ + U_-);\ U_2 = \tfrac{1}{2i}(U_+ - U_-) \qquad \text{(B-22)}$$

and similarly for V.

We can write in the matrix form

$$I_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix};\quad Y = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \text{(B-23)}$$

$$I_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\quad U_+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};\quad V_+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

---

[4] A subgroup S of group G is an invariant subgroup if gSg^{-1} is in S for every g in G and s in S.

and

$$I_- = I_+^\dagger \; ; \; U_- = U_+^\dagger \; ; \; V_- = V_+^\dagger \tag{B-24}$$

or

$$\begin{aligned} I_1 &= \tfrac{1}{2}\lambda_1 \qquad U_1 = \tfrac{1}{2}\lambda_6 \qquad V_1 = \tfrac{1}{2}\lambda_4 \\ I_2 &= \tfrac{1}{2}\lambda_2 \qquad U_2 = \tfrac{1}{2}\lambda_7 \qquad V_2 = -\tfrac{1}{2}\lambda_5 \\ I_3 &= \tfrac{1}{2}\lambda_3 \qquad U_3 = \tfrac{1}{4}(-\lambda_3 + \lambda_8\sqrt{3}) \; ; \; V_3 = -\tfrac{1}{4}(\lambda_3 + \lambda_8\sqrt{3}) \end{aligned} \tag{B-25}$$

The traceless 3×3 Hermitian matrices λ_a were used to introduce the set of real coefficients f_{abc} and d_{abc}, and by the definition (B-6) the connection between f_{abc} and the structure coefficients of the group was established. Now we introduce the adjoint representation.[5] By using the antisymmetry property and the Jacoby identity for the structure constants we can get

$$C^d_{ac} C^e_{bd} - C^d_{bc} C^e_{ad} = - C^d_{ab} C^e_{dc} \tag{B-26}$$

If we now define a set of 8 matrices R_a by

$$(R_a)_{bc} \equiv - C^c_{ab} \tag{B-27}$$

the equation (B-26) can be written as

---

[5]Sometimes instead of the adjoint representation the word regular representation is used.

$$(R_a)_{cd}(R_b)_{de} - (R_b)_{cd}(R_a)_{de} = C_{ab}^{d}(R_d)_{ce} \qquad \text{(B-28)}$$

or

$$[R_a, R_b] = i f_{abc} R_c = -R_{ab}^{c} R_c \qquad \text{(B-29)}$$

(The writing of superscripts and subscripts on these matrices is only a matter of a notational convenience.)

So, with the help of the structure constants we have introduced the Hermitian 8×8 matrices $(R_a)_{bc}$.

We are able now to define in the usual way a vector operator in SU(3).

Suppose we have a set (of operators or N×N matrices) V_a whose commutators with the generators T_a (operators or N×N matrices) satisfy

$$[T_a, V_b] = i f_{abc} V_c = -R_{abc} \qquad \text{(B-30)}$$

Then V_b is said to be a vector operator. This definition shows that under the transformation

$$U(\Theta) = e^{i\Theta_a T_a} \qquad \text{(B-31)}$$

V_b transforms according to the adjoint representation. We consider infinitesimal transformation

$$U(\Theta) V_b U^{-1}(\Theta) \cong V_b + i\delta\Theta_a [T_a, V_b] = \qquad \text{(B-32)}$$

$$= V_b - i\delta\Theta_a R_{abc} V_c = (\delta_{bc} + i\delta\Theta_a R_{bac}) V_c$$

i.e.,

$$V_{a'} \longrightarrow e^{i\Theta_a R_a} V_{a'} \qquad \text{(B-33)}$$

APPENDIX C - LIE GROUPS

An abstract group G is a set of elements G={a,b,c,...} with respect to which a single law of composition (or multiplication) is defined. The ordered product a b of any two elements must satify the following conditions

1. Closure:

$$\forall\ a,b \in G\ ;\ a \circ b \in G$$

2. Associative law:

$$\forall\ a,b,c \in G\ ,\ a \circ (b \circ c) = (a \circ b) \circ c$$

3. Unit element:

$$\exists!\ e \in G,\ \forall\ a \in G\ ,\ a \circ e = e \circ a = a$$

4. Inverse element:

$$\forall a \in G\ ,\ \exists!\ a^{-1} \in G\ ,\ a \circ a^{-1} = a^{-1} \circ a = e$$

A group is called Abelian if all its elements commute with each other.

The order of G is the number of elements in G. A group is said to be finite if it has a finite number of elements. Otherwise the group is said to be infinite.

A number of elements of G are independent if none of them can be expressed in terms of the others. If a set of independent elements exists so that for any $x \in G$, x can be expressed in terms of the elements of the set we say that the latter is a set of independent generators for the group.

The subgroup H of G is invariant if it commutes with any element x of G. In other words, all conjugates of the elements of H defined by

$$x^{-1} h x \; ; \quad h \in H \; , \quad x \in G \tag{C-1}$$

are elements in H. Thus

$$x^{-1} h x \in H \tag{C-2}$$

A simple group has no invariant subgroups (except the unit element). A semi-simple group may have invariant subgroups but has no Abelian invariant subgroup.

We endow the set G (an infinite group) with a topology in such a way that we define a system of subsets of G such that every element of G is in at least one subset of G, that the null set and G itself are in the system, the intersection (set of elements in common) of two subsets and their union are also in the system of subsets. Then we say that G is a topological space and the elements of G are called the points of the space.

But G has also group properties, which mean that with any two of its elements is associated another element that is

their product. Thus, with any two points a and b of G we associate another point $\varphi(a,b)$. Keeping a fixed, to every b corresponds $\varphi(a,b)$. Also to every b corresponds another point that is its inverse. If these mappings of G into itself induced by group operations are continuous we say that G forms a topological group. Therefore the set G has two kinds of structure: a geometrical one which makes it a topological space and also a group structure which induces continuous mapping of G into itself.

The simplest kind of topological groups are those which locally have the properties of an r dimensional Euclidean space E_r, so that the neighbourhood of a point can be continuously mapped into a one-to-one way (homeomorphic mapping) to the neighbourhood of a point in E_r. Such a topological space is called an r-dimensional manifold. A topological group which is manifold is called a Lie group. It is also stated that a Lie group is a topological group whose underlying space (space formed by its elements) is an analytic manifold.

All the concepts of abstract group theory continue to hold for topological groups and in particular for Lie groups. In addition we define compactness for any topological group by the following: a topological group G is compact if its underlying space S is compact in the topological sense, that is, if any infinite subset of S contains a sequence which converges to an element of S.

Another important topological notion is connectivity.To define connectivity take an arbitrary point p in the group space S of G. Consider two closed curves $L_1(t)$ and $L_2(t)$ that both begin and end at p. Here t is a parameter that parametrizes the closed curves such that one point on the curve is associated with only one value of t. Further t=0 at the beginning and t=1 at the end of the curves. If there exist a function h(s,t) continuous in both s and t such that

$$h(0,t) = L_1(t) \; ; \quad h(1,t) = L_2(t) \tag{C-3}$$

with $0<s<1$ representing intermediate curves between $L_1(t)$ and $L_2(t)$, then the two curves are said to be homotopic. This means that they can be continuously deformed into each other by changing the parameter s from 0 to 1. If all closed curves from an arbitrary point p in a space can be deformed to zero ("are homotopic to zero"), then the space is simply conected. If there are m closed curves that cannot be deformed into one another, then the space is m-fold connected.

Lie proved some remarkable theorems concerning the relations between the generators and the group.

In the first place, the <u>commutator</u>

$$[T_a, T_b] = T_a T_b - T_b T_a \tag{C-4}$$

of two generators is always a linear combination of generators

$$[T_a, T_b] = C_{abc} T_c \qquad (C-5)$$

The real constants C_{abc} are called the structure constants of the Lie group.

The relation (C-5) leads to the concept of the Lie algebra associated with a given Lie group. This algebra is composed of all linear combinations of generators

$$c_a T_a \qquad (a = 1, \ldots\ldots, N) \qquad (C-6)$$

with real coefficients c_a. This is an algebra if we take for the "product" of two elements their commutator.

It is clear that the Lie group completely determines the structure of the associated Lie algebra. The converse is also essentially true: the local structure (that is, the structure in some neighbourhood of the identity) of a Lie group is completely determined by its Lie algebra, that is, by the structure constants C_{abc}.

Bibliography

Abers, E.S. and Lee, B.W. 1973, Phys.Rep. 9C, 1.

Abraham R. and Marsden J.E. 1967, Foundations of Mechanics, Benjamin, New York.

Adler S.L. 1979, Phys.Rev.D 19, 2997.

Akhoury R., Jung-Hwan J. and Goldhaber A.S. 1980, Phys.Rev. D 21, 454.

Ascher U., Christiansen J. and Russell D. 1979, Math.Comp. 33, 659.

Ascher U., Christiansen J. and Russell D. 1979, in Lecture Notes in Computer Science vol. 76 (Goos G. and Hartmanis J. eds.) Springer Verlag, Berlin.

Bacry H. 1977, Lectures on Group Theory and Particle Theory, Gordon and Breach, New York.

Bogoliubov N.N and Shirkov D.V. 1976, Introduction to the Theory of Quantized Fields, (in Russian) Nauka, Moscow.

Cahill K. 1978, Phys.Rev.Lett. 41, 599.

Carruthers P.A. 1966, Introduction to Unitary Symmetry, Interscience, New York.

Chakrabarti A. 1980, private communication.

Chakrabarti A. and Viswanathan K.S. 1980, Linear Deformations of Static, Self-dual de Sitter Solutions and their Prasad-Sommerfield Limit (to appear in Phys.Rev. D).

Corrigan E.F., Olive D.I., Fairlie D.B. and Nuyts J. 1976, Nucl.Phys. B106, 475.

Deser S. 1976, Phys.Lett. 64B, 463.

Feynman R.P. and Hibbs A.R. 1965, Quantum Mechanics and Path Integrals, McGraw-Hill, New York.

Gasiorowicz S. 1966, Elementary Particle Physics, Wiley, New York.

Gell-Mann M. 1962, Phys.Rev. 125, 1067.

't Hooft G. 1971a, Nucl.Phys. B33, 173.

't Hooft G. 1971b, Nucl.Phys. B35, 167.

't Hooft G. 1974, Nucl.Phys. B79, 276.

Ince E.L. 1956, Ordinary Differential Equations, Dover, New York.

Jackiw R., Jacobs L. and Rebbi C. 1979, Phys.Rev.D 20, 474.

Jackiw R. and Rossi P. 1980, Phys.Rev.D 21, 426.

Krein S.G. (ed.) 1972, Functional Analysis, Wolters-Noordhoff, Groningen.

Kurepa S. 1967, Finite-dimensional Vector Spaces and Applications (in Croatian), Tehnička knjiga, Zagreb.

Landau L.D. and Lifshitz E.M. 1973, Teorija polja (in Russian) Nauka, Moscow.

Li Ling-Fong 1974, Phys.Rev. D9, 1723.

Magg M. 1978a, Phys.Lett. 74B, 246.

Magg M. 1978b, Phys.Lett. 77B, 199.

Magg M. 1978c, Phys.Lett. 78B, 481.

Magg M. 1979, Nucl.Phys. B158, 154.

Mandula J.E. 1976, Phys.Rev. D14, 3497.

Mandula J.E. 1977a, Phys.Lett.67B, 175.

Mandula J.E. 1977b, Phys.Lett.69B, 495.

Marciano W. and Pagels H. 1978, Phys.Rep.36C , 137.

Marshak R.E., Riazuddin and Ryan C.P. 1969, Theory of Weak Interactions in Particle Physics, Wiley, New York.

Pagels H. 1980, private communication.

Polyakov A.M. 1974, Zh.Eksp.Teor.Fiz.Pis'ma Red. 20, 430.

Popov V.N. 1977, CERN Report No. Cern-TH.2424 (unpublished).

Racah G. 1964, in Group Theoretical Concepts and Methods in Elementary Particle Physics, (F.Gursey ed.) Gordon and Breach, New York.

Ragiadakos Ch. and Viswanathan K.S. 1979, Phys.Lett.86B, 288.

Rohrlich F. 1965, Classical Charged Particles, Addison-Wesley, Reading, Massachusetts.

Salam A. 1968, In Elementary Particle Theory (N.Svartholm, ed.) Almquist and Wiksell, Stockholm.

Schwinger J.(ed.) 1958, Quantum Electrodynamics , Dover, New York.

Sciuto S. 1977, Lectures at Cern, CERN report (unpublished).

Siegel C.L. and Moser J.K. 1971, Lectures on Celestial Mechanics, Springer-Verlag, Berlin.

Sikivie P. and Weiss N. 1978a, Phys.Rev.Lett. 40, 1411.

Sikivie P. and Weiss N. 1978b, Phys.Rev.D 18, 3809.

Speiser D. 1964, in Group Theoretical Concepts and Methods in Elementary Particle Physics, (F.Gursey ed.) Gordon and Breach, New York.

Weinberg S. 1967, Phys.Rev.Lett.19, 1264.

Werle J. 1966, Relativistic Theory of Reactions, North-Holland, Amsterdam.

Yang C.N. and Mills R.L. 1954, Phys.Rev.96, 191.

Yates R.G. 1977, Nucl.Phys.B123, 333.