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LINEAR FUNCTIONALS ON THE SPACE OF REAL BOUNDED SEQUENCES

by

Pushpa Kumari Jain
M.A., Panjab University, Chandigarh, 1965

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
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ABSTRACT

The accent in this thesis is on the structure of linear functionals on the space M of bounded sequences of real numbers. A relation between the value of a continuous linear functional (defined on M) at a convergent sequence and the limit of the sequence is established. This forms the foundation for the structure theorems which follow. Ultimately, it is shown that any continuous linear functional on M can be written as a linear combination of at most two non-negative regular linear functionals and a linear functional of another type, i.e., an ℓ_1-multiplier.

The existence of several types of linear functionals on M is also discussed. This involves an application of the Hahn-Banach extension theorem and an infinite dimensional Hamel base argument.

DEDICATION

to my parents

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CHAPTER 1

INTRODUCTION

This preliminary chapter covers some fundamental concepts. Its purpose is to introduce notation and elementary concepts and to give the reader a survey of the material which will be used later in the thesis. A versed reader may omit it and proceed to the next chapter on structure theorems and use Chapter 1 only for reference.

We take for granted that the reader is familiar with the concepts of a set, a subset and a sequence. We also presuppose a familiarity with the basic operations involving sets. Now we introduce some notation.

The set of all natural numbers (positive integers) will be denoted by the letter N and the set of all real numbers, by R^1. We shall write, for a sequence x, of real numbers,

$$x = (x_n) = (x_1, x_2, x_3, \ldots),$$

using round brackets to avoid confusion with a mere set. The sequence $(1,1,1,\ldots)$ will be denoted by the letter e. For each $i = 1,2,3,\ldots$, the sequence $(0,0,\ldots,0,1,0,0,\ldots)$, where the 1 comes in the i-th place, will be denoted by e^i. The sequence $(0,0,0,\ldots)$ will be denoted by $\overline{0}$. The sum of two sequences $a = (a_i)$ and $b = (b_i)$ is the sequence $(a_1+b_1, a_2+b_2, a_3+b_3, \ldots)$. It will be denoted by $a+b = (a_i+b_i)$. Similarly, the difference of two sequences is defined. Furthermore, for a real number λ , $\lambda x = \lambda(x_i) = (\lambda x_i)$. Consequently, for a real number k, the constant sequence

$$\overline{k} = (k,k,k,\ldots) = k(1,1,1,\ldots) = ke$$

and

$$\begin{aligned}\overline{k}-x = (k-x_i) &= (k-x_1,k-x_2,k-x_3,\ldots)\\ &= (k,k,k,\ldots) - (x_1,x_2,x_3,\ldots)\\ &= k(1,1,1,\ldots) - (x_i)\\ &= ke-x .\end{aligned}$$

We presume a familiarity with the concepts of bounded sequences and convergent sequences. The set of all <u>bounded sequences of real numbers</u> will be denoted by M . For a sequence $x = (x_n)$ of real numbers, which is convergent to a limit ℓ (where ℓ is a real number, of course), we shall write $x_n \to \ell$ as $n \to \infty$ or $\lim x = \ell$ or $\lim_{n \to \infty} x_n = \ell$. The set of all convergent sequences of real numbers will be denoted by c. Moreover, the set of all sequences of real numbers that are convergent to 0 will be denoted by c_0. It is easy to see that if a sequence (x_n) of real numbers is convergent, then it is bounded. Thus, we have the following inclusions:

$$c_0 \subsetneqq c \subsetneqq M .$$

Now we acquaint the reader with the concepts of limit superior and limit inferior of a sequence, which we will need in the third chapter of our thesis. We, therefore, state the following definition.

<u>Definition 1.1</u>. Let (x_n) be a sequence of real numbers that is bounded. Let $M_n = \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Then the sequence (M_n) converges and we define $\lim \sup x_n$ to be $\lim_{n \to \infty} M_n$.

It is not hard to see that the sequence (M_n) is monotonically decreasing. Hence,

$$\lim \sup x_n = \lim_{n \to \infty} [\sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}]$$

$$= \inf_n [\sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}].$$

Similarly, we define for a sequence (x_n), which is bounded,

$$\liminf x_n = \lim_{n\to\infty} [\inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}]$$

$$= \sup_n [\inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}].$$

Evidently, for any bounded sequence x, of real numbers, $\limsup x_n$ and $\liminf x_n$ are finite real numbers.

Next, we list a few properties of limit superior and limit inferior of a sequence, which will be used later:

(a) If (x_n) is a convergent sequence of real numbers, then $\liminf x_n = \limsup x_n = \lim x$, and conversely.

(b) If (x_n) and (y_n) are bounded sequences of real numbers, then $\limsup (x_n+y_n) \le \limsup x_n + \limsup y_n$.

(c) For a real number $\alpha \ge 0$, $\limsup (\alpha x_n) = \alpha \limsup x_n$.

(d) For a sequence (x_n) of real numbers, $\liminf x_n = -\limsup (-x_n)$ and $\limsup (x_n) = -\liminf (-x_n)$.

We assume that the reader is familiar with the concept of a linear space. We will denote the zero vector by $\overline{0}$. One type of linear space, which we will be concerned with, is that consisting of bounded sequences of real numbers.

<u>Definition 1.2.</u> A non-empty set of bounded sequences of real numbers is called a <u>linear space of bounded sequences</u> over the reals

if it is closed under co-ordinatewise addition and scalar multiplication of sequences as defined earlier.

For example, M , c and c_0 are linear spaces over the reals.

For the moment, we shall be concerned with some general definitions and properties.

<u>Definition 1.3</u>. A <u>norm</u> $\|\cdot\|$, on a linear space X , is a function $\|\cdot\| : X \to R^1$ such that

(i) $\|x\| = 0$ if, and only if, $x = \bar{0}$,

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$,

(iii) $\|x+y\| \le \|x\| + \|y\|$

<u>Definition 1.4</u>. A <u>normed linear space</u> $(X, \|\cdot\|)$ is a linear space X , with a norm defined on it.

It is not hard to see that M , c and c_0 become normed linear spaces over the reals with norm defined on them as follows:

$$\|x\| = \sup_n |x_n| \text{ , for all } x \text{ .}$$

<u>Definition 1.5</u>. A non-empty subset S of a linear space X is called a <u>subspace</u> of X if $\lambda x + \mu y \in S$, whenever $x, y \in S$, for all $\lambda, \mu \in R^1$.

As an illustration, both c and c_0 are subspaces of M. Moreover, if $\{S_\alpha\}$ is a family of subspaces, then $\cap S_\alpha$ is also a subspace.

We now introduce the important concept of a linear operator.

Definition 1.6. Let X, Y be linear spaces. Then a function $f : X \to Y$ is called a linear operator (or map, transformation) if, and only if, for all $x_1, x_2 \in X$, and all scalars λ, μ,

$$f(\lambda x_1 + \mu x_2) = \lambda f(x_1) + \mu f(x_2).$$

It is easy to see that the composition of two linear operators is again a linear operator.

We are now in a position to define the concept of a linear functional.

Definition 1.7. f is called a linear functional on a linear space X if $f : X \to R^1$ is a linear operator, i.e., a linear functional is a real-valued linear operator.

The zero linear functional will be denoted by θ. Thus, $\theta : X \to R^1$ is such that $\theta(x) = 0$, for all $x \in X$.

Linear operators on normed spaces, which are continuous, are of special interest in functional analysis. They form the primary subject matter of our thesis.

Definition 1.8. Let X, Y be normed linear spaces. Let $f : X \to Y$ be a linear operator. Then f is called continuous at $x_0 \in X$ if, and only if, for every $\varepsilon > 0$, there exists $\delta > 0$, $\delta(x_0, \varepsilon)$ such that $\|f(x) - f(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$.

Definition 1.9. The function f in the above definition is called continuous on X if f is continuous at each point in X.

Another type of operator on a normed space, which actually turns out to be the same thing as a continuous linear operator, is a bounded linear operator.

Definition 1.10. Let X, Y be normed linear spaces. Then a linear operator $f : X \to Y$ is called bounded if, and only if, there exists a constant M_1 such that

$$\|f(x)\| \leq M_1 \|x\|, \quad \text{for all } x \in X .$$

Definition 1.11. Let X be a normed linear space. A linear functional $f : X \to R^1$ is called bounded if, and only if, there exists a constant M_1 such that

$$|f(x)| \leq M_1 \|x\|, \text{ for all } x \in X .$$

Now we give a very useful and well-known property of continuous linear operators.

Theorem 1.12. Let X, Y be normed linear spaces. Let $f : X \to Y$ be a linear operator. Then f is continuous on X if, and only if, it is bounded.

Proof. Let f be bounded. Therefore, there exists a constant M_1 such that $\|f(x)\| \le M_1 \|x\|$ for all $x \in X$. Now,

$$\|f(x)-f(y)\| = \|f(x-y)\| \le M_1 \|x-y\| < \varepsilon ,$$

if $\|x-y\| < \frac{\varepsilon}{M_1}$. Hence f is uniformly continuous on X .

Conversely, let f be continuous on X . Then it is continuous at $\overline{0}$, in particular. Hence, there exists $\delta > 0$, $\delta = \delta(1)$ such that $\|f(x)\| < 1$ whenever $\|x\| < \delta$. Take any $x \neq \overline{0}$. Then

$$\left\|\frac{\delta \cdot x}{2\|x\|}\right\| = \left|\frac{\delta}{2\|x\|}\right| \|x\| = \frac{\delta}{2} < \delta ,$$

and so

$$\left\|f\left(\frac{\delta \cdot x}{2\|x\|}\right)\right\| < 1 .$$

That is, $\frac{\delta}{2\|x\|} \|f(x)\| < 1$ and consequently,

$$\|f(x)\| < \frac{2\|x\|}{\delta} = \frac{2}{\delta}\|x\| \le \frac{2}{\delta}\|x\| .$$

If $x = \overline{0}$, then $\|f(x)\| = 0 = \frac{2}{\delta} \cdot 0 = \frac{2}{\delta}\|x\| \le \frac{2}{\delta}\|x\|$.

Thus, in both cases, $\|f(x)\| \le \frac{2}{\delta}\|x\|$, whence f is bounded. This completes the proof.

In particular, taking $Y = R^1$, we have the following important and commonly used theorem.

<u>Theorem 1.13</u>. A linear functional $f : X \to R^1$ is continuous on X if, and only if, it is bounded.

<u>Definition 1.14</u>. A non-empty set of linear functionals on the same space X, which is closed under addition and scalar multiplication, is called a <u>linear space of linear functionals</u>.

<u>Definition 1.15</u>. The set of all linear functionals on M is a linear space (under the usual operations) and it will be denoted by M'.

<u>Definition 1.16</u>. The set of all continuous (bounded) linear functionals on M is also a linear space under the usual operations and it will be denoted by M^*. Obviously, M^* is a subspace of M'.

We now discuss a few more concepts concerning linear spaces in general, which we will apply later in the thesis.

<u>Definition 1.17</u>. Let S be a subset of a linear space X. The <u>linear hull of S</u> is the intersection of all subspaces containing S. It will be denoted by $[S]$. Symbolically,

$$[S] = \cap \{V \mid V \text{ a subspace of } X \text{ and } S \subset V\}$$

$$= \text{smallest subspace of } X \text{ containing } S .$$

We shall also use the terms 'span of S' or 'subspace generated by S' for linear hull of S. We now present the following interesting theorem.

<u>Theorem 1.18</u>. Let S be a non-empty subset of a linear space X. Then the linear hull of S is the set of all finite linear combinations of elements of S.

<u>Proof</u>. Let $S' = \{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \mid x_i \in S, 1 \leq i \leq n, n \in N\}$. It suffices to show that $[S] = S'$.

It is easy to see that S' is a subspace of X containing S. Since $[S]$ is the smallest subspace of X containing S, it follows that $[S] \subseteq S'$. On the other hand, since $[S]$ is a subspace of X, finite linear combinations of elements of $[S]$ belong to $[S]$. But as $[S]$ contains S, finite linear combinations of elements of S also belong to $[S]$. Consequently, $S' \subseteq [S]$, and we are done.

Definition 1.19. A finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is called a linearly independent set if, and only if, a relation of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \overline{0}$$

implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.

If a finite subset of a linear space is not linearly independent, it will be called linearly dependent.

Definition 1.20. An arbitrary subset (not necessarily finite) of X is called linearly independent if, and only if, every one of its finite subsets is linearly independent.

Definition 1.21. A subset B of X is called a Hamel base (or basis) for X if, and only if, B is a linearly independent set and [B] = X, i.e., B generates the linear space X .

Theorem 1.22. Every linear space X has a Hamel basis.

For a proof see, e.g., Maddox [3], p. 78.

It is well-known that any basis for a subspace of X is contained in some basis for X . (The proof of this requires an application of Zorn's lemma and is similar to the proof of Theorem 1.22).

Definition 1.23. A linear space X is called finite dimensional if, and only if, X has a finite Hamel base B, i.e., B is a finite set which is a Hamel base. If X is not finite dimensional, it is called infinite dimensional.

We remark that M, c, c_0, M^* and M' are all infinite dimensional linear spaces. Furthermore, if X is any infinite dimensional space with a Hamel base $\{b_\alpha | \alpha \in \Delta\}$, then for each $x \in X$, there exist unique scalars $\lambda_\alpha, \alpha \in \Delta$ such that $x = \sum_{\alpha \in \Delta} \lambda_\alpha b_\alpha$, where $\lambda_\alpha \neq 0$ for at most finitely many α.

Definition 1.24. If X is a finite dimensional space, then its dimension is defined to be the number of elements in any of its Hamel bases. (By Theorem 1.26 below, the dimension of X is well-defined).

We conclude this chapter by stating a couple of theorems that will be useful later on. For proofs, see, e.g., Maddox [3], pp. 76, 77.

Theorem 1.25. Let X have a Hamel base with n elements. Then any set of $n + 1$ elements in X is linearly dependent.

Theorem 1.26. Let X be finite dimensional. Then all the Hamel bases for X have the same number of elements.

CHAPTER 2

STRUCTURE THEOREMS

This chapter is concerned with the structure of continuous linear functionals defined on the linear space, M, of all bounded sequences, over the field of real numbers. We recall that M is a normed linear space over the reals, with norm defined on it as follows: for $x \in M$, $\|x\| = \sup_n |x_n|$. Moreover, the class of all continuous linear functionals on M is itself a linear space, denoted by M^*.

Non-negative linear functionals and regular linear functionals on M will be defined. It will, furthermore, be shown that every non-negative linear functional on M is continuous on M (Theorem 2.13). A special class, L, of continuous (and non-regular) linear functionals on M will be defined and it will turn out to be a subspace of M^*. A relation between the value of a continuous linear functional at a convergent sequence and the limit of the sequence will be established (Theorem 2.10). Then it will be shown that every continuous linear functional on M can be expressed either in terms of a continuous and regular linear functional, and a linear functional from L (Theorem 2.15) or as a difference of two continuous and regular linear functionals, and a linear functional from L (Theorem 2.16). Moreover, it will be shown that every continuous linear functional on M can be written as a difference of two non-negative linear functionals on M (Lemma 2.17). In addition, it will be demonstrated that every continuous and regular linear functional on M can be expressed as a particular linear combination of two non-negative and regular linear functionals on M.

The set of all continuous and regular linear functionals on M will be denoted by R and the linear hull of R will be denoted by $[R]$. Furthermore, it will be exhibited that M^* is the direct sum of its subspaces $[R]$ and L. The set of all non-negative and regular linear functionals on M will be denoted by R^+ and its linear hull, by $[R^+]$. It will then be shown that $[R] = [R^+]$. Consequently, M^* will become the direct sum of its subspaces $[R^+]$ and L.

Finally, it will be demonstrated that every continuous linear functional on M can be expressed as a linear combination of at most two non-negative, regular linear functionals and a linear functional from L. This gives an upper bound to the number of linear functionals to be taken from R^+, in the preceding result.

We begin our discussion with a few important definitions.

Definition 2.1. A sequence $a = (a_i)$ is called absolutely convergent if

$$\sum_{i=1}^{\infty} |a_i| < \infty .$$

The set of all absolutely convergent sequences will be called ℓ_1 and ℓ_1 forms a linear space under the usual operations. Evidently, ℓ_1 is a subspace of M. In fact, ℓ_1 is a subspace of c_0 so that we have the following inclusions:

$$\ell_1 \subsetneqq c_0 \subsetneqq c \subsetneqq M .$$

Definition 2.2. A sequence $c = (c_i)$ is called the term by term product of two sequences $a = (a_i)$ and $b = (b_i)$ if

$$c_i = a_i b_i, \; i = 1,2,3,\ldots .$$

The following theorem establishes the absolute convergence of the term by term product of an absolutely convergent sequence and a bounded sequence.

Theorem 2.3. If $a \in \ell_1$ and $x \in M$, then the series $\sum_{i=1}^{\infty} |a_i x_i|$ converges.

Proof.
$$\sum_{i=1}^{\infty} |a_i x_i| = \lim_{n \to \infty} \sum_{i=1}^{n} |a_i x_i|$$

$$\leq \lim_{n \to \infty} \sum_{i=1}^{n} |a_i| \sup_i |x_i|$$

$$= \|x\| \sum_{i=1}^{\infty} |a_i| < \infty ,$$

as required.

The above theorem permits the following corollary.

Corollary 2.4. If $a \in \ell_1$ and $x \in M$, then the series $\sum_{i=1}^{\infty} a_i x_i$ converges.

Proof. An absolutely convergent series of real numbers is convergent.

We now introduce the concept of a regular linear functional on M.

Definition 2.5. A linear functional f on M is called regular if it extends 'lim'. In other words, for $x \in c$,

$$f(x) = \lim x .$$

The following result is an indispensable tool in working with continuous linear functionals on M.

Theorem 2.6. Let $a \in \ell_1$. Then the function f defined on M by

$$f(x) = \sum_{i=1}^{\infty} a_i x_i, \text{ for all } x \in M,$$

is a continuous linear functional on M. Furthermore, f is not regular.

<u>Proof</u>. It follows, from Corollary 2.4, that $f(x) = \sum_{i=1}^{\infty} a_i x_i$ converges. Therefore, $f : M \to R^1$ and so f is defined for all x in M.

Now we show that f is a linear map. To this end, let α, β be any scalars and $x, y \in M$. Then

$$f(\alpha x + \beta y) = \sum_{i=1}^{\infty} a_i (\alpha x_i + \beta y_i)$$

$$= \sum_{i=1}^{\infty} (a_i \alpha x_i + a_i \beta y_i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} (a_i \alpha x_i + a_i \beta y_i)$$

$$= \alpha \lim_{n \to \infty} \sum_{i=1}^{n} a_i x_i + \beta \lim_{n \to \infty} \sum_{i=1}^{n} a_i y_i$$

$$= \alpha \sum_{i=1}^{\infty} a_i x_i + \beta \sum_{i=1}^{\infty} a_i y_i$$

$$= \alpha f(x) + \beta f(y) .$$

In order to show that f is continuous on M, it is sufficient (by Theorem 1.13) to show that f is bounded on M. Thus, for each $x \in M$,

$$|f(x)| = \left|\sum_{i=1}^{\infty} a_i x_i\right|$$

$$\leq \sum_{i=1}^{\infty} |a_i|\,|x_i|$$

$$\leq \|x\| \sum_{i=1}^{\infty} |a_i|$$

$$= A\|x\| ,$$

where $A = \sum_{i=1}^{\infty} |a_i|$. Thus, there exists a constant A such that $|f(x)| \leq A\|x\|$, for all $x \in M$. Therefore, f is bounded and hence continuous on M.

It remains to show that f is not regular. For this, it is enough to prove the existence of $x \in c$ such that $\lim x = \ell$, but $f(x) \neq \ell$. We consider the following two cases.

<u>Case I</u>. There exists i such that $a_i \neq 0$. Then $e^i = (0,0,\ldots,0,1,0,0,\ldots)$, where the 1 comes in the i-th place, is such that $\lim e^i = 0$, but $f(e^i) = a_i \neq 0 = \lim e^i$.

Case II. $a_i = 0$, for all i. Then $e = (1,1,1,\ldots)$ is such that $\lim e = 1$, but $f(e) = 0 \neq 1 = \lim e$. This proves the theorem.

Definition 2.7. The linear functionals on M, of the type $f(x) = \sum_{i=1}^{\infty} a_i x_i$, where $a = (a_i) \in \ell_1$ and $x \in M$, are called the ℓ_1-multipliers. The set of all such linear functionals will be denoted by L.

In view of Theorem 2.6, for $f(x) = \sum_{i=1}^{\infty} a_i x_i$, it follows that the ℓ_1-multipliers are continuous and non-regular linear functionals on M.

It is instructive to observe that the correspondence $a \leftrightarrow f$, where $a \in \ell_1$ and $f(x) = \sum_{i=1}^{\infty} a_i x_i$ for all $x \in M$, is an isomorphism. Consequently, L is a subspace of M^*.

Definition 2.8. The signum of a real number a, denoted by sgn a, is defined as follows:

$$\operatorname{sgn} a = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

Thus, for a non-zero real number a, sgn a is $+1$ or -1 according as a is positive or negative. Moreover, for all a, we have

$$a \operatorname{sgn} a = |a| .$$

The following theorem gives a very important property of the sequence formed by the values of a continuous linear functional evaluated at the bounded sequences e^i, $i = 1,2,3,\ldots$.

Theorem 2.9. Let $f \in M^*$ and $a_i = f(e^i)$, $i = 1,2,3,\ldots$. Then $a \in \ell_1$.

Proof. $$\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} |f(e^i)|$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} |f(e^i)|$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} (\operatorname{sgn} f(e^i))\, f(e^i)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f((\operatorname{sgn} f(e^i))e^i)$$

$$= \lim_{n \to \infty} f\Big(\sum_{i=1}^{n} (\operatorname{sgn} f(e^i))e^i\Big)$$

$$= \lim_{n \to \infty} f(E^n) ,$$

where $E^n = (\text{sgn } f(e^1), \text{sgn } f(e^2), \ldots, \text{sgn } f(e^n), 0, 0, \ldots)$. Note that $\|E^n\| \le 1$. Now, since f is bounded, there exists a constant M_1 such that $|f(E^n)| \le M_1 \|E^n\| \le M_1$, which implies that $f(E^n) \le M_1$. Thus, $\lim_{n \to \infty} f(E^n) \le M_1 < \infty$; that is, $\sum_{i=1}^{\infty} |a_i| < \infty$, whence a is an element of ℓ_1.

The following theorem provides a relation between the value of a continuous linear functional at a convergent sequence and the limit of the sequence. It represents a slight generalization of the well-known characterization of continuous linear functionals on c (see, e.g., Maddox [3], p. 109).

Theorem 2.10. Let $f \in M^*$, $f(e^i) = a_i$ $(i = 1,2,3,\ldots)$, and $g(x) = \sum_{i=1}^{\infty} a_i x_i$ for all $x \in M$. Let $s = g(e)$ and $\ell = f(e)$. Then for $x \in c$,

$$f(x) = (\ell - s) \lim x + g(x).$$

Proof. Note that by Theorem 2.9, $a \in \ell_1$ and by Theorem 2.6, $g \in L$. Also, observe that

$$s = g(e) = \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} f(e^i).$$

Now, let $x \in c$ and let $\lim x = L$. We shall first show that

$$x = Le + \sum_{i=1}^{\infty} (x_i - L)e^i ,$$

that is,

$$\|x - [Le + \sum_{i=1}^{n} (x_i - L)e^i]\| \to 0 \quad \text{as} \quad n \to \infty . \qquad (1)$$

Now,

$$\|x - [Le + \sum_{i=1}^{n} (x_i - L)e^i]\| = \|x - Le - \sum_{i=1}^{n} (x_i - L)e^i\|$$

$$= \|(x_1, x_2, \ldots) - (L, L, \ldots) - (x_1 - L, x_2 - L, \ldots, x_n - L, 0, 0, \ldots)\|$$

$$= \|(0, 0, \ldots, 0, x_{n+1} - L, x_{n+2} - L, \ldots)\|$$

$$= \sup_{i \geq n+1} |x_i - L| .$$

Since $\lim x = L$, the last expression tends to zero as n tends to infinity. Therefore, (1) has been proved.

Further, using the linearity and the continuity of f, we have

$$f(x) = f(Le) + f[\lim_{n\to\infty} \sum_{i=1}^{n} (x_i - L)e^i]$$

$$= f(Le) + \lim_{n\to\infty} f[\sum_{i=1}^{n}{}' (x_i - L)e^i]$$

$$= Lf(e) + \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - L)f(e^i)$$

$$= L\ell + \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - L)a_i$$

$$= L\ell + \lim_{n\to\infty} [\sum_{i=1}^{n} x_i a_i - L \sum_{i=1}^{n} a_i]$$

$$= L\ell + \lim_{n\to\infty} \sum_{i=1}^{n} x_i a_i - L \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

$$= L\ell + \sum_{i=1}^{\infty} x_i a_i - L \sum_{i=1}^{\infty} a_i$$

$$= L\ell + g(x) - Lg(e)$$

$$= L\ell + g(x) - Ls$$

$$= (\ell - s)L + g(x)$$

$$= (\ell - s) \lim x + g(x).$$

This completes the proof.

Definition 2.11. A sequence $x = (x_i)$ is called non-negative if, and only if, $x_i \geq 0$ for $i = 1,2,3,\ldots$. We shall write $x \geq \overline{0}$. The set of all non-negative and bounded sequences of real numbers will be denoted by M^+ , i.e., $M^+ = \{x \in M \mid x \geq \overline{0}\}$.

We now introduce the important concept of a non-negative linear functional on M .

Definition 2.12. A linear functional f, defined on M , is called non-negative if, and only if, for all $x \in M$, $x \geq \overline{0}$ implies that $f(x) \geq 0$. The set of all non-negative linear functionals on M will be denoted by N . Occasionally, for convenience,we shall write $f \geq 0$ for $f \in N$.

The following theorem presents an interesting and a very useful property of non-negative linear functionals defined on M , see, e.g., Schaefer [6], p. 228.

Theorem 2.13. Any non-negative linear functional on M is continuous, i.e., $N \subsetneq M^*$.

Proof. Let f be a non-negative linear functional on M . Let $f(e) = \ell$. For any constant sequence $\overline{c} = (c,c,c,\ldots)$, we have

$$f(c,c,c,\ldots) = f(ce) = cf(e) = c\ell . \tag{2}$$

Let $x \in M$ and let $b = \|x\|$. We have, for each i,

$$|x_i| \leq \sup_i |x_i| = b .$$

Therefore, $-b \leq x_i \leq b$ for all i, which implies that $b - x_i \geq 0$ for all i, and $b + x_i \geq 0$ for all i.

Thus, the sequences $(b-x_i) = be-x$ and $(b+x_i) = be+x$ are bounded and non-negative. Since f is a non-negative linear functional on M, we have

$$0 \leq f(be-x) = f(be) - f(x) = b\ell - f(x) ,$$

which implies that $f(x) \leq b\ell = \|x\|\ell$. Similarly, $0 \leq f(be+x) = b\ell + f(x)$, which implies that $-f(x) \leq b\ell = \|x\|\ell$.

Therefore, it follows that $|f(x)| \leq \ell\|x\|$ for all $x \in M$ and hence f is bounded on M, which is what we wished to show.

It is worth remarking that there exist linear functionals on M which are continuous but are not non-negative (viz. Chapter 3, Theorems 3.4 and 3.6) so that the converse of the preceding theorem is not true.

Definition 2.14. We shall denote the set of all continuous and regular linear functionals on M by R. Likewise, the set of all non-negative (hence continuous) and regular linear functionals on M will be denoted by R^+.

We now prove some of the most important theorems of this chapter which demonstrate the structure of continuous linear functionals on M .

<u>Theorem 2.15</u>. Let $f \in M^*$, $\ell = f(e)$ and $s = \sum_{i=1}^{\infty} f(e^i)$. If $\ell \neq s$, then there exist $h \in R$, $g \in L$ and a non-zero real constant A such that

(i) $f(x) = A\,h(x) + g(x)$, for all $x \in M$;

(ii) A, h, g are unique.

<u>Proof</u>. (i) From Theorem 2.10, we have, for $f \in M^*$ and $x \in c$,

$$f(x) = (\ell - s) \lim x + g(x) \ , \tag{3}$$

where $g \in L$. Since $\ell \neq s$, $\frac{1}{\ell - s}$ exists. Therefore, from (3) we obtain, for $x \in c$,

$$\frac{1}{\ell - s}\,[f(x) - g(x)] = \lim x \ . \tag{4}$$

Now $f, g \in M^*$ implies that $\frac{1}{\ell-s}(f-g) \in M^*$. Let $h = \frac{1}{\ell-s}(f-g)$. Then $h \in M^*$. Moreover, in view of (4), h is regular. Hence $h \in R$. Furthermore, for each $x \in M$,

$$f(x) = [f(x) - g(x)] + g(x) = (\ell-s)h(x) + g(x)$$

$$= A\, h(x) + g(x) ,$$

where $A = \ell-s$ is a constant. This proves (i) .

(ii) We first prove that g is unique. It has already been shown in (i) that for each $x \in M$, $f(x) = A\, h(x) + g(x)$, where $h \in R$, $g \in L$ and A is a non-zero real constant. Let $f(x) = A'h'(x) + g'(x)$, be another representation of $f(x)$, where $h' \in R$, $g' \in L$ and A' is a non-zero real constant. Putting $x = e^i$ in both representations of f and using the regularity of h and h', we have, for each i,

$$f(e^i) = A\, h(e^i) + g(e^i) = A \lim e^i + g(e^i) = 0 + g(e^i) = g(e^i),$$

and similarly, $f(e^i) = g'(e^i)$. Hence $g(e^i) = g'(e^i)$, for all i. Now let (a_i) and (b_i) be the sequences in ℓ_1 corresponding to g and g' respectively. Then since $g(e^i) = g'(e^i)$ for all i, it follows that $a_i = b_i$ for all i. Consequently, $g = g'$.

Next, we show that A is unique. To this end, let $x = e$ in the two representations of $f(x)$. Then

$$f(e) = A\, h(e) + g(e) = A \lim e + g(e) = A + g(e).$$

Similarly, $f(e) = A' + g(e)$. Thus, $A = A'$.

Finally, since $A = A'$ and $g = g'$, we conclude that $h = h'$ as $A \neq 0$.

We can formulate a theorem analogous to Theorem 2.15, which handles the case $\ell = s$, as follows:

<u>Theorem 2.16.</u> Let $\mathcal{D} = \mathcal{R} - \mathcal{R} = \{\sigma - \tau \mid \sigma, \tau \in \mathcal{R}\}$. Let $f \in \mathcal{M}^*$, $\ell = f(e)$ and $s = \sum_{i=1}^{\infty} f(e^i)$. If $\ell = s$, then there exist $t \in \mathcal{D}$ and $g \in \mathcal{L}$ such that

(i) $f(x) = t(x) + g(x)$, for all $x \in \mathcal{M}$;

(ii) t and g are unique.

<u>Proof.</u> The argument is essentially the same as that in the proof of Theorem 2.15.

(i) Observe that from Theorem 2.10, for $f \in \mathcal{M}^*$ and $x \in c$,

$$f(x) = (\ell - s)\lim x + g(x),$$

where $g \in \mathcal{L}$. Here $\ell = s$ and so

$$f(x) = g(x), \text{ for all } x \in c. \tag{5}$$

Let τ be any continuous and regular linear functional[†] on $\mathcal{M}$, i.e., $\tau \in \mathcal{R}$. Define $\sigma : \mathcal{M} \to R^1$ as follows:

$$\sigma(x) = f(x) - g(x) + \tau(x) . \tag{6}$$

We claim that $\sigma \in \mathcal{R}$. Since $f, g, \tau \in \mathcal{M}^*$, we have $\sigma \in \mathcal{M}^*$. Moreover, from (5) and the regularity of τ, we get, for $x \in c$,

$$\sigma(x) = f(x) - g(x) + \tau(x) = 0 + \tau(x) = \lim x ,$$

which shows that σ is regular. This establishes our claim.

Finally, from (6), we have, for $x \in \mathcal{M}$, $\sigma(x) = f(x) - g(x) + \tau(x)$. This means that $f(x) = \sigma(x) - \tau(x) + g(x)$, where $\sigma, \tau \in \mathcal{R}$ and $g \in \mathcal{L}$. Let $t = \sigma - \tau$. Then for $x \in \mathcal{M}$,

$$f(x) = t(x) + g(x), \tag{7}$$

where $t \in \mathcal{D}$ and $g \in \mathcal{L}$. This completes the proof of (i).

(ii) Let $f(x) = t'(x) + g'(x)$, where $t' \in \mathcal{D}$ and $g' \in \mathcal{L}$,

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[†]The question of the existence of τ arises here. This is treated in the next chapter. The reader is assured that such τ do exist, e.g., τ a Banach limit (see Theorem 3.1).

be another representation of f . Let $t' = \sigma' - \tau'$, where $\sigma', \tau' \in R$. Putting $x = e^i$ in both representations of f , it follows exactly as in the proof of Theorem 2.15 that $g(e^i) = g'(e^i)$, for every i and that $g = g'$.

Now, we proceed to prove that t is unique. Again, considering the above two representations of f , we have, for $x \in M$,

$$f(x) = t(x) + g(x) = t'(x) + g(x) .$$

Consequently,

$$t(x) = f(x) - g(x) = t'(x),$$

for all $x \in M$ and hence t is unique. This completes the proof of (ii), and the theorem follows.

It is interesting to note that σ and τ are not unique in the above theorem.

We recall that the set of all continuous and regular linear functionals on M is denoted by R and the set of all non-negative (hence continuous) and regular linear functionals on M is denoted by R^+ . The letter N denotes the set of all non-negative (hence continuous) linear functionals on M . Furthermore, M^+ denotes the set of all non-negative and bounded sequences of real numbers.

Our next venture is to prove a very important lemma, which is a corollary of a result in the theory of Topological Vector Spaces, see, e.g., Schaefer [6], p. 218. We are going to present an independent proof. The importance of this lemma lies in the fact that it is a powerful tool in determining the structure of a continuous and regular linear functional on M.

Lemma 2.17. $M^* = N - N$, i.e., for each $f \in M^*$, there exist $g, h \in N$ such that $f = g - h$.

In order to establish the proof of this lemma, we first present the necessary background material.

Definition 2.18. Let α be a real number. Then α^+, the positive part of α, is defined by $\alpha^+ = \max\{\alpha, 0\}$, and α^-, the negative part of α, is defined by $\alpha^- = \max\{-\alpha, 0\}$. Thus,

$$\alpha^+ = \begin{cases} \alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leq 0 ; \end{cases}$$

and

$$\alpha^- = \begin{cases} -\alpha & \text{if } \alpha < 0 , \\ 0 & \text{if } \alpha \geq 0 . \end{cases}$$

Clearly, $\alpha = \alpha^+ - \alpha^-$, $|\alpha| = \alpha^+ + \alpha^-$ and $\alpha^+ = \alpha$ if, and only if, $\alpha \geq 0$.

Definition 2.19. Let x be any sequence of real numbers. We define the positive and the negative parts of x as follows:

$$x^+ = (x_i^+),\ x^- = (x_i^-) .$$

Then $x = x^+ - x^-$. It is clear that x^+ and x^- are both non-negative sequences.

Proposition 2.20. For any sequence x and any real number λ, we have

(a) $(\lambda x)^+ = \lambda x^+ \quad (\lambda \geq 0)$,

(b) $(\lambda x)^- = \lambda x^- \quad (\lambda \geq 0)$,

(c) $(\lambda x)^+ = -\lambda x^- \quad (\lambda < 0)$,

(d) $(\lambda x)^- = -\lambda x^+ \quad (\lambda < 0)$.

Proof. (a) We first establish the result for a real number α. Let $\alpha \geq 0$. By Definition 2.18, $(\lambda\alpha)^+ = \max(\lambda\alpha, 0) = \lambda\alpha = \lambda \max(\alpha, 0) = \lambda\alpha^+$. Now let $\alpha < 0$. Again, by Definition 2.18, $(\lambda\alpha)^+ = \max(\lambda\alpha, 0) = 0 = \lambda \cdot 0 = \lambda \max(\alpha, 0) = \lambda\alpha^+$. Thus, in either case, $(\lambda\alpha)^+ = \lambda\alpha^+$.

Now, for any sequence x, by Definition 2.19, we have $(\lambda x)^+ = ((\lambda x_i)^+) = (\lambda x_i^+) = \lambda(x_i^+) = \lambda x^+$. This proves (a).

The rest of the cases can also be handled similarly.

Corollary 2.21. For any sequence x, $(-x)^+ = x^-$ and $(-x)^- = x^+$.

Proof. Take $\lambda = -1$ in parts (c) and (d) of Proposition 2.20.

We insert a few more definitions and propositions which will be used in the proof of Lemma 2.17.

Definition 2.22. We say that a sequence x is less than or equal to a sequence y and write $x \leq y$ if, and only if, $x_i \leq y_i$, for all i. Equivalently, $x \leq y$ if, and only if, $y - x \geq \overline{0}$.

The following proposition is essentially found in the 'Decomposition Lemma' of [2], p. 230.

Proposition 2.23. Let x, y, z be non-negative sequences. Then $\overline{0} \leq z \leq x + y$ if, and only if, $z = u + v$, where $\overline{0} \leq u \leq x$, $\overline{0} \leq v \leq y$.

Proof. Let $z = u + v$, where $\overline{0} \leq u \leq x$, $\overline{0} \leq v \leq y$. Then by Definitions 2.11 and 2.22, $0 \leq u_i \leq x_i$ for all i and $0 \leq v_i \leq y_i$ for all i. This implies that $0 \leq u_i + v_i \leq x_i + y_i$ for all i. Therefore, $\overline{0} \leq u + v \leq x + y$, i.e., $\overline{0} \leq z \leq x + y$.

Conversely, let $\overline{0} \le z \le x + y$. We wish to show that there exist sequences u and v such that $\overline{0} \le u \le x$, $\overline{0} \le v \le y$ and such that $z = u + v$. We first prove the result for non-negative real numbers. It suffices to show that if $0 \le r \le s + t$, where r, s, t are non-negative real numbers, then there exist real numbers a and b such that $0 \le a \le s$, $0 \le b \le t$ and $r = a + b$. For this, let

$$a = \inf \{r, s\} \text{ and } b = r - a .$$

Clearly, $0 \le a = \inf \{r, s\} \le s$. It remains to show that $0 \le b \le t$. Since $b = r - a$, if $a = r$, then $b = 0$. Thus, $0 \le b \le t$. If $a = s$, then $r \ge s$ (since $s = \inf \{r, s\} \le r$) and $b = r - s \ge 0$. Also, since $r \le s + t$, we have $b = r - s \le t$. Therefore, in both cases, $0 \le b \le t$.

Now we prove the desired result for non-negative sequences. That is, we intend to show that given x, y, z non-negative and $\overline{0} \le z \le x + y$, there exist sequences u and v such that $\overline{0} \le u \le x$, $\overline{0} \le v \le y$ and such that $z = u + v$. To this end, let

$$u_i = \inf \{z_i, x_i\}, \; v_i = z_i - u_i, \; i = 1,2,3,\ldots .$$

Proceeding as above, it can be seen that $0 \le u_i \le x_i$ and $0 \le v_i \le y_i$ for all i. This means $\overline{0} \le u \le x$ and $\overline{0} \le v \le y$, where $z = u + v$, which is what we wished to show.

Proposition 2.24. Let x be a non-negative sequence. Then for $\lambda \geq 0$,

$$\overline{0} \leq z \leq \lambda x \text{ if, and only if, } z = \lambda y, \text{ where } \overline{0} \leq y \leq x.$$

Proof. Let $\overline{0} \leq z \leq \lambda x$. Then for $\lambda = 0$, the proof is obvious. If $\lambda > 0$, then on dividing by λ, we have $\overline{0} \leq \frac{1}{\lambda} z \leq x$. Let $\frac{1}{\lambda} z = y$. Then $z = \lambda y$, where $\overline{0} \leq y \leq x$.

Conversely, assume $z = \lambda y$, where $\overline{0} \leq y \leq x$. In view of Definitions 2.11 and 2.22, $0 \leq y_i \leq x_i$ for all i. Therefore, $0 \leq \lambda y_i \leq \lambda x_i$ for all i, and so $\overline{0} \leq \lambda y \leq \lambda x$. That is, $\overline{0} \leq z \leq \lambda x$. This proves the proposition completely.

Finally, we come to the long-awaited proof of Lemma 2.17, which states that every continuous linear functional on M can be expressed as a difference of two non-negative linear functionals on M. Symbolically, $M^* = N - N$.

Proof (of Lemma 2.17). Let $f \in M^*$. Define g on M^+ as follows:

$$g(x) = \sup \{f(y) \mid \overline{0} \leq y \leq x\}. \tag{8}$$

We first show that if $\lambda \geq 0$ and $x \in M^+$, then $g(\lambda x) = \lambda g(x)$.

Since $\lambda \geq 0$ and $x \in M^+$ imply that $\lambda x \geq \overline{0}$ (i.e., $\lambda x \in M^+$), we have by virtue of (8) and Proposition 2.24,

$$\begin{aligned} g(\lambda x) &= \sup \{f(z) \mid \overline{0} \leq z \leq \lambda x\} \\ &= \sup \{f(\lambda y) \mid \overline{0} \leq \lambda y \leq \lambda x\} \\ &= \sup \{\lambda f(y) \mid \overline{0} \leq y \leq x\} \\ &= \lambda \sup \{f(y) \mid \overline{0} \leq y \leq x\} \\ &= \lambda\, g(x) \; . \end{aligned} \tag{9}$$

Now we show that for $x, y \in M^+$, $g(x+y) = g(x) + g(y)$. By (8) and Proposition 2.23,

$$\begin{aligned} g(x+y) &= \sup \{f(z) \mid \overline{0} \leq z \leq x+y\} \\ &= \sup \{f(u+v) \mid \overline{0} \leq u+v \leq x+y, \overline{0} \leq u \leq x, \overline{0} \leq v \leq y\} \\ &= \sup \{f(u) + f(v) \mid \overline{0} \leq u \leq x, \overline{0} \leq v \leq y\} \\ &= \sup \{f(u) \mid \overline{0} \leq u \leq x\} + \sup \{f(v) \mid \overline{0} \leq v \leq y\} \\ &= g(x) + g(y) \; . \end{aligned} \tag{10}$$

In order to extend g to M, we define g as follows. For $x \in M$,

$$g(x) = g(x^+) - g(x^-). \tag{11}$$

We wish to prove that g is a linear functional on M. Clearly, g is a real-valued function on M. Therefore, it is enough to verify that

(a) $g(\lambda x) = \lambda\, g(x)$, for all scalars λ and for all $x \in M$;

(b) $g(x+y) = g(x) + g(y)$, for all $x, y \in M$.

We split (a) into the following two cases.

<u>Case I.</u> $\lambda \geq 0$, $x \in M$.

Using (11) and Proposition 2.20, we have

$$\begin{aligned} g(\lambda x) &= g((\lambda x)^+) - g((\lambda x)^-) \\ &= g(\lambda x^+) - g(\lambda x^-) \\ &= \lambda g(x^+) - \lambda g(x^-) \\ &= \lambda [g(x^+) - g(x^-)] \\ &= \lambda g(x) . \end{aligned} \tag{12}$$

<u>Case II</u>. $\lambda < 0,\ x \in M$.

Since $\lambda < 0$, it follows that $-\lambda > 0$. Using (11), Proposition 2.20 and Case I, we get

$$g(\lambda x) = g((\lambda x)^+) - g((\lambda x)^-)$$

$$= g(-\lambda x^-) - g(-\lambda x^+)$$

$$= -\lambda g(x^-) - (-\lambda) g(x^+)$$

$$= (-\lambda)\ [g(x^-) - g(x^+)]$$

$$= \lambda g(x) \ . \tag{13}$$

In both cases, $g(\lambda x) = \lambda g(x)$. This completes the proof of (a).

Now we prove (b). We show that $g(x+y) = g(x) + g(y)$ for all $x, y \in M$ in the following two steps.

<u>Step I</u>. We wish to prove that if $\overline{0} \le x \le y$, then

$$g(y-x) = g(y) - g(x).$$

Since $x \le y$, it follows that $y - x \ge \overline{0}$. Also, $y = (y-x) + x$. Therefore, $g(y) = g[(y-x) + x] = g(y-x) + g(x)$, by (10). Thus, $g(y-x) = g(y) - g(x)$.

Step II. We wish to show that if x, y be arbitrary in M, then

$$g(x-y) = g(x) - g(y).$$

To prove the above assertion, we introduce four sequences u, v, w, z as follows. For $x_i \geq y_i$, let

$$u_i = x_i, \; w_i = 0,$$

$$v_i = y_i, \; z_i = 0.$$

For $x_i < y_i$, let

$$w_i = x_i, \; u_i = 0,$$

$$z_i = y_i, \; v_i = 0.$$

We observe that $u - v \geq \overline{0}$, $z - w \geq \overline{0}$, $u^+ + v^- \geq u^- + v^+$ and $z^+ + w^- \geq z^- + w^+$. Furthermore, we note the following six identities:

$$(x-y)^+ = u-v, \; (x-y)^- = z-w,$$

$$x^+ = u^+ + w^+, \; x^- = u^- + w^-,$$

$$y^+ = v^+ + z^+, \; y^- = v^- + z^-.$$

Now, in view of the above observations, (11), Step I, result (10) and Definition 2.19, we have

$$g(x-y) = g((x-y)^+) - g((x-y)^-)$$

$$= g(u-v) - g(z-w)$$

$$= g[(u^+-u^-) - (v^+ - v^-)] - g[(z^+-z^-) - (w^+-w^-)]$$

$$= g[(u^++v^-) - (u^-+v^+)] - g[(z^++w^-) - (z^-+w^+)]$$

$$= g(u^++v^-) - g(u^-+v^+) - g(z^++w^-) + g(z^-+w^+)$$

$$= g(u^+) + g(v^-) - g(u^-) - g(v^+) -g(z^+) - g(w^-) + g(z^-) + g(w^+)$$

$$= [g(u^+) + g(w^+)] - [g(u^-) + g(w^-)] - [g(v^+) + g(z^+)] + [g(v^-) + g(z^-)]$$

$$= [g(u^++w^+)] - [g(u^-+w^-)] - [g(v^++z^+)] + [g(v^-+z^-)]$$

$$= g(x^+) - g(x^-) - g(y^+) + g(y^-)$$

$$= [g(x^+) - g(x^-)] - [g(y^+) - g(y^-)]$$

$$= g(x) - g(y) .$$

This proves the assertion in Step II.

Now for $x, y \in M$, we have $g(y) = g[(x+y)-x] = g(x+y) - g(x)$, by Step II. This yields

$$g(x+y) = g(x) + g(y) \tag{14}$$

and the proof of (b) is complete. Hence g is a linear functional on M, which is what we wished to show.

Next, we turn to the task of showing that g is non-negative. For this purpose, let $x \in M^+$. From (8), $g(x) = \sup \{f(y) \mid \overline{0} \leq y \leq x\}$. Since $f(\overline{0}) = 0$, 0 is a member of the set $\{f(y) \mid \overline{0} \leq y \leq x\}$. Hence $g(x) \geq 0$, which implies that g is non-negative. Thus, $g \in N$.

The proof of Lemma 2.17 will be complete if we show that $g - f \in N$. Since g and f are linear functionals on M, so is $g - f$. It remains to show that $g - f$ is non-negative. To this end, let x be a non-negative element of M. By (8), $g(x) = \sup \{f(y) \mid \overline{0} \leq y \leq x\}$. Obviously, $f(x)$ is a member of the set $\{f(y) \mid \overline{0} \leq y \leq x\}$, and this leads to the conclusion that $\sup \{f(y) \mid \overline{0} \leq y \leq x\} \geq f(x)$. Equivalently, $g(x) \geq f(x)$. Consequently, $g(x) - f(x) \geq 0$ and so $(g-f)(x) \geq 0$. Therefore, $g - f$ is non-negative. Hence $g - f \in N$. Finally,

$$f = g - (g - f)$$

$$= g - h ,$$

where $h = g - f$ and both g and $h \in N$. This ends the proof.

In the proof of Theorem 2.26, we use the following lemma which is itself an interesting result.

<u>Lemma 2.25</u>. If h is any regular linear functional on M, then for $x \in M$,

$$h(x_1,x_2,x_3,\ldots) = h(0,x_2,x_3,\ldots) = h(0,0,x_3,\ldots) = \ldots .$$

<u>Proof</u>.

$$h(x_1,x_2,x_3,\ldots) = h[(x_1,0,0,\ldots) + (0,x_2,x_3,\ldots)]$$

$$= h(x_1,0,0,\ldots) + h(0,x_2,x_3,\ldots)$$

$$= \lim(x_1,0,0,\ldots) + h(0,x_2,x_3,\ldots)$$

$$= h(0,x_2,x_3,\ldots) .$$

It is now very easy to see that a repeated application of the regularity leads to the desired conclusion.

The following theorem is a consequence of Theorems 2.15, 2.16 and Lemma 2.17. It asserts that any continuous and regular linear functional on $\mathcal{M}$ can be expressed as a particular linear combination of two non-negative and regular linear functionals on $\mathcal{M}$.

More precisely,

Theorem 2.26. Let $f \in \mathcal{R}$. Then there exist $g, h \in \mathcal{R}^+$ and a real number c such that

$$f = cg + (1-c)h .$$

Proof. Since $f \in \mathcal{R}$, it follows that $f \in \mathcal{M}^*$ (Definition 2.14). Therefore, by Lemma 2.17, there exist $\sigma, \tau \in \mathcal{N}$ such that

$$f = \sigma - \tau . \tag{15}$$

In view of Theorems 2.15 and 2.16, we proceed to enumerate the following four cases:

Case I. Both σ and τ have the representation given by Theorem 2.15.

Case II. Both σ and τ have the representation given by Theorem 2.16.

Case III. σ and τ have the representations given by Theorems 2.15 and 2.16 respectively.

Case IV. σ and τ have the representations given by Theorems 2.16 and 2.15 respectively.

Now we discuss the theorem in each case.

Case I. By virtue of Theorem 2.15, there exist $h_1, h_2 \in \mathcal{R}$, $g_1, g_2 \in L$ and non-zero real constants A_1, A_2 such that $\sigma = A_1h_1+g_1$ and $\tau = A_2h_2+g_2$. Consequently, from (15),

$$f = \sigma - \tau = (A_1h_1+g_1) - (A_2h_2+g_2)$$

$$= A_1h_1 - A_2h_2 + g_1 - g_2 . \tag{16}$$

In order to prove the desired result, we first intend to prove that A_i, h_i $(i = 1,2)$ are non-negative. For this, we consider $x = (0,0,\ldots,0,1,1,\ldots) \in M^+$, where the first 1 is in the k+1st position. Since $\sigma \in N$, it follows that $\sigma(x) \geq 0$ (Definition 2.12). Also, for $g_1, g_2 \in L$, there exist $(a_i), (b_i) \in \ell_1$ such that $g_1(x) = \sum_{i=1}^{\infty} a_ix_i$ and $g_2(x) = \sum_{i=1}^{\infty} b_ix_i$, for all $x \in M$. Hence,

$$0 \leq \sigma(x) = (A_1h_1 + g_1)(x)$$

$$= A_1 h_1(x) + g_1(x)$$

$$= A_1 h_1(0,0,\dots,0,1,1,\dots) + g_1(0,0,\dots,0,1,1,\dots)$$

$$= A_1 \cdot 1 + \sum_{i=k+1}^{\infty} a_i .$$

The last expression tends to A_1 as $k \to \infty$ [since $(a_i) \in \ell_1$ implies that $\sum_{i=1}^{\infty} a_i$ also converges and so $\lim_{k \to \infty} \sum_{i=k+1}^{\infty} a_i = 0$]. This implies that $A_1 \geq 0$. But by Theorem 2.15, A_1 is non-zero, hence $A_1 > 0$. A similar argument reveals that $A_2 > 0$.

Finally, we show that h_i $(i = 1,2)$ are non-negative. Let x be a non-negative sequence in M. By virtue of the non-negativity of σ and Lemma 2.25, we have

$$0 \leq \sigma(0,0,\dots,0,x_{n+1},x_{n+2},\dots)$$

$$= A_1 h_1(0,0,\dots,0,x_{n+1},x_{n+2},\dots) + g_1(0,0,\dots,0,x_{n+1},x_{n+2},\dots)$$

$$= A_1 h_1(x_1,x_2,x_3,\dots) + g_1(0,0,\dots,0,x_{n+1},x_{n+2},\dots)$$

$$= A_1 h_1(x) + \sum_{i=n+1}^{\infty} a_i x_i .$$

By Corollary 2.4, $\sum_{i=1}^{\infty} a_i x_i$ converges, this implies that

$\lim_{n \to \infty} \sum_{i=n+1}^{\infty} a_i x_i = 0$. Therefore, it follows that on taking limits, we obtain $0 \le A_1 h_1(x)$. But since $A_1 > 0$, it follows that $h_1(x) \ge 0$, i.e., h_1 is non-negative. A similar argument shows that h_2 is non-negative. Thus, $h_1, h_2 \in R^+$.

Our next attempt is to show that $g_1 = g_2$. This can be done by evaluating f at e^i and using the regularity of f, h_1 and h_2 . Thus, from (16), for each i ,

$$0 = f(e^i) = A_1 h_1(e^i) - A_2 h_2(e^i) + g_1(e^i) - g_2(e^i) = a_i - b_i ,$$

whence $a_i = b_i$, for each i . It now follows that $g_1(x) = \sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} b_i x_i = g_2(x)$, for each $x \in M$; that is, $g_1 = g_2$.

The proof of Case I will be complete if we prove that $A_2 = A_1 - 1$. For this objective, we evaluate f at e in (16), keeping in view that $g_1 = g_2$. This gives $f(e) = A_1 h_1(e) - A_2 h_2(e)$, which implies that $1 = A_1 - A_2$ or $A_2 = A_1 - 1$. The theorem now follows on replacing A_1, A_2, h_1, h_2 by $c, c-1, g, h$ respectively and taking $g_2 = g_1$ in (16).

<u>Case II</u>. We wish to show that this case is impossible, that is, both σ and τ cannot be chosen as in Theorem 2.16. Thus,

Theorem 2.26 holds vacuously in this case. Suppose, on the contrary, that $\sigma = t_1 + g_1$ and $\tau = t_2 + g_2$, where $\sigma, \tau \in N$, $t_1, t_2 \in R - R$, $g_1, g_2 \in L$. In addition, $g_1, g_2 \in L$ implies that there exist $(a_i), (b_i) \in \ell_1$ such that $g_1(x) = \sum_{i=1}^{\infty} a_i x_i$ and $g_2(x) = \sum_{i=1}^{\infty} b_i x_i$, for all $x \in M$. Therefore, by (15),

$$f = \sigma - \tau = t_1 - t_2 + g_1 - g_2 . \tag{17}$$

Evaluating f at e^i and using the regularity of f, we get for each i, $a_i = b_i$, as before. Thus, $g_1 = g_2$.

Again, evaluating f at e in (17), keeping in mind that $g_1 = g_2$, we get

$$1 = f(e) = t_1(e) - t_2(e) = 0 - 0 = 0 ;$$

this absurdity leads to the conclusion that Case II is impossible.

<u>Case III</u>. By Theorem 2.15, there exist $h_1 \in R$, $g_1 \in L$ and a non-zero constant A_1 such that $\sigma = A_1 h_1 + g_1$. Likewise, by Theorem 2.16, there exist $k_1, k_2 \in R$ and $g_2 \in L$ such that $\tau = (k_1 - k_2) + g_2$. Now, from (15), we have

$$f = \sigma - \tau = (A_1 h_1 + g_1) - (k_1 - k_2) - g_2 . \tag{18}$$

Let (a_i) and (b_i) be the sequences in ℓ_1 corresponding to g_1 and g_2 respectively. We have already shown in Case I that $A_1 > 0$ and $h_1 \geq 0$. Our next task is to show that $k_1 - k_2$ is non-negative. For this, take $x \in M^+$. Now using the non-negativity of τ and the regularity of k_1, k_2, we obtain

$$0 \leq \tau(0,0,0,\ldots,0,x_{n+1},x_{n+2},\ldots)$$

$$= [(k_1-k_2) + g_2](0,0,\ldots,0,x_{n+1},x_{n+2},\ldots)$$

$$= k_1(0,0,\ldots,0,x_{n+1},x_{n+2},\ldots) - k_2(0,0,\ldots,0,x_{n+1},x_{n+2},\ldots) + g_2(0,0,\ldots,0,x_{n+1},x_{n+2},\ldots)$$

$$= k_1(x) - k_2(x) + \sum_{i=n+1}^{\infty} b_i x_i$$

$$\to k_1(x) - k_2(x) \quad \text{as} \quad n \to \infty .$$

Thus, $0 \leq k_1(x) - k_2(x) = (k_1-k_2)(x)$, whence k_1-k_2 is non-negative.

Next, we claim that if k_1, k_2 are regular and $k_1-k_2 \geq 0$, then $k_1 = k_2$. We first establish our claim in the case when x is non-negative (i.e., $x \in M^+$). Noting that $x \in M^+$ implies that $k_1(x) \geq k_2(x)$, we assume that there exists some $x \in M^+$ such that

$k_1(x) > k_2(x)$. Let $\|x\| = \sup_i |x_i| = b$. Therefore, the sequence $\bar{b} - x$ is non-negative. Since k_1-k_2 is non-negative, $(k_1-k_2)(\bar{b} - x) \geq 0$. That is,

$$0 \leq (k_1-k_2)(\bar{b} - x)$$

$$= k_1(\bar{b} - x) - k_2(\bar{b} - x)$$

$$= k_1(\bar{b}) - k_1(x) - k_2(\bar{b}) + k_2(x)$$

$$= b - k_1(x) - b + k_2(x)$$

$$= -k_1(x) + k_2(x) ,$$

which shows that $k_1(x) \leq k_2(x)$. This is a contradiction to our assumption. Thus, $k_1(x) = k_2(x)$ for all $x \in M^+$.

Now, we accomplish our claim in the case when $x \in M$ is arbitrary. We know that $x = x^+ - x^-$ and both x^+ and x^- are non-negative (Definition 2.19). Therefore, in the light of the previous result, $k_1(x^+) = k_2(x^+)$ and $k_1(x^-) = k_2(x^-)$. Thus, we have

$$k_1(x) = k_1(x^+-x^-) = k_1(x^+) - k_1(x^-)$$

$$= k_2(x^+) - k_2(x^-)$$

$$= k_2(x^+ - x^-) = k_2(x).$$

This establishes our claim.

We now continue with Case III. From (18), we have

$$f = A_1h_1 + g_1 - g_2 , \tag{19}$$

where $A_1 > 0$, $h_1 \in R^+$ and $g_1, g_2 \in L$. In order to get the required form, we wish to show that $g_1 = g_2$. For this purpose, we employ our usual technique. Then from (19), for each i,

$$0 = f(e^i) = A_1h_1(e^i) + g_1(e^i) - g_2(e^i) = a_i - b_i .$$

Hence $a_i = b_i$ for each i, whence it follows that $g_1 = g_2$. Consequently, from (19),

$$f = A_1h_1 , \tag{20}$$

where $A_1 > 0$ and $h_1 \in R^+$. The proof of Case III will be complete if we determine the value of A_1. We proceed as follows. From (20),

$$1 = f(e) = A_1h_1(e) = A_1 \lim e = A_1 \cdot 1 = A_1 .$$

Now let h be any non-negative regular linear functional on M. Then letting $c = 1 = A_1$ and $h_1 = g$ in (20), we obtain

$$f = cg = cg + 0 = cg + 0\cdot h = cg + (1-1)h = cg + (1-c)h \ .$$

This evidently shows that the theorem holds in Case III also.

Case IV. We intend to demonstrate that this case is impossible, that is, σ and τ cannot be chosen respectively as in Theorem 2.16 and Theorem 2.15. Thus, Theorem 2.26 holds vacuously in this case. Suppose, on the contrary, that σ and τ have the representations given by Theorems 2.16 and 2.15 respectively. Then it follows that $\sigma = (k_1-k_2) + g_1$, where $k_1, k_2 \in R$, $g_1 \in L$, and $\tau = A_1h_1 + g_2$, where A_1 is a non-zero constant, $h_1 \in R$, $g_2 \in L$. Therefore, in view of (15),

$$f = \sigma - \tau = (k_1-k_2) + g_1 - A_1h_1 - g_2 \ . \tag{21}$$

Let (a_i), (b_i) be the sequences in ℓ_1, corresponding to g_1, g_2 respectively. We have already shown, in Case I, that $A_1 > 0$ and $h_1 \geq 0$. Also, in Case III, we showed that $k_1 = k_2$. Thus, from (21), we get

$$f = g_1 - g_2 - A_1h_1 \ , \tag{22}$$

where $g_1, g_2 \in L, A_1 > 0, h_1 \in R^+$.

As before, $g_1 = g_2$. Now from (22), we have

$$f = -A_1 h_1 , \tag{23}$$

where $A_1 > 0$ and $h_1 \in R^+$. Evaluating f at e , we have $A_1 = -1$. This absurdity leads to the conclusion that Case IV is impossible.

Thus the theorem is proved.

It is easy to see that c, g and h are not unique in Theorem 2.26.

From now on, we shall be concerned mainly with the direct sum of two subspaces. We, therefore, state the following definition.

Definition 2.27. A linear space X is called the direct sum of two of its subspaces M and N if

$$\text{(i)} \quad X = M + N ,$$

$$\text{(ii)} \quad M \cap N = \{\overline{0}\} ,$$

where $\overline{0}$ is the zero vector. In this case, we write $X = M \oplus N$.

We recall that $[R]$ and $[R^+]$ denote the linear hulls of R and R^+ respectively (Definition 1.17).

It is natural to ask how M^*, $[R]$ and L are related with one another. An answer is given in the following theorem.

Theorem 2.28. $M^* = [R] \oplus L$.

Proof. We know (by the earlier remarks) that $[R]$ and L are subspaces of M^* . Therefore, to prove the required assertion, we first show that $M^* = [R] + L$. It suffices to show that $M^* \subseteq [R] + L$.

To this end, let $f \in M^*$ and let f have the representation given by Theorem 2.15. Then, there exist $h \in R$, $g \in L$ and a non-zero constant A such that $f = Ah + g$. Obviously, $Ah \in [R]$. This shows that $f \in [R] + L$, which leads to the conclusion that

$$M^* \subseteq [R] + L .$$

Now, let f be given by Theorem 2.16. Then there exist $k_1, k_2 \in R$ and $g \in L$ such that $f = k_1 - k_2 + g$. Since $k_1 - k_2 = 1 \cdot k_1 + (-1)k_2 \in [R]$, it follows that $f \in [R] + L$, which shows that in this case also,

$$M^* \subseteq [R] + L .$$

Our next task is to ensure that $[R] \cap L = \{\theta\}$, where θ is the zero linear functional. Of course, $\theta \in [R] \cap L$. The proof will be complete if we show that no continuous linear functional, other than θ, is in both $[\mathcal{R}]$ and L. Assume the contrary. Then there exists a continuous linear functional f such that $f \in [\mathcal{R}] \cap L$ and $f \neq \theta$. Now, since $f \in [\mathcal{R}]$, there exist $f_j \in \mathcal{R}$ and scalars α_j $(1 \leq j \leq n)$ such that $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$. Consequently, for each $i \in N$,

$$f(e^i) = \alpha_1 f_1(e^i) + \ldots + \alpha_n f_n(e^i)$$

$$= \alpha_1 \cdot 0 + \ldots + \alpha_n \cdot 0 = 0 . \tag{24}$$

Again, since $f \in L$, there exists $(a_i) \in \ell_1$ such that $f(x) = \sum_{i=1}^{\infty} a_i x_i$ for all $x \in M$. Since $f \neq \theta$, we have $a_i \neq 0$, for some i. Then

$$f(e^i) = a_i \neq 0 . \tag{25}$$

From (24) and (25), we arrive at a contradiction. Therefore, we conclude that no continuous linear functional, other than θ, belongs to $[\mathcal{R}] \cap L$. Hence, $[\mathcal{R}] \cap L = \{\theta\}$ and the proof is complete.

The following lemma establishes a relation between $[\mathcal{R}]$ and $[\mathcal{R}^+]$.

<u>Lemma 2.29</u>. $[\mathcal{R}] = [\mathcal{R}^+]$.

Proof. We need only show that $[R] \subseteq [R^+]$. Consider $f \in [R]$. Then $f = a_1 f_1 + \ldots + a_n f_n$, where $f_i \in R$ and a_i scalars, $1 \leq i \leq n$. But, by virtue of Theorem 2.26, for each f_i $(1 \leq i \leq n)$, there exist $g_i, h_i \in R^+$ and a real number c_i such that $f_i = c_i g_i + (1-c_i)h_i$. Consequently,

$$f = a_1[c_1g_1+(1-c_1)h_1] + a_2[c_2g_2+(1-c_2)h_2] + \ldots + a_n[c_ng_n+(1-c_n)h_n]$$

$$= a_1c_1g_1 + a_1(1-c_1)h_1 + a_2c_2g_2 + a_2(1-c_2)h_2 + \ldots + a_nc_ng_n + a_n(1-c_n)h_n,$$

where $g_i, h_i \in R^+$ $(1 \leq i \leq n)$. This exhibits that $f \in [R^+]$, and we are done.

The following theorem gives a stronger result than Theorem 2.28.

Theorem 2.30. $M^* = [R^+] \oplus L$.

Proof. The proof is an immediate consequence of Theorem 2.28 and the preceding lemma.

We conclude this chapter with a brief discussion of the structure of a continuous linear functional on M. This discussion takes the form of the following theorem.

Theorem 2.31. Every continuous linear functional on M can be written as a linear combination of at most two non-negative regular linear functionals and an ℓ_1-multiplier.

Proof. Let $f \in M^*$. Then in view of Theorem 2.30, there exist $f_i \in R^+$, scalars $\lambda_i (1 \le i \le n)$ and $g \in L$ such that

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n + g , \tag{26}$$

where each $\lambda_i \neq 0$. The following cases can arise:

Case I. $\lambda_i > 0$ for each i.

Case II. $\lambda_i < 0$ for each i.

Case III. $\lambda_{i_1} < 0$ and $\lambda_{i_2} > 0$ for some i_1 and i_2.

In Case I, the expression $\lambda_1 f_1 + \dots + \lambda_n f_n$ can be replaced by

$$(\lambda_1 + \lambda_2 + \dots + \lambda_n)\sigma , \tag{27}$$

where

$$\sigma = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} f_1 + \dots + \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} f_n . \tag{28}$$

Moreover, $\sigma \in R^+$ due to the following reason. σ is a continuous linear functional as it is a finite linear combination of continuous linear functionals. Also, since all λ_i and all f_i are positive, it follows that $\sigma \in N$. Furthermore, for $x \in c$,

$$\sigma(x) = \left(\frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} + \dots + \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} \right) \lim x = \lim x .$$

Thus,

$$f = (\lambda_1 + \lambda_2 + \dots + \lambda_n)\sigma + g = A\sigma + g , \tag{29}$$

where $\sigma \in R^+$ and $g \in L$.

In Case II, the expression $\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n$ can be replaced by

$$(\lambda_1 + \dots + \lambda_n)\tau , \tag{30}$$

where

$$\tau = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} f_1 + \dots + \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} f_n . \tag{31}$$

We now show that $\tau \in R^+$. Clearly, $\tau \in M^*$. Also, $\tau \geq 0$, because $\frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$ is positive for each j $(j = 1,2,\ldots,n)$ and $f_i \geq 0$ for each i $(i = 1,2,\ldots,n)$. Furthermore, reasoning as in Case I, we can show that τ is regular. Hence $\tau \in R^+$. Thus,

$$f = (\lambda_1 + \ldots + \lambda_n)\tau + g = B\tau + g \,, \tag{32}$$

where $\tau \in R^+$ and $g \in L$.

Finally, in Case III, without loss of generality, we can assume that $\lambda_i > 0$ for $i = 1,2,\ldots,p$ and $\lambda_i < 0$ for $i = p+1,\ p+2,\ \ldots,\ n$. Then the expression $(\lambda_1 f_1 + \ldots + \lambda_n f_n) = (\lambda_1 f_1 + \ldots + \lambda_p f_p) + (\lambda_{p+1} f_{p+1} + \ldots + \lambda_n f_n)$ can be replaced by

$$(\lambda_1 + \ldots + \lambda_p)k_1 + (\lambda_{p+1} + \ldots + \lambda_n)k_2 \,, \tag{33}$$

where

$$k_1 = \frac{\lambda_1}{\sum_{i=1}^{p} \lambda_i} f_1 + \ldots + \frac{\lambda_p}{\sum_{i=1}^{p} \lambda_i} f_p \,,$$

and

$$k_2 = \frac{\lambda_{p+1}}{\sum_{j=p+1}^{n} \lambda_j} f_{p+1} + \ldots + \frac{\lambda_n}{\sum_{j=p+1}^{n} \lambda_j} f_n .$$

As before, $k_1, k_2 \in R^+$. Thus,

$$f = (\lambda_1 + \ldots + \lambda_p)k_1 + (\lambda_{p+1} + \ldots + \lambda_n)k_2 + g$$

$$= A_1 k_1 + A_2 k_2 + g , \tag{34}$$

where $k_1, k_2 \in R^+$ and $g \in L$.

From the above discussion, it is evident that in representing f in Theorem 2.30, the number of linear functionals from R^+ need not exceed two, as asserted.

CHAPTER 3

EXISTENCE THEOREMS

In the last chapter, the existence of a continuous and regular linear functional τ on $\mathcal{M}$ was required in Theorem 2.16. In this chapter, we prove the existence of such a τ, using the Hahn-Banach extension theorem. In fact, the present chapter is devoted to a systematic study of the existence of various types of linear functionals on $\mathcal{M}$ and finally, of Banach limits.

Our first theorem shows that $R^+ \neq \phi$.

Theorem 3.1. There exists a linear functional on $\mathcal{M}$ which is continuous, regular and non-negative, i.e., $R^+ \neq \phi$.

Before turning to the proof of the above theorem, we need a few preliminary lemmas.

Lemma 3.2. Let f be any linear functional on $\mathcal{M}$ such that $f(x) \leq \limsup x_n$ for all $x \in \mathcal{M}$. Then

$$\liminf x_n \leq f(x) \leq \limsup x_n .$$

Proof. Since $f(x) \leq \limsup x_n$ for all $x \in \mathcal{M}$, replacing x by $-x$, we have $f(-x) \leq \limsup (-x_n)$. Therefore, $-f(-x) \geq -\limsup (-x_n)$, which yields $f(x) \geq -\limsup (-x_n) = \liminf x_n$, which is what we wished to show.

<u>Lemma 3.3</u>. Let f be a linear functional on $\mathcal{M}$. Then $f \in \mathcal{R}^+$ if, and only if,

$$(*) \quad \text{for all } x \in \mathcal{M}, \ \liminf x_n \le f(x) \le \limsup x_n .$$

<u>Proof</u>. Suppose that f satisfies (*). We wish to prove that $f \in \mathcal{R}^+$. It suffices to show that f is non-negative and regular. To this end, consider $x \in \mathcal{M}$ such that $x \ge \overline{0}$. Then $\liminf x_n \ge 0$. Therefore, in view of (*), $f(x) \ge 0$. This means that f is non-negative. Now, let x be a convergent sequence. Then $\liminf x_n = \lim x = \limsup x_n$. Again, in view of (*), $f(x) = \lim x$, whence f is regular.

To prove the reverse implication, we observe that $f \in \mathcal{R}^+$ implies that f is non-negative and regular. We wish to prove (*). The result is trivially true, in case x is a convergent sequence. Let $x \in \mathcal{M}$ be arbitrary. Then since x is bounded, we can choose convergent sequences y and z in $\mathcal{M}$ such that $y \le x \le z$ and such that $\lim y = \liminf x_n$, $\lim z = \limsup x_n$. To accomplish this, let

$$y_i = \inf \{x_i, x_{i+1}, x_{i+2}, \ldots\},$$

$$z_i = \sup \{x_i, x_{i+1}, x_{i+2}, \ldots\} \quad (i = 1,2,3,\ldots).$$

Then $y_i \le x_k$ for $k \ge i$ and $z_i \ge x_k$ for $k \ge i$, where $i = 1,2,3,\ldots$. This implies that $y_i \le x_i \le z_i$ for all i. That is, $y \le x \le z$. By the definition of lim inf and lim sup, we have that (y_i) and (z_i) converge and therefore, $\lim y = \liminf x_n$, $\lim z = \limsup x_n$.

Now $y \le x$ implies that $x - y \ge \bar{0}$. Also, since f is non-negative, $f(x-y) \ge 0$, whence $f(x) \ge f(y)$, which shows that f is monotone. But f regular implies that $f(y) = \lim y$. Hence,

$$\liminf x_n = \lim y = f(y) \le f(x) . \tag{1}$$

A similar argument reveals that

$$f(x) \le f(z) = \lim z = \limsup x_n . \tag{2}$$

The required result now follows from (1) and (2). This completes the proof of Lemma 3.3.

<u>Proof</u>. (Of Theorem 3.1). By virtue of Lemmas 3.2 and 3.3, we have, for any linear functional f on M,

$$f(x) \le \limsup x_n \Leftrightarrow f \in R^+ . \tag{3}$$

We shall show that there is a linear functional f on M satisfying the property on the left hand side of (3). To this end, for $x \in M$, let $p(x) = \limsup x_n$. Then from (3), we have, for any linear functional f on M,

$$f(x) \le p(x) \Leftrightarrow f \in R^+ . \tag{4}$$

Now p is subadditive on M, because, for $x, y \in M$, $p(x + y) = \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n = p(x) + p(y)$. Moreover, for $\alpha \geq 0$ and $x \in M$, $p(\alpha x) = \limsup (\alpha x_n) = \alpha \limsup x_n = \alpha p(x)$. Now let $\ell = \lim : c \to R^1$, where c is the subspace of M of all convergent sequences. It is easy to see that ℓ is a linear functional on c. Furthermore, $\ell(x) \leq p(x)$ on c. [In fact, $\ell(x) = p(x)$ on c]. Hence, by the Hahn-Banach extension theorem (see, e.g., Maddox [3], p. 121), there exists a linear extension g of ℓ to M such that $g(x) \leq p(x)$ on M. It now follows from (4) that $g \in R^+$. Hence the theorem.

<u>Theorem 3.4</u>. There exists a linear functional on M which is continuous and regular but not non-negative.

<u>Proof</u>. Let $f : M \to R^1$ such that $f = 2g-h$, where $g, h \in R^+$. We claim that f is a continuous and regular linear functional on M. That f is continuous follows from the fact that M^* is a linear space. Now take $x \in c$. The regularity of g and h yields that $f(x) = 2g(x) - h(x) = 2 \lim x - \lim x = \lim x$. This means that f is also regular.

Our next task is to find $g, h \in R^+$ so that $f = 2g-h$ is not non-negative. To this end, we define $\theta_1, \theta_2 : M \to M$ as follows:

$$\theta_1(x_1, x_2, x_3, \ldots) = (x_1, x_3, x_5, \ldots) ,$$

$$\theta_2(x_1, x_2, x_3, \ldots) = (x_2, x_4, x_6, \ldots) .$$

Then for $x, y \in M$ and for any scalars α, β ,

$$\theta_1(\alpha x + \beta y) = \theta_1(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots)$$

$$= (\alpha x_1 + \beta y_1, \alpha x_3 + \beta y_3, \ldots)$$

$$= (\alpha x_1, \alpha x_3, \ldots) + (\beta y_1, \beta y_3, \ldots)$$

$$= \alpha(x_1, x_3, \ldots) + \beta(y_1, y_3, \ldots)$$

$$= \alpha\ \theta_1(x) + \beta\ \theta_1(y) .$$

It follows that θ_1 is a linear operator on M . Similarly, it can be shown that θ_2 is a linear operator on M . Now we intend to prove that θ_1 and θ_2 are bounded. For this, let $x \in M$ be arbitrary. Then

$$\|\theta_1(x)\| = \|\theta_1(x_1,x_2,x_3,\ldots)\| = \|(x_1,x_3,\ldots)\|$$

$$= \sup_{i = 1,3,5,\ldots} |x_i|$$

$$\leq \sup_{j = 1,2,3,\ldots} |x_j|$$

$$= \|x\| = 1\cdot\|x\|.$$

A similar argument reveals that $\|\theta_2(x)\| \leq 1 \cdot \|x\|$, which is what we wished to show.

Now let σ be a continuous, regular and non-negative linear functional on M, the existence of which is assured by Theorem 3.1. Define g on M as follows:

$$g = \sigma \circ \theta_1 .$$

Then clearly, $g : M \to R^1$. Also, for $x, y \in M$ and for any scalars α, β, we have

$$\begin{aligned} g(\alpha x + \beta y) = \sigma \circ \theta_1(\alpha x + \beta y) &= \sigma(\theta_1(\alpha x + \beta y)) \\ &= \sigma(\alpha\theta_1(x) + \beta\theta_1(y)) \\ &= \alpha\sigma(\theta_1(x)) + \beta\sigma(\theta_1(y)) \\ &= \alpha\sigma \circ \theta_1(x) + \beta\sigma \circ \theta_1(y) \\ &= \alpha g(x) + \beta g(y); \end{aligned}$$

this shows that g is a linear functional on M. Moreover, g is continuous, being a composition of two continuous functions. Next, we check the non-negativity of g. We consider any $x \in M^+$. Then using the non-negative property of σ, we obtain

$$g(x) = \sigma \circ \theta_1(x) = \sigma(\theta_1(x_1,x_2,x_3,\ldots))$$

$$= \sigma(x_1,x_3,x_5,\ldots) \geq 0 \ ,$$

which reveals that g is non-negative. Now in order to verify the regularity of g, we consider any sequence $x \in c$. We note that $\lim \theta_1(x) = \lim \theta_2(x) = \lim x$. Then using the regularity of σ we have

$$g(x) = \sigma \circ \theta_1(x) = \sigma(\theta_1(x)) = \lim \theta_1(x)$$

$$= \lim(x) \ .$$

Thus we have shown that $g \in R^+$. In a similar fashion, we can show that if τ is a continuous, regular and non-negative linear functional on M, then $h = \tau \circ \theta_2 \in R^+$. It has already been established (in the beginning of the proof) that $g, h \in R^+$ implies that $f = 2g-h$ is a continuous and regular linear functional on M. Now we wish to prove that f is not non-negative. To do this, let us consider the bounded sequence $(0,1,0,1,\ldots)$, which is evidently non-negative. Then using the regularity of σ and τ, we get

$$f(x) = (2g-h)(x) = 2g(x)-h(x) = 2(\sigma \circ \theta_1(x)) - \tau \circ \theta_2(x)$$

$$= 2\sigma(\theta_1(x)) - \tau(\theta_2(x))$$

$$= 2\sigma(0,0,0,\ldots) - \tau(1,1,1,\ldots)$$

$$= 2\lim(0,0,0,\ldots) - \lim(1,1,1,\ldots)$$

$$= 2 \times 0 - 1 = -1 < 0 .$$

This proves that f is not non-negative, and the theorem follows.

<u>Theorem 3.5</u>. There exists a linear functional on M which is continuous and non-negative but not regular.

<u>Proof</u>. Let us consider the function $f : M \to R^1$ defined as follows:

$$f(x) = \sum_{i=1}^{\infty} \frac{x_i}{i^2} .$$

Since the sequence $(\frac{1}{i^2}) \in \ell_1$, $f \in L$ (Definition 2.7). Hence f is a linear functional on M that is continuous but not regular (Theorem 2.6). It remains to show that f is non-negative. To do this, consider $x \in M$ such that $x \geq \overline{0}$. Then

$$f(x) = \sum_{i=1}^{\infty} \frac{x_i}{i^2} = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{x_i}{i^2} \right) \geq 0 ,$$

since $\sum_{i=1}^{n} \frac{x_i}{i^2} \geq 0$ for all n. This completes the proof.

Theorem 3.6. There exists a linear functional on M which is continuous, but is neither regular nor non-negative.

Proof. By virtue of Theorem 3.1, there exists a linear functional g on M that is continuous, regular and non-negative. Define

$$f = -g .$$

Then f is a continuous linear functional on M (since M^* is a linear space). It is not hard to see that f is neither regular nor non-negative.

Theorem 3.7. There exists no linear functional on M which is regular and non-negative but not continuous.

Proof. A non-negative linear functional on M is continuous (Theorem 2.13).

The proof of the following theorem is quite lengthy and involves vector space terminology. Thus, sequences in M will occasionally be called vectors. Furtheremore, all bases are Hamel bases.

Theorem 3.8. There exists a linear functional on M which is regular but is neither continuous nor non-negative.

Proof. Let $c_1 = \{c_\alpha | \alpha \in \Delta\}$ be a basis for c and $m_1 = \{b_\alpha | \alpha \in \Delta'\}$ be a basis for M. We can assume that $\Delta \subset \Delta'$ and $c_1 \subset m_1$, because, we can always extend a basis for c to a basis for M. We normalize the basis vectors as follows. Let $b_\alpha' = \frac{b_\alpha}{\|b_\alpha\|}$ and

$c'_\alpha = \frac{c_\alpha}{\|c_\alpha\|}$. Let $m' = \{b'_\alpha | \alpha \in \Delta'\}$ and $c' = \{c'_\alpha | \alpha \in \Delta\}$. We claim that $m' - c'$ is an infinite set.

Assume, on the contrary, that $m' - c'$ is a finite set. Then there exist $n - 1$ linearly independent vectors $x_1, x_2, \ldots, x_{n-1}$ such that $m' - c' = \{x_1, x_2, \ldots, x_{n-1}\}$, whence $m' = c' \cup \{x_1, x_2, \ldots, x_{n-1}\}$ is a basis for $\mathcal{M}$. Obviously, $\dim [x_1, x_2, \ldots, x_{n-1}] = n - 1$, where $[x_1, x_2, \ldots, x_{n-1}]$ is the linear span of $x_1, x_2, \ldots, x_{n-1}$.

We first find vectors (these are sequences, of course) $y^1, y^2, \ldots, y^n$ in $\mathcal{M}$ such that they are linearly independent and so that no linear combination of them is convergent, except the trivial one (i.e., when all scalars are zero), as follows. For $j = 1, 2, \ldots, n$, define

$$y_i^j = \begin{cases} 1, \text{ for } i = (n+1)\,k + j, \text{ where } k = 0, 1, 2, \ldots; \\ 0, \text{ otherwise.} \end{cases}$$

Now, for any scalars λ_j, $j = 1, 2, \ldots, n$,

$$\sum_{j=1}^{n} \lambda_j\, y^j = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots),$$

so that the sequences $y^1, y^2, \ldots, y^n$ evidently fulfil our requirements.

Now, since $y^j \in M$ $(j = 1,2,\ldots,n)$ and m' is a basis for M, for each j there exist scalars $\delta_{j\alpha}$, $\alpha \in \Delta$ and scalars λ_{ji}, $i = 1,2,\ldots,n-1$ such that

$$y^j = \sum_{\alpha\in\Delta} \delta_{j\alpha} c'_\alpha + \sum_{i=1}^{n-1} \lambda_{ji} x_i , \quad j = 1,2,\ldots,n ,$$

where the first sum is finite. Equivalently,

$$y^1 = \sum_{\alpha\in\Delta} \delta_{1\alpha} c'_\alpha + \lambda_{11}x_1 + \lambda_{12}x_2 + \ldots + \lambda_{1,n-1}x_{n-1} ,$$

$$\vdots$$

$$y^n = \sum_{\alpha\in\Delta} \delta_{n\alpha} c'_\alpha + \lambda_{n1}x_1 + \lambda_{n2}x_2 + \ldots + \lambda_{n,n-1}x_{n-1} .$$

Let $c^j = \sum_{\alpha\in\Delta} \delta_{j\alpha} c'_\alpha$, $j = 1,2,\ldots,n$. Then

$$y^1 = c^1 + \sum_{i=1}^{n-1} \lambda_{1i}x_i ,$$

$$\vdots$$

$$y^n = c^n + \sum_{i=1}^{n-1} \lambda_{ni}x_i .$$

That is,

$$y^1 - c^1 = \sum_{i=1}^{n-1} \lambda_{1i} x_i \ ,$$

$$\vdots$$

$$y^n - c^n = \sum_{i=1}^{n-1} \lambda_{ni} x_i \ .$$

Note that each $y^j - c^j \in [x_1,\ldots,x_{n-1}]$.

Next, we intend to prove that the vectors $y^1-c^1, y^2-c^2, \ldots, y^n-c^n$ are linearly independent. To do this, consider

$\lambda_1(y^1-c^1) + \ldots + \lambda_n(y^n-c^n) = \overline{0}$, where $\lambda_j (j = 1,2,\ldots,n)$ are scalars. This yields

$$\lambda_1 y^1 + \lambda_2 y^2 + \ldots + \lambda_n y^n = \lambda_1 c^1 + \lambda_2 c^2 + \ldots + \lambda_n c^n \ .$$

But since $c^1, c^2, \ldots, c^n$ are all convergent sequences, so is $\lambda_1 c^1 + \lambda_2 c^2 + \ldots + \lambda_n c^n$. Hence the left hand side is also convergent. It follows, by the property of the y^j's, that $\lambda_j = 0$ for $j = 1,2,\ldots,n$. Thus, we have been able to find a linearly independent set $y^1-c^1, y^2-c^2, \ldots, y^n-c^n$ of n vectors in $[m'-c'] = [x_1,x_2,\ldots,x_{n-1}]$, which is a space of dimension $n-1$. This is a contradiction and so $m'-c'$ is an infinite set.

Now let $\{b'_{\alpha_1}, b'_{\alpha_2}, \ldots, b'_{\alpha_n}, \ldots\}$ be a countably infinite subset of $m'-c'$. We note that for each $x \in M$, there exist unique scalars

λ_α, $\alpha \in \Delta'$ such that $x = \sum_{\alpha\in\Delta'} \lambda_\alpha b'_\alpha$, where $b'_\alpha \in m'$ and at most finitely many $\lambda_\alpha \neq 0$. Define f on M as follows:

$$f(x) = f(\sum_{\alpha\in\Delta'} \lambda_\alpha b'_\alpha) = \sum_{\alpha\in\Delta'} \lambda_\alpha \phi(b'_\alpha), \qquad (5)$$

where

$$\phi(b'_\alpha) = \begin{cases} \lim b'_\alpha & \text{if } b'_\alpha \in c', \\ 1 \text{ if } b'_\alpha \in m'-c' & \text{and } \alpha \neq \alpha_i, \\ i \text{ if } \alpha = \alpha_i . \end{cases}$$

We wish to prove that f is a linear functional on M, f is regular, f is not continuous and f is not non-negative.

Obviously, $f : M \to R^1$ [since from (5), $\phi : m' \to R^1$]. To verify the linearity of f, consider $x, y \in M$. Then there exist unique scalars $\lambda_\alpha, \delta_\alpha (\alpha \in \Delta')$ such that $x = \sum_{\alpha\in\Delta'} \lambda_\alpha b'_\alpha$ and $y = \sum_{\alpha\in\Delta'} \delta_\alpha b'_\alpha$, where at most finitely many λ_α and $\delta_\alpha \neq 0$. Now if β and γ are any scalars, then using (5) we have

$$f(\beta x + \gamma y) = f(\beta \sum_\alpha \lambda_\alpha b'_\alpha + \gamma \sum_\alpha \delta_\alpha b'_\alpha)$$

$$= f(\sum_\alpha (\beta\lambda_\alpha + \gamma\delta_\alpha) b'_\alpha)$$

$$= \sum_{\alpha} (\beta\lambda_{\alpha} + \gamma\delta_{\alpha}) \; \phi(b'_{\alpha})$$

$$= \sum_{\alpha} \beta\lambda_{\alpha} \; \phi(b'_{\alpha}) + \sum_{\alpha} \gamma\delta_{\alpha} \; \phi(b'_{\alpha})$$

$$= \beta \sum_{\alpha} \lambda_{\alpha} \; \phi(b'_{\alpha}) + \gamma \sum_{\alpha} \delta_{\alpha} \; \phi(b'_{\alpha})$$

$$= \beta \; f(x) + \gamma \; f(y) \; ,$$

which evidently shows that f is a linear functional on M .

Next we show that f is regular. For this, we consider any $x \in c$. Then there exist unique scalars λ_{α}, $\alpha \in \Delta$ such that $x = \sum_{\alpha \in \Delta} \lambda_{\alpha} c'_{\alpha}$. Then keeping in mind that only finitely many $\lambda_{\alpha} \neq 0$, we have

$$f(x) = \sum_{\alpha \in \Delta} \lambda_{\alpha} \; \phi(c'_{\alpha}) = \sum_{\alpha \in \Delta} \lambda_{\alpha} \lim c'_{\alpha}$$

$$= \lim \Big(\sum_{\alpha \in \Delta} \lambda_{\alpha} \; c'_{\alpha} \Big) = \lim x \; .$$

Our next task is to show that f is not continuous. We consider $f(b'_{\alpha_i})$, $i = 1,2,3,\ldots$. Using (5) we have

$$\left| f(b'_{\alpha_i}) \right| = \left| \phi(b'_{\alpha_i}) \right| = |i| = i = i \cdot 1 = i \cdot \| b'_{\alpha_i} \| .$$

Since $i \to \infty$, we cannot find a real number M_1 such that $|f(x)| \le M_1 \|x\|$ for all $x \in M$. Hence f is not continuous. Furthermore, it follows from Theorem 2.13 that f is not non-negative. This completes the proof.

<u>Theorem 3.9</u>. There exists <u>no</u> linear functional on M which is non-negative but is neither continuous nor regular.

<u>Proof</u>. Every non-negative linear functional on M is continuous (Theorem 2.13).

<u>Theorem 3.10</u>. There exists a linear functional on M which is neither continuous, nor regular, nor non-negative.

<u>Proof</u>. In view of Theorem 3.8, there exists a linear functional h on M that is regular but is neither continuous nor non-negative. Define

$$f = \lambda h, \ \lambda > 0, \ \lambda \neq 1 .$$

We observe the following:

(i) f is not continuous, for if it were, then so would be $\frac{1}{\lambda} f = \frac{1}{\lambda} \lambda h = h$, which is not the case.

(ii) f is not regular, because, for $x \in c$, $f(x) = \lambda h(x) = \lambda \lim x \neq \lim x$ (since $\lambda \neq 1$).

(iii) f is not non-negative, since h not non-negative implies that there exists a non-negative sequence y in M such that $h(y) < 0$. Then $f(y) = \lambda h(y) < 0$.

This proves our assertion and the theorem follows.

Thus, it is evident from the above discussion that we have considered eight different possibilities for the existence of a linear functional on M . The reader may find them summarized in Table I.

TABLE I*

	Continuous	Regular	Non-negative	Existence	Theorem
1.	+	+	+	+	3.1
2.	+	+	-	+	3.4
3.	+	-	+	+	3.5
4.	+	-	-	+	3.6
5.	-	+	+	-	3.7
6.	-	+	-	+	3.8
7.	-	-	+	-	3.9
8.	-	-	-	+	3.10

*In the above table, '+' indicates the property holding and '-' indicates the absence of the property.

In the next few pages of this chapter, we shall be concerned mainly with the existence of a Banach limit on M. We, therefore, state the following definition.

Definition 3.11. A Banach limit is any linear functional L defined on M such that

(a) $L(x) \geq 0$ if $x_n \geq 0$ for all n ,

(b) $L(x) = L(\sigma x)$ where σ denotes the shift

$\sigma(x) = \sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$,

(c) $L(x) = 1$ if $x = (1,1,1,\ldots)$.

It is evident that L is non-negative and hence continuous (Theorem 2.13). Now we state an important property possessed by a Banach limit. The proof can be found in [1], p. 64.

Theorem 3.12. If L is a Banach limit, then

$$\liminf x_n \leq L(x) \leq \limsup x_n \quad \text{for all } x \in M .$$

As a consequence of Theorem 3.12 and Lemma 3.3, we have a very useful corollary.

Corollary 3.13. $L \in R^+$.

In view of the above corollary, we are in a position to say that L is regular. Hence a Banach limit is a continuous, regular and non-negative linear functional on M. Moreover, it is shift-invariant [Definition 3.11(b)].

Our next venture is to demonstrate the existence of a Banach limit. The existence of continuous, regular and non-negative linear functionals on M has already been ensured by Theorem 3.1. We note that not all members of R^+ are shift-invariant, e.g., the functional g of Theorem 3.4. We now wish to exhibit that some such linear functionals are shift-invariant also. The following theorem accomplishes the desired purpose.

Theorem 3.14. Let s be a function defined on M by $s(x) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \ldots)$. Then

(i) $s : M \to M$.

(ii) s is a linear operator.

(iii) s is continuous.

(iv) s is non-negative.

(v) $x \to \ell \Rightarrow s(x) \to \ell$.

(vi) $f \in R^+ \Rightarrow f \circ s \in R^+$.

(vii) $f \in R^+ \Rightarrow f \circ s$ is a Banach limit.

Proof. (i) Since $x \in M$, x is a bounded sequence. Therefore, there exists a real number $M_1 > 0$ such that $|x_n| \leq M_1$ for all $n \in N$. Now, the general term of the sequence $s(x)$ is $\frac{x_1+x_2+\ldots+x_n}{n}$, and

$$\left|\frac{x_1+x_2+\ldots+x_n}{n}\right| \le \frac{|x_1|+|x_2|+\ldots+|x_n|}{n} \le \frac{nM_1}{n} = M_1 .$$

Hence $\left|\frac{x_1+x_2+\ldots+x_n}{n}\right| \le M_1$ for all $n \in N$. This implies that the sequence $s(x)$ is bounded, i.e., $s(x) \in M$.

(ii) Let $x, y \in M$ and α, β any scalars. Then

$$s(\alpha x+\beta y) = s(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \ldots)$$

$$= (\alpha x_1+\beta y_1, \frac{\alpha x_1+\beta y_1+\alpha x_2+\beta y_2}{2}, \ldots)$$

$$= (\alpha x_1, \frac{\alpha x_1+\alpha x_2}{2}, \ldots) + (\beta y_1, \frac{\beta y_1+\beta y_2}{2}, \ldots)$$

$$= \alpha(x_1, \frac{x_1+x_2}{2}, \ldots) + \beta(y_1, \frac{y_1+y_2}{2}, \ldots)$$

$$= \alpha\, s(x) + \beta\, s(y) ,$$

and we are done.

(iii) It suffices to show that there exists a constant M_1 such that $\|s(x)\| \le M_1 \|x\|$ for all $x \in M$. Let $s(x) = y$. Then

$$\|s(x)\| = \|y\| = \sup_n |y_n| = \sup_n \left|\frac{x_1+x_2+\ldots+x_n}{n}\right|$$

$$\leq \sup_n \left(\frac{|x_1|+|x_2|+\ldots+|x_n|}{n} \right)$$

$$\leq \sup_n \left(\frac{\|x\|+\|x\|+\ldots+\|x\|}{n} \right)$$

$$= \sup_n \|x\| = \|x\| = 1 \cdot \|x\| .$$

(iv) The sequence formed by the arithmetic means of a non-negative sequence will also be non-negative.

(v) We first show that if $z = (z_1, z_2, z_3, \ldots)$ is any sequence of real numbers such that z converges to 0, then so does the sequence $\sigma = (z_1, \frac{z_1+z_2}{2}, \frac{z_1+z_2+z_3}{3}, \ldots)$. Since $z_n \to 0$ as $n \to \infty$, given any $\varepsilon > 0$, there exists $N_1 \in N$ (where N is the set of natural numbers) such that

$$|z_n - 0| < \frac{\varepsilon}{2} \text{ for all } n \geq N_1 . \tag{6}$$

If we let $M_1 = \max(|z_1|, |z_2|, \ldots, |z_{N_1-1}|)$, then we have, for $n \geq N_1$,

$$|\sigma_n| = \left| \frac{z_1 + z_2 + \ldots + z_{N_1-1} + z_{N_1} + z_{N_1+1} + \ldots + z_n}{n} \right|$$

$$\leq \frac{\left(|z_1| + |z_2| + \ldots + |z_{N_1-1}|\right) + \left(|z_{N_1}| + \ldots + |z_n|\right)}{n}$$

$$\leq \frac{(N_1-1)M_1 + (n-N_1+1)\frac{\varepsilon}{2}}{n} \leq \frac{(N_1-1)M_1}{n} + \frac{\varepsilon}{2} . \qquad (7)$$

Choose $N_2 \in N$, such that for $n \geq N_2$, $\frac{(N_1-1)M_1}{n} < \frac{\varepsilon}{2}$. This implies that $\frac{(N_1-1)M_1}{n} \leq \frac{(N_1-1)M_1}{N_2} < \frac{\varepsilon}{2}$. Now let $N_3 = \max(N_1,N_2)$. Then (7) gives $|\sigma_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq N_3$, which is the desired conclusion.

Now we show that $\lim x = \ell \neq 0 \Rightarrow \lim s(x) = \ell$. Since x converges to ℓ, $x-\ell$ converges to 0, i.e., $x_n - \ell \to 0$ as $n \to \infty$, and hence $|x_n-\ell| \to 0$ as $n \to \infty$, i.e., $\lim_{n \to \infty} |x_n - \ell| = 0$. Therefore, by what we have just now shown, the sequence $\tau = (\tau_n)$, where

$$\tau_n = \frac{|x_1-\ell| + |x_2-\ell| + \ldots + |x_n-\ell|}{n},$$ also converges to 0. Now

$$\left|\frac{x_1 + \ldots + x_n}{n} - \ell\right| = \left|\frac{x_1 + \ldots + x_n - n\ell}{n}\right|$$

$$= \left|\frac{(x_1-\ell) + (x_2-\ell) + \ldots + (x_n-\ell)}{n}\right|$$

$$\leq \frac{|x_1-\ell| + |x_2-\ell| + \ldots + |x_n-\ell|}{n}$$

$$\to 0 \text{ as } n \to \infty,$$

which shows that $s(x) \to \ell$.

(vi) Since $s : M \to M$ and $f : M \to R^1$, $f \circ s : M \to R^1$. Moreover, composition of two linear transformations is a linear transformation, whence it follows that $f \circ s$ is a linear functional on M. In addition, $f \circ s$ is continuous (since it is the composition of two continuous mappings). That $f \circ s$ is non-negative is a trivial consequence of the facts that f and s are both non-negative. Next, we show that $f \circ s$ is regular. Let $x \in c$ be such that $\lim x = \ell$. Then using the regularity of f and result (v), we get

$$f \circ s\ (x) = f(s(x)) = \lim s(x) = \ell ,$$

which shows that $f \circ s$ is regular and hence $f \circ s \in R^+$.

(vii) That $f \circ s$ is a continuous, regular and non-negative linear functional on M has already been established in (vi). It remains to show that $f \circ s$ is shift-invariant (Definition 3.11). It suffices to show that $f \circ s\ (x) = f \circ s\ (\sigma x)$, where σ denotes the shift $\sigma x = \sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. Equivalently, it suffices to show that $f \circ s\ (x - \sigma x) = 0$. Now

$$f \circ s\ (x-\sigma x) = f \circ s\ ((x_1,x_2,x_3,\ldots) - (x_2,x_3,x_4,\ldots))$$

$$= f(s(x_1-x_2, x_2-x_3, \ldots, x_n-x_{n+1}, \ldots))$$

$$= f\left(x_1-x_2, \frac{x_1-x_3}{2}, \frac{x_1-x_4}{3}, \ldots, \frac{x_1-x_{n+1}}{n}, \ldots\right)$$

$$= \lim\left(x_1-x_2, \frac{x_1-x_3}{2}, \frac{x_1-x_4}{3}, \ldots, \frac{x_1-x_{n+1}}{n}, \ldots\right).$$

Note that $\lim\left(x_1-x_2, \frac{x_1-x_3}{2}, \frac{x_1-x_4}{3}, \ldots, \frac{x_1-x_{n+1}}{n}, \ldots\right)$ exists and is 0. It now follows that $f \circ s$ is a Banach limit.

In the light of the above discussion, we can say that there exist shift-invariant members of $\mathcal{R}^+$. Equivalently, there exist linear functionals on $\mathcal{M}$ which are continuous, regular, non-negative and shift-invariant. This ensures the existence of Banach limits on $\mathcal{M}$.

CHAPTER 4

ANOTHER CHARACTERIZATION OF CONTINUOUS LINEAR FUNCTIONALS ON M

In Chapter 2, it was demonstrated that M^* is the direct sum of its subspaces $\mathcal{S}[R^+]$ and L and that in representing an element of M^*, the number of linear functionals from R^+ need not exceed two. In this chapter, we give another characterization of continuous linear functionals on M in terms of 'charges', the concept of which is due mainly to P.L. Rosenbloom [5].

We begin our discussion with the following definition.

Definition 4.1. Let 2^N be the class of all subsets of N, where N is the set of all natural numbers. Then a charge on 2^N is a function $\mu : 2^N \to R^1$ such that it satisfies the following postulates:

(i) If $A, B \in 2^N$ such that $A \cap B = \phi$, then

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

(ii) There exists a real number $b > 0$ such that

$$|\mu(A)| \leq b ,$$

for all $A \in 2^N$.

The postulate (i) is also called finite additivity of μ . Thus, a charge μ is a real-valued finitely additive and bounded set function on 2^N .

As an immediate consequence of (i), we have

$$\mu(\phi) = 0.$$

Furthermore, proceeding inductively, one can extend (i) in Definition 4.1 to give the result that for any finite disjoint sequence $E_1, E_2, \ldots, E_n$ of sets from 2^N ,

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i). \tag{1}$$

Now we give a couple of examples of a charge.

Example 1. Consider $a = (a_i) \in \ell_1$. Then the function $\mu : 2^N \to R^1$, defined by

$$\mu(A) = \sum_{i \in A} a_i \quad \text{for all } A \in 2^N ,$$

is a charge on 2^N .

Proof. Let $A, B \in 2^N$ such that $A \cap B = \phi$. Then

$$\mu(A \cup B) = \sum_{i \in A \cup B} a_i = \sum_{i \in A} a_i + \sum_{i \in B} a_i = \mu(A) + \mu(B) .$$

Moreover,

$$|\mu(A)| = \left| \sum_{i \in A} a_i \right| \le \sum_{i \in A} |a_i| \le \sum_{i \in N} |a_i| < \infty ,$$

as required.

Before we proceed to the next example, we introduce the concept of the characteristic function of a subset of N .

Definition 4.2. Let $A \subseteq N$. We define a function $\chi_A : N \to R^1$ as follows:

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A , \\ 0 & \text{otherwise} . \end{cases}$$

χ_A is called the characteristic function of the set A .

Observe that χ_A has the following properties:

(a) χ_A is a sequence of 0's and 1's.

(b) $\chi_N = e$, $\chi_\phi = \overline{0}$.

(c) $\chi_A \geq \overline{0}$, for all $A \subset N$.

(d) $\chi_A \in M^+$, for all $A \subset N$.

(e) If A and B are two disjoint subsets of N , then $\chi_{A \cup B} = \chi_A + \chi_B$.

(f) $\|\chi_A\| = 1$, except when $A = \phi$, in which case $\|\chi_A\| = 0$. Thus, $\|\chi_A\| \leq 1$.

Example 2. Let $f \in M^*$. Define $\mu : 2^N \to R^1$ as follows:

$$\mu(A) = f(\chi_A) , \quad \text{for all } A \in 2^N ,$$

where χ_A is the characteristic function of A . Then μ is a charge on 2^N .

Proof. Let $A, B \in 2^N$ such that $A \cap B = \phi$. Then

$$\mu(A \cup B) = f(\chi_{A \cup B}) = f(\chi_A + \chi_B) = f(\chi_A) + f(\chi_B) = \mu(A) + \mu(B) .$$

Since f is bounded,

$$|\mu(A)| = |f(\chi_A)| \leq k\|\chi_A\| \leq k .$$

Thus, μ is a charge on 2^N.

The following theorem is analogous to a Jordan decomposition theorem for signed measures.

<u>Theorem 4.3</u>. Each charge μ on 2^N has a decomposition into the difference of two non-negative charges so that

$$\mu = \mu^+ - \mu^- ,$$

where μ^+ and μ^- are defined on 2^N as follows. For each $A \in 2^N$,

$$\mu^+(A) = \sup\{\mu(B) \mid B \in 2^N , \ B \subset A\} ,$$

$$\mu^-(A) = -\inf\{\mu(B) \mid B \in 2^N , \ B \subset A\} .$$

<u>Proof</u>. We shall first show that μ^+ is a non-negative charge on 2^N. Obviously, $\mu(\phi) = 0$. Consequently, μ^+ is a non-negative set function. That μ^+ is bounded,is a trivial consequence of the fact that μ is bounded. It remains to show that μ^+ is finitely additive. To this end, let $A_1, A_2 \in 2^N$ such that $A_1 \cap A_2 = \phi$. We wish to prove that

$$\mu^+(A_1 \cup A_2) = \mu^+(A_1) + \mu^+(A_2). \tag{2}$$

For this, we consider any $B \subset A_1 \cup A_2$ such that $B \in 2^N$. Clearly, $(B \cap A_1) \cap (B \cap A_2) = \phi$. Then by the finite additivity of μ we have

$$\mu(B) = \mu((B \cap A_1) \cup (B \cap A_2)) = \mu(B \cap A_1)$$

$$+ \mu(B \cap A_2) \leq \mu^+(A_1) + \mu^+(A_2).$$

Since the above inequality holds for every $B \subset A_1 \cup A_2$, we get

$$\mu^+(A_1 \cup A_2) \leq \mu^+(A_1) + \mu^+(A_2). \tag{3}$$

Now, to prove the reverse inequality, observe that $\mu^+(A_n)$, $n = 1,2$, is always finite. Therefore, given $\varepsilon > 0$, there exists for $n = 1,2$, a set $B_n \subset A_n$ such that

$$\mu(B_n) \geq \mu^+(A_n) - \frac{\varepsilon}{2}.$$

Note that $B_1 \cap B_2 = \phi$ (since $A_1 \cap A_2 = \phi$). Consequently,

$$\mu^+(A_1 \cup A_2) \geq \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) \geq \mu^+(A_1) + \mu^+(A_2) - \varepsilon .$$

Since the above inequality holds for every $\varepsilon > 0$, we have

$$\mu^+(A_1 \cup A_2) \geq \mu^+(A_1) + \mu^+(A_2). \tag{4}$$

Thus, (2) has been proved and it follows that μ^+ is a non-negative charge on 2^N.

Next, we consider μ^-. Clearly, $\mu(\phi) = 0$, whence $-\mu^-(A) = \inf\{\mu(B) | B \subset A \in 2^N\} \leq 0$ for every $A \in 2^N$. Consequently, μ^- is a non-negative set function. Moreover, proceeding as in the case of μ^+, we can show that μ^- is a charge on 2^N.

Our next venture is to show that $\mu = \mu^+ - \mu^-$. It suffices to show that $\mu(A) = \mu^+(A) - \mu^-(A)$ for every $A \in 2^N$. To this end, let $B \subset A$ be arbitrary. Then, by the finite additivity of μ, we have

$$\mu(A) = \mu(B \cup (A - B))$$

$$= \mu(B) + \mu(A - B) .$$

Since μ is bounded, $\mu(A - B)$ is finite. Therefore,

$$\mu(A) - \mu(A - B) = \mu(B) ,$$

that is,

$$\mu(B) = \mu(A) - \mu(A - B) \begin{cases} \leq \mu(A) + \mu^-(A) \\ \geq \mu(A) - \mu^+(A) \end{cases}$$

because of the facts that $\mu(A - B) \geq \inf\{\mu(E) | E \subset A\}$ and $\mu(A - B) \leq \sup\{\mu(E) | E \subset A\}$. Since the above inequalities are true for all $B \subset A$, we have

$$\mu^+(A) \leq \mu(A) + \mu^-(A) \tag{5}$$

and

$$-\mu^-(A) \geq \mu(A) - \mu^+(A). \tag{6}$$

Now as $\mu(A)$, $\mu^-(A)$ and $\mu^+(A)$ are finite, we can transpose in these inequalities and get

$$\mu^+(A) - \mu^-(A) \leq \mu(A) \leq \mu^+(A) - \mu^-(A) .$$

This completes the proof of Theorem 4.3.

<u>Definition 4.4</u>. Let $|\mu| : 2^N \to R^1$ be a function such that

$$|\mu| = \mu^+ + \mu^- .$$

It is easy to see that $|\mu|$ is a non-negative charge on 2^N.

We claim that the charges can be used to represent members of M^*. To this end, we wish to introduce the notion of the integral of a bounded sequence with respect to a charge μ . This involves the concept of a partition.

Definition 4.5. By a partition of N we mean a finite collection $E_1, E_2, \ldots, E_n$ of non-empty subsets of N such that

(i) $E_i \cap E_j = \phi$, for $i \neq j$,

(ii) $\bigcup_{i=1}^{n} E_i = N$.

Definition 4.6. Let $\pi_1 = (E_1, E_2, \ldots, E_m)$ and $\pi_2 = (F_1, F_2, \ldots, F_n)$ be two partitions of N . Then π_2 is called a refinement of π_1 if each F_j is a subset of some E_k . We shall write $\pi_1 \leq \pi_2$.

Note that the relation of refinement gives a partial ordering of partitions and every pair of partitions has an upper bound, e.g., the "superposition" of two partitions, where by superposition we mean a refinement of the two partitions consisting of all non-empty sets of the form $E_i \cap F_j$, where $E_i \in \pi_1$ $(1 \leq i \leq m)$ and $F_j \in \pi_2$ $(1 \leq j \leq n)$.

Now we are in a position to define the integral of a bounded sequence of real numbers with respect to a charge μ on 2^N .

Definition 4.7. Let $x \in M$. Let μ be a charge on 2^N. We say that the integral of x with respect to the charge μ is ℓ (where ℓ is a real number, of course) and write $\int x\, d\mu = \ell$, if, for every $\varepsilon > 0$, there exists some partition π of N such that if

(i) $\pi_1 = (F_1, F_2, \ldots, F_m)$ is any refinement of π and

(ii) $t_i \in F_i$, $i = 1,2,\ldots,m$,

then

$$\left|\left(\sum_{i=1}^{m} x_{t_i}\, \mu(F_i)\right) - \ell\right| < \varepsilon .$$

For a proof of the existence of the integral, see, e.g., Taylor [7], p. 402.

Definition 4.8. For a sequence $x \in M$, we define

$$|x| = (|x_i|) .$$

Note that for each i, $|x_i| \le \sup_i |x_i| = \|x\|$. Thus,

$$|x| \le \|x\| e \text{ , for all } x \in M ,$$

where e is the sequence $(1,1,1,\ldots)$.

Now we state without proof a few standard properties of the integral which we shall need later:

(i) $\int(x+y)d\mu = \int x d\mu + \int y d\mu$ and $\int x d\mu_1 + \int x d\mu_2 = \int x d(\mu_1+\mu_2)$.

(ii) $\int \lambda x d\mu = \lambda \int x d\mu$, λ any scalar.

(iii) $|\int x d\mu| \leq \int |x| d|\mu|$.

(iv) $x \leq y \Rightarrow \int x d\mu \leq \int y d\mu$, $\mu \geq 0$.

(v) $\int \bar{c} d\mu = c\mu(N)$, where $\bar{c}$ is the constant sequence $(c,c,c,\ldots)$.

The following two theorems give a characterization of M^* in terms of the charges on 2^N .

Theorem 4.9. Let $f \in M^*$. Then there exists a charge μ on 2^N such that

$$f(x) = \int x d\mu ,$$

for all $x \in M$.

Proof. Define a function $\mu : 2^N \to R^1$ by

$$\mu(A) = f(\chi_A), \; A \in 2^N , \tag{7}$$

where χ_A is the characteristic function of A. It was already shown in Example 2 that μ is a charge on 2^N.

Next, we intend to prove that $f(x) = \int x d\mu$, for all $x \in M$. To this end, consider any $x \in M$. Since x is a bounded sequence and $f \in M^*$, there exist constants B and k such that

$$-B \le x_i \le B (i \in N) \text{ and } |f(x)| \le k \|x\| \; (x \in M).$$

Let $\varepsilon > 0$. Subdivide the interval $[-B, B]$ into n equal subintervals, each of width $h < \frac{\varepsilon}{k}$, by taking equally spaced points $-B, -B+h, -B+2h, \ldots, -B+(n-1)h, -B+nh(=B)$. Then clearly, $nh = 2B$. Furthermore, let the subintervals be $I_1, I_2, \ldots, I_n$ where $I_1 = [-B, -B+h]$, $I_2 = (-B+h, -B+2h]$, ..., $I_n = (-B+(n-1)h, -B+nh]$, so that the subintervals are disjoint. Now, define

$$E_i = \{j | x_j \in I_i\}, \quad i = 1,2,\ldots,n .$$

Then $\pi = (E_1, E_2, \ldots, E_n)$ is a partition of N. Let $\pi_1 = (F_1, F_2, \ldots, F_m)$ be any refinement of π. Further, let $t_i \in F_i$, $i = 1,2,\ldots,m$. Observe that

$$|x_j - x_{t_i}| \le h < \frac{\varepsilon}{k} \tag{8}$$

for each $j \in F_i$ (since j and t_i belong to the same F_i and hence to the same E_s); this implies that x_j and x_{t_i} lie in the same subinterval I_s. Also, note that if $y = \sum_{i=1}^{m} x_{t_i} \chi_{F_i}$, then $y_j = x_{t_i}$ where $j \in F_i$. Therefore, for each $j \in N$,

$$|x_j - y_j| = |x_j - x_{t_i}| < \frac{\varepsilon}{k}, \tag{9}$$

whence

$$\|x-y\| < \frac{\varepsilon}{k}. \tag{10}$$

Hence, in view of the above observations and the linearity of f, we have

$$\begin{aligned} \left|f(x) - \sum_{i=1}^{m} x_{t_i} \mu(F_i)\right| &= \left|f(x) - \sum_{i=1}^{m} x_{t_i} f(\chi_{F_i})\right| \\ &= \left|f(x) - \sum_{i=1}^{m} f(x_{t_i} \chi_{F_i})\right| \\ &= \left|f(x) - f\left(\sum_{i=1}^{m} x_{t_i} \chi_{F_i}\right)\right| \\ &= |f(x) - f(y)| \\ &= |f(x-y)| \end{aligned}$$

$$\leq \|x-y\| \cdot k$$

$$< \frac{\varepsilon}{k} \cdot k = \varepsilon .$$

This completes the proof.

Theorem 4.10. Let μ be a charge on 2^N . Then the function f , defined on M by

$$f(x) = \int x d\mu$$

for all $x \in M$, is in M^* .

Proof. Obviously, $f : M \to R^1$. Let $x, y \in M$ and λ any scalar. Then using the standard properties of the integral, we have

$$f(x+y) = \int (x+y) d\mu = \int x d\mu + \int y d\mu = f(x) + f(y) ,$$

$$f(\lambda x) = \int \lambda x d\mu = \lambda \int x d\mu = \lambda f(x) ;$$

this shows that f is a linear functional on M .

It remains to show that f is bounded on M . For this, let $x \in M$. Then in view of the standard properties of the integral, we have

$$|f(x)| = \left|\int x d\mu\right| \leq \int |x| \, d|\mu| \leq \int \|x\| \, e d|\mu|$$

$$= \|x\| \int e d|\mu| = \|x\| \cdot |\mu|(N).$$

Consequently, $f \in M^*$, as required.

Thus, we have shown that each charge μ defines an $f \in M^*$ by the formula $f(x) = \int x d\mu$, $x \in M$. On the other hand, if we start with $f \in M^*$ and define μ by the formula $\mu(A) = f(\chi_A)$, $A \in 2^N$, then μ is a charge and $f(x) = \int x d\mu$, $x \in M$.

We now reach the centre of our discussion. We know that the members of R^+ and L played a significant role in the characterization of continuous linear functionals on M in Chapter 2. It is now natural to ask whether properties of a charge μ can be determined which are necessary and sufficient to cause the linear functional $f(x) = \int x d\mu$ to lie in R^+ or in L. The answers are given in Theorems 4.12 and 4.19.

Theorem 4.11. Let $\mu \geq 0$, i.e., $\mu(A) \geq 0$ for all $A \in 2^N$. Then $f \in N$ and conversely.

Proof. By virtue of Theorem 4.10, the function f defined on M by $f(x) = \int x d\mu$, for all $x \in M$, is in M^*. It remains to show that f is non-negative. To this end, let $x \in M^+$. Then using property (iii) of the integrals, and the non-negativity of x and μ, we have

$$0 \le \left|\int x d\mu\right| \le \int |x| d|\mu| = \int x d\mu = f(x) .$$

Hence, $f \in \mathcal{N}$, as required.

Conversely, let $f \in \mathcal{N}$. Consider any $A \in 2^N$. Then using the facts that $\chi_A \in \mathcal{M}^+$ and f is non-negative, we have

$$\mu(A) = f(\chi_A) \ge 0 .$$

This completes the proof.

<u>Theorem 4.12</u>. Let $\mu \ge 0$, $\mu(N) = 1$ and $\mu(F) = 0$ for all finite $F \subset N$. Then the function f defined on $\mathcal{M}$ by $f(x) = \int x d\mu$ is in $\mathcal{R}^+$. Conversely, if $f \in \mathcal{R}^+$ and $f(x) = \int x d\mu$, then μ has the stated properties.

<u>Proof</u>. By Theorem 4.11, $f \in \mathcal{N}$. It remains to show that f is regular. To this end, consider $x \in c$. Let $\lim x = \ell$ and let $\varepsilon > 0$. Then there exists n_0 such that

$$|x_i - \ell| < \varepsilon , \quad \text{for all } i \ge n_0 . \tag{11}$$

Let $\pi = (E_1, E_2)$ be the partition of N such that

$$E_1 = \{1, 2, \ldots, n_0 - 1\} , \quad E_2 = \{n_0, n_0+1, n_0+2, \ldots\} .$$

Let $\pi_1 = (F_1, F_2, \ldots, F_m)$ be a refinement of π. Without loss of generality, we may suppose that $F_1, F_2, \ldots, F_s \subset E_1$ and $F_{s+1}, F_{s+2}, \ldots, F_m \subset E_2$. Note that $F_1, F_2, \ldots, F_s$ are finite sets (since E_1 is finite), so that

$$\mu(F_i) = 0 \quad \text{for } i = 1, 2, \ldots, s. \tag{12}$$

Moreover, since $(F_1, F_2, \ldots, F_m)$ is a partition of N, it follows that $N = \bigcup_{i=1}^{m} F_i$ and $F_i \cap F_j = \phi$ for $i \neq j$. Consequently,

$$1 = \mu(N) = \mu\left(\bigcup_{i=1}^{m} F_i\right) = \sum_{i=1}^{m} \mu(F_i). \tag{13}$$

Now, let $t_i \in F_i$, $i = 1, 2, \ldots, m$. Then using (11), (12) and (13), we have

$$\left|\ell - \sum_{i=1}^{m} x_{t_i} \mu(F_i)\right| = \left|\ell \cdot \sum_{i=1}^{m} \mu(F_i) - \sum_{i=1}^{m} x_{t_i} \mu(F_i)\right|$$

$$= \left|\sum_{i=1}^{m} (\ell - x_{t_i}) \mu(F_i)\right|$$

$$= \left|\sum_{i=s+1}^{m} (\ell - x_{t_i}) \mu(F_i)\right|$$

$$\leq \sum_{i=s+1}^{m} |\ell - x_{t_i}| \mu(F_i)$$

$$< \varepsilon \sum_{i=s+1}^{m} \mu(F_i)$$

$$= \varepsilon \sum_{i=1}^{m} \mu(F_i)$$

$$= \varepsilon . \qquad (14)$$

Hence $\ell = \int x d\mu = f(x)$, whence f is regular.

Conversely, let $f \in R^+$. Then by the last theorem, $\mu \geq 0$. Furthermore,

$$\mu(N) = f(\chi_N) = f(e) = \lim e = 1 .$$

Finally, if F is a finite subset of N, then

$$\mu(F) = f(\chi_F) = \lim (\chi_F) = 0 ,$$

since χ_F is a finite sequence in M

Thus, the theorem is proved.

In order to establish the proof of the next theorem, we first present the necessary background material.

<u>Definition 4.13</u>. A charge $\mu : 2^N \to R^1$ is called <u>completely additive</u> provided it satisfies the following postulate:

If $\{E_n\}$ is a sequence of disjoint sets from 2^N, then

$$\sum_{n=1}^{\infty} \mu(E_n)$$

converges and

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) .$$

Definition 4.14. Let $x \in M$ and $A \subset N$. We define

$$\int_A x d\mu = \int x \cdot \chi_A d\mu ,$$

where $x \cdot \chi_A = (x_1\chi_A(1), x_2\chi_A(2), \ldots) \in M$.

Thus, $\int_N x d\mu = \int x \cdot \chi_N d\mu = \int x \cdot e d\mu = \int x d\mu$.

Theorem 4.15. Let μ be a completely additive charge on 2^N. Let $\{E_n\}$ be a sequence of disjoint subsets of N and $E_0 = \bigcup_{n=1}^{\infty} E_n$. Further, let $x \in M^+$. Then

$$\int_{E_0} x d\mu = \sum_{n=1}^{\infty} \int_{E_n} x d\mu .$$

For a proof, in a more general setting, see, e.g., Munroe [4], p. 134.

Proposition 4.16. Let μ be a charge on 2^N. Then for every $A \in 2^N$,

$$|\mu(A)| \leq |\mu|(A) .$$

Proof.

$$|\mu(A)| = |\mu^+(A) - \mu^-(A)| \leq |\mu^+(A)| + |\mu^-(A)|$$

$$= \mu^+(A) + \mu^-(A)$$

$$= |\mu| (A) ,$$

as required.

Proposition 4.17. $|\mu|$ is monotone.

Proof. Let $A \subset B \subset N$. Then

$$|\mu| (B) = |\mu| (A \cup (B - A))$$

$$= |\mu| (A) + |\mu| (B - A)$$

$$\geq |\mu| (A) ,$$

since $|\mu|$ is non-negative.

<u>Lemma 4.18</u>. Let μ be a charge on 2^N and let $x \in M$. Then

$$\int_{\{i\}} x d\mu = x_i \, \mu(\{i\}) .$$

<u>Proof</u>. In view of Theorems 4.8 and 4.10, there exists $f \in M^*$ such that $f(x) = \int x d\mu$, $x \in M$ and $\mu(A) = f(\chi_A)$, $A \subset N$. Therefore, using Definition 4.14, we have

$$\int_{\{i\}} x d\mu = \int x \chi_{\{i\}} d\mu$$

$$= f(x \chi_{\{i\}})$$

$$= f(0,0,\ldots,0,x_i,0,\ldots)$$

$$= f(x_i(0,0,\ldots,0,1,0,\ldots))$$

$$= x_i \, f(0,0,\ldots,0,1,0,\ldots)$$

$$= x_i \, f(\chi_{\{i\}}) = x_i \, \mu(\{i\}) ,$$

as required.

Finally, we come to Theorem 4.19.

<u>Theorem 4.19</u>. (i) Let $\mu : 2^N \to R^1$ be such that it is completely additive and bounded (i.e., a completely additive charge on 2^N). Then the linear functional f on $\mathcal{M}$, defined by

$$f(x) = \int x d\mu \ , \quad \text{for all} \quad x \in \mathcal{M} \ ,$$

is such that it is an element of $\mathcal{L}$ (i.e., f is an ℓ_1-multiplier on $\mathcal{M}$).

(ii) Conversely, let $f \in \mathcal{L}$. Then the function $\mu : 2^N \to R^1$ defined by

$$\mu(A) = f(\chi_A) \ , \quad \text{for all} \quad A \in 2^N \ ,$$

is completely additive and bounded.

<u>Proof</u>. (i) It suffices to show that $f(x) = \sum_{i=1}^{\infty} a_i x_i$, for all $x \in \mathcal{M}$, where $a = (a_i) \in \ell_1$. Now,

$$f(x) = \int x d\mu = \int x \, d(\mu^+ - \mu^-)$$

$$= \int x \, d\mu^+ - \int x \, d\mu^-$$

$$= \int_{\cup\{i\}, i=1,2,\ldots} x \quad d\mu^+ - \int_{\cup\{i\}, i=1,2,\ldots} x \quad d\mu^-$$

$$= \sum_{i=1}^{\infty} \int_{\{i\}} x d\mu^{+} - \sum_{i=1}^{\infty} \int_{\{i\}} x d\mu^{-}$$

$$= \sum_{i=1}^{\infty} x_i \, \mu^{+}(\{i\}) - \sum_{i=1}^{\infty} x_i \, \mu^{-}(\{i\})$$

$$= \sum_{i=1}^{\infty} x_i \, (\mu^{+} - \mu^{-})(\{i\})$$

$$= \sum_{i=1}^{\infty} x_i \, \mu(\{i\}) = \sum_{i=1}^{\infty} x_i a_i .$$

The proof of (i) will be complete if we show that $(a_i) = (\mu(\{i\})) \in \ell_1$. To this end, we consider the sequence $(t_k) = (\sum_{i=1}^{k} |a_i|)$, of partial sums of $\sum_{i=1}^{\infty} |a_i|$. Then using Definition 4.13, and Propositions 4.16 and 4.17, we have

$$t_k = \sum_{i=1}^{k} |a_i| = \sum_{i=1}^{k} |\mu(\{i\})| \leq \sum_{i=1}^{k} |\mu| (\{i\})$$

$$= |\mu|(\bigcup_{i=1}^{k} \{i\})$$

$$= |\mu|(\{1,2,\ldots,k\})$$

$$\leq |\mu| (N) < \infty .$$

This shows that (t_k) is a non-decreasing and bounded sequence of real numbers, which implies that (t_k) is a convergent sequence.

Consequently,

$$\sum_{i=1}^{\infty} |\mu(\{i\})| = \sum_{i=1}^{\infty} |a_i| = \lim_{k\to\infty} \sum_{i=1}^{k} |a_i| = \lim_{k\to\infty} t_k < \infty .$$

Thus, $f \in L$.

(ii) Let $f \in L$. Therefore, $f(x) = \sum_{i=1}^{\infty} a_i x_i$ for all $x \in M$, where $(a_i) \in \ell_1$. Let (A_j) be a sequence of disjoint sets from 2^N . Then using a rearrangement theorem for absolutely convergent series of real numbers, we have

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = f\left(\chi_{\bigcup_{j\geq 1} A_j}\right) = \sum_{i=1}^{\infty} \chi_{\bigcup_{j\geq 1} A_j}(i)\, a_i$$

$$= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \chi_{A_j}(i) a_i \right)$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \chi_{A_j}(i)\, a_i$$

$$= \sum_{j=1}^{\infty} f(\chi_{A_j}) = \sum_{j=1}^{\infty} \mu(A_j).$$

Now, since $f \in L$, it follows that f is a bounded linear functional on M (Theorem 2.6). Therefore, for any $A \in 2^N$,

$$|\mu(A)| = |f(\chi_A)| \leq M_1 \|\chi_A\| \leq M_1 .$$

This completes the proof of the theorem.

Thus, it is evident from the above discussion that the members of M^* can be characterized in terms of the charges on 2^N. It is also clear that there exists a one-to-one correspondence between the set M^* of all continuous linear functionals on M and the set C of all charges on 2^N. The correspondence between a continuous linear functional f and its associated charge μ being indicated by the two formulas

$$f(x) = \int x d\mu \ , \quad x \in M \ , \tag{15}$$

$$\mu(A) = f(\chi_A) \ , \quad A \in 2^N \ . \tag{16}$$

The above correspondence is, in fact, an isomorphism between the spaces M^* and C. Consequently, our structure theorems of Chapter 2 can be formulated in terms of the charges on 2^N. As an illustration, Theorem 2.31 can be stated as follows. Every charge μ on 2^N can be written as a linear combination of at most two non-negative charges which satisfy the conditions of Theorem 4.12 and a charge which satisfies the conditions of Theorem 4.19.

Furthermore, the spaces M^* and C become normed linear spaces (in fact, Banach spaces) if the norms on them are defined as follows:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}, \text{ for all } f \in M^* \tag{17}$$

and

$$\|\mu\| = |\mu|(N), \text{ for all } \mu \in C. \tag{18}$$

Consequently, the correspondence between M^* and C given by formulas (15) and (16) becomes an isometry. This means that if f and μ correspond to each other, then

$$\|f\| = \|\mu\|. \tag{19}$$

We conclude our discussion by proving this fact.

Theorem 2.20. If $f \leftrightarrow \mu$, where $f(x) = \int x d\mu$ for all $x \in M$, then $\|f\| = \|\mu\|$.

Proof. Using the standard properties of the integral, we have, for each $x \in M$,

$$|f(x)| = \left|\int x d\mu\right| \leq \int |x| \, d|\mu| \leq \|x\| \int e d|\mu| = |\mu|(N) \cdot \|x\|,$$

whence

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq |\mu|(N) = \|\mu\|. \tag{20}$$

Now we prove the reverse inequality. Let $\varepsilon > 0$. Then, by the definitions of μ^+ and μ^- (see Theorem 4.3), there exist $B_1, B_2 \in 2^N$ such that

$$\mu(B_1) > \mu^+(N) - \varepsilon/2 ,$$

$$-\mu(B_2) > \mu^-(N) - \varepsilon/2 .$$

Let $E = B_1 \cap B_2$. Therefore, $B_1 = (B_1 - E) \cup E = A_1 \cup E$ and $B_2 = (B_2 - E) \cup E = A_2 \cup E$. Clearly, $A_1 \cap A_2 = \phi$. Hence,

$$\mu(A_1) - \mu(A_2) = \mu(A_1) + \mu(E) - \mu(A_2) - \mu(E)$$

$$= \mu(B_1) - \mu(B_2)$$

$$> |\mu|(N) - \varepsilon = \|\mu\| - \varepsilon . \tag{21}$$

Let $x = \chi_{A_1} - \chi_{A_2}$ so that $\|x\| = 1$. Consequently,

$$\|f\| = \|f\| \cdot \|x\| \geq |f(x)| = |\int x d\mu|$$

$$= |\int \chi_{A_1} d\mu - \int \chi_{A_2} d\mu|$$

$$= |\mu(A_1) - \mu(A_2)| > \|\mu\| - \varepsilon .$$

Since ε is arbitrary, we have

$$\|f\| \geq \|\mu\|, \tag{22}$$

as required.

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