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TITLE OF THESIS/*TITRE DE LA THÈSE* Application of the Inverse Scattering Method to the Three-Wave Problem

UNIVERSITY/*UNIVERSITÉ* Simon Fraser University

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* Master of Science

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1980

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DATED/*DATÉ* Aug. 27, 1979 SIGNED/*SIGNÉ*

PERMANENT ADDRESS/*RÉSIDENCE FIX*[redacted]

APPLICATION OF THE INVERSE SCATTERING METHOD TO THE THREE-WAVE PROBLEM

by

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B.Sc., Simon Fraser University, 1976

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Physics

SIMON FRASER UNIVERSITY

August 1979

APPROVAL

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Title of Thesis: Application of the Inverse Scattering Method to the Three-Wave Problem

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ABSTRACT

The inverse scattering method is applied to the nonlinear resonant interaction of two oppositely directed laser beams passing through each other in a dielectric fluid (an example of the three-wave problem). Assuming that the laser profiles are initially rectangular, analytic closed form expressions are obtained for the time dependent transformed scattering coefficients of the associated linear (direct) eigenvalue problem. The ~~linear~~ inverse problem, which involves two coupled Marchenko integral equations, is dealt with numerically. The effect of the nonlinear interaction on one of the laser beams is investigated. The solution obtained is seen to exhibit many of the features announced in the literature by authors who have numerically solved the three-wave problem directly.

ACKNOWLEDGEMENTS

I am indebted to my supervisor Dr. Richard Enns for suggesting the thesis topic and assisting me on some of the more vexatious aspects of the problem. I am also indebted to Dr. Sadat Rangnekar who meticulously checked the validity of the algebra.

Jo-Ann Murphy typed the thesis. Jo-Ann Murphy typed the corrections. Jo-Ann Murphy typed the corrections of the corrections. (Jo-Ann Murphy typed this page.) Thanks, Jo-Ann.

I wish to acknowledge the support of the Natural Sciences and Engineering Research Council Canada and thank Dr. A. Curzon who urged me to apply for this award.

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CHAPTER 1

INTRODUCTION

In the beginning man created out of the chaos of the universe the linear differential equation, and he saw that life was good; good as long as he did not look too closely at his world, a world inherently nonlinear.

Today's physicists are entering regimes in which the results of their experiments and theories can be modeled accurately only by utilizing nonlinear equations. Examples of this would include experiments in nonlinear optics such as Stimulated Brillouin Scattering (an example of a three-wave problem) and theoretical calculations in gravitation such as the dynamics of rotating dense stars. As a result, exploration into the field of nonlinear partial differential equations by mathematicians and physicists has grown tremendously.

In 1967 Gardner, Greene, Kruskal and Miura[1] solved the nonlinear Korteweg-deVries (KdV) equation by an approach now termed the inverse scattering method (ISM). Because comprehensive techniques for solving nonlinear problems, whether they be numerical or analytic, are difficult to find, great interest was taken in a subsequent paper by Lax[2] in which he showed that the ISM was not restricted to solving the KdV equation but could be applied to a larger class of nonlinear differential equations. Later theoretical physicists working in the field of nonlinear optics raised their eyebrows to a paper by Zakharov and co-workers[3] in 1973 which demonstrated, that in principle, the three-wave problem could be solved using the ISM, i.e. a non-trivial analytic solution to the three-wave problem may exist. As a demonstration of the difficulty of the ISM, it took these leaders in the field two years (1975) to realize an asymptotic ($t \to \infty$) N-soliton solution to the forward scattering case of the three-wave problem.[4] Another group, [Bers, Kaup and Reiman] were able to give a more general treatment of the three-wave problem by using numerical techniques along with the ISM.[5] In particular, they used a WKB approximation to find the soliton regimes for the explosive interaction case of the three-wave problem; they used the ISM to realize a soliton solution to the decay interactions case, forward scattering of the three-wave problem in the asymptotic limit $t \to \infty$; and they used both the ISM and numerical techniques to investigate the non-soliton solution to the decay interaction case, backscattering, of the three-wave problem where they assumed the initial envelopes to be Gaussian (the ISM was only used to find a "closed form" expression for the reflection coefficient, numerical techniques provided the actual solution). Of the two papers that presently exist, one by Zakharov [6] and the other by Kaup,[7] which describe in detail the general method of applying the ISM to the three-wave problem, it is the latter we have chosen to follow.

The ISM is based upon finding, associated to the nonlinear differential equation, a linear differential equation and an evolution operator. The difficult, if not impossible, step of solving the nonlinear equation directly is avoided by executing a series of linear steps which include solving directly and in an inverse manner the associated linear differential equation.

This method will be used to find solutions to the three-wave problem as it is associated with stimulated Brillouin scattering (SBS). The laser pulses will be assumed to be rectangular in profile passing through each other, in opposite directions (backscattering) in an initially undisturbed medium. The asymptotic solution obtained for one of the beams will clearly show the effects of nonlinear interaction betweeen the two laser pulses beating together and driving the dielectric medium at its Brillouin frequency.

The text that follows will begin with a description of the physical basis of the three-wave problem as it relates to SBS, followed by an explanation of the ISM, and concluding with details of the path taken to obtain wave envelope solutions.

CHAPTER 2

THE THREE WAVE PROBLEM IN NONLINEAR OPTICS

Consider three wave packets with wave vectors $\underline{k}_1$, $\underline{k}_2$, $\underline{k}_3$ and characteristic frequencies ω_1, ω_2, ω_3 satisfying the resonance conditions

$$\begin{aligned} \omega_1 &= \omega_2 + \omega_3 \\ \underline{k}_1 &= \underline{k}_2 + \underline{k}_3 \end{aligned} \qquad 2.1$$

In nonlinear optics the first and third wave packets represent two intense coherent plane-polarized laser beams beating together at the resonant frequency, ω_2, of a dielectric medium, through which the beams are passing while the second wave packet represents variations from equilibrium of the dielectric constant of the medium.

The electric fields of the two laser beams cause fluctuations in the dielectric constant which in turn couples the two laser beams so that energy from one of the beams may scatter into the other. The "predator" beam grows in intensity as it eats energy from its "prey",† the other laser beam. The roles of predator and prey are determined by the sign of the fluctuations in the dielectric constant and the phase of the two beams.

There is another three-wave interaction, possible only in an unstable plasma medium, referred to as the explosive instability process.[7] For this process the resonant conditions show the creation of three-wave packets from a vacuum:

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 &= 0 \\ \underline{k}_1 + \underline{k}_2 + \underline{k}_3 &= 0 \end{aligned} \qquad 2.2$$

†The nonlinear terms in equations 2.6 describing SBS are analogous to Volterra's competing species equations.

This class of three-wave interactions will not be dealt with in this thesis, hence restricting our attention to the so-called decay instability process given in 2.1.

The one dimensional form of the three-wave resonant interaction equations as given in the notation of Kaup[7] are,

$$
\begin{aligned}
Q_{1,t} + c_1 Q_{1,x} &= i\gamma_1 Q_2^* Q_3^* \\
Q_{2,t} + c_2 Q_{2,x} &= i\gamma_2 Q_1^* Q_3^* \qquad 2.3 \\
Q_{3,t} + c_3 Q_{3,x} &= i\gamma_3 Q_1^* Q_2^*
\end{aligned}
$$

where $Q_j(x,t)$ is (in general) the complex envelope of the j^{th} wave packet; ,x and ,t, represent partial differentiation with respect to x and t; * is complex conjugate; $\gamma_j = \pm 1$ are constants; and c_j is the group velocity of the j^{th} wave packet. When all the γ_j's are of the same sign, equations 2.3 describe the explosive instability process;[7] if any one of the γ_j's has a sign different from the other two, equations 2.3 describe the decay instability process. It is the latter case that we shall solve using the inverse scattering method.

Let's now see how the physical problem we wish to investigate, namely Stimulated Brillouin Scattering (SBS), is related to equations 2.3, first, "What is SBS?"

In a transparent isotropic homogeneous dielectric liquid it is possible to generate acoustic phonons in the region of overlap of two plane polarized monochromatic laser beams. The frequency difference between the high frequency beam, called the laser (or pump) beam with frequency and wavevector, ω_L, $\underline{k}_L$ and the lower frequency beam, called the signal (or scattered) beam with frequency and wavevector ω_S, $\underline{k}_S$ is adjusted to equal the Brillouin frequency $\omega_B = v_\xi k$ of the dielectric medium (v_ξ is the sound velocity in the dielectric and k is the phonon's wavevector). In the backscattering case the laser beams are sent through each other in opposite directions in which case one can show, by conservation of momentum, $k = k_L + k_S$. When subjected to electric fields a dielectric feels internal stresses (see 2.4) due to the interaction of the electric field and the polarized dielectric. The dielectric mechanically deforms to equalize these stresses, stretching along the direction of the electric field, as a result the density and hence the dielectric constant ε change.

The electrostrictive force density $\underline{F}$ produced by the electric field $\underline{E}$ is related to the dielectric constant ε by

$$\underline{F} = \underline{P} \cdot \nabla \underline{E}$$
$$= \frac{\varepsilon - 1}{4\pi} \underline{E} \cdot \nabla \underline{E} \quad (c.g.s) \qquad 2.4$$

where $\underline{P}$ is the polarization vector.

In the literature the electrostrictive pressure, P_{el} is calculated using 2.4 and is given as

$$P_{el} = \gamma^e E^2 / 8\pi \qquad 2.5$$

where $\gamma^e = \rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T \cong \frac{1}{3}(n^2-1)(n^2+2)$ (the Lorentz-Lorenz Law)

Here γ^e is the electrostrictive coefficient and ρ, T, n are respectively the density, temperature and the refractive index of the dielectric medium.

It is the nonlinear term E^2 in 2.5 that mixes the harmonics of the light beams together to give Brillouin Scattering. When the light beams have very high intensities the signal beam photons stimulates the decay of laser photons into more signal photons and accoustic phonons. This amplification continues until either damping or the finite energies of the beams limits the process. This is called Stimulated Brillouin Scattering.

Starting with Maxwell's wave equations describing the electromagnetic fields, the hydrodynamic equations describing the liquid dielectric medium and the resonant conditions 2.1 the following equations can be obtained. (Their derivation is lengthy and non-trivial and may be found in the literature.[8])

$$A_{s\partial t} + V_s A_{s\partial z} + \alpha_s A_s = -\beta_s A_L \xi^* \qquad 2.6a$$

$$A_{L\partial t} + V_L A_{L\partial z} + \alpha_L A_L = \beta_L A_s \xi \qquad b$$

$$\xi_{\partial t} + V_\xi \xi_{\partial z} + (\alpha_\xi + i\Delta\omega)\xi = -\beta_\xi A_L A_s^* \qquad c$$

where

$A_L(s)$ is the laser (signal) electric field amplitude,

$\xi = -i\,\varepsilon_1$: ε_1 is the amplitude of the change of the dielectric constant

α's are damping coefficients

$\Delta\omega = \omega - \omega_B$: ω is frequency of the dielectric equal to $\omega_L - \omega_s$ and ω_B is the Brillouin frequency of the dielectric

$\beta_{L(s)} = \omega_{L(s)}/4\varepsilon_o$: ε_o is the equilibrium value of the dielectric constant.

$\beta_\xi = \dfrac{(\gamma^e)^2 k}{16\pi\rho_o V_\xi}$: ρ_o is the density of the dielectric in equilibrium, and V_ξ is the velocity of sound in the dielectric. β_ξ is the coupling constant for SBS.

Setting the right hand side of equations 2.6 equal to zero turns off all nonlinear coupling. In fact, all one has to do is remove the electrostrictive coupling term in 2.6 c, then since ξ is initially zero (i.e. the dielectric constant is at its equilibrium value) it will be identically zero for all z and t leaving two uncoupled equations describing the evolution of the laser envelopes.

By changing the coupling constant β_ξ, equations 2.6 may be used to describe other phenomena in nonlinear optics such as Stimulated Rayleigh Scattering, Stimulated Thermal Brillouin Scattering, Stimulated Thermal Rayleigh Scattering, Stimulated Raman Scattering.[8] When the pulses are of very short duration (picosecond laser pulses) or when $\Delta\omega$ is large, equation 2.6c does not apply to any of the above processes, including SBS, and a more complete thermodynamic description is needed.[9]

A considerable effort was spent in trying to modify the inverse scattering method so that it could be applied to equations 2.6, with one or more damping coefficients non-zero. The attempts were unsuccessful. (See appendix A) We therefore set $\alpha_S = \alpha_L = \alpha_\xi = 0$, and also $\Delta\omega = 0$.

Making the following change of variables

$$
\begin{aligned}
q_1 &= i\,(\beta_\xi \beta_L)^{\frac{1}{2}} A_s^* \\
q_2 &= -i\,(\beta_s \beta_L)^{\frac{1}{2}} \xi \\
q_3 &= -i\,(\beta_\xi \beta_s)^{\frac{1}{2}} A_L \\
V_1 &= V_s \\
V_2 &= V_\xi \\
V_3 &= V_L
\end{aligned}
\qquad 2.7
$$

equations 2.6 become

$$q_{1,t} + V_1 q_{1,z} = -i q_2 q_3^*$$

$$q_{2,t} + V_2 q_{2,z} = i q_1 q_3 \qquad 2.8$$

$$q_{3,t} + V_3 q_{3,z} = i q_1^* q_2$$

Equations 2.8 are equivalent to equations 2.3 with $\gamma_1 = \gamma_2 = -\gamma_3 = -1$. To see this, set

$$Q_1 = q_1$$

$$Q_2 = q_2^*$$

$$Q_3 = q_3$$

in 2.3.

We will be applying the inverse scattering method (described in the next two chapters) to equations 2.8 (SBS) with $v_1 = -v_3$ (back-scattering) and $v_2 = 0$.† We will take the initial pulse profiles to be rectangular so that the first step of the inverse scattering technique can be solved in closed form. Realistic laser pulses are distorted Gaussian shapes, but rectangular pulses yield experimentally verifiable features (see Ref. [8]).

† For realistic laser pulse durations, since $|v_2| << |v_1|, |v_3|$ one can ignore the propagation of the fluctuations in the time that it takes the two pulses to travel through each other.

CHAPTER 3

INVERSE SCATTERING METHOD: PART I

The development of the Inverse Scattering Method (ISM) began with Gardner, Greene, Kruskal and Miura's paper "Method for solving the Korteweg-de-Vries (KdV) equation" published in Phys. Rev. Lett. in 1967.[1] Later in 1968 P.D. Lax explained in a mathematically rigorous manner, the workings of the method.[2] In so doing he was, also, able to generalize the method, thus demonstrating that the relationship between the KdV equation and the inverse method was not a fluke. Since that time many more equations have been discovered that may be solved by the ISM, including the three-wave problem.[6][10]

In this chapter we will introduce the ideas and workings of the inverse method, using the KdV equation as an example. The next chapter will deal with the complicated marriage of the three-wave problem to the ISM.

The KdV equation describing the propagation of shallow water waves of amplitude $\phi(x,t)$, viz.,

$$i\phi_t = -i\phi_{xxx} + i6\phi\phi_x \tag{3-1}$$

like most nonlinear differential equations is extremely difficult to solve directly. The ISM allows one to solve, in sequence, a series of linear problems which when completed yields the solution to the KdV equation.

The series of linear steps are all formulated in terms of the following two linear operators L and B which depend on $\phi(x,t)$

$$L(\phi(x,t)) = -\frac{\partial^2}{\partial x^2} + \phi(x,t) \qquad 3.2$$

$$B(\phi(x,t)) = -4i\frac{\partial^3}{\partial x^3} + 3i\left[\phi(x,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\phi(x,t)\right] \qquad 3.3$$

By forcing these two operators to satisfy the operator equation

$$iL_t = BL - LB \qquad 3.4$$

one indirectly forces $\phi(x,t)$ to satisfy the KdV equation. Equation 3.4 is, in fact, the KdV equation when L and B are given by 3.2 and 3.3. (i times the time derivative of L equals $i\phi_t$ and the commutator of B and L equals $-i\phi_{xxx} + i6\phi\phi_x$).

The following theorem quoted from Scott, ref. [11], will be the key that will enable the linear operators L and B to be used to solve the KdV equation.

<u>Theorem 0</u>

We are interested in a general nonlinear wave equation represented abstactly by

$$\phi_t = K(\phi) \qquad 3.5$$

where K denotes a nonlinear operator on some suitable space of functions. Suppose we can find linear operators L and B which depend on ϕ, a solution of PDE 3.5, and satisfy the operator equation

$$iL_t = BL - LB \qquad 3.6\,(3.4)$$

When B is self-adjoint, 3.6 automatically implies that the eigenvalues ξ of L, which appear in

$$L(\phi(x,t))\,\psi(x,t) = \xi\,\psi(x,t) \qquad 3.7$$

are independent of time. Furthermore, the eigenfunctions ψ may be shown to evolve in time according to

$$i\,\psi_t = B\psi \qquad 3.8$$

The proof of this theorem may be found in ref. [11].

To solve the KdV equation 3.1 with the initial condition

$$\phi(x,t=0) \equiv \phi_0(x) \qquad 3.9$$

by the ISM we begin by solving the linear eigenvalue equation 3.7 at time t = 0, i.e.

$$-\psi_{xx}(x,0) + \phi_0(x)\,\psi(x,0) = \xi\,\psi(x,0) \qquad 3.10$$

Note that the initial wave shape $\phi_0(x)$ takes on the role of the scattering potential in 3.10. The next step of the ISM involves theorem 0, whereby the time dependence of the solution $\psi(x,o)$ is calculated giving $\psi(x,t)$ (a solution of 3.7). To use theorem 0 we must show L and B satisfy 3.6 and that L has time independent eigenvalues. The first requirement has already been established and the latter requirement is proven in ref. [11].

When the ISM is applied to the three-wave problem in the next chapter the constraint that ξ is time independent is assumed before B is derived following a method outlined by Ablowitz [11]. It should be noted that B obtained is not self-adjoint thus demonstrating that the self-adjointness of B is a sufficient, but not necessary, condition for the eigenvalues ξ to be time independent.

The time evolution equation for the KdV problem becomes

$$i\Psi_t(x,t) = -4i\Psi_{xxx}(x,t) + 3i\left[\phi(x,t)\Psi_x(x,t) + \left(\phi(x,t)\Psi(x,t)\right)_x\right] \qquad 3.11$$

We would now solve 3.11 for $\psi(x,t)$ using the initial condition $\psi(x,o)$, obtained from the previous linear step, if it were not for the fact that $\phi(x,t)$ is unknown. We impose the realistic constraint that

$$\lim_{x\to\pm\infty} \phi(x,t) = 0 \qquad 3.12$$

which specifies that the amplitude of the waves must go zero as $x \to \pm\infty$.

Taking the limit $x \to \pm\infty$ in equation 3.11 the $\phi(x,t)$ dependence is eliminated enabling the resulting equation

$$i\Psi_t(x,t) = -4i\Psi_{xxx}(x,t) \quad \text{at } x \to \pm\infty \qquad 3.13$$

to be solved for the time dependent "scattered" eigenfunctions of $L(\phi(x,t))$, $\lim\limits_{x\to\pm\infty} \Psi(x,t)$.

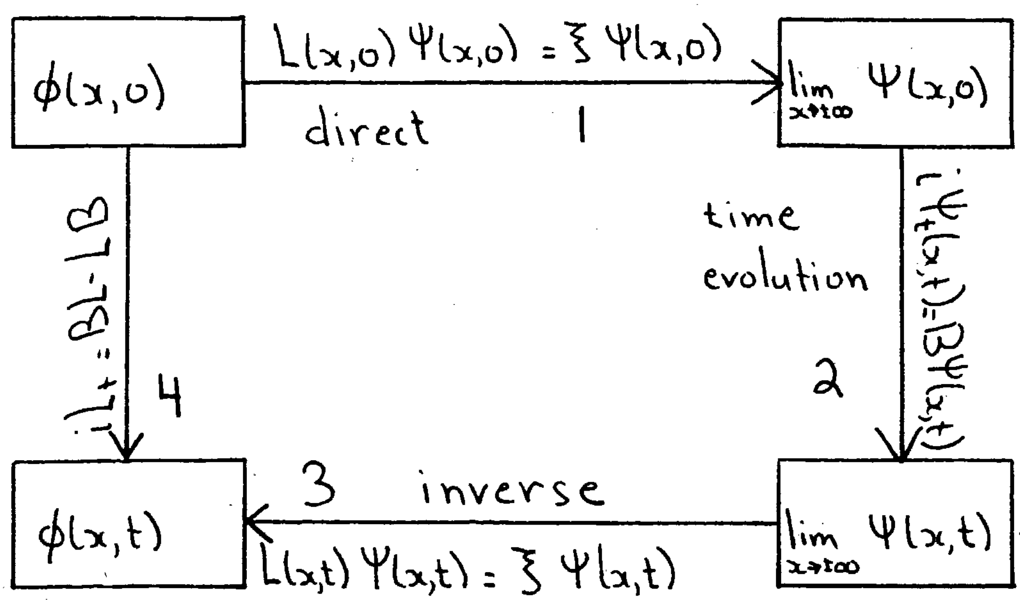
Knowing the scattering data i.e. $\lim\limits_{x\to\pm\infty} \psi(x,t)$ the next and final step is to solve the inverse scattering problem. That is to say the scattering potential $\phi(x,t)$ (solution to the KdV equation)

$$-\Psi_{xx}(x,t) + \phi(x,t)\,\Psi(x,t) = \xi\,\Psi(x,t) \qquad 3\cdot14$$

is found from the scattered solutions $\lim_{x\to\pm\infty}\Psi(x,t)$. This "inverse scattering" step is done by solving the linear Gel'fand-Levitan integral equations which are given in ref. [11].

With the aid of the following figure 1.1 depicting diagramatically the steps of the ISM we will run through the three linear steps (of the ISM) once more.

Figure 1.1



If we are given a nonlinear differential equation which may be expressed in the form

$$iL_t = BL - LB$$

where L and B are linear differential operators dependent on the solution $\phi(x,t)$ of the nonlinear equation and if the eigenvalues of L are time independent then we may solve the nonlinear problem by traversing linear steps 1, 2 and 3 (in figure 1.1).

Step 1: With the initial condition $\phi(x,o)$, representing the scattering potential at $t = o$ for the operator L we solve the $t = o$ linear eigenvalue problem for $\psi(x,o)$. From this solution we extract the $t = o$ scattering data i.e.

$$\lim_{x \to \pm\infty} \psi(x,o).$$

Step 2: The time dependent scattering data $\lim\limits_{x \to \pm\infty} \psi(x,t)$ is now calculated using the time evolution operator B in the asymptotic limit $x \to \pm\infty$. This limit is taken so that the $\phi(x,t)$ dependence in B will vanish as we will always assume, in physical problems, that $\lim\limits_{x \to \pm\infty} \phi(x,t) = o$.

Step 3: The inverse scattering problem is solved - the full time dependent scattering potential $\phi(x,t)$ of the eigenvalue equation for $L(\phi(x,t))$ is determined from the scattered solutions $\lim\limits_{x \to \pm\infty} \psi(x,t)$. This step usually involves solving a system of coupled linear integral equations.

$\phi(x,t)$, the solution of the nonlinear differential equation, is known after completing step 3 of the ISM.

At this point one may wonder, how general is the ISM? The ISM is severely limited in the sense that it cannot be applied to any arbitrary nonlinear differential equation. Given a nonlinear differential equation one is immediately limited in that one must guess the operators L and B if they even exist. Also one must be able to solve the inverse scattering problem (step 3) for the time dependent L eigenvalue equation. In general Gel'fand-Levitan type integral equations, which enable us to carry out this step, do not exist. At present there exist only a handful of nonlinear equations of interest to the physicist which are solveable by the ISM (see ref. [11]).

To avoid the guessing game involved in finding L and B authors Zakharov and Manakov[3][4] chose a particular class of linear operators L, for which the inverse problem, step 3, is solveable, and a general class of time evolution operators B and from this starting point generated a large set of nonlinear equations, solveable by the ISM. Both the KdV equation and the three-wave probelm were included in this set of nonlinear equations.

In the next chapter we will assume the existence and form of the L operator for the three-wave problem and from this starting point we will derive the inverse problem integrals (Marchenko type) and derive the time evolution operator B (the method for finding B when L is known is found in ref. [13]).

CHAPTER 4

INVERSE SCATTERING METHOD: PART II

In this chapter we shall bring together the three-wave problem and the inverse scattering method. The theory given here is an expanded, and hopefully much clearer, version of Kaup's paper.[7]

Following Kaup we begin with an investigation of the linear differential operator, L, and its associated eigenvalue problem. Its relationship with the three-wave problem will be discussed later.

The linear eigenvalue equations associated with the three-wave problem that must be solved directly in step 1 and in an inverse manner in step 3 (figure 1.1) are as follows:

$$Lv = \xi v \tag{4.1}$$

where

$$L \equiv A(-i I \partial_x + V) \tag{4.2}$$

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

ξ is the eigenvalue of L; v is a rank-3 column vector; V is a 3×3 potential matrix with $V_{ii} = 0$; A is a diagonal matrix with constant and real eigenvalues α_1, α_2, α_3; I is the identity matrix; and ∂_x represents the partial differential operator $\frac{\partial}{\partial x}$. The α_i's are ordered as follows:

$$\frac{1}{\alpha_1} > \frac{1}{\alpha_2} > \frac{1}{\alpha_3} \qquad 4.3$$

where none of the α_i's are equal to zero.

On the condition that $V \to o$ sufficiently rapidly as $x \to \pm \infty$, two sets of three linear independent eigenfunctions may be defined. The first set of eigenfunctions, $\phi^{(j)}(\xi,x)$, $j = 1,2,3$, ξ real, are obtained from the left hand boundary conditions,

$$\phi_n^{(j)}(\xi,x) \to \delta_n^j e^{i\xi x/\alpha_j} \quad \text{as} \quad x \to -\infty \qquad 4.4$$

where n designates the n^{th} component of $\phi^{(j)}$, $n = 1,2,3,$. Similarly the second set of eigenstates, $\psi^{(j)}$ are defined using the right hand boundary conditions

$$\psi_n^{(j)}(\xi,x) \to \delta_n^j e^{i\xi x/\alpha_j} \quad \text{as} \quad x \to \infty \qquad 4.5$$

To see that the boundary conditions 4.4 and 4.5 are consistent with the linear eigenvalue equations, we solve 4.1 in the asymptotic limit $x \to \pm \infty$. It is important to realize that $\phi^{(j)}$ and $\psi^{(j)}$ are solutions of 4.1 satisfying their respective boundary conditions. They do not represent the solutions of 4.1 in the limit $x \to \pm \infty$.

The two sets $\{\phi^{(j)}\}$ and $\{\psi^{(j)}\}$ are each a set of linear independent solutions spanning the same solution space, hence, the two sets can be related as follows

$$\phi^{(j)}(\xi,x) = \sum_{k=1}^{3} a(\xi)_{jk} \psi^{(k)}(\xi,x) \qquad 4.6$$

$[a_{jk}(\xi)]$ is in fact the scattering matrix for $\phi^{(j)}$. To see this, consider $\phi_n^{(1)}$ on the left written as

$$\delta_n^1 e^{i\xi x/\alpha_1}$$

This eigenstate will be scattered into some general state on the right which is represented by a linear combination of $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ on the left i.e.

$$\lim_{x\to\infty} \phi_n^{(1)} = \sum_m a_{1m} \lim_{x\to -\infty} \phi_n^{(m)}$$

$$= a_{1n} e^{i\xi x/\alpha_n}$$

It is seen from 4.4 and 4.5 that $\phi^{(j)}$ as $x \to -\infty$ is identical to $\psi^{(j)}$ as $x \to \infty$. We are therefore able to write 4.6 with $[a_{jk}]$ representing the scattering matrix for $\phi^{(j)}$.

Without loss of generality we take

$$\mathrm{Det}\left[a_{jk}(\xi)\right] = 1 \qquad 4.7$$

where the inverse of $[a_{jk}]$ exists and is defined by

$$\sum_{k=1}^{3} a_{jk} b_{kl} = \delta_{jl} \qquad 4.8$$

The inverse relation of 4.6 can now be defined as follows

$$\psi^{(j)}(\xi, x) = \sum_{k=1}^{3} b(\xi)_{jk} \phi^{(k)}(\xi, x) \qquad 4.9$$

where $[b_{jk}(\xi)]$ represents the scattering matrix, from right to left, of $\psi^{(j)}$.

It is more convenient to deal with the scattering matrix (reflection and transmission coefficients in the case of the KdV equation) when passing through steps 1,2 and 3 of the ISM than with the actual scattered solutions. With this in mind expressions will be obtained relating the scattering potentials V_{ij} to the scattering matrix coefficients a_{ij}. What follows will be the necessary theory to achieve this goal.

An integral equation is obtained for $\phi^{(j)}$ by taking the linear eigenvalue equation 4.1.

$$-i\alpha_n \frac{\partial}{\partial x}\phi_n^{(j)} + \sum_m \alpha_n V_{nm}\phi_m^{(j)} = \xi\phi_n^{(j)}, \qquad 4{\cdot}10$$

multiplying by the integrating factor $e^{-i\xi y/\alpha_n}$ and then integrating $\int_{-\infty}^{x} dy$ $(x \to y$ in 4.10), the result being

$$-i\alpha_n\left[\phi_n^{(j)}(y)e^{-i\xi y/\alpha_n}\Big|_{-\infty}^{x} + \frac{i\xi}{\alpha_n}\int_{-\infty}^{x}\phi_n^{(j)}(y)e^{-i\xi y/\alpha_n}dy\right] +$$

$$+\sum_m \alpha_n\int_{-\infty}^{x}V_{nm}(y)\phi_m^{(j)}(y)e^{-i\xi y/\alpha_n}dy = \xi\int_{-\infty}^{x}\phi_n^{(j)}(y)e^{-i\xi y/\alpha_n}dy \qquad 4{\cdot}11$$

Using 4.4, 4.11 can be rewritten as

$$\phi_n^{(j)}(x)e^{-i\xi x/\alpha_j} = \delta_n^j - i\int_{-\infty}^{x}dy\, e^{i\xi(x-y)\beta_{nj}}\sum_{m=1}^{3}V_{nm}(y)\phi_m^{(j)}(y)e^{-i\xi y/\alpha_j}$$

$$4{\cdot}12$$

where

$$\beta_{ij} = \frac{1}{\alpha_i} - \frac{1}{\alpha_j} \tag{4.13}$$

It will now be shown that the integral equation for $\phi_n^{(1)}$ has an absolutely convergent Neumann series solution in the lower half ξ-plane. This will allow us to analytically extend $\phi_n^{(1)}$ into the complex ξ-plane where contour integral techniques can be used. Define

$$U(x) \equiv \max_{k,l} |V_{kl}(x)| \tag{4.14}$$

$$R(x) \equiv \int_{-\infty}^{\infty} U(y)\, dy \tag{4.15}$$

$$\eta \equiv \operatorname{Im}(\xi) \tag{4.16}$$

$$W_n(x) \equiv \left| \phi_n^{(1)}(x) e^{-i\xi x/\alpha_1} \right| \tag{4.17}$$

and take $\eta < 0$. Then substituting the integral equation for $\phi_n^{(1)}(x)$ into 4.17 it follows that

$$W(x) \leq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + M \int_{-\infty}^{x} dy\, Q(y) w(y) \tag{4.18}$$

where

$$Q_{nm} \equiv |V_{nm}| \tag{4.19}$$

and

$$M \equiv \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{4.20}$$

We have for the first approximations of the Neumann series solution

$$W^{(0)} \leqslant \delta'_k \tag{4.21 a}$$

$$W^{(1)} \leqslant \delta'_k + M \int_{-\infty}^{x} Q\, W^{(0)}\, dy \tag{b}$$

$$W^{(2)} \leqslant \delta'_k + M \int_{-\infty}^{x} Q\, W^{(1)}\, dy \tag{c}$$

$$\vdots$$

Substituting 4.21 a into 4.21b and then this result into 4.21c gives

$$W^{(1)} \leqslant \delta'_k + M \int_{-\infty}^{x} Q\, \delta'_k\, dy = \delta'_k \left[I + M \int_{-\infty}^{x} Q\, dy \right] \tag{4.22a}$$

$$W^{(2)} \leqslant \delta'_k + M \int_{-\infty}^{x} Q \left[\delta'_k \left(I + M \int_{-\infty}^{y} Q\, dy' \right) \right] dy \tag{b}$$

$$W^{(2)} \leqslant \delta'_k \left[I + M \int_{-\infty}^{x} Q(y)\, dy + M^2 \int_{-\infty}^{x} Q(y) \int_{-\infty}^{y} Q(y')\, dy\, dy' \right] \tag{c}$$

$$W^{(2)} \leqslant \delta'_k \left[I + MR + M^2 \int_{-\infty}^{x} Q(y)\, R(y)\, dy \right] \tag{d}$$

Integrating the last term of 4.22d by parts will give

$$\int_{-\infty}^{x} Q(y)R(y)\,dy \leq R^2(x) - \int_{-\infty}^{x} Q(y)R(y)\,dy \,,$$

hence,

$$\int_{-\infty}^{x} Q(y)\,R(y)\,dy \leq \frac{R^2}{2} \qquad 4.23$$

Substituting 4.23 into 4.22d yields the following expression for $w^{(2)}$.

$$W^{(2)} \leqslant \delta'_k \left[I + MR + \frac{M^2R^2}{2} \right] \qquad 4.24$$

which may be generalized to higher order solutions as follows

$$w(x) \leq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + M \int_{-\infty}^{x} Q(y)w(y)\,dy \leq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \left[I + \sum_{p=1}^{\infty} \frac{1}{p!} M^p R^p(x) \right] \qquad 4.25$$

The Cauchy ratio test may be used to prove that the series in 4.25 is absolutely convergent provided R is finite. (Note that R is associated with the area of the laser pulse.) Provided the potential (i.e. laser pulse) goes to zero sufficiently rapidly as $x \to \pm\infty$ the Neumann series for $\phi_n^{(1)}$ is absolutely convergent.

In a similar manner $\phi_n^{(3)}$, $\psi_n^{(1)}$, $\psi_n^{(3)}$ can be shown to have absolutely convergent Neumann series solutions in their respective ξ-half planes. The results are summarized in the following theorem taken from Kaup.

Theorem 1 If $\int_{-\infty}^{\infty} dx\, \mathrm{Max}|V_{jk}(x)| < \infty$ then $\phi^{(1)}\, e^{-i\xi x/\alpha_1}$, $\psi^{(3)} e^{-i\xi x/\alpha_3}$, a_{11} and b_{33} are analytic functions of ξ in the lower half ξ-plane $(\eta < 0)$; $\phi^{(3)} e^{-i\xi x/\alpha_3}$, $\psi^{(1)} e^{-i\xi x/\alpha_1}$, a_{33}, b_{11} are analytic functions of

ξ in the upper half ξ-plane ($\eta < 0$); and $\phi^{(j)}e^{-i\xi x/\alpha_j}$, $\psi^{(j)}e^{-i\xi x/\alpha_j}$, a_{jk}, b_{jk} are bounded functions of ξ on the real axis ($\eta = 0$).

Using the result that

$$\lim_{x\to\infty} \phi_n^{(j)} = a_{jn} e^{i\xi x/\alpha_n}$$

in the integral equation for $\phi_n^{(j)}$ one obtains the following integral equation for a_{ij}

$$a_{jn} = \delta_n^j - i\int_{-\infty}^{\infty} dy\, e^{-i\xi y/\alpha_n} \sum_m V_{nm}\phi_m^{(j)} \qquad 4.26$$

Writing out the expression for a_{11} in 4.26

$$a_{11} = 1 - i\int_{-\infty}^{\infty} dy\, e^{-i\xi y/\alpha_1} \sum_m V_{1m}\phi_m^{(1)}$$

We see that since the term $\phi_m^{(1)}e^{-i\xi y/\alpha_1}$ is analytic in the lower half ξ-plane, so is a_{11}. In a similar manner the analyticity of a_{33}, b_{33} and b_{11} may be shown.

Theorem 1 will enable us to analytically extend the functions $\phi^{(j)}(\xi)$, $\psi^{(j)}(\xi)$, $a_{ij}(\xi)$ and $b_{ij}(\xi)$ into the appropriate complex ξ half plane. Once extended, many of the integrals to be encountered will be easily evaluated using contour integration techniques. Unfortunately $\overset{(2)}{\psi}(\xi,x)$ and $\overset{(2)}{\phi}(\xi,x)$ cannot be analytically extended since parts of these functions diverge in the upper half ξ-plane and other parts diverge in the lower half ξ-plane. To overcome this difficulty, two analytically extendable functions will be constructed by considering the adjoint problem of 4.1. One function, independent of $\psi^{(1)}$ and $\psi^{(3)}$, will take on the role of $\psi^{(2)}$ and the other function, independent of $\phi^{(1)}$ and $\phi^{(3)}$, will take on the role of $\phi^{(2)}$.

The adjoint problem of 4.1 is taken to be

$$L^A V^A = \xi V^A \qquad 4.27^\dagger$$

$$L^A = i A \partial_x + \tilde{V} A \qquad 4.28$$

where $\tilde{V}$ is the matrix transpose of V.

Solutions to the adjoint problem 4.27 can be generated from solutions to the regular linear eigenvalue problem, 4.1, by

$$V_n^A = \frac{1}{\alpha_n} \sum_{m,p} \varepsilon_{nmp} u_m w_p e^{-i\xi x/\alpha} \qquad 4.29$$

with $\alpha^{-1} = \alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}$

where ε_{nmp} is the usual antisymmetric tensor; u_m and w_p are two linear independent solutions of 4.1. 4.29 may be verified by substituting it into 4.28. In a similar manner solutions to the regular linear eigenvalue equation can be expressed in terms of the adjoint solutions as follows

---

† $L^A u_n = i\alpha_n \frac{\partial u_n}{\partial x} + \sum_m \alpha_m V_{mn} u_m = \xi u_n$

$$V_n = e^{i\xi x/\alpha} \sum_{m,p} \varepsilon_{nmp} U_m^A W_p^A \alpha_m \alpha_p \qquad 4.30$$

Eigenstates $\phi^{(j)A}$ and $\psi^{(j)A}$ are defined as solutions of the adjoint equations satisfying the boundary conditions given below.

$$\phi_n^{(j)A}(\xi, x) \rightarrow \frac{1}{\alpha_j} \delta^j{}_n e^{-i\xi x/\alpha_j} \quad \text{as} \quad x \rightarrow -\infty \qquad 4.31$$

$$\psi_n^{(j)A}(\xi, x) \rightarrow \frac{1}{\alpha_j} \delta^j{}_n e^{-i\xi x/\alpha_j} \quad \text{as} \quad x \rightarrow \infty \qquad 4.32$$

A warning is needed at this stage: **the adjoint** solutions $\phi^{(j)A}$, $\psi^{(j)A}$, v^A are not the same as the adjoints of the regular solutions $\phi^{(j)}$, $\psi^{(j)}$, v. i.e. $\phi^{(j)A} \neq [\phi^{(j)}]^A$.

The analytic properties of $\phi_n^{(j)A}$ and $\psi_n^{(j)A}$ are summarized in the following theorem, taken from Kaup, and verifiable in a manner similar to theorem 1.

<u>Theorem 2</u> If $\int_{-\infty}^{\infty} dx \mathrm{Max}|V_{jk}(x)| < \infty$ then $\phi^{(3)A} e^{i\xi x/\alpha_3}$ and $\psi^{(1)A} e^{i\xi x/\alpha_1}$ are analytic functions of ξ in the lower half ξ-plane ($\eta < 0$); $\phi^{(1)A} e^{i\xi x/\alpha_1}$ and $\psi^{(3)A} e^{i\xi x/\alpha_3}$ are analytic functions of ξ in the upper half ξ-plane ($\eta < 0$) and $\psi^{(j)A} e^{i\xi x/\alpha_j}$, $\phi^{(j)A} e^{i\xi x/\alpha_j}$ are bounded functions of ξ on the real axis ($\eta = 0$).

The integral equations for ψ^A, ϕ^A, ψ, b_{ij} are derived in a similar manner to the integral equations for ϕ and a_{ij}. These are required in the proof of theorem 2. These together with equations for ϕ and a_{ij} are listed below.

$$\phi_n^{(j)} e^{-i\xi x/d_j} = \delta_n^j - i\int_{-\infty}^{x} dy\, e^{i\xi(x-y)\beta_{nj}} \sum_m V_{nm}\, \phi_m^{(j)} e^{-i\xi y/d_j} \qquad 4{\cdot}33$$

$$\psi_n^{(j)} e^{-i\xi x/d_j} = \delta_n^j + i\int_{x}^{\infty} dy\, e^{i\xi(x-y)\beta_{nj}} \sum_m V_{nm}\, \psi_m^{(j)} e^{-i\xi y/d_j} \qquad 4{\cdot}34$$

$$\phi_n^{(j)A} e^{i\xi x/d_j} = \delta_n^j + i\int_{-\infty}^{x} dy\, e^{-i\xi(x-y)\beta_{nj}} \sum_m V_{mn}\, \phi_m^{(j)A} e^{i\xi y/d_j} \qquad 4{\cdot}35$$

$$\psi_n^{(j)A} e^{i\xi x/d_j} = \delta_n^j - i\int_{x}^{\infty} dy\, e^{-i\xi(x-y)\beta_{nj}} \sum_m V_{mn}\, \psi_m^{(j)A} e^{i\xi y/d_j} \qquad 4{\cdot}36$$

$$a_{jn} = \delta_n^j - i\int_{-\infty}^{\infty} dy\, e^{-i\xi y/d_n} \sum_m V_{nm}\, \phi_m^{(j)} \qquad 4{\cdot}37$$

$$b_{jn} = \delta_n^j + i\int_{-\infty}^{+\infty} dy\, e^{-i\xi y/d_n} \sum_m V_{nm}\, \psi_m^{(j)} \qquad 4{\cdot}38$$

Two column vectors χ, analytic in the upper half ξ-plane and $\bar{\chi}$, analytic in the lower half ξ-plane, are constructed as follows:

$$\chi_n \equiv \sum_{jk} \varepsilon_{njk}\, \phi_j^{(1)A} d_j\, e^{i\xi x/d_1}\, \psi_k^{(3)A} d_k\, e^{i\xi x/d_3} \qquad 4{\cdot}39$$

$$\bar{\chi}_n \equiv \sum_{jk} \varepsilon_{njk}\, \phi_j^{(3)A} d_j\, e^{i\xi x/d_3}\, \psi_k^{(1)A} d_k\, e^{i\xi x/d_1} \qquad 4{\cdot}40$$

Now from equation 4.29 in the asymptotic limits $x \to \pm \infty$ it is not difficult to show that

$$\varepsilon_{ijk}\,\phi_n^{(i)} = e^{i\xi x/d} \sum_{m,p} \varepsilon_{nmp}\,\phi_m^{(j)A}\,\phi_p^{(k)A}\,d_m\,d_p \qquad 4.41$$

$$\varepsilon_{ijk}\,\psi_n^{(i)} = e^{i\xi x/d} \sum_{m,p} \varepsilon_{nmp}\,\psi_m^{(j)A}\,\psi_p^{(k)A}\,d_m\,d_p \qquad 4.42$$

From which it follows that

$$\phi^{(l)A} = \sum_n b_{nl}\,\psi^{(n)A} \qquad 4.43$$

$$\psi^{(l)A} = \sum_n a_{nl}\,\phi^{(n)A} \qquad 4.44$$

The derivation of equations 4.43 and 4.44 is sketched below.

In matrix form equation 4.6 is written as

$$[\phi] = [a][\psi] \qquad 4.45$$

where

$$[\phi]_{ij} \sim \phi_j^{(i)} \quad \text{etc.}$$

The inverse of 4.45 is written as

$$[\phi]^{-1} = [\psi]^{-1}[a]^{-1} \qquad 4.46$$

As $[b] = [a]^{-1}$ 4.46 can be rewritten as

$$[\phi]^{-1} = [\psi]^{-1}[b] \qquad 4.47$$

The inverse of $[\phi]$ $([\psi])$ is the transpose of the cofactor matrix of $[\phi]$ $([\psi])$ divided by determinant of $[\phi]$ $([\psi])$ i.e.

$$[\phi]^{-1} = \frac{\widetilde{[\text{cofactor } \phi]}}{\text{Det}[\phi]} \quad \text{and} \quad [\psi]^{-1} = \frac{\widetilde{[\text{cofactor } \psi]}}{\text{Det}[\psi]} \qquad 4.48$$

Using expressions* similar to 4.41 and 4.42 obtained from 4.30 it can be shown that

$$[\phi^A] = [\text{cofactor } \phi]k \quad , \; k \equiv \text{const.} \qquad 4.49$$

$$[\psi^A] = [\text{cofactor } \psi]k' \quad , \; k' \equiv \text{const.} \qquad 4.50$$

which when substituted in 4.48 gives (constant terms can be shown to cancel)

$$\widetilde{[\phi^A]} = \widetilde{[\psi^A]}\,[b] \qquad 4.51$$

Equation 4.51 is equation 4.43 in matrix form.

Using 4.6 to 4.9 and 4.39 to 4.44 expressions for χ and $\bar{\chi}$ can be obtained in terms of $\psi^{(i)}$.

$$\chi = e^{-i\xi x/\alpha_2}\left(b_{21}\psi^{(1)} - b_{11}\psi^{(2)}\right) \qquad 4.52$$

$$\bar{\chi} = e^{-i\xi x/\alpha_2}\left(b_{33}\psi^{(2)} - b_{23}\psi^{(3)}\right) \qquad 4.53$$

The following results are now obtained using the integral equations for ϕ, ψ, and a_{ij} and b_{ij}:

For $|\xi| \to \infty$ and $\eta > 0$,

$$\phi_k^{(3)} e^{-i\xi x/\alpha_3} \to \delta_k^3 + O[1/\xi] \qquad 4.54\text{a}$$

$$\psi_k^{(1)} e^{-i\xi x/\alpha_1} \to \delta_k^1 + O[1/\xi] \qquad \text{b}$$

$$\chi_k \to -\delta_k^2 + O[1/\xi] \qquad \text{c}$$

$$a_{33} \to 1 + O[1/\xi] \qquad \text{d}$$

$$b_{11} \to 1 + O[1/\xi] \qquad \text{e}$$

and for $|\xi| \to \infty$ and $\eta < 0$,

$$\phi_k^{(1)} e^{-i\xi x/\alpha_1} \to \delta_k^1 + O[1/\xi] \qquad 4.55\text{a}$$

$$\psi_k^{(3)} e^{-i\xi x/\alpha_3} \to \delta_k^3 + O[1/\xi] \qquad \text{b}$$

$$\bar{\chi}_k \to \delta_k^2 + O[1/\xi] \qquad \text{c}$$

$$a_{11} \to 1 + O[1/\xi] \qquad \text{d}$$

$$b_{33} \to 1 + O[1/\xi] \qquad \text{e}$$

Coupled integral expressions for $\psi^{(j)}$ in terms of the scattering matrix coefficients will now be obtained using contour integration techniques.

Define the contour C in the complex ξ-plane to extend from $-\infty + i\varepsilon$ to $+\infty + i\varepsilon$ such that C lies above all the zeros of a_{33} and b_{11}. Define contour $\bar{C}$ in the complex ξ-plane to extend from $-\infty - i\varepsilon$ to $+\infty - i\varepsilon$ such that $\bar{C}$ lies below all the zeros of a_{11} and b_{33}†

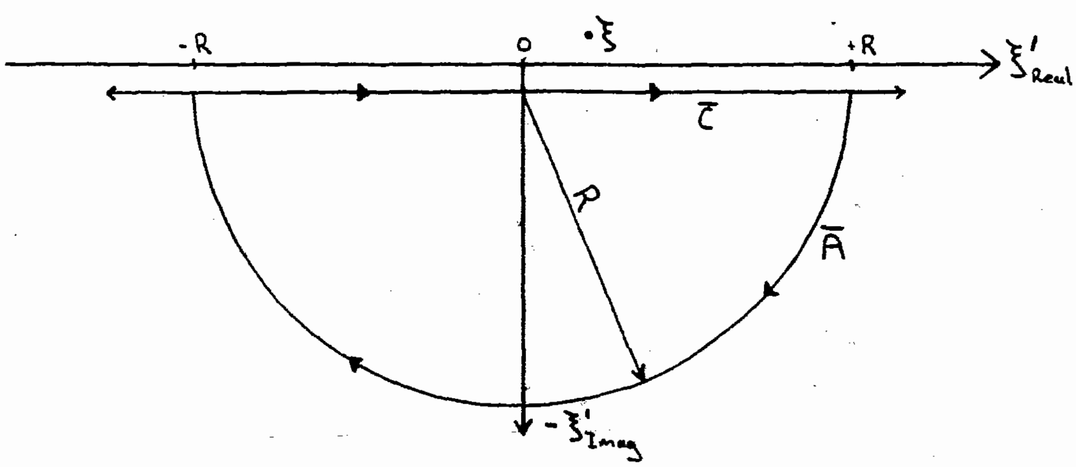
Consider the contour integral

$$I = \int_{\bar{C}} \frac{\phi^{(1)}(\xi', x)\, e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')(\xi' - \xi)}\, d\xi' \qquad 4.56$$

where ξ lies above $\bar{C}$ (and below C).

The contour $\bar{C}$ is closed by adding a semicircle in the lower half ξ-plane, $\bar{A}$.

Table 4.1



† a_{11}, b_{11}, a_{33} and b_{33} may in general have zeros anywhere in the complex plane. The restriction here limits our calculation to the continuous spectrum solution. In the next chapters we will show for SBS back-scattering the zeros of a_{11} etc. are consistant with the above definitions of C and $\bar{C}$.

As there are no poles inside the closed contour $\bar{A}U\bar{C}$,

$$\lim_{R\to\infty} \int_{\bar{A}U\bar{C}} \frac{\phi^{(1)} e^{-i\xi' x/\alpha_1}}{a_{11}(\xi'-\xi)} d\xi' = 0$$

$$\therefore \quad I + \lim_{R\to\infty} \int_{\bar{A}} \frac{\phi^{(1)} e^{-i\xi' x/\alpha_1}}{a_{11}(\xi'-\xi)} d\xi' = 0 \qquad 4.57$$

Using 4.55 it can be shown that the integrand in 4.57 approaches 1 as $R \to \infty$ where $\xi^1 \equiv Re^{i\theta}$. Hence,

$$\underset{\sim}{I} \equiv \int_{\bar{C}} \frac{\phi_k^{(1)}(\xi') e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')(\xi'-\xi)} d\xi' = i\pi \delta_k^1 \qquad 4.58$$

Expressing $\phi^{(1)}$ in terms of $\psi^{(j)}$ and a_{ij} (using 4.6), 4.58 becomes

$$\int_{\bar{C}} \frac{\Psi_k^{(1)}(\xi') e^{-i\xi' x/\alpha_1}}{(\xi'-\xi)} d\xi' + \int_{\bar{C}} \frac{a_{12}(\xi')\Psi_k^{(2)}(\xi') e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')(\xi'-\xi)} d\xi' +$$

$$+ \int_{\bar{C}} \frac{a_{13}(\xi')\Psi_k^{(3)}(\xi') e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')(\xi'-\xi)} d\xi' = i\pi \delta_k^1 \qquad 4.59$$

The first integral in 4.59 is evaluated by closing the contour $\bar{C}$ with a semicircle, A, in the upper half ξ-plane. The integrand has a simple pole at ξ. We have

$$\lim_{R\to\infty} \int_{\bar{C}UA} \frac{\Psi^{(1)}(\xi') e^{-i\xi' x/\alpha_1}}{(\xi'-\xi)} d\xi' = 2\pi i \Psi^{(1)}(\xi) e^{-i\xi x/\alpha_1} \qquad 4.60$$

The integrand of 4.60 can be shown to approach 1 as $R \to \infty$ hence the contribution along A is $i\pi\delta^1_k$. 4.60 may now be written in the following form

$$\psi^{(1)}_k(\xi,x)e^{-i\xi x/\alpha_1} = \delta^1_k - \frac{1}{2\pi i}\int_C \frac{a_{12}(\xi')\,\psi^{(2)}_k(\xi')e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')\,(\xi'-\xi)}\,d\xi'$$

$$-\frac{1}{2\pi i}\int_C \frac{a_{13}(\xi')\,\psi^{(3)}_k(\xi')e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')\,(\xi'-\xi)}\,d\xi' \qquad 4\cdot 61$$

In a similar manner by considering the contour integral

$$\int_C \frac{\phi^{(3)}(\xi',x)\;e^{-i\xi' x/\alpha_3}}{a_{33}(\xi')(\xi'-\xi)}\,d\xi' \qquad 4\cdot 62$$

it can be shown that

$$\psi^{(3)}_k(\xi,x)e^{-i\xi x/\alpha_3} = \delta^3_k + \frac{1}{2\pi i}\int_C \frac{a_{31}(\xi')\,\psi^{(1)}_k(\xi')e^{-i\xi' x/\alpha_3}}{a_{33}(\xi')\,(\xi'-\xi)}\,d\xi'$$

$$+\frac{1}{2\pi i}\int_C \frac{a_{32}(\xi')\psi^{(2)}_k(\xi')e^{-i\xi' x/\alpha_3}}{a_{33}(\xi')(\xi'-\xi)}\,d\xi' \qquad 4\cdot 63$$

An integral expression for $\psi^{(2)}$ is obtained as follows. We see from 4.54 that $\chi(\xi')/b_{11}(\xi')$ is analytic inside CUA, hence

$$\lim_{R\to\infty}\int_{CUA} \frac{\chi(\xi')}{b_{11}(\xi')(\xi'-\xi)}\,d\xi' = 0$$

In the limit $R \to \infty$ the integrand along contour R approaches $-\delta^2_k$ (use 4.54), therefore

$$\int_C \frac{\chi_k(\xi')\,d\xi'}{b_{11}(\xi')(\xi'-\xi)} = i\pi\delta_k^2 \qquad 4.64$$

Using the contour $\overline{\text{C}}\text{U}\overline{\text{A}}$ one can show in a similar manner that

$$\int_{\bar{C}} \frac{\bar{\chi}_k(\xi')\,d\xi'}{b_{33}(\xi')(\xi'-\xi)} = i\pi\delta_k^2 \qquad 4.65$$

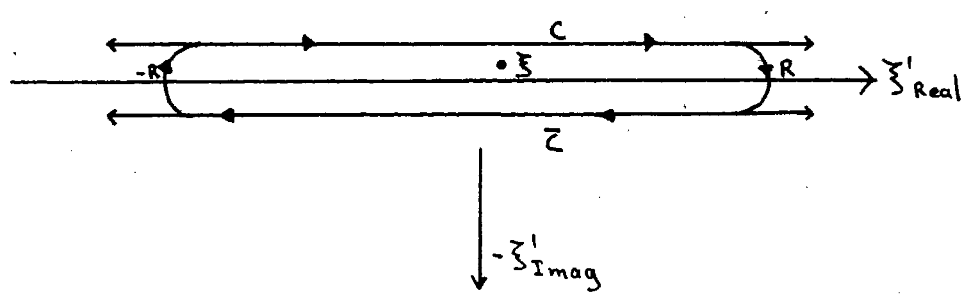
Combining 4.64 and 4.65 using 4.52 and 4.53 yields

$$\int_C \frac{b_{21}\psi_k^{(1)}}{b_{11}(\xi'-\xi)} e^{-i\xi'x/\alpha_2}\,d\xi' - \int_C \frac{\psi_k^{(2)}e^{-i\xi'x/\alpha_2}}{(\xi'-\xi)}\,d\xi' +$$

$$+ \int_{\bar{C}} \frac{\psi_k^{(2)}}{(\xi'-\xi)} e^{-i\xi'x/\alpha_2}\,d\xi' - \int_{\bar{C}} \frac{b_{23}\psi_k^{(3)}e^{-i\xi'x/\alpha_2}}{b_{33}(\xi'-\xi)}\,d\xi' = 2\pi i\delta_k^2 \qquad 4.66$$

The second and third integrals at 4.66 are evaluated by considering the following closed contour

Figure 4·2



We have

$$\int_{c\cup\bar{c}} \frac{\psi^{(2)}(\xi')}{(\xi'-\xi)} e^{-i\xi' x/\alpha_2} d\xi' = -2\pi i \, \psi^{(2)} e^{-i\xi x/\alpha_2} \qquad 4.67$$

An integral expression for $\psi^{(2)}$ can now be written as follows

$$\psi_k^{(2)} e^{-i\xi x/\alpha_2} = \delta_k^2 - \frac{1}{2\pi i}\int_c \frac{b_{21}\,\psi^{(1)} e^{-i\xi' x/\alpha_2}}{b_{11}(\xi'-\xi)} d\xi' +$$

$$+ \frac{1}{2\pi i}\int_{\bar{c}} \frac{b_{23}\,\psi^{(3)} e^{-i\xi' x/\alpha_2}}{b_{33}(\xi'-\xi)} d\xi' \qquad 4.68$$

The "fundamental" scattering coefficients are defined in 4.69 and substituted into the integral expressions for $\psi^{(i)}$ below.

$$\bar{\rho}_2 = \frac{a_{12}}{a_{11}} \qquad 4.69\,a$$

$$\bar{\rho}_3 = \frac{a_{13}}{a_{11}} \qquad b$$

$$\rho_1 = \frac{a_{31}}{a_{33}} \qquad c$$

$$\rho_2 = \frac{a_{32}}{a_{33}} \qquad d$$

$$\sigma = \frac{b_{21}}{b_{11}} \qquad 4.69e$$

$$\bar{\sigma} = \frac{b_{23}}{b_{33}} \qquad f$$

$$\psi_k^{(1)}(\xi,x)e^{-i\xi x/\alpha_1} = \delta_k^1 - \frac{1}{2\pi i}\int_{\bar{c}} \frac{d\xi' \; e^{-i\xi' x/\alpha_1}}{(\xi'-\xi)}\bar{\rho}_2(\xi')\psi_k^{(2)}(\xi',x)$$

$$-\frac{1}{2\pi i}\int_{\bar{c}} \frac{d\xi' \; e^{-i\xi' x/\alpha_1}}{(\xi'-\xi)}\bar{\rho}_3(\xi')\psi_k^{(3)}(\xi',x) \qquad 4.70$$

$$\psi_k^{(2)}(\xi,x)e^{-i\xi x/\alpha_2} = \delta_k^2 + \frac{1}{2\pi i}\int_{\bar{c}} \frac{d\xi' \; \bar{\sigma}(\xi')}{(\xi'-\xi)}e^{-i\xi' x/\alpha_2}\psi_k^{(3)}(\xi',x)$$

$$-\frac{1}{2\pi i}\int_{c} \frac{d\xi' \; \sigma(\xi')}{(\xi'-\xi)}e^{-i\xi' x/\alpha_2}\psi_k^{(1)}(\xi',x) \qquad 4.71$$

$$\psi_k^{(3)}(\xi,x)e^{-i\xi x/\alpha_3} = \delta_k^3 + \frac{1}{2\pi i}\int_{c} \frac{d\xi' \; e^{-i\xi' x/\alpha_3}}{(\xi'-\xi)}\rho_1(\xi')\psi_k^{(1)}(\xi',x)$$

$$+\frac{1}{2\pi i}\int_{c} \frac{d\xi' \; e^{-i\xi' x/\alpha_3}}{(\xi'-\xi)}\rho_2(\xi')\psi_k^{(2)}(\xi',x) \qquad 4.72$$

We are now ready to derive the inverse equations, i.e. the Marchenko linear integral equations.

Assume that $\psi^{(1)}$ can be written in the following form

$$\psi_k^{(1)}(\xi,x)\,e^{-i\xi x/\alpha_1} = \delta_k^1 + \int_x^\infty K_k^{(1)}(x,s)\,e^{i\xi(s-x)\beta_{12}}\,ds \qquad 4.73$$

Requiring that $\psi^{(1)}$, in this form, satisfy the linear eigenvalue equations 4.1 will put several constraints on the kernel $K^{(1)}(x,y)$ in 4.73. After some elaboration we have

$$\left[\frac{A\xi}{\alpha_1} + AV - \xi\right]\begin{pmatrix}1\\0\\0\end{pmatrix} e^{i\xi x/\alpha_1} + iAK^{(1)}(x,x)\,e^{i\xi x/\alpha_1} +$$

$$+\int_x^\infty ds\left[-iA\frac{\partial}{\partial x}K^{(1)}(x,s) + \frac{A\xi K^{(1)}(x,s)}{\alpha_2} + AVK^{(1)}(x,s) + \right.$$

$$\left. -\,\xi K^{(1)}(x,s)\right] e^{i\xi s\beta_{12}}\,e^{i\xi x/\alpha_2} = 0 \qquad 4.74$$

Note that the two terms

$$\frac{A\xi K^{(1)}}{\alpha_2} \qquad -\xi K^{(1)} \qquad 4.75$$

are equal to zero for the second component $K^{(1)}(x,s)$.

These two terms can be integrated by parts to give the following alternate expression

$$\int_x^\infty K_n^{(1)}(x,s)\, e^{i\xi s\beta_{12}}\, ds = \frac{i K_n^{(1)}}{\xi\beta_{12}}(x,x)\, e^{i\xi x\beta_{12}}$$

$$+ \int_x^\infty \frac{i}{\xi\beta_{12}} \frac{\partial K_n^{(1)}}{\partial s}(x,s) e^{i\xi s\beta_{12}}\, ds \qquad 4.76$$

where the following constraint is introduced

$$\lim_{s\to\infty} K_n^{(1)}(x,s) = 0 \qquad \text{for} \quad n = 1,3 \qquad 4.77$$

Substituting 4.76 into 4.74 we get

$$\left[\frac{A\xi}{\alpha_1} + AV - \xi\right]\begin{pmatrix}1\\0\\0\end{pmatrix} e^{i\xi x/\alpha_1} + \left[iA + \frac{iA}{\alpha_2\beta_{12}} - \frac{i}{\beta_{12}}\right] K^{(1)}(x,x) e^{i\xi x/\alpha_1} +$$

$$+ \int_x^\infty \left\{\left[-iA\partial_x + AV + \frac{iA}{\alpha_2\beta_{12}}\partial_s - \frac{iI}{\beta_{12}}\partial_s\right] K^{(1)}(x,s)\right\} e^{i\xi s\beta_{12}} e^{i\xi x/\alpha_2}\, ds =$$

$$= 0 \qquad 4.78$$

A second constraint is now imposed on $K^{(1)}(x,y)$ by requiring that the integrand of 4.78 be equal to zero. This constraint may be written as

$$\left[A(\partial_x + \partial_s) + \frac{1}{\beta_{12}}(I - \alpha_i^{-1}A)\partial_s\right] K^{(1)}(x,s) = -iAV(x)K^{(1)}(x,s) \qquad 4.79$$

It is this constraint that assures the existence and uniqueness of $K^{(1)}$. After a little algebra what remains in 4.78 constitutes another constraint viz.

$$K_n^{(1)}(x,x) = i\frac{\beta_{12}}{\beta_{1n}} V_{n1}(x) \qquad n = 2,3 \tag{4.80}$$

This constraint on $K^{(1)}$ will be very useful later on when the potentials $V_{ij}(x)$ are to be recovered from the kernels $K_j^{(i)}$.

Writing $\psi^{(3)}$ in the form

$$\Psi_k^{(3)}(\xi,x)e^{-i\xi x/\alpha_3} = \delta_k^3 + \int_x^\infty K_k^{(3)}(x,s)e^{-i\xi(s-x)\beta_{23}}\,ds \tag{4.81}$$

and proceeding along the same lines as above the following constraint on $K^{(3)}$ is obtained

$$K_n^{(3)}(x,x) = i\frac{\beta_{23}}{\beta_{n3}} V_{n3}(x) \qquad n = 1,2 \tag{4.82}$$

Our original goal was to obtain some equations relating the scattering coefficients to the scattering potential. We are finally ready to do so.

Eliminate the $\psi^{(2)}$ term in the integral expressions for $\psi^{(1)}$ and $\psi^{(3)}$ by substituting 4.71 into 4.70 and 4.72. Next, eliminate $\psi^{(1)}$ and $\psi^{(3)}$ by using expressions 4.73 and 4.81. Finally take the Fourier Transform of the resulting equations. The resultant integral equations (Marchenko type equations) are given below followed by the straightforward but lengthy derivation of the first integral equation.

$$K^{(1)}(x,y) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \bar{F}(y) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \bar{G}(x,y) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bar{H}(x,y) +$$

$$\int_x^\infty \left[K^{(1)}(x,s)\,\bar{G}(s,y) + K^{(3)}(x,s)\,\bar{H}(s,y) \right] ds = 0 \qquad 4.83$$

$$K^{(3)}(x,y) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} F(y) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} G(x,y) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} H(x,y) +$$

$$\int_x^\infty \left[K^{(1)}(x,s)\,H(s,y) + K^{(3)}(x,s)\,G(s,y) \right] ds = 0 \qquad 4.84$$

where

$$\bar{F}(x) = \frac{\beta_{12}}{2\pi} \int_{\bar{c}} \bar{\rho}_2(\xi)\, e^{-i\xi\beta_{12}x}\, d\xi \qquad 4.85$$

$$F(x) = \frac{\beta_{23}}{2\pi} \int_c \rho_2(\xi)\, e^{i\xi\beta_{23}x}\, d\xi \qquad 4.86$$

$$\bar{G}(x,y) = \frac{\beta_{12}}{4\pi^2 i} \int_c d\xi\, \sigma(\xi)\, e^{i\xi\beta_{12}x} \int_{\bar{c}} \frac{d\xi'}{\xi'-\xi}\, \bar{\rho}_2(\xi')\, e^{-i\xi'\beta_{12}y} \qquad 4.87$$

$$G(x,y) = -\frac{\beta_{23}}{4\pi^2 i} \int_{\bar{c}} d\xi\, \bar{\sigma}(\xi)\, e^{-i\xi\beta_{23}x} \int_c \frac{d\xi'}{\xi'-\xi}\, \rho_2(\xi')\, e^{i\xi'\beta_{23}y} \qquad 4.88$$

$$\bar{H}(x,y) = \frac{\beta_{12}}{2\pi}\int_{\bar{c}} d\xi\, \bar{\rho}_3(\xi)\, e^{-i\xi(\beta_{12}y + \beta_{23}x)}$$

$$-\frac{\beta_{12}}{4\pi^2 i}\int_{\bar{c}} d\xi\, \bar{\sigma}(\xi) e^{-i\xi\beta_{23}x}\int_{\bar{c}} \frac{d\xi'}{\xi'-\xi+i\varepsilon}\bar{\rho}_2(\xi')e^{-i\xi'\beta_{12}y} \qquad 4.89$$

$$H(x,y) = \frac{\beta_{23}}{2\pi}\int_{c} d\xi\, \rho_1(\xi)\, e^{i\xi(\beta_{12}x + \beta_{23}y)}$$

$$+\frac{\beta_{23}}{4\pi^2 i}\int_{c} d\xi\, e^{i\xi\beta_{12}x}\sigma(\xi)\int_{c} \frac{d\xi'}{\xi'-\xi-i\varepsilon}\rho_2(\xi')e^{i\xi'\beta_{23}y} \qquad 4.90$$

The term $\pm\, i\varepsilon$ in 4.89 and 4.90 is required to push the pole off the contour. The sign is chosen so that the integrals do not diverge at $\pm\,\infty$. (This will become clearer in the next chapter.)

Derivation of the first integral equation proceeds as follows. Substituting 4.71 into 4.70 gives

$$\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{2\pi i}\begin{pmatrix}0\\1\\0\end{pmatrix}\int_{\bar{c}} \frac{d\xi'}{\xi'-\xi} e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi') \;+$$

$$-\frac{1}{2\pi i}\int_{\bar{c}} \frac{d\xi'}{\xi'-\xi} e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi')\frac{1}{2\pi i}\int_{\bar{c}} \frac{d\xi''}{\xi''-\xi'}\bar{\sigma}(\xi'')e^{-i\xi'' x/\alpha_2}\psi^{(3)}(\xi'') \;+$$

$$-\frac{1}{2\pi i}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi')\left[\frac{-1}{2\pi i}\right]\int_{c}\frac{d\xi''}{\xi''-\xi'}\sigma(\xi'')e^{-i\xi'' x/\alpha_2}\psi^{(1)}(\xi'')$$

$$-\frac{1}{2\pi i}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x/\alpha_1}\bar{\rho}_3(\xi')\psi^{(3)}(\xi') = \psi^{(1)}(\xi)e^{-i\xi x/\alpha_1} \qquad 4.91$$

The first term on the LHS and the RHS of 4.91 may be expressed in terms of $K^{(1)}$ using 4.73.

Equation 4.91 then becomes

$$I_1 + I_2 + (I_{3a} + I_{3b}) + (I_{4a} + I_{4b}) + (I_{5a} + I_{5b}) = 0 \qquad 4.92$$

where

$$I_1 = \int_x^\infty K^{(1)}(x,s)\, e^{i\xi(s-x)\beta_{12}}\, ds \qquad 4.93a$$

$$I_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}\frac{1}{2\pi i}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi') \qquad b$$

$$I_{3a} + I_{3b} = \frac{1}{(2\pi i)^2}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi')\int_{\bar{c}}\frac{d\xi''}{\xi''-\xi'}\bar{\sigma}(\xi'')e^{-i\xi'' x/\alpha_2}\psi^{(3)}(\xi'') \qquad c$$

$$I_{4a} + I_{4b} = \frac{-1}{(2\pi i)^2}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x\beta_{12}}\bar{\rho}_2(\xi')\int_{c}\frac{d\xi''}{\xi''-\xi'}\sigma(\xi'')e^{-i\xi'' x/\alpha_2}\psi^{(1)}(\xi'') \qquad d$$

$$I_{5a} + I_{5b} = \frac{1}{2\pi i}\int_{\bar{c}}\frac{d\xi'}{\xi'-\xi}e^{-i\xi' x/\alpha_1}\bar{\rho}_3(\xi')\psi^{(3)}(\xi') \qquad e$$

Multiplying each term in 4.92 by

$$e^{i\xi(x-y)\beta_{12}}$$

and integrating with respect to ξ from $-\infty$ to ∞ gives

$$\int_{-\infty}^{\infty} I_1 e^{i\xi(x-y)\beta_{12}} d\xi = K^{(1)}(x,y)\frac{2\pi}{\beta_{12}} \tag{4.94}$$

$$\int_{-\infty}^{\infty} I_2 e^{i\xi(x-y)\beta_{12}} d\xi = \begin{pmatrix}0\\1\\0\end{pmatrix}\int_{\bar{C}} d\xi' e^{-i\xi' y\beta_{12}}\bar{\rho}_2(\xi') \tag{4.95}$$

Before Fourier transforming $I_{3a} + I_{3b}$ we first use 4.81 to express $\psi^{(3)}(\xi'')$ in terms of $K^{(3)}$. The I_{3a} term corresponds to the $\delta^3{}_k$ part and the I_{3b} term to the rest.

$$\int_{-\infty}^{\infty} I_{3a} e^{i\xi(x-y)\beta_{12}} d\xi = \frac{1}{2\pi i}\begin{pmatrix}0\\0\\1\end{pmatrix}\int_{\bar{C}} d\xi' e^{-i\xi' y\beta_{12}}\bar{\rho}_2(\xi')\int_{\bar{C}}\frac{d\xi''\,\bar{\sigma}(\xi'')}{\xi''-\xi'} e^{-i\xi'' x\beta_{23}} \tag{4.96}$$

It is necessary to push the pole ξ'' (or ξ') off the contour $\bar{C}$. This is done as follows

$$\int_{-\infty}^{\infty} I_{3a} e^{i\xi(x-y)\beta_{12}} d\xi = \frac{-1}{2\pi i}\begin{pmatrix}0\\0\\1\end{pmatrix}\int_{\bar{C}} d\xi\,\bar{\sigma}(\xi) e^{-i\xi x\beta_{23}}\int_{\bar{C}}\frac{d\xi'\,\bar{\rho}_2(\xi')}{\xi'-\xi+i\varepsilon} e^{-i\xi' y\beta_{12}} \tag{4.97}$$

$$\int_{-\infty}^{\infty} I_{3b} e^{i\xi(x-y)\beta_{12}} d\xi = \int_{x}^{\infty} ds\, K^{(3)}(x,s) \left[\frac{-1}{2\pi i} \int_{\bar{c}} d\xi\, \bar{\sigma}(\xi) e^{-i\xi x \beta_{23}} \int_{\bar{c}} \frac{d\xi' \bar{\rho}_2(\xi')}{\xi' - \xi + i\varepsilon} e^{-i\xi' y \beta_{12}} \right] \qquad 4\cdot 98$$

I_{4a} and I_{4b} follow in a similar manner.

$$\int_{-\infty}^{\infty} I_{4a} e^{i\xi(x-y)\beta_{12}} d\xi = \frac{1}{2\pi i} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \int_{c} d\xi\, \sigma(\xi) e^{i\xi x \beta_{12}} \int_{\bar{c}} \frac{d\xi' \bar{\rho}_2(\xi')}{\xi' - \xi} e^{-i\xi' \beta_{12} y} \qquad 4\cdot 99$$

$$\int_{-\infty}^{\infty} I_{4b} e^{i\xi(x-y)\beta_{12}} d\xi = \int_{x}^{\infty} ds\, K^{(1)}(x,s) \left[\frac{1}{2\pi i} \int_{c} d\xi\, \sigma(\xi) e^{i\xi \beta_{12} s} \int_{\bar{c}} \frac{d\xi'}{\xi' - \xi} \bar{\rho}_2(\xi') e^{-i\xi' y \beta_{12}} \right] \qquad 4\cdot 100$$

$I_{5a} + I_{5b}$ is split up by using 3.81 to express $\psi^{(2)}$.

$$\int_{-\infty}^{\infty} I_{5a} e^{i\xi(x-y)\beta_{12}} d\xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_{\bar{c}} d\xi' \bar{\rho}_3(\xi') e^{-i\xi'(\beta_{12} y + \beta_{23} x)} \qquad 4\cdot 101$$

$$\int_{-\infty}^{\infty} I_{5b} e^{i\xi(x-y)\beta_{12}} d\xi = \int_{x}^{\infty} ds\, K^{(3)}(x,s) \left[\int_{\bar{c}} e^{-i\xi' s \beta_{23}} e^{-i\xi' y \beta_{12}} \bar{\rho}_3(\xi') d\xi' \right] \qquad 4\cdot 102$$

Combining the above terms gives 4.83, 4.85, 4.87 and 4.89.

Due to the lengthy calculations that have taken place since the beginning of this chapter the reader may not realize that at this point equations for the direct linear problem and the inverse linear problem have been established. In the direct problem the scattering coefficients are found by solving equations 4.1. In the inverse problem the scattering potentials are found using the Marchenko integral equations, 4.83 and 4.84.

All that remains to be done is to calculate the time evolution operator and to show the connection between the scattering potentials, V_{ij}, and the three waves Q_i of the original nonlinear problem.

We begin by guessing the time evolution equation to be of the following form. (The following method is found in ref. [13].)

$$iv_t = \left[B_0 + B_1\xi\right] v \tag{4.103}$$

The choice of going only to first order in ξ is a result of the fact that the problem has already been worked through, previous to writing this thesis, finding that coefficients of higher orders are zero.

By differentiating the time evolution equation 4.103 with respect to x and the linear eigenvalue equation 4.1 with respect to t, the resulting two equations can be matched and then compared to the three wave equation, thereby B_0 and B_1, ~~and~~ the relation between V_{ij} and Q_i will be determined.

Differentiating 4.103 with respect to x and 4.1 with respect to t gives

$$iv_{tx} = \left[B_0 + B_1\xi\right]_x v + \left[B_0 + B_1\xi\right] v_x \tag{4.104}$$

$$-iAv_{xt} + AV_t v + AVv_t = \xi v_t \tag{4.105}$$

In 4.104 use 4.1 to express v_x in terms of v. Similarly use 4.103 in 4.105 to express v_t in terms of v. Eliminate v_{xt} from the two equations and set the coefficients of ξ^0, ξ^1 and ξ^2 equal to zero. The resulting equations are

$$\xi^0 : -AV_t v + iAVB_0 v + AB_{0,x} v - iAB_0 V v = 0 \qquad 4.106$$

$$\xi^1 : -iB_0 v + iAVB_1 v + AB_{1,x} v + iAB_0 A^{-1} v - iAB_1 V v = 0 \qquad 4.107$$

$$\xi^2 : iB_1 v - iAB_1 A^{-1} v = 0 \qquad 4.108$$

Equation 4.108 implies that B_1 is diagonal (recall that A is diagonal) i.e.

$$B_1 \equiv b_i \delta^i_j \qquad 4.109$$

Using 4.109 in 4.107 and looking at diagonal elements only it can be shown that,

$$b_{i,x} = 0 \qquad 4.110$$

Equation 4.107 now gives (using 4.109 and 4.110)

$$B_{0ij} = \alpha_i \alpha_j \left(\frac{b_i - b_j}{\alpha_i - \alpha_j} \right) V_{ij} \qquad 4.111$$

Define c_i such that

$$\frac{1}{\alpha_i} - \frac{1}{\alpha_j} = \beta_{ij} \equiv c_j - c_i \qquad 4.112$$

Without loss of generality and consistent with the ordering of α_i's, the c_i's are ordered as follows

$$c_1 < c_2 < c_3 \tag{4.113}$$

Rewriting 4.111 as

$$B_{oij} = \left(\frac{b_i - b_j}{c_i - c_j}\right) V_{ij} \tag{4.114}$$

and using it in 4.106 gives

$$V_{ij,t} - \left(\frac{b_i - b_j}{c_i - c_j}\right) V_{ij,x} = i \sum_m V_{im} V_{mj} \left[\left(\frac{b_m - b_j}{c_m - c_j}\right) - \left(\frac{b_m - b_i}{c_m - c_i}\right)\right] \tag{4.115}$$

We wish to compare equations 4.115 with equations 2.3 rewritten below

$$\begin{aligned} Q_{1,t} + c_1 Q_{1,x} &= i\gamma_1 Q_2^* Q_3^* \\ Q_{2,t} + c_2 Q_{2,x} &= i\gamma_2 Q_1^* Q_3^* \\ Q_{3,t} + c_3 Q_{3,x} &= i\gamma_3 Q_1^* Q_2^* \end{aligned} \tag{4.116 (2.3)}$$

The potentials V_{ij} have the following symmetry

$$V_{ij} = V_{ji}^* \varepsilon_i \varepsilon_j \tag{4.117}$$

where $\varepsilon_i = \pm 1$ depending on the type of three wave problem being solved.

Comparison of 4.116 and 4.115 using 4.117 yields

$$b_1 = -c_2c_1 - c_1c_3 + d$$

$$b_2 = -c_3c_2 - c_2c_1 + d \qquad 4.118$$

$$b_3 = -c_1c_3 - c_3c_2 + d$$

$$V_{23} = (\beta_{12}\beta_{13})^{-\frac{1}{2}} Q_1$$

$$V_{31} = (\beta_{12}\beta_{23})^{-\frac{1}{2}} Q_2 \qquad 4.119$$

$$V_{12} = (\beta_{13}\beta_{23})^{-\frac{1}{2}} Q_3$$

and therefore

$$B_{oij} = \frac{-c_1c_2c_3}{c_ic_j} V_{ij} + d_1 \delta^i_j \qquad 4.120$$

$$B_{1ij} = \delta^i_j \left(\frac{c_1c_2c_3}{c_j} + d_0 \right) \qquad 4.121$$

d_0 and d_1 are arbitrary constants.

We are esentially done. The inverse method has been established for the three wave problem. The results that will now be established are designed to make the inverse method easier to work with.

As all the Marchenko integral equations are in terms of the scattering coefficients it is useful to have equations for the time evolution of these coefficients. Recall the relation between the left hand and right solutions of $\phi^{(j)}$, i.e.

$$\lim_{x\to\infty} \phi_n^{(j)}(x) = \sum_k a_{jk} \lim_{x\to-\infty} \phi_n^{(k)}(x) \qquad 4.122$$

$$= a_{jn} e^{i\xi x/d_n}$$

where

$$\lim_{x\to-\infty} \phi_n^{(j)}(x) = \delta^j_n e^{i\xi x/d_j}$$

From the time evolution equation 4.103 with 4.120 and 4.121 we see that in the limit $x \to -\infty$ the time evolution of $\phi_n^{(j)}(x,t)$ is given by

$$\lim_{x\to-\infty} \phi_n^{(j)}(x,t) = \delta^j_n e^{i\xi x/d_j} e^{-i\xi c_1 c_2 c_3 t/c_j} \qquad 4.123$$

We define the time dependence of the scattering matrix $a_{ij}(t)$ so that in the limit $x \to \infty$, $\phi_n^{(j)}$ becomes,

$$\lim_{x\to\infty} \phi_n^{(j)}(x,t) = a_{jn}(t) e^{i\xi x/d_n} e^{-i\xi c_1 c_2 c_3 t/c_j} \qquad 4.124$$

This choice is arbitrary. Using 4.123 in 4.124 we have

$$\lim_{x\to\infty} \phi_n^{(j)}(x,t) = a_{jn}(0)\, e^{i\xi x/d_n}\, e^{-i\xi c_1c_2c_3t/c_n} \qquad 4.125$$

4.125 and 4.124 immediately give the time dependence of the scattering coefficients, i.e.

$$a_{ij}(t) = a_{ij}(0)\, e^{i\xi c_1c_2c_3t\left(\frac{1}{c_i}-\frac{1}{c_j}\right)} \qquad 4.126$$

By using 4.117 and comparing regular solutions of the linear eigen-value equation, 4.1, with solutions to the adjoint equation 4.27 we see that

$$W_n^A = d_n^{-1}\,\varepsilon_n\left[v_n(\xi^*)\right]^* \qquad 4.127$$

from which it can be shown that the scattering matrices satisfy the symmetry property. (Use 4.37 and 4.38.)

$$b_{ij}(\xi) = \varepsilon_i\varepsilon_j\left[a_{ji}(\xi^*)\right]^* \qquad 4.128$$

It follows from 4.128 that

$$\sigma(\xi) = \varepsilon_1\varepsilon_2\, \bar{\rho}_2^*(\xi^*) \qquad 4.129$$

$$\bar{\sigma}(\xi) = \varepsilon_2\varepsilon_3\, \rho_2^*(\xi^*) \qquad 4.130$$

$$\varepsilon_1 \rho_1(\xi) + \varepsilon_3 \bar{\rho}_3^*(\xi^*) + \varepsilon_2 \bar{\rho}_2^*(\xi^*)\, \rho_2(\xi) = 0 \qquad 4.131$$

$$\bar{G}(x,y)^* = \bar{G}(y,x) \qquad 4.132$$

$$G(x,y)^* = G(y,x) \qquad 4.133$$

$$H(x,y)^* = -\varepsilon_1 \varepsilon_3\, \beta_{23}^{-1} \beta_{12} H(y,x) \qquad 4.134$$

The time dependence of the scattering coefficients as defined in 4.126 yields

$$\bar{F}(x;t) = \bar{F}(x - c_3 t;0) \qquad 4.135$$

$$F(x;t) = F(x - c_1 t;0) \qquad 4.136$$

$$\bar{G}(x,y;t) = \bar{G}(x - c_3 t, y - c_3 t;0) \qquad 4.137$$

$$G(x,y;t) = G(x - c_1 t, y - c_1 t;0) \qquad 4.138$$

$$\bar{H}(x,y;t) = \bar{H}(x - c_1 t, y - c_3 t;0) \qquad 4.139$$

$$H(x,y;t) = H(x - c_3 t, y - c_1 t;0) \qquad 4.140$$

It is of interest to note the lack of dispersion in the scattering transforms unlike other investigated systems where so-called non-soliton or background solutions decay exponentially. Solitons and their non-occurrence in the three wave problem, backscattering case, will be discussed briefly in the final chapters.

This concludes the derivation of the inverse scattering method as applied to the three wave problem. A summary is in order.

To solve the nonlinear three wave equations 4.116 one begins by solving the associated linear eigenvalue problem 4.1 with asymptotic boundary conditions given by 4.4. The scattering potentials V_{ij} at $t = 0$ are determined from the initial three wave shapes Q_j using 4.119 and 4.117. The scattering coefficients a_{ij} at $t = 0$ can be determined from the solutions $\phi_n^{(j)}$ and 4.122. Fourier transforms of the fundamental scattering coefficients ρ_2, $\bar{\rho}_2$, σ, $\bar{\sigma}$, etc. are calculated using 4.85 to 4.90. Next, time dependence is introduced with the help of 4.135 to 4.140. The Marchenko integral equations, 4.83 and 4.84 are solved for the kernels $K_j^{(i)}(x,y,t)$. Finally using 4.80 and 4.82 the potentials V_{ij} and hence $Q_i(x,t)$ are determined.

The three wave problem is solved in principle. A solution is carried through as far as possible in the next chapters.

CHAPTER 5

STEPS 1 AND 2 - THE DIRECT LINEAR PROBLEM

The equations for SBS given in 2.6 and rewritten below, with the damping coefficients set equal to zero, will now be solved.

$$
\left.
\begin{aligned}
A_{s,t} + v_s A_{s,z} &= -\beta_s A_L \xi^* \\
A_{L,t} + v_L A_{L,z} &= \beta_L A_s \xi \\
\xi_{,t} + v_\xi \xi_{,z} &= -\beta_\xi A_L A_s^*
\end{aligned}
\right\} \quad 5.1
$$

Equations 5.1, transformed to the usual form of the three wave problem, 2.3, are also rewritten here.

$$
\begin{aligned}
Q_{1,t} + c_1 Q_{1,x} &= i\gamma_1 Q_2^* Q_3^* \\
Q_{2,t} + c_2 Q_{2,x} &= i\gamma_2 Q_1^* Q_3^* \\
Q_{3,t} + c_3 Q_{3,x} &= i\gamma_3 Q_1^* Q_2^*
\end{aligned}
\quad 5.2
$$

where for SBS we have

$$
\gamma_1 = \gamma_2 = -\gamma_3 = -1 \quad 5.3
$$

By definition the potentials, V_{ij} satisfy the following symmetry

$$
V_{ij} = \varepsilon_i \varepsilon_j V_{ji}^* \quad 5.4
$$

where

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \equiv (\gamma_1, -\gamma_2, \gamma_3) \qquad 5\cdot5$$

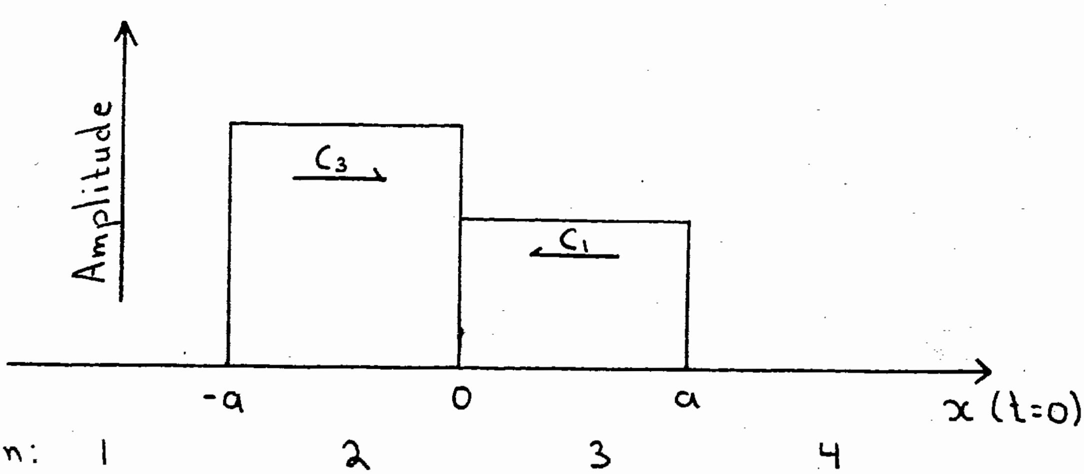
We take

$$-c_1 = c_3 = c/n \qquad 5\cdot6$$

i.e. the backscattering case in which the "pump" envelope, Q_3, travels to the right and the "signal" envelope, Q_1, passes through it going to the left.

The initial shapes and positions of the two laser beams are depicted below. The medium is assumed to be initially undisturbed. This is a good approximation to reality since the spontaneous fluctuations tend to be small compared to the laser induced ones (see Enns and Rangnekar).

Figure 5·1



As can be seen from the diagram the pulses are rectangular in profile, with arbitrary heights and identical lengths. Rectangular pulses were chosen because the linear eigenvalue problem (step 1 in the procedure) can be solved <u>exactly</u>. Furthermore, from earlier calculations of Enns and Rangnekar on stimulated thermal scattering, one can expect to obtain many of the qualitative features of nonlinear light scattering using rectangular pulses. Taking the velocity, v_2, characteristic of the fluctuations produced in the medium to be zero as already mentioned, is a very good approximation to physical reality. (The ordering of the velocities is consistent with 4.113.)

The initial pulse shapes (t = 0) are defined mathematically as follows

$$V_{31} = -V_{13} = 0 \tag{5.7a}$$

$$V_{12} = -V_{21} = \begin{Bmatrix} 0 & \text{for} \quad x < -a \quad \text{and} \quad x > 0 \\ H & \text{for} \quad -a < x < 0 \end{Bmatrix} \equiv H(x) \tag{b}$$

$$V_{23} = V_{32} = \begin{Bmatrix} 0 & \text{for} \quad x < 0 \quad \text{and} \quad x > a \\ h & \text{for} \quad 0 < x < a \end{Bmatrix} \equiv h(x) \tag{c}$$

where

$$Q_1 = (\beta_{12}\beta_{13})^{\frac{1}{2}} V_{23}$$

$$Q_2 = (\beta_{12}\beta_{23})^{\frac{1}{2}} V_{31} \tag{5.8}$$

$$Q_3 = (\beta_{13}\beta_{23})^{\frac{1}{2}} V_{12}$$

To solve the linear eigenvalue equations in step 1, 4.1 is written out

$$i u_{1x} + \frac{\xi}{\alpha_1} u_1 = H(x) u_2 \qquad 5.9\ a$$

$$i u_{2x} + \frac{\xi}{\alpha_2} u_2 = h(x) u_3 - H(x) u_1 \qquad b$$

$$i u_{3x} + \frac{\xi}{\alpha_3} u_3 = h(x) u_2 \qquad c$$

The general solution to equations 5.9 will be found by solving these equations in the four distinct regions defined by h(x) and H(x). Then continuity conditions will be imposed to eliminate the integration constants. Finally the boundary conditions will be used to define three linearly independent solutions of 5.9.

In region 1, $x < -a$, the solution is immediately found since the RHS of equations 5.9 are identically zero. The solution is

(1) $x < -a$

$$u_1^1 = u_{10}^1 e^{i\xi x/\alpha_1} \qquad 5.10\ a$$

$$u_2^1 = u_{20}^1 e^{i\xi x/\alpha_2} \qquad b$$

$$u_3^1 = u_{30}^1 e^{i\xi x/\alpha_3} \qquad c$$

where u_i^1 are the general solutions of equations 5.9 in region 1 and u_{io}^1 are arbitrary constants that will be determined when continuity and boundary conditions are imposed.

In region 2, -a < x < o, equations 5.9 become

(2) $-a < x < 0$

$$i u^2_{1,x} + \frac{\xi}{\alpha_1} u^2_1 = H u^2_2 \qquad 5.11\ a$$

$$i u^2_{2,x} + \frac{\xi}{\alpha_2} u^2_2 = -H u^2_1 \qquad b$$

$$i u^2_{3,x} + \frac{\xi}{\alpha_3} u^2_3 = 0 \qquad c$$

Equations 5.11 a and 5.11 b are easily decoupled and the resulting second order linear differential equation easily solved. The solution to equation 5.11 c, of course, is similar to the solution in region 1. The solutions are

(2) $-a < x < 0$

$$u^2_1 = a^2_1 e^{A_1 x} + b^2_1 e^{A_2 x} \qquad 5.12\ a$$

$$u^2_2 = a^2_2 e^{A_1 x} + b^2_2 e^{A_2 x} \qquad b$$

$$u^2_3 = u^2_{30} e^{i\xi x/\alpha_3} \qquad c$$

where

$$A_{\frac{1}{2}} = \frac{i\xi}{2}\left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right] \pm \frac{i\xi}{2}\left[\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right)^2 - \frac{4H^2}{\xi^2}\right]^{\frac{1}{2}} \qquad 5.13$$

and a^2_1, b^2_1, a^2_2, b^2_2 are arbitrary constants related as follows

$$a^2_2 = \left[i A_1 + \frac{\xi}{\alpha_1}\right] \frac{a^2_1}{H} \qquad 5.14a$$

or equivalently

$$a_1^2 = -\left[iA_1 + \frac{\xi}{\alpha_2}\right]\frac{a_2^2}{H} \qquad 5{\cdot}14b$$

and

$$b_2^2 = \left[iA_2 + \frac{\xi}{\alpha_1}\right]\frac{b_1^2}{H} \qquad 5{\cdot}15a$$

or equivalently

$$b_1^2 = -\left[iA_2 + \frac{\xi}{\alpha_2}\right]\frac{b_2^2}{H} \qquad 5{\cdot}15b$$

The solution in region 3, 0 < x < a, obtained in a similar manner to that in region 2 is

(3) $0 < x < a$

$$u_1^3 = u_{10}^3 \, e^{i\xi x/\alpha_1} \qquad 5{\cdot}16a$$

$$u_2^3 = a_2^3 e^{B_1 x} + b_2^3 e^{B_2 x} \qquad b$$

$$u_3^3 = a_3^3 e^{B_1 x} + b_3^3 e^{B_2 x} \qquad c$$

where

$$B_{1,2} \equiv \frac{i\xi}{2}\left[\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right] \pm \frac{i\xi}{2}\left[\left(\frac{1}{\alpha_2} - \frac{1}{\alpha_3}\right)^2 + \frac{4h^2}{\xi^2}\right]^{\frac{1}{2}} \qquad 5{\cdot}17$$

and the arbitrary constant a_2^3, b_3^3, a_3^3 and b_3^3 satisfy

$$a_3^3 = \left[iB_1 + \frac{\xi}{\alpha_2}\right]\frac{a_2^3}{h} \qquad 5{\cdot}18a$$

or equivalently

$$a_2^3 = \left[iB_1 + \frac{\xi}{\alpha_3} \right] \frac{a_3^3}{h} \qquad 5.18b$$

and

$$b_3^3 = \left[iB_2 + \frac{\xi}{\alpha_2} \right] \frac{b_2^3}{h} \qquad 5.19a$$

or equivalently

$$b_2^3 = \left[iB_2 + \frac{\xi}{\alpha_3} \right] \frac{b_3^3}{h} \qquad 5.19b$$

Finally in region 4, x < a, our solution is

(4) $\quad x < a$

$$u_1^4 = u_{10}^4 \, e^{i\xi x/\alpha_1} \qquad 5.20a$$

$$u_2^4 = u_{20}^4 \, e^{i\xi x/\alpha_2} \qquad b$$

$$u_3^4 = u_{30}^4 \, e^{i\xi x/\alpha_3} \qquad c$$

Of the 16 integration constants; 4 are determined from equations 5.14, 5.15, 5.18 and 5.19; 9 from the continuity conditions written below; and the remaining 3 from the boundary conditions.

Imposing continuity on the solutions gives the following equations for these constants.

at x = -a

$$u^1_{10}\, e^{-i\xi a/\alpha_1} = a^2_1\, e^{-A_1 a} + b^2_1\, e^{-A_2 a} \qquad 5.21a$$

$$u^1_{20}\, e^{-i\xi a/\alpha_2} = a^2_2\, e^{-A_1 a} + b^2_2\, e^{-A_2 a} \qquad b$$

$$u^1_{30}\, e^{-i\xi a/\alpha_3} = u^2_{30}\, e^{-i\xi a/\alpha_3} \qquad c$$

at x = 0

$$a^2_1 + b^2_1 = u^3_{10} \qquad d$$

$$a^2_2 + b^2_2 = a^3_2 + b^3_2 \qquad e$$

$$u^2_{30} = a^3_3 + b^3_3 \qquad f$$

at x = a

$$u^3_{10}\, e^{i\xi a/\alpha_1} = u^4_{10}\, e^{i\xi a/\alpha_1} \qquad g$$

$$a^3_2\, e^{B_1 a} + b^3_2\, e^{B_2 a} = u^4_{20}\, e^{i\xi a/\alpha_2} \qquad h$$

$$a^3_3\, e^{B_1 a} + b^3_3\, e^{B_2 a} = u^4_{30}\, e^{i\xi a/\alpha_3} \qquad i$$

Solving equations 5.14, 5.15, 5.18, 5.19 and 5.21 in terms of u^1_{10}, u^1_{20} and u^1_{30}, one gets

$$b^2_1 = \frac{e^{A_2 a}}{i(A_2 - A_1)}\left[H u^1_{20}\, e^{-i\xi a/\alpha_2} - \left(iA_1 + \frac{\xi}{\alpha_1}\right) u^1_{10}\, e^{-i\xi a/\alpha_1}\right] \qquad 5.22a$$

$$b^2_2 = \frac{e^{A_2 a}}{i(A_2 - A_1)}\left[\left(iA_2 + \frac{\xi}{\alpha_1}\right) u^1_{20}\, e^{-i\xi a/\alpha_2} - H u^1_{10}\, e^{-i\xi a/\alpha_1}\right] \qquad b$$

$$a^2_1 = \frac{e^{A_1 a}}{i(A_2 - A_1)}\left[\left(iA_2 + \frac{\xi}{\alpha_1}\right) u^1_{10}\, e^{-i\xi a/\alpha_1} - H u^1_{20}\, e^{-i\xi a/\alpha_2}\right] \qquad c$$

$$a_2^2 = \frac{e^{A_1 a}}{i(A_2 - A_1)}\left[H u_{10}^1 e^{-i\xi a/d_1} - \left(iA_1 + \frac{\xi}{d_1}\right) u_{20}^1 e^{-i\xi a/d_2}\right] \qquad 5.22d$$

$$u_{30}^2 = u_{30}^1 \qquad e$$

$$u_{10}^3 = a_1^2 + b_1^2 \qquad f$$

$$b_2^3 = \frac{1}{i(B_2 - B_1)}\left[h u_{30}^1 - \left(iB_1 + \frac{\xi}{d_2}\right)(a_2^2 + b_2^2)\right] \qquad g$$

$$b_3^3 = \frac{1}{i(B_2 - B_1)}\left[\left(iB_2 + \frac{\xi}{d_2}\right) u_{30}^1 + h(a_2^2 + b_2^2)\right] \qquad h$$

$$a_2^3 = \frac{1}{i(B_2 - B_1)}\left[-h u_{30}^1 + \left(iB_2 + \frac{\xi}{d_2}\right)(a_2^2 + b_2^2)\right] \qquad i$$

$$a_3^3 = \frac{1}{i(B_2 - B_1)}\left[-\left(iB_1 + \frac{\xi}{d_2}\right) u_{30}^1 - h(a_2^2 + b_2^2)\right] \qquad j$$

$$u_{10}^4 = a_1^2 + b_1^2 \qquad k$$

$$u_{20}^4 = (a_2^3 e^{B_1 a} + b_2^3 e^{B_2 a})\, e^{-i\xi a/d_2} \qquad l$$

$$u_{30}^4 = (a_3^3 e^{B_1 a} + b_3^3 e^{B_2 a})\, e^{-i\xi a/d_3} \qquad m$$

We continue now by imposing boundary conditions 4.4 to determine the three linearly independent solutions $\phi^{(j)}$. The asymptotic boundary conditions 4.4 are rewritten below.

$$\phi_n^{(j)}(\xi, x) \to \delta_n^j e^{i\xi x/d_j} \quad \text{as} \quad x \to -\infty \qquad 5.23$$

Setting j = 1 in 5.23, $u^1_{10} = 1$, $u^1_{20} = 0$ and $u^1_{30} = 0$ hence $\phi^{(1)}$ is given as follows

$$\phi_1^{(1)} = \begin{cases} e^{i\xi x/\alpha_1} & x<-a \quad 5.24\,a \\ a_1^2 e^{A_1 x} + b_1^2 e^{A_2 x} & -a<x<0 \quad b \\ u_{10}^3 e^{i\xi x/\alpha_1} & 0<x<a \quad c \\ u_{10}^4 e^{i\xi x/\alpha_1} & a<x \quad d \end{cases}$$

$$\phi_2^{(1)} = \begin{cases} 0 & x<-a \quad 5.25\,a \\ a_2^2 e^{A_1 x} + b_2^2 e^{A_2 x} & -a<x<0 \quad b \\ a_2^3 e^{B_1 x} + b_2^3 e^{B_2 x} & 0<x<a \quad c \\ u_{20}^4 e^{i\xi x/\alpha_2} & a<x \quad d \end{cases}$$

$$\phi_3^{(1)} = \begin{cases} 0 & x<-a \quad 5.26\,a \\ u_{30}^2 e^{i\xi x/\alpha_3} & -a<x<0 \quad b \\ a_2^3 e^{B_1 x} + b_3^3 e^{B_2 x} & 0<x<a \quad c \\ u_{30}^4 e^{i\xi x/\alpha_3} & a<x \quad d \end{cases}$$

$$b_1^2 = -\left(iA_1 + \frac{\xi}{\alpha_1}\right)\frac{e^{(A_2 - i\xi/\alpha_1)a}}{i(A_2 - A_1)} \qquad 5.27\,a$$

$$b_2^2 = -\frac{H e^{(A_2 - i\xi/a_1)a}}{i(A_2 - A_1)} \qquad b$$

$$a_1^2 = \frac{e^{(A_1 - i\xi/\alpha_1)a}}{i(A_2 - A_1)}\left(iA_2 + \frac{\xi}{\alpha_1}\right) \qquad c$$

$$a_2^2 = \frac{H e^{(A_1 - i\xi/a_1)a}}{i(A_2 - A_1)} \qquad d$$

$$u_{30}^2 = 0 \qquad e$$

$$u^3_{10} = \frac{1}{i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_1)a}-\left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_1)a}\right] \qquad 5.27\ f$$

$$b^3_2 = \frac{-1}{i(B_2-B_1)}\left(iB_1+\frac{\xi}{\alpha_1}\right)\frac{H}{i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right] \qquad g$$

$$b^3_3 = \frac{hH}{i(B_2-B_1)\,i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right] \qquad h$$

$$a^3_2 = \frac{1}{i(B_2-B_1)}\left(iB_2+\frac{\xi}{\alpha_2}\right)\frac{H}{i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right] \qquad i$$

$$a^3_3 = \frac{-h}{i(B_2-B_1)}\left(iB_2+\frac{\xi}{\alpha_2}\right)\frac{H}{i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right] \qquad j$$

$$u^4_{10} = \frac{1}{i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_1)a}-\left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_1)a}\right] \qquad k$$

$$u^4_{20} = \frac{H}{i(B_2-B_1)\,i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right]$$

$$\left[\left(iB_2+\frac{\xi}{\alpha_2}\right)e^{(B_1-i\xi/\alpha_2)a}-\left(iB_1+\frac{\xi}{\alpha_1}\right)e^{(B_2-i\xi/\alpha_2)a}\right] \qquad l$$

$$u^4_{30} = \frac{-hH}{i(B_2-B_1)\,i(A_2-A_1)}\left[e^{(A_1-i\xi/\alpha_1)a}-e^{(A_2-i\xi/\alpha_1)a}\right]$$

$$\left[e^{(B_1-i\xi/\alpha_3)a}-e^{(B_2-i\xi/\alpha_3)a}\right] \qquad m$$

In a similar manner $\phi^{(2)}$ and $\phi^{(3)}$ are found to be

$$
\phi_1^{(2)} = \begin{cases} 0 & x < -a \\ a_1^2 e^{A_1 x} + b_1^2 e^{A_2 x} & -a < x < 0 \\ u_{10}^3 e^{i\xi x/\alpha_1} & 0 < x < a \\ u_{10}^4 e^{i\xi x/\alpha_1} & a < x \end{cases}
\qquad
\begin{matrix} 5.28\ a \\ b \\ c \\ d \end{matrix}
$$

$$
\phi_2^{(2)} = \begin{cases} e^{i\xi x/\alpha_2} & x < -a \\ a_2^2 e^{A_1 x} + b_2^2 e^{A_2 x} & -a < x < 0 \\ a_2^3 e^{B_1 x} + b_2^3 e^{B_2 x} & 0 < x < a \\ u_{20}^4 e^{i\xi x/\alpha_2} & a < x \end{cases}
\qquad
\begin{matrix} 5.29\ a \\ b \\ c \\ d \end{matrix}
$$

$$
\phi_3^{(2)} = \begin{cases} 0 & x < -a \\ u_{30}^2 e^{i\xi x/\alpha_3} & -a < x < 0 \\ a_3^3 e^{B_1 x} + b_3^3 e^{B_2 x} & 0 < x < a \\ u_{10}^4 e^{i\xi x/\alpha_3} & a < x \end{cases}
\qquad
\begin{matrix} 5.30\ a \\ b \\ c \\ d \end{matrix}
$$

$$b_1^2 = \frac{H}{i(A_2 - A_1)} e^{(A_2 - i\xi/\alpha_2)a} \tag{5.31 a}$$

$$b_2^2 = \left(iA_2 + \frac{\xi}{\alpha_1}\right) \frac{e^{(A_2 - i\xi/\alpha_2)a}}{i(A_2 - A_1)} \tag{b}$$

$$a_1^2 = \frac{-H}{i(A_2 - A_1)} e^{(A_1 - i\xi/\alpha_2)a} \tag{c}$$

$$a_2^2 = -\left(iA_1 + \frac{\xi}{\alpha_1}\right) \frac{e^{(A_1 - i\xi/\alpha_2)a}}{i(A_2 - A_1)} \tag{d}$$

$$u_{30}^2 = 0 \tag{e}$$

$$u_{10}^{3} = \frac{H}{i(A_2-A_1)}\left[-e^{(A_1-i\xi/\alpha_2)a} + e^{(A_2-i\xi/\alpha_2)a}\right] \quad 5.31\,f$$

$$b_2^3 = -\left(iB_1+\frac{\xi}{\alpha_2}\right)\frac{1}{i(B_2-B_1)i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_2)a} - \left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_2)a}\right] \quad g$$

$$b_3^3 = \frac{h}{i(B_2-B_1)i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_2)a} - \left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_2)a}\right] \quad h$$

$$a_2^3 = \left(iB_2+\frac{\xi}{\alpha_2}\right)\frac{1}{i(B_2-B_1)i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_2)a} - \left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_2)a}\right] \quad i$$

$$a_3^3 = -b_3^3 \quad j$$

$$u_{10}^4 = u_{10}^3 \quad k$$

$$u_{20}^4 = \frac{1}{i(B_2-B_1)i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_2)a} - \left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_2)a}\right]\left[\left(iB_2+\frac{\xi}{\alpha_2}\right)e^{(B_1-i\xi/\alpha_2)a} - \left(iB_1+\frac{\xi}{\alpha_2}\right)e^{(B_2-i\xi/\alpha_2)a}\right] \quad l$$

$$u_{30}^4 = \frac{h}{i(B_2-B_1)i(A_2-A_1)}\left[\left(iA_2+\frac{\xi}{\alpha_1}\right)e^{(A_2-i\xi/\alpha_2)a} - \left(iA_1+\frac{\xi}{\alpha_1}\right)e^{(A_1-i\xi/\alpha_2)a}\right]\left[-e^{(B_1-i\xi/\alpha_3)a} + e^{(B_2-i\xi/\alpha_3)a}\right] \quad m$$

$$\phi_1^{(3)} = \begin{cases} 0 & x < -a & 5.33\,a \\ a_1^2 e^{A_1 x} + b_1^2 e^{A_2 x} & -a < x < 0 & b \\ u_{10}^3 e^{i\xi x/d_1} & 0 < x < a & c \\ u_{10}^4 e^{i\xi x/d_1} & a < x & d \end{cases}$$

$$\phi_2^{(3)} = \begin{cases} 0 & x < -a & 5.34\,a \\ a_2^2 e^{A_1 x} + b_2^2 e^{A_2 x} & -a < x < 0 & b \\ a_2^3 e^{B_1 x} + b_2^3 e^{B_2 x} & 0 < x < a & c \\ u_{20}^4 e^{i\xi x/d_2} & a < x & d \end{cases}$$

$$\phi_3^{(3)} = \begin{cases} e^{i\xi x/d_3} & x < -a & 5.35\,a \\ u_{30}^2 e^{i\xi x/d_3} & -a < x < 0 & b \\ a_3^3 e^{B_1 x} + b_3^3 e^{B_2 x} & 0 < x < a & c \\ u_{30}^4 e^{i\xi x/d_3} & a < x & d \end{cases}$$

$$b_1^2 = b_2^2 = 0 \qquad 5.35\,a,b$$

$$a_1^2 = a_2^2 = 0 \qquad c,d$$

$$u_{30}^2 = 1 \qquad e$$

$$u_{10}^3 = 0 \qquad f$$

$$b_2^3 = \frac{h}{i(B_2 - B_1)} \qquad g$$

$$b_3^3 = \left(iB_2 + \frac{\xi}{d_2}\right)\frac{1}{i(B_2 - B_1)} \qquad h$$

$$a_2^3 = \frac{-h}{i(B_2 - B_1)} \qquad i$$

$$a_3^3 = -\left(iB_1 + \frac{\xi}{d_2}\right)\frac{1}{i(B_2 - B_1)} \qquad j$$

$$u^4_{10} = 0 \qquad 5.35k$$

$$u^4_{20} = \frac{-h}{i(B_2 - B_1)}\left[e^{(B_1 - i\xi/\alpha_2)a} - e^{(B_2 - i\xi/\alpha_2)a}\right] \qquad l$$

$$u^4_{30} = \frac{1}{i(B_2 - B_1)}\left[-\left(iB_1 + \frac{\xi}{\alpha_2}\right)e^{(B_1 - i\xi/\alpha_3)a} + \left(iB_2 + \frac{\xi}{\alpha_2}\right)e^{(B_2 - i\xi/\alpha_3)a}\right] \qquad m$$

Using (see 4.122)

$$\lim_{x\to\infty} \phi_j^{(i)} = a_{ij}\, e^{i\xi x/\alpha_j} \qquad 5.36$$

it is a simple matter to find a_{ij} from our solutions $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$. These scattering coefficients are given as

$$a_{11} = \frac{e^{-i\xi\beta_{12}a/2}}{\Delta_H}\left[\beta_{12}\, i \sin(\xi\Delta_H a/2) + \Delta_H \cos(\xi\Delta_H a/2)\right] \qquad 5.37a$$

$$a_{12} = \frac{e^{-i\xi\beta_{12}a/2}\, e^{-i\xi\beta_{23}a/2}\, 2H\left(i\sin(\xi\Delta_H a/2)\right)}{\Delta_H \Delta_h \xi}\left[\beta_{23}\, i\sin(\xi\Delta_h a/2) + \Delta_h \cos(\xi\Delta_h a/2)\right] \qquad b$$

$$a_{13} = \frac{+4hH\, e^{-i\xi\beta_{12}a/2}}{\xi^2 \Delta_H \Delta_h}\, e^{i\xi\beta_{23}a/2}\left[i\sin(\xi\Delta_H a/2)\right]\left[i\sin(\xi\Delta_h a/2)\right] \qquad c$$

$$a_{21} = \frac{-2H\, e^{i\xi\beta_{12}a/2}}{\xi\Delta_H}\left[i\sin(\xi\Delta_H a/2)\right] \qquad d$$

$$a_{22} = \frac{-e^{i\xi\beta_{12}a/2}}{\Delta_H \Delta_h} e^{-i\xi\beta_{23}a/2}\left[\beta_{12} i \sin(\xi\Delta_H a/2) - \Delta_H \cos(\xi\Delta_H a/2)\right]$$

$$\left[\beta_{23} i \sin(\xi\Delta_h a/2) + \Delta_h \cos(\xi\Delta_h a/2)\right] \qquad 5.37e$$

$$a_{23} = \frac{2h\ e^{i\xi\beta_{12}a/2}}{\xi\Delta_h\Delta_H} e^{i\xi\beta_{23}a/2}\left[\beta_{12} i \sin(\xi\Delta_H a/2) - \Delta_H \cos(\xi\Delta_H a/2)\right]$$

$$\left[i \sin(\xi\Delta_h a/2)\right] \qquad f$$

$$a_{31} = 0 \qquad g$$

$$a_{32} = -\frac{2h\ e^{-i\xi\beta_{23}a/2}}{\xi\Delta_h}\left[i \sin(\xi\Delta_h a/2)\right] \qquad h$$

$$a_{33} = \frac{-e^{i\xi\beta_{23}a/2}}{\Delta_h}\left[\beta_{23} i \sin(\xi\Delta_h a/2) - \Delta_h \cos(\xi\Delta_h a/2)\right] \qquad i$$

From 5.37 the fundamental scattering coefficients are readily determined to be

$$\bar{\rho}_2 = \frac{2He^{-i\xi\beta_{23}a/2}\left[i\sin(\xi\Delta_H a/2)\right]\left[\beta_{23} i \sin(\xi\Delta_h a/2) + \Delta_h \cos(\xi\Delta_h a/2)\right]}{\xi\Delta_H\left[\beta_{12} i \sin(\xi\Delta_H a/2) + \Delta_H \cos(\xi\Delta_H a/2)\right]} \qquad 5.38$$

$$\bar{\rho}_3 = \frac{4hHe^{i\xi\beta_{23}a/2}\left[i\sin(\xi\Delta_H a/2)\right]\left[i \sin(\xi\Delta_h a/2)\right]}{\xi^2\Delta_h\left[\beta_{12} i \sin(\xi\Delta_H a/2) + \Delta_H \cos(\xi\Delta_H a/2)\right]} \qquad 5.39$$

$$\rho_1 = 0 \qquad 5.40$$

$$\rho_2 = \frac{2h e^{-i\xi\beta_{23}a}\left[i \sin(\xi\Delta_h a/2)\right]}{\xi\left[\beta_{23} i \sin(\xi\Delta_h a/2) - \Delta_h \cos(\xi\Delta_h a/2)\right]} \qquad 5.41$$

$$\sigma = \frac{-2He^{i\xi\beta_{23}\frac{a}{2}}\left[-i\sin(\xi\Delta_H a/2)\right]\left[-i\beta_{23}\sin(\xi\Delta_h a/2) + \Delta_h\cos(\xi\Delta_h a/2)\right]}{\xi\Delta_h\left[-i\beta_{12}\sin(\xi\Delta_H a/2) + \Delta_H\cos(\xi\Delta_H a/2)\right]} \qquad 5.42$$

$$\bar{\sigma} = \frac{2h e^{i\xi\beta_{23}a}\left[-i\sin(\xi\Delta_H a/2)\right]}{\xi\left[i\beta_{23}\sin(\xi\Delta_h a/2) + \Delta_h\cos(\xi\Delta_h a/2)\right]} \qquad 5.43$$

where in 5.37 to 5.43 I have used the following notation.

$$A_{\frac{1}{2}} \equiv \frac{i\xi}{2}\left[-\beta_{12} \pm \Delta_H\right] + \frac{i\xi}{d_1} \qquad 5.44a$$

$$B_{\frac{1}{2}} \equiv \frac{i\xi}{2}\left[-\beta_{23} \pm \Delta_h\right] + \frac{i\xi}{d_2} \qquad b$$

$$\Delta_H = \left[\beta_{12}^2 - \frac{4H^2}{\xi^2}\right]^{\frac{1}{2}} \quad \text{(principle value)} \qquad 5.45a$$

$$\Delta_h = \left[\beta_{23}^2 + \frac{4h^2}{\xi^2}\right]^{\frac{1}{2}} \quad \text{(principle value)} \tag{5.45b}$$

Step 1 is now completed, the linear eigenvalue equation has been solved and the fundamental scattering coefficients have been determined. Before proceeding to step 2, the time evolution step, Fourier transforms of the scattering coefficients will be taken, since our theory was developed to deal with these transforms, i.e. we must evaluate $\bar{F}$, F, $\bar{G}$, G, etc. In the spirit of this thesis we shall try to push our analytical approach through as far as possible, trying to obtain exact expressions for $\bar{F}$, etc, using contour integration.

We begin by calculating the Fourier transform of $\bar{\rho}_2(\xi)$ i.e. $\bar{F}(x)$. From 4.85 we have

$$\bar{F}(x) = \frac{\beta_{12}}{2\pi}\int_{\bar{C}} \bar{\rho}_2(\xi)\, e^{-i\xi\beta_{12}x}\, d\xi \tag{5.46}$$

where $\bar{C}$ is the contour extending from $-\infty - i\varepsilon$ to $\infty - i\varepsilon$. Define I_1, J_1 and J_1' as follows

$$I_1 = \frac{\bar{\rho}_2}{2H} \tag{5.47}$$

$$J_1 = \frac{I_1\, e^{i\xi\Delta_H a/2}}{2i\sin(\xi\Delta_H a/2)} \tag{5.48a}$$

$$J_1' = \frac{-I_1\, e^{-i\xi\Delta_H a/2}}{2i\sin(\xi\Delta_H a/2)} \tag{b}$$

where, of course, $J_1 + J_1' = I_1$. $\overline{F}$ may then be rewritten as

$$\overline{F} = \frac{\beta_{12} H}{\pi} \int_{\overline{C}} (J_1 + J_1') e^{-i\xi\beta_{12}x} \, d\xi \qquad 5.49$$

$\overline{F}$ will be calculated using contour integration techniques. Do we complete the contour $\overline{C}$ with a semicircle in the upper half ξ-plane, A, or with a semicircle in the lower half ξ-plane, $\overline{A}$? The answer to this question, as will now be shown, depends on the variable x.

Consider the integrand of $\overline{F}$ in the limit $R \to \infty$ where $\xi = Re^{i\theta}$. Within constant factors the integrand goes as

$$\text{Integrand} \sim \lim_{R \to \infty} \frac{e^{-i\xi\beta_{12}x}}{\xi} \left[1 - e^{-i\xi\beta_{12}a} \right] \qquad 5.50$$

To have convergence of the integral for $\overline{F}$ we require

$$\mathrm{Re}\left[-i\xi\beta_{12}x\right] < 0 \qquad \text{and} \qquad \mathrm{Re}\left[-i\xi\beta_{12}(x+a)\right] < 0 \qquad 5.51a$$

$$\text{ie} \quad x \sin\theta < 0 \qquad \text{and} \qquad (x+a)\sin\theta < 0 \qquad b$$

The first term of 5.51 comes from J_1 and the second term from J_1'. From 5.51 the following table is constructed indicating the half ξ-plane and the corresponding range of x for which the integral converges.

Table 1

	J_1	J_1'
$x < -a$	U	U
$-a < x < 0$	U	L
$0 < x$	L	L

where U ≡ "upper half ξ-plane" indicating θ in 5.51 between 0 and π and L ≡ lower half ξ-plane" indicating $\theta \epsilon (\pi, 2\pi)$. Note that it is necessary to split up the integrand for $\overline{F}$ into two parts when $-a < x < 0$.

It can be seen from 5.50 that the integrand of $\overline{F}$ is of the form

$$\text{Integrand} \sim \frac{f(R)}{R} \qquad 5.52$$

where

$$f(R) \to 0 \quad \text{as} \quad R \to \infty \qquad 5.53$$

hence the contour integral on A (or $\overline{A}$) goes to zero as R goes to infinity. The mathematical demonstration is not totally correct as I have ignored the existence of poles in which case the limit in 5.53 must jump around the poles.

There are in fact an infinite number of poles associated with the integrand.

To find the poles, look at the zeros of the denominator of the integrand. The denominator, D, is

$$D(\xi) = \xi \Delta_h \left[\beta_{12} i \sin(\xi \Delta_H a/2) + \Delta_H \cos(\xi \Delta_H a/2) \right] \qquad 5.54$$

Upon inspection of 5.54 one might guess that $\xi = 0$ and $\Delta_h = 0$ and $\Delta_H = 0$ are zeros. More careful examination will show $\xi = 0$ is not a zero of $D(\xi)$ ($\lim_{\xi \to 0} \xi\Delta_h$ = finite number). ξ such that $\Delta_h = 0$ and ξ such that $\Delta_H = 0$ are true zeros of $D(\xi)$ but it can be shown that the numerator also goes to zero at these points. Next, using L 'Hospital's rule it can be shown that $\Delta_h = 0$ and $\Delta_H = 0$ are not poles. The question asked, "does the integrand have a branch cut"? is answered in the negative. Series expansion of the integrand about $\xi\Delta_h$ or $\xi\Delta_H$ will show that only terms of order $(\xi\Delta_h)^{2n}$ or $(\xi\Delta_H)^{2n}$ where n is an integer survive hence no branch cut terms like $\xi\Delta_H = \xi\left[\beta_{12}^2 - \frac{4H^2}{\xi^2}\right]^{\frac{1}{2}}$ occur.

At this point one can use the computer to find the zeros of the integrand. But there will be no assurance that all the poles have been obtained. Further examination of the denominator is called for.

We wish to find the roots of the following equation

$$i\beta_{12}\sin(\xi\Delta_H a/2) + \Delta_H\cos(\xi\Delta_H a/2) = 0 \qquad 5.55$$

where
$$\Delta_H = \left[\beta_{12}^2 - \frac{4H^2}{\xi^2}\right]^{\frac{1}{2}}$$

After several algebraic manipulations equation 5.55 can be written as

$$\frac{\beta_{12}^2\xi^2}{4H^2} = \cos^2\left[\left(\frac{\beta_{12}^2\xi^2 a^2}{4} - a^2H^2\right)^{\frac{1}{2}}\right] \qquad 5.56$$

Introducing z such that

$$z = \frac{\beta_{12}\xi}{2H} \qquad 5.57$$

5.56 becomes

$$z^2 = \cos^2\left(aH(z^2-1)^{\frac{1}{2}}\right) \qquad 5.58$$

Now introduce w such that

$$z = \left[1 + \frac{w^2}{a^2H^2}\right]^{\frac{1}{2}} \qquad 5.59$$

or

$$\xi = \frac{2H}{\beta_{12}}\left[1 + \frac{W^2}{a^2H^2}\right]^{\frac{1}{2}} \qquad 5\text{-}60$$

so that equation 5.58 becomes

$$W = \pm iaH \sin W \qquad 5\text{-}61$$

By separating 5.61 into real and imaginary parts the following two coupled transcendental equations are obtained.

$$\pm x = aH \cos x \sinh y \qquad 5\text{-}62a$$

$$\mp y = aH \sin x \cosh y \qquad b$$

where

$$W \equiv x + iy$$

Solving 5.62 b for x we have

$$x = \sin^{-1}\left[\frac{\mp y}{aH \cosh y}\right] \qquad 5\text{-}63$$

where

$$\sin^{-1}\theta = \begin{cases} \arcsin\theta + 2n\pi \\ (2n+1)\pi - \arcsin\theta \end{cases} \qquad n = 0,1,2,\ldots \qquad 5\text{-}64$$

and arcsin is defined as the principal value of $\sin^{-1}$. For each $n \geq 1$†, or branch of the $\sin^{-1}$ function, two roots of equation 5.63 and 5.62 a may be found (one for each sign -, + in 5.63). It can also be seen that for relatively large y

$$x \sim m\pi \qquad m = 1,2,3,\ldots$$

$$y \sim \ln\left(\frac{m\pi}{aH}\right) \qquad 5\text{-}65$$

† for n = 0 only 1 root is used as $\xi = 0$ is not a pole.

Both x and y, roots of 5.62 and 5.63, go to infinity, but y approaches infinity much slower than x. This will create a problem numerically. When the residues are summed their convergence will be very slow, as more poles are included.

It is not difficult to show that the zeros are simple poles of the integrand, lying in the upper half ξ-plane and occurring in pairs $\xi_1^i, -\xi_1^{i*}$. Since the poles are simple the residue of $I_1(\xi)$ is given by

$$P_1(\xi) \equiv \frac{2\Delta_H e^{-i\xi\beta_{23}a/2}\left[i\sin(\xi\Delta_H a/2)\right]\left[\beta_{23} i\sin(\xi\Delta_h a/2) + \Delta_h\cos(\xi\Delta_h a/2)\right]}{\xi\Delta_h\left[ia\beta_{12}^3\cos(\xi\Delta_H a/2) - a\beta_{12}^2\Delta_H\sin(\xi\Delta_H a/2) + (8H^2/\xi^3)\cos(\xi\Delta_H a/2)\right]} \quad 5.66$$

The residues of J_1 and J_1' are defined in a similar manner. The residue of $J_1(J_1')$ is $Q_1(Q_1')$ where

$$Q_1(\xi) = \frac{e^{i\xi\Delta_H a/2}\, P_1(\xi)}{2i\sin(\xi\Delta_H a/2)} \quad 5.67a$$

and

$$Q_1'(\xi) = -\frac{e^{-i\xi\Delta_H a/2}\, P_1(\xi)}{2i\sin(\xi\Delta_H a/2)} \quad b$$

With the aid of table 1, $\overline{F}(x)$ is evaluated, using contour integration techniques, to be: (all poles are in upper half ξ-plane).

$$x < -a \quad \overline{F}(x) = \sum_{\xi_1^i} 2i\beta_{12} H\left[P_1(\xi_1^i)e^{-i\xi_1^i\beta_{12}x} + cc\right] \quad 5.68a$$

$$= \sum_{\xi_1^i} 2i\beta_{12} H e^{y_i\beta_{12}x}\left[P_1(\xi_1^i)e^{-ix_i\beta_{12}x} + cc\right] \quad b$$

$$-a<x<0 \quad \bar{F}(x) = \sum_{\xi_i} 2i\beta_{12} H\left[Q_1(\xi_1^i)e^{-i\xi_1^i\beta_{12}x} + cc\right] \qquad 5.69a$$

$$= \sum_{\xi_i} 2i\beta_{12} H e^{y_1^i\beta_{12}x}\left[Q_1(\xi_1^i)e^{-ix_1^i\beta_{12}x} + cc\right] \qquad b$$

$$0<x \quad \bar{F}(x) = 0 \qquad 5.70$$

where c.c $\equiv$ complex conjugate; $\xi_1^i \equiv x_1^i + iy_1^i$ is the i^{th} pole of $I_1(\xi)$ in the 1st quadrant of the complex ξ-plane; $\sum_{\xi_i}$ is sum over all poles ξ_1^i.

Evaluation of F(x), rewritten below, is similar to $\bar{F}(x)$.

$$F(x) = \frac{\beta_{23}}{2\pi}\int_C \rho_2(\xi)e^{i\xi\beta_{23}x}\,d\xi \qquad 5.71$$

where C is the contour $-\infty + i\varepsilon$ to $+\infty + i\varepsilon$.

We define I_2, J_2 and J_2' is a manner similar to I_1, J_1 and J_1' as follows

$$I_2 = \frac{\rho_2}{2h} \qquad 5.72a$$

$$J_2 = \frac{I_2 e^{i\xi\Delta_h a/2}}{2i\sin(\xi\Delta_h a/2)} \qquad b$$

$$J_2' = -\frac{I_2 e^{-i\xi\Delta_h a/2}}{2i\sin(\xi\Delta_h a/2)} \qquad c$$

to write F as

$$F = \frac{\beta_{23}h}{\pi}\int_C (J_2 + J_2')e^{i\xi\beta_{23}x}\,d\xi \qquad 5.74$$

The integrand in the limit $R \to \infty$, $\xi = Re^{i\theta}$, goes as

$$\text{Integrand} \sim \frac{e^{i\xi\beta_{23}x}}{\xi}\left[1 - e^{-i\xi\beta_{23}a}\right] \qquad 5.75$$

Hence for convergence we require

$$-x\sin\theta < 0 \qquad \text{and} \qquad -(x-a)\sin\theta < 0 \qquad 5.76$$

The contours will be completed in the half plane indicated by the following table

Table 2

	J_2	J_2'
$x < 0$	L	L
$0 < x < a$	U	L
$a < x$	U	U

The poles of I_2 are found by looking for the zeros of

$$\beta_{23} i \sin(\xi\Delta_h a/2) - \Delta_h \cos(\xi\Delta_h a/2) = 0 \qquad 5.77$$

where

$$\Delta_h = \left[\beta_{23}^2 + \frac{4h^2}{\xi^2}\right]^{\frac{1}{2}}$$

In a manner similar to that used in finding the poles of the integrand of $\overline{F}$ we transform 5.77 using

$$\xi = -\frac{2h}{\beta_{23}}\left[\frac{w^2}{a^2h^2} - 1\right]^{\frac{1}{2}} \qquad 5.78$$

to get

$$w = \pm ah \sin w \qquad 5.79$$

or when rewritten in imaginary and real parts

$$x = \pm ah \sin x \cosh y \qquad 5.80a$$

$$y = \pm ah \cos x \sinh y \qquad b$$

Analysis of 5.80 a and 5.80 b is similar to 5.62 a and 5.62 b.

There are an infinite number of isolated simple poles ξ_2^i for $I_2(\xi)$, lying in the lower half ξ-plane and occurring in pairs $\xi_2^{(i)}$ and $-\xi_2^{(i)*}$. For each sign of 5.80 there exists a zero which for large y has the asymptotic form (in the 4^{th} quadrant)

$$x = \left(\frac{2n+1}{2}\right)\pi \quad , \quad y = -\ln\left[\left(\frac{2n+1}{2}\right)\frac{\pi}{ah}\right] \quad n = 0, 1, 2 \ldots$$

$$P_2(\xi) = \frac{2\Delta_h e^{-i\xi\beta_{23}a}\left[i\sin(\xi\Delta_h a/2)\right]}{\xi\left[ia\beta_{23}^3\cos(\xi\Delta_h a/2) + a\beta_{23}^2\Delta_h\sin(\xi\Delta_h a/2) + (8h^2/\xi^3)\cos(\xi\Delta_h a/2)\right]}$$

$$Q_2(\xi) = \frac{P_2(\xi)e^{i\xi\Delta_h a/2}}{2i\sin(\xi\Delta_h a/2)}$$

$$Q_2'(\xi) = -\frac{P_2(\xi)e^{-i\xi\Delta_h a/2}}{2i\sin(\xi\Delta_h a/2)}$$

Evaluation of F(x) using table 2 gives

$$x<0 \quad F(x) = \sum_{\xi_2^i} -2i\beta_{23}h\left[P_2(\xi_2^i)\,e^{i\xi_2^i\beta_{23}x} + cc\right] \qquad 5\cdot81a$$

$$= \sum_{\xi_2^i} -2i\beta_{23}h\,e^{y_2^i\beta_{23}x}\left[P_2(\xi_2^i)\,e^{ix_2^i\beta_{23}x} + cc\right] \qquad b$$

$$0<x<a \quad F(x) = \sum_{\xi_2^i} -2i\beta_{23}h\left[Q_2'(\xi_2^i)\,e^{i\xi_2^i\beta_{23}x} + cc\right] \qquad 5\cdot82a$$

$$= \sum_{\xi_2^i} -2i\beta_{23}h\,e^{y_2^i\beta_{23}x}\left[Q_2'(\xi_2^i)\,e^{ix_2^i\beta_{23}x} + cc\right] \qquad b$$

$$a<x \quad F(x) = 0 \qquad 5\cdot83$$

where $\xi_2{}^i = x_2{}^i - iy_2{}^i$ is the i^{th} pole of $I_2(\xi)$.

We now move onto $\bar{G}(x,y)$ and its integral expression rewritten here as

$$\bar{G}(x,y) = \frac{\beta_{12}}{4\pi^2 i}\int_C d\xi\,\sigma(\xi)\,e^{i\xi\beta_{12}x}\,\ell(\xi,y) \qquad 5\cdot84$$

where

$$\ell(\xi,y) = 2H\int_{\bar{C}} \frac{e^{-i\xi'\beta_{12}y}}{\xi'-\xi}\,I_1(\xi')\,d\xi' \qquad 5\cdot85$$

Evaluation of $\ell(\xi,y)$ is the same as $\bar{F}$ except for the additional pole at ξ which is in the upper half ξ-plane on the contour C. We have for $\ell(\xi,y)$

$$y<-a \quad \ell(\xi,y) = \sum_{\xi_1^i} 4\pi i H\left[\frac{e^{-i\xi_1^i\beta_{12}y}\,P_1(\xi_1^i)}{\xi_1^i-\xi} + \frac{e^{i\xi_1^{i*}\beta_{12}y}\,P_1^*(\xi_1^i)}{-\xi_1^{i*}-\xi}\right. +$$

$$+ e^{-i\xi\beta_{12}y} I_1(\xi)\Big] \qquad 5{\cdot}86$$

$$-a<y<0 \quad \ell(\xi,y) = \sum_{\xi_i} 4\pi i H\left[\frac{e^{-i\xi_i^i\beta_{12}y} Q_1(\xi_i^i)}{\xi_1^i - \xi} + \frac{e^{i\xi_i^{i*}\beta_{12}y} Q_1^*(\xi_1^i)}{-\xi_1^{i*} - \xi} + \right.$$

$$\left. + e^{-i\xi\beta_{12}y} J_1(\xi)\right] \qquad 5{\cdot}87$$

$$0<y \qquad \ell(\xi,y) = 0 \qquad 5{\cdot}88$$

We define $I_3(\xi)$ such that

$$I_3(\xi) = -\frac{\sigma(\xi)}{2H} \qquad 5{\cdot}89$$

From the definition $I_1(\xi)$ and 4.128 it can be shown that

$$I_1^*(\xi^*) = I_3(\xi) \qquad 5{\cdot}90$$

Also, splitting I_3 up into J_3 and J_3' where

$$J_3(\xi) = \frac{I_3(\xi)\, e^{i\xi\Delta_H a/2}}{2i\sin(\xi\Delta_H a/2)} \qquad 5{\cdot}91a$$

$$J_3'(\xi) = -\frac{I_3(\xi)\, e^{-i\xi\Delta_H a/2}}{2i\sin(\xi\Delta_H a/2)} \qquad b$$

it can be shown that

$$J_1'^*(\xi^*) = J_3(\xi) \qquad 5{\cdot}92a$$

$$J_1^*(\xi^*) = J_3'(\xi) \qquad b$$

With these relations, the poles and residues are easily determined. The poles of $I_3(\xi)$ are $-\xi_1^i$ and ξ_1^{i*}, found in the lower half ξ-plane and the residues $P_3(\xi)$, $Q_3(\xi)$ and $Q_3'(\xi)$ are determined from the following identities

$$P_3(\xi) = P_1^*(\xi^*) \qquad 5.93$$

$$Q_3(\xi) = Q_1'^*(\xi^*) \qquad 5.94a$$

$$Q_3'(\xi) = Q_1^*(\xi^*) \qquad b$$

We now continue our evaluation of $\overline{G}(x,y)$.

In the region $y < -a$, $\overline{G}(x,y)$ becomes

$$\overline{G}(x,y) = \sum_{\xi_1^i} -\frac{2\beta_{12}H^2}{\pi}\left[\int_c I_3 I_1 e^{i\xi\beta_{12}(x-y)}d\xi + P_1(\xi_1^i)e^{-i\xi_1^i\beta_{12}y}\int_c \frac{I_3 e^{i\xi\beta_{12}x}}{\xi_1^i-\xi}d\xi + e^{i\xi_1^{i*}\beta_{12}y}P_1^*(\xi_1^i)\int_c \frac{I_3 e^{i\xi\beta_{12}x}}{-\xi_1^{i*}-\xi}d\xi\right] \qquad 5.95$$

After examination of the integrand of the first integral in the limit $R \to \infty$ one finds

$$\text{Integrand} \sim \frac{e^{i\xi\beta_{12}(x-y)}}{\xi^2}\left[e^{i\xi\beta_{12}a} + e^{-i\xi\beta_{12}a} - 2\right] \qquad 5.96$$

where the first term in the square brackets comes from the J_3J_1 combination of I_3I_1, the second term from $J_3'J_1'$ and the last term from J_3J_1' + J_1J_3'. From 5.96 we see that the integral is convergent whenever

$$Re\left[i\xi\beta_{12}(x-y+a)\right]<0 \qquad \text{and} \qquad 5.97a$$

$$Re\left[i\xi\beta_{12}(x-y-a)\right]<0 \qquad \text{and} \qquad b$$

$$Re\left[i\xi\beta_{12}(x-y)\right]<0 \qquad c$$

from which the following table is constructed.

Table 3

	$J_3 J_1$	$J_3' J_1'$	$J_3 J_1' + J_1 J_3'$
$x-y<-a$	L	L	L
$-a<x-y<0$	U	L	L
$0<x-y<a$	U	L	U
$a<x-y$	U	U	U

If we define

$$L_1 \equiv \int_C e^{i\xi\beta_{12}(x-y)}\,(J_3+J_3')(J_1+J_1')\,d\xi \qquad 5.98$$

then L_1 in the above mentioned four regions becomes

$y<-a \qquad x-y<-a$

$$L_1 = \sum_{\xi_i} -2\pi i\left[P_1(\xi_i)I_1(-\xi_i)\,e^{-i\xi_i\beta_{12}(x-y)} - cc\right] \qquad 5.99$$

$y<-a \qquad -a<x-y<0$

$$L_1 = \sum_{\xi_i} -2\pi i\left\{\left[\left(P_1(\xi_i)J_1'(-\xi_i) + Q_1(\xi_i)J_1(-\xi_i)\right)e^{-i\xi_i\beta_{12}(x-y)} - cc\right] + \right.$$

$$+\left[J_1'(-\xi_1^i)\,Q_1(\xi_1^i)\,e^{i\xi_1^i\beta_{12}(x-y)} - cc\right]\Big\}\qquad 5{\cdot}100$$

$y < -a \qquad 0 < x-y < a$

$$L_1 = \sum_{\xi_1^i} -2\pi i\Big\{\Big[\Big(J_1'(-\xi_1^i)P_1(\xi_1^i) + J_1(-\xi_1^i)Q_1(\xi_1^i)\Big)e^{i\xi_1^i\beta_{12}(x-y)} - cc\Big]$$

$$+\left[Q_1(\xi_1^i)\,J_1'(-\xi_1^i)\,e^{-i\xi_1^i\beta_{12}(x-y)} - cc\right]\Big\}\qquad 5{\cdot}101$$

$y < -a \qquad a < x-y$

$$L_1 = \sum_{\xi_1^i} -2\pi i\left[I_1(-\xi_1^i)\,P_1(\xi_1^i)\,e^{i\xi_1^i\beta_{12}(x-y)} - cc\right]\qquad 5{\cdot}102$$

The next two integrals in 5.95 are

$$L_2 \equiv \int_c \frac{I_3\, e^{i\xi\beta_{12}x}}{\xi_1^i - \xi}\,d\xi \qquad 5{\cdot}103$$

and

$$L_3 \equiv \int_c \frac{I_3\, e^{i\xi\beta_{12}x}}{-\xi_1^{i*} - \xi}\,d\xi \qquad 5{\cdot}104$$

The integrands of these integrals are similar in their asymptotic behaviour, $R \to \infty$ where $\xi = Re^{i\theta}$. In this limit the integrand goes as

$$\text{Integrand} \sim \frac{e^{i\xi\beta_{12}x}}{\xi^2}\left[e^{i\xi\beta_{12}a} - 1\right]\qquad 5{\cdot}105$$

Therefore, our requirement for the convergence of the terms J_3 and J_3' is respectively

$$\mathrm{Re}\left[i\xi\beta_{12}(x+a)\right] < 0 \quad \text{and} \quad \mathrm{Re}\left[i\xi\beta_{12}'x\right] < 0 \qquad 5{\cdot}106$$

which gives the following table

Table 4

	J_3	J_3'
$x < -a$	L	L
$-a < x < 0$	U	L
$a < x$	U	U

Upon integration, L_2 and L_3 are given as

$x < -a$

$$L_2 = \sum_{\xi_1^j} -2\pi i \left[\frac{P_1(\xi_1^j)\, e^{-i\xi_1^j \beta_{12} x}}{\xi_1^i + \xi_1^j} + \frac{P_1^*(\xi_1^j)\, e^{i\xi_1^{j*} \beta_{12} x}}{\xi_1^i - \xi_1^{j*}} \right] \qquad 5\cdot107$$

$$L_3 = \sum_{\xi_1^j} -2\pi i \left[\frac{P_1(\xi_1^j)\, e^{-i\xi_1^j \beta_{12} x}}{-\xi_1^{i*} + \xi_1^j} + \frac{P_1^*(\xi_1^j)\, e^{i\xi_1^{j*} \beta_{12} x}}{-\xi_1^{i*} - \xi_1^{j*}} \right] \qquad 5\cdot108$$

$-a < x < 0$

$$L_2 = \sum_{\xi_1^j} \left\{ 2\pi i\, J_1'(-\xi_1^i)\, e^{i\xi_1^i \beta_{12} x} - 2\pi i \left[\frac{Q_1(\xi_1^j)\, e^{-i\xi_1^j \beta_{12} x}}{\xi_1^i + \xi_1^j} + \frac{Q_1^*(\xi_1^j)\, e^{i\xi_1^{j*} \beta_{12} x}}{\xi_1^i - \xi_1^{j*}} \right] \right\} \qquad 5\cdot109$$

$$L_3 = \sum_{\xi_1^j} \left\{ -2\pi i\, J_1'^*(-\xi_1^i)\, e^{-i\xi_1^{i*} \beta_{12} x} - 2\pi i \left[\frac{Q_1(\xi_1^j)\, e^{-i\xi_1^j \beta_{12} x}}{-\xi_1^{i*} + \xi_1^j} + \frac{Q_1^*(\xi_1^j)\, e^{i\xi_1^{j*} \beta_{12} x}}{-\xi_1^{i*} - \xi_1^{j*}} \right] \right\} \qquad 5\cdot110$$

$0 < x$

$$L_2 = 2\pi i \, I_1(-\xi_1^i) e^{i\xi_1^i \beta_{12} x} \qquad 5\cdot 111$$

$$L_3 = -2\pi i \, I_1^*(-\xi_1^i) e^{-i\xi_1^{i*} \beta_{12} x} \qquad 5\cdot 112$$

There is no restriction indicated for the y values because integrals L_2 and L_3 are identical in both the $y < -a$ region and the $-a < y < 0$ region (they do not occur in the region $y > 0$).

The only other integral that must be evaluated to completely determine $\overline{G}(x,y)$ occurs in the $-a < y < 0$ region. In this region L_1 is given as

$$L_1 = \int_c I_3(\xi)^* J_1(\xi) e^{i\xi\beta_{12}(x-y)} d\xi \qquad 5\cdot 113$$

As before the asymptotic behaviour of the integrand is examined to determine the half plane in which the contour is to be extended. The following table is thereby obtained

Table 5

	$J_3 J_1'$	$J_3' J_1'$
$x-y < -a$	L	L
$-a < x-y < 0$	U	L
$0 < x-y$	U	U

With the aid of table 5 L_1 is evaluated, yielding

$-a < y < 0 \qquad x - y < -a$

$$L_1 = \sum_{\xi_1^i} -2\pi i \left[P_1(\xi_1^i)\, J_1(-\xi_1^i)\, e^{-i\xi_1^i \beta_{12}(x-y)} - cc \right] \tag{5-114}$$

$-a < y < 0 \qquad -a < x - y < 0$

$$L_1 = \sum_{\xi_1^i} \Big\{ -2\pi i \left[J_1'(-\xi_1^i)\, Q_1(\xi_1^i)\, e^{i\xi_1^i \beta_{12}(x-y)} - cc \right]$$

$$- 2\pi i \left[Q_1(\xi_1^i)\, J_1(-\xi_1^i)\, e^{-i\xi_1^i \beta_{12}(x-y)} - cc \right] \Big\} \tag{5-115}$$

$-a < y < 0 \qquad 0 < x - y$

$$L_1 = \sum_{\xi_1^i} -2\pi i \left[I_1(-\xi_1^i)\, Q_1(\xi_1^i)\, e^{i\xi_1^i \beta_{12}(x-y)} - cc \right] \tag{5-116}$$

We our now ready to write out $\bar{G}(x,y)$, by combining the expressions for L_1, L_2 and L_3 in their respective regions. $\bar{G}(x,y)$ is

$y < -a \qquad x - y < -a \qquad x < -a$

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \left[P_1(\xi_1^i)\, I_1(-\xi_1^i)\, e^{-i\xi_1^i \beta_{12}(x-y)} - cc \right]$$

$$+ \left[\frac{P_1(\xi_1^i)\, P_1(\xi_1^j)}{\xi_1^i + \xi_1^j}\, e^{-i\xi_1^i \beta_{12} y}\, e^{-i\xi_1^j \beta_{12} x} - cc \right]$$

$$+ \left[\frac{P_1(\xi_1^i)\, P_1^*(\xi_1^j)}{\xi_1^i - \xi_1^{j*}}\, e^{-i\xi_1^i \beta_{12} y}\, e^{i\xi_1^{j*} \beta_{12} x} - cc \right] \Big\} \tag{5-117}$$

$y < -a \quad -a < x - y < a \quad x < -a$ 2

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \Big[J_1'(-\xi_1^i) Q_1(\xi_1^i) e^{i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\Big(P_1(\xi_1^i) J_1'(-\xi_1^i) + Q_1(\xi_1^i) J_1(-\xi_1^i) \Big) e^{-i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i) P_1(\xi_1^j)}{\xi_1^i + \xi_1^j} e^{-i\xi_1^i \beta_{12} y} e^{-i\xi_1^j \beta_{12} x} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i) P_1^*(\xi_1^j)}{\xi_1^i - \xi_1^{j*}} e^{-i\xi_1^i \beta_{12} y} e^{i\xi_1^{j*} \beta_{12} x} - cc \Big] \Big\} \qquad 5.118$$

$y < -a \quad 0 < x - y < a \quad x < -a$ 3

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \Big[Q_1(\xi_1^i) J_1'(-\xi_1^i) e^{-i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\Big(J_1'(-\xi_1^i) P_1(\xi_1^i) + J(-\xi_1^i) Q_1(\xi_1^i) \Big) e^{i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i) P_1(\xi_1^j)}{\xi_1^i + \xi_1^j} e^{-i\xi_1^i \beta_{12} y} e^{-i\xi_1^j \beta_{12} x} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i) P_1^*(\xi_1^j)}{\xi_1^i - \xi_1^{j*}} e^{-i\xi_1^i \beta_{12} y} e^{i\xi_1^{j*} \beta_{12} x} - cc \Big] \Big\} \qquad 5.119$$

$y < -a \qquad 0 < x-y < a \qquad -a < x < 0$ 4

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \Big[J_1(-\xi_1^i) Q_1(\xi_1^i)\, e^{i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[Q_1(\xi_1^i) J_1'(-\xi_1^i)\, e^{-i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i)\, Q_1(\xi_1^j)}{\xi_1^i + \xi_1^j}\, e^{-i\xi_1^i \beta_{12} y}\, e^{-i\xi_1^j \beta_{12} x} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i)\, Q_1^*(\xi_1^j)}{\xi_1^i - \xi_1^{j*}}\, e^{-i\xi_1^i \beta_{12} y}\, e^{i\xi_1^{j*} \beta_{12} x} - cc \Big] \Big\} \qquad 5\cdot120$$

$y < -a \qquad a < x-y \qquad x < -a$ 5

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \Big[I_1(-\xi_1^i) P_1(\xi_1^i)\, e^{i\xi_1^i \beta_{12}(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i)\, P_1(\xi_1^j)}{\xi_1^i + \xi_1^j}\, e^{-i\xi_1^i \beta_{12} y}\, e^{-i\xi_1^j \beta_{12} x} - cc \Big]$$

$$+ \Big[\frac{P_1(\xi_1^i)\, P_1^*(\xi_1^j)}{\xi_1^i - \xi_1^{j*}}\, e^{-i\xi_1^i \beta_{12} y}\, e^{i\xi_1^{j*} \beta_{12} x} - cc \Big] \Big\} \qquad 5\cdot121$$

$y < -a \qquad a < x-y \qquad -a < x < 0$ 6

$$\bar{G}(x,y) = \sum_{\xi_1^i} \sum_{\xi_1^j} 4\beta_{12} i H^2 \Big\{ \Big[J_1(-\xi_1^i) P_1(\xi_1^i)\, e^{i\xi_1^i \beta_{12}(x-y)} - cc \Big] +$$

$$+\left[\frac{P_1(\xi_1^i)\,Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i\beta_{12}y}e^{-i\xi_1^j\beta_{12}x}-cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)\,Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i\beta_{12}y}e^{i\xi_1^{j*}\beta_{12}x}-cc\right]\Bigg\} \qquad 5.122$$

$y<-a \qquad a<x-y \qquad 0<x$ 7

$$\bar{G}(x,y)=0 \qquad 5.123$$

$-a<y<0 \qquad x-y<-a \qquad x<-a$ 8

$$\bar{G}(x,y)=\sum_{\xi_1^i}\sum_{\xi_1^j}4\beta_{12}iH^2\Bigg\{\left[P_1(\xi_1^i)\,J(-\xi_1^i)\,e^{-i\xi_1^i\beta_{12}(x-y)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)\,P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i\beta_{12}y}e^{-i\xi_1^j\beta_{12}x}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)\,P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i\beta_{12}y}e^{i\xi_1^{j*}\beta_{12}x}-cc\right]\Bigg\} \qquad 5.124$$

$-a<y<0 \qquad -a<x-y<0 \qquad x<-a$ 9

$$\bar{G}(x,y)=\sum_{\xi_1^i}\sum_{\xi_1^j}4\beta_{12}iH^2\Bigg\{\left[J_1'(-\xi_1^i)\,Q_1(\xi_1^i)\,e^{i\xi_1^i\beta_{12}(x-y)}-cc\right]$$

$$+\left[Q_1(\xi_1^i)\,J_1(-\xi_1^i)\,e^{-i\xi_1^i\beta_{12}(x-y)}-cc\right]+$$

$$+\left[\frac{Q_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i\beta_{12}y}e^{-i\xi_1^j\beta_{12}x}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i\beta_{12}y}e^{i\xi_1^{j*}\beta_{12}x}-cc\right]\Bigg\} \qquad 5\cdot125$$

$-a<y<0 \qquad -a<x-y<0 \qquad -a<x<0$ 10

$$\bar{G}(x,y)=\sum_{\xi_1^i}\sum_{\xi_1^j}4\beta_{12}iH^2\Bigg\{\left[Q_1(\xi_1^i)J_1(-\xi_1^i)e^{-i\xi_1^i\beta_{12}(x-y)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i\beta_{12}y}e^{-i\xi_1^j\beta_{12}x}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i\beta_{12}y}e^{i\xi_1^{j*}\beta_{12}x}-cc\right]\Bigg\} \qquad 5\cdot126$$

$-a<y<0 \qquad 0<x-y \qquad -a<x<0$ 11

$$\bar{G}(x,y)=\sum_{\xi_1^i}\sum_{\xi_1^j}4\beta_{12}iH^2\Bigg\{\left[J_1(-\xi_1^i)Q_1(\xi_1^i)e^{i\xi_1^i\beta_{12}(x-y)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i\beta_{12}y}e^{-i\xi_1^j\beta_{12}x}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i\beta_{12}y}e^{i\xi_1^{j*}\beta_{12}x}-cc\right]\Bigg\} \qquad 5\cdot127$$

$-a < y < 0 \qquad 0 < x - y \qquad 0 < x$ 12

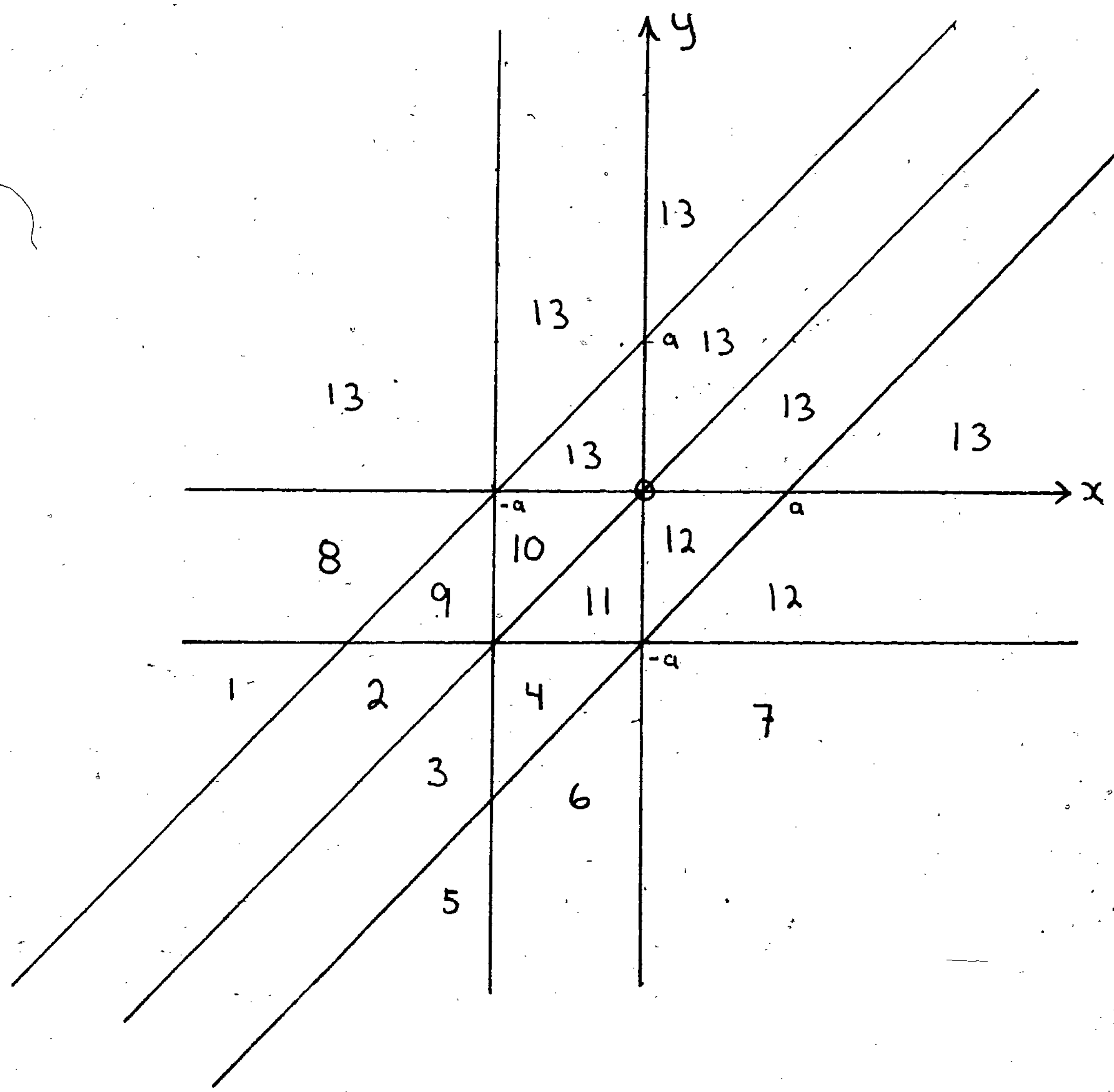
$\bar{G}(x,y) = 0$ 5.128

$0 < y$ 13

$\bar{G}(x,y) = 0$ 5.129

The regions of $\overline{G}(x,y)$ are summarized in the following diagram.

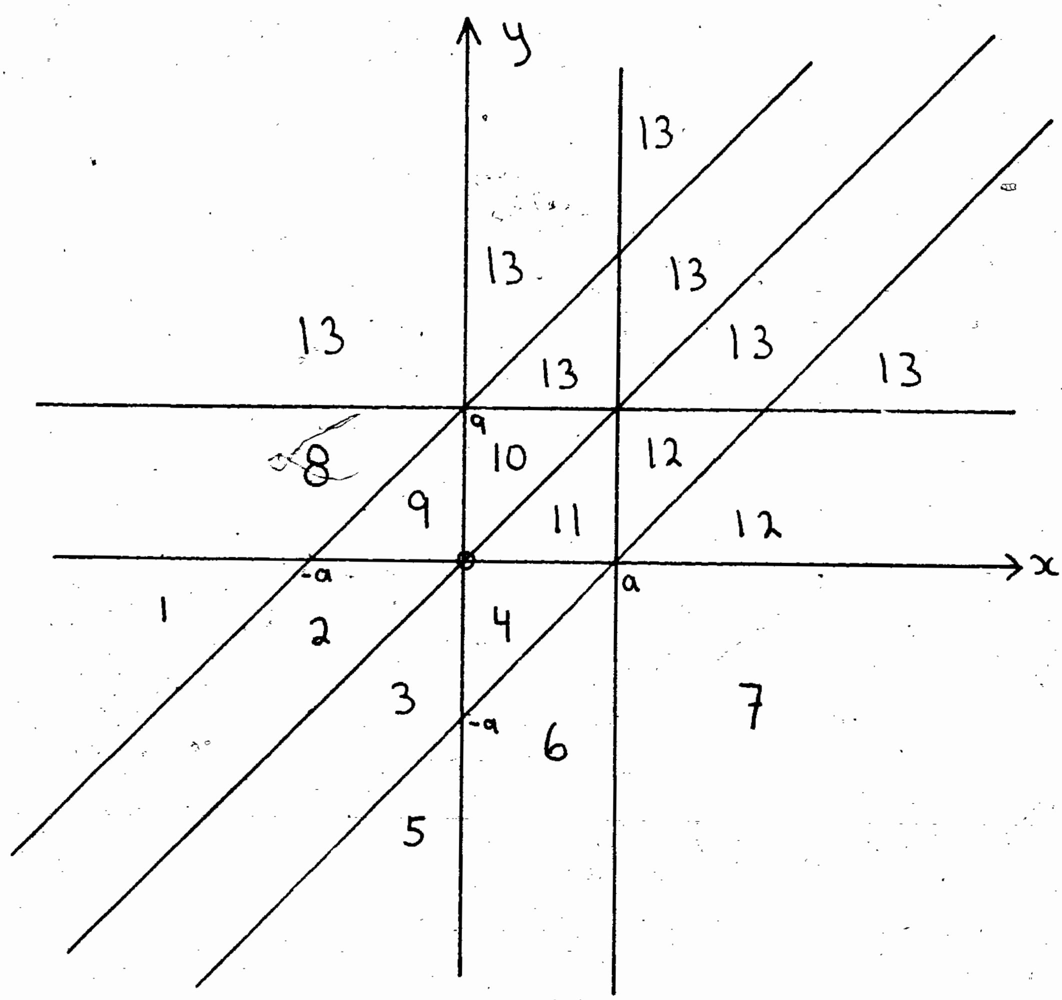
Figure 5·2



The evaluation of $G(x,y)$ proceeds in exactly the same manner as $\overline{G}(x,y)$. The solution for $G(x,y)$ may be found in Appendix B.

The following diagram summarizes the regions of G(x,y).

Figure 5·3



Continuing, we calculate H(x,y). Then using the symmetry relation 4.134, $\overline{H}(x,y)$ may be written. H(x,y) is given by

$$H(x,y) = \frac{\beta_{23}}{4\pi^2 i}\int_c d\xi\, e^{i\xi\beta_{12}x}\sigma(\xi)\int_c \frac{d\xi'\ \rho_2(\xi')}{\xi'-\xi-i\varepsilon}e^{i\xi'\beta_{23}y} \qquad 5\cdot143$$

The evaluated expression for $H(x,y)$ is given in Appendix C.

Time dependence is introduced at this stage using 4.135 to 4.146. We set $c_1 = -c_3 \equiv c$ with $c_2 = 0$. $\overline{F}(x;t)$, $F(x;t)$, $\overline{G}(x,y;t)$, $G(x,y;t)$, $\overline{H}(x,y;t)$ and $H(x,y;t)$ are given in Appendix D.

The next step in the ISM is to solve the Marchenko integral equations for the kernels $k_j^{(i)}(x,y;t)$.

CHAPTER 6

STEP 3 - THE INVERSE PROBLEM-SOLUTION OF THE MARCHENKO INTEGRAL EQUATIONS

The final step consists of attempting to solve the Marchenko integral equations

$$K_2^{(1)}(x,y;t)+\bar{F}(y;t)+\int_x^\infty K_2^{(1)}(x,s;t)\,\bar{G}(s,y;t)+K_2^{(3)}(x,s;t)\,\bar{H}(s,y;t)\,ds=0 \qquad 6.1$$

$$K_2^{(3)}(x,y;t)+F(y;t)+\int_x^\infty K_2^{(3)}(x,s;t)\,G(s,y;t)+K_2^{(1)}(x,s;t)\,H(s,y;t)\,ds=0 \qquad 6.2$$

for $K^{(\frac{1}{2})}$ and $K^{(\frac{3}{2})}$ (hence Q_1 and Q_3). Because the kernels $\bar{G}$, $\bar{H}$, G and H, found in the previous chapter, are split into many regions, an exact analytical solution of these coupled Volterra-type integral equations appears impossible. However, an iterative solution may be feasible as series expressions exist for the functions $\bar{F}$, $\bar{G}$, $\bar{H}$, F, G, and H.

First, however, let's examine the structure of the Marchenko equations when the signal beam is turned off, i.e. $h = o$, thus allowing the pump beam to travel through the medium undisturbed. In this case, $\rho_2 = \bar{\rho}_3 = \bar{\sigma} = 0$ so that H, G, F and $\bar{H}$ are all identically zero. (See Appendices B, C,D.) The Marchenko integral equations then reduce to

$$K_2^{(1)}(x,y;t)+\bar{F}(y;t)+\int_x^\infty K_2^{(1)}(x,s;t)\,\bar{G}(s,y;t)\,ds=0 \qquad 6.3$$

$$K_2^{(3)}(x,y;t)=0 \qquad 6.4$$

We now investigate the term $\bar{F}$ which is much simpler in structure than the kernel $\bar{G}$. To explicitly evaluate $\bar{F}$, the poles which contribute to this term must be determined. A numerical evaluation of the first 50 poles ξ_1^i was carried out for two different sets of input parameters, the results being summarized in Tables 6 and 7. The medium was chosen to be transparent liquid CCl_4 with the following room temperature characteristics: density $\rho = 1.594$ gm/cm^3, sound velocity $v_s = 9.3 \times 10^5$ cm/s and refractive index $n = 1.46$. In the first table, the laser intensity was taken to be $I_L = 1 \times 10^{14}$ ergs/s-cm^2 and the pulse duration $\tau = 10^{-9}$ s, yielding $H = 8.7 \times 10^{-3}$ statvolts/cm, $a = 20.5$ cm and $aH = 0.18$ statvolts. In the second table, $I_L = 5 \times 10^{14}$ ergs/s-cm^2, $\tau = 10^{-8}$ s so that $H = 1.95 \times 10^{-2}$ statvolts/cm, $a = 205$ cm and $aH = 4.0$ statvolts. These laser input parameters are typical of a Q-switched ruby laser (operating frequency $\omega = 2.7 \times 10^{15}$ Hz). With double precision specified in the program the first fifty poles, ξ_1^i, were calculated in fifty seconds on the IBM /370-155 (Fortran H compiler).

Several features of the poles are immediately apparent: (1) the imaginary part of ξ_1^i is increasing very slowly; (2) the poles lie in the first quadrant, hence, upper half ξ-plane (there are also poles at $-\xi_1^{i*}$, (i.e. in the second quadrant) which we have not bothered to show); and (3) the rate of increase of the imaginary part of ξ_1^i is faster for $aH = 4.0$ statvolts than it is for $aH = .18$ statvolts.

The second feature leads immediately to a very important conclusion. Kaup has shown that if a_{11} has a zero (i.e. pole of $\overline{\rho}_2$) in the lower half ξ-plane, which is possible in the forward scattering case, then this zero will give rise to a soliton† wave solution. To see where the terms which give rise to solitons come from we must go back to Chapter 3.

†Soliton is a term coined by Zabusky and Kruskal to describe a localized travelling wave or a travelling wave whose transition from one constant asymptotic state to another is localized (e.g. tanh (x-ct)) which asymptotically preserves its shape and velocity upon collision with other solitons. [15]

TABLE 6

Zeros of $a_{11}(\xi)$, ξ_1^i, with $I_L = 10^{14}$ erg/cm^2-s,

$\tau = 10^{-8}$ s, $H = 8.72366 \times 10^{-3}$ statvolts/cm, $h = 0$,

$a = 2.05336 \times 10^1$ cm, $aH = .179128$ statvolts.

$\xi_1^1 = 9.76543 \times 10^{-12} + i\ 1.84623 \times 10^{-11}$ s/cm^2

$\xi_1^2 = 2.66759 \times 10^{-11} + i\ 2.07529 \times 10^{-11}$

$\xi_1^3 = 4.24064 \times 10^{-11} + i\ 2.24167 \times 10^{-11}$

$\xi_1^4 = 5.77696 \times 10^{-11} + i\ 2.36682 \times 10^{-11}$

$\xi_1^5 = 7.29693 \times 10^{\ 11} + i\ 2.46655 \times 10^{-11}$

$\xi_1^6 = 8.80807 \times 10^{-11} + i\ 2.54929 \times 10^{-11}$

$\xi_1^7 = 1.03139 \times 10^{-10} + i\ 2.61993 \times 10^{-11}$

$\xi_1^8 = 1.18162 \times 10^{-10} + i\ 2.68153 \times 10^{-11}$

$\xi_1^9 = 1.33161 \times 10^{-10} + i\ 2.73613 \times 10^{-11}$

$\xi_1^{10} = 1.48143 \times 10^{-10} + i\ 2.78515 \times 10^{-11}$

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$\xi_1^{41} = 6.10724 \times 10^{-10} + i\ 3.44961 \times 10^{-11}$

$\xi_1^{42} = 6.25632 \times 10^{-10} + i\ 3.46102 \times 10^{-11}$

$\xi_1^{43} = 6.40540 \times 10^{-10} + i\ 3.47216 \times 10^{-11}$

$\xi_1^{44} = 6.55447 \times 10^{-10} + i\ 3.48305 \times 10^{-11}$

$\xi_1^{45} = 6.70354 \times 10^{-10} + i\ 3.49369 \times 10^{-11}$

$\xi_1^{46} = 6.86261 \times 10^{-10} + i\ 3.50410 \times 10^{-11}$

$\xi_1^{47} = 7.00167 \times 10^{-10} + i\ 3.51429 \times 10^{-11}$

$\xi_1^{48} = 7.15074 \times 10^{-10} + i\ 3.52426 \times 10^{-11}$

$\xi_1^{49} = 7.29980 \times 10^{-10} + i\ 3.53402 \times 10^{-11}$

$\xi_1^{50} = 7.44887 \times 10^{-10} + i\ 3.54359 \times 10^{-11}$

TABLE 7

Zeros of $a_{11}(\xi)$, ξ_1^i, with $I_L = 5 \times 10^{14}$ erg/cm^2-s, $\tau = 10^{-8}$ s, H = 1.95067×10^{-2} statvolts/cm, h = 0, a = 2.05336×10^2 cm, aH = 4.00543 statvolts.

$\xi_1^1 = 2.35966 \times 10^{-12} + i\ 1.99994 \times 10^{-13}$ s/cm^2

$\xi_1^2 = 3.45406 \times 10^{-12} + i\ 4.86700 \times 10^{-13}$

$\xi_1^3 = 4.78042 \times 10^{-12} + i\ 6.92397 \times 10^{-13}$

$\xi_1^4 = 6.18658 \times 10^{-12} + i\ 8.40435 \times 10^{-13}$

$\xi_1^5 = 7.62667 \times 10^{-12} + i\ 9.54039 \times 10^{-13}$

$\xi_1^6 = 9.08392 \times 10^{-12} + i\ 1.04574 \times 10^{-12}$

$\xi_1^7 = 1.05509 \times 10^{-11} + i\ 1.12249 \times 10^{-11}$

$\xi_1^8 = 1.20240 \times 10^{-11} + i\ 1.18844 \times 10^{-12}$

$\xi_1^9 = 1.35011 \times 10^{-11} + i\ 1.24626 \times 10^{-12}$

$\xi_1^{10} = 1.49810 \times 10^{-11} + i\ 1.299771 \times 10^{-12}$

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$\xi_1^{41} = 6.11133 \times 10^{-11} + i\ 1.97440 \times 10^{-12}$

$\xi_1^{42} = 6.26031 \times 10^{-11} + i\ 1.98586 \times 10^{-12}$

$\xi_1^{43} = 6.40930 \times 10^{-11} + i\ 1.99705 \times 10^{-12}$

$\xi_1^{44} = 6.55828 \times 10^{-11} + i\ 2.00798 \times 10^{-12}$

$\xi_1^{45} = 6.70727 \times 10^{-11} + i\ 2.01867 \times 10^{-12}$

$\xi_1^{46} = 6.85625 \times 10^{-11} + i\ 2.02912 \times 10^{-12}$

$\xi_1^{47} = 7.00524 \times 10^{-11} + i\ 2.03934 \times 10^{-12}$

$\xi_1^{48} = 7.15423 \times 10^{-11} + i\ 2.04935 \times 10^{-12}$

$\xi_1^{49} = 7.30323 \times 10^{-11} + i\ 2.05915 \times 10^{-12}$

$\xi_1^{50} = 7.45222 \times 10^{-11} + i\ 2.06875 \times 10^{-12}$

To obtain an integral expression for $\psi^{(1)}$ we considered the integral 4.56.

$$\int_{\bar{C}} \frac{\phi^{(1)}(\xi',x)\, e^{-i\xi' x/\alpha_1}}{a_{11}(\xi')(\xi'-\xi)}\, d\xi' \qquad 6.5$$

where $\bar{C}$ was defined to pass under all zeros of $a_{11}(\xi)$. In the forward scattering case this assumption, about $\bar{C}$, is not true. In addition to the pole at ξ there will be a number of additional poles in the lower half ξ-plane which will add discrete spectra terms to expression 4.60. It is these additional terms which, Kaup has shown, give rise to solitons. In the problem that we are solving (backscattering case with rectangular pulses) a_{11} has no zeros in the lower half ξ-plane hence no solitons solutions, only a continuous spectrum. Similar remarks apply to a_{33} where in this case poles in the upper half ξ-plane produce solitons.

We now continue with our evaluation of $\bar{F}$. The output of the pole calculation was used in another computer program designed to sum the first fifty residues of the series expression for $\bar{F}$. The rather surprising output is plotted in Figures 6.1 and 6.2, where Figure 6.1 corresponds to aH = .18 statvolts and Figure 6.2 to aH = 4.0 statvolts. The computer time was of the order of 60s per graph of twenty data points. In the asymptotic limit $x \to \infty$ we initially thought that the integral term, containing $\bar{G}$ in 6.3 would be equal to zero and we, therefore, expected $\bar{F}(x)$ to approximate the initial rectangular pulse shape of height H. To explain the unexpected results we conjectured that the pole representation of $\bar{F}(x)$ must be converging very slowly. (We had already checked and re-checked the algebra and computer program).

To check this conjecture the integral expression for $\overline{F}(x)$ in 4.85 (5.46) was (directly) numerically integrated using the trapezoidal rule. Because the integrand of $\overline{F}$ is a rapidly oscillating function with a period on the order of $\xi_1^{(i)}$ a step size of 10^{-14} s/cm^2 was adopted. Integration limits (lower and upper, respectively) were cutoff at -10^{-10} s/cm^2 and 10^{-10} s/cm^2. Results for aH = .18 statvolt and aH = 4.0 statvolts are plotted in Figs. 6.3 and 6.4 respectively. The error bars around the data points indicate the maximum uncertainty due to the oscillatory nature of the integrand and the fact that the limits were cut off at finite values. For convenience in viewing, a smooth curve has been drawn through the center of the error bars. In Fig. 6.3 (aH = .18 statvolts) a nearly rectangular pulse is shown with the initial wave height $H = 8.7 \times 10^{-3}$ statvolts/cm. In Fig. 6.4 (aH = 4.0 statvolts) we see an oscillatory function with amplitudes on the order of the initial rectangular pulse height, $H = 1.95 \times 10^{-2}$ statvolts/cm. To aid in the interpretation of the $\overline{F}$ diagrams, note that $\overline{F}$ is purely imaginary and hence only this component is plotted. Also note that since the contribution of $\overline{F}$ to the final pump wave envelope Q_3 is given by $Q_3 = -i\overline{F}$ + integral terms, the contribution to Q_3 can be read directly from these diagrams.

Comparison of Figs. 6.1 and 6.2 to 6.3 and 6.4, respectively, reveals an interesting point. Including the first 50 poles in the calculation of $\overline{F}$ (diagrams 6.1 and 6.2) does not bring $\overline{F}$'s series representation anywhere near the actual value of $\overline{F}$ (Figs. 6.3 and 6.4) evaluated directly from its integral form. It appears that the slow increase of the imaginary part of the poles is directly related to the convergence of the $\overline{F}$ pole representation. To see this examine the x dependent portion of the pole representation for $\overline{F}$. This term goes as

$$e^{-i\xi_1^i cx} \qquad 6.6$$

The magnitude of this term and hence $\overline{F}$ depends upon y_1^i, the imaginary part of ξ_1^i. Taking any x value between -a and 0 and evaluating this term at the 1st and 50th pole for both aH = .18 statvolts and aH = 4.0 statvolts will reveal two points of interest. First, the 50th term is not that much smaller than the 1st term. Second, for aH = 4.0 statvolts the ratio between the 1st and 50th term is greater, i.e. the terms get smaller faster, than for aH = .18 statvolts. The first point indicates that because the terms in the infinite series representation for $\overline{F}$ decrease slowly in magnitude, these terms will necessarily be small in magnitude and therefore a great number of these terms must be summed in order to obtain an accurate representation of $\overline{F}$. The second point explains why for aH = 4.0 statvolts the series representation of $\overline{F}$ is qualitatively a much better fit than for aH = .18 statvolts.

We conclude that for the problem we are investigating the pole representation of $\overline{F}$ converges too slowly for this method of evaluating $\overline{F}$ to be viable. Future investigation by numerical experts may change this situation but in this thesis we will use the direct numerical integration method to evaluate $\overline{F}(x)$.

An obvious question arises when one compares Fig. 6.3 and 6.4, "Why does $-i\overline{F}(x)$ approximate a rectangular envelope in the case aH = .18 statvolts and does not in the case aH = 4.0 statvolts"?

To answer this question, let us return to the full Marchenko integral equations, 6.1 and 6.2.

Take $aH << 1$ and $ah << 1$. We will determine the order of the individual terms that occur in the Marchenko equations. First note that the poles ξ_1^1 and ξ_2^1 are of the order $1/ac$ hence Δ_H and Δ_h are of the order c. It is not difficult to show that

$$I_1(\xi_1^1) \sim I_1(\xi_2^1) \sim a \qquad 6.7\ a$$

$$P_1(\xi_1^1) \sim 1/c \qquad b$$

$$I_2(\xi_1^1) \sim I_2(\xi_2^1) \sim a \qquad c$$

$$P_2(\xi_2^1) \sim 1/c \qquad d$$

where the symbol "$\sim$" stands for "of the order of". Next we find the order of the terms $\overline{F}$, $\overline{G}$, $\overline{H}$, F, G and H by using 6.7 in 5.169 - 5.217.

We include only the first residue since incorporating residues evaluated at poles ξ_1^i $i \geq 2$, ξ_2^i $i \geq 2$ complicates matters and does not alter the validity of the demonstration.

$$\overline{F} \sim H \qquad 6.8\ a$$

$$\overline{G} \sim aH^2 \qquad b$$

$$\overline{H} \sim ahH \qquad c$$

$$F \sim h \qquad d$$

$$G \sim ah^2 \qquad e$$

$$H \sim ahH \qquad f$$

The order of each term in the Marchenko integral equations can now be calculated. From 6.1, 6.2 and 6.8 we have,

$$aK_2^{(1)} + O[aH] + \int_x^\infty K_2^{(1)}\, O[(aH)^2] + K_2^{(3)}\, O[aH]\, O[ah]\, ds = 0 \qquad 6.9$$

or $$aK_2^{(1)} + O[aH] + aK_2^{(1)}\, O[(aH)^2] + aK_2^{(3)}\, O[aH]\, O[ah] = 0 \qquad 6.10$$

also $$aK_2^{(3)} + O[ah] + aK_2^{(3)}\, O[(ah)^2] + aK_2^{(1)}\, O[aH]\, O[ah] = 0 \qquad 6.11$$

In obtaining 6.10 from 6.9 we have replaced the integral with a times the integrand since $K_2^{(1)}$ and $K_2^{(3)}$ are approximately rectangular pulses of width a.

We see that for aH < < 1 and ah < < 1 $\overline{F}$ provides the major contribution to $K_2^{(1)}$ and F to $K_2^{(3)}$. In particular, for h = 0 and aH = 0.179, Fig. 6.11 tells us that $-i\overline{F}(x)$ should approximate the initial rectangular shape of height H = 8.7 × 10^{-3} statvolts/cm to within about 3% since the integral term provides a correction $\sim(aH)^2$. The magnitude of the "ripple" in Fig. 6.3 is indeed less than 3%!

On the other hand, carrying out a similar analysis for aH > > 1, it can be shown that the terms in Eq. 6.1, for example, are of the following order.

$$aK_2^{(1)} + O[(aH)^0] + aK_2^{(1)}\, O[(aH)^0] + aK_2^{(3)}\, O[ah] = 0 \qquad 6.12$$

Thus, for h = 0 and aH = 4.0 statvolts/cm, one would expect that the integral contribution must not be neglected if one wants to regain the initial rectangular shape. This is why Fig. 6.4 does not look at all rectangular. Or to put in another way, if aH > > 1, one has to solve an integral equation even for h = 0 to regain the rectangular shape. The rectangular shape does not drop out in a trivial way in this case. Furthermore one needs to first evaluate the kernel $\overline{G}$ which is not an easy task.

For h non-zero, we must return to the coupled equations 6.1 and 6.2. First we will try to obtain some relatively simple results (avoiding a brute force approach) which might be relevant to an experimentalist. Let's try to find the asymptotic shape of the pump beam long after the interaction with the signal beam has taken place. First consider the limit $t \to +\infty$, holding x-ct=constant. As t increases so must x. Returning to the definitions in the previous chapter, in this limit we find that F, G, H, $\overline{H}$ all go to zero. Then Eqs. 6.1 and 6.2 reduce to (recall $y \geq x$)

$$K_2^{(1)}(x,y;t) + \overline{F}(y-ct) + \int_x^\infty K_2^{(1)}(x,s;t)\,\overline{G}(s-ct,y-ct)\,ds \approx 0 \qquad 6.13$$

$$K_2^{(3)}(x,y;t) = 0 \qquad 6.14$$

The second equation simply confirms that there is no signal beam in this limit since it has travelled off towards $x = -\infty$ as $t \to \infty$.

The term $\overline{F}$ now depends on h as well as H. Provided that aH << 1, it can be shown that for ah >1 the ah dependence cancels out of the integral term in 6.13 and the integral is again negligible as it was for ah<<1. We have

$$K_2^{(1)}(x,x;t) \approx -\overline{F}(x-ct)$$

$$\text{or} \quad V_{21}(x,t) \approx i\overline{F}(x-ct) \quad \text{as} \quad t \to \infty \qquad 6.15$$

Using direct numerical integration, $\overline{F}$ is calculated for aH = .179 statvolts/cm and varying values of h.

In Figs. 6.5 - 6.7, we see the output for $h = \sqrt{10}\,H$, $h = 10H$, and $h = \sqrt{250}\,H$, respectively.

Fig. 6.5 shows the first effects of the signal beam's existence. A slight depletion of the pump beam's amplitude is apparent near its tail. It is perturbutions in the medium which couples the two laser beams together. Initially the medium is undisturbed, but when the two beams overlap they drive the medium, gradually building up significant perturbations in the medium's dielectric constant. By the time the tails of the two beams are overlapping the medium's dielectric constant has been driven to sufficient amplitudes to make the two beams' predator, prey roles visible. In this case the signal beam is taking energy from the pump beam.

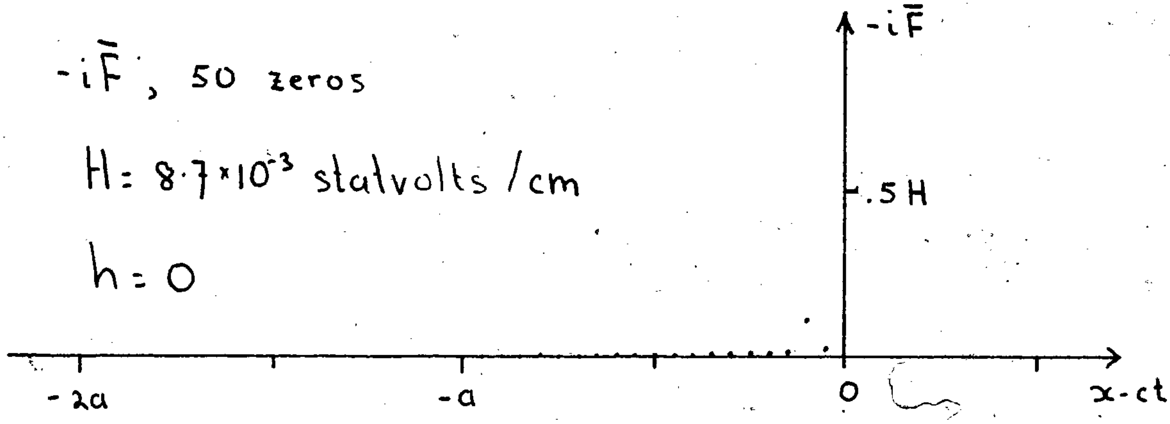
Increasing the amplitude of the initial signal beam to h = 10 H, in Fig. 6.6, we see the effects of even stronger nonlinear coupling. Depletion is still visible in the pump envelope but in addition oscillations in the amplitude and pulse stretching occur. The oscillations in pulse amplitude agrees qualitatively with the numerical results of Bers, Kaup and Reiman [5] using Gaussian shapes† (they numerically solved the three wave equations directly). The oscillations occur as the roles of predator and prey reverse. Pulse stretching was predicted and explained by Enns and Rangnekar.

After the laser pulses have passed through each other the medium's dielectric constant is still highly perturbed (damping of the fluctuation has been neglected). Because of the continued coupling of the medium

† Note added in proof: See also [14].

with each of the laser beams there is an exponential decay rather than a sharp cut off in the trailing edges of the pulses. By increasing the amplitude of singal pulse $\sqrt{250}$ H, Fig. 6.7, this effect becomes quite dramatic. Increasing the amplitude of the signal beam beyond these values is physically unrealistic because electric field gradients of the laser envelope will then be greater than the ionization potential of CCl_4 and dielectric breakdown will result.

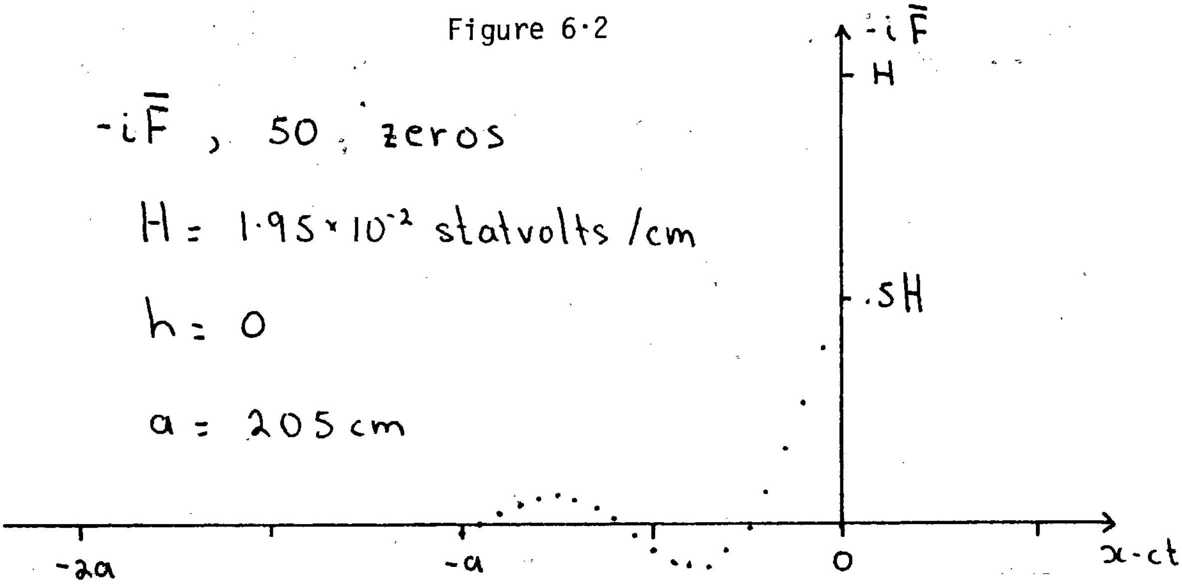
Figure 6·1



$a = 20.5$ cm

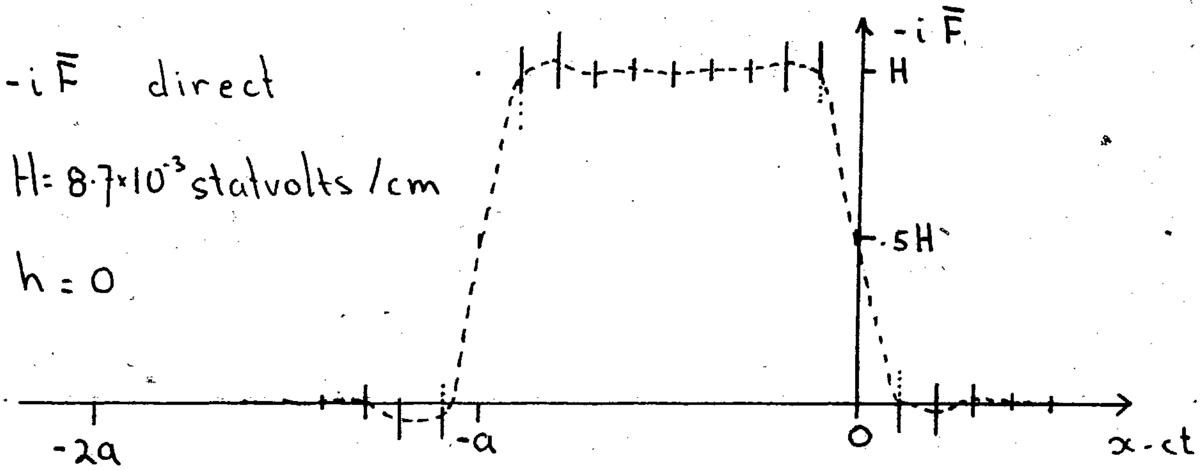
$aH = .18$ statvolts

Figure 6·2



$aH = 4.0$ statvolts

Figure 6·3



$-i\bar{F}$ direct

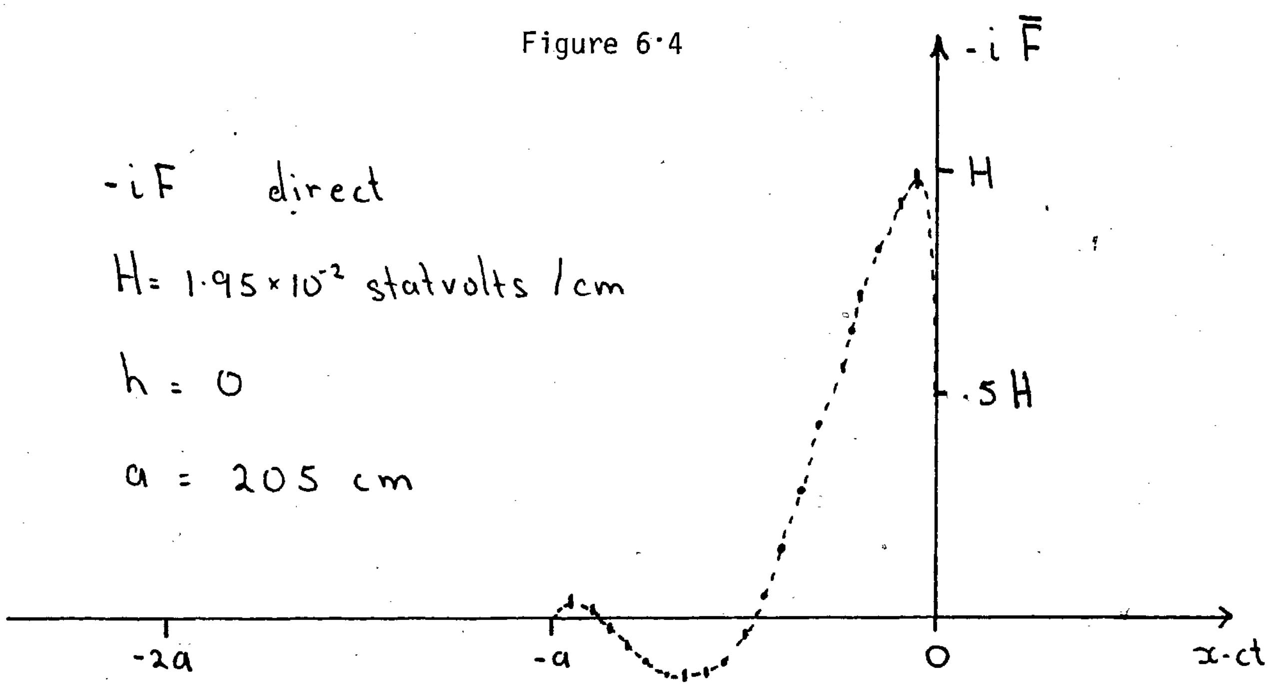
$H = 8.7 \times 10^{-3}$ statvolts /cm

$h = 0$

$a = 20.5$ cm

$aH = .18$ statvolts

Figure 6·4



$-iF$ direct

$H = 1.95 \times 10^{-2}$ statvolts /cm

$h = 0$

$a = 205$ cm

$aH = 4.0$ statvolts

Figure 6·5

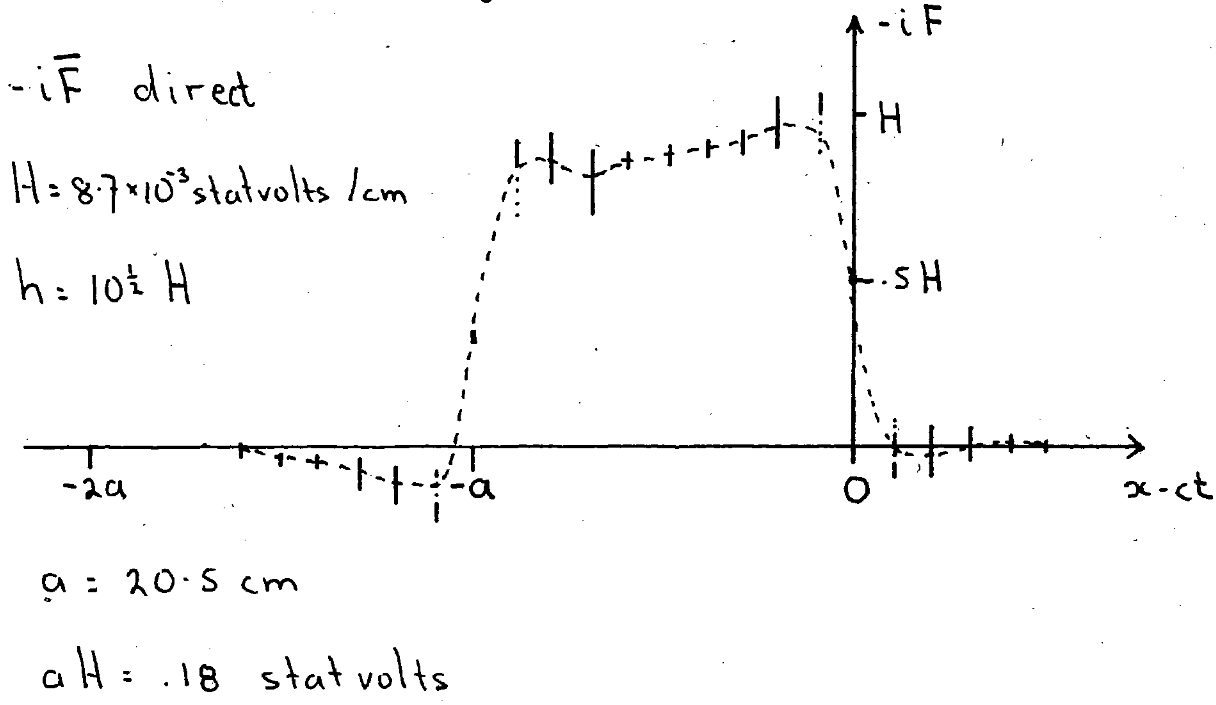


Figure 6·6

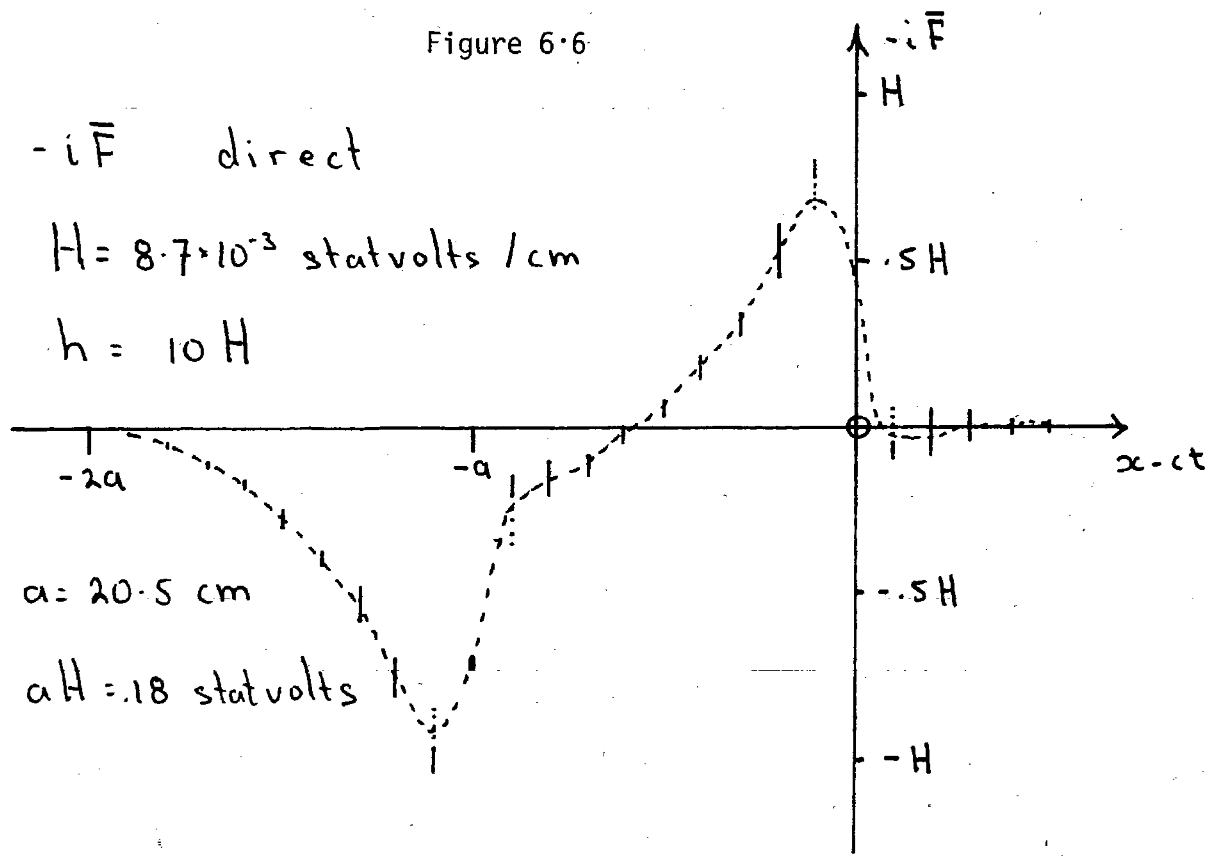
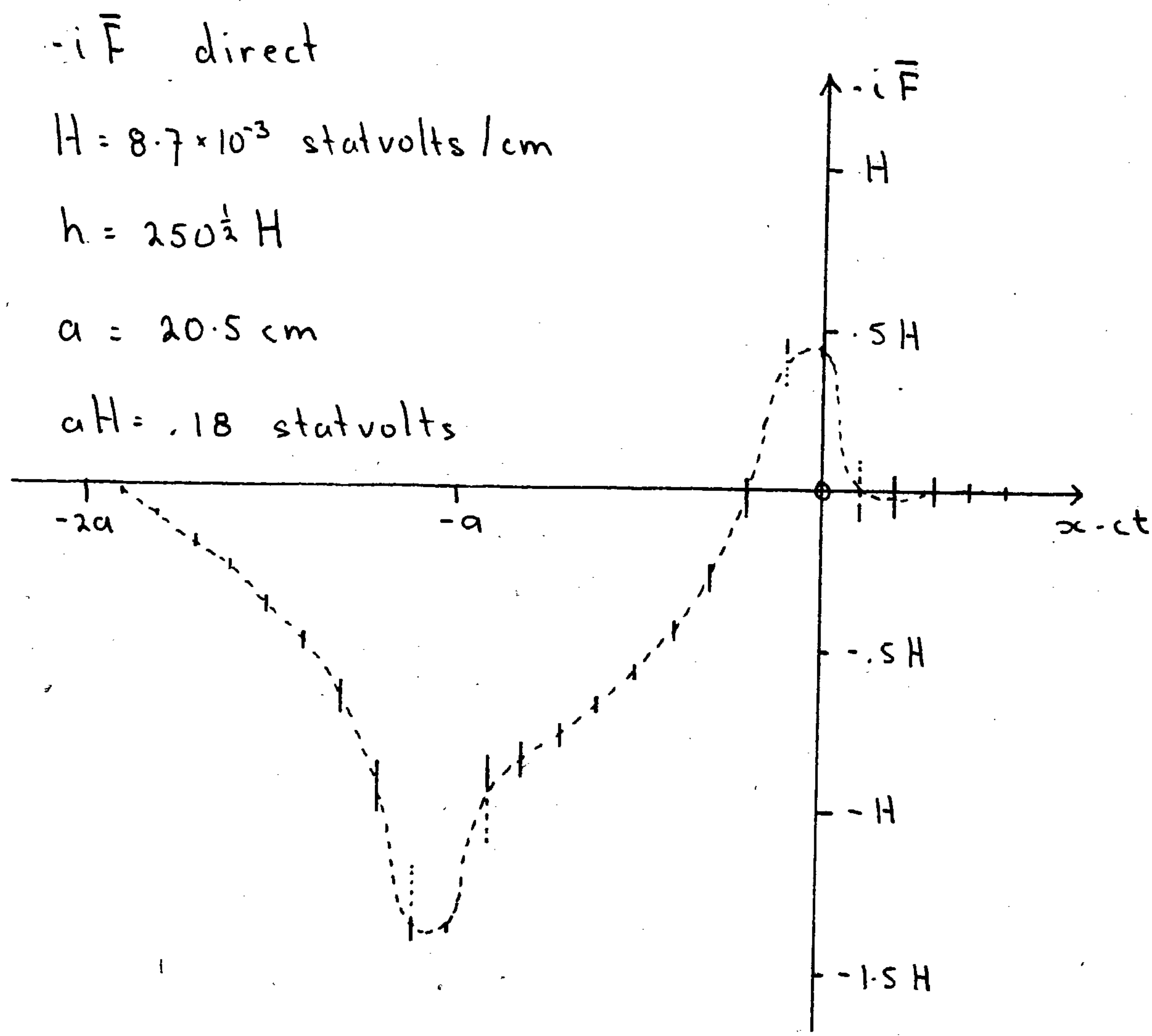


Figure 6·7


$-i\bar{F}$ direct
$H = 8{\cdot}7 \times 10^{-3}$ statvolts/cm
$h = 250^{\frac{1}{2}} H$
$a = 20{\cdot}5$ cm
$aH = .18$ statvolts
$-i\bar{F}$
H
.5 H
-.5 H
- H
-1·5 H
-2a
-a
$x - ct$

To more easily understand the occurrence of these nonlinear effects see Fig. 6.8 a, b, c, and d, depicting the spatial positions of the two laser pulses and the disturbed medium.

Figure 6·8

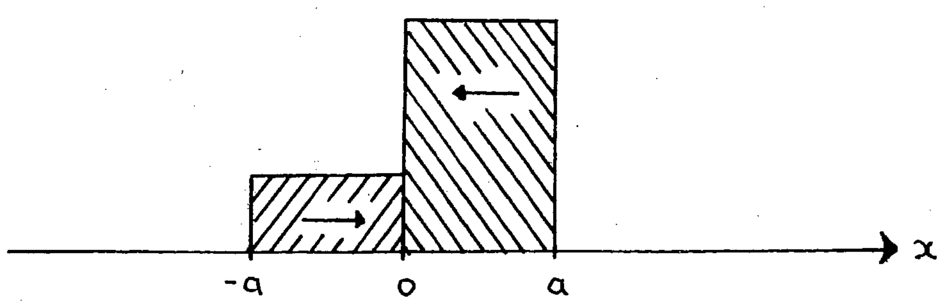
Key to figures

pump envelope

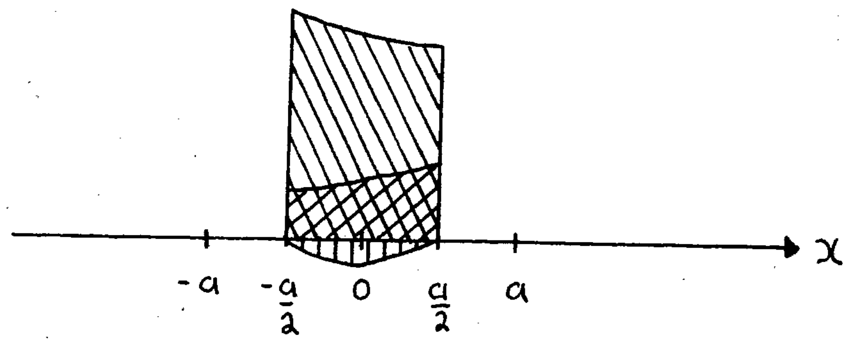
signal envelope

–(medium's fluctuations envelope)

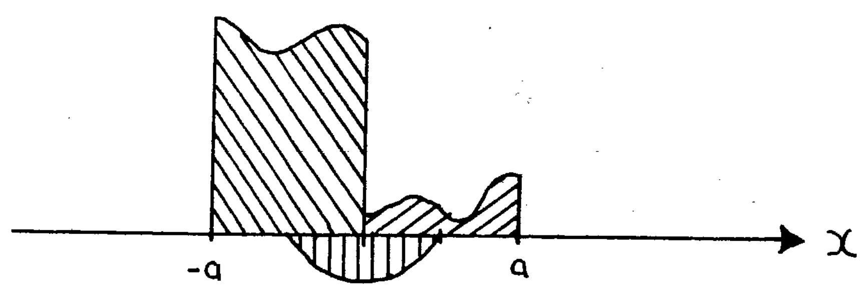
(a) $t = 0$



(b) $t = a/2c$



(c) $t = a/c$



(d) $t = 3a/2c$

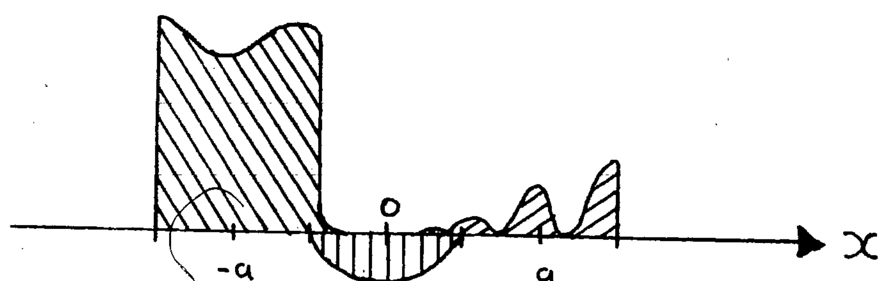


Fig. 6.8(a) schematicaly depicts the initial situation of an undisturbed medium, with two rectangular laser beams about to pass through each other. As the pulses pass through each other they perturb the dielectric constant of the medium. The maximum spatial extent of this disturbance of medium occurs when the two laser pulses completely overlap as shown in diagram 6.8(b). The pulses continue to drive the medium until they no longer overlap, diagram 6.8(c). Once the medium is perturbed it can couple the two laser pulses together so that nonlinear interactions between the two beams can take place. In diagram 6.8(c) we see that the signal beam still overlaps the dielectric perturbation, causing pulse stretching of the pump beam and, similarly, the pump beam overlapping the dielectric perturbation causes the signal beam to stretch. All appreciable nonlinear effects end when the two laser pulses trailing edge reach the edge of the dielectric perturbation, as depicted in the last diagram of the series. The pulses, now in their asymptotic form travel undamped in their respective directions. Since the fluctuation has a negligible velocity (which we have approximated as zero) compared to the speed of light, the asymptotic form of either pulse is actually achieved when the pulses move out of the interaction region. The interaction region does not propagate but remains stationary between $-\frac{a}{2}$ & $+\frac{a}{2}$. Physically this disturbance should die away due to damping but this feature was not included in the three wave equations that we have attempted to solve. If damping of the medium were included the only effect that this would have on the laser pulse solution is to decrease the magnitude of the pulse stretching effect since the magnitude of the medium's disturbance would begin to die away at $t = a/c$ shown in diagram 6.8(c).

What about the asymptotic form of the signal beam? Unfortunately F, in equ. 6.2, can never be an accurate representation of $K_2^{(3)}$ after the interaction so one is forced to attempt to solve the complete integral equation. The reason for this is that F structurally differs from $\bar{F}$ in a very significant way. Unlike $\bar{F}$, F contains no H dependence so it can not possibly account for the interaction of the signal beam with the pump beam.

One also notices that the regions for $\bar{F}$, $\bar{G}$ when compared to F, G are structurally different. The terms $\bar{F}$ and $\bar{G}$, which are associated with the pump beam travelling to the right, are identically zero in front of the Q_3 envelope, $x - ct > 0$. F and G, which are associated with the signal beam, are not zero in front of the leading edge of Q_1, $x + ct < 0$. This suggests that, in the second integral equation 6.2 both the F term and the kernel G term are required to play an equally important role in determining $K_2^{(3)}$, thereby cancelling out in the region $x + ct < 0$. Otherwise, the physically unrealistic situation of the leading edge of a light pulse travelling faster than the speed of light in the medium would occur. It is a feature of the method that the solution for the signal beam is harder to obtain.

Of course if G could be evaluated we would attempt to solve Eq. 6.2 by an iterative procedure. Direct integration of G will involve considerably more computer time than F since a double integral is involved (numerical evaluation of $\overline{F}$'s integral representation took about 15 s/data point). No attempt will be made to carry this out. Obviously, the shape of light pulses for "intermediate times" involves the same problem, i.e., one needs $\overline{G}$ as well.

CHAPTER 7

SUMMARY AND CONCLUSIONS

In this thesis the inverse scattering method has been described and applied to the three wave problem as related to SBS, the backscattering case. Because rectangular initial pulse shapes were chosen steps 1 and 2 proceeded without difficulty and exact expressions were obtained for the time dependent scattering transforms. Numerical techniques were incorporated in the third step to obtain asymptotic solutions for the pump wave envelope.

It was found that the pole representation of $\bar{F}$ converged very slowly; so slowly, in fact, that $\bar{F}$ was subsequently evaluated numerically using its integral form. We have shown that $-i\bar{F}(x)$ represents, to first order in aH when aH < < 1, the amplitude of the pump beam. No corresponding relation was obtained for signal beam as it was shown that the function F and the integral term containing G played an equal role in the determination of the amplitude of the signal beam.

In agreement with results obtained by direct numerical integration of the three wave problem found in the literature our solutions for the pump beam's envelope, using $\bar{F}$, showed the nonlinear effects of: depletion, pulse stretching and amplitude oscillation.

With regard to the poles of $\bar{\rho}_2$ and ρ_2, i.e. zeros of a_{11} and a_{33} we confirmed that a_{11} has all its zeros in the upper half ξ-plane and a_{33} has all its zeros in the lower half ξ-plane†, hence no soliton solutions exist in the backscattering case of SBS. There exists an infinite number of these poles, the real component increasing linearly

†Note added in proof: Kaup[14] has found that a_{33} can have zeros on the positive imaginary axis provided that $ah \gtrsim 1.5$.

while the imaginary component increases logarithmically.

All of the above mentioned results were obtained without having to solve the Marchenko integral equations. It was origianlly hoped that a closed form analytic solution could be carried through steps 1, 2 and 3 of the inverse scattering method. This goal was not achieved. Restricting the problem to a numerical evaluation of the $\bar{F}$ integral to get out an asymptotic solution to the pump laser envelope is of little consequence to the experimentalist who usually only investigates the scattered (i.e. asymptotic) laser pulse, anyway. Numerical evaluation of the original three-wave equations is conceptually simpler but definitely not as efficient as the numerical evaluation of the one integral corresponding to $\bar{F}$.

The problem is by no means wrapped up; there are many unanswered questions. Is it possible to choose a continuous function to represent the initial shapes for which steps 1, 2 and 3 may be solved? Is the slow convergence of the pole representation of $\bar{F}$, $\bar{G}$, etc. a general feature of the continuous background solution? Can we devise an efficient programme, with respect to computing time to evaluate those terms involving double integrals, e.g. G, and thus solve for the final form of the signal beam? Is it possible to apply the ISM when damping is included?

It has been stated in the literature, as early as 1976, that a closed form solution to the three-wave problem, backscattering case, with rectangular initial profiles is obtainable. However, no solution has ever appeared.† This fact and the attempted closed form solution here indicate that such a solution is in fact more difficult to obtain than as first anticipated.

† Even ref.[14] which only appeared when this thesis was being completed

APPENDIX A

Strictly speaking, damping of the laser induced fluctuation should not be neglected in the backscattering case. However, all ISM treatments of the three-wave problem ignore damping. Examination of the relevant operators suggests a simple modification which would lead to the inclusion of damping terms. We go back to chapter 3 where we derived the form of the time evolution operator for the three-wave problem. Inspection of the calculation suggests that a modification of L or B_0 will introduce the desired damping term. Either take

$$L = -iA\partial_x + AV + AE \qquad \text{A·1}$$

where

$$E \equiv \eta_i \delta^i_j$$

or

$$B_0 = -(c_1c_2c_3/c_ic_j)V_{ij} + d_i \delta^i_j \qquad \text{A·2}$$

If L is chosen as above then the resulting three-wave equation obtained from cross differentiating the time evolution equation and the modified linear eigenvalue equation in terms of V_{ij} is

$$V_{ij,t} + \frac{c_1c_2c_3}{c_ic_j}V_{ij,x} + i(\eta_i - \eta_j)V_{ij}\frac{c_1c_2c_3}{c_ic_j} =$$

$$= i\sum_k c_1c_2c_3 V_{ik}V_{kj}\left(\frac{1}{c_ic_k} - \frac{1}{c_kc_j}\right) \qquad \text{A·3}$$

The following conditions are imposed upon η_i when the symmetry of V_{ij} is introduced.

$$
\begin{aligned}
i(\eta_1 - \eta_2) &= i(\eta_1^* - \eta_2^*) \\
i(\eta_1 - \eta_3) &= i(\eta_1^* - \eta_3^*) \qquad \text{A.4} \\
i(\eta_2 - \eta_3) &= i(\eta_2^* - \eta_3^*)
\end{aligned}
$$

We see immediately from A4 that η_i must be real but this leads to <u>purely imaginary damping</u> coefficients in the three-wave equations, i.e. in A3. This is physically unacceptable.

APPENDIX B

$y<0 \qquad x-y<-a \qquad x<0$ 1

$$G(x,y)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4\beta_{23}h^2 i\Bigg\{\Big[P_2(\xi_2^i)I_2(-\xi_2^i)e^{i\xi_2^i\beta_{23}(x-y)}-cc\Big]$$

$$+\Big[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i+\xi_2^j}e^{i\xi_2^i\beta_{23}y}e^{i\xi_2^j\beta_{23}x}-cc\Big]$$

$$+\Big[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}e^{i\xi_2^i\beta_{23}y}e^{-i\xi_2^{j*}\beta_{23}x}-cc\Big]\Bigg\}$$

$y<0 \qquad -a<x-y<0 \qquad x<0$ 2

$$G(x,y)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4\beta_{23}h^2 i\Bigg\{\Big[J_2(-\xi_2^i)Q_2'(\xi_2^i)e^{-i\xi_2^i\beta_{23}(x-y)}-cc\Big]$$

$$+\Big[\Big(P_2(\xi_2^i)J_2(-\xi_2^i)+Q_2'(\xi_2^i)J_2'(-\xi_2^i)\Big)e^{i\xi_2^i\beta_{23}(x-y)}-cc\Big]$$

$$+\Big[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i+\xi_2^j}e^{i\xi_2^i\beta_{23}y}e^{i\xi_2^j\beta_{23}x}-cc\Big]$$

$$+\Big[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}e^{i\xi_2^i\beta_{23}y}e^{-i\xi_2^{j*}\beta_{23}x}-cc\Big]\Bigg\}$$

$y<0 \qquad 0<x-y<a \qquad x<0$ 3

$$G(x,y)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4\beta_{23}h^2 i\Bigg\{\Big[Q_2'(\xi_2^i)J_2(-\xi_2^i)e^{i\xi_2^i\beta_{23}(x-y)}-cc\Big]+$$

$$+\left[\left(J_2(-\xi_2^i)P_2(\xi_2^i) + J_2'(-\xi_2^i)Q_2'(\xi_2^i)\right)e^{-i\xi_2^i\beta_{23}(x-y)} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i + \xi_2^j} e^{i\xi_2^i\beta_{23}y} e^{i\xi_2^j\beta_{23}x} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}} e^{i\xi_2^i\beta_{23}y} e^{-i\xi_2^{j*}\beta_{23}x} - cc\right]\Big\}$$

$y<0 \qquad 0<x-y<a \qquad 0<x<a$ 4

$$G(x,y) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4\beta_{23}h^2 i\Big\{\left[Q_2'(\xi_2^i)J_2(-\xi_2^i)e^{i\xi_2^i\beta_{23}(x-y)} - cc\right]$$

$$+\left[J_2'(-\xi_2^i)Q_2'(\xi_2^i)e^{-i\xi_2^i\beta_{23}(x-y)} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)Q_2'(\xi_2^j)}{\xi_2^i + \xi_2^j} e^{i\xi_2^i\beta_{23}y} e^{i\xi_2^j\beta_{23}x} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)Q_2'^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}} e^{i\xi_2^i\beta_{23}y} e^{-i\xi_2^{j*}\beta_{23}x} - cc\right]\Big\}$$

$y<0 \qquad a<x-y \qquad x<0$ 5

$$G(x,y) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4\beta_{23}h^2 i\Big\{\left[I_2(-\xi_2^i)P_2(\xi_2^i)e^{-i\xi_2^i\beta_{23}(x-y)} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i + \xi_2^j} e^{i\xi_2^i\beta_{23}y} e^{i\xi_2^j\beta_{23}x} - cc\right] +$$

$$+\left[\frac{P_2(\xi_2^i)\,P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}\,e^{i\xi_2^i\beta_{23}y}\,e^{-i\xi_2^{j*}\beta_{23}x}-cc\right]\Bigg\}$$

$y<0 \qquad a<x-y \qquad 0<x<a$ 6

$$G(x,y)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4\beta_{23}h^2 i\Bigg\{\left[J_2'(-\xi_2^i)\,P_2(\xi_2^i)\,e^{-i\xi_2^i\beta_{23}(x-y)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)\,Q_2'(\xi_2^j)}{\xi_2^i+\xi_2^j}\,e^{i\xi_2^i\beta_{23}y}\,e^{i\xi_2^j\beta_{23}x}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)\,Q_2'^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}\,e^{i\xi_2^i\beta_{23}y}\,e^{-i\xi_2^{j*}\beta_{23}x}-cc\right]\Bigg\}$$

$y<0 \qquad a<x-y \qquad a<x$ 7

$$G(x,y)=0$$

$0<y<a \qquad x-y<-a \qquad x<0$ 8

$$G(x,y)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4\beta_{23}h^2 i\Bigg\{\left[P_2(\xi_2^i)\,J_2'(-\xi_2^i)\,e^{i\xi_2^i\beta_{23}(x-y)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^i)\,P_2(\xi_2^j)}{\xi_2^i+\xi_2^j}\,e^{i\xi_2^i\beta_{23}y}\,e^{i\xi_2^j\beta_{23}x}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^i)\,P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}\,e^{i\xi_2^i\beta_{23}y}\,e^{-i\xi_2^{j*}\beta_{23}x}-cc\right]\Bigg\}$$

$0<y<a \qquad -a<x-y<0 \qquad x<0$ 9

$$
\begin{aligned}
G(x,y) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4\beta_{23}h^2 i \Big\{ & \Big[Q_2'(\xi_2^i)\, J_2'(-\xi_2^i)\, e^{i\xi_2^i\beta_{23}(x-y)} - cc \Big] \\
& + \Big[J_2(-\xi_2^i)\, Q_2'(\xi_2^i)\, e^{-i\xi_2^i\beta_{23}(x-y)} - cc \Big] \\
& + \Big[\frac{Q_2'(\xi_2^i)\, P_2(\xi_2^j)}{\xi_2^i + \xi_2^j}\, e^{i\xi_2^i\beta_{23}y}\, e^{i\xi_2^j\beta_{23}x} - cc \Big] \\
& + \Big[\frac{Q_2'(\xi_2^i)\, P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}\, e^{i\xi_2^i\beta_{23}y}\, e^{-i\xi_2^{j*}\beta_{23}x} - cc \Big] \Big\}
\end{aligned}
$$

$0<y<a \qquad -a<x-y \qquad 0<x<a$ 10

$$
\begin{aligned}
G(x,y) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4\beta_{23}h^2 i \Big\{ & \Big[Q_2'(\xi_2^i)\, J_2'(-\xi_2^i)\, e^{i\xi_2^i\beta_{23}(x-y)} - cc \Big] \\
& + \Big[\frac{Q_2'(\xi_2^i)\, Q_2'(\xi_2^j)}{\xi_2^i + \xi_2^j}\, e^{i\xi_2^i\beta_{23}y}\, e^{i\xi_2^j\beta_{23}x} - cc \Big] \\
& + \Big[\frac{Q_2'(\xi_2^i)\, Q_2'^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}\, e^{i\xi_2^i\beta_{23}y}\, e^{-i\xi_2^{j*}\beta_{23}x} - cc \Big] \Big\}
\end{aligned}
$$

$0<y<a \qquad 0<x-y \qquad 0<x<a$ 11

$$
G(x,y) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4\beta_{23}h^2 i \Big\{ \Big[J_2'(-\xi_2^i)\, Q_2'(\xi_2^i)\, e^{-i\xi_2^i\beta_{23}(x-y)} - cc \Big] +
$$

$$+\left[\frac{Q_2'(\xi_2^i)\,Q_2'(\xi_2^j)}{\xi_2^i+\xi_2^j}\, e^{i\xi_2^i\beta_{23}y}\, e^{i\xi_2^j\beta_{23}x} - cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^i)\,Q_2'^{*}(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}\, e^{i\xi_2^i\beta_{23}y}\, e^{-i\xi_2^{j*}\beta_{23}x} - cc\right]\Bigg\}$$

$0<y<a \qquad 0<x-y \qquad a<x$ 12

$G(x,y)=0$

$y<a$ 13

$G(x,y)=0$

APPENDIX C

Evaluation of $H(x,y)$ is straightforward, proceeding along similar lines to $G(x,y)$ and $\bar{G}(x,y)$. The term $-i\varepsilon$ pushes the pole $\xi + i\varepsilon$ above the contour C (the poles of $\rho_2(\xi')$ are below C). After evaluating the first integral (with respect to ξ'), ε is set to zero from the right. Although, in the beginning of this chapter I choose $c_1 = -c_3$, I have not introduced this restriction into the problem, and will not do so until time evolution of the scattering data is calculated. Therefore β_{12} is in general not assumed to be equal to β_{23}. I take $\beta_{12} > \beta_{23}$ as this corresponds to having the velocity characteristic of the fluctuations $v_2 \geq 0$.

Define X such that

$$X = \beta_{12} x + \beta_{23} y$$

$H(x,y)$ is

$y < 0 \qquad x < -a$

$$H(x,y) = \sum_{\xi_1^i} \sum_{\xi_2^j} -4\beta_{23} h H i \left\{ \left[P_2(\xi_2^j) I_1(-\xi_2^j)\, e^{i\xi_2^j X} - cc \right] \right.$$

$$+ \left[\frac{P_2(\xi_2^j) P_1(\xi_1^i)}{\xi_2^j + \xi_1^i} e^{i\xi_2^j \beta_{23} y}\, e^{-i\xi_1^i \beta_{12} x} - cc \right]$$

$$\left. + \left[\frac{P_2(\xi_2^j) P_1^*(\xi_1^i)}{\xi_2^j - \xi_1^{i*}} e^{i\xi_2^j \beta_{23} y}\, e^{i\xi_1^{i*} \beta_{12} x} - cc \right] \right\}$$

---

* Equ. 5.145 - 5.154 do not exist

$y<0 \qquad -a<x<0$ 2

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\left\{\left[P_2(\xi_2^j)\,\mathcal{F}_1(-\xi_2^j)\,e^{i\xi_2^j x}-cc\right]\right.$$

$$+\left[\frac{P_2(\xi_2^j)\,Q_1(\xi_1^i)}{\xi_2^j+\xi_1^i}\,e^{i\xi_2^j\beta_{23}y}\,e^{-i\xi_1^i\beta_{12}x}-cc\right]$$

$$\left.+\left[\frac{P_2(\xi_2^j)\,Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\,e^{i\xi_2^j\beta_{23}y}\,e^{i\xi_1^{i*}\beta_{12}x}-cc\right]\right\}$$

$y<0 \qquad 0<x$ 3

$$H(x,y)=0\,.$$

$0<y<a \qquad X<-\beta_{12}a \qquad x<-a$ 4

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\left\{\left[Q_2(\xi_2^j)\,I_1(-\xi_2^j)\,e^{i\xi_2^j x}-cc\right]\right.$$

$$-\left[J_2(-\xi_1^i)\,P_1(\xi_1^i)\,e^{-i\xi_1^i x}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)\,P_1(\xi_1^i)}{\xi_2^j+\xi_1^i}\,e^{i\xi_2^j\beta_{23}y}\,e^{-i\xi_1^i\beta_{12}x}-cc\right]$$

$$\left.+\left[\frac{Q_2'(\xi_2^j)\,P_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\,e^{i\xi_1^i\beta_{23}y}\,e^{i\xi_1^{i*}\beta_{12}x}-cc\right]\right\}$$

$$0<y<a \qquad -\beta_{12}a<X<0 \qquad x<-a \qquad\qquad 5$$

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\Big\{-\Big[J_2(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i x}-cc\Big]$$

$$+\Big[\Big(Q_2(\xi_2^j)J_1(-\xi_2^j)+Q_2'(\xi_2^j)I_1(-\xi_2^j)\Big)e^{i\xi_2^j x}-cc\Big]$$

$$+\Big[\frac{Q_2'(\xi_2^j)P_1(\xi_1^i)}{\xi_2^j+\xi_1^i}e^{i\xi_2^j\beta_{23}\overset{x}{y}}e^{-i\xi_1^i\beta_{12}x}-cc\Big]$$

$$+\Big[\frac{Q_2'(\xi_2^j)P_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}e^{i\xi_2^j\beta_{23}y}e^{i\xi_1^{i*}\beta_{12}x}-cc\Big]\Big\}\ .$$

$$0<y<a \qquad -\beta_{12}a<X<0 \qquad -a<x<0 \qquad\qquad 6$$

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\Big\{-\Big[J_2(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i x}-cc\Big]$$

$$+\Big[\Big(Q_2(\xi_2^j)J_1(-\xi_2^j)+Q_2'(\xi_2^j)J_1(-\xi_2^j)\Big)e^{i\xi_2^j x}-cc\Big]$$

$$+\Big[\frac{Q_2'(\xi_2^j)Q_1(\xi_1^i)}{\xi_2^j+\xi_1^i}e^{i\xi_2^j\beta_{23}y}e^{-i\xi_1^i\beta_{12}x}-cc\Big]$$

$$+\Big[\frac{Q_2'(\xi_2^j)Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}e^{i\xi_2^j\beta_{23}y}e^{i\xi_1^{i*}\beta_{12}x}-cc\Big]\Big\}\ .$$

$0 < y < a \qquad 0 < X \qquad -a < x < 0$ 7

$$H(x,y) = \sum_{\xi_1^i} \sum_{\xi_2^j} -4\beta_{23} h H i \left\{ \left[Q_2'(\xi_2^j) J_1(-\xi_2^j) e^{-i\xi_2^j X} - cc \right] \right.$$

$$+ \left[\frac{Q_2'(\xi_2^j) Q_1(\xi_1^i)}{\xi_2^j + \xi_1^i} e^{i\xi_2^j \beta_{23} y} e^{-i\xi_1^i \beta_{12} x} - cc \right]$$

$$\left. + \left[\frac{Q_2'(\xi_2^j) Q_1^*(\xi_1^i)}{\xi_2^j - \xi_1^{i*}} e^{i\xi_2^j \beta_{23} y} e^{i\xi_1^{i*} \beta_{12} x} - cc \right] \right\}$$

$0 < y < a \qquad 0 < X \qquad 0 < x$ 8

$$H(x,y) = 0$$

$a < y \qquad X < -\beta_{12} a$ 9

$$H(x,y) = \sum_{\xi_1^i} \sum_{\xi_2^j} -4\beta_{23} h H i \left\{ \left[P_2(\xi_2^j) I_1(-\xi_2^j) e^{i\xi_2^j X} - cc \right] \right.$$

$$\left. - \left[I_2(-\xi_1^i) P_1(\xi_1^i) e^{-i\xi_1^i X} - cc \right] \right\}$$

$a < y \qquad -\beta_{12} a < X < \beta_{23} a - \beta_{12} a$ 10

$$H(x,y) = \sum_{\xi_1^i} \sum_{\xi_2^j} -4\beta_{23} h H i \left\{ \left[\left(P_2(\xi_2^j) J_1(-\xi_2^j) + Q_2'(\xi_2^j) J_1'(-\xi_2^j) \right) e^{i\xi_2^j X} - cc \right] + \right.$$

$$-\left[\left(I_2(-\xi_1^i)Q_1(\xi_1^i)+J_2'(-\xi_1^i)Q_1'(\xi_1^i)\right)e^{-i\xi_1^i X}-cc\right]\Big\}$$

$a<y \qquad \beta_{23}a-\beta_{12}a<X<0$ 11

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\Big\{\left[P_2(\xi_2^j)J_1(-\xi_2^j)e^{i\xi_2^j X}-cc\right]$$

$$-\left[I_2(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i X}-cc\right]\Big\}$$

$a<y \qquad 0<X<\beta_{23}a$ 12

$$H(x,y)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4\beta_{23}hHi\Big\{\left[Q_2'(\xi_2^j)J_1(-\xi_2^j)e^{i\xi_2^j X}-cc\right]$$

$$-\left[J_2'(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i X}-cc\right]\Big\}$$

$a<y \qquad \beta_{23}a<X$ 13

$H(x,y)=0$

$\bar{H}(x,y)$ not written out here, is easily found using

$$\bar{H}(x,y) = \frac{\beta_{12}}{\beta_{23}} H^*(y,x)$$

APPENDIX D

$x - ct < -a$

$$\bar{F}(x;t) = \sum_{\xi_1^i} 2cHi\left[P_1(\xi_1^i)e^{-i\xi_1^i c(x-ct)} + c.c.\right]$$

$-a < x - ct < 0$

$$\bar{F}(x;t) = \sum_{\xi_1^i} 2cHi\left[Q_1(\xi_1^i)e^{-i\xi_1^i c(x-ct)} + c.c.\right]$$

$0 < x - ct$

$$\bar{F}(x;t) = 0$$

$x + ct < 0$

$$F(x;t) = \sum_{\xi_2^i} -2chi\left[P_2(\xi_2^i)e^{i\xi_2^i c(x+ct)} + c.c.\right]$$

$0 < x + ct < a$

$$F(x;t) = \sum_{\xi_2^i} -2chi\left[Q_2'(\xi_2^i)e^{i\xi_2^i c(x+ct)} + c.c.\right]$$

$a < x + ct$

$$F(x;t) = 0$$

$y-ct<-a \qquad x-y<-a \qquad x-ct<-a$ 1

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j}4H^2ci\left\{\left[P_1(\xi_1^i)\,I_1(-\xi_1^i)\,e^{-i\xi_1^i c(x-y)}-cc\right]\right.$$

$$+\left[\frac{P_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}\,e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)}-cc\right]$$

$$\left.+\left[\frac{P_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}\,e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*} c(x-ct)}-cc\right]\right\}$$

$y-ct<-a \qquad -a<x-y<0 \qquad x-ct<-a$ 2

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j}4H^2ci\left\{\left[Q_1(\xi_1^i)\,J_1'(-\xi_1^i)\,e^{i\xi_1^i c(x-y)}-cc\right]\right.$$

$$+\left[\left(P_1(\xi_1^i)\,J_1'(-\xi_1^i)+Q(\xi_1^i)\,J_1(-\xi_1^i)\right)e^{-i\xi_1^i c(x-y)}-cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}\,e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)}-cc\right]$$

$$\left.+\left[\frac{P_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}\,e^{-i\xi_1^i c(y-ct)}e^{+i\xi_1^{j*} c(x-ct)}-cc\right]\right\}$$

$y-ct<-a \qquad 0<x-y<a \qquad x-ct<-a$ 3

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j}4H^2ci\left\{\left[Q_1(\xi_1^i)\,J_1'(-\xi_1^i)\,e^{-i\xi_1^i c(x-y)}-cc\right]\right.$$

$$+\left[\left(J_1'(-\xi_1^i)\,P_1(\xi_1^i)+J_1(-\xi_1^i)\,Q_1(\xi_1^i)\right)e^{i\xi_1^i c(x-y)}-cc\right]+$$

$$+\left[\frac{P_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)} - cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*} c(x-ct)} - cc\right]\Bigg\}$$

$y-ct< -a \qquad 0<x-y<a \qquad -a<x-ct<0$ 4

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j} 4H^2 ci\Bigg\{\left[J_1(-\xi_1^i)Q_1(\xi_1^i)e^{i\xi_1^i c(x-y)} - cc\right]$$

$$+\left[Q_1(\xi_1^i)J_1'(-\xi_1^i)e^{-i\xi_1^i c(x-y)} - cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)Q_1(\xi_1^i)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)} - cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^j c(x-ct)} - cc\right]\Bigg\}$$

$y-ct<-a \qquad a<x-y \qquad x-ct<-a$ 5

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j} 4H^2 ci\Bigg\{\left[I_1(-\xi_1^i)P_1(\xi_1^i)e^{i\xi_1^i c(x-y)} - cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)} - cc\right]$$

$$+\left[\frac{P_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*} c(x-ct)} - cc\right]\Bigg\}$$

$y-ct<-a \qquad a<x-y \qquad -a<x-ct<0$ 6

$$\bar{G}(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_1^j} 4H^2ci\Big\{\Big[J_1(-\xi_1^i)P_1(\xi_1^i)e^{i\xi_1^i c(x-y)} - cc\Big]$$

$$+\Big[\frac{P_1(\xi_1^i)Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)} - cc\Big]$$

$$+\Big[\frac{P_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*}c(x-ct)} - cc\Big]\Big\}$$

$y-ct<-a \qquad a<x-y \qquad 0<x-ct$ 7

$$\bar{G}(x,y;t) = 0$$

$-a<y-ct<0 \qquad x-y<-a \qquad x-ct<-a$ 8

$$\bar{G}(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_1^j} 4H^2ci\Big\{\Big[P_1(\xi_1^i)J_1(-\xi_1^i)e^{-i\xi_1^i c(x-y)} - cc\Big]$$

$$+\Big[\frac{Q_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)} - cc\Big]$$

$$+\Big[\frac{Q_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*}c(x-ct)} - cc\Big]\Big\}$$

$-a<y-ct<0 \qquad -a<x-y<0 \qquad x-ct<-a$ 9

$$\bar{G}(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_1^j} 4H^2ci\Big\{\Big[J_1'(-\xi_1^i)Q(\xi_1^i)e^{i\xi_1^i c(x-y)} - cc\Big]$$

$$+\Big[Q_1(\xi_1^i)J_1(-\xi_1^i)e^{-i\xi_1^i c(x-y)} - cc\Big] +$$

$$+\left[\frac{Q_1(\xi_1^i)P_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)P_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*} c(x-ct)}-cc\right]\Bigg\}$$

$-a<y-ct<0 \qquad -a<x-y<0 \qquad -a<x-ct<0$ 10

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j}4H^2ci\Bigg\{\left[Q_1(\xi_1^i)J_1(-\xi_1^i)e^{-i\xi_1^i c(x-y)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*} c(x-ct)}-cc\right]\Bigg\}$$

$-a<y-ct<0 \qquad 0<x-y \qquad -a<x-ct<0$ 11

$$\bar{G}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_1^j}4H^2ci\Bigg\{\left[J_1(-\xi_1^i)Q_1(\xi_1^i)e^{i\xi_1^i c(x-y)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1(\xi_1^j)}{\xi_1^i+\xi_1^j}e^{-i\xi_1^i c(y-ct)}e^{-i\xi_1^j c(x-ct)}-cc\right]$$

$$+\left[\frac{Q_1(\xi_1^i)Q_1^*(\xi_1^j)}{\xi_1^i-\xi_1^{j*}}e^{-i\xi_1^i c(y-ct)}e^{i\xi_1^{j*}(x-ct)}-cc\right]\Bigg\}$$

$-a<y-ct<0 \qquad 0<x-y \qquad 0<x-ct$ 12

$$\bar{G}(x,y;t)=0$$

$0 < y - ct$ 13

$$\bar{G}(x,y;t) = 0$$

$y + ct < 0 \qquad x - y < -a \qquad x + ct < 0$ 1

$$G(x,y;t) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4h^2ci\left\{\left[P_2(\xi_2^i)I_2(-\xi_2^i)e^{i\xi_2^i c(x-y)} - cc\right]\right.$$

$$+\left[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i + \xi_2^j}e^{i\xi_2^i c(y+ct)}e^{i\xi_2^j c(x+ct)} - cc\right]$$

$$\left.+\left[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}e^{i\xi_2^i c(y+ct)}e^{-i\xi_2^{j*}c(x+ct)} - cc\right]\right\}$$

$y + ct < 0 \qquad -a < x - y < 0 \qquad x + ct < 0$ 2

$$G(x,y;t) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4h^2ci\left\{\left[J_2(-\xi_2^i)Q_2'(\xi_2^i)e^{-i\xi_2^i c(x-y)} - cc\right]\right.$$

$$+\left[\left(P_2(\xi_2^i)J_2(-\xi_2^i) + Q_2'(\xi_2^i)J_2'(-\xi_2^i)\right)e^{i\xi_2^i c(x-y)} - cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i + \xi_2^j}e^{i\xi_2^i c(y+ct)}e^{i\xi_2^j c(x+ct)} - cc\right]$$

$$\left.+\left[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}e^{i\xi_2^i c(y+ct)}e^{-i\xi_2^{j*}c(x+ct)} - cc\right]\right\}$$

$y+ct<0 \qquad 0<x-y<a \qquad x+ct<0$ 3

$$G(x,y;t) = \sum_{\xi_2^i} \sum_{\xi_2^j} -4h^2 ci \Big\{ \Big[Q_2'(\xi_2^i) J_2(-\xi_2^i) e^{i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\Big(J_2(-\xi_2^i) P_2(\xi_2^i) + J_2'(-\xi_2^i) Q_2'(\xi_2^i) \Big) e^{-i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_2(\xi_2^i) P_2(\xi_2^j)}{\xi_2^i + \xi_2^j} e^{i\xi_2^i c(y+ct)} e^{i\xi_2^j c(x+ct)} - cc \Big]$$

$$+ \Big[\frac{P_2(\xi_2^i) P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}} e^{i\xi_2^i c(y+ct)} e^{-i\xi_2^{j*} c(x+ct)} - cc \Big] \Big\}$$

$y+ct<0 \qquad 0<x-y<a \qquad 0<x+ct<a$ 4

$$G(x,y;t) = \sum_{\xi_2^i} \sum_{\xi_2^j} -4h^2 ci \Big\{ \Big[Q_2'(\xi_2^i) J_2(-\xi_2^i) e^{i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[J_2'(-\xi_2^i) Q_2'(\xi_2^i) e^{-i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\frac{P_2(\xi_2^i) Q_2'(\xi_2^j)}{\xi_2^i + \xi_2^j} e^{i\xi_2^i c(y+ct)} e^{i\xi_2^j c(x+ct)} - cc \Big]$$

$$+ \Big[\frac{P_2(\xi_2^i) Q_2'^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}} e^{i\xi_2^i c(y+ct)} e^{-i\xi_2^{j*} c(x+ct)} - cc \Big] \Big\}$$

$y+ct<0 \qquad a<x-y \qquad x+ct<0$ 5

$$G(x,y;t) = \sum_{\xi_2^i} \sum_{\xi_2^j} -4h^2 ci \Big\{ \Big[I_2(-\xi_2^i) P_2(\xi_2^i) e^{-i\xi_2^i c(x-y)} - cc \Big] +$$

$$+\left[\frac{P_2(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i+\xi_2^j}e^{i\xi_2^i c(y+ct)}e^{i\xi_2^j c(x+ct)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}e^{i\xi_2^i c(y+ct)}e^{-i\xi_2^{j*}c(x+ct)}-cc\right]\Bigg\}$$

$y+ct<0 \qquad a<x-y \qquad 0<x+ct<a$ 6

$$G(x,y;t)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4h^2ci\Bigg\{\left[J_2'(-\xi_2^i)P_2(\xi_2^i)e^{-i\xi_2^i c(x-y)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)Q_2'(\xi_2^j)}{\xi_2^i+\xi_2^j}e^{i\xi_2^i c(y+ct)}e^{i\xi_2^j c(x+ct)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^i)Q_2'^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}e^{i\xi_2^i c(y+ct)}e^{-i\xi_2^{j*}c(x+ct)}-cc\right]\Bigg\}$$

$y+ct<0 \qquad a<x-y \qquad a<x+ct$ 7

$$G(x,y;t)=0$$

$0<y+ct<a \qquad x-y<-a \qquad x+ct<0$ 8

$$G(x,y;t)=\sum_{\xi_2^i}\sum_{\xi_2^j}-4h^2ci\Bigg\{\left[P_2(\xi_2^i)J_2'(-\xi_2^i)e^{i\xi_2^i c(x-y)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^i)P_2(\xi_2^j)}{\xi_2^i+\xi_2^j}e^{i\xi_2^i c(y+ct)}e^{i\xi_2^j c(x+ct)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^i)P_2^*(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}e^{i\xi_2^i c(y+ct)}e^{-i\xi_2^{j*}c(x+ct)}-cc\right]\Bigg\}$$

$0 < y+ct < a \qquad -a < x-y < 0 \qquad x+ct < 0$ 9

$$G(x,y;t) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4h^2ci \Big\{ \Big[Q_2'(\xi_2^i)\, J_2'(-\xi_2^i)\, e^{i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[J_2(-\xi_2^i)\, Q_2(\xi_2^i)\, e^{-i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\frac{Q_2'(\xi_2^i)\, P_2(\xi_2^j)}{\xi_2^i + \xi_2^j}\, e^{i\xi_2^i c(y+ct)}\, e^{i\xi_2^j c(x+ct)} - cc \Big]$$

$$+ \Big[\frac{Q_2'(\xi_2^i)\, P_2^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}\, e^{i\xi_2^i c(y+ct)}\, e^{-i\xi_2^{j*} c(x+ct)} - cc \Big] \Big\}$$

$0 < y+ct < a \qquad -a < x-y < 0 \qquad 0 < x+ct < a$ 10

$$G(x,y;t) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4h^2ci \Big\{ \Big[Q_2'(\xi_2^i)\, J_2'(-\xi_2^i)\, e^{i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\frac{Q_2'(\xi_2^i)\, Q_2'(\xi_2^j)}{\xi_2^i + \xi_2^j}\, e^{i\xi_2^i c(y+ct)}\, e^{i\xi_2^j c(x+ct)} - cc \Big]$$

$$+ \Big[\frac{Q_2'(\xi_2^i)\, Q_2'^*(\xi_2^j)}{\xi_2^i - \xi_2^{j*}}\, e^{i\xi_2^i c(y+ct)}\, e^{-i\xi_2^{j*} c(x+ct)} - cc \Big] \Big\}$$

$0 < y+ct < a \qquad 0 < x-y \qquad 0 < x+ct < a$ 11

$$G(x,y;t) = \sum_{\xi_2^i}\sum_{\xi_2^j} -4h^2ci \Big\{ \Big[J_2'(-\xi_2^i)\, Q_2'(\xi_2^i)\, e^{-i\xi_2^i c(x-y)} - cc \Big]$$

$$+ \Big[\frac{Q_2'(\xi_2^i)\, Q_2'(\xi_2^j)}{\xi_2^i + \xi_2^j}\, e^{-i\xi_2^i c(y+ct)}\, e^{i\xi_2^j c(x+ct)} - cc \Big] +$$

$$+\left[\frac{Q_2'(\xi_2^i)\,Q_2'^{*}(\xi_2^j)}{\xi_2^i-\xi_2^{j*}}\,e^{i\xi_2^i c(y+ct)}\,e^{-i\xi_2^{j*}c(x+ct)}-cc\right]\Big\}$$

$0<y+ct<a$ $\quad$ $0<x-y$ $\quad$ $a<x+ct$ 12

$G(x,y;t)=0$

$y+ct<a$ 13

$G(x,y;t)=0$

$x+ct<0$ $\quad$ $y-ct<-a$ 1

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4hHci\Big\{\left[P_2(\xi_2^j)I_1(-\xi_2^j)e^{i\xi_2^j c(x+y)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^j)\,P_1(\xi_1^i)}{\xi_2^j+\xi_1^i}\,e^{i\xi_2^j c(x+ct)}\,e^{-i\xi_1^i c(y-ct)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^j)\,P_1^{*}(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\,e^{i\xi_2^j c(x+ct)}\,e^{i\xi_1^{i*}c(y-ct)}-cc\right]\Big\}$$

$x+ct<0$ $\quad$ $-a<y-ct<0$ 2

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4hHci\Big\{\left[P_2(\xi_2^j)J_1(-\xi_2^j)e^{i\xi_2^j c(x+y)}-cc\right]$$

$$+\left[\frac{P_2(\xi_2^j)\,Q_1(\xi_1^i)}{\xi_2^j+\xi_1^i}\,e^{i\xi_2^j c(x+ct)}\,e^{-i\xi_1^i c(y-ct)}-cc\right]+$$

$$+\left[\frac{P_2(\xi_2^j)Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}e^{i\xi_2^j c(x+ct)}e^{i\xi_1^{i*}c(y-ct)}-cc\right]\Bigg\}$$

$x+ct<0 \qquad 0<y-ct$ 3

$$\bar{H}(x,y;t)=0$$

$0<x+ct<a \qquad x+y<-a \qquad y-ct<-a$ 4

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4hHci\Bigg\{\left[Q_2(\xi_2^j)I_1(-\xi_2^j)e^{i\xi_2^j c(x+y)}-cc\right]$$

$$+\left[J_2(-\xi_1^i)P_1(\xi_1^i)e^{-i\xi_1^i c(x+y)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)P_1(\xi_1^i)}{\xi_2^j+\xi_1^i}e^{i\xi_2^j c(x+ct)}e^{-i\xi_1^i c(y-ct)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)P_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}e^{i\xi_2^j c(x+ct)}e^{i\xi_1^{i*}c(y-ct)}-cc\right]\Bigg\}$$

$0<x+ct<a \qquad -a<x+y<0 \qquad y-ct<-a$ 5

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j}-4hHci\Bigg\{\left[\left(Q_2(\xi_2^j)J_1(-\xi_2^j)+Q_2'(\xi_2^j)I_1(-\xi_2^j)\right)e^{i\xi_2^j c(x+y)}-cc\right]$$

$$-\left[J_2(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i c(x+y)}-cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)P_1(\xi_1^i)}{\xi_2^j+\xi_1^i}e^{i\xi_2^j c(x+ct)}e^{-i\xi_1^i c(y-ct)}-cc\right]\Bigg\}$$

$$+\left[\frac{Q_2'(\xi_2^j)P_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\, e^{i\xi_2^j c(x+ct)}\, e^{i\xi_1^{i*}c(y-ct)} - cc\right]\Bigg\}$$

$0<x+ct<a \qquad -a<x+y<0 \qquad -a<y-ct<0$ 6

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j} -4hHci\Bigg\{\left[\left(Q_2(\xi_2^j)J_1(-\xi_2^j)+Q_2'(\xi_2^j)J_1(-\xi_2^j)\right)e^{i\xi_2^j(x+y)} - cc\right]$$

$$-\left[J_2(-\xi_1^i)Q_1(\xi_1^i)e^{-i\xi_1^i c(x+y)} - cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)Q_1(\xi_1^i)}{\xi_2^j+\xi^i}\, e^{i\xi_2^j c(x+ct)}\, e^{-i\xi_1^i c(y-ct)} - cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\, e^{i\xi_2^j c(x+ct)}\, e^{i\xi_1^{i*}c(y-ct)} - cc\right]\Bigg\}$$

$0<x+ct<a \qquad 0<x+y \qquad -a<y-ct<0$ 7

$$\bar{H}(x,y;t)=\sum_{\xi_1^i}\sum_{\xi_2^j} -4hHci\Bigg\{\left[Q_2'(\xi_2^j)J_1(-\xi_2^j)e^{i\xi_2^j c(x+y)} - cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)Q_1(\xi_1^i)}{\xi_2^j+\xi_1^i}\, e^{i\xi_2^j c(x+ct)}\, e^{-i\xi_1^i c(y-ct)} - cc\right]$$

$$+\left[\frac{Q_2'(\xi_2^j)Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}\, e^{i\xi_2^j c(x+ct)}\, e^{i\xi_1^{i*}c(y-ct)} - cc\right]\Bigg\}$$

$0<x+ct<a \qquad 0<x+y \qquad 0<y-ct$ 8

$$\bar{H}(x,y;t)=0$$

$a < x+ct \qquad x+y < -a$ 9

$$\bar{H}(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_2^j} -4hHci\left\{\left[P_2(\xi_2^j)I_1(-\xi_2^j)e^{i\xi_2^j c(x+y)} - cc\right] - \left[I_2(-\xi_1^i)P_1(\xi_1^i)e^{-i\xi_1^i c(x+y)} - cc\right]\right\}$$

$a < x+ct \qquad -a < x+y \le 0$ 10

$$\bar{H}(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_2^j} -4hHci\left\{\left[\left(P_2(\xi_2^j)J_1(-\xi_2^j)+Q_2'(\xi_2^j)J_1'(-\xi_2^j)\right)e^{i\xi_2^j c(x+y)} - cc\right] - \left[\left(I_2(-\xi_2^i)Q(\xi_1^i) + J_2'(-\xi_1^i)Q_1'(\xi_1^i)\right)e^{-i\xi_1^i c(x+y)} - cc\right]\right\}$$

$a < x+ct \qquad 0 < x+y < a$ 11

$\bar{H}(x,y;t) = 0$

$a < x+ct \qquad a < x+y$ 12

$\bar{H}(x,y;t) = 0$

H(x,y;t) is exactly the same as $\overline{H}(x,y;t)$ given in the preceding equations except that y and x are interchanged. As an example in region 7, H(x,y;t) is given as

$0 < y+ct < a \qquad 0 < x-y \qquad -a < x-ct < 0 \qquad\qquad 7$

$$H(x,y;t) = \sum_{\xi_1^i}\sum_{\xi_2^j} -4hHci\left\{\left[Q_2'(\xi_2^j)J_1(-\xi_2^j)e^{i\xi_2^j c(x+y)} - cc\right]\right.$$

$$+\left[\frac{Q_2'(\xi_2^j)Q_1(\xi_1^i)}{\xi_2^j+\xi_1^i}e^{i\xi_2^j c(y+ct)}e^{-i\xi_1^i c(x-ct)} - cc\right]$$

$$\left.+\left[\frac{Q_2'(\xi_2^j)Q_1^*(\xi_1^i)}{\xi_2^j-\xi_1^{i*}}e^{i\xi_2^j c(y+ct)}e^{+i\xi_1^{i*}c(x-ct)} - cc\right]\right\}$$

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For a more complete set of references on th ISM see the review papers by Scott et al [11] and Kaup et al [14].