

44962



National Library of Canada

Bibliothèque nationale du Canada

CANADIAN THESES ON MICROFICHE

THÈSES CANADIENNES SUR MICROFICHÉ

NAME OF AUTHOR/NOM DE L'AUTEUR John Edward Rowcroft

TITLE OF THESIS/TITRE DE LA THÈSE The Production Function in the Neo-Classical Theory of the Firm: An Axiomatic Critique

UNIVERSITY/UNIVERSITÉ Simon Fraser University

DEGREE FOR WHICH THESIS WAS PRESENTED/ GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE Ph.D.

YEAR THIS DEGREE CONFERRED/ANNÉE D'OBTENTION DE CE GRADE 1979

NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE Larry Boland

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de microfilmer cette thèse et de prêter ou de vendre des exemplaires du film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/DATE 13th September 79 SIGNED/SIGNÉ _____

PERMANENT ADDRESS/RÉSIDENCE FIXE _____



National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS, L'AVONS REÇUE?**

THE PRODUCTION FUNCTION IN
THE NEO-CLASSICAL THEORY OF THE FIRM:
AN AXIOMATIC CRITIQUE.

by

John Edward Rowcroft

B.Sc.(Hons.), Manchester University (U.K.), 1966

M.Sc., Manchester University (U.K.), 1967

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in the Department
of
Economics and Commerce

© John Edward Rowcroft 1979

SIMON FRASER UNIVERSITY

April 1979

All rights reserved. This thesis may not be reproduced in whole or in part, by photocopy or other means, without permission of the author.

PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis or dissertation (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this thesis for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis:

The Production Function in the Neo-Classical Theory of the
Firm: an Axiomatic Critique.

Author: _____

(signature)

John E. Rowcroft

(name)

April 30, 1979

(date)

APPROVAL

Name: John Edward Rowcroft

Degrée: Doctor of Philosophy

Title of Essay: The Production Function in the Neo-Classical Theory
of the Firm: An Axiomatic Critique

Examining Committee:

Chairperson: S.T. Easton

L. Boland
Senior Supervisor

D.J. DeVoretz

P.E. Kennedy

K. Nagatani
External Examiner
Department of Economics, U.B.C.

Date Approved: April 30, 1979

ABSTRACT

This thesis is an analysis of the structure of the Neo-Classical theory of the firm conducted primarily by examining the constraints which the theory places on the selection of a production or profit function. The logical requirements for a complete theory of the firm are summarised and an axiomatic framework developed which embraces the essential features common to most treatments of the firm within the Neo-Classical tradition. In particular, the short, intermediate and long run distinctions made in the two input analysis are generalised to a sequence of periods in which progressively fewer inputs are constrained.

Conditions are derived for the existence of unique profit maxima in each run and for the possibility of zero maximum profit in the long run as a result of the predicted exit of firms from industries making negative profit and entry into industries with positive profits. Two results of particular interest emerge from this analysis. The first is the construction of a Neo-Classical Framework for the theory of the firm which includes several hitherto largely neglected models as well as the familiar cases of perfect and imperfect competition. Secondly the analysis shows that within this framework the concept of the perfectly competitive firm having no influence on input and output prices is internally inconsistent, that is, a 'perfectly competitive production function cannot be found'.

ACKNOWLEDGEMENTS

I would like to thank Professor L.A. Boland for suggesting the topic of this thesis and for his help in its execution.

TABLE OF CONTENTS

Approval Page.....ii
Abstract.....iii
Acknowledgements.....iv
Table of Contents.....v
List of Tables.....ix
List of Figures.....x

Chapter

I Introduction

1. Purpose.....1
2. The Neo-Classical Theory of the Firm.....1
3. Particular Aspects of the Theory.....3
4. The Form of the Production Function.....4
5. The Axiom Structure.....6
6. Differentiability and Divisibility.....7
7. The Form of the Analysis.....7
8. Conclusion.....10

II Methodology

1. Introduction.....11
2. The Purpose of Theory.....11
3. Testing Conventions.....12
4. The Model.....12
5. Model and Observation.....13
6. Summary.....15
Footnotes.....17

III The Theory of the Firm

1. Introduction.....	18
2. The Technological 'Black Box'.....	18
3. Constrained Optimisation.....	19
4. The General Model.....	20
5. The Neo-Classical Model.....	21
6. Profit and Production Functions.....	23
7. An Alternative Theory Set.....	25
8. Summary.....	31
Footnotes.....	34

IV Model Structure and the Intermediate Runs

1. Introduction.....	37
2. The Neo-Classical Profit Function.....	37
3. Input Value Theory.....	41
4. Distinguishing Observations.....	42
5. Representability.....	44
6. Requirements on the Profit Function.....	47
7. The Non-Differentiable Profit Function.....	48
8. Summary.....	51
Footnotes.....	53

V Existence of Long Run Zero Maximum Profit

1. Introduction.....	54
2. Homotheticity.....	54
3. An Example.....	55
4. Existence of Zero Maximum Profit.....	57
5. Summary.....	60

	Footnotes.....	61
VI	The Neo-Classical Firm in the Long Run	
	1. Introduction.....	62
	2. Profit and Production Functions.....	62
	3. Long Run Behaviour of the Perfectly Competitive Firm.....	63
	4. Output Price Behaviour in the Long Run.....	64
	5. Long Run Zero Profit.....	65
	6. Properties of the Zero Profit Point.....	66
	7. Summary.....	68
	Footnotes.....	72
VII	The Neo-Classical Production Function	
	1. Introduction.....	73
	2. Requirements on the Production Function in Perfect Competition.....	73
	3. The Positive (Net) Profit Region.....	74
	4. The Non-Positive Profit Region: An Illustration.....	75
	5. The Negative Profit Region: A Difficulty.....	77
	6. Neo-Classical Profit Maximisation.....	79
	7. Summary.....	80
	Footnotes.....	83
VIII	Conclusion	
	1. Complete Explanation.....	84
	2. Returns to Scale.....	85
	3. Further Research.....	86

Appendix: Definitions and Proofs.....98
List of References.....137

LIST OF TABLES

Table

page

- 1: Summary of Predicted Behaviour of
Perfectly Competitive Firm.....33

LIST OF FIGURES

Figure	page
1: Logic of Theory, Model and Observations.....	16
2: Singularity of the Production Function: Isoquants.....	70
3: Singularity of the Production Function: Product Curves.....	71
4: Returns to Scale in the Positive Profit Region.....	81
5: Converging Isoquants in the Negative Profit Region.....	81
6: Properties of Elliptical Isoquants.....	82
7: Generalisations of Elliptical Isoquants.....	82

CHAPTER ONE

Introduction

1. PURPOSE: This thesis is an analysis of the structure of the Neo-Classical Theory of the firm conducted primarily by examining the constraints which the theory places on the form of the technological relationship between inputs and outputs, usually called the 'production function'. By constructing a logical framework for the theory of the firm expanding on the usual text-book expositions attention is drawn to the non-trivial problem of devising a production function which is consistent with all the assertions which the theory makes about the firm's behaviour. Thus a postulated technical relation which satisfactorily explains the actions of a profit-maximising firm subject to a limited supply of capital may fail to justify the same firm's behaviour when the constraint is removed. Similarly a production function which agrees with the predicted 'long run equilibrium' of the firm and industry is not necessarily consistent with the theoretical explanation of how this equilibrium is attained.

2. THE NEO-CLASSICAL THEORY OF THE FIRM: In its simplest form the Neo-Classical theory deals with a firm which converts quantities of two inputs ('labour' and 'capital') into a single output, subject to fixed unit prices over which it has no control. The firm's decisions are limited to the selection of input and output quantities subject to technical

capability (the 'production function') and are made on the basis of the maximisation of the excess of sales revenue over input costs ('profit'). (See Ferguson [26], for example).

The conventional text book explanation of this choice process is incomplete in that it treats only the 'short run' and the 'long run'. In the first case the firm is obliged to pay for a fixed amount of capital and chooses only the quantity of labour it will use. By contrast in the long run the firm is free to select not only the optimum combination of labour and capital but also the industry to which it will belong.

Completeness of the theory requires some specification of the 'intermediate run' in which input quantities are variable but the product type or industry is not. This extension is provided by Boland [9], and it is this completed version of the Neo-Classical Theory which is considered in this thesis. Neo-Classical Theory predicts that the independent long run decisions of profit maximising firms will, through their combined effect on selling prices, produce an equilibrium situation in which the rate of profit is the same for all industries, and hence for all firms. However those versions of the Neo-Classical Theory which extol the virtues of competition on this basis are intellectually inadequate since, as Boland argues [14], competition is unnecessary for the achievement of a uniform profit level. This point is addressed in Chapter Six.

Generalisations of the above description are possible by increasing the number of inputs as in Henderson and Quandt [36] for example, and by relaxing the inflexibility of output prices as in the theory of Imperfect Competition [50]. However, where an explicit account is rendered the behavioural pattern is essentially the same.

3. PARTICULAR ASPECTS OF THE THEORY: In empirical work it is common to assume a particular form of the production relationship which meets certain theoretical requirements (such as decreasing returns to an input) and to attempt to estimate the parameters of this function [64, for example]. Theoretical studies such as [62] have also taken a particular family of production functions as given. This presupposes that the functional form is consistent with the assumed theory and confounds the implications of the results for the theory and for the choice of production function. It is therefore appropriate here to treat the production relationship as an integral part of the Neo-Classical theory limiting its form only in so far as this is necessary for consistency with that theory. In short, the characteristics of the production function are to be inferred from the combination of theory and observation rather than imposed a priori.

The uniform rate of profit prevailing at the end of the long run is often referred to as 'normal profit' or 'long run zero profit' [26]. Interpreted literally this leaves unanswered the question of why a profit maximising firm

chooses to remain in operation in equilibrium rather than earning the same profit from closing down. Friedman [28] has suggested that the 'guiding spirit' or 'entrepreneur' earns a return for his administrative services in the long run whereas he would be obliged to seek employment elsewhere were the firm to be liquidated. However this raises the problem of a perpetually constrained and largely unquantifiable input (as Friedman notes) and introduces a second argument into the firm's objective function, that of entrepreneurial preference. Profit maximisation may be retained as the sole choice criterion if long run profit is a positive constant and the analysis conducted as usual by defining profit net of this amount.

For a profit maximising firm to change industries within the scope of the Neo-Classical Theory an opportunity must appear to exist for profit to be increased at the end of the intermediate run. Either profits are uniform throughout all industries in which case long run equilibrium is established immediately or a distribution of profit rates exists between firms such that immigration into above normal profit industries occurs together with emmigration from lower than equilibrium profit situations. If the long run is to be a distinct period with the entry and exit phenomena described by the theory, varying levels of profit in the intermediate run must be possible.

4. THE FORM OF THE PRODUCTION FUNCTION: Whenever the form

of a production function is specified in Economic analysis it is almost invariably linearly homogeneous, usually of the Cobb-Douglas form [18]. Lancaster [42] understated the case when he remarked that "Homogeneous functions are of considerable importance...especially in Neo-Classical production theory". He notes that the more general class of homothetic functions (which includes the only other functional form regularly represented in the literature, the Constant Elasticity of Substitution function), are "used from time to time in various contexts".

Functions of this type are mathematically tractable both directly and in additive logarithmic form for regression analysis [44]. Moreover the linear homogeneous function exhibits decreasing marginal productivity for all inputs and generates constrained and unconstrained maxima when these marginal products are in the same ratio as prices. Similarly the existence of long run equilibrium is ensured by the identity of unconstrained maximum output with zero profit. However this raises the problem discussed by Boland [14] and referred to in the preceding section. If maximum profit is necessarily zero in the intermediate run, this period is indistinguishable from the long run and any entry or exit behaviour is purely gratuitous in terms of the theory. A full account of the Neo-Classical firm's choices requires a production function which yields zero and non-zero maximum (net) profit in the intermediate run depending on the prevailing prices. Such a function is not apparent

from the literature and it is the purpose of this thesis to examine the form of a production function which meets this condition whilst retaining the other attributes required by the Neo-Classical theory (such as constrained profit maxima).

5. THE AXIOM STRUCTURE: Considerable attention has been given to the axiomatisation of microeconomic theory when applied to the consumer (see Walsh [63], for example), and to general equilibrium as exemplified by the work of Debreu [21]. With the exception of Boland [9], little attention seems to have been given to reducing the theory of the firm to a similar logical structure. Work has concentrated on the representation of the technological relationship divorced from the wider requirements of the underlying theory. Linear programming [23] and activity analysis [42] provide obvious examples although the more recent enunciation and applications of Shephard's lemma [57] have expanded upon the intrinsic duality of the technical and 'economic' aspects of production first noted in the linear case [8].

Since an inconsistency appears to exist between the usual text book theory of the firm and the most commonly used forms of production function it is appropriate to restate the theory in a more rigorous form. Hence it becomes apparent what 'demands' are placed on the form of the technological relationship by the structure of the theory and the extent to which different functions meet these requirements.

6. DIFFERENTIABILITY AND DIVISIBILITY: Traditional presentations of the Neo-Classical theory of the firm frequently assume the differentiability of the production function to permit the use of calculus maximisation techniques as in [36], for example. However the work of Debreu [21] on general equilibrium and of Shephard [57] and Jorgenson and Lau [40] has stressed the adaptation of results in convex analysis to avoid differentiability. Attempts have also been made to incorporate non-convexities, notably by Frank [27]. Linear technologies necessarily also include points of non-differentiability [8].

The extent to which the various relaxations of the differentiability of the production function are consistent with the Neo-Classical theory of the firm (as opposed to general equilibrium theory) do not appear to have been widely discussed in the literature. It is apparent from the present analysis that there is little difficulty in replacing a differentiable production function by a continuous strictly concave one with the appropriate 'subdifferentials' but that linearity and indivisibility necessarily generate ambiguity within the traditional explanation of the firm's behaviour.

7. THE FORM OF THE ANALYSIS: In Chapter Two the logical requirements for a complete specification of the Neo-Classical Theory of the firm are discussed and the distinction made between a general, irrefutable, theory and an auxiliary

model by which the premises of the theory are related to observable phenomena or data. It is noted that an essential characteristic of a complete theory-model combination is that there exist conceivable data which would enable it to be distinguished from competing explanations of the same behaviour, in this case, the firm's choice of an input-output combination.

Neo-Classical Production Theory does not appear in the literature as a clearly defined structure. Consequently in Chapter Three a framework is described which embraces the essential features common to most treatments of the theory whilst attempting to avoid unnecessary restrictions peculiar to particular versions. Thus the short, intermediate, and long run distinctions made in the usual two input analysis are generalised to a sequence of runs in which progressively fewer inputs are constrained. This sequence culminates in the penultimate run in which no inputs are constrained and the firm is restricted only to remaining in the same industry, corresponding to the intermediate run, and the long run in which the firm may change industries as in the simpler case.

The consequences of requiring the theory to be a complete explanation of the firm's behaviour when restricted to one industry are examined in Chapter Four by expressing the framework of Chapter Three in mathematical terms. Hence it is shown that the restriction of the relationship between inputs and profit to strict concavity suffices to ensure

the existence of the required unique input-output combinations in all runs.

Since the theory stipulates that profit achieve a uniform or 'normal' level in all industries in the long run, sufficient conditions to guarantee the existence of normal profit for the firm are derived in Chapter Five. These conditions prove the possibility of the existence of normal profit and in Chapter Six attention is turned to the necessary consequences of postulating this existence. Hence the perfectly competitive firm is shown to possess a unique input-output combination which generates this level of profit regardless of input prices, and the consequences of the singularity are illustrated for the familiar two input case.

In Chapter Seven the requirements which the theory makes on the production function of the perfectly competitive firm are summarised. It is proved that if all other requirements are met, the production function cannot be consistent with the Neo-Classical explanation of the transition from penultimate (or intermediate) to long run. Thus the Neo-Classical theory of perfect competition is necessarily inconsistent with itself. The significance of this and the other results derived in the analysis are discussed in Chapter Eight.

Mathematical definitions are repeated in the Appendix which contains proofs of the results mentioned in the text.

8. CONCLUSION: Two results of particular interest emerge from this thesis. The first is the construction of what may be termed a 'Neo-Classical framework' for the theory of the firm covering a broad class of models of technical and market behaviour. Secondly the analysis shows that within this framework the concept of the perfectly competitive firm having no influence on input and output prices is internally inconsistent, that is, the perfectly competitive production function does not exist.

The most significant restrictions placed on the analysis are those of uniqueness of the profit maxima for any given input and output market situations and of generality with respect to all positive input prices. However the first condition is a logical requirement of a complete explanation rather than a peculiarity of the Neo-Classical approach. To explain why a firm chooses the input-output combination which it does it is necessary (and sufficient) to explain why it does not choose any other attainable combination [9]. The absence of uniqueness requires a supplementary theory to explain why the firm chooses one optimum rather than another. Ad hoc limitations on which input price vectors are admissible (i.e. 'work') also need justification in the form of subsidiary assumptions about which prices can exist or how the firm reacts to an inadmissible situation in an input market. As received the Neo-Classical theory of the firm provides no indication of the form either of these modifications should take.

CHAPTER TWO

Methodology

1. INTRODUCTION: The purpose of this chapter is to summarise the job which a theory is intended to do, and the criteria by which its performance is to be judged.

Adopting the approach taken by Boland [12] leads to a consideration of the correspondence between the specification of a model of a theory and the data available with which to compare it. Thus every model is noted to require a supplementary related theory regarding observations. In the following chapter these conclusions are applied to the specification of the Neo-Classical model of the firm which is to be examined in the remainder of the thesis.

2. THE PURPOSE OF THEORY: Discussion of the purpose of economic theory has centred on three viewpoints:

Instrumentalism, of which Friedman [29] is the best known proponent; Descriptivism which is espoused by Samuelson [55] and a Logical Positivist approach derived from the writings of Karl Popper [49]. The shortcomings of the first two attitudes have been detailed in the literature by Boland [10], Wong [56], and others and the methodology adopted here is a modification of the views of Popper and Agassi [2] expressed by Boland [12]¹. Thus theory is seen as a device for understanding and explaining (economic) phenomena in terms of existential, and hence, irrefutable premises

(assumptions).

3. TESTING CONVENTIONS: The various criteria for the acceptability of a theory are discussed by Boland [10] who follows Popper in suggesting that the primary requirements are logical consistency and the existence of conceptually false, but hitherto unrefuted, implications. Contradiction of one such implication by an observation, or 'fact', implies the falsity of the set of basic assumptions although it does not directly indicate the culprit(s). However Boland also argues [12] that the implications of a theory cannot be compared with received data without the introduction of subsidiary assumptions specifying the manner in which theory and observation interact. Thus contradiction of an implication permits only the deduction that the augmented assumption set is false when considered as a compound statement. Theories alone cannot be empirically refuted.²

4. THE MODEL: The model serves to provide the interface between (empirically untestable) theory and fact. By supplementing the existential premises of the theory with specifications of functional forms the possibility of a directly falsifiable implication is introduced. Therefore choice of an appropriate model from the non-denumerable set which can be derived from a single theory hinges on the data available. The construction of the model must be such that there exist conceivable observations which

would contradict as least one implication of the theory-model combination. Hence each model of a theory implies or requires assumptions on the data which are available (conceptually) for its refutation.

Where competing theories exist to explain the same phenomena from the same events³, a further requirement is placed on the model. As noted in [14], at least one 'distinguishing observation' must be possible which refutes one theory but not the other.⁴

5. MODEL AND OBSERVATION: An observation is defined as a true statement that an event has or has not occurred.⁵

Three categories of events are of interest:

- (i) events which the theory seeks to explain, such as the combination of goods which a consumer buys.
- (ii) events which are used by the theory to explain those in (i), for example, income and prices.
- (iii) events which are implied by the theory-model conjunction such as reactions to price changes.

Sets (i) and (ii) correspond to the statistical division between independent and explanatory variables respectively, and must be assumed to be observable for the theory to be of interest.⁶ As noted earlier, set (iii) must be non-empty and contain events described by universal rather than existential statements, to permit refutation. For consumer theory the statement 'all reactions to a price change can be divided into a substitution effect and an

income effect' describes an event in (iii) but is existential and irrefutable whereas 'the substitution effect always dominates the income effect' is general and could be known false (Giffen good).

By definition the theory-model must explain any particular conjunction of events in categories (i) and (ii) which, as Boland has noted [9], requires the explanation of why no other events in (i) can occur with the same observations in (ii). Hence two types of implications would appear to permit refutation:

- a) a sequence of conjunctions of observations on (i) and (ii)
- b) the observation of separate events about which the theory-model makes predictions.

The simplest example of (a) is the failure of replication in which the same observations in (ii) do not correspond to the same observations in (i) on different occasions.⁷ Such situations are rare and gratuitous in social and astronomical science although traditionally the mainstay of refutation in physics and chemistry, for example.

Events of type (b) fail to satisfy the requirements of refutation since they are necessarily defined within the theory-model under test.⁸ Otherwise they are either explanatory (category (ii)), or to be explained (category (i)). Thus the statement that the failure of consistency of preference in consumer theory is due to 'taste change'

implies acceptance of that theory and cannot provide refutation unless the theory is augmented to explain how tastes change, placing tastes in set (i) above.

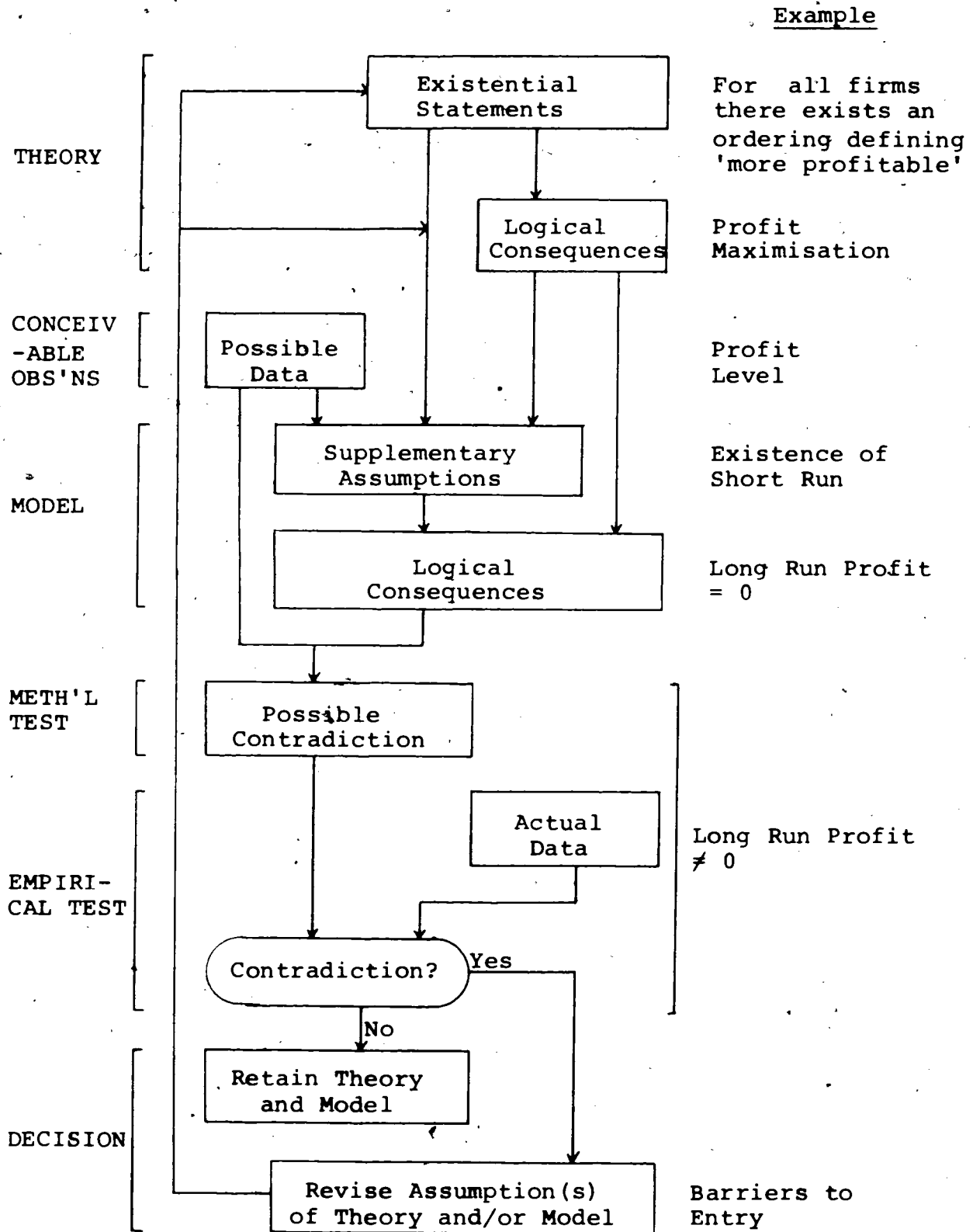
Hence the intrinsic theory of observation which accompanies a theory-model combination is one of a sequence of observations on the independent and explanatory variables and may be considered analagous to the process by which a statistical model is successively 'identified' and 'tested' by sets of data.

6. SUMMARY: From the preceeding discussion the characteristics required of a theory-model combination are that

- (i) it be logically consistent,
- (ii) it be complete in the sense that any combination of explanatory and independent events is explained to the exclusion of any other set of independent events, and
- (iii) sequences of independent and explanatory events exist, the observation of which would
 - a) distinguish the theory-model from other postulated theory-model explanations of the same sets of events, or
 - b) prove the theory-model false.

In the next chapter these requirements are applied to a theory of the firm.

FIGURE 1: LOGIC OF THEORY, MODEL AND OBSERVATIONS



Footnotes

1. A much fuller treatment of this topic is provided in the writings of Boland, and by R. Trumper in Economic Methodology and the Methodology of Economists, unpublished M. A. report supervised by the author, University of New Brunswick, Fredericton, New Brunswick, 1977.
2. Hence a theory might be considered as analagous to Weber's "ideal type" [65].
3. For example, Lancaster's characteristic analysis has been suggested to be an alternative explanation of consumer choice based on prices and income as is the traditional theory.
4. If only one theory-model possesses a distinguishing observation it is sometimes described as having 'greater explanatory power', and may be preferred. If distinguishing observations exist for both theories, they remain in contention until one such observation is made.
5. Thus false information is excluded as are probabilistic statements since the description of, say the result of a penny toss as 'an event of probability 0.5' is properly a compound of theory, model, and observation. See the discussion of stochasticism in Boland [13] and of the theory of probability in, for example, Carnap [16].
6. Thus the celebrated Oxford conundrum of the existence of the unobserved does not arise since an unobserved event requires no explanation, except perhaps in terms of other unobserved phenomena as Chesterton suggested.
7. This is implied by Samuelson's Revealed Preference argument in consumer theory.
8. They could, however, constitute distinguishing observations for two theory-model combinations which defined them in the same way.

CHAPTER THREE

The Theory of the Firm

1. INTRODUCTION: The broad outlines of the economic theory of the firm are largely consistent in the literature in specifying the maximisation of a function of revenue and cost subject to varying constraints.¹ However the models derived from this basis vary considerably in detail from the diagrammatic two factor approach of, say, Ferguson [26] through the linear activity analysis models. (see Lancaster, [42], for example) to the integer programming discrete goods analysis by Frank [27]. In this chapter a model is developed which it is hoped captures the important characteristics of what may be termed the 'Neo-Classical Theory of the firm' without introducing restrictions peculiar to any particular treatment of it.

2. THE TECHNOLOGICAL 'BLACK BOX': Economic theory has almost invariably treated the technical relationship between goods purchased and goods available for sale by a firm as exogenously determined at any particular time.² Thus the 'state of the art' is represented by Marshall [46], Hicks [37], Samuelson [55] and others as an engineering 'black box' by which quantities of certain goods, inputs, become quantities of, usually, different goods, outputs. Such a transformation relationship is referred to as a production function. Constraints are placed on the form of the production function

by the economic theory in which it is used, as has been noted by Hicks [37] in his treatment of returns to scale. These constraints will be the subject of further examination in the ensuing analysis.

3. CONSTRAINED OPTIMISATION: Like the theory of the consumer, the economic explanation of the firm's behaviour is a specialisation of the general theory of choice [63]. Thus the firm is described (or defined) as selecting a combination of input and output quantities from an attainable set in accordance with a preference ordering. This preference ordering is commonly, but not universally, referred to as 'increasing profit'.³ Following Boland [9] the theory of the firm may be represented by five existential statements which justify an answer to the question 'Why does the firm produce at the input-output combination that it does?'

The traditional answer is characterised by Boland as follows:

'Because, given prices, that is the input-output combination which is the "most profitable" of all the combinations at which it "can produce"'. [9]

This answer may be justified by the following statements (assumptions) which serve to represent the economic theory of the firm:

- (i) for each firm there exists an ordering based on a maximum output (or revenue) for each input combination which defines 'technical capabilities'.
- (ii) for each firm there exists a measure on input-output combinations based on prices which

defines "more profitable".

- (iii) for each firm there exists a behavioural pattern such that it seeks to improve by producing at the more profitable of any two input-output combinations at which it can produce.
- (iv) for each firm, given at least one input, there exists a limit on technical capabilities which defines "can produce".
- (v) for each firm, given prices, there exists an input-output combination which is more profitable than all other combinations at which it can produce.⁴

In the terminology of Chapter Two, the set (i) of independent events consists of all input-output combinations which are to be explained by prices which make up set (ii).

4. THE GENERAL MODEL: Supplementary assumptions are required to relate the theory to the set of conceivable observations of a firm. Since the process by which inputs become outputs is treated exogenously, attention is focussed on the firm's selection of input quantities. To explain the choice of each input they are separated temporally. Thus in the 'short period' or 'short run' the firm is able to change the quantity it owns of one input only (the most flexible).⁵ By contrast, in the 'long run' the firm is free to change not only the quantities of all inputs which it owns, but also the industry to which it belongs, that

is the type(s) of output goods which it produces. Between these two extremes lie time periods in which the purchased quantities of some, but not all inputs may be varied. The additional assumption of the General Model is that:

- (vi) all inputs used by a firm may be ordered in such a way that a distinct time period exists for each input in which the firm may vary its ownership only of that input and all those ordered below it.⁶

Hence a sequence of 'runs' is established such that in the j th run only the first j inputs are traded by the firm. If there are m inputs, the $(m+1)$ th run corresponds to the long run described above.

5.. THE NEO-CLASSICAL MODEL: A representation of the Neo-Classical Model of the firm within an industry requires one assumption in addition to those of the General Model.

- (vii) . That the firm's preference function is determined by ordering one attainable input-output combination above another if, and only if, it yields a greater excess of sales revenue over costs. (or a smaller deficit).

This is the profit maximisation assumption, in its usual form.

However, Assumption (v) refers to fixed prices and Assumption (vi) applies to a firm within a particular industry and are thus insufficient to justify the explanation of the firm's behaviour in the long run when it is free to change

industries and output prices are considered flexible.⁷

Thus further assumptions are required to justify the profit maximisation answer to the question: 'Why does the firm choose those outputs (that industry) which it does?' The Neo-Classical explanation is that firms change industries seeking higher profits (Assumption (i)) until profit is the same in every industry. Such an equality implies that the return to entrepreneurial endeavour over and above managerial salary (which is a real or imputed cost) is uniform throughout the economy. (cf. Friedman [28]) It does not embrace a return to capital per se which the Neo-Classical model includes as a further cost.⁸ This explanation requires that:

(viii) in the long run, any firm is able to adopt the technology of any other firm and operate on the same economic terms,⁹ i.e. there are no "barriers to entry (or exit)".

(ix) for all industries there exists a mechanism whereby the price of any fixed quantity of an output good decreases (increases) as the total quantity produced increases (decreases).¹⁰

(x) for any firm in any industry maximum attainable profit increases (decreases) when the price of a fixed quantity of an output of that firm increases (decreases).¹¹

(xi) for all firms in all industries with any input

prices there exists a set of output-price combinations for which maximum attainable profit is the same positive value, called "normal profit".

Thus a profit-seeking firm is able to change industries (Assumption (viii)), increasing price in the industry it leaves (Assumption (ix)) and consequently increasing maximum attainable profit for the firms remaining (Assumption (x)). Conversely price and maximum attainable profit are reduced in the industry entered. Assumption (xi) ensures that normal profit is attainable in every industry and under the behaviour described each firm remains in an industry when all firms are earning normal profits.¹² This normal profit level must be positive otherwise the theory fails to explain why a firm which bases its decision purely on profit level would not close down (earning zero profit) at the end of the long run.¹³ Since normal profit is a fixed quantity it will be convenient, and in accordance with common usage, to consider profit net of this quantity and hence at the end of the long period net profit will be zero for all firms.

6. PROFIT AND PRODUCTION FUNCTIONS: The Neo-Classical theory explains the behaviour of the firm directly in terms of fixed input markets, that is invariant price-quantity relationships, and similarly fixed output markets in all but the long run. Once these are specified Assumption (i)

postulates the existence of a relationship between inputs and outputs which has already been noted as the production function. Combining market structures with this function yields the profit function which transforms a set of price-quantity relationships into a difference between revenue and sales which the firm 'can produce'. Thus a constraint imposed by the assumptions on the form of the profit function may make different requirements on the production relationship according to the character of the markets for the goods concerned.

In the ensuing analysis results are therefore obtained on the profit function where appropriate and then specialised to the production function using a particular market structure. The structure referred to throughout is that of 'perfect competition' represented by the following three assumptions:

- (xii) each firm produces one output
- (xiii) each firm may purchase any quantity of any input at a fixed price, independent of quantity.
- (xiv) each firm may sell any quantity of output at or below a fixed maximum price.¹⁴

Reference is also made to 'imperfect competition' in which Assumption (xiv) is replaced by

- (xiv') the maximum price for which a firm may sell its output is inversely related to the quantity of that output.

One further restriction is often placed directly on the

profit or production function, namely the requirement of differentiability twice with respect to all inputs. Whilst many inferences may be made from the theory-model without this assumption (See Debreu [21], and Jorgenson and Lau [40], for example) its heuristic appeal and central place in common treatments of the Neo-Classical model (e.g. Henderson and Quandt [36]) make its total omission inappropriate. Thus the present analysis will be conducted in terms of differentiable functions with corresponding results for non-differentiable relationships noted where applicable.

7. AN ALTERNATIVE THEORY SET: Alternatives to the Neo-Classical form of the General Model may be generated by varying one or more of Assumptions (vi) to (ix) above. For example the replacement of (viii) by prohibition of entry to an industry can generate a model of monopoly in which the firm never enters the long run [26]. Similarly the firm's preference function may be defined in terms of different measures of desirability either by re-defining profit or admitting alternative objectives. As Nordquist [47] remarks

Although it is not easy to classify the many suggestions which have been made to revise or reconstruct the theory of the firm, it is convenient to group them roughly according to how the motivational assumption is to be modified.

Since 'profit' is mentioned in the traditional theory it may be argued that the second case is an instance of a competing theory rather than a different model. However as 'profit'

is primitive or undefined in the theory it may be set equal to any quantity as a substitute for Assumption (vii). Thus an uncountable number of alternative models exist from which it is desirable that the Neo-Classical version be distinguished. It is therefore logically impossible to ensure that this model or any other is fully 'identified'.

Nevertheless it is minimally necessary to provide for distinction between the Neo-Classical model and such competing economic explanations of the firm's behaviour as have appeared in the literature. These may be summarised as follows:

- (i) Maximisation of long run, rather than short run profit: Neo-Classical theory generates long run maximisation through a sequence of short-run optimising decisions and it is a requirement on the theory-model that the two be consistent with each other. No alternative explanation emerges in the literature perhaps due to the 'difficulties of advancing a concrete and objectively satisfying definition of long-run profit' [42] distinct from the Neo-Classical one.
- (ii) Maximisation of Utility. Cyert and March [19] suggest that 'entrepreneurs...have a host of personal motives' and Papandreou [48] extends the argument by claiming that,

'organisational objectives grow out of interaction among the various participants in the organisation. This produces a general preference function.'

Whilst such an approach increases the similarity between the theory of consumer behaviour and that of the firm, it does not, of itself, provide an explanation of the firm's behaviour without additional, and, hitherto, unspecified, statements about the firm's utility.¹⁵ Moreover, in response to Papandreou, it is not clear 'whether or not it makes sense to speak of a well-ordered set of preferences for the firm in view of the size and complexity of most business operations' [47]. This difficulty arises from the problems of preference aggregation noted by Arrow [5], and others, and which present similar objections to the concept of household decision-making in consumer theory.

- (iii) Equilibrium and Survival. Rothschild [52] has suggested that the firm's primary goal is long run survival and accordingly that its objective is to maximise security. In its received form this appears to be a reiteration of the importance of competition and suffers from the Darwinian problem that success is perceived in terms of survival. Extinct firms are, by definition, unsuccessful.

An alternative biological analogy was proposed by Boulding [15] as "Balance sheet homeostasis" in which firms seek a desired

set of accounting ratios and react to re-instate this equilibrium.

In the perfectly competitive case this reduces to utility maximisation in asset space and is formally identical to profit maximisation regardless of the precise form of the firm's preferences, as Boulding notes. He also shows that the two models diverge in the presence of imperfect markets for which the production transformation itself takes sufficient time to permit prices changes between the purchase of inputs and the sale of the resultant outputs. A thorough analysis along these lines requires the separation of the buying, producing and selling decisions and a characterisation of time which are at variance with the general choice theory enunciated earlier. It does not appear to offer any competing model of the same theory.

- (iv) Maximisation of Revenue: If sales revenue is maximised subject to a fixed cost constraint the model is immediately distinguished from the Neo-Classical version by its inability to explain any change in the firms behaviour not accompanied by a change in input or output prices. A varying cost constraint requires a subsidiary explanation of the way in which

it changes unless it is dependent solely on the ability to alter the amounts of particular inputs owned by the firm. In this latter case the model becomes formally identical to the Neo-Classical one.

The absence of a cost constraint renders the revenue maximisation model incapable of explaining any finite output level in perfect competition (Assumption (xiv)).¹⁶

- (v) Maximisation of Market Share: A firm's market share is defined as the ratio of its sales revenue (or output quantity) to that of the industry. The remarks addressed to revenue maximisation are also appropriate in this case with the additional difficulty of explaining industry size and inter-industry transfers in the long run, since the firm's choice criterion is defined in terms of the size and type of industry to which it currently belongs.
- (vi) Maximisation of Growth or Firm Size: No clear definition of size emerges from the literature although sales revenue and market share are used. These cases fall directly under (i) and (ii) above. Alternatively size may be determined by the amount of certain inputs (such as 'capital') which leaves the choice of other input quantities

(such as 'labour') unexplained. See the discussion of Average Net Product, below, however.

(vii) Satisfactory Performance. Gordon [32], Simon [58] and Margolis [45] have suggested that the firm seeks only satisfactory profit and this approach has been formalised by Day [20]. Whilst 'satisficing' violates the general choice theory as well as the model it also fails to provide an explanation of why the firm chooses one satisfactory position rather than another.¹⁷

The same objection arises from the contention of Cyert and March [19] that firm's do not consistently maximise any quantity. Baumol, [7] however suggests that satisfactory profit may be a preliminary objective beyond which firms maximise sales revenue, treating the minimum profit level as a constraint.

Revenue maximisation subject to a minimum profit objective is also identical to the Neo-Classical model unless it is possible for the firm to exceed its objective. The observation of any unconstrained firm with a finite output earning more than the specified profit level serves to discredit the revenue maximisation model.¹⁸

(viii) Maximisation of Average Net Product: The maximisation of the Average Net Product¹⁹ of one of the inputs, in each of the runs defined for the Neo-Classical Model appears to be the only distinct proposal articulated in the literature which is consistent with the remaining model structure. It is shown subsequently that a Model of Perfect Competition deduced from one of the set of ANP_i models (for some input i) can be distinguished from the Neo-Classical version by very few observations and may hinge crucially on Assumption (x).

8. SUMMARY: The Neo-Classical model of the firm is one in which a firm maximises profit subject to constraints on input quantities which are successively relaxed in a sequence of runs. In the penultimate run all inputs are variable and in the long (or longest) run the firm is able to pursue greater profit by adopting a different technology to produce different outputs (i.e. change industry). A normal level of positive profit exists for all industries at which all firms, given input markets, are unable to increase profit by further change.

An alternative explanation of the firm's behaviour within the same framework is provided by postulating the maximisation of the average net product of any one of the firm's inputs. In subsequent chapters the requirements

imposed on the profit and production relationship by the assumptions of the theory and the necessity to distinguish it from the average net product explanation are examined.

TABLE 1: SUMMARY OF PREDICTED BEHAVIOUR OF PERFECTLY COMPETITIVE FIRM

$$\text{Profit, } \pi = P \cdot X - \sum_{i=1}^n p_i x_i,$$

where p_i , P are input and output prices respectively and x_i , X are input and output quantities respectively.

At time zero prices are p_i , P and input quantities $x_i = \bar{x}_i \quad i = 1, \dots, n$.

End of Run:	Neo-Classical Model	Average Net Product Model	Comments
1	π maximised s.t. $x_j \leq \bar{x}_j, j=2, \dots, n$	$ANP_i = (X - \sum_{k \neq i} x_k) / x_i$ maximised s.t. $x_j \leq \bar{x}_j, j=2, \dots, n$	P_i, P constant maximisation of π \leftrightarrow maximisation of $X - \sum p_i x_i / P$
2	π maximised s.t. $x_j \leq \bar{x}_j, j=3, \dots, n$	ANP_i maximised s.t. $x_j \leq \bar{x}_j, j=3, \dots, n$	
n	$\pi \max = \pi(n)$ $\exists x_j > 0 \quad j=1, \dots, n$ ($x_j = 0 \quad \forall j \rightarrow$ industry exit \rightarrow run n+1)	$ANP_i \max$	
n+1	$\pi \max = \text{NORMAL } \pi$ $= \max \bar{P} X - \sum_{i=1}^n p_i x_i$	as n	$\pi(n) \succ$ NORMAL π $\rightarrow \bar{P} \succ P$ $\prec P$

Footnotes

1. Day's treatment in terms of 'satisficing' [20] is a notable exception, but fails to explain a firm's behaviour completely except in the limiting case when it converges to a maximisation solution.
2. It has been suggested that in the macro context 'technical progress' may be a function of capital or profit levels. However at the level of the individual firm any change in technology dependent upon input quantities is necessarily embodied in the description of the input-output relationship defined below.
3. Such a reference may be no more than labelling as was suggested in Levine [43] and Rowcroft [54]. Efforts to interpret 'profit' empirically have generated considerable controversy. See, for example, Hall and Hitch [33] and Cyert and March [19].
4. Boland [9] combines (iv) and (v) in the assumption that 'for each firm there is a limit on technical capabilities which defines "can produce"'. However this precludes the possibility that indefinitely large inputs might generate indefinitely large outputs which is sufficiently common in the literature (e.g. constant returns to scale) not to be excluded a priori. Nevertheless it should be noted that in the Boland Model all production decisions are implicitly short run. Intermediate run and long run decisions concern changes of the short run constraints. Thus Assumption (iv) in the text simply acknowledges the existence of a constraint explicitly.
5. Ownership of other inputs is fixed although utilisation of such inputs is merely bounded. See Boland [9], for example.
6. Algebraically, if there are m inputs, x_j , $j = 1, \dots, m$, there exists an ordering $k = 1, \dots, m$ such that there are m distinct time periods where, in period k , inputs x_1, \dots, x_k (suitably re-numbered) are variable and inputs x_{k+1}, \dots, x_m are fixed. In the two-input case, input 1 is usually associated with labour, L , which alone is variable in period 1 (the short run) and input 2 with capital, K . Both L and K are variable in period 2 (the "intermediate run").
 Since input x_k is variable in all runs subsequent to k , each period must be at least as long as that preceding it to allow re-adjustment of the preceding

variables. This implies a type of "nesting" process for input contracts or durability analogous to that described by Alexander [3] in the context of consumer goods.

7. This is not only usual in the specification of the long run (See Ferguson [26], for example) but also essential to the theory of General Equilibrium (Debreu [21]).
8. A difficulty arises here from the imprecision of the Neo-Classical definition of a firm. It would seem unnecessarily restrictive to assume that a production unit of any size should earn the same return to entrepreneurial effort when the size and type of commitment may vary considerably. Whilst this is in part recognised by the imputing of costs to managerial input (as well as to capital), (the atomistic concept of the perfect competitor also appears to play an implicit role.
 If varying levels of entrepreneurial effort were acknowledged in different industries, it would be appropriate to define the industry choice in terms of the rate of return to the entrepreneur. However since this effort is unobservable except perhaps as the managerial labour already "netted out", its inclusion as a variable renders the Neo-Classical explanation irrefutable in the long run. Any spectrum of positive inter-industry profits may be considered uniform by presuming appropriate degrees of entrepreneurial input in the different industries. Thus it is necessary to assume or define the firm to embody a fixed "quantum" of entrepreneurial effort [28], and 'profit' and 'profit rate' become synonymous.
9. Specifically every firm faces the same markets for inputs and outputs as every other.
10. This is the 'downward sloping aggregate demand curve' which is consistent with the requirements of consumer theory (downward sloping individual demand curve) and of general equilibrium theory (Debreu [21]).
11. Thus an upward shift in the aggregate demand curve increases maximum attainable profit for a firm producing the good concerned.
12. Neo-Classical theory offers no indication of the adjustment mechanism beyond that specified which suffices to explain the absence of industry changes after the long period.

13. A firm earning zero profit in the long run might continue to operate if the controlling interest (entrepreneur) preferred its source of income as an input to the alternatives available. However this introduces a second preference function not normally found (at least explicitly) in the traditional theory.
14. Assumption (vi) ensures that the firm in fact charges the maximum price.
15. Such a theory might be developed by stipulating only that each firm's preferences be such as to generate an upward sloping supply curve for its product, although general equilibrium requires only upward sloping aggregate supply curves and the stronger condition is sufficient but unnecessary. Compare the analysis of consumer preferences in Boland [9].
16. In imperfect competition such a model requires output at the point of unit elasticity on the demand schedule. If this schedule is observable any other elasticity constitutes a distinguishing observation.
17. If there exists only one satisfactory position in all situations, this theory converges to utility or profit maximisation depending on the definition of 'satisfactory'.
18. Since output price is constant and independent of output quantity in perfect competition (Assumption (xiv)) the existence of a finite output implies positive marginal costs for a firm under either explanation. Hence the revenue maximiser will necessarily increase output and hence revenue until profit is reduced to its minimum acceptable level.
19. The Average Net Product of input i is defined in the case of a single output, X , as $(X - \sum_{j \neq i} x_j) / x_i$. See Marshall [46], Robinson [50], and Boland [9].



CHAPTER FOUR

Model Structure and the Intermediate Runs

1. INTRODUCTION: Discussion in the remainder of this thesis centres on requirements for the mathematical structure of the specification of the firm's technology such that the requirements of the Neo-Classical Model described in Chapter Three are met. In this chapter the structure is defined consistent with the assumptions and the consequences of the theory are examined for every run except the longest which is analysed in Chapter Five.

Results are generated on the profit function with multiple outputs and imperfect markets in most cases. Whilst not essential to the argument in later chapters, this provides further generality without complicating the analysis unduly. For convenience this analysis is conducted in terms of net profit as defined in Chapter Three, that is total profit less the assumed level of uniform profit prevailing at the end of the long run. Thus the expression ' $\pi = 0$ ' denotes the situation in which the firm is earning exactly normal profit.

To preserve continuity only the results of theorems are noted and proofs are provided in the Appendix.

2. THE NEO-CLASSICAL PROFIT FUNCTION:

Definition 1: A firm is defined as a set of mappings from an input space S into the real line R such that for all

$x = (x^1, \dots, x^n) \in S$ where S is of finite dimension, n , and $x^j \geq 0$ for all $j = 1, \dots, n$, there exists $\pi \in R$ such that

$$\pi = \sum_{i=1}^m p^i x^i - \sum_{j=1}^n p^j x^j, \text{ the } \underline{\text{net profit function}}, \text{ where}$$

$x^i = F^i(x^1, \dots, x^n)$, the production function for good i ,

$p^i = p^i(x^i)$, the output price function for good i ,

$p^j = p^j(x^j)$, the input price function for good j .

$i = 1, \dots, m; j = 1, \dots, n$.

This distinction between inputs and outputs and the omission of joint products is consistent with most Neo-Classical analysis¹ and facilitates specialisation of the results to common cases. However it is not crucial to the main results of this chapter which are couched in terms of π rather than the production function(s), F^i . A similar justification may be made for the independence of the price functions from each other.

The sequence of runs described in Chapter Three presupposes that the firm reacts only to the successive relaxation of input constraints in the first n runs and to changes in the output market(s) in the long run. Thus it is convenient to assume that all price functions are fixed throughout runs 1 to n .² In the long run the set of mappings, given the p^j is indexed by the changing p^i .³

As noted in Chapter Three the main body of the analysis is conducted in terms of differentiable functions, hence

Definition 2: A Neo-Classical firm is defined as a firm

such that F^i , p^j are twice differentiable with respect to all x^k and P^i is twice differentiable with respect to X^i , $i = 1, \dots, m$; $j, k = 1, \dots, n$.

As an immediate consequence, P^i and π are both twice differentiable with respect to x^k , $i = 1, \dots, m$, $k = 1, \dots, n$.

[Lemma 1].

Definition 3: A shift in demand is defined as a mapping α which takes $P^i(X^i)$ to $\alpha P^i(X^i)$, $i = 1, \dots, m$, where $P^i(X^i)$ is specified in Definition 1 and α is an (unspecified) operator.

The introduction of demand shift permits allowance for the changes in output market conditions associated with the long run entry and exit of firms (Assumption (ix)). In particular a firm shall achieve an unconstrained optimum (net) profit level of zero for some possible level of market demand (Assumption (xi)).

Definition 4: A firm is described as being in its jth run if it is free to vary x^1, \dots, x^j and x^k is constrained by $x^k \leq \bar{x}^k$, \bar{x}^k finite, for all $k = j+1, \dots, n$.

This is simply a restatement of the concept of runs used in the General Model and Assumption (vi). The requirements of the Neo-Classical Model may now be restated as

Definition 5: The Neo-Classical Model of the firm is defined by the following statements about the behaviour of a Neo-Classical firm:

(a) In the j th run the firm's choice of inputs and outputs is uniquely determined by specifying that the firm

maximises profit subject to the appropriate input constraints,
 $j = 1, \dots, n$.

(b). In the long (($n+1$)th) run the entry and exit of competing firm's causes a shift in demand such that the maximum net profit which the firm can earn is zero.

Statement (a) embraces Assumption (i) to (vii) and (b) corresponds to Assumptions (viii) and (xi). The requirements of Assumptions (ix) and (x) are considered in Chapter Six on long run behaviour.

Definition 6: A run j is well-defined by the Neo-Classical Model if, and only if, $\max \pi$ in the j th run $\neq \max \pi$ in ($j+1$)th run, almost everywhere.

No change in profit from run j to run ($j+1$) implies that the constraint $x^{j+1} \leq \bar{x}^{j+1}$ was not binding and hence no change in x nor in x^i will be perceived. Unless this is an exceptional case, the possibility exists that the run cannot be identified, implying that x^j and x^{j+1} are complementary in production for the firm concerned and should be treated as a composite good.⁴ Henceforth we assume that run j is well-defined for all $j = 1, \dots, n$.

If a Neo-Classical firm has (net) profit function $\pi(x) = \pi(x^1, \dots, x^n)$ it may be shown that a sufficient condition for the extreme points of π in runs 1 to n to be maxima subject to the appropriate constraints is that the matrix of second order partial derivatives of π , with respect to x , (π_{jk}) be negative definite. [Theorem 2]. Thus (π_{jk})

negative definite ensures that the familiar marginal conditions (see [36], for example) generate constrained profit maxima in the first $(n-1)$ runs and an unconstrained maximum in run n . Moreover if each run remains well defined the result is true for any ordering of the x^j [Corollary 2.1] and provides unique maxima by Lemma 2.2.

One consequence of the condition on (π_{jk}) is that

$\sum_{j=1}^k (\partial^2 \pi / \partial x^j)^2 < 0$ for all $k = 1, \dots, n$ [Lemma 2.3] which is a weaker version of the common specification that the profit from any input be increasing at a decreasing rate as the quantity of the input increases.⁵

By virtue of the constrained maximisation it should also be noted [Lemma 2.4] that profit in run j cannot be greater than profit in run $j+1$, for all $j = 1, \dots, n-1$.⁶

3. INPUT VALUE THEORY: The alternative model (or theory) introduced in Chapter Three may be generalised slightly to correspond with the non-specific treatment of markets in this chapter. Thus

Definition 7: The Average Revenue Product of an input x^k for a firm is defined as

$$ARP_k \equiv \left(\sum_{i=1}^m p^i x^i - \sum_{j \neq k}^n p^j x^j \right) / x^k \quad \text{for } x^k > 0$$

Since ARP_k is undefined for $x^k = 0$, it is useful to make reference to the case when some input is always positive.

Definition 8: A j -integrated Neo-Classical Firm is a

Neo-Classical firm for which there exists a sequence of well-defined runs $1, \dots, n$ such that $x^i > 0$ implies $x^j > 0$ for all $i = 1, \dots, n$. Thus x^j is an input to every non-trivial production.

Hence:

Definition 9: The Input Value Model of the firm is defined by the following statement about the behaviour of a k -integrated Neo-Classical firm:

In the j th run the firm's choice of inputs and output(s) is uniquely determined by specifying that the firm maximises the Average Revenue Product in input x^k subject to the appropriate constraints, $j = 1, \dots, n$.⁷

The first order conditions for the maximisation of ANP_k are identical to those for profit maximisation for all inputs except x^k . Consequently the two models are indistinguishable in runs 1 to $k-1$. The existence of possible observations on subsequent runs which permit the input value model to be rejected in favour of the Neo-Classical model is examined in the next section.

4. DISTINGUISHING OBSERVATIONS: The necessity for the existence of distinguishing observations was noted in Chapter Two and an example was provided in Chapter Three using revenue maximisation subject to a minimum profit. For a k -integrated Neo-Classical firm, a sufficient condition for it to be possible that exactly one of the Neo-Classical and Input Value models be known false is that either

$\partial\pi/\partial x^k$ is known for run r , $r > k$ or that the values of $\partial^2\pi/\partial x^{k^2}$ and $\partial^2/\partial x^{k^2} \sum_{j=1}^n p^j x^j \neq 0$ are known for the same run, $r > k$ [Theorem 3].

Whilst the behaviour of the cost expression may be presumed known since market conditions for inputs are known (see section 2 above), the observation of (temporary) equilibria at the end of each run does not permit the deduction of values for the derivatives of π . Unless it may be shown that the structure of the theory imposes sufficiently strict conditions on the derivatives of π to generate a contradiction, the Input Value Model cannot be distinguished on this basis. Thus for a 1-integrated firm it would be possible to discredit the ANP₁ model if

$$\partial^2/\partial x^{1^2} \sum p^j x^j + \partial^2\pi/\partial x^{1^2} > 0 \quad [\text{Lemma 3.1}]$$

This may be expressed as the requirement that the marginal cost of good 1 be increasing faster than the marginal profit of the same good is falling when x^1 is increased. However this would not refute another model ANP_k, $k \neq 1$, if the firm were k-integrated. [Compare Lemma 3.2.]

The argument that the value of $\partial\pi/\partial x^k$ is known since p^k is observed and $\partial\pi/\partial x^k = p^k$ in run j for all $j > k$ is invalid since it depends on the assumption of profit maximisation which is part of only one of the models under scrutiny.

If the price of the indispensable factor j is invariant with respect to x the second derivative condition fails but

may be replaced by the requirement that maximum $\pi \neq 0$ in run n . [Lemma 3.3]. This raises the question of the observability of π . Since π here is net of the constant (but unknown) normal profit, a direct calculation from prices and quantities is inadequate. However it is possible to compute changes in π . Both models indicate zero (net) profit in the long run, whence a change in π from run n to run $(n+1)$ suffices to indicate $\max \pi \neq 0$ in run n and provide the required distinguishing observation in Lemma 3.3. Thus, drawing on the analysis of legitimate observations in Chapter Two, Definition 10: A set of acceptable observations consists of the following information for each run k of a Neo-Classical firm, $k = 1, \dots, (n+1)$:

- (i) the values of $p^i(x^i), p^j(x^j)$ for all i, j
- (ii) the values of x^j, x^i for all i, j
- (iii) by deduction, the value of $\pi(r) - \pi(s)$ for all $r, s = 1, \dots, n+1$, where $\pi(k)$ denotes the (maximum net) value of π in run k . Hence the truth or falsity of the statement $\pi(n) = 0$.

It follows immediately that for a j -integrable firm a distinguishing observation exists within the set of acceptable observation if p^j is constant [Theorem 4].

5. REPRESENTABILITY: The questions of explanation and distinguishing observations raised in Chapter Two are sufficiently important in the requirements they place upon the Neo-Classical Model to justify the amplification of the

definitional structure to accommodate them specifically.

Definition 11: A firm is representable by a theory if, and only if, there exists a proper subset A of the set of acceptable observations B , such that, for all $b \in B$,

- (i) $b \in A$ implies that the theory explains why the firm is not using any other input-output combination, and
- (ii) $b \notin A$ implies that the theory is false.

This is simply the requirement of "complete" explanation. The necessity for a distinguishing observation is expressed as

Definition 12: A firm is Neo-Classically representable if, and only if, it is

- (i) Neo-Classical
- (ii) representable by the Neo-Classical Model with proper subset $A_N \subset B$, and
- (iii) representable by the Input Value Model with proper subset $A_I \subset B$ only if $A_N \neq A_I$.

By condition (iii) B must contain at least one observation which is not common to A_N and A_I and hence refutes one of the models. Although not admissible for a complete theory it is common in the literature (see [42] for example) to make less stringent requirements on the Neo-Classical Model. These may be represented as follows:

Definition 13: A firm is weakly representable by a theory if, and only if, there exists a proper subset A of acceptable

observations B , such that, for all $b \in B$,

- (i) the observation $b \in A$ is explained by the theory,
- and
- (ii) $b \notin A$ implies that the theory is false.

Thus the theory may indicate that $b \in A$ is one of a number of possible observations between which it does not distinguish as in the case of an isoquant with a linear segment in the traditional analysis (e.g. [8]).

Definition 14: A firm is weakly Neo-Classically representable if, and only if, it is

- (i) Neo-Classical
- (ii) weakly representable by the Neo-Classical Theory with proper subset $A_N \subset B$, and
- (iii) weakly representable by the Input Value Model with proper subset A_I only if $A_I \cap A_N = \emptyset$.

By Definition 13 a given price set does not necessarily permit a single input-output combination to be inferred whence the existence of a distinguishing observation is not ensured unless A_I and A_N have no common element.

Drawing on Definition 11 it may be shown that a Neo-Classical firm is representable by the Neo-Classical Theory if (π_{jk}) is negative definite, $\pi(n) \neq 0$ and there exists a shift in demand such that the unconstrained maximum for π is zero [Theorem 5]. If, in addition it is not k -integrable for any $k = 1, \dots, n$, the firm is Neo-Classically representable [Lemma 5.1]. This remains

true if the firm is integrable for $k = 1$ only and either

$$(a) \quad (\partial^2 / \partial x^1)^2 \left[\sum_{i=1}^m p^i x^i \right] > 0$$

or $(b) \quad (\partial^2 / \partial x^1)^2 \left(\sum_{j=1}^n p^j x^j \right) = 0$ [Theorem 6]

Condition (a) requires that the marginal revenue of x^1 be increasing with x^1 whilst marginal profit is decreasing if (π_{jk}) is negative definite [Lemma 2.3]. However condition (b) leads directly to the result that if input prices are constant, as in perfect and imperfect competition, the requirements of Theorem 5 suffice for the firm to be Neo-Classically representable [Corollary 6.1].

6. REQUIREMENTS ON THE PROFIT FUNCTION:

Definition 15: A Neo-Classical Profit Function is a function π such that a firm with profit function π is Neo-Classically representable.

For any Neo-Classical firm, a necessary condition for maximum profit to be identically zero in run n is that for all $x^k > 0$ at the maximum, $\partial^2 \pi / \partial x_k^2 = 0$ [Theorem 7]. If the firm is 1-integrable this condition is inconsistent with the requirement that (π_{jk}) be negative definite [Theorem 8]. Consequently the following requirements are sufficient for π to be a Neo-Classical profit function by Theorem 9.

$$(i) \quad x^i(0, x^2, \dots, x^n) = 0 \text{ for all } x^2, \dots, x^n > 0, \\ \text{for all } i = 1, \dots, m$$

$$(ii) \quad (\pi_{jk})_{n \times n} \text{ is negative definite}$$

(iii) there exists a shift in demand, α , such that

$$\max_x [\pi = \sum_{i=1}^m \alpha p^i x^i - \sum_{j=1}^n p^j x^j] = 0, \text{ and}$$

(iv) either

$$\text{a) } (\partial^2 / \partial x^1)^2 (\sum_{i=1}^m p^i x^i) > 0$$

$$\text{or b) } (\partial^2 / \partial x^1)^2 \sum_{j=1}^n p^j x^j = 0.$$

If input prices are constant then condition (iv) is satisfied and conditions (i) to (iii) suffice for imperfect and perfect competition [Corollary 9.1].

7. THE NON-DIFFERENTIABLE PROFIT FUNCTION: The differentiability of the profit function is a direct consequence of Definition 2. If this requirement is removed, the stipulation that π_{jk} be negative definite [Theorem 2] may not be meaningful. It may however be replaced by the condition that π be a concave function of x [Theorem 10] with uniqueness of the maxima ensured by strict concavity [Lemma 10.2]. These two results include Theorem 2 as a special case since if π_{jk} exists for a concave function, π , it is negative semi-definite and negative definite for a strictly concave function. An analogue of the maximisation condition in the differentiable case is noted in Corollary 10.1, namely that the (non-differentiable) function π will attain a (possibly constrained) maximum at x when zero belongs to the appropriate subdifferential at x .

If the convex conjugate of $\pi(x)$ is defined as

$$\pi^*(x^*) = \sup_x \{ \langle x, x^* \rangle + \pi(x) \}$$

where $\langle x, x^* \rangle$ is the inner product of x and x^* , $\sum_{j=1}^n x^j x^{*j}$,

uniqueness conditions may be obtained which do not require strict concavity. Thus a finite, concave π has a unique unconstrained maximum at \bar{x} if π^* is differentiable at $x^* = 0$ and $\bar{x} = \nabla \pi^*(0)$ [Theorem 11]. If π is also closed, this condition is both necessary and sufficient [Corollary 11.1]. It may also be demonstrated [Theorem 12] that under the same conditions, the constrained maxima are unique.

Since the concavity of π is central to the assurance that the appropriate maxima exist, a number of conditions which ensure this concavity are noted below:

- a) a sufficient condition for π to be concave is that $\sum_{j=1}^n p^j x^j$ and $\sum_{i=1}^m p^i x^i$ be convex and concave functions

of x respectively [Lemma 12.1]

- b) if $C = \sum_{j=1}^n p^j x^j$, a necessary condition for C to

be convex is

- (i) $[C(x+\lambda y) - C(x)]/\lambda$ be a non-decreasing function of $\lambda > 0$ for all $y = y^1, \dots, y^n$, i.e. the cost function does not exhibit "decreasing returns to scale" with any origin,

- (ii) the one-sided directional derivative of

$$C \text{ at } x, C'(x;y) = \inf_{\lambda > 0} \{ [C(x+\lambda y) - C(x)]/\lambda \}$$

for all y , a type of "upper semi-differentiability" analogous to upper semi-continuity, and

- (iii) $C'(x:y)$ is a positively homogeneous convex function of y with $C'(x:0) = 0$ and $-C'(x:-y) \leq C'(x:y) \quad \forall y$. [Lemma 12.2].

- c) Analogous conditions to b) apply for $R = \sum_{i=1}^m P^i X^i$

to be a concave function of x , including the absence of increasing returns [Lemma 12.3].

- d) It is sufficient for C and R to be convex and concave respectively that $p^j x^j$ and $P^i X^i$ be convex and concave in x respectively for all $i = 1, \dots, m$; $j = 1, \dots, n$ [Lemma 12.4 and Corollary 12.5].

- e) Thus similar conditions to those in b) and c) may be placed directly on $c^j = p^j x^j$ and $r^i = P^i X^i$ [Lemma 12.6 and Corollary 12.7].

If either cost or revenue are differentiable, necessary conditions are placed on the derivatives by the requirement of convexity or concavity. Thus

- f) if c^j is finite and differentiable convexity requires that the marginal cost of input j be non-decreasing [Lemma 12.8]
- g) if r^i is finite and differentiable concavity requires that the marginal revenue product of each input in output i be non-increasing [Corollary 12.9].
- h) the result in f) may be generalised if C is finite

and c^j differentiable for all j to the requirement that all marginal costs be non-decreasing, i.e. $\nabla C \geq 0$ [Corollary 12.10]. Similarly if R is finite and r^i differentiable for all i , the marginal revenue product of any input in any output must be non-increasing, i.e. $\nabla R \leq 0$ [Corollary 12.11].

The assumption of differentiability is significant in two aspects of Theorem 9 which provides sufficient conditions for π to be a Neo-Classical Profit Function. Firstly condition (i) that π_{jk} be negative definite may be replaced by strict concavity [Lemma 10.2] or by the conditions of Theorem 12. However condition (iv) requires the differentiability of R or C to generate the distinguishing observation. Referring to Theorem 5, this condition may be replaced by the double requirement that input prices be fixed (perfect or imperfect competition) and maximum π in the n th run be non-zero [Theorem 13]. From Corollary 11.1 this may be ensured if $\pi^*(x^*)$ is non-differentiable when $\pi(x) = 0$ [Theorem 14]. This result will be remarked upon in Chapter 6.

8. SUMMARY: In this chapter the assumption set in Chapter Three has been used to formulate a definitional structure for the examination of the Neo-Classical profit and production functions. From these definitions constraints were deduced on the form of the profit function, particularly

that it be concave (strictly so if unbounded) and that for imperfect and perfect competition it must not achieve an unconstrained maximum of zero. The conditions necessary to guarantee an acceptable long run profit situation are examined in the next chapter.

Footnotes

1. Activity Analysis [42] is a notable exception.
2. The necessary condition is that all p^i and p^j do not change in run k for all $i = 1, \dots, m$ and $j \leq k$ since a change in price of a constrained variable does not alter the firm's 'maximisation calculation' subject to that constraint.
3. The Neo-Classical model does have implications for the firm's reaction to changes in input prices. However these are most readily examined in the context of the properties of the production function in Chapter Seven.
4. Compare Samuelson's Composite Commodity Theorem in consumer theory [42]. This point is reconsidered in Chapter Seven.
5. For the perfectly competitive firm this becomes the same condition on marginal products.
6. Profit may fall from run $n-1$ to run n due to the behaviour of all firms seeking long run equilibrium, as noted in Chapter Six.
7. Often labour is considered to be an indispensable input which with Definition 9 yields a primitive Labour Theory of Value applied to the behaviour of an individual firm.

CHAPTER FIVE

Existence of Long Run Zero Maximum Profit

1. INTRODUCTION: The existence of a position of zero maximum (net) profit is crucial to the explanation of the firm's long run behaviour. (Assumption (xi)) and is in contrast with the requirement of non-zero profit in the penultimate (nth) run. This potential conflict is resolved by the concept of a demand shift (Definition 3) by which the firm moves from a position of net gain or loss under a set of output market conditions to one of normal profit under different, but related conditions. In this chapter sufficient conditions are examined for the existence of zero maximum profit under some set of market relations. Consideration of necessity and integration with the requirements of the preceding chapter is deferred until Chapter Six.

2. HOMOTHETICITY: If the profit function is zero whenever the maximising conditions are effective, it will have zero maximum value in the long run, regardless of the demand conditions which the firm faces. This is commonly achieved for the perfectly competitive firm by specifying that the production function be linearly homogeneous. (Compare Boland [14] and Lancaster [42], for example.)¹ However it may be shown that this is a particular case of the result that almost any homothetic profit function necessarily has zero maxima [Theorem 15]. Consequently if revenue and cost are homogeneous

of the same degree, other than zero², maximum profit will be zero. [Theorem 15].

Whilst the existence of a demand shift such that the profit function becomes homothetic provides a sufficient condition it degenerates for the perfectly competitive firm to the trivial case of linear homogeneous production. An obvious recourse in this case is to construct a production function of a linear combination of two homogeneous functions such that increasing and decreasing returns to scale "cancel out" to produce zero maximum profit at a particular production level. This procedure is illustrated in the next section.

3. AN EXAMPLE: Let the production function of a perfectly competitive firm be the sum of two Cobb-Douglas functions such that the isoquants of one function are a (non-linear) translation of the isoquants of the other. Thus the isoquants for the compound production function will have the same concave contour as those of its component functions. If the production is given by

$$X = AL^{\alpha}K^{\theta-\alpha} + BL^{\beta}K^{\vartheta-\beta}$$

A, B positive constants, $\vartheta > 1 > \theta > \alpha > 0$; $\vartheta > \beta > 0$,

it consists of one function with decreasing returns to scale and one with increasing returns to scale. If, in addition,

$$\alpha\vartheta = \beta\theta$$

the slopes of the isoquants of both functions will be identical for any values of L and K.

First order derivatives of X yield

$$X_L = A\alpha L^{\alpha-1} K^{\theta-\alpha} + B\beta L^{\beta-1} K^{\vartheta-\beta}$$

$$X_K = A(\theta-\alpha)L^\alpha K^{\theta-\alpha-1} + B(\vartheta-\beta)L^\beta K^{\vartheta-\beta-1}$$

both of which are positive for all positive L, K.

Zero Maximum profit occurs where

$$X = LX_L + KX_K, \text{ i.e.}$$

$$A(1-\theta)L^\alpha K^{\theta-\alpha} = B(\vartheta-1)L^\beta K^{\vartheta-\beta}$$

Together with the first order conditions, the following are necessary and sufficient for a maximum:

$$X_{LL} < 0; \quad X_{KK} < 0; \quad X_{LK} > 0$$

and

$$X_{LL}X_{KK} - X_{LK}^2 > 0.$$

The second derivatives have the required signs for all positive L, K if $\beta < 1$ and $\vartheta < 1 + 2$. However the determinantal condition becomes

$$\begin{aligned} & A^2 \alpha(\theta-\alpha)(1-\theta)L^{2\alpha-2} K^{2\theta-2\alpha-2} + B^2 \beta(\vartheta-\beta)(1-\vartheta)L^{2\beta-2} K^{2\vartheta-2\beta-2} \\ & + AB\{\alpha(\vartheta-\beta)[(\alpha-1)(\vartheta-\beta-1) - \beta(\theta-\alpha)] + \beta(\theta-\alpha) \\ & \quad [(\beta-1)(\theta-\alpha-1) - \alpha(\vartheta-\beta)]\} \\ & > 0 \end{aligned}$$

At zero maximum profit this condition is

$$\begin{aligned} & \frac{A^2 L^{2\alpha-2} K^{2\theta-2\alpha-2}}{(\vartheta-1)} \{ \alpha(\theta-\alpha)(1-\theta)(\vartheta-1) - (1-\theta)^2 \beta(\vartheta-\beta) \\ & \quad + \alpha(\vartheta-\beta)[(\alpha-1)(\vartheta-\beta-1) - \beta(\theta-\alpha)] \\ & \quad + \beta(\theta-\alpha)[(\beta-1)(\theta-\alpha-1) - \alpha(\vartheta-\beta)] \} > 0 \end{aligned}$$

and, noting that $\alpha\vartheta = \theta\beta$, we have, after some simplification,

$$(1-\theta)(\alpha-\beta)^2 [1 + \frac{\vartheta-\theta}{\alpha-\beta}] > 0$$

i.e. $(\vartheta-\theta)/(\alpha-\beta) > -1$

Since $\theta - \theta > 0$ by specification, the condition is met if $\alpha > \beta$. However $\alpha > \beta > 0$ implies $\alpha\theta > \beta\theta$ since $\theta > 0$.

whence $\beta\theta > \beta\theta$

and thus $\theta > \theta$ since $\beta > 0$,

a contradiction.

Therefore $0 < \alpha < \beta$ and the second-order condition becomes

$$\theta - \theta < -(\alpha - \beta)$$

i.e. $\theta^2 - \theta\theta < \beta\theta - \alpha\theta$ since $\theta > 0$

Hence $\theta(\theta - \theta) < \beta(\theta - \theta)$

i.e. $\theta < \beta$ since $\theta - \theta > 0$, contradicting the specification of β .

Thus the second order conditions for a maximum do not hold at the point of singularity where profit is zero³.

The problem of ensuring the existence of zero maximum profit is therefore addressed in greater generality in the next section.

4. EXISTENCE OF ZERO MAXIMUM PROFIT: For a perfectly competitive firm to attain a point of zero maximum profit its production function $F(x)$ must be such that when it is maximised subject to some set of price ratios p^j/P , factor payments just absorb output, i.e.

$$F(x) = x \cdot \nabla F(x).$$

Such an expression is always true if the production function is homogeneous of degree 1, and hence for any $F(x)$, the zero maximum profit point may be considered as one at which $F(x)$ is "locally linearly homogeneous".⁴

Equivalently if $F(x)$ attains zero maximum profit when $x = x^*$, then there exists a linearly homogeneous function $G(x)$ such that $G(x^*) = F(x^*)$ and both functions have the same first derivatives at x^* , i.e. $\nabla G(x^*) = \nabla F(x^*)$.

Let $\psi: x \rightarrow y$ represent a mapping from input space into itself such that $F(x) = G(y)$ and $\nabla F(x) = \nabla G(y)$, then the existence of a zero maximum profit point for $F(x)$ is clearly equivalent to the existence of a fixed point for ψ such that $x = y$.

A general result on mappings from one function to another in which an operation on the function is preserved (such as ∇) is provided by Theorem 16. Applied to a Neo-Classical Production Function, $F(x)$, the theorem requires that for any x, p and some P it be possible to find a y such that $F(x) = G(y)$, $p/P = \nabla F(x) = \nabla F(y)$ and that the set of such y be closed and bounded [Lemma 16.1]. Under these conditions either the required fixed point exists or there is a pair of values $x^* \neq y^*$ such that x^* maps to y^* and vice versa, which we may call a "two-cycle".

The possibility of a two-cycle may be removed by placing further conditions on the production function. By Theorem 17, if $F(x)$ is concave and $\nabla F(x)$ non-zero, no two-cycle exists if

$$[F(y) - F(x)] \cdot x \cdot [\nabla F(y) - \nabla F(x)] < 0$$

for all $x \neq y$ with equality included if $F(x)$ is strictly concave. This is a type of "diminishing marginal product" rule which asserts that if output is, say, increased by

changing factor inputs from x to y , the total cost of the old inputs at the new prices ($\nabla F(y)$) must be less than the cost of the old inputs at the old prices ($\nabla F(x)$).

Requirements for a (not necessarily differentiable) production function to generate zero maximum profit for some output price P are summarised in Theorem 18. It should be noted however that the restriction of finiteness, whilst useful in Theorem 11 to avoid the necessity for strict concavity, and in Theorems 16 and 17 to ensure the existence of the fixed point, is hard to justify a priori. Without input restrictions, which are excluded by definition in the long run, there would appear to be no reason why a firm might not produce any output, although not necessarily for maximum profit.⁵

Conversely if $F(x)$ is differentiable on an open convex set, such as $x \geq 0$, strict concavity implies that if $F(x) \geq 0$, maximum profit must always be positive [Theorem 19]. This result illustrates two difficulties concerning the origin. Whilst a zero maximum profit with zero inputs will not permit the Neo-Classical Model to explain the (non-trivial) behaviour of the firm in the long run, this possibility has not been specifically excluded in the preceding theorems. Moreover by Theorem 22 the natural supposition that zero input generates zero output leads to difficulties with strict concavity. Recalling the discussion of normal profit in Chapter Three it may be noted that $F(x)$ corresponds to the net profit function $\pi(x)$. If the level of normal profit π_N

and the long run price level \bar{p} were known this difficulty might be resolved by setting $F(0) = -\pi_N/\bar{P} < 0$ although this introduces the further complication of a region of positive inputs for which 'output' is less than zero.⁶

It seems more appropriate to define $F(\bar{x})$ only on $\bar{x} > 0$ and to consider $\bar{x} = 0$ as a special case in which the production function is no longer applicable since the firm has 'exited from the industry'. Thus $\{x\}$ remains an open convex set but the argument in Theorem 19 ceases to apply. Moreover any point of zero maximum profit must necessarily be distinct from the origin as required by the Neo-Classical Model.

5. SUMMARY: In this chapter sufficient conditions for a general profit or production function to generate an unconstrained maximum profit of zero have been explored. Homotheticity provides maximum zero profit to the exclusion of any other value, whereas an approach using fixed point techniques imposes somewhat stringent requirements on the production function. It was also noted that the origin should be excluded from the domain of this function.

Since the theory requires the zero maximum profit point it is also appropriate to assume its existence and hence deduce constraints on the profit or production function described in Chapter Four. The derivation of such necessary conditions is the subject of the next chapter.

Footnotes

1. The most common example is the Cobb-Douglas function $X = A.L^\alpha K^\beta$ where $\alpha + \beta = 1$, [14,42]. If, in particular $\alpha = \beta = 1/2$, then in terms of the familiar short-run diagrammatic analysis, minimum average cost will occur when (average) variable cost is equal to (average) fixed cost.
2. Zero would imply that costs and revenue were invariant with respect to scale. The input-output combination would be undetermined by the Neo-Classical Model in this case.
3. This example is an instance of a function which is concave-contoured but neither concave nor convex (c.f. Lancaster [42]).
4. This expression was suggested by W. J. Baumol.
5. Conditions which limit the domain of the mapping in Theorem 19 to points of maximum profit (for some p, P), or to a closed, bounded set containing them but excluding values for which $F(x)$ is infinite, appear to amount to an assumption that the result of the theorem holds. Consequently they are omitted.
6. If otherwise desirable this might be rationalised as being 'below minimum operating scale' or as the opportunity cost to the firm of being in the 'wrong' position with respect to industry or input constraints.

CHAPTER SIX

The Neo-Classical Firm in the Long Run

1. INTRODUCTION: In the preceding chapter sufficient conditions were deduced for the existence of a zero maximum profit point for the firm. Here the requirements of the Neo-Classical Model described in Chapter Four are assumed to apply and are used to derive necessary conditions on the production relationship. To avoid confusion between requirements on production and market conditions, this analysis is conducted in terms of the perfectly competitive firm defined in Chapter Three. Thus all prices are invariant with respect to quantity¹ and the long run is distinguished by changes in output price in response to the entry and exit of firms. Analysis is facilitated by tentatively retaining Definition 2 that the profit and hence production function is twice differentiable for all positive input vectors.

2. PROFIT AND PRODUCTION FUNCTIONS: For any fixed set of input prices it may readily be shown that concavity of the profit function (strict or otherwise) imposes the same degree of concavity on the revenue function, $R(x)$. [Lemma 20.1]. The perfectly competitive firm produces one output sold at a quantity-invariant price, P , (Assumption (xiv)), and hence the concavity applies directly to the (single) production function $F(x)$. [Lemma 20.3]. If π is twice differentiable, it follows directly that the requirement

for a Neo-Classical Profit Function that (π_{jk}) be negative definite given in Theorem 9 is equivalent to the stipulation that $(F_{jk}) \equiv \nabla^2 F(x)$ be negative definite [Lemma 20.4]. It was noted in Chapter Four that this condition ensures that the extreme points of $\pi(x)$ are unique maxima subject to the appropriate constraints, if any.

3. LONG RUN BEHAVIOUR OF THE PERFECTLY COMPETITIVE FIRM:

Assumptions (viii) to (xi) describe the behaviour of Neo-Classical firms in the long run as a process of adjustment to changing output market conditions resulting from firms changing industries in search of greater profit. Ultimately the situation is static at the end of the long run when all firms earn the same normal profit, or net profit, $\pi = 0$.²

In perfect competition, Assumptions (xii) to (xiv) permit long run behaviour to be stated more fully as follows:

- a) the firm faces a set of positive input prices $p = (p^1, \dots, p^n)$ which do not change throughout runs 1 to $n+1$.
- b) the firm faces an output price, P , which does not change throughout runs 1 to n .
- c) in run $(n+1)$, the long run, the entry and exit of firms causes P to change to P^* such that the maximum net profit a firm can earn with prices p , P^* and given technology $F(x)$ is zero.
- d) if $F(x)$ is twice differentiable the firm

maximises profit for prices \underline{p} , P by choosing an input vector x such that

$$\nabla F(x) \equiv \left(\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^n} \right) = \frac{\underline{p}}{P} \text{ [See [36], for example]}$$

$$\begin{aligned} \text{Thus (net) profit, } \pi(x) &= PF(x) - x \cdot \underline{p} \\ &= P[F(x) - x \cdot \nabla F(x)] \\ &\text{when } \pi \text{ is maximised.} \end{aligned}$$

Consequently $\max \pi(x^*) = 0$ if, and only if,

$$F(x^*) = x^* \cdot \nabla F(x^*)$$

Since \underline{p} is constant, but unrestricted beyond $\underline{p} > 0$ in this process, the explanation of the firm's behaviour must be consistent with any choice of (finite) positive input prices.³ Thus the theory requires that for any $\underline{p} > 0$, and positive $P \neq P^*$, maximum $\pi(x)$ in run n is non-zero but that P changes to P^* in run $(n+1)$ in a manner consistent with Assumptions (viii) to (xi).

4. OUTPUT PRICE BEHAVIOUR IN THE LONG RUN: If consideration is limited to input combinations, x^0 , for which profit is maximised for some P and a given input price vector, \underline{p} , the first order condition in the previous section indicates that the components of $\nabla F(x^0)$ must be positive and inversely proportional to P . Consequently $F(x^0)$ is an increasing function of the components of x^0 and all are non-decreasing functions of P [Theorem 21]. This is a particular case of the positive "output substitution effect" [Theorem 22] which corresponds to the negative demand effect specified

by Assumption (ix) or deducible from consumer theory.⁴ Thus the output of a perfectly competitive industry increases with price for two reasons: individual firms increase output as they maximise profit and new firms are drawn into the industry by the (hypothesised) increase in profits. This increase in output quantity tends to reduce output price by the assumed action of market demand. The significance of entry and exit is consequently to initiate an output price change and sustain this change until (net) profit is zero. Assumption (x) that maximum attainable profit is an increasing function of P is crucial for the required entry and exit behaviour to be consistent with (explained by) the firm's profit maximising objective.

5. LONG RUN ZERO PROFIT: For a given input price vector p the Neo-Classical Model requires the existence of an input vector, x^* , such that, for some $P^* > 0$, profit is maximised by setting $x = x^*$ and $\pi(x^*) = 0$. Equivalently, for the perfectly competitive firm,

$$F(x^*) = x^* \cdot \nabla F(x^*) = x^* (p/P^*)$$

Moreover from the preceding section,

$$P \lesssim P^* \rightarrow \pi(x^0) \lesssim 0, \text{ i.e.}$$

$$F(x^0) \lesssim x^0 \cdot \nabla F(x^0) = x^0 \cdot (p/P)$$

Since an input-point with these properties must exist for any choice of a positive input price vector, p , it is appropriate to examine the locus of such 'zero profit points'. However, the requirement that $\nabla^2 F(x)$ negative

definite suffices to make this locus degenerate, that is only one point x^* can exist, independent of the choice of p . [Theorem 23]. Furthermore, no other input combination will produce the same output $F(x^*)$. [Theorem 24]. Two dimensionally, there is no isoquant through x^* .

The singularity of the input combination x^* which must be finite and non-zero to fulfill the requirements of the Neo-Classical Model, provides a direct contradiction to Definition 2 which specified the differentiability of $F(x)$. An (ad hoc) relaxation of this definition permits the convenience of the calculus treatment to be retained for most situations:

Definition 2': A Neo-Classical firm is defined as a firm such that F^i , p^j and P^i are twice differentiable with respect to x^k , $i = 1, \dots, m$; $j, k = 1, \dots, n$ almost everywhere.

For consistency, and as a heuristic device, $\nabla F(x^*)$, may be defined as taking on the values of all positive vectors with a specified norm. This ensures that for some P^* dependent on p , and for all $p > 0$, the usual profit maximisation calculation will lead the firm to use inputs x^* to produce $F(x^*)$ with consequent zero profit [Theorem 25].

6. PROPERTIES OF THE ZERO PROFIT POINT: Under Definition 2' strict concavity of the production function implies that $\nabla^2 F(x)$ is negative definite almost everywhere [Theorem 26].

If x^* is the only singular point, the isoquant surfaces must be smooth and concave contoured through all other input

points [Lemma 27]. As P changes, given p , the expansion path of any firm must pass through x^* but remain distinct from that of the same firm with a different set of input price ratios.⁵ Thus x^* may be conceptualised as a 'switching' point at which the firm changes from one mode of operation to another. Specifically, the relationship between profit maximising x (x^0) and P described in Section 4 above, implies that profit is positive for all $x^0 > x^*$ and negative for all $x^0 < x^*$ [Lemma 28].⁶ This phenomenon may be incorporated in the explanation of the firm's behaviour in a number of ways, of which three are illustrated for the two input situation in Figure 2. Here the 'isoquant-space' is divided into two disjoint regions such that for high values of p/P the firm's maximum profit is negative and is achieved at an input point in the area labelled ' $\pi_{\max} < 0$ '. Conversely for a sufficiently large output price P the firm will optimise by choosing an input combination in the ' $\pi_{\max} > 0$ ' set with consequent positive profit. Corresponding marginal product curves are depicted in Figure 3.

The behaviour shown in Figure 2(a) requires discontinuity at all points for which $x^j = x^{j*}$, any $j = 1, \dots, n$ such that a small change in the quantity of one input, from $x^{j*} - \Delta$ to $x^{j*} + \Delta$ say, could produce large changes in the profit maximising input quantities.⁷ Figure 2(b) indicates a situation which avoids this type of general discontinuity at the expense of creating a subset of input vectors the choice

of which is proscribed by the Model a priori, and hence unexplained.⁸ Both difficulties are resolved by requiring the isoquants to converge on x^* as $F(x)$ increases to $F(x^*)$ as shown in Figure 2(c). In this case the only excluded input combinations are those which are "technically inefficient" and lie outside 'ridge lines' defined by the production function in the manner of standard Neo-Classical analysis (see [9], for example).

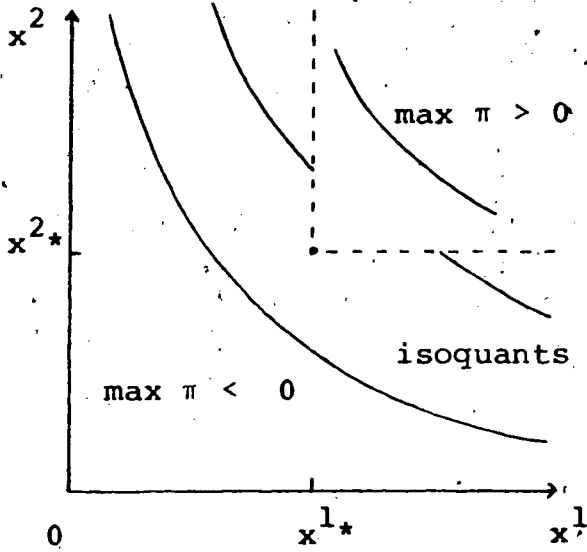
The specification of a production function which meets the requirements derived here will be considered in the next Chapter. However it should be noted that the necessary existence of the singularity x^* provides a means by which the Neo-Classical Model may be refuted by direct observation of input combinations. Thus if the long run input combination has been observed, any observation of any input vector x such that $x^j = x^{j*}$ and $x^j \geq x^*$ in any run, and at any prices, necessarily implies that the theory-model is false [Theorem 29].

7. SUMMARY: In perfect competition the zero maximum profit position required by the Neo-Classical model must occur with the same input combination, irrespective of input prices. This input point generates a singularity in the production function which makes the model inconsistent with the requirement that this function be differentiable everywhere. If this singularity is admitted by revising the specification of differentiability, it indicates a set of observations of input quantities which refute the theory model irrespective

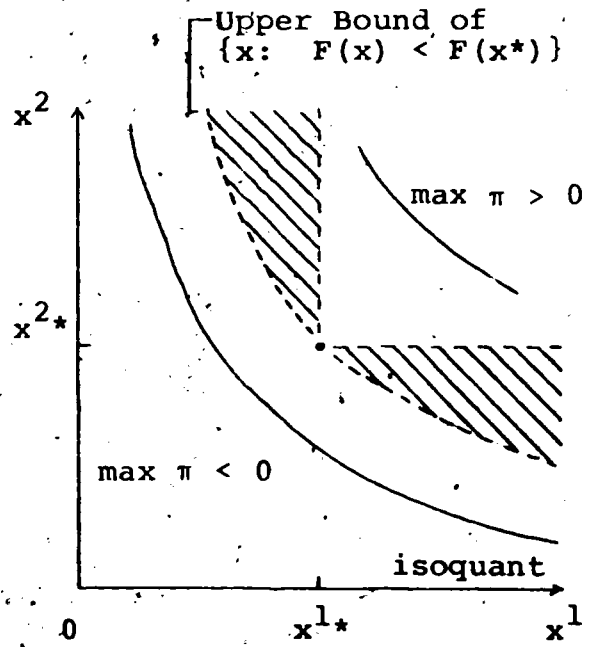
of prices or input constraints.

In Chapter Seven a class of production functions is constructed which appears to meet the conditions derived from the Neo-Classical Model of Perfect Competition and the properties of these functions are examined.

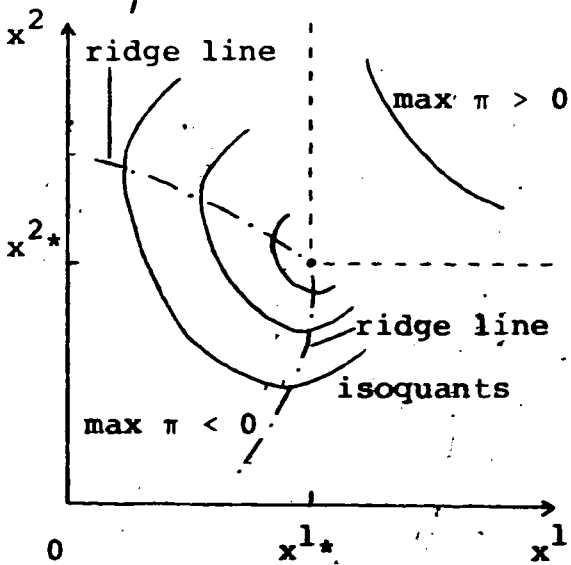
FIGURE 2: SINGULARITY OF THE PRODUCTION FUNCTION: ISOQUANTS



(a)

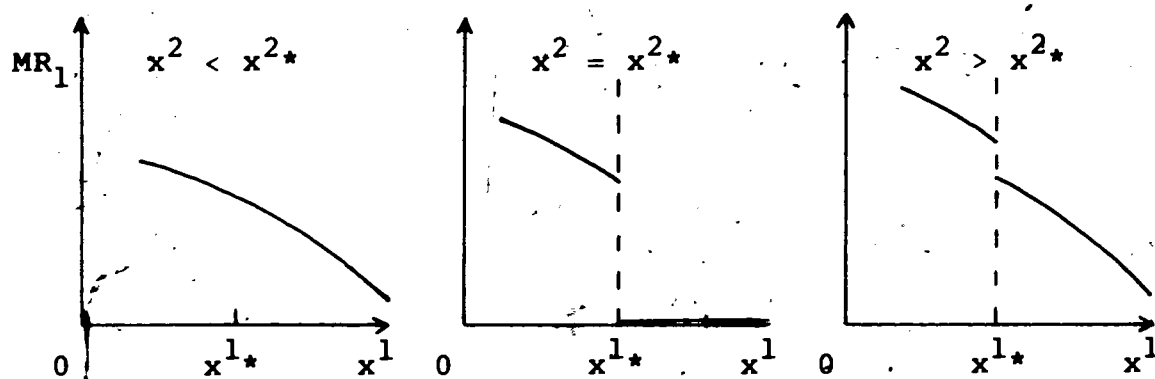


(b)

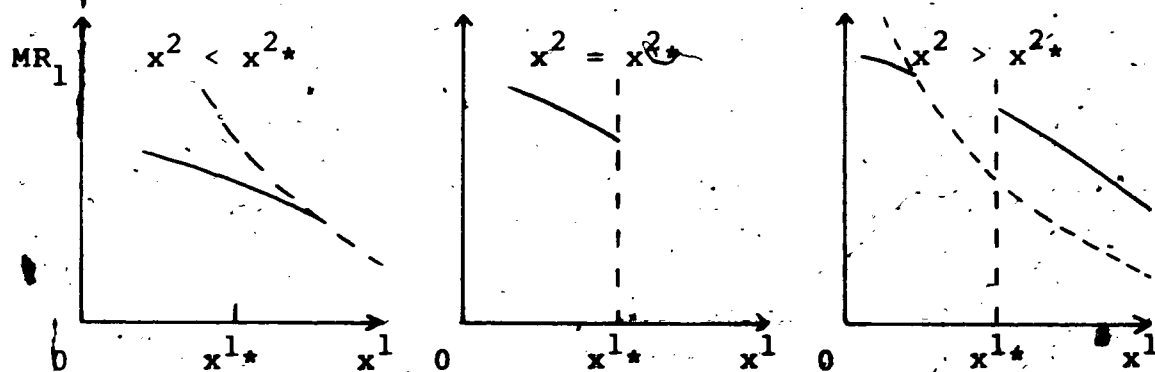


(c)

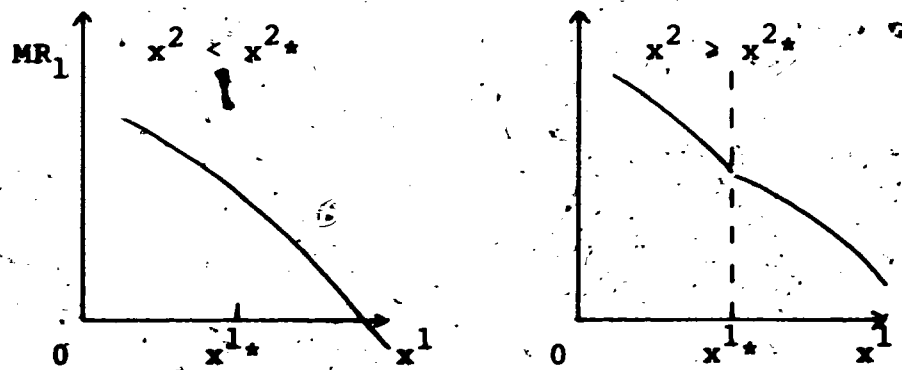
FIGURE 3: SINGULARITY OF THE PRODUCTION FUNCTION: MARGINAL PRODUCT CURVES



(a)



(b)



(c)

Footnotes

1. Equivalently marginal revenue is identical to average revenue for the firm [9].
2. As noted in Chapter Three only the end of the run is relevant to the consideration of the Neo-Classical Model which generates no statement concerning behaviour between static equilibria.
3. Whilst this is a requirement of the Neo-Classical Model itself, it is also necessary for most general equilibrium proofs such as that given in Debreu [21]. Compare [4], however.
4. Assumption (ix) is the aggregate analogue of the 'own price substitution effect' obtainable from the Generalised Substitution Theorem for the individual consumer (see [42], Section 8.3, for example).
5. λp , $\lambda > 0$, generates the same expansion path as p since it is equivalent to dividing P by λ and retaining P .
6. The firm's behaviour is analogous to that of Relaxation Phenomena encountered in Engineering and Electronics. Compare [52], for example.
7. Whilst not a contradiction, this would make the trial-and-error profit seeking behaviour described by Boland [9] and Day [20], somewhat implausible.
8. An observation of inputs in the interior of this region would not refute the model but simply imply that the production function for $F(x) < F(x^*)$ was misspecified, i.e. the shaded region was incorrectly defined.

CHAPTER SEVEN

The Neo-Classical Production Function

1. INTRODUCTION: The analysis of the preceding chapters has generated constraints on the form of the Neo-Classical Profit or Production function. Here the requirements are summarised and a family of production functions is sought which fulfills them for the perfectly competitive firm. As before the emphasis is on functions which are twice differentiable almost everywhere although mention is made of the consequences of relaxing this stipulation.

In Chapter Six it was noted that the perfectly competitive firm experiences a switching phenomenon or "phase change" at the point of zero maximum profit. Thus the production function is treated separately in its two modes in this chapter before examining behaviour in the neighbourhood of the relaxation point. For ease of exposition a specific two input model of the production function is described which fulfills most of the requirements of the model before proceeding to considerations of the general, n-input form.

2. REQUIREMENTS ON THE PRODUCTION FUNCTION IN PERFECT

COMPETITION: Since the conditions on the Neo-Classical Production function are developed in the preceding chapters and the related theorems, they are summarised here without further justification:

- (a) Under Definition 2' $F(x)$ is differentiable almost everywhere. Hence the existence of unique profit maxima requires that $\nabla^2 F(x)$ be negative definite almost everywhere.
- (b) The Neo-Classical description of long run behaviour requires that there exist a unique input point, x^* , such that for any given set of positive input prices p there exists a positive output price, P^* dependent on p for which x^* maximises profit and $\pi(x^*) = P^*[F(x^*) - x^*.p] = 0$.
- (c) For all output prices $P > P^*$ profit is maximised at a point x^0 such that $x^0 > x^*$ and $\pi(x^0) > 0$. Conversely for all positive $P < P^*$.
- (d) $F(x)$ is defined over all inputs $x \geq 0$ and must generate a unique profit maximising input-output combination for all positive prices p, P .

3. THE POSITIVE (NET) PROFIT REGION: By condition (a) above the production function is strictly concave in this region and hence cannot exhibit increasing or constant returns between any two input combinations $x, \theta x, \theta > 0$, within the region [Lemma 30]. Hence, a function exhibiting constant returns to scale within this region may be found such that the production function lies beneath it for all

$x > x^*$ [Theorem 31]. Denoting the production function in its "positive profit phase" by $H(x)$ it may be shown that

$$H(x) < H(x^*) \cdot \prod_{j=1}^n (x^j/x^{j*})^{\alpha_j} \quad \sum_{j=1}^n \alpha_j = 1, \quad \alpha_j > 0$$

$j = 1, \dots, n$

where $H(x^*)$ is the output which would be produced by applying $H(x)$ to the zero profit input combination x^* , and the α_j are arbitrary positive indices summing to unity. Thus decreasing returns 'to scale' apply to any two input combinations $x_1, x_2 > x^*$ lying on the same ray through x^* . 'Scale' in this context refers to expansion along a ray through x^* rather than through the origin as is more usual. Hence the usual property of decreasing returns to scale of generating negative profit for all positive inputs is avoided.¹ This requirement is illustrated for a two input case in Figure 4.

4. THE NON-POSITIVE PROFIT REGION: AN ILLUSTRATION

From Chapter Six a two input production function should imply concave isoquants which converge to a single point, x^* , in positive input space.² A family of such curves may be generated by considering the appropriate portions of ellipses centred on x^* such that the foci approach x^* as x increases to x^* . This process is shown in Figure 5.

The general formula for an ellipse centred on (x^*, y^*) with major axis parallel to the x axis is

$$(x-x^*)^2/a^2 + (y-y^*)^2/b^2 = 1 \quad 0 < b < a$$

and the foci are given by $(x^* \pm a, y^*)$. For concreteness let $b = a/2$, whence both axes of the ellipse approach zero simultaneously. If $F(x)$ is chosen such that

$F(x) = A - a$, A a positive constant, $F(x)$ will increase as a approaches zero and A may be selected to ensure that maximum profit,

$$P\{F(x) - x \nabla F(x)\} \rightarrow 0 \text{ as } a \rightarrow 0.$$

Combining this condition with substitution for a yields

$$F(x,y) = [x^{*2} + 4y^{*2}]^{1/2} - [(x^* - x)^2 + 4(y^* - y)^2]^{1/2}$$

which has the following properties for $0 < (x,y) < (x^*,y^*)$:

- a) $\partial F/\partial x > 0$; $\partial F/\partial y > 0$
- b) $\partial^2 F/\partial x^2 < 0$; $\partial^2 F/\partial y^2 < 0$; $\partial^2 F/\partial x \partial y > 0$
- c) $\text{Lt}_{(x,y) \rightarrow 0} F(x,y) = 0$
- d) $\text{Lt}_{(x,y) \rightarrow (x^*,y^*)} F(x,y) = [x^{*2} + 4y^{*2}]^{1/2} > 0$
- e) $\text{Lt}_{(x,y) \rightarrow (x^*,y^*)} \nabla F(x,y)$ is indeterminate.

These results follow directly from the specification of the production function and are in Theorem 32. Property (b) is the two dimensional expression of the general requirement that $\nabla^2 F(x)$ be negative definite. The limiting value of output as inputs approach (x^*,y^*) is positive from property (d) and is the obvious choice for $F(x^*,y^*)$. This may then be substituted for $H(x^*)$ in the condition derived in the previous section, establishing continuity of the production function as inputs expand through x^* .

Property (e) confirms the validity of allowing $\nabla F(x)$ to assume any positive slope at x^* as suggested in Chapter Six.³

As shown in Figure 6, the ridge lines for this production function are parallel to the x axes or coincide with them which is a result of the linear homogeneity of a family of ellipses about a common centre. Similarly each expansion path is a straight line through x^* for positive inputs.⁴ However, even in this form, the specified production function appears to mimic conventional text book treatments, (e.g. [26]), whilst converging to the required singularity at x^* . The function fails in the vital requirement that it generate negative profit since throughout the region contained by the ridge lines, maximum profit is non-negative, becoming zero on the ray from the origin through x^* [Theorem 33]. No variation in the functional form which preserves a) to e) and $F(x) > 0$ for (some) inputs less than x^* appears to produce negative profit throughout the region. This leads to the consideration of the requirements for this region in general in the next section.

5. THE NEGATIVE PROFIT REGION: A DIFFICULTY: Strict concavity of the profit function, which in the differentiable case corresponds to (π_{jk}) being negative definite, is required to ensure the existence and uniqueness of the appropriate profit maximising input combinations. It implies that decreasing returns must apply between any

two input combinations [Lemma 34.1]. However this requires that if profit is negative for a particular set of inputs it must be decreasing as inputs are increased along the appropriate expansion path [Lemma 34.2] if a path exists.⁵ The consequent contradiction may be expressed in at least two ways:

- (a) With constant input prices a firm experiencing an increase in output price from P to P' is always able to use the same input-output combination under P' as it did under P . If the expansion path leads to lower (more negative) profit, the profit maximising firm will prefer its old input-output combination thus rejecting the "tangential solution" which should generate a maximum if (π_{jk}) negative definite.
- (b) As noted in Chapter Six maximum profit is required to increase with P which it clearly does not. Moreover the decline in profit invalidates the limiting procedure illustrated in the same chapter, generating regions of discontinuity and/or sets of input values which are arbitrarily proscribed.

Weak concavity may be used in conjunction with auxiliary conditions for non-differentiable functions as shown in Chapter Four. However this admits only the extra possibility

that profit be constant throughout the negative profit region and does not meet the objections described in (a) and (b) above. Hence no expansion path in the usual sense exists in the negative profit region and the Neo-Classical Model can only explain the behaviour of a firm making negative profit by postulating input constraints, that is in runs 1 to $n-1$ ⁶ [Theorem 35]. The consequences of this restriction are examined in Section 6 below.

6. NEO-CLASSICAL PROFIT MAXIMISATION: The absence of negative (or 'sub-normal') profit in the penultimate (n th) run does not directly invalidate the Neo-Classical explanation of the entry and exit of firms to and from different industries since a firm may change output type in search of higher than normal profit.⁷ However the assymetry of this process linked with the output price adjustment mechanism of Chapter Three implies that if an industry exists in which firms are making more than normal profit, no long run equilibrium is possible. If a firm leaves a normal profit industry it must, according to the model enter a higher profit industry and hence depress price. Simultaneously the output price in the industry left will rise. Either this process results in a uniform level of profit above normal profit (a contradiction)⁸ or the profit level in the entered industry must sink below normal profit. Hence a profit maximising firm will move in the reverse direction and the cycle repeats indefinitely [Theorem 36].

Long run equilibrium may be achieved only by ensuring that no industry is making excess profit at the beginning of the long run. However as noted earlier this effectively "collapses" the long run into run n since they cannot be distinguished either by changes in profit or industry (compare Boland[14]). Any change in industry is unexplained by the model and must be assumed to be either whimsical (if the firm maximises only profit) or a refutation of the Neo-Classical model (if it doesn't). The class of homothetic profit functions which provide zero (net) profit for all maxima was defined in Theorem 15.

7. SUMMARY: For the perfectly competitive firm two sets of input vectors were delineated, those which produced negative maximum profit for given input prices and some output prices and those which resulted in positive maximum profit. The requirements of concavity generated an upper bound on the production function in the positive profit region, but resulted in a contradiction for the area of negative profit. Exclusion of this region in the penultimate and long runs causes the two runs to be indistinguishable, and long run equilibrium cannot exist unless it is made automatic by requiring unconstrained maximum net profit to be zero at all times. Thus the Neo-Classical Model fails to provide any explanation for the movement of a firm from one industry to another.

FIGURE 4: RETURNS TO SCALE IN THE POSITIVE PROFIT REGION

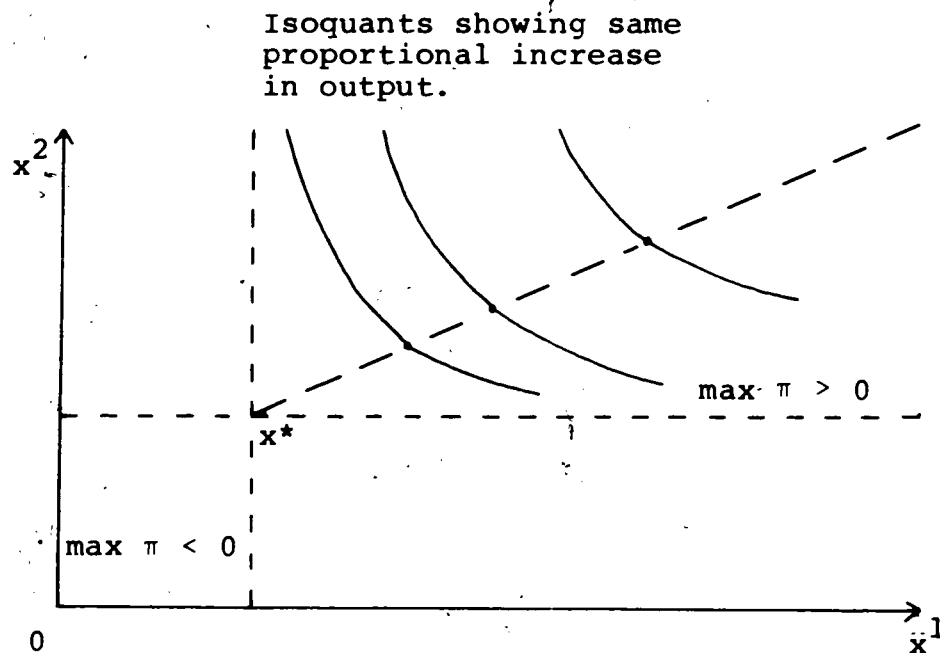


FIGURE 5: CONVERGING ISOQUANTS IN THE NEGATIVE PROFIT REGION

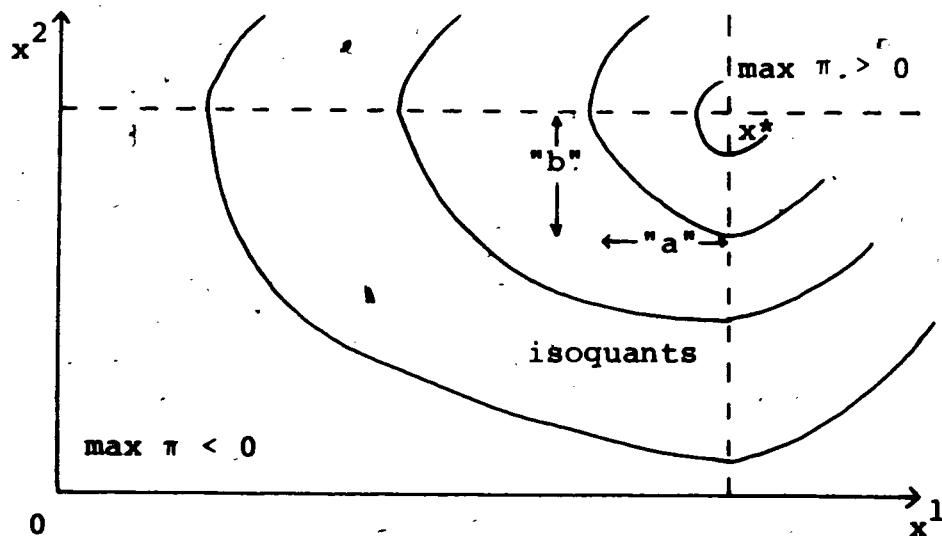
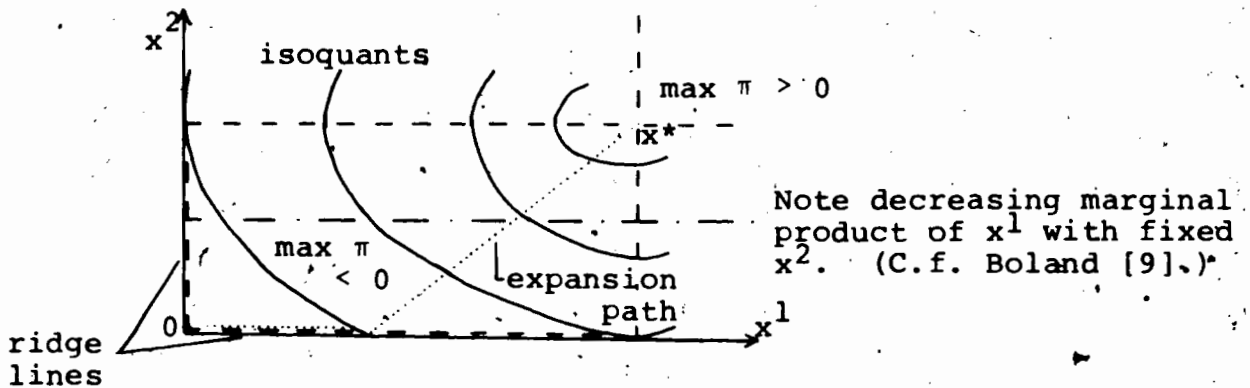
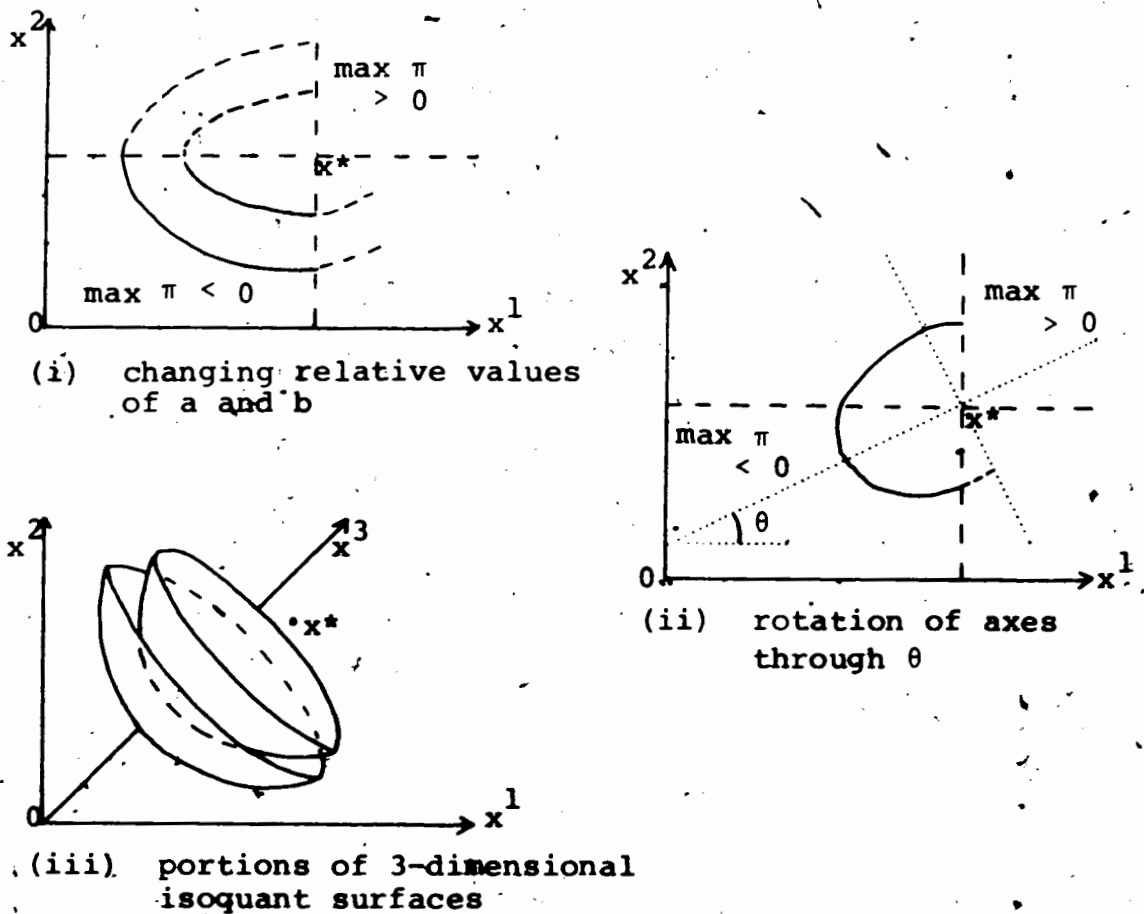


FIGURE 6: PROPERTIES OF ELLIPTICAL ISOQUANTSFIGURE 7: GENERALISATIONS OF ELLIPTICAL ISOQUANTS

Footnotes

1. The existence of decreasing returns of some kind for sufficiently large inputs has long been noted in the literature (see [26]; for example).
2. The presence of a zero component in X^* is not excluded per se although it leads in this case to the absence of that input in optimum production throughout the negative profit region.
3. Since, by inspection, it is apparent that $F(x)$ is neither 1 nor 2-integrated the refutation of ARP models is superfluous since they are not defined. However the zero profit point remains an integral part of the Neo-Classical explanation of the long-run.
4. These characteristics may be avoided, at the cost of some complexity in the production function by varying the relationship between the major and minor axes (a and b) and by rotation of the axes of the ellipses. See Figure 7.
5. This may explain the common depiction of a production function in text books (e.g. [26]) as having a region of increasing returns to scale followed by a region of decreasing returns to scale as inputs increase. As often noted the profit maximising firm will not produce in a region of increasing returns unless constrained to do so. Thus its presence in such a region in the penultimate run in which there are no input constraints (other than $x \geq 0$) would contradict the Neo-Classical model.
6. Constraining $\nabla F(x) > 0$ implies that constrained inputs will be used to the full since the cost of their purchase must be incurred regardless of use. See Boland [9].
7. All that is required here is a non-trivial 'ordering' of profits analagous to that of consumer preference. C.f. Takayama [61], for example.
8. If the other conditions of the model are met this would imply an upward movement of normal profit and output prices over time, i.e. a kind of "technical inflation" against which the model could be tested.

CHAPTER EIGHT

Conclusion

1. COMPLETE EXPLANATION: The foregoing analysis indicates that the received Neo-Classical Theory does not provide an adequate explanation of the behaviour of a firm which exercises no control over its input and output prices. This inadequacy may be manifested in one (or more) of three ways: failure to distinguish between different input-output combinations which generate the same maximum profit; restrictions on the input price vectors for which the firms response can be explained; and inability to justify a long run equilibrium resulting from the entry and exit of firms to and from different industries. As noted in Chapter One the first two cases were excluded from the analysis on methodological grounds. Both ad hoc price restrictions and failure of the long run equilibrating mechanism raise difficulties for general equilibrium analysis. If the firm's technology is dependent on input price ratios, additional assumptions must be made (and justified) to guarantee the appropriate market conditions for overall equilibrium. This problem is non-trivial as has been noted in similar attempts to introduce the influence of prices on consumer preference. (See Arrow and Hahn [6]): An alternative solution is the abandonment of the entry and exit aspect of the theory and the merging of the 'intermediate' and long runs. Whilst this provides no explanation of the

firms change of industry, it permits a general equilibrium solution with the required uniform profit rate generated automatically by the production functions. However it implies that the distribution of firms between industries is gratuitous and must be taken as given together with the initial distribution of wealth.

2. RETURNS TO SCALE: In Chapter One it was remarked that whenever the form of a production function is specified in the literature it is almost invariably homothetic and most commonly linearly homogeneous. It now appears that this choice is not solely a matter of algebraic convenience but necessary to reconcile profit maximisation with the long run "zero" profit equilibrium required by general equilibrium analysis. Apart from the failure to justify industry (or output type) changes, the use of such a function also renders the level of the firms output under unconstrained profit maximisation dependent on absolute price levels. Thus the familiar general equilibrium assumption of homogeneity of degree zero in prices renders every perfectly competitive firm's behaviour inexplicable in both the intermediate and long runs. As long as the real prices of inputs are in the same ratios as their marginal products, maximum (net) profit is zero.

Introductions to microeconomics often depict the production function (graphically) as an S-shaped curve (or surface) with a region of increasing returns to scale

followed for larger inputs by decreasing returns to scale. The large scale portion agrees with the analysis presented in Chapter Seven, for the positive profit region, and serves to ensure a limit on the size of the unconstrained firm [26]. At the boundary the point of inflection might be considered analogous to the zero maximum net profit singularity derived in Chapter Six. Moreover the region of increasing returns corresponds to the difficulty discussed in connection with the negative profit region, namely, that no profit maximising firm will produce in this area unless constrained to do so.

Under constrained optimisation it appears from Chapter Four that any concave function with strictly concave contours or isoquants will 'do the job' the theory requires by yielding the appropriate (unique) profit maxima. However the long run equilibrium of the firm described by Neo-Classical Theory requires the imposition of a homothetic (e.g. linearly homogeneous) production function which is logically equivalent to the direct assumption of zero long run profit. No inferences about the firm's unconstrained behaviour may be drawn from the theory with such a function which do not follow from postulating a direct transition from constrained optimisation to long run equilibrium at input-output combinations dependent on the absolute price level.

Referring to the discussion in Chapters Two and Three, for the perfectly competitive case at least the Neo-Classical

Theory cannot do the job which it sets out to do. Thus the theory fails to explain why the firm chooses that input-output combination which it does by providing no justification for the type of output good produced, i.e. the industry to which the firm belongs. By the same token the automatic attainment of zero net profit in the 'intermediate-long run' makes the shutting down of any firm inexplicable within the theory.

3. FURTHER RESEARCH: Further investigation is indicated for the case of imperfect competition in which the firm exercises some influence over the price of its output. Since the basic framework of this thesis is couched in terms of profit rather than production functions it remains appropriate for this analysis. However the traditional description of changes in industry is less clearly delineated here than in perfect competition. A priori it is not clear whether a change in the number of competing firms, however defined, results simply in a lateral shift of the imperfect competitor's demand correspondence, or variations in the price elasticity of demand. Text book graphical analysis tends to blur these points and the existence of the depicted equilibrium requires more justification than is usually given. See, however, Boland [9].

In addition to imperfect competition, the framework developed in this discussion permits the examination of a number of other models of the firm's behaviour scantily

treated in the usual analyses. Specifically, 'imperfections' in the markets for inputs received little attention except in the celebrated example of indeterminacy resulting from the confrontation of a monopolist and a monopsonist.

Thus the relevance of the Neo-Classical framework to firms which are 'perfect sellers' and 'imperfect buyers' or imperfect buyers and sellers remains to be explored.

APPENDIX: Definitions and Proofs.

Definition 1: A firm is defined as a set of mappings from an input space S into the real line \mathbb{R} such that $\forall x = (x^1, \dots, x^n) \in S$ where S is a subset of the non-negative orthant of n -dimensional Euclidean space, E^n , n finite

$\exists \pi \in \mathbb{R}$ s.t.

$$\pi = \sum_{i=1}^m p^i x^i - \sum_{j=1}^n p^j x^j$$

where

$$x^i = F^i(x^1, \dots, x^n), \quad F^i(x^1, \dots, x^n) > 0 \quad \forall x > 0$$

$$\forall i = 1, \dots, m, \quad m \text{ finite}$$

$$p^i = p^i(x^i) \quad \forall i = 1, \dots, m$$

$$p^j = p^j(x^j) \quad \forall j = 1, \dots, n.$$

Definition 2: A Neo-Classical firm is defined as a firm such that S is compact.

Definition 2a): A Differentiable Neo-Classical firm is defined as a Neo-Classical firm for which F^i, p^j are twice differentiable with respect to all x^k and p^i is twice differentiable with respect to x^i , for $i = 1, \dots, m$; $j, k = 1, \dots, n$.

LEMMA .1

Given: $p^i(x^i)$ a twice differentiable function of x^i and $x^i = F^i(x)$ a twice differentiable function of x^j , $\forall j = 1, \dots, n$, $x = (x^1, \dots, x^n)$.

To Prove: $p^i(F^i(x))$ is a twice differentiable function of x^j , $j = 1, \dots, n$.

Proof: Immediate by the Composite Function Theorem

[61, Theorem 1.C.2, for example].

Definition 3: A shift in demand is defined as a mapping α which takes $P^i(x^i)$ to $\alpha P^i(x^i)$, $i = 1, \dots, m$, where $P^i(x^i)$ is as specified in Definition 1.

Definition 4: A firm is described as being in its j th run if it is free to vary x^1, \dots, x^j and x^k is constrained by $x^k \leq \bar{x}^k$, $\forall k = j+1, \dots, n$.

Definition 5: The Neo-Classical Model of the firm is defined by the following statements about the behaviour of a Neo-Classical firm:

(a) In the j th run the firm's choice of inputs and outputs is uniquely determined by specifying that the firm maximises profit subject to the appropriate input constraints, $j = 1, \dots, n$.

(b) In the long (or $(n+1)$ th) run the entry and exit of competing firms causes a shift in demand such that the maximum profit which the firm can earn is zero.

Definition 6: A run j is well-defined by the Neo-Classical model if, and only if, $\max \pi$ in the j th run $\neq \max \pi$ in the $(j+1)$ th run almost everywhere.

THEOREM 2

Given: A differentiable Neo-Classical firm with profit function $\pi(x) = \pi(x^1, \dots, x^n)$.

To Prove: A sufficient condition for the extreme points of π in runs 1 to n to be maxima subject to the appropriate constraints is that the matrix of second order partial derivatives of π , (π_{jk}) be negative definite.

Proof: We may consider the r th run, $1 < r < n$ as the maximisation of π as a function of x^1, \dots, x^r, x^{r+1} subject to the single constraint $x^{r+1} - \bar{x}^{r+1} = 0$ since this is necessary and sufficient for the r th run to be well-defined by Definition 6.

If $\pi_j = \partial\pi/\partial x^j = 0$ for $j = 1, \dots, r$, we have from [42, p. 53] that a sufficient condition for this extreme point to be a maximum is that the determinant of \hat{L} have sign $(-1)^{r+1}$, the largest principal minor should have a sign opposite to this and successively smaller minors should alternate in sign down to the principle minor of order 2, where

$$\hat{L} = \begin{bmatrix} 0, & 0, & \dots, & 1 \\ 0, & & & \\ \cdot & \cdot & L & \\ 1 & & & \end{bmatrix}; \quad L = \begin{bmatrix} \partial^2 L(x, \lambda) \\ \partial x^j & \partial x^k \end{bmatrix} = \begin{bmatrix} \pi_{jk} - \lambda \frac{\partial^2}{\partial x^j \partial x^k} (x^{r+1} - \bar{x}^{r+1}) \\ (\pi_{jk}) \end{bmatrix}$$

where

(π_{jk}) is $(r+1) \times (r+1)$

Now $\det \hat{L}$ has the opposite sign to that of $(\pi_{jk})_{rxr}$. Hence we require that $(\pi_{jk})_{rxr}$ have sign $(-1)^r$ and that $(\pi_{jk})_{sxs}$ have sign $(-1)^s$ for $s = 2, \dots, r$.

By induction " $(\pi_{jk})_{sxs}$ has sign $(-1)^s$, $s = 1, \dots, n$ " is a sufficient condition for maxima in all n runs, since $\pi_{11} < 0$ is sufficient for run 1.

From [42, p. 299], a matrix is negative definite if and only if all its principal minors of order r have the sign $(-1)^r$. Thus $(\pi_{jk})_{n \times n}$ negative definite is sufficient for the extreme points of π to be maxima in runs 1 to n .

COROLLARY 2.1

Provided each run remains well-defined, Theorem 2 holds for any ordering of the x^j . This result follows immediately from:

"A matrix is negative definite if and only if the principle minors of successively higher order alternate in sign. This condition must hold for all possible sequences of principle minors" [42, p. 300] (Emphasis added).

LEMMA 2.2

Given: A differentiable Neo-Classical firm with profit function $\pi(x^1, \dots, x^n)$ such that π_{jk} is negative definite.

To Prove: All extreme points of π in runs 1 to n are unique maxima.

Proof: Direct by Theorem 2 and [42: Section R 8.4].

LEMMA 2.3

Given: A differentiable Neo-Classical firm with profit function $\pi(x) = \pi(x^1, \dots, x^n)$ with runs 1 to n-1 well-defined.

To Prove: A necessary condition for (π_{jk}) to be negative definite is that $\sum_{j=1}^r (\partial^2 \pi / \partial x^j)^2 < 0 \quad \forall r = 1, \dots, n$.

Proof: From [42, p. 298], "a real symmetric matrix is negative definite if, and only if, all its characteristic roots are negative". Now $(\pi_{jk})_{rxr}$ is real and symmetric $\forall r = 1, \dots, n$ and if $(\pi_{jk})_{n \times n}$ is negative definite so are $(\pi_{jk})_{n-r}$, $r = 0, 1, \dots, n-1$. But, from [67, p. 376], $\text{tr} A = \sum \lambda$ where $\{\lambda\}$ are the characteristic roots of A.

$$\begin{aligned} \therefore \text{tr}(\pi_{jk})_{r \times r} < 0 \quad \forall r = 1, \dots, n, \\ \text{i.e. } \sum_{j=1}^r (\pi_{jj}) < 0 \quad \forall r = 1, \dots, n. \end{aligned}$$

LEMMA 2.4.

Given: A Neo-Classical firm with profit function $\pi(x^1, \dots, x^n)$ with runs 1 to n well-defined.

To Prove: $\max \pi$ in the run $j \leq \max \pi$ in run $(j+1)$,
 $j = 1, \dots, n-1$.

Proof: Maximisation in run j is subject to the constraints $x^k \leq \bar{x}^k$, $k = j+1, \dots, n$ whereas maximisation in run $(j+1)$ is subject to the constraints $x^k \leq \bar{x}^k$, $k = j+2, \dots, n$. $\therefore \max \pi(j) > \max \pi(j+1)$ implies π not maximised in run $j+1$ contrary to assumption.
 $\therefore \max \pi(j) \leq \max \pi(j+1)$.

In fact, by Definition 6, $\max \pi(j) < \max \pi(j+1)$ almost everywhere.

Definition 7: The Average Revenue Product of an input x^k for a firm is defined as $ARP_k \equiv \left(\sum_{i=1}^m p^i x^i - \sum_{j \neq k}^n p^j x^j \right) / x^k$.

Definition 8: A j -integrated Neo-Classical firm is a Neo-Classical firm for which there exists a sequence of well-defined runs $1, \dots, n$ such that $\exists j \in \{1, \dots, n\}$ for which $x^i > 0 \rightarrow x^j > 0 \quad \forall i = 1, \dots, m$. Thus x^j is an input to every non-trivial production.

Definition 9: The Input Value Model of the firm is defined by the following statement about the behaviour of a k -integrated Neo-Classical firm:

In the j th run the firm's choice of inputs and outputs is uniquely determined by specifying that the firm maximises the Average Revenue Product of input k subject to the appropriate constraints, $j = 1, \dots, n$.

THEOREM 3

Given: A k -integrated differentiable Neo-Classical firm.

To Prove: A sufficient condition that it be possible for exactly one of the two theories of the firm in Definitions 5 and 9 to be known false from the equilibrium behaviour of the firm is that one of the following statements be true:

(i) The value of the function π_k is known for run j where $j > k$

(ii) the values of the function π_{kk} and $\frac{\partial^2}{\partial x^k} \left(\sum_{\ell=1}^n p^\ell x^\ell \right) \neq 0$ are known for run j

where $j > k$.

Proof: From the Neo-Classical theory of the firm, in run r ,

π is maximised when

$$\frac{\partial \pi}{\partial x^j} = 0 \quad \forall j = 1, \dots, r; r = 1, \dots, n$$

where the second order conditions are functions of

$$\pi_{js}, \quad j, s = 1, \dots, r$$

From the definition of ARP_k ,

$$\frac{\partial}{\partial x^j} (ARP_k) = \pi_j / x^k \quad \forall j \neq k$$

Since $x^k > 0$ by assumption, $\frac{\partial ARP_k}{\partial x^j} = 0 \leftrightarrow \pi_j = 0$

$$\frac{\partial}{\partial x^k} (ARP_k) = 0 \rightarrow ARP_k = \pi_k + \frac{\partial}{\partial x^k} (p^k x^k) = \pi_k + \frac{\partial}{\partial x^k} \sum_{i=1}^m p^i x^i \quad \forall k \neq j$$

$$\begin{aligned} \frac{\partial^2(\text{ARP}_k)}{\partial x^r \partial x^s} &= \frac{\pi_{rs}}{x^k} \text{ when } \frac{\partial \text{ARP}_k}{\partial x^r} = 0 \text{ or} \\ \frac{\partial \text{ARP}_k}{\partial x^s} &= 0 \quad \forall r \neq k, \forall s \\ \frac{\partial^2(\text{ARP}_k)}{\partial x^k^2} &= 1/x^k [\pi_{kk} + \frac{\partial^2}{\partial x^k^2} (\sum_{\ell=1}^n p^\ell x^\ell)] \text{ when} \\ \frac{\partial \text{ARP}_k}{\partial x^k} &= 0 \end{aligned}$$

Thus the marginal and second order conditions for each theory fulfill those of the other theory with two exceptions:

- (i) $\pi_k \neq 0$ under the input value theory for all runs $k, k+1, \dots, n$ whereas the Neo-Classical theory implies $\pi_k = 0$ in the same runs.
- (ii) For run k and subsequent runs $(\text{ARP}_k)_{kk}$ may be of a different sign to that of π_{kk} and hence it is possible that the second order conditions for a maximum might be fulfilled for one theory and not for the other.

LEMMA 3.1

Given: A 1-integrable differentiable Neo-Classical firm.

To Prove: A sufficient condition for the input value model of the firm using input 1 to be known to be false when the Neo-Classical model is not known to be false is that

$$\frac{\partial^2}{\partial x^1^2} (\sum_{j=1}^n p^j x^j) > -\pi_{11} > 0$$

Proof: From the proof of Theorem 3, the condition specifies that a necessary condition for ARP_1 to be maximum fails whilst the same condition for π does not.

LEMMA 3.2

Given: A differentiable Neo-Classical firm which is j -integrable for $j = 1$ only.

To Prove: A sufficient condition for all input value models of the firm to be known false when the Neo-Classical is not known false is that

$$\frac{\partial^2}{\partial x^j} \sum_{j=1}^n p^j x^j > -\pi_{11} > 0$$

Proof: Direct by noting that ARP_j is only defined for all x for $j = 1$ by Definition 8 and can be distinguished from the Neo-Classical model by Lemma 3.1.

LEMMA 3.3

Given: A j -integrated Neo-Classical firm with the price of factor j invariant with respect to x .

To Prove: A sufficient condition that it be possible for exactly one of the two theories of the firm in definitions 5 and 9 to be known false from the equilibrium behaviour of the firm is that one of the following statements be true:

- (i) the value of the function π_j is known for run k where $k \geq j$
- (ii) maximum $\pi \neq 0$ in run n .

Proof: (i) follows immediately from Theorem 3 by noting that in this case $\partial/\partial x^j (ARP_j) = 0 \rightarrow ARP_j = \pi_j + p^j$.

(ii) from Theorem 3 max. ARP_j in run n occurs when

$$ARP_j = \frac{\partial}{\partial x^j} \left(\sum_{i=1}^m X^i P^i \right)$$

Hence, by definition of ARP_j ,

$$1/x^j \left[\sum_{i=1}^m X^i P^i - \sum_{k=1}^n p^k x^k \right] = \frac{\partial}{\partial x^j} \sum_{i=1}^m X^i P^i$$

$$\text{i.e. } \sum_{i=1}^m x^i p^i = \sum_{k=1}^n x^k \partial / \partial x^k \left(\sum_{i=1}^m x^i p^i \right)$$

But under this condition, the Neo-Classical model leads to the conclusion that maximum profit is zero in run n .

Definition 10: A set of acceptable observations consists of the following information for each run k of a Neo-Classical firm, $k = 1, \dots, (n+1)$:

- (i) the values of $P^i(x^i), p^j(x^j) \quad \forall i, j$
- (ii) values of $x^j, x^i \quad \forall i, j$
- (iii) by deduction, the value of $\pi(r) - \pi(s)$ for all $r, s = 1, \dots, n+1$, where $\pi(k)$ denotes the (maximum net) value of π in run k . Hence the truth or falsity of the statement $\pi(n) = 0$.

THEOREM 4

Given: A j -integrable Neo-Classical firm and a set of acceptable observations.

To Prove: The possibility of the input value theory of the firm being shown false and the Neo-Classical theory not being shown false by a set of acceptable observations exists if the price of factor j is constant.

Proof: Referring to Definition 10, item (ii) permits the definition of a sequence of runs and if the number of constraints observed to be binding is $c \leq n$, the observations $\pi(c) \neq 0$, and $\pi(c+1) = 0$ suffice to discredit the input value theory by Lemma 2.4 and Lemma 3.1.

Definition 11: A firm is representable by a theory if, and only if, \exists a proper subset A of the set of acceptable observations B , such that $\forall b \in B$, (i) $b \in A \rightarrow$ the theory explains why the firm is not using any other input-output point, and (ii) $b \notin A \rightarrow$ the theory is false.

Definition 12: A firm is Neo-Classically representable if, and only if, it is

- (i) Neo-Classical
- (ii) representable by the Neo-Classical theory with proper subset A_N , and
- (iii) representable by the Input Value Model with proper subset A_I only if $A_N \neq A_I$.

Definition 13: A firm is weakly representable by a theory if, and only if, \exists a proper subset A of the set of acceptable observations B , such that $\forall b \in B$, (i) $b \in A$ is explained by the theory and (ii) $b \notin A \rightarrow$ the theory is false.

Definition 14: A firm is weakly Neo-Classically representable if, and only if, it is

- (i) Neo-Classical
- (ii) weakly representable by the Neo-Classical theory with proper subset A_N , and
- (iii) weakly representable by the Input Value Model with proper subset A_I only if $A_I \wedge A_N = \emptyset$.

THEOREM 5

Given: A differentiable Neo-Classical firm such that:

- (i) (π_{jk}) is negative definite.
- (ii) \exists a shift in demand α such that $\max \pi(x) = 0$.

(iii) $\max \pi$ in run $n \neq 0$.

To Prove: The firm is representable by the Neo-Classical theory.

Proof: The set A of Definition 11 consists of any acceptable observations in runs 1 to $(n-1)$, the observation $\pi \neq 0$ in run n with $\pi = 0$ in run $(n+1)$.

$A \neq \emptyset$ by condition (ii)

$A \neq B$ by condition (iii)

Thus A is a proper subset of B .

For all $b \in A$ the Neo-Classical theory is a complete explanation by condition (i), Theorem 2 and Lemma 2.2. $b \notin A \rightarrow$ the Neo-Classical theory is false by Definition 5.

LEMMA 5.1

Given: A Neo-Classical firm which is representable by the Neo-Classical theory and not j -integrable, for any $j \in \{1, \dots, n\}$.

To Prove: The firm is Neo-Classically representable.

Proof: Conditions (i) and (ii) of Definition 12 are fulfilled by assumption.

Since the firm is not j -integrable for any $j \in \{1, \dots, n\}$, $A_I = \emptyset$ and Condition (iii) is satisfied trivially.

THEOREM 6

Given: A firm such that

- (i) it is representable by the Neo-Classical theory,
- (ii) it is 1-integrable

(iii) it is differentiable and either

$$a) \quad \partial^2 / \partial x^1{}^2 \left(\sum_{j=1}^n p^j x^j \right) > -\pi_{11} > 0 \quad \text{i.e.}$$

$$\partial^2 / \partial x^1{}^2 \left(\sum p^i x^i \right) > 0$$

or

$$b) \quad \partial^2 / \partial x^1{}^2 \left(\sum_{j=1}^n p^j x^j \right) = 0$$

To Prove: The firm is Neo-Classically representable.

Proof: It suffices to show that \exists an acceptable observation which is not common to A_N and A_I .

If condition (iii) a) is met this follows immediately from Lemma 3.1.

If condition (iii) b) is met the result follows from Lemma 3.3.

COROLLARY 6.1

A 1-integrable differentiable Neo-Classical firm facing constant factor prices is Neo-Classically representable if

- (i) (π_{jk}) is negative definite
- (ii) \exists a shift in demand α such that $\max \pi(x) = 0$, and
- (iii) $\max \pi$ in run $n \neq 0$.

This follows directly from Theorem 5 and Theorem 6.

Definition 15: A Neo-Classical profit function is a function π such that a firm with profit π is Neo-Classically representable.

Theorems 7 and 8 provide necessary preliminary results.

THEOREM 7

Given: A differentiable Neo-Classical firm

To Prove: A necessary condition for $\pi \max \equiv 0$ in run n is that for all k such that $x^k \neq 0$ when $\pi = \pi \max.$,

$$\pi_{kk} = 0.$$

Proof: $\pi = \sum_{i=1}^m p^i x^i - \sum_{j=1}^n p^j x^j$ and for a maximum

$$\pi_k = \sum_{i=1}^m X_k^i [P_i^i X^i + P^i] - p^k - p_k^k x^k = 0 \quad \forall k = 1, \dots, n$$

$$\text{i.e. } p^k = \sum_{i=1}^m X_k^i [P_i^i X^i + P^i] - p_k^k x^k$$

$$\text{Let } ARP_k = \left(\sum_{i=1}^m p^i x^i - \sum_{j \neq k} p^j x^j \right) / x^k = p^k \quad \text{when } \pi = 0$$

where k is such that $x^k \neq 0$ when $\pi = \pi \max$. Thus

$$\left(\sum_{i=1}^m p^i x^i - \sum_{j \neq k} p^j x^j \right) / x^k = \sum_{i=1}^m X_k^i [P_i^i X^i + P^i] - p_k^k x^k$$

Differentiating w.r.t. x^k :

$$\{ x^k \left[\sum_{i=1}^m X_k^i (P_i^i X^i + P^i) \right] - \left[\sum_{i=1}^m p^i x^i - \sum_{j \neq k} p^j x^j \right] \} / (x^k)^2$$

$$= \sum_{i=1}^m \{ (X_k^i)^2 [P_{ii}^i X^i + 2P_i^i] + P_{kk}^i X_{kk}^i \} - p_{kk}^k x^k - p_k^k$$

$$\therefore p_k^k = \sum_{i=1}^m \{ (X_k^i)^2 [P_{ii}^i X^i + 2P_i^i] + P_{kk}^i X_{kk}^i \} - p_{kk}^k x^k - p_k^k$$

$$\therefore \pi_{kk} = 0.$$

THEOREM 8

Given: A 1-integrable differentiable Neo-Classical firm with profit function $\pi(x) = \pi(x^1, \dots, x^n)$.

To Prove: A sufficient condition for the extreme points of π in runs 1 to n to be maxima subject to the appropriate constraints, and for it to be possible that maximum $\pi \neq 0$ in run n is that (π_{jk}) be negative definite.

Proof: The sufficiency of (π_{jk}) negative definite for maxima is proved in Theorem 2.

(iv) a) holds, the firm is Neo-Classically representable by Theorem 7.

Similarly $\pi_{\max} \neq 0$ by Theorem 8, and hence if (iv) b) holds, the firm is Neo-Classically representable by Theorem 7.

COROLLARY 9.1

A function $\pi(x)$ is a Neo-Classical profit function if x^i, p^i are twice differentiable w.r.t. $x^k \quad \forall i = 1, \dots, m$ and $k = 1, \dots, n$, $p = (p^1, \dots, p^n)$ are constant, and finite, and

$$(i) \quad x^i(0, x^2, \dots, x^n) = 0 \quad \forall x^2, \dots, x^n \geq 0, \\ \forall i = 1, \dots, m$$

(ii) $(\pi_{jk})_{n \times n}$ is negative definite, and

(iii) \exists a shift in demand α such that $\max \pi(x) = 0$.

Proof is immediate by Theorem 9 using condition (iv) b).

LEMMA 9.2

Given: A differentiable Neo-Classical firm with input prices p independent of inputs x , and profit function $\pi(x)$.

To Prove: If profit is maximised by setting $\nabla \pi(x) = 0$ for unconstrained x , and the set of all profit maximising input vectors for any positive p is open and convex, (π_{jk}) is negative semi-definite over this set. If the maxima are unique (π_{jk}) is negative definite over this set.

Proof: Let $\pi(x) \equiv \sum_{i=1}^m x^i p^i(x^i) - \sum_{j=1}^n x^j p^j \equiv R(x) - x \cdot p$

and let x, y be distinct profit maximising vectors.

By assumption x maximises profit when $\nabla \pi(x) = 0$, i.e.

$$\nabla R(x) = p, \quad \text{whence}$$

$$R(x) - x \cdot p \geq R(y) - y \cdot p \rightarrow$$

$$R(x) - x \cdot \nabla R(x) \geq R(y) - y \cdot \nabla R(x) \rightarrow$$

$$(y-x) \cdot \nabla R(x) \geq R(y) - R(x) \rightarrow$$

$R(x)$ concave by [61, Theorem 1.C.3]

$\rightarrow \pi(x)$ concave by Lemma 20.1 below.

$\rightarrow (\pi_{jk})$ negative semi-definite from [51: Theorem 4.5].

Uniqueness implies strict inequality hence strict concavity and (π_{jk}) negative definite from the same Theorems.

THEOREM 10 (A generalisation of Theorem 2)

Given: A Neo-Classical firm such that $\pi(x) < \infty \quad \forall x$ and $x \gg 0$.

To Prove: A sufficient condition that π attain a global maximum in run k , $k = 1, \dots, n-1$ is that π be a concave function of x , or, equivalently, if π is twice differentiable, π_{jk} be negative semi-definite.

Proof: Maximising π in run k is equivalent to the problem $\min(-\pi)$ over \mathbb{R}^n s.t. $x^j - \bar{x}^j \leq 0 \quad j = k+1, \dots, n$. π concave $\rightarrow (-\pi)$ convex, \mathbb{R}^n is a convex set and all constraints are affine, hence convex and concave. Thus the problem is an ordinary convex program [51]. By assumption $\bar{x}^j > 0 \quad \forall j = k+1, \dots, n$ so a feasible x exists, and again by assumption $-\pi > -\infty$, whence a Kuhn-Tucker vector $\lambda = (\lambda_{k+1}, \dots, \lambda_n) \geq 0$ exists for this problem. [51: Corollary 28.2.1]. Thus there exists a minimum for $-\pi$ and hence a maximum π .

π_{jk} negative semi-definite \leftrightarrow concave is a direct implication of [51: Theorem 4.5].

The local minima of $-\pi(x)$ are global minima by the convexity of $-\pi(x)$.

COROLLARY 10.1

If $\pi(\bar{x}) = \pi_{\max}$ in the k th run in Theorem 10,

$$0 \in [-\partial\pi(\bar{x}) + \sum_{j=k+1}^n \lambda_j dx^j]$$

where $\partial\pi(\bar{x})$ is the subdifferential of π at \bar{x} . This result follows directly from [51: Theorem 28.3]. From the same theorem we have

$$\lambda_j \geq 0, x^j - \bar{x}^j \leq 0, \lambda_j (x^j - \bar{x}^j) = 0 \quad \forall j = k+1, \dots, n$$

LEMMA 10.2

Given: A Neo-Classical firm

To Prove: A sufficient condition for $\pi(x)$ to have unique maxima at its extreme points for each run $k, k = 1, \dots, (n+1)$ is that $\pi(x)$ be a strictly concave function of (x^1, \dots, x^j) $\forall j = 1, \dots, n$.

Proof: Uniqueness follows from the properties of a strictly concave function, and from [42, p. 333] we have that all extreme points of a strictly concave function are maxima.

THEOREM 11

Given: A Neo-Classical firm such that $\pi(x)$ finite and $\pi(x)$ concave $\forall x$.

To Prove: If $\pi^*(x^*)$ is defined as $\inf\{x \cdot x^* - \pi(x)\}$, a sufficient condition for π^* to have a unique (unconstrained) global maximum at \bar{x} is that $\pi^*(x^*)$ be differentiable at $x^* = 0$ and $\bar{x} = \nabla\pi^*(0)$.

Proof: $\pi^*(0) = -\pi(\bar{x})$ which is finite by assumption. $\pi^*(x^*)$ differentiable at $x^* = 0 \rightarrow \pi^*(x)$ subdifferentiable at $x^* = 0$. $\therefore -\pi^*(x^*)$ is proper by [51: Theorem 23.3]. $\therefore -\pi(x)$, the convex conjugate of $-\pi^*(x^*)$ is closed and proper by [51: Theorem 12.2]. $\bar{x} = \nabla\pi^*(0)$ by assumption, \therefore the minimum set of $(-\pi(x))$ contains a unique vector \bar{x} by [51: Theorem 27.1(e)]. $-\pi(\bar{x})$ is a global minimum by the convexity of $-\pi(x)$ [51, § 27].

COROLLARY 11.1

If $\pi(x)$ fulfills the conditions of Theorem 11 and is, in addition closed, then the existence of a unique maximum at $x = \bar{x}$ implies that $\pi^*(x^*)$ is differentiable at $x^* = 0$ and $\bar{x} = \nabla\pi^*(0)$. The proof is direct by noting π finite $\rightarrow -\pi$ proper and applying [51: Theorem 27.1(e)].

THEOREM 12

Given: A Neo-Classical firm such that $\pi(x) < \infty$ and $\pi(x)$ is closed and concave $\forall x$.

To Prove: If the condition of Theorem 11 is satisfied, the constrained maxima of Theorem 10 are unique.

Proof: $\{\pi(x)\}$ closed $\rightarrow \{-\pi(x)\}$ closed.

$x^j - \bar{x}^j \leq 0$ are closed functions since $x^j \geq 0$,

$\forall j = k+1, \dots, n$

Let $h = -\pi(x) + \sum_{j=k+1}^n \lambda_j (x^j - \bar{x}^j)$

where $\lambda = (\lambda_{k+1}, \dots, \lambda_n)$ is a Kuhn-Tucker vector.

By Corollary 10.1, $\inf.h = \inf[-\pi(x)] = \sup\pi(x)$.

By Theorem 11 $\sup \pi(x)$ is unique. Thus $\inf h$ is unique. Therefore by [51: Corollary 28.1.1], the minimum problem in Theorem 10 has a unique solution for each $k = 1, \dots, n-1$.

LEMMA 12.1

Given: A firm with profit function $\pi(x)$

To Prove: A sufficient condition for $\pi(x)$ to be concave is that $\sum_{j=1}^n p^j x^j$ be convex and that $\sum_{i=1}^m p^i x^i$ be a concave function of $x = (x^1, \dots, x^n)$.

Proof: $\sum_{i=1}^m p^i x^i$ concave $\rightarrow -\sum_{i=1}^m p^i x^i$ convex

$\therefore -\pi(x) = -\sum_{i=1}^m p^i x^i + \sum_{j=1}^n p^j x^j$ is convex by [51:

Theorem 5.2] $\therefore \pi(x)$ is concave.

LEMMA 12.2

Given: A firm with total cost function, $C = \sum_{j=1}^n p^j x^j$
 $\forall x = (x^1, \dots, x^n)$.

To Prove: A necessary condition for $\sum_{j=1}^n p^j x^j$ to be convex is that

- (i) $\forall y = (y^1, \dots, y^n)$, $[C(x + \lambda y) - C(x)]/\lambda$ be a non-decreasing function of $\lambda > 0$,
- (ii) the one-sided directional derivative of C at x , $C'(x;y) = \inf_{\lambda > 0} \left[\frac{C(x + \lambda y) - C(x)}{\lambda} \right]$, $\forall y$ and
- (iii) $C'(x;y)$ is a positively homogeneous convex function of y with $C'(x;0) = 0$ and $-C'(x;-y) \leq C'(x;y) \quad \forall y$.

Proof: Since $C(x)$ is finite for all x , results follow directly from [51: Theorem 23.1].

COROLLARY 12.3

If the firm has a total revenue function $R = \sum_{i=1}^m p^i x^i$

which is finite $\forall x = (x^1, \dots, x^n)$, a necessary condition for R to be concave is that

- (i) $\forall y = (y^1, \dots, y^n)$, $[R(x+y) - R(x)]/\lambda$ be a non-increasing function of $\lambda > 0$,
- (ii) the one-sided directional derivative of R at x ,
 $R'(x:y) = \sup_{\lambda > 0} \frac{R(x+\lambda y) - R(x)}{\lambda}$, $\forall y$, and
- (iii) $R'(x:y)$ is a concave function of y with
 $R'(x:0) = 0$ and $-R'(x:-y) > R'(x:y) \quad \forall y$.

LEMMA 12.4

Given: A firm with total cost function $C = \sum_{j=1}^n p^j x^j$

To Prove: A sufficient condition for $\sum_{j=1}^n p^j x^j$ to be convex is

that $p^j x^j$ be convex $\forall j = 1, \dots, n$.

Proof: Immediate by [51: Theorem 5.2].

COROLLARY 12.5

A sufficient condition for $\sum_{i=1}^m p^i x^i$ to be concave in $x = (x^1, \dots, x^n)$ is that $p^i x^i$ be concave in $x \quad \forall i = 1, \dots, m$.

LEMMA 12.6

Given: A firm with total cost function $C = \sum_{j=1}^n p^j x^j$ such that

$c^j = p^j x^j$ is finite $\forall j = 1, \dots, n$ and $\forall x = (x^1, \dots, x^n)$.

To Prove: A necessary condition for c^j to be convex is that

- (i) $\forall y = (y^1, \dots, y^n)$, $[c^j(x^j + \lambda y) - c^j(x^j)]/\lambda$ be a non-decreasing function of $\lambda > 0$.
- (ii) the one-sided directional derivative of c^j at x^j , $c^{j'}(x^j:y) = \inf_{\lambda > 0} \left[\frac{c^j(x^j + \lambda y) - c^j(x^j)}{\lambda} \right]$, $\forall y$, and
- (iii) $c^{j'}(x^j:y)$ is a positively homogeneous convex function of y with $c^{j'}(x^j:0) = 0$ and $-c^{j'}(x^j:-y) \leq c^{j'}(x^j:y)$, $\forall y$.

Proof: As in Lemma 12.2.

COROLLARY 12.7

If the firm has a total revenue function, $R = \sum_{i=1}^m p^i x^i$ such that $r^i = p^i x^i$ is finite $\forall i = 1, \dots, m$ and $\forall x = (x^1, \dots, x^n)$, a necessary condition for r^i to be concave in x , is that

- (i) $\forall y = (y^1, \dots, y^n)$, $[r^i(x + \lambda y) - r^i(x)]/\lambda$ be a non-increasing function of $\lambda > 0$,
- (ii) the one-sided directional derivative of r^i at x , $r^{i'}(x:y) = \sup_{\lambda > 0} \left[\frac{r^i(x + \lambda y) - r^i(x)}{\lambda} \right]$, $\forall y$, and
- (iii) $r^{i'}(x:y)$ is a concave function of y with $r^{i'}(x:0) = 0$ and $-r^{i'}(x:-y) \geq r^{i'}(x:y)$ $\forall y$.

LEMMA 12.8

Given: A firm with total cost function $C = \sum_{j=1}^n p^j x^j$ such that

$c^j = p^j x^j$ is finite and differentiable for some $j \in \{1, \dots, n\}$ and $\forall x$.

To Prove: A necessary condition for c^j to be convex is that the marginal cost, $\partial/\partial x^j(c^j)$, of the j th input be

non-decreasing.

Proof: Immediate by Lemma 12.6.

COROLLARY 12.9

If the firm has total revenue function $R = \sum_{i=1}^m p^i x^i$ such that $r^i = p^i x^{i-1}$ is finite and differentiable for some $i \in \{1, \dots, m\}$ and $\forall x$ a necessary condition for r^i to be concave in x is that the marginal revenue product $\partial/\partial x^j (r^i)$ of the j th input in the i th output be non-increasing, $\forall j = 1, \dots, n$.

COROLLARY 12.10

If the firm has total cost function $C = \sum_{j=1}^n p^j x^j$ such that C is finite and c^j is differentiable $\forall j = 1, \dots, n$, $\forall x$, a necessary condition for C to be convex is that $\nabla C \geq 0$, i.e. the marginal cost of input j is non-decreasing $\forall j = 1, \dots, n$.

COROLLARY 12.11

If the firm has total revenue function $R = \sum_{i=1}^m p^i x^i$ such that R is finite and r^i is differentiable $\forall i = 1, \dots, m$, $\forall x$, a necessary condition for R to be concave is that $\nabla R \leq 0$, i.e. the marginal revenue product of input j in total output is non-increasing $\forall j = 1, \dots, n$.

THEOREM 13

Given: A function $\pi(x) \equiv \pi(x^1, \dots, x^n) \equiv \sum_{i=1}^m x^i p^i (x^i) - \sum_{j=1}^n x^j p^j (x^j)$ defined on $x \geq 0$.

To Prove: A sufficient condition that $\pi(x)$ meet all the requirements of a Neo-Classical Profit Function

without necessarily being differentiable, i.e. is a "non-differentiable Neo-Classical Profit Function", is that all of the following hold:

- (i) Either π is strictly concave or π is concave, closed, finite and the concave conjugate $\pi^*(x^*)$ fulfills the conditions of Theorem 12.
- (ii) p^j is independent of x^j , $\forall j = 1, \dots, n$.
- (iii) \exists a set of functions $\{P^{i*}(x^i) > 0, i = 1, \dots, m\}$ s.t. $\max \pi = 0$
- (iv) \forall functions $P^i(x^i) > 0$ s.t. $P^i(x^i) \neq P^{i*}(x^i)$, $i = 1, \dots, m$, $\max \pi \neq 0$.

Proof: By (i) π has unique maxima in all runs. If the firm is j -integrated for some $j = 1, \dots, n$, the Neo-Classical model is distinguishable from the ARP_j model by conditions (iii) and (iv). Hence by direct analogy with Theorem 5, π is a non-differentiable Neo-Classical Profit Function.

THEOREM 14

Given: A function $\pi(x) \equiv \pi(x^1, \dots, x^n) \equiv \sum_{i=1}^m x^i P^i(x^i) - \sum_{j=1}^n x^j p^j(x^j)$ defined on $x \geq 0$.

To Prove: $\pi(x)$ is a non-differentiable Neo-Classical Profit Function if all the following conditions hold:

- (i) π is concave, closed, finite and $\pi^*(x^*)$ fulfills the conditions of Theorem 12.
- (ii) p^j is independent of x^j , $\forall j = 1, \dots, n$.
- (iii) \exists a set of functions $\{P^{i*}(x^i) > 0, i = 1, \dots, m\}$

s.t. $\max \pi = 0$

- (iv) $\pi(x) = 0$ implies that if $\pi(x^*)$ is differentiable at $x^* = 0$, $\nabla \pi^*(0) \neq x$ or
- $$P^i(X^i) = P^{i*}(X^i) \quad \forall i = 1, \dots, m$$

Proof: From Theorem 13 it suffices to show that conditions

(i) and (iv) imply that $\max \pi \neq 0$ unless $P^i(X^i) = P^{i*}(X^i) \quad \forall i = 1, \dots, m$. Suppose the contrary:

$\max \pi = 0 \rightarrow \pi^*(x^*)$ differentiable at $x^* = 0$ by Corollary 11.1. Thus if $\pi^*(x^*)$ is not differentiable at $x^* = 0$ the contradiction is immediate, otherwise $\max \pi(x) = 0 \rightarrow \nabla \pi^*(0) = x$ also by Corollary 11.1, contrary to assumption.

THEOREM 15

Given: A differentiable Neo-Classical firm with unconstrained outputs $x = (x^1, \dots, x^n)$

- To Prove:
- (i) A sufficient condition for $\pi = 0$ at all extreme points of π is that π be homothetic in x , but $\pi(tx) \neq (t-1)^n \pi(x)$ for any $n \neq 1$.
 - (ii) A sufficient condition that π be homothetic is that Total Revenue ($\sum_{i=1}^m P^i X^i$) and Total Cost ($\sum_{j=1}^n p^j x^j$) be homogeneous functions of the same degree.
 - (iii) If Total Revenue and Total Cost are both homogeneous of degree $r \neq 0$, every extreme point of π has $\pi = 0$.

Proof: (i) By definition π is homothetic in x iff $\pi(tx) =$

$\phi(t)\pi(x)$, $\forall t > 0$ and some function ϕ . Since π is differentiable for a Neo-Classical firm, ϕ is differentiable and by the Generalised Euler Theorem [42]

$$\phi'(1)\pi(x) = \nabla\pi(x) \cdot x$$

$$\pi(tx) \neq (t-1)^n \pi(x) \text{ unless } n = 1.$$

$$\phi'(1) \neq 0. \text{ At an extreme point } \nabla\pi(x) = 0,$$

$$\therefore \pi(x) = 0.$$

$$(ii) \text{ Since } \pi(x) = \sum_{i=1}^m p^i x^i - \sum_{j=1}^n p^j x^j, \pi \text{ is homogeneous}$$

of the same degree as total revenue and total cost, and hence homothetic.

(iii) This result follows immediately from (i) and (ii).

THEOREM 16

Given: Two families of correspondences, $X = F(x|\alpha)$,

$Y = G(y|\beta)$ where α and β are parameters such that for some $\alpha = \alpha^*$, and for all β :

- (i) $\text{domain}(Y|\beta) \subseteq \text{domain}(X|\alpha^*)$
- (ii) $\text{range}(Y|\beta) \supseteq \text{range}(X|\alpha^*)$
- (iii) $D.F(x|\alpha^*)$, $D.G(y|\beta)$ are defined for all $x \in \text{domain}(X|\alpha^*)$ and for all $y \in \text{domain}(Y|\beta)$, where D is an operator such that $\text{range}(D.F(x|\alpha^*)) \subseteq \text{range}(D.G(y|\beta))$

To Prove: Let $\Phi(x) = \{y: [y \in \text{dom}(Y|\beta)] \& [F(x|\alpha^*) \subseteq G(y|\beta)] \& [D.F(x|\alpha^*) \wedge D.G(y|\beta) \neq \emptyset]\}$

If \exists an ordering on (Y, β) , $\forall \beta$ such that $\{\min \Phi(x); x \in \text{domain}(X|\alpha^*)\}$ is closed, then there exist x^*

$\in \text{domain } (Y|\beta_1)$ and $y^* \in \text{domain } (Y|\beta_2)$ for some β_1, β_2 , such that

$$x^* \in \phi(y^*) \ \& \ y^* \in \phi(x^*).$$

x^*, y^* are not necessarily distinct but are necessarily finite if $\min\phi(x)$ is bounded for all x .

Proof: Let $\alpha = \alpha^*$: $\phi(x)$ is non-empty by specification.

Define the point-set mapping θ : $\text{domain } (X|\alpha^*) \rightarrow \text{domain } (X|\alpha^*)$ such that $\theta(x) = \min\phi(x)$ which is well-defined by assumption. The composite mapping $\theta^n(x)$ is defined recursively by

$$\theta^n(x) = \theta(y), \quad \forall y \in \theta^{n-1}(x) \quad \forall n > 1.$$

θ generates a relation \odot on $\text{domain } (X|\alpha^*)$ such that $x_1 \odot x_2 \leftrightarrow \exists n > 0$ s.t. $[x_1 \in \theta^n(x_2)] \ \& \ [x_2 \in \theta^m(x_1)] \rightarrow m > n$.

Hence the sequences $Cr(x) = x, \theta(x), \dots, \theta^r(x)$ may be defined $\forall x$ such that

$$r \geq i > j \geq 0 \rightarrow y^i \odot y^j \\ \forall y^i \in \theta^i(x) \ \& \ \forall y^j \in \theta^j(x).$$

Three cases may be distinguished:

(i) If $\exists x \in \text{domain } (X|\alpha^*)$ for which $Cr(x)$ is defined only for $r = 0$, $y \in \theta(x) \rightarrow y \not\odot x$

But $n = 1$ yields $y \in \theta^n(x)$, hence $\exists m \leq n = 1$

$$\text{s.t.} \quad x \in \theta^m(y) \quad \forall y \in \theta(x).$$

$$\text{i.e.} \quad x \in \theta(x) \text{ or } x \in \theta(y) = \theta^2(x)$$

which is the required result.

(ii) If $\exists x \in \text{domain } (X|\alpha^*)$ for which $Cr(x)$ is defined only for finite r ; \exists a largest integer

\bar{r} for which $C\bar{r}(x)$ is defined. Hence $\exists i$,
 $0 \leq i \leq \bar{r}$ s.t. $y^{\bar{r}+1} \in \phi^{\bar{r}+1}(x)$ and
 $y^i \in \phi^i(x)$ for which $y^{\bar{r}+1} \oplus y^i$.
 But $y^{\bar{r}} \in \phi^{\bar{r}}(x) \rightarrow y^{\bar{r}} > y^i \quad \forall y^j \in \phi^j(x)$,
 $0 \leq j < \bar{r}$ by construction. Whence, since \otimes
 is transitive within a particular chain by
 construction,

$$y^{\bar{r}+1} \oplus y^{\bar{r}} \quad \forall y^{\bar{r}} \in \phi^{\bar{r}}(x).$$

Setting $x = y^{\bar{r}}$ in (i) the required result
 follows.

- (iii) If an infinite sequence $C(x)$ exists for
 every x , $C(x) \subseteq \{\min\phi(x) : x \in \text{domain}(X|\alpha^*)\}$
 which is closed by assumption. Since $x \oplus x$,
 an oscillating sequence must be finite, whence
 $C(x)$ approaches a limit in $\{\min\phi(x)\}$. Let
 this limit be $L(x)$. By construction
 $L(x) \subseteq \text{domain}(Y|\beta) \subseteq \text{domain}(X|\alpha^*)$, and by
 the construction of $C(x)$,
 $y^L \in L(x) \rightarrow y^L \otimes y^n(x) \quad \forall y^n \in \phi^n(x), \forall n$ and
 $y^{L+1} \in \phi(L(x)) \rightarrow y^{L+1} \oplus L(x)$
 Setting $x = y^L$ in (i) the required result
 follows.

[The approach used in this proof is derived
 from that of Abian and Brown [1] in their
 work on partially ordered sets].

Definition 16: A Perfectly Competitive firm or Perfect
Competitor is defined as a Neo-Classical firm such that

$m = 1$ and P, p^j are positive and independent of x, X ,
 $\forall j = 1, \dots, n$.

LEMMA 16.1

Given: A perfectly competitive firm with production function
 $F(x)$.

To Prove: If $F(x)$ is concave, closed and bounded and \exists a
 finite, positive vector $z \in [\partial F]_x, \forall x$, then

(i) \exists two finite vectors x^*, y^* , not necessarily
 distinct, such that

$$F(x^*) = \alpha \prod_{j=1}^n (y^{j*})^{\alpha^j}; \quad F(y^*) = \gamma \prod_{j=1}^n (x^{j*})^{\gamma^j}$$

$$\alpha > 0, \gamma > 0, \alpha^j > 0, \gamma^j > 0 \quad j = 1, \dots, n;$$

$$\sum_{j=1}^n \alpha^j = \sum_{j=1}^n \gamma^j = 1$$

$$F(x^*) (\alpha^1/y^{1*}, \dots, \alpha^n/y^{n*}) \in [\partial F]_{x^*}$$

$$F(y^*) (\gamma^1/x^{1*}, \dots, \gamma^n/x^{n*}) \in [\partial F]_{y^*} \quad \text{where}$$

$[\partial F]_x$ is the subdifferential of $F(x)$
 evaluated at x .

(ii) If $x^* = y^*$, \exists positive prices p, P such that
 $F(x) - x \cdot p/P \leq 0$ with equality if $x = x^*$.

Proof: Result (i) follows from Theorem 16 by setting $X|\alpha \equiv F(x)$
 $Y|\beta \equiv \beta \prod_{j=1}^n (y^j)^{\beta^j}$ any $\beta^j > 0$ s.t. $\sum_{j=1}^n \beta^j = 1$.

$$D\phi(x) \equiv \{z: z \in [\partial\phi(x)]_x \text{ \& } z \text{ finite \& } z > 0\}$$

and noting that the domain and range requirements are
 met. F closed and bounded implies that Y is closed and
 bounded whence $\phi(x)$ is closed and x^*, y^* are finite.

Result (ii) follows by setting

$p/P = (\alpha^1/x^{1*}, \dots, \alpha^n/x^{n*}) F(x^*)$ in (i) and noting that for $F(x)$ concave, $p/P \in [\partial F]_{x^*}$ implies that $F(x) - x \cdot p/P$ is maximised at $x = x^*$ from [51: Theorem 28.3].

Definition 17: An Imperfectly Competitive firm or Imperfect Competitor is defined as a Neo-Classical firm such that $m = 1$ and p^j is positive and independent of x , $\forall j = 1, \dots, n$.

COROLLARY 16.2

Lemma 16.1 may be applied to an imperfect competitor by substituting the revenue function $R(x) = P(x) \cdot F(x)$ for $F(x)$ and making the requirements of $R(x)$.

THEOREM 17

Given: A perfect competitor with once-differentiable concave production function $F(x)$ such that $\nabla F(x) > 0$, F closed and finite for all x .

To Prove: If $F(x)$ meets the conditions of Theorem 16 when

$$Y|\beta = \beta \prod_{j=1}^n y_j^{1/n}, \text{ a sufficient condition for the}$$

existence of a zero maximum profit point is that,

$$\forall x, y,$$

$$[F(y) - F(x)] \cdot x \cdot [\nabla F(y) - \nabla F(x)] < 0 \text{ with equality}$$

permitted if $F(x)$ strictly concave.

Proof: By Lemma 16.1 the result follows immediately unless a

two-cycle exists. Let x^* , y^* constitute a two-cycle

and w.l.o.g. let $F(y^*) \geq F(x^*)$. Since $F(x)$ concave:

$$(y-x) \nabla F(x) \geq F(y) - F(x) \quad \forall x, y \text{ [61, Theorem 1.C.3]}$$

$\therefore (y^* - x^*) \nabla F(x^*) \geq F(y^*) - F(x^*) > 0$ by assumption

$$\therefore y^* \nabla F(x^*) \geq x^* \nabla F(x^*) \dots \dots \dots (1)$$

Choosing $y|\beta = \beta \prod_{j=1}^n y_j^{1/n}$ in Theorem 16 yields

$\phi(x) = y$ when $y_j = F(x)/(n \partial F/\partial x^j)$ which exists by assumption and Lemma 16.1

$$\therefore F(x^*) = n y_j^* \partial F/\partial x^j = y^* \nabla F(x^*) \dots \dots \dots (2)$$

$$\text{Similarly } F(y^*) = x^* \nabla F(y^*) \dots \dots \dots (3)$$

(2) in (1) yields $F(x^*) \geq x^* \nabla F(x^*)$

Hence from (3) $x^* \nabla F(y^*) \geq F(x^*) \geq x^* \nabla F(x^*)$

$$\text{i.e. } x^* [\nabla F(y^*) - \nabla F(x^*)] \geq 0 \dots \dots \dots (4)$$

which contradicts the presumption

$$[F(y) - F(x)] \times [\nabla F(y) - \nabla F(x)] < 0, \quad \forall x, y.$$

If $F(x)$ is strictly concave (1) and hence (4) become strict inequalities, contradicting equality in the presumption.

THEOREM 18

Given: A perfect competitor with production function $F(x)$, not necessarily differentiable, $x > 0$.

To Prove: A sufficient condition for the existence of a zero maximum profit point is that all the following be true:

- (a) $F(x)$ finite and concave
- (b) $\partial F(x)$, the subdifferential of $F(x)$ contains a finite positive vector quantity for all $x > 0$
- (c) either $F(x)$ is strictly concave or $\pi(x) = PF(x) - p \cdot x$ meets the conditions of Theorem 11 for all $P > 0$

(d) $[F(y) - F(x)] \cdot x [z(y) - z(x)] < 0$ where $z(\theta)$ is any positive element of $\partial F(\theta)$ and equality is permissible for $F(x)$ strictly concave.

Proof: Given input prices p . From Corollary 10.1 and conditions (a) and (c), $\pi(x)$ achieves an unconstrained maximum when $0 \in \{-\partial\pi(x)\}$ i.e. $0 \in \{-\partial F(x) + p/P\}$. Since p and P are both positive this implies that $\{\partial F(x)\}$ contains a positive vector which may be mapped into $\beta \prod_{j=1}^n y_j^{1/n}$ as in Lemma 16.1, for any P . As $\partial F(x)$ always contains a finite positive vector, the mapping is always well defined. Hence Theorem 16 may be applied directly since $F(x)$ finite, and the result follows by an analogous process to Lemma 16.1, with 2 cycles eliminated by Theorem 17.

THEOREM 19

Given: A perfect competitor with strictly concave production function $F(x)$ such that $F(0) > (0)$

To Prove: $F(x)$ is once differentiable on an open convex set in \mathcal{R}^n , $\max \pi(x) = P[F(x) - x \cdot p] = P[F(x) - x \cdot \nabla F(x)] > 0$ for all positive p, P .

Proof: $F(x)$ strictly concave and differentiable on an open convex set X in \mathcal{R}^n implies

$$(y - x) \nabla F(x) > F(y) - F(x) \quad \forall x, y \in X, x \neq y$$

[61, Theorem 1.C.3]

Letting $y = 0$, $-x \nabla F(x) > -F(x)$

i.e. $F(x) - x \nabla F(x) > 0$.

LEMMA 20.1

Given: A profit function with fixed input prices

To Prove: If $\pi(x) \equiv R(x) - x \cdot p$, $\pi(x)$ is (strictly) concave if, and only if, $R(x)$ is (strictly) concave.

Proof: $\pi(x)$ is strictly concave iff

$$\pi(\theta x + (1-\theta)y) > \theta \pi(x) + (1-\theta)\pi(y) \quad 0 < \theta < 1$$

$$\begin{aligned} \text{i.e. } R[\theta x + (1-\theta)y] - \theta xp - (1-\theta)yp &> \theta R(x) - \theta xp \\ &+ (1-\theta)R(y) - (1-\theta)yp \end{aligned}$$

$$\text{i.e. } R[\theta x + (1-\theta)y] > \theta R(x) + (1-\theta)R(y)$$

i.e. $R(x)$ strictly concave.

Similarly for weak concavity.

COROLLARY 20.2

If $R(x)$ is the revenue generated by the sale of one output, X , at price $P(X)$ and $X = F(x)$, $\pi(x)$ is (strictly) concave iff $P(X)F(x)$ is a (strictly) concave function of x .

LEMMA 20.3

Given: A profit function $\pi(x)$ with fixed input prices and single output price, P , independent of output quantity, x .


To Prove: $\pi(x)$ is a (strictly) concave function of x iff

$X = F(x)$ is a (strictly) concave function of x .

Proof: Since $P(X) \equiv P$ the proof is direct by Corollary 20.2.

LEMMA 20.4

Given: A profit function $\pi(x)$ with fixed input prices and single output price, P , independent of output quantity, x .



To Prove: If $X = F(x)$ is twice differentiable, (π_{jk}) is negative (semi) definite if, and only if, (F_{jk}) is negative (semi) definite.

Proof: $\pi(x) = PF(x) - x \cdot p$, p fixed, hence if $F(x)$ is twice differentiable so is $\pi(x)$. Whence (π_{jk}) negative (semi) definite $\leftrightarrow \pi(x)$ strictly (weakly) concave by [51: Theorem 4.5] where the proof given extends directly to strict concavity. Hence the result follows by Lemma 20.3.

LEMMA 20.5

Given: A concave function $F(x)$, $x = (x^1, \dots, x^n)$.

To Prove: If $z_1 \in \partial F$ evaluated at x_1 and $z_2 \in \partial F$ evaluated at $x_2 = x_1 + \xi$ where $\xi = (0, \dots, 0, \xi^j, 0, \dots, 0)$, $\xi^j > 0$ $z_2^j < z_1^j$, $j = 1, \dots, n$ where ∂F is the subdifferential of F , and $z = (z^1, \dots, z^n)$.

Proof: By definition ∂F is the set of all z such that

$$F(y) \leq F(x) + (y - x) \cdot z \quad \forall y \quad [51: \S 30]$$

$$\text{Let } y_1 = x_1 + \xi, \text{ whence } F(x_1 + \xi) - F(x_1) \leq z_1^j \cdot \xi^j$$

$$\text{If } y_2 = x_1, \quad F(x_1) - F(x_1 + \xi) \leq -z_2^j \xi^j$$

$$\text{i.e. } z_1^j \xi^j \geq F(x_1 + \xi) - F(x_1) \geq -z_2^j \xi^j \quad \forall \xi^j > 0$$

$$\text{i.e. } z_1^j \geq z_2^j$$

THEOREM 21

Given: A perfectly competitive firm with concave production function $F(x)$, and fixed input prices $p \equiv (p^1, \dots, p^n)$.

To Prove: If $x(P)$ is the profit maximising input vector for output price P , $x(P)$ and $F[x(P)]$ are increasing

functions of P and $z^j(P)$ is a decreasing function of P where $Pz^j(P) = p^j$, $j = 1, \dots, n$ and $z(P) = (z^1(P), \dots, z^n(P)) \in [\partial F]_{x(P)}$, the subdifferential of $F(x)$ evaluated at $x(P)$.

Proof: Since x maximises profit, z exists from [51: Theorem 28.3] \therefore for p constant and positive, $z^j(P) \propto 1/P$ $\forall j = 1, \dots, n$. z^j is a non-increasing function of x^j by Lemma 20.5, hence x^j is an increasing function of P , and $F(x)$ is an increasing function of x^j , $\forall j = 1, \dots, n$. Thus $F(x)$ is an increasing function of P .

THEOREM 22

Given: A strictly concave profit function $\pi(x) = \underline{X} \cdot \underline{P}(X) - \underline{x} \cdot \underline{p}(x)$

To Prove: If prices change in such a way that $\underline{P} \rightarrow \underline{P}'$, and maximum profit is unchanged;

$(X' - X)(P' - P) \geq 0$ where X', X are the profit maximising outputs at the two price levels, and $(x' - x)(p' - p) < 0$ or $x' = x$, where x', x are the profit maximising inputs.

These results might be termed the "generalised output and input substitution effects" respectively.

Proof: Let $S = \{x: \pi(x) \geq \pi^0\}$ and $T = \{X: x \in S\}$

Since $\pi(x)$ is strictly concave, S is a strictly convex set [42]. If $\max \pi(x)$ for \underline{P} , \underline{p} is π^0 , let x' be such that $\pi(x')$ just attains this level for \underline{P}' , \underline{p}' , in the sense that $\underline{x} \cdot \underline{p}$ and $\underline{x}' \cdot \underline{p}'$ are minimised over S .

Consider the hyperplane H defined by $y \cdot \underline{p} = x \cdot \underline{p}$

and let $H^+ = \{y: y_p \geq x.p\}$

$$H^- = \{y: y_p \leq x.p\}$$

By the minimum property of S , $S \subset H^+$ and H supports S at \underline{x} . \therefore Either $\underline{x}' = \underline{x}$ or $\underline{x}'p > \underline{x}p$.

Similarly H' defined by $y_{p'} = \underline{x}'p'$ yields either $\underline{x}' = \underline{x}$ or $\underline{x}p' > \underline{x}'p'$

whence $\underline{x}' = \underline{x}$ or $(p' - p)(\underline{x}' - \underline{x}) < 0$

Now let $X' = X(\underline{x}')$. By the strict concavity of $\pi(x)$, XP and $X'P'$ are maximised over T and hence S when regarded as functions of x, x' . Thus if the hyperplane G is defined by

$$\underline{y}.P = \underline{X}P, \text{ and}$$

$$G^+ = \{Y: YP \geq XP\}$$

$$G^- = \{Y: YP \leq XP\},$$

$T \subset G^-$ by the maximum property of T .

$$\text{i.e. } X'P \leq XP$$

$$\text{Similarly } XP' \leq X'P'$$

whence $(X' - X)(P' - P) \geq 0$.

THEOREM 23

Given: A perfectly competitive firm with strictly concave production function $F(x)$.

To Prove: If there exists a non-zero input vector x such that

$$F(x) - x.z = 0 \text{ where } z \in [\partial F]_x, z > 0, F(x) \neq 0, x \text{ is unique.}$$

Proof: Suppose the contrary: Let $y \neq x$ be such that

$$F(y) - y.w = 0, w \in [\partial F]_y, w > 0, F(y) \neq 0.$$

Choose $p_x, p_{x'}, p_y, p_{y'}$ positive such that

$$P_x \cdot z = P_x \text{ and } P_y \cdot w = P_y.$$

If $P_x = P_y$, $P_x = P_y$, otherwise by Theorem 21

maximum profit must change from x to y . Hence

$x = y$ by the uniqueness of maxima. Thus $P_x \neq P_y$.

Let $u = \theta x + (1-\theta)y$ for some θ , $0 < \theta < 1$ such that $u \cdot P_x / P_x$

$= u \cdot P_y / P_y$ which exists by Theorem 21. Thus

$$u \cdot P_x / P_x = \theta x P_x / P_x + (1-\theta) y P_x / P_x \dots \dots \dots (1)$$

$$= \theta F(x) + (1-\theta) y P_x / P_x \text{ since } \pi(x) = 0$$

$$< F(\theta x) + (1-\theta) y P_x / P_x \text{ by strict concavity of } F(x)$$

$$\text{and } u \cdot P_y / P_y = \theta x P_y / P_y + (1-\theta) y P_y / P_y < \theta x P_y / P_y + F[(1-\theta)y] \dots (2)$$

Combining (1) and (2)

$$\theta x P_x / P_x + (1-\theta) y P_x / P_x < \theta P_y / P_y + F[(1-\theta)y] \dots \dots \dots (3)$$

and similarly

$$\theta x P_y / P_y + (1-\theta) y P_y / P_y < F(\theta x) + (1-\theta) y P_x / P_x \dots \dots \dots (4)$$

Adding (3) and (4)

$$\{F[(1-\theta)y] - (1-\theta) y P_y / P_y\} + \{F(\theta x) - \theta x P_x / P_x\} > 0 \dots (5)$$

At least one of the terms in {} must be positive and hence either $(1-\theta)y$ or θx yield positive profit at y or x prices respectively, contrary to assumption. $\therefore x = y$.

THEOREM 24

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: There does not exist a $y \neq x^*$ such that $F(y) = F(x^*)$, i.e. there is no "isoquant" through x^* .

Proof: Suppose the contrary:

Let $F(y) = F(x^*)$, $y \neq x^*$ and choose $p, P > 0$ such

that $P \cdot w = p$ and $w \in [\partial F]_y$, i.e. y is the unique profit maximising input for p, P .

$$\therefore \max \pi(y) = P \cdot F(y) - y \cdot p > P \cdot F(x^*) - x^* \cdot p$$

$$\therefore x^* \cdot p > y \cdot p.$$

Let $P^* > 0$ be such that $P^* F(x^*) - x^* p = 0$.

Since x^* is unique by Theorem 23, and every firm has a zero maximum profit point for some P and every p , x^* maximises profit for P^*, p .

However $x^* \neq y$, whence by Theorem 21,

$P^* \neq P$ and $F(x^*) \neq F(y)$ contrary to assumption.

Definition 2b): An Almost Differentiable Neo-Classical Firm is defined as a firm such that F^i, p^j and P^i are twice differentiable with respect to x^k almost everywhere, $i = 1, \dots, m; j, k = 1, \dots, n$.

Definition 2': A Neo-Classical Firm with Concave Production Functions is defined as a firm such that F^i, p^j and P^i are continuous functions of x^k and F^i is a concave function of x^k such that the subdifferential of $F^i(x)$ contains a finite positive vector for all $x, i = 1, \dots, m; j, k = 1, \dots, n$.

THEOREM 25

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: If $[\partial F]_{x^*} = \{ \underline{a} : \underline{a} > 0 \ \& \ \sum_{j=1}^n \alpha_j^2 = N^2 \}$ for some

$N \neq 0, \exists P^*(p) > 0$ such that x^* is the profit maximising point for prices p, P^* .

Proof: For prices p, P the firm maximises profit at x where

$p/P \in [\partial F]_x$ from [51: Theorem 28.3]. It suffices to show that $\forall p \exists P^*$ s.t. $p/P^* \in [\partial F]_{x^*}$.

$$\text{Let } \sum_{j=1}^n p^j{}^2 = M^2 > 0$$

$$\therefore \text{ if } P^*(p) = M^2/N^2 > 0,$$

$$p/P^* > 0 \text{ and } \sum_{j=1}^n p^j{}^2 / P^* = M^2 \cdot N^2 / M^2 = N^2$$

as required.

THEOREM 26

Given: A concave function $\pi(x)$ which is twice differentiable almost everywhere in its domain.

To Prove: (π_{jk}) is negative semi-definite almost everywhere and negative definite almost everywhere if $\pi(x)$ is strictly concave.

Proof: Let $\pi(x)$ be defined on the domain $x \in X$ and let Y be the (denumerable) subset of points $Y \subset X$ at which $\pi(x)$ is not twice differentiable. Now $X \setminus Y$ contains no boundary points not in X whence $\pi(x)$ is twice differentiable at all points in $X \setminus Y$ by construction and since $X \setminus Y \subset X$, $\pi(x)$ is concave for $x \in X \setminus Y$. Whence (π_{jk}) negative semi-definite $\forall x \in X \setminus Y$ from [51: Theorem 4.5].

The proof for strict concavity is directly analagous.

LEMMA 27

Given: A concave function $F(x)$.

To Prove: The family of loci, $F(x) = \text{constant}$ are concave

contoured, i.e. $F(x) = c$ is the lower boundary of a convex set.

Proof: Immediate by [42: Section R8.5].

LEMMA 28

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: $x \begin{matrix} < \\ > \end{matrix} x^* \rightarrow F(x) - x \cdot z \begin{matrix} < \\ > \end{matrix} 0$
 $\forall z > 0$ s.t. $z \in [\partial F]_x$.

Proof: By Theorem 23 x^* lies on the expansion path of the firm for any input prices, i.e. $\forall \underline{p} > 0 \exists P^* > 0$ s.t. $P^*F(x) - x \cdot \underline{p}$ is maximised at $x = x^*$.

Let $x < x^*$, and let \underline{p}, P be such that $\underline{p} > 0$, $P > 0$ and $\underline{p}/P = z \in [\partial F]_x$, i.e. x maximises profit for prices \underline{p}, P .

By Theorem 21 $x < x^*$ implies that $P < P^*$ and hence maximum profit at $x^* >$ maximum profit at x by Assumption (x).

$$\therefore F(x) - x \cdot z = F(x) - x \cdot \underline{p}/P < F(x^*) - x^* \underline{p}/P^* < 0$$

Similarly for $x > x^*$.

THEOREM 29

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: If x is a profit maximisation point, $x \geq x^*$ implies $x > x^*$.

Proof: Let $\underline{p}, P > 0$ be such that $\underline{p}/P \in [\partial F]_x$, i.e. x maximises profit for prices \underline{p}, P .

$$x \geq x^* \rightarrow \exists j \in \{1, \dots, n\} \text{ s.t. } x^j > x^{j*}$$

- ∴ by Theorem 21, $P > P^*$ where $p/P^* \in [\partial F]_{x^*}$
- ∴ by the same theorem $x_i^k > x_i^{k*} \quad \forall k = 1, \dots, n$
- ∴ $x > x^*$.

LEMMA 29.1

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: $x > x^*$ if, and only if, $F(x) - x.z > 0$
 $\forall z > 0$ s.t. $z \in [\partial F]_x$.

Proof: Sufficiency is proved in Lemma 28

Necessity: Suppose $\exists j \in \{1, \dots, n\}$ s.t. $x^j < x^{j*}$

$\exists z > 0$ s.t. $z \in [\partial F]_x$ by Definition 2', ∴ x is profit maximising for prices p, P s.t. $Pz = p$.

∴ not $(x > x^*)$ requires not $(x \geq x^*)$ by Theorem 29

∴ $x^j < x^{j*}$ for some j

∴ $P < P^*$ where $p/P^* \in [\partial F]_{x^*}$ by Theorem 21.

∴ $F(x) - x.z = F(x) - x.p/P < F(x^*) - x^*p/P^* = 0$

i.e. $F(x) - x.z > 0$ for some $z \in [\partial F]_x \rightarrow x > x^*$.

COROLLARY 29.2

By an analogous proof to that of Lemma 29.1 it may be shown that $x < x^* \leftrightarrow F(x) - x.z < 0 \quad \forall z > 0$ s.t. $z \in [\partial F]_x$.

COROLLARY 29.3

By Lemma 29.1 and Corollary 29.2 the observation of $x^j = x^{j*}$ and $x^k \neq x^{k*}$ necessarily implies from the Neo-Classical Model that input x^j is constrained, i.e. the firm is in run $r < j$.

THEOREM 30.

Given: A perfectly competitive firm with concave production function $F(x)$ defined over $x \in S$.

To Prove: If S is not compact, every expansion path (set) must have at least one common point with all others, i.e. $x^* \in S$ such that $\forall p > 0, \exists P(p)$ for which x^* is the profit maximising input vector.

Proof: If S is not compact, $\exists x^* \in S, \theta x^* + (1+\theta)y \in S, 0 < \theta < 1, y \in S \rightarrow \theta = 0$. Consider $y \neq x^*$ such that $F(y) = F(x^*)$: By supposition $\exists z > 0, z \in [\partial F]_{x^*}$ and $w > 0, w \in [\partial F]_y$. Choose $\underline{p}/P = z$ and $\underline{p}'/P' = w$, i.e. x^* maximises profit for prices \underline{p}, P and y maximises profit for prices \underline{p}', P' . Thus

$$(\underline{p}/P) (y - x^*) > 0 \quad \forall y \text{ and } (\underline{p}'/P') (x - y) > 0 \quad \forall x$$

Three cases are possible: a) $y \not\geq x^*$ & $x^* \not\geq y$;

b) $y \geq x^*$; c) $y < x^*$.

a) Let G be the hyperplane through x^*, y defined by $v(\underline{p}''/P'') = y(\underline{p}''/P'') = x^*(\underline{p}''/P''), \underline{p}'', P'' > 0$ and let u maximise profit for prices \underline{p}'', P'' . Now

(i) $(\underline{p}/P) (u - x^*) < 0 \rightarrow F(u) > F(x) \rightarrow x$ does not maximise profit for \underline{p}/P , contrary to assumption, $\therefore (\underline{p}/P) (u - x^*) \geq 0$.

(ii) similarly $(\underline{p}'/P') (u - y) < 0 \rightarrow F(u) > F(y) \rightarrow y$ does not maximise profit for \underline{p}'/P' , contrary to assumption.

$\therefore (\underline{p}'/P') (u - y) \geq 0$.

Hence $u = \theta x^* + (1-\theta)y$, $0 \leq \theta \leq 1$

which implies $u = x^*$ or y

$\therefore F(u) = F(x^*) = F(y)$. Thus x^* and y both maximise profit for prices p'' , P'' , contradicting uniqueness.

b) $y \geq x^* \rightarrow (p'/P')(x - y) \leq 0 \quad \forall p'/P' > 0$,
contrary to assumption.

c) similarly $y \leq x^* \rightarrow (p/P)(y - x) \leq 0 \quad \forall p/P > 0$,
contrary to assumption.

Hence no y exists such that $y \neq x^*$ and $F(y) = F(x^*)$. Moreover from Theorem 21, if y is a profit maximising point $y \geq x^* \rightarrow y > x^*$ and $y \leq x^* \rightarrow y < x^*$, (proof directly analogous to Theorem 29) and hence all expansion sets contain x^* , irrespective of p .

COROLLARY 30.1

If all inputs are indivisible, that is the input set consists of distinct points x , the firm maximises profit by choosing an x from a sequence of input combinations x_1, \dots, x_k, \dots such that $x_k > x_{k-1}$. This choice is made entirely on the basis of the output price and is independent of the input price ratios generated by $p > 0$. Hence no 'input substitution' occurs.

LEMMA 30.2

Given: A perfectly competitive firm with concave production function $F(x)$ and zero maximum profit point x^* .

To Prove: If $x > x^*$, $F(\theta x) < \theta F(x) \quad \forall \theta > 1$.

Proof: Suppose $F(\theta x) > \theta F(x)$ for some $x > x^*$, $\theta > 1$.

Choose $p, P > 0$ s.t. $p/P \in [\partial F]_x$, i.e. x maximises profit for prices p, P , whence

$$F(x) - x \cdot p/P > F(y) - y \cdot p/P \quad \forall y \neq x.$$

Let $y = \theta x \neq x$ since $\theta > 1$:

$$\begin{aligned} F(x) - x \cdot p/P &> F(\theta x) - \theta x \cdot p/P \\ &> \theta F(x) - \theta x \cdot p/P \text{ by assumption.} \\ &= \theta [F(x) - x \cdot p/P] \end{aligned}$$

But $x > x^*$ implies $F(x) - x \cdot p/P > 0$ by Lemma 28

$\therefore 1 > \theta$ contrary to assumption.

COROLLARY 30.3

If $x < x^*$, maximum profit is negative by Lemma 28 which implies $F(\theta x) < \theta F(x) \quad \forall \theta < 1$.

THEOREM 31

Given: A perfectly competitive firm with concave production function $F(x)$ and unique zero maximum profit point x^* .

To Prove: If $F(x) \equiv H(x)$ for $x > x^*$, $H(x) < (x^*) \prod_{j=1}^n (x_j/x_j^*)^{\alpha_j}$

$$\forall \alpha_j > 0 \text{ such that } \sum_{j=1}^n \alpha_j = 1.$$

Proof: Let $H(x) \triangleq Y(x) = A \prod_{j=1}^n (x_j)^{\alpha_j}$, $A > 0$, $\sum \alpha_j = 1$ for some $x > x^*$.

$$\text{Now } Y(\theta x) = A \prod_{j=1}^n (\theta x_j)^{\alpha_j} = \theta Y(x) = \theta H(x) \quad \forall \theta$$

But $H(\theta x) < \theta H(x) \quad \forall \theta > 1$ by Lemma 30.2

$$\therefore H(\theta x) < Y(\theta x) \quad \forall \theta > 1$$

Since $x > x^*$, $\exists \theta > 1$ such that $x > \theta x^*$

\therefore if $Y(x^*) = \lim_{x \rightarrow x^*} H(x)$, $H(x) < Y(x) \quad \forall x > x^*$

i.e. $A = \prod_{j=1}^n (x_j^*)^{\alpha_j} = H(x^*)$, say.

$\therefore A = H(x^*) \prod_{j=1}^n (x_j^*)^{-\alpha_j}$

$\therefore Y(x) = H(x^*) \prod_{j=1}^n (x_j/x_j^*)^{\alpha_j}$ as required.

THEOREM 32

Given: A function $F(x, y) \equiv \left| [x^{*2} + 4y^{*2}]^{1/2} \right| - \left| [(x^*-x)^2 + 4(y^*-y)^2]^{1/2} \right|$

defined on $0 < (x, y) < (x^*, y^*)$.

- To Prove:
- $\partial F / \partial x > 0$; $\partial F / \partial y > 0$
 - $\partial^2 F / \partial x^2 < 0$; $\partial^2 F / \partial y^2 < 0$; $\partial^2 F / \partial x \partial y > 0$
 - $\lim_{(x, y) \rightarrow 0} F(x, y) = 0$
 - $\lim_{(x, y) \rightarrow (x^*, y^*)} F(x, y) = \left| [x^{*2} + 4y^{*2}]^{1/2} \right| > 0$
 - $\lim_{(x, y) \rightarrow (x^*, y^*)} \nabla F(x, y)$ is indeterminate.

Proof: a) $\partial F / \partial x = -\left(\frac{1}{2}\right) \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-1/2} 2(x^*-x)(-1)$
 $= \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-1/2} (x^*-x) > 0$

for $x < x^*$.

Similarly $\partial F / \partial y = 4 \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-1/2} (y^*-y) > 0$

for $y < y^*$.

b) $\partial^2 F / \partial x^2 = -\left(\frac{1}{2}\right) \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-3/2} 2(x^*-x)^2(-1)$
 $- \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-1/2}$
 $= \left[(x^*-x)^2 + 4(y^*-y)^2 \right]^{-3/2} \left[(x^*-x)^2 - \right.$
 $\left. (x^*-x)^2 - 4(y^*-y)^2 \right] < 0$

Similarly for $\partial^2 F / \partial y^2$.

$$\begin{aligned} \partial^2 F / \partial x \partial y &= -2[(x^*-x)^2 + 4(y^*-y)^2]^{-3/2} (y^*-y) 2(x^*-x)(-1) \\ &> 0 \text{ for } 0 < (x,y) < (x^*,y^*) \end{aligned}$$

c), d) follow directly by substitution.

$$e) \quad \partial F / \partial x = (x^*-x) / \{[(x^*-x)^2 + 4(y^*-y)^2]\}$$

Lt $\partial F / \partial x$ as $(x,y) \rightarrow (x^*,y^*)$ depends on the relation between x and y . Similarly for $\partial F / \partial y$. In particular,

$$\lim_{(x,y) \rightarrow (x^*,y^*)} (\partial F / \partial x) (\partial F / \partial y)^{-1} = \lim_{(x,y) \rightarrow (x^*,y^*)} (x^*-x) / y^*-y.$$

THEOREM 33

Given: A production function $F(x,y) = [x^2 + 4y^2]^{1/2}$
 $- [(x^*-x)^2 + 4(y^*-y)^2]^{1/2}$, $0 < (x,y) < (x^*,y^*)$.

To Prove: Maximum profit is non-negative and equal to zero
 when (x,y) lies on the ray through (x^*,y^*) .

Proof: Maximum profit is proportional to $F(x) - x \cdot \nabla F(x)$ and

$$\begin{aligned} F(x) - x \cdot \nabla F(x) &= [x^2 + 4y^2]^{1/2} - [(x^*-x)^2 + 4(y^*-y)^2]^{1/2} \\ &- [x(x^*-x) + 4y(y^*-y)] / \{[(x^*-x)^2 + 4(y^*-y)^2]\}^{1/2} \\ &= \{[(x^*-x)^2 + 4(y^*-y)^2]\}^{-1/2} \{[x^2 + 4y^2]^{1/2} \\ &- [(x^*-x)^2 + 4(y^*-y)^2]^{1/2} - [(x^*-x)x + 4(y^*-y)y]\}. \end{aligned}$$

Since $\{[(x^*-x)^2 + 4(y^*-y)^2]\} > 0$, taking positive roots,

this function has the same sign as

$$\begin{aligned} &[x^2 + 4y^2] \{[(x^*-x)^2 + 4(y^*-y)^2]\} - [(x^*-x)x + 4(y^*-y)y]^2 \\ &= 4(x^*y - y^*x)^2 \geq 0. \end{aligned}$$

This expression is zero when $x^*/y^* = x/y$, i.e.

(x,y) lies on the ray through (x^*, y^*) .

LEMMA 34

Given: A perfectly competitive firm with concave production

function $F(x)$ and zero maximum profit point x^* .

To Prove: Maximum profit is non-negative for all positive prices p, P .

Proof: Let x maximise profit for prices p, P and suppose $F(x) - x \cdot z < 0$ where $p/P = z \in [\partial F]_x$.

By Corollary 30.3 $F(\theta x) < \theta F(x) \quad \forall \theta, 0 < \theta < 1$

Let $y = \theta x$: $F(y) < \theta F(\theta^{-1}y)$

i.e. $\theta F(y) < F(\theta y) \quad \forall \theta = \theta^{-1} > 1$

contradicting the assumed concavity of $F(x)$.

THEOREM 35

Given: A perfectly competitive firm with concave production function $F(x)$ and unique zero maximum profit point x^* .

To Prove: $x^j < x^{j*}$ for some $j = 1, \dots, n$ implies that the firm is in run k , $k < n-1$.

Proof: Since $\text{Max } \pi(x) > 0 \rightarrow x \geq x^*$ by Lemma 28, x cannot be an unconstrained profit maximisation point by Lemma 34. Therefore x must be constrained, i.e. if the firm is in run k , $k < n-1$ by Definition 4.

THEOREM 36

Given: A perfectly competitive firm.

To Prove: The Neo-Classical Model is inconsistent with long run equilibrium.

Proof: By Theorem 35 every firm has non-negative net profit in run n . From Assumption (*) the Neo-Classical firm does not have zero net profit identically.

Hence there exists an industry for which $\max \pi(x) = \pi_1 > 0$. If all firms have profit π_1 , Assumption (xi) is

contradicted since π_1 defines a different level of normal profit from $\pi = 0$. Thus there exists another industry with maximum profit $\pi_2 > 0$ such that $\pi_2 \neq \pi_1$ W.l.o.g. let $\pi_1 > \pi_2$.

By Assumption (iii) a firm earning profit π_2 will leave industry 2 and enter industry 1 in the long run causing π_1 to decrease and π_2 to increase by Assumptions (ix) and (x). Let the new levels of π_1, π_2 be $\pi_1(1)$ and $\pi_2(1)$.

If $\pi_1(1) > \pi_2(1)$ the process repeats until $\pi_1(t) < \pi_2(t)$

When $\pi_1(t) < \pi_2(t)$ the process is repeated with reversed indices, i.e. $\pi_2(t) > \pi_1(t) > 0$.

Equilibrium requires that $\pi_1(t) = \pi_2(t)$, but this must be the limit of a sequence of decreasing values of π_j and increasing values of π_k , $j, k = 1, 2$, $j \neq k$, i.e.

$$\pi_j(t-1) > \pi_j(t) = \pi_k(t) > \pi_k(t-1) > 0$$

Hence the equilibrium defines a level of normal profit above that defined by $\pi = 0$, contradicting Assumption (xi).

LIST OF REFERENCES

1. Abian, S., and Brown, A. B. "A Theorem on Partially Ordered Sets with Applications to Fixed Point Theorems." Canadian Journal of Mathematics 13 (1961): 78-83.
2. Agassi, J. Towards an Historiography of Science, History and Theory. The Hague: Mouton, 1963.
3. Alexander, J. A. "Some Aspects of Consumer Behaviour when Goods are Interrelated." Ph.D. dissertation, Simon Fraser University, 1974.
4. Alexander, J. A., and Rowcroft, J. E. "Exchange, Equilibrium with Indivisible Goods and Discrete Prices." Unpublished, 1977.
5. Arrow, K. J. Social Choice and Individual Values. 2nd ed. New York: Wiley, 1963.
6. Arrow, K. J., and Hahn, F. H. General Competitive Analysis. San Francisco: Houlden-Day, 1971.
7. Baumol, W. J. Business Behavior, Value and Growth. New York: Macmillan, 1959.
8. _____ . Economic Theory and Operations Analysis. 4th ed. Englewood Cliffs, N. J.: Prentice-Hall, 1977.
9. Boland, L.A. "Introduction to Price Theory: A Dialectical Approach." Mimeographed. Burnaby, B. C.: Simon Fraser University, 1967.
10. _____ . "Axiomatic Analysis and Economic Understanding." Australian Economic Papers 9 (1970): 62-75.
11. _____ . "The Law of Demand, Weak Axiom of Revealed Preference and Price Consumption Curves." Australian Economic Papers 14 (1975): 104-119.
12. _____ . "Testability in Economic Science." South African Journal of Economics 45 (1977): 93-105.
13. _____ . "Model Specification and Stochasticism in Economic Methodology." South African Journal of Economics 45 (1977): 182-189.
14. _____ . "Linear-Homogeneous Production Functions with Profit Maximisation." Paper read at the Canadian Economics Association Meetings, 1977, at the University of New Brunswick. Mimeographed.

15. Boulding, K.E. A Reconstruction of Economics. New York: Wiley, 1950.
16. Carnap, R. Logical Foundations of Probability. Chicago: University of Chicago Press, 1950.
17. Christensen, L.R.; Jorgenson, D.W.; and Lau, L.J. "Transcendental Logarithmic Production Frontiers." Review of Economics and Statistics 55 (1973): 28-45.
18. Cobb, C.W., and Douglas, P.H. "A Theory of Production." American Economic Review Supplement 18 (1928): 139-165.
19. Cyert, R., and March, J. A Behavioural Theory of the Firm. Englewood Cliffs, N. J.: Prentice-Hall, 1973.
20. Day, R.H. "Profits, Learning and the Convergence of Satisficing to Marginalism." Quarterly Journal of Economics 31 (1967): 302-311.
21. Debreu, G. Theory of Value: An Axiomatic Analysis of Economic Equilibrium. New York: Wiley, 1959.
22. Diewert, W.E. "An Application of the Shephard Duality Theorem: A Generalised Leontief Production Function." Journal of Political Economy 79 (1971): 481-507.
23. Dorfman, R.; Samuelson, P.A.; and Solow, R.M. Linear Programming and Economic Analysis. New York: McGraw-Hill, 1958.
24. Fama, E.F., and Laffer, A.B. "The Number of Firms and Competition." American Economic Review 62 (1972): 670-674.
25. _____ . "The Number of Firms and Competition: Reply." American Economic Review 64 (1974): 806.
26. Ferguson, C.E. Microeconomic Theory. Homewood, Ill.: Irwin, 1966.
27. Frank, C.R. Production Theory and Indivisible Commodities. Princeton: Princeton University Press, 1969.
28. Friedman, M. Price Theory: A Provisional Text. Rev. ed. Chicago: Aldine, 1966.
29. _____ . Essays in Positive Economics. Chicago: Chicago University Press, 1953.
30. Frisch, R. Maxima and Minima: Theory and Economic

- Applications. Chicago: Rand McNally, 1966.
31. Georgescu-Roegen, N. "Fixed Coefficients of Production and the Marginal Productivity Theory." Review of Economic Studies 3 (1935): 40-49.
 32. Gordon, R.A. "Short-Period Price Determination." American Economic Review 38 (1948): 265-288.
 33. Hall, R.L., and Hitch, C.J. "Price Theory and Business Behaviour." Oxford Economic Papers 2 (1939): 12-45.
 34. Hanoch, G. "Generation of New Production Functions through Duality." Discussion Paper, Harvard Institute of Economic Research, 1970.
 35. Hansen, T., and Jaskold-Gabszewicz, J. "Collusion of Factor Owners and Distribution of Social Output." Journal of Economic Theory 4 (1972): 1-18.
 36. Henderson, J.M., and Quandt, R.E. Microeconomic Theory: A Mathematical Approach. 2nd ed. New York, McGraw-Hill, 1971.
 37. Hicks, J. Value and Capital. Oxford: Clarendon Press, 1939.
 38. Hildenbrand, W. "Pareto Optimality for a Measure Space of Economic Agents." International Economic Review 10 (1969): 363-372.
 39. Hori, H. "The Number of Firms and Competition: Comment." American Economic Review 64 (1974): 805.
 40. Jorgenson, D.W., and Lau, L.J. "The Duality of Technology and Economic Behaviour." Review of Economic Studies 41 (1974): 181-200.
 41. Karlin, S. Mathematical Methods and Theory in Games, Programming and Economics, vol. 1. Reading, Mass.: Addison-Wesley, 1959.
 42. Lancaster, K. Mathematical Economics. New York: Macmillan, 1968.
 43. Levine, A.L. "Profit Maximisation: The End of the Affair?" Paper read at the Canadian Economics Association Meetings, 1974, at the University of Toronto. Mimeographed.
 44. Malinvaud, E. Statistical Methods of Econometrics, tr. A. Silvey. Chicago: Rand McNally, 1966.

45. Margolis, J. "The Analysis of the Firm: Rationalism, Conventionalism and Behaviorism." Journal of Business 31 (1958): 187-199.
46. Marshall, A. Principles of Economics. 8th ed. London: Macmillan, 1920.
47. Nordquist, G.L. "The Breakup of the Maximisation Principle." Quarterly Review of Economics and Business 5 (1965): 33-46.
48. Papandreou, A.G. "Some Basic Problems in the Theory of the Firm." In A Survey of Contemporary Economics, vol. 2, ed. B.F. Hayley, pp. 183-222. Homewood, Ill.: Irwin, 1952.
49. Popper, K. Logic of Scientific Discovery. London: Science Editions, 1959.
50. Robinson, J. The Economics of Imperfect Competition. London: Macmillan, 1933.
51. Rockafellar, R.T. Convex Analysis. Princeton: Princeton University Press, 1970.
52. Rothschild, K.W. "Price Theory and Oligopoly." Economic Journal 42 (1947): 297-320.
53. Rowcroft, J.E. "Relaxation Phenomena." Mimeographed. Burnaby, B. C.: Simon Fraser University, 1970.
54. . "A Note on 'Profit Maximisation: The End of the Affair?' by A.L. Levine." Mimeographed. Fredericton, N. B.: University of New Brunswick, 1974.
55. Samuelson, P.A. Foundations of Economic Analysis. Cambridge, Mass.: Harvard University Press, 1948.
56. Shaikh, A. "Laws of Production and Laws of Algebra: the Humbug Production Function." Review of Economics and Statistics 41 (1974): 115-120.
57. Shephard, R.W. Theory of Cost and Production Functions. Princeton: Princeton University Press, 1970.
58. Simon, H.A. "A Behavioural Model of Rational Choice." Quarterly Journal of Economics 69 (1952): 99-118.
59. Solow, R.M. "Laws of Production and Laws of Algebra: the Humbug Production Function: Comment." Review of Economics and Statistics 41 (1974): 121.

60. Solow, R.M., and Stiglitz, J.E. "Output, Employment and Wages in the Short Run." Quarterly Journal of Economics 82 (1968): 537-560.
61. Takayama, A. Mathematical Economics. Hinsdale, Ill.: The Dryden Press, 1974.
62. Theil, H. Economics and Information Theory. Amsterdam: North-Holland, 1967.
63. Walsh, V.C. Introduction to Contemporary Microeconomics. New York: McGraw-Hill, 1970.
64. Walters, A.A. "Production and Cost Functions: An Econometric Survey." Econometrica 31 (1963): 1-66.
65. Weber, M. The Methodology of the Social Sciences. New York: The Free Press, 1949.
66. Wong, S. "The F-twist and the Methodology of Paul Samuelson." American Economic Review 63 (1973): 312-325.
67. Yamane, T. Mathematics for Economists. Englewood Cliffs, Ill.: Prentice-Hall, 1962.

8