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NAME OF AUTHOR/*NOM DE L'AUTEUR* Badri N. Varma

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SOME DECOMPOSITION PROBLEMS FOR COMPLETE GRAPHS

by

Badri N. Varma

M.Sc., Aligarh Muslim University, (India), 1961

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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APPROVAL

Name: Badri N. Varma

Degree: Doctor of Philosophy

Title of Thesis: Some decomposition problems for complete graphs

Examining Committee:

Chairman: Dr. S.K. Thomason

____
B.R. Alspach
Senior Supervisor

____
T.C. Brown

____
B.S. Thomson

____
K.A. Heinrich

____
Pavol Hell
External Examiner
Associate Professor
Rutgers, The State University of New Jersey
New Brunswick, New Jersey

Date Approved: August 9, 1979

ABSTRACT

Given a graph G and graphs $H_1, H_2, \ldots, H_s$, if there exists a partition of the edge set $E(G)$ such that the resulting subgraphs of G are isomorphic to $H_1, H_2, \ldots,$ or H_s we say that the graph G can be *decomposed* into the graphs $H_1, H_2, \ldots, H_s$. In particular if $H_1 \cong H_2 \cong H_s \cong H$ (say), the decomposition is called an *isomorphic factorization* of G. In this case we also say that H *divides* G.

Similar definitions hold for directed graphs.

In Chapter 1, decompositions of some complete multipartite graphs and some special graphs into cycles of different lengths are obtained. Most of the graphs considered here often appear as factors in the decomposition of complete graphs and complete symmetric digraphs considered in the subsequent chapters. Thus the results of this chapter are not only used in obtaining results in other chapters, but can be used as important tools in many other decomposition problems.

In Chapter 2 it is shown that for $n \equiv 5 \pmod 6$ the complete graph K_n can be decomposed into C_3's and one K_5 where C_r denotes a cycle of length r. This together with the fact that a Steiner triple system is known to exist for $n \equiv 1$ or $3 \pmod 6$ establishes the existence of a pairwise balanced design $(n;5,3;1)$ for any odd n.

In Chapter 3 it is shown that the necessary conditions for the cycle $C_{2 \cdot p^\alpha}$, p prime and α any positive integer, to divide K_n are also sufficient.

In Chapter 4 it is shown that necessary and sufficient conditions for each orientation of C_5 to divide a complete symmetric digraph DK_n are $n \equiv 0$ or $1 \pmod 5$ and $n \geq 5$.

In Chapter 5 sufficient conditions are found for a self-converse orientation of C_k, k odd, to divide a complete symmetric digraph DK_n. This result is used to show that the necessary conditions for any self-converse orientation of C_7 to divide DK_n are also sufficient.

To Kamal, my wife, who while driving
the car once discovered that $8 + 6 = 14$.

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TABLE OF SYMBOLS

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BIBD (n,k,λ)	balanced incomplete block design	27
C_k^*	directed cycle of length k	90
D'	converse of digraph D	7
DG	digraph associated with G	6
DK_n	complete symmetric digraph	6
$G_1 \cup G_2$	union of G_1 and G_2	4
$G_1 + G_2$	join of G_1 and G_2	4
$G_1[G_2]$	composition of G_1 and G_2	5
$H\|G$	H divides G	8
$K_n^{(\lambda)}$	complete graph with n vertices and λ edges connecting every pair of vertices.	39
$K_{\|A\|,\|B\|}$	complete bipartite graph with A and B as maximal independent sets of vertices.	4
λ-PBD	pairwise balanced design of index λ	27
nG	graph with n components isomorphic to G	5
nK_x	complete n-partite graph	4

CHAPTER 0

INTRODUCTION

It all began in 1736 when Leonhard Euler initiated the theory of graphs in his famous paper [12], though people had been using combinatorial ideas to solve entertaining puzzles earlier. Thus Oystein Ore has rightly remarked in his book *Graphs and their uses* [27] that the theory of graphs is one of the few fields of mathematics which has a definite birth date. Euler himself started his paper with the discussion of a puzzle, known as *Königsberg problem* (see [27, page 23]). In the initial stage, since it dealt largely with recreational problems, graph theory could not draw much attention from the scholars in mathematics until 1847 when Kirchhoff [24] counted the number of spanning trees of a labeled graph and applied it to electrical networks. After almost four decades, there came another startling result in graph theory when Cayley [11] gave a count for the labeled trees with n vertices. He applied this result to the problem of counting chemical isomers. In spite of all these early promising applications, graph theory could not sever its relationship with entertaining puzzles. In 1890, Heawood [18] in an attempt to solve the famous *Four color problem** proved the *five color theorem*, which is regarded as another landmark in the history of graph theory.

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*The Four color problem after its long history was finally solved by Appel, Haken and Koch [1] in the summer of 1976.

For many years to come, because of its study of topological properties such as connectedness, planarity and so on, graph theory was regarded as a branch of topology. Thus, when J.H.C. Whitehead once described graph theory as 'the slums of topology', he was expressing the general consensus which prevailed among the mathematicians at that time. Denés König's book *Theorie der endlichen und unendlichen Graphen* published in 1936 was the first book ever written on graph theory. Though it was an attempt to establish graph theory as a unitary discipline, it failed to satisfy a section of mathematicians who subscribed to the above stated view. We are indeed indebted to Turán who discovered a pioneering theorem in 1941. It not only opened a new avenue in the study of graphs known as *extremal graph theory* but settled the issue that graph theory has its own entity and can not be looked upon merely as a branch of topology. Turán [36] found the maximum possible number of edges for all graphs on n vertices which do not contain a triangle. He also showed that there is a unique graph which realizes this number.

Graph theory today is like an old banyan tree with its many branches extending friendly help to the other branches of mathematics, sciences and humanities. To cite a few examples: Menger's *separation theorem* of 1927 has led to the development of *network theory* formulated by Ford and Fulkerson in their work *Flows in network* (1962). Graph theory is used in the study of certain engineering and structural systems which involve interrelated components. It has found applications in many areas like assignment problems, timetabling problems, storage problems and so on. The study of directed graphs has successfully been applied to the problem of making a road system one-way, ranking participants in a tournament or that of designing an efficient computer drum. Graph Theory has also found applications in economics, psychology and biology.

The problem considered in this dissertation come under the branch called *combinatorial designs* which has been growing at a fast rate. These designs are used as powerful tools in statistical experiments. Starting with *Latin squares*, *Hadamard matrices*, *Steiner triple systems* and more recently *coding theory*, it has generated much enthusiasm. Some of these designs such as *balanced incomplete block designs*, *pairwise block designs*, *balanced cycle* or *circuit designs* or more generally the G-*designs* are defined in the subsequent chapters. The third chapter deals with the *balanced cycle* designs which are useful in serology, a science of virus research. To conduct the *Ouchterlony gel diffusion test*, samples from a number of antigens (virus preparations) are to be arranged around an antiserum on a plate so that every antigen has two others as its neighbours. There are n kinds of antigens to be arranged on b plates each containing k antigens. (see Rees [28] and Hwang [22]).

DEFINITIONS AND NOTATION

We shall use here the terminology commonly used in any standard text on graph theory, for example, by Bondy and Murty [8], by Behzad and Chartrand [2] or by Harary [15]. However, we give here a few definitions which are not in common usage.

By a *graph* here we mean an ordinary graph, that is, a finite graph with no multiple edges or loops.

We shall use the same notation as used in the above books except for a *complete* n-*partite graph* $K_{x,x,\ldots,x}$, which is a graph whose vertex set may be partitioned into sets $X_1, X_2, \ldots, X_n$ where $|X_i| = x$ for each $i = 1,2,\ldots,n$ and two vertices u and v are adjacent if and only if they belong to distinct sets X_i and X_j of the partition. We shall denote this graph by ${}_nK_x$.

We shall use the notation $K_{|A|,|B|}$ for the complete bipartite graph with maximal independent sets A and B.

DEF.: The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is a graph with

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \cup G_2) = E(G_1) \cup E(G_2) .$$

DEF.: The *join* $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets is their union $G_1 \cup G_2$ together with all edges joining the vertices of G_1 and G_2.

(Note: Bondy and Murty [8] use the notation $G_1 + G_2$ for the disjoint union of the graphs G_1 and G_2). It is easy to see that

$$K_{m,n} = \overline{K}_m + \overline{K}_n$$

and

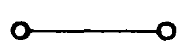
$$K_m + K_n = K_{m+n} = K_m \cup K_n \cup K_{m,n} ,$$

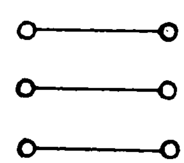
where $\overline{K}_m$ represents the *complement* of the complete graph K_m , that is, a set of m isolated vertices.

DEF.: For a connected graph G, nG is a graph with n components each isomorphic to G .

Note that nG itself is not connected if $n > 1$.

Example:

G: 

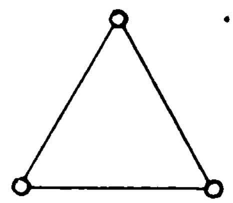
3G: 

DEF.: The *composition* $G = G_1[G_2]$ of two graphs G_1 and G_2 is a graph with $V(G) = V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$

and $v = (v_1, v_2)$ of G are adjacent if and only if u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2.

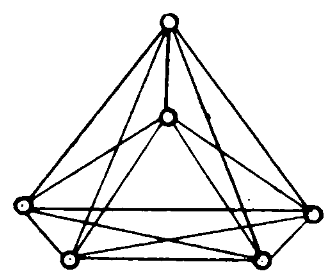
Example:

G_1:



G_2:

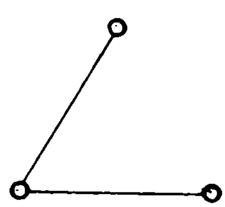
$G_1[G_2]$:



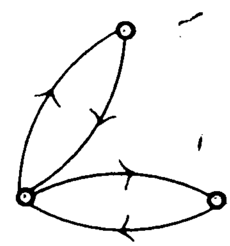
DEF.: For any graph G, DG is a directed graph with $V(G) = V(DG)$ and for each edge uv of G there are arcs uv and vu in DG.

Example:

G:



DG:

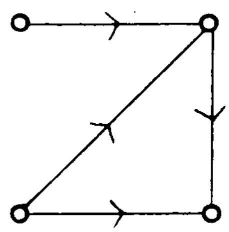


The digraph DK_n in particular is called a *complete symmetric digraph*.

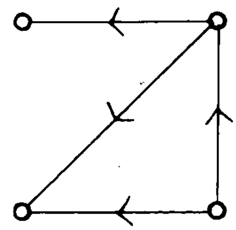
DEF.: The *converse* D' of a digraph D is a digraph with $V(D') = V(D)$ and the arc uv is in D' if and only if the arc vu is in D .

Example:

D:

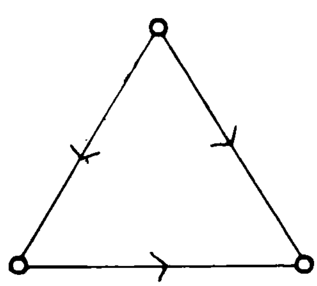


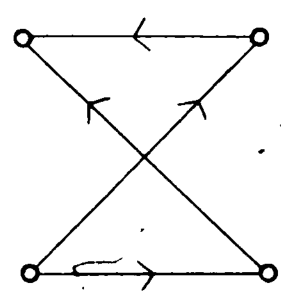
D':



DEF.: A digraph is said to be *self-converse* if and only if it is isomorphic to its converse digraph.

Example: The digraph D of the above example is not self-converse, while the following digraphs are self-converse.





DEF.: A *decomposition* of a graph G into graphs $H_1, H_2, \ldots, H_s$ is a partition of $E(G)$ so that the resulting subgraphs of G are isomorphic to $H_1, H_2, \ldots,$ or H_s and we say that G can be *decomposed* (or G is *decomposable*) into the graphs $H_1, H_2, \ldots, H_s$.

In particular, if $H_1 \cong H_2 \cong \ldots \cong H_s \cong H$, the decomposition is called an *isomorphic factorization* of G and we write $H|G$ and read it as H *divides* G.

Isomorphic factorizations of digraphs are defined analogously.

In the proof of Lemma 2.3 we use the following concept.

DEF.: A graph G with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ is called a *circulant* if for $0 \leq i, j \leq n-1$, $i \neq j$, $v_i v_j$ is an edge if and only if $v_{i+k} v_{j+k}$, $1 \leq k \leq n-1$, is an edge of G. The set $S = \{i : v_0 v_i \in E(G)\}$ is called the *symbol* of G.

A circulant graph is described completely by its symbol. The vertices in the neighbourhood of one vertex determine the neighbourhoods of all the other vertices.

CHAPTER 1

In this chapter we consider the decomposition of some complete multipartite graphs and some other graphs into cycles of different lengths. The graphs considered here generally appear as factors in the decomposition of complete graphs and complete symmetric graphs. These results thus can be used as important tools in other decomposition problems. The main results of the succeeding chapters will illustrate this fact.

1.1 PROPOSITION: $C_3 | {}_3K_x$ *for any positive integer* x .

Proof: Let $\{u_1, u_2, \ldots, u_x\}$, $\{v_1, v_2, \ldots, v_x\}$ and $\{w_1, w_2, \ldots, w_x\}$ be the three maximal independent subsets of $V({}_3K_x)$. Consider the triangles

$$u_i, v_j, w_k, u_i \qquad \ldots (1.1.1)$$

where $1 \leq i,j,k \leq x$ and $i + j + k \equiv 0 \pmod{x}$. Since for any pair of positive integers i,j such that $1 \leq i, j \leq x$, the relation $i + j + k \equiv 0 \pmod{x}$ determines a unique positive integer k, $1 \leq k \leq x$, the set of triangles in (1.1.1) gives a desired decomposition of ${}_3K_x$.□

Prop. 1.1 also follows from a result of Bermond [3].

1.2 PROPOSITION: $C_3 | {}_5K_{3x}$ *for any positive integer* x .

<u>Proof</u>: First let $x = 1$. Also let $\{a_1,a_2,a_3\}$, $\{b_1,b_2,b_3\}$, $\{c_1,c_2,c_3\}$, $\{d_1,d_2,d_3\}$ and $\{e_1,e_2,e_3\}$ be the five maximal independent subsets of $V({}_5K_3)$. Following is a decomposition of ${}_5K_3$ into disjoint triangles. With subscripts reduced modulo 3,

$$a_i, b_i, d_{i+1}, a_i$$

$$a_i, b_{i+1}, e_{i+1}, a_i$$

for $i = 1,2,3$, give six triangles. The rest are obtained by rotating a,b,c,d,e in the cyclic order a to b, b to c, c to d, d to e and e to a .

Now for any $3x$ in general, let A_1, A_2, A_3, A_4 and A_5 be the maximal independent subsets of $V({}_5K_{3x})$, each of cardinality $3x$. For $i = 1,2,3,4,5$, let

$$A_i = A_{i1} \cup A_{i2} \cup A_{i3} ,$$

where $|A_{i1}| = |A_{i2}| = |A_{i3}| = x$. With ${}_5K_{3x}$ we associate a ${}_5K_3$ so that

$$V({}_5K_3) = \{A_{ij} : 1 \le i \le 5, 1 \le j \le 3\} ,$$

and an edge incident with $A_{\ell m}$ and A_{rs} corresponds to the set of all edges of the complete bipartite graph $K_{|A_{\ell m}|,|A_{rs}|}$ which is a subgraph of ${}_5K_{3x}$. By the first part ${}_5K_3$ is decomposable into triangles, which amounts to the fact that the graph ${}_5K_{3x}$ is decomposable into ${}_3K_x$'s and the result then follows from Prop. 1.1.□

1.3 PROPOSITION: *For* $n \equiv 0$ *or* $2 \pmod 6$, $n \geq 6$, $C_3 | (K_n - I)$ *where* I *is a 1-factor of* K_n.

Proof: Adjoin a vertex u to K_n. The graph $K_n + u$ is a complete graph K_{n+1} with $n + 1 \equiv 1$ or $3 \pmod 6$ and hence can be decomposed into triangles by the existence of Steiner triple system [14, Thm. 15.4.3]. The graph $K_n = K_{n+1} - u$ can be decomposed into one 1-factor and the rest triangles.□

1.4 PROPOSITION: $C_3 \mid {}_nK_{2x}$ *where* $n \equiv 0$ *or* $4 \pmod 6$ *and* x *is any positive integer.*

Proof: With ${}_nK_{2x}$ we associate a graph G as follows. Let

$$V_i = \{u_1^i, u_2^i, \ldots, u_{2x}^i\}, \quad 1 \leq i \leq n$$

be n maximal independent subsets of $V({}_nK_{2x})$. For $i = 1,2, \ldots, n$, let

$$S_i = \{u^i_1, u^i_2, \ldots, u^i_x\},$$

$$S_{n+i} = \{u^i_{x+1}, \ldots, u^i_{2x}\}.$$

Then we define

$$V(G) = \{S_1, S_2, \ldots, S_{2n}\},$$

and an edge between S_i and S_j corresponds to the set of all edges of the complete bipartite subgraph $K_{|S_i|,|S_j|}$ of ${}_nK_{2x}$. We observe that any two vertices S_i, S_j are adjacent except when $j = n + i$ $(1 \le i \le n)$. Moreover, the edges S_iS_{n+i} $(1 \le i \le n)$ form a 1-factor of $\overline{G}$. Thus the graph G is $K_{2n} - I$ where I denotes a 1-factor of K_{2n}. Since $2n \equiv 0$ or $2 \pmod 6$, by Prop. 1.3, $K_{2n} - I$ can be decomposed into C_3's, which amounts to the fact that ${}_nK_{2x}$ can be decomposed into ${}_3K_x$'s and the result then follows by Prop. 1.1.□

1.5 PROPOSITION; $C_5 | C_5[\overline{K}_x]$ *for any positive integer* x.

<u>Proof</u>: Let $V(C_5[\overline{K}_x]) = \bigcup_{i=1}^{5} A_i$ where $|A_i| = x$ for $i = 1,2,3,4,5$ and the vertices u and v are adjacent if and only

if $\mu \in A_i$ and $v \in A_{i+1}$, $i = 1,2,3,4,5$ and subscripts are to be taken modulo 5 . For $i = 1,2,3,4,5$ let

$$A_i = \{a_{i1}, a_{i2}, \ldots, a_{ix}\} .$$

Consider the 5-cycles

$$a_{1i}, a_{2j}, a_{3i}, a_{4j}, a_{5k}, a_{1i} \qquad \cdots (1.5.1)$$

where $i = 1,2,\ldots,x$; $j = 1,2,\ldots,x$ and $i + j + k \equiv 0 \pmod{x}$. Once i and j are fixed, the integer k is uniquely determined by the relation $i + j + k \equiv \pmod{x}$. This implies that the C_5's in 1.5.1 are all disjoint, and that they cover all the edges of $C_5[\overline{K}_x]$. Thus 1.5.1 gives a decomposition of $C_5[\overline{K}_x]$ into C_5's.□

1.6 PROPOSITION: $C_5 | {}_5K_x$ *for any positive integer* x .

<u>Proof:</u> Let $V({}_5K_x) = \bigcup_{i=1}^{5} A_i$ where for $i = 1,2,3,4,5$, A_i is a maximal independent subset of vertices and $|A_i| = x$. With ${}_5K_x$ we associate a complete graph K_5 so that

$$V(K_5) = \{A_1, A_2, A_3, A_4, A_5\} ,$$

and an edge A_iA_j, $1 \le i, j \le 5$, corresponds to the set of all edges of the complete bipartite subgraph $K_{|A_i|,|A_j|}$ of ${}_5K_x$. We know that K_5 can be decomposed into two cycles of length five, which amounts to the fact that the graph ${}_5K_x$ can be decomposed into two isomorphic factors $C_5[\bar{K}_x]$ which can be further decomposed into 5-cycles by Prop. 1.5.□

1.7 PROPOSITION: $C_5 \mid {}_3K_{5x}$ *for any positive integer* x .

Proof: First let $x = 1$. Also let $\{u_1,u_2,u_3,u_4,u_5\}$, $\{v_1,v_2,v_3,v_4,v_5\}$ and $\{w_1,w_2,w_3,w_4,w_5\}$ be the three maximal independent subsets of $V({}_3K_5)$. The following gives one decomposition of ${}_3K_5$ into C_5's . The cycles

u_1,v_1,u_5,v_5,w_3,u_1; u_2,v_2,u_4,v_4,w_3,u_2; u_3,v_3,u_1,w_2,v_5,u_3;

u_4,v_1,w_5,v_2,w_3,u_4; u_4,v_5,w_4,v_1,w_2,u_4

are five C_5's of the decomposition. The rest then are obtained by rotating u_i,v_i,w_i in the cyclic order u_i to v_i, v_i to w_i and w_i to u_i.

Now for any $5x$ in general, let A_1, A_2 and A_3 be the maximal independent subsets of $V({}_3K_{5x})$, each of cardinality $5x$.

For $i = 1,2,3$, let

$$A_i = A_{i1} \cup A_{i2} \cup A_{i3} \cup A_{i4} \cup A_{i5} ,$$

where $|A_{i1}| = |A_{i2}| = |A_{i3}| = |A_{i4}| = |A_{i5}| = x$. With ${}_3K_{5x}$ we associate a ${}_3K_5$ so that

$$V({}_3K_5) = \{A_{ij} : 1 \le i \le 3,\ 1 \le j \le 5\},$$

and an edge incident with $A_{\ell m}$ and A_{rs} , $1 \le \ell, r \le 3$, $1 \le m, s \le 5$, corresponds to the set of all edges of the complete bipartite subgraph $K_{|A_{\ell m}|,|A_{rs}|}$ of ${}_3K_{5x}$. By the first part, ${}_3K_5$ can be decomposed into C_3's, which amounts to the fact that the graph ${}_3K_{5x}$ can be decomposed into $C_5[\overline{K}_x]$ and the result then follows from Prop. 1.5.□

1.8 PROPOSITION: *For any odd* $n \ge 3$, K_n *can be decomposed into 3-cycles and 5-cycles.*

<u>Proof:</u> If $n \equiv 1$ or $3 \pmod 6$, we know that there exists a Steiner triple system on n objects, which is equivalent to the fact that the complete graph K_n can be decomposed into disjoint triangles. In case $n \equiv 5 \pmod 6$, Spencer [35] has shown that K_n can be decomposed into triangles and one C_4 . Let a, b, c, d, a be the 4-cycle of one such decomposition. Let u

and v (need not be distinct) be two vertices of K_n such that u, b, d, u and v, a, c, v are two triangles of the decomposition.

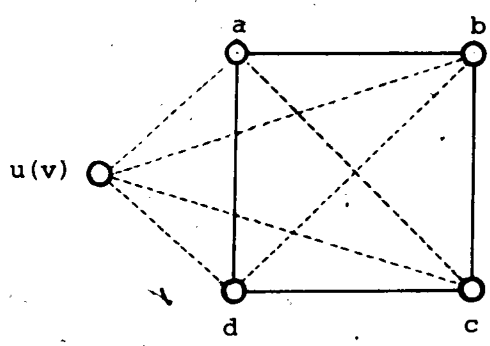


FIGURE 1(A)

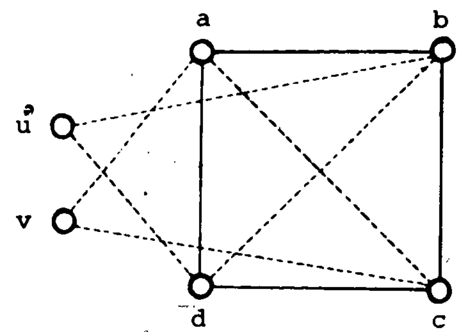


FIGURE 1(B)

If the two vertices u and v are the same (see Figure 1(A)), we get a decomposition of K_n into triangles and one K_5 whose vertex set is $\{a,b,c,d,u\}$. But K_5 can be decomposed into two C_5's, giving us the desired decomposition. In case the two vertices u and v are distinct (see Figure 1(B)) we get a decomposition of K_n into triangles and the two 5-cycles

u, b, a, c, d, u and v, a, d, b, c, v.□

1.9 PROPOSITION: $C_5 | {}_nK_5$ *for any odd positive integer* n .

Proof: With ${}_nK_5$ we associate a complete graph K_n as follows: Let $V_1, V_2, \ldots, V_n$ be the maximal independent subsets of $V({}_nK_5)$.

Then $V(K_n) = \{V_1, V_2, \ldots, V_n\}$ and an edge V_iV_j, $1 \le i, j \le n$, corresponds to the set of all edges of the complete bipartite subgraph $K_{|V_i|,|V_j|}$ of ${}_nK_5$. By Prop. 1.8, K_n can be decomposed into 3-cycles and 5-cycles, which amounts to the fact that the graph ${}_nK_5$ can be decomposed into graphs isomorphic to either ${}_3K_5$ or $C_5[\overline{K}_5]$ and the result then follows from Prop. 1.1 and Prop. 1.5.□

1.10 PROPOSITION (Bermond and Faber [5]): *For any odd positive integer* $n \ge 5$, *the complete symmetric digraph* DK_n *can be decomposed into directed cycles of length* $n-1$.

<u>Proof</u>: Let $V(DK_n) = \{0,1,2,\ldots,n-1\}$. Also let ϕ be a permutation on the vertices of DK_n with cycle representation

$$\phi = (0\ 1\ 2\ .\ .\ .\ n-1).$$

<u>Case 1</u>. First let $n \equiv 1 \pmod 4$, that is, $n = 4m + 1$ for some positive integer m. Consider the directed cycle C of length $4m$ (see Figure 1(C)) as given below.

$$C: 0,\ 1,\ 4m,\ 2,\ 4m-1,\ 3,\ \ldots,\ m,\ 3m+1,\ m+2,\ 3m,\ m+3,\ \ldots,\ 2m,\ 2m+2,\ 2m+1,\ 0\ .$$

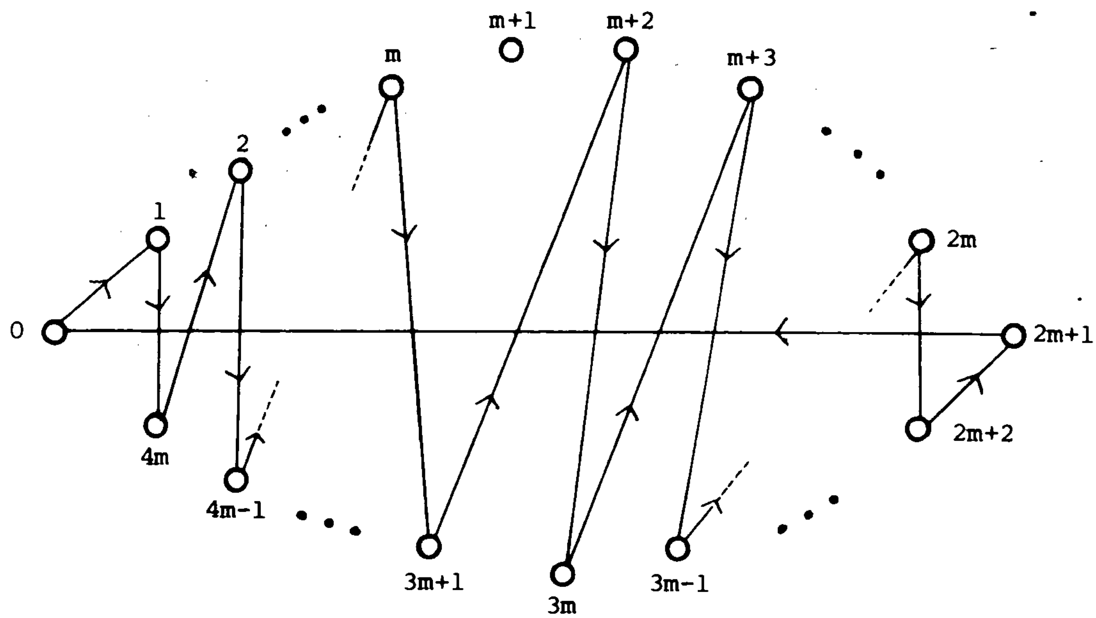


FIGURE 1(C)

We say that a vertex j is at *distance* d from a vertex i, $0 \le i, j \le 4m$, if and only if $j - i \equiv d$ and d is a residue 1, 2, ..., or 4m mod (4m + 1). We observe that the vertices in the neighbourhood of a vertex i , $0 \le i \le 4m$, are at distances 1, 2, 3, ..., and 4m from i . Moreover, if we consider the distances between the successive vertices of the directed cycle C , each of the distances 1, 2, ..., 4m occur exactly once. Hence, the directed cycles $\phi^k C$, k = 0, 1, 2, ..., 4m are all disjoint and cover all the arcs of DK_{4m+1}. This gives a desired decomposition of DK_{4m+1} .

Case 2: If $n \equiv 3 \pmod 4$, let $n = 4m+3$ for some positive integer m. Consider the directed cycle C' of length $4m+2$ (see Figure 1(D))

$$C' : 0,\ 1,\ 4m+2,\ 2,\ 4m+1,\ \ldots,\ 3m+3,\ m+1,\ 3m+1,\ m+2,\ 3m,\ \ldots$$
$$\ldots,\ 2m,\ 2m+2,\ 2m+1,\ 0\ .$$

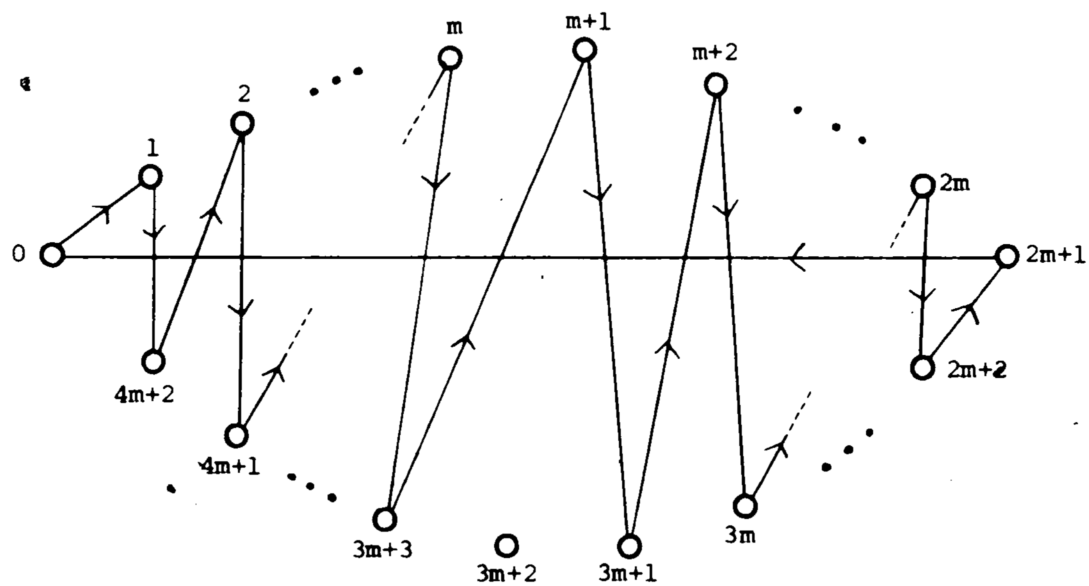


FIGURE 1(D)

As in the case of the directed cycle C, here also the distances $1, 2, \ldots, 4m+2$ occur exactly once among the distances between the successive vertices of the directed cycle C'. Hence the directed cycles $\phi^k C'$, $k = 0, 1, \ldots, 4m+2$ are all disjoint and cover all the arcs of DK_{4m+3}. Moreover, by an argument similar to the one we used in the case of C, the rotation under the permutation ϕ

preserves the orientation. Thus the directed cycles $\phi^k C'$, $k = 0, 1, 2, \ldots, 4m+2$, give a decomposition of $\dot{D}K_{4m+3}$. □

1.11 PROPOSITION: $C_n \mid {}_5K_n$ *for any odd positive integer* $n \geq 3$.

Proof: Let the five maximal independent subsets of $V({}_5K_n)$ be

$A = \{a_0, a_1, \ldots, a_{n-1}\}$, $B = \{b_0, b_1, \ldots, b_{n-1}\}$,

$C = \{c_0, c_1, \ldots, c_{n-1}\}$, $D = \{d_0, d_1, \ldots, d_{n-1}\}$,

$E = \{e_0, e_1, \ldots, e_{n-1}\}$.

On the 5n symbols in the union of the sets A, B, C, D and E , we define a permutation σ whose cycle representation is the product of n 5-cycles as given below.

$$\sigma = (a_0\ b_0\ c_0\ d_0\ e_0)(a_1\ b_1\ c_1\ d_1\ e_1) \ \ldots \ (a_{n-1}\ b_{n-1}\ c_{n-1}\ d_{n-1}\ e_{n-1})$$

Also, let τ be a permutation on the n numbers 0, 1, 2, ..., n-1 whose cycle representation is

$$\tau = (0\ 1\ 2\ \ldots\ n-1).$$

We shall show that the 10n n-cycles of the decomposition are given by

$$\sigma^i \tau^j Z \quad \text{and} \quad \sigma^i \tau^j Z' \qquad \ldots(1.11.1)$$

for $0 \le i \le 4$ and $0 \le j \le n-1$, where Z and Z' are two n-cycles constructed with the help of the directed cycles C and C' of length $n-1$ of Prop. 1.10, in the following manner. We recall that in the case $n = 4m+1$, m a positive integer, C is the cycle

$$0,\ 1,\ 4m,\ 2,\ 4m-1,\ 3,\ \ldots,\ m,\ 3m+1,\ m+2,\ 3m,\ m+3,\ \ldots,\ 2m,\ 2m+2,\ 2m+1,\ 0$$

and in the case $n = 4m+3$; m a positive integer, C' is the cycle

0, 1, 4m+2, 2, 4m[illegible], ..., 3m+3, m+1, 3m+1, m+2, 3m, ..., 2m, 2m+2, 2m+1, 0 .

If we regard the sets A, B, C, D and E as the vertices of a complete graph K_5, this complete graph can be decomposed into two disjoint 5-cycles

$$H_1 : A,\ B,\ C,\ D,\ E,\ A$$

and H_2 : A, C, E, B, D, A .

Let α and β be mappings of the set $\{0,1,\ldots,n-2\}$ into the set $\{0,1,\ldots,n-1\}$ defined as follows. For any $0 \le i \le n-2$

$\alpha(i)$ = label of the i^{th} vertex of C and

$\beta(i)$ = label of the i^{th} vertex of C' .

Let us first consider $n \not\equiv 1 \pmod{10}$. In each case given below the two n-cycles Z and Z' are constructed as follows. In the case of the cycle Z we first construct a path of length n by starting from the vertex a in A and moving through the sets A,B,C,D and E according to the 5-cycle H_1 while changing the subscripts according to the cycle C or C' . The cycle Z is then obtained by joining the end-vertices of this path. The cycle Z' is also constructed in a similar manner with the difference that this time we use the 5-cycle H_2 instead of H_1 .

<u>Case 1</u>: If $n \equiv 3 \pmod{20}$, the cycle Z is

$$a_0 = a_{\beta(0)},\ b_{\beta(1)},\ c_{\beta(2)},\ d_{\beta(3)},\ e_{\beta(4)},\ a_{\beta(5)},\ \ldots,\ a_{\beta(n-2)},\ b_{\beta(n-1)},$$
$$c_{\beta(n)} = c_0,\ a_0$$

and the cycle Z' is

$$a_0 = a_{\beta(0)},\ c_{\beta(1)},\ e_{\beta(2)},\ b_{\beta(3)},\ d_{\beta(4)},\ a_{\beta(5)},\ \ldots,\ a_{\beta(n-2)},\ c_{\beta(n-1)},$$
$$e_{\beta(n)} = e_0,\ a_0$$

Case 2: If $n \equiv 5 \pmod{20}$, the cycle Z is

$$a_0 = a_{\alpha(0)}, b_{\alpha(1)}, c_{\alpha(2)}, d_{\alpha(3)}, e_{\alpha(4)}, a_{\alpha(5)}, \ldots, e_{\alpha(n)} = e_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\alpha(0)}, c_{\alpha(1)}, e_{\alpha(2)}, b_{\alpha(3)}, d_{\alpha(4)}, a_{\alpha(5)}, \ldots, d_{\alpha(n)} = d_0, a_0 .$$

Case 3: If $n \equiv 7 \pmod{20}$, the cycle Z is

$$a_0 = a_{\beta(0)}, b_{\beta(1)}, c_{\beta(2)}, d_{\beta(3)}, e_{\beta(4)}, a_{\beta(5)}, \ldots, a_{\beta(n-1)}, b_{\beta(n)} = b_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\beta(0)}, c_{\beta(1)}, e_{\beta(2)}, b_{\beta(3)}, d_{\beta(4)}, a_{\beta(5)}, \ldots, a_{\beta(n-1)}, c_{\beta(n)} = c_0, a_0 .$$

Case 4: If $n \equiv 9 \pmod{20}$, the cycle Z is

$$a_0 = a_{\alpha(0)}, b_{\alpha(1)}, c_{\alpha(2)}, d_{\alpha(3)}, e_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-3)}, b_{\alpha(n-2)}, c_{\alpha(n-1)}, d_{\alpha(n)} = d_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\alpha(0)}, c_{\alpha(1)}, e_{\alpha(2)}, b_{\alpha(3)}, d_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-3)}, c_{\alpha(n-2)}, e_{\alpha(n-1)}, b_{\alpha(n)} = b_0, a_0 .$$

Case 5: If $n \equiv 13 \pmod{20}$, the cycle Z is

$$a_0 = a_{\alpha(0)}, b_{\alpha(1)}, c_{\alpha(2)}, d_{\alpha(3)}, e_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-2)}, b_{\alpha(n-1)}, c_{\alpha(n)} = c_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\alpha(0)}, c_{\alpha(1)}, e_{\alpha(2)}, b_{\alpha(3)}, d_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-2)}, c_{\alpha(n-1)}, e_{\alpha(n)} = e_0, a_0 .$$

<u>Case 6</u>: If $n \equiv 15 \pmod{20}$, the cycle Z is

$$a_0 = a_{\beta(0)}, b_{\beta(1)}, c_{\beta(2)}, d_{\beta(3)}, e_{\beta(4)}, a_{\beta(5)}, \ldots, e_{\beta(n)}, e_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\beta(0)}, c_{\beta(1)}, e_{\beta(2)}, b_{\beta(3)}, d_{\beta(4)}, a_{\beta(5)}, \ldots, d_{\beta(n)} = d_0, a_0 .$$

<u>Case 7</u>: If $n \equiv 17 \pmod{20}$, the cycle Z is

$$a_0 = a_{\alpha(0)}, b_{\alpha(1)}, c_{\alpha(2)}, d_{\alpha(3)}, e_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-1)}, b_{\alpha(n)} = b_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\alpha(0)}, c_{\alpha(1)}, e_{\alpha(2)}, b_{\alpha(3)}, d_{\alpha(4)}, a_{\alpha(5)}, \ldots, a_{\alpha(n-1)}, c_{\alpha(n)} = c_0, a_0 .$$

<u>Case 8</u>: If $n \equiv 19 \pmod{20}$, the cycle Z is

$$a_0 = a_{\beta(0)}, b_{\beta(1)}, c_{\beta(2)}, d_{\beta(3)}, e_{\beta(4)}, a_{\beta(5)}, \ldots, a_{\beta(n-3)}, b_{\beta(n-2)}, c_{\beta(n-1)}, d_{\beta(n)} = d_0, a_0$$

and the cycle Z' is

$$a_0 = a_{\beta(0)}, c_{\beta(1)}, e_{\beta(2)}, b_{\beta(3)}, d_{\beta(4)}, a_{\beta(5)}, \ldots, a_{\beta(n-3)}, c_{\beta(n-2)}, e_{\beta(n-1)}, b_{\beta(n)} = b_0, a_0$$

In the case $n \equiv 1 \pmod{10}$, if we use the same construction, the path of length n ends with the last vertex being a_0 and thus gives a cycle of length n-1. To avoid this we make a different jump at the next to last vertex of the path. Thus in the case of the cycle Z, we first construct a path of length n-1 moving along the 5-cycle H_1 as follows

$$a_0 = a_{\alpha(0)}, b_{\alpha(1)}, c_{\alpha(2)}, d_{\alpha(3)}, e_{\alpha(4)}, a_{\alpha(5)}, \ldots, e_{\alpha(n-2)} \quad \text{if } n \equiv 1 \pmod{20}$$

or

$$a_0 = a_{\beta(0)}, b_{\beta(1)}, c_{\beta(2)}, d_{\beta(3)}, e_{\beta(4)}, a_{\beta(5)}, \ldots, e_{\beta(n-2)} \quad \text{if } n \equiv 11 \pmod{20}$$

and then complete the cycle by joining $e_{\alpha(n-2)}$ or $e_{\beta(n-2)}$ to b_0 and b_0 to a_0 according as $n \equiv 1$ or $11 \pmod{20}$, respectively. The cycle Z' is constructed in the similar manner by first constructing a path of length n-1 moving along the 5-cycle H_2 as

$$a_0 = a_{\alpha(0)}, c_{\alpha(1)}, e_{\alpha(2)}, b_{\alpha(3)}, d_{\alpha(4)}, a_{\alpha(5)}, \ldots, d_{\alpha(n-2)} \quad \text{if } n \equiv 1 \pmod{20}$$

or

$$a_0 = a_{\beta(0)}, c_{\beta(1)}, e_{\beta(2)}, b_{\beta(3)}, d_{\beta(4)}, a_{\beta(5)}, \ldots, d_{\beta(n-2)} \quad \text{if } n \equiv 11 \pmod{20}$$

and then completing the cycle by joining $d_{\alpha(n-2)}$ or $d_{\beta(n-2)}$ to c_0 and c_0 to a_0 according as $n \equiv 1$ or $11 \pmod{20}$, respectively.

Noting that the permutation τ acts on the vertex subscripts and the permutation σ on the vertices, it follows then that

$$\sigma^i \tau^j Z \quad \text{and} \quad \sigma^i \tau^j Z'$$

for $0 \le i \le 4$ and $0 \le j \le n-1$, are all edge-disjoint, have length n and thus cover all the edges of K_n. This completes the proof.□

CHAPTER 2

In Prop. 1.8 of the previous chapter we have shown that for any odd positive integer $n \geq 3$, a complete graph K_n can be decomposed into cycles of length 3 and 5. For $n \equiv 1$ or $3 \pmod 6$ the result follows immediately from the existence of Steiner triple systems. For the case $n \equiv 5 \pmod 6$, in the proof, using Spencer's [35] result, we show that K_n can be decomposed into either one K_5 and the rest triangles or two disjoint C_5's and the rest triangles. Since K_5 itself is a union of two disjoint C_5's, it is natural to ask: For $n \equiv 5 \pmod 6$, is it possible to decompose K_n into exactly one K_5 and the rest triangles? In this chapter we answer this question in the affirmative. Moreover, such a decomposition is a pairwise balanced design, which gives another direction to look at this problem.

Pairwise balanced designs were introduced by R.C. Bose and S.S. Shrikhande [9] and applied by them [9,10] to construct sets of pairwise orthogonal Latin squares and thus prove the falsity of an old conjecture of Euler [13] on Latin squares.

DEF.: *A pairwise balanced design* $(v;k_1,k_2,\ldots,k_m;\lambda)$ *of index* λ is an arrangement of v elements into subsets (called *blocks*), such that each block contains either $k_1,k_2,\ldots,$ or k_m distinct elements $(k_i \leq v)$, and such that every pair of distinct elements occurs in precisely λ blocks of the design.

Unless the size of the blocks is important, we shall denote a pairwise balanced design of index λ, by λ-PBD or simply by PBD. In graph theory the existence of a pairwise balanced design $(v;k_1,k_2,\ldots,k_m;1)$ of index unity is equivalent to partitioning the edge set of K_v so that the resulting subgraphs are isomorphic to a complete graph $K_{k_1}, K_{k_2}, \ldots,$ or K_{k_m}. In geometry 1-PBD's have also been known as *linear spaces* (in which case the blocks are called *lines*). A λ-PBD on v elements in which all blocks have the same size k is traditionally known as a BIBD(v,k,λ) (*balanced incomplete block design: see* [14]). Thus, pairwise balanced designs of index λ are generalizations of BIBD(v,k,λ)-designs.

R.M. Wilson has considered the problem of existence of balanced incomplete block designs and pairwise balanced designs. He has shown [37,38,40] that the necessary conditions for the existence of BIBD's or that of PBD's on v objects are sufficient for sufficiently large values of v. He calls such conditions 'necessary and "asymptotically sufficient" conditions'. The problem of constructing balanced incomplete block designs or pairwise balanced designs has generated a great deal of interest. Rosa, Hell, Huang, Bermond, Mendelsohn and many others have considered the problem of constructing these designs for different values of the parameters v, k and λ.

Spencer's result [35] that for $n \equiv 5 \pmod 6$, the edge set of K_n can be partitioned into triangles and one C_4, the cycle of length 4, corresponds to the existence of a pairwise balanced design if each edge of C_4 is taken as a block of size two. However, if trivial blocks of size two are not allowed, the decomposition is not a pairwise balanced design and simple arithmetic shows that the best one can obtain is one block of size 5 and the rest triangles. Thus the main result of this chapter is equivalent to showing the existence of a PBD(n;3,5;1) for any odd positive integer n. We have learned that the existence of this design also follows as a special case from a result of C. Huang, E. Mendelsohn and A. Roşa which shall appear in a paper under preparation. However, the result proved here shows something more than the existence of a PBD(n;3,5;1). It shows that there exists a PBD(n;3,5;1) in which the blocks of size 5 are at most one in number. This result as such also follows from a result due to R. Wilson [39]. His proof is based on group theoretic concepts. Here we give an independent proof which is consistent with the unified proof techniques used in obtaining other results of this dissertation.

2.1 LEMMA: *If* K_m, m *odd can be decomposed into one* K_5 *and* C_3*'s, then so can* $tK_{m+1} + K_m$ *where* t *is any non-negative integer.*

Proof. We write

$$tK_{m+1} + K_m = tK_{m+1} \cup K_m \cup tK_{m,m+1}$$

where the vertex set of each of the complete bipartite graphs $K_{m,m+1}$ is chosen appropriately. Since K_m can be decomposed into one K_5 and C_3's, it is enough to show that the graph

$$tK_{m+1} \cup tK_{m,m+1} = t(K_{m+1} \cup K_{m,m+1})$$

can be decomposed into C_3's. Moreover, since this graph is a union of t copies of the graph

$$K_{m+1} \cup K_{m,m+1} \; ,$$

it is sufficient to show that $K_{m+1} \cup K_{m,m+1}$ can be decomposed into C_3's.

Since $m+1$ is even, we know that K_{m+1} can be decomposed into m 1-factors [15, Thm. 9.1]. Take one such 1-factor of K_{m+1} and one vertex, say u of $\overline{K}_m$. The edges of this 1-factor of K_{m+1} together with the edges of $K_{m,m+1}$ which are incident with the vertices of K_{m+1} and the vertex u of $\overline{K}_m$ give triangles. This is shown in the Figure 2(A).

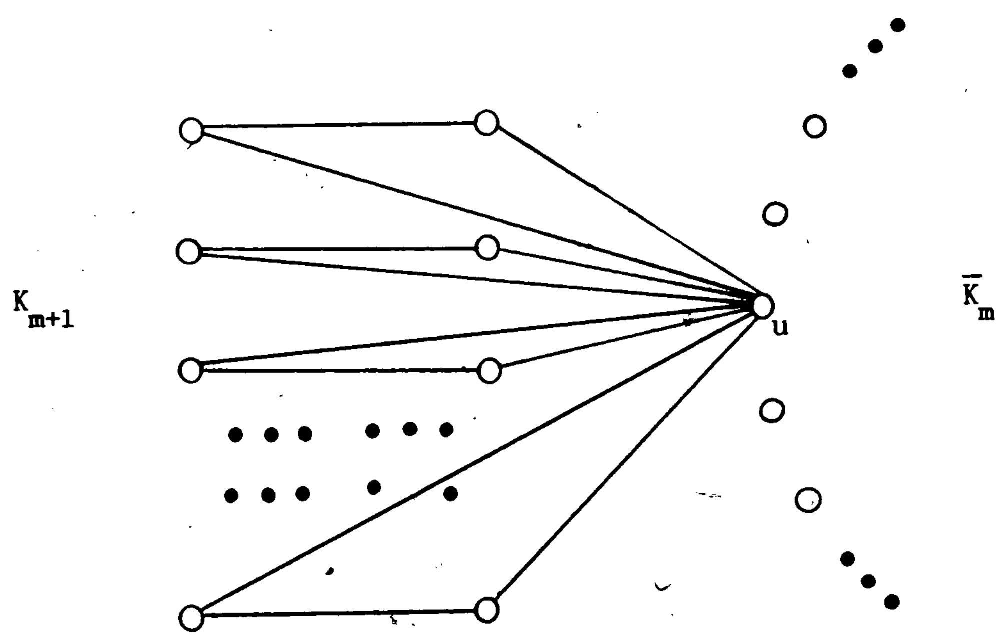


FIGURE 2(A)

Since the number of 1-factors of K_{m+1} equals m, we can associate each 1-factor with a distinct vertex of $\overline{K}_m$. From the above it then follows that $C_3 | K_m \cup K_{m+1,m}$ and hence the result.□

2.2 **THEOREM:** *If* $n \equiv 0$ *or* $4 \pmod 6$, K_{6n+5} *can be decomposed into one* K_5 *and* C_3*'s.*

<u>Proof</u>: We write

$$K_{6n+5} = (nK_6 + K_5) \cup {}_nK_6 ,$$

where the vertex set of the complete multipartite graph ${}_nK_6$ is chosen appropriately. By Lemma 2.1, $nK_6 + K_5$ can be decomposed into one K_5 and C_3's. Also, since $n \equiv 0$ or $4 \pmod 6$, ${}_nK_6$ can be decomposed into C_3's by Prop. 1.4.□

To prove the result for K_{6n+5}, $n \equiv 2 \pmod 6$ we need the following lemmas.

2.3 **LEMMA:** K_{17} *can be decomposed into one* K_5 *and* C_3*'s.*

<u>Proof</u>: We write $K_{17} = K_5 + K_{12}$. We know K_{12} can be decomposed into eleven 1-factors. Moreover, K_{12} can be regarded as a circulant graph with symbol $\{\pm1,\pm2,\pm3,\pm4,\pm5,6\}$. The circulant subgraph with symbol $\{\pm1,\pm2,6\}$ is the union of five 1-factors of K_{12}, which together with the edges joining the vertices of K_{12} and the vertices

of K_5 give C_3's. The union of the remaining six 1-factors of K_{12} is a circulant subgraph with symbol $\{\pm3,\pm4,\pm5\}$. We now show that it can be decomposed into C_3's. Let $V(K_{12}) = \{v_1,v_2,\ldots,v_{12}\}$, let $C = \{v_1,v_4,v_8\}$, and let ϕ be the permutation whose cycle decomposition is $(v_1\ v_2\ \ldots\ v_{12})$. Then C and $\phi^k C$ $(1 \le k \le 11)$ are the disjoint 3-cycles of the decomposition.□

2.4 **LEMMA**: *Let* K_{16} *be the complete graph with*

$$V(K_{16}) = \{u_i : 1 \le i \le 16\} ,$$

and $I = \{u_i u_{i+8} : 1 \le i \le 8\}$ *a 1-factor of* K_{16} . *Then* $K_{16} - I$ *can be decomposed into four* K_5*'s and* C_3*'s.*

Proof: A desired decomposition of $K_{16} - I$ can be constructed as follows: Let $A = \{u_1,u_4,u_7,u_{10},u_{13}\}$ and ϕ be the permutation whose cycle decomposition is $(u_1\ u_2\ .\ .\ .\ u_{16})$. Then A, $\phi^4 A$, $\phi^8 A$ and $\phi^{12} A$ are the vertex sets of the four K_5's. The twenty-four triangles in the remainder of the graph are given by

$$C_1 = \{u_1,u_2,u_6\} , \qquad C_4 = \{u_2,u_4,u_8\} ,$$

$$C_2 = \{u_1,u_3,u_{12}\} , \qquad C_5 = \{u_3,u_8,u_{10}\} ,$$

$$C_3 = \{u_1,u_{15},u_{16}\} , \qquad C_6 = \{u_9,u_{10},u_{14}\} ,$$

and

$$\phi^4 C_i \ , \ \phi^8 C_i \ , \ \phi^{12} C_i \qquad \text{for} \qquad i = 2,3,4,5 \ ,$$

$$\phi C_i \ , \ \phi^4 C_i \ , \ \phi^5 C_i \qquad \text{for} \qquad i = 1,6. \square$$

2.5 **LEMMA**: $C_3 \mid {}_8K_{6x}$ *for any positive integer* x .

Proof: Using the method used in the proof of Prop. 1.4., we associate with ${}_8K_{6x}$ a $K_{16} - I$ where the vertices of K_{16} are independent subsets of $V({}_8K_{6x})$ each of cardinality $3x$ and I is a 1-factor of K_{16} . Moreover, by construction this 1-factor is the same as that of K_{16} in Lemma 2.4. Hence, by Lemma 2.4 $K_{16} - I$ can be decomposed into C_3's and four K_5's which amounts to the fact that ${}_8K_{6x}$ can be decomposed into four ${}_5K_{3x}$'s and the rest ${}_3K_{3x}$'s. The result then follows from Prop. 1.1 and Prop. 1.2.□

2.6 **LEMMA**: $K_{6 \cdot 3^n - 1}$ *can be decomposed into one* K_5 *and* C_3*'s* .

Proof: We shall use induction on n . The result is true for $n = 0$ and 1 using Lemma 2.3. Suppose that $n \geq 2$ and that the result is true for all $k < n$. Consider $K_{6 \cdot 3^n - 1}$ and write

$$K_{6 \cdot 3^n - 1} = K_{8 \cdot 6 \cdot 3^{n-2} + 6 \cdot 3^{n-2} - 1}$$

$$= (8K_{6 \cdot 3^{n-2}} + K_{6 \cdot 3^{n-2} - 1}) \cup {}_8K_{6 \cdot 3^{n-2}} \ ,$$

where the vertex set of the complete multipartite graph is chosen appropriately. By Lemma 2.5, ${}_{8}K_{6\cdot 3^{n-2}}$ can be decomposed into C_3's. By the induction hypothesis $K_{6\cdot 3^{n-2}-1}$ is decomposable into one K_5 and C_3's and since $6\cdot 3^{n-2} - 1$ is odd, by Lemma 2.1 $8K_{6\cdot 3^{n-2}} + K_{6\cdot 3^{n-2}-1}$ is decomposible into one K_5 and C_3's.□

2.7 THEOREM: *If* $n \equiv 2$ (*mod* 6), K_{6n+5} *can be decomposed into one* K_5 *and* C_3*'s* .

<u>Proof</u>: Let $n = 6m_1 + 2$ for some non-negative integer m_1 . We write

$$K_{6n+5} = K_{36m_1+17} = (2m_1K_{18} + K_{17}) \cup {}_{2m_1}K_{18} .$$

By Lemma 2.3 and Lemma 2.1, $2m_1K_{18} + K_{17}$ can be decomposed into one K_5 and C_3's. Also, if $2m_1 \equiv 0$ or 4 (mod 6), by Prop. 1.4 ${}_{2m_1}K_{18}$ can be decomposed into all C_3's, and hence the result. In case $2m_1 \equiv 2$ (mod 6), we write $2m_1 = 6m_2 + 2$ for some non-negative integer m_2 and

$$K_{6n+5} = K_{54\cdot 2m_2+53} = (2m_2K_{54} + K_{53}) \cup {}_{2m_2}K_{54} .$$

Since $53 = 6 \cdot 3^2-1$, by Lemma 2.6, K_{53} can be decomposed into one K_5 and C_3's and, hence, by Lemma 2.1, $2m_2K_{54} + K_{53}$ also. Moreover,

$2m_2 K_{54}$ can be decomposed into C_3's by Prop. 1.4, provided $2m_2 \equiv 0$ or 4 (mod 6). In case $2m_2 \equiv 2$ (mod 6) we write $2m_2 = 6m_3 + 2$ for some non-negative integer m_3 and

$$K_{6n+5} = K_{162 \cdot 2m_3 + 161} = (2m_3 K_{162} + K_{161}) \cup 2m_3 K_{162} .$$

Repeating the above argument, at the p^{th} stage we get

$$K_{6n+5} = K_{6 \cdot 3^p \cdot 2m_p + 6 \cdot 3^p - 1} = (2m_p K_{6 \cdot 3^p} + K_{6 \cdot 3^p - 1}) \cup 2m_p K_{6 \cdot 3^p} ,$$

and as long as $2m_p \equiv 0$ or 4 (mod 6) we get the desired decomposition of K_{6n+5} into one K_5 and C_3's. In case $2m_p \equiv 2$ (mod 6) we write $2m_p = 6m_{p+1} + 2$ and

$$K_{6n+5} = K_{6 \cdot 3^{p+1} \cdot 2m_{p+1} + 6 \cdot 3^{p+1} - 1}$$

$$= (2m_{p+1} K_{6 \cdot 3^{p+1}} + K_{6 \cdot 3^{p+1} - 1}) \cup 2m_{p+1} K_{6 \cdot 3^{p+1}} .$$

Continuing like this, at the final stage, say the t^{th}, we have

$$K_{6n+5} = (2m_t K_{6 \cdot 3^t} + K_{6 \cdot 3^t - 1}) \cup 2m_t K_{6 \cdot 3^t} ,$$

where $2m_t$ is either 0, 2 or 4 . In case $2m_t = 4$, Lemmas 2.6, 2.1 and Prop. 1.4 give the result.

If $2m_t = 0$ or 2 then K_{6n+5} is $K_{6\cdot 3^t-1}$ or $K_{6\cdot 3^{t+1}-1}$, respectively, and the result then follows from Lemma 2.6.□

2.8 THEOREM: K_{6n+5}, n *odd, can be decomposed into one* K_5 *and* C_3*'s.*

Proof: We shall use induction on n. For $n = 1$ we write

$$K_{11} = K_{6+5} = K_6 + K_5 \quad ,$$

and the result follows from Lemma 2.1. Suppose that the result is true for all odd $k < n$. Consider K_{6n+5} and write

$$K_{6n+5} = (nK_6 + K_5) \cup {}_nK_6 \; .$$

By Lemma 2.1, $nK_6 + K_5$ can be decomposed into one K_5 and C_3's. Let $V_1, V_2, \ldots, V_n$ be n maximal independent subsets of $V({}_nK_6)$. With ${}_nK_6$ we associate a complete graph K_n with

$$V(K_n) = \{V_1, V_2, \; . \; . \; . \; , V_n\} \quad ,$$

and for all $1 \le i, j \le n$, the edge V_iV_j corresponds to the set of all edges of the complete bipartite subgraph $K_{|V_i|,|V_j|}$ of ${}_nK_6$.

Now if $n \equiv 1$ or $3 \pmod 6$, K_n can be decomposed into C_3's and hence, ${}_nK_6$ can be decomposed into ${}_3K_6$'s and the result then follows from Prop. 1.1. If $n \equiv 5 \pmod 6$, let $n = 6m + 5$ for some m. We claim K_n can be decomposed into one K_5 and C_3's. In case m is even, our claim follows from Theorem 2.2 and Theorem 2.7. If m is odd, the claim follows from the induction hypothesis, as $m < n$. Thus the graph ${}_nK_6$ can be decomposed into one ${}_5K_6$ and ${}_3K_6$'s. The result then follows from Prop. 1.1 and Prop. 1.2.□

Finally the results of Theorems 2.2, 2.7 and 2.8 can be put together into a single theorem.

2.9 THEOREM: *If* $n \equiv 5 \pmod{6}$, $n \geq 5$, K_n *can be decomposed into one* K_5 *and* C_3*'s.*

2.10 COROLLARY: *For any odd integer* $n \geq 3$, K_n *can be decomposed into all triangles with the exception of at most one* K_5.

<u>Proof</u>: For $n \equiv 1$ or $3 \pmod{6}$, we know that a Steiner triple system exists, which is equivalent to the fact that K_n can be decomposed into triangles. In case $n \equiv 5 \pmod{6}$ the result follows from Theorem 2.9.□

As we have stated in the beginning of the chapter, Corollary 2.10 can be restated in the following way.

2.11 COROLLARY: *For any odd integer* $n \geq 3$, *there exists a pairwise balanced design* $(n;3,5;1)$.

2.12 COROLLARY: *For* $n \geq 10$ *and* $n \equiv 4 \pmod 6$, $K_n - I$ *can be decomposed into one* K_5 *and the rest* C_3*'s where* I *denotes a 1-factor of* K_n.

Proof: Let $u \notin V(K_n)$ be any point. The graph $K_n + u$ is a complete graph K_{n+1} with $n + 1 \equiv 5 \pmod 6$ and hence by Theorem 2.9 can be decomposed into one K_5 and the rest C_3's. We can relabel the vertices of K_{n+1} so that u is not one of the vertices of K_5. This implies that the graph $K_n = K_{n+1} - u$ can be decomposed into one 1-factor, one K_5 and the rest C_3's.□

The results of the Prop. 1.3, Corollary 2.12 and the fact that K_5 is a union of two disjoint C_5's, give the following result.

2.13 COROLLARY: *For any even* $n \geq 6$, $K_n - I$ *can be decomposed into 3-cycles and 5-cycles where* I *denotes a 1-factor of* K_n.

In Prop. 1.11 we proved that for any odd integer $n \geq 3$, $C_n \mid {}_5K_n$. Cor. 2.10 enables us to extend this result.

2.14 THEOREM: $C_n \mid {}_mK_n$ *for any odd integers* $m, n \geq 3$.

Proof: In case $m \geq 3$, let $V_1, V_2, \ldots, V_m$ be m maximal independent subsets of $V({}_mK_n)$ and $|V_i| = n$, $1 \leq i \leq m$. With ${}_mK_n$ we associate a complete graph K_m as follows:

$$V(K_m) = \{V_1, V_2, \ldots, V_m\}$$

and an edge incident with the vertices V_i and V_j , $1 \leq i, j \leq m$, corresponds to all the edges of the complete bipartite subgraph $K_{|V_i|,|V_j|}$ of the original graph ${}_mK_n$. Since m is odd by Cor. 2.10 K_m can be decomposed into at most one K_5 and the rest triangles, which amounts to the fact that the graph ${}_mK_n$ can be decomposed into at most one ${}_5K_n$ and the rest ${}_3K_n$. Sotteau [33] has shown that $C_n | {}_3K_n$ for any odd n . That $C_n | {}_5K_n$ follows from Prop. 1.11. This completes the proof.□

CHAPTER 3

In the beginning of Chapter 2 we have described BIBD's (*balanced incomplete block designs*). These designs are special cases of G-designs introduced by Hell and Rosa [19].

DEF.: Let $K_n^{(\lambda)}$ denote a graph with n vertices and any two distinct vertices joined by exactly λ edges. An (n,k,λ) *G-design* is a partition of the edge set of the graph $K_n^{(\lambda)}$, so that the resulting subgraphs are isomorphic to a given graph G where $|V(G)| = k$.

In the particular case where G is the complete graph K_k, an (n,k,λ) K_k-design is nothing but a BIBD (n,k,λ). For a short account of G-designs we refer to a survey article [7] by Bermond and Sotteau.

In case the graph G is C_k, a cycle of length k, an (n,k,λ) C_k-design is also called a BCD (n,k,λ) (*balanced cycle design,* see [20]).

Similar definitions hold for directed graphs. If G is a directed graph with k vertices, an (n,k,λ) G-*design* is a partitioning of the *arcs* of $DK_n^{(\lambda)}$ so that the resulting directed subgraphs are isomorphic to G. In case G is a circuit C_k^*, a C_k^*-design is called a *balanced circuit design*. A number of people have considered the problem of existence of BCD's for directed and undirected graphs. We mention here a few of them: Hung and

Mendelsohn [21] , Schönheim [32] , Merriell [26] , Bermond [3] , Bermond and Faber [5] , Sotteau [33], Bermond and Sotteau [6] and Bermond, Huang and Sotteau [4], Rosa [29] and Kotzig [23].

For the existence of a BCD (n,k,λ) it is necessary that $n \geq k$, k divides the number of edges of λK_n , that is, $k \mid \frac{\lambda n(n-1)}{2}$ and the degree $\lambda(n-1)$ of each vertex of K_n is even. Thus we have the following result.

3.1 THEOREM: *The necessary conditions for the existence of a* BCD (n,k,λ) *are*

(i) $n \geq k$,

(ii) $\lambda n(n-1) \equiv 0 \ (mod\ 2k)$ *and*

(iii) $\lambda(n-1) \equiv 0 \ (mod\ 2)$.

We wish to know whether the conditions (i), (ii) and (iii) of Theorem 3.1 are also sufficient for the existence of a BCD (n,k,λ).

In this Chapter we consider the case for k even. The most complete results in this case are due to Bermond, Huang and Sotteau [4] . They have shown that the necessary conditions of Theorem 3.1 are also sufficient for even values of k satisfying $4 \leq k \leq 16$.

We shall extend the result for an infinite number of even positive integers $k \geq 4$ where we restrict ourselves to the case $\lambda = 1$ and prove that for $k = 2p^{\alpha}$ where p is a prime and α is a positive integer, a BCD $(n,k,1)$ exists if and only if the conditions (i), (ii) and (iii) of Theorem 3.1 hold.

We shall use the following result of Sotteau [34].

3.2 LEMMA: $K_{m,n}$ *can be decomposed into cycles of length* k, k *even, if and only if* m *and* n *are even,* $m \geq \frac{k}{2}$, $n \geq \frac{k}{2}$ *and* k *divides* mn.

In case $k \equiv 0 \pmod 4$, Kotzig [23] has shown that K_{2mk+1} can be decomposed into C_k's for all positive integers m. A similar result has been obtained by Rosa [29] for $k \equiv 2 \pmod 4$. We give here an independent proof to cover both the cases together.

3.3 LEMMA: *For any even positive integer* k *and any positive integer* m, $C_k | K_{2mk+1}$.

Proof: We shall use induction on m. For $m = 1$, we use a construction to show that the result holds. In case $k \equiv 0 \pmod 4$, let $k = 2\ell$, ℓ even. Consider a complete graph K_{2k+1} with $V(K_{2k+1}) = \{u_0, u_1, \ldots, u_\ell, u_{\ell+1}, \ldots, u_{2k}\}$. Let ϕ be the permutation with cycle representation $(u_0\ u_1\ u_2\ \ldots\ u_{2k})$. Consider a cycle C of length k as given below (see Figure 3(A))

$$C: u_0, u_1, u_{2k-1}, u_3, \ldots, u_{2k-\ell+3}, u_{\ell-1}, u_{k+\ell+1}, u_{k-\ell+3}, \ldots$$

$$\ldots, u_{k+5}, u_{k-1}, u_{k+3}, u_{k+1}, u_0$$

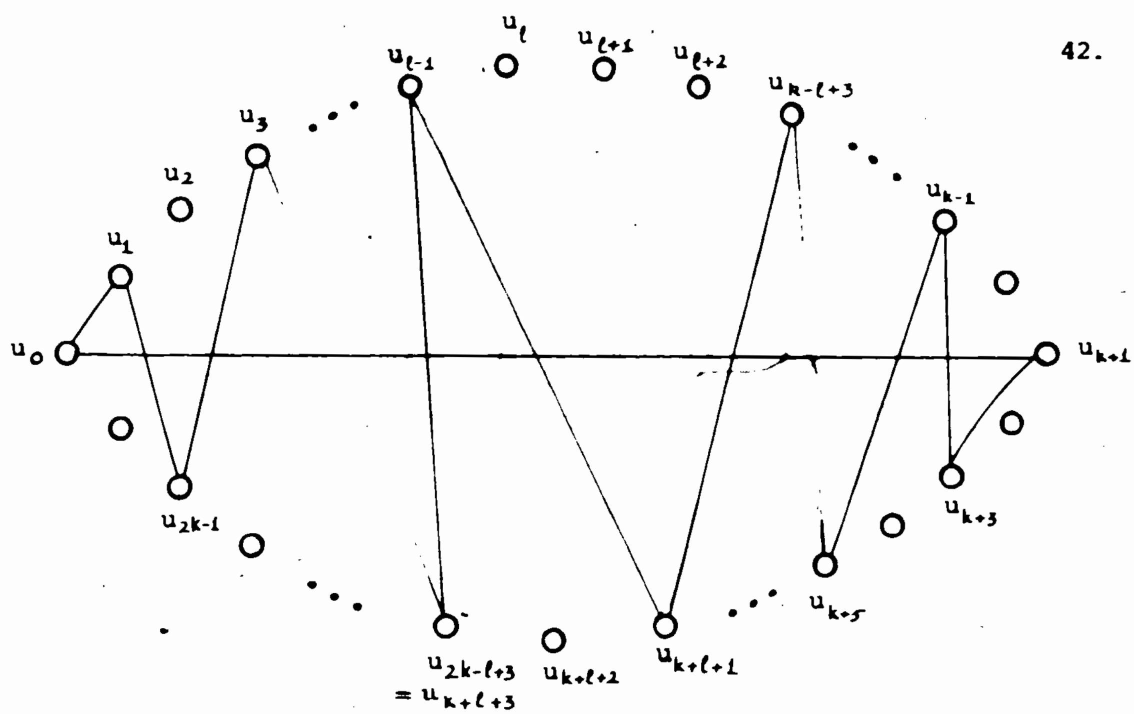


FIGURE 3(A)

If we say that the two vertices u_i and u_j are at a *distance* d if and only if $|i-j| = d$, we observe that the vertices in the neighbourhood in K_{2k+1} of a vertex u_i for all i are at distances $1,2,3,\ldots,k$ from u_i. Moreover, if we consider the distances between the successive vertices of the cycle C, the distances $1,2,\ldots,k$ occur exactly once. Hence the cycles $\phi^j C$, $j = 0,1,\ldots,2k$ are all disjoint and cover all the edges of K_{2k+1}.

In case $k \equiv 2 \pmod 4$, that is, $k = 2\ell$, ℓ odd, $\ell \geq 3$, consider the cycle C' given below (see Figure 3(B))

$$C': u_0, u_1, u_4, u_{2k}, u_6, u_{2k-2}, \ldots, u_{2k-\ell+5}, u_{\ell+1}, u_{k+\ell+1}, u_{k-\ell+3}, \ldots,$$

$$u_{k+6}, u_{k-2}, u_{k+4}, u_k, u_{k+2}, u_0$$

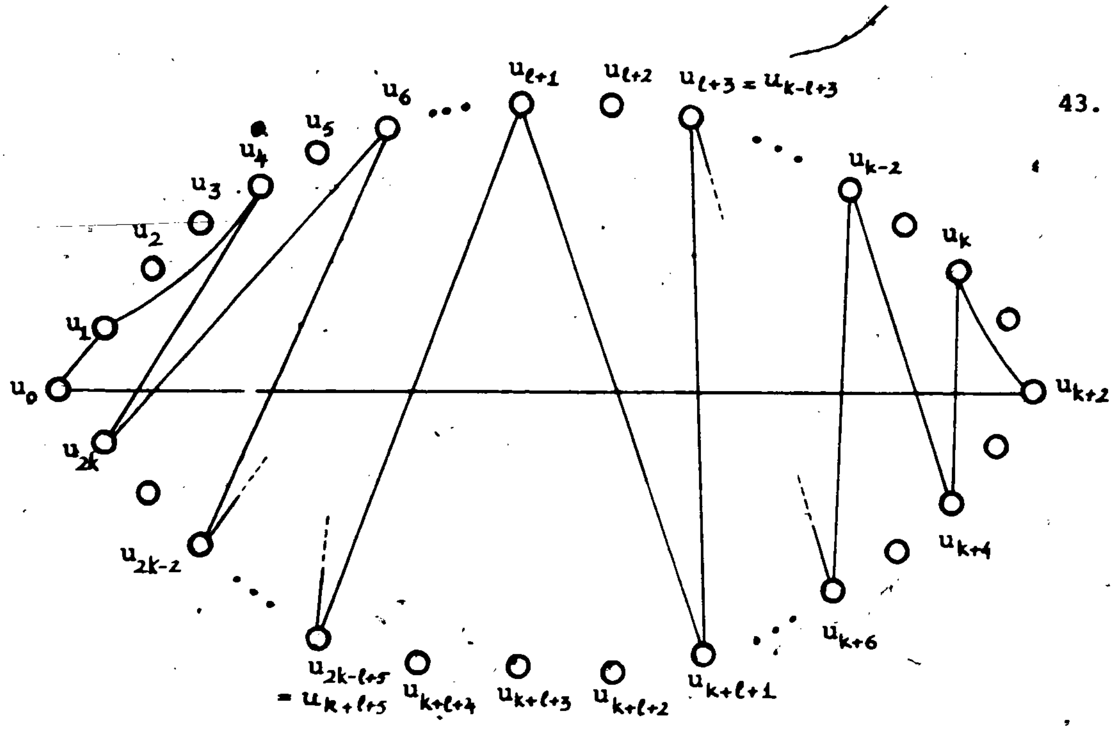


FIGURE 3(B)

Here again among the distances between vertices of the cycle C', the distances $1,2,\ldots,k$ occur exactly once. Hence the cycles $\phi^j C'$, $j = 0,1,2,\ldots,2k$ are all disjoint and cover all the edges of K_{2k+1}.

This shows that the result is true for $m = 1$. Next assume that the result is true for m, that is, K_{2mk+1} can be decomposed into C_k's. Consider the graph $K_{2(m+1)k+1}$. We write it as

$$K_{2(m+1)k+1} = K_{2mk+2k+1} = (K_{2mk} + K_1) \cup (K_{2k} + K_1) \cup K_{2k,2mk}$$

$$= K_{2mk+1} \cup K_{2k+1} \cup K_{2k,2mk} ,$$

where the vertex set of each graph is chosen appropriately. Since K_{2mk+1}, by the induction hypothesis, and K_{2k+1}, by the first part for $m = 1$, can be decomposed into cycles C_k, it is enough to show that $C_k | K_{2k,2mk}$. Since $2k$ and $2mk$ are both even and each is greater than $\frac{k}{2}$ and also $2k \cdot 2mk \equiv 0 \pmod{k}$, the result then follows from Lemma 3.2.□

3.4 LEMMA: *If* $C_k | K_r$, k *even and* r *odd, then* $C_k | K_{2mk+r}$ *for all positive integer* m.

<u>Proof</u>: We write

$$K_{2mk+r} = K_{2mk+(r-1)+1} = (K_{2mk} + K_1) \cup (K_{r-1} + K_1) \cup K_{2mk,(r-1)}$$

$$= K_{2mk+1} \cup K_r \cup K_{2mk,(r-1)} ,$$

where the vertex set of each graph is chosen appropriately. Now K_{2mk+1} can be decomposed into C_k's by Lemma 3.3. Also C_k divides K_r by the hypothesis of the Lemma. Thus it is enough to show that C_k divides $K_{2mk,(r-1)}$, which follows from Lemma 3.2, as $2mk$ and $r-1$ are even, both are greater than $\frac{k}{2}$ and k divides $2mk(r-1)$.□

The following result follows immediately from Lemma 3.4.

3.5 **THEOREM:** *Given* k *even let* $n = 2mk+r$ *where* m *is a positive integer and* $0 < r < 2k$. *Also, let* n, k *and* $\lambda = 1$ *satisfy conditions* (i), (ii) *and* (iii) *of Theorem* 3.1. *Then there exists a BCD* $(n,k,1)$ *if there exists*

(a) *a BCD* $(r,k,1)$ *if* $k < r < 2k$ *or*

(b) *a BCD* $(2k+r,k,1)$ *if* $0 < r < k$.

Now and henceforth let $n = 2mk+r$, $0 < r < 2k$, for some integer m . Furthermore, let $k = 2 \cdot p^{\alpha}$ where p is a prime and α is a positive integer. The conditions (ii) and (iii) of Theorem 3.1 are equivalent to

$$r(r-1) \equiv 0 (\text{mod } 2k) \quad \text{and} \quad r-1 \equiv 0 (\text{mod } 2) \quad \ldots(*)$$

In case $p = 2$ we have the following theorem.

3.6 **THEOREM:** *For any integer* n *and* $k = 2^{\alpha}$, *satisfying the conditions* (i), (ii) *and* (iii) *of Theorem* 3.1 *with* $\lambda = 1$ *there exists a BCD* $(n, 2^{\alpha}, 1)$ *where* α *is a positive integer.*

<u>Proof:</u> The conditions (*) in this case become

$$r(r-1) \equiv 0 (\text{mod } 2^{\alpha+1}) \quad \text{and} \quad r-1 \equiv 0 (\text{mod } 2).$$

That is, $r \equiv 1 (\text{mod } 2^{\alpha+1})$. Since $0 < r < 2^{\alpha+1}$ holds, it follows that $r = 1$. Thus in view of Theorem 3.5, it is enough to show that there exists a BCD $(2^{\alpha+1}+1, 2^{\alpha}, 1)$. The result then follows from Lemma 3.3.□

In case $k = 2\cdot p^{\alpha}$ where p is an odd prime and α is a positive integer, the conditions (*) become

$$r(r-1) \equiv 0 (\text{mod } 4p^{\alpha}) \quad \text{and} \quad r-1 \equiv 0 (\text{mod } 2).$$

That is either $r \equiv 1(\text{mod } p^{\alpha})$ and $r \equiv 1(\text{mod } 4)$

or $r \equiv 0(\text{mod } p^{\alpha})$ and $r \equiv 1(\text{mod } 4)$.

But $r \equiv 1(\text{mod } p^{\alpha})$ and $r \equiv 1(\text{mod } 4)$ imply that $r \equiv 1(\text{mod } 4\cdot p^{\alpha})$. Since $0 < r < 4p^{\alpha}$ we have $r = 1$. In view of Theorem 3.5, it suffices to show the existence of a BCD $(4\cdot p^{\alpha}+1, 2\cdot p^{\alpha}, 1)$ which follows from Lemma 3.3.

If $r \equiv 0(\text{mod } p^{\alpha})$ and $r \equiv 1(\text{mod } 4)$ we claim:

$$r = 3p^{\alpha} \quad \text{if} \quad p^{\alpha} \equiv 3(\text{mod } 4)$$

$$\text{and} \quad r = p^{\alpha} \quad \text{if} \quad p^{\alpha} \equiv 1(\text{mod } 4).$$

To show this let $r = q\cdot p^{\alpha}$ for some integer q, so that $r \equiv 1(\text{mod } 4)$ implies that $r-1 = q\cdot p^{\alpha}-1 \equiv 0(\text{mod } 4)$. Hence,

$$q \equiv 3(\text{mod } 4) \quad \text{if} \quad p^{\alpha} \equiv 3(\text{mod } 4)$$

$$\text{or} \quad q \equiv 1(\text{mod } 4) \quad \text{if} \quad p^{\alpha} \equiv 1(\text{mod } 4).$$

But $0 < r < 4\cdot p^{\alpha}$ implies that $q = 3$ if $p^{\alpha} \equiv 3(\text{mod } 4)$ or $q \equiv 1$ if $p^{\alpha} \equiv 1(\text{mod } 4)$.

In view of Theorem 3.5 and the above discussion, to prove that the conditions (i), (ii) and (iii) of Theorem 3.1 are necessary and sufficient for the existence of a BCD $(n, 2\cdot p^{\alpha}, 1)$ where p is an odd prime and α a positive integer, it is enough to show the existence of a BCD $(3\cdot p^{\alpha}, 2\cdot p^{\alpha}, 1)$ if $p^{\alpha} \equiv 3 \pmod 4$ and that of a BCD $(5\cdot p^{\alpha}, 2\cdot p^{\alpha}, 1)$ if $p^{\alpha} \equiv 1 \pmod 4$.

Thus we have proved the following lemma.

3.7 **LEMMA:** *Let* n *be a positive integer where* $n, 2\cdot p^{\alpha}$ *and* $\lambda = 1$ *satisfy the conditions* (i), (ii) *and* (iii) *of Theorem* 3.1. *Then there exists a BCD* $(n, 2\cdot p^{\alpha}, 1)$ *for* p *an odd prime and* α *a positive integer if there exists*

(a) *a BCD* $(3\cdot p^{\alpha}, 2\cdot p^{\alpha}, 1)$ *if* $p^{\alpha} \equiv 3 \pmod 4$ *or*

(b) *a BCD* $(5\cdot p^{\alpha}, 2\cdot p^{\alpha}, 1)$ *if* $p^{\alpha} \equiv 1 \pmod 4$.

3.8 **LEMMA:** *There exists a BCD* $(3\cdot p^{\alpha}, 2\cdot p^{\alpha}, 1)$ *where* p *is a prime,* α *is a positive integer and* $p^{\alpha} \equiv 3 \pmod 4$.

Proof: Partition the vertex set $V(K_{3\cdot p^{\alpha}})$ into p^{α} subsets each of size three. Label them with

$$T_0: \{u_{00}, u_{01}, u_{02}\},\ T_1: \{u_{10}, u_{11}, u_{12}\},\ \ldots,\ T_{p^{\alpha}-1}: \{u_{p^{\alpha}-1,0}, u_{p^{\alpha}-1,1}, u_{p^{\alpha}-1,2}\}.$$

With $K_{3\cdot p^{\alpha}}$ we associate a complete graph $K_{p^{\alpha}}$ whose vertices are the vertex sets $T_0, T_1, \ldots, T_{p^{\alpha}-1}$ and an edge T_iT_j, $0 \le i, j \le p^{\alpha}-1$, corresponds to all the edges of the bipartite subgraph $K_{|T_i|,|T_j|}$ of $K_{3\cdot p^\alpha}$. Since p^α is odd, we know that K_{p^α} can be decomposed into $\frac{(p^\alpha-1)}{2}$ disjoint hamiltonian cycles. We obtain one such decomposition using Walecki's construction (see [25], p 162-3) as follows. Let

$$\zeta : T_0, T_1, T_2, T_{p^\alpha-1}, T_3, T_{p^\alpha-2}, \ldots, T_{\frac{(p^\alpha+3)}{2}}, T_{\frac{(p^\alpha+1)}{2}}, T_0$$

be one hamiltonian cycle of the decomposition. (see Figure 3(C))

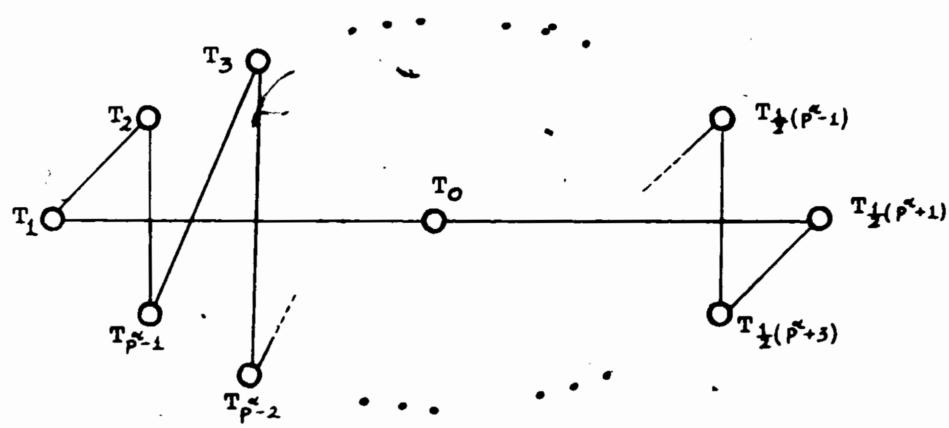


FIGURE 3(C)

Also let ϕ be a permutation on p^α symbols $T_0, T_1, \ldots, T_{p^\alpha-1}$ whose cycle representation is

$$\phi = (T_0)(T_1\ T_2\ T_3\ \cdots\ T_{p^\alpha-1}).$$

Then the $\frac{(p^\alpha-1)}{2}$ disjoint hamiltonian cycles of the decomposition are given by $\phi^k\zeta$, $k = 0,1,\ldots, \frac{(p^\alpha-3)}{2}$.

Using the cycle ζ we define two cycles C and C' of length $2\cdot p^\alpha$ in the graph $K_{3\cdot p^\alpha}$ as follows.

$$C: u_{00}, u_{01}, u_{11}, u_{10}, u_{20}, u_{21}, u_{p^\alpha-1,1}, u_{p^\alpha-1,0}, u_{30}, u_{31}, \ldots$$

$$\ldots, u_{\frac{(p^\alpha-1)}{2},1}, u_{\frac{(p^\alpha+3)}{2},1}, u_{\frac{(p^\alpha+3)}{2},2}, u_{\frac{(p^\alpha+1)}{2},2}, u_{\frac{(p^\alpha+1)}{2},0}, u_{00}$$

and

$$C': u_{00}, u_{11}, u_{20}, u_{p^\alpha-1,1}, u_{30}, \ldots, u_{\frac{(p\ +1)}{2},0}, u_{01}, u_{10}, \ldots$$

$$\ldots, u_{\frac{(p^\alpha-1)}{2},1}, u_{\frac{(p^\alpha+3)}{2},0}, u_{\frac{(p^\alpha+1)}{2},1}, u_{00}$$

These cycles are also shown in Figures 3(D) and 3(E) respectively.

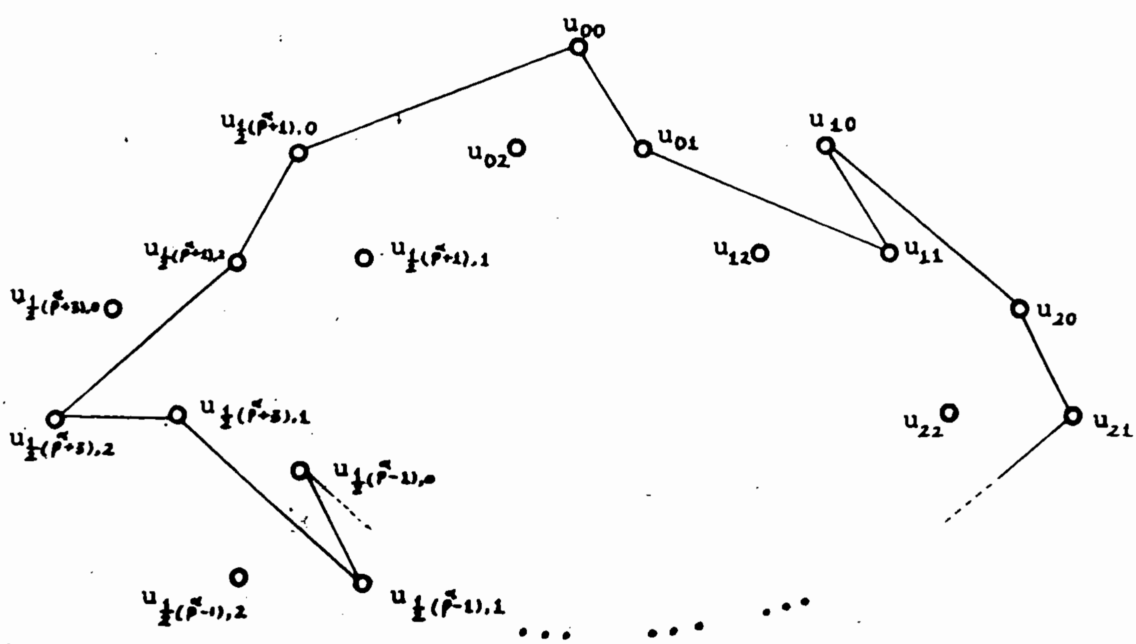


FIGURE 3(D)

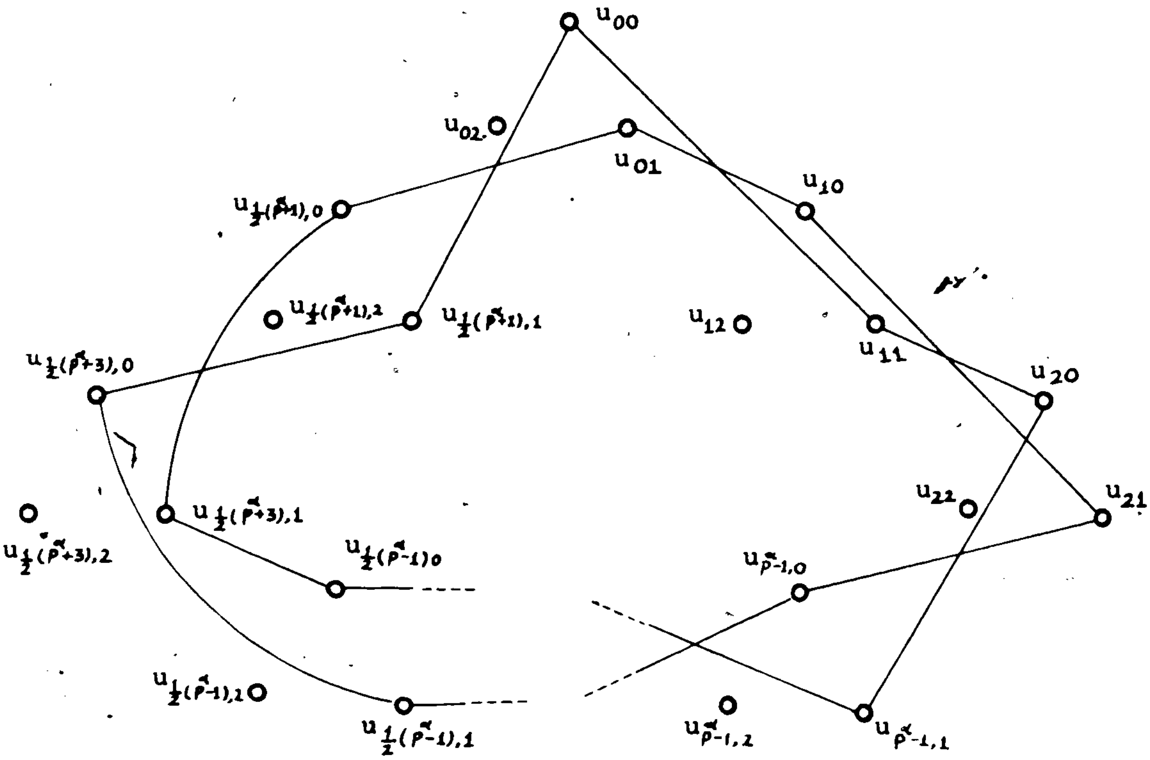


FIGURE 3(E)

Let σ be a permutation on $3p^{\alpha}$ symbols $u_{00}, u_{01}, u_{02}, \ldots,$ $u_{p^{\alpha}-1,0}, u_{p^{\alpha}-1,1}, u_{p^{\alpha}-1,2}$ whose cycle representation is

$$\sigma = (u_{00}\ u_{01}\ u_{02})(u_{10}\ u_{11}\ u_{12}) \ \cdots \ (u_{p^{\alpha}-1,0}\ u_{p^{\alpha}-1,1}\ u_{p^{\alpha}-1,2}).$$

The cycles C, σC and $\sigma^2 C$ are three cycles of length $2p^{\alpha}$ that use all internal edges of the p^{α} triangles on the vertex sets $T_0, T_1, \ldots, T_{p^{\alpha}-1}$ and all edges $u_{ij}\ u_{kj}$, $0 \le j \le 2$ between the successive vertex sets T_i and T_k of the cycle ζ.

Also C', $\sigma C'$ and $\sigma^2 C'$ are three more cycles of length $2p^{\alpha}$ and they use the remaining edges $u_{ir}u_{ks}$, $0 \le r, s \le 2, r \ne s$, between the successive vertex sets T_i and T_k of the cycle ζ.

Thus we have obtained six cycles $C, \sigma C, \sigma^2 C, C', \sigma C'$ and $\sigma^2 C'$ of length $2 \cdot p^{\alpha}$ in $K_{3 \cdot p^{\alpha}}$ and we have used:

edges of $K_{3 \cdot p^{\alpha}}$ which are the internal edges of the vertex sets $T_0, T_1, \ldots, T_{p^{\alpha}-1}$ and

edges of $K_{3 \cdot p^{\alpha}}$ which corresponds to the edges of the cycle ζ.

In case $p = 3$ and $\alpha = 1$ this gives a desired decomposition of K_9 into C_6's. Otherwise, we are left with the edges of $K_{3 \cdot p^{\alpha}}$ which correspond to the edges of the remaining cycles $\phi^k \zeta$, $k = 1, 2, \ldots, \frac{(p^{\alpha}-3)}{2}$ of $K_{p^{\alpha}}$. Since $p^{\alpha} \equiv 3 \pmod 4$, the number $\frac{(p^{\alpha}-3)}{2}$ is even. We pair these cycles as

$$\phi^{(2m-1)}\zeta, \quad \phi^{2m}\zeta \qquad 1 \le m \le p^{\alpha}-3 .$$

To complete the proof of the lemma, because of the rotation under the permutation ϕ, it is enough to show that the edges of $K_{3 \cdot p^{\alpha}}$ in the union of one such pair can be decomposed into cycles of length $2 \cdot p^{\alpha}$. Further, again because of the action of ϕ there is no loss in assuming this pair is ζ, $\phi\zeta$. We relabel the vertices of the complete graph $K_{p^{\alpha}}$ so that the cycle ζ is written as

$$\zeta' : T_0, T_1, T_2, \ldots, T_{p^\alpha - 1} .$$

Under this new labeling the cycle $\phi\zeta$ then becomes

$$\zeta'' : T_0, T_2, T_4, T_1, \ldots, T_{p^\alpha-1}, T_{p^\alpha-4}, T_{p^\alpha-2}, T_0 .$$

Let Z be a cycle of length $2 \cdot p^\alpha$ obtained by joining the vertices with labels 0 and 1 alternately between the successive triangles of the cycle ζ', that is

$$Z: \; u_{00}, u_{11}, u_{20}, u_{31}, \ldots, u_{p^\alpha-1,0}, u_{01}, u_{10}, \ldots, u_{p^\alpha-1,1}, u_{00} .$$

The three cycles Z, σZ and $\sigma^2 Z$ use all the edges between the successive vertex sets of the cycle ζ' except the edges of the form $u_{ij}\, u_{i+1,j}$ for $0 \le i \le p^\alpha-1$ and $0 \le j \le 2$ which form a union of three disjoint cycles of length p^α. In other words, the set of edges between the successive triangles of the cycle ζ' can be so partitioned that after pulling out three disjoint cycles of length p^α, the remaining edges can be partitioned into three disjoint cycles of length $2 \cdot p^\alpha$. Moreover, this is also true for the other hamiltonian cycles in the decomposition of K_{p^α}. Thus, in general, for any hamiltonian cycle of K_{p^α}, if from the edges between its successive triangles we pull out three disjoint cycles of length p^α then the remaining edges can be decomposed into three disjoint cycles of length $2 \cdot p^\alpha$.

If we can find three edge-disjoint cycles of length $2 \cdot p^\alpha$ which use only the edges of $K_{3 \cdot p^\alpha}$ along the cycle ζ' or ζ'' and show that their union can be decomposed into three disjoint cycles of length p^α along ζ' and three disjoint cycles of length p^α along ζ'', then we are done. We regroup the vertices of $K_{3 \cdot p^\alpha}$ into three groups

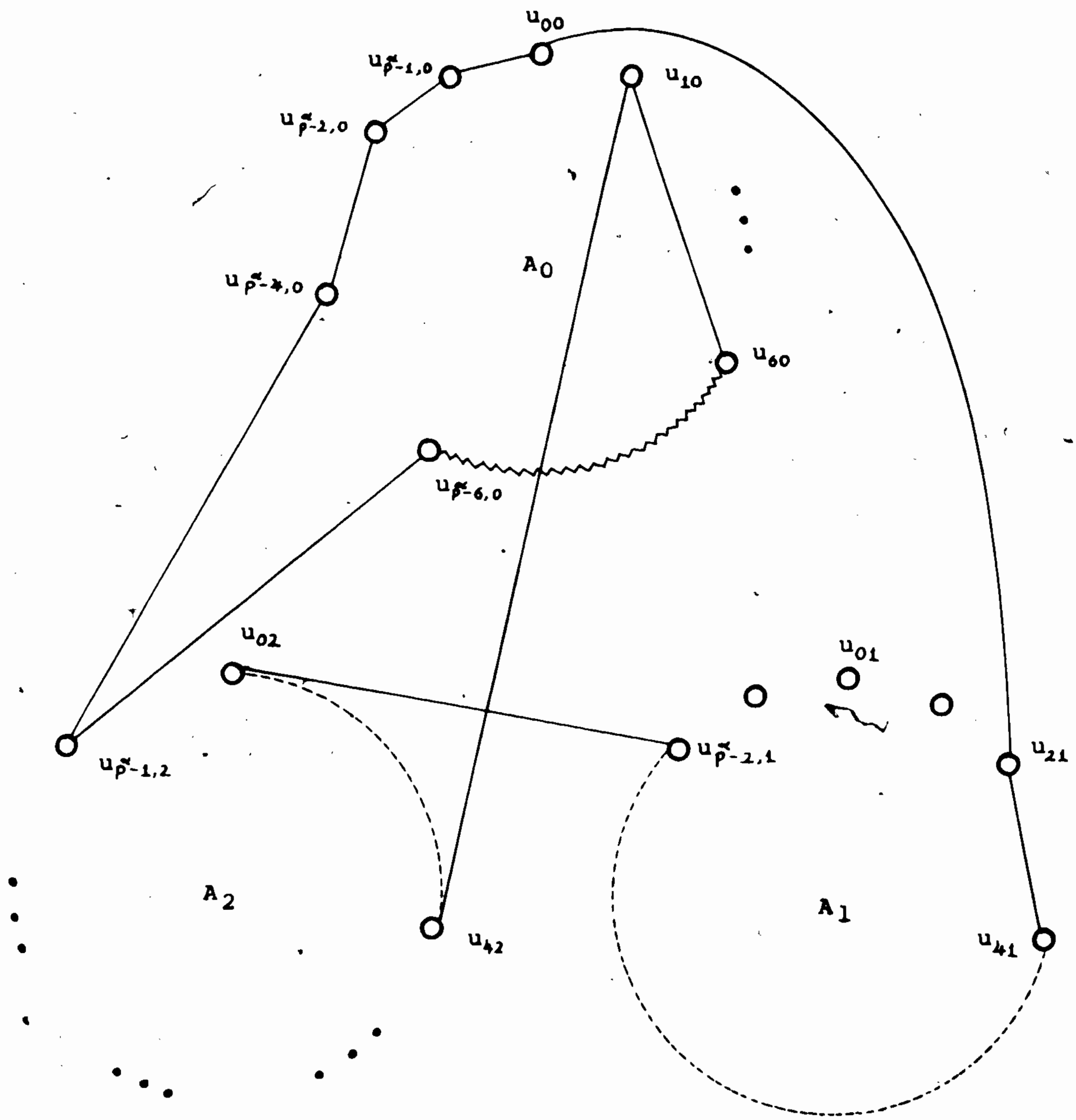
$$A_0 = \{u_{00}, u_{10}, u_{20}, \ldots, u_{p^\alpha-1,0}\}$$

$$A_1 = \{u_{01}, u_{11}, u_{21}, \ldots, u_{p^\alpha-1,1}\}$$

and $$A_2 = \{u_{02}, u_{12}, u_{22}, \ldots, u_{p^\alpha-1,2}\}$$

We construct a cycle $\hat{C}$ of length $2 \cdot p^\alpha$ illustrated in Figure 3(F) and described below.

A solid line joining two vertices, say u and v, represents an edge incident with those vertices. A dotted line



u_{00}
$u^{\alpha}_{\rho-1,0}$
u_{10}
$u^{\alpha}_{\rho-2,0}$
A_0
$u^{\alpha}_{\rho-4,0}$
u_{60}
$u^{\alpha}_{\rho-6,0}$
u_{02}
u_{01}
$u^{\alpha}_{\rho-1,2}$
$u^{\alpha}_{\rho-2,1}$
u_{21}
A_2
A_1
u_{42}
u_{41}

FIGURE 3(F)

between two vertices, say u and v, represents a path obtained by joining the successive vertices, starting at u and ending at v along the cycle ζ'. The jagged line between the two vertices u_{60} and $u_{p^\alpha-6,0}$ represents a path obtained by joining the vertices in succession along the cycle ζ''. A forward jump from one block to another, for example, from A_0 to A_1, A_1 to A_2 or A_2 to A_0 corresponds to a change in labeling in the second coordinate of the vertices in the cyclic order 0 to 1, 1 to 2 or 2 to 0.

We start at the vertex u_{00} of the vertex set T_0 in the block A_0 and join it to the vertex u_{21} in block A_1. Note that this is an edge between the vertex sets T_0 and T_2 of cycle ζ''. Next we join u_{21} to u_{41} (this is an edge between the vertex sets T_2 and T_4 of ζ'') and then remain in block A_1 joining edges along the path

$$u_{41}, u_{51}, u_{61}, \ldots, u_{p^\alpha-2,1}.$$

We then join the last vertex $u_{p^\alpha-2,1}$ of this path to the vertex u_{02} of the block A_2. We remain in block A_2 joining edges along the path

$$u_{02}, u_{12}, u_{22}, u_{32}, u_{42}.$$

We join the vertex u_{42} to the vertex u_{10} of the block A_0. We remain in the same block joining edges along the path

$$u_{10}, u_{60}, u_{30}, \ldots, u_{p^\alpha-6,0}$$

and these are the vertices belonging to the successive vertex sets of the cycle ζ'' . We join the vertex $u_{p^\alpha-6,0}$ to $u_{p^\alpha-1,2}$ and then $u_{p^\alpha-1,2}$ to $u_{p^\alpha-4,0}$. We complete the cycle by joining $u_{p^\alpha-4,0}$ to $u_{p^\alpha-2,0}$, $u_{p^\alpha-2,0}$ to $u_{p^\alpha-1,0}$ and finally $u_{p^\alpha-1,0}$ to u_{00} .

In the construction of $\hat{C}$, since we have picked $p^\alpha-1$ vertices from the block A_0 , $p^\alpha-4$ vertices from the block A_1 and five vertices from the block A_2 , $\hat{C}$ is a cycle of length $2\cdot p^\alpha$. We rotate the blocks A_0, A_1 and A_2 two times in cyclic order, that is, A_0 to A_1 , A_1 to A_2 and A_2 to A_0 and get two more cycles of length $2\cdot p^\alpha$. In other words these cycles are nothing but $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$. In view of what we have said earlier, it is enough to show that the union of these three cycles from two sets of three disjoint cycles of length p^α .

Let us first consider the edges along the cycle ζ . In the construction of $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$ we have used

the path	$u_{00}, u_{10}, u_{20}, u_{30}, u_{40}$	in	$\sigma\hat{C}$
the path	$u_{40}, u_{50}, u_{60}, \ldots, u_{p^\alpha-2,0}$	in	$\sigma^2\hat{C}$
and the path	$u_{p^\alpha-2,0}$, $u_{p^\alpha-1,0}$, u_{00}	in	$\hat{C}$

Therefore the cycles $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$ contain all the edges $u_{ij}\, u_{i+1,j}$, $0 \le i \le p^\alpha-1$, $0 \le j \le 2$, which form three disjoint cycles of length p^α .

As regard to the edges along the cycle ζ'' , in the construction of $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$ we have used

the path	u_{00}, u_{21}, u_{41}	in	$\hat{C}$
the edge	$u_{41}\, u_{12}$	in	$\sigma^2\hat{C}$
the path	$u_{12}, u_{62}, \ldots, u_{p^\alpha-6,2}$	in	$\sigma^2\hat{C}$
the edge	$u_{p^\alpha-6,2}\, u_{p^\alpha-1,1}$	in	$\sigma^2\hat{C}$
the edge	$u_{p^\alpha-1,1}\, u_{p^\alpha-4,2}$	in	$\sigma^2\hat{C}$
the edge	$u_{p^\alpha-4,2}\, u_{p^\alpha-2,2}$	in	$\sigma^2\hat{C}$
and the edge	$u_{p^\alpha-2,2}\, u_{00}$	in	$\sigma\hat{C}$

Therefore, the cycles $\hat{C}$, $\sigma\hat{C}$ and $\sigma^2\hat{C}$ contain three disjoint cycles of length p^α whose edges lie along the cycle ζ'' . This completes the proof. □

3.9 **LEMMA**: *There exists a BCD* $(5\cdot p^\alpha, 2\cdot p^\alpha, 1)$ *where* p *is a prime,* α *is a positive integer and* $p^\alpha \equiv 1$ *(mod 4)*.

<u>Proof</u>: The proof is similar to that of Lemma 3.8. We divide the vertices of $K_{5 \cdot p^\alpha}$ into p^α groups each of size five and label them with

$$S_0: \{u_{00}, u_{01}, u_{02}, u_{03}, u_{04}\}, \quad S_1 : \{u_{10}, u_{11}, u_{12}, u_{13}, u_{14}\}, \ \ldots$$

$$\ldots, \ S_{p^\alpha-1} : \{u_{p^\alpha-1,0}, u_{p^\alpha-1,1}, u_{p^\alpha-1,2}, u_{p^\alpha-1,3}, u_{p^\alpha-1,4}\} .$$

As in the proof of Lemma 3.8, with $K_{5 \cdot p^\alpha}$ we associate a complete graph K_{p^α}, so that $V(K_{p^\alpha}) = \{S_0, S_1, \ldots, S_{p^\alpha-1}\}$ and an edge $S_i S_j$, $0 \le i, j \le p^\alpha-1$ corresponds to all the edges between the vertices of the sets S_i and S_j in the original graph. We decompose K_{p^α} into $\frac{(p^\alpha-1)}{2}$ disjoint hamiltonian cycles $\psi^k \eta$, $k = 0, 1, \ldots, \frac{(p^\alpha-3)}{2}$ where

$$\eta : S_0, S_1, S_2, S_{p^\alpha-1}, S_3, S_{p^\alpha-2}, \ldots, S_{\frac{(p^\alpha-1)}{2}}, S_{\frac{(p^\alpha+3)}{2}}, S_{\frac{(p^\alpha+1)}{2}}, S_0$$

and ψ is a permutation on p^α symbols $S_0, S_1, \ldots, S_{p^\alpha-1}$ with cycle representation

$$\psi = (S_0)(S_1 \ S_2 \ S_3 \ \ldots \ S_{p^\alpha-1}) .$$

Each K_5 with the vertex set S_i, $0 \le i \le p^\alpha-1$ can be decomposed into two disjoint 5-cycles which are obtained by joining the vertices at distance 1 and vertices at distance 2 respectively. The five cycles $\tau^k C$, $0 \le k \le 4$ of length $2 \cdot p^\alpha$ where we define C as

$$C: \; u_{00}, u_{01}, u_{11}, u_{10}, u_{20}, u_{21}, u_{p^\alpha-1,1}, u_{p^\alpha-1,0}, \ldots, u_{\frac{1}{2}(p^\alpha+5),1}, u_{\frac{1}{2}(p^\alpha+5),2},$$

$$u_{\frac{1}{2}(p^\alpha-1),2}, u_{\frac{1}{2}(p^\alpha-1),3}, u_{\frac{1}{2}(p^\alpha+3),3}, u_{\frac{1}{2}(p^\alpha+3),4}, u_{\frac{1}{2}(p^\alpha+1),4}, u_{\frac{1}{2}(p^\alpha+1),0}, u_{00}$$

and τ is the permutation

$$\tau = (u_{00}\; u_{01}\; u_{02}\; u_{03}\; u_{04}) \cdots (u_{p^\alpha-1,0}\; u_{p^\alpha-1,1}\; u_{p^\alpha-1,2}\; u_{p^\alpha-1,3}\; u_{p^\alpha-1,4})$$

use all the internal edges of K_5's which are at distance one and all edges of the form $u_{ij}\, u_{kj}$, $0 \le j \le 4$ between the successive pentagons S_i and S_j of the cycle η . Further we construct ten more cycles $\tau^k C'$, $\tau^k C''$, $0 \le k \le 4$, of length $2 \cdot p^\alpha$ where

$$C': \; u_{00}, u_{11}, u_{20}, u_{p^\alpha-1,1}, \ldots, u_{\frac{1}{2}(p^\alpha+1),0}, u_{01}, u_{10}, \ldots, u_{\frac{1}{2}(p^\alpha+1),1}, u_{00}$$

and

$$C'': \; u_{00}, u_{12}, u_{20}, u_{p^\alpha-1,2}, \ldots, u_{\frac{1}{2}(p^\alpha+1),0}, u_{02}, u_{10}, \ldots, u_{\frac{1}{2}(p^\alpha+1),2}, u_{00} \, .$$

These cycles use the remaining edges between the successive pentagons of the cycle η . Next using the internal edges at distance 2 of each of the K_5's with vertex set S_i, $0 \le i \le p^\alpha-1$ and the edges between the successive vertex sets of the cycle $\psi\eta$ we get fifteen more cycles of length $2\cdot p^\alpha$ in the same way.

In case $p = 5$ and $\alpha = 1$ this gives a desired decomposition of K_{25} into cycles of length ten. Otherwise we are left with the edges of $K_{5\cdot p^\alpha}$ which correspond to the edges of the remaining cycles $\psi^k\eta$, $k = 2,3,\ldots,\frac{(p^\alpha-3)}{2}$ of K_{p^α}. Since $p^\alpha \equiv 1 (\mathrm{mod}\ 4)$, these cycles are even in number. We pair these cycles as

$$\psi^{2m}\eta\ ,\ \psi^{2m+1}\eta \quad \text{for} \quad 1 \le m \le \frac{p^\alpha-5}{4}\ .$$

To prove the lemma it suffices to show that the edges of $K_{5\cdot p^\alpha}$ in the union of any such pair can be decomposed into cycles of length $2\cdot p^\alpha$. Because of the rotation under the permutation ψ, it is enough to show this for the pair η, $\psi\eta$. We relabel the vertices of the complete graph K_{p^α} so that the cycle η is written as

$$\eta' : S_0, S_1, S_2, \ldots, S_{p^\alpha-1}\ .$$

Under this new labeling the cycle $\psi\eta$ then becomes

$$\eta'' : S_0, S_2, S_4, S_1, \ldots, S_{p^\alpha-1}, S_{p^\alpha-4}, S_{p^\alpha-2}, S_0\ .$$

Using the edges between the successive vertex sets of the cycle η', we obtain ten cycles of length $2 \cdot p^{\alpha}$ and we are left with the edges of the form $u_{ij} u_{i+1,j}$ for $0 \le i \le p^{\alpha}-1$ and $0 \le j \le 4$ which form a union of five disjoint cycles of length p^{α}. This, in general, is true for the edges of $K_{5 \cdot p^{\alpha}}$ along any hamiltonian cycle of $K_{p^{\alpha}}$. Moreover, this fact can be restated as follows. The edges of $K_{5 \cdot p^{\alpha}}$, between the successive vertex sets along any hamiltonian cycle can be decomposed into five disjoint cycles of length p^{α} and five cycles of length $2 \cdot p^{\alpha}$.

Our next attempt, therefore, is to pull out five disjoint cycles of length p^{α} along the cycle η' and five disjoint cycles of length p^{α} along the cycle η'' such that their union gives five cycles of length $2 \cdot p^{\alpha}$. In view of what we have said earlier the remaining edges of $K_{5 \cdot p^{\alpha}}$ along the cycles η' and η'' then can be decomposed into twenty cycles of length $2 \cdot p^{\alpha}$. We regroup the vertices of $K_{5 \cdot p^{\alpha}}$ into five groups

$$A_i = \{u_{0i}, u_{1i}, u_{2i}, \ldots, u_{p^{\alpha}-1,i}\}$$

for $i = 0,1,2,3,4$.

Using the notations of Figure 3(F), we give here in Figure 3(G) a method of constructing a cycle $\bar{C}$ of length $2 \cdot p^{\alpha}$.

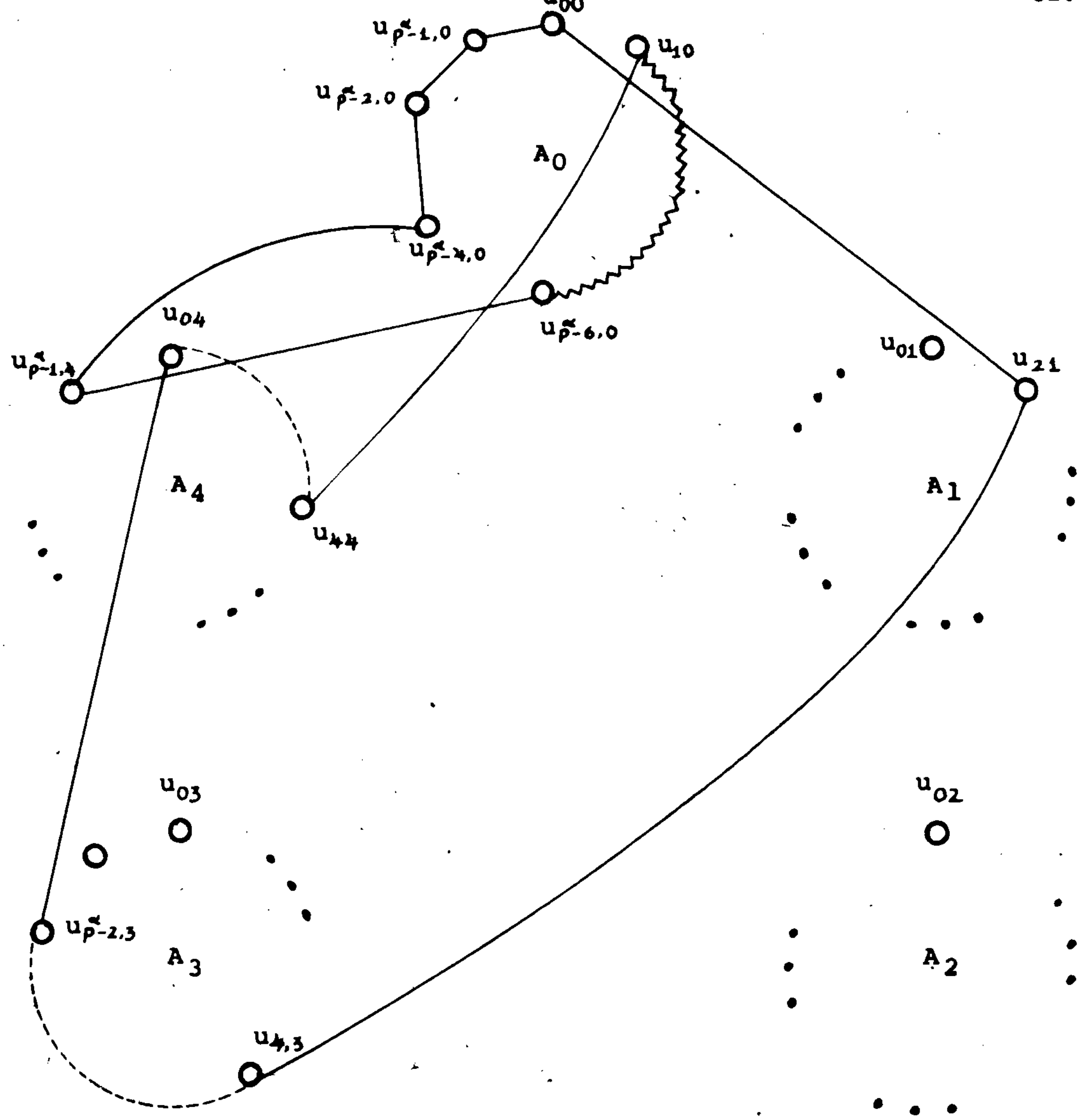


FIGURE 3(G)

We rotate the blocks A_0, A_1, ..., A_4 four times in cyclic order, that is, A_0 to A_1, A_1 to A_2, ..., A_4 to A_0 and obtain four more cycles of length $2 \cdot p^{\alpha}$. These cycles are nothing but $\tau^k \bar{C}$, $0 \le k \le 4$.

Now to check that $\tau^k\bar{C}$, $0 \le k \le 4$, contain five disjoint cycles of length p^α whose edges lie along the cycle η' we observe that in the construction of $\tau^k\bar{C}$, $0 \le k \le 4$, we have used:

the path $u_{00}, u_{10}, u_{20}, u_{30}, u_{40}$ in $\tau\bar{C}$,

the path $u_{40}, u_{50}, \ldots, u_{p^\alpha-2,0}$ in $\tau^2\bar{C}$,

and the path $u_{p^\alpha-2,0}, u_{p^\alpha-1,0}, u_{00}$ in $^4\bar{C}$.

Their union gives a cycle of length p^α whose edges lie along the cycle η' . Under the rotations we obtain five disjoint such cycles of length p^α .

Similarly, we have used the following edges and the paths along the cycle η'' :

the edge $u_{00}\, u_{21}$ in $\bar{C}$,

the edge $u_{21}\, u_{43}$ in $\bar{C}$,

the edge $u_{43}\, u_{14}$ in $\tau^4\bar{C}$,

the path $u_{14}, u_{64}, \ldots, u_{p^\alpha-6,4}$ in $\tau^4\bar{C}$,

the edge $u_{p^\alpha-6,4}\, u_{p^\alpha-1,3}$ in $\tau^4\bar{C}$,

the edge $u_{p^\alpha-1,3}\, u_{p^\alpha-4,4}$ in $\tau^4\bar{C}$,

the edge $u_{p^\alpha-4,4}\, u_{p^\alpha-2,4}$ in $\tau^4\bar{C}$,

and the edge $u_{p^\alpha-2,4}\, u_{00}$ in $\tau\bar{C}$

Their union gives a cycle of length p^{α}. Under the rotations we obtain five disjoint cycles of length p^{α} whose edges lie along the cycle η''. This completes the proof. □

3.10 **THEOREM:** *There exists a BCD* $(n, 2\cdot p^{\alpha}, 1)$ *where* p *is an odd prime and* α *a positive integer if and only if* n *is odd*, $n > 2\cdot p^{\alpha}$ *and* $n(n-1) \equiv 0(\mathit{mod}\ 4\cdot p^{\alpha})$.

Proof: In view of Lemmas 3.7, the proof follows from the Lemmas 3.8 and 3.9. □

Theorems 3.6 and 3.10 can be put together to give the main result.

3.11 **THEOREM:** *There exists a BCD* $(n, 2\cdot p^{\alpha}, 1)$ *where* p *is any prime and* α *a positive integer if and only if* $n > 2\cdot p^{\alpha}$, n *odd and* $n(n-1) \equiv 0(\mathit{mod}\ 4\cdot p^{\alpha})$.

CHAPTER 4

As we have mentioned in the beginning of the previous chapter, the problem of decomposing complete symmetric digraphs DK_n into directed cycles has been of equal interest to that of decomposing complete graphs K_n into cycles of different lengths. But much less work has been done in the direction of decomposing complete symmetric digraphs DK_n into orientations of a cycle, other than the directed cycle. Hung and Mendelsohn [21] have found a necessary and sufficient condition for the partitioning of the set of arcs of a complete symmetric digraph into each of the oriented triangles. In a recent paper [16], Harary, Heinrich and Wallis have considered the same problem for each of the four oriented quadrilaterals. In doing so they have made strong use of the fact that each orientation of a triangle and that of a quadrilateral is self-converse. Also, Harary, Palmer and Smith [17] had earlier shown that the only graphs for which every orientation is self-converse are the two smallest complete graphs K_1 and K_2 and the three smallest cycles C_3, C_4 and C_5. In their paper [16], Harary, Heinrich and Wallis made a concluding remark that it remains to investigate the case of oriented pentagons. In this chapter we consider this problem and show that the necessary conditions for each of the oriented pentagon to divide complete symmetric digraphs DK_n are also sufficient.

A cycle of length five can be oriented in only four different ways and these are shown in Figure 4(A).

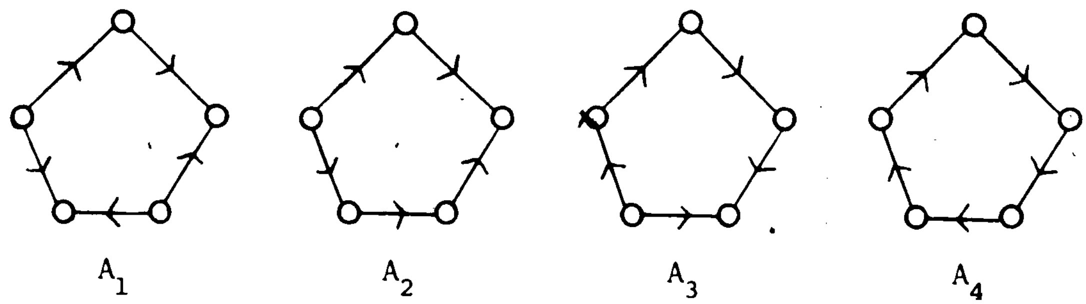


FIGURE 4(A)

The orientations will be described as

	A_1	contains a path of maximum length two,
	A_2	contains a path of maximum length three,
	A_3	contains a path of maximum length four,
and	A_4	is the directed cycle of length five.

4.1 THEOREM: *If* $A_i | DK_n$, $i = 1,2,3,4$, *then* $n \equiv 0, 1, 5$ *or* 6 (*mod* 10).

Proof: For $i = 1,2,3,4$ $A_i | DK_n$ implies that $5 | n(n-1)$. That is, $n \equiv 0$ or 1 (mod 5) or equivalently $n \equiv 0, 1, 5$ or 6 (mod 10).□

Since each oriented pentagon is self-converse, we have the following result.

4.2 LEMMA: *If* $C_5|K_n$, *then* $A_i|DK_n$ *for* $i = 1,2,3,4$.

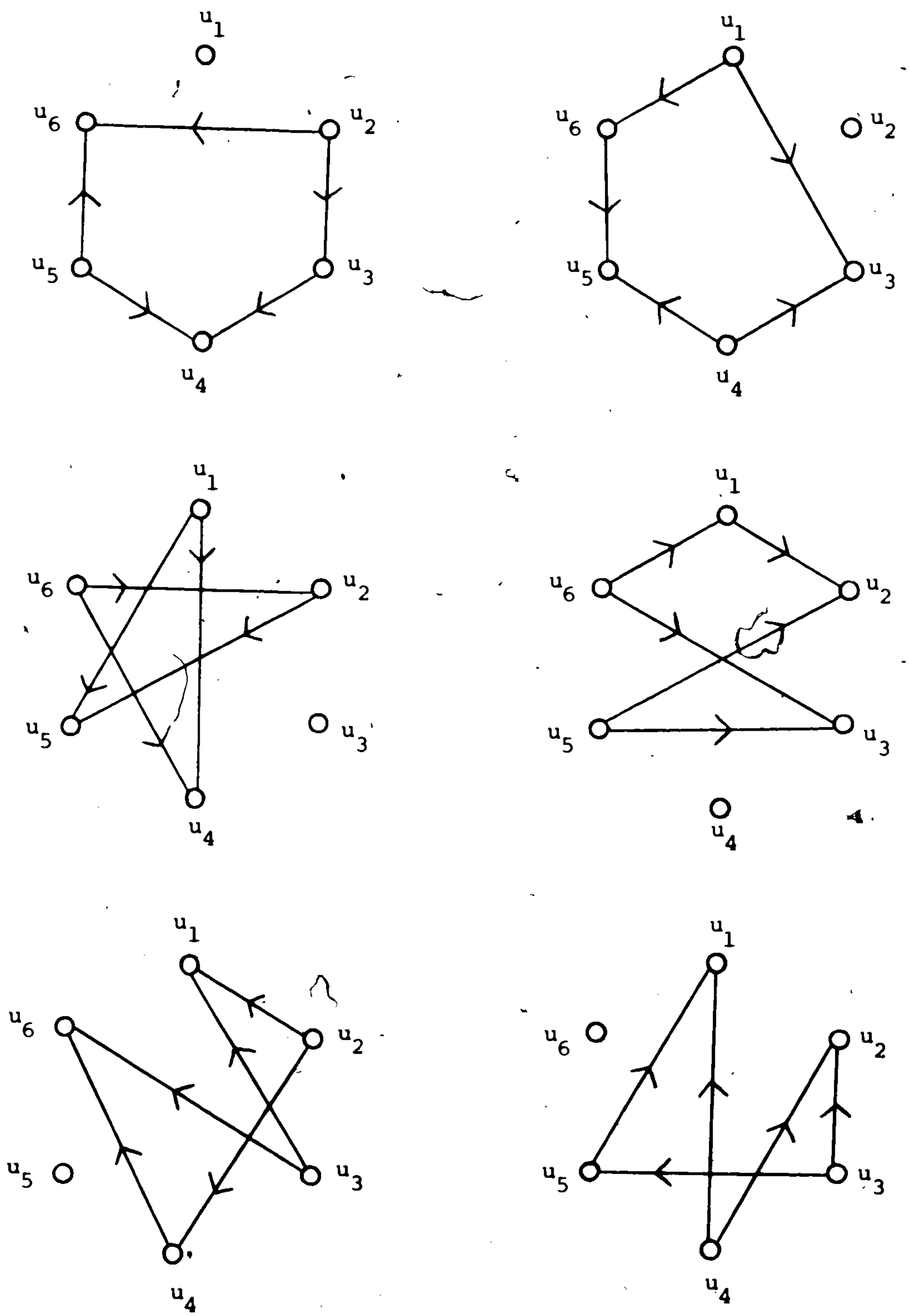
Rosa and Huang [31] and Bermond and Sotteau [6] have shown independently that K_n can be decomposed into C_5's if and only if $n \equiv 1$ or 5 (mod 10). This result together with Lemma 4.2 gives the following result.

4.3 THEOREM: *If* $n \equiv 1$ *or* 5 (mod 10), *then* $A_i|DK_n$ *for* $i = 1,2,3,4$.

To prove the result for $n \equiv 0$ or 6 (mod 10), first we show it for $n = 6$ and $n = 10$.

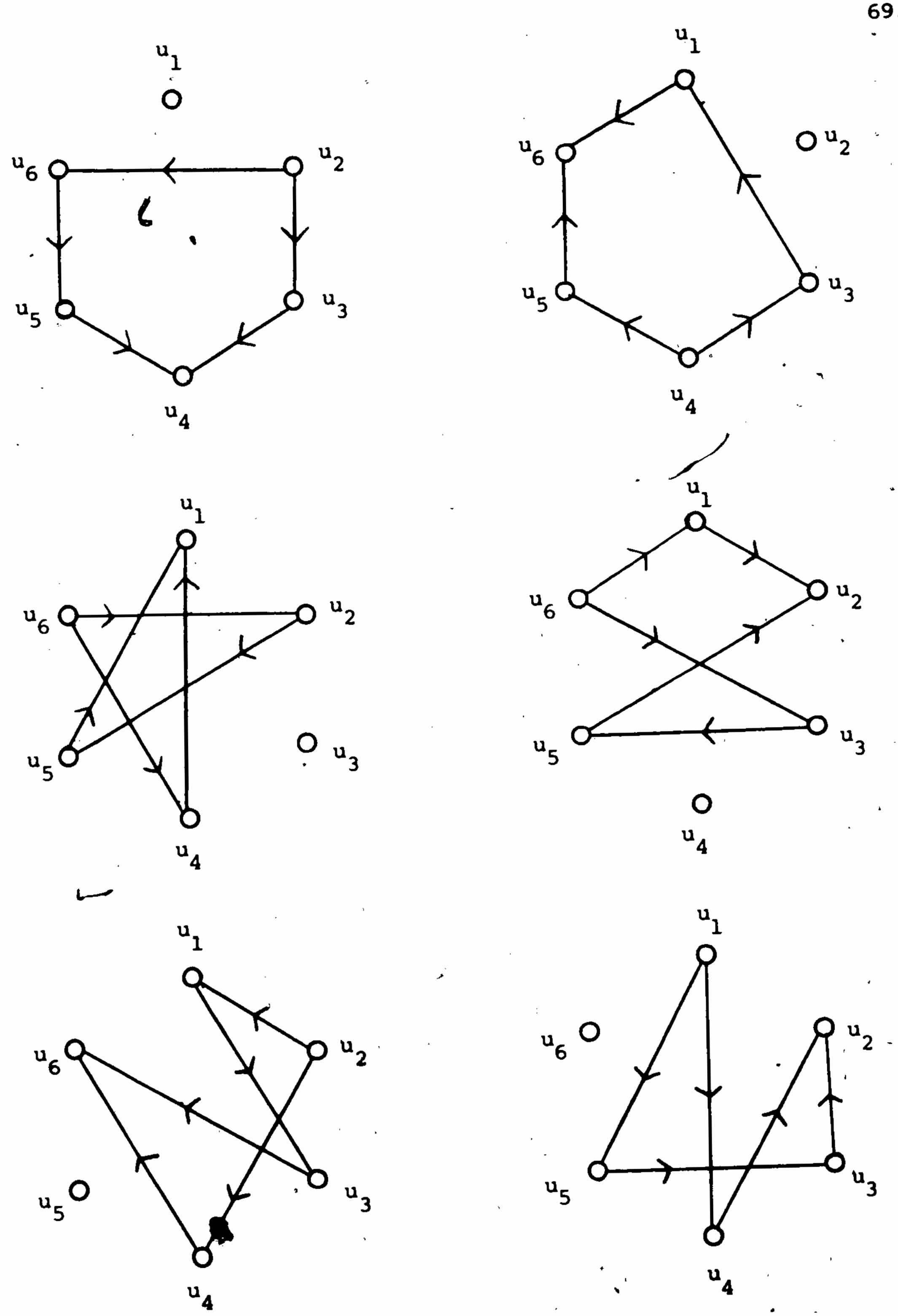
4.4 LEMMA: $A_i|DK_6$ *for* $i = 1,2,3,4$.

<u>Proof:</u> Let $V(DK_6) = \{u_1,u_2,u_3,u_4,u_5,u_6\}$. Following is the decomposition of DK_6 into each of A_i, $i = 1,2,3,4$ by construction.



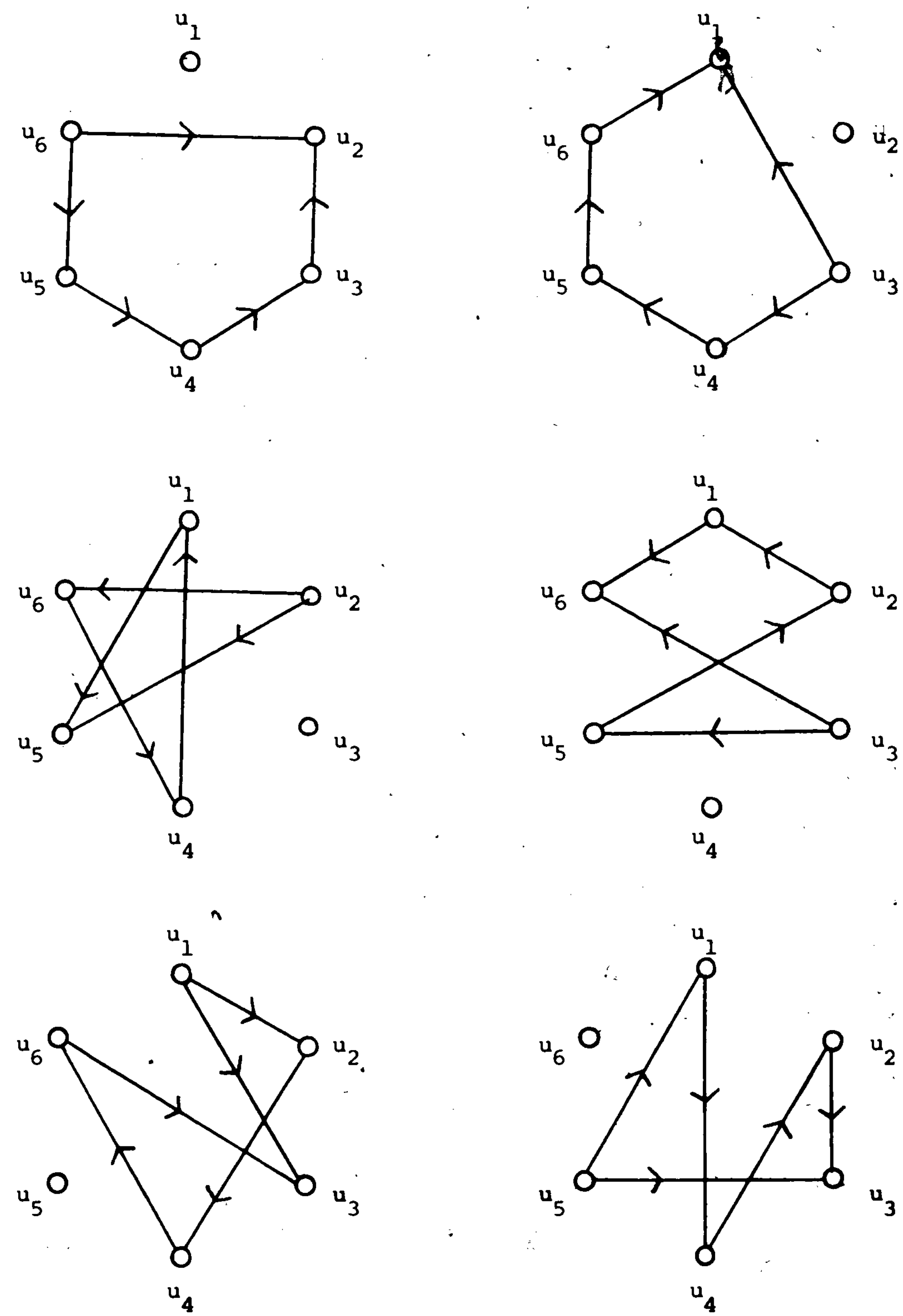
$A_1 | DK_6$

FIGURE 4(B)



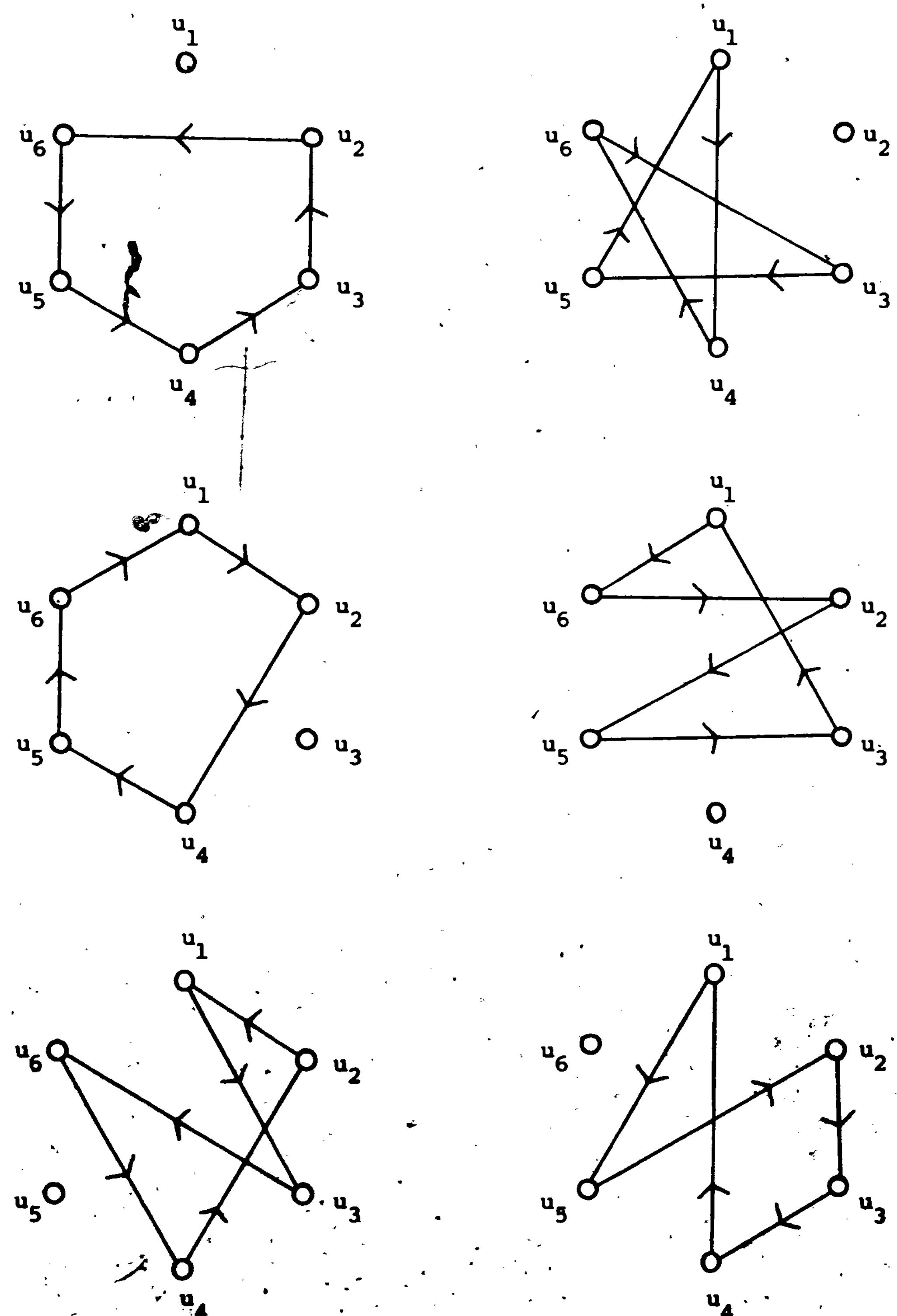
$A_2 | DK_6$

FIGURE 4(C)



$A_3 | DK_6$

FIGURE 4(D)



$A_4 | DK_6$

FIGURE 4(E)

4.5 LEMMA: $A_i | DK_{10}$ *for* $i = 1,2,3,4$.

<u>Proof:</u> We write

$$K_{10} = K_6 + K_4 = K_6 \cup K_4 \cup K_{4,6}$$

Let $V(K_6) = \{u_0, u_1, u_2, u_3, u_4, u_\infty\}$ and $V(K_4) = \{v_1, v_2, v_3, v_4\}$ where $V(K_{4,6}) = V(K_4) \cup V(K_6)$, $V(K_4)$ and $V(K_6)$ are the maximal independent subsets of $K_{4,6}$.

The graph $K_4 \cup K_{4,6}$ can be decomposed into four C_5's ,

$$u_1, v_1, v_4, u_3, v_2, u_1 \quad , \quad u_2, v_2, v_1, u_4, v_3, u_2 \quad ,$$

$$u_3, v_3, v_4, u_2, v_1, u_3 \quad , \quad u_4, v_4, u_1, v_3, v_2, u_4 \quad ,$$

and a graph H shown in the Figure 4(F)

H :

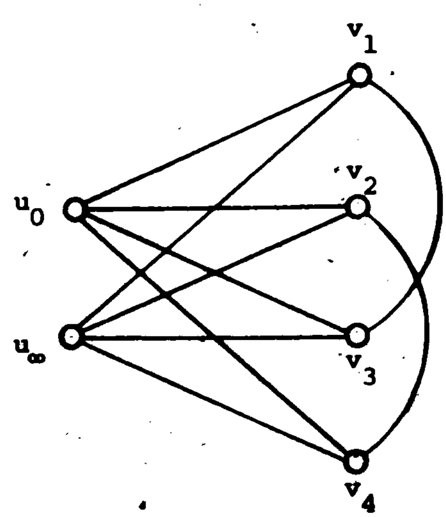
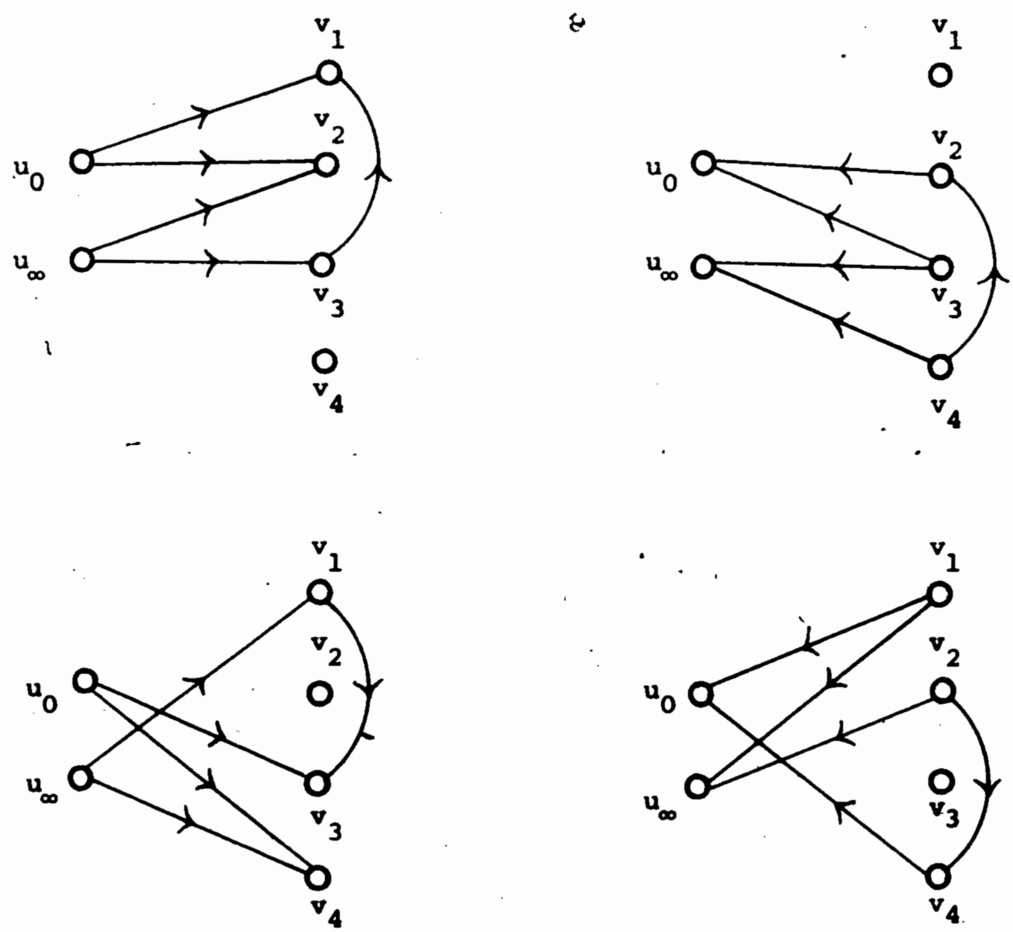


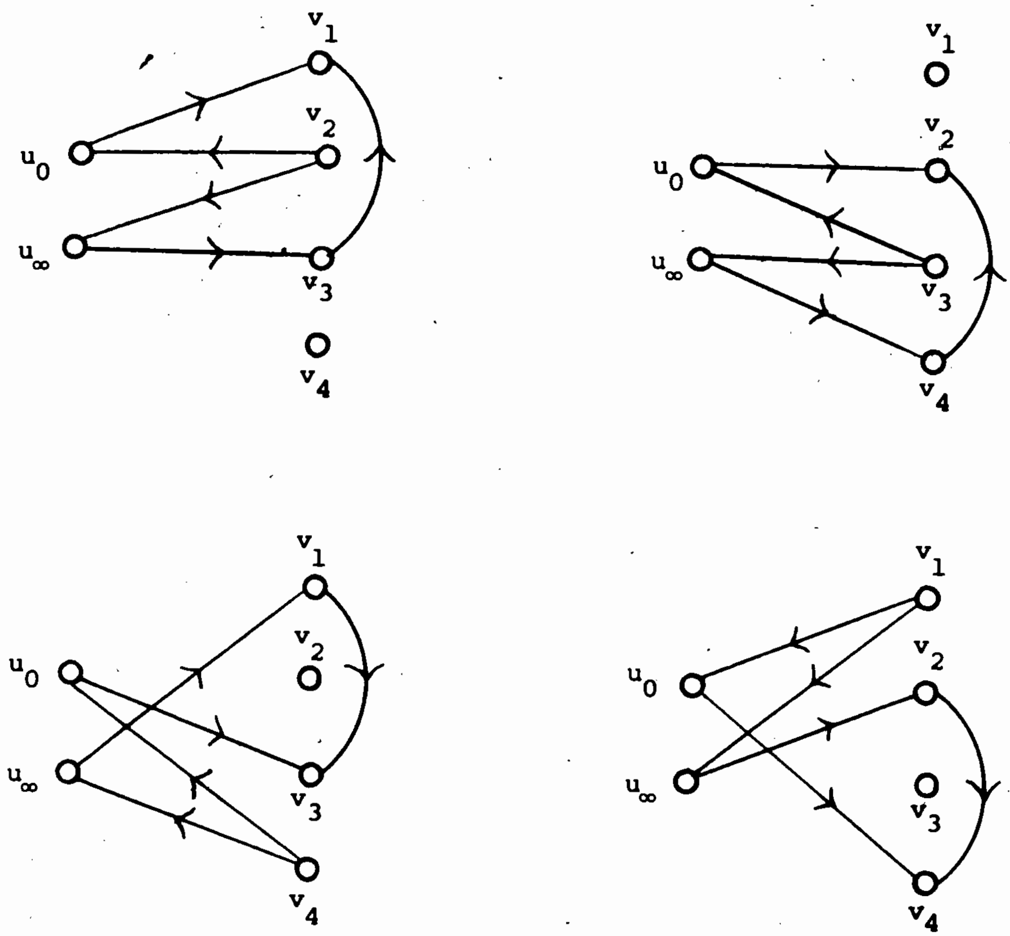
FIGURE 4(F)

Since, for $i = 1,2,3,4$, A_i is self-converse, $A_i|DC_5$. Moreover, $A_i|DK_6$, $i = 1,2,3,4$, by Lemma 4.4. Thus to prove that $A_i|DK_{10}$, $i = 1,2,3,4$, it is enough to show that $A_i|DH$ for $i = 1,2,3,4$. Here we give a decomposition of DH into each of A_i, $i = 1,2,3,4$.



$A_1|DH$

FIGURE 4(G)



$A_2|DH$

FIGURE 4(H)

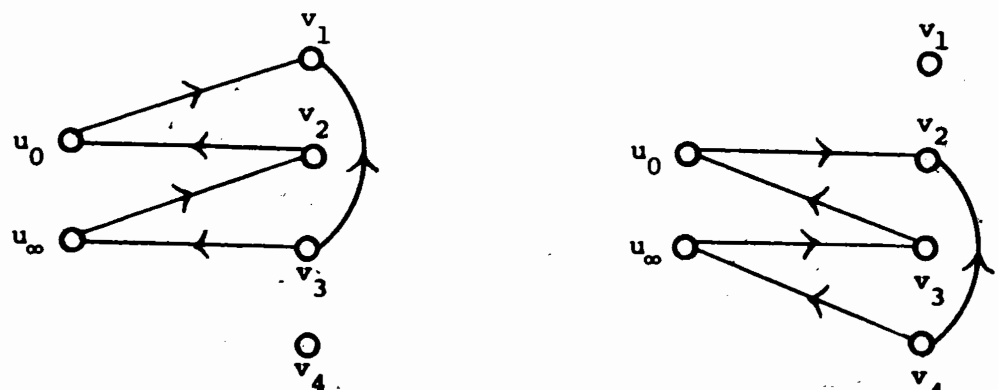
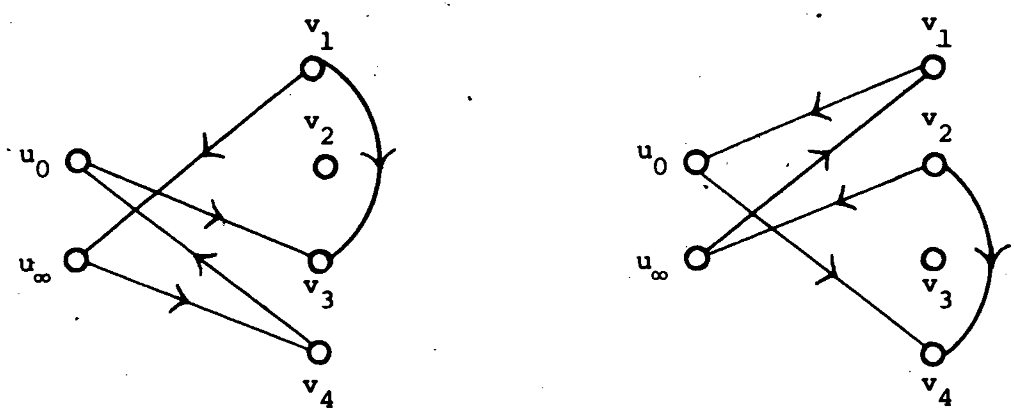
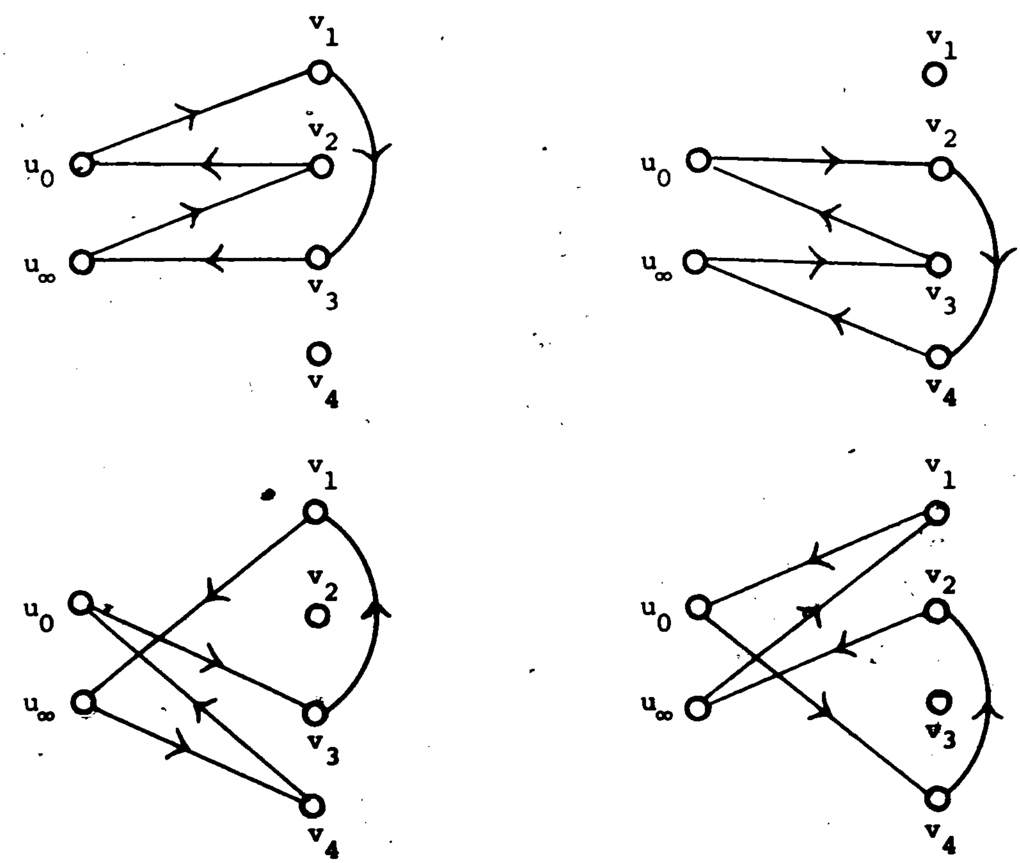


Figure continued.



$A_3|DH$

FIGURE 4(I)



$A_4|DH$

FIGURE 4(J) □

4.6 THEOREM: *If* $n \equiv 6 \pmod{10}$, $A_i | DK_n$ *for* $i = 1,2,3,4$.

Proof: Let $n = 10k + 6$ for some positive integer k. We write

$$K_n = K_{10k+6} = K_{5(2k+1)+1} = [(2k+1)K_5 + K_1] \cup {}_{(2k+1)}K_5$$

$$= (2k+1)K_6 \cup {}_{(2k+1)}K_5 \quad ,$$

where the vertex set of the complete multipartite graph ${}_{(2k+1)}K_5$ is chosen appropriately. By Prop. 1.9 of Chapter 1, ${}_{(2k+1)}K_5$ can be decomposed into C_5's and hence the complete multipartite digraph $D_{(2k+1)}K_5$ can be decomposed into each of A_i, $i = 1,2,3,4$. Moreover, by Lemma 4.4 $A_i | DK_6$, $i = 1,2,3,4$ and hence the result. □

4.7 THEOREM: *If* $n \equiv 0 \pmod{10}$ *and* $n \neq 20$, $A_i | DK_n$ *for* $i = 1,2,3,4$.

Proof: The result for $n = 10$ has been proved in Lemma 4.5. Hence, let $n = 10k$, $k > 2$. We write

$$K_n = K_{10k} = kK_{10} \cup {}_kK_{10} \ ,$$

where the vertex set of the complete multipartite graph ${}_kK_{10}$ is chosen appropriately. In view of Lemma 4.5, it is enough to show that $A_i | D\,{}_kK_{10}$ for $i = 1,2,3,4$. With ${}_kK_{10}$ we associate a graph G as follows.

Let $V_i = \{u_1^i, u_2^i, \ldots, u_{10}^i\}$, $1 \le i \le k$, be the maximal independent subsets of vertices in ${}_kK_{10}$. Also, for $i = 1,2,\ldots,k$ let

$$S_i = \{u_1^i, u_2^i, u_3^i, u_4^i, u_5^i\} ,$$

$$S_{i+k} = \{u_6^i, u_7^i, u_8^i, u_9^i, u_{10}^i\} .$$

We define $V(G) = \{S_1, S_2, \ldots, S_{2k}\}$ and an edge incident with S_i and S_j corresponds to the set of all edges of the complete bipartite subgraph $K_{|S_i|,|S_j|}$ of ${}_kK_{10}$. We observe that any two vertices S_i, S_j are adjacent except when $j = k + i$. Moreover, the edges S_iS_{i+k}, $1 \le i \le k$, form a 1-factor of $\overline{G}$. Thus $G \cong K_{2k} - I$ where I is a 1-factor of K_{2k}. By Cor. 2.13 of Chapter 2, G can be decomposed into 3-cycles and 5-cycles, which amounts to the fact that the graph ${}_kK_{10}$ can be decomposed into the factors ${}_3K_5$ and $C_5[\overline{K}_5]$. Both these factors can be decomposed into C_5's by Prop. 1.7 and Prop. 1.5 of Chapter 1, respectively. Thus $C_5 | {}_kK_{10}$ and the result then follows from the fact that each orientation A_i, $i = 1,2,3,4$ of C_5 is self-converse.□

4.8 THEOREM: $A_i \mid DK_{20}$ *for* $i = 1,2,3,4$.

Proof: We write

$$K_{20} = 2K_{10} \cup {}_2K_{10}$$

with the vertex set of the complete bipartite graph ${}_2K_{10}$ chosen appropriately. We shall show that $A_i \mid D(K_{10} \cup {}_2K_{10})$ for $i = 1,2,3,4$. This together with Lemma 4.5 will prove the result. Let the vertex sets of the two K_{10}'s be $\{u_1,u_2,\ldots,u_{10}\}$ and $\{v_1,v_2,\ldots,v_{10}\}$. The graph $K_{10} \cup {}_2K_{10}$ can be decomposed into a graph Λ , as shown in the Figure 4(K)

Λ :

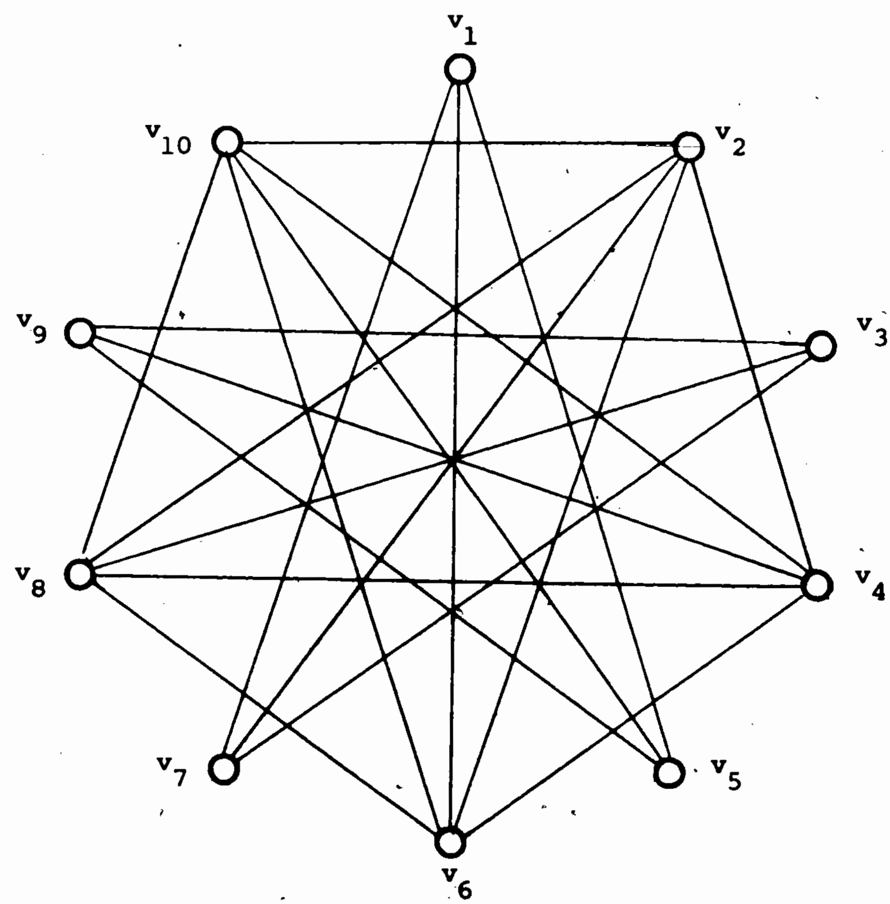


FIGURE 4(K)

and twenty five 5-cycles given by

$$C:\ u_1, v_1, u_2, v_3, v_4, u_1 \quad , \qquad C':\ u_1, v_3, u_7, v_2, v_5, u_1 \ ,$$

$$C'':\ u_1, v_8, u_{10}, v_7, v_9, u_1 \ ,$$

$$\phi^k C \ , \ \phi^k C' \ (1 \le k \le 9) \text{ and } \phi^{2k} C'' \ (1 \le k \le 4)$$

where $\phi = (1\ 2\ 3\ \ldots\ 10)$ is a cyclic permutation acting on the ten subscripts.

Here we give a decomposition of $D\Lambda$ into each of A_2, A_3 and A_4 .

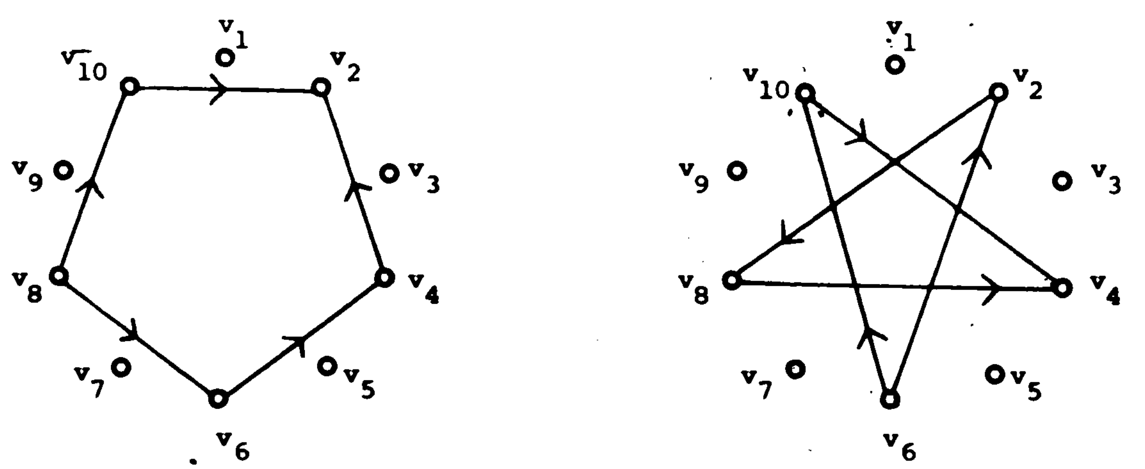
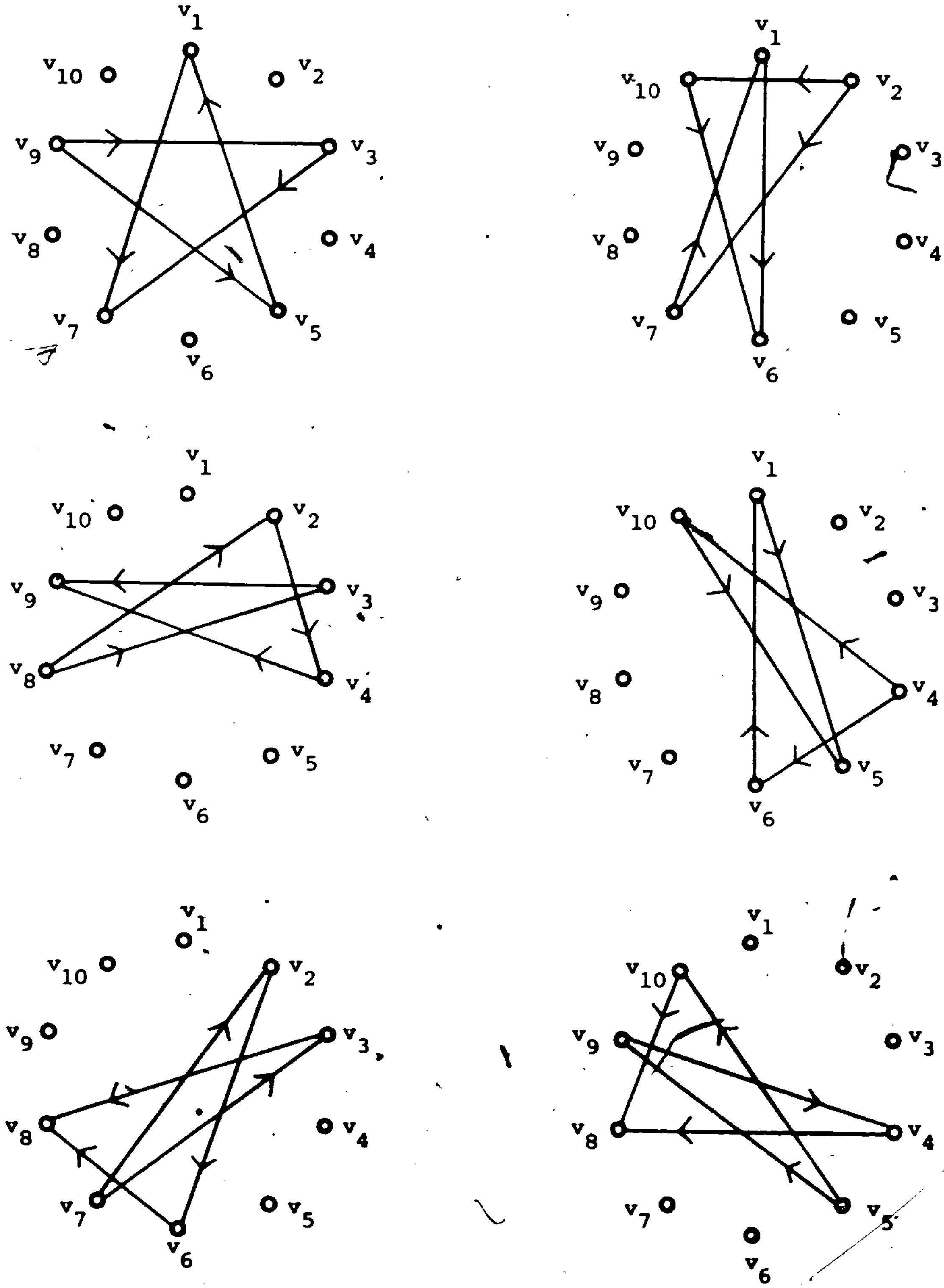


Figure continued .



$A_2 \mid DA$

FIGURE 4(L)

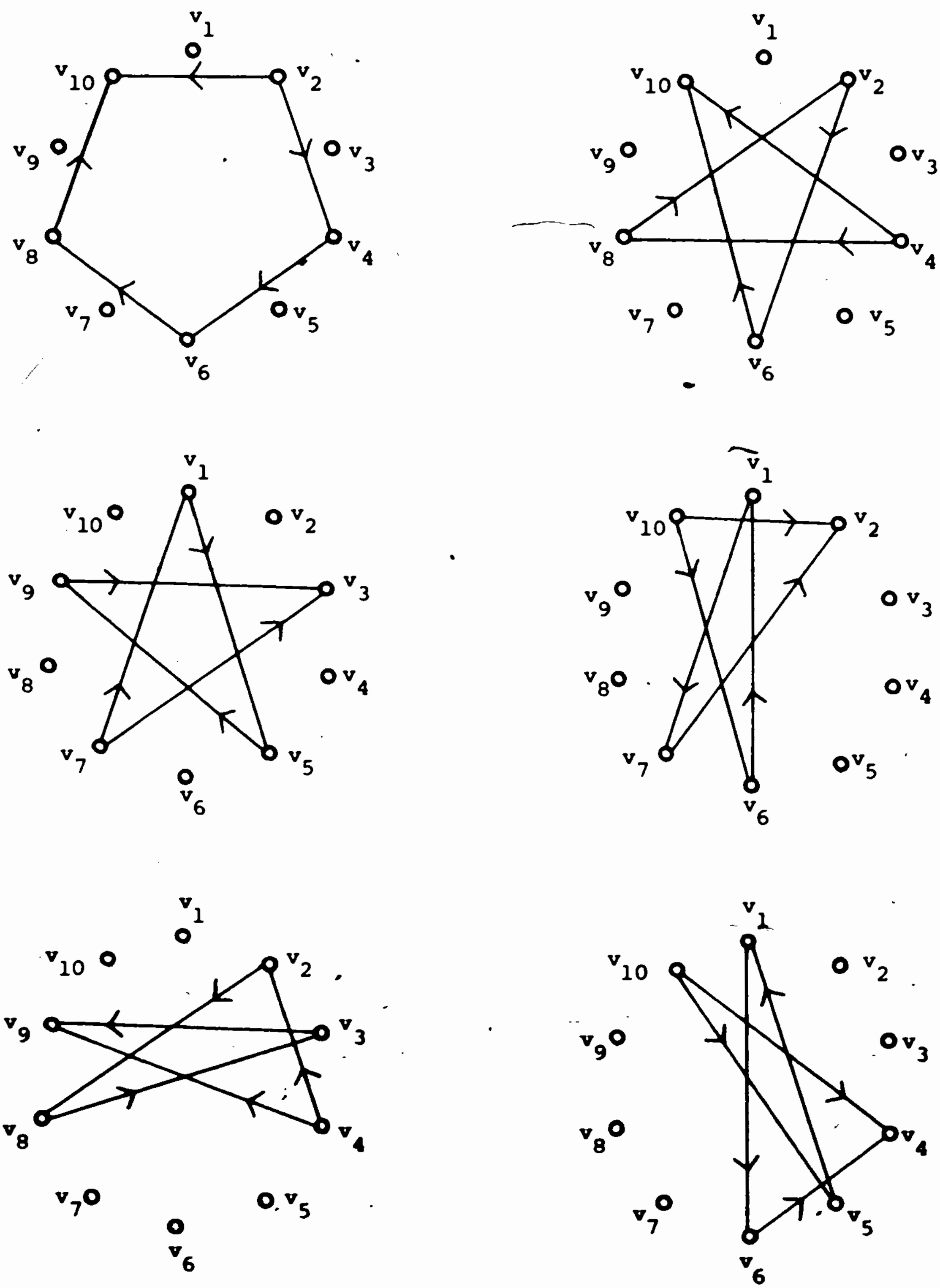
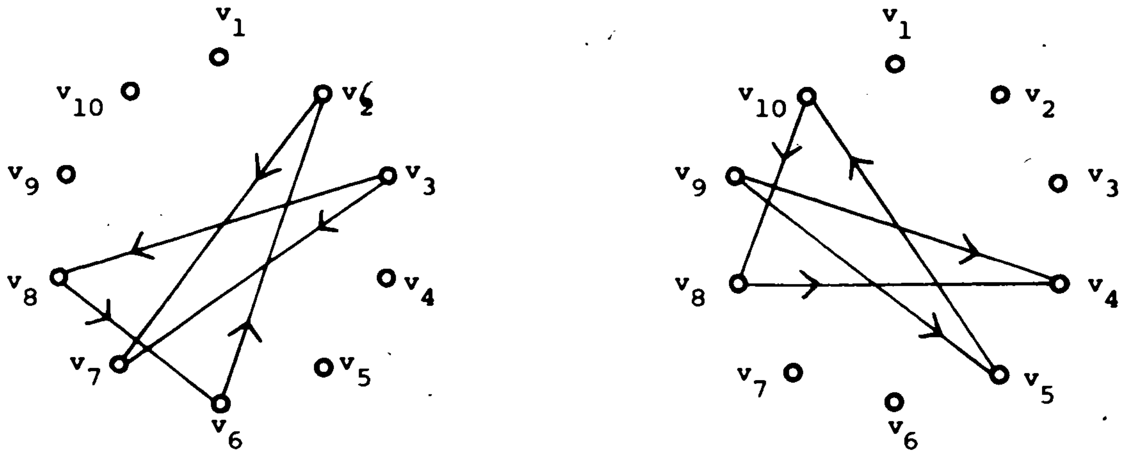


Figure continued.



$A_3 | D\Lambda$

FIGURE 4(M)

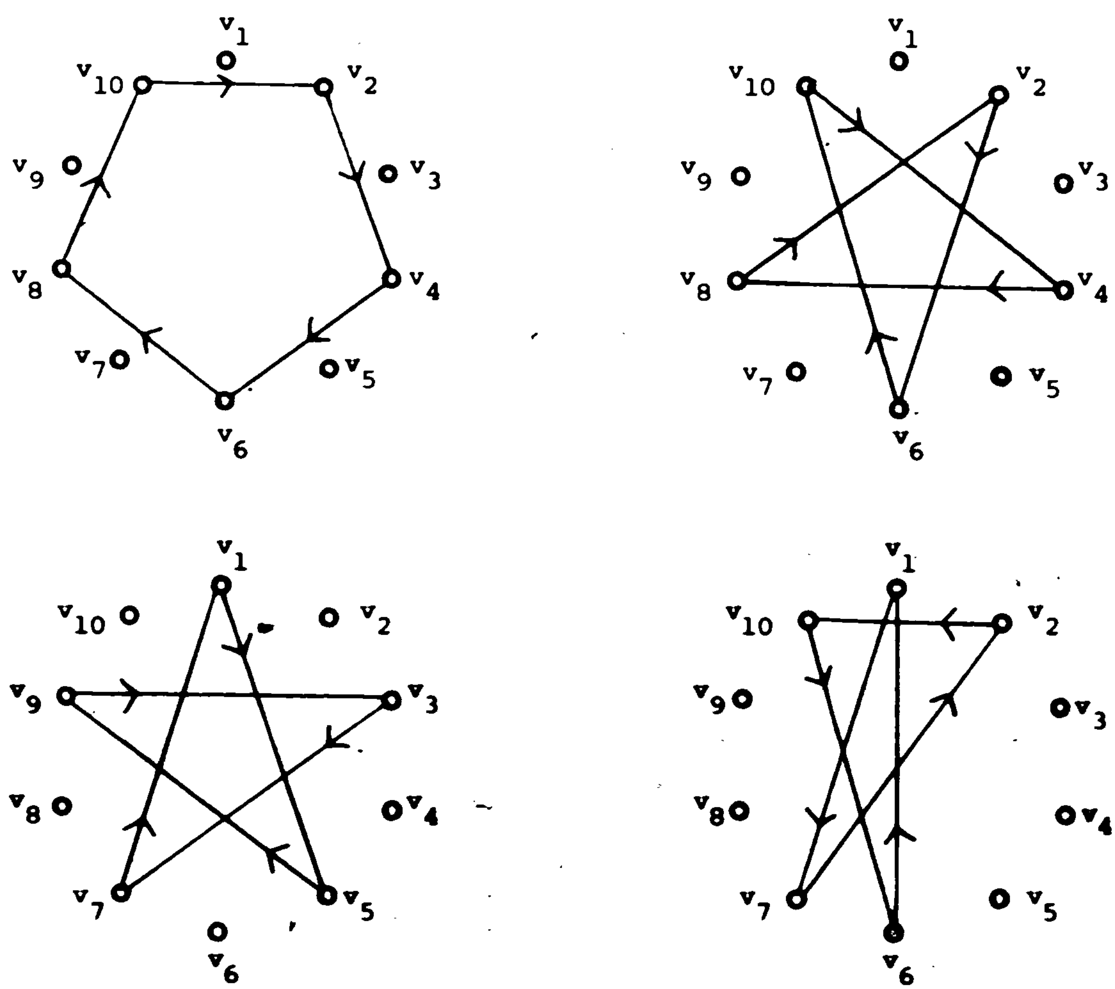
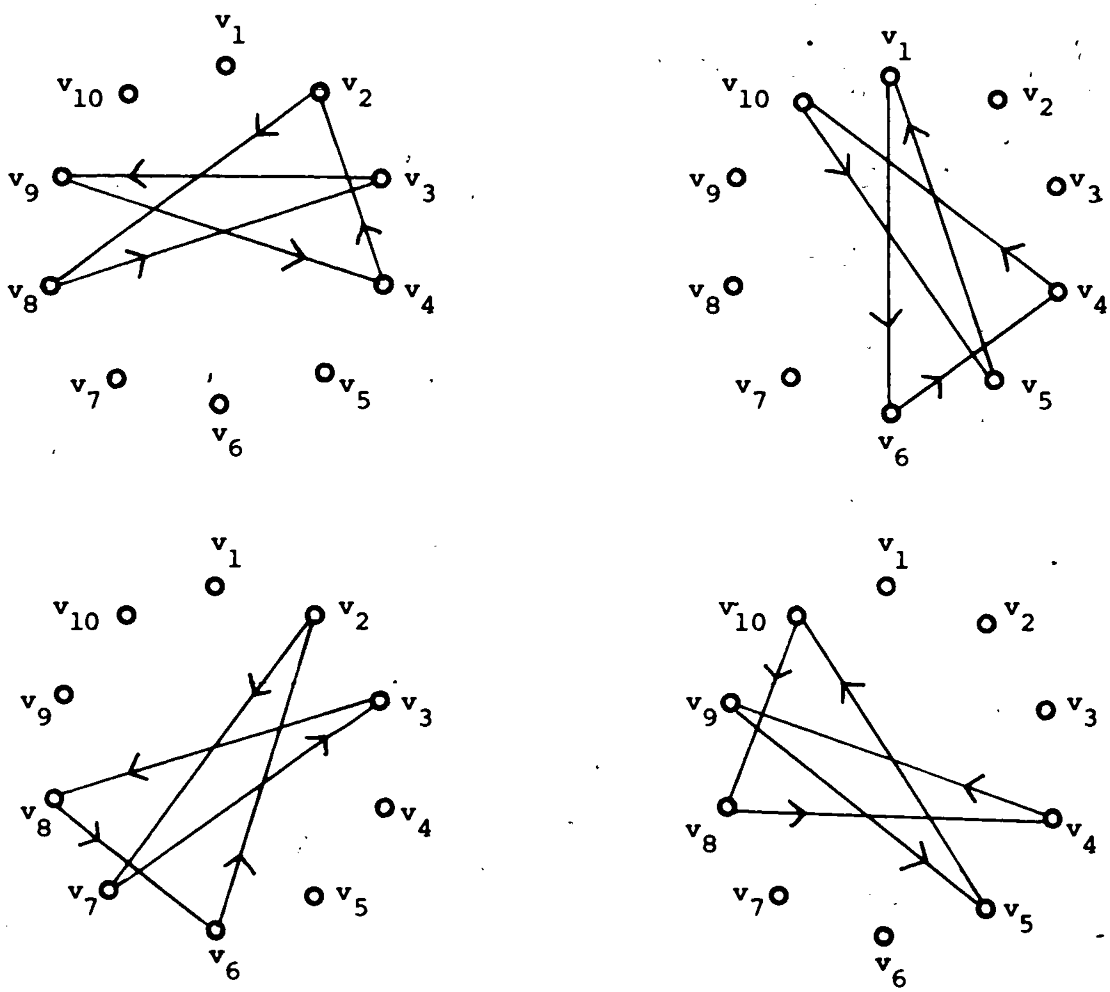


Figure continued.



$A_4 \mid D\Lambda$

FIGURE 4(N)

Now to show that $A_1 \mid D(K_{10} \cup {}_2K_{10})$, let Γ be a graph defined by

$$\Gamma = \Lambda - C^* + C' + \phi^4 C'' ,$$

where C^* is the 5-cycle $v_2, v_6, v_{10}, v_4, v_8, v_2$. Then $K_{10} \cup {}_2K_{10}$

can be decomposed into the graph Γ and twenty four disjoint 5-cycles:

$$C^*,\ C,\ C'',\ \phi^k C,\ \phi^k C'\ (1 \le k \le 9),\ \phi^2 C'',\ \phi^6 C'' \text{ and } \phi^8 C'' .$$

Since A_1 is self-converse, it is enough to show that $A_1 | D\Gamma$. Following is one such decomposition:

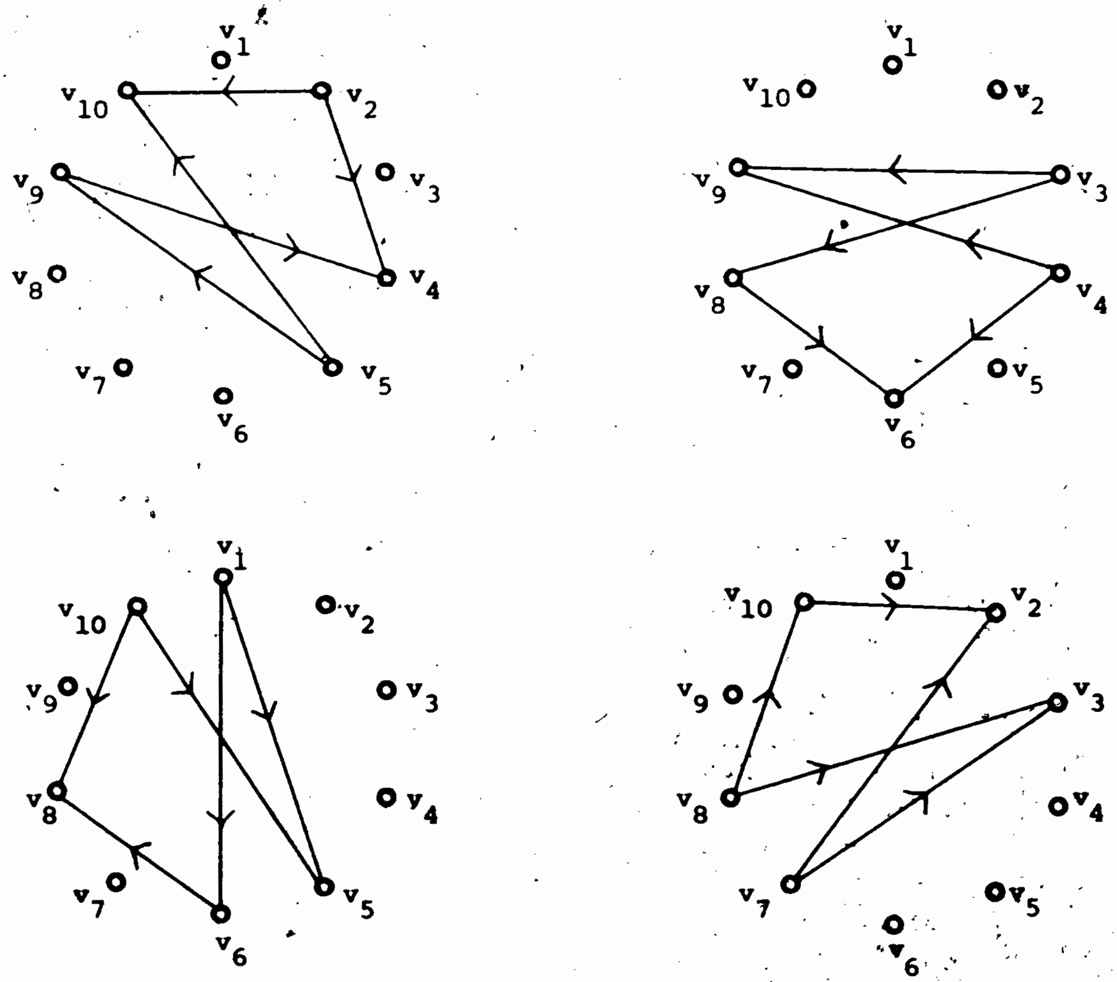
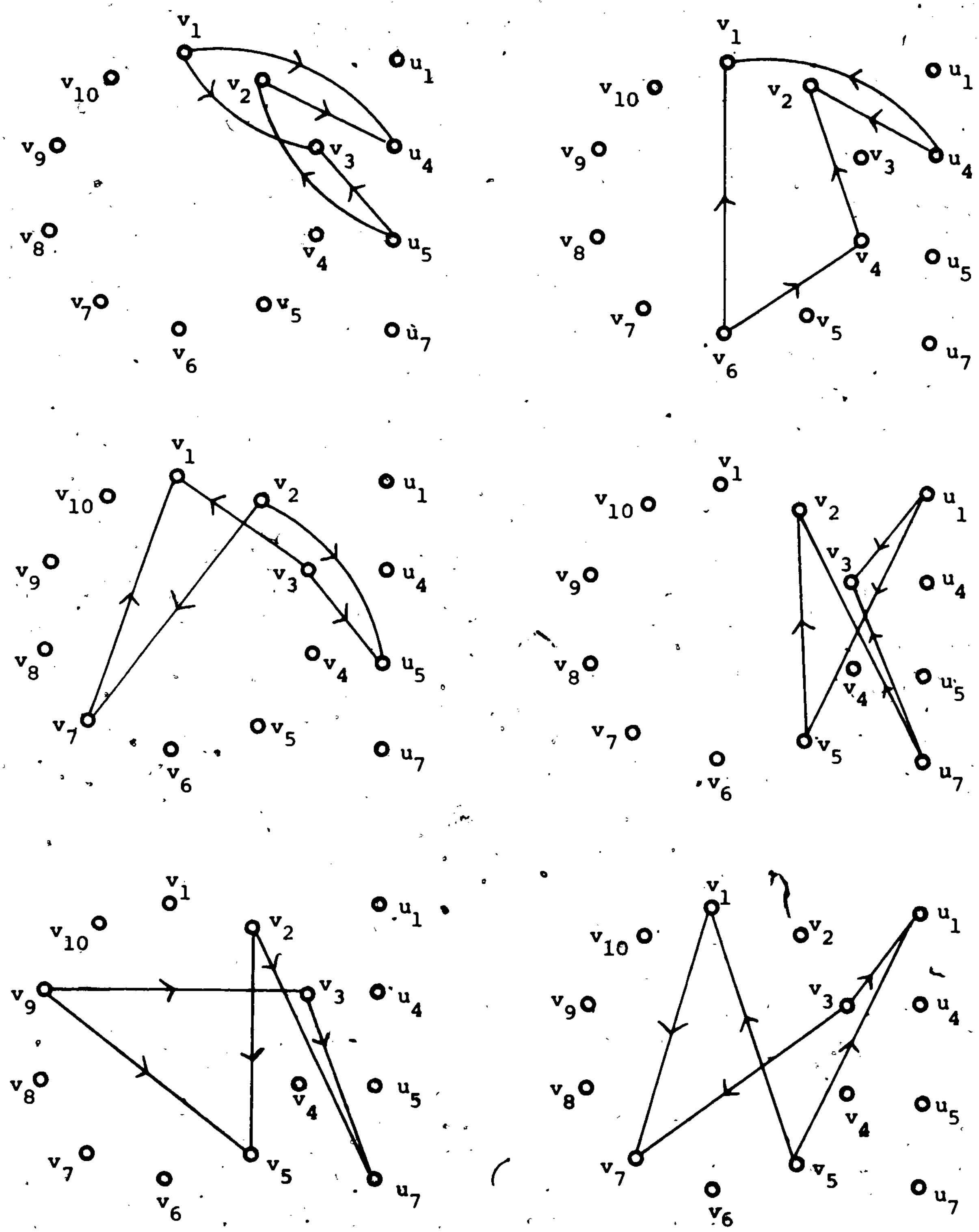


Figure continued.



$A_1 | D\Gamma$

FIGURE 4(O)

□

Finally, the results of Theorems 4.1, 4.3, 4.6, 4.7 and 4.8 can be put together into a single theorem.

4.9 THEOREM: $A_i \mid DK_n$, $n \geq 5$, *for* $i = 1,2,3,4$ *if and only if* $n \equiv 0$ *or* $1 \pmod 5$.

CHAPTER 5

In Chapter 4 we have seen that each orientation of C_5, the cycle of length five divides the complete symmetric digraph DK_n for all n's which satisfy the necessary conditions. In this chapter we consider the case of the cycle C_k where k is any odd positive integer and give some sufficient conditions so that any self-converse orientation X of C_k divides DK_n for $n \equiv 0$ or $1 \pmod{k}$. As an application of these results, we show that the necessary conditions are also sufficient for any self-converse orientation of C_7, the cycle of length seven, to divide DK_n.

Theorem 5.2 does not include the case for $k = 3$, as we know that the directed cycle C_3^* does not divide DK_6 (see [21]) which is one of the conditions of the hypothesis. We prove the result for odd $k > 3$ and for the case $k = 3$ we refer to Hung and Mendelsohn [21] who have shown that each orientation X of C_3 divides DK_n if and only if $n \equiv 0$ or $1 \pmod{3}$ and $n \geq 3$ with the exception of $n = 6$ in the case when X is the directed cycle C_3^*.

5.1 **LEMMA:** *Let* X *be a self-converse orientation of* C_k, *the cycle of length* k, *and* G *be any graph. If* $C_k | G$, *then* $X | DG$.

Proof: The proof follows from the fact that the orientation X of the cycle C_k is self-converse.□

5.2 **THEOREM:** *Let* X *be a self-converse orientation of* C_k, *a cycle of length* k *where* k *is an odd positive integer greater than* 3, *such that*

(i) $X|DK_{k+1}$

(ii) $X|DK_{2k}$

and (iii) $X|DK_{k,k,2k}$. *Then*

$X|DK_n$ *if* $n \equiv 0$ *or* 1 *(mod* k) *and* $n \geq k$.

Proof: To prove that the conditions $n \equiv 0$ or $1 \pmod{k}$ and $n \geq k$ are sufficient, we consider the three cases $n \equiv 1, k \pmod{2k}$, $n \equiv k+1 \pmod{2k}$ and $n \equiv 0 \pmod{2k}$ separately.

Case 1: Let $n \equiv 1$ or $k \pmod{2k}$. In this case Rosa [30] has shown that $C_k|K_n$. Since X is a self-converse orientation of C_k , the result then follows from Lemma 5.1.

Case 2: Let $n \equiv k+1 \pmod{2k}$, that is, $n = 2mk + (k+1)$ for some non-negative integer m . We write

$$K_n = K_{(2m+1)k+1} = [(2m+1)K_k + K_1] \cup {}_{2m+1}K_k$$

$$= (2m+1)K_{k+1} \cup {}_{2m+1}K_k$$

where the vertex set of the complete multipartite graph ${}_{2m+1}K_k$ is appropriately chosen. Since $X|DK_{k+1}$ by hypothesis, it is enough to show that $X|D({}_{2m+1}K_k)$. By Theorem 2.14, $C_k|{}_{2m+1}K_k$ and the result then follows from Lemma 5.1.

<u>Case 3</u>: Finally let $n \equiv 0 \pmod{2k}$, that is, $n = 2mk$ for some non-negative integer m. For $m = 1$, the result is true by hypothesis as $X|DK_{2k}$. In case $m = 2$, we write

$$K_{4k} = 2K_k \cup K_{2k} \cup K_{k,k,2k} ,$$

where the vertex set of each graph is appropriately chosen. Since k is odd, we know K_k can be decomposed into hamiltonian cycles, that is, $C_k|K_k$ and hence by Lemma 5.1 $X|DK_k$. Moreover, $X|DK_{2k}$ and $X|DK_{k,k,2k}$ by hypothesis and hence the result.

Now let $m \geq 3$. We write

$$K_n = K_{2mk} = mK_{2k} \cup {}_mK_{2k}$$

where the vertex set of the complete multipartite graph is appropriately chosen. Since $X|DK_{2k}$ by hypothesis, it is enough to show that $X|D({}_mK_{2k})$. We will show that $C_k|{}_mK_{2k}$ and the result then will follow from Lemma 5.1. With the graph ${}_mK_{2k}$ we associate a graph G as follows.

Let $V_1, V_2, \ldots, V_m$ be m maximal independent subsets of $V({}_mK_{2k})$. Clearly $|V_i| = 2k$ for $1 \leq i \leq m$. Also let $V_i = \{u_1^i, u_2^i, \ldots, u_k^i, u_{k+1}^i, \ldots, u_{2k}^i\}$ for $1 \leq i \leq m$.

We define

$$S_i = \{u_1^i, u_2^i, \ldots, u_k^i\}$$

$$\text{and} \quad S_{m+i} = \{u_{k+1}^i, u_{k+2}^i, \ldots, u_{2k}^i\}$$

for $1 \le i \le m$. Then

$$V(G) = \{S_1, S_2, \ldots, S_{2m}\}$$

and an edge S_iS_j corresponds to the set of all edges in the original graph ${}_mK_{2k}$ which join vertices in S_i with vertices in S_j . We observe S_iS_j is an edge for all $1 \le i, j \le 2m$, $i \ne j$ except when $j = m + i$ for $1 \le i \le m$. But S_iS_{m+i} , $1 \le i \le m$ are not edges of G , and form a 1-factor of $\overline{G}$, the complement of G . Thus, G is isomorphic to $K_{2m}-I$ where I is a 1-factor of K_{2m} . If $2m \equiv 0$ or $2 \pmod 6$, then $2m \ge 6$, as $m \ge 3$ and hence by Prop. 1.3, G can be decomposed into C_3's. If $2m \equiv 4 \pmod 6$, then $2m \ge 10$, as $m \ge 3$ and hence by Cor. 2.12 G can be decomposed into one K_5 and the rest C_3's. Hence for $m \ge 3$, G can be decomposed into C_3's and at most one K_5 , which amounts to the fact that the graph ${}_mK_{2k}$ can be decomposed into ${}_3K_k$'s and at most one ${}_5K_k$. Since k is odd, $C_k | {}_3K_k$ and $C_k | {}_5K_k$ by Theorem 2.14. This completes the proof.□

Let C_k^* , k odd, denote the directed cycle of length k . Bermond and Faber [5] have shown the existence of a balanced circuit design $(k+1,k,1)$ for any odd $k \ge 3$. Also Bermond and Sotteau [6] have shown the existence of a balanced circuit design $(2k,k,1)$ for $k \ge 5$. In other words, $C_k^* | DK_{k+1}$ for any odd $k \ge 3$ and $C_k^* | DK_{2k}$ for any odd $k \ge 5$. Moreover, Sotteau [33] has shown that $C_k^* | DK_{k,k,2k}$ for any odd $k \ge 5$. In view of the fact that the directed cycle C_k^* is a self-converse orientation of C_k, the cycle of length k, Theorem 5.2 gives the following result of Sotteau [33].

5.3 THEOREM: DK_n *can be decomposed into* k-*circuits (directed cycles of length* k), k *odd*, $k \geq 5$ *if* $n \equiv 0$ *or* 1 (*mod* k) *and* $n \geq k$.

In case $k = p^\alpha$ where p is an odd prime and α a positive integer, the necessary conditions for a self-converse orientation X of C_{p^α} to divide DK_n are $n \equiv 0$ or $1 \pmod{p^\alpha}$ and $n > p^\alpha$. Thus Theorem 5.2 gives the following in this case.

5.4 THEOREM: *Let* X *be a self-converse orientation of* C_{p^α}, *where* p *is an odd prime and* α *a positive integer, such that*

(i) $X | DK_{p^\alpha+1}$

(ii) $X | DK_{2\cdot p^\alpha}$

and (iii) $X | DK_{p^\alpha, p^\alpha, 2\cdot p^\alpha}$. *Then*

$X | DK_n$ *if and only if* $n \equiv 0$ *or* 1 (*mod* p^α) *and* $n > p^\alpha$.

As an application of Theorem 5.4, we show that a self-converse orientation X of C_7 divides DK_n if and only if $n \equiv 0$ or $1 \pmod 7$ and $n \geq 7$. We know that C_7 has ten orientations and they are shown in Figure 5(A).

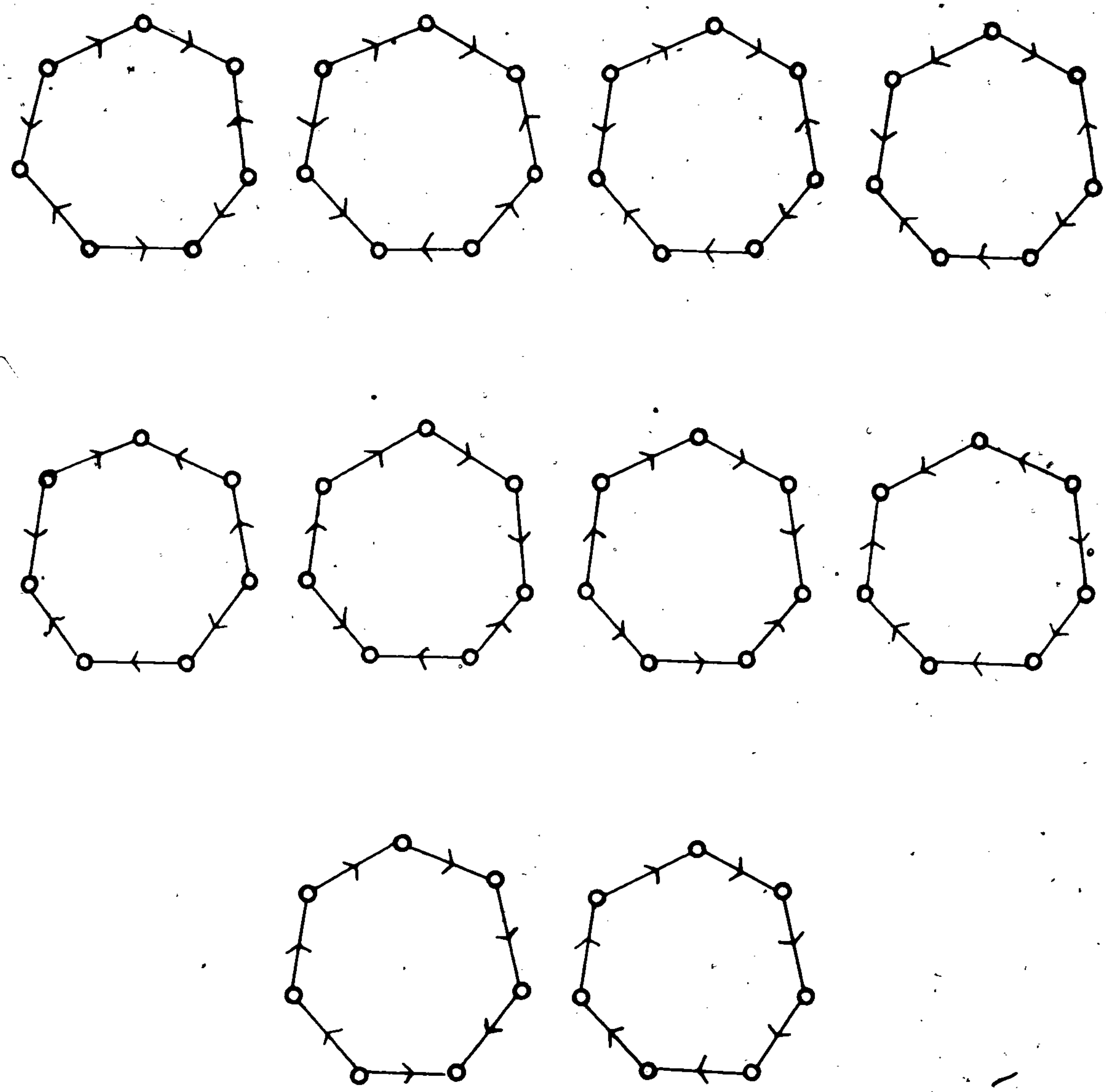


FIGURE 5(A)

The orientations A_{21} and A_{22} have a path of maximum length 2; A_{31}, A_{32} and A_{33} have a path of maximum length 3; A_{41} and A_{42} have a path of maximum length 4; A_{51} and A_{61} have paths of maximum length 5 and 6 respectively and A_{71} is the directed cycle.

Out of these ten orientations of C_7, A_{21}, A_{22}, A_{31}, A_{41}, A_{42}, A_{51}, A_{61} and A_{71} are the only self-converse orientations.

5.5 THEOREM: *Let* X *be one of the eight self-converse orientations of* C_7, *then* $X|DK_n$ *if and only if* $n \equiv 0$ *or* $1 \pmod 7$ *and* $n \geq 7$.

<u>Proof</u>: In view of the Theorem 5.4, it is enough to show that $X|DK_8$, $X|DK_{14}$ and $X|DK_{7,7,14}$. Henceforth we shall write the eight self-converse orientations as:

If a, b, c, d, e, f, g, a is a 7-cycle then

$$A_{21} : a \to b \to c \leftarrow d \to e \leftarrow f \to g \leftarrow a$$

$$A_{22} : a \to b \to c \leftarrow d \leftarrow e \to f \to g \leftarrow a$$

$$A_{31} : a \to b \to c \to d \leftarrow e \to f \to g \leftarrow a$$

$$A_{41} : a \to b \to c \to d \to e \leftarrow f \to g \leftarrow a$$

$$A_{42} : a \to b \to c \to d \to e \leftarrow f \leftarrow g \leftarrow a$$

$$A_{51} : a \to b \to c \to d \to e \to f \leftarrow g \leftarrow a$$

$$A_{61} : a \to b \to c \to d \to e \to f \to g \leftarrow a$$

$$A_{71} : a \to b \to c \to d \to e \to f \to g \to a$$

(i) Let $V(DK_8) = \{u_1, u_2, u_3, \ldots, u_8\}$. We list below a decomposition of DK_8 into each X . The direction of an edge is as given by the top cycle.

A_{21}: a → b → c ← d → e ← f → g ← a

u_2	u_3	u_5	u_8	u_7	u_4	u_6	u_2
u_5	u_8	u_1	u_6	u_4	u_3	u_7	u_5
u_8	u_4	u_5	u_7	u_2	u_1	u_6	u_8
u_3	u_2	u_7	u_1	u_5	u_6	u_8	u_3
u_7	u_6	u_2	u_8	u_3	u_4	u_1	u_7
u_2	u_5	u_4	u_1	u_3	u_7	u_8	u_2
u_3	u_1	u_8	u_4	u_2	u_5	u_6	u_3
u_6	u_7	u_4	u_2	u_1	u_5	u_3	u_6

A_{22}: a → b → c ← d ← e → f → g ← a

u_2	u_3	u_5	u_8	u_7	u_4	u_6	u_2
u_7	u_5	u_8	u_1	u_6	u_4	u_3	u_7
u_4	u_5	u_7	u_2	u_1	u_6	u_8	u_4
u_1	u_7	u_2	u_3	u_8	u_6	u_5	u_1
u_6	u_7	u_1	u_4	u_3	u_8	u_2	u_6
u_3	u_1	u_4	u_5	u_2	u_8	u_7	u_3
u_8	u_1	u_3	u_6	u_5	u_2	u_4	u_8
u_5	u_3	u_6	u_7	u_4	u_2	u_1	u_5

A_{31}: a → b → c → d ← e → f → g ← a

u_6	u_8	u_1	u_2	u_4	u_7	u_3	u_6
u_5	u_3	u_2	u_1	u_7	u_4	u_8	u_5
u_8	u_2	u_3	u_4	u_6	u_1	u_5	u_8
u_7	u_5	u_4	u_3	u_1	u_6	u_2	u_7
u_2	u_4	u_5	u_6	u_8	u_3	u_7	u_2
u_1	u_7	u_6	u_5	u_3	u_8	u_4	u_1
u_4	u_6	u_7	u_8	u_2	u_5	u_1	u_4
u_3	u_1	u_8	u_7	u_5	u_2	u_6	u_3

A_{41}: a → b → c → d → e ← f → g ← a

u_4	u_7	u_8	u_5	u_3	u_2	u_6	u_4
u_1	u_6	u_4	u_3	u_7	u_5	u_8	u_1
u_6	u_1	u_2	u_7	u_5	u_4	u_8	u_6
u_3	u_8	u_6	u_5	u_1	u_7	u_2	u_3
u_8	u_3	u_4	u_1	u_7	u_6	u_2	u_8
u_5	u_2	u_8	u_7	u_3	u_1	u_4	u_5
u_2	u_5	u_6	u_3	u_1	u_8	u_4	u_2
u_7	u_4	u_2	u_1	u_5	u_3	u_6	u_7

A_{42}: $a \to b \to c \to d \to e \leftarrow f \leftarrow g \leftarrow a$

u_6	u_3	u_7	u_4	u_2	u_1	u_8	u_6
u_3	u_5	u_8	u_4	u_7	u_1	u_2	u_3
u_8	u_5	u_1	u_6	u_4	u_3	u_2	u_8
u_5	u_7	u_2	u_6	u_1	u_3	u_4	u_5
u_2	u_7	u_3	u_8	u_6	u_5	u_4	u_2
u_7	u_1	u_4	u_8	u_3	u_5	u_6	u_7
u_4	u_1	u_5	u_2	u_8	u_7	u_6	u_4
u_1	u_3	u_6	u_2	u_5	u_7	u_8	u_1

A_{51}: $a \to b \to c \to d \to e \to f \leftarrow g \leftarrow a$

u_4	u_7	u_8	u_5	u_3	u_2	u_6	u_4
u_5	u_8	u_1	u_6	u_4	u_3	u_7	u_5
u_6	u_1	u_2	u_7	u_5	u_4	u_8	u_6
u_7	u_2	u_3	u_8	u_6	u_5	u_1	u_7
u_8	u_3	u_4	u_1	u_7	u_6	u_2	u_8
u_1	u_4	u_5	u_2	u_8	u_7	u_3	u_1
u_2	u_5	u_6	u_3	u_1	u_8	u_4	u_2
u_3	u_6	u_7	u_4	u_2	u_1	u_5	u_3

A_{61}: $a \to b \to c \to d \to e \to f \to g \leftarrow a$

u_4	u_7	u_8	u_5	u_3	u_2	u_6	u_4
u_5	u_8	u_1	u_6	u_4	u_3	u_7	u_5
u_6	u_1	u_2	u_7	u_5	u_4	u_8	u_6
u_7	u_2	u_3	u_8	u_6	u_5	u_1	u_7

$a \to b \to c \to d \to e \to f \to g \leftarrow a$

u_8	u_3	u_4	u_1	u_7	u_6	u_2	u_8
u_1	u_4	u_5	u_2	u_8	u_7	u_3	u_1
u_2	u_5	u_6	u_3	u_1	u_8	u_4	u_2
u_3	u_6	u_7	u_4	u_2	u_1	u_5	u_3

A_{71}: The following directed cycles give a decomposition of DK_8 into A_{71}'s.

$$u_1 \to u_5 \to u_2 \to u_6 \to u_3 \to u_7 \to u_4 \to u_1$$

and

$$u_8 \rightarrow u_i \leftarrow u_{i+6} \rightarrow u_{i+1} \rightarrow u_{i+4} \rightarrow u_{i+2} \rightarrow u_{i+3} \rightarrow u_8$$

for $i = 1,2,\ldots,7$ and the subscripts are taken modulo 7 .

(ii) Let $V(DK_{14}) = \{u_0,u_1,\ldots,u_{13}\}$. The following gives a decomposition of DK_{14} into A_{21}

$$u_0 \rightarrow u_i \leftarrow u_{i+1} \rightarrow u_{i+12} \leftarrow u_{i+2} \rightarrow u_{i+11} \leftarrow u_{i+3} \rightarrow u_0$$

and

$$u_i \rightarrow u_{i+1} \leftarrow u_{i+12} \rightarrow u_{i+2} \leftarrow u_{i+10} \rightarrow u_{i+3} \leftarrow u_{i+9} \rightarrow u_i$$

for $i = 1,2,\ldots,13$ where the subscripts are taken modulo 13.

To get the decomposition of DK_{14} into the other self-converse orientations of C_7 we write

$$K_{14} = K_8 + K_6$$

where $V(K_8) = \{u_0,u_1,u_2,u_3,u_4,u_5,u_6,u_\infty\}$

and $V(K_6) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$

using all the edges of the form u_iv_j , $1 \le i,\ j \le 6$ and the six edges of the 6-cycle

$$v_1, v_2, v_3, v_4, v_5, v_6, v_1$$

we construct the following six 7-cycles

$$C_i: \quad v_i, u_i, v_{i+1}, u_{i+2}, v_{i+4}, u_{i+1}, v_{i+5}, v_i$$

for $1 \le i \le 6$ where the subscripts are taken modulo 6 . Then K_{14} can be decomposed into six 7-cycles C_i, $1 \le i \le 6$, and a graph H shown in Figure 5(B).

H:

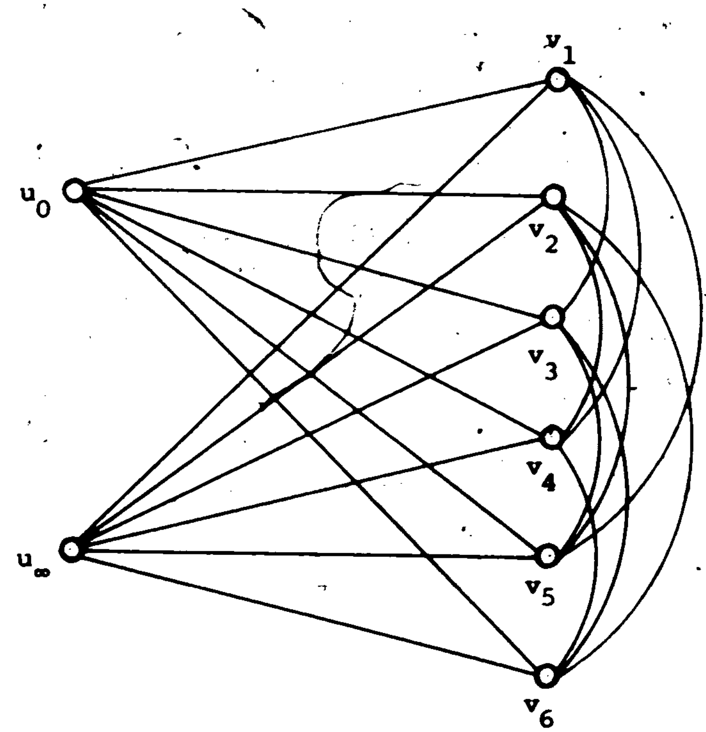


FIGURE 5(B)

In view of Lemma 5.1, it is enough to show that each self-converse orientation X divides DH. Here we list a decomposition of DH into, A_{22}, A_{31}, A_{41}, A_{42}, A_{51}, A_{61} and A_{71}.

In each decomposition below the subscripts are to be taken modulo 6 except the subscript 0 and ∞ which remain the same in each orientation. The cycles

$$v_i \to u_\infty \to v_{i+5} \leftarrow v_{i+1} \leftarrow v_{i+4} \to u_0 \to v_{i+2} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{22}. The cycles

$$v_i \to u_\infty \to v_{i+3} \to v_{i+1} \leftarrow v_{i+4} \to u_0 \to v_{i+2} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{31}. The cycles

$$v_i \to u_\infty \to v_{i+2} \to u_0 \to v_{i+3} \leftarrow v_{i+1} \to v_{i+4} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{41}. The cycles

$$v_i \to u_0 \to v_{i+2} \to v_{i+5} \to v_{i+1} \leftarrow u_\infty \leftarrow v_{i+4} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{42}. The cycles

$$v_i \to u_0 \to v_{i+4} \to v_{i+2} \to u_\infty \to v_{i+5} \leftarrow v_{i+3} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{51}. The cycles

$$v_i \to u_\infty \to v_{i+3} \to v_{i+1} \to v_{i+4} \to u_0 \to v_{i+2} \leftarrow v_i$$

for $1 \le i \le 6$ give a decomposition of DH into A_{61} and finally, the cycles

$$u_0 \to v_i \to v_{i+4} \to v_{i+1} \to v_{i+3} \to u_\infty \to v_{i+5} \to u_0$$

for $1 \le i \le 6$ give a decomposition of DH into A_{71}.

(iii) Let $V(DK_{7,7,14}) = \{u_1,u_2,\ldots,u_7\} \cup \{v_1,v_2,\ldots,v_7\} \cup$

$$\{w_1,w_2,\ldots,w_7,z_1,z_2,\ldots,z_7\}$$

where the three sets in the union are the maximal independent subsets of $V(DK_{7,7,14})$. The undirected graph $K_{7,7,14}$ with $V(K_{7,7,14}) = V(DK_{7,7,14})$ can be decomposed into twenty-eight 7-cycles given below and graph G, which is the subgraph induced by the remaining edges of $K_{7,7,14}$. The cycles are

$$u_i,w_i,u_{i+1},w_{i+2},u_{i+4},v_{i+5},z_i,u_i$$

$$v_i,z_i,v_{i+1},z_{i+2},v_{i+4},u_{i+5},w_i,v_i$$

$$u_i,v_{i+2},u_{i+2},v_{i+5},w_{i+6},v_{i+4},z_{i+1},u_i$$

$$u_i,z_{i+2},u_{i+4},z_{i+3},v_i,w_{i+3},v_{i+4},u_i$$

for $1 \le i \le 7$ where the subscripts are taken modulo 7. In view of Lemma 5.1, to prove that a self-converse orientation X divides

$DK_{7,7,14}$, it is enough to show that X divides DG. Here we list a decomposition of DG into the orientations A_{21}, A_{22}, A_{31} and A_{41}.

In each decomposition below the subscripts are to be taken modulo 7. The cycles

$$w_i \to u_{i+3} \to z_i \leftarrow u_{i+4} \to w_{i+1} \leftarrow u_{i+5} \to v_{i+3} \leftarrow w_i$$

and

$$w_i \to v_{i+2} \to w_{i+6} \leftarrow v_{i+1} \to u_{i+3} \leftarrow z_i \to u_{i+4} \leftarrow w_i$$

for $1 \le i \le 7$ give a decomposition of DG into A_{21}. The cycles

$$z_i \to u_{i+3} \to w_i \leftarrow v_{i+3} \leftarrow u_{i+5} \to w_{i+1} \to u_{i+4} \leftarrow z_i$$

and

$$w_i \to u_{i+4} \to z_i \leftarrow u_{i+3} \leftarrow v_{i+1} \to w_{i+6} \to v_{i+2} \leftarrow w_i.$$

for $1 \le i \le 7$ give a decomposition of DG into A_{22}. The cycles

$$v_{i+3} \to u_{i+5} \to w_{i+1} \to u_{i+4} \leftarrow z_i \to u_{i+3} \to w_i \leftarrow v_{i+3}$$

and

$$u_{i+3} \to v_{i+1} \to w_{i+6} \to v_{i+2} \leftarrow w_i \to u_{i+4} \to z_i \leftarrow u_{i+3}$$

for $1 \leq i \leq 7$ give a decomposition of DG into A_{31} . The cycles

$$v_{i+3} \to u_{i+5} \to w_{i+1} \to u_{i+4} \to z_i \leftarrow u_{i+3} \to w_i \leftarrow v_{i+3}$$

and

$$z_i \to u_{i+3} \to v_{i+1} \to w_{i+6} \to v_{i+2} \leftarrow w_i \to u_{i+4} \leftarrow z_i$$

for $1 \leq i \leq 7$ give a decomposition of DG into A_{41} .

Now we give a decomposition of $DK_{7,7,14}$ into each of the remaining orientations A_{42}, A_{51}, A_{61} and A_{71} . This time we decompose the undirected graph $K_{7,7,14}$ into fourteen 7-cycles given below and a graph G', which is the subgraph induced by the remaining edges of $K_{7,7,14}$. The cycles are

$$v_i, w_{i+6}, v_{i+1}, w_{i+5}, v_{i+2}, w_{i+4}, u_{i+4}, v_i$$

and

$$w_i, u_{i+6}, w_{i+1}, u_{i+5}, w_{i+2}, u_{i+4}, v_{i+6}, w_i$$

for $1 \leq i \leq 7$ where the subscripts are taken modulo 7 . To prove that a self-converse orientation X divides $DK_{7,7,14}$, because of Lemma 5.1 it is enough to show that X divides DG'. Here we list a decomposition of DG' into each of A_{42}, A_{51}, A_{61} and A_{71}. In each decomposition the subscripts are to be taken modulo 7. The cycles

$$v_i \to z_1 \to v_{i+1} \to z_2 \to v_{i+2} \leftarrow z_3 \leftarrow u_{i+3} \leftarrow v_i$$

$$u_i \to v_i \to u_{i+2} \to z_1 \to u_{i+6} \leftarrow v_{i+5} \leftarrow z_7 \leftarrow u_i$$

$$v_{i+1} \to z_6 \to v_i \to w_i \to u_{i+1} \leftarrow z_2 \leftarrow u_i \leftarrow v_{i+1}$$

$$v_i \to z_4 \to v_{i+1} \to z_5 \to v_{i+2} \leftarrow u_{i+5} \leftarrow z_3 \leftarrow v_i$$

$$u_{i+6} \to z_4 \to u_{i+2} \to v_i \to u_i \leftarrow z_7 \leftarrow v_{i+5} \leftarrow u_{i+6}$$

$$u_{i+2} \to z_5 \to u_{i+1} \to w_i \to v_i \leftarrow u_{i+6} \leftarrow z_6 \leftarrow u_{i+2}$$

for $1 \le i \le 7$ give a decomposition of DG' into A_{42}. The cycles

$$v_i \to z_1 \to v_{i+1} \to z_2 \to v_{i+2} \to z_3 \leftarrow u_{i+3} \leftarrow v_i$$

$$u_i \to v_i \to u_{i+2} \to z_1 \to u_{i+6} \to v_{i+5} \leftarrow z_7 \leftarrow u_i$$

$$v_i \to z_6 \to v_{i+1} \to u_i \to z_2 \to u_{i+1} \leftarrow w_i \leftarrow v_i$$

$$z_3 \to v_i \to z_4 \to v_{i+1} \to z_5 \to v_{i+2} \leftarrow u_{i+5} \leftarrow z_3$$

$$v_{i+5} \to u_{i+6} \to z_4 \to u_{i+2} \to v_i \to u_i \leftarrow z_7 \leftarrow v_{i+5}$$

$$u_{i+1} \to z_5 \to u_{i+2} \to z_6 \to u_{i+6} \to v_i \leftarrow w_i \leftarrow u_{i+1}$$

for $1 \le i \le 7$ give a decomposition of DG' into A_{51}. The cycles

$$v_i \to z_1 \to v_{i+1} \to z_2 \to v_{i+2} \to z_3 \to u_{i+3} \leftarrow v_i$$

$$u_i \to v_i \to u_{i+2} \to z_1 \to u_{i+6} \to v_{i+5} \to z_7 \leftarrow u_i$$

$$v_i \to z_6 \to v_{i+1} \to u_i \to z_2 \to u_{i+1} \to w_i \leftarrow v_i$$

$$u_{i+5} \to z_3 \to v_i \to z_4 \to v_{i+1} \to z_5 \to v_{i+2} \leftarrow u_{i+5}$$

$$z_7 \to v_{i+5} \to u_{i+6} \to z_4 \to u_{i+2} \to v_i \to u_i \leftarrow z_7$$

$$w_i \to u_{i+1} \to z_5 \to u_{i+2} \to z_6 \to u_{i+6} \to v_i \leftarrow w_i$$

for $1 \le i \le 7$ give a decomposition of DG' into A_{61}. The cycles

$$v_i \to z_1 \to v_{i+1} \to z_2 \to v_{i+2} \to z_3 \to u_{i+3} \to v_i$$

$$u_i \to v_i \to u_{i+2} \to z_1 \to u_{i+6} \to v_{i+5} \to z_7 \to u_i$$

$$u_i \to z_2 \to u_{i+1} \to w_i \to v_i \to z_6 \to v_{i+1} \to u_i$$

$$z_3 \to v_i \to z_4 \to v_{i+1} \to z_5 \to v_{i+2} \to u_{i+5} \to z_3$$

$$u_i \to z_7 \to v_{i+5} \to u_{i+6} \to z_4 \to u_{i+2} \to v_i \to u_i$$

$$w_i \to u_{i+1} \to z_5 \to u_{i+2} \to z_6 \to u_{i+6} \to v_i \to w_i$$

for $1 \le i \le 7$ give a decomposition of DG' into A_{71}. This completes the proof of the theorem. □

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