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THE EQUILIBRIUM VALUATION OPERATOR

AND FINANCIAL MARKET EFFICIENCY

by

William John Heaney

B.Sc., University of Saskatchewan, 1967

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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ABSTRACT

A valuation operator is developed that prices risky income streams when no opportunities for arbitrage profits exist. The properties of the valuation operator are investigated under alternate market environments.

It is shown that when firm values follow a diffusion process, the equilibrium valuation operator can be expressed as a unique function of the present and future firm values, and the time. This allows for the determination of the prices of a wide variety of financial instruments contingent upon the present firm values.

When the firm values are endogenous, it is further shown that, if financial markets are capable of exhausting the gains from exchange, the valuation operator is related to future aggregate resource constraints in a simple manner such that it permits the development of explicit expressions for the operator associated with many of the discrete time models that appear in the literature of financial theory.

Finally, it is shown that, when beliefs are contingent on prevailing prices, the continuous time capital asset pricing model emerges.

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Pareto Efficient Capital Markets

In an Arrow-Debreu world any efficient (i.e. Pareto-Optimal allocation of risk bearing can be achieved by a system of competitive markets in claims on state contingent commodities: the claims being traded are (iron clad) contracts for the delivery of each and every commodity in each and every future state of the world. If individuals maximize expected utility the same allocation can be achieved by competitive exchange, first in a complete securities market, and then in the (spot) commodity markets of whatever future state of the world occurs (Arrow, 1964). The securities are claims to money, the amount depending on which state of the world actually occurs. In general, it is required (for efficiency) that there be as many linearly independent securities as there are states of nature and also that the money prices of commodities in each state are known to the individuals purchasing the securities.

The significance of securities is that they allow for economizing on markets; but in the absence of any costs associated with opening and operating markets, securities play no essential role since any desired allocation can be achieved in the market for contingent commodity claims alone. Furthermore, the requirement, for the efficiency of security markets, that individuals correctly foresee future spot commodity prices, is so stringent, we can be sure that in practice a security market in combination with future spot

markets is not equivalent to a market for contingent commodity claims.

However, abstracting from the diversity of commodities, and assuming that utility depends only on wealth [or on consumption for lifetime decision makers] in each state of nature, then security markets can accomplish the same allocations as contingent commodity markets, provided the security market is complete.

While, under the conditions stated in the last paragraph, a complete security market is sufficient to guarantee a Pareto optimal allocation of risk bearing, it is not necessary, there being the well known cases when only two securities, one riskless the other risky, are required for efficiency:

a) The case where investors have homogeneous otherwise arbitrary beliefs, and have utility functions belonging to the same linear risk tolerance class;

b) The case where investors have homogeneous Gaussian beliefs and arbitrary utility functions [so long as the utility functions are compatible with Gaussian beliefs, i.e. as long as expected utility is defined][1].

In these cases all desired patterns of returns across future states of the world can be constructed from securities issued by firms, which are claims to the wealth generated by the firms. Such securities are called primary securities.

---

[1] There is also the (probably less well known) case of homogeneous separating distributions of Ross [1975].

However, in general, Pareto efficient allocations cannot be achieved in the primary security markets alone. If this is the case, markets in secondary securities, issued by individuals or financial intermediaries, can be expected to open up. In the absence of costs associated with opening and operating markets secondary securities can be expected to proliferate until the gains from issuing them have been exhausted.

Perhaps the most investigated of any secondary security is the option. The role of the option in attaining efficiency in security markets has been studied by Ross [1976]. The return on an option depends upon the return on the underlying primary securities, and options are capable of distinguishing two states of the world if the returns on the primary assets are different in the two states. In fact, Ross [1976] has shown that, provided the pattern of returns on primary assets are not identical in any two states of the world, a fully efficient market can be achieved by supplementing the primary securities with a simple call option written on a <u>single</u> portfolio of primary securities.

In general, identifying the particular portfolio of interest is not a simple manner, but it is possible to identify the conditions under which, for example, the market portfolio plays this role.

Thus, in a one period model, if the levels of aggregate social wealth are different in each future state of the world, then a full set of (simple) call options written on the market

portfolio can distinguish all states[2]. On the other hand,
if aggregate wealth is the same in some future states, options
written on the market portfolio cannot distinguish these
states. However, if all individuals desire the same payoffs
in those states with the same aggregate wealth then the
inability to distinguish the states is not important.
Hakansson [1978] shows that this is the case when individuals
beliefs, conditional on aggregate wealth are homogeneous. In
this case options written on the market portfolio, or super-
shares, (securities which pay one dollar contingent on a given
level of aggregate wealth, zero otherwise), together with
riskless borrowing and lending are sufficient to ensure a
Pareto efficient market. An analagous result holds
when utility is defined on consumption, for the lifetime decision
maker, and options are written on aggregate consumption with
exercise prices equal to the various possible levels of aggregate
consumption, Breeden and Litzenberger [1978]

Pricing of Securities

Valuation of the primary assets involves solving the
aggregation problem, which according to Rubinstein [1974], is
"the chief difficulty befouling the analysis of securities
market equilibrium". On the other hand the valuation of
secondary securities need not involve the aggregation problem
provided that primary security prices be taken as given, as for

---

[2] Choose the exercise prices of the options to correspond to
the various levels of aggregate wealth.

example, is the case with the option pricing formula of Black-Scholes [1971].

Models which value the primary assets are classified into two types, discrete time and continuous time models, and this distinction is generally viewed as being an important one. On the other hand, secondary securities in discrete time are, under certain conditions, priced according to the Black-Scholes option pricing formula, (Rubinstein [1976], Breeden and Litzenberger [1978]), despite the fact that it was initially derived in a continuous time framework.

Furthermore, it has recently been claimed by Garman [1977], that within the continuous time framework it is possible to value primary securities without solving the aggregation problem. In fact, it is claimed that no expected utility maximization is required. (See Chapter II.)

In this thesis we present a unified approach for the valuation of securities based on the "Single Price Law of Markets", Rubinstein [1976]. This law implies the existence of a linear operator or discount factor that prices all securities.

In Chapters I and II we consider the problem of pricing secondary securities contingent on primary security prices. The chapters develop the following idea: Cox, Ross and Rubinstein [1978] reduce the uncertainty in the primary securities to discrete binomial movements. In so doing they observe that the discount rates for the two states are determined in terms of the probabilities for the two outcomes.

This motivates us to ask the following question: having specified the stochastic process for the primary securities are the discount rates determined for all times in terms of the parameters of the process. The answer is yes under certain conditions.

In Chapter I we show how the valuation operator, or discount factor, can be explicitly constructed for all times, from the basic two step process that leads to a log normal distribution of security prices at time t, given their values at time zero.

In Chapter II we generalize the results to the case of general diffusion processes. We discuss the connection between our approach and that of Garman, deriving his equation, and show that it is not possible to price the primary securities without more information.

In Chapter III we find the linear operator that prices the primary assets when capital markets are Pareto efficient. We show how, by using Pareto optimal sharing rules, the valuation operator can be obtained for linear risk tolerance economies in continuous or discrete time, and how the Black-Scholes option pricing formula can result in either case.

Finally, we show how, when beliefs are contingent on present prices, the continuous time capital asset pricing model emerges.

Chapter I

The Discount Factor in Continuous Time: Some Basic Ideas

Recently new insights have been obtained in the theory of option pricing by assuming that the price of the asset on which the option is written follows a binomial stochastic process through time (Cox, Ross and Rubinstein [1978]). In this chapter we use this approach to investigate the properties of a random variable, called a discount factor, which can be used to price contingent claims written on the asset.

In section 1 we use a moment generating function to show that if the price of an asset follows a binomial stochastic process then its price at time t, contingent on its price at time zero, is log normally distributed.

In section 2 we introduce a random discount factor that relates the price of the asset at time t, to its price at time zero. We show that if the asset price is log normally distributed at time t, then the discount factor is also log normally distributed. This result allows us to derive an explicit expression for the discount factor.

In section 3 we utilize this expression for the discount factor to price a simple call option written on an asset which has the stochastic properties discussed in section 2. We obtain the Black-Scholes option pricing formula.

Finally, in section 4, we derive the discount factor for the multiple asset case and demonstrate that the price of a simple call option wirtten on a single asset is still given by the Black-Scholes formula.

1. Use of the Moment Generating Function to Describe a Binomial Stochastic Process

Consider one step of the discrete, binomial, stochastic process illustrated in figure I. Let p be the beginning of the period price. Denote the end of the period price by $p(1 + \tilde{\Delta})$ where $\tilde{\Delta}$ is a binomial random variable.

$$\tilde{\Delta} = +\Delta \text{ with probability } q$$
$$= -\Delta \text{ with probability } 1 - q$$

and $\Delta > 0$.
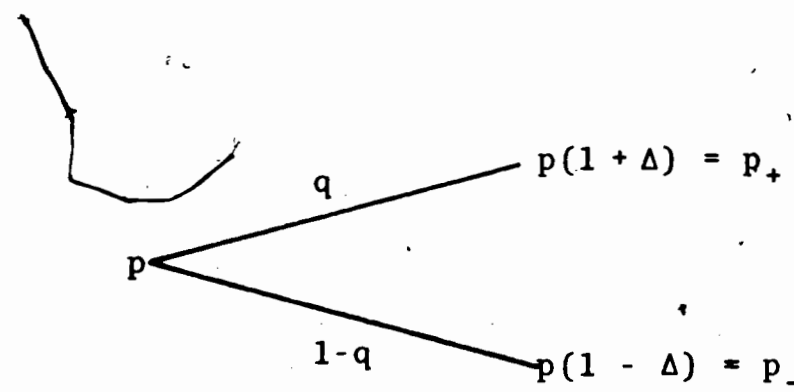
FIGURE I

Let the period be of length τ. If we imagine small changes in price, taking place in small intervals of time, then a series of discrete steps such as that shown in Figure I, can be converted into a continuous process by taking the limit as the interval of time goes to zero. In particular, we choose Δ and q such that as $\tau \to 0$ $\frac{p(t)}{p(0)}$ is

distributed log normal with[1]

$$E\left\{\log\frac{p(t)}{p(0)}\right\} = (\alpha - \frac{1}{2}\sigma^2)t$$

$$Var\left\{\log\frac{p(t)}{p(0)}\right\} = \sigma^2 t$$

This is the distribution considered by Merton [1971, 1973] where the asset follows a random walk with return per unit of time α, and variance per unit of time σ^2.

Beginning at time zero, the asset price at time t, after n periods of the type shown in figure I, is

$$p(t) = p(0) \prod_{i=1}^{n} [1 + \tilde{\Delta}_i] \qquad (1)$$

where $t = n\tau$.

Consider the moment generating function of $\log (p(t)/p(0))$

$$E\left\{e^{-\theta \log(p(t)/p(0))}\right\} = E\left\{e^{-\theta \sum_i \log(1 + \tilde{\Delta})_i}\right\}$$

$$= E\left\{\prod_{i=1}^{n} e^{-\theta \log(1 + \tilde{\Delta}_i)}\right\}$$

$$= \left[E\left\{e^{-\theta \log(1 + \tilde{\Delta})}\right\}\right]^n$$

$$= \left[E\left\{1 + \tilde{\Delta}\right\}^{-\theta}\right]^n \qquad (2)$$

---

[1] The choices made by Cóx, Ross and Rubinstein [1978] are such that $p(t)/p(0)$ is distributed log normally with mean αt and variance $\sigma^2 t$. That is, the return per unit of time is $\alpha + 1/2 \sigma^2$. This does not affect the option pricing formula since it is independent of the return per unit of time on the stock.

In obtaining equation (2) we have used equation (1) and assumed that the price changes are independent and identically distributed in each period so that $\tilde{\Delta}_i = \tilde{\Delta}$ for all i. Now since $\tilde{\Delta}$, is the binomial random variable illustrated in figure I, we have from equation (2) that

$$E\left\{e^{-\theta \log(p(t)/p(0))}\right\} = \left[q(1+\Delta)^{-\theta} + (1-q)(1-\Delta)^{-\theta}\right]^n$$
$$= \left[(1-\Delta)^{-\theta} + q((1+\Delta)^{-\theta} - (1-\Delta)^{-\theta})\right]^n$$

$$(3)$$

Choose

$$\Delta \equiv \sigma \sqrt{\tau} \tag{4}$$

$$q \equiv \frac{1}{2}\left\{1 + \frac{\alpha}{\sigma}\sqrt{\tau}\right\} \tag{5}$$

so that

$$(1 \pm \Delta)^{-\theta} = 1 \mp \theta\Delta + \frac{\theta(\theta+1)}{2!}\Delta^2 + 0(\tau) \tag{6}$$

where the remainder is $0(\tau)$. Our choice of Δ in (4) implies that the remainder has the property that $\lim 0(\tau)/\tau = 0$.

Substituting equations (4), (5) and (6) into equation (3) we obtain

$$E\left\{e^{-\theta \log(p(t)/p(0))}\right\} = \left\{1 - \theta\left(\alpha - \frac{\sigma^2}{2}\right)\tau + \frac{\theta^2\sigma^2}{2} + 0(\tau)\right\}^{t/\tau}$$

$$(7)$$

since $t = n\tau$.

In the limit as $\tau \to 0$ ($n \to \infty$ so that $t = n\tau$ is finite) we have from equation (7)

$$\lim_{\tau \to 0} E \left\{ e^{-\theta \log(p(t)/p(0))} \right\}$$

$$= \lim_{\tau \to 0} \left[1 - \theta(\alpha - \frac{\sigma^2}{2})\tau + \frac{\theta^2 \sigma^2}{2}\tau + 0(\tau) \right]^{t/\tau}$$

$$= e^{\left[-\theta(\alpha - \frac{\sigma^2}{2}) + \frac{1}{2}\theta^2 \sigma^2 \right] t} \tag{8}$$

using the definition of e. The expression (8) is the moment generating function for a normally distributed variable with mean $(\alpha - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$.

---

[2] Cox, Ross and Rubinstein [1978] choose the end of the period price to be $p\,e^{\sigma\sqrt{\tau}}$ with probability q and $p\,e^{-\sigma\sqrt{\tau}}$ with probability $1-q$. Substituting these into the expression for the moment generating function we obtain

$$\lim_{\tau \to 0} E \left\{ e^{-\theta \log(p(t)/p(0))} \right\} = \lim_{\tau \to 0} \left[qe^{-\theta\sigma\sqrt{\tau}} + (1-q)e^{\theta\sigma\sqrt{\tau}} \right]^{t/\tau}$$

$$= \lim_{\tau \to 0} \left[1 - \left[\alpha\theta + \frac{1}{2}\theta^2\sigma^2 \right]\tau + 0(\tau) \right]^{t/\tau}$$

$$= e^{\left[-\alpha\theta + \frac{1}{2}\theta^2\sigma^2 \right] t} \tag{9}$$

which is the moment generating function for a normally distributed variable with mean αt and variance $\sigma^2 t$ (see footnote 1).

2. The Discount Factor

Cox, Ross and Rubinstein [1978] give a "complete markets"[3] interpretation to the binomial approach of section 1. They introduce state-contingent discount rates π_+ and π_-, where π_+ is the current price of one dollar received at the end of the period, of length τ, if and only if the + state occurs (see figure I). Using these discount rates, the beginning and end of the period prices are related by

$$p = \pi_+ p_+ + \pi_- p_- \qquad (10)$$

However, we are more interested in the 'discount factors' Z_+ and Z_- which are defined in terms of the discount rates as follows

$$q Z_+ \equiv \pi_+ \qquad (11)$$

$$(1-q) Z_- \equiv \pi_- \qquad (12)$$

In terms of the discount factors, the beginning and end of the period prices are related by (from (10), (11) and (12))

$$p = q Z_+ p_+ + (1-q) Z_- p_- \qquad (13)$$

---

[3] The term "complete markets" as used by Cox, Ross and Rubinstein in this context has nothing to do with efficiency.

Moreover, if we assume the existence of a riskless rate of interest, r per unit of time, then

$$1 = q\,Z_+(1 + r\tau) + (1-q)Z_-(1 + r\tau) \qquad (14)$$

Solving for Z_+ and Z_- and using equations (4) and (5) we obtain

$$Z_+ = 1 + \left(\frac{r-\alpha}{\sigma}\right)\sqrt{\tau} + \left(\frac{\alpha^2}{\sigma^2} - \frac{r\alpha}{\sigma^2} - r\right)\tau + 0(\tau) \qquad (15)$$

and

$$Z_- = 1 - \left(\frac{r-\alpha}{\sigma}\right)\sqrt{\tau} + \left(\frac{\alpha^2}{\sigma^2} - \frac{r\alpha}{\sigma^2} - r\right)\tau + 0(\tau) \qquad (16)$$

Consider the two steps of the binomial process shown in figure II
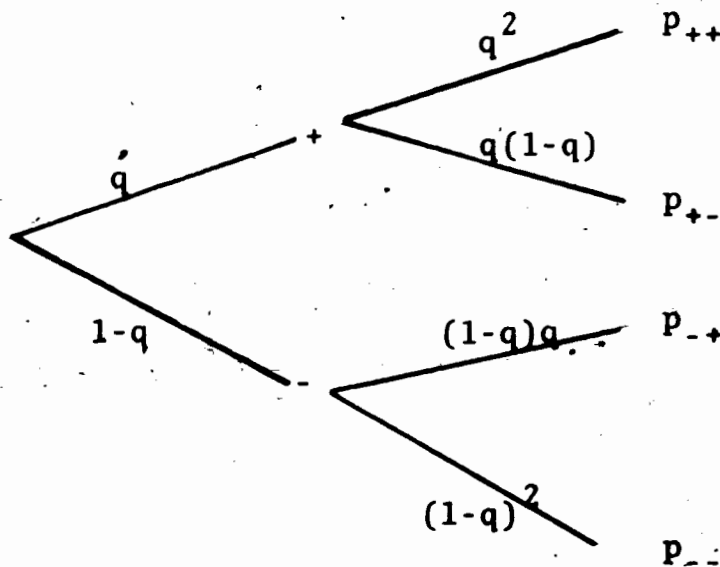
FIGURE II

Using the 'complete markets' interpretation we can introduce the four discount factors Z_{++}, Z_{+-}, Z_{-+}, Z_{--}, where present prices, and two period later prices are related by

$$p = q^2 Z_{++} p_{++} + q(1-q) Z_{+-} p_{+-}$$
$$+ (1-q)q Z_{-+} p_{-+} + (1-q)^2 Z_{--} p_{--} \tag{17}$$

where we use the notion of figure II. However, we also have

$$p_+ = q Z_+ p_{++} + (1-q) Z_- p_{+-} \tag{18}$$

and

$$p_- = q Z_+ p_{-+} + (1-q) Z_- p_{--} \tag{19}$$

where Z_+ and Z_- are given by equations (15) and (16). Substituting (18) and (19) into

$$p = q Z_+ p_+ + (1-q) Z_- p_- \tag{20}$$

we obtain

$$p = q^2 Z_+^2 p_{++} + q(1-q) Z_+ Z_- p_{+-}$$
$$+ (1-q)q Z_- Z_+ p_{-+} + (1-q)^2 Z_-^2 p_{--} \tag{21}$$

Comparing equations (17) and (21) we notice that $Z_{++} = Z_+^2$, $Z_{+-} = Z_{-+} = Z_+ Z_-$, and $Z_{--} = Z_-^2$. Hence, we conclude:

if there exists a time invariant riskless rate of interest then Z follows a discrete independently distributed binomial process whenever p does. This conclusion suggests that we use the method of section I to construct the moment generating function for Z.

$$E\left\{e^{-\theta \ \log Z(t)}\right\} = \left[qZ_+^{-\theta} + (1-q)Z_-^{-\theta}\right]^{t/\tau} \qquad (22)$$

Substituting (15) and (16) into equation (22) and using our choice of q given by equation (5), we have for the moment generating function of Z

$$E\left\{e^{-\theta \ \log Z(t)}\right\} = \left\{1 + \theta\left[r + \frac{1}{2}\left(\frac{r-\alpha}{\sigma}\right)^2\right]\tau \right.$$
$$\left. + \frac{\theta^2}{2}\left(\frac{r-\alpha}{\sigma}\right)^2 \tau + 0(\tau)\right\}^{t/\tau} \qquad (23)$$

Taking the limit of (23) as $\tau \to 0$, converts the discrete process into a continuous process and yields:

$$\lim_{\tau \to 0} E\left\{e^{-\theta \ \log Z(t)}\right\} = e^{\left[\theta\left(r + \frac{1}{2}\left(\frac{r-\alpha}{\sigma}\right)^2\right)+ \frac{\theta^2}{2}\left(\frac{r-\alpha}{\sigma}\right)^2\right]t}$$
$$\qquad (24)$$

Expression (24) is the moment generating function for a normally distributed variable with mean $-rt - \frac{1}{2}\left(\frac{r-\alpha}{\sigma}\right)^2 t$ and variance $\left(\frac{r-\alpha}{\sigma}\right)^2 t$. As we shall see, however, it is fruitful to express Z(t) as a function of p(t). Now, since Z(t) and p(t) are both log normally distributed random variables, the most general functional relationship between these variables

can be written in the form

$$\log Z(t) = a + \beta \log p(t) \tag{25}$$

where a and β are non random. From the moment generating function for Z, equation (24), we have

$$E \left\{ \log Z(t) \right\} = -rt - \frac{1}{2} \left(\frac{r-\alpha}{\sigma} \right)^2 t \tag{26}$$

and

$$Var \left\{ \log Z(t) \right\} = \left(\frac{r-\alpha}{\sigma} \right)^2 t \tag{27}$$

Substituting (25) into (27) we find that

$$\beta = \frac{r-\alpha}{\sigma^2} \tag{28}$$

Substituting (25) into (26) we obtain

$$a = -rt - \beta \left[\alpha - \frac{1}{2} \sigma^2 + \frac{1}{2} \beta \sigma^2 \right] t - \beta \log p(0) \tag{29}$$

Thus $Z(t)$ can be written

$$Z(t) = e^{-rt} e^{-\beta \left\{ \alpha - \frac{1}{2} \sigma^2 + \frac{1}{2} \beta \sigma^2 \right\} t} \left[\frac{p(t)}{p(0)} \right]^{\beta} \tag{30}$$

where β is given by equation (28).

3. Option Pricing

In section 2 we found the discount factor $Z(p(t),t)$ for any time t. This was accomplished by factoring the discount rates π_+ and π_- for one step of the discrete binomial process into two parts, a probability or belief part and a discount factor Z, and then constructing the moment generating function for Z after N steps of the binomial process. By taking the continuous limit of the discrete process, we developed an explicit expression for $Z(p(t),t)$. The discount rate $\pi(p(t),t)$ is just the product of $Z(p(t),t)$ and the probability (density) that the asset price assumes the value $p(t)$ at time t, given the value at time zero. However, it is convenient to focus attention on the discount factor Z rather than the discount rate and write the price of the asset at time zero in terms of its price at time t as

$$p(0) \;=\; E\left\{ Z(t)\, p(t) \right\} \qquad (31)$$

As an illustration, consider the pricing of a simple call option which has the striking price K. The value of this call option at time zero given that it be exercised at time t is;

$$C(0) \;=\; E\left\{ Z(t)\, (p(t) - K) \,\big|\, p(t) \geq K \right\} \qquad (32)$$

Substituting for $Z(t)$ from equation (30)

$$C(0) = e^{-rt} e^{-\beta\left\{\alpha - \frac{\sigma^2}{2} + \frac{1}{2}\beta\sigma^2\right\}t} \left[p(0) \; E\left(\left\{\frac{p(t)}{p(0)}\right\}^{\beta+1} \;\middle|\; \frac{p(t)}{p(0)} \geq \frac{K}{p(0)} \right) \right.$$

$$\left. - K \; E\left(\left\{\frac{p(t)}{p(0)}\right\}^{\beta} \;\middle|\; \frac{p(t)}{p(0)} \geq \frac{K}{p(0)} \right) \right] \tag{33}$$

Using the standard integral (see for example Rubinstein [1976])

$$\int_a^\infty e^{sx} f(x)dx = e^{s\mu + \frac{1}{2}s^2\sigma^2} F\left(\frac{-a+\mu}{\sigma} + \sigma s \right) \tag{34}$$

where $f(x)$ is the normal density function, with mean μ and variance σ^2, and $F(x)$ is the normal distribution function; we can evaluate expression (33) to find

$$C(0) = p(0) \; F(a + \sigma\sqrt{t}) - K \; e^{-rt} F(a) \tag{35}$$

where $a = \left[\log(p(0)/K) + rt \right] \Big/ \sigma\sqrt{t} - \frac{1}{2}\sigma\sqrt{t}$ \tag{36}

Expression (35) is the Black-Scholes option pricing formula. The present approach is compared with that of Black-Scholes in appendix A.

4.   The Discount Factor in the Multiple Asset Case

Consider N primary assets, whose values at time t are log normally distributed. Assume that the discount factor Z is log normal[4]. Then we have

$$\log Z(t) = a + \sum_{i=1}^{N} \beta_i \log p_i(t) \tag{37}$$

or

$$Z(t) = A \prod_{i=1}^{N} p_i^{\beta_i}(t), \qquad A = e^a \tag{38}$$

In equations (37) and (38) there are N+1 unknowns to be determined in terms of quantities known at time t = 0. We have N relationships of the form

$$p_i(0) = E\left\{ Z(t)\, p_i(t) \right\} \qquad i = 1, 2, \ldots N \tag{39}$$

and assuming that one dollar invested at t = 0, at the riskless rate, grows to e^{rt} dollars at time t, we have the additional relationship,

$$1 = E\left\{ Z(t)\, e^{rt} \right\} \tag{40}$$

---

[4]One could prove this by a method analogous to that of section 2. But this is tedious. In any event, the result follows from the more general approach of Chapter II.

The N+1 equations (39) and (40) are sufficient to determine the N+1 unknowns A and β_i (i = 1, 2, ... N). Substituting (38) into (40) we obtain

$$1 = A\ e^{rt}\ \prod_{i=1}^{N} p_i(0)^{\beta_i}\ E\left\{ \prod_{i=1}^{N} \left[\frac{p_i(t)}{p_i(0)}\right]^{\beta_i} \right\} \qquad (41)$$

Define the vectors β, with components β_i, μ, with components $(\alpha_i - \sigma_i^2/2)$ and the variance covariance matrix Σ. In terms of these quantities

$$E\left\{ \prod_{i=1}^{N} \left[\frac{p_i(t)}{p_i(0)}\right]^{\beta_i} \right\} = e^{\beta'\mu t + \frac{1}{2}\beta'\Sigma\beta t} \qquad (42)$$

Equation (42) is the generalization of equation (24) to the multivariate case. Substituting (42) into (41) we obtain

$$1 = A\ e^{rt}\ \prod_{i=1}^{N} p_i(0)^{\beta_i}\ e^{\beta'\mu t + \frac{1}{2}\beta'\Sigma\beta t} \qquad (43)$$

Similarly substituting (38) into the N equations (39) we obtain

$$1 = A\ \prod_{j=1}^{N} p_j(0)^{\beta_j}\ e^{\left\{\beta'\mu + \frac{1}{2}\beta'\Sigma\beta + [\alpha + \Sigma\beta]_i\right\}t} \qquad (44)$$

Together equations (43) and (44) imply that

$$-r\iota + \alpha + \Sigma\beta = 0$$

or that

$$\beta = -\sum{}^{-1} (\alpha - r\iota) \tag{45}$$

where ι is a column vector of ones. Putting (43) into (38) we obtain:

$$Z(t) = e^{-rt} \, e^{-\beta'\left[\mu + \frac{1}{2}\sum\beta\right]t} \, \prod_{i=1}^{N} \left\{\frac{p_i(t)}{p_i(0)}\right\}^{\beta_i} \tag{46}$$

where β is given by (45).

Expression (46) can be used to price a wide variety of contingent claims on the primary assets. In particular, one can use it to price a simple call option written on a single asset. A cursory examination of (46) suggests that the price of this call option depends upon the covariance of the underlying asset with other assets (through $\sum$). Surprisingly, however, this is not the case.

The price of a simple call option on the ith asset with striking price K is

$$C_i = E\left\{ Z(p_i - K) \, \middle| \, p_i \geq K \right\} \tag{47}$$

Substituting (46) into (47) and making use of the fact that

$$E\left\{ \left[\frac{p_i(t)}{p_i(0)}\right]^{\beta_i+1} \prod_{j\neq i} \left[\frac{p_j(t)}{p_j(0)}\right]^{\beta_j} \, \middle| \, \frac{p_i(t)}{p_i(0)} > \frac{K}{p_i(0)} \right\}$$

$$= e^{\left[\beta\mu + \frac{1}{2}\beta'\sum\beta\right]t} \, e^{[\alpha + \sum\beta]_i t} \, F\left(\frac{\ln p_i(0)/K + [\mu+\sum\beta]_i t}{\sigma_i\sqrt{t}} + \sigma_i\sqrt{t}\right) \tag{48}$$

(equation 48) is a generalization of (34) to the multivariate case), we obtain upon substituting for β_i from (45), the Black-Scholes formula. This implies, of course, that the price of the option does not depend upon the covariance of the underlying asset returns with other assets returns.

APPENDIX I

Black-Scholes Approach to Option Pricing

Black-Scholes [1973] assumed that the price of an option is a function of the underlying asset price,

$$C = C(p,t) \tag{1}$$

Hence, if the price of the primary asset follows a binary stochastic process then so does the option with

$$C_+ = C(p + \Delta p, t + \tau) \text{ with probability } q \tag{2}$$

and

$$C_- = C(p - \Delta p, t + \tau) \text{ with probability } 1-q \tag{3}$$

Moreover, if we assume that expression (1) is differentiable, then from (2) and (3) we obtain

$$C_+ = C(p,t) + \frac{\partial C}{\partial p} \Delta p + \frac{1}{2} \frac{\partial^2 C}{\partial p^2} \Delta^2 p^2 + \frac{\partial C}{\partial t} \tau + 0(\tau) \tag{4}$$

and

$$C_- = C(p,t) - \frac{\partial C}{\partial p} \Delta p + \frac{1}{2} \frac{\partial^2 C}{\partial p^2} \Delta^2 p^2 + \frac{\partial C}{\partial t} \tau + 0(\tau) \tag{5}$$

We can evaluate the expected value of the call after time τ, using (4) and (5)

$$E_C = q\,C_+ + (1-q)\,C_-$$

$$= C + \frac{1}{2}\frac{\partial^2 C}{\partial p^2}\sigma^2 p^2 \tau + \frac{\partial C}{\partial t}\tau + \frac{\partial C}{\partial p}p\tau + 0(\tau) \qquad (6)$$

Hence, the expected return per unit of time on the call option is

$$\alpha_C \equiv \lim_{\tau \to 0} \frac{1}{\tau}\frac{E_C - C}{C}$$

$$\equiv \frac{1}{C}\left[\alpha\,p\,\frac{\partial C}{\partial p} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 C}{\partial p^2}\right] \qquad (7)$$

Similarly we can evaluate the variance of the call option

$$V_C = \sigma^2 p^2 \left[\frac{\partial C}{\partial p}\right]^2 \tau + 0(\tau) \qquad (8)$$

and hence the variance of the call option, per dollar, per unit of time, using (8),

$$\sigma_C^2 \equiv \lim_{\tau \to 0} \frac{1}{\tau}\frac{V_C}{C^2}$$

$$= \sigma^2\,\frac{p^2}{C^2}\left[\frac{\partial C}{\partial p}\right]^2 \qquad (9)$$

Finally, the covariance between the price of the call and the price of the asset

$$\mathrm{Cov}(c,p) = \sigma^2 p^2 \frac{\partial C}{\partial p}\tau + 0(\tau) \qquad (10)$$

and thus the covariance between the option and the asset per unit of time is

$$\sigma_{pc} \equiv \lim_{\tau \to 0} \frac{1}{\tau} \frac{1}{pC} \text{Cov}(C,p)$$

$$= \sigma^2 \frac{p}{C} \frac{\partial C}{\partial p} \quad \text{using (10)}$$

$$= \sigma \, \sigma_C \quad \text{using (9)}$$

Hence, we obtain the crucial result that the returns per dollar invested, per unit of time on the option and on the asset are perfectly correlated. This implies that it is possible to construct a risk free asset from a linear combination of the risky asset and its associated option (see for example Fama and Miller [1972]) see figure (AI).

From figure (AI) we see that the risk free rate of interest is given by the intersection of the straight line joining (α, σ) with (α_C, σ_C) and the vertical axis. From consideration of the slope of this line

$$\frac{\alpha - r}{\sigma} = \frac{\alpha_C - r}{\sigma_C} \tag{11}$$
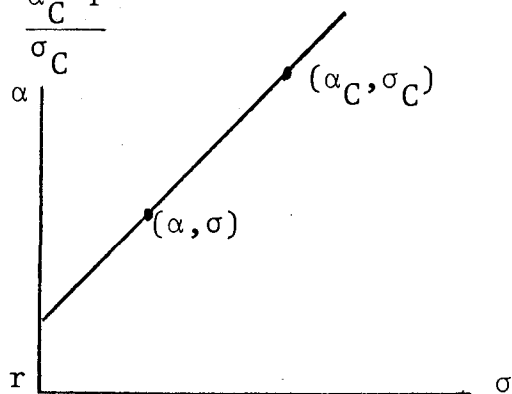
FIGURE AI

Substituting for α_C and σ_C in (11) from (7) and (9) we have

$$\frac{\alpha - r}{\sigma} = \left[\alpha \, p \, \frac{\partial C}{\partial p} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 C}{\partial p^2} - rC \right] \Bigg/ \sigma p \, \frac{\partial C}{\partial p} \tag{12}$$

Simplifying we obtain

$$\frac{\partial C}{\partial t} + r \, p \, \frac{\partial C}{\partial p} - r \, C + \frac{1}{2} \sigma^2 p^2 \frac{\partial^2 C}{\partial p^2} = 0 \tag{13}$$

Given the boundary condition that at time t,

$$C(p,t) = p - K \qquad p \geq K$$

$$C(p,t) = 0 \qquad p < K$$

then the value of the option at time zero is given by equation (36) (Chapter I).

To summarize: according to the Black-Scholes approach, equation (1) imples the existence of a riskless rate of interest. However, according to the approach adopted here the assumption of a riskless rate imples a log normal[1] discount factor which in turn implies a functional relationship between the option price and the underlying asset price of the form (1).

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[1] When the underlying asset price is distributed log normal.

CHAPTER II

Development of a Theory of the Discount Factor

When Asset Prices Follow a Diffusion Process

In Chapter I we derived an expression for the discount factor when asset prices follow a particular diffusion process known as geometric Brownian motion. In this chapter we generalize these results and develop the theory of the discount factor when asset prices follow any diffusion process.

By definition diffusion processes are those for which the probability density function of asset prices at time t, contingent on their prices at time zero, obeys the Fokker-Planck equation. This partial differential equation, together with the assumption that riskless arbitrage opportunities are absent, implies that the discount factor obeys a set of partial differential equations. When there exists a riskless rate of interest there are enough equations to determine the discount factor contingent on time zero primary asset prices.

In section 1, we show that the absence of riskless arbitrage opportunities implies the existence of a linear operator, that prices all risky cash flows. We show the relationship of this operator to the discount factor Z. In section 2 we develop the general theory of the discount factor. Finally, in section 3, we compare the theory of asset valuation under diffusion processes presented here, with that presented by Garman [1977].

1. - Zero Arbitrage and the Valuation Operator

The fact that the zero riskless arbitrage condition implies the existence of a linear operator which prices all risky assets was utilized by Beja [1970], Rubinstein [1976], Garman [1977], and by others. Ross proved it in [1978]. Here we outline the proof and develop some notation.

Consider an economy consisting of N firms, and facing S possible final states of the world. Assume that to each final state $i = 1, 2, \ldots S$ there corresponds a realization of the vector of firm values. Consider an N dimensional Euclidean space spanned by a Cartesian co-ordinate system. Along one of the axes plot the possible realizations of the end of the period values of the first firm, p_1^i, $i = 1, \ldots S$. Along a second axis plot those for the second firm. Continue this process for all N firms. Then construct the S vectors, p^i $i = 1, \ldots S$ in the N dimensional space. Each vector ends at a point representing a possible realization of firm values in some future state of the world. Assume that all of these points lie in the positive orthant[1].

Assume that primary assets are issued by the firms and no riskless arbitrage profits can be made from dealing in these assets. Then no portfolio of primary assets will have a price ≤ 0.

Let $p(0)$ represent the vector of firm values. Then a necessary condition for the absence of riskless arbitrage

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[1]This is not an essential condition, e.g. see footnote 2.

opportunities is that there does not exist a vector X in the N dimensional space ($\iota'X = 1$) such that

$$X'p(0) < 0 \text{ and } X'p^i \geq 0 \qquad \text{for all } i \qquad (1)$$

According to the Farkes-Minkowski lemma (c.f. Takayama [1974]), given S points p^i and a vector $p \neq 0$ in R^N, then either there exists a vector X in R^N satisfying (1) or p(0) is a positive linear combination of the vectors p^i, but not both[2]. Since by hypothesis there does not exist an X satisfying (1) we have

$$p(0) = \sum_{i=1}^{S} a_i p^i \qquad (2)$$

where $\quad a_i \geq 0 \qquad (3)$

Now, even though the state dependent firm values are discrete points in price space, we would like to treat them as being distributed continuously in the N dimensional space. For this purpose we introduce the density function $\Gamma(p)$ such that $\Gamma(p)d\Omega$, is the number of possible realizations of end of the period firm values contained in an N dimensional volume which encloses the point p and is of size $d\Omega$,

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[2]If we allow for the possibility that end of period values may be negative [absence of limited liability] then we can state a zero arbitrage condition as: a portfolio whose payoffs are zero or less in each future state cannot sell for a positive price. This also implies (2).

$$d\Omega = \prod_{i=1}^{N} dp_i \tag{4}$$

enclosing the point p. Hence $\int \Gamma(p) d\Omega$ is the number of possible realizations of the end of the period firm values contained in whatever volume of price space is integrated over. The density is actually discrete (for the purposes of proving the Farkas-Minkowski lemma) so that $\Gamma(p)$ is zero everywhere except where $p = p^i$. Therefore $\Gamma(p)$ can be expressed with the help of the δ function[3] as follows:

$$\Gamma(p) = \sum_{i=1}^{S} \delta(p_1 - p_1^i) \delta(p_2 - p_2^i) \cdots \delta(p_N - p_N^i) \tag{5}$$

Equation (5) can be written more compactly as

$$\Gamma(p) = \sum_{i=1}^{S} \delta(p - p^i) \tag{6}$$

Integrating (6) over all price space we obtain,

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3 The δ function is defined as follows: $\delta(X) = 0$ for all non-zero values of X. $\delta(X) = \infty$ when $X = 0$ in such a way that $\int_{-\infty}^{\infty} dX \delta(X) = 1$.

The δ function can be manipulated algebraically [except for dividing] as if it were a normal function. It is however only well defined underneath the integral sign where its properties can be derived from its definition. Its most imporant property is

$$\int_{-\infty}^{\infty} f(X) \delta(X) dX = f(0).$$

$$\int\limits_{\text{all space}} \Gamma d\Omega = \int\limits_{\text{all space}} \sum_{i=1}^{S} \delta(p-p^i) d\Omega$$

$$= \sum_{i=1}^{S} \quad \text{from the definition of } \delta(X)$$

$$= S \tag{7}$$

Consider the normalized density function

$$\rho(p) \equiv \frac{\Gamma(p)}{S} \tag{8}$$

so that $\int\limits_{\text{all space}} \rho(p) dp = 1.$

We assume that each realization of the vector of future firm values has an equal likelihood of occurring. Then $\rho(p)$ as defined by equation (8) is the probability density function for a particular realization of end of the period firm values.

We further define the "discount rate"

$$\pi(p) \equiv \sum_i a_i \delta(p-p^i) \tag{9}$$

Using (9) we can rewrite equation (2) as follows:

$$p(0) = \int \pi(p) p d \Omega \tag{10}$$

where we have integrated over all of price space. Substituting (9) into (10) and using the properties of the δ function (c.f.

footnote 3) we obtain equation (2).

With the aid of the probability density function we can define the discount factor Z(p):

$$Z(p) \; \rho(p) \equiv \pi(p)$$

and rewrite equation (10) as

$$p(0) = \int \rho(p) Z(p) p \, d\Omega$$

$$= E \left\{ Z(p) p \right\} \tag{11}$$

which defines the expected value operator.

In the price space that we have constructed the valuation operator is an integral operator $\int d\Omega \, \pi(p)$; as in Chapter I, we split π into two parts $\rho(p)$ and $Z(p)$. In the next section, again as in Chapter 1, we postulate a particular form of ρ and find a particular form for Z.

2. General Theory of Discount Factor

We consider the class of stochastic processes for which the probability density function for the distribution of firm values at time t, contingent on their values at time zero, denoted by $\rho(p, t, \, p(0), 0)$, obeys the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} + \sum\frac{\partial}{\partial p_i}(\alpha_i p_i \rho) + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial p_i \partial p_j}(p_i p_j \sigma_{ij}\rho) = 0 \qquad (12)$$

where the α_i and σ_{ij} are in the most general case functions of the prices and time. Equation (12) is first order in time, and we can write the required initial condition using the δ function

$\rho(p,t; p(0),0)$ evaluated at $t = 0$

$$= \rho(p,0;p(0),0) = \delta(p-p(0)) \qquad (13)$$

Recall that we are assuming that no dividends are paid so that equation (11) holds for all time $t \geq 0$. Hence,

$$p(0) = \int d\Omega p \ Z(p,t)\rho(p,t;p(0),0) \qquad (14)$$

where $t \geq 0$. In particular when $t = 0$ we can substitute (13) into (14) to obtain

$$p(0) = Z(p(0),0)p(0) \qquad (15)$$

which implies

$$Z(p(0),0) = 1 \qquad (16)$$

Now for all times t, the right hand side of equation (14) is a constant. Differentiating equation (14) with respect to

time

$$0 = \int d\Omega p \, \frac{\partial}{\partial t} \, Z\rho \quad , \tag{17}$$

and since p is a vector (17) represents N equations. Assuming a riskless rate of interest so that one dollar invested at time zero grows with certainty to R(t) dollars at time t, we obtain from (11)

$$1 = \int d\Omega \, Z(p,t)\rho(p,t;p(0),0)R(t) \tag{18}$$

Differentiating (18) with respect to time

$$0 = \int d\Omega \, \frac{\partial}{\partial t} \, (Z\rho R) \tag{19}$$

It is shown in appendix IIA that when ρ obeys the Fokker-Planck equation (12), as well as appropriate boundary conditions, then equation (17) implies that Z obeys the following equations:

$$\frac{\partial Z}{\partial t} + \alpha_i Z + \sum_j (\alpha_j + \sigma_{ij}) p_j \frac{\partial Z}{\partial p_j}$$

$$+ \frac{1}{2} \sum_j \sum_k \frac{\partial^2 Z}{\partial p_j \partial p_k} \, p_j p_k \sigma_{jk} = 0$$

$$i = 1 \ldots N \tag{20}$$

Further, equation (19) implies

$$\frac{\partial Z}{\partial t} + \frac{1}{R} \frac{\partial R}{\partial t} Z + \sum_j \alpha_j p_j \frac{\partial Z}{\partial p_j} + \frac{1}{2} \sum_j \sum_k \sigma_{jk} p_j p_k \frac{\partial^2 Z}{\partial p_j \partial p_k}$$

$$= 0 \qquad\qquad (21)$$

Equation (21) can be used to simplify the N equations (20). Substituting (21) into (20) we obtain

$$\left(\alpha_i - \frac{1}{R} \frac{\partial R}{\partial t} \right) Z + \sum_j \sigma_{ij} p_j \frac{\partial Z}{\partial p_j} = 0$$

$$i = 1 \ldots N \qquad\qquad (22)$$

Multiplying (22) by the inverse of the variance covariance matrix, $\sum^{-1}$ we obtain the N first order partial differential equations satisfied by Z

$$p_i \frac{\partial Z}{\partial p_i} = \beta_i Z, \quad i = 1, \ldots N \qquad\qquad (23)$$

where β_i are the elements of the vector

$$\beta \equiv - \sum^{-1} \left(\alpha - \frac{1}{R} \frac{dR}{dt} \mathbf{1} \right) \qquad\qquad (24)$$

The N equations (23) determine Z as a function of the price vector p, except for a constant of integration. The constant of integration [meaning a function independent of prices] is a function of time and is determined by equation (21) using

the initial condition (16).

Example 1: Geometric Brownian Motion.

Firm values are growing "exponentially" but stochasticly with time. By analogy we choose $R(t) = e^{rt}$. In this case Σ and α are both constants, independent of p and t. In appendix IIB Z is obtained by integrating the N+1 equations (21) and (23). As expected, the expression for $Z(p,t)$ obtained in this fashion is identical to that given by equation (46) of chapter I.

Example 2:   Arithmetic Brownian Motion.

Firm values, in this case, grow "linearly" but stochasticly with time. By analogy we choose $R(t) = 1 + rt$. If, instead, we choose $R(t) = e^{rt}$ there is no solution for Z (see appendix IIC). Assuming only one risky asset, we have for arithmetic Brownian motion $\alpha = \alpha_o/p$ and $\sigma^2 = \sigma_o^2/p$ where α and σ_o are constants independent of price and time. It is shown in appendix IIB, by integrating equations (21) and (23) that $Z(p,t)$ is given by

$$Z(p,t) = \frac{1}{[1+rt]^{3/2}} \, e^{\frac{1}{2}\alpha_o^2/\sigma_o^2 t} \, e^{-\frac{1}{\sigma_o^2}\left[\alpha_o(p-p(0)) - \frac{1}{2}\left(\frac{rp^2}{1+rt} - rp^2(0)\right)\right]}$$

$$(25)$$

In the case of arithmetic Brownian motion, the probability density function is

$$\rho(p,t) = \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{t}} \, e^{-\frac{1}{2\sigma_o^2 t}\left[p - p(0) - \alpha_o t\right]^2} \qquad (26)$$

From (25) and (26) we obtain the result that the discount rate

$$\pi(p,t) \equiv Z(p,t)\rho(p,t)$$

is independent of the return on the asset, α_o. Hence, arithmetic Brownian motion in continuous time gives rise to what Brennan [1978] has termed a risk neutral valuation relationship. Brennan [1978] investigated these relationships in discrete time.

3. Comparison with the Approach of Garman

Garman [1977] presented "A General Theory of Asset Valuation under Diffusion State Processes". His approach has many formal similarities to the theory presented in this chapter. There are, however, significant differences.

Garman states that his theory implies the existence of "... a single partial differential equation which is satisfied by all existing marketable assets in a world governed by diffusion state processes".

The differential equation referred to by Garman is written in terms of prevailing market prices (as is the differential equation of Black-Scholes). In this chapter, however, the differential equations hold for all points in price space for all times $t \geq 0$. At $t = 0$, setting $\frac{1}{R}\frac{\partial R}{\partial t} = r$, with $p = p(0)$, equation (22) becomes

$$(\alpha_i - r) + \sum_j \sigma_{ij} p_j(0) \left. \frac{\partial Z}{\partial p_j} \right|_{p_j = p_j(0)} = 0 \qquad (27)$$
$$i = 1, \ldots N$$

Equations (27) are identical, ignoring differences in notation, with Garman's differential equations for primary assets (c.f. Garman [1977] equations (11)). Garman calls (27) the capital asset pricing model.

Equations (22) are satisfied by Z at all points p of price space to $t \geq 0$. We cannot "solve" (22) for the prices, we solve it for $Z(p,t)$ contingent on $p = p(0)$ at $t = 0$. Thus the $t = 0$ form of (22), equation (27), is not the Capital Asset Pricing Model.

Suppose, however, we have an alternate theory for Z. If, for example, as in Chapter III, we solve the aggregation problem, and obtain an expression for Z independent of prevailing prices, then when beliefs are described by a continuous diffusion process, the Z thus obtained must satisfy (27). In this way we can obtain the C.A.P.M. (see Chapter III).

Garman also states that "one convenience of the diffusion assumption turns out to be that (given the diffusion belief) exactly three quantities completely determine all asset prices. These quantities are seen to be identifiable via simple linear regression against the current interest rate".

The three determinants of prices referred to by Garman are, $\frac{\partial Z}{\partial t}$, $\frac{\partial Z}{\partial p}$ and $\frac{\partial^2 Z}{\partial p^2}$, all evaluated at prevailing market prices ($t = 0$). We have shown however that whenever the riskless term structure is given, specification of diffusion beliefs determines Z, and hence the three quantities.

APPENDIX IIA

Partial Differential Equations for Z

Rewriting equation (17) of chapter II

$$\int d\Omega p_i \rho \frac{\partial Z}{\partial t} + \int d\Omega p_i Z \frac{\partial \rho}{\partial t} = 0$$

$$i = 1, \ldots N \tag{1}$$

Consider the second term of equation (1). Substituting for $\frac{\partial \rho}{\partial t}$ from equation 12, Chapter II, we obtain

$$\int d\Omega p_i Z \frac{\partial \rho}{\partial t}$$

$$= \int d\Omega p_i Z \left\{ -\sum_i \frac{\partial}{\partial p_i} (\alpha_i p_i \rho) + \frac{1}{2} \sum_i \sum_j \frac{\partial^2}{\partial p_i \partial p_j} p_i p_j \sigma_{ij} \rho \right\}$$

$$\tag{2}$$

We can evaluate the first term in (2) by integrating by parts, and assuming that ρ vanishes at the boundary of integration. For example, when the possible realization of price values lie in the positive orthant the boundary of integration is the surface of the N dimensional cube bounding the positive orthant. When prices are log normally distributed, ρ vanishes on this boundary. Other boundary conditions can be chosen to suit the economics of the situation.

Hence, since by assumption vanishes on the boundary of integration

$$\int d\Omega p_i \ Z \left\{ - \sum_i \frac{\partial}{\partial p_i} (\alpha_i p_i \rho) \right\}$$

$$= \int d\Omega \ \rho \ p_i \left[\alpha_i Z + \sum_j \alpha_j p_j \frac{\partial Z}{\partial p_j} \right] \tag{3}$$

Similarly, we can evaluate the second term in (2) by integrating by parts twice, and assuming ρ and its derivatives vanish on. the boundary. Substituting the resulting expressions into equation (1) we obtain

$$\int d\Omega p_i \rho(p, t; p(0), 0) \left\{ \frac{\partial Z}{\partial t} + \alpha_i Z \right.$$

$$\left. + \sum_j \alpha_j p_j \frac{\partial Z}{\partial p_j} \right| + \sum_j \sigma_{ij} p_j \frac{\partial Z}{\partial p_j} + \frac{1}{2} \sum_{jk} p_j p_k \sigma_{jk} \frac{\partial^2 Z}{\partial p_j \partial p_k} \right\}$$

$$= 0 \tag{4}$$

Since equation (4) holds for arbitrary choices of the vector $p(0)$ and thus for the corresponding choices of ρ, we must have

$$\frac{\partial Z}{\partial t} + \alpha_i \ Z + \sum_j p_j \alpha_j \frac{\partial Z}{\partial p_j} + \sum_j \sigma_{ij} p_j \frac{\partial Z}{\partial p_j}$$

$$+ \frac{1}{2} \sum_i \sum_j p_i p_j \sigma_{ij} \frac{\partial^2 Z}{\partial p_i \partial p_j} = 0 \tag{5}$$

Equation (5) is identical with equation (20) of Chapter II. In a similar fashion equation (21) can be obtained from equation (19).

APPENDIX IIB

Derivation of Z in Specific Cases

Geometric Brownian Motion:

With $\sum$ and α constants and $R(t) = e^{rt}$, we can easily integrate the N first order partial differential equations to obtain

$$Z(p,t) = A(t) \prod_j p_j^{\beta_j} \tag{1}$$

where $A(t)$ is an unknown function of time. Substituting (1) into (21) we obtain

$$\beta'\alpha \, Z(t) + \frac{1}{2} \beta'\sum \beta \, Z(t) - \frac{1}{2} \beta' \sum_D Z(t)$$

$$+ \frac{\partial Z}{\partial t} + r \, Z = 0 \tag{2}$$

where we have used the fact that

$$\sum_j \alpha_j p_j \frac{\partial Z}{\partial p_j} = Z(t) \sum_j \alpha_j \beta_j = Z(t) \, \beta'\alpha \tag{3}$$

and

$$\frac{1}{2} \sum_j \sum_k p_j p_k \sigma_{jk} \frac{\partial^2 Z}{\partial p_j \partial p_k} = \left[\frac{1}{2} \beta' \sum \beta - \frac{1}{2} \beta' \sum_D \right] Z \tag{4}$$

where

$$\sum_D \equiv \text{a column vector with elements } \sigma_i^2 \quad i = 1, \ldots N.$$

Dividing equation (2) by $\prod\limits_{j} p_j^{\beta_j}$ we obtain

$$\frac{\partial A}{\partial t} + \left\{ r + \beta' \left[\alpha - \frac{1}{2} \sum_D + \frac{1}{2} \sum \beta \right] \right\} A = 0 \qquad (5)$$

Thus

$$A(t) = A(0) \; e^{-rt} e^{-\beta'[(\alpha - \frac{1}{2} \sum_D) + \frac{1}{2} \sum \beta]t} \qquad (6)$$

But since

$$Z(p(0),0) = 1 = A(0) \prod\limits_{j} p_j^{\beta_j}(0) \qquad (7)$$

we obtain using (1), (6) and (7)

$$Z(t) = e^{-rt} e^{-\beta'[(\alpha - \frac{1}{2} \sum_D) + \frac{1}{2} \sum \beta]t} \prod\limits_{j} \left[\frac{p_j}{p_j(0)} \right]^{\beta_j} \qquad (8)$$

Arithmetic Brownian Motion:

Assuming $\alpha = \alpha_o/p$ and $\sigma = \sigma_o/p$ and the compatible choice (see appendix IIC) $R(t) = 1 + rt$ we obtain using (22) and assuming the existence of a single asset

$$\left(\alpha_o - \frac{rp}{1+rt} \right) Z + \sigma_o^2 \frac{\partial Z}{\partial p} = 0 \qquad (9)$$

The solution of (9) is

$$Z(p,t) = A(t) \; e^{-\frac{1}{\sigma_o^2} \left[\alpha p - \frac{1}{2} \frac{rp^2}{1+rt} \right]} \qquad (10)$$

where A(t) is an unknown function of time. Substituting (10) into (21) of Chapter II and rearranging terms we obtain

$$- \frac{1}{2} \frac{\alpha_o^2}{\sigma_o^2} A(t) + \frac{3}{2} \frac{r}{r+rt} A(t) + \frac{\partial A(t)}{\partial t} = 0 \qquad (11)$$

$$\therefore \quad A(t) = A(0) e^{\frac{1}{2} \alpha_o^2 t / \sigma_o^2} \frac{1}{[1+rt]^{3/2}} \qquad (12)$$

using the condition that

$$Z(p(0),0) = 1$$

we obtain equation (25) of Chapter II.

APPENDIX IIC

Example of Conditions Under Which $Z(p,t)$ Does Not Exist

Assume a riskless dollar grows exponentially and the risky asset grows linearly with time. Substituting equation (1) from appendix IIB, into equation (21) of Chapter II we obtain

$$- \frac{1}{2} \frac{\alpha_o^2}{\sigma_o^2} A(t) + r\, A(t) + \frac{1}{2} \frac{1}{\sigma_o^2} r^2 p^2\, A(t) + \frac{\partial A}{\partial t} = 0 \qquad (1)$$

Now, since $\frac{\partial A}{\partial p} = 0$, equation (1) has an acceptable solution when

$$- \frac{1}{2} \frac{\alpha_o^2}{\sigma_o^2} + r + \frac{1}{2} \frac{r^2}{\sigma_o^2} p^2 = g(t) \qquad (2)$$

where $g(t)$ is a pure function of time (i.e. $\frac{\partial g}{\partial p} = 0$). But this implies a determinate value for p from equation (2), which violates the assumption of the existence of a risky asset.

Chapter III

The Valuation Operator for Efficient Capital Markets

In the first two chapters we treated the problem of finding the discount factor contingent on the present prices of the primary securities. In this chapter we consider the problem of finding the discount factor that prices the primary securities themselves.

The approach is based on the following considerations. If we assume that financial markets are capable of exhausting gains from exchange then each individual's opportunity set is governed by the market discount factor Z and his choices are Pareto optimal. As a result, individual choices, which depend on Z, will correspond to some Pareto optimal allocation which depends on aggregate resource constraints, but not on Z. Hence, we would expect that Z could be expressed as a function of tastes, beliefs and aggregate resource constraints, in such a market.

In section 1 we show how this idea can be applied to a one period model using Pareto optimal sharing rules (Wilson [1968]). In section 2 we apply the results of section 1 to the one period linear risk tolerance economies investigated by Rubinstein [1974]. In section 3 we obtain the discount factor for an economy comprised of lifetime decision makers. The interesting result emerges that the expression for Z does not depend in any important way on whether the problem is formulated in discrete or continuous time.

1. The Discount Factor for an Efficient Market with a One
    Period Horizon.

Let p be a possible realization of the vector of end of
the period values of the firms and W be the end of the period
aggregate social wealth:

$$W = \iota'p \tag{1}$$

Let $S_k(p)$ be the amount received by the kth individual when
a state corresponding to the outcome p occurs.

$$\sum_k S_k(p) = W \tag{2}$$

Assuming that individuals maximize the expected utility
of end of the period wealth, a Pareto optimal allocation $S_k(p)$
maximizes

$$\int_{-\infty}^{\infty} d\Omega \left[\sum_k 1_k U_k(S_k(p) f_k(p) - \lambda(p) \left\{ \sum_k S_k(p) - W \right\} \right] \tag{3}$$

for some positive values of 1_k. In (3) $f_k(p)$ is the subjective
probability density for the occurrence of a state corresponding
to the realization p, while U_k is the utility function of the
k^{th} individual. The conditions for a maximum are (c.f. Wilson
[1968])

$$1_k \frac{\partial U_k}{\partial S_k} f_k(p) - \lambda(p) = 0 \tag{4}$$

Solving (4) we obtain the sharing rule

$$S_k(p) = U_k'^{-1} \left[\frac{\lambda}{1_k f_k(p)} \right] \tag{5}$$

The $\lambda(p)$'s are the Lagrange multipliers for the wealth constraint associated with every possible realization of the vector p. Applying the constraint (2) to equation (5) we obtain

$$W = \sum_k U_k'^{-1} \left[\frac{\lambda}{1_k f_k(p)} \right] \tag{6}$$

From (6) we obtain λ (perhaps implicitly) as a function of p and W. Denote the function thus obtained by

$$\lambda^* = \lambda(p,W) \tag{7}$$

Substituting (7) into (5) we obtain the Pareto optimal sharing rule for a given set of 1_k.

$$S_k^*(p,W,1_k) = U_k'^{-1} \left[\frac{\lambda^*}{1_k f_k(p)} \right] \tag{8}$$

The Individual's Problem

Having looked at the allocation problem from the point of view of what is socially optimal, consider the problem from the point of view of an individual in a market in which there is an absence of riskless arbitrage opportunities. Assume that the capital market provides sufficient exchange opportunities

(for a discussion of this point see appendix IIIA) to ensure that for any realization of the end of the period vector of firm values, the individual can choose an end of the period wealth $W_k(p)$. Because of the absence of riskless aribtrage opportunities the initial wealth of the kth individual is given by

$$W_{o_k} = E\left\{Z \; W_k\right\} \tag{9}$$

where the expectation operator is defined in section 1, Chapter II, and Z is the discount factor. The kth individual chooses $W_k(p)$ that maximizes

$$\int d\Omega \left[f_k(p) U_k(W_k) - \lambda_k^{\cdot}(\rho(p) Z(p) W_k - W_{o_k}) \right] \tag{10}$$

For a maximum we require

$$f_k(p) U_k'(W_k) - \lambda_k \rho(p) Z(p) = 0 \tag{11}$$

or

$$W_k = U_k'^{-1} \left[\frac{\lambda_k \rho(p) Z(p)}{f_k(p)} \right] \tag{12}$$

where λ_k is determined by

$$\int d\Omega \rho(p) Z(p) W_k = W_{o_k} \tag{13}$$

Assuming that the capital markets are capable of exhausting gains from exchange (12) must correspond to one of the possible Pareto optimal allocations given by (8). Hence, there exists a set of 1_k such that

$$W_k^*(p) = S_k^*(p, 1_k, W) \tag{14}$$

From equations (8), (12) and (14) we have:

$$1_k \lambda_k \rho(p) Z(p) = \lambda^*(p, W) \tag{15}$$

Since $\rho(p)$, Z and $*$ are market quantities independnet of k, $1_k \lambda_k$ is independent of k. Choose 1_k, such that $1_k \lambda_k = 1$. Then

$$\rho(p) Z(p, W) = \lambda^*(p, W) \tag{16}$$

Thus the discount factor in an efficient market is directly proportional to the marginal social evaluation of an extra dollar of aggregate future wealth.

Homogeneous Beliefs

With homogeneous beliefs $f_k(p) = f(p)$ for all k. Then from equation (6) we obtain

$$W = \sum_k U_k^{'-1} \left[\frac{\lambda}{1_k f(p)} \right] \tag{17}$$

If we solve (17) for $\lambda / f(p)$ we see that

$$\frac{\lambda}{f(p)} = \nu(W) \tag{18}$$

where ν is a function of W alone, and not p separately.
Together (16) and (18) imply

$$\rho(p)Z(p,W) = f(p)\nu(W) \tag{19}$$

Making the natural identification

$$\rho(p) = f(p)$$

we obtain

$$Z(W) = \nu(W) \tag{20}$$

With homogeneous beliefs Z is a function of aggregate wealth
alone.

2.   The Discount Factor for Single Period Linear Risk
      Tolerance Economy.

Exponential Utility

     For the case of investors who have exponential utility
functions we obtain from (16) and appendix IIIA (equation (6)),

$$\rho(p)Z(p,w) = \prod_k \left[1_k f_k(p) \right]^{A_k/A} e^{-W/A} \tag{21}$$

When there are homogeneous beliefs, we make the natural identification

$$\rho(p) = f(p) \tag{22}$$

where $f_k = f$ for all k. Then from (21) and (22) we obtain

$$Z(W) = \sum_k 1_k^{A_k/A} e^{-W/A} \tag{23}$$

Power Utility

From appendix IIIA (equation (10)), and equation (16) we obtain

$$\rho(p)Z(p,W) = \left\{ \sum_k \left[1_k f_k(p) \right]^B \right\}^{1/B} [A + BW]^{-1/B} \tag{24}$$

With homogeneous beliefs we have

$$Z(W) = \left\{ \sum_k 1_k^B \right\}^{1/B} [A + BW]^{-1/B} \tag{25}$$

Example

In this example, we assume homogeneous normal probability assessments and negative exponential utility functions. We use Z to price the primary securities. This enables us to eliminate taste parameters from the expression for Z and obtain the price of a simple call option written on a primary security, contingent on primary security prices.

Assuming homogeneous normal beliefs

$$f_k = N(\bar{p}, C) \tag{26}$$

where $\bar{p}$ is the vector of expected values of end of the period firm prices and C is the variance covariance matrix. Assuming riskless borrowing and lending

$$\frac{1}{1+r_f} = E\{Z\} \tag{27}$$

where Z is given by (23). From (23) and (27) we obtain

$$\frac{1}{1+r_f} = \pi_k \, 1_k^{A_k/A} \, e^{-1/A\iota'\bar{p} + 1/2A^2 \, \iota'C\iota} \tag{28}$$

Solving for $\pi_k \, 1_k^{A_k/A}$ and substituting into (23) we obtain

$$Z(p) = \frac{1}{1+rf} \, e^{\frac{1}{A}\iota'\bar{p} - \frac{1}{2A^2}\iota'C\iota} \, e^{-W/A} \tag{29}$$

The vector of beginning of the period values of the firms, p_o, is, using (29), (and results of appendix IIIB)

$$p_o = E\left\{Z(p)p\right\}$$

$$= \frac{1}{1+r_f}\left[\bar{p} - \frac{1}{A}C\iota\right] \tag{30}$$

which is the well known result.

The price of a call option with striking price K_i on the ith firm is given by

$$E\left\{ Z(p)\,[p_i - K_i] \;\middle|\; p_i \geq K_i \right\} \tag{31}$$

From appendix IIIB we obtain

$$E\left\{ e^{-W/A} \;\middle|\; p_i \geq K_i \right\}$$
$$= \frac{1}{1+r_f}\, e^{-\frac{\iota''\bar{p}}{A} + \frac{1}{2A^2}\,\iota'C\iota}\; F\left(\frac{-K_i + \bar{p}_i - \frac{1}{A}\,\iota'C}{\sqrt{C_{ii}}} \right) \tag{32}$$

and

$$E\left\{ Z\,p_i \;\middle|\; p_i \geq K_i \right\} = p_i(0)\, F\left(\frac{-K_i + \bar{p}_i - \frac{1}{A}\,\iota'C}{\sqrt{C_{ii}}} \right)$$
$$+ \frac{1}{1+r}\,\sqrt{C_{ii}}\; f\left(\frac{-K_i + \bar{p}_i - \frac{1}{A}\,\iota'C}{\sqrt{C_{ii}}} \right) \tag{33}$$

Substituting (32) and (33) into (31) and making use of (30) we obtain the call option price, contingent on present prices

$$\left[p_i(0) - \frac{K_i}{1+r_f} \right] F\left(\frac{-K_i + (1+r_f)p_{oi}}{\sqrt{C_{ii}}} \right)$$
$$+ \frac{1}{1+r_f}\,\sqrt{C_{ii}}\; f\left(\frac{-K_i + (1+r_f)p_{oi}}{\sqrt{C_{ii}}} \right) \tag{34}$$

Equation (34) was obtained by Brennan, by a somewhat different

approach, [1979].

In the absence of transactions costs options are redundant for the economy discussed above since the efficient allocations of future wealth can be achieved in the primary security markets alone, Rubinstein [1974], (c.f. appendix IIIA).

3. The Discount Factor for an Efficient Market with Lifetime Decision Makers.

Pareto Optimal Allocations

In continuous time the Pareto optimal consumption allocations $C_k(p,t)$ maximize

$$\int_0^\infty dt \int d\Omega \left[\sum_k l_k U_k \left\{ C_k(p,t) \right\} f_k(p,t) \right.$$

$$\left. - \lambda(p,t) \left\{ \sum_k C_k(p,t) - C(p,t) \right\} \right] \tag{35}$$

Where $U_k(C_k(p,t))$ is the utility per unit of time of the kth individual, from consuming $C_k(p,t)$ dollars worth of goods per unit of time, in state p at time t. For simplicity we assume that individuals are infinitely long lived and their utility functions do not depend explicitly on the state or the time.

In a discrete time framework the Pareto optimal consumption allocations $C_k^o(p,t)$ maximize

$$\sum_{i=1}^\infty \int d\Omega \left[\sum_k l_k U_k^o \left\{ C_k^o(p,t_i) \right\} f_k(p,t_i) \right.$$

$$\left. - \lambda^o(p,t_i) \left\{ \sum_k C_k^o(p,t) - C^o(p,t) \right\} \right] \tag{36}$$

where U_k^o $C_k^o(p,t)$ is the utility of the kth individual from consuming $C_k^o(p,t)$ dollars worth of goods in state p at time t.

The continuous time formalism deals with flows while the discrete time formalism deals with stocks[1]. The Pareto optimal allocations in continuous time satisfy

$$1_k U_k' \left\{ C_k(p,t) \right\} f_k(p,t) = \lambda(p,t) \tag{37}$$

while in discrete time they satisfy

$$1_k U_k^o \left\{ C_k^o(p,t) \right\} f_k(p,t) = \lambda^o(p,t) \tag{38}$$

Thus the discrete and continuous time approaches are formally identical and there is no need to develop them separately. From (37)

$$C_k(p,t) = U_k'^{-1} \left\{ \frac{\lambda(p,t)}{1_k f_k(p,t)} \right\} \tag{39}$$

where λ is determined by

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[1] The two approaches can be related by use of the δ function as follows

$$C(p,t) = \sum_i C^o(p,t_i) \delta(t - t_i)$$

$$\sum_k C_k(p,t) = C(p,t)$$

$$= \sum_k U_k^{'-1}\left\{\frac{\lambda}{I_k f_k(p,t)}\right\} \tag{40}$$

Equation (40) gives λ (implicitly or explicitly), as a function of $C(p,t)$, p, and t

$$\lambda^* = \lambda(C(p,t),p,t) \tag{41}$$

Substituting (41) into (39) we obtain the optimal consumption allocations

$$C_k^*(p,C(p,t),t) = U_k^{'-1}\left\{\frac{\lambda^*}{I_k f_k(p,t)}\right\} \tag{42}$$

Restrictions on Beliefs

With homogeneous beliefs, $f_k = f$ for all k, equation (40) determines λ/f as a function of aggregate consumption alone, and not of W and p separately. Thus $\lambda/f = \nu(C)$, where $\nu(C)$ is implied by (40), which gives the sharing rule

$$C_k^*(C,t) = U_k^{'-1}\left\{\frac{\nu(C)}{I_k}\right\} \tag{43}$$

When beliefs contingent on aggregate consumption are homogeneous [i.e. $f_k(p,t) = h(p/C)g_k(C)$] then equation (40) determines $\lambda/h(p/C)$ as a function of $C(p,t)$ alone, i.e., $\lambda/h(p/C) = \eta(C)$, which gives the sharing rule

$$C_k^*(C,t) = U_k^{'-1} \left\{ \frac{\eta(C,t)}{I_k J_k(C,t)} \right\} \tag{44}$$

Once again the sharing rule is a function of aggregate consumption alone and hence, Pareto optimal allocations can be achieved with the aid of financial instruments which distinguish these states, such as call options, written on aggregate consumption, Breeden and Litzenberger [1978], or supershares Hakansson [1978].

The Individuals Problem

With a rich enough capital market structure so that the individual can choose $C_k(p,t)$, then in the absence of riskless arbitrage opportunities $C_k(p,t)$ maximizes

$$\int_0^\infty dt \int d\Omega \, U_k\{C_k(p,t)\} f_k(p,t)$$
$$- \lambda_k \left[\int_0^\infty dt \int d\Omega \, Z(p,t) C(p,t) \rho(p,t) - W_{o_k} \right] \tag{45}$$

where W_{o_k} is the initial wealth of the kth individual. From (45) we obtain

$$C_k(p,t) = U_k^{'-1} \left\{ \frac{\lambda_k \, Z(p,t) \rho(p,t)}{f_k(p,t)} \right\} \tag{46}$$

There is an analagous discrete time expression obtained by replacing the flow variable by stock variables. In a market where the gains from exchange are exhausted, the choices (46), correspond to one of the Pareto optimal allocations (42). Thus

choosing $l_k \lambda_k = 1$, we obtain from (42) and (46)

$$Z(p,t)\rho(p,t) = \lambda^*(C(p,t),p,t) \qquad (47)$$

When beliefs are homogeneous we make the identification $\rho(p,t) = f(p,t)$, and obtain from (43) and (46)

$$Z(C) = \nu(C(p,t)) \qquad (48)$$

Equations similar to (47) and (48) hold in a discrete time framework, the only difference being that the flow variable $C(p,t)$ is replaced by the stock variable $C^o(p,t)$. Hence, the valuation formula for discrete and continuous time framework, are for all practical purposes identical, when markets exhaust gains from exchange.

In the multiperiod discrete time approach as typified by the work of Rubinstein [1974, 1976]. Breeden and Litzenberger [1978] and Hakansson [1978], $C(p,t)$ is exogenous, i.e. either aggregate consumption in each state, or its distribution are exogenously given.

For example, Rubinstein [1976] and Breeden and Litzenberger [1978] consider the case of a market where individuals have identical relative risk aversion and homogeneous beliefs. In this case from (48) and appendix IIIA

$$Z(C) = \sum_k [l_k^{B}]^{1/B} B^{-1/B} C^{-1/B} \qquad (49)$$

From (49) when C is log normal, Z(C) is log normal. This is analagous to the situation in Chapters I and II where log normal firm values and log normal Z lead to the Black-Scholes option pricing formula. In a similar fashion (48) leads to the Black-Scholes formula for options written on aggregate consumption when C is log normal[2].

As a further example, consider the case of individuals with exponential utility functions in consumption and homogeneous beliefs. From appendix IIIA we obtain

$$Z(C) = \pi_k [1_k]^{A_k/A} e^{-C/A} \qquad (50)$$

Assuming $C(p,t)$ to be exogenous, choose it such that

$$C(p,t) = \sum_{i=1}^{N} \beta_i \ln p_i \qquad (51)$$

where β_i are as yet unknown constants. Substituting (51) into (50) we obtain

$$Z = \pi_k [1_k]^{A_k/A} \prod_{i=1}^{N} p_i^{\beta_i} \qquad (52)$$

Equation (52) has the same form as equation (38) of Chapter I, and thus leads to the same form for Z contingent on present firm values, equation (46) Chapter I. Thus when p_i are log normal

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[2] Since beliefs are homogeneous, a market in these options, together with riskless borrowing and lending lead to a Pareto optimal allocation of risk bearing.

(52) leads to the Black-Scholes formula for options written
on the firm. This result is independent of whether or not
the problem is formulated in discrete or continuous time.

Connection with the Model of Merton

Despite the continuous time character of most of this
thesis, the approach is closer in spirit to the discrete
multiperiod models of finance, rather than to the continuous
time model of Merton [1971, 1973]. This is because of the
"forward looking" nature of the analysis. The differential
equations of Chapter II hold for all future time and all of
price space. In this chapter individuals are (explicitly)
concerned with making choices for all future times.

In Merton's model, individuals are indirectly concerned
about the future through the induced or derived utility of
wealth function. The partial differential equations for this
function are in terms of prevailing market prices and the
present time, as are the differential equations of Garman
and Black-Scholes (c.f. discussion at the end of Chapter II.).
Similarly, the Ito description of the diffusion process
involves derivatives evaluated at prevailing market prices
only. As a result this approach does not appear to be
useful for obtaining the differential equations of Chapter II.

In Merton's model, (this is the important point) beliefs
are contingent on prevailing market prices. In this respect,
the primary asset valuation model has something in common
with the approach of Chapter II. According to equation (22)
Chapter II, at t=0, Z satisfies

$$\left[\alpha_i - \frac{1}{R} \frac{\partial R}{\partial t} \right] + \sum_j \sigma_{ij} p_j(0) \frac{\partial Z}{\partial p_j} \Bigg|_{p_j = p_j(0)} = 0 \qquad (53)$$

Assuming homogeneous beliefs, Z is a function of aggregate consumption alone. Assuming a constant propensity to consume out of wealth, $C = aW$ we have

$$\frac{\partial Z}{\partial p_j} = \frac{\partial Z}{\partial C} \frac{\partial C}{\partial p_j} = a \frac{\partial Z}{\partial C} \qquad (54)$$

Substituting (54) into (53) we obtain

$$\dot{p}(0) = \left[\frac{1}{a \frac{\partial Z}{\partial C(0)}} \right] \sum^{-1} \left(\alpha - \frac{1}{R} \frac{\partial R}{\partial t} \right) \qquad (55)$$

Choosing $\frac{1}{R} \frac{\partial R}{\partial t} = r$, equation (55) has an obvious resemblance to the pricing formula of Merton [1973b]. If $a \frac{\partial Z}{\partial C(0)}$, α and $\sum$ are all independent of prices, then (55) is an explicit expression for prevailing market prices. Otherwise (55) represents N equations that must be solved for N market prices.

From (55) it follows that the vector of weights on the market portfolio is

$$\omega = \frac{1}{\iota' \sum^{-1} (\alpha - r)} \sum^{-1} (\alpha - r) \qquad (56)$$

Equation (56) implies the capital asset pricing model (c.f. Merton [1973b] for example).

APPENDIX IIIA

Sharing Rules for Linear Risk Tolerance Utility Functions

Sharing rules for linear risk tolerance tuility functions in a complete market context have been developed by Rubinstein [1974]. We develop the sharing rules here without recourse to the idea of a complete market; the approach is similar to that of Wilson [1968].

In section 1 of Chapter III we obtained the Pareto optimal sharing rules as

$$S_k(p,W,1_k) = U_k'^{-1}\left[\frac{\lambda^*}{I_k f_k(p)}\right] \tag{1}$$

where λ^* is a function of p and W determined by the constraint

$$\sum_k S_{k'} = W \tag{2}$$

We assume that for the kth investor

$$U_k' \equiv \frac{\partial U_k}{\partial S_k} = e^{-S_k/A_k} \tag{3}$$

or

$$U_k' = (A_k + B_k S_k)^{-1/B_k} \qquad B_k \neq 0 \tag{4}$$

Exponential Utility

Substituting (3) into (1) we obtain

$$S_k = A_k \ln \frac{1_k f_k}{\lambda *} \qquad (5)$$

Using the constraint (2) we obtain

$$\lambda * = e^{-W/A} \prod_k [1_k f_k]^{A_k/A} \qquad (6)$$

where $\quad A = \sum_k A_k \qquad (7)$

Substituting (6) into (5) we obtain the Pareto optimal sharing rule

$$S_k(p,W,1_k) = A_k \ln \frac{1_k}{\prod\limits_k 1_k^{A_k/A}} + A_k \ln \frac{f_k(p)}{\prod\limits_k f_k(p)^{A_k/A}}$$

$$+ \frac{A_k}{A} W \qquad (8)$$

An interpretation of the terms in (8) has been given by Wilson and Rubinstein. We outline this interpretation briefly here.

The first term in (8) is called a side payment; it is independent of p and W and vanishes when summed over all individuals. The allocations corresponding to this term can be achieved by a market for riskless borrowing and lending.

The second term in (8) is called a side bet; the amount of the payment depends on which realization of p occurs, and vanishes when summed over all individuals. When beliefs are not homogeneous the allocation corresponding to the term cannot

in general be achieved by a market in primary securities[1].
[Opportunities for side bets are not required when beliefs
are homogeneous.]

The last term in (8) is called a dividend, and the
allocation corresponding to this term can be achieved by the
market for primary securities.

When individuals have homogeneous beliefs we can see
from (8) that only two markets are required to achieve a Pareto
otpimal allocation of claims to future wealth. A market for
riskless borrowing and lending, and one for the shares of a
mutual fund that owns the market portfolio.

However, with heterogeneous beliefs, riskless borrowing
and lending, together with the markets for primary securities
are, in general, incapable of securing the allocation (8).
In this case if markets are costless to open and operate,
markets in secondary securities will open up. As long as the
allocation (8) has not been achieved there still remains
unexploited gains from exhange and secondary securities
continue to proliferate until gains have been exhausted.

Substituting (4) into (1) we obtain

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[1] An exception occurs when probability assessments are
multivariate normal, with disagreement about the expected
value of p, but agreement about the variance covariance
matrix.

$$S_k = -\frac{A_k}{B_k} + \frac{1}{B_k}\left[\frac{\lambda}{1_k f_k}\right]^{-B_k} \qquad (9)$$

Assuming $B_k = B$ for all k we can use the constraint (2) to show for

$$\lambda^* = \left\{\sum_k [1_k f_k]^B\right\}^{1/B} [A + BW]^{-1/B} \qquad (10)$$

Substituting (10) into (9) we obtain

$$S_k(p,W,1_k) = -\frac{A_k}{B} + \frac{[1_k f_k(p)]^B}{\sum_i [1_i f_i(p)]^B}\left[\frac{A}{B} + W\right] \qquad (11)$$

Equation (11) can be given an interpretation similar to that given to (8). With homogeneous beliefs we obtain the sharing rule

$$S_k(W,1_k) = -\frac{A_k}{B} + \frac{A}{B}\frac{1_k^B}{\sum_i 1_i^B} + \frac{1_i^B}{\sum_i 1_i^B}W \qquad (12)$$

APPENDIX IIZB

A· Useful Integral

Let the vector $X \sim N(\mu, \sum)$ and let $h(X)$ be the joint density function for X.

$$\int_{\alpha_i}^{\infty} dX_i \int_{\infty}^{\infty} \sum_{j \neq i} dX_j \ e^{\beta'X} h(X)$$

$$= e^{\beta'\mu} e^{\frac{1}{2}\beta'\sum\beta} \ F\left\{ \frac{-\alpha_i + [\mu + \sum\beta]_i}{\sigma_i} + \sigma_i \right\} \quad (1)$$

To obtain

$$\int_{\alpha_i}^{\infty} X_i dX_i \int_{-\infty}^{\infty} \prod_{j \neq i} dX_j e^{\beta'X} h(X)$$

differentiate (1) with respect to β_i and divide by β_i.

CHAPTER IV

Conclusion

We have presented a unified approach to security valuation based on a linear operator. We have found that when capital markets are capable of exhausting gains from exchange the valuation operator that prices all securities is related in a simple fashion to aggregate resource constraints, and that the relationship does not depend in any essential way on whether the problem is formulated in discrete or continuous time. We have derived explicit forms of the operator for linear risk tolerance economies, and shown how the Black-Scholes option pricing formula for secondary securities might arise.

We further demonstrated that when primary asset prices follow a diffusion process, the valuation operator contingent on present market prices is determined by the parameters of the stochastic process.

Finally, when beliefs are contingent on prevailing market prices, we have shown how the two approaches to the valuation operator can be combined to produce the Capital Asset Pricing Model in continuous time.

Observations

We have been able to reproduce many of the models of financial theory within a unified framework that assumes aggregate resource constraints at time t can be exogenously specified. However, consumption today, affects the amount available tomorrow; the models should be extended to endogenize

the aggregate constraints.

The valuation operator contingent on present prices can be used to price a wide variety of secondary securities. The analysis of Chapter II has to be extended in some of these cases to include the effects of boundary terms (see Appendix IIA).

Finally, beliefs do not emerge from a vacuum, and we have left unanalyzed the discrete time model that prices primary securities when beliefs are contingent on present prices. In the discrete time, continuous time debate, it seems that it is only this model that can be called the competitor of the continuous time C.A.P.M.

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